On Thinking

Albrecht von Müller
Elias Zafiris

Concept and Formalization of Constellatory Self-Unfolding

A Novel Perspective on the Relation between Quantum and Relativistic Physics

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On Thinking

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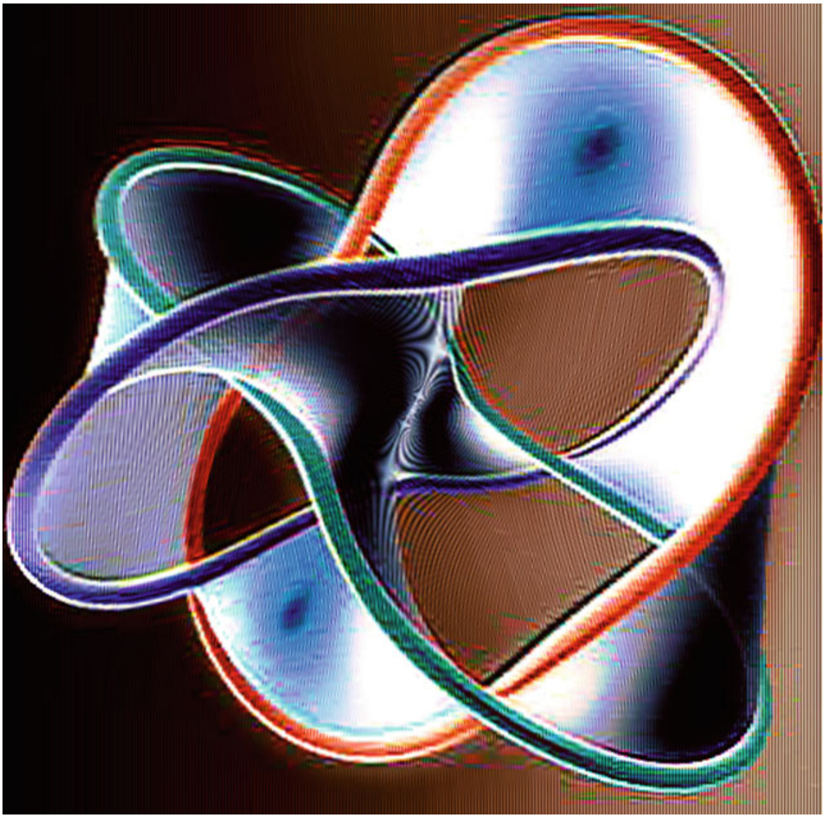


τό γάρ ἀυτό νοεῖν
ἐστίν τε καί εῖναι

Dedicated to Tiziana, Raphael, Maximilian, and Constantin from A.v.M., and to Johanna and Szofi from E.Z. Hoping that Roger Penrose is right in his expectation: "It is quite likely that the 21st century will reveal even more wonderful insights than those we have been blessed with in the 20th. But for this to happen, we shall need powerful new ideas, which shall take us in directions significantly different from those currently being pursued. Perhaps what we mainly need is some subtle change in perspective – something that we all have missed…".

Preface

Modern physical science is erected on categorial foundations that are incompatible with the phenomenon of autogenetic unfolding and the associated logical structure of strong self-referentiality. Autogenesis means that something unfolds out of itself, within itself, and towards itself. In order to overcome these structural pitfalls, we need (a) to unearth the categorial foundations of our present theories, (b) to rethink our notions of time and reality, and that, in turn, allows us (c) to reconceptualize some of the big, unresolved issues in modern science in a fundamentally novel perspective. One of the most prominent among them refers to the recently formulated conjecture "ER = ERP," around which there emerged a very interesting and productive debate about the pivotal challenge of modern physics, the relation of general relativity theory (GRT) and quantum physics (QP). "ER = ERP" is a shorthand that joins two ideas proposed by Einstein in 1935, namely quantum entanglement (EPR entanglement, named after Einstein, Podolsky, and Rosen) and wormholes (ER, for Einstein-Rosen bridges).

Both the enigmatic character of QP and its incompatibility with GRT are rooted in a historically grown one-sidedness of the categorial underpinnings on which modern science is erected.[1] Metaphorically speaking this deprivation could be characterized as a "facticity imprisonment" of our thinking. By this we inadvertently reduce reality to its factual footprints and time to its sequential structure. Both

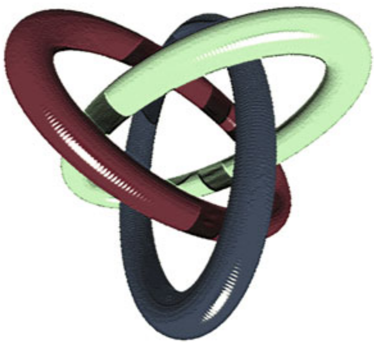
[1]In the following papers this topic has been addressed already in a preliminary form:
Filk T, von Müller A (2010) A categorical framework for quantum theory. Ann. Phys. 522(11), 783–801;
von Müller A (2010) Thought and reality. In: Towards a theory of thinking. Springer, Heidelberg, pp 59–70;
von Müller A (2012) On the emergence and relativity of the local spacetime portrait of reality. In: Welt der Gründe - Deutsches Jahrbuch für Philosophie, vol 4. Felix Meiner, Hamburg, pp 1233–1245
von Müller A (2015) The forgotten present. In: von Müller A, Filk T (eds.) Re-thinking time at the interface of physics and philosophy. Springer, Heidelberg;
von Müller A (2011) The logic of constellations. In: Culture and neural frames of cognition and communication. Springer, Heidelberg, pp 199–213.

are correct and important, but only partial aspects of time and reality. In order to overcome the rift between the two foundational theories of modern physics we need to unearth their different, hitherto overlooked categorial underpinnings and develop a richer, overarching categorial framework.

In the novel account, facts turn out to be just the traces of the actual taking place of reality, left behind on the co-emerging canvas of local spacetime. The actual taking place of reality, instead, occurs still in a primordial form of time, the nonlocal "time-space of the present" (TSP). Interestingly enough, already Albert Einstein complained vis-à-vis Rudolf Carnap, in their discussions in Princeton between 1952 and 1954, about the "painful, but inevitable abandonment" of the present in physics. The necessity of this abandonment, however, exists only as long as time is reduced to its linear-sequential aspect, and, directly related, the notion of the present being reduced to a point-like now. A nonlocal TSP as an aspect of time in its own right and even as its primordial form (from which the sequential structure of time emerges as a derivative feature) is, instead, fully compatible with GRT. One can even argue that GRT implicitly contains such a richer notion of time and reality, e.g., for what there remains once the local space-time fabric unravels in singularities. The TSP provides the primordial "stage" or "platform" on which reality can occur in the first place. Only by "taking place" (!) there, reality gains the chrono-ontological format of facticity. In the novel account, our human experience of a present needs no longer to be derogated as just a subjective confabulation. In the new framework our experience of a present turns out to be the hitherto most advanced adaptation of cognitive evolution to the actual taking place of reality, as it occurs in this primordial form of time, the TSP. Obviously, this richer notion of time changes also our notion of reality. Without the sequential structure of time there is no causality. In the TSP, reality occurs as a constellatory self-unfolding: Out of itself, within itself, and towards itself. Based on this richer notion of time and reality, QP can now be understood as addressing the "reduction" of reality to the format of facticity, respectively the "crystallization" of time to its linear-sequential format. Classical and relativistic physics, instead, turn out to be focused on the resulting factual portrait. But, the singularities of GRT, an integral part of the factual portrait, indicate the possibility of an inverse transition: They are the instance where the fabric of local spacetime, and with it the chrono-ontological format of facticity, dissolves again. Quantum physical reduction and the singularities of GRT, thus, turn out to describe inverse transitions: Into and out of the chrono-ontological format of facticity, respectively spacetime locality.

Fundamental for the new theory is to overcome our implicit fixation on a "monolithic ontology." In the novel framework, three chrono-ontological portraits are united like Borromean rings, i.e., every two of them are linked only via the third. There is (a) the—today erroneously generalized—factual portrait, painted on the canvas of local space-time, (b) the statu nascendi portrait, covering the actual taking place of reality, as it occurs in the TSP, and (c) an aspect of inseparable, eventually impredicable unity—with reference to Anaximander, called the "apeiron portrait." In this new, three-faceted chrono-ontological framework, it becomes possible to unite QP and GRT as they stand, i.e., without subduing one to the other—just by recognizing that they address different, but complementary aspects of time and reality. Convergence is achieved by adding a "third step" to both, QP and GRT, in which spacetime locality itself is put into perspective.

The two perennial problems in this context, namely the quantum state reduction or quantum measurement problem in quantum physics, and the problem of singularities in general relativity, may be considered as targeting precisely the issue of transition into and out of a local space-time event structure respectively, pertaining to the factual layer of reality. This naturally generates the question, if there exists a universal mechanism of a topological or logical nature, which would manifest appropriately these two inverse types of transition, and concurrently provide a concrete mathematical modeling of the categorial apparatus characteristic of a "statu-nascendi" layer, according to the autogenetic theory. If such a universal mechanism is actually functioning, then the autogenetic theory, beyond its philosophical impact, acquires significant interpretative power in relation to the resolution of these pestilential problems of physics. Here, we propose to explore the viable possibility that this universal mechanism is based on the logical and topological characteristics of the "Borromean link," displayed below:



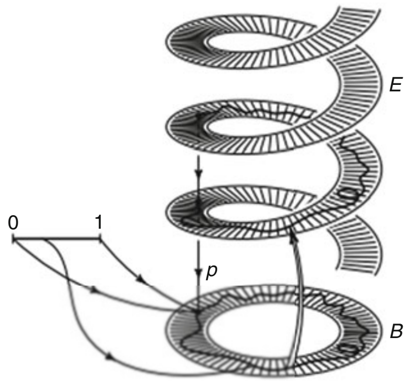
The "Borromean link" consists of an interlocking family of three rings, thought of as topological circles, such that if any one of them is cut at a point and removed, then the remaining two become completely unlinked. The "Borromean link" can be encoded algebraically in terms of the structure of the noncommutative free group in two generators. Its unique ubiquity lies on seven distinctive roles that constitute the main focus of this treatise:

1. The "Borromean link" is threefold symmetric and can be iterated self-referentially ad infinitum by replacing simultaneously each one of the rings by a "Borromean triad" of rings.

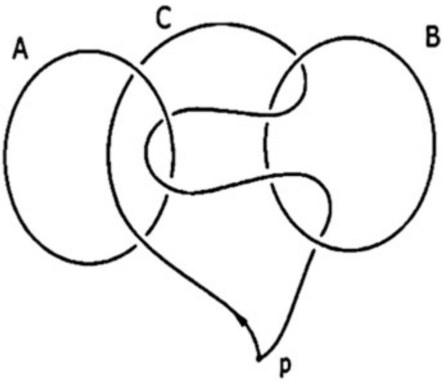
2. All other topological links can be constructed and expressed algebraically in terms of two simple algebraic operations within the same noncommutative group-theoretic model, namely the operations of forming "Borromean stacks" and "Borromean chains" out of "Borromean stacks."
3. It serves as a universal singular locus in the algebraic-topological theory of branched covering spaces.
4. The "Borromean link" can be characterized topologically by means of a higher-order homological invariant pertaining to the complement of the rings.
5. It provides the simplest model of nonlocal linkage in 3-d space independently of metrical distance.
6. This nonlocal topological linkage can be extended to 4-d spacetime by adjoining a temporal symmetry axis of rotation perpendicular to the rings, which is linked once with each of them.
7. The noncommutative group-theoretic model of the "Borromean link" admits irreducible representations in both the Lorentz group (local symmetry group in general relativity) and the unitary group (local symmetry group in quantum mechanics).

The connection between the "Borromean link" and the dynamics of autogenesis, i.e., the dynamics of constellatory, self-referential unfolding, emanates from the adjunction of an observer, as referent of the "time-space of the present," located inside a 3-d sphere (compactification of 3-d Euclidean space), where the "Borromean link" may be realized. We consider that each one of the three rings surrounds a puncture on the 3-d sphere, assuming a well-defined physical semantics, and thus it gives rise to a nonbounding cycle. The existence of each single puncture is associated with the topological property of multiple connectivity.

First, it is instructive to consider the case of a single puncture together with the corresponding ring. The internal observer perceives multiple connectivity by means of the universal covering space of this ring. The concept of a universal covering space is rooted in algebraic topology and is formulated to depict precisely the process of dynamic unfolding of a multiply connected space. The term universal refers to the property that the unfolding space becomes eventually simply connected. In other words, the perception of the internal observer is dynamically completed when the unfolding space becomes simply connected. The semantics of the universal covering space, in the considered case, is that the multiple connectivity induced by a ring is being dynamically unfolded as a helix, which is spiraling around the surface of a cone based on this ring and extended to infinity. With reference to a single ring, we may easily visualize the first steps of this spiral unfolding, where the emerging levels are indexed in terms of the integers.



According to the above, in the case of three rings interconnected topologically in the form of the "Borromean link," the respective helically unfolding spirals are not independent, but each one of them functions as a nonlocal gluing helical staircase for the other two. In particular, if we consider a snapshot of this unfolding type, the gluing helical staircase involves four crossing points, i.e., two for each of the indirectly linked rings with opposite orientation, concatenated in an alternating manner.



A gluing helical unfolding of the prescribed form is constellatory, since the nonlocal connectivity function of a helix can take place only in the context of two other not directly linked, and thus, paratactically placed helices. Moreover, it is also strongly self-referential, since any of the three helical unfoldings in the universal covering space functions as a gluing datum for the other two, by the defining property of the "Borromean link."

The present treatise, intended to offer a novel perspective on the relation between quantum and relativistic physics according to the preceding introductory remarks, comprises seven chapters. Chapter 1 has been written by Albrecht von Müller, and Chaps. 2–7 have been written by Elias Zafiris. Each chapter is designed to have an autonomous structure and can be read independently from the other ones. In this way, the access to all different conceptual and technical constituting elements elaborating the main argument of this book is facilitated for readers of diverse backgrounds, who may be only selectively interested in some particular aspect of the whole schema. Nevertheless, all the chapters are connected—metaphorically speaking—in the form of a nonlinear circuit that conceptualizes time and reality in a fundamentally different way from the standard one that a scholar is usually trained to think about.

The first chapter serves as a brief and sketchy introduction to the theory of an autogenetic universe, targeting, in particular, the relation between general relativity theory and quantum physics. It is placed in the context of the debate surrounding the "ER = ERP" conjecture, and it elaborates the substantiation of this conjecture in the refined version of arguing that the singularities of general relativity and quantum reduction can be seen as inverse transitions into and out of the chrono-ontological format of factual spacetime. The second chapter develops a precise mathematical model of the autogenetic universe theory, targeting in particular the following: (a) The notion of autogenetic constellatory unfolding together with the associated notion of strong self-referentiality; (b) The notion of the "time-space of the present" and the precise form of the relation with the standard notion of spacetime; (c) The connectivity among the three chrono-ontological formats of reality and the role of the Borromean topological link in this respect. A deeper understanding of the arguments presented in this chapter requires a certain degree of familiarization with the technical apparatus developed in detail in the sequel chapters, so the interested reader is invited to reflect back on this chapter after completing the reading of the whole treatise. The third chapter focuses on the algebraic encoding of the Borromean topological link, culminating in the remarkable theorem that an arbitrarily complex topological link can be constructed solely in terms of "Borromean connectivity units." The fourth chapter develops the logical anatomy of the Borromean topological link based on the strategy of conjugation. In this manner it enunciates a metaperspective on algorithmic information theory pointing out essential connections with quantum logic, quantum information, and the theory of generic sets. The fifth chapter questions the smooth spacetime event manifold of general relativity theory from a precise sheaf-theoretic rendering of Einstein's field equations. The leading idea is that in a theory with intrinsically dynamic variables, like general relativity, it should be the pertinent physical conditions or the sources of the field themselves that determine the type of the admissible extensions over singularities as distributional solutions to the field equations. In this context, the Borromean link admits a physical gravitational realization as a higher order wormhole solution of the field equations. The sixth chapter illustrates the realization of the Borromean link in quantum mechanics in terms of one-parameter unitary groups. This sheds new light on the phenomenon of quantum entanglement, the notion of localization in the quantum domain, the aspect of objective indistinguishability pertaining to quantum interference, and the ubiquitous concept of quantum topological and geometric phases. The seventh chapter attempts to provide a critical evaluation and substantiation of the "ER = EPR" conjecture, which, in the absence of an exact quantum gravity theory, establishes a precise relation between spacetime geometry and quantum theory. The two fundamental and still imperishable issues in the interface between quantum theory and general relativity, namely the quantum state reduction and the problem of singularities, can be thought of as targeting the issue of transition into and out of a space-time event domain respectively. Given that the quantum state reduction is necessitated in virtue of entanglement between the quantum system and the measurement means, the latter being in this way the conceptual inverse of the former, the "ER = EPR" conjecture may be refined by thinking of it in the categorial context of a universal topological mechanism by means of which the folding out of a local space-time event domain takes place. It is proposed and demonstrated that the Borromean topological link provides the sought for universal mechanism to qualify and understand the relation between entanglement and wormholes, and thus addresses effectively the validity of the "ER = EPR" conjecture.

In a nutshell, the present treatise argues in favor of a fundamentally different way of conceptualizing time and reality. In the new conceptual framework, both the sequentially ordered aspect of time and the factual aspect of reality are emergent phenomena that come into being only when the actual taking place of reality is over. In the new view, facts are just the "traces" that the actual taking place of reality leaves behind on the co-emergent "canvas" of local spacetime. Local spacetime itself emerges only as facts come into being—and only facts can be adequately localized in it. But, how does reality then actually occur in the first place? This "taking place" (in a most literal sense) is conceived as a "constellatory self-unfolding." This self-unfolding is characterized by strong self-referentiality, and it occurs still in the primordial form of time, i.e., in the not yet sequentially structured "time-space of the present." In its primordial form, time is the "ontophainetic platform",[2] i.e., the "stage," on which reality can occur in the first place.

In the novel framework quantum reduction and singularities can be addressed as inverse transitions: In quantum physical state reduction reality "gains" the chrono-ontological format of facticity, and the sequentially ordered aspect of time becomes applicable. In singularities, instead, the inverse happens: Reality losses its local spacetime formation and gets back into its primordial, pre-local shape—making also the use of causality relations, Boolean logic, and the dichotomization of subject and object obsolete. For our understanding of the relation between quantum and relativistic physics this new view opens up fundamentally new perspectives:

[2]In ancient Greek, "ta onta" means "that, what is." "Phainesthai" means that "something is bringing itself into appearance" (this is a so-called middle voice, i.e., a grammatical construction in-between active and passive voice). The neologism "ontophainetic" is meant to indicate the quality of allowing something that eventually is to appear in the first place.

Both the quantum physical and general relativistic picture are internally consistent and legitimate views of time and reality—they just address very different chrono-ontological portraits. This means that all trials to subjugate one view under the other, i.e., trying to find hidden variables "beneath" quantum physics, or trying to quantize gravity, are profoundly erroneous and lead nowhere.

The task of the book is to provide a formal framework in which this categorially richer view of time and reality can be addressed properly. The mathematical approach is based on the logical and topological features of the Borromean rings. It draws upon concepts and methods of algebraic and geometric topology—especially the theory of sheaves and links, group theory, logic and information theory, in relation to the standard constructions employed in quantum mechanics and general relativity, shedding new light on the pestilential problems of their compatibility.

Pullach, Germany Albrecht von Müller
Pullach, Germany Elias Zafiris

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Chapter 1
The Autogenetic Universe Theory Quantum Reduction and Singularities as Inverse Transitions: Into and Out of the Chrono-Ontological Format of Facticity

1.1 Introduction

Around the conjecture "ER = ERP" there emerged a very interesting and productive debate about the pivotal challenge of modern physics, the relation of general relativity theory (GRT) and quantum physics. In the following we introduce a new—and not so new—conceptual framework that has been developed quietly over the last three decades. It allows to substantiate the "ER = ERP" conjecture in the refined version of arguing that the singularities of GRT and quantum reduction can be seen as *inverse transitions* into and out of the chrono-ontological format of *facticity*, respectively the applicability of *local spacetime* and *causal accounts*. In addition, an algebraization of Borromean topologies will be introduced as a new mathematical tool for elaborating this approach.

The novel approach is rooted in a philosophical analysis of the incompleteness of a purely sequential notion of time and in the development of a richer notion of time in which a *nonlocal time-space of the present* moves to the center. The linear-sequential structure turns out to be an important, but derivative aspect of time that is applicable only for the "traces" of the actual self-unfolding of reality, the facts it leaves behind on the co-emergent canvas of local spacetime. It follows directly from this modified conceptual framework that quantum reduction describes the transition of reality into the state of facticity. Causal account become available only there, i.e. they are not yet available for the transition itself. Pari passu, it allows to see the singularities of GRT, i.e. the meltdown of the local spacetime, as the inverse transition by which reality returns into its primordial, pre-factual and pre-causal state.

In the following the gist of the novel conceptual framework will be given in four reasoning steps. Thereafter, the new mathematical approach is outlined.

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1.2 Step I: With Kant Beyond Kant: Discovering the Existence and Role of Categorial Apparatus as Enabling and Constraining All Further Theorizing

Immanuel Kant drew our attention to the fact that all further thinking about reality is based on initial "symmetry breakings" in our appreciation of time and reality. Based on the physics of his times, Kant saw Newtonian space and Newtonian time as the inevitable prerequisites of any consistent account of reality. Today, we enjoy a much richer notion of space—but, in general we work with a still rather narrow notion of time that limits it—even in GRT—essentially to its linear-sequential structure. This deprived notion of time prevents us from understanding (a) the crucial role of singularities, (b) what happens in quantum reduction, and (c) that and how GRT and quantum theory describe complementary aspects of the taking place of reality.

In order to overcome this pitfall and develop a richer notion of time, it is, however, necessary to go a philosophical extra mile—so to say "with Kant beyond Kant"—and to recognize the existence and role of underlying categorial apparatus (pl.) which enable but also constrain all subsequent thinking.

A categorial apparatus consists of four interrelated components:

– a basic form of connecting predications,
– a basic aspect of time,
– a basic relation between events,
– a basic epistemological setting.

The four constituents of the "classical" categorial apparatus are

– Boolean logic (implementing the principle of "tertium non datur"),
– the linear-sequential aspect of time (i.e., as the ability to order events),
– the principle of causal closure (historically called "causa sufficiens"),
– full separability of subject and object (resp. observer and observandum).

This set of underlying pre-configurations constitutes the *factual aspect of reality*. It is a very important and powerful *portrait of reality*. But this portrait alone does not yet give us a comprehensive *picture of reality*—as we know, e.g. from quantum physics or Gödel's incompleteness theorem of 1931.

The main structural deficit of the classical apparatus, respectively the factual portrait of reality, is that it is incompatible with the twin phenomena of *strong self-referentiality* and *autogenetic unfolding* (in which something unfolds in and out of itself, i.e. in the absence of external causal drivers).

The critical, hitherto unknown features of a categorial apparatus is that is contains four elements and that these are fully interdependent, i.e. one cannot abandon or substantially modify any of them without also affecting the others. This explains e.g. why giving up causality, or formulating a richer notion of time, does not work in isolation. It requires a comprehensive rethinking.

1.3 Step II: The Need and the Possibility to Develop a Second Categorial Apparatus

Since 100 years, theoretical physicists are banging their heads against the wall in order to “understand” quantum physics and to overcome the rift between GRT and quantum physics.

Seen from the novel conceptual framework, the reason why all these trials essentially failed becomes clear: As long as one does not recognize the apparatus character of the underlying categorial setup, one tries to change its components in isolation—and this leads inevitably to inconsistencies.

In order to *think* what happens in quantum reduction and for better understanding of the relation between GRT and quantum physics, we need to dig still one layer deeper in our analysis and to unearth the different categorial underpinnings, situated “beneath” the two foundational theories of physics. Only “down there” we can recognize the fundamental differences in their portrait of reality, and develop a richer, overarching conceptual framework.

Having discovered the apparatus character of the underlying categorial setup, one can formulate a second (and eventually even a third, but for scientific concerns less important) apparatus. It consists again of four interdependent constituents which, so to say, “fill the four slots” of a categorial apparatus:

– a *constellatory logic* (i.e., a predication space in which different, and even contradicting, propositions unfold their full meaning only mutually, and the overall significance emerges only in the constellation of all of them),
– a *nonlocal time-space of the present* (as the temporal platform on which the primordial self-unfolding of reality actually occurs; only once this “taking place” (!) has occurred, spatiotemporal locality is available),
– the phenomenon of *autogenesis* (resp. the principle of constellatory self-unfolding by which something unfolds out of, within, and toward itself, i.e. in the absence of external causal drivers),
– the structure of *strong self-referentiality* (respectively the phenomenon of a rich identity, like a person, in which something refers to itself in its entirety, thus further unfolding what existed before this self-reference).

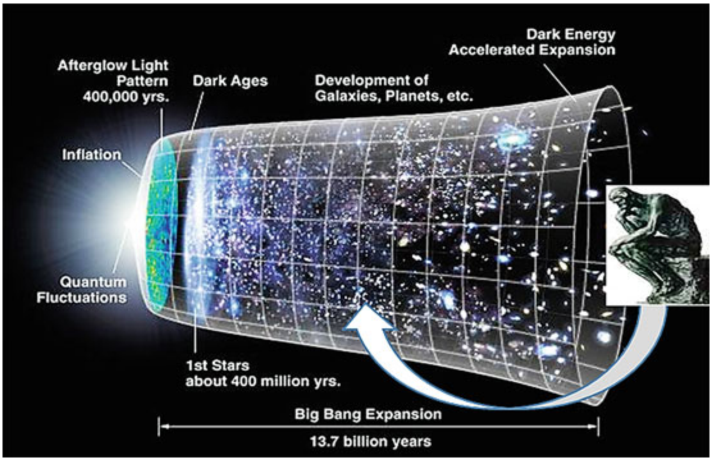
Each of these four constituents may initially seem quite strange, especially if we project them—as we almost automatically do—into the rest of the classical categorial framework. But, taken together, they form a full-fledged second categorial apparatus in its own right. This apparatus does not give us a comparably precise portrait of reality like the first one, nor does it allow for formal conclusions or far-reaching predictions. But, it allows us to appreciate and address the actual taking place of reality, i.e. its ongoing self-unfolding.

By (a) recognizing the existence and role of categorial apparatus, (b) understanding the inherent limitations of the classical apparatus, and (c) complementing it with a second one, capable to address “reality in the making” respectively the “statu-nascendi aspect” of reality, we have fundamentally expanded the space of possible theories.

1.4 Step III: The Idea of an Autogenetic Universe, the Three Ways of Portraying It, and Constellatory Self-Unfolding as Its Most Fundamental Principle

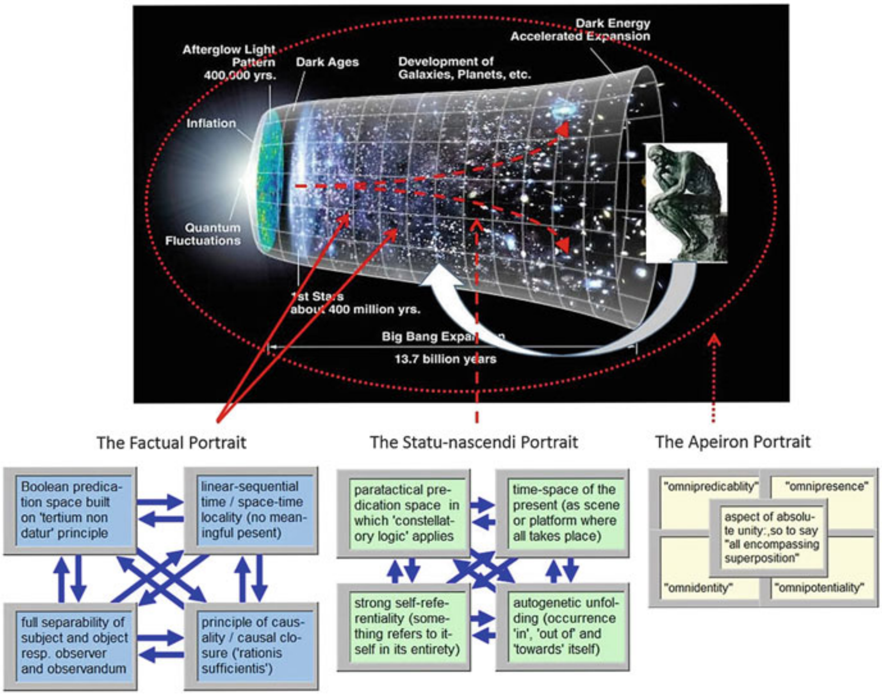
Facts can now be seen as the traces that the actual taking place leaves behind on the co-emergent canvas of local spacetime. Metaphorically speaking one can compare the self-unfolding of reality with somebody walking through fresh snow: As long as we only focus on the traces, we never get to see the wanderer. Or, in other words, the "facticity imprisonment" of our thinking made us take the "exhaust of reality" for the actual taking place of reality.

Putting the—now thinkable—phenomenon of constellatory self-unfolding at the center of our appreciation of reality, we start to see that we might live in an "autogenetic universe" that unfolds 'out of', 'within', and 'towards itself'.



The qualification of the unfolding as 'out of itself' refers to the absence of external drivers. 'Within itself' refers to the fact that an autogenetic universe does not unfold within local spacetime, but the emergence of the latter is part of its unfolding. The qualification as 'towards itself', finally, refers to the phenomenon that at a certain stage there have emerged entities which became aware of themselves in an explicit (i.e., language-based) manner. As they are part and parcel of the overall unfolding of reality, this very process starts—in them, i.e. in every single human being—to become aware of itself.

An autogenetic universe has three complementary portraits that are related to each other in the topology of Borromean rings, i.e. taking one of them away leaves the other two in unmitigated duality.



– The factual portrait focuses on reality that has already taken place and, thus, exists in the chrono-ontological format of fact in local spacetime. This is the only portrait in which coercive, formal proofs are possible (and even here, this possibility exists only in asymptotic approximation).
– The status-nascendi portrait depicts "reality in the making", i.e. the actual taking place of reality that precedes all facts and occurs in the time-space of the present. Already here, coercive proof is no longer available and "the convincingness of the more plausible argument", respectively authentic experience become the—significantly weaker—"criteria of truth."
– In the apeiron portrait, finally, reality is addressed in its entirety, as typically in religious belief systems or in some parts of philosophy and art. This is a fully legitimate and respectable way of addressing our universe—as long as one respects the insurmountable limitations of this portrait and all that can be claimed within it: Nobody can ever prove anything or is ever entitled to force others to believe what oneself has chosen to believe. (All religio-ideological intolerance is based on the dramatic epistemological mistake of treating impressions and beliefs that belong to the apeiron portrait as if they were propositions from the factual portrait.)

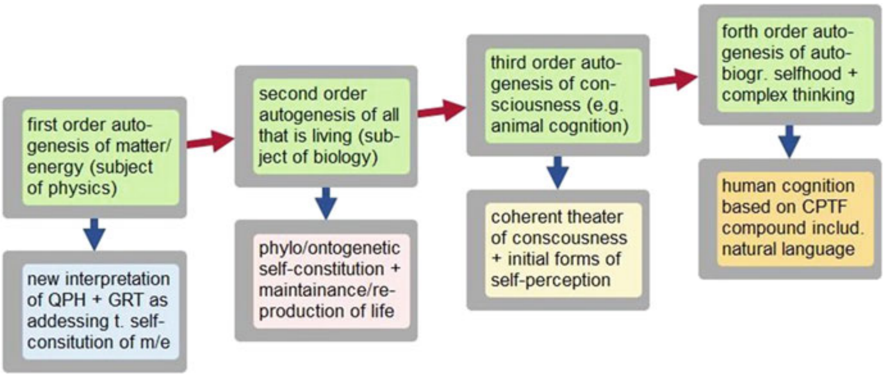
Only all three portraits together allow for an adequate appreciation of an autogenetic universe in its essential self-unfolding. By reducing our notion of reality to facts, and our notion of time to its linear-sequential structure, we deprive our appreciation of both, the world in which we live and ourselves in a most dramatic way.

To overcome this "facticity imprisonment" of our thinking is the prerequisite for overcoming the present obstacles in understanding matter, life, Consciousness, and mind. But before discussing at least the implications of the new approach for the foundations of physics in some more detail, I would like to make still a few remarks on the idea of an autogenetic universe.

The notion 'autopoietic' refers to processes in which an entity uses existing material and configures it in a way that the system reproduces itself. In an 'autogenetic' process also the material—and even the framework in which all takes place—emerges as part of the overall self-unfolding.

A universe that starts to become aware of itself is completely different from one which just "drags on blindly". By starting to become aware of itself, the whole universe gains a fundamentally novel quality—in every single instance where this happens. This new quality of the whole is the reason for the infinite and nonnegotiable dignity of every single human being.

With constellatory self-unfolding as the most fundamental and most cross-cutting principle a radically novel way to appreciate our universe becomes feasible. The self-constitution of physical matter/energy can be seen as "first order autogenesis," which is addressed in quantum physics. Out of this emerges life as a kind of "second order autogenesis," characterized already by a higher degree of self-referentiality, i.e. of self-constitution and self-unfolding. The emergence of consciousness, and eventually even mind, can again be interpreted as still higher orders of autogenesis respectively self-unfolding.



Taken together, these thoughts result in what could be called a modest "ToE". "Modest" means that there is no claim to understand all, not to speak of being able to pre- or retrodict all in the sense of a Laplacean demon.

The notion of a ToE indicates that, despite this irreducible incompleteness, there is one coherent conceptual framework—allowing to see both, the unity and the diversity of the emergent reality. In philosophy there is an age-old controversy, whether unity or diversity is the ultimate principle of reality, closely related to the debate between monistic and dualistic world views.

The novel thought pattern of constellatory self-unfolding transcends this controversy. If something unfolds *itself*, an underlying coherence and thus unity is given. At the same time, the notion *unfolding* indicates that genuinely novel features emerge, i.e. there is permanently increasing diversity. The notion of *richness* brings the two features quite well together. If something is rich, it (i.e., something that belongs inherently together) is *rich*, meaning that it has a multitude of different facets and components, thus being characterized also by fundamental diversity.

In the new way of appreciating reality we draw on three instead of only one categorial framework, respectively "apparatus". These three apparatus constitute three different but complementary chrono-ontological portraits of reality. Their Borromean interrelatedness is a self-confirming aspect of the novel, above described, dynamic combination, Integration, and mutual deepening of unity and diversity.

The thought pattern of an autogenetic universe, thus, offers a new way of describing our world that combines openness for genuine novelty with conceptual coherence, i.e. it constitutes what has been characterized as a "modest ToE". In an autogenetically unfolding universe also, explainability and wonderfulness are no longer at the detriment of each other—they, too, deepen mutually.

In practical terms, all this leads to a fundamentally richer appreciation of the world we live in, of all other human beings and of ourselves: It leads in a natural way to a new basic tenor via-a-vis the ongoing taking place of reality that could be characterized as "thankful attentiveness".

1.5 Step IV: What All This Means for the Understanding of Quantum Physics, General Relativity, and the Relation Between the Two Theories

In quantum physics the actual taking place of physical reality, i.e. its ongoing self-constitution is addressed. Relativistic physics, instead, focus mainly on the factual portrait of reality—with the important exemption of singularities which can now be seen as the fascinating instance of *de-factization*, respectively the meltdown of local spacetime.

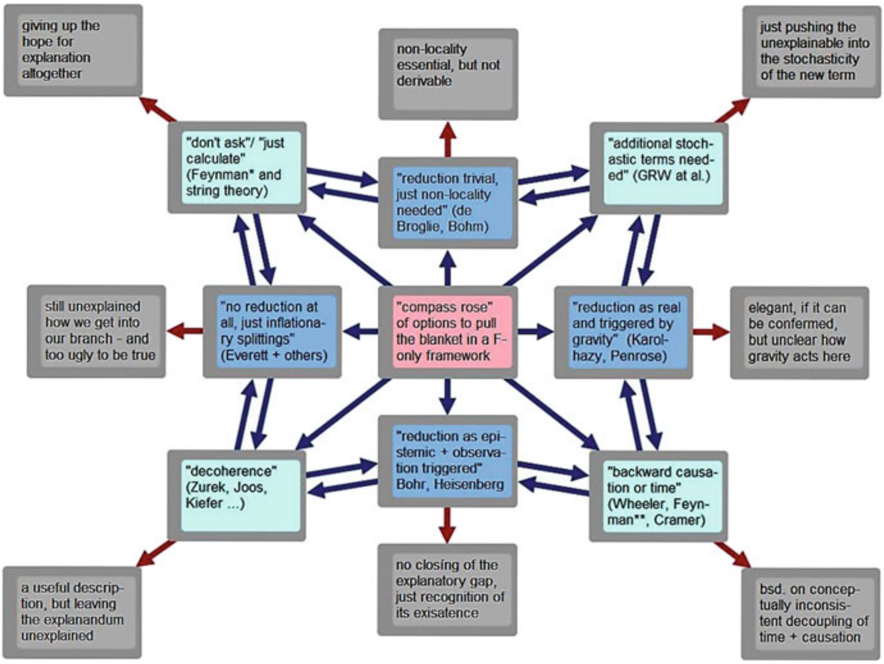
As already mentioned, trying to subjugate one approach under the other, i.e. trying to quantize gravity or to find hidden causal mechanisms beneath quantum physics is neither needed nor adequate. The two theories address different portraits of reality, complementing each other *because* of their fundamental difference.

The two theories can and should remain as they are—understanding their relation, however, requires (a) to go the extra mile and unearth the different categorial foundations of the two theories and (b) to make the transition from a monolithic to multiple chrono-ontology that comprises all three, the factual, the statu-nascendi, and the apeiron portrait of reality.

All the essential features of quantum physics fit exactly with the statu-nascendi portrait of reality: non-locality, superposition, entanglement, genuine indeterminacy, and the a-causal, inherently constellatory nature of the reduction. All of them require the second categorial apparatus for thinking of them in a consistent way and as a complementary aspect of reality in its own right.

As long as we have only the factual portrait at our disposal, quantum physics will inevitably remain mysterious. The situation is a bit similar to trying to cover oneself with a blanket that is inherently too small. One can cover feet and upper body, but not both at the same time. By covering one, one bares the other.

With the best brains in physics trying restlessly for almost 100 years, all positions to place the blanket have been tested—and one can represent the failing efforts as a kind of compass rose indicating all possible positions of the blanket, i.e. what is covered and what is left unexplained.

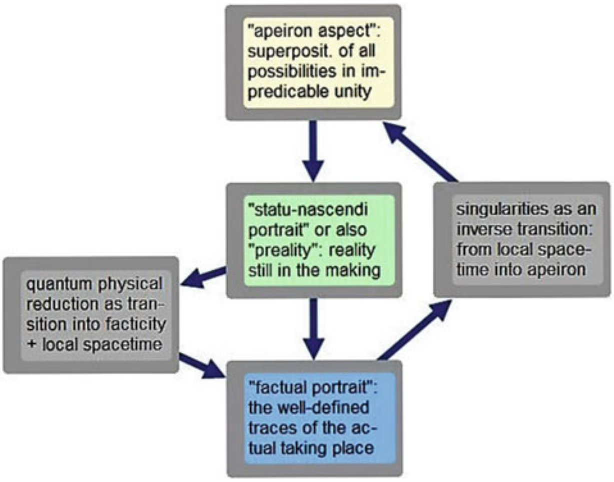


Real progress can only be made by "enlarging the blanket", i.e. by a fundamentally novel conceptual framework—or as Einstein put it wisely: "We cannot solve our problems with the same thinking we used when we created them". With GRT the situation is equally fascinating. The curvature of spacetime by what it contains, mass/energy, constitutes the phenomenon of strong self-referentiality and it ensures the unity of our universe, despite its diversity.

The appearance of singularities has often been considered a fundamental weakness of GRT. In the here offered conceptual framework of an autogenetic universe they turn out to be one of the deepest insights of GRT and the crucial bridge between quantum physics and relativity theory: Singularities are the instances of de-factization, i.e. the points where reality (driven by the strong self-referentiality of gravity) leaves again the factual portrait, bringing itself back into the primordial statu-nascendi format of time and reality.

But, in order to see this, one must have a richer categorial framework, and based on this, the notion of a self-unfolding universe with three complementary chrono-ontological portraits.

The phenomenon of reduction in quantum physics and the singularities of GRT can now be understood as inverse transitions of reality: into and out of the chrono-ontological format of facticity, respectively, the realm in which the classical categorial apparatus can be applied properly and legitimately.

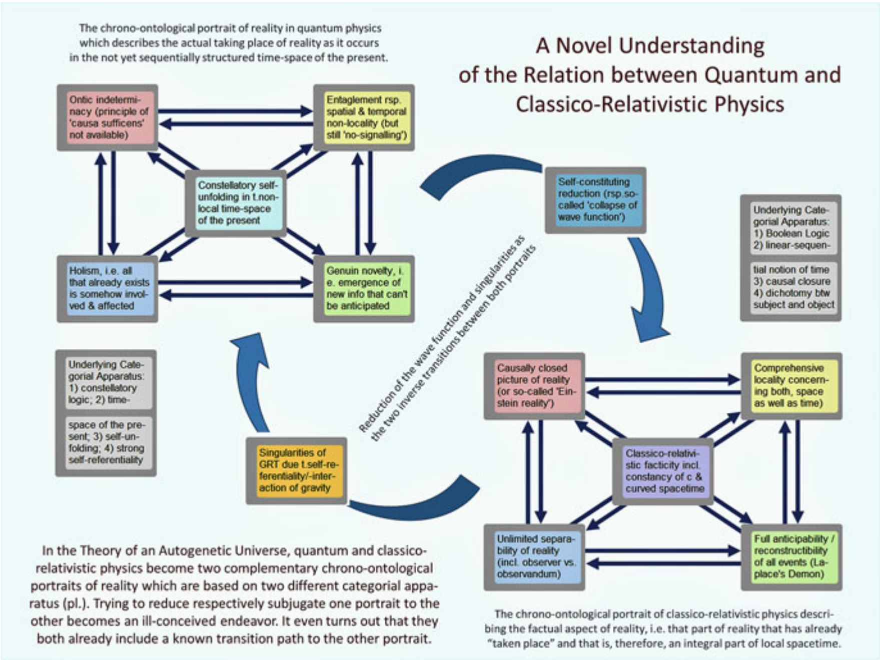


This interpretation of quantum physical reduction and the singularities of GRT as inverse transitions is also the point where the theory of the autogenetic universe allows to support and to substantiate the "ER = ERP" conjecture—and to explain why this is so.

In the following section it will be sketched out why and how Borromean topologies can play a pivotal role in formalizing and proving what has been introduced here in more philosophical terms.

In closing the first part of this very brief and sketchy introduction of the theory of an autogenetic universe, I would like to stress again that both, quantum physics and GRT, confirm and require the novel conceptual framework and how they both articulate some of its crucial points in the most elegant way:

– the coincidence of unity and diversity,
– the mutual deepening of explainability and wonderfulness,
– the key role of constellatory self-unfolding as the underlying principle of our autogenetic universe that unfolds out of, within, and toward itself.



Chapter 2
Model of an Autogenetic Universe Constellatory Self-Unfolding: A Novel Syntaxis of Time in the Time-Space of the Present

2.1 Reflecting on the Basic Premises of the Theory of an "Autogenetic Universe"

The philosophical theory of an "autogenetic universe" (von Müller 2011, 2012, 2015) proposes new "categorial foundations" for science aiming to overcome the inherent limitations, incompatibilities and structural pitfalls of the current scientific paradigm. The basic premise of the proposed new theory is that we live in an autogenetic universe, meaning that we live in a self-unfolding and strongly self-referential universe. In relation to this hypothesis, the theory of an "autogenetic universe" proposes a novel account of *time* and *reality*, which aims at a deeper re-conceptualization of these fundamental notions going beyond or underneath the structural reduction of the former to its linear-sequential aspect and the concurrent related reduction of the latter to its factual or event-like aspect. This is of particular significance in relation to the frontier area of theoretical physics aiming at a unification of quantum mechanics and general relativity, where it is argued that a key conceptual element for this purpose requires the relativization of facticity, namely of the event structures pertaining to a local space-time description capturing exclusively the factual portrait of reality. It is instructive to note that the notion of an unfolding universe has been also explored by means of a different approach in the work of Kafatos and Nadeau (2013).

The "autogenetic universe" theory proposes a *triality account* constituted in the form of three interdependent layers, which are connected together in the form of the "Borromean rings" topology, that is if any one of the layers is removed, then there remain two unlinked layers. Each layer captures a different aspect of reality, namely the "apeiron aspect," the "statu-nascendi," and the "factual aspect" correspondingly. The "apeiron aspect" is inherently without any structure and expresses the irreducible global unity or non-separability of reality at this layer, which acts as a source for "the actual taking place," to be thought of as a kind of logical disclosure

A. von Müller, E. Zafiris, *Concept and Formalization of Constellatory Self-Unfolding*, On Thinking,
https://doi.org/10.1007/978-3-319-89776-9_2

topos pertaining to the time-space of the present. The observed traces of this process, viz. the events embedded within a local space-time background constitute the "factual aspect" of reality. Whereas the "apeiron aspect" is not amenable to any direct structural predicative determination, both the "statu-nascendi" and the "factual aspect" constitute layers whose respective characteristic function can be depicted in the terms of distinctive underlying "categorial frameworks."

Each "categorial framework" stands for an integral apparatus consisting of four interrelated and bidirectionally interdependent components:

(a) a *logical structure* of a predication space,
(b) a related notion of a *spatio-temporal background*,
(c) a *causal scheme* accounting for linkages, and
(d) a corresponding *epistemological setting*.

In this way, the "factual aspect" of reality is captured by means of a categorial apparatus, which consists of the following components respectively:

(a) a Boolean logical predication space,
(b) a local metrical space-time continuum,
(c) a classical scheme of efficient causality, and
(d) an epistemological setting based on the notion of absolute separability between observer and observandum.

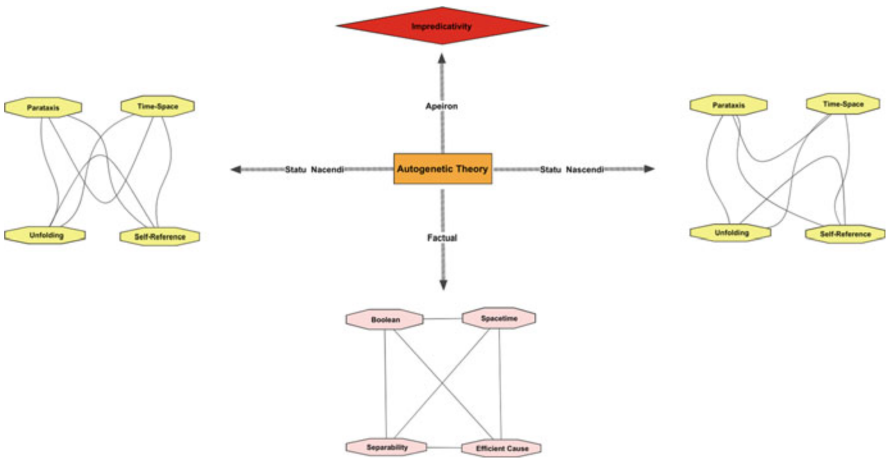
The intrinsic necessity of introducing another categorial apparatus constituting the "statu-nascendi" layer of reality is based on the inability of the former one to account for the logical structural phenomenon of *strong self-referentiality* and its concomitant operational manifestation as *autogenesis*, meaning a process of self-referential *folding/unfolding* without any separable external cause.

From this perspective, the issue of quantum state reduction or quantum measurement problem in quantum physics and the problem of singularities in general relativity are considered as artifacts caused by focusing exclusively on the categorial apparatus attached to the factual aspect of reality, while ignoring completely the categorial apparatus fitting into the "statu-nascendi" layer. In particular, the quantum reduction problem targets the emergence of a local space-time event continuum from the fundamental non-spatio-temporal quantum theoretic description of nature, whereas the singularities problem targets the global breakdown of the metrical smooth space-time point-event-manifold model of the general theory of relativity. Thus, both problems viewed from an extended perspective as targeting the issue of transition into and out of the local space-time event continuum pertaining to the factual layer of reality point to the conclusion that their resolution requires the explicit consideration of the *categorial apparatus characteristic of the "statu-nascendi" layer together with the "Borromean rings"-type of topology* interconnecting the three reality layers. Consequently, the resolution of these problems, which may be both considered as different types of self-referentiality, the first as a self-referential folding into a local space-time point-event stratum and the second as a self-referential folding out of this point-event stratum, poses the necessity of a higher-order relativization of facticity targeting the very notion of a local perspective on reality.

We claim that the nature of this notion, that is of a “local perspective on reality” should not refer to the concept of metrical/geometrical locality in a point-event set-theoretic space-time manifold but should be of a *logical/topological origin* to be thought of as a *local logical disclosure topos* demarcating the logical structural pre-conditions of reduction from the global to the local and inversely extension from the local to the global. This higher-order logical/topological relativization of facticity, called “categorial relativity,” requires a careful qualification of the categorial apparatus characteristic of the “statu-nascendi” layer of reality. The constituent interrelated components of this layer are the following:

(a) a *paratactical predication space* on which some form of “constellatory logic” becomes applicable,
(b) a *local logical disclosure topos* pertaining to the time-space of the present,
(c) a causal scheme of *autogenetic folding/unfolding*, and
(d) an epistemological setting of *strong self-referentiality*.

The notion of parataxis refers to a mode of logical coherence of a multiplicity which is independent of linear sequential organization. This is captured by the functional role of a “constellatory logic,” where an individuated component of such a multiplicity can be evaluated only in the context of all other components being compatible with it in an appropriate manner.



The “autogenetic universe” theory based on the triality account constituted by the Borromean-type of interconnections of the three layers capturing the “apeiron aspect,” the “statu-nascendi,” and the “factual aspect” of reality correspondingly, sheds new light on the old problem of time, together with the concurrent problem of unfixing the conception of reality from its exclusive reference to the facticity stratum, which disregards completely even the necessary logical pre-conditions for the manifestation of events. In this way, it becomes important for the autogenetic theory to specify more concretely the appropriate conceptual and technical bridges which bind together the three layers forming the triality account, as well as to refine the modeling of this triadic inter-relational scheme to a Borromean structural type of topological linking. This would be particularly significant for the elucidation and technical formulation of the principle of "categorial relativity" as a higher-order relativization of facticity, which would create a common ground for the resolution of both the problem of quantum reduction and the problem of singularities as inverse-type of transitions into and out of a local space-time event stratum. The specification of these bridges would be ultimately necessary for the consistent formulation of a strong self-referentiality scheme, which would involve the triality account as a whole, and would give rise to a form of "constellatory logic" functioning at the "statu-nascendi" with respect to the factual layer.

For this purpose, we propose and develop a precise mathematical model of the "autogenetic universe" theory, targeting in particular the following:

(a) The notion of autogenetic constellatory unfolding together with the associated notion of strong self-referentiality;
(b) The notion of the "time-space of the present" and the precise form of the relation with the standard notion of spacetime.
(c) The connectivity among the three chrono-ontological formats of reality and the role of the Borromean topological link in this respect.

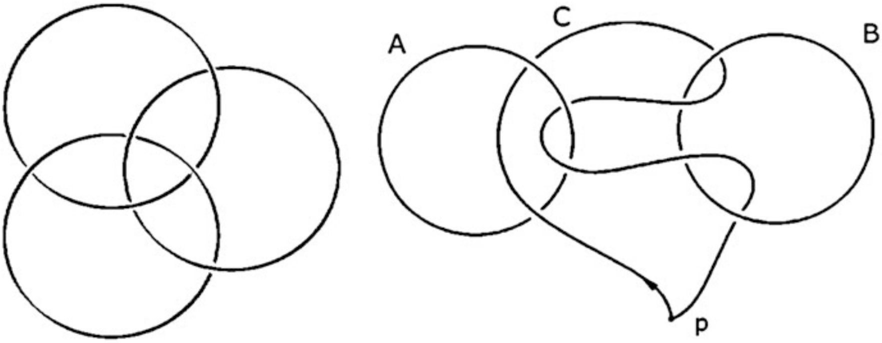
2.2 Chrono-Topological Binding in the Time-Space of the Present

According to the major premise of the "autogenetic universe" theory, reality exists in three different chrono-ontological formats, facticity, statu nascendi, and apeiron. Because of this, all parts of one, coherent reality must somehow be mutually interrelated, otherwise there would be no point in speaking of one reality. In this respect, the autogenetic conception of reality requires that a precise meaning has to be given to the crucial notion of the "time-space of the present," which has to be distinguished from the standard spacetime of events-facts. Given that reality is characterized by the three different chrono-ontological formats the "time-space of the present" has to be understood in its potential to bind "the past" with "the future" in relation to "the present," not in the sequential event temporal order of the "continuum of the real line" that models only the factual aspect of reality, but in another chrono-topological form. The principal argument that is put forward in this regard is that the sought-after chrono-topological binding form is characterized by the Borromeanicity property, i.e. it is not chain-like, such that there exists the possibility that "the very far past" can be glued together with the "very near future" through the "time-space of the present" if they form a "Borromean temporal bond." As a consequence, the Borromean bond pertains to the chrono-ontological domain, i.e. the "time-space of the present" becomes the temporal topos of the process of topological historic unfolding. In this way, and interestingly enough, "the past" and "the future" exist paratactically in their potential to convey meaning with respect to the "time-space of the present," and not hypotactically as in the sequential-chain model. In turn, this justifies the need for characterizing reality in "statu-nascendi" via a different categorial framework. From this conceptual perspective, the interpretation of the "Borromean link" (Zafiris 2016a,b) as a "temporal-historic bond" requires the following:

1. Reconciliation of the static three-dimensional spatial representation of the "Borromean link" with the dynamic constellatory unfolding and self-referentiality characteristics of reality in "statu-nascendi." This issue can be resolved by realizing that the static representation of Borromeanicity is just the spatial image, or more precisely, the "epiphany" of the temporal bond. Equivalently, we consider a cross-section of the bond projected spatially and giving rise to the standard spatial non-local Borromean-rings-type of linkage in 3-d space. This admits a concrete mathematical formulation via the algebraic-topological notion of a "covering space" (Hatcher 2002), which is literally the concept of a "self-referentially unfolding temporal dimensionality";
2. Interpretation of the algebraic model of the "Borromean rings" as a "Borromean temporal bond" among "past," "present," and "future," from the standpoint of the "time-space of the present," taking place in "statu-nascendi" with respect to the connectivity potential of the apeiron. Here, it is proposed that the concepts of "memory" and "anticipation" play a key role in order to give meaning to the algebraic model of "strongly self-referential Borromean gluing," developed in detail in Chap. 3. This is symbolically represented by the "commutator gluing,"

$$[\alpha, \beta^{-1}] = \alpha\beta^{-1}\alpha^{-1}\beta,$$

   where the irreducible formula $\alpha\beta^{-1}\alpha^{-1}\beta$ represents the third ring of the "Borromean rings" as a product loop, composed by the ordered composition of the four based oriented loops α, β^{-1}, α^{-1}, and β. It is important to stress that these loops bear an orientation from the standpoint of the topos characteristic of "the time-space of the present" with respect to the "past A" and the "future B" conceptualized as topological circles in a cross-section of the unfolding enunciated by the "present." The algebraic irreducibility of $\alpha\beta^{-1}\alpha^{-1}\beta$ encodes precisely the non-reducible form of binding generated by the Borromean 3-link in the cross-section of the "time-space of the present" at p. Note that A and B are in relation of parataxis before the "temporal binding" and the distance between them is immaterial for the effectuation of the bond;



3. Distillation of the deeper meaning encoded in the "ER = ERP" conjecture (Maldacena and Susskind 2013) in relation to the notion of a "Borromean temporal bond," where "ER = ERP" is a short-hand joining together two ideas proposed by Einstein in 1935, namely quantum entanglement (EPR entanglement, named after Einstein, Podolsky and Rosen) and wormholes (ER, for Einstein–Rosen bridges). It is shown explicitly that the "Borromean-type of temporal binding" can be interpreted as a kind of "gravity effect" in the global complement of this bond. This is in a precise topological sense a type of a "global curvature effect," as it will be demonstrated in the sequel, since it is not localizable anywhere. A cross-section depicted by the "time-space of the present" and projected spatially, leading to the effect of "spatial non-locality" in 3-d space, can be interpreted quantum-mechanically as a form of entanglement in 3-d space. In other words, quantum entanglement is the epiphenomenon observed in 3-d space, playing the role of a "cross-sectional holographic boundary hypersurface" with respect to the 4-d of the "gravity effect" being caused upon the establishment of a "Borromean temporal bond." Equivalently, entanglement is the "epiphany in cross-sectional 3-d" (in the ancient Greek meaning of epiphaino) of the "bulk gravity effect in 4-d" due to the action of a "Borromean temporal bond." This "bulk gravity effect" is conceptualized as a wormhole and constitutes the crux of the "ER = EPR" correspondence, which actually pertains precisely to the effect of a "Borromean temporal bond."

This higher-level abstraction (capturing the essence of the "ER = EPR" correspondence) necessitates a re-thinking of the notion of "time" in Special Relativity (SR) and General Relativity (GR) as the 4-th dimension of a "spatiotemporal continuum (Einstein 1956, Hawking and Ellis 1973, Misner et al. 1970)." Not only this, but the "3-d spatial epiphany" in the form of entanglement is meaningful only if the "epiphany = 3-d cross-sectional spatial hypersurface" is actually a "holographic boundary" of 4-d, since the "gravity effect" of the "Borromean temporal bond" is global (i.e., not localizable anywhere). This necessitates the conceptual and technical differentiation between the notions of "dimension" and "dimensionality." Given that the notion of dimension pertains to the standard notion of spatial dimension, the treatment of time as a kind of 4-th dimension comes only after the imposition of metrical chrono-geometric relations. The “pre-metrical topological notion of time” (from the standpoint of the “time-space of the present”) should be thought of in terms of physical dimensionality, meaning an “unfolding dimension” coming about via a process of temporal division (i.e., in the form of the ancient Greek notion of dia-stasis). The notion of an “unfolding dimension” is captured precisely by the algebraic-topological concept of a “covering space” or a “covering scheme,” which is considered indispensable for the explication of the process of “self-referential autogenetic unfolding.”

2.3 Multiple-Connectivity in the Time-Space of the Present

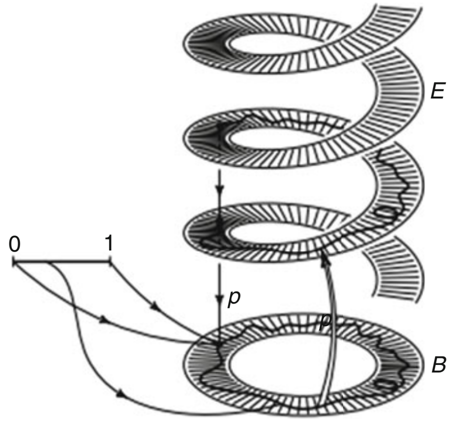
The existence of the three different chrono-ontological formats constituting reality, i.e. facticity, statu nascendi, and apeiron, from the standpoint of the “time-space of the present” bears a distinguishing quality as a whole, only if “the past” can be connected to “the future” in a multiplicity of possible ways according to some scheme of “temporal division” or “temporal partition.” This should be thought of in contradistinction to the sequential simply-connected ab initio connectivity pattern of the “standard real-line event continuum,” which is based on the totally ordered sequential structure of the real numbers.

In mathematical terms, this is described by the topological notion of “multiple-connectivity” in the “time-space of the present.” In this understanding, multiple-connectivity pertains to temporal binding according to some temporal division scheme. It will be shown in the sequel how the temporal division scheme is implemented in SR and GR based on the constancy of the speed of light and the induced chrono-geometric relations, respectively. The important thing is that the topological property of multiple-connectivity has a temporal connotation, whereas the potential appearance of “alleged non-locality” in 3-d space, for instance, when quantum entanglement effects are considered, has a spatial connotation, which is precisely the spatial cross-sectional projection, i.e. the “epiphany” of the particular form of “temporal binding” in the “time-space of the present.” As a consequence, time cannot be treated as a dimension bearing the same status like the spatial dimensions, but has to be conceptualized as a dimensionality, or equivalently, as an “unfolding dimension” according to some temporal division scheme from the standpoint of the “time-space of the present.” The basic claim, which will be presented and elaborate in what follows is that the mathematical consistency of the autogenetic theory is based on the conception of time as an “unfolding dimension” according to the topological theory of “covering schemes” and not on “epi-cyclic notions” like “probabilistic Bayesian updating” or “randomly evolving connectivity graphs.” At best, the latter become meaningful only at the cross-sectional spatial projection as epi-phenomena following a fundamental scheme of “temporal binding.” In other words, the epi-phenomenon of randomness in a spatial connectivity graph is traced back to the chrono-topological property of multiple-connectivity as distinguished from a priori simple-connectivity, which leads to a classical deterministic model.

Beyond the physical applications, referring to the higher-level abstraction of interpreting the Borromean topology in terms of a “temporal bond” in the “time-space of the present” and the concomitant notion of an “unfolding dimension” via the theory of “covering schemes” or “covering spaces,” it is expected that these notions can find very important applications in the following: Study of thinking and understanding, language evolution, human and artificial intelligence, and also strategic decision making. For instance, the process of acquiring meaning and understanding via reading a book does not conform to the sequential order of time. It is more natural that the brain establishes “Borromean temporal bonds” at each and every “cross-sectional present,” which due to topological multiple-connectivity allow information amalgamations irrespective of any notion of distance proximity in the text. This is also important in decision making where things in the “very far past” can form a “Borromean temporal bond” with anticipated things in the “very near future” in the “time-space of the present.” It will be described later how this “temporal binding” can be visualized by means of a “kind of a gravitational field” inducing a “curvature” that makes things roll to a particular slope without any direct external cause beyond this bond operating on a higher level.

2.4 The Notion of “Unfolding Temporal Dimension” and “Covering Schemes”

The notion of an “unfolding temporal dimension” (“dia-stasis”) pertains to all situations that the “past” can be connected to the “future” in a multiplicity of possible ways according to some scheme of “temporal division” or “temporal partition” with respect to the “time-space of the present.” *The fundamental example of an “unfolding temporal dimension” is provided by a spiral or helix that is unfolding in a “snake-like manner.”* This can be visualized either as an “Archimedean screw-type” of unfolding or as a “logarithmic screw-type” of unfolding, depending on the periodic rule of temporal division, with two possible orientations. Alternatively, we may simply think of a “topological chord” wrapped around a cone that is extended to infinity, such that the particular type of wrapping is subordinate to a specific rule of temporal division. In this case, the cone represents the time-space of the present in “statu-nascendi” where the “temporal chords” are unfolding with respect to the multiple potential connectivities appearing at the spatial epiphany of the present. The latter is a *spatial cross-sectional projection of the spirally unfolding dia-stasis, and clearly bears the topology of a circle. In this manner, an unfolding spiral constitutes a “covering space” or a “covering scheme” of the epiphenomenal spatial circle*. The simplest example is demonstrated below, where a spiral in “statu-nascendi,” unfolding according to a constant periodic rule of temporal division from the perspective of the “time-space of the present,” covers evenly the epiphenomenal spatial circle.



It is going to be described with all details in the sequel, how this type of an "epiphenomenal spatial circle" arises in the context of our fundamental physical theories. For example, in the case of SR it "takes place" by the metrical spatialization of a "temporal unfolding dimension" through the rotational periodic rule determined by the finitude and constancy of the speed of light. The important thing to keep in mind at this stage is that *a "spirally or helically unfolding temporal dimension" always gives rise to an epiphenomenal spatialized dimension that bears the topology of the circle and not of a line*. An epiphenomenal spatialized circle can be coordinatized by means of the unit circle in the complex domain (in two spatial dimensions) or the quaternion domain (in four spatial dimensions) giving rise to what may be called as an "imaginary dimension." Since the unit of this "imaginary dimension" is interpreted as a rotation by 90° in the counterclockwise orientation on this spatially epiphenomenal circle with respect to the real spatial dimension, whose extension is depicted horizontally in the complex domain, the "imaginary dimension" cannot be separated from any "real-number coordinatized spatial dimension." This is precisely what gives rise to a "2-d inseparable spacetime" (if we consider just one "real spatial dimension") or a "4-d inseparable spacetime" (if we consider all three spatial dimensions).

The crucial idea is that "an imaginary dimension" constitutes the epiphenomenal spatialized cross-sectional form of a genuine "temporal unfolding dimension" according to the above, and this is precisely the *major characteristic that distinguishes the notion of the "time-space of the present" from the notion of "spacetime."* It is a category mistake to treat an "imaginary dimension" as a "temporal dimension" in the same footing like the spatial ones. An "epiphenomenal spatialized imaginary dimension" bears an "imaginary unit" inducing "circular action by rotation" in contradistinction to a "real spatial dimension" which bears a unit inducing "linear extension in a specified direction."

2.5 Autogenetic Perspective on Special Relativity

The main distinguishing features of SR, according to the standard presentation, are the following:

(a) The maximal speed of signal transmission is defined by the speed of light c;
(b) The speed of light is constant in all directions;
(c) The laws of physics are the same in all inertial frames;
(d) Time is treated as the 4-th dimension of an inseparable flat spacetime;
(e) The spatiotemporal metric relations are constant at every point-event of spacetime giving rise to the group of Lorentz transformations as the kinematical symmetry group of the theory;
(f) The relation between energy and mass $E = mc^2$.

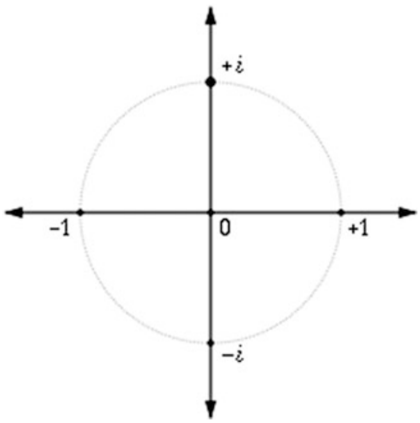
The challenge is to think of SR from the perspective of a "temporal unfolding dimension," and according to the previous arguments, interpret it via the premises of the autogenetic theory. The clue comes from the form of the "spacetime metric," where the spatialized temporal coordinate comes with a minus sign. For simplicity, the argument will be presented in the case of "2-d spacetime" (involving one spatial and one spatialized temporal dimension), since it extends to the case of "4-d spacetime" in a straightforward manner. In particular, the "spacetime metric" reads $dS^2 = dx^2 - c^2dt^2$, which is equivalently written as $dS^2 = dx^2 + (icdt)^2$, where the imaginary unit i has been used in the conversion of the temporal factor into the spatialized form, where the metric relation refers to. In other words, the temporal metric factor is brought into a spatialized form by the use of the speed of light c via the intervention of the imaginary unit i (which acts as a conversion factor).

Hence, we are in the case of an "imaginary dimension" conceived as the epiphenomenal spatialized cross-sectional form of a genuine "temporal unfolding dimension" projected at the factual level. This "temporal unfolding dimension" with respect to the "time-space of the present" in "statu-nascendi" is brought about by the upper bound in information signaling defined by the speed of light c, and thus, it pertains to phenomena approximating that speed. Since the finitude of the speed of light affects the metrical chrono-geometric relations at very high speeds, and not the chrono-topological ones, the cone of unfolding of the "temporal chords" is actually a "metrical light-cone." In effect, this means that the "spirally unfolding temporal dimension" is *degenerate topologically*, in the sense that the "winding stairs of the spiral" are not distinguishable metrically, and thus, the potential of multiple-connectivity between the "past" and the "future" is reduced only to the possibility of branching with respect to the "time-space of the present."

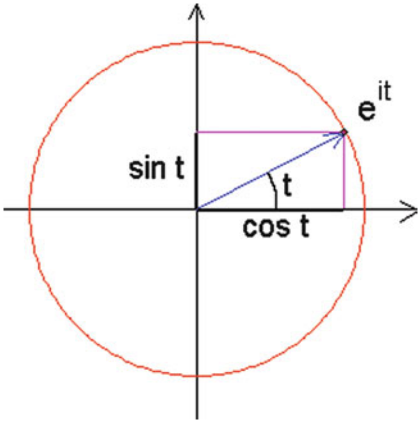
From the higher-abstraction level of the "degenerate spiral covering scheme" this type of temporal unfolding at very high speeds takes place at a constant rate given by the speed of light c. A conceptual parenthesis will be opened in the sequel in relation to a possible autogenetic interpretation of this constant rate. But currently, it is important to examine how the connectivity between the "past" and the "future" should be thought of with respect to the "time-space of the present" in the case of

SR. The crucial thing here is not only that the speed of light is constant, but that *the chronogeometric relations induced by this upper bound are constantly the same at every point-event*, since the metric is not variable. This means that the light-cone at the "time-space of the present" is isomorphic to the light-cone at any past event and will be isomorphic to the light-cone at any future event. Thus, they can be both isomorphically rooted at the same point-event in the "time-space of the present," but in an inverted relation with respect to each other so that the "past one" can be distinguished from the "future one." Epigrammatically, it suffices to consider the same "degenerate spirally unfolding temporal dimension" for both the "past" and the "future," since the periodic unfolding rate is constantly the same, differing only in orientation, and thus, considered as rooted at the same point-event in the "time-space of the present."

If we consider the corresponding "epiphenomenal spatialized imaginary dimension" as the cross-sectional projection of the genuine "unfolding temporal dimension" at the factual level, according to the analysis of the previous section, it bears an "imaginary unit" inducing "circular action by rotation." Hence, if we consider the rooting at the same point-event in the "time-space of the present" of both the "past" and the "future" differing in orientation, at the "epiphenomenal spatialized imaginary dimensional level," which can be precisely thought of as the "{*imaginarily spatialized time*}-{*real space*} *of the present rooted at the same point-event* 0," the "future" is represented by "circular action in the counterclockwise orientation" induced by rotation via the imaginary unit i, whereas the "past" is represented by "circular action in the clockwise orientation" induced by rotation via the complex conjugate imaginary unit $i^* = -i$.



Simply put, at the epiphenomenal level, *change of time in SR amounts to change of phase*, and this is *the same* for both the "past" and the "future," differing only in orientation with respect to the rooting at a point-event in the present. An immediate consequence of this is what is usually called "Lorentz contraction of lengths in the direction of motion." More concretely, if we consider motion in the horizontal spatial direction at a high speed below the speed of light, then spatial extension in the real horizontal linear dimension by 1 unit of length will appear contracted with respect to "0," since it amounts to a change of phase on the unit circle equal to the passage of spatialized time, making an angle with respect to the real horizontal dimension. Hence, the length contraction (depending on the speed of motion) with respect to "0" is just *the projection* on the horizontal linear spatial dimension of the corresponding phase change on the unit circle.



Let us return now to the higher-abstraction level of the "degenerate spiral covering scheme" pertaining to genuine temporal unfolding at very high speeds, which in SR takes place always at a *constant rate* given by the speed of light, e.g. "past" and "future" are distinguished only by opposite orientations and not by any difference at the unfolding rate. The issue is if we can derive an *autogenetic interpretation of this constant rate*. The starting point will be the energy-mass relation in SR, $E = mc^2$. The standard interpretation of this relation as indicative of the energy/mass equivalence is not adequate from an autogenetic perspective. The autogenetics problematics arises through the conception of normal matter as corresponding to the reality that has already taken place and is, thus, "full member" of local spacetime.

Following this conception, if we think of normal matter as corresponding to the reality that has already taken place in relation to the total energy of apeiron reality, then the equation $E = mc^2$, or equivalently, $c = \pm\sqrt{E/m}$, gives the *constant rate of genuine temporal unfolding* following the conversion of *apeiron reality (energy)* to *factual reality (relativistic mass)* via the "statu-nascendi" where the unfolding is conceptualized. The $\pm$ sign is meaningful with respect to the time-space of the present as the only distinguishing element between the "past" and the "future" in SR, since the magnitude itself giving the unfolding rate is always constant. This perspective turns SR into its head, because the interpretation of the speed of light as a constant magnitude specifying the rate of temporal unfolding, determined in turn, by the conversion of apeiron reality to factual reality, requires to take into account *all three portraits of reality*. The autogenetic perspective on the energy-mass relation in SR is also indicative of the conceptual shift involved in the transition from SR to GR.

Namely, it is the possibility of the *non-constancy of the rate of temporal unfolding* between the "past" and the "future" from the standpoint of the "time-space of the present," determined as previously, that constitutes the crucial difference from an autogenetic perspective. As it will be shown in the following section, this is not at odds with the standard spacetime metric curvature interpretation of gravity caused by uneven matter distributions. The difference is that the latter is the epiphenomenal spatialized appearance of a change in the rate of temporal unfolding between the "past" and the "future" from the standpoint of the "time-space of the present."

2.6 Autogenetic Perspective on General Relativity

According to Einstein's principle of equivalence, GR is reduced to SR in the infinitesimal vicinity of every point-event in spacetime. This is usually referred to as metric locality, which has to be distinguished from the notion of topological locality. GR is also a metric theory and topological considerations start to enter the scene only at the appearance of singularities where the spacetime event geometry breaks down. The major difference from SR in this respect is that the spacetime metric becomes variable from point to point in spacetime depending on the distribution of matter in its vicinity. Thus, the spacetime metric, and therefore, *the chrono-geometric relations are not constant as the case of SR but become variable*. In turn, the variability of the spacetime metric gives rise to the *observable spacetime curvature* through which Einstein's field equations are formulated. The important thing is that due to the variability of the metric a standard of comparison is required at each spacetime point. This is called the infinitesimal process of parallel transport (technically called a connection) involving small round trips around each point according to a prescribed rule of parallelism (usually referred to as the metric-compatibility of the connection). These round trips detect the change of orientation (called the metric anholonomy of the connection) due to local curvature associated with uneven matter distributions.

The challenge is again to think of GR from the perspective of a "temporal unfolding dimension" in analogy to the case of SR treated before. For simplicity, let us consider again the case of a "2-d spacetime" (involving one spatial and one spatialized temporal dimension) as a model, whence the arguments can be extended to the case of "4-d spacetime." In the infinitesimal vicinity of any spacetime point the metric can get the SR form, $dS^2 = dx^2 + (icdt)^2$, but in this case, this form is *not retained constantly* as we move from point to point.

Therefore, in the infinitesimal vicinity of a point-event at present (i.e., locally in a metric sense), we consider an "imaginary dimension" conceived as the epiphenomenal spatialized cross-sectional form of a genuine "temporal unfolding dimension" projected at the factual level. Again, since this "imaginary dimension" pertains to *the metrical chrono-geometric relations, and not to the chrono-topological ones (the latter become relevant only around singularities)*, the cone of unfolding is a "metrical light-cone." Consequently, as in the case of SR the "spirally unfolding temporal dimension" is *degenerate topologically*, in the sense that the "winding stairs of the spiral" are not distinguishable metrically, and thus the potential of multiple-connectivity between the "past" and the "future" is reduced only to the possibility of branching with respect to the "time-space of the present." The important subtlety in comparison to the SR case is that *the rate of unfolding is not constant between the "past" and the "future" with respect to the "time-space of the present."* As a consequence, if we consider the rooting at the same point-event in the "time-space of the present" of both the "past" and the "future" differing in orientation, due to the differing rates of unfolding, the light-cone structure may twist or tilt. At the "epiphenomenal spatialized imaginary dimensional level," which can be thought of as the "{*imaginarily spatialized time*}-{*real space*} *of the present rooted at the same point-event*, *this discrepancy in the temporal rate of unfolding between the "past" and the "future" appears as spacetime curvature*.

Conclusively, in the case of GR at the epiphenomenal level, *change of time amounts to change of phase, but the rate of change is not the same for both "past" and "future."* Equivalently, "past" and "future" are not differing only in orientation with respect to the rooting at a point-event in the present, *but they also differ in relative phase that epiphenomenally appears as local metric curvature*.

2.7 Autogenetic Perspective on Singularities, Quantization, Entanglement and the "ER = EPR" Correspondence

The limits of GR as a metrical theory arise when the epiphenomenal curvature blows up, i.e. at the appearance of "spacetime singularities." At singularities the smooth metrical spacetime structure breaks down and global topological changes may take place, like ER bridges or wormholes. From the perspective of a "temporal unfolding dimension" in this case, *the difference between "past" and "future" cannot be captured by the use of a single "imaginarily spatialized time dimension" adjoined to 3-d space*. More precisely, a relative phase difference with respect to an imaginary dimension, bearing the topology of the circle as previously, cannot account for the difference and connectivity between the "past" and the "future" at a singular point. Singularities open up *the multiple connectivity possibilities* from the perspective of genuine temporal unfolding at the "statu-nascendi" level, and thus, pertain to chrono-topological relations in contradistinction to chrono-geometrical ones.

In order that these multiple-connectivity possibilities can take place, giving rise to a different higher type of "temporal bonds," there are two inter-related conditions that need to be fulfilled:

First, a "spirally unfolding temporal dimension" may be characterized by a more elaborate type of cross-sectional projection in the "time-space of the present," in the sense that *change of time at the spatialized epiphenomenal level does not correspond to change of phase with respect to a single imaginary dimension, but corresponds to change of circle*. This happens when the "past" and the "future" do not differ merely by a change in the rate of unfolding, which can be realized as a relative phase difference within the same "imaginary dimension," but require *complementary or conjugate "imaginary dimensions" in the "time-space of the present." In this case, change of time at the epiphenomenal level requires an appropriate process of circle change, which can be interpreted as a higher-order connectivity or "temporal bond."*

Second, the potential of multiple-connectivity between the "past" and the "future" with respect to the "time-space of the present" can be actualized only if the "spirally unfolding temporal dimension" is *non-degenerate topologically*, in the sense that the "different winding stairs of the spiral" can be *distinguished spectrally*.

We conclude that none of the above two conditions are fulfilled at the metrical spacetime event-level of GR. In this manner, the issue of spacetime singularities forces the *transition from the factual to the statu-nascendi level* without the metrical resource provided by an "imaginary spatialized time dimension" *adjoined* to 3-d space, since the metric breaks down at singularities. Therefore, chrono-topological relations become prevalent necessitating the *spectral distinguishability* of the winding stairs of an unfolding temporal dimension by means of *quantization*. In the simplest case, upon quantization, the winding stairs become distinguishable spectrally by means of the discrete algebraic structure of the integers.

The subtlety is now that the *inverse transition from the statu-nascendi to the factual level* does not happen in an unqualified manner, but requires measurement processes of quantum observables, not all of which are simultaneously compatible with respect to the "time-space of the present." From the viewpoint of the previous analysis, instead of an "imaginary spatialized time dimension" adjoined to 3-d space metrically, what is required is *a multiplicity of non-simultaneously applicable "contextual imaginary dimensions" adjoined non-metrically to 3-d space* (i.e., not as additional spatialized time dimensions) via spectral orthonormal bases (or equivalently, spectral frames of projection operators) for the measurement of observables. These "contextual imaginary dimensions" are in the relation of *parataxis* with respect to each other. Each one of them instantiates the demarcation of a non-metrical locality (i.e., a locality not based on the notion of distance) in the "time-space of the present." *It is precisely this independence from spatial proximity and distance that allows the emergence of syntaxis and cohesion at a higher connectivity level, i.e. the formation of "temporal bonds."*

Thus, upon entering the quantum domain of discourse for dealing with the chrono-topological relations pertaining to the singularities of GR in the transition from the metricized event spacetime to the statu-nascendi, the inverse transition can only take place *locally* or *contextually* by means of an arsenal of non-simultaneously applicable spectral frames for measurement.

The main claim in this interpretational framework of the autogenetic theory is that *singularities open up multiple connectivity interfaces between the "past" and the "future" at the "time-space of the present" in "statu-nascendi."* Since the realization of such a temporal connectivity interface becomes effective only on the condition of topological non-degeneracy of the genuine temporal unfolding, and therefore upon quantization according to the preceding, it can take place by the non-metrical adjunction of “contextual imaginary dimensions” to 3-d space, i.e. *the adjunction of local spectral frames* at the “time-space of the present.”

Note that the notion of a “contextual imaginary dimension” now is not playing the role of an “imaginary spatialized time dimension,” but plays the role of an “event horizon,” since the transition from “statu-nascendi” to the factual happens always only via a spectral frame of measurement. In a nutshell, what appears as a singularity at the metrical level of 4-d spacetime, forcing the transition to the “statu-nascendi,” where quantization is invoked to account for the pertinent chrono-topological relations, requires *the instantiation of an “event horizon” via the adjunction of a “contextual imaginary dimension”* to facilitate the inverse transition from the “statu-nascendi” to the factual level.

Following the understanding of a “contextual imaginary dimension” via the notion of an “event horizon,” it is important to examine now how two singularities can open up a “higher connectivity interface” between the “past” and the “future” at the “time-space of the present” in “statu-nascendi.” A necessary condition for such a type of “connectivity interface,” non-dependent on metrical proximity, is that *the “two induced contextual imaginary dimensions” of the singularities are “relationally conjugate” in the “time-space of the present,” so that they can be cohesively glued together not in absolute pair-wise fashion, but only in modular relation to the “present.”*

This modularity dependence on the “present” implies that the corresponding “event horizons” can be amalgamated homologically in relation to the “present.” In chrono-topological terms this type of “modular gluing” pertaining to the “present” (in the “time-space of the present”) can be instantiated by means of a “holographic boundary” adjoined to 3-d space at “present,” demarcating the “imaginary oriented surface of cohesion” of the two corresponding “contextual imaginary dimensions.” *It must be emphasized that the compatible fusion of the pertaining “contextual imaginary dimensions” does not happen in spacetime, but refers to their modular amalgamation with and with respect to the “present” in the “time-space of the present.”* Taking into account the association of the former with quantum theoretical spectral “event horizons” at the “statu-nascendi” level, it becomes transparent that the “modular gluing” of these event horizons pertaining to the “present” is precisely a process of quantum entanglement. In this manner, the “holographic cohesive boundary” adjoined to 3-d space at “present” by this “modular gluing” constitutes the *topological manifestation* of quantum entanglement.

Put equivalently, from an inverse viewpoint, quantum entanglement is the expression of *modular amalgamation with and with respect to the “present”* of two “relationally conjugate event horizons” (in the “time-space of the present” and independently of any metrical proximity) in the form of a “holographic boundary” adjoined to 3-d space at “present.” The crucial point here is that this “holographic boundary” can function as a “higher connectivity interface” between the “past” and the “future” with respect to their modular relation to the “present,” if and only if it is *oriented*. It is precisely the orientation on the so demarcated “imaginary boundary surface at present,” adjoined to 3-d space, that makes it a “temporally synectic boundary” or a “holographic boundary of cohesion” between the “past” and the "future" in their "modular gluing" capacity to the "present." In the context of GR, reminding the association of singularities with the opening up of "contextual imaginary dimensions" to 3-d space, the capacity of "modular gluing" of two "relationally conjugate singularities," e.g. of a "black hole" with a "white hole," according to the preceding, is interpreted at the factual level of spacetime as a *wormhole* that is impossible to pass through (non-traversable wormhole). Therefore, we obtain a conceptual grasp of the meaning of the "ER = EPR" correspondence from an autogenetic standpoint, which in the context of its initial conception and formulation is a conjecture pertaining to the quantum-gravity theoretical domain.

2.8 Syntaxis and Cohesion of Temporal Unfolding at the Time-Space of the Present

The major objective of grasping conceptually this correspondence is not only to demonstrate the potency of the implications associated with the notion of a "genuine unfolding temporal dimension" understood autogenetically, but also to pave the way for applying this framework to a novel theory of thinking, in particular, to a novel approach to "decision making." For this reason, it is worth attempting to transfer these notions metaphorically in the field of "decision making" taking place at the "time-space of the present."

The conceptual grasp of the autogenetic notion of a "genuine temporal unfolding dimension" via the algebraic-topological theory of "covering schemes," together with the crystallization of the idea that a "spirally or helically unfolding temporal dimension" in the "time-space of the present" always gives rise, either, to an epiphenomenal spatialized-time imaginary dimension at the metrical level, or, to an arsenal of non-simultaneously applicable contextual imaginary dimensions at the non-metrical level, provides an *optimal starting point* for this application. The abstraction required to perform the metaphor properly is based, on the one hand, in the preservation of the distinction among the *three chrono-ontological formats of reality*, and on the other hand, in the appropriate utilization of the notion of an "imaginary dimension" metrically or non-metrically, i.e. as a means of getting adjoined to 3-d space and induce observable effects at the epiphenomenal level.

In the course of this problematics, we realize that the "backbones" of the crucial ideas pertaining to SR, GR, and QG (quantum gravity), from the unifying autogenetic perspective of a "genuine temporal unfolding dimension," refer to particular constraints imposed on "imaginary dimensions" at the "time-space of the present." In the first two cases, the constraints are of a metrical kind, whereas in the latter case, the constraint is of a topological kind that forces the necessity of quantization. To be more precise, the important idea is always to consider *a cross-sectional projection of a "spirally or helically unfolding temporal dimension" in the "time-space of the present," according to a metrical constraint (being constant as in SR or variable as in GR) or a topological constraint*. Then, *this constraint induces* *observable effects at the spatialized epiphenomenal level*, depending on what meaning is conveyed to the notion of "change of time" with respect to the "time-space of the present." This notion of "change of time" is fundamental, because it pertains to the connectivity between the "past" and the "future" from the standpoint of the "present." What has been shown using the notion of applicable "imaginary dimensions" arising through the pertinent constraints are the following:

(α) "Change of time" in SR amounts to "change of phase," and this is the same for both the "past" and the "future" differing only in orientation with respect to the rooting at a point-event in the present. At the epiphenomenal spatialized level this induces the non-trivial observable effect of "length contraction" in the direction of motion;

(β) "Change of time" in GR amounts to "change of phase," but the rate of change is not the same for both the "past" and the "future." Equivalently, "past" and "future" are not differing only in orientation with respect to the rooting at a point-event in the present, but they also differ in "relative phase." At the epiphenomenal spatialized level this induces the non-trivial observable effect of "local metric curvature" associated with some "matter source," and thus, geometrizes the effect of gravity;

(γ) "Change of time" in QG does not amount to "change of phase" with respect to a single imaginary dimension, but amounts to "change of circle" with respect to two complementary imaginary dimensions in connection with the "present." This is the case because the "past" and the "future" do not differ merely by a change in the rate of unfolding, which can be realized as a relative phase difference within the same "imaginary dimension," but require "relationally conjugate contextual imaginary dimensions" in the "time-space of the present." In this case, due to the capacity of "multiple-connectivity" between the "past" and the "future" with respect to the "present," "change of time" amounts to a "synectic circle change" instantiated by the novel conceptualized process of "modular gluing" with and with respect to the "present." At the epiphenomenal spatialized level, this induces the non-trivial observable effect of "quantum entanglement" taking place at a "holographic boundary of cohesion" adjoined to 3-d space at "present." From then on, in order to distinguish the metrical from the topological semantics of an "imaginary dimension" we will refer to the QG-type of "change of time" as a "synectic cycle change."

The aim of recapitulating the above differences among SR, GR, and QG, from the unifying perspective pertaining to the *distinctive applied notions of "change of time" via the adjunction of "imaginary dimensions" to 3-d space at the "time-space of the present"* is the underlying realization that these notions can be transferred outside the strict technical contexts of these theories by *abstracting the content of the relevant constraints*. For instance, in the simplest case of SR, the constraint emanates from the constancy of the speed of light, in its function as a universal metrical factor for spatializing time in a projected cross-section of the temporal unfolding in the "time-space of the present" in terms of a metrical imaginary dimension. Bringing into mind that the speed of light demarcates the upper bound in the propagation of electromagnetic signals, there is *nothing that prevents the applicability of the SR-analogous notion of metrical imaginary dimension, adjoined to 3-d space, by another type of constant spatializing time by the demarcation of another upper bound pertaining to a different sort of propagation*. The crucial thing here is that the latter type of upper bound as a metrical constraint would also amount to "change of time" as a "change of phase" (true, with a different period or frequency) in analogy to the SR-case, and most important, would induce the non-trivial observable effect of "length contraction" in the direction of propagation at the epiphenomenal level.

To be more concrete, consider the exemplary case of propagation of an army in the battlefield. Initially, the notion of an armored vehicle was conceived as a means of protecting the infantry following it. The strategic transmutation of this conception into a unit of armored vehicles moving independently of the infantry amounts to *a change in the syntaxis of time* in the battlefield. This is because the speed of propagation is altered by the upper bound set by the unit of armored vehicles, and consequently, at the epiphenomenal level, "change of time" amounts to "change of phase" in the battlefield, caused by the adjunction of the SR-type of "spatialized time imaginary dimension" to 3-d space (as a metricized constraint of upper bound in the speed of propagation). Again, the observable effect at the epiphenomenal spatial level, caused by this decision, is the "length contraction" in the direction of motion of the army in the battlefield. In the same stream of ideas, it is clear how GR-type of observable effects appear in the battlefield, i.e. effects of local curvature analogous to the gravitational ones, in the strategic decision of "metallaxis" of a cavalry unit in the "past" into an "armored unit" in the "future" that took place at the "time-space of some pertinent present." "Change of time" in both cases amounts to "change of phase" in the battlefield, but the rate of unfolding is different with reference to the "cavalry unit" in the "past" in comparison to the "armored unit" in the "future." This is qualified as a "relative phase" in the "variable metric-spatialized imaginary time dimension" adjoined to 3-d space, that epiphenomenally results in a local metric curvature effect in the battlefield, i.e. a bending or twisting the battlefield in analogy to the "geometrized gravitational effect."

We will focus our attention now to the QG-type of "change of time," i.e. to what is called a "synectic cycle change" and scrutinize in detail its implications for "decision making" in the "time-space of the present." The claim is that this type of "change of time" bears a particular significance in relation to our reading and historic understanding of "international treaties" for instance, as well as for guiding "high strategy decision making" when the different historical stages of a genuine temporal unfolding are not suppressed or eradicated, but on the contrary, are utilized as a resource for higher types of connectivity interfaces between the "past" and the "future", not visible and not comprehensible from the factual portrait of reality. In the course of this synthesis, it will be attempted to delineate the major constraints required for the realization of this type of "change of time" as a "synectic cycle change," i.e. as instantiating a "higher connectivity interface or cohesion" between the "past" and the "future" with and with respect to the "present," and not as destroying any type of connectivity that a "random cycle change" would result to. This is an important difference and epitomizes the notion of historicity as a process of unfolding via "temporal chords" that bear the capacity to resonate at "present" with "elicited seeds," directed from the "past," through "memory," "tradition" and "value," but also from the "future," through "anticipation," "innovation," "expectation," "insight" and "vision."

2.9 Change of Time as a Synectic Cycle Change

From the standpoint of the "time-space of the present" in the context of the autogenetic theory, the *first constraint* for comprehending "change of time" as a "synectic cycle change" is the non-annihilation of the "past" and the non-repulsion of the "future," both conceived in "seed-like-form" in their capacity to form a "temporal bond" with and with respect to the "present." The notion of a "temporal bond" is a fundamental one for this purpose and is characterized by the following premises:

1. A "temporal bond" is *not conditioned by relations of metrical proximity* of the "elicited seeds" from the "past" and the "future" at "present" in the "time-space of the present";
2. A "temporal bond" between the "elicited seeds" from the "past" and the "future" always bears a *modular relation with and with respect to the "present,"* i.e. it is not tantamount to a "pair-wise gluing" of the "past" with the "future," but to a "modular gluing" in relation to "the present" and together with the "present";
3. A "temporal bond" induces a "synectic cycle change" if and only if the pertinent "elicited seeds" from the "past" and the "future," in their capacity to get glued together in a "modular manner" with and with respect to the "present," are both "relatively prime" with respect to the "present," i.e. not analyzable and not localizable to any other common factors with respect to the "present";
4. A "temporal bond" as a "synthetic unit" modulo the "present," characterized by the quality that the "elicited seeds" from the "past" and the "future" amenable to amalgamation, are both "relatively prime" in relation to the "present," defines a *division scheme of the temporal unfolding according to this unit*, which, in turn, specifies the "syntaxis of time change" at the epiphenomenal spatial level in the form of a "synectic cycle change";
5. A "temporal bond" is a bond of "least action" in the "time-space of the present." This is because it constitutes an inseparable tripartite correlation, which cannot be analyzed to any pairwise correlations.

It is especially worth to highlight the quality of "relative primeness" with respect to the "present," characterizing non-metrically proximal seeds from the "past" and "the future" in their capacity to establish a "modular gluing relation" with and with respect to the "present," if and only if they enter into a "temporal bond." The notion of being a "relatively prime" is analogous to the corresponding notion in *integer modular algebra* conceived first by Gauss, where the notion of an "absolute integer prime" is relativized with respect to *a modulus*. The significance of this generalization in the case of integer modular systems is that any integer can assume the role of a prime only in relation to another one acting as a modulus. The idea here is that the quality of "relative primeness" is crucial for the realization of a "temporal bond" inducing, according to the preceding, a "synectic cycle change" from the "past" to the "future" in the "time-space of the present." Intuitively, the underlying conception is that "relatively to the present," a "seed" from the "past" becomes "spectrally spontaneously visible" and thus, "spontaneously elicited" in "the present," without being factorizable through anything else. The same symmetrically holds for a seed in the "future" in its capacity to enter into a "temporal bond" with a seed from the "past" in a "modular way" with and with respect to the "present."

A natural issue arising in this setting pertains to the explication of the roots on which the analogy with integer modular algebra is based on in the present case. The indirect resolution of this issue comes from the *second constraint* required for the realization of this type of "change of time" as a "synectic cycle change" effected by a "temporal bond." It is instructive to remind that the realization of such a temporal connectivity interface becomes effective only on the condition of *topological non-degeneracy* of the genuine temporal unfolding between the "past" and the "future" at the "time-space of the present" in "statu-nascendi." Equivalently, the potential of multiple-connectivity between the "past" and the "future" in the form of "seeds" entering into a "temporal bond" with respect to the "time-space of the present" can be actualized only if a "spirally unfolding temporal dimension" is non-degenerate topologically, in the sense that the "different winding stairs of the spiral" can be distinguished spectrally.

Physically, the *spectral distinguishability* of the winding stairs of an unfolding temporal dimension takes place by means of *quantization*. More precisely, upon quantization, the winding stairs become distinguishable spectrally by means of the *group of the integers*, which physically count "quanta of action." Notice that this constitutes a form of absolute distinguishability. According to the fundamental quality characterizing a "temporal bond," spectral distinguishability should be always *relativized* with respect to the pertinent "present" in the "time-space of this present," and not be considered in absolute terms. From the autogenetic perspective, a "spirally unfolding temporal dimension" may unfold *outwards, inwards, and multi-directionally*. Most important, it can be *subdivided* according to the "synthetic unit" established by the formation of a "temporal bond" *modulo* the "present." Thus, the subdivision property, considered together with the quality of "relative primeness" with respect to the "present," characterizing seeds from the "past" and "the future" entering into a "temporal bond," leads to the conclusion that spectral distinguishability relativized with respect to the pertinent "present" takes place in the fashion of modular integer algebra, i.e. by the *residue modular system* determined by "relative primeness" with respect to the "present" playing the role of the "modulus."

The most important consequence of "relative primeness" in this respect is that the pertinent seeds of the "past" and the "future" entering into a "temporal bond," as described previously, become "relationally inverse with respect to the present" and "relationally conjugate with respect to each other" in the "time-space of the present."

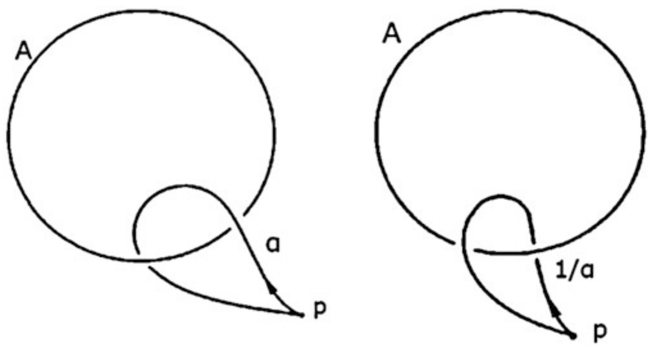
Is there any way to visualize these relations at the epiphenomenal spatial level referring to the "present"? For this purpose, we remind that we have to utilize the device of "imaginary dimensions." More precisely, we have to consider some seed from the "past" and some seed in the "future" (in their capacity to enter into a "temporal bond" at "present") in their respective contexts of two non-simultaneously applicable "imaginary dimensions" adjoined non-metrically to 3-d space. In this manner, a seed from the "past" with a seed in the "future" entering into a "temporal bond" at present, and thus being "relationally conjugate with respect to each other" due to "relative primeness" at "present," can be visualized in terms of the corresponding "contextual imaginary dimensions" being *transverse*, and thus *complementary* at "present." Then, their "modular gluing" with respect to the "present," upon establishment of the "temporal bond," gives rise to a "holographic boundary" adjoined to 3-d space at "present." This "temporal synectic boundary" of cohesion of the "past" with the "future" at "present" demarcates the "imaginary oriented surface of cohesion" of those "contextual imaginary dimensions."

What is required for understanding more deeply this "holographic boundary of cohesion" is to describe and visualize the action of "eliciting seeds" from the "past" and the "future" at "present" in view of their power or capacity to enter into a "temporal bond" in the "time-space of the present." Since a "temporal bond" is tantamount to gluing the pertinent "eliciting seeds" from the "past" and the "future" in a "modular manner" with and with respect to the "present," "the present" should be thought of as an "Archimedean fulcrum" relative to these seeds, or more precisely, relative to their respective "contextual imaginary dimensions" in the "time-space of the present." It is important to keep in mind that these "contextual imaginary dimensions" of the "eliciting seeds" from the "past" and the "future" should be thought *topologically* as *cycles*.

The quality of being "relatively prime" with respect to the "present," characterizing "eliciting seeds" from the "past" and the "future" entering into a "temporal bond" is *the key* for the sought for topological representation, set up as a task for enhancing our understanding, in the preceding paragraph. First, it implies that the corresponding "contextual imaginary dimensions" are *non-mutually inclusive and transverse*, thus, complementary with respect to the fulcrum. Second, it implies that seeds from the "past" and the "future" become "eliciting seeds" in their power to enter into a temporal bond at "present" only if they can be "leveraged to the present" relationally to each other with respect to the "fulcrum."

2.10 Genuine Novelty: Relative Primeness as the Key for Synectic Cycle Change

It is significant to amplify the implications derived in the previous section in order to establish the sought-after topological representation referring to a "synectic cycle change" via the notion of a "temporal bond." For this purpose, it is instructive to start from the clear intuitive idea that the quality of "relative primeness" with respect to the "present" means that a pertinent "seed" from the "past" or the "future" becomes "spectrally spontaneously visible" from the "present" in the "time-space of the present." Therefore, if we consider a seed either in the "past" or in the "future" in the context of its "imaginary dimension," it becomes "spectrally spontaneously visible" from the fulcrum, i.e. *not factorizable* through any other simpler common factor, by means of a *loop (simple tame closed curve) that is based at the fulcrum*, i.e. it starts and ends at the fulcrum, and passes through the cycle (non-metrical, deformable circle) representing spatially the corresponding "imaginary dimension." More precisely, since we refer to a seed, it is appropriate to consider the whole equivalence class of such loops that can be continuously deformed to each other. It is enough to make visible a single representative of this class, which is recognized reflectively, by means of a based loop at the fulcrum as previously. We may think of it as a "reflexive recognition principle" (relatively to the fulcrum) of a seed in the "past" in its *power to enter into a temporal bond* with a seed in the "future" (or the other way round) at "present." If we denote the relevant cycle by A, then it is important to notice that a based loop at the fulcrum passing through A may admit *two distinct orientations*: If the loop passes through the cycle A with direction away from the fulcrum it is denoted by α, whereas if it passes with direction toward the fulcrum it is denoted by α^{-1}. Thus, in a 2-d spatial representation a seed can be recognized reflexively by means of *a "cycle crossing" directed away from the fulcrum* $(+)$*, or directed toward the fulcrum* $(-)$.



After having established the "reflexive recognition principle" of a seed relative to the fulcrum, in terms of a loop based at the fulcrum and crossing the corresponding cycle with a $(+)$ or $(-)$ orientation, we need to examine the meaning of "relative primeness" in this representation setting.

"Relative primeness" is a quality characterizing both a seed from the "past" and a seed from the "future" in relation to the "present," upon entering into a "temporal bond" at "present." In this case, the pertinent seeds from the "past" and the "future" are qualified as "elicited seeds" at "present," meaning that they are "leveraged to the present" relationally to each other with respect to the "fulcrum." In other words, reflexive recognition of a seed in the "past" or in the "future" in their power to enter into a "temporal bond" is not enough for the establishment of the bond. What is required additionally is "leveraging" these seeds to the "present" using the quality of "relative primeness," so that they become "elicited seeds" at present. The idea is that an "elicited seed" from the "past" can be *fused together* with an "elicited seed" from the "future" with respect to the "present"-"fulcrum," so that a "temporal bond" is formed with the "present."

It is worth elaborating in more detail how "temporal leveraging" takes place with respect to the fulcrum. This introduces the novel concept of a "temporal chord" at "present" via which the expression of a "temporal bond" becomes explicit.

"Temporal leveraging" of reflexively recognized seeds with respect to the fulcrum is enunciated by decoding their quality of being "relatively prime" with respect to the fulcrum. This means that they are *(i) "relationally inverse with respect to the fulcrum," and (ii) "relationally conjugate with respect to each other" in the "time-space of the present."* Consider a recognized seed from the "past," identified either with the based oriented loop at the fulcrum, $\alpha^{+1} := \alpha$, or with α^{-1}, by means of "cycle A crossing," depending on the orientation. Analogously, consider a recognized seed in the "future," identified either with the based oriented loop at the fulcrum, β^{-1}, or with β, by means of "cycle B crossing," depending again on the orientation. For instance, if α and β^{-1} are recognized, they both become "elicited seeds" at "present" by "temporal leveraging" with respect to the fulcrum. Being relationally conjugate with respect to each other means that β and β^{-1} play the role of bidirectional bridges for the leveraging of α, and also that α and α^{-1} play the role of bidirectional bridges for the leveraging of β^{-1}. "Elicited seeds" give rise to "temporal chords" at "present." Equivalently, a "temporal chord" at "present" is formed by interpolating a recognized seed from the "past," for example, α, between the bridges β and β^{-1}, i.e. by "temporal leveraging" the seed α with respect to the fulcrum, e.g. $\beta\alpha\beta^{-1}$.

The significance of "temporal chords" at "present" is that they can form "resonances." More precisely, a "temporal chord" from a recognized seed in the "past" can be "fused" together with a "temporal chord" from a recognized seed in the "future" by forming a "resonance" at "present." *This "fusion" of "temporal chords" resulting into a "resonance" at "present" takes place if and only if a "new cycle" is instantiated at "present" gluing together the cycles A and B in a non-pairwise manner.*

2.11 Autogenetic Fusion and Synectic Cycle Change: A Temporal Bond Links in the Borromean Topology

The process of modular gluing epitomizing the establishment of a "temporal bond" and giving rise to a "synectic cycle change" can be described in detail as follows:

Consider a recognized seed from the "past," identified with the based oriented loop at the fulcrum, α, by means of "cycle A crossing" in the prescribed orientation, and analogously, a recognized seed in the "future," identified with the based oriented loop at the fulcrum, β, by means of "cycle B crossing," in the prescribed orientation as well. These oriented fulcrum-based loops can be *composed*, either in the order $\alpha\beta$ or in the order $\beta\alpha$, and these compositions are *non-commutative*. Let's consider the composition in the order $\alpha\beta$. The first objective is to extend this composition in consecutive stages so as to form "temporal chords" at the fulcrum. If we adjoin by composition α^{-1} to $\alpha\beta$, we obtain the "temporal chord" $\alpha\beta\alpha^{-1}$, which amounts to leveraging β with respect to the fulcrum, utilizing the bridges α and α^{-1} for this recognized seed in the future. Next, we adjoin β^{-1} to the "temporal chord" $\alpha\beta\alpha^{-1}$, to obtain $\alpha\beta\alpha^{-1}\beta^{-1}$, which can be read either as the composition of the "temporal chord" $\alpha\beta\alpha^{-1}$ with β^{-1} or as the composition of α with the "temporal chord" $\beta\alpha^{-1}\beta^{-1}$, due to the associativity property of non-commutative composition. In this way, continuing the process of adjoining as above, i.e. keep leveraging with respect to the fulcrum, the second, and most important objective is to generate a cycle based at the fulcrum. A cycle of this form is generated when the leveraging process ends with the composition $\alpha\beta$ that has been utilized at the initial stage. A cycle based at the fulcrum is generated by the resonance of a "temporal chord" from a recognized seed in the "past" with a "temporal chord" from a recognized seed in the "future" as follows:

$$\alpha\beta \to \alpha\beta\alpha^{-1} \to \alpha\beta\alpha^{-1}\beta^{-1} \to \alpha\beta\alpha^{-1}\beta^{-1}\alpha \to \alpha\beta\alpha^{-1}\beta^{-1}\alpha\beta.$$

In the above process, the cycle generated is given by

$$C = \alpha\beta\alpha^{-1}\beta^{-1},$$

since starting from the ordered non-commutative composition $\alpha\beta$ and leveraging with respect to the fulcrum, we arrived at:

$$(\alpha\beta\alpha^{-1}\beta^{-1})(\alpha\beta) = C(\alpha\beta).$$

The cycle

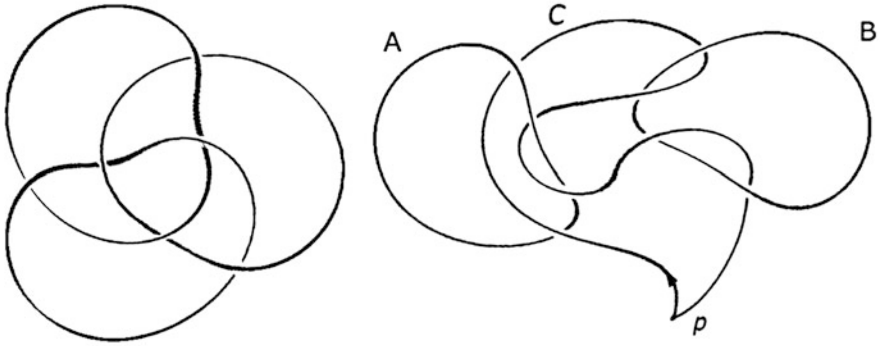
$$C = (\alpha\beta\alpha^{-1}\beta^{-1}) := [\alpha, \beta],$$

i.e. the commutator of α and β, is generated by the resonance of the “temporal chord” $\alpha\beta\alpha^{-1}$ with the “temporal chord” $\beta\alpha^{-1}\beta^{-1}$ leading to their autogenetic fusion by means of the cycle $C = (\alpha\beta\alpha^{-1}\beta^{-1})$, at the fulcrum. This provides a physical visual interpretation of the “novel cycle” C based at the fulcrum, which effects the “modular gluing” of the cycles A and B.

In other words, it provides the visual representation of the abstract algebraic process of “modular gluing” of the “past” and the “future” with the “present” and with respect to the “present,” which we have described in the preceding. Visually, the cycle $C = (\alpha\beta\alpha^{-1}\beta^{-1}) := [\alpha, \beta]$ involves *four crossings* of the cycles A and B, namely two of cycle A and two of cycle B, with *opposite orientations and in an alternating order*. Notice that the formation of a cycle of the form C, i.e. of a resonance pertaining to a “temporal chord” from a recognized seed in the “past” with a “temporal chord” from a recognized seed in the “future” does not depend on what we consider as an initial composition, like $(\alpha\beta)$ in the case we presented. If we consider any other initial composition from all possible ones, we will again arrive at a cycle of the form C, i.e. to a resonance of a “temporal chord” from the “past” with a “temporal chord” in the “future” with respect to the fulcrum. In other terms, the formation of a cycle of the form C gluing A and B modularly with respect to the fulcrum is the *invariant* of “resonance” between a “temporal chord” from the “past” with a “temporal chord” in the “future.” *It is precisely this invariant that characterizes “change of time” as a “synectic cycle change”* in this case.

The significant thing to highlight is that *a “temporal bond” induces a particular type of topological linking of the cycles A, B, and C, which is described by the “Borromean rings” topology*. Equivalently, “synectic cycle change” is tantamount to the “Borromean topological link” of the cycles *A, B, and C at the epiphenomenal spatial level*, as a consequence of the autogenetic analysis. This means that if any one of the cycles is removed from the “Borromean link” the remaining two come completely apart, and leads us to a complete understanding that this type of “change of time,” according to the above, as a “cycle change” which is synectic.

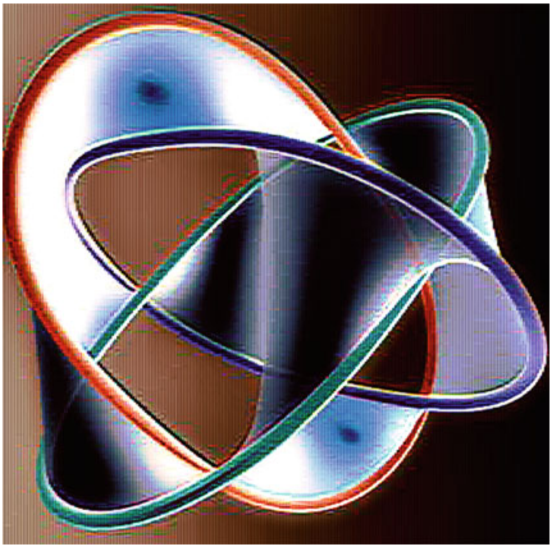
It is enough to state briefly here that the algebraic representation of the "Borromean rings," which is developed in full detail in the next chapter, a cycle of the form $C = (\alpha\beta\alpha^{-1}\beta^{-1}) := [\alpha, \beta]$ based at the fulcrum, and involving four crossings of the cycles A and B, encodes algebraically the modular gluing condition of this non-splittable 3-link as well as the complete splittability of all 2-sublinks due to the absence of any pair-wise gluing.



2.12 Minimal Surface of Cohesion in the Time-Space of the Present

The representation of a "temporal bond" in the form of the "Borromean rings" is of the utmost significance for strategic decision theory, because it enables us to visualize the process of "synectic cycle change," effected by the "modular gluing" of a seed from the "past" with a seed in the "future" with respect to the "present," upon establishment of this "temporal bond." More precisely, it provides the *topological means to elucidate how a "holographic boundary of cohesion" is adjoined to 3-d space at "present" as the epiphenomenal spatial reflection of a "temporal bond."* This "synectic boundary" connecting holographically a seed from the "past" with a seed in the "future" at "present," independently of their proximal distance, demarcates an "imaginary oriented compact and connected surface of cohesion" in 3-d space at "present," which is adjoined to it as a "holographic boundary" at "present."

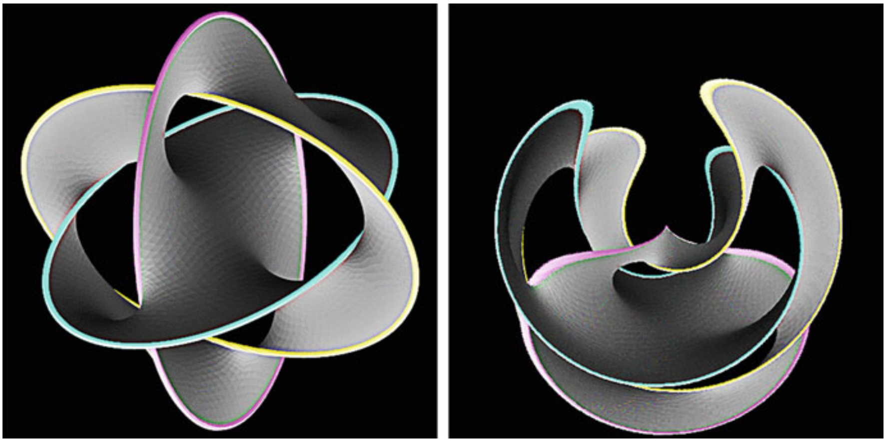
The adjunction of this "holographic boundary of cohesion" to 3-d space at the epiphenomenal spatial level of the "present" takes place as follows: We consider the compact, connected, and oriented surface with boundary the "Borromean rings." This surface is visualized as follows:

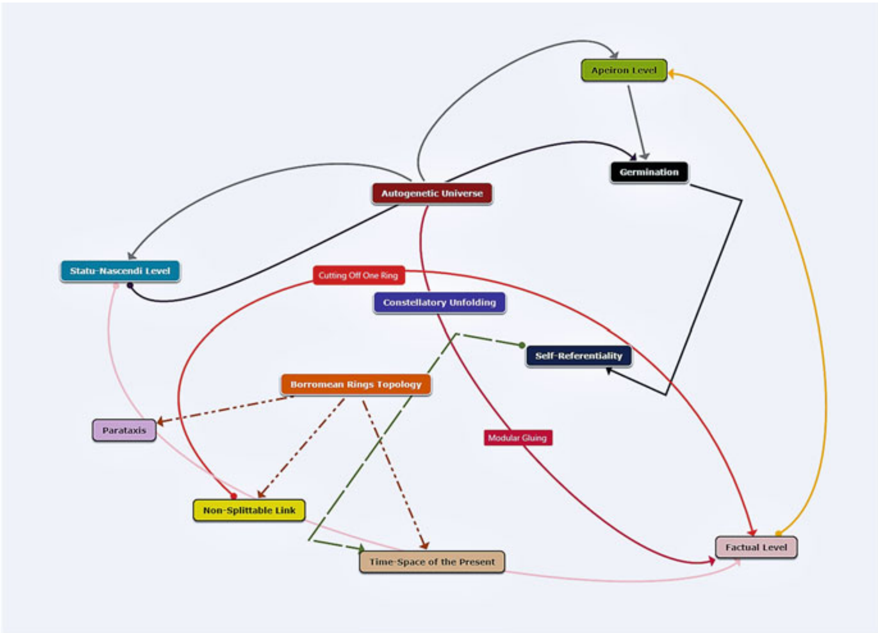


Thus, the "imaginary surface of cohesion" at the epiphenomenal spatial level is equivalent to a torus bearing three punctures (corresponding to the aphaeresis of three disks). This is a surface of genus one playing the role of the "holographic boundary of cohesion" adjoined to 3-d space in the complement of the three topologically linked cycles instantiating the "Borromean rings" at "present."

The significance of this "imaginary surface of cohesion" caused by a "temporal bond" is that physically it can be interpreted as a *"global curvature topological effect" in analogy to the "local curvature metrical effect" associated with gravity due to matter sources in the case of GR*. Not only this, but additionally, this "global curvature effect" is the "least-action solution" to any physical or strategic problem that requires a "higher connectivity interface" to glue modularly the "past" with the "future" at "present." How can we think of a simple way to visualize at the epiphenomenal spatial level the instantiation of a "temporal bond," implemented as a "least-action solution," and giving rise to such a "global curvature effect"?

The proposed visualization is to consider the "minimal surface" formed by a soap film, when three wire rings linked together as the "Borromean rings" are immersed into a solution of soapy water and then taken out. This surface is a "least-action" solution to the shape that a soap film acquires in this case, since it minimizes the area. Interestingly enough, every point in this "surface of cohesion" is locally similar to a saddle, i.e. its local curved geometry is of the hyperbolic type, whereas its global topology is of the toroidal type.





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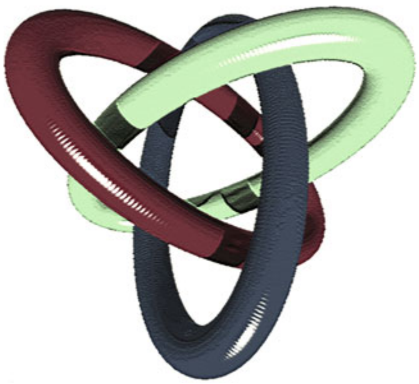
Chapter 3
Borromean Link in Algebraic Form Group-Theoretic Encoding: The Borromean Rings as Prime Connectivity Units of All Topological Links

3.1 The "Borromean Link" Topological Type

The term "Borromean rings" originates from their display at the coat of arms of the Borromeo family in Northern Italy. Mathematically speaking, the "Borromean rings" (Brunn 1892; Cromwell 1998) consist of three topological circles, which are linked together in such a way that each of the rings (topological circles) lies completely over one of the other two, and completely under the other, as it is shown at the picture below:



This particular type of topological linking displayed by the "Borromean rings" is called the "Borromean link," and is characterized by the following distinguishing property: If any one of the rings is removed from the "Borromean link" the remaining two come completely apart. It is important to emphasize that the rings should be thought of as *topological circles and not as perfectly circular geometric circles* (Brunn 1892; Cromwell 1998; Debrunner 1961; Hatcher 2002; Kawauchi

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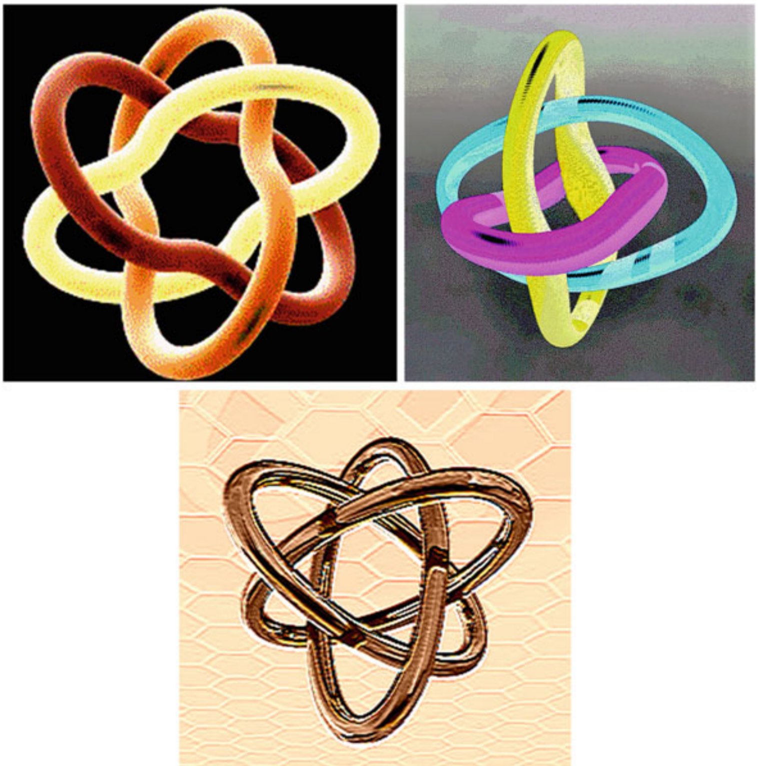
1996; Lindström and Zetterström 1991; Milnor 1954). The adjective topological means that they can be deformed continuously under the constraint that the particular type of linkage forming the Borromean configuration is preserved. Thus, a convenient way to imagine them is in terms of deformable elastic closed strings, which for ease of visualization can be considered as embedded in ordinary three-dimensional space.



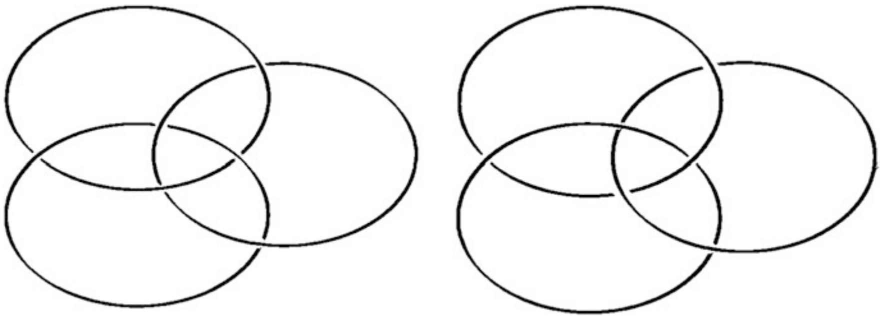
3.2 The "Borromean Link" in Terms of Loops

The notion of a deformable elastic closed string, which we use to model each of the three rings forming the "Borromean-link," is abstracted in topological terms by means of the concept of a tame closed curve or loop. Thus, topologically speaking the "Borromean-link" is considered as an interlocking family of three loops, such that if any one of them is cut, then the remaining two become completely unlinked. The modeling of the "Borromean-link" in terms of loops is important because it provides the possibility for an appropriate algebraic expression of the topological gluing conditions, which express the particular manifestation of the Borromean configuration. In this way, our objective is an algebraic encoding of the nature of interlocking of three loops constituting the "Borromean-link."

First, it is instructive to specify precisely the meaning of the notion of a tame closed curve or loop. The property of tameness means that the closed curves considered can be deformed continuously and without self-intersections into polygonal curves, viz. these ones formed by a finite collection of straight-line segments. Given this qualification, a loop is characterized by the following properties:

1. No point separates a loop, viz. no single scissors-cut can separate a loop into two pieces.
2. Each set of two points does separate a loop, viz. two scissors-cuts separates a loop into two pieces.
3. A loop is an one-dimensional object.
4. A loop is bounded, viz. it is contained in some sphere of sufficiently large radius.

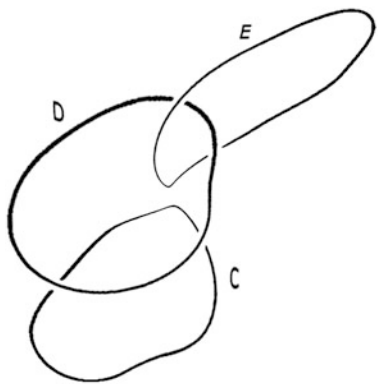
We stress the fact that all four of the above listed properties are essential for the characterization of the notion of a loop, thought of as a figure in three-dimensional space. Moreover, a loop is called knotted if it cannot be continuously deformed into a circle in three-dimensional space without self-intersection. Therefore, each one of the three interlocking rings of the "Borromean link" should be considered as an unknotted tame closed curve. We refer to them simply as loops keeping in mind that each one of them is unknotted. In terms of loops, the "Borromean link" is depicted as the configuration displayed on the left below, which is to be contrasted with a different type of configuration consisting of three interconnected loops displayed on the right.



The "Borromean link" configuration of loops on the left is such that if any of the loops is cut at a point and removed, then the remaining two loops become completely unlinked. In contradistinction, the configuration on the right is such that each loop actually links each of the other two.

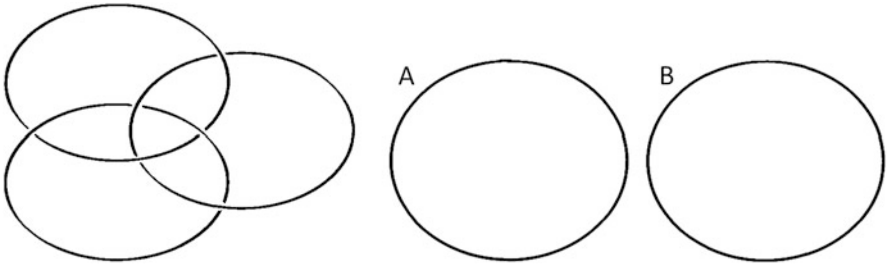
Up to now, the notion of a topological link has been used quite informally, so it is necessary to specify it in more precise terms, based on the underlying idea of connectivity among a collection of loops. Hence, we begin by defining an N-link as a collection of N loops in three-dimensional space, where N is a natural number. Regarding the connectivity of a collection of N loops, the crucial property is the property of splittability of the corresponding N-link. We say that an N-link is splittable if it can be deformed continuously in three-dimensional space, such that part of the link lies within B and the rest of the link lies within C, where B, C denote mutually exclusive solid spheres (balls) in three-dimensional space.

Intuitively, the property of splittability of an N-link means that the link can come at least partly apart without cutting. Complete splittability means that the link can come completely apart without cutting. On the other side, non-splittability means that not even one of the involved loops, or any pair of them, or any combination of them, can be separated from the rest without cutting. As an illustration, we consider the following 3-link:



The above 3-link consists of three loops, denoted by C, D, and E. Clearly, this is a splittable 3-link, which is not completely splittable. As it can be easily seen in the above figure, the loops D and E cannot be split apart without cutting. Notwithstanding this fact, it is a splittable 3-link because the loop C can be separated from the rest without cutting. Thus, the above 3-link can come at least partly apart, and therefore is splittable.

The property of splittability of a topological link as defined previously, is adequate to characterize completely the particular type of the "Borromean link." First, the "Borromean link" is a 3-link, since it is consists of three loops. Second, the connectivity of this 3-link in terms of the splittability property is formulated as follows: *The "Borromean link" is a non-splittable 3-link, such that every 2-sublink of this 3-link is completely splittable*. It is clear that it is a non-splittable 3-link because not even one of the three loops, or any pair of them, can be separated from the rest without cutting. A 2-sublink is simply any sub-collection of two loops obtained by erasing the loop that does not belong to this sub-collection. Since the "Borromean link" is characterized by the property that if we erase any one of the three interlocking loops, then the remaining two loops become unlinked, it is obvious that every 2-sublink of the initial 3-link is completely splittable, according to the figure below:

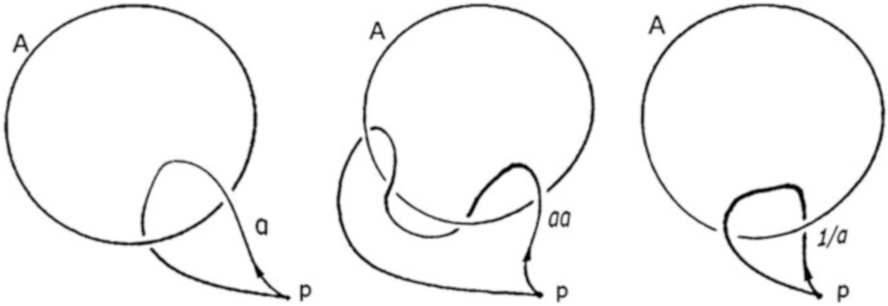


3.3 The Group Structure of Based Oriented Loops in 3-d Space

In the previous section we characterized the "Borromean link" in terms of a topological property, namely the splittability or not of a 3-link and its 2-sublinks, which describes completely the connectivity type of the Borromean configuration. In this section, we shift our perspective and seek for an algebraic structure that will be able to encode the above topological information. For this purpose we proceed as follows: First, we consider an unknotted tame closed curve in three-dimensional space. Since any such curve can be continuously deformed to a circle it is enough to think of a circle in three-dimensional space, denoted by A. Second, we consider a *based oriented loop* in three-dimensional space, which may pass through this circle a finite number of times, each one with a prescribed *orientation*. A based loop means simply that it starts and ends at a fixed point p of the three-dimensional space. The orientation of the loop can be thought of in terms of an observer, which is fixed at the point p, such that: If the loop passes through the circle one time with direction away from the observer, it is denoted by α^{+1}, whereas if it passes one time with direction toward the observer, it is denoted by α^{-1}.

Note that in the symbols of the generic algebraic form "χ" it is encoded the following information: First, the passage or not of a based loop through a circle A, which qualifies or not the naming of the loop by the corresponding symbol α. Second, the number of times that this based loop passes through the circle A, which is encoded as a power of the symbol α. Third, the orientation of the loop with respect to the fixed observer at the base point of the loop, which is encoded by a "$+$" sign if a passage through the circle takes place away from the observer and by a "$-$" sign if a passage takes place toward the observer.

We may illustrate schematically the above as follows:



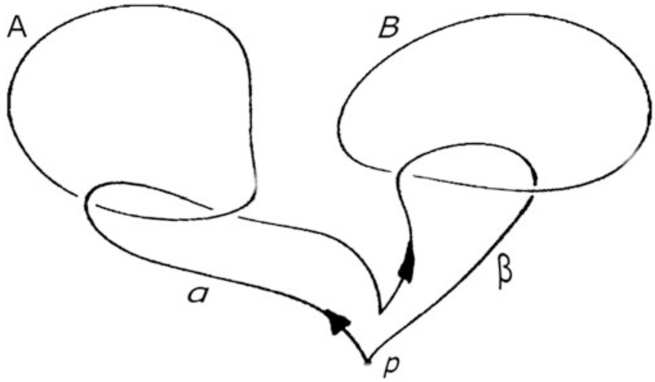
In the first figure from the left, it is depicted a loop in 3-d space, which is based at the point p, such that: It starts at the point p, then it passes through the circle A once directed away from the fixed observer at p, then it curves around the circle A, and finally it returns to the point p. According to the above, this loop in relation to the circle A should be denoted by α^{+1}, which we write simply as α.

An important observation is that any other loop with the same properties can be continuously deformed to the loop α. Thus, the algebraic symbol "α" should actually stand for the equivalence class $[\alpha]$ of all loops of the kind α, passing through the circle A once with the prescribed orientation. Any loop in the class $[\alpha]$ can be continuously deformed to an equivalent one in the same class. Under this understanding, we may still keep using the symbol α as above, where α is thought of as a representative of the whole equivalence class $[\alpha]$.

In the middle figure, it is depicted a loop in 3-d space, which is based at the point p, such that: It starts at the point p, then it passes through the circle A twice directed away from the fixed observer at p, then it curves around the circle A, and finally it returns to the point p. According to the above, this loop in relation to the circle A should be denoted by $\alpha^{+2} = \alpha \circ \alpha$, which we write simply as α^2.

In the last figure from the left, it is depicted a loop in 3-d space, which is based at the point p, such that: It starts at the point p, then it curves around the circle A, then it passes through the circle A once directed toward the fixed observer at p, and finally it returns to the point p. According to the above, this loop in relation to the circle A should be denoted by $\alpha^{-1} = 1/\alpha$.

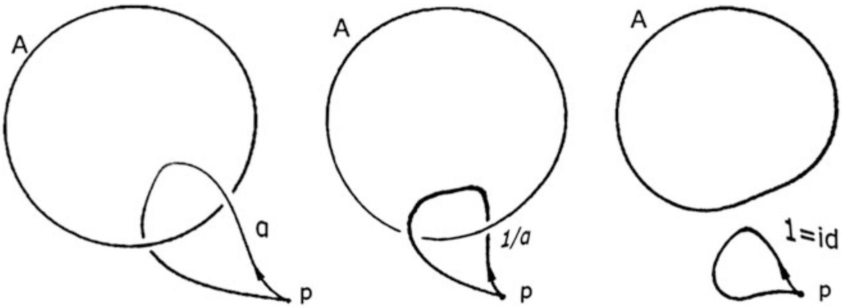
Taking for granted the algebraic encoding of based oriented loops in relation to circles in 3-d space, according to the above, we may proceed by thinking of an appropriate algebraic structure having the capacity to express symbolically these relations. The first step in this direction is to consider the possibility of composition of based oriented loops of the type "χ" in relation to circles of the type "X" in 3-d space. Clearly, the possibility of defining the composition of two loops of the above form is viable if both of the loops are based on the same point p. Then, the composed based oriented loop should be also a loop of the same form in relation to the two circles of the composing ones. We may illustrate the proposed composition rule schematically as follows:



In the above figure, we consider two based oriented loops, which are both based at the same point p, where as usual we imagine an observer fixed at this point. Taking into account the orientations, we denote the first loop by α (in relation to the circle A) and the second loop by β (in relation to the circle B). Then, we can define their composition denoted by $\alpha \circ \beta$ respecting the order of tracing the loops, viz. we first trace α, and then we trace β. Thus, the rule of composition produces a based oriented loop $\alpha \circ \beta$ in 3-d space in relation to the circles A and B, which is interpreted as follows: It starts at the point p, then it passes through the circle A once directed away from the fixed observer at p, then it passes through the circle B once directed away from the fixed observer at p, and finally it returns to the point p. We note that it is allowed to remove the end of α and the beginning of β from the base point p, and then join them together at a nearby point as it is illustrated in the above figure. We think of the composition rule $\alpha \circ \beta$ as the product of the oriented loops α and β based at the same point in 3-d space, which we may denote simply as $\alpha\beta$.

Now, using this product operation we wish to define a suitable algebraic structure, where this product would play the role of multiplication of the elements of this algebraic structure. A significant observation is that the sought algebraic structure cannot be a commutative one, since the multiplication operation of the elements is not a commutative operation, viz. $\alpha\beta \neq \beta\alpha$. This is clear by the fact that the rule of composition of based oriented loops at a point is order dependent, such that $\alpha \circ \beta \neq \beta \circ \alpha$. This means that the based oriented loop $\alpha \circ \beta$ cannot be continuously deformed to the based oriented loop $\beta \circ \alpha$. In other words, the order dependence of the composition rule makes the corresponding multiplication operation a non-commutative operation. Besides, it is immediate to show that multiplication is an associative operation, viz. that $(\alpha \circ \beta) \circ \gamma = \alpha \circ (\beta \circ \gamma)$, so that we may skip parentheses altogether in multiple compositions of based oriented loops.

Having established the closure of the elements of the generic type "χ" under non-commutative associative multiplication as previously, we look for the existence of an identity element, as well as for the existence of inverses with respect to this operation. There is an obvious candidate for each based oriented loop α, namely the loop α^{-1}, whose encoding meaning in terms of orientation has been already explained. If we consider the compositions $\alpha \circ \alpha^{-1}$ and $\alpha^{-1} \circ \alpha$, we obtain in both cases as a multiplication product the based loop at the same point, which does not pass through any circle at all. Thus, we name the latter loop as the multiplicative identity 1 in our algebraic structure, such that $\alpha\alpha^{-1} = \alpha^{-1}\alpha = 1$. It is also easy to verify that $1\alpha = \alpha 1 = \alpha$. Notice that the equality sign is interpreted as an equivalence of based oriented loops under continuous deformation, according to our previous remarks referring to the meaning of the equivalence class $[\alpha]$.



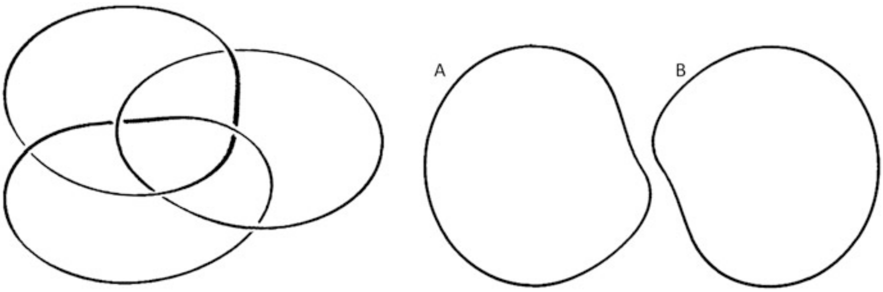
We conclude that the set of symbols of the generic type "χ" representing based oriented loops in relation to topological circles X, endowed with the non-commutative multiplication operation of composition product of loops based at the same point, form the algebraic structure of a non-commutative group, denoted by Θ. This group structure will be our means to encode algebraically the connectivity of the "Borromean link" in the sequel.

3.4 The Algebraic Encoding of the "Borromean Link"

We have already explained previously that the equality in the non-commutative multiplicative group Θ encodes the topological relation of equivalence of based oriented loops under continuous deformation. In other words, using the multiplication operation we may form any permissible string of symbols in the group Θ, which can be shortened into an irreducible form by using the group-theoretic relations $\alpha\alpha^{-1} = \alpha^{-1}\alpha = 1$, $1\alpha = \alpha 1 = \alpha$, $\alpha\alpha = \alpha^2$, and so on. Thus, two arbitrary strings of symbols in the group Θ are equal if they can be brought into the same irreducible form in Θ. Concomitantly, this means that the corresponding composed based oriented loops, or simply product loops, are equivalent under continuous deformation.

The property of irreducibility of a string of symbols in the group Θ, or equivalently of a product loop in Θ, is the guiding idea for the algebraic encoding of the "Borromean link" in terms of the structure of Θ. First, it is instructive to remind that a product loop in Θ is always considered with respect to the corresponding circles it passes through with a prescribed orientation. For example, as we have seen the product loop $\alpha\beta$ is considered in relation to the circles A and B in this specific order. Moreover, it is clear that both the based oriented loops of the form α and α^{-1} and any power of them in the group Θ with a prescribed orientation, are considered with respect to a circle of the form A. Thus, any multiplicative concatenation of symbols in the group Θ, when translated in product loop terms is thought of in relation to corresponding circles. These form the collection of all circles that a product loop is associated with.

Having clarified this fact, we proceed by trying to understand what kind of topological information the property of irreducibility of a string of symbols in the group Θ translates in algebraic terms. The crucial idea is that algebraic irreducibility encodes the topological property of non-splittability of a link. In order to motivate this idea, we remind that a link has been defined as a collection of loops, whence the topological connectivity of a link has been captured by the property of splittability. In particular, the "Borromean link" is a non-splittable 3-link, such that every 2-sublink formed by erasing one of the three loops of this 3-link is completely splittable.



Because of the fact that all three loops of the Borromean configuration are unknotted tame closed curves, we can equivalently think of this configuration in terms of a product loop in the group Θ, which is associated with two circles A and B in a precise way characteristic of the "Borromean link." The first hint comes from the realization that the cutting and removal of this hypothetical product loop would leave the two circles alone. This phenomenon corresponds to the complete splittability of the 2-sublink obtained by erasing this loop. In algebraic terms, this situation depicted by the above figure on the right is described by the identity element 1 of the group Θ. Thus, complete splittability of this 2-sublink is encoded by the identity 1 of Θ. For symmetry reasons, we expect that the same phenomenon will take place if we erase any of the circles A or B, since the identity element of Θ is unique. Nevertheless, in order to prove it algebraically we need the explicit formula describing the product loop in the terms of elements of Θ.

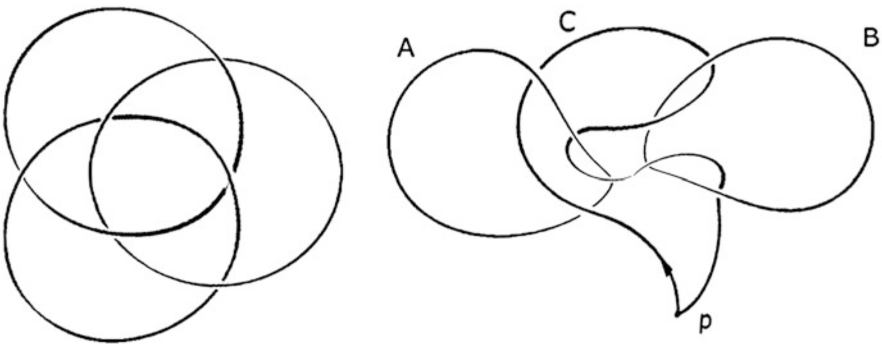
The second hint comes from the realization that, since the product loop should be expressed in relation to the circles A and B, it would certainly involve at least a string of symbols consisting of α, β and their group inverses α^{-1}, β^{-1} in some specific order, which does not involve any consecutive appearance of $\alpha\alpha^{-1}$, $\alpha^{-1}\alpha$, $\beta\beta^{-1}$, $\beta^{-1}\beta$, because all of them are reduced to the identity 1. The reason for the appearance of both α, β, and their group inverses α^{-1}, β^{-1}, lies on our expectation that removal of any of the circles A or B would collapse the product loop to the identity 1. This is the desired case referring to the "Borromean link" because every 2-sublink is completely splittable. It is obvious that if the circle A is erased, for instance, then in the sought-after product loop formula both instances of α and α^{-1} should be deleted, since both α and α^{-1} have a meaning with respect to A. The same holds symmetrically for β and β^{-1} in relation to the circle B.

The last ingredient before arriving at the product loop formula in terms of the group-theoretic structure Θ is the requirement of *irreducibility*. This is quite clear from the preceding discussion already. Since the fact that every 2-sublink of the "Borromean link" is completely splittable is encoded algebraically by reducibility to the identity of Θ, the natural requirement is that the non-splittability of the total 3-link should be encoded by the irreducibility of the product loop formula.

Taking into account all the above considerations and assuming the order from left to right, we conclude that there exists only one combination of symbols that fulfills our requirements, namely:

$$\gamma = \alpha\beta^{-1}\alpha^{-1}\beta.$$

Thus, the irreducible formula $\alpha\beta^{-1}\alpha^{-1}\beta$ represents the loop γ as a *product loop* composed by the *ordered composition of the four based oriented loops* $\alpha \circ \beta^{-1} \circ \alpha^{-1} \circ \beta$. We call the product loop γ the "Borromean loop" and the formula or multiplicative string $\alpha\beta^{-1}\alpha^{-1}\beta$ the "Borromean loop formula." *The algebraic irreducibility of* $\alpha\beta^{-1}\alpha^{-1}\beta$ *in the group* Θ *encodes the non-splittability of the 3-link in the "Borromean rings" configuration*. We notice that deletion of both α and α^{-1} (corresponding to removal of the circle A) reduces the formula to the identity 1 (and the same happens symmetrically for both β and β^{-1} in relation to the circle B). Thus, every 2-sublink of the Borromean 3-link is completely splittable. We represent schematically the above as follows:



In the above figure, we imagine that we pull continuously apart the two upper rings of the "Borromean link" displayed on the left. Then, we obtain the configuration on the right, which is interpreted in group-theoretic terms as a product loop, viz. the irreducible “Borromean loop” associated with these two circles. Hence, we have a geometric representation of the “Borromean loop formula.” The algebraic irreducibility of this formula $\alpha\beta^{-1}\alpha^{-1}\beta$ in the group Θ encodes the non-splittability of the 3-link in the Borromean configuration. Clearly, if we cut the “Borromean loop,” or remove any of the circles A or B, we obtain a completely splittable 2-sublink.

In geometric terms, the “Borromean loop formula” reads as follows: First, it passes away from the fixed observer at p through A (represented by α). Second it passes toward the observer at p through B (represented by β^{-1}). Third it passes again toward the observer through A (represented by α^{-1}). Fourth, it passes away from the observer through B (represented by β).

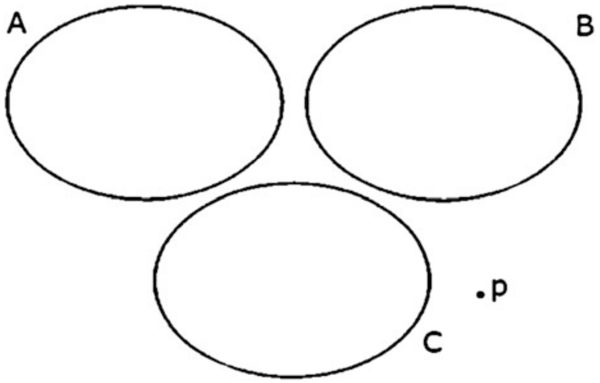
Thus, the topological information of the “Borromean link” has been completely encoded in terms of the algebraic structure of the non-commutative multiplicative group Θ. In this way, we have obtained a bi-directional bridge between the topological connectivity model of the “Borromean rings” expressed in terms of links and the algebraic algorithmic information model expressed in terms of the structure of the group Θ. This is of fundamental significance because it allows the translation of a hard topological problem into algebraic terms, viz. the encoding of the problem in group-theoretic terms, where it can be solved quite easily, and then inversely, the decoding of this solution into topological terms, which provides the solution of the initially posed topological problem. An illustration of this powerful method, which generalizes the case of the “Borromean rings” to higher non-splittable links whose all sublinks are completely splittable, in analogy to the Borromean case, will be presented in the sequel.

3.5 The “Borromean Link” as a Building Block for Generalized Topological Links

It is instructive to clarify that the algebraic structure of the group Θ is not only restricted to the typical Borromean configuration, explained in the previous section, but it can encode the topological information of higher links since we are free to construct product loops composed of any number of factors according to the composition rule we have defined. This presents the challenge of using the group Θ in order to solve the harder topological problem of identifying a non-splittable 4-link whose all 3-sublinks are completely splittable. Clearly, this problem constitutes the immediate higher generalization of the “Borromean link,” which involves a non-splittable 3-link whose all 2-sublinks are completely splittable. The main interest in such a generalization lies in the intuition that the “Borromean link” acts as a kind of a building block for the substantiation of higher order links of this type.

The method we will follow in order to attack this topological problem is the use of the bi-directional bridge between topology and algebra we have established in this context. Namely, we will translate the problem in terms of the algebraic structure of the group Θ, we will try to solve it in group-theoretic terms, and then decode the solution back into topological terms. Intuitively, the notion of a link involves the gluing conditions among its constituents. It is precisely these gluing conditions that are expressed algebraically in terms of the group Θ, as the fundamental case of the "Borromean link" has revealed by means of the "Borromean loop formula" $\gamma = \alpha\beta^{-1}\alpha^{-1}\beta$ in relation to the circles A and B.

The starting point is the analogous one to the standard "Borromean link" case. Namely, since all 3-sublinks of the sought-after non-splittable 4-link are completely splittable we will consider three circles A, B, C and look for a product loop composed of the products of α, β, γ and their group inverses α^{-1}, β^{-1}, γ^{-1}, in some specific order, which does not involve any consecutive appearance of $\alpha\alpha^{-1}$, $\alpha^{-1}\alpha$, $\beta\beta^{-1}$, $\beta^{-1}\beta$, $\gamma\gamma^{-1}$, $\gamma^{-1}\gamma$, because all of them are reduced to the identity 1. The crucial point again is that the product loop formula should reduce to 1 in the group Θ in case of removal of any of the circles A, B, or C, which is encoded algebraically by the deletion of all instances of both α, α^{-1}, or β, β^{-1}, or γ, γ^{-1}, depending on erasing A, or B or C, respectively. This is again the algebraic encoding of the fact that every 3-sublink of the total non-splittable 4-link should be completely splittable. Clearly, the non-splittability of the 4-link is again encoded by means of irreducibility of the product formula describing this 4-link.



Algebraically, this problem can be solved quite easily. The most elegant solution, which also trivializes the algebraic encoding of even higher links of this type, is to use the "Borromean link," viz. the algebraic "Borromean loop formula" $\alpha\beta^{-1}\alpha^{-1}\beta$ in the group Θ as a building block and iterate it self-referentially. For our purposes, we will explain how this works for the case at issue. First, by inspecting the "Borromean loop formula" $\alpha\beta^{-1}\alpha^{-1}\beta$ we realize that it can be written as the commutator in the group Θ, that is defined as follows:

$$[\alpha, \beta^{-1}] = \alpha\beta^{-1}\alpha^{-1}\beta.$$

This means that the commutator $[\alpha, \beta^{-1}]$ of the elements α and β^{-1} in the group Θ producing the "Borromean loop formula" encodes algebraically both the gluing condition of the non-splittable 3-link and of the completely splittability of all 2-

sublinks, according to the preceding analysis. We may also re-define the element β^{-1} as b, viz. $\beta^{-1} := b$ in the group Θ in order to obtain the commutator:

$$[\alpha, b] = \alpha b \alpha^{-1} b^{-1}$$

in the group Θ equivalently. Thus, the idea of using the "Borromean link" as a building block for analogous links of a higher type means employing the group commutator iteratively as an encoding device for these higher links of the same type. Therefore, in the case of a total non-splittable 4-link all 3-sublinks of which are completely splittable that involves the gluing of the three circles A, B, and C of the above figure by a "higher Borromean loop" we proceed as follows:

First, we glue the circles A and B by the standard "Borromean loop" and then we glue analogically this product with C. Algebraically speaking, the first step is simply the commutator $\xi = [\alpha, b] = \alpha b \alpha^{-1} b^{-1}$. The first iteration of this procedure, which involves the gluing of the product ξ with γ (in relation to the circle C), reads simply as the commutator of ξ with γ. We conclude that a "higher Borromean loop" that solves the problem is given in the structural terms of the group Θ simply as follows:

$$\delta = [\xi, \gamma] = [[\alpha, b], \gamma].$$

If we expand this formula, by using the definition of the group commutator as well as the group theoretic relation

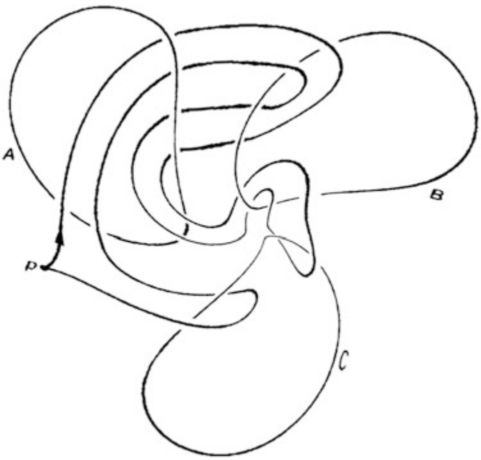
$$(\chi \psi)^{-1} = \psi^{-1} \chi^{-1},$$

where χ, ψ may stand for arbitrary strings of elements of the group Θ, we obtain the following unfolded expression for the "higher Borromean loop formula":

$$\delta = [\xi, \gamma] = [[\alpha, b], \gamma] = \{\alpha b \alpha^{-1} b^{-1}\} \gamma \{b \alpha b^{-1} \alpha^{-1}\} \gamma^{-1}.$$

From the above expanded "higher Borromean loop formula" it also becomes clear how the Borromean link becomes a building block via terms of the form $\lambda \mu \lambda^{-1} \mu^{-1} = [\lambda, \mu]$ for expressing higher order links of the Borromean type. We can also see that deletion of all incidences of any of the symbols (which involves the simultaneous deletion of the inverse symbol as well, according to the preceding) reduces the formula to the identity 1 in the group Θ.

As a final step, we decode back the obtained algebraic solution in topological terms by using the inverse bridge, and the obtained topological solution of the problem of finding a non-splittable 4-link whose all 3-sublinks are completely splittable by means of "Borromean building blocks" is illustrated as follows:



We conclude by noticing that although the topological solution of the problem is quite hard to obtain in a straightforward manner, as it is evidenced by the above figure, the same problem can be solved quite easily by using the algebraic structure of the group Θ, and in particular, the notion of the group commutator and its iterations. It is a remarkable fact that the "Borromean link" is encoded in terms of the commutator of Θ. In this way, the "Borromean link" can be efficiently used as a building block for the encoding of higher-order links of the type described above, by iterating the formation of commutators for product loops.

3.6 Borromean Extension in Depth: The Self-Referential Unfolding of Commutators and "Borromean Stacks"

In the previous section we proposed the idea of using the "Borromean link" as a building block for analogous links of a higher type by making higher order iterations of the group Θ commutator. We have explained already how this method works in the case of a total non-splittable 4-link all 3-sublinks of which are completely splittable. The crucial insight is that the group commutator acts as an encoding device for these higher links of the same type in two ways: First, the commutator provides the gluing scheme of link-formation by means of "Borromean loops." Second, due to the fact that deletion of all incidences of any of the involved symbols reduces the commutator to the identity 1 in the group Θ, the commutator also encodes the information of complete splittability of any remaining sublink after removing any of the constituents of the total non-splittable link.

In order to proceed more efficiently, we need to systematize our terminology as follows: The notion of the commutator of the simple oriented based loops a, b, that is $[a, b]$, is used as synonymous to the algebraic "Borromean loop formula" in the group Θ and it is decoded in topological terms as the concept of a "Borromean link." We denote the latter by $\Sigma(3, 2)$ meaning that it is a total non-splittable 3-link all 2-sublinks of which are completely splittable. In this way, the symbol $\Sigma(4, 3)$ denotes a total non-splittable 4-link all 3-sublinks of which are completely splittable. By induction, the symbol $\Sigma(N, N-1)$, where $N \geq 3$, denotes a total non-splittable N-link all $(N-1)$-sublinks of which are completely splittable. We have shown that a $\Sigma(4, 3)$ link can be constructed in terms of "Borromean link building blocks" simply by iterating once the commutator formation. This means that starting with three symbols a, b, c, we first glue a with b together by means of the commutator $[a, b]$, and then we glue their glued product $[a, b]$ with c to obtain the stacked commutator $[[a, b], c]$. This final glued product gives the required fourth symbol in the group Θ_2, which decodes topologically as a $\Sigma(4, 3)$ link. In an analogous manner, by iterating twice the commutator formation starting with four symbols a, b, c, d, we obtain a $\Sigma(5, 4)$ link. The same process can be clearly repeated inductively, so that we finally can construct any $\Sigma(N, N-1)$ link by means of Borromean building blocks, or more precisely, Borromean connectivity units, where $N \geq 3$. We may summarize this process in the following table:

Borromean Link	Σ(3,2)	[a, b]	Gluing of a with b	3-link
1st Iteration	Σ(4,3)	[[a, b], c]	Gluing of [a, b] with c	4-link
2nd Iteration	Σ(5,4)	[[[a, b], c], d]	Gluing of [[a, b], c] with d	5-link
By Induction N≥3	Σ(N,N-1)	Repeat process with (N-1)-symbols.	Gluing via commutator stacking	N-link

We note that the process of iterating the commutator formation in the group Θ, so as to obtain any link of the form $\Sigma(N, N-1)$, can be realized as an algorithmic procedure of commutator stacking in consecutive nested levels. Semantically, this procedure may be thought of as an operation of self-referential unfolding. The reason is that if we start iterating the commutator formation from level-0 ("Borromean link" $\Sigma(3, 2)$) which involves simple loops, then already at level-1 (link $\Sigma(4, 3)$), the symbol $[a, b]$ in the composite stacked commutator $[[a, b], c]$ plays a dual role: First, it is the symbol of a loop, namely the product "Borromean loop" of a and b, and second, it is the symbol of a gluing operator acting on a and b. Thus, the unfolding from level-0 to level-1 takes place self-referentially by identifying a loop as an argument of the stacked commutator at level-1 with the result of a gluing operator at the previous level-0. Clearly, the same phenomenon repeats at all higher levels.

It is instructive to explain in more detail the algebraic operation of commutator stacking. Recall that a commutator of two symbols a and b produces a new symbol $[a, b]$ in the group Θ, where $[a, b]$ denotes the gluing of a and b together to produce a new symbol, such that the triad of symbols a, b, and $[a, b]$ constitute a "Borromean link" of the type $\Sigma(3, 2)$. Thus, a $\Sigma(3, 2)$ link involves a commutator in two symbols standing for the gluing operator of these two symbols according to the Borromean constraint. Similarly, a $\Sigma(4, 3)$ link involves a stacked commutator in three symbols. The commutator is stacked because first we have to glue a with b, and then we have to glue their product $[a, b]$ with c in order to produce a new symbol $[[a, b], c]$, such that the tetrad of symbols a, b, c, and $[[a, b], c]$ constitute a $\Sigma(4, 3)$ link.

We stress again that deletion of any of the involved symbols in the stacked commutator collapses it to the unity of the group Θ, meaning that erasing any one of them causes the rest to come apart. Thus, by induction a $\Sigma(N, N-1)$ link involves a stacked commutator in $(N-1)$ symbols, where $N \geq 3$. For convenience, we call it a stacked commutator of order $(N-1)$. Note that the order of the stacked commutator in any link of the form $\Sigma(N, N-1)$ coincides with the number of symbols that separate if we remove any symbol from the total non-splittable N-link. For example, a $\Sigma(7, 6)$ link is expressed via a stacked commutator of order 6, meaning that it should be a commutator in six symbols of the form $[[[[[a, b], c], d], e], f]$. For reasons of simplicity, we define a stacked commutator of order $(N-1)$ as a "Borromean stack" of order $(N-1)$.

3.7 Borromean Extension in Length: The Formation of "Borromean Chains"

First, we introduce another definition to the series of the previous ones for terminological convenience. This refers to the characterization of a link of the general form $\Sigma(N, K)$. A link of the form $\Sigma(N, K)$ is defined as a link of N loops in 3-d space, such that each K-sublink is completely splittable, but each $(K+1)$-sublink, $(K+2)$-sublink, ..., $(N-1)$-sublink up to the N-link itself is non-splittable. For example, a $\Sigma(7, 3)$ link is a link of seven loops, such that each 3-sublink is completely splittable, but each 4-sublink, 5-sublink, 6-sublink, and the 7-link itself is non-splittable. The natural question emerging in this context is if it is possible to express a general link $\Sigma(N, K)$ in terms of "Borromean building blocks," or equivalently "Borromean functional units" encoded algebraically by the gluing operator of symbols, viz. by the commutator in the group Θ. We already know the answer in case that $K = (N-1)$. Namely, we have shown that the algebraic operation of commutator stacking of order $(N-1)$ is enough to express any $\Sigma(N, N-1)$ link. In other words, an arbitrary $\Sigma(N, N-1)$ link is simply a "Borromean stack" of order $(N-1)$. So we need to consider what happens in the general case, where $K \neq (N-1)$.

We will show in the sequel that there exists another natural operation on "Borromean building blocks," which is described by taking an appropriate product of commutators in the group Θ. Intuitively speaking, this natural operation should express a procedure of Borromean extension in length, or simply the formation of a "Borromean chain" of some appropriate length. In order to motivate the notion of a "Borromean chain" it is necessary to start with the simplest example of this type, namely the $\Sigma(4, 2)$ link. This is a link of four loops, such that each 2-sublink is completely splittable, but each 3-sublink and the 4-link itself is non-splittable. From this definition, we immediately deduce that if we remove any loop from a $\Sigma(4, 2)$ link we obtain a 3-sublink which is non-splittable. Moreover, since each 2-sublink is completely splittable, we deduce that if we remove any loop from a $\Sigma(4, 2)$ link we actually obtain a $\Sigma(3, 2)$ link, viz. a "Borromean link." Furthermore, if we remove any two loops from a $\Sigma(4, 2)$ link the remaining two fall completely apart because again each 2-sublink of a $\Sigma(4, 2)$ link is completely splittable. Therefore, by encoding this information in the group Θ, we attack the problem as follows: Consider three symbols a, b, and c. We seek a formula expressing the fourth symbol, such that deletion of all incidences of any of the symbols a or b or c causes the formula to reduce to the "Borromean loop formula" (that is the commutator of the remaining two symbols), whereas deletion of all incidences of any two of the three symbols, viz. (a, b), or (a, c), or (b, c) causes the formula to reduce to the unity 1.

It is instructive to emphasize that the algebraic encoding of the problem referring to a $\Sigma(4, 2)$ link paves the way to its solution. The problem is if it is possible to express a $\Sigma(4, 2)$ link in terms of "Borromean building blocks," viz. in terms of suitable operations on commutators in the group Θ. By the defining properties of a $\Sigma(4, 2)$ link, if a formula in three symbols a, b, c actually existed fulfilling the two requirements of the previous paragraph, and also expressed exclusively in terms of commutators built from these three symbols, then it would be true that the $\Sigma(4, 2)$ link can be constructed in terms of "Borromean building blocks." Now, considering the symbols a, b, and c, we may construct the "Borromean stack" of order 3, viz. the stacked commutator formula $[[a, b], c]$. Clearly, although this expresses a $\Sigma(4, 3)$ link as we have seen in the previous section, it is not an appropriate formula to express a $\Sigma(4, 2)$ link because deletion of any of the three symbols causes the formula to reduce to 1. What we need is another operation, which hopefully can involve only commutators and have the desired properties. A simple observation is that given three symbols a, b, and c, we may construct out of them three distinct commutators, namely $[a, b]$, $[a, c]$, and $[b, c]$. Since each of these commutators gives a new symbol in the group Θ, we may take their product which is also a new symbol in the group Θ.

Notice that each of the commutators $[a, b]$, $[a, c]$, $[b, c]$ gives separately a "Borromean link." Thus, their product $[a, b][a, c][b, c]$ is actually a composition of three separate "Borromean links" in the group Θ:

$$\rho = [a, b] \circ [a, c] \circ [b, c],$$

which gives rise to a "Borromean chain" of length 3. The formation of this "Borromean chain" ρ provides the sought-after operation on "Borromean building blocks" to express a $\Sigma(4, 2)$ link, and therefore solve the posed problem. We can immediately see this as follows: First, we notice that deletion of any one of the symbols a, b, c, in the "Borromean chain" ρ of length 3, $[a, b] \circ [a, c] \circ [b, c]$, reduces this chain to a "Borromean link." For instance, if we delete the symbol a, what remains is the "Borromean link" $[b, c]$, and analogously for the other two cases. Second, we notice that deletion of any two of the symbols a, b, c, reduces this chain to unity. Hence, we conclude that the "Borromean chain" of length 3, defined by the product of commutators $[a, b][a, c][b, c]$, provides the formula for the fourth symbol ρ in the group Θ, such that the defining properties of a $\Sigma(4, 2)$ link are satisfied, and moreover, this link is expressed in terms of "Borromean building blocks." An interesting observation that we will use in the sequel is that the length of the "Borromean chain" solving the problem is given by the number of combinations of two symbols out of three, where a combination is simply the formation of the commutator of two symbols in this case.

3.8 The Fundamental Theorem: The "Borromean Link" as the Prime Connectivity Unit in the Universe of All Links

Regarding the possibility of expressing arbitrary links in 3-d space of the general form $\Sigma(N, K)$ in terms of "Borromean building blocks," or equivalently "Borromean connectivity units" we have proved up to present the following: First, the algebraic operation of commutator stacking of order $(N - 1)$ is enough to express any $\Sigma(N, N - 1)$ link. In other words, an arbitrary $\Sigma(N, N - 1)$ link is simply a "Borromean stack" of order $(N - 1)$. For instance, a $\Sigma(4, 3)$ link is simply a "Borromean stack" of order 3. Second, we have shown that the expression of a $\Sigma(4, 2)$ link requires the consideration of another operation on "Borromean building blocks," which is interpreted as the operation of extension of length 3, called the formation of a "Borromean chain" of length 3. Based on these findings, the next question posing itself naturally in this context is if these two operations on "Borromean building blocks," namely the formation of "Borromean stacks" of some suitable order and the formation of "Borromean chains" of some suitable length are adequate in order to express any arbitrary link in 3-d space of the general form $\Sigma(N, K)$.

This would be certainly of significance in our understanding of the whole universe of links, because it would prove that any $\Sigma(N, K)$ link can be constructed by means of "Borromean connectivity units" via the combinatorial formation of "Borromean stacks" and "Borromean chains." Moreover, due to the algebraic modelling scheme instantiated structurally by the non-commutative group Θ, the process of analysis and synthesis of arbitrary links in terms of prime elements, viz. in terms of "Borromean connectivity units" would be implementable algorithmically, and thus used as a valuable tool for making evaluations and predictions.

Before we consider the general case of a $\Sigma(N, K)$ link, it is instructive for our intuition to examine the case of a $\Sigma(5, 3)$ link. The reason is that a $\Sigma(5, 3)$ link has enough complexity so as to pave the way for the treatment of the general case of a $\Sigma(N, K)$ link. From the definition of a $\Sigma(5, 3)$ link, the crucial observation is that if we remove any of the loops what remains is a $\Sigma(4, 3)$ link, which we already know that is expressed by means of a "Borromean stack" of order 3, viz. by the stacked commutator formula $[[a, b], c]$ in three symbols. Thus, in order to express the formula of a $\Sigma(5, 3)$ link, if we consider four symbols a, b, c, d, we require a formula such that deletion of any of them causes the formula to reduce to the one of a $\Sigma(4, 3)$ link, viz. to a "Borromean stack" of order 3.

The important concept solving this problem is based on the observation that we can form *"Borromean chains" of arbitrary length using "Borromean stacks."* In the particular case of a $\Sigma(5, 3)$ link considered, since we require that deletion of any of the four involved symbols a, b, c, d, reduces the formula to a "Borromean stack" of order 3, we just need to form a "Borromean chain" of "Borromean stacks" of order 3, where the length of the chain should be 4. This is explained easily by the fact that the length of the "Borromean chain" is given by the number of combinations of three symbols (which is the number of symbols involved in a "Borromean stack" of order 3) out of four symbols a, b, c, d. We immediately conclude that the sought-after formula expressing a $\Sigma(5, 3)$ link is given by the "Borromean chain" of length 4, composed by "Borromean stacks" of order 3, and described explicitly by the following formula:

$$\chi = [[a, b], c] \circ [[a, b], d] \circ [[a, c], d] \circ [[b, c], d].$$

In more detail, we see that the above formula is given by the composition of four "Borromean stacks" of order 3 (since they involve three symbols each), and thus produces a "Borromean chain" of length 4, such that deletion of any of the four involved symbols a, b, c, d reduces this chain to a "Borromean stack" of order 3 as required. Thus, we have completely resolved the problem of a $\Sigma(5, 3)$ link in terms of prime "Borromean connectivity units."

Now, having understood in detail the case of a $\Sigma(5, 3)$ link, we are ready to state the central theorem of this treatise:

Fundamental Theorem *An arbitrarily complex link of the general form $\Sigma(N, K)$, where $1 \leq K \leq N$, can be constructed solely in terms of "Borromean building blocks," by means of forming Borromean stacks and Borromean chains out of Borromean stacks of appropriate order and length, respectively.*

Proof We consider an arbitrarily complex link of the general form $\Sigma(N, K)$, where $1 \leq K \leq N$, and prove that it can be constructed solely in terms of "Borromean building blocks" within the group Θ. For any K, we already know that the link $\Sigma(K + 1, K)$ is expressed by means of a "Borromean stack" of order K. Next, we consider $(K + 1)$ symbols in Θ, and we wish to construct a $\Sigma(K + 2, K)$ link. The crucial observation is that if we remove any topological circle from a $\Sigma(K + 2, K)$ link, what remains is a $\Sigma(K + 1, K)$ link. Thus, we treat this case in complete analogy to the case of a $\Sigma(5, 3)$ link, discussed previously. More precisely, we form a "Borromean chain" out of "Borromean stacks" of order K, where the length of this chain is given by the number of combinations of K symbols out of $(K + 1)$ symbols. The formula expressing this "Borromean chain" provides the sought-after $(K + 2)$ symbol. Now, we consider $(K + 2)$ symbols, and we wish to construct a $\Sigma(K + 3, K)$ link. We just have to form a “Borromean chain” out of “Borromean stacks” of order K, where the length of this chain is given by the number of combinations of K symbols out of $(K + 2)$ symbols. The formula expressing this new “Borromean chain” provides the sought-after $(K + 3)$ symbol in Θ. We continue the same process of formation of new “Borromean chains” of appropriate combinatorial length composed by “Borromean stacks” of order K, stage by stage, until we reach N. This completes the proof of the theorem that an arbitrarily complex link of the general form $\Sigma(N, K)$ can be constructed solely in terms of “Borromean building blocks,” or equivalently, “Borromean connectivity units.”

We may consider as an application of this theorem the case of a $\Sigma(7, 4)$ link. The link $\Sigma(5, 4)$ is expressed by means of a “Borromean stack” of order 4. Next, we consider five symbols, and we wish to construct a $\Sigma(6, 4)$ link. Let us call these symbols a, b, c, d, e. Next, we form a “Borromean chain” of “Borromean stacks” of order 4, where the length of this chain is given by the number of combinations of four symbols out of five symbols, which is 5. Let us denote by f the new symbol provided by this “Borromean chain” of length 5. Thus, we have constructed a $\Sigma(6, 4)$ link. Now, we consider these six symbols a, b, c, d, e, f, and we wish to construct a $\Sigma(7, 4)$ link. We just have to form a “Borromean chain” of “Borromean stacks” of order 4, where the length of this chain is given by the number of combinations of four symbols out of six symbols, which is 15. The product formula expressing this new “Borromean chain” of length 15 provides the sought-after 7th symbol. Therefore, we have constructed a $\Sigma(7, 4)$ link by means of prime “Borromean connectivity units” using only the combinatorial formation of “Borromean stacks” and “Borromean chains.”

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Chapter 4
Borromean Link in Logic
A Metaperspective on Algorithmic Information: Logical Conjugation Strategy and the Role of the Borromean Topology

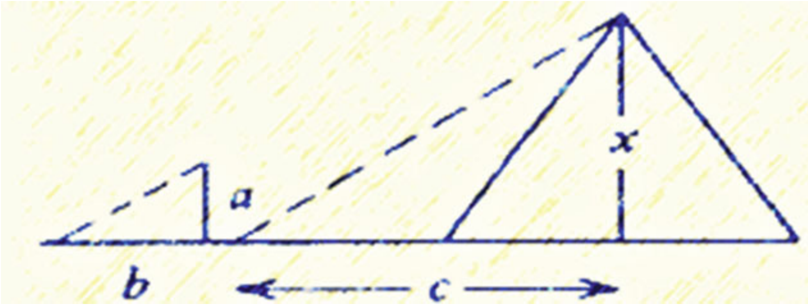
4.1 On the Notion of Analogical Relations and Metaphors

The notion of analogy will be considered in its broadest possible sense, namely as a mode of reasoning or problem-solving in which a phenomenon, or a quantity, or an object, or a class of objects, or even a category of objects, is intentionally compared to another in order to establish similarity of relationship. Moreover, of the two particular instances between which a resemblance (similarity of relationship) is established, one is generally not directly comprehensible, while the other is assumed to be better or more easily tractable. It is important to clarify that according to the above, an analogical relation bears the semantics of a resemblance not between instances, but between the relations of instances. Thus, an analogy is a resemblance relation, involving (at least) two terms, each of which is itself a relation.

Hence, if assumed temporarily that the latter are binary relations between objects (conceived set-theoretically), then, we obtain four terms constituting an analogical relation. The four terms are being distributed in two distinct levels, two of the four on each level. Furthermore, three of the four terms are assumed to be known or directly measurable, or accessible, or more generally, determinable by some method, and the purpose is to determine the fourth.

The primary examples of analogies emanate from Thales' paradigm on the theory of homothesis or proportionality. It is important to emphasize that the purpose of Thales' theory of proportions had been the measurement of not directly accessible magnitudes. More concretely, the objective of Thales was to find the height x of an inaccessible pyramid, given the length c of its accessible shadow, as well as the height a and shadow length b of an accessible object, functioning as a measurement rod. The analogical relation devised by Thales for the resolution of this problem reads as follows:

A. von Müller, E. Zafiris, *Concept and Formalization of Constellatory Self-Unfolding*, On Thinking,
https://doi.org/10.1007/978-3-319-89776-9_4



(*a* to *b*) is as (*x* to *c*)

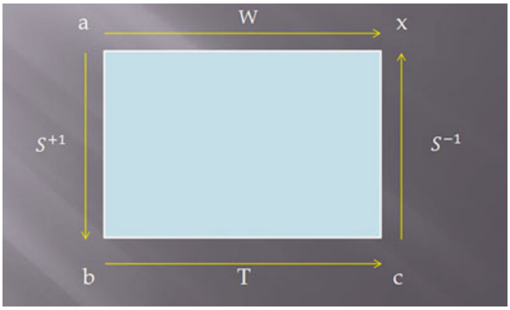
Symbolically, the above analogy is depicted by the equation $\frac{a}{b} = \frac{x}{c}$, from which the not directly accessible magnitude x can be determined indirectly as $x = \frac{ac}{b}$. Note that the four terms of this proportion between magnitudes are arranged into two distinct levels according to some qualifying characteristic. More concretely, in Thales' theory a and x occupy one level as vertical heights, whereas b and c occupy the other level as horizontal shadows.

It is important to remind the fact that Thales has provided a geometric solution to the posed problem, since the algebraic solution presented above formulated by means of an equation involving the operations of multiplication and division of positive integer magnitudes was not known in his days. Hence, it is not an exaggeration to claim that the geometric theory of proportions contained the seeds of conception of modern algebraic structures (closed under the action of operations on their elements), together with the notion of setting up algebraic equations for the determination of unknowns. In this mode of thinking, the geometric resolution of the Thalesian problem, in terms of proportionality (analogy of magnitudes), implicitly anticipates the discovery of the multiplicative monoid structure of positive integers, and subsequently, the multiplicative group structure of the rationals and the real numbers.

The meaning of this assertion boils down to the realization that the determination of an unknown magnitude in the Thalesian setting, by analogical reasoning, interpreted now algebraically, requires the introduction of the multiplicative group structure of the rational numbers or the real numbers (standing for magnitudes) in order to provide a solution to the associated equation expressing that analogical relation. In a suggestive manner, we can rewrite the solution of this equation in the following form:

$$x = M_a c {M_b}^{-1}$$

meaning that to obtain the not directly accessible magnitude x, "multiply by a" (denoted by M_a) the magnitude c, and then, divide by b (denoted by ${M_b}^{-1}$). Thus, the determination of inaccessible magnitudes by means of analogy algebraically necessitates the introduction of the group-theoretic closure structure on magnitudes, equipped with the operation of multiplication and possessing an inverse, which is division.



By extrapolating, we may assume that the resolution of a more general problem, based on analogical reasoning (not restricted to the situation of proportionality of magnitudes) implicitly requires for its algebraic manipulation the following:

Firstly, the distribution of the four terms of an analogical relation into two distinct levels, two of the four on each level, where, three of the four terms are assumed to be directly determinable, and the purpose is to determine the fourth.

Secondly, the introduction of an appropriate closed algebraic structure with respect to a process that connects the two distinct levels, playing a similar role to the operation of multiplication (between magnitudes at different levels). This multiplicative adjunctive process can be thought of as a directed bridge which connects the upper level with the lower one, where each level is occupied by things belonging to the same class or universe of discourse.

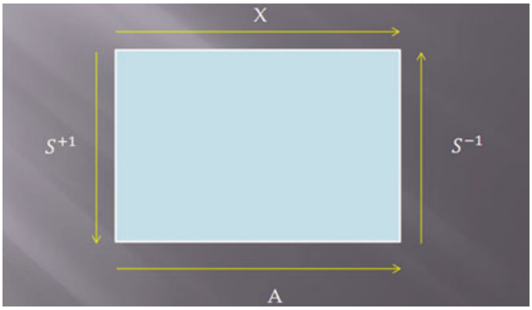
Thirdly, the possible determination of the inverse to the multiplicative adjunctive process, called the division process. In many of the cases an exact inverse process (being suggestive of the global schematism of reversibility via another level) may not be attainable, and thus, partially or locally inverse processes should be employed, satisfying appropriate conditions.

According to the above, in case that an exact inversion process is available or globally constructible, facilitating an effective exact round-trip between two delineated levels, we call the analogical relation a metaphor. This conception has an Aristotelian origin and captures the relevant qualifying statement in Poetics (Aristotle), according to which: "Metaphor is the substitution of the name of something else, and this may take place from genus to species, or from species to genus, or from species to species, or according to proportion." Projecting this statement back to the general environment of analogical relations, we conclude that a general analogy between instances may be concerned with class membership or class characterization.

Thus, an analogy, formulated as a relation among four terms distributed at two distinguished levels, expresses a resemblance between two instances at the same level, only within the context of totalities, or reference frames, or networks of relations, conceived as corresponding individual instances at the other level. Note that the unifying conceptual thread on all different manifestations of analogical relations is the following: Starting from a term at some level the determination of an inaccessible term with respect to the first, at the same level, via a cyclical global round-trip process through another level, involving three stages:

First, setting up an encoding multiplicative adjunctive bridge of correspondence of the initial term with a reference domain, or gnomon, conceived individually at another level. Second, processing or resolving the task at this other level. Finally, devising a decoding bridge of correspondence, inverse to the multiplicative one, that facilitates the return at the initial level and simultaneously resolves indirectly the problem of direct inaccessibility.

Subject to the above observations characterizing the essence of an analogical relation, resembling the algebraic transcription of the Thalesian theory of proportions of magnitudes in a generalized conceptual setting, we may attempt to formulate an analogy in the form of the following symbolic relation:



$$X = SAS^{-1}$$

where the unknown X is determined by some three-stage resolution process of the form described above, through some quite easily determinable A at another level mediated via the opposite pointing bridges S and S^{-1} connecting the two levels. In case that the bridges S and S^{-1} are exact inverses and A is considered to be noise-free, we say that the analogy is effective, characterized as a metaphor. In the general case, where the bridges S and S^{-1} are not exact inverses to each other, but only conceptually inverse, they are called adjoint. This type of resolution process making up an analogy has been proposed independently in relation to the reduction of complexity, and as a means of modelling complex systems, in Melzak (1983) and Rosen (1978). The above characterization of a metaphor subsumes the category-theoretic notion of a functorial duality (Awodey 2006). In this way, an analogical relation pertaining to different categorical levels of structure may also be considered from the viewpoint of the theory of adjunctions, where the conceptually inverse bridges form a pair of adjoint functors (Zafiris 2012).

From a general interpretative standpoint, the symbolic relation $X = SAS^{-1}$ admits a dualistic interpretation, namely one in terms of substances and another in terms of operations. In a general context, the operational interpretation is preferable for our purposes, since it stresses the emphasis on the process devised for overcoming a direct inaccessibility. In this sense, the indirectly determinable by analogy symbol X, followed by the sign of equality, may be interpreted as signifying the total ordered series of the three actions needed for its effective determination via another level, connected to its own by two inversely directed bridges. It is also instructive to notice that the meaning of the operational interpretation can be captured even from its dual substantive viewpoint, under the convention that the symbolic relation of analogy can be extended in the notational form:

$$X(l_1, l_2) = S(l_1)A(l_1, l_2)S^{-1}(l_2)$$

where the symbols l_1 and l_2 denote some kind of base locality or base indexing parameter.

4.2 Logical Conjugation and Its Properties

In general mathematical terms, the presentation of an effective analogical relation, or metaphor, in the symbolic form

$$X = SAS^{-1}$$

defines X to be conjugate to A under S, where S^{-1} is considered to be the conceptual inverse of S. This is a useful observation because it associates the principle of conjugation with the semantics of a metaphor. Let us now examine briefly the structure of an analogical relation presented in the above form by means of logical conjugation.

First of all, we realize that the relation $X = SAS^{-1}$ consists of two basic semantic parts: The first part is constituted by the conceptually inverse vertical processes S and S^{-1}, forming the outer part of the analogical relation, and signifying a bidirectional bridge of information encoding/decoding between two different levels. The second part is constituted by the horizontal process A, forming the inner part of the analogy, and signifying a directed process of information transfer, or even information storage, within the level specified by the functioning of the first vertical directed bridge, performed previously. Note that the functionality of an analogical relation is being crucially dependent on the interpolation of some appropriate inner part A between the succession of the actions of the inversely pointing bridges. More precisely, if the inner part A is absent, then the outer part simply does not have any functionality since it cancels out. Based on this fact, we can formulate the basic properties of logical conjugation as pertaining to effective analogical relations as follows:

1. Logical Conjugation or Metaphor Extension in Length: This means that two metaphors sharing the same bridges can be combined horizontally simply by

juxtaposing one with another as follows: if $X_1 = SA_1S^{-1}$ and $X_2 = SA_2S^{-1}$, then $X_1X_2 = SA_1A_2S^{-1}$;
2. Logical Conjugation or Metaphor Extension in Depth (Metaphor Stacking): This means that the inner part of a metaphor can be substituted by another metaphor, such that the initial metaphor can be accomplished via a splitting into a deeper level of hypostasis, and so on, as follows: if $X = SAS^{-1}$ and $A = TBT^{-1}$, so that, $X = STBT^{-1}S^{-1}$, then $X = [ST]B[ST]^{-1}$;
3. Logical Conjugation or Metaphor Inversion: This means that if a process X is conjugate to a process A at another level under the action of a bridge S, then A is conjugate to X under S^{-1}, as follows: if $X = SAS^{-1}$, then $A = S^{-1}XS$.

Due to the properties listed above, an effective analogical relation constituted by means of logical conjugation can be presented in the form of an equivalence relation, namely as:

$$X \sim_S A$$

stating that X is conjugate to A under S. This is an equivalence relation because it is reflexive, transitive, and symmetric: First, due to the property of metaphor extension in length if $X_1 \sim_S A_1$ and $X_2 \sim_S A_2$, then $X_1X_2 \sim_S A_1A_2$. Second, due to the property of metaphor extension in depth, the transitivity condition is established since, if $X \sim_S A$ and $A \sim_T B$, then, $X \sim_{ST} B$. Finally, due to the property of metaphor inversion, the symmetry condition is established since, if $X \sim_S A$, then $A \sim_{S^{-1}} X$.

Now, suppose that $M \subseteq K \times K$ is the equivalence relation induced by logical conjugation on a set of processes K. We may consider a category (K, M) in which K is the set of objects (standing for processes), M is the set of arrows, and the source and target maps $M \to K$ are given by the first and second projection. Then given X and A in K, there is precisely one arrow (X, A) if X and A are in the same equivalence class, viz. they are metaphorically related by conjugation, while there is none if they are not. Then transitivity assures us that we can compose arrows, while reflexivity tells us that over each process X in K there is a unique arrow (X, X), which is the identity. Finally symmetry tells us that any arrow (X, A) has an inverse (A, X). Thus, (K, M) is a groupoid (category in which all arrows are isomorphisms) such that from a given object of this category (process) to another there is at most one arrow (if they are metaphorically related). Conversely, given a groupoid, such that from a given object to another there is at most one arrow, if we denote by K the set of objects and by M the set of arrows, the source and target maps induce an injective morphism $M \hookrightarrow K \times K$, which gives an equivalence relation on K with the desired semantics.

An interesting type of logical conjugation arises in case that a bridge S equals its own inverse, that is $S = S^{-1}$. An immediate consequence is that if the inter-level transformation S is repeated twice in succession, then it gives the identity, viz. $S^2 = 1$. In this case the bridge S is called an involution bridge. The most well-known example of an involution bridge is provided by any device operating strictly between two states, represented by the simplest Boolean algebra containing two truth values

(True and False, or 0 and 1). Then, if the bridge S represents the transformation from the one state to the other (acting like a Boolean negation operator between the levels of truth and falsity), its repeated application for a second time brings us back to the original state. In logical terms, the negation of negation is equivalent to the identity, and therefore, an involution bridge functioning between two states distributed in two distinct levels is a picturesque way of expressing the law of excluded middle in Boolean logic.

4.3 Logical Conjugation and Extension of Algebraic Structure

Let us now examine the functionality of logical conjugation from a structural algebraic standpoint. We have already claimed previously that the resolution of the Thalesian problem of determination of an inaccessible magnitude by the method of proportions implicitly contains the seeds of discovery of the multiplicative group structure of (positive) rationals. More precisely, multiplication is an essential operation that can be performed on integers endowing them with the closed structure of a multiplicative monoid. Division, the inverse operation to multiplication, is nevertheless not a total operation on integers, and thus, the determination of inaccessible magnitudes on the basis of proportion cannot be effectively performed within the reference domain of integers.

The resolution of the problem of making the operation of division total, and thus resolving the Thalesian problem, requires the extension of this domain into a new domain of numbers, where the required inverse operation can be always implemented. This means that the resolution of the problem requires an appropriate extension of the initial closed structure (integers) with respect to the operation of multiplication into a new structure (rationals) being closed with respect to both multiplication and its inverse operation of division. This is a recurring theme in universal algebra and thus it deserves a closer analysis in order to explain the way of its implementation by means of the logical conjugation strategy.

For this purpose, it is necessary to state explicitly the ordered series of three processes that have to be performed, according to the general pattern characterizing metaphors, for the construction of the field of rationals from the ring of integers. We remind that the rationals constitutes the set of all fractions a/b, a, and b integers, $b \neq 0$ with the usual relation $a/b \equiv c/d$ if $ad = bc$, which makes invertible every non-zero element of the integers.

The basic ingredient of the construction of the field of fractions is the fact that the set of non-zero elements of the integers is multiplicatively closed (Atiyah and MacDonald 1969). The structural metaphor characterizing completely this construction is technically called the process of localization of the commutative unital ring of the integers $\mathbb{Z}$ with respect to the multiplicative closed subset of the non-zero integers. The whole purpose of this structural metaphor by conjugation is to make every element of the multiplicative closed subset of non-zero integers invertible, such that the new structure of numbers obtained in this manner fulfills the following objectives: First, it bears a structural similarity to the initial domain of numbers, viz. it is also a commutative unital ring with respect to addition and multiplication. Second, the operation of division (inverse to multiplication) can be performed by the existence of inverses of non-zero integers, which have been incorporated in the new extended closure domain of numbers. Third, as a consequence of the above, the initial domain of numbers together with their arithmetic can be embedded in the new one.

We consider the commutative unital ring of integers $\mathbb{Z}$ and let $S \subseteq \mathbb{Z}$ be the multiplicative closed subset of non-zero integers. The first step is to set up a directed bridge from the level of commutative unital rings to the level of sets, encoding the process of extending the underlying set-theoretic domain of integers $\mathbb{Z}$ into a new domain formed by the cartesian product of sets $\mathbb{Z} \times S$. Note that the ordered pairs of integers (a, s) with $s \neq 0$ are not supposed to have any a priori structure, since their existence is required at the level of sets by means of the encoding directed bridge connecting the involved structural levels. In this extended new set-theoretic domain the initial task can be facilitated by imposing the homological equivalence criterion, according to which the ordered pair of integers (va, vs) should be equivalent to (a, s) for any non-zero integer v. Technically this condition is described in the following way:

In the set $\mathbb{Z} \times S$ we define the following binary relation: $(a, s) \diamond (b, t)$ if and only if there exists $v \in S$ such that: $v(at - bs) = 0$. The relation $\diamond$ is an equivalence relation, partitioning the set $\mathbb{Z} \times S$ into equivalence classes. We will denote the quotient set by $\mathbb{Z}_S$, and the equivalence class of (a, s) by the fraction symbol a/s. Thus, the quotient set $\mathbb{Z}_S$ contains elements which can be interpreted as fractions, bearing the semantics of numbers allowing division by non-zero integers.

The structural metaphor is completed via logical conjugation by setting up an inversely directed decoding bridge from the level of sets to the level of commutative unital rings, effectuating the indirect round-trip as follows: We set $a/s + b/t := (ta + sb)/st$, $(a/s)(b/t) = (ab/st)$ for every $a/s, b/t \in \mathbb{Z}_S$. The operations are well defined and endow $\mathbb{Z}_S$ with the structure of a ring. The zero and unit elements are, respectively, $0/s$ and s/s, for every $s \in S$. Finally, we define the canonical morphism of rings $h : \mathbb{Z} \to \mathbb{Z}_S$, given by $h(a) = a/1$, for every $a \in \mathbb{Z}$. Note that for any $s \in S$ we have that $1/s$ is the inverse of $h(s)$ in $\mathbb{Z}_S$. Hence, $\mathbb{Z}_S$ is the smallest ring containing $\mathbb{Z}$, in which every element of the multiplicative closed subset of non-zero integers S is invertible.

Thus, the extension of scalars of the commutative unital ring of integers $\mathbb{Z}$ by means of algebraic localization, with respect to the multiplicative closed subset of non-zero integers, is understood as a structural algebraic metaphor implemented by logical conjugation. The structural effect of this metaphor by conjugation is the addition of multiplicative inverses to the elements of the multiplicative closed subset $S \subseteq \mathbb{Z}$, such that the extended ring $\mathbb{Z}_S$ consists of fractions a/s, where $a \in \mathbb{Z}, s \in \mathcal{S}$. Moreover, the conceptualization of algebraic localization as a structural metaphor for the resolution of the general problem of making division a total operation by congruent extension of structure via the logical process of conjugation permits its application in generalized structural environments as we shall see in the sequel.

4.4 Structural Logical Conjugation in Relation to Homothesis

It is instructive to explicate in more detail the conjugation strategy related with the efficient functioning of the above structural metaphor. First, we observe that the encoding process of the underlying set-theoretic domain of $\mathbb{Z}$ into the new domain formed by the cartesian product of sets $\mathbb{Z} \times S$ takes place by means of extending the scalars of $\mathbb{Z}$ with respect to the scalars of the multiplicative closed subset S of $\mathbb{Z}$. This means that the extension of scalars of the set-theoretic domain of $\mathbb{Z}$ is effectuated by adjoining to $\mathbb{Z}$ the scalars of a well-defined internal algebraic part S of $\mathbb{Z}$ distinguished by its anticipated operational role.

Second, the level of sets can be thought of as a temporary underlying scaffolding via which logical conjugation can be effectively applied. More precisely, at the level of sets the operational role of the distinguished part S of $\mathbb{Z}$ can be implemented by the imposition of an appropriate homological equivalence relation on the previously extended set-theoretic domain $\mathbb{Z} \times S$. The conceptual underpinning of this process is the identification of those elements of the extended domain $\mathbb{Z} \times S$, which exhibit a certain resemblance of behavior, which we symbolize by the relation R. Any suitable criterion of homological indiscernibility must lead to a partition of $\mathbb{Z} \times S$ into disjoint classes of elements bearing the imposed relation of resemblance R, and hence R must be an equivalence relation. Since the imposition of such a relation R effectuates a classification of the elements of $\mathbb{Z} \times S$ into disjoint classes of equivalent elements, partitioning it in the particular way determined by R, the latter can be thought of as a resemblance perspective. In this manner, an equivalence class modulo the resemblance perspective R consists of all the elements of $\mathbb{Z} \times S$, indiscernible with respect to R, and thus homologically identical.

More specifically, the resemblance perspective R imposed on $\mathbb{Z} \times S$ requires that the ordered pair of integers (va, vs) should be homologically identical as (a, s) for any non-zero integer v, under the intended interpretation of the resemblance class of (a, s) by the fraction symbol a/s. Note that the resemblance classes (a, s) are metaphorically interpreted as elements a/s, being assigned a new name, viz. fractions, of a new set, namely of the quotient set $\mathbb{Z}_S$. It is important to notice that consequent to the transition from $\mathbb{Z} \times S$ to $\mathbb{Z}_S$ is the replacement of equivalence modulo R, viz. R-perspective resemblance, by equality (identity) of elements in the quotient $\mathbb{Z}_S$.

Third, the structural metaphor realizing the result of the applied logical conjugation is completed by means of the inversely directing bridge from the level of sets back to the initial level of commutative unital rings. The semantic aspect of this bridge amounts to a re-casting of the elements of the quotient set $\mathbb{Z}_S$, as elements of a new ring, viz. as elements of the same closed structural genus as the initial $\mathbb{Z}$. This is accomplished by modifying appropriately the addition and multiplication operations referring to these new elements (fractions). This modification takes place according to the principle that the new operations should incorporate and reproduce the effect of the old ones, when restricted to the old elements, being dressed in the new form imposed by the adopted resemblance perspective.

The important thing to notice is that the completion of the structural metaphor according to the logical conjugation strategy described above accomplishes the task of making the operation of division total, and thus, resolves the geometric problem of homothesis in a structural way. In this way, from the standpoint of the ring of integers, the structural metaphor permitting the unconstrained action of the division operation on magnitudes, belonging now to an extended closed partially congruent structure of the same algebraic genus (ring of rationals), accomplishes the interpretation of division as an emergent well-defined total operation. This is due to the fact that the operation of division acts properly on this new kind of species (fractions), which remains closed with respect to its action. The logical conjugation resolves the original Thalesian problem structurally because fractions are formed at the set-theoretic level, and then lifted at the ring-theoretic level by means of encoding/decoding bridges. In particular, fractions are formed by the inverse processes of extending the set-theoretic domain $\mathbb{Z}$ to the larger one $\mathbb{Z} \times S$ with respect to the part S, and then restricting this extended domain by collapsing it, viz. by partitioning it homologically into disjoint classes, with respect to the imposed internal resemblance perspective subsumed.

In more general terms, the above algebraic localization structural metaphor is a particular application of the logical conjugation strategy designed for the resolution of a specific problem involving (at least) two delineated structural levels, and based on the existence of a pair of inversely pointing bridges connecting these two levels, as follows: First, by means of an extension bridge, encoding the information of a structural domain into a new extended one assuming existence at a different level. Second, performing the required task at that level by realizing an appropriate equivalence relation, and subsequently forming the associated quotient structure. Finally, by means of a reciprocal bridge, decoding the acquired information in a structural form congruent to the form of the structural domain we started with, according to the specification of the initial level.

4.5 Self-referential Structural Metaphors via Logical Conjugation

At a further stage of development of these ideas, we realize that the successful epistemological implementation of the conjugation strategy, concerning structural metaphors, necessitates primarily the investigation of the meaning of an effective analogical relation within the same algebraic structural genus. This task is important, because it clarifies the nature of an indirect analogical self-referential relation taking place within a certain closed structural genus. From the general context of the preceding analysis, it has become clear that at least, referring to the set-theoretic level of magnification, a set can be related to a distinguished part of it by the imposition of an equivalence relation on their jointly formed cartesian product with respect to a resemblance perspective, which reciprocally necessitated the delineation of that distinguished part in the first place. The total process can be cast into the pattern assumed by a self-referential structural metaphor as follows:

Initially, we assume that a set of elements, considered as an individual object within the genus of sets (characterized by the membership relation), can relate to itself by separation of a well-defined part of it, viz. a subset bearing the functional role subsumed by a particular resemblance perspective. In turn, this resemblance perspective can be applied to the extended object obtained from the initial object by adjoining the distinguished part. Finally, using the quotient construction, we collapse the extended object into a new partitioned object belonging to the same genus. Of course, this is only possible if all of the following conditions can be fulfilled: First, if the initial object can split its substance between two internal levels or hypostases within the same genus, such that the latter, formed by extension with respect to a part, is also an object of the same genus encoding the former. Second, if the application of the resemblance perspective on the extended object partitions it into equivalence classes, forcing in this way a homological criterion of identity, or equivalently an indiscernibility relation with respect to this resemblance perspective, at the same level. Thirdly, if the equivalence classes of the quotient can be re-interpreted as elements of a new object of the same genus, being formed at the initial level by identifying equivalent elements with respect to the resemblance perspective.

It is significant to realize that an indirect self-referential relation, implicated by logical conjugation within the same genus, accomplishes precisely the satisfaction of the above conditions. This is possible by means of two inverse internal bridges connecting these two separate levels of hypostasis into a non-contradictory circular pattern as follows: the first bridge carries out the extension process of an object to another level of hypostasis, being formed by adjoining to it a distinguished part, delineated by the functional role subsumed under a resemblance perspective. At the new level, an appropriate equivalence relation on the extended object implements the functional role of the resemblance perspective, viz. implements a homological criterion of identity. As a result, we end up with a partitioning of the extended object into a set of equivalence classes constituted by indiscernible elements with respect to the imposed criterion. Finally, an inverse bridge performs the transition back to the initial level, by collapsing the extended object with respect to the resemblance perspective, and thus, transforming the resemblance relation into an equality (identity) of elements in the quotient set, formed back at the initial level. Notice the crucial point that the quotient structure formed by returning to the initial level has to be again a set-theoretic object, that is, it must be congruent to the structural specification of the initial object we started with.

After this series of remarks, there arises the natural problem of applying the logical conjugation strategy realizing indirect self-referential metaphors into the context of objects belonging to some algebraic structural genus, like groups, rings, and algebras. This becomes possible, if we formalize the notion of a resemblance perspective as an equivalence kernel of a comparison morphism (homomorphism) between structures of the same algebraic genus. Notice that the functional role subsumed by a resemblance perspective elevates the relation of equivalence among elements belonging into the same equivalence class at the level reached at by descending the first bridge, to a relation of equality (homological identity) at the initial level regained by ascending back through the inverse bridge. In turn, this constitutes the precise implementation of what we call a homological criterion of identity.

Set-theoretically speaking, this amounts to the implication that if two elements α and β of the extended set, at the new internal level of hypostasis, are equivalent with respect to a perspective R, viz. $\alpha R\beta$, then their images inside the quotient set, interpreted as new elements, at the initial level, are identical, viz. $[\alpha]_R = [\beta]_R$. Based on this argument, we can deduce the modelling of the notion of a resemblance perspective between structures of the same algebraic genus, by passing into some appropriately restricted type of equivalence relation by means of logical conjugation, depending on the algebraic genus considered.

4.6 Self-Referential Algebraic Kernels of Resemblance

In a general context, the minimum requirements for an algebraic system include the existence of a set S with an equality relation for which there is defined a binary law of composition, viz., a single-valued function of pairs α, β such that $\alpha\beta$ is in S for α, β in S (Bourbaki 1990). Adopting this as our starting point, we superimpose an equivalence relation R on S in order to investigate how a desired restricted type of equivalence relation arises. Namely, denoting by Σ the set of equivalence classes $C_\alpha mod R$, we raise the following question: Can an operation $\bigodot$ be defined in Σ based upon the composition operation in S?

We proceed along the lines of what might be a first attempt to investigate this question by defining:

$$C_\alpha \bigodot C_\beta = C_{\alpha\beta}$$

The above apparently makes the product dependent upon the choice of class representatives. This deficiency can be amended by requiring that, if $C_{\acute{\alpha}} = C_\alpha$ and $C_{\acute{\beta}} = C_\beta$, then $C_{\acute{\alpha}} C_{\acute{\beta}} = C_\alpha C_\beta$. This amounts to the assertion, if $\acute{\alpha} R\alpha$ and $\acute{\beta} R\beta$, then $\acute{\alpha}\acute{\beta} R\alpha\beta$. Equivalently stated, we obtain the condition: $\acute{\alpha} R\alpha$ implies that $\acute{\alpha}x R\alpha x$ and $x\acute{\alpha} Rx\alpha$ for all x. We call regular those equivalence relations which satisfy the condition above. The latter constitutes a necessary and sufficient condition upon R in order that $C_\alpha \bigodot C_\beta = C_{\alpha\beta}$ stands for a well-defined operation. Then, we can easily deduce that the correspondence φ of S onto Σ defined by:

$\varphi(x) = C_\alpha$ if and only if $x \in C_\alpha$ is an algebraic homomorphism, called the natural homomorphism. Essentially, from a reciprocal standpoint, Σ should be a homomorphic image of S under a correspondence, mapping all elements of S belonging to an equivalence class onto an element of Σ. But the existence of such a homomorphism immediately implies the existence of one mapping the class containing α upon C_α, and the homomorphism property then requires that $C_\alpha \bigodot C_\beta = C_{\alpha\beta}$ holds true.

The central idea explained previously can be now easily applied to structures of some algebraic genus, for example, to groups. In this case, we consider a group S together with a regular equivalence relation R. Then, defining an operation in $\Sigma = \{C_\alpha, C_\beta, ...\}$ according to the composition rule $C_\alpha \bigodot C_\beta = C_{\alpha\beta}$, we obtain a homomorphic image of S. Since a homomorphic image of a group is necessarily a group, we deduce that Σ is actually a group whose identity element is C_e, where e is the identity element of the group S.

The above construction shows that the process of shrinking a group S with the aid of a regular equivalence R produces a homomorphic image Σ of S being also a group, and thus, preserving the structural specification of its algebraic genus. Conversely, given a homomorphic image Σ of S, there is defined a partition, and therefore, an equivalence relation R on S. Moreover, the homomorphism property implies that R is a regular equivalence relation.

In a nutshell, we conclude that in the case of groups, the problem of finding all homomorphic images of S reduces to that of finding all regular equivalence relations over S. For this purpose, we make use of the coset decomposition of a group S with respect to a subgroup H. More precisely, we define $\alpha R \beta$ if and only if $\alpha = h\beta$, where $h \in H$. We can easily show that R is actually an equivalence relation, such that the equivalence class $C_\alpha = H\alpha$, called right coset of H. Moreover, since $\acute{\alpha} R \alpha$ implies that $\acute{\alpha} x R \alpha x$, the equivalence relation R is right regular. But conversely, starting with a right regular equivalence R in S we find that C_e is a subgroup and $C_\alpha = C_e\alpha$, since $\beta R \alpha$ implies that $\beta\alpha^{-1} Re$; hence $\beta\alpha^{-1} \in C_e$, or, $\beta \in C_e\alpha$ and conversely. Thus, the problem of finding the various right regular equivalence relations in S is reduced to the problem of determination of the right coset decompositions of S with respect to its subgroups.

Precisely analogous considerations establish that the various left regular equivalence relations in S are completely determined by the left coset decompositions of S with respect to its subgroups. Thus, we conclude that, if R is a regular equivalence relation, then, on the one side, it defines a left coset decomposition with respect to the subgroup H of all elements x such that xRe, and on the other side, it defines a right coset decomposition with respect to the same subgroup. Hence R stems from a subgroup for which the left cosets are identical with its right cosets. Such a subgroup N is called a normal subgroup of S, satisfying: $xN = Nx$ for all x in S. Thus, a regular equivalence relation R in S stems from a normal subgroup N of S, viz., a subgroup remaining invariant under logical conjugation, meaning that $N = xNx^{-1}$ for all x in S. Conversely, a normal subgroup of S defines a regular equivalence relation on S. Now, if N is a normal subgroup of S, then its cosets $C_x = xN$ form a group with the following composition rule of closure: $\alpha N \bigodot \beta N = \alpha\beta N$, or equivalently, $C_\alpha \bigodot C_\beta = C_{\alpha\beta}$ holds. The resulting quotient $\Sigma = S/N$ is a group homomorphic to S and constitutes that group, which collapses the normal subgroup N of S to the identity element of Σ. Conversely, every homomorphic image of S can be duplicated by, viz. it becomes isomorphic to such a quotient group.

The completely analogous analysis for the case of rings yields the corresponding homomorphism theorem with the same efficiency. Thus, we have deduced the modelling of the notion of a resemblance perspective between structures of the same algebraic genus, by the concept of regular equivalence relations. Consequently, the implementation of self-referential metaphors within the context of objects belonging to some algebraic structural genus becomes possible if we formalize the notion of a resemblance perspective precisely as a regular equivalence kernel of a comparison morphism (homomorphism) between structures of the same algebraic genus.

More concretely, in the case of groups, we have the following: Let S and T be groups and let ϕ be a group homomorphism from S to T. If e_T is the identity element of T, then the kernel of ϕ is the subset of S consisting of all those elements of S which are being mapped by ϕ to the element e_T:

$$Ker(\phi) = \{x \in S : \phi(x) = e_T\}$$

Since a group homomorphism preserves identity elements, the identity element e_S of S must belong to $Ker(\phi)$. By the preceding analysis, it turns out that $Ker(\phi)$ is actually a normal subgroup of S. Thus, we can form the quotient group $S/Ker(\phi)$, which is naturally isomorphic to $Im(\phi)$, viz. the image of ϕ (which is a subgroup of T).

Analogously, in the case of rings with a unit element we have the following: Let S and T be rings and let ϕ be a ring homomorphism from S to T. If 0_T is the zero element of T, then the kernel of ϕ is the subset of S consisting of all those elements of S which are being mapped by ϕ to the element 0_T:

$$Ker(\phi) = \{x \in S : \phi(x) = 0_T\}$$

Since a ring homomorphism preserves zero elements, the zero element 0_S of S must belong to the kernel. It turns out that, although $Ker(\phi)$ is generally not a subring of S, since it may not contain the multiplicative identity, it is nevertheless a two-sided ideal of S. Thus, we can form the quotient ring $S/Ker(\phi)$, which is naturally isomorphic to $Im(\phi)$, viz. the image of ϕ (which is a subring of T).

The effective generation of self-referential structural metaphors via logical conjugation in the context of some algebraic genus, implicated by the action of regular equivalence relations within this genus, provides a powerful methodological device for the resolution of a wide range of problems. Moreover, a self-referential structural metaphor may be combined with another type of metaphor, for instance a genus to species metaphor.

As an example, we may consider the case of the genus of (finite) groups. We have already seen previously that a regular equivalence relation on a group corresponds to a normal subgroup of this group, interpreted as an internal resemblance perspective.

More precisely, this resemblance perspective constitutes the regular equivalence kernel of the homomorphism from the group to its corresponding quotient group. A natural problem arising in this context refers to the possibility of decomposition of a group into a finite series of non-further decomposable groups (simple groups) using the method of division with respect to internal resemblance perspectives, namely, with respect to normal subgroups. This is the problem of solvability of a group-theoretic structure, which has been first posed in the context of Galois theory (Cox 2004). If solvability is attainable, then the initial group can be thought of as being decomposed into a finite series of irreducible group layers (factor groups) being adjoined to each other in a proper way.

This problem can be successfully tackled by means of the conjugation strategy, if we combine the previously explained self-referential structural metaphor with a genus to species metaphor between the genus of multiplicative groups and the species of the integers. In the context of the latter metaphor, if a group corresponds to an integer, then a normal subgroup corresponds to a divisor of this integer and the associated quotient group corresponds to the quotient of the integer by the divisor. Furthermore, a non-further decomposable group (simple group) corresponds to a prime integer number, and finally, the notion of decomposition of a group into a finite series of simple groups using the method of division with respect to normal subgroups corresponds to the Euclidean algorithm for divisibility of the integers.

4.7 Logical Conjugation via a Gnomon: Homological Criterion of Identity

It is instructive to emphasize that the appropriate operational implementation of all different manifestations of the logical conjugation strategy rests only on two prerequisites:

First, the ability to induce a meaningful stratification into different levels which can be connected by means of encoding and decoding bridges. In the general case, we may think of these levels as structural ones. The stratification may even involve substructures of an initially given structure, delineated according to a specific characteristic and adjoined to the initial structure, as separate levels. The latter is particularly suited to the resolution of self-referential problems through a cyclical conjugation process by means of the reciprocal and reflexive techniques of descending and ascending.

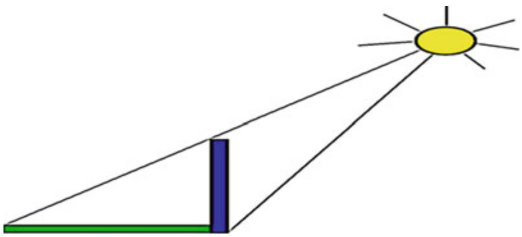
Second, the ability to establish a relation of homology among the stratified levels. It is precisely the ingenuity of a homological criterion that provides the seed for the successful implementation of the logical conjugation strategy. Put differently, an effective analogical relation or metaphor subsumed by logical conjugation requires an appropriate criterion of homology among stratified levels in order to operate. We point out that the notion of metaphor literally means transport. Thus, logical conjugation can be conceived as a logical transport process involving at least two separate levels according to a specific criterion of homology among these levels. We also note that metaphor may refer to transport of information or structure or matter or energy or whatever else this notion can refer to, whereas the logical conjugation strategy via which it takes place is indifferent to its particular qualifications. This provides the sought for universality in the application of logical conjugation in different fields.

From the above, we deduce that what is crucial for the logical conjugation method is the establishment of some appropriate homological criterion operating among the stratified levels. Then, based on this homological criterion it becomes more tractable to devise appropriate encoding and decoding bridges connecting reciprocally all different levels and effectuating a metaphor process. It is interesting to note that from the present viewpoint the notion of homology bears a logical function although it is usually introduced and implemented via topological means. At least, it is important to stress that a homological criterion is independent of local metrical spatiotemporal distance notions. For this reason, it can operate non-locally or among different scales. The ubiquity of a homological criterion is that it establishes some particular measure of invariance among the stratified levels. This measure can be expressed as an arithmetic invariant, like a ratio or a fraction, or even in structural terms like a group or groupoid. The essential thing is that inter-level connectivity, or simply a process of metaphor, requires a homological criterion in order to be expressed via the logical conjugation strategy and conversely.

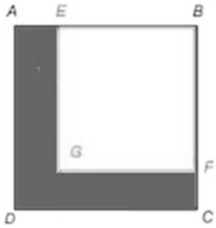
In standard mathematical terminology, what we call a homological criterion appears in a variety of different formulations, which are unified conceptually from our perspective. This unification is facilitated by means of logical conjugation and its net effect, which is metaphor according to some qualification, and ultimately as an effective means of copying with complexity and self-reference. For instance, a homological criterion may be expressed in the simplest possible manifestation as a relation of homothesis or proportionality of integer magnitudes as in the original Thalesian conception. It may also be expressed as a relation of similarity between two square matrices, where the homological criterion is the representation of the same linear transformation with respect to two different basis of a vector space. In this case, the logical conjugation strategy resolves the problem of diagonalization via the method of eigenvalues. In the field of differential topology and differential geometry a homological criterion is provided by the notion of a local homeomorphism or local diffeomorphism correspondingly (Bredon 1997). We may note parenthetically that from the perspective of logical conjugation the notions of topological or differential manifolds defined by descending to simpler spaces like the Euclidean ones and then ascending back via the method of gluing from the local to the global level are solely needed for the formulation of the metaphor process of differentiation, called covariant transport, and giving rise to the invariants of curvature (Mallios and Zafiris 2016). Finally, a homological criterion may be literally expressed in standard algebraic topological terms, viz. in terms of homology and cohomology theory (Hatcher 2002; Mac Lane 1998). In broad terms, homology theory establishes invariant measures of topological similarity in terms of a series of groups stratified into different scales or dimensions. The topological similarity is defined by means of classifying chains of connectivity into two classes, called cycles and boundaries correspondingly. More precisely, two cycles are homologically equivalent if they differ by a boundary. The dual theory, called cohomology theory, is based correspondingly on the notion of cochains of connectivity, which are classified respectively into cocycles and coboundaries. In this case, two cocycles are cohomologically equivalent if they differ by a coboundary. For example, in the case of de Rham cohomology theory, the cocycles are represented as closed differential forms and the coboundaries as exact differential forms.

A natural question arising in this context is the following: Notwithstanding the technicalities involved, for example in the setting of homology and cohomology theories of various forms, is there a guiding concept that lends itself to a proper and efficient depiction of a homological criterion? In other words, what is the common thread between the homothesis equivalence relation and the more sophisticated algebraic-topological homology equivalence relation making them both amenable by means of the logical conjugation strategy?

We argue that the common conceptual thread for establishing a proper homological criterion is provided by the use of a gnomon. The intuitive idea of a gnomon also makes more easily conceptualized the quite abstract notion of an algebraic kernel of resemblance, developed in the previous section. The best definition of the notion of a gnomon has been given by the great mathematician Heron of Alexandria in the following terms: A gnomon is that form which, if it is adjoined to some originally given form, results in a new extended form being similar or homologous to the original one. In order to understand the depth of this simple looking definition of a gnomon it is necessary to start from its initial conception in the context of the Thalesian theory of homothesis. In this context, the gnomon is literally speaking the part of the sundial that casts the shadow.

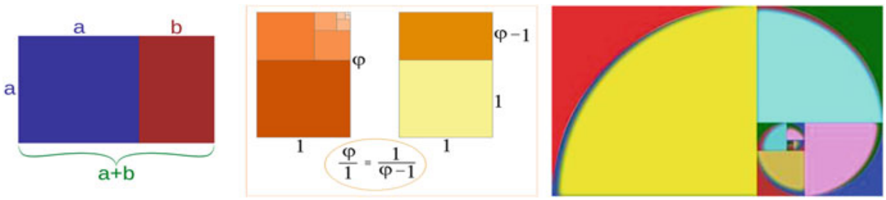


We can easily see that it is exactly the adjunction of the gnomon to the pyramid, which induces a homothetic equivalence relation between the level of objects and the level of their shadows with reference to their magnitudes at the same time of the day, and consequently makes logical conjugation operative for the determination of the not directly accessible magnitude of the height of the pyramid in terms of proportion. In its simplest possible form the general process of adjoining a gnomon in order to obtain a relation of homothesis may be visualized as follows:

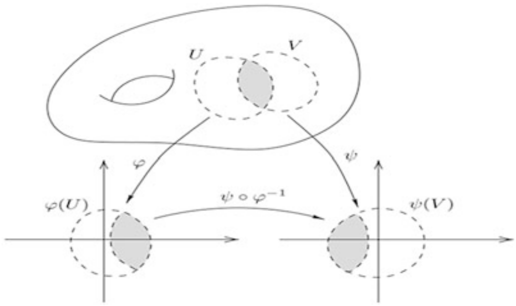


Formally, the relation of homothesis is an equivalence relation, and thus induces a partition into equivalence classes standing for the blocks or cells of this partition. The quotient structure obtained by factoring out this equivalence relation incorporates a new criterion of logical identity in comparison to the initial one, which is precisely characterized in terms of the chosen gnomon of homothesis. In other words, the notion of logical identity is relativized with respect to the gnomon, such that the unit element of the quotient structure expresses equivalence modulo the gnomon.

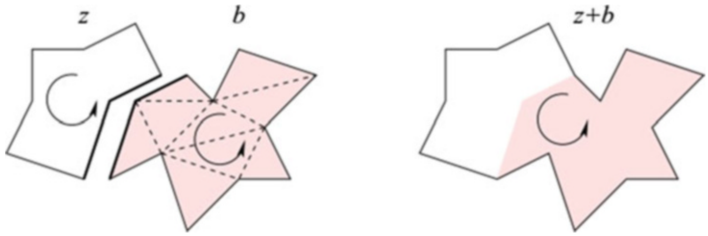
In the case of homothesis or proportionality of magnitudes, the metaphoric aspect of logical conjugation may be easily visualized in terms of a recursive or periodic application of a gnomon. This leads naturally to the dynamical notions of gnomonic growth or unfolding and reciprocally gnomonic subdivision or folding by means of logical conjugation. A particular well-known example is provided by the function of the golden mean gnomon, depicted graphically as follows:



The conclusion obtained from the analysis of the notion of a homothetic gnomon can be extrapolated to more complex situations, where a more general homological criterion is required for the effective application of logical conjugation. The abstraction consists in thinking of a gnomon as a means to indicate, or discern, or distinguish, or to set a boundary. The function of a gnomon is again to induce a certain type of modularity incorporating a logical criterion of identity, which is effectuated homologically. For instance, in the case of a manifold, the gnomon is a local Euclidean space and the homological criterion is subsumed by the notion of a local homeomorphism. The modularity type is expressed by the gluing conditions of local Euclidean patches adjoined homologically to a globally intractable space endowing it with the structure of a manifold. The logical conjugation strategy is used as a means to resolve a difficult problem for manifolds in terms of simpler problems, which can be solved at the level of local Euclidean patches and their amalgamations. Equivalently put, this logical method conjugates a complex problem at the manifold level to a simpler problem at the local Euclidean level where it can be directly resolved. The efficiency of logical conjugation rests on the fact that we are able to descend and ascend between these levels due to the homological criterion enforced by the associated gnomon.



Finally, it is worth explaining the notion of gnomon employed in standard homology theory, as it is conceptualized in algebraic topology. In this case, the role of a gnomon is played by the notion of a boundary. We remind that chains of connectivity in homology theory are classified in terms of cycles and boundaries. Intuitively, a boundary at some dimension is a bounding chain of a higher dimensional topological form, whereas a cycle stands for a non-bounding chain. Visually, non-bounding chains may be thought of in terms of holes or punctures or higher dimensional cavities, whereas boundaries may be thought of in terms of filled, and thus bounding chains. The basic idea of a boundary as a gnomon, establishing a homological criterion such that logical conjugation can operate, is that adjoining a boundary to a cycle gives a topologically similar or homologous cycle. Thus, two cycles differing by a boundary belong to the same homology equivalence class as it is depicted visually below.



In this sense, homology equivalence classes, which are actually abelian groups due to the algebraic operations involved in composing chains and orienting boundaries, enfold the invariant information of holes and cavities of topological forms. We emphasize again that these group invariants are obtained solely by the logical conjugation strategy on the basis of the homological criterion of identity set up by the notion of a gnomonic boundary.

4.8 Logical Conjugation Applied to Incompleteness: Gödel's Gnomon

In this section we will attempt to understand the formulation of Gödel's first incompleteness theorem (Gödel 1992), see also Franzen (2005), from the perspective of the logical conjugation strategy. Stated concisely, the essence of this theorem says that if an arithmetic structure endowed with the operations of addition and multiplication is consistent, then it contains undecidable propositions, viz. propositions whose truth or false valuation cannot be proved within this arithmetic structure.

The key to understanding Gödel's argument from our perspective consists in delineating the stratification of the argument into levels and uncovering the gnomon which induces an appropriate homological criterion and permits the metaphor by conjugation or descent and ascent between these levels. Gödel's argument requires a stratification into two levels, one of which is called the mathematical level and the other the metamathematical level. Intuitively, the mathematical level involves general propositions about numbers and the metamathematical level involves general propositions about general propositions about numbers. The argument refers to a true proposition at the metamathematical level, whose truth is established by logical conjugation through the mathematical level. It is clear that Gödel's argument involves an indirect self-reference, which is absolutely legitimate since it is implemented via descending to and ascending from the mathematical level. The crucial thing to realize is that Gödel used a gnomon to express his theorem, which induces a process of metaphor based on a homological symmetry between the metamathematical and mathematical levels. The Gödelian homological criterion between these two levels is a gnomon of numbering or ordering. Gödel's gnomon is used to establish encoding and decoding reciprocal translation bridges between the two levels, such that a particular argument obtained at the mathematical level by means of a process called Cantor's diagonalization (Smullyan 1991, 1994) is transferred by logical conjugation to the metamathematical level in order to prove the theorem.

In more detail, it is significant to explain the function of Gödel's gnomon. Since the alphabet of arithmetic is countable, it is possible to instantiate a fixed schema of numbering or ordering, which assigns a unique positive integer to every legitimate arithmetic formula. The same schema can be extended to order finite strings of arithmetic formulas. Of course, there exist many such appropriate schemas of ordering or numbering, but the essential idea is that by fixing any one of them the function of ordering or numbering can be carried out. For example, we may fix the ordering gnomon provided by the natural numbers' sequence, such that every arithmetic formula and every finite string of arithmetic formulas is assigned a unique number in this sequence, called its Gödel number. It is immediate to realize that in the way described the structure of natural numbers may be adjoined to the structure of an arithmetic. In particular, the proof of an arithmetic formula K constitutes a finite string ending with K itself, and thus proofs are naturally assigned Gödel numbers in the ordering. Gödel considers the proposition $p(x, y)$ at the metamathematical level stating the following: "x is the Gödel number of an arithmetic formula whose proof has Gödel number y". Then, still at the metamathematical level, considers the associated proposition $\forall y \neg p(x, y)$, which reads as follows: "No number y is the Gödel number of a proof of the arithmetic formula whose Gödel number is x". The last proposition simply means that the x-th formula in our ordering schema is not provable.

The crucial think to notice is that in the last proposition the variable x is a free variable. Then, the natural question to ask is the following: Is the proposition $\forall y \neg p(x, y)$ at the metamathematical level Gödel-numberable itself, viz. does the Gödel gnomon applies to this proposition? This is the crux of the matter because, as we already know, a gnomon is effective if it enforces a homological criterion to the structure it is adjoined to. Clearly, such a homological criterion is feasible in the present case, only if the Gödel gnomon actually assigns a unique number to the proposition $\forall y \neg p(x, y)$, where x is a free variable. It is now clarified why the major part of Gödel's paper (Gödel 1992) is devoted to show that the aforementioned proposition is indeed Gödel-numberable. Let us denote the Gödel number of the metamathematical level proposition $\forall y \neg p(x, y)$, where x is free, by the number ξ at the mathematical level. The homological criterion can now be implemented using Gödel's gnomon by applying Cantor's diagonalization process at the mathematical level in order to achieve closure. This simply amounts to substituting the free variable x in the proposition $\forall y \neg p(x, y)$ by the definite number ξ to obtain now at the mathematical level the proposition $\forall y \neg p(\xi, y)$, which means that the ξ-th formula in our ordering schema is not provable.

A little moment of reflection convinces us about the role of Gödel's gnomon: Note that by applying this gnomon the metamathematical level proposition $\forall y \neg p(x, y)$, where x is free, is precisely mirrored at the number ξ at the mathematical level. This means that the above metamathematical level proposition is homologically identical to a certain arithmetic formula at the mathematical level whose sequential number is ξ modulo the gnomon employed. Equivalently, the metamathematical level proposition $\forall y \neg p(x, y)$, where x is free, is symmetrical modulo the gnomon, and thus homologically identical, with the ξ-th arithmetic formula in the ordering induced by the gnomon at the mathematical level. Now, the process of Cantor diagonalization at the mathematical level involves a reflexive action, since we feed this fixed ordering number ξ as an argument in the place of the free variable x of $\forall y \neg p(x, y)$. In this manner, we obtain a legitimate mathematical level proposition $\forall y \neg p(\xi, y)$, which states that the ξ-th formula in our ordering schema is not provable, since no number y is the Gödel number of a proof of the arithmetic formula whose Gödel number is ξ. Finally, using the homological criterion of identity established by Gödel's gnomon in reverse, we ascend back at the metamathematical level, where we finally obtain a proposition that ascertains its own unprovability. It is precisely this proposition that expresses Gödel's incompleteness theorem itself, since this proposition is undecidable given the consistency of our arithmetic.

It is instructive to highlight that Gödel's gnomon and the previously described logical conjugation strategy between the metamathematical and mathematical levels is operative with respect to the whole structure of an arithmetic, viz. with respect to both the additive and multiplicative structure of an arithmetic system. In case that only the additive structure is considered, Gödel's gnomon does not induce by adjunction a homological criterion between the metamathematical and mathematical levels, and it can be shown that the incompleteness theorem is not valid.

A final remark to also be stressed refers to the observation that Gödel's gnomon, from the perspective of the logical conjugation strategy, effects an indirect self-reference at the metamathematical level via descending to and ascending back from the mathematical level. The metaphorical process of indirect self-reference is actually conducted at the mathematical level by means of an infinite closure operation substantiated by Cantor's diagonalization process. In other words, employing Gödel's gnomon we become able to make indirect self-reference feasible by conjugating it to an infinite closure operation. This also leads to the conclusion that Turing's argument, according to which the halting problem by means of a universal Turing machine is undecidable, should be viewed as the computational variant of Gödel's first incompleteness theorem (Chaitin 2007). The reason is that Turing's argument can be also considered as a logical conjugation argument of the same form, meaning that indirect self-reference at the level of a universal Turing machine is feasible by conjugating it to the infinite closure operation of Cantor's diagonalization. Turing's gnomon is similarly a gnomon of ordering or numbering programs by means of the natural numbers' sequence.

4.9 Logical Conjugation Applied to Algorithmic Complexity and Generic Forcing Conditions: Relating Chaitin's with Cohen's Gnomons

A significant refinement of Gödel's first incompleteness theorem is provided by Chaitin's incompleteness theorem in the context of algorithmic or program-size complexity theory (Chaitin 1987, 2007). The algorithmic complexity of a string is essentially defined by the length of the shortest program that generates this string and then halts. In this sense, a finite string is characterized as random if its complexity is equal approximately to its length. There are strings with arbitrarily large algorithmic complexity and the problem of program-size complexity is undecidable. In this context, Chaitin's incompleteness theorem states that given a consistent arithmetic, there exists a number C depending upon the given arithmetic, such that any proposition of the form "the program-size complexity of the string s is greater than C" is not provable. Thus, since there are true such propositions, it follows that there are propositions of the above form being undecidable within the context of the given arithmetic. From our perspective of logical conjugation, Chaitin's argument is a refinement of Gödel's first incompleteness theorem because it involves a metaphor extension in depth or stacking. First, Chaitin's gnomon is based on counting the number of bits in a program whence the homological criterion is applied for self-delimiting programs, viz. strings having the property that one can tell where they end. Second, Chaitin's program-size counting gnomon is modified probabilistically, by a deeper stage conjugation via the measure-theoretic level involving the probability $P(x)$ that a program will give a number x at the higher level, preserving nevertheless the same homological criterion as applied to self-delimiting programs. This is called the algorithmic probability of x, and a summation of probabilities over all possible outputs x yields the halting probability $\Omega = \sum_x P(x)$, where Ω is interpreted as a random infinite sequence of bits (Chaitin 2006).

Chaitin's incompleteness theorem constitutes a refinement of Gödel's first incompleteness theorem because it involves a deeper stage logical conjugation via the measure-theoretic level. In particular, the halting probability Ω is a random real number (Chaitin 2006; Calude 2007). The most intuitive conception of randomness is tied to the notion of non-predictability. In other words, if one knows the first n-bits of a random sequence, it is not possible to predict the next $n + 1$-bit. Here, the central objects of our attention are elements of the continuum $\{0, 1\}^N := 2^N$.

Elements of 2^N may be viewed either as infinite sequences of bits (infinite strings) or as sets of natural numbers, which can be identified with their characteristic functions. We denote the set of finite binary strings as $2^{[N]}$. The set $2^{[N]}$ can be canonically identified with N, so that subsets of N may be thought of as sets of strings. We also denote the length of a finite string σ by $\mid \sigma \mid$. Using finite binary strings, we may define a topology on 2^N as follows: First, we define the extension of a finite string σ by the clopen set $E(\sigma) = \{x \in 2^N : \sigma = [x]_{|\sigma|}\}$, where $[x]$ denotes the operation of restriction. Second, we consider clopen sets of the form $E(\sigma)$, where σ is a finite binary string, as the base of a topology on 2^N, where each $E(\sigma)$ is a basic clopen set, to be thought of as an interval in the continuum. In particular, we may identify 2^N with the interval of real numbers $[0, 1]$ by associating each real number with its usual binary representation. If we regard μ as the Lebesgue measure on $[0, 1]$, then we have that $\mu(E(\sigma)) = 2^{-|\sigma|}$. Now, we expect that non-random sequences form a set of measure zero. Intuitively, using the above defined topology, we require that the extensions of longer and longer initial segments σ of a string $x \in 2^N$ become arbitrarily small. In this manner, random sequences are defined from a complementary viewpoint measure-theoretically on the basis of the fact that non-random sequences should form sets of measure zero.

Now, if we come back to the intuitive conception of randomness as related with non-predictability, we may require that there is no algorithm α which can ever compute, and thus uniformly measure, $[x]_{|\sigma|}$ from any sorter string. Here an algorithm is considered as a function $\alpha : 2^{[N]} \to \{0, 1\}$. The above idea constitutes, in effect, a complexity measure based on program-size. The notion of program-size complexity introduced by Chaitin to this effect regards σ as a self-delimiting program, viz. as a program delimited by an end-marker (Chaitin 2006). Clearly, no extension ϱ of a self-delimiting program can be a self-delimiting program, since the end-marker will not be in the right place. If $\psi : 2^{[N]} \rightarrow 2^{[N]}$ is a partial recursive function with prefix-free domain, viz. computable by a self-delimiting reference universal Turing machine, the Chaitin complexity of σ, or the algorithmic information content of σ is defined by $I(\sigma) = \min\{| \tau |: \psi(\tau) = \sigma\}$. This is the length of the shortest program τ of the self-delimiting universal Turing machine that outputs σ. Then, we define an infinite sequence $x \in 2^N$ to be random if all its extensions have high Chaitin complexity, capturing in this way the above intuitive conception of randomness. More precisely, an infinite sequence x is random if and only if there exists a constant k, such that $(\forall n)[I([x]_n) \geq (n - k)]$. The infinite sequences that satisfy this condition form a set of measure one, and thus random sequences form a set of measure one. This result is in good compatibility with the measure theoretic characterization of non-random sequences as sets of measure zero derived in the previous paragraph. In this sense, the characterization of random sequences according to Chaitin or program-size complexity is in agreement with the measure-theoretic characterization completing the logical conjugation.

Chaitin's incompleteness theorem constitutes not only a refinement of Gödel's first incompleteness theorem due to the deeper stage logical conjugation via the measure-theoretic level, or equivalently via the program-size complexity level, but it also contains the germs of two powerful generalizations: The first comes from a deeper level conjugation via the level of generic sets and Cohen's forcing conditions (Cohen 2008) based on an analogical type of relation between the notions of random sets and generic sets. The second comes from an interpretation of the constant involved in the definition of random sequences in terms of an uncertainty relation between two conjugate domains in the spirit of Heisenberg's uncertainty principle in quantum mechanics (Heisenberg 1949). Both of these generalizations will be examined in detail in another place. Currently, it is important to explain the concepts involved in the interrelations between Chaitin's gnomon with Cohen's gnomon on the one hand, and Chaitin's gnomon with Heisenberg's gnomon on the other hand. According to the knowledge of the author the proposed interrelations have not been considered in the literature before. Not surprisingly both of them involve the notion of logical conjugation.

Let us start to explain the logical conjugation strategy involving a deeper level metaphor stacking via the level of generic sets related with Cohen's method of forcing (Cohen 2008), or equivalently the level of Boolean-valued sets (Bell 1988). This conjugation is based on the analogical relation between random sets and generic sets. Both of them can be formulated as Boolean-valued models of set theory, or equivalently as variable sets, called sheaves, over a Boolean algebra. In the first case the Boolean algebra is identified with the Borel algebra of clopen sets defined above modulo the sets of measure zero (non-random sequences), whence in the second is identified with a Boolean algebra of Cohen forcing conditions. In this manner, the proposed deeper level logical conjugation stacking views random sequences as Cohen forcing conditions with respect to a Boolean measure algebra in the context of a Boolean-valued model of set theory containing a consistent arithmetic. Intuitively stated, the sets in this Boolean-valued model, or equivalently the sheaves over the Boolean measure algebra, are to be thought of as sets, whose elements are not evaluated to the two-valued Boolean algebra 2, but are evaluated on the clopen sets of the Boolean measure algebra.

For the sake of completeness, it is instructive to explain in more detail the logical conjugation via the deeper level of Boolean-valued sets. We may think that we start from a standard model of set theory, which we agree to call constant sets. The elements of constant sets are characterized by valuations in the two-valued Boolean algebra 2. Then we adjoin a multiplicative encoding bridge from the level of constant sets to the level of variable sets, which in this case are the sets varying over a Boolean algebra. From Stone's representation theorem for Boolean algebras (Johnstone 1986), we construct a totally disconnected compact Hausdorff space (Stone space) and we think of the variation in terms of the function space of measurable functions over this space.

If we arrest the variation at a point of this space, viz. at a principal ultrafilter of the associated Boolean algebra, then we define a homological criterion of identity by the stipulation that two functions are equivalent if their measurable values agree at this point. Thus, after having identified the equivalence classes induced by this criterion, we can ascend back to the level of constant sets. In other words, the quotient set obtained is a standard set and we have come back full circle via Stone duality. Of course, if we decide to arrest the variation at an ideal point of the Stone space, viz. at a non-principal ultrafilter of the Boolean algebra, then a new possibility arises. More concretely, if we apply the same homological criterion for ideal points, we obtain a new quotient set at the level of constant sets, which is an extension of the constant set we started with, called a Boolean ultrapower of this set. The Boolean ultrapower is a new constant set, which is internally indistinguishable from the initial set we started with.

Cohen's forcing method via the gnomon of generic sets is a refinement of the method of evaluation at ideal points aiming to the construction of new constant sets internally distinguishable from the set we started with. Instead of ideal points, one considers a partially ordered set P of forcing conditions. Arresting the variation with respect to these forcing conditions, one obtains a generic distinguishable extension of the initial set at the level of constant sets, such that a proposition is true in the generic extension if and only if it is forced by some generic forcing condition in P. Note that the generic set of forcing conditions is not contained in the initial constant set, and thus Cohen's forcing requires logical conjugation via the deeper level of variable sets. Moreover, Cohen's method of forcing via some generic set is equivalent to forcing with respect to a Boolean algebra, which in the present case is identified with a Boolean measure algebra. In this manner, we propose that the notion of random sets involved in applying Chaitin's gnomon may be interpreted by logical conjugation via the notion of generic sets involved in Cohen's gnomon.

4.10 Logical Conjugation Applied to Uncertainty Relations: Relating Chaitin's with Heisenberg's Gnomons

Let us now explain the second germ of generalization involved in Chaitin's gnomon in relation to Heisenberg's gnomon in quantum mechanics. We remind that Heisenberg's uncertainty relation involves a limit, defined by Planck's constant, pertaining to the simultaneous determination of two conjugate observables, for example, position and momentum of a quantum system (Heisenberg 1949).

We note that observables in quantum mechanics are defined as self-adjoint operators, bearing thus a spectral resolution in terms of projection operators. In this way, each observable is associated with a complete Boolean algebra of projection operators obtained by its spectral decomposition (Epperson and Zafiris 2013; Davis 1977; Heelan 1970; Selesnick 2004). If two observables commute, then they can be resolved by means of a common Boolean algebra of projectors. In other words, a commutative algebra of observables is logically characterized by means of the Boolean algebra of projectors (idempotent elements of the commutative algebra), which simultaneously resolve all the observables belonging in this algebra. The non-commutativity of observables like the position and the momentum of a quantum system, quantified by means of Heisenberg's uncertainty principle, signifies the fact that there does not exist a universal Boolean algebras of projectors resolving all the observables in quantum mechanics. Thus, the internal logic of a quantum system is not a Boolean logic of projection operators, but a globally non-Boolean amalgam of local Boolean patches, where each patch covers the manifestation of a maximal commutative algebra of simultaneously measurable observables (Zafiris 2006a,b, 2007; Zafiris and Karakostas 2013). Non-commutative observables like position and momentum belong to two different Boolean patches, which cannot be amalgamated together simultaneously under a bigger Boolean patch. Notwithstanding this fact, a position observable can be transformed to a momentum observable by means of a unitary transformation and conversely, viz. the well-known Fourier and inverse Fourier transform. In this manner, the position and momentum Boolean patches constitute two conjugate logical domains, which cannot be subsumed under a universal Boolean domain, and thus are complementary in Bohr's terminology (Bohr 1958).

These conjugate Boolean domains correspond to conjugate Boolean projection-valued measure algebras. Note that each Boolean algebra of projectors gives rise, using Cohen's gnomon in this context, viz. logical conjugation via the level of variable sets as above, to a generic set of forcing conditions. Then, a proposition is true in the generic extension, obtained as explained previously, if and only if it is forced by some generic forcing condition. This is suited in understanding the measurement process of an observable in quantum mechanics, where a proposition refers to the result of a measurement on this observable and the generic forcing condition corresponds to the projection operator of a measurement device which clicks upon registration of this result.

The difference in comparison to the previous case, appearing for the first time in quantum mechanics, is that there exist distinct local generic sets of forcing conditions corresponding to conjugate observables, which cannot be subsumed under a universal global generic set. Hence, in a well-defined sense, which can be made precise using the theory of sheaf-theoretic localization of observables (Mallios and Zafiris 2016), the logical treatment of quantum mechanics requires a localization of Cohen's gnomon of forcing, with respect to local Boolean domains, and thus giving rise to generalized local models of set theory called topoi. In turn, this logical localization with respect to conjugate Boolean valued sets gives rise to the phenomena of contextuality in quantum theory. We interpret Heisenberg's uncertainty principle as setting the bound (in terms of Planck's constant) of the simultaneous determination of two conjugate observables with respect to the same Boolean domain of measurement. This is expressed in terms of the standard deviations in the expectation values of conjugate observables in the form $\delta x \cdot \delta p \geq \hbar/2$, where $\hbar := h/2\pi$ in the case of position and momentum observables. Each of these observables is considered as a Boolean homomorphism from the Borel measure algebra of the real line (where the results of measurements are recorded) to the corresponding Boolean patch containing the respective projections in the spectral resolution of these observables.

Let us now examine if Chaitin's gnomon can be presented in a form giving rise to an uncertainty relation between two conjugate Boolean domains. The first Boolean domain we consider is the domain of random real numbers in the continuum $[0, 1]$. We remind that we identify 2^N with the interval of real numbers $[0, 1]$ by associating each real number with its binary representation. Moreover, if we regard μ as the Lebesgue measure on $[0, 1]$, we have that $\mu(E(\gamma)) = 2^{-|\gamma|}$, where γ is a finite binary string, to be thought of as a program of a self-delimiting universal Turing machine ψ. For an output χ of this machine, we have immediately that the probability of χ is given by:

$$P(\chi) := \mu(\chi) = \sum_{\gamma:\psi(\gamma)=\chi} 2^{-|\gamma|}$$

Chaitin's $\Omega = \sum_\chi P(\chi)$ is a random infinite sequence of bits, and thus a random real in $[0, 1]$ of Lebesgue measure one. It is interpreted as the halting probability of ψ, viz. the probability that ψ halts when its binary input is chosen randomly bit by bit, such as by flipping a coin. In practice, we may only compute finitely many digits of Ω.

The second Boolean domain we consider is the domain of program-size complexity. If $\psi : 2^{[N]} \to 2^{[N]}$ is a partial recursive function with prefix-free domain, viz. computable by a self-delimiting universal Turing machine, the Chaitin or program-size complexity of χ, or even the algorithmic information content of χ is defined by:

$$I(\chi) = \min\{| \gamma |: \psi(\gamma) = \chi\}$$

The complexity measure $I(\chi)$ is the length of the shortest program γ of the self-delimiting universal Turing machine that outputs χ. Moreover, an infinite sequence x is random if and only if there exists a constant k, such that:

$$(\forall n)[I([x]_n) \geq (n-k)]$$

The infinite random sequences that satisfy this condition form a set of measure one, and thus for Chaitin's Ω we obtain:

$$(\forall n)[I([\Omega]_n) \geq (n-k)]$$

The above inequality is interpreted clearly as an uncertainty relation pertaining to the conjugate Boolean domains of random real numbers in [0, 1] and program-size complexity length measures. Since it is an uncertainty relation between two conjugate Boolean domains, these domains cannot be embedded in a universal Boolean domain simultaneously subsuming both of them. Thus, the constant k is interpreted as setting the bound of the simultaneous determination of two conjugate observables, viz. the random real Ω in [0, 1] and the program-size complexity length measure I.

4.11 Composite Logical Conjugation and Galois Solvability

We have analyzed previously that both Heisenberg's and Chaitin's logical conjugation methods give rise to uncertainty relations between two conjugate or complementary Boolean domains which cannot be subsumed under a common universal Boolean domain simultaneously with absolute precision. Moreover, if we consider each Boolean domain separately we may interpret it as a Boolean algebra of generic forcing conditions, descend to the level of Boolean valued sets, then apply a Cohen-type criterion of homological identity with respect to these forcing conditions, and finally ascend back to the initial level of constant sets, obtaining in this manner a generalized model internally distinguishable from the one we started with. The latter reflects the intervention of a suitable measurement procedure for obtaining information with respect to an observable being logically classified by this Boolean domain. The logical classification takes place via the procedure of spectral resolution in terms of a Boolean algebra of projectors in the context of operator functional analysis, or more generally, via the procedure of measurability in terms of a Borel measure algebra, which can even be projection-valued. The important point to be emphasized is that the Cohen-type strategy of logical conjugation cannot be implemented simultaneously with respect to two complementary Boolean domains.

A natural question arising in this context is if it possible to implement the strategy of logical conjugation in such a way that circumvents the above obstacle. We may think of each logical Boolean domain as giving rise to a separate gnomon of conjugation. If we consider two complementary Boolean domains, we cannot apply the method of logical conjugation with respect to both of them simultaneously, but there exists the possibility of composing these two gnomons in an appropriate way. Since we consider these two gnomons as complementary in a precise sense, justified by the existence of an uncertainty relation as above, then the most economical hypothesis is to assume that each gnomon may conjugate the complementary one. In other words, the levels between each gnomon operates should function as the encoding/decoding bridges of the complementary gnomon. We may explain this idea in more detail as follows: We remind that the method of logical conjugation requires a stratification into levels and the delineation of encoding/decoding bridges between these levels in order to be able to descend and ascend back. Now each Boolean domain of discourse provides a natural stratification as well as a natural descending/ascending bridging between the strata, which can be conceptualized via Cohen's gnomon. But, what if there is no intrinsic way of distinguishing between strata and bridges? Reciprocally put, the distinction between strata and bridges is meaningful only under the specification of a Boolean domain. If two complementary gnomons pertaining to two complementary Boolean domains are utilized simultaneously the only way that logical conjugation can function is by reversing the role of strata and bridges with respect to these two gnomons, such that a closure is achieved. Algebraically, the only way that these two complementary gnomons may be glued together simultaneously is by temporarily suspending the rigid distinctions between strata and bridges, and just iterating the process of logical conjugation with respect to the composition of these two gnomons until we reach a closure. The closure corresponds to a non-trivial cycle of compositions. It turns out that the formation of this cycle is equivalent to composite logical conjugation where the levels of one gnomon correspond to the bridges of its complementary gnomon. We present this simple algebraic argument as follows:

A logical conjugation is generally expressed in the symbolic form

$$X = S \circ A \circ S^{-1}$$

which defines X to be conjugate to A under S, where S^{-1} is considered to be the conceptual inverse of S. Now we consider the first two symbols of the conjugation $S \circ A \circ S^{-1}$, viz. $S \circ A$, as a string, and extend this string by adding new symbols at the end, such that every three consecutive symbols form a logical conjugation or equivalently a metaphor. We iterate this operational procedure until we generate a cycle, viz. until the last two symbols are $S \circ A$ again that we started with. In more detail we obtain successively:

$$\begin{aligned} &S \circ A \dashrightarrow S \circ A \circ S^{-1} \dashrightarrow S \circ A \circ S^{-1} \circ A^{-1} \dashrightarrow \\ &\qquad \dashrightarrow S \circ A \circ S^{-1} \circ A^{-1} \circ S \dashrightarrow S \circ A \circ S^{-1} \circ A^{-1} \circ S \circ A \end{aligned}$$

Since the iteration has produced the string $S \circ A \circ S^{-1} \circ A^{-1} \circ S \circ A$, where the last two symbols are $S \circ A$ again, that we started with, we have generated a closure, viz.

a nontrivial conjugation cycle that in linear form reads as follows:

$$S \circ A \circ S^{-1} \circ A^{-1} := [S, A] :=\circlearrowright (S, A) := S \circlearrowright A$$

By a slight abuse of notation we may identify the complementary gnomons by the symbols S, A correspondingly, whence their composition or gluing is denoted by the conjugation cycle $S \circlearrowright A$. Note that the order of composition cannot be reverted, viz. $S \circlearrowright A \neq A \circlearrowright S$, viz. the operation of composition of complementary gnomons is a non-commutative operation. Thus, it is significant to impose an orientation on the conjugation cycle, which reflects the specified cyclic order of composition.

In the case that S, A are elements of a non-commutative group, the composition $[S, A]$ is referred to as the commutator of S, A. In this case the symbols S^{-1} and A^{-1} stand for the group-theoretic inverses of S, A, respectively. This observation leads to the conjecture that the complementarity of conjugate Boolean domains pertains to their Boole group theoretic structures, or else it is of a group-theoretic origin. A Boole group is a group-structure on the topological spectrum of a Boolean algebraic domain. Thinking of two complementary Boolean group domains as local patches of a non-abelian global structure the notion of a conjugation cycle provides a natural method of logically gluing them together. Before we examine the aspects of this logical gluing by conjugation cycles of complementary gnomons it is instructive to start from a reciprocal viewpoint and leverage the existing knowledge about the structure of groups. This will provide the method to locate the existence of complementary gnomons from a group-theoretic perspective. The central notion of significance for our problem has to do with the Galoisian notion of solvability of a group, which we have introduced from our perspective in Sect. 4.6. In particular, the understanding of Galois theory of groups by the strategy of logical conjugation uses the gnomon of solvability. This will be explained in more detail in the sequel, but for the time being it is enough to convey the basic idea.

The triumph of Galois theory (Cox 2004) is based on the theorem that a polynomial equation is solvable by radicals if and only if the corresponding Galois group of the equation is solvable. Now a general group is solvable if it can be derived by the method of group extensions of abelian (commutative) groups. Reciprocally, a solvable group is a group whose derived series terminates in the trivial subgroup. Intuitively, the derived series is a stratification into group levels together with a descending staircase among these strata formed by identifying each subgroup in the descending series with the commutator subgroup of the previous one. In turn, the commutator subgroup of a group is the group generated by all the commutators of this group. The importance of the commutator subgroup of a group rests on the fact that it provides the most economical way (technically it is the smallest normal subgroup) such that the quotient of the initial group by the commutator subgroup is an abelian group. Thus, a group is solvable if by descending into lower and lower subgroup strata by division with the commutator subgroup we end up with the trivial subgroup.

It is well-known that all abelian groups are solvable, as well as that all nilpotent groups are solvable. The first is trivial, but the second is very important, for example, in quantum mechanics. It is worth explaining the latter in more detail. A nilpotent group is a group that may be thought of as an almost-abelian group, in the sense that the commutator subgroup is almost trivial. For instance, we know that in quantum mechanics we have complementary Boolean algebraic domains, like the position and momentum ones. The bounded form of these conjugate observables (or Weyl form) are constrained to obey the canonical commutation relations expressed by means of the infinitesimal Planck's constant, and hence almost commute. These give rise to a nilpotent group, called the Heisenberg group (Weyl 1950). The Heisenberg group is of fundamental importance in quantum mechanics and essentiality constitutes the solvability of the theory in group-theoretic terms. In other words, the non-commutativity induced by any two conjugate or complementary Boolean domains in quantum mechanics is circumvented in an almost-commutative manner by the nilpotency of the Heisenberg group, and thus the solvability of this group. This circumvention is possible in all cases that we have a vector space structure equipped with a symplectic form (Mallios and Zafiris 2016). In other words, the structure of a nilpotent group, induced symplectically, transforms an intrinsic unsolvability of two conjugate domains into a solvable case. From the perspective of logical conjugation this amounts to considering conjugation cycles as infinitesimally small, and thus behaving like covariant differentials or connections in a precise differential geometric sense. This will be treated in detail in a separate paper.

The above analysis invites for a search of the source of intrinsic unsolvability. It is enough to consider the case of finitely generated linear groups, viz. matrix groups which are used as a concrete representation of abstract groups. There we find the astonishing result, called Tits alternative, that a finitely generated linear group is either virtually solvable, meaning that it contains a solvable subgroup involving a finite descending staircase, or it contains a non-abelian (non-commutative) free subgroup in two generators (de la Harpe 2000). Thus, we are able to locate the free group in two generators, denoted by Θ_2, as the actual source of intrinsic unsolvability. Intuitively speaking, the existence of Θ_2 is associated with non-trivial and non-reducible logical conjugation cycles between two complementary Boolean domains. The only way that non-solvability can be traded with or circumvented is by nilpotency, like in the case of the Heisenberg group in quantum mechanics. We remind that there always going to exist uncertainty relations between the observables of two complementary Boolean domains. If the associated constant of interrelation can be made either infinitesimally small or reciprocally very big, then the formed logical conjugation cycles vanish in higher order iterations and the complexity is reducible. It is not an accident that both of our fundamental physical theories, viz. the theory of relativity and quantum mechanics involve this type of constants between conjugate Boolean domains. Thus, from the perspective of logical conjugation, the free group in two generators is the source of logical conjugation cycles and the group-theoretic property of nilpotency is the golden mean between non-commutativity and commutativity.

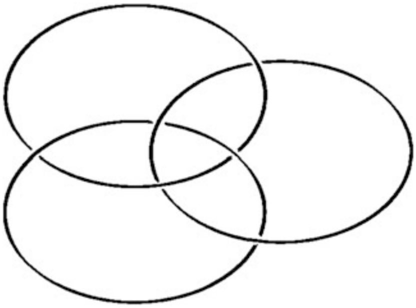
Therefore, it is of high priority to examine in detail the non-abelian free group Θ_2 and provide particular examples of the manifestation of its action. It will turn out that the notion of a conjugation cycle exemplified by the group Θ_2 has a particularly simple representation in three-dimensional space by means of the link topology of the Borromean rings (Zafiris 2016). This is going to be our anchor for the development of our geometric intuition in relation to the action of the group Θ_2. Conversely, difficult problems in the topological theory of links escaping our geometric grasp can be resolved by utilizing the structure of the group Θ_2. This has been already accomplished in the previous chapter, where it has been proved that the Borromean rings and their higher order generalizations, called Brunnian links, constitute the primes in realizing all possible types of topological links. A very interesting aspect of this treatise is that all the constructions can be performed inside the free group Θ_2. The reason is due to the surprising and counterintuitive result that the non-abelian free group in two generators contains copies of all other non-abelian free groups in any finite number of generators as finite index subgroups! Thus, the complexity of non-reducible logical conjugation cycles and their iterations generated by two complementary (in some appropriate sense) gnomons subsumes the whole complexity we may get from any number of gnomons.

We also argue that the group Θ_2 is important in relation to algorithmic information theory. There are two reasons on which we base our claim. The first is based on the fact that elements of Θ_2 can be assigned complexity lengths. Since every element of Θ_2 can be uniquely expressed as a freely reduced word in the generators and their inverses, we may simply define the length of an element as the number of terms in this freely reduced expression. This notion of length has the property that the length of an element equals the length of its inverse element in this group. The second is related with the fact that the group Θ_2 has exponential growth rate (de la Harpe 2000). A deep result of Gromov shows that a nilpotency circumvention reduces the growth rate to a polynomial one.

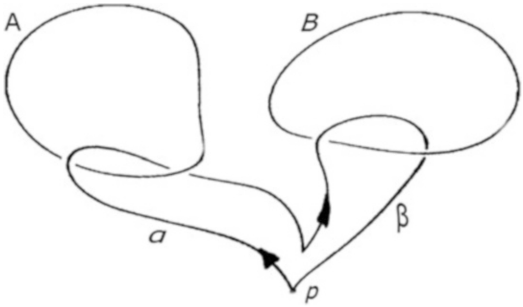
The simplest way to describe the concept of a free group is the following: We consider the set of elements $S = \{x_1, x_2, \dots x_n\}$ in a group G. A word or string $w \in \{S \cup S^{-1}\}^\star$ is said to be freely reduced if it does not contain a substring consisting of an element adjacent to its formal inverse. For instance, the string $w = xyx^{-1}y^{-1} = [x, y]$ is freely reduced, while $z = xy^{-1}yxy$ is not. The group G is a non-abelian free group with basis S if S is a set of generators for G and no freely reduced string in the x_i and their inverses represent the identity of the group. The rank of a free group with basis S is the number of elements of S. We denote a free group of rank 2 by Θ_2. It can be easily shown that all free groups of the same rank are isomorphic replicas of each other. So we may identify all of them and talk of the non-abelian free group on two generators Θ_2. In the sequel, we will uncover the topological semantics implicated by the action of the group Θ_2. In this way, the notion of a logical conjugation cycle can be associated and implemented by a precise topological link, which turns out to be the Borromean rings. The significance of this result is that whenever a copy of the group Θ_2 is identified within an algebraic structure, there always exists a Borromean type of topological connectivity between the represented elements, which gives rise to a non-trivial logical conjugation cycle.

4.12 Representation of Logical Conjugation Cycles by the Borromean Link Topology

From a topological viewpoint the Borromean rings constitutes a link formed by an interlocking family of three loops (tame closed curves), such that if any one of them is cut at a point and removed, then the remaining two loops become completely unlinked (Kawauchi 1996; Lindström and Zetterström 1991; Debrunner 1961; Cromwell 1998). In more precise terms, the Borromean link is characterized topologically by the property of splittability as follows: The Borromean link is a non-splittable 3-link (because it consists of three loops), such that every 2-sublink of this 3-link is completely splittable. It is clear that it is a non-splittable 3-link because not even one of the three loops, or any pair of them, can be separated from the rest without cutting. A 2-sublink is simply any sub-collection of two loops obtained by erasing the loop that does not belong to this sub-collection. Since the Borromean link is characterized by the property that if we erase any one of the three interlocking loops, then the remaining two loops become unlinked, it is clear that every 2-sublink of the non-splittable 3-link is completely splittable.



The topological information incorporated in the characterization of the Borromean link can be encoded algebraically by unfolding carefully the non-commutative group-structure of based oriented loops in a 3-d space representation. First, we consider an unknotted tame closed curve in three-dimensional space. Since any such curve can be continuously deformed to a topological circle, it is enough to think of a circle in 3-d space, denoted by A. Second, we consider a based oriented loop in 3-d space, which may pass through this circle A a finite number of times, each one with a prescribed orientation. A based loop means simply that it starts and ends at a fixed point p of the 3-d space. The orientation of the loop can be thought of in terms of an observer, which is fixed at the point p, such that: If the loop passes through the circle one time with direction away from the observer, it is denoted by α^1, whereas if it passes one time with direction toward the observer, it is denoted by α^{-1}. We note that any other loop with the same properties can be continuously deformed to the loop α. Thus, the algebraic symbol α actually denotes the equivalence class $[\alpha]$ of all loops of kind α, passing through the circle A once with the prescribed orientation. Taking into account the algebraic encoding of based oriented loops in relation to circles in 3-d space, we can define the composition of two oriented loops under the proviso that they are based on the same point p in 3-d space. Notice that the composition operation $\alpha \circ \beta$ of the p-based oriented loops α and β in relation to circles A and B correspondingly is not a commutative operation, meaning that the order of composition is not allowed to be reversed. Clearly, the rule of composition produces a based oriented loop $\alpha \circ \beta$ in 3-d space in relation to the circles A and B in the prescribed order. We think of the composition rule $\alpha \circ \beta$ as the non-commutative multiplicative product of the oriented loops α and β based at the same point p in 3-d space, which we denote simply as $\alpha\beta$. It is immediate to verify that the above defined multiplication is an associative operation.



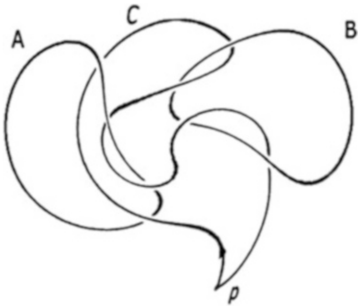
Having established the closure of the elements of the generic form χ under non-commutative associative multiplication as previously, we look for the existence of an identity element, as well as for the existence of inverses with respect to this operation. There is an obvious candidate for each based oriented loop α, namely the loop α^{-1}, where the orientation has been reversed. If we consider the compositions $\alpha \circ \alpha^{-1}$, $\alpha^{-1} \circ \alpha$ we obtain in both cases as a multiplication product the based loop at the same point, which does not pass through any circle at all. Thus, we name the latter loop as the multiplicative identity 1 in our algebraic structure, such that $\alpha\alpha^{-1} = \alpha^{-1}\alpha = 1$. It is also easy to verify that $1\alpha = \alpha 1 = 1$. We conclude that the set of symbols of the generic form χ representing based oriented loops in relation to circles X, endowed with the non-commutative multiplication operation of composition product of loops based at the same point, form the algebraic structure of a non-commutative group, denoted by Θ.

It is instructive to emphasize that the equality sign in the non-commutative group Θ is interpreted topologically as an equivalence relation of p-based oriented loops under continuous deformation. By making use of the multiplication operation in Θ we may form any permissible string of symbols in this group, which can be reduced into an irreducible form by using only the group-theoretic relations $\alpha\alpha^{-1} = \alpha^{-1}\alpha = 1$, $\alpha\alpha = \alpha^2$, and so on. Thus, if we consider only two p-based oriented loops, denoted by the symbols α and β respectively with the prescribed orientation, we form a free group in two generators, denoted by Θ_2.

The property of irreducibility of a string of symbols in the non-commutative group Θ_2 is the guiding idea for the algebraic encoding of the Borromean link in terms of the structure of Θ_2. The crucial observation is that algebraic irreducibility of strings in Θ_2 can be used to model the topological property of non-splittability of a 3-link, where complete splittability of all 2-sublinks is encoded by the unique identity element of Θ_2. In particular, the group-theoretic commutator induced by the generators of Θ_2:

$$[\alpha, \beta^{-1}] = \alpha\beta^{-1}\alpha^{-1}\beta$$

produces an irreducible non-commutative string of symbols in Θ_2. This string represents a new based loop γ as a product loop composed by the ordered composition of the based oriented loops $\alpha \circ \beta^{-1} \circ \alpha^{-1} \circ \beta$. We call the product loop γ the Borromean loop and the formula or multiplicative string $\alpha\beta^{-1}\alpha^{-1}\beta$ in Θ_2 the Borromean loop formula. Thus, we have obtained a topological representation of a logical conjugation cycle!



The algebraic irreducibility of the commutator $[\alpha, \beta^{-1}]$ in the group Θ_2 encodes the topological non-splittability property of the Borromean 3-link. We notice that deletion of both α and α^{-1} (corresponding to removal of the circle A) reduces the formula to the identity 1 (and the same happens symmetrically for both β and β^{-1}). This fact models algebraically in the terms of Θ_2 that every 2-sublink of the Borromean 3-link is completely splittable. We conclude that the topological information of the Borromean 3-link can be completely encoded in terms of the algebraic structure of the non-commutative multiplicative free group in two generators Θ_2. In particular, the group-theoretic commutator $[\alpha, \beta^{-1}]$ in Θ_2, encodes algebraically the gluing condition of the based oriented loops α and β^{-1} (with respect to the circles A and B respectively in the prescribed orientation), and therefore the non-splittability of the Borromean 3-link, together with the complete splittability of all 2-sublinks of this 3-link.

4.13 Logical Conjugation Cycles and Computability on the Sphere: Revisiting Chaitin's Gnomon

The starting point of this investigation is based on the following profound remark of Chaitin (2007, p. 67): "A key technical point that must be stipulated in order for Ω to make sense is that an input program must be self-delimiting: its total length (in bits) must be given within the program itself. (This seemingly minor point, which paralyzed progress in the field for nearly a decade, is what entailed the redefinition of algorithmic randomness.) Real programming languages are self-delimiting, because they provide constructs for beginning and ending a program. Such constructs allow a program to contain well-defined subprograms, which may also have other subprograms nested in them. Because a self-delimiting program is built up by concatenating and nesting self-delimiting subprograms, a program is syntactically complete only when the last open subprogram is closed. In essence the beginning and ending constructs for programs and subprograms function respectively like left and right parentheses in mathematical expressions.

If programs were not self-delimiting, they could not be constructed from subprograms, and summing the halting probabilities for all programs would yield an infinite number. If one considers only self-delimiting programs, not only is Ω limited to the range between 0 to 1 but also it can be explicitly calculated in the limit from below."

Our main interest in this section focusses on the metaphor considering the beginning and ending constructs of self-delimiting programs and subprograms in analogy to the left and right parentheses in mathematical expressions. It is true that our linear representation of strings or words implicates the self-delimiting property by means of left and right parentheses. A natural generalization would be to complete each such pair of parentheses in the 1-dim line to a circle in the 2-dim plane, or equivalently the 1-d complex line. This extremely simple generalization generates immediately two conjugate domains, where each one of them corresponds to the choice of orientation on the circle. If we do not impose any orientation on a circle, it is like we work in the modular arithmetic $\mathbb{Z}_2$, viz. we recover the bit representation of linear strings. Even better, we may complete each pair of parentheses in the 1-dim line to a circle in the one-point compactification of the 1-dim complex line, viz. on the 1-dim complex projective space, or equivalently the Riemann sphere S^2. Can we imagine representing self-delimiting programs by means of circular strings on the sphere S^2?

The choice of the sphere S^2 is not accidental. Without loss of generality we may consider the unit sphere S^2, that is imply the normalization according to which all points lying on the sphere are of distance 1 from the origin. The unit 2-sphere S^2 constitutes the space of pure states, or equivalently rays, of a 2-level quantum mechanical system, called currently a qubit. The unit 2-sphere may be thought of as embedded in the usual three-dimensional space $\mathbb{R}^3$. The Hilbert space of normalized unit state vectors of a qubit is the 3-sphere S^3, and thus the unit 2-sphere is considered as the base space of the topological Hopf fibration (Urbantke 2003):

$$S^1 \hookrightarrow S^3 \twoheadrightarrow S^2$$

We note that each pair of antipodal points of S^2 corresponds to mutually orthogonal state vectors. The north and south poles are chosen to correspond to the standard orthonormal basis vectors $|0\rangle$ and $|1\rangle$ correspondingly. In the case of a spin-$\frac{1}{2}$ system, these simply correspond to the spin-up and spin-down states of this system.

We consider the unit sphere S^2 as the set of points of three-dimensional space $\mathbb{R}^3$ that lie at distance 1 from the origin. Then, the non-commutative group $SO(3)$ denotes the group of rotation operators on $\mathbb{R}^3$ with center at the origin, viz. linear transformations from $\mathbb{R}^3$ to $\mathbb{R}^3$ represented as 3×3 matrices with determinant one. These are called orthogonal matrices, characterized by the fact that their columns form an orthonormal basis of $\mathbb{R}^3$. Rotations around an axis going through the origin are the isometries of three-dimensional Euclidean space $\mathbb{R}^3$ leaving the origin fixed. Note that a 3×3 orthogonal transformation preserves the inner product for any pair of vectors in $\mathbb{R}^3$, and moreover it is an isometry of $\mathbb{R}^3$ that takes the unit sphere S^2 to itself.

In this context, we ask the following question: Does there exist a representation of the non-abelian free group in two generators Θ_2 on the unit sphere S^2, which lifts to a unitary representation on S^3? We remind that the existence of such a representation would imply the action of non-trivial logical conjugation cycles on S^2 and S^3, respectively. Moreover, these logical conjugation cycles would be representable by means of the Borromean link topology. Such a representation definitely exists if we are able to locate a subgroup of the non-commutative group $SO(3)$, which is isomorphic to Θ_2.

We will show in the sequel that this is indeed the case. The proof is based on the observation that there actually exist rotation operators A and B about two independent axes through the origin in $\mathbb{R}^3$ generating a non-commutative subgroup of $SO(3)$, which is isomorphic to the free group Θ_2. In other words, there exists an isomorphic copy of Θ_2 in $SO(3)$ generated by two independent rotations A and B. The term independent refers to the requirement that all rotations performed by sequences of A and B and their inverses are distinct strings in Θ_2.

Actually, we realize that most pairs of rotations in $SO(3)$ are independent in the above sense, so that even by picking A and B randomly would do. For instance, one could consider two counterclockwise rotations A and B about the z-axis and the x-axis respectively of the same angle arccos(3/5). The proof is based on showing that no reduced string in the symbols A and B and their inverses collapses to the identity transformation (3×3 identity matrix). Intuitively, if we choose two counterclockwise rotations A and B about the z-axis and the x-axis of the same angle, then this specific angle needs to be an irrational number of degrees. More precisely, given an initial orientation, if the specified angle is an irrational number of degrees, then none of the distinct strings of rotations in Θ_2 performed by sequences of A and B and their inverses can give back the initial orientation. Thus, no reduced word in Θ_2 collapses to the identity transformation.

The existence of an isomorphic copy of Θ_2 in $SO(3)$ has the following consequence: Each rotation belonging to the non-commutative free subgroup Θ_2 of $SO(3)$ fixes two points in the unit sphere S^2, namely the intersection of S^2 with the axis of rotation passing through the origin. If we take the union of all these points, they form a countable set of points. Then, not only there exists an action of Θ_2 on the unit sphere S^2 (as a subgroup of $SO(3)$ generated by A and B) but this action is actually free on S^2 modulo the countable set of fixed points K.

Thus, we can partition $S^2\backslash K$ into a disjoint union of orbits for the action of Θ_2. If we choose a base point for an orbit, we may identify this orbit with Θ_2 due to the freeness of the action. Moreover, if a countable collection K of points as above is removed from S^2, they can be restored by rotations around an axis through the origin which has zero overlap with K. In this way, the action of the group Θ_2 via strings of rotation operators allows to resolve the whole unit sphere S^2. The crucial point again is that the algebraic irreducibility of the commutator $[A, B]$ of the rotations A and B generating an isomorphic copy of Θ_2 in $SO(3)$ expresses a non-trivial logical conjugation cycle. In turn, such a logical conjugation cycles express the fundamental property of topological Borromean non-splittability or non-separability of these three rotations belonging to the subgroup of $SO(3)$ that is isomorphic with Θ_2.

Most important, this interpretation provides a topological justification of the fact that one cannot specify a finitely additive rotation-invariant probability measure on all subsets of the unit sphere S^2 simultaneously. In the same vein of ideas, if we consider S^2 embedded in 3-dim space $\mathbb{R}^3$, we deduce that it is not possible to specify a finitely additive measure on $\mathbb{R}^3$ that is both translation and rotation invariant, which can measure every subset of $\mathbb{R}^3$, and which gives the unit ball a non-zero measure. This explains why the Lebesgue measure, which is countably additive and both translation and rotation invariant, and additionally, gives the unit ball a non-zero measure, cannot measure every subset of $\mathbb{R}^3$. Thus, it has to be carefully restricted to only measuring subsets that can be Lebesgue measurable.

According to the analysis presented in Sect. 4.11, since the group of rotation operators $SO(3)$ contains an isomorphic copy of the free non-commutative group Θ_2 is unsolvable.

An immediate consequence of the above is that the group of 2×2 complex unitary matrices with unit determinant $SU(2)$ is also unsolvable, that is it also contains an isomorphic copy of Θ_2. The reason is that topologically, the simply-connected special unitary group $SU(2)$ is a covering space of the non-simply connected group of rotations $SO(3)$, and in particular it is a double cover. More concretely, there exists a two-to-one surjective homomorphism of groups:

$$\Delta : SU(2) \twoheadrightarrow SO(3)$$

whose kernel is given by $Ker\Delta = \mathbb{Z}_2 = \{+1, -1\}$.

Hence, it follows that there exists an isomorphic copy of Θ_2 in $SU(2)$. More precisely, if A and B are rotations generating an isomorphic copy of Θ_2 in $SO(3)$, and $\Delta : SU(2) \twoheadrightarrow SO(3)$ is the covering projection, then $\overline{A}$ and $\overline{B}$ generate a free subgroup of the form Θ_2 in $SU(2)$, for any $\overline{A}$ and $\overline{B}$ with $\Delta\overline{A} = A$ and $\Delta\overline{B} = A$. Since $SU(2)$ is a double cover of $SO(3)$ there exist exactly two elements of the form $\overline{A}$, namely U and $-U$ such that $\Delta U = \Delta(-U) = A$ (the same holds for $\overline{B}$ respectively).

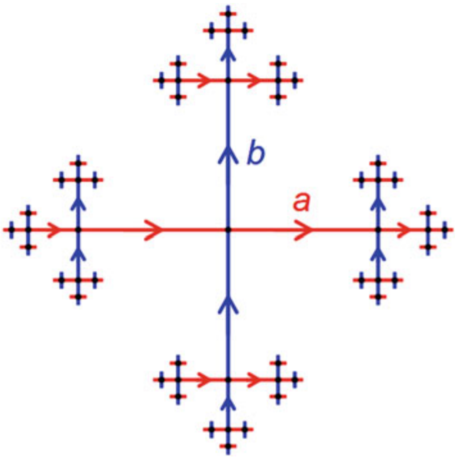
We conclude that there exists a representation of the group Θ_2 on the unit sphere S^2, which lifts to a unitary representation on S^3. The representation of the group Θ_2 on the unit sphere S^2 is given by the free subgroup of rotations of $SO(3)$ generated by A and B according to the above. Concomitantly, this representation lifts to a unitary representation on S^3 by the free subgroup of unitary operators of $SU(2)$ generated by $\overline{A}$ and $\overline{B}$. Thus, the Hilbert space of normalized unit state vectors of a qubit or of a spin-$\frac{1}{2}$ system carries a unitary representation of the group Θ_2. This means that the algebraic irreducibility of the commutator $[\overline{A}, \overline{B}]$ of the unitary operators $\overline{A}$ and $\overline{B}$ generating an isomorphic copy of Θ_2 in $SU(2)$ expresses non-trivial conjugation cycles. Moreover, since the action of the group Θ_2 by strings of rotations in two generators allows to resolve S^2, such that the same lifted action resolves S^3 as well, by strings of corresponding unitary operators, we conclude that the Borromean link topological connectivity by means of conjugation cycles is transferred via these actions to the space of rays S^2 and the space of unit state vectors S^3 of a qubit. This is the crux of the non-classical behavior of a qubit and the problem is if the existence of non-trivial conjugation cycles can be turned to a new computational possibility.

We will outline the first steps towards implementing such a computational paradigm. For this purpose, our guiding principle will be the implementation of Chaitin's uncertainty relation, formulated in Sect. 4.10 in the present context. We remind that the form of Chaitin's uncertainty relation reads:

$$(\forall n)[I([\Omega]_n) \geq (n - k)]$$

where the constant k is interpreted as setting the bound of the simultaneous determination of two conjugate observables, viz. the random real Ω and the program-size complexity length measure I.

We have shown that there exist isomorphic copies of the group Θ_2 in the groups of rotation operators $SO(3)$ and the group of unitary operators $SU(2)$ realizing logical conjugation cycles, or equivalently Borromean loops, on S^2 and S^3 respectively. Since the group $SU(2)$ is a subgroup of the group $GL(2, \mathbb{C})$, viz. the matrix group of 2×2 matrices with complex coefficients, and of the group $SL(2, \mathbb{C})$, viz. the group of 2×2 matrices with complex coefficients and unit determinant, they also contain a copy of the group Θ_2. So we are going to identify two complex matrices acting as the generators of this copy of Θ_2 using Chaitin's uncertainty relation in the present setting. For this purpose, we assume that there exists a positive integer N playing the role of string length measure, such that for all $m \geq N$, the powers G^m and H^m, where G, H are complex matrices, generate a copy of Θ_2. This is possible using the method of dominant eigenvalues and dominant eigenvectors of matrices. We remark that for this purpose we have to diagonalize these matrices, a technique which is also based on logical conjugation. In particular, we look for two matrices G and H, such that: G has the dominant eigenvalue μ corresponding to a dominant eigenvector u. This means that the eigenspace of $G - \mu I$ is one-dimensional and all other eigenvalues of G have modulus less than $|\mu|$. Similarly, let H have the dominant eigenvalue ν corresponding to a dominant eigenvector v. Finally, we denote the dominant eigenvalues and corresponding dominant eigenvectors of G^{-1} and H^{-1} by ρ, w, and υ, z, respectively. Next, we consider the dominant eigenvectors as points on the 1-dim complex projective space, viz. equivalently on S^2. Then, the dominant eigenvalues/eigenvectors implement the requirement that there exist disjoint open sets containing the points u, v, w, z, denoted by U, V, W, Z, respectively, such that: There is some $m \geq N$ with the property that G^m sends each of these open sets to U, and correspondingly for the others, viz. H^m to V, G^{-m} to W and H^{-m} to Z. Now, we think of a finite state computer, with four states labelled by U, V, W, Z and an alphabet $G^m := a$, $H^m := b$, $G^{-m} := a^{-1}$, $H^{-m} := b^{-1}$ and transitions rules as described above. It is clear that the matrices a and b generate now a copy of the free group Θ_2, and thus we obtain logical conjugation cycles for the formation of strings using our alphabet with the prescribed transition rules.



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Chapter 5
Borromean Link in Relativity Theory What Is the Validity Domain of Einstein's Field Equations? Sheaf-Theoretic Distributional Solutions over Singularities and Topological Links in Geometrodynamics

5.1 Centennial Perspective on General Relativity

One hundred years after Einstein's initial conception and formulation of the General Theory of Relativity, it still remains a vibrant subject of intense research and formidable depth. In this way, during all these years our understanding of gravitation in differential geometric terms is being continuously refined. We believe that one of the highest priorities of a centennial perspective on General Relativity should be a careful re-examination of the validity domain of Einstein's field equations. These equations constitute the irreducible kernel of General Relativity and the possibility of retaining the form of Einstein's equations, while concurrently extending their domain of validity, is promising for shedding new light to old problems and guiding toward their effective resolution. These problems are primarily related with the following perennial issues: (a) the smooth manifold background of the theory, (b) the existence of singular loci in spacetime where the metric breaks down or the curvature blows up, and (c) the non-geometric nature of the second part of Einstein's equations involving the energy-momentum tensor. It turns out that these problems are intrinsically related to each other and require a critical re-thinking of the initial assumptions referring to the domain of validity of Einstein's equations.

In this communication, first of all, we would like to consider the problem of constructing distinguishable extensions of the smooth spacetime manifold solution space of Einstein's equations incorporating singularities by taking into account recent developments in differential geometry. These developments pertain to the possible generalization of the technical framework of differential manifolds, on which the formalism and interpretation of General Relativity is based on, to non-smooth or singular topological spaces by applying concepts and methods of sheaf theory and sheaf cohomology. In a nutshell, it turns out that all the usual local constructions of differential geometry, re-interpreted sheaf-theoretically, do not require the notion of a global smooth manifold, but are based on much weaker

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conditions of an essentially cohomological nature. The physical interpretation of these findings, referring to appropriate extensions of Einstein's equations over singular domains, is tantamount to the viable possibility of extending the covariant formulation of Einstein's equations using continuous distribution-like or even non-smooth sheaves of coefficients for all the involved tensorial physical quantities.

Second, we would like to show explicitly how certain generalized distribution-like solutions of partial differential equations, which fit appropriately in the above-mentioned sheaf-theoretic framework of differential geometry, bear significance in relation to obtaining singularity-free solutions of Einstein's equations in extended domains. We scrutinize the generation of these distribution-like algebra sheaves of coefficients from a physical perspective and explain the means of their construction in terms of residue classes of sequences of smooth functions modulo the information of singular loci encoded in suitable ideals.

Finally, we consider the application of these distribution-like solution sheaves in Geometrodynamics. The geometrodynamical formalism is very instructive in relation to the proposed extensions because it leads to the conclusion that active positive gravitational mass may emerge from purely topological considerations taking into account the constraints imposed by Einstein's field equations in the vacuum. In this manner, we may re-access fruitfully Wheeler's insights referring to "mass without mass" and "charge without charge" as well as re-evaluate the notion of wormhole solutions from a cohomological point of view. In this context, we propose to model topologically circular boundaries of singular loci in three-dimensional space in terms of topological links. It turns out that there exists a universal topological link bearing the connectivity property of the Borromean rings. The cohomological expression of the Borromean link points to its physical interpretation as a higher order wormhole solution of the field equations.

5.2 General Relativity from the Perspective of Sheaf Theory

In the standard formulation of General Relativity, the spacetime event structure is represented by means of a connected, four-dimensional real smooth manifold X. The chronogeometric relations on the event manifold X are expressed in terms of a pseudo-Riemannian metric of Lorentzian signature, called the spacetime metric. The chronogeometric relations are not fixed kinematically a priori, like in all predecessor classical field theories, but they should be obtained dynamically in terms of the metric as a solution of Einstein's field equations depending on the energy-momentum matter field distributions. In this manner, all the pertinent chronogeometric relations defined on a four-dimensional smooth manifold, endowing it with the structure of a spacetime manifold, become variable. The dynamical constitution of these relations by means of the field equations requires the imposition of a compatibility requirement relating the metric tensor, which represents the spacetime geometry, with the affine connection, which represents the differential evolution of the gravitational field. A spacetime manifold is considered to be without singularities if the coefficients of the metric tensor field are smooth and the manifold X is geodesically complete with respect to the metric. In this case, all timelike geodesic curves can be extended to arbitrary length in the smooth spacetime manifold X. From a physical viewpoint, according to the above requirements, the notion of localization at a spacetime point-event is sensible only if the coefficients of the metric tensor field are smooth in an open neighbourhood of this point.

Algebraically speaking, a real smooth manifold X can be reconstructed entirely from the $\mathbb{R}$-algebra $\mathbb{C}^{\infty}(X)$ of smooth real-valued functions on it, and in particular, the points of X are derived from the algebra $\mathbb{C}^{\infty}(X)$ as the $\mathbb{R}$-algebra homomorphisms $\mathbb{C}^{\infty}(X) \rightarrow \mathbb{R}$. This important observation in relation to General Relativity has been first proposed and explicated by Geroch in the form of Einstein algebras (Geroch 1972). From a modern mathematical perspective, it is a consequence of the Gelfand representation theorem applied to the case of smooth manifolds (Mallios 1993a,b). In this way, manifold points constitute the $\mathbb{R}$-spectrum of the algebra of smooth functions $\mathbb{C}^{\infty}(X)$, being isomorphic with the maximal ideals of this algebra. Notice that the $\mathbb{R}$-algebra $\mathbb{C}^{\infty}(X)$ is a commutative topological algebra that contains the field of real numbers $\mathbb{R}$ as a distinguished subalgebra, encapsulating the predominant physical assumption that our means of characterizing events is conducted by evaluations in the field of real numbers $\mathbb{R}$.

The algebraic viewpoint is instructive because it makes clear that in the standard differential geometric setting of General Relativity, all the tensorial physical quantities are coordinatized by means of the commutative $\mathbb{R}$-algebra of globally defined smooth real-valued functions $\mathbb{C}^{\infty}(X)$. Hence, the background of the theory remains fixed as the $\mathbb{R}$-spectrum of the commutative topological algebra $\mathbb{C}^{\infty}(X)$, supplying smooth coefficients for the coordinatization of physical quantities. The points of the manifold X, although not dynamically localizable degrees of freedom in General Relativity, serve as the semantic information carriers of spacetime events. More precisely, the points are marked on a smooth manifold in terms of global evaluations of the smooth algebra $\mathbb{C}^{\infty}(X)$ in the field of real numbers. The subtlety of General Relativity is exactly that manifold points are not dynamically localizable entities in the theory. More precisely, manifold points assume an indirect reference as indicators of spacetime events, only after the dynamical specification of chronogeometrical relations among them, as particular solutions of the generally covariant field equations. Clearly, the existence of singular loci in spacetime where the metric breaks down in terms of smooth function coefficients forbids the association of smooth manifold points with spacetime events. What remains is an emergent notion of an event horizon of a singular locus where spacetime information may be encoded appropriately.

The dynamical variability of the coefficients coordinatizing all tensorial physical quantities requires the action of a covariant differential operator to be applied upon them. This takes place via the notion of an affine connection, which is expressed as a covariant derivative acting on these smooth coefficients. The result of differentiation is encoded in $\mathbb{C}^{\infty}(X)$-modules over the algebra $\mathbb{C}^{\infty}(X)$, called modules of differential forms Ω and their duals $\Xi = Hom(\Omega, \mathbb{C}^{\infty}(X))$, as well as their higher powers constructed by means of exterior algebra.

In the same algebraic context, the role of a metric geometry on a smooth manifold, as related with the above modules of differential forms and their duals in General Relativity, pertains to the representability of spacetime events by points of the manifold, which in turn necessitates their coordinatization in terms of real numbers. This is tantamount to the requirement that all types of differentially variable quantities should possess uniquely defined dual types, such that their point-event representability can be made possible by means of real numbers. This is precisely the role of a geometry induced by a metric. Concretely, the spacetime metric assigns a unique dual to each differentially variable quantity, by effecting an isomorphism between the modules Ω and $\Xi := Hom(\Omega, \mathbb{C}^{\infty}(X))$, that is $g : \Omega \simeq \Xi$, such that $df \mapsto v_f := g(df)$.

The important thing to notice is that all these constructions can be performed strictly locally, that is by using only sections defined in the neighborhood of points. This is an implication that differential geometric constructions should be expressed not in terms of global algebra coefficients, but in terms of sheaves of coefficients defined locally. Then, the task is to study the maximal extendibility of these constructions from the local to the global level, which is technically expressed via the theory of sheaf cohomology.

In the context of General Relativity, the modelling of the dynamical variability, caused by the gravitational field by means of the Levi-Civita connection, from a local sheaf-theoretic perspective, is becoming even more relevant in view of the spacetime metric compatibility of this connection and the associated solution space of the theory. Einstein's equations are formulated in terms of non-linear partial differential equations involving smooth functions, playing the role of local coefficients coordinatizing the metric tensor, the Ricci tensor, and the scalar curvature. The solution of these equations in terms of the spacetime metric determines the local metrical properties of the spacetime manifold around any point, depending on the energy-momentum tensor. Notwithstanding this, all the global cosmological predictions of the theory are obtained not from these local solutions of the field equations per se, but from the possibility of continuation of some local solution to an extended region. The method of continuation or extension of some solution from the local to the global level is mathematically of a sheaf-theoretic nature.

In view of the problem of singularities in General Relativity, this is a clear warning that distribution-like sheaves of coefficients may be more appropriate for the continuation of some local solution over extended regions when the smooth ones become ill-defined over singular loci. It is a natural requirement that these sheaves of coefficients contain the standard smooth ones as a subalgebra, or equivalently there is an algebra sheaf embedding of the smooth coefficients into the generalized ones. It is expected that distribution-like sheaves of coefficients can prevent the breaking down of the metric at singularities, and therefore, provide the means to extend the domain of validity of the field equations, under the proviso that the same tensorial equations can be re-expressed covariantly in terms of these generalized sheaves of coefficients.

5.3 Cohomological Conditions for Extending the Smooth Sheaf of Coefficients in General Relativity

Cohomology theory constitutes a sophisticated algebraic-topological method of assigning global invariants to a topological space in a homotopy-invariant way. The cohomology groups measure the global obstructions for extending sections from the local to the global level, for instance, extending local solutions of a differential equation to a global solution. The differential geometric mechanism of smooth manifolds is essentially based on the setup of the de Rham complex in terms of locally defined smooth coefficients. In particular, de Rham cohomology measures the extent that closed differential forms fail to be exact, and thus the obstruction to integrability. In this context, the central role is played by the lemma of Poincaré, according to which every closed differential form is locally exact in terms of smooth coefficients. The de Rham theorem asserts that the homomorphism from the de Rham cohomology ring to the differentiable singular cohomology ring, given by integration of closed forms over differentiable singular cycles, is a ring isomorphism. The sheaf-theoretic understanding of this deep result came after the realization that both the de Rham cohomology and the differentiable singular cohomology are actually special isomorphic cases of sheaf cohomology. In particular, it has been also crystalized that the de Rham cohomology of a differential manifold depends only on the property of paracompactness of the underlying topological space. In turn, the paracompactness property, which is required in the definition of a differential manifold can also be characterized cohomologically via the acyclic behavior of soft sheaves, like the sheaf of smooth functions. In other words, soft sheaves, namely sheaves whose sections over any closed subset can be extended to a global section, are acyclic over a paracompact topological space.

The re-interpretation and generalization of the standard de Rham cohomology theory on manifolds in sheaf-cohomological terms is physically significant, because it provides an intrinsic way to set up and solve differential equations expressing the dynamical variability of physical quantities. The concepts and technical tools of sheaf cohomology have been developed through the groundbreaking work of Grothendieck in geometry (Grothendieck 1957, 1958). What should be initially kept in mind for physical applications is that the natural argument of a cohomology theory is a pair consisting of a topological space together with a sheaf of commutative algebras defined over it, rather than just a space.

It is instructive to include the basic definition characterizing the notion of a sheaf of sets on a topological space X, which also gives rise in a direct way to the notion of a sheaf of commutative algebras over X that we will employ in the sequel:

A presheaf $\mathbb{F}$ of sets on a topological space X consists of the following information:

1. For every open set U of X, a set denoted by $\mathbb{F}(U)$, and
2. For every inclusion $V \hookrightarrow U$ of open sets of X, a restriction morphism of sets in the opposite direction:

$$r(U|V) : \mathbb{F}(U) \to \mathbb{F}(V) \tag{5.1}$$

such that:

(a) $r(U|U)$ = identity at $\mathbb{F}(U)$ for all open sets U of X.
(b) $r(V|W) \circ r(U|V) = r(U|W)$ for all open sets $W \hookrightarrow V \hookrightarrow U$. Usually, the following simplifying notation is used: $r(U|V)(s) := s|_V$.

A presheaf $\mathbb{F}$ of sets on a topological space X is defined to be a sheaf if it satisfies the following two conditions, for every family V_a, $a \in I$, of local open covers of V, where V open set in X, such that $V = \cup_a V_a$:

1. Local identity axiom of sheaf:
   Given $s, t \in \mathbb{F}(V)$ with $s|_{V_a} = t|_{V_a}$ for all $a \in I$, then $s = t$.
2. Gluing axiom of sheaf:
   Given $s_a \in \mathbb{F}(V_a)$, $s_b \in \mathbb{F}(V_b)$, $a, b \in I$, such that:

$$s_a|_{V_a \cap V_b} = s_b|_{V_a \cap V_b}, \tag{5.2}$$

   for all $a, b \in I$, then there exists a unique $s \in \mathbb{F}(V)$, such that: $s|_{V_a} = s_a \in F(V_a)$ and $s|_{V_b} = s_b \in F(V_b)$.

As a basic example, if $\mathbb{F}$ denotes the presheaf that assigns to each open set $U \subset X$, the commutative algebra of all real-valued continuous functions on U, then $\mathbb{F}$ is actually a sheaf. This is intuitively clear since the specification of a topology on X is solely used for the definition of the continuous functions on X. Thus, the continuity of each function can be determined locally. This means that continuity respects the operation of restriction to open sets, and moreover that continuous functions can be amalgamated together in a unique manner, as it is required for the satisfaction of the sheaf condition.

The realization that the natural argument of a cohomology theory is not only a space, but it is actually a pair consisting of a topological space together with a sheaf of commutative algebras localized over it, has given rise to the notion of a commutative locally $\mathbb{R}$-algebraized space, defined by means of a pair $(X, \mathbb{A})$ consisting of a topological space X and a sheaf of commutative $\mathbb{R}$-algebras $\mathbb{A}$ on X, such that the restriction $\mathbb{A}_x$ is a local commutative $\mathbb{R}$-algebra for any point $x \in X$. Regarding the possibility of extending consistently all the standard local differential geometric constructions in the context of smooth manifolds to singular spaces, in terms of locally $\mathbb{R}$-algebraized spaces, where a suitable sheaf of commutative $\mathbb{R}$-algebras $\mathbb{A}$ on X substitutes the smooth sheaf of $\mathbb{R}$-algebras $\mathbb{C}^\infty(X)$), a full-grown theory has been recently developed, called Abstract Differential Geometry (ADG). This theory has shown that the standard differential-analytic tools of locally Euclidean spaces and smooth manifolds leading to the formulation and solution of differential equations can be actually re-produced and generalized to non-smooth or singular topological spaces by means of sheaf cohomology. Equivalently, the suitability of a sheaf of commutative $\mathbb{R}$-algebras $\mathbb{A}$ on an abstract topological space X for expressing the differential geometric mechanism in terms of these coefficients instead of the smooth ones is entirely determined only by the satisfaction of precise cohomological conditions pertaining to the characterization of the algebra sheaf $\mathbb{A}$. We note, in passing, that for economy of symbols we denote algebra sheaves by the same symbols like we used for algebras before, since the difference is clear from the context.

The mathematical theory of ADG has been built rigorously by Mallios (1998a,b), see also Vassiliou (2004) and Fragoulopoulou and Papatriantafillou (2014), based on critical prior work of Selesnick (1976). The significance of ADG for physics has been also shown by an explicit reconstruction and generalization of the framework of the Maxwell and Yang-Mills gauge field theories in sheaf cohomological terms (Mallios 2006c, 2009), see also Mallios (2006a,b, 2008). An exposition of the basic didactics of ADG in relation to its physical applications has been presented by Raptis (2007). The basic method introduced for the generalization of the standard analytic tools of Classical Differential Geometry (CDG) consists in the following: Initially, a concept of CDG is suitable for extension to a broader differential context (beyond the context of smooth manifolds) if it is liable to a process of sheaf-theoretic localization (Mallios 2004). In CDG all the differential geometric constructions require that the base space is a smooth manifold. The underlying reason is that the means of differentiation are lifted locally from the structure of a Euclidean space. In this way, the de Rham complex is fixed with respect to smooth coefficients and all tensorial quantities are coordinatized in smooth terms. In ADG the base space provides merely a topological basis of sheaf-theoretic localization, such that all the pertinent differential geometric constructions can take place locally, whereas the latter are not subordinate to this topological basis, meaning that they are not dependent on any particular localization basis. Thus, the object of primary significance in ADG is not the base space itself but the algebra sheaf of coefficients localized over it. The differentiation structure is built in the algebra sheaf of coefficients by means of the notion of a connection defined independently of any locally Euclidean considerations. In this way, the associated de Rham complex can be satisfied by various possible algebra sheaves of coefficients modulo some well-defined cohomological conditions. We emphasize that the prominent role in the context of ADG is played by the algebra sheaf of coefficients, interpreted as a "functional coordinate arithmetic" (Mallios 2007, 2008), see also Zafiris (2004a,b, 2007), Epperson and Zafiris (2013), Mallios and Zafiris (2016), meaning that all geometric objects involved in the formalism are locally expressed in terms of its sections. In this way, an algebra sheaf of coefficients is not constrained ab initio to be a smooth one, restricting the geometric solution space within the spectrum of a smooth manifold. More generally, a suitable algebra sheaf of coefficients turns out to be an algebra sheaf of generalized functions including distributions, defined by Rosinger in the context of solutions to non-linear partial differential equations (Mallios and Rosinger 1999, 2001).

Concerning General Relativity, which is formulated using the CDG of smooth manifolds, the possibilities offered by ADG bear a remarkable physical significance. In particular, there arises the possibility of re-assessing the global problems of General Relativity related with the existence of singularities, where the metric breaks down, from the perspective of appropriate generalized algebra sheaves of coefficients. In this manner, the validity of Einstein's equations may be extended beyond differential manifolds, under the condition that the covariance properties of all tensorial physical quantities are maintained under these extensions, expressed in terms of the new sheaves of coefficients. From a physical viewpoint, this approach would allow to obtain solutions in terms of distribution-like sheaves corresponding to non-punctual localization properties, which would nevertheless still satisfy the field equations. This clearly vindicates the following critical remark of Weyl (2009): "While topology has succeeded fairly well in mastering continuity, we do not yet understand the inner meaning of the restriction to differential manifolds. Perhaps one day physics will be able to discard it."

The possibility of obtaining extended admissible solution spaces in terms of generalized algebra sheaves of coefficients is based on the fact that the validity of the de Rham complex, in its sheaf-theoretic guise, is not restricted exclusively to coordinatization of the tensorial physical quantities by smooth coefficients $\mathbb{C}^{\infty}$, as it is actually the case when the $\mathbb{R}$-spectrum of the coefficients is a smooth manifold. Thus, we may consider distribution-like sheaves of coefficients satisfying the validity of the de Rham complex, and therefore, formulate and solve the field equations in terms of these distribution coefficients instead of the smooth ones. More precisely, this is the case if the following sequence of $\mathbb{R}$-linear sheaf morphisms:

$$\mathbb{A} \to \Omega^1(\mathbb{A}) \to \ldots \to \Omega^n(\mathbb{A}) \to \ldots \tag{5.3}$$

is a complex of $\mathbb{R}$-vector space sheaves, identified as the sheaf-theoretic de Rham complex of $\mathbb{A}$.

In this case, if the cohomological condition expressing the Poincaré Lemma, $Ker(d^0) = \mathbb{R}$ is satisfied with respect to the algebra sheaf $\mathbb{A}$, and requiring that $\mathbb{A}$ is a soft algebra sheaf, viz. any section over any closed subset of X can be extended to a global section, we obtain that the sequence:

$$\mathbf{0} \to \mathbb{R} \to \mathbb{A} \to \Omega^1(\mathbb{A}) \to \ldots \to \Omega^n(\mathbb{A}) \to \ldots \tag{5.4}$$

is an exact sequence of $\mathbb{R}$-vector space sheaves. Thus, the sheaf-theoretic de Rham complex of the algebra sheaf $\mathbb{A}$ constitutes an acyclic resolution of the constant sheaf $\mathbb{R}$.

The physical interpretation of this fact is the following: First of all, the essential feature of the localization method, utilizing coefficients from algebra sheaves instead of global algebras, is that the sheaf-theoretic de Rham complex is actually an acyclic resolution of the constant sheaf of the reals coordinatizing the events. For instance, referring to the CDG of smooth manifolds, the de Rham complex, expressed in terms of local smooth coefficients and their differential forms of higher orders, provides such an acyclic resolution of the constant sheaf $\mathbb{R}$. What has been uncovered by ADG is that the smooth algebra sheaf $\mathbb{C}^{\infty}(X))$ is not unique in this respect. More concretely, any other soft algebra sheaf $\mathbb{A}$ constituting an acyclic resolution of the constant sheaf $\mathbb{R}$ is a viable source of coefficients for the coordinatization of the tensors, maintaining at the same time all their covariance properties in terms of the new local coefficients. This crucial fact essentially questions the uniqueness of the role of local smooth coefficients for formulating the means of dynamical variability. In other words, it questions the unique role of smooth manifold geometric spectrums as domains of validity of the field equations.

The idea to address the problem of singularities from the perspective of ADG has been proposed already, for instance in Mallios (2009, 2006a). More concretely, in particular relation to the issue of spacetime singularities, Mallios and Rosinger (1999, 2001) have applied ADG using as an algebra sheaf of coefficients a variety of the so-called "spacetime foam algebras," and by Raptis (2006), building up on prior work by Mallios and Raptis (2003), using as a sheaf of coefficients "differential incidence algebras" defined over a locally finite poset substitute of a continuous manifold.

Our present proposal constitutes a twist of perspective in comparison to these works, which is actually implemented by physical criteria of suitability going beyond the satisfaction of the cohomological conditions. Our quest is related with the possibility of using a particular type of a "spacetime foam algebra" as a kind of a distribution-like sheaf of coefficients, distinguished on physical grounds, for extending the domain of validity of Einstein's field equations. For this purpose, from the whole variety of "spacetime foam algebras" we distinguish only the "nowhere dense generalized function algebra" as bearing physical significance in relation to the field equations of General Relativity. This is based on a physical criterion determining which properties should be characterized as intrinsic to the gravitational field and eventually deciding what should be generic with respect to its function or not. This physical criterion refers to the viable possibility of expressing the gravitational field sources via the instantiation of these generalized algebra sheaves of coefficients. Our rational is based on the idea that in an intrinsically dynamically variable theory, like General Relativity, it should be the pertinent physical conditions or the sources of the field themselves that determined the type of these extensions as solutions to the field equations.

5.4 Coping with Spacetime Singularities: Conceptual and Technical Aspects

In the classical differential geometric formulation of General Relativity, spacetime is represented as a connected, paracompact, and Hausdorff four-dimensional $\mathbb{C}^{\infty}$ manifold X, endowed with a pseudo-Riemannian metric of Lorentzian signature, which is obtained as a solution of Einstein's field equations. A spacetime manifold is considered to be without singularities if the coefficients of the metric tensor field are at least of class $\mathbb{C}^2$ and X is geodesically complete with respect to the metric, meaning that all timelike geodesic curves can be extended to arbitrary length (Clarke 1993; Hawking and Ellis 1973). Consequently, a spacetime manifold is considered to be singular if there exist incomplete geodesic curves, or equivalently finite affine length geodesics that cannot be extended. A spacetime singularity delimits a locus where the behavior of the metric tensor coefficients become ill-defined with respect to the smooth characterization of the manifold. Usually the singular locus is identified as a locus where the spacetime curvature blows up. We note that the localization at a spacetime point-event is meaningful if the metric coefficients are smooth, or at least of class $\mathbb{C}^2$ in a neighborhood of this point.

The usual way to cope with a spacetime singularity is to consider it as a singular spacetime boundary rather than a locus within spacetime. For instance, a spacetime boundary may be defined in terms of a set of incomplete curves S. This takes place by the imposition of an appropriate equivalence relation $\sim$ on the set S, such that the quotient set $S/\sim := \partial X$ is interpreted as the singular boundary of X. The criterion of equivalence is determined by the choice of those equivalence classes, which are forced to play the role of ideal points in the extension of X by ∂X. There have been proposed various possible choices, for example Geroch's "g-boundary" or Schmidt's "b-boundary," but it is always assumed that X is topologically dense in $X \bigsqcup \partial X$ (Bosshard 1976; Schmidt 1971). Following this approach, Heller and Sasin have shown that Einstein's field equations can be formulated in the extension of X by ∂X, that is on $X \bigsqcup \partial X$ defined as an "Einstein structured space" (Heller and Sasin 1995). Actually, this is the Gelfand spectrum of a sheaf of Einstein algebras, which constitutes the sheaf-theoretic localization of an Einstein algebra, a notion proposed initially by Geroch in his attempt to re-formulate General Relativity in algebraic terms without invoking directly a spacetime manifold background (Geroch 1972). In particular, it has been proved that the closed Friedmann world model and the Schwarzschild solution, combined with Schmidt's "b-boundary" construction, fit nicely in the sheaf-theoretic context of an "Einstein structured space." In turn, this has been a first indication that the validity of Einstein's equations may be extended to bigger domains incorporating singular loci, which are not smooth manifolds anymore. It has been also pointed out that some sorts of singularities can also appear when there exists a transition to the quantum gravity regime. More concretely, the smooth manifold structure of spacetime can break down and the possible validity of Einstein's equations should be sought for in further extended and generalized non-smooth spectrums of appropriate sheaves of algebras, where the singularities are not necessarily forced to some type of spacetime boundary.

From a broader conceptual perspective, the issue of singularities in General Relativity as impossibilities of extending smooth metric solutions of Einstein's equations necessitates the coordinatization of all the tensorial quantities by distributional coefficients effecting a type of topological coarse-graining over singular loci, and thus localizing the point-event stratum in their terms. Under the proviso that these distributional coefficients form algebra sheaves fulfilling all the required cohomological conditions, the means of extending local distributional solutions generalizes the standard method of extending timelike geodesic curves in a smooth manifold. The physical significance of this generalization is that the domain of validity of the field equations can be extended beyond the notion of a smooth manifold. Not only this, but additionally, these distinguishable extensions may be associated intrinsically with the gravitational field, under the constraint that sources of the field itself giving rise to singularities can be expressed topologically in the terms of distribution-like algebra sheaves.

In this state of affairs, the smoothness assumption can be retained, at best, only locally and certainly far from singular loci. Mathematically, there should exist an embedding of the algebra sheaf of smooth functions into a distribution-like algebra sheaf of coefficients qualified as a solution of the extended field equations. An illuminating way to think of the proposed approach in non-technical terms is that coping efficiently with singularities requires a process of folding out of the smooth point-event manifold background. This viewpoint has been emphasized by von Müller (2015), according to whom, the process of folding out into a "statu-nascendi" level should be considered in the context of a whole new categorial apparatus qualifying its intrinsic characteristics in contradistinction to the event stratum. In this manner, we suggest that the existence of a distribution-like sheaf of coefficients as a solution of the field equations within an appropriately extended domain characterized by some generic gravitational criterion paves the way for understanding the precise nature of this folding out of the smooth point-event stratum.

The possibility of extending the formulation of Einstein's equations in the case of non-smooth spectrums using the sheaf-theoretic technique of localization in the context of ADG is of major significance. We note that non-smooth spectrums of algebra sheaves do not require the consideration of singularities as ideal points on the boundary of a smooth manifold. In other words, singular loci are allowed to be located, according to specific topological criteria, within a manifold. Of course, a natural requirement should be that the exclusion of singular loci would recast Einstein's equation in the familiar form in terms of smooth coefficients. But, clearly in case that Einstein's equations become meaningfully extended over singular loci, then the coefficients of the metric and curvature tensors cannot be smooth any more. Therefore, from a smooth perspective, a singularity functions as an obstruction to the extension of a local solution to the field equations expressed in terms of smooth coefficients. Thus, more precisely, a singular locus plays the role of a cohomological obstruction to the extendibility of a local smooth solution. This criterion incorporates and generalizes sheaf-cohomologically the initial definition of singular behavior in terms of non-extendibility of geodesics. Essentially the reason is that the notion of extendibility of local solutions is of a sheaf-theoretic nature, recalling for instance the well-known procedure of analytic continuation.

There are two important physical consequences emanating from the possibility of formulating Einstein's equations in terms of generalized non-smooth sheaves of coefficients. The first is related with the natural question concerning the criterion of depicting a particular sheaf of algebras for this purpose. The second is related with a possible re-evaluation of the status of the energy-momentum source term in Einstein's equations, which currently is not implemented by any process of geometrization.

Regarding the first, the required physical condition is the following: Since the formulation of Einstein's equations can be extended over singular loci, it should precisely be the nature and specification of these singular loci that would determine the appropriate sheaf of coefficients, such that a solution can be expressed eventually in terms of these coefficients. In the non-singular case, we know already that a solution can be expressed in terms of smooth coefficients. In other words, we already know that if no singularity is present, the spacetime metric—obtained as a particular solution of the vacuum Einstein equations, for example—is always expressible in terms of smooth coefficients, i.e. in terms of the sheaf of algebras $\mathbb{C}^\infty(X)$. Hence, we expect that in the presence of a particular type of a singular locus over which Einstein's equations hold in terms of a distribution-like sheaf of coefficients, there exists a metric solution expressed in terms of these coefficients. Not only this, but additionally, since the knowledge of the metric solution is completely expressible in terms of these coefficients—considered as unknowns when plugged into the equations—the specification of a singular locus should force a corresponding algebra sheaf as the solution. In other words, the nature of a singular locus should determine the differentiability properties of a metric solution in case that Einstein's equations can be extended over this locus. As we stressed previously, the physical association of singular loci with sources of the gravitational field itself, giving rise to distinguishable extensions of the standard smooth manifold spacetime model of General Relativity, implies that sources can be expressed topologically after all, if solutions of the field equations are expressed in terms of appropriate distribution-like algebra sheaves.

In this context, the physical significance of ADG is that it determines rigorously the criteria that these algebra sheaves of coefficients have to satisfy, such that Einstein's equations can be satisfied over various sorts of singular loci, expressed in terms of these coefficients. Not surprisingly these criteria are of a cohomological nature. Essentially, they determine viable algebra sheaves of coefficients by the requirement that they are soft, and thus acyclic, such that the validity of the de Rham complex remains intact. In turn, the basic idea is that the Poincaré Lemma should remain in force, viz. closed differential forms expressed in these generalized coefficients should be locally exact as in the smooth case, so that the differential geometric mechanism can be extended over singularities without breaking down. We will present a general form of these algebra sheaves consisting of distribution-like coefficients in the sequel. According to Clarke, the answer to many of the problems related with singularities "involve detailed considerations of distributional solutions to Einstein's equations, leading into an area that is only starting to be explored ..." (Clarke 1993). We propose that the extension of validity of Einstein's equations over singular loci in terms of appropriate sheaves of algebras, which are generally non-smooth, sheds new light on the problem of singular behavior in General Relativity.

Regarding the second physical consequence, it is instructive to remind that the energy-momentum source term in the smooth formulation of Einstein's equations is not of any geometric nature. The energy-momentum tensor attributes the source of curvature entirely to matter (including the cosmological dark energy), as it does not incorporate the stress-energy associated with the gravitational field itself. There is an underlying assumption that spacetime is somehow empty unless it is filled in by matter, expressed in terms of the smooth coefficients of the energy-momentum tensor. This is the reason that when the energy-momentum part is zero, then the equations are called vacuum equations. Now the validity of Einstein's equations over singular domains in terms of generalized non-smooth algebra sheaves casts serious doubts on this assumption. Namely, the form of Einstein's equations with vanishing non-geometric second part may turn out to be the fundamental form of these equations. The reason is that sources of the gravitational field itself might be implemented in terms of non-smooth algebra sheaves, and thus what is called a vacuum is not empty at all, precisely because it engulfs these sources. This idea is not actually as controversial as it sounds, if we take seriously into account that all classical experimental tests of General Relativity involve a vanishing energy-momentum tensor, and thus what they really verify is the equation $\mathbb{R}_{\mu\nu} = 0$. This issue has been also pointed out and argued for extensively, from a non-sheaf-theoretic point of view, by Vishwakarma (2014), who conducted a careful analysis based on the observational tests of the theory. In the sequel, we will discuss this issue in more detail from a geometrodynamical perspective in the light of the particular form of distribution-like algebra sheaves.

5.5 Spacetime Extensions in Terms of Singularity-Free Distributional Algebra Sheaves

It is physically reasonable to expect that an admissible commutative algebra sheaf of coefficients in term of which Einstein's equations may be extended over a singular locus should be distribution-like. For example, we may think of a matter distribution confined to a submanifold of spacetime whose density is integrable over this submanifold. In the context of a linear field theory this should be naturally modelled in terms of a linear distribution. Unfortunately, this is not possible in the context of General Relativity, which is a non-linear theory. In other words, Schwarz's linear distributions are not suitable candidates for expressing the information of singular loci.

The unsuitability of linear distributions rests on the fact that the space $\mathbb{D}'$ they form is only a linear space, but it is not an algebra. This is characterized as the "Schwarz Impossibility," and may be formulated as follows: There is no symmetric bilinear morphism:

$$\circ : \mathbb{D}'(V) \times \mathbb{D}'(V) \ni (S, T) \to S \circ T \in \mathbb{D}'(V)$$

so that $S \circ T$ is the usual point-wise product of continuous functions, when $S, T \in \mathbb{C}^0(V)$. Equivalently, $\mathbb{D}'(V)$ is not closed under any multiplication that extends the usual multiplication of continuous functions, where V is an open subset X. Since all the involved arguments are of a local character, without loss of generality, we may simply consider V as an open subset of $\mathbb{R}^4$.

A physically natural way to bypass "Schwarz Impossibility" is to assume the existence of an embedding morphism $\mathbb{D}'(V) \hookrightarrow \mathbb{A}(V)$, which embeds the vector space of distributions $\mathbb{D}'(V)$ as a vector subspace in $\mathbb{A}(V)$, where $\mathbb{A}(V)$ is the quotient algebra:

$$\mathbb{A}(V) = \mathbb{K}(V)/\mathbb{I}, \tag{5.5}$$

and $\mathbb{K}(V)$ is a subalgebra in $\mathbb{C}^\infty(V)^\Lambda$, for some index set Λ, whereas $\mathbb{I}$ is an ideal in $\mathbb{K}(V)$. This approach was initiated by Rosinger (1978, 1980), and developed further in Rosinger (1987, 1990, 2001, 2007).

We will restrict ourselves to a certain subclass of this type of algebras, namely the unital, associative, and commutative algebras of generalized functions, whose suitably defined ideals can engulf algebraically the information of singular loci. These algebras, introduced by Rosinger (1980), have been formed in such a way as to express generalized solutions of non-linear partial differential equations. We may describe the generation of these algebras locally as follows:

Let $V \subseteq \mathbb{R}^4$ be an open set, and $L = (\Lambda, \leq)$ be a right directed partial order on some specified index set Λ. That is, for all $\lambda, \mu \in \Lambda$, there exists $\nu \in \Lambda$ such that $\lambda, \mu \leq \nu$. With respect to the usual componentwise operations, $\mathbb{C}^\infty(V)^\Lambda$ is a unital and commutative algebra over the reals. We define the following ideal $\mathbb{I}_L$ in $\mathbb{C}^\infty(V)^\Lambda$ whose physical meaning will be described in the sequel:

$$\mathbb{I}_L(V) = \left\{ \phi = (\phi_\lambda)_{\lambda \in \Lambda} \;\middle|\; \begin{array}{l} \exists\, \Gamma \subset V \text{ closed nowhere dense:} \\ \forall\, x \in [V \setminus \Gamma] \text{ being dense:} \\ \exists\, \lambda \in \Lambda: \\ \forall\, \mu \in \Lambda,\ \mu \geq \lambda: \\ \quad \phi_\mu(x) = 0,\ \partial^p \phi_\mu(x) = 0 \end{array} \right\} \tag{5.6}$$

In the above definition, we think of Γ as a singular locus in $\mathbb{R}^4$, characterized as a closed and nowhere dense subset relative to the open set $V \subseteq \mathbb{R}^4$, such that its complement $V \setminus \Gamma$ in V is dense. The unital and commutative algebra $\mathbb{C}^\infty(V)^\Lambda$ contains smooth functions ϕ_λ indexed by the set Λ and defined over V, to be thought of as diagrams or sequences of Λ-indexed smooth functions. The requirement of the right directed partial order on the specified index set Λ, which is denoted by $L = (\Lambda, \leq)$, is technically necessary in order that the above set forms actually an ideal in $\mathbb{C}^\infty(V)^\Lambda$. Now, the ideal $\mathbb{I}_L(V)$ in $\mathbb{C}^\infty(V)^\Lambda$ includes all these sequences of smooth functions ϕ_λ that vanish asymptotically outside the singular locus Γ together with all their partial derivatives. Therefore, intuitively speaking, the ideal of the form $\mathbb{I}_L(V)$ incorporates all these sequences of smooth functions indexed by Λ whose support covers the singular locus Γ, whereas they vanish outside it. In this manner, the information of the singular locus Γ is encoded in the ideal $\mathbb{I}_L(V)$ in $\mathbb{C}^\infty(V)^\Lambda$. Hence, the quotient commutative algebra $\mathbb{A}_L(V) = \mathbb{C}^\infty(V)^\Lambda / \mathbb{I}_L(V)$ is an algebra of residues of sequences of smooth functions modulo the singular information ideal $\mathbb{I}_L(V)$.

A natural question in the above context refers to the requirement that the complement $V \setminus \Gamma$ of the singular locus Γ in V should be dense. The necessity of this requirement can be understood by the fact that we wish to obtain an embedding ι of the algebra of smooth functions $\mathbb{C}^\infty(V)$ into the algebra of generalized functions $\mathbb{A}_L(V)$:

$$\iota : \mathbb{C}^\infty(V) \hookrightarrow \mathbb{A}_L(V) = \frac{\mathbb{C}^\infty(V)^\Lambda}{\mathbb{I}_L(V)} \tag{5.7}$$

such that:

$$\varphi \hookrightarrow \iota(\varphi) = \Delta(\varphi) + [\mathbb{I}_L(V)] \tag{5.8}$$

where $\Delta_\Lambda \mid_V : \mathbb{C}^\infty(V) \to \mathbb{C}^\infty(V)^\Lambda$ is the diagonal morphism with respect to Λ, defined for an open set V as follows:

$$\Delta_\Lambda(\varphi) \mid_V = \left\{\Delta(\varphi) = (\varphi_\lambda)_{\lambda\in\Lambda} \mid \varphi_\lambda = \varphi,\ \forall\lambda \in \Lambda, \varphi \in \mathbb{C}^\infty(V)\right\}.$$

Hence, for every smooth function φ in $\mathbb{C}^\infty(V)$, the diagonal image $\Delta(\varphi)$ of φ in $\mathbb{C}^\infty(V)^\Lambda$ is a sequence of smooth functions all identical to φ, indexed by Λ. The embedding ι is feasible according to the above, if and only if the ideal $\mathbb{I}_L(V)$ satisfies the off diagonality condition:

$$\mathbb{I}_L(V) \cap \Delta_\Lambda \mid_V = \{0\}. \tag{5.9}$$

Therefore, it remains to show that if the complement $V \setminus \Gamma$ of the singular locus Γ in V is dense, according to the specification in (5.6), then the ideal $\mathbb{I}_L(V)$ actually satisfies the above off diagonality condition. So we suppose that $V \setminus \Gamma$ is dense in V, and consider a smooth function χ in $\mathbb{C}^\infty(V)$. If $\Delta_\Lambda(\chi) \mid_V := \Delta(\chi)$ belongs to the ideal $\mathbb{I}_L(V)$, then the asymptotic vanishing condition in (5.6) implies that $\chi = 0$ in $V \setminus \Gamma$, and therefore, we must have $\chi = 0$ in V because $V \setminus \Gamma$ is dense in V by hypothesis. Thus, it follows that the ideal $\mathbb{I}_L(V)$ satisfies the off diagonality condition (5.9), as required.

Conclusively, there exists a canonical injective homomorphism of commutative algebras, or equivalently, an embedding ι of the algebra of smooth functions $\mathbb{C}^\infty(V)$ into the algebra of generalized functions $\mathbb{A}_L(V)$:

$$\iota : \mathbb{C}^\infty(V) \hookrightarrow \mathbb{A}_L(V) = \frac{\mathbb{C}^\infty(V)^\Lambda}{\mathbb{I}_L(V)} \tag{5.10}$$

Furthermore, in view of (5.6), it follows immediately that the partial differential operators:

$$\partial^p : \mathbb{C}^\infty(V)^\Lambda \ni \phi = (\phi_\lambda) \mapsto \partial^p \phi = (\partial^p \phi_\lambda) \in \mathbb{C}^\infty(V)^\Lambda$$

satisfy the inclusion:

$$\partial^p \left(\mathbb{I}_L\left(V\right)\right) \subseteq \mathbb{I}_L\left(V\right). \tag{5.11}$$

Thus, the standard partial derivative operators on $\mathbb{C}^\infty\left(V\right)$ extend to $\mathbb{A}_L\left(V\right)$:

$$\partial^p : \mathbb{A}_L\left(V\right) \ni \left[\phi + \mathbb{I}_L\left(V\right)\right] \mapsto \left[\partial^p \phi + \mathbb{I}_L\left(V\right)\right] \in \mathbb{A}_L\left(V\right), \tag{5.12}$$

We conclude that the embedding of commutative algebras (5.10) extends to an embedding of differential algebras. Therefore, the following diagram commutes:

$$\begin{tikzcd}
\mathbb{C}^\infty\left(V\right) \arrow[r, "\partial^p"] \arrow[d] & \mathbb{C}^\infty\left(V\right) \arrow[d] \\
\mathbb{A}_L\left(V\right) \arrow[r, "\partial^p"] & \mathbb{A}_L\left(V\right)
\end{tikzcd}$$

We emphasize that the embedding (5.10) preserves not only the algebraic structure of $\mathbb{C}^\infty\left(V\right)$, but also its differential structure. The off diagonality condition (5.9) implies also the existence of an injective, linear morphism:

$$\mathbb{D}'\left(V\right) \hookrightarrow \mathbb{A}_L\left(V\right). \tag{5.13}$$

Therefore, the differential algebra $\mathbb{A}_L\left(V\right)$ contains the space of distributions as a linear subspace, see Rosinger (1990, pp. 234–244), where those algebras that admit linear embeddings of distributions are characterized in terms of such off diagonality conditions. However, in contradistinction with (5.10), the embedding (5.13) does not commute with partial derivatives, and thus, the partial derivatives on $\mathbb{A}_L\left(V\right)$ do not, in general, coincide with distributional derivatives, when restricted to $\mathbb{D}'\left(V\right)$.

Finally, it is crucial to observe that a subset of a topological space is closed and nowhere dense if and only if it satisfies this condition locally. This is the key idea used to prove that the algebras of generalized functions $\mathbb{A}_L\left(V\right)$ form actually sheaves of commutative algebras, which additionally are soft and flasque or flabby (Mallios and Rosinger 1999, 2001). Thus, they are characterized as cohomologically appropriate sheaves of coefficients according to ADG. More precisely, the distribution-like soft algebra sheaves of the form $\mathbb{A}_L$ constitute an acyclic resolution of the constant sheaf of the reals coordinatizing the events. Thus, we conclude that the de Rham complex can be rigorously expressed in terms of these coefficients instead of the smooth ones, and consequently Einstein's equations can be formulated with respect to coefficients from the algebra sheaf $\mathbb{A}_L$ instead of the smooth ones from $\mathbb{C}^\infty$. Consequently, the validity of Einstein's equations can be extended over singular loci in a covariant manner by utilizing coefficients from the sheaf $\mathbb{A}_L$ for expressing all involved differential geometric tensorial quantities. Reciprocally, according to the intended physical interpretation of these algebra sheaves, pertaining to expressing sources of the gravitational field in terms of closed and nowhere dense subsets, the presence of a singular locus forces an algebra sheaf of the form $\mathbb{A}_L$ as coefficients with respect to which Einstein's equations retain their validity over this locus and do not break down, like in the case of insisting to use indiscriminately smooth coefficients.

For the sake of completeness, it is instructive to remind that the softness property of the sheaves of the form $\mathbb{A}_L$ means that any section over any closed subset can be extended to a global section. Thus, these types of sheaves characterize cohomologically the topological property of paracompactness by means of acyclicity. Equivalently, soft sheaves are acyclic over a paracompact topological space. Moreover, sheaves of the form $\mathbb{A}_L$ are not only soft, but they are flasque or flabby as well, which is a local property. This means that the restriction morphism of sections in the sheaf definition is an epimorphism. Hence, in this case, we can always extend any local section by zero to obtain a global section of $\mathbb{A}_L$.

We may recapitulate by pointing out that the first basic idea involved in the construction of distribution-like algebra sheaves of coefficients, in their role to coordinatize solutions of non-linear partial differential equations, is to model a singular locus Γ in $\mathbb{R}^4$ as a closed and nowhere dense subset relative to an open set $V \subseteq \mathbb{R}^4$, such that its complement $V \setminus \Gamma$ in V is dense. The second basic idea is to express such a closed and nowhere dense singular locus as an ideal in an algebra sheaf constructed as an extension of the smooth one over a partially ordered set. In this manner, the ideal expressing algebraically a singular locus contains diagrams of locally defined smooth functions indexed by Λ whose support covers the singular locus Γ, whereas they vanish outside it. Then, it can be shown that the quotient commutative algebra sheaf $\mathbb{A}_L(V) = \mathbb{C}^\infty(V)^\Lambda / \mathbb{I}_L(V)$ is an algebra sheaf of residues of diagrams of smooth functions modulo the closed nowhere dense singular ideal $\mathbb{I}_L(V)$.

It is instructive to emphasize that the algebra of global sections of the sheaf $\mathbb{A}_L(V)$ contains the space of Schwarz distributions $\mathbb{D}'(V)$ only as a linear subspace and not as a commutative subalgebra. For example, Dirac's delta, considered as a distribution, is represented in terms of a generalized function whose pertinent closed and nowhere dense set is an one-point set. It is well known that the square of the delta distribution is not a distribution itself, since the operation of pointwise multiplication of distributions is not well-defined in $\mathbb{D}'(V)$. Notwithstanding this fact, the representative generalized function may be unproblematically squared providing a legitimate generalized function without being a linear distribution itself.

Clearly, by the rules of construction of these commutative algebras of generalized functions, arbitrary nonlinear continuous operations may be applied to a generalized function giving another generalized function in the same algebra. In passing, it is also worth to point out that the linear space of Schwarz distributions does not give rise to a flasque vector sheaf in contradistinction to the case of the embedding sheaf $\mathbb{A}_L\,(V)$, a property which is crucial for the global extendibility of all standard local differential geometric constructions.

In the sequel, we are going to propose a concrete class of closed and nowhere dense sets modelling the boundaries of singular loci and forming a topological link in 3-d space. Conceptually, this essentially means that the semantics of folding out of a local smooth event stratum into a singular domain may be associated with the formation of some topological link configuration and its concomitant algebraic expression in terms of an algebra sheaf of the type $\mathbb{A}_L$. At the final stage, we have to examine if this algebra sheaf satisfies the cohomological conditions necessary for expressing the differential geometric mechanism of General Relativity in these terms instead of the globally smooth ones. This turns out to be actually the case, and therefore, algebra sheaves of the type $\mathbb{A}_L$ can be used legitimately to express the metric solution of Einstein's field equations extended now over singularities.

The important consequence is that we can retain not only the validity, but the form and covariance property of Einstein's equations even over singular loci. The reason is that all physical quantities can still be transformed according to a tensor law for any arbitrary admissible coordinate transformation. The difference in comparison to the smooth case is that the coordinates are allowed to be non-standard or non-smooth, while at the same time all the machinery of differential geometry can be applied with respect to them. In particular, while the coefficients of the tensorial physical quantities are non-smooth, all the usual differential-geometric constructions can be carried out as in the smooth case. The only price to be paid for this generalization is the rejection of the fixed absolute smooth manifold background of the theory. We consider this fact as physically nondisturbing, since the essence of General Relativity is in the covariant formulation and validity of Einstein's equations and not on the existence of a smooth background manifold. In particular, what we gain from such a generalization is not only that Einstein's equations can be extended covariantly over singular loci, but also that the solution of these equations in terms of coefficients from a sheaf of the form $\mathbb{A}_L$ is free of singularities.

5.6 Topological Links in Geometrodynamics

According to the paradigm of Geometrodynamics (Misner et al. 1970), we may foliate a spacetime manifold X into three-dimensional spacelike leaves Σ_t by utilizing a one-parameter family of embeddings $\varepsilon_t : \Sigma \hookrightarrow X$, such that $\varepsilon_t(\Sigma) = \Sigma_t$. In the geometrodynamical formulation, the three-dimensional Riemannian manifold (Σ, h) is thought of as dynamically evolving, where the corresponding metric at time t, $h_t = {\varepsilon_t}^* g$, is derived by pulling back the spacetime metric g via ε_t. It is implicitly assumed that all three-dimensional spacelike leaves Σ_t are mutually disjoint, such that the Lorentzian manifold $(\mathbb{R} \times \Sigma, \varepsilon^* g)$ represents X, where the leaves of the considered foliation correspond to the constant time hypersurfaces.

The geometrodynamical picture is instructive for our purposes because it shows that active gravitational mass may emerge from purely topological considerations taking into account the constraints imposed by Einstein's field equations in the vacuum (Arnowitt et al. 1962). From a physical perspective, this may be interpreted in a novel way according to Wheeler's insight referring to "mass without mass" (Misner and Wheeler 1957; Wheeler 1957) as follows: Localized configurations of topologically singular loci in open sets of a spacetime manifold restricted to closed nowhere dense subsets amount to active gravitational mass/energy in their complementary open dense subsets. In particular, if we consider that the Lorentzian manifold $(\mathbb{R} \times \Sigma, \varepsilon^* g)$ represents X, the singular loci may be localized within the three-dimensional manifold Σ. In this context, if Σ has a non-trivial topology, Gannon's theorem (Gannon 1975) implies that spacetime is geodesically incomplete, and thus singular. The simplest way to implement a non-trivial topology on Σ is via the hypothesis of non-simple connectivity. More precisely, the existence of singular loci in Σ, localized in closed nowhere dense subsets, make Σ a multiple-connected topological space, and thus topologically different from $\mathbb{R}^3$. We may recapitulate our conclusion up to now by asserting that the existence of singular loci in closed nowhere dense subsets of Σ, making it a multiply-connected topological space, implies active gravitational mass/energy in the complementary open dense subsets. Moreover, according to the "positive mass theorem" considered in the vacuum case, this gravitational mass/energy is non-zero and strictly positive. In passing, we would like to stress that Gannon's theorem should be conceived as a significant generalization of the Penrose-Hawking singularity theorems (Hawking and Ellis 1973), in the sense of replacing the usual geometric hypothesis of closed trapped surfaces in Σ by the more general applicable topological hypothesis of non-simple connectivity of Σ.

In the same vein of ideas, we may also consider the system of Einstein-Maxwell equations without sources for the Maxwell field, and in this way address from our perspective the alternative Wheeler's insight referring to "charge without charge" (Misner and Wheeler 1957; Wheeler 1957). This has been originally tied to the assumption that Σ is orientable and bears the standard wormhole topology, that is homotopically equivalent to $S^1 \times S^2 - \{point\}$, such that the magnetic flux lines thread through the wormhole. In this case, the homology class of all 2-spheres containing both of the wormhole mouths has zero charge, whereas the two individual wormhole mouths may be considered as having equal and opposite charges. In this context, a wormhole may be thought of in terms of a one-dimensional homology class in spacetime. From general results of low-dimensional geometric topology (Scorpan 2005), we know that every homology class of a four-dimensional spacetime can be represented by an embedded submanifold. Using the geometrodynamic foliation, we may restrict this representation to Σ. In this manner, we can instantiate a higher-order wormhole solution, for example, by considering an appropriate two-dimensional homology class.

We are going to outline a general method of generating these types of solutions guided by the form of the algebra sheaves $\mathbb{A}_L$ incorporating gravitational properties defined on dense open sets of X and by restriction to dense open sets of Σ. For this purpose, we may consider a singular locus with boundary in $\mathbb{R}^3$ or in its compactification S^3, which is excised from $\mathbb{R}^3$ or S^3. We consider a singular locus as a singular disk cut-off from S^3, which may be visualized in terms of a cone whose apex is at infinity and whose base lies at the boundary of the singular locus. A singular disk of this form excised from S^3 gives rise to a two-dimensional relative homology class of S^3, which may be interpreted according to the above as a two-dimensional embedded compact submanifold. The circular boundary of this singular disk is a closed and nowhere dense subset with respect to an open set of S^3. Analogously, we may consider the excision of more than one singular disks from S^3, such that their circular boundaries collectively define a closed and nowhere dense subset of an open set of S^3. We propose to think of these circular singular boundaries as giving rise to topological links.

The notion of a topological link is based on the underlying idea of connectivity among a collection of topological circles, called simply loops (Kawauchi 1996). We consider that a loop is a tame closed curve. The property of tameness means that a closed curve can be deformed continuously and without self-intersections into a polygonal one, that is a closed curve formed by a finite collection of straight-line segments. Given this qualification, a loop is characterized by the following properties: First, it is a one-dimensional object. Second, it is bounded, meaning that it is contained in some sphere of sufficiently large radius. Third, a single cut at a point cannot separate a loop into two pieces, whereas any set of two cuts at two different points does separate a loop into two pieces. Moreover, a loop is called knotted if it cannot be continuously deformed into a circle without self-intersection. We only consider unknotted tame closed curves. A topological N-link is a collection of N loops, where N is a natural number. Regarding the connectivity of a collection of N loops, the crucial property is the property of splittability of the corresponding N-link. We say that a topological N-link is splittable if it can be deformed continuously, such that part of the link lies within B and the rest of the link lies within C, where B, C denote mutually exclusive solid spheres (balls). Intuitively, the property of splittability of an N-link means that the link can come at least partly apart without cutting. Complete splittability means that the link can come completely apart without cutting. On the other side, non-splittability means that not even one of the involved loops, or any pair of them, or any combination of them, can be separated from the rest without cutting.

According to our hypothesis, a collection of circular singular boundaries defining a closed and nowhere dense subset of an open set of S^3 gives rise to a topological link in S^3. We may now replace the loop components of such a topological link by open non-intersecting tubular neighborhoods such that the complement of the link in S^3 can be given the structure of a three-dimensional compact and oriented manifold with boundary. Clearly, this space is homologically equivalent to the original one since it is just its deformation retract. Next, we may consider an ordering of the loops $l_1, l_2, \ldots l_N$ constituting the link, or equivalently an ordering of their tubular neighborhoods $\lambda_1, \lambda_2, \ldots \lambda_N$. Then, if we take λ_i, λ_j, together with their ordering, we define the relative homology class σ_{ij} that is represented by the compact oriented embedded submanifold whose two boundary components lie on the total boundary, that is the first one in $\partial\lambda_i$ and the second in $\partial\lambda_j$. The orientation is defined as being negative on the first boundary component and positive on the second, so that we have a path from λ_i to λ_j in this case.

5.7 The Borromean Rings as a Universal Nowhere Dense Singular Link

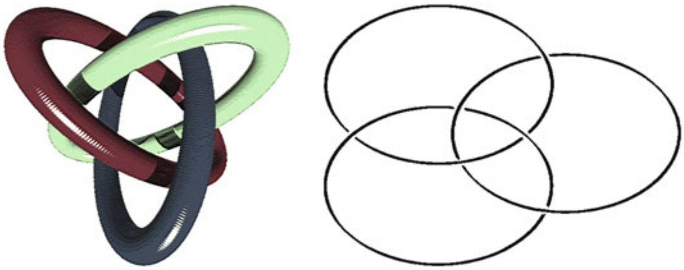
According to the formalism of Geometrodynamics, we consider the Lorentzian manifold $(\mathbb{R} \times \Sigma, \varepsilon^* g)$ as a representative of X, where the singularities are localized within the three-dimensional manifold Σ. We remind that, according to Gannon's theorem, if Σ is multiple-connected as a topological space, then spacetime is geodesically incomplete. According to our previous analysis, a collection of circular singular boundaries defining a closed and nowhere dense subset of an open set of S^3 gives rise to a topological link in S^3. Moreover, this implies the existence of active gravitational mass/energy in the complementary open dense subsets, which is non-zero and strictly positive.

In this context, it is important to examine if there exists a universal way via which we can obtain the three-dimensional manifold Σ by the information incorporated in a topological link in S^3 representing the singular boundaries, forming collectively a closed and nowhere dense subset. This sheds more light on the role of the algebra sheaves $\mathbb{A}_L$ utilized to express gravitational properties defined on dense open sets of X and by restriction to dense open sets of Σ, and is guiding in our quest of exploring generalized wormhole-types of solutions based on topological links and their associated homology classes.

It turns out that a universal way to obtain Σ by using a topological link in S^3 representing the singular boundaries, according to the above, actually exists and is based on the notion of a universal topological link. In view of the type of solutions we are interested in, such a universal link is defined by the Borromean rings. In particular, using methods of geometric topology, it can be shown that any compact oriented three-dimensional manifold Σ without boundary can be obtained as the branched covering space of the 3-sphere S^3 with branch set the Borromean rings (Hilden et al. 1987). In this manner, the Borromean rings constitute a universal topological link.

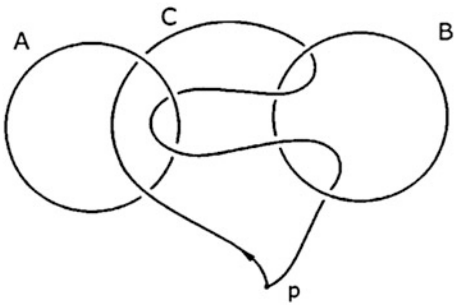
The notion of a branched covering space is a generalization of the standard notion of a covering space, characterized as a local homeomorphism bearing the unique path lifting and homotopy lifting property (Hatcher 2002). More precisely, a branched covering space of the 3-sphere S^3 is considered as a map from Σ to S^3 such that this map is a covering space after we delete or exclude a locus of points, called the branched locus. The universality property says that Σ can be obtained in this way if the branched locus is formed by the Borromean rings, considered as a closed and nowhere dense set with respect to an open set in S^3 in our setting. In a well-defined sense, this branched covering space provides the geometric representation of an algebra sheaf of the form $\mathbb{A}_L$ restricted to the three spatial dimensions, where the closed and nowhere dense subset formed by the Borromean rings is localized. We may extend this closed and nowhere dense subset to four dimensions by considering a timelike axis perpendicular to the Borromean rings, which plays the role of a threefold symmetry axis of rotation.

The Borromean rings consist of three rings localized in S^3, which are linked together in such a way that each of the rings lies completely over one of the other two, and completely under the other, as it is shown at the pictures below:



This particular type of topological linking displayed by the Borromean rings is called the Borromean link, and is characterized by the following distinguishing property: If any one of the rings is removed from the Borromean link, the remaining two come completely apart. It is important to emphasize that the rings should be modelled in terms of unknotted tame closed curves and not as perfectly circular geometric circles. The adjective topological means that they can be deformed continuously under the constraint that the particular type of linkage forming the Borromean configuration is preserved.

From the viewpoint of the theory of topological links, the Borromean link constitutes an interlocking family of three loops, such that if any one of them is cut at a point and removed, then the remaining two loops become completely unlinked (Cromwell et al. 1998; Debrunner 1961; Hatcher 2002; Lindström and Zetterström 1991; Kawauchi 1996). In more precise terms, the Borromean link is characterized topologically by the property of splittability as follows: The Borromean link is a non-splittable 3-link (because it consists of three loops), such that every 2-sublink of this 3-link is completely splittable. It is clear that it is a non-splittable 3-link because not even one of the three loops, or any pair of them, can be separated from the rest without cutting. A 2-sublink is simply any sub-collection of two loops obtained by erasing the loop that does not belong to this sub-collection. Since the Borromean link is characterized by the property that if we erase any one of the three interlocking loops, then the remaining two loops become unlinked, it is clear that every 2-sublink of the non-splittable 3-link is completely splittable.



In our context, we conclude that if a triad of circular singular boundaries defining a closed and nowhere dense subset of an open set of S^3 are connected in the form of the Borromean topological link, then Σ as a compact oriented three-dimensional manifold can be obtained as the branched covering space of the 3-sphere S^3 with branch set these Borromean-linked boundaries. Based on these findings, we would like to explore their semantics in relation to the instantiation of a higher-order wormhole solution. For this purpose, we remind that the standard wormhole solution is thought of in terms of a one-dimensional homology class in a space homotopically equivalent to $S^1 \times S^2 - \{point\}$. In our framework, we do not need to impose a particular topology on Σ ab initio, since it can now be derived universally as the branched covering space of S^3 over the branch nowhere dense subset of singular boundaries forming a Borromean link. The fact that the Borromean link is a non-splittable 3-link, such that every 2-sublink of this 3-link is completely splittable, is characterized in homology theory by a non-vanishing triple Massey product, where all pairwise intersection products of one-dimensional homology classes vanish, reflecting the fact that the components of the Borromean link are not pairwise linked. If we denote the components of the Borromean link $\mathcal{B}$ by $\lambda_1, \lambda_2, \lambda_3$, the triple Massey product (Hatcher 2002) is expressed as a two-dimensional cohomology class in the dense complement of $\mathcal{B}$ in S^3, that is it defines a non-trivial class in $H^2(S^3 \backslash (\lambda_1 \sqcup \lambda_2 \sqcup \lambda_3))$.

5.8 Revisiting Einstein's Insights on Field Theory

The main purpose of this communication, 100 years after Einstein's formulation of the General Theory of Relativity, has been an invitation to re-think about the validity domain of the field equations. The primary motivations emanate from three distinct sources: The first comes from Clarke's assertion concerning the problem of singularities, according to which the answers "involve detailed considerations of distributional solutions to Einstein's equations, leading into an area that is only starting to be explored ...." The second comes from Weyl's critical remark regarding the role of a background differential manifold, according to which "while topology has succeeded fairly well in mastering continuity, we do not yet understand the inner meaning of the restriction to differential manifolds. Perhaps one day physics will be able to discard it." The third comes from Wheeler's ideas regarding the notions of "mass without mass" and "charge without charge" in the vacuum, which can be given a more precise mathematical formulation in topological terms.

The sheaf-theoretic re-formulation of the usual differential geometric framework of smooth manifolds points to the conclusion that there exist distinguishable extensions of the standard smooth manifold spacetime model of General Relativity, which are utilized by appropriate extensions of the sheaf of coefficients parameterizing all tensorial physical quantities of the theory. The criteria of suitability of these extensions are determined by sheaf-cohomological means and maintain the standard covariance properties of the theory in domains including singular loci. We have presented and discussed in detail a concrete distribution-like sheaf of coefficients incorporating singularities in closed and nowhere dense subsets of an open set of a four-dimensional spacetime. An instructive way to think of these generalized algebra sheaves of coefficients refers to the role of a singularity as an obstruction to the existence of a solution to the field equations, expressed in terms of smooth coefficients. Thus, more generally, a singular locus may be thought of as a cohomological obstruction to the extendibility of a local smooth solution. This criterion incorporates and generalizes sheaf-cohomologically the initial definition of singular behavior in terms of non-extendibility of geodesics. Essentially the reason is that the notion of extendibility of some local solution is of a sheaf-theoretic nature.

At a further stage involving the formalism of Geometrodynamics, the existence of singular loci in closed and nowhere dense subsets of a spatial hypersurface, making it a multiply-connected topological space, implies active gravitational mass/energy in the complementary open dense subsets. Moreover, according to the "positive mass theorem" considered in the vacuum case, this gravitational mass/energy is non-zero and strictly positive. We show that it is enough for this purpose to consider singular boundaries forming closed and nowhere dense subsets and forcing a multiple-connected topology, which in turn implies that spacetime is geodesically incomplete. In view of expressing generalized wormhole solutions in this context, we propose that closed singular boundaries may form topological links. In this manner, using results of geometric topology, we point out that the Borromean topological link is characterized as a universal link. Since this link is characterized cohomologically by a higher order invariant, it may be associated with a generalized wormhole model, which reinforces Wheeler's ideas in Geometrodynamics.

Finally, we express the hope that the proposed approach paves the way for a further technical and semantical refinement of the following two Einstein's fundamental insights in building up General Relativity, which have not been addressed in satisfactory completeness up to present:

> Under the influence of the ideas of Faraday and Maxwell the notion developed that the whole of physical reality could perhaps be represented as a field whose components depend on four space-time parameters. If the laws of this field are in general covariant, that is, are not dependent on a particular choice of coordinate system, then the introduction of an independent (absolute) space is no longer necessary. That which constitutes the spatial

> character of reality is then simply the four-dimensionality of the field. There is then no "empty" space, that is, there is no space without a field. (Jammer 1993)

> A field theory is not yet completely determined by the system of field equations. Should one admit the appearance of singularities? ... It is my opinion that singularities must be excluded. It does not seem reasonable to me to introduce into a continuum theory points (or lines etc.) for which the field equations do not hold ... (Einstein 1956)

In a nutshell, regarding the first, the utilization of distribution-like sheaves of coefficients extending the smooth one over singularities, and thus extending the domain of validity of the field equations beyond globally smooth manifolds, shows in agreement with Geometrodynamics that active gravitational mass/energy may emerge from purely topological considerations taking into account the constraints imposed by the field equations in the vacuum. These topological considerations pertain to the modelling of singularities in terms of closed and nowhere dense sets, such that their complements who bear the induced active gravitational mass/energy are open and dense. In this manner, the vacuum can be legitimately considered as a structural quality of the field itself. Regarding the second, it is indeed unreasonable to consider singular loci in a continuum theory, where the field equations do not hold. The existence of distribution-like sheaves of coefficients provides precisely the means to bypass this problem by coordinatizing all the tensorial quantities in their terms, extending the smooth ones, and therefore extending the domain of validity of the field equations.[1]

5.9 Addressing the Resolution of Singularities in the Interpretation Context of Autogenetic Theory

5.9.1 The Problem of Singularities in General Relativity

A spacetime manifold is considered to be without singularities if the coefficients of the metric tensor field are smooth and the manifold X is geodesically complete with respect to the metric. In this case, all timelike geodesic curves can be extended to arbitrary length in the smooth spacetime manifold X. Thus, the notion of localization at a spacetime point-event is sensible only if the coefficients of the metric tensor field are smooth in an open neighborhood of this point.

A spacetime singularity may be thought of as a locus where the behavior of the metric tensor field coefficients become irregular. This irregularity is expressed by the

[1] A similar version to the above has appeared in *Zafiris, E. "What is the Validity Domain of Einstein's Equations? Distributional Solutions over Singularities and Topological Links in Geometrodynamics."* Invited paper for the Centennial Volume on Albert Einstein's 1915 Paper on the General Theory of Relativity: 100 Years of Chronogeometrodynamics: The Status of the Einstein's Theory of Gravitation in Its Centennial Year. *Universe*, **2** (3), 17; doi:10.3390/universe2030017 (2016).

impossibility of expressing the metric tensor field in terms of smooth coefficients in the vicinity of such a singular locus. Since the metric should belong to the solution space of Einstein's field equations, there naturally raises the physical concern if the validity of Einstein's equations may be extended appropriately over singular domains. Clearly, this is not possible by retaining the global smoothness requirement in the determination of the metric coefficients as solutions of the field equations. In particular, this issue targets the assumption of the global smooth point-event manifold substratum of the theory in relation to the existence of singular loci in spacetime where the metric breaks down or the curvature blows up.

We remind that a real smooth manifold X can be reconstituted completely from the global $\mathbb{R}$-algebra $\mathbb{C}^{\infty}(X)$ of smooth real-valued functions on it, and in particular, the point-events of X can be expressed in terms of the algebra $\mathbb{C}^{\infty}(X)$ as the $\mathbb{R}$-algebra homomorphisms $\mathbb{C}^{\infty}(X) \to \mathbb{R}$. Equivalently formulated, the manifold points-events constitute the $\mathbb{R}$-spectrum of the global algebra of smooth functions $\mathbb{C}^{\infty}(X)$.

Moreover, all the tensorial physical quantities, like the spacetime curvature, are coordinatized by means of the commutative $\mathbb{R}$-algebra of globally defined smooth real-valued functions $\mathbb{C}^{\infty}(X)$. In this manner, the background reference scaffolding of the theory remains fixed as the $\mathbb{R}$-spectrum of the algebra $\mathbb{C}^{\infty}(X)$, which supplies smooth coefficients for the coordinatization of all physical quantities. The points of the manifold X, although not dynamically localizable degrees of freedom in General Relativity, serve as the semantic information carriers of the factual level of reality. The subtlety of General Relativity is exactly that manifold points are not dynamically localizable entities in the theory. More precisely, manifold points assume an indirect reference as indicators of spacetime events, only after the dynamical specification of chronogeometrical relations among them, as particular solutions of the generally covariant field equations. Clearly, the existence of singular loci in spacetime where the metric breaks down in terms of smooth function coefficients forbids the association of smooth manifold points with spacetime events. What remains is an emergent notion of an event horizon of a singular locus where spacetime information may be encoded appropriately.

The conceptual underpinning of singularities in General Relativity is that the physical content of the theory, imprinted in the field equations, cannot be exhausted by looking exclusively at the factual level of reality, idealized by means of the point-events of a global smooth spacetime manifold. Even worse, the factual level cannot be accessed globally, but only in terms of local descriptions and their interconnections. It is a standard practice in General Relativity to look for local solutions of the field equations and study their maximal extendibility. Thus, the solutions of the field equations in terms of the spacetime metric actually form sheaves of smooth algebra coefficients in the process of extension from the local to the global level. The existence of a singular locus signifies the non-extendibility of a local solution expressed in terms of smooth coefficients. This does not a priori exclude the possibility that there exist other distribution-like sheaves of metric coefficients satisfying the field equations and being extendible over the former singular loci as well! This can be possible under the proviso that the domain of validity of the field equations goes beyond the notion of a smooth manifold. At best, the smoothness assumption should be employed only locally and certainly far from singular loci. In other words, there should exist an embedding of the algebra sheaf of smooth functions into a distribution-like sheaf of coefficients qualified as a solution of the field equations. In this way, even if we retain locally the association of the factual level of reality with the point-events of a smooth manifold, the existence of singularities signifies a process of folding out of this smooth stratum. In this manner, the existence of a distribution-like sheaf of coefficients as a solution of the field equations paves the way for understanding the precise nature of this folding out of the factual level into a statu-nascendi level. We argue that the mathematical specification of such a sheaf solution sheds light on the nature of singularities, determines what should be considered as a generic property of the gravitational field, and ultimately proves that the interpretation of General Relativity is logically incomplete without delving into the statu-nascendi level.

From a broad philosophical perspective, the issue of singularities in General Relativity as impossibilities of extending smooth metric solutions of Einstein's equations necessitates a higher logical order of relativization of facticity, going beyond the standard sequential order of extending timelike geodesic curves in a smooth manifold. In particular, there should be a statu-nascendi, level with its intrinsic logical order, via which the process of folding out of the factual level can be cast into a meaningful form. More concretely, there should be a whole categorial apparatus qualifying the intrinsic characteristics of the statu-nascendi level in comparison to the factual one. The necessity of a categorial apparatus and the indispensable role of an appropriately qualified statu-nascendi level for understanding the role of singularities in General Relativity has been emphasized by the conceptual framework of Autogenesis, developed into a fully-fledged philosophical theory with many applications by von Müller.

5.9.2 The Role of Singularities from the Viewpoint of Autogenetic Theory

It is instructive to present the basic premises of von Müller's theory of an autogenetic universe in a nutshell, since this would facilitate a better understanding of the proposed strategy to tackle the issue of singularities in General Relativity. The qualification autogenetic pertains to the two predominant characteristics of the universe, which are self-unfoldment and strong self-referentiality. Both of these characteristics cannot be comprehended by an exclusive restriction to a world of facts. What is required is a relativization of facticity, which leads inevitably to a novel account of time and reality. More precisely, the structural reduction of time to its linear-sequential aspect and the concurrent reduction of reality to its factual or event-like aspect is inadequate to account for critical processes related with folding into or folding out of the factual portrait of reality. Thus, the problem of singularities in General Relativity may be accessed effectively from this conceptual angle under the proviso that these enfolding/unfolding processes can be qualified by suitable means, enforcing a relativization of the factual level with respect to a statu-nascendi level. These means give rise to distinctive categorial frameworks distinguishing the statu-nascendi level from the factual level.

In more detail, the theory of autogenesis introduces a threefold scheme constituted in the form of three interdependent layers, which are connected together in the form of the linking properties of the Borromean rings, that is if any one of the layers is removed, then there remain two unlinked layers. Each layer captures a different aspect of reality, namely the apeiron aspect, the statu-nascendi and the factual aspect correspondingly. The apeiron aspect is inherently without any structure and expresses the irreducible global unity or non-separability of reality at this layer, which acts as a potential source for the actual taking place. The latter should involve both the statu-nascendi and the factual layers. The statu-nascendi should be better considered as a kind of a non-Boolean logical disclosure topos pertaining to the time-space of the present. As such it incorporates the logical or topological pre-conditions for relativizing the semantics of events at the factual level. It becomes visually informative to think of this relativization of facticity in terms of some self-referential process which either folds into or inversely folds out of the factual layer. In this manner, the factual aspect of reality is constituted by the observed traces of this process, viz. the events embedded within a local space-time context. Whereas the apeiron aspect is not amenable to any direct structural predicative determination, both the statu-nascendi and the factual aspect constitute layers whose respective characteristic function can be depicted in the terms of distinctive underlying categorial frameworks.

Each categorial framework stands for an integral apparatus consisting of four interrelated and bidirectionally interdependent components: (a) a logical structure of a predication space, (b) a related notion of a spatiotemporal context, (c) a causal scheme accounting for linkages, and (d) a corresponding epistemological setting. In this way, the factual aspect of reality is captured by means of a categorial apparatus, which consists of the following components respectively: (a) a Boolean logical predication space, (b) a local metrical space-time continuum, (c) a classical scheme of efficient causality, and (d) an epistemological setting based on the notion of absolute separability between observer and observandum. The intrinsic necessity of introducing another categorial apparatus constituting the statu-nascendi layer of reality is based on the inability of the former one to account for the logical structural phenomenon of strong self-referentiality and its concomitant operational manifestation as autogenesis, meaning a process of self-referential folding/unfolding without any separable external cause.

The constituent bidirectionally interrelated components of the statu-nascendi layer are the following: (a) a paratactical predication space on which some appropriate form of constellatory logic becomes applicable, (b) a local logical disclosure topos pertaining to the time-space of the present, (c) a causal scheme of autogenetic folding/unfolding, and (d) an epistemological setting of strong self-referentiality. The notion of parataxis refers to a mode of logical coherence of a multiplicity which is independent of linear sequential organization. This is captured by the functional role of a constellatory logic, where an individuated component of such a multiplicity can be evaluated only in the context of all other components being compatible with it in a suitable manner.

Therefore, from the perspective of the theory of autogenesis, the problem of singularities in General Relativity targets exactly the global breakdown of the metrical smooth space-time point-event-manifold model of this physical theory. Thus, it proposes to understand the means of folding out of the local space-time event continuum pertaining to the factual layer of reality via consideration of the categorial apparatus pertaining to the statu nascendi level. We stress again that the categorial apparatus of this level is indispensable for enforcing a higher-order relativization of facticity, which addresses the very notion of a local perspective on reality.

It is clear from the preceding that the nature of this notion, that is of a local perspective on reality, should not refer to the concept of geometrical locality in a global point-event manifold. In contradistinction, it should be of a logical/topological origin demarcating the logical structural pre-conditions that will allow us to perform indirect self-reference via the statu-nascendi associated with the signification of folding into and out of the factual level. This higher-order logical/topological relativization of facticity provides legitimate mathematical modeling means to exemplify the notion of categorial relativity, related to the function of the categorial apparatus of the statu nascendi level in the context of the theory of autogenesis.

5.9.3 *Outline of the General Mathematical Framework*

The construction of a concrete mathematical model consistent with the principle of categorial relativity in relation to the problem of singularities is not straightforward and requires a combination of methods leveraging recent advances in differential geometry, algebraic topology, theory of non-linear partial differential equations and mathematical logic together with the physical and mathematical constraints posed by Einstein's field equations. Notwithstanding the mathematical complexity of this endeavor, there are five basic axons in the intertwined network of mathematical concepts and tools.

The first is that the most natural way to describe the process of folding/unfolding in precise terms is provided by the algebraic notion of a sheaf and its concomitant topological manifestation in terms of the concept of a local homeomorphism as exemplified in the theory of etale spaces and covering spaces. Conceptually, we may think of a covering space as the mathematical model of a folding/unfolding process according to some periodic rule which is expressed by a discontinuous group action, for example in terms of the group of the integers. In the case of covering spaces this periodic rule is constant, viz. the folding/unfolding takes place according to a constant ratio, whereas in the general case of etale spaces it is variable. There exists a significant intermediate sheaf-theoretic notion between these two extremes, which bears the name of a branched covering space. The importance of this notion is that we may consider the folding/unfolding as taking place according to a constant periodic rule with the exception of a finite set constituting a locus where a type of branching behavior becomes manifest. Intuitively, we may think of this locus as an interface characterized by some potency index capturing the kind of branching.

The second is that the algebraic notion of a sheaf encapsulating the above types of local homeomorphisms is naturally associated with a calculus, called sheaf cohomology, via which all the localized constructions of differential geometry can be efficiently carried out without the intervention of any smoothness assumption, like in the classical theory of differential manifolds. This essentially means that a sheaf of smooth algebra coefficients is dethroned from its unique absolute role to express the differential geometric mechanism. The physical significance of this advancement in differential geometry for physics is that distribution-like solution sheaves turn out to be more suitable than the smooth one in cases where field sources or singular domains need to be incorporated in the field equations.

The third is the universal role being played by a specific topological link in three-dimensional space. This is the Borromean link, depicted algebraically by means of a non-commutative free group in two generators. Its significance lies on five distinctive roles: (a) it is threefold symmetric and can be iterated self-referentially ad infinitum. (b) All other topological links can be expressed algebraically in terms of simple algebraic operations within the same group-theoretic model. (c) It serves as a universal singular locus in the theory of branched covering spaces. (d) The components of the Borromean link serve as basis elements in homological vector spaces and the link itself can be characterized homologically by means of a third-order topological invariant. (e) It provides the simplest model of non-local linkage in 3-d space independently of metrical distance.

The fourth is the role of a particular class of sets, called topologically nowhere dense sets, in relation to establishing solutions of non-linear partial differential equations. It turns out that closed nowhere dense sets provide the analytic key for extending solutions over singularities.

The fifth is the logical notion of genericity in mathematical logic crystalized by the logical method of forcing conditions with respect to a partially ordered set. This method can be reformulated in sheaf-theoretic terms via the notion of a non-classical topos, conceived as a generalized and localized model of a set-theoretic universe of discourse, where indirect self-reference can be unproblematically performed with respect to the standard absolute model. In a well-defined sense, the notion of a logical topos bears a semantic logical role complementary to the topological or geometric role of branched covering spaces in qualifying the categorial apparatus necessary to access the precise form of a folding/unfolding process.

The rationale of applying the mathematical framework outlined concisely above to attack the problem of singularities in General Relativity is the following: In an intrinsically dynamically variable theory like General Relativity, where the properties of matter determine the gravitational field, it should be the precise physical conditions themselves that specified the type of the metric tensor field sheaf coefficients as solutions to the field equations, instead of fixing the coefficients ab initio to the smooth ones, and eventually face the breaking down of the metric or the curvature at singular loci. On the basis of this insight, it turns out that the domain of validity of Einstein's field equations can be rigorously extended beyond smooth real manifolds by admitting other distribution-like algebra sheaves of coefficients, instead of the smooth one, with respect to which all the differential geometric constructions can still be performed unambiguously subject only to some well-understood cohomological conditions. This approach vindicates the following critical remark of Hermann Weyl: "While topology has succeeded fairly well in mastering continuity, we do not yet understand the inner meaning of the restriction to differential manifolds. Perhaps one day physics will be able to discard it."

5.9.4 Generic Gravitational Properties via Closed Nowhere Dense Singular Loci

The first basic idea involved in the construction of distribution-like sheaves of coefficients as solutions of a non-linear partial differential equation is to model a singular locus Γ in $\mathbb{R}^4$ as a closed and nowhere dense subset relative to an open set $V \subseteq \mathbb{R}^4$, such that its complement $V \setminus \Gamma$ in V is dense. The second basic idea is to express such a closed and nowhere dense singular locus as an ideal in an algebra sheaf constructed as an extension of the smooth one over a partially ordered set. In this manner, the ideal expressing algebraically a singular locus contains diagrams of locally defined smooth functions indexed by Λ whose support covers the singular locus Γ, whereas they vanish outside it. Then, we prove that the quotient commutative algebra sheaf $\mathbb{A}_L(V) = \mathbb{C}^{\infty}(V)^{\Lambda} / \mathbb{I}_L(V)$ is an algebra sheaf of residues of diagrams of smooth functions modulo the closed nowhere dense singular ideal $\mathbb{I}_L(V)$. All the examples of such closed and nowhere dense sets can be obtained by some knot or link in 3-d space. Among them there exists a universal link, namely the Borromean rings, and thus if we make use of their universality property all possible closed and nowhere dense loci serving as models of singularities may be obtained through the Borromean link! Conceptually, this essentially means that the semantics of folding out of a local smooth event stratum into a singular domain can be associated with the formation of the Borromean link configuration and its concomitant algebraic expression in terms of an algebra sheaf of the type $\mathbb{A}_L$. At the final stage, we have to examine if this algebra sheaf satisfies the cohomological conditions necessary for expressing the differential geometric mechanism of General Relativity in these terms instead of the globally smooth ones. This turns out to be actually the case, and therefore, algebra sheaves of the type $\mathbb{A}_L$ can be used legitimately to express the metric solution of Einstein's field equations extended now over singularities.

In this manner, we conclude that it is possible to maintain the differential geometric mechanism, used in setting up General Relativity, by using a distribution-like sheaf of coefficients, and most important, extend it over singularities, such that the obtained solutions do not break down as in the smooth case. Philosophically, we may think of these solutions as extending into the statu-nascendi level, and thus they carry the information of unfolding subsumed in the construction of the sheaves of the type $\mathbb{A}_L$. Furthermore, the extended sheaf solutions admit a precise formulation over closed and nowhere dense loci serving as models of singularities, which can be obtained in a universal way via the Borromean topological link.

Consequently, we are able to retain not only the validity, but the form and covariance property of Einstein's field equations even over singular loci via solution sheaves of the type $\mathbb{A}_L$, whence these singular loci bear the role of closed and nowhere dense sets relative to an open set $V \subseteq \mathbb{R}^4$. The reason is that all physical quantities can still be transformed according to a tensor law for any arbitrary admissible coordinate transformation. The difference in comparison to the smooth case is that the coordinates are allowed to be non-standard or non-smooth, while at the same time all the machinery of differential geometry can be applied with respect to them. In particular, while the coefficients of the tensorial physical quantities are non-smooth, all the usual differential-geometric constructions can be carried out as in the smooth case. The only price to be paid for this generalization is the rejection of the fixed absolute global smooth manifold background of the theory. We consider this fact as physically nondisturbing, since the essence of General Relativity is in the covariant formulation and validity of Einstein's equations and not on the existence of a smooth background manifold. In particular, what we gain from such a generalization is not only that Einstein's equations can be extended covariantly over singular loci, but also that the solution of these equations in terms of coefficients from a sheaf of the form $\mathbb{A}_L$ is free of singularities!

In the sequel, we propose a physical gravitational interpretation of algebra sheaves of the form $\mathbb{A}_L$, containing the smooth $\mathbb{C}^\infty$ as a subalgebra and the Schwarz distributions $\mathbb{D}'$ as a linear subspace. For this purpose, we start by naming algebras of the form $\mathbb{A}_L$ as generic gravitational algebras. The task is to explain the notion of a property being gravitationally generic and on the basis of this characterization to probe the structure of $\mathbb{A}_L$ from the perspective of General Relativity.

We define a property to be gravitationally generic if it occurs and holds on a dense open set. In this way, a gravitationally non-generic property should appear only on a closed nowhere dense subset. We propose that this notion of genericity sheds light on the structure of algebras of the form $\mathbb{A}_L$ if exemplified in a gravitational context. In this manner, it is instructive to think of the notion of topological density in physical terms, viz. as an indicator of gravitational energy density caused by sources. In this context, the notion of genericity should be implemented by forcing conditions. More concretely, a condition forces a gravitational property if this property holds on a dense open set. A forcing condition forces every gravitational property either to hold or not in relation to the criterion of density, and thus, a forcing condition is generic in this sense.

Now using the criterion of gravitational genericity we attempt to explain the structure of the algebras $\mathbb{A}_L$. In particular, we point out that their construction is based on the notion of gravitationally generic properties. We focus our attention on the fact that the definition of these algebras is based on the extension of the algebra of smooth functions with respect to a partially ordered set L. Thus, the setting up of these algebras involves the extension of $\mathbb{C}^\infty$ to $\mathbb{C}^{\infty\Lambda}$, where Λ is the indexing set of the right directed partial order $L = (\Lambda, \leq)$. We also stress that this partial order is necessary in order that the set $\mathbb{I}_L(V)$ is qualified as an ideal in $\mathbb{A}_L$. A set of this form is characterized precisely as a closed and nowhere dense subset relative to an open set V, which remarkably can be obtained in a universal manner by the Borromean topological link. Since the off-diagonal ideal $\mathbb{I}_L(V)$ subsumes algebraically the information of some singular locus Γ, characterized as a closed and nowhere dense set, and thus as a bearer of a gravitationally non-generic property, the quotient algebra of the form $\mathbb{A}_L(V) = \mathbb{C}^\infty(V)^\Lambda/\mathbb{I}_L(V)$ incorporates only properties defined on dense open sets. Hence, according to our definition, $\mathbb{A}_L(V)$ incorporates gravitationally generic properties. This is possible if the partial order $L = (\Lambda, \leq)$ is actually a partial order of generic forcing conditions.

We note that the generic set of forcing conditions should not be contained in the initial standard model we started with. In summary, if we start from a standard smooth manifold model of spacetime, we can construct distinguishable extensions incorporating gravitational sources implemented by generic sets of forcing conditions of the form $L = (\Lambda, \leq)$. Most important, these extensions can be obtained as singularity-free solutions of Einstein's equations if all the tensorial physical quantities are expressed in terms of coefficients from generic gravitational algebras of the form $\mathbb{A}_L(V)$.

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Chapter 6
Borromean Link in Quantum Theory Loops, Projective Invariants and the Realization of the Borromean Topological Link in Quantum Mechanics

6.1 Introduction

The notion of a topological or geometric phase has been introduced in quantum mechanics by Berry in 1984 (Berry 1984; Simon 1983), and generalized by Aharonov and Anandan 2 years later in 1986 (Aharonov and Anandan 1987). The conceptual precursors of this astonishing discovery, which has been unnoticed in the foundations of quantum theory for more than 60 years, is the work of Pancharatnam in polarization optics (Wilczek and Shapere 1989) and the Aharonov–Bohm effect (Aharonov and Bohm 1959, 1961) in electromagnetism. In 1956 Pancharatnam realized that in order to understand interference phenomena it is not required to know the absolute phase, but only the relative phase difference between light beams in different states of polarization. For two light beams this relative phase is given by the phase argument of their complex-valued scalar inner product. Actually all the typical global quantum mechanical observables are relative phases obtained by interference phenomena. These phenomena involve various splitting and recombination processes of beams whose global coherence is measured precisely by some relative phase difference. If we consider an external time parametrization of interference phenomena, then the relative phase global observable can be thought of as the physical attribute measuring the coherence between two histories of events sharing a common initial and final temporal point.

For example, we may think of the simplest case of a beam which is split into two beams propagating for a period of time and finally recombined. Their interference is always measured by a global relative phase difference. We may summarize the discussion up to this point by stressing the fact that although quantum mechanics may be locally interpreted in terms of probabilities of events, so that complex phases do not play any role and can be gauged away, globally it is the relative phase differences between histories of events that bear the major physical significance.

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Equivalently, a global geometric complex phase may be thought of as the "memory" of a quantum system undergoing a "cyclic evolution" after coming back to its original physical state. The "cyclic evolution" is considered with respect to a loop in an underlying space of control parameters upon which the Hamiltonian depends on. Given that the temporal dependence of the driving Hamiltonian is only implicit via the control parameters, tracing a loop in the space of control parameters is naturally associated with a periodicity property of the state vector.

Beyond the local phase invariance of the probability assignment in quantum mechanics, there is another type of invariance stemming from the fact that probability amplitudes are complex numbers. In general, the complex-valued inner product $\langle\phi|\psi\rangle := z$ is interpreted as the transition amplitude from the unit state vector $|\psi\rangle$ to $\langle\phi|$. The complex-valued transition amplitude z is used to calculate the corresponding real-valued, transition probability by squaring, that is $z^2 = z^* z$. The underlying symmetry of the transition probability is that it remains invariant under the operation of complex conjugation. Most important, geometric phase factors distinguish between unitary and antiunitary transformations in terms of complex conjugation. The implications of this invariance are far reaching because they target the semantics of the temporal parameter in comparison to its classical connotation. The necessity for a profoundly different notion of time in the quantum regime, which encompasses a crucial role for the present, has been emphasized by von Müller and Filk (von Müller 2015; Filk and von Müller 2010) from the perspective of the categorical conceptual frame underlying quantum physics. We propose to use these two interrelated types of invariance, pertaining to transition probabilities, as an anchor point from which we unravel the role of loops, based loops, and their projective invariants in the state space of a quantum system. The fact that based oriented loops can be composed by means of a non-commutative group law carries the seed for the realization of the Borromean link in quantum mechanics via its representation in terms of one-parameter unitary groups acting on a specified state vector. These unitary groups are parameterized by some parameter bearing a temporal or spatial semantics that plays a distinctively different role in comparison to the classical counterparts.

The motivation of this work arose from a quite different, although closely related, perspective concerning the utilization of topological links to probe the nature of quantum entanglement. More precisely, Aravind proposed to investigate the correspondence between topological and quantum entanglement based on the following analogy: If the state of a simple quantum system is to be thought of as a loop, then the state of an entangled quantum system should be thought of as a topological link. In the context of this analogy, there emerged the striking result that the entangled GHZ (Greenberger–Horne–Zeilinger) state of the composite 3-qubit system is analogous to a Borromean link (Aravind 1997). Notwithstanding this finding, the weak point of the analogy is that it is dependent on the measurement basis.

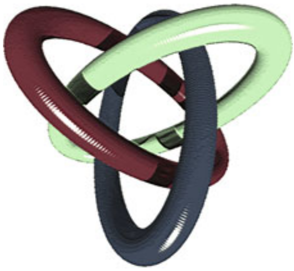
From our viewpoint, the success of an analogy is based on the initial modelling hypothesis, which in the present case refers to the interesting conception of a quantum state in terms of a loop. In this sense, although the analogy between topological entanglement and quantum entanglement turns out to be basis-dependent, there arises the possibility to test the initial hypothesis in another contingency, namely the one of quantum interference. In particular, given the invariance of the transition probability under complex conjugation, it seems more natural to think of a quantum transition amplitude in terms of a loop instead of Aravind's initial hypothesis. In this context, the emergence of a specific link would refer to the specific form of interference among transition amplitudes. For this purpose, we consider loops in the space of rays and calculate their projective invariants. We show that arbitrary transition probabilities can be calculated by means of these projective invariants. We also deduce the expression of global geometric phase factors and study their properties. Then, we show that they lead to an invariant distinction between unitary and antiunitary transformations. Next, we apply the method of projective invariants of loops in ray space for the calculation of the transition probabilities involved in the double slit experiment. From the analysis of the double slit experiment in terms of loops, we realize that we can represent the action of one-parameter unitary groups in terms of pairs of oppositely oriented based loops at a fixed reference ray. In this context, we explain the relation among observables, local Boolean frames of projectors, and one-parameter unitary groups. This leads to the criterion of differentiation among pairs of based loops in terms of the localization properties of Boolean frames formed by spectral families of orthogonal projectors. In the sequel, we exploit the non-commutative group structure of based oriented loops in 3-d space and demonstrate that it carries the topological semantics of a Borromean link. In particular, we show that the topological information incorporated in the specification of the Borromean link can be encoded algebraically by means of the non-commutative group-structure of the free group Θ_2 generated by two oriented loops, which are based at the same fixed point. Finally, we prove that there exists a representation of this group structure in terms of one-parameter unitary groups acting on a quantum state space that realizes the topological linking properties of the Borromean link.

6.2 Tame Closed Curves and the Borromean Link

The notion of a topological link is based on the underlying idea of connectivity among a collection of loops. We consider that a loop is a tame closed curve. The property of tameness means that a closed curve can be deformed continuously and without self-intersections into a polygonal one, that is a closed curve formed by a finite collection of straight-line segments. Given this qualification, a loop is characterized by the following properties: First, it is a one-dimensional object. Second, it is bounded, meaning that it is contained in some sphere of sufficiently large radius. Third, a single cut at a point cannot separate a loop into two pieces, whereas any set of two cuts at two different points does separate a loop into two pieces. Moreover, a loop is called knotted if it cannot be continuously deformed into a circle without self-intersection. We only consider unknotted tame closed curves.

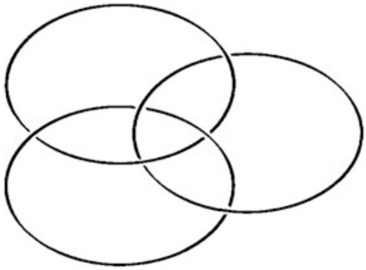
A topological N-link is a collection of N loops, where N is a natural number. Regarding the connectivity of a collection of N loops, the crucial property is the property of splittability of the corresponding N-link. We say that a topological N-link is splittable if it can be deformed continuously, such that part of the link lies within B and the rest of the link lies within C, where B, C denote mutually exclusive solid spheres (balls). Intuitively, the property of splittability of an N-link means that the link can come at least partly apart without cutting. Complete splittability means that the link can come completely apart without cutting. On the other side, non-splittability means that not even one of the involved loops, or any pair of them, or any combination of them, can be separated from the rest without cutting.

The "Borromean rings" consist of three rings, which are linked together in such a way that each of the rings lies completely over one of the other two, and completely under the other, as it is shown at the picture below:



This particular type of topological linking displayed by the "Borromean rings" is called the "Borromean link," and is characterized by the following distinguishing property: If any one of the rings is removed from the "Borromean link" the remaining two come completely apart. It is important to emphasize that the rings should be modelled in terms of unknotted tame closed curves and not as perfectly circular geometric circles. The adjective topological means that they can be deformed continuously under the constraint that the particular type of linkage forming the Borromean configuration is preserved.

From the viewpoint of the theory of topological links, the Borromean link constitutes an interlocking family of three loops, such that if any one of them is cut at a point and removed, then the remaining two loops become completely unlinked (Cromwell et al. 1998; Debrunner 1961; Hatcher 2002; Lindström and Zetterström 1991; Kawauchi 1996). In more precise terms, the Borromean link is characterized topologically by the property of splittability as follows: The Borromean link is a non-splittable 3-link (because it consists of three loops), such that every 2-sublink of this 3-link is completely splittable. It is clear that it is a non-splittable 3-link because not even one of the three loops, or any pair of them, can be separated from the rest without cutting. A 2-sublink is simply any sub-collection of two loops obtained by erasing the loop that does not belong to this sub-collection. Since the Borromean link is characterized by the property that if we erase any one of the three interlocking loops, then the remaining two loops become unlinked, it is clear that every 2-sublink of the non-splittable 3-link is completely splittable.



6.3 Aravind's Analogy: GHZ Entangled State as a Borromean Link

The existence of topological links, like the Borromean link, may be thought of as a form of topological entanglement. From the other side, one of the basic distinguishing features between classical and quantum systems is the phenomenon of quantum entanglement. Thus, there arises the natural question if there exists any type of correspondence between the forms of topological and quantum entanglement. In the context of this, Aravind (1997) proposed to investigate the correspondence between topological and quantum entanglement based on the following analogy: If the state of a simple quantum system is to be thought of as a ring (topological circle or loop), then the state of an entangled quantum system should be thought of as a topological link. Moreover, the measurement of a subsystem of an entangled system should be thought of as the process of cutting of the corresponding loop. The caveat of this approach is that there are many possible measurements on a subsystem of a composite entangled system, and consequently the proposed correspondence should depend on the choice of the measurement basis.

It is well known that the state space of a composite quantum system is given by the tensor product of the state spaces of the component subsystems. In the case of two subsystems A and B, if the state vector of the composite system can be written as $|\psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$, where $|\psi_A\rangle$, $|\psi_B\rangle$ denote some state vector of the subsystem A and B, respectively, then the state vector $|\psi\rangle$ is called separable. Otherwise, if the pure state $|\psi\rangle$ of the composite system cannot be written in the above form it is called entangled. In the simplest case, we may consider qubits, that is quantum systems whose state space is 2-d. Let us consider a basis of the 2-d state space consisting of the state vectors $|0\rangle$ and $|1\rangle$. So we may consider a composite quantum system consisting of two qubits. It is immediate to see that there are states of the composite qubit system, for example:

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

where we follow the general convention:

$$|\psi_1\rangle \otimes |\psi_2\rangle := |\psi_1\rangle|\psi_2\rangle := |\psi_1\psi_2\rangle$$

which are not separable, and thus they are entangled.

If we follow Aravind's analogy, then an entangled state of a composite two qubit system corresponds to a non-splittable topological 2-link, whereas a separable state corresponds to splittable 2-link. Let us now consider the case of three qubit systems denoted by A, B, and C correspondingly. The composite quantum system of these three qubits is characterized by the state space given by the tensor product of the state spaces of the three component subsystems. We consider the so-called GHZ state (Greenberger–Horne–Zeilinger) of the composite system (Greenberger et al. 1990) defined by:

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$$

It is clear that the GHZ state $|\psi\rangle$ of the composite system is entangled. Moreover, the GHZ state is a symmetric state under permutations of the states of the three component subsystems. Thus, it can be considered as representing a non-splittable 3-link. Now, we may consider a measurement basis of the composite system given by the projection operators $P_0 := |0\rangle\langle 0| \otimes Id \otimes Id$ and $P_1 := |1\rangle\langle 1| \otimes Id \otimes Id$. These projections correspond to potential measurements only on qubit A. After a measurement is performed, the composite system is either in the state $|000\rangle$ or in the state $|111\rangle$. Both of these states are separable. Therefore, a measurement carried on the qubit A can be thought of as a process of cutting the corresponding loop. Consequently, the remaining 2-sublink of the initial non-splittable 3-link becomes completely splittable. Clearly, due to the permutation symmetry of the GHZ state, one may consider a potential measurement only on qubit B or only on qubit C without affecting the argument. Hence, the entangled GHZ state of the composite 3-qubit system is analogous to a Borromean link. The weak point of this analogy is that it is dependent on the measurement basis.

From the theoretic perspective of our work, although Aravind's analogy is instructive for thinking about a possible bridge between the notions of topological and quantum entanglement, in particular with reference to the Borromean link, it suffers from the unjustifiable initial assumption that the state of a quantum system may be thought of in terms of a loop. This poses the problem of investigating, in the first place, the possible role of loops in the foundations of quantum mechanics.

6.4 Loop Symmetry of Quantum Transition Probabilities

In quantum mechanics the transition amplitude from the physical state $|\psi\rangle$ to the state $\langle\phi|$ is given by the complex-valued inner product $\langle\phi|\psi\rangle$ in Dirac's notation. This notation is justified by the fact that the Hilbert space complex-valued inner product induces a conjugate isomorphism between the Hilbert space and its dual. Moreover, the double dual is isomorphic to the original Hilbert space. Equivalently, there exists a bijective correspondence between the covectors of the dual space and the conjugate vectors of the state space, which represent physical states. In this way, a vector is transformed to a covector by Hermitian conjugation, that is $|\psi\rangle^\dagger = \langle\psi|$, which in the one-dimensional case is reduced simply to complex conjugation.

The complex-valued transition amplitude $\langle\phi|\psi\rangle := z$ is used to calculate the corresponding real-valued, normalized transition probability by squaring. For simplicity, if $|\psi\rangle$ and $|\phi\rangle$ are unit vectors, the transition probability is given by the normalized real number $z^2 = z^* z$. It is clear that the transition probability, which represents a physical magnitude, remains invariant under complex conjugation. This innocent looking fact can have far-reaching implications. In particular, it implies that every set of equations in quantum mechanics, which are used for calculating transition probabilities, may be written in two equivalent and physically indistinguishable forms differing by complex conjugation.

The whole issue arises from the arbitrary choice of the positive root of $\sqrt{-1} := i$ for the formulation of Schrödinger's equation:

$$i\hbar\frac{\partial\psi}{\partial t} = \hat{H}\psi$$

instead of the opposite convention involving the negative root of $\sqrt{-1}$, that is $-\sqrt{-1} := -i$, which would give:

$$-i\hbar\frac{\partial\psi}{\partial t} = \hat{H}\psi$$

Notice that both formulations are physically indistinguishable, since they provide identical transition probabilities.

After choosing the first convention, as it is usually the case, the issue of invariance of the transition probabilities under complex conjugation is treated as follows: We note that Schrödinger's equation is not invariant under complex conjugation, but it remains invariant under the combined action of complex conjugation and parameter t reversal. The parameter t is treated as a classical temporal parameter and is referred to as the parameter of "unitary time evolution," and thus t-reversal is interpreted as "time-reversal." We will refrain from the interpretation of t as a classical temporal parameter for reasons that will be explained in the sequel, but presently, in order to comply with the current usage of this parameter we will refer to it as "time."

The conclusion we obtain from the above may be summarized as follows: For each description of a quantum system in terms of a state vector "evolving" along the positive direction of the real parameter t ("time"), there exists a physically indistinguishable description in terms of the conjugate state vector "evolving" along the negative direction of the real parameter t ("time"). Equivalently, in terms of transition amplitudes, we may say that z is associated with a process "evolving" forward in "time," whereas z^* with a process "evolving" backward in "time." From this viewpoint, the transition probability may be associated with a process "evolving" around a loop in "time." The first reaction is to deliberately cease to consider these loops because they are associated with causality paradoxes.

We will argue that the problem is not associated with the loops themselves, expressing the symmetry in the calculation of transition amplitudes under complex conjugation, but with the interpretation of the parameter t as a classical "time" parameter in analogy to the situation in classical mechanics. Up to present, acknowledgement of this issue in the foundations of quantum mechanics has not led to questioning the status of the real parameter t as a classical "time" parameter, but to a view most commonly referred to as the "double inferential state-vector formalism" after Watanabe (1955), or the "two-state vector formalism" after Aharonov, Bergmann and Lebowitz (Aharonov et al. 1964, 2014). In these formulations, the two-state vector is represented by the pair $\langle\phi||\psi\rangle$, where the parameter t denotes "time," and $\langle\phi|$ "evolves" backward in "time," whereas $|\psi\rangle$ "evolves" forward in "time." These formalisms are "time"-reversal invariant and have been used for describing pre-selected or post-selected quantum systems, which has led recently to the theory of weak measurements (Vaidman 1996). Under this view, causality is "time"-symmetric, since there exists an apparent combination of (forward) causality and retrocausality.

From our perspective, although these formalisms have contributed significantly in unraveling important temporal and information-theoretic notions in the foundations of quantum physics, they have been especially designed to preserve the interpretation of the parameter t as a classical time parameter, together with the associated concept of "evolution in time," paying the price of introducing into the formalism notions of forward and backward causality (with reference to the complex conjugation symmetry discussed previously). Our strategy will be to discard the interpretation of t as a classical time parameter and simultaneously to utilize loops—expressing the symmetry intrinsic in the calculation of quantum transition probabilities—in an appropriate manner. In this way, it is necessary, first of all, to differentiate the meaning of the words "transition" and "evolution."

The main problem is intrinsically associated with the notion of localization in the quantum domain. For instance, in the position representation of a quantum system in one dimension, where the position operator is multiplicative and the momentum operator is differential, the complex valued state vector is expressed as a continuous and differentiable function of two parameters, where the first is interpreted as a spatial coordinate position parameter, whereas the other as a temporal parameter. This induces the inaccurate mental picture of a system whose wavefunction evolves in an a priori differentiated spatiotemporal continuum, and thus its localization is thought of in classical spacetime terms irrespective of the decisive role of measurement processes and the actualization of events.

A mental picture of a similar type is induced by Feynman's "spacetime approach to quantum mechanics," where the "time evolution" of a quantum system is considered to take place along all possible continuous paths connecting two classical spacetime points (Feynman et al. 2010). In this formulation, there is clearly implicated a notion of spatiotemporal localization, in the sense that the "evolution" is conceived to take place in an a priori given classical spacetime continuum. The quantal aspect of Feynman's approach does not consist in rejecting the classical spacetime continuum as a means of localizing quantum systems, but in considering all possible continuous spacetime paths and weighting each one of them by a transition amplitude z proportional to the complex exponential factor $z \sim \exp(iS/\hbar)$, where S is the classical action associated with the depicted path. Although Feynman's path integral method is an invaluable calculation tool of transition probabilities, it subscribes to a concept of localization in the quantum domain, which is formulated within an a priori differentiated classical spacetime substratum. Actually, the operational description of the path integral in the double slit experiment by Feynman's approach makes matters more complicated and we are going to return to this point later.

A natural question emerging at this stage is if the transition probability calculated by the path integral method bears the property of invariance under complex conjugation. Again it is easy to see that the real-valued transition probability for a path, obtained by the product of the associated transition amplitude z with its complex conjugate amplitude z^*, remains invariant under complex conjugation if "time" is reversed simultaneously. In other words, the calculation of the transition probability for a path bears the symmetry of a process "evolving" around a loop in "time." Hence, the total transition probability may be obtained by summing all contributions emanating from all possible loops connecting the initial and final points (in the forward "time" direction). We are going to exploit this observation—acquired by a simple symmetry argument—later on, although we will refrain from interpreting loops of this form as loops in "time."

The symmetry of a process "evolving" around a loop in "time" in relation to the calculation of transition probabilities also figures out predominantly in Schwinger's "closed time path formalism," which is extensively used in quantum field theory (Schwinger 2000). The basic idea of Schwinger's formalism is the following: The transition amplitude from a spacetime point to another one may be considered as a matrix element. Concomitantly, if we consider that the state at an initial fixed "time" $t = 0$ is described by a diagonal matrix element, we may insert a complete orthonormal basis of states into this matrix element at a different later time t'. In this manner, it is possible to express the original matrix element at fixed time $t = 0$ as a product of the transition amplitude (matrix element) from $t = 0$ to t' with a "time"-reversed complex conjugate transition amplitude from t' to $t = 0$. Notice that in the simple case that we associate the transition amplitude with a path from $t = 0$ to t', the "time"-reversed complex conjugate transition amplitude from t' to $t = 0$ does not have to refer necessarily to the same path connecting these two points. In other words, if the boundary conditions referring to the "forward evolution" are different from the boundary conditions referring to the "backward evolution" a non-trivial and non-reducible product of a transition amplitude with a "time"-reversed complex conjugate transition amplitude is obtained.

6.5 Projective Invariants and the Emergence of Geometric Phases

The mystery around the symmetry of processes "evolving" around a loop in "time" can be partially resolved by a careful consideration of the notion of geometric phase in quantum mechanics. A general generation schema of an experimentally observable global phase factor, which is of a purely geometric origin has been discovered by Berry (Berry 1984; Simon 1983; Wilczek and Shapere 1989) and generalized by Aharonov and Anandan (1987). It has been shown that a quantum system undergoing a slowly evolving (adiabatic) cyclic evolution retains a trace of its motion after coming back to its original physical state. This trace is expressed by means of a complex phase factor in the state vector of the system, called Berry's phase or the geometric phase. The "cyclic evolution," which is better to be thought of as a periodicity property of the state vector of a quantum system, is driven by a Hamiltonian bearing an implicit time dependence through a set of control variables. For instance, we may think of external electric or magnetic fields which define the Hamiltonian parametric dependence of a charged system. The adiabatic condition defines a constraint of parallel transport, or equivalently a connection, specified by the requirement that the implicit "time" dependence of the Hamiltonian is sufficiently slow so that the state vector stays in the eigenspace of the same instantaneous eigenvalue of the Hamiltonian. Intuitively, once the state vector is prepared in an instantaneous eigenstate of the Hamiltonian with an eigenvalue which is separated from the neighboring eigenstates by a finite energy gap, then it remains there during its transport within a finite period.

We may think of the space of control variables as a slowly varying environment with respect to which a state vector (eigenvector of the Hamiltonian localized at the corresponding eigenspace) displays a "history" dependent geometric effect: When the environment returns to its original state, the system also does, but for an additional global geometric phase factor. Due to the implicit "temporal" dependence imposed by the "time parameterization" of a closed path in the environmental parameters of the control space, this global geometric phase factor is thought of as a trace or memory of the motion encoding the global geometric features of the control space. The Berry phase is a complex number of modulus one and is experimentally observable. The two most important features regarding the experimental detection of a quantum global phase are (1) that it is a statistical object, and (2) it can be measured only relatively. Thus it becomes observable by comparing the "historical evolution" of two distinct statistical ensembles of systems through their interference pattern. The Berry phase is geometric because it depends solely on the geometry of the control space pathway along which the state vector is transported. It depends neither on the "temporal metric" duration of the "evolution," nor on the particular dynamics that is applied to the system. In more precise differential geometric terms, a geometric phase factor is expressed in group-theoretic terms as the holonomy of the associated Chern–Berry connection determining the rule of transport.

The differential-geometric qualification of a global geometric phase factor, interpreted physically as a memory of a quantum state vector under a global cyclic transition in a base space of control parameters, is instructive for many reasons:

First, it requires to take explicitly into account the local phase invariance in the specification of the state vector of a quantum system. The root of this gauge-type symmetry is the invariance of the probability assignment under local complex phase transformations of the state vector. In other words, the state vector of a quantum system is determined locally only up to an arbitrary complex phase factor.

Second, the base space of control parameters is not spacetime. It is thought of as an environment of control variables on which the Hamiltonian depends. In this manner, the "time" dependence of the Hamiltonian is only implicit via the control variables. Subsequently, loops in the base space of control variables are not "loops in time," but they merely signify a periodicity property of the state vector.

Third, the global geometric phase factor is obtained as the holonomy transformation of the Chern–Berry connection with respect to a loop in the base space of control parameters. Thus, the geometric phase constitutes the global manifestation of the curvature of this connection and is totally independent of the parameterization of this loop by a "temporal parameter."

A suitable differential geometric model of understanding the notion of global geometric phase factors involves a homotopy fibration over a base topological space of control variables, which can be presented either as a principal fiber bundle with local structure group the group of complex phases, or as a line bundle of states associated with the former, together with the parallel transport rule imposed by the Chern–Berry connection (Zafiris 2015). In order to avoid these technicalities at this point, we will follow a simplified approach which retains all the essential aspects.

We start from the observation that a normalized state vector $|\psi\rangle$ describes the state of a quantum system by the set of its expectation values with respect to an observable, represented by means of a self-adjoint operator $\hat{V}$. These expectation values are real numbers obtained by the assignment:

$$\hat{V} \longmapsto \langle\psi|\hat{V}\psi\rangle$$

The symmetry of this assignment leads to the conclusion that two different state vectors actually specify the same state if and only if they are linearly dependent. For example, the unit state vectors $|\psi\rangle$ and $e^{i\varphi}|\psi\rangle$ specify the same state for any $0 \leq \varphi \leq 2\pi$. If we identify linear dependent unit state vectors, that is for any fixed unit state vector $|\psi\rangle$ consider the equivalence class of all unit vectors related by phases, given by $\Psi = \{e^{i\varphi}|\psi\rangle, 0 \leq \varphi \leq 2\pi\}$, then we obtain a 1-1 correspondence between a physical state (pure state) and the ray Ψ generated by $|\psi\rangle$.

We conclude that if we consider the set of all normalized unit state vectors in a Hilbert space $\mathcal{H}$, that is the subspace $\mathfrak{U} = \{|\psi\rangle \in \mathcal{H} | \langle\psi|\psi\rangle = 1\}$, then for each unit state vector $|\psi\rangle \in \mathfrak{U}$ there exists a definite ray $\Psi \in \mathbf{P}\mathcal{H}$, where $\mathbf{P}\mathcal{H}$ denotes the set of all rays, to which it belongs to. It is important to notice that each ray Ψ is an equivalence class of physically indistinguishable unit state vectors under the action of the group of complex phases $\mathcal{U}(1) := \mathbb{S}^1$. It is clear that each ray spans a one-dimensional linear subspace of $\mathcal{H}$ (we exclude the zero vector from $\mathcal{H}$). Thus, it can be identified with the one-dimensional projection operator $P_{\psi}^2 = P_{\psi} := |\psi\rangle\langle\psi|$ that projects $\mathcal{H}$ onto this one-dimensional linear subspace.

Therefore, the space of physical states (pure states) $\mathbf{P}\mathcal{H}$ is identified with the space of one-dimensional projection operators of $\mathcal{H}$. We note that although $\mathcal{H}$ is a linear space, neither $\mathfrak{U}$ nor $\mathbf{P}\mathcal{H}$ are linear spaces. Nevertheless, there exists a well-defined projection mapping pr from $\mathfrak{U}$ to $\mathbf{P}\mathcal{H}$, that is $pr : \mathfrak{U} \to \mathbf{P}\mathcal{H}$, such that:

$$|\psi\rangle \longmapsto pr(|\psi\rangle) := P_{\psi} = |\psi\rangle\langle\psi|$$

For instance, in the case of a two-level quantum system, where the Hilbert space is 2-complex dimensional, the space of rays is the one-dimensional complex projective space, which is isomorphic to the Riemann sphere $\mathbb{S}^2$. Thus, the projection mapping gives rise to the Hopf fibration $\mathbb{S}^1 \hookrightarrow \mathbb{S}^3 \to \mathbb{S}^2$ (Urbantke 2003).

We note that for any unit state vector $|\phi\rangle$ in $\mathcal{H}$, the action of the one-dimensional projection operator or filter $P_{\psi} := |\psi\rangle\langle\psi|$ is given by:

$$|\psi\rangle\langle\psi|\phi\rangle = \lambda_{\phi}|\psi\rangle$$

where $\lambda_{\phi} := \langle\psi|\phi\rangle$ is valued in the complex numbers $\mathbb{C}$, such that $\lambda_{\phi}^2 = \lambda_{\phi}\lambda_{\phi}{}^*$ is the probability to find the system described by the unit state vector $|\phi\rangle$ in the state vector $|\psi\rangle$ under the action of the filter P_{ψ}. Notice that the probability assignment is invariant under the action of a complex phase transformation on the unit state vector $|\phi\rangle$, defined by $|\phi\rangle \longmapsto e^{i\varphi}|\phi\rangle$. Clearly, it is also invariant under the action $|\psi\rangle \longmapsto e^{i\xi}|\psi\rangle$, since $\langle\psi| \longmapsto e^{-i\xi}\langle\psi|$, and thus $P_{\psi} = |\psi\rangle\langle\psi|$ remains invariant under complex phase transformations. Thus, the probability $\lambda_{\phi}^2 = \lambda_{\phi}\lambda_{\phi}{}^*$ although it is formulated in terms of state vectors, it actually refers to their corresponding rays.

Then, it is straightforward to generalize the above argument as follows: For any two non-orthogonal unit state vectors $|\psi_1\rangle$ and $|\psi_2\rangle$ the 2-vertex projective invariant quantity:

$$\begin{aligned}\mathbf{I}_2(|\psi_1\rangle, |\psi_2\rangle) &= \mathbf{I}_2(\Psi_1, \Psi_2) = \mathbf{I}_2(P_{\psi_1}, P_{\psi_2}) \\ &= |\langle\psi_1|\psi_2\rangle|^2 = (\langle\psi_1|\psi_2\rangle) \cdot (\langle\psi_1|\psi_2\rangle)^* \\ &= Tr(P_{\psi_1}P_{\psi_2})\end{aligned}$$

is a real non-negative normalized quantity, which gives the transition probability from $|\psi_1\rangle$ to $|\psi_2\rangle$.

Now, for any three unit pairwise non-orthogonal state vectors $|\psi_1\rangle$, $|\psi_2\rangle$ and $|\psi_3\rangle$, we may define the 3-vertex projective invariant quantity:

$$\begin{aligned}\mathbf{I}_3(|\psi_1\rangle, |\psi_2\rangle, |\psi_3\rangle) &= \mathbf{I}_3(\Psi_1, \Psi_2, \Psi_3) = \mathbf{I}_3(P_{\psi_1}, P_{\psi_2}, P_{\psi_3}) \\ &= (\langle\psi_1|\psi_2\rangle) \cdot (\langle\psi_2|\psi_3\rangle) \cdot (\langle\psi_3|\psi_1\rangle) = \langle\psi_1|\psi_2\rangle\langle\psi_2|\psi_3\rangle\langle\psi_3|\psi_1\rangle \\ &= Tr(P_{\psi_1}P_{\psi_2}P_{\psi_3})\end{aligned}$$

The 3-vertex invariant $\mathbf{I}_3(\Psi_1, \Psi_2, \Psi_3)$ is complex-valued, and depends only on the relative ordering of the projection operators P_{ψ_1}, P_{ψ_2}, and P_{ψ_3}. Clearly, it remains invariant under independent complex phase transformations of the state vectors and it is cyclically symmetric. The latter means that the projective invariant $\mathbf{I}_3(\Psi_1, \Psi_2, \Psi_3)$ bears the symmetry of a loop in the space of rays. In turn, this means that it does not matter which projection operator comes first if we trace the loop in a way that respects their relative ordering. Thus, in case we make an odd permutation of the projection operators the sign of the phase representing the complex number $\mathbf{I}_3(\Psi_1, \Psi_2, \Psi_3)$, defined modulo $2\kappa\pi$, $\kappa \in \mathbb{Z}$, changes.

In a similar manner we may define higher order projective invariants, for instance the 4-vertex invariant $\mathbf{I}_4(\Psi_1, \Psi_2, \Psi_3, \Psi_4)$, and so on, but it is immediate to realize that they are actually reduced to the information contained in $\mathbf{I}_3(\Psi_1, \Psi_2, \Psi_3)$ and $\mathbf{I}_2(\Psi_1, \Psi_2)$. So it is worth examining more carefully the complex valued projective invariant $\mathbf{I}_3(\Psi_1, \Psi_2, \Psi_3)$, since the real valued non-negative invariant $\mathbf{I}_2(\Psi_1, \Psi_2)$ has been already interpreted as the transition probability from $|\psi_1\rangle$ to $|\psi_2\rangle$. A first observation is that for any two non-orthogonal state vectors (not necessarily unit ones) $|\psi_1\rangle$ and $|\psi_2\rangle$ the polar expression of the complex-valued transition amplitude:

$$\langle\psi_1|\psi_2\rangle = |\langle\psi_1|\psi_2\rangle| \cdot e^{-i\delta\phi_{12}}$$

provides the relative angle:

$$\delta\phi_{12} = -Im[ln\langle\psi_1|\psi_2\rangle]$$

which is uniquely defined modulo $2\kappa\pi$, $\kappa \in \mathbb{Z}$. However, it does not bear any physical meaning since the complex phases of $|\psi_1\rangle$ and $|\psi_2\rangle$ may be arbitrarily re-gauged, and thus $\delta\phi_{12}$ can get any arbitrary value in the allowed range. In contradistinction, this is not the case if we choose any three pairwise non-orthogonal state vectors $|\psi_1\rangle$, $|\psi_2\rangle$ and $|\psi_3\rangle$ and form the 3-vertex invariant $\mathbf{I}_3(\Psi_1, \Psi_2, \Psi_3)$ by tracing a loop in the space of their corresponding rays in a given relative order, for instance $\Psi_1 \to \Psi_2 \to \Psi_3 \to \Psi_1$. In this case, we obtain a total relative angle θ as follows:

$$\langle\psi_1|\psi_2\rangle\langle\psi_2|\psi_3\rangle\langle\psi_3|\psi_1\rangle = |\langle\psi_1|\psi_2\rangle\langle\psi_2|\psi_3\rangle\langle\psi_3|\psi_1\rangle| \cdot e^{-i\theta}$$

$$\mathbf{I}_3(\Psi_1, \Psi_2, \Psi_3) = \langle\psi_1|\psi_2\rangle\langle\psi_2|\psi_3\rangle\langle\psi_3|\psi_1\rangle = r \cdot e^{-i\theta}$$

where $r := |\mathbf{I}_3(\Psi_1, \Psi_2, \Psi_3)|$, and

$$\theta = \delta\phi_{12} + \delta\phi_{23} + \delta\phi_{31}$$

The total relative angle θ is a complex phase invariant quantity, is uniquely defined modulo $2\kappa\pi$, $\kappa \in \mathbb{Z}$, and depends only on the relative order of the corresponding rays as we trace a loop in the space of rays. Note that total relative angles differing by an integer cannot be distinguished experimentally. Thus, the corresponding phase factor $\exp(-i\theta) \in \mathcal{U}(1) \cong \mathbb{R}/\mathbb{Z}$ is the physically meaningful gauge-invariant and thus observable factor, called the geometric phase factor. It is instructive to notice that the derivation of the geometric phase factor did not require any argument of "temporal evolution", but is solely based on the relative cyclic order of the involved rays, which induces an orientation on the traced loop. In this way, an odd permutation of these rays leads to complex conjugation of the geometric phase factor.

6.6 Invariant Distinction of Unitary from Antiunitary Transformations

At the next stage of development of the ideas related with the projective invariants $\mathbf{I}_3(\Psi_1, \Psi_2, \Psi_3)$ and $\mathbf{I}_2(\Psi_1, \Psi_2)$ we examine the relation of $\mathbf{I}_3$ with the symmetries of $\mathbf{I}_2$. We remind that in the case of two unit non-orthogonal state vectors $|\psi_1\rangle$ and $|\psi_2\rangle$, the real valued non-negative invariant $\mathbf{I}_2(\Psi_1, \Psi_2)$ expresses the transition probability from $|\psi_1\rangle$ to $|\psi_2\rangle$:

$$\mathbf{I}_2(\Psi_1, \Psi_2) = \langle\psi_1|\psi_2\rangle \cdot \langle\psi_1|\psi_2\rangle^* = Tr(P_{\psi_1} P_{\psi_2})$$

A symmetry of $\mathbf{I}_2(\Psi_1, \Psi_2)$ is defined as a bijective mapping, that is an automorphism of $\mathbf{P}\mathcal{H}$:

$$\omega : \mathbf{P}\mathcal{H} \to \mathbf{P}\mathcal{H}$$

with action $P_{\psi_1} \mapsto \omega \star P_{\psi_1}$, $P_{\psi_2} \mapsto \omega \star P_{\psi_2}$, such that:

$$\mathbf{I}_2(\Psi_1', \Psi_2') = \mathbf{I}_2(\omega(\Psi_1), \omega(\Psi_2)) = \mathbf{I}_2(\Psi_1, \Psi_2)$$

$$Tr(\omega \star P_{\psi_1} \omega \star P_{\psi_2}) = Tr(P_{\psi_1} P_{\psi_2})$$

The last expression is identical with the notion of invariance of transition probability under a Wigner transformation ω (Bargmann 1964). All automorphisms of $\mathbf{P}\mathcal{H}$ leaving the transition probability invariant form a group of automorphisms, which is identified as the symmetry group of the projective invariant $\mathbf{I}_2(\Psi_1, \Psi_2)$. According to Wigner's theorem any such automorphism of the base space of rays is lifted to the linear space of unit vectors (and thus by extension to all vectors of the Hilbert space) either as a linear unitary transformation or as a conjugate linear antiunitary transformation. If we consider the case of a linear unitary transformation U, we obtain that: $\omega \star P_{\psi_1} = U P_{\psi_1} U^{-1}$, $\omega \star P_{\psi_2} = U P_{\psi_2} U^{-1}$,

$$Tr(\omega \star P_{\psi_1} \omega \star P_{\psi_2}) = Tr(U P_{\psi_1} U^{-1} U P_{\psi_2} U^{-1}) = Tr(P_{\psi_1} P_{\psi_2})$$

The symmetry of $\mathbf{I}_2(\Psi_1, \Psi_2)$ basically says that given any element ω in the symmetry group of automorphisms of $\mathbf{P}\mathcal{H}$, such that $\Psi \mapsto \Psi' = \omega(\Psi)$, then the corresponding state vector $|\psi'\rangle$ is determined by the state vector $|\psi\rangle$ either by means of a unitary transformation or by means of an antiunitary transformation. The only physically viable antiunitary transformation on a state vector is complex conjugation combined with "time" reversal in the usual terminology.

A natural question arising in this context is if there exists any intrinsic way to distinguish between unitary and antiunitary transformations leaving the transition probability invariant as previously. For this purpose, we consider the 3-vertex invariant $\mathbf{I}_3(\Psi_1, \Psi_2, \Psi_3)$. A straightforward calculation shows that $\mathbf{I}_3(\Psi_1, \Psi_2, \Psi_3)$ remains invariant under unitary transformations, whereas it is complex conjugated under antiunitary transformations. Thus, $\mathbf{I}_3(\Psi_1, \Psi_2, \Psi_3)$, or equivalently the geometric phase factor, provides an intrinsic way to distinguish between unitary and antiunitary transformations leaving the transition probability invariant. Using this criterion, we also understand the interpretation of the geometric phase factor as a memory built up after tracing a loop on the space of rays with a prescribed orientation.

We close this section by calculating for any four unit non-orthogonal state vectors the 4-vertex invariant $\mathbf{I}_4(\Psi_1, \Psi_2, \Psi_3, \Psi_4)$ that we are going to use later on. We first show that it is reduced on calculations of 3-vertex invariants. The general expression of $\mathbf{I}_4$ reads as follows:

$$\begin{aligned}
\mathbf{I}_4(|\psi_1\rangle, |\psi_2\rangle, |\psi_3\rangle, |\psi_4\rangle) &= \mathbf{I}_4(\Psi_1, \Psi_2, \Psi_3, \Psi_4) = \mathbf{I}_4(P_{\psi_1}, P_{\psi_2}, P_{\psi_3}, P_{\psi_4}) \\
&= (\langle\psi_1|\psi_2\rangle) \cdot (\langle\psi_2|\psi_3\rangle) \cdot (\langle\psi_3|\psi_4\rangle) \cdot (\langle\psi_4|\psi_1\rangle) \\
&= \langle\psi_1|\psi_2\rangle\langle\psi_2|\psi_3\rangle\langle\psi_3|\psi_4\rangle\langle\psi_4|\psi_1\rangle \\
&= Tr(P_{\psi_1} P_{\psi_2} P_{\psi_3} P_{\psi_4})
\end{aligned}$$

By inspecting the above general expression of $\mathbf{I}_4$, we realize that its argument remains invariant by insertion of a real-valued non-negative invariant (transition probability) of the form $\mathbf{I}_2$. In particular, we insert $\mathbf{I}_2(\Psi_3, \Psi_1) = \langle\psi_3|\psi_1\rangle \cdot \langle\psi_3|\psi_1\rangle^*$, and we obtain:

$$\begin{aligned}
&\mathbf{I}_4(|\psi_1\rangle, |\psi_2\rangle, |\psi_3\rangle, |\psi_4\rangle) \\
&\quad = \mathbf{I}_4(\Psi_1, \Psi_2, \Psi_3, \Psi_4) \\
&\quad = \langle\psi_1|\psi_2\rangle\langle\psi_2|\psi_3\rangle\langle\psi_3|\psi_4\rangle\langle\psi_4|\psi_1\rangle \\
&\quad = \big(\langle\psi_1|\psi_2\rangle\langle\psi_2|\psi_3\rangle\langle\psi_3|\psi_1\rangle\langle\psi_1|\psi_3\rangle\langle\psi_3|\psi_4\rangle\langle\psi_4|\psi_1\rangle\big)/\mathbf{I}_2(\Psi_3, \Psi_1) \\
&\quad = \frac{\mathbf{I}_3(\Psi_1, \Psi_2, \Psi_3) \cdot \mathbf{I}_3(\Psi_1, \Psi_3, \Psi_4)}{\mathbf{I}_2(\Psi_3, \Psi_1)}
\end{aligned}$$

We may provide a simple geometric interpretation of the above calculation. We think of a 2-vertex invariant (transition probability) as a line connecting the two vertices in the space of rays, whereas we think of a 3-vertex invariant $\mathbf{I}_3$ as an oriented triangular loop connecting the three involved vertices in the space of rays. Then, we consider a 4-vertex invariant $\mathbf{I}_4$ as an oriented four-sided polygonal loop connecting the four vertices. This oriented polygonal loop may be triangulated by subdivision into 2 oriented triangular loops. Clearly, this is the case emerging after connecting vertices Ψ_3 and Ψ_1 by a line.

6.7 General Transition Probabilities via Projective Invariants: The Double Slit Experiment

We have already interpreted the 2-vertex invariant in the space of rays as the transition probability from $|\psi_1\rangle$ to $|\psi_2\rangle$, where $|\psi_1\rangle$, $|\psi_2\rangle$ denote non-orthogonal unit state vectors:

$$\mathbf{I}_2(\Psi_1, \Psi_2) = (\langle\psi_1|\psi_2\rangle) \cdot (\langle\psi_1|\psi_2\rangle)^*$$

Moreover, using the method of triangulation, we have shown that a 4-vertex invariant $\mathbf{I}_4$ in the space of rays is expressed in terms of the product of two 3-vertex invariants of the form $\mathbf{I}_3$. A natural question arising in this setting is if it is possible to calculate the transition probability from a state vector $|\psi_1\rangle$ to a state vector $|\psi_2\rangle$ when the transition involves intermediate state vectors, in terms of invariants in the space of rays. Usually the intermediate state vectors arise from the intervention of some complete orthonormal basis of vectors diagonalizing a corresponding observable, and thus giving rise to a Boolean frame of projection operators. From our previous analysis, it is evident that the crucial role is to be played by 3-vertex invariants. We remind that for 3 unit non-orthogonal state vectors the corresponding 3-vertex invariant has been interpreted as a geometric phase factor. If we remove the unicity condition, the 3-vertex complex-valued quantity $\mathbf{I}_3$ is still an invariant in the space of rays, whereas the angle of the corresponding geometric phase factor is given by:

$$\theta = -Im[\ln(\mathbf{I}_3(\Psi_1, \Psi_2, \Psi_3))]$$

Notice again that θ is defined modulo $2\kappa\pi$, $\kappa \in \mathbb{Z}$, and depends only on the relative order of the corresponding rays as we trace a loop along the associated oriented triangle in the space of rays.

The basic idea boils down to using these 3-vertex invariants of oriented loops in the space of rays for calculating transition probabilities when the transition involves intermediate state vectors. In order to demonstrate the proposed method, we start from the simplest example that involves the transition from a state vector $|\psi_a\rangle$ to a state vector $|\psi_c\rangle$ via a state vector $|\psi_b\rangle$. If we follow Feynman's formulation, the transition probability is given by squaring the corresponding transition amplitude.

More precisely, the transition amplitude is given by:

$$\langle\psi_c|\psi_a\rangle = \langle\psi_c|\psi_b\rangle\langle\psi_b|\psi_a\rangle$$

so that the corresponding transition probability is given by:

$$p_{ac}^{b} = |\langle\psi_c|\psi_b\rangle\langle\psi_b|\psi_a\rangle|^2$$

Alternatively, we work in the space of rays and consider the oriented loop obtained by tracing Ψ_a, Ψ_b, Ψ_c, Ψ_b, Ψ_a in the prescribed order.

$$\begin{aligned}
\mathbf{I}_4(|\psi_a\rangle, |\psi_b\rangle, |\psi_c\rangle, |\psi_b\rangle) &= \mathbf{I}_4(\Psi_a, \Psi_b, \Psi_c, \Psi_b) \\
&= \langle\psi_a|\psi_b\rangle\langle\psi_b|\psi_c\rangle\langle\psi_c|\psi_b\rangle\langle\psi_b|\psi_a\rangle \\
&= \frac{\mathbf{I}_3(\Psi_a, \Psi_b, \Psi_c)\cdot\mathbf{I}_3(\Psi_a, \Psi_c, \Psi_b)}{\mathbf{I}_2(\Psi_c, \Psi_a)} \\
&= \frac{\mathbf{I}_3(\Psi_a, \Psi_b, \Psi_c)\cdot\mathbf{I_3}^*(\Psi_a, \Psi_b, \Psi_c)}{\mathbf{I}_2(\Psi_c, \Psi_a)} \\
&= |\langle\psi_c|\psi_b\rangle\langle\psi_b|\psi_a\rangle|^2 = p_{ac}^{b}
\end{aligned}$$

We conclude that the transition probability from the state vector $|\psi_a\rangle$ to the state vector $|\psi_c\rangle$ via a state vector $|\psi_b\rangle$ is obtained in terms of the product of the 3-vertex invariant $\mathbf{I}_3(\Psi_a, \Psi_b, \Psi_c)$ with its complex conjugate $\mathbf{I_3}^*(\Psi_a, \Psi_b, \Psi_c)$. Thus, it is completely described in terms of the associated complex-valued 3-vertex invariants modulo $\mathbf{I}_2(\Psi_c, \Psi_a)$.

We move on to examine the case of the double slit experiment, which involves the transition probability from the state vector $|\psi_a\rangle$ to the state vector $|\psi_c\rangle$ via two different possible state vectors $|\psi_1\rangle$, $|\psi_2\rangle$ corresponding to the action of the one-dimensional projection operators or filters $P_{\psi_1} := |\psi_1\rangle\langle\psi_1|$ and $P_{\psi_2} := |\psi_2\rangle\langle\psi_2|$ associated with each one of the two slits, respectively. If we follow Feynman's formulation, the transition probability is given by squaring the corresponding transition amplitude. More precisely, the transition amplitude is given by summing over the two potential transition amplitudes

$$\langle\psi_c|\psi_a\rangle = \langle\psi_c|\psi_1\rangle\langle\psi_1|\psi_a\rangle + \langle\psi_c|\psi_2\rangle\langle\psi_2|\psi_a\rangle$$

so that the corresponding transition probability is given by:

$$p_{ac}^{1,2} = |\langle\psi_c|\psi_1\rangle\langle\psi_1|\psi_a\rangle + \langle\psi_c|\psi_2\rangle\langle\psi_2|\psi_a\rangle|^2$$

or equivalently:

$$p_{ac}^{1,2} = |\langle\psi_c|\psi_1\rangle\langle\psi_1|\psi_a\rangle|^2 + |\langle\psi_c|\psi_2\rangle\langle\psi_2|\psi_a\rangle|^2$$
$$+(\langle\psi_c|\psi_1\rangle\langle\psi_1|\psi_a\rangle)\cdot(\langle\psi_c|\psi_2\rangle\langle\psi_2|\psi_a\rangle)^*$$
$$+(\langle\psi_c|\psi_1\rangle\langle\psi_1|\psi_a\rangle)^*\cdot(\langle\psi_c|\psi_2\rangle\langle\psi_2|\psi_a\rangle)$$

$$p_{ac}^{1,2} = p_{ac}^1 + p_{ac}^2 + z_{ac}^1 \cdot z^{*2}_{ac} + z^{*1}_{ac} \cdot z_{ac}^2$$

Feynman's calculation of the transition probability by squaring the sum of the two potential transition amplitudes (if no measurement is actually performed at the slits) is given as a rule under the assumption of assigning these transition amplitudes to two potential paths from the initial to the final state. In this way, the appearance of the mixed interference terms $z_{ac}^1 \cdot z^{*2}_{ac}$ and $z^{*1}_{ac} \cdot z_{ac}^2$ has no clear conceptual explication.

Alternatively, we work in the space of rays and consider all potential oriented loops involving Ψ_a, Ψ_1, Ψ_2, and Ψ_c. In this manner, we end up with four potential mutually exclusive and jointly exhaustive oriented loops in the space of rays obtained as follows in the prescribed order:

Loop 1 $\equiv l_{11}$: $\Psi_a \to \Psi_1 \to \Psi_c \to \Psi_1 \to \Psi_a$,
Loop 2 $\equiv l_{22}$: $\Psi_a \to \Psi_2 \to \Psi_c \to \Psi_2 \to \Psi_a$,
Loop 3 $\equiv l_{12}$: $\Psi_a \to \Psi_1 \to \Psi_c \to \Psi_2 \to \Psi_a$,
Loop 4 $\equiv l_{21}$: $\Psi_a \to \Psi_2 \to \Psi_c \to \Psi_1 \to \Psi_a$.

To each one of the above potential loops we assign the following projective invariants, respectively:

$$\mathbf{I}_4(\Psi_a, \Psi_1, \Psi_c, \Psi_1) = \frac{\mathbf{I}_3(\Psi_a, \Psi_1, \Psi_c)\cdot \mathbf{I_3}^*(\Psi_a, \Psi_1, \Psi_c)}{\mathbf{I}_2(\Psi_c, \Psi_a)} = |\langle\psi_c|\psi_1\rangle\langle\psi_1|\psi_a\rangle|^2 = p_{ac}^1$$

$$\mathbf{I}_4(\Psi_a, \Psi_2, \Psi_c, \Psi_2) = \frac{\mathbf{I}_3(\Psi_a, \Psi_2, \Psi_c)\cdot \mathbf{I_3}^*(\Psi_a, \Psi_2, \Psi_c)}{\mathbf{I}_2(\Psi_c, \Psi_a)} = |\langle\psi_c|\psi_2\rangle\langle\psi_2|\psi_a\rangle|^2 = p_{ac}^2$$

$$\mathbf{I}_4(\Psi_a, \Psi_1, \Psi_c, \Psi_2) = \frac{\mathbf{I}_3(\Psi_a, \Psi_1, \Psi_c)\cdot \mathbf{I_3}^*(\Psi_a, \Psi_2, \Psi_c)}{\mathbf{I}_2(\Psi_c, \Psi_a)} = z_{ac}^1 \cdot z^{*2}_{ac}$$

$$\mathbf{I}_4(\Psi_a, \Psi_2, \Psi_c, \Psi_1) = \frac{\mathbf{I}_3(\Psi_a, \Psi_2, \Psi_c)\cdot \mathbf{I_3}^*(\Psi_a, \Psi_1, \Psi_c)}{\mathbf{I}_2(\Psi_c, \Psi_a)} = z^{*1}_{ac} \cdot z_{ac}^2$$

Thus, the transition probability from the state vector $|\psi_a\rangle$ to the state vector $|\psi_c\rangle$ via two different potential state vectors $|\psi_1\rangle$, $|\psi_2\rangle$ is obtained by summation over all the above invariants in the space of rays:

$$p_{ac}^{1,2} = p_{ac}^{1} + p_{ac}^{2} + z_{ac}^{1} \cdot {z^{*}}_{ac}^{2} + {z^{*}}_{ac}^{1} \cdot z_{ac}^{2}$$

$$\begin{aligned} p_{ac}^{1,2} &= \mathbf{I}_4(\Psi_a, \Psi_1, \Psi_c, \Psi_1) + \mathbf{I}_4(\Psi_a, \Psi_1, \Psi_c, \Psi_2) \\ &\quad + \mathbf{I}_4(\Psi_a, \Psi_1, \Psi_c, \Psi_2) + \mathbf{I}_4(\Psi_a, \Psi_2, \Psi_c, \Psi_1) \end{aligned}$$

$$\begin{aligned} p_{ac}^{1,2} = \big(&\mathbf{I}_3(\Psi_a, \Psi_1, \Psi_c) \cdot \mathbf{I_3}^{*}(\Psi_a, \Psi_1, \Psi_c) \\ &+ \mathbf{I}_3(\Psi_a, \Psi_2, \Psi_c) \cdot \mathbf{I_3}^{*}(\Psi_a, \Psi_2, \Psi_c) \\ &+ \mathbf{I}_3(\Psi_a, \Psi_1, \Psi_c) \cdot \mathbf{I_3}^{*}(\Psi_a, \Psi_2, \Psi_c) \\ &+ \mathbf{I}_3(\Psi_a, \Psi_2, \Psi_c) \cdot \mathbf{I_3}^{*}(\Psi_a, \Psi_1, \Psi_c)\big) / \mathbf{I}_2(\Psi_c, \Psi_a) \end{aligned}$$

We consider the expression of the total transition probability $p_{ac}^{1,2}$ in terms of 3-vertex invariants as the fundamental one. The underlying reason is that a 3-vertex invariant of an oriented triangular loop in the space of rays remains also invariant under unitary transformations, whereas it is complex conjugated under antiunitary transformations. Thus, we may consider a 3-vertex invariant as the scalar invariant of an oriented triangular loop l in the space of rays realized by means of a unitary transformation under the choice of an initial vertex. Analogously, we may consider the complex-conjugate 3-vertex invariant as the scalar invariant of an oppositely oriented loop $l^{\dagger}$ in the space of rays realized by means of an antiunitary transformation under the choice of the final vertex in the previous case as the initial vertex in the present one. Note that the latter loop $l^{\dagger}$ can be rotated by an even permutation of the vertices T, such that $T \cdot l^{\dagger} = l^{-1}$ is also based at the initial vertex of the former, leaving the invariant unchanged. This simply means that we may consider oriented loops, which are now based at a single vertex, for instance Ψ_a, and represent the actions of unitary/antiunitary transformations on the corresponding state vector in terms of them. We are going to explore this viewpoint in the sequel. Before this, it is important to understand better the role of unitary/antiunitary transformations in relation to utilization of Wigner's theorem on symmetries of the space of rays, together with the function of Boolean frames.

6.8 One-Parameter Unitary Transformation Groups and Boolean Frames

We have already demonstrated that if ω is a $\mathbf{I}_2$ symmetry, then it can be interpreted as a Wigner-type automorphism of the space of rays $\mathbf{P}\mathcal{H}$, such that $\Psi \mapsto \Psi' = \omega(\Psi)$. Consequently, the corresponding state vector $|\psi'\rangle$ is determined by the state vector

$|\psi\rangle$ either by means of a unitary transformation or by means of an antiunitary transformation. Note that this determination is unique up to a multiplicative constant phase factor of absolute value 1. We remind again that these two types of transformations are intrinsically distinguished by means of the 3-vertex invariant $\mathbf{I}_3$.

If the symmetry group of automorphisms of $\mathbf{I}_2$ is connected, then $|\psi'\rangle$ is determined by the state vector $|\psi\rangle$ by means of a unitary transformation. More precisely, if we take the symmetry group to be the group of translations on the real line $\mathbb{R}$, then the unitary transformation is obtained by the unitary representation of $\mathbb{R}$ on the Hilbert space $\mathcal{H}$ of state vectors:

$$s \mapsto e^{-is\Gamma/\hbar}$$

where Γ is an observable, or equivalently a self-adjoint operator. The existence and uniqueness of the observable Γ is given by Stone's theorem on one-parameter unitary groups. Given, the one-parameter group of unitary operators $e^{-is\Gamma/\hbar} = U(s) = U_s$, s in $\mathbb{R}$, on the Hilbert space of state vectors, there is one uniquely defined observable Γ, which acts as the infinitesimal generator of U_s by means of $-i\Gamma/\hbar$. Now, if $|\psi_0\rangle$ corresponds to $s = 0$, we obtain:

$$|\psi_s\rangle = e^{-is\Gamma/\hbar}|\psi_0\rangle$$

Infinitesimal differentiation with respect to s in $\mathbb{R}$ gives the following:

$$\frac{d|\psi_s\rangle}{ds} = \frac{-i\Gamma}{\hbar}|\psi_s\rangle$$

In the standard terminology, the group of translations on the real line $\mathbb{R}$ is identified with the group of "time" translations and the variable s is denoted by t. Then, the uniquely defined observable Γ is identified with the energy observable, or else Hamiltonian H of the quantum system. Then, the relation:

$$|\psi_t\rangle = e^{-itH/\hbar}|\psi_0\rangle$$

is interpreted as unitary "time evolution" of the system, where $|\psi_0\rangle$ is the state at "time" $t = 0$. Moreover, the differential equation:

$$\frac{d|\psi_t\rangle}{dt} = \frac{-iH}{\hbar}|\psi_t\rangle$$

is immediately identified with the Schrödinger equation.

We note that we may alternatively identify the group of translations on the real line $\mathbb{R}$ with the group of "space" translations in some specified direction. Then, the uniquely defined observable Γ is identified with the momentum self-adjoint operator with respect to this direction. For instance, if a "space" translation in the direction y

is denoted by L, we obtain the following unitary representation of $\mathbb{R}$ on the Hilbert space $\mathcal{H}$ of state vectors:

$$L \mapsto e^{-iLp_y/\hbar}$$

where $p_y = -i\hbar\partial_y$ in the position representation. Then, the corresponding unitary transformation is given by the y-displacement unitary operator.

A concrete application of the above unitary transformation can be given in the case of the double slit experiment. We assume that the two slits are separated by a distance L in the y-direction and the quantum system is propagating in the x-direction being normal to the two slits. If no measurement is performed at any of the slits, the state vector is described by the superposition of state vectors corresponding to the projections or filters $P_{\psi_1} := |\psi_1\rangle\langle\psi_1|$, and $P_{\psi_2} := |\psi_2\rangle\langle\psi_2|$. Then, at the screen the normalized state vector can be expressed as follows:

$$|\psi_c\rangle = \frac{1}{\sqrt{2}}(|\psi_1\rangle + e^{i\theta}|\psi_2\rangle)$$

Thus, in the position representation we obtain:

$$\psi_c(y) = \frac{1}{\sqrt{2}}(\psi_1(y - L) + e^{i\theta}\psi_2(y))$$

where we assume that the two wavefunctions are the same modulo $e^{i\theta}$ denoting the total geometric relative phase difference between $|\psi_1\rangle$ and $|\psi_2\rangle$. Using the y-displacement unitary operator we obtain:

$$|\psi_1\rangle = e^{-iLp_y/\hbar}|\psi_2\rangle$$

Moreover, the expectation value of the y-displacement unitary operator is the following:

$$\langle\psi_c|e^{-iLp_y/\hbar}|\psi_c\rangle = \frac{e^{i\theta}}{2}$$

By the previous analysis, it has been crystallized the following: If ω is an $\mathbf{I}_2$ symmetry induced by the group of translations on the real line $\mathbb{R}$, that is a Wigner-type automorphism of the space of rays $\mathbf{P}\mathcal{H}$, such that $\Psi \mapsto \Psi' = \omega(\Psi)$, the corresponding state vector $|\psi'\rangle$ is determined by the state vector $|\psi\rangle$ by means of a unitary transformation. Concrete unitary transformations are given by unitary representations of $\mathbb{R}$ on the Hilbert space $\mathcal{H}$ of state vectors, under the qualification of the translations in the real line as "time" translations or "space" translations. The connection of this viewpoint with the notion of observables is provided by the utilization of Stone's theorem establishing a bijective correspondence between continuous one-parameter groups of unitary operators and observables (Stone

1932). We saw in the examples explained previously that the involved parameter is interpreted as a parameter of "time" translations or "space" translations in the real line. This fact requires a more careful conceptual understanding of the role of unitary transformations in quantum mechanics (if a measurement does not take place) as well as their subtle connection with the notion of observables.

The crucial distinguishing feature of quantum mechanics in relation to all classical theories is that the totality of all physical observables constitutes a global non-commutative algebra, and thus quantum observables are not theoretically compatible. This simply means that not all observables are simultaneously measurable with respect to a single universal global logical Boolean frame as is the case in all classical theories of physics. Thus, there exists a multiplicity of potential local Boolean frames, where each one of them stands for a context of co-measurable observables. Technically speaking, each Boolean frame is a complete Boolean algebra of orthogonal projection operators obtained by the simultaneous spectral resolution of a family of compatible observables—represented as self-adjoint operators—with respect to a complete orthonormal basis of eigenstates. Such a family of compatible observables forms a commutative observable algebra whose orthogonal idempotent elements (orthogonal projections) constitute a logical Boolean frame (Zafiris 2006a; Epperson and Zafiris 2013), see also Omnés (1994) and Selesnick (2003). In this way, each local Boolean frame signifies the local logical precondition predication space for the probabilistic evaluation of all the observables belonging to the corresponding commutative observable algebra (Zafiris 2006b). Thus, the manifestation of every single observed event in the quantum regime requires taking explicitly into account the specific local Boolean frame with respect to which it is contextualized. Since there does not exist a single, unique, global Boolean frame, due to the non-commutativity of the totality of quantum observables, there appears the necessity to consider all possible local Boolean frames and their interrelations.

The remarkable fact is that each observable instantiates a Boolean algebra of orthogonal projection operators, which is utilized for the expression of a state vector as a linear superposition with respect to the associated complete orthonormal basis of eigenstates of this observable. In this way, a Boolean frame functions as a means of inducing differentiations in the initially objectively indistinguishable state of a quantum system in terms of the orthogonal projection operators of this algebra. In other words, orthogonal projections induce potential differentiations in a quantum state, which are realized only if a measurement is actually performed. Thus, observables through their spectral resolution in terms of orthogonal projectors can be thought of as potential distinguishability filters acting on a quantum state. In this way, a measurement process creates information by actualizing differentiations with reference to the associated filters, or else refines the grain of resolution associated to a quantum state (Zafiris and Karakostas 2013).

What is then the role of the bijective correspondence between observables and one-parameter groups of unitary transformations, where the parameter is considered to by varying continuously on the real line? A unitary transformation is an automorphism of the Hilbert space of state vectors preserving the inner product structure, and thus realized by means of a unitary operator as we have shown previously. The inner product between two state vectors, interpreted as the transition amplitude from one to the other, if viewed from the perspective of the space of rays can be thought of as the degree of overlap between the corresponding rays or projection operators. Given that a projection operator functions as a distinguishability filter, the overlap provides the degree of indistinguishability between the associated states. Thus, a unitary transformation is simply a transformation which preserves the degree of indistinguishability between states of a quantum system (if a measurement does not take place). In this sense, the real-valued varying parameter in a one-parameter group of unitary operators associated bijectively with an observable is simply a parameter indexing continuously the preservation of the degree of indistinguishability between quantum states. This is the crucial aspect that distinguishes the temporal or spatial meaning of such a parameter in comparison to the classical semantics of these terms.

6.9 Pairs of Based Oriented Loops and Action of One-Parameter Unitary Groups

In order to unfold the consequences of this distinction we are going to investigate in detail the role of based oriented loops in relation to our analysis of the double slit experiment in terms of projective invariants. More precisely, we have concluded that oriented loops, which are based at a single reference vertex, for instance Ψ_a, represents the actions of unitary/antiunitary transformations on the corresponding state vector. In particular, with respect to a reference vertex it is sufficient to specify pairs of oppositely oriented loops based at this vertex, for example the pair denoted by l_1 and l_1^{-1} if we choose as a reference vertex the ray Ψ_a.

A natural question is what distinguishes different pairs of such based loops after specifying a reference vertex. In the guiding case of the double slit experiment the differentiation is clearly defined with respect to the non-simultaneously realizable potential filters instantiated by the projection operators $P_{\psi_1} := |\psi_1\rangle\langle\psi_1|$ and $P_{\psi_2} := |\psi_2\rangle\langle\psi_2|$. Equivalently, the projection operators P_{ψ_1} and P_{ψ_2} belong to two different disjoint local Boolean frames of potential position measurement that cannot be embedded into a single global Boolean frame of projections simultaneously. Conceptually, if such an embedding was possible, then there would not be any interference effect at the detection screen if no measurement had taken place at any of the two slits. In this setting, the ray Ψ_c can be thought of as corresponding to a momentum measurement, and thus, it belongs to the spectral resolution of the momentum operator. In a nutshell, from the perspective of Boolean frames the interference effect is simply a consequence of the fact that the Boolean frame of the momentum is not reducible to the disjoint union of the position Boolean frames at the slits.

Therefore, in the case of the double slit experiment, what distinguishes different pairs of oppositely oriented based loops at the specified reference vertex Ψ_a is the existence of two potential filters P_{ψ_1} and P_{ψ_2}, which cannot be simultaneously realizable. Equivalently, the differentiation may be considered by means of their respective two-valued local Boolean frames of potential position measurement according to the above. Thus, if we take into account the bijection between an observable (or its associated Boolean frame) and its corresponding continuous one-parameter unitary group of transformations, we reach the following conclusion: A pair of oppositely oriented based loops at a specified reference vertex should represent the action of a continuous one parameter unitary group at this vertex. Moreover, since Boolean frames are solely used for localization, the representation of an observable as a self-adjoint operator should be considered locally, that is with respect to the local Boolean frame it refers to. In particular, the position observable is resolved differently with respect to the local two-valued Boolean frames generated by the filters P_{ψ_1} and P_{ψ_2} correspondingly, such that these resolutions cannot be simultaneously realizable. Hence, the action of the position observable in relation to the potential filters P_{ψ_1} and P_{ψ_2} at the specified reference vertex Ψ_a gives rise to two different pairs of oppositely oriented based loops at Ψ_a, where each one of them represent the action of a continuous one parameter unitary group at this vertex in relation to the distinguishability induced by the corresponding filter or its associated local two-valued Boolean frame.

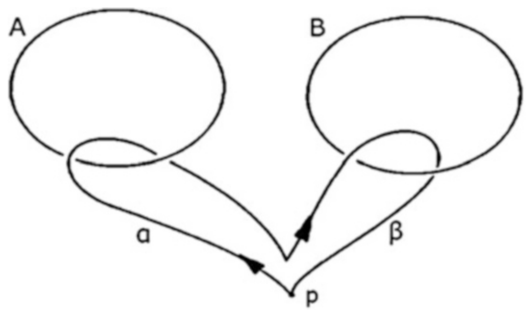
The previous discussion, in relation to the double slit experiment, has served the purpose of introducing the proposed representation of a continuous one parameter unitary group action at a vertex by a pair of oppositely oriented loops, which are based at this vertex, as well as the criterion of differentiation among such pairs of based loops according to the localization properties of local Boolean frames. What is particularly interesting by this change of perspective is that there immediately appears the possibility of composition of different oriented loops based at the same vertex. Moreover, since we also have an inverse for each based loop, namely the based loop with opposite orientation, we can obtain a group structure. This group structure, without any further constraints, is free but non-commutative. It is precisely the natural symmetry of the double slit experiment that suggests to think of the free non-commutative group generated by the based oriented loops l_1 and l_2, if we choose as a reference vertex the ray Ψ_a, where these based loops are distinguished by means of the two potential filters P_{ψ_1} and P_{ψ_2} according to the above. We claim that this group structure with respect to a reference ray emanates from the non-commutative group structure of based oriented loops at a point of 3-d space, and more precisely, it constitutes its representation in the Hilbert space of state vectors. Therefore, initially it is necessary to focus our attention on the latter group structure and unfold its semantics.

6.10 Non-commutative Group Structure of Based Oriented Loops in 3-d Space

First, we consider a loop in three-dimensional space as an unknotted tame closed curve. Since any such closed curve can be continuously deformed to a topological circle it is enough to think of such a circle in 3-d space, denoted by A. Second, we consider a based oriented loop in 3-d space, which may pass through this circle A a finite number of times, each one with a prescribed orientation.

A based loop means simply that it starts and ends at a fixed reference point p of the 3-d space. The orientation of the loop can be thought of in terms of an observer, which is fixed at the point p, such that: If the loop passes through the circle one time with direction away from the observer, it is denoted by α^1, whereas if it passes one time with direction toward the observer, it is denoted by α^{-1}. We note that any other loop with the same properties can be continuously deformed to the loop α. Thus, the algebraic symbol α actually denotes the equivalence class $[\alpha]$ of all loops of kind α, passing through the circle A once with the prescribed orientation.

Taking into account the algebraic encoding of based oriented loops in relation to topological circles in 3-d space, we can define the composition of two oriented loops under the proviso that they are based on the same point p in 3-d space. Notice that the composition operation $\alpha \circ \beta$ of the p-based oriented loops α and β in relation to circles A and B correspondingly is not a commutative operation, meaning that the order of composition is not allowed to be reversed. Clearly, the rule of composition produces a based oriented loop $\alpha \circ \beta$ in 3-d space in relation to the circles A and B in the prescribed order. We think of the composition rule $\alpha \circ \beta$ as the non-commutative multiplicative product of the oriented loops α and β based at the same point p in 3-d space, which we may simply denote as $\alpha\beta$. It is immediate to verify that the above defined multiplication is an associative operation.



Having established the closure of the elements of the generic form χ under non-commutative associative multiplication as previously, we look for the existence of an identity element, as well as for the existence of inverses with respect to this operation. There is an obvious candidate for each based oriented loop α, namely the loop α^{-1}, where the orientation has been reversed. If we consider the compositions

$\alpha \circ \alpha^{-1}$, $\alpha^{-1} \circ \alpha$ we obtain in both cases as a multiplication product the based loop at the same point, which does not pass through any circle at all. Thus, we name the latter loop as the multiplicative identity 1 in our algebraic structure, such that $\alpha\alpha^{-1} = \alpha^{-1}\alpha = 1$. It is also easy to verify that $1\alpha = \alpha 1 = 1$. We conclude that the set of symbols of the generic form χ representing based oriented loops in relation to circles X, endowed with the non-commutative multiplication operation of composition product of loops based at the same point, form the algebraic structure of a non-commutative group, denoted by Θ.

It is instructive to emphasize that the equality sign in the non-commutative group Θ is interpreted topologically as an equivalence relation of p-based oriented loops under continuous deformation. By making use of the multiplication operation in Θ we may form any permissible string of symbols in this group, which can be reduced into an irreducible form by using only the group-theoretic relations $\alpha\alpha^{-1} = \alpha^{-1}\alpha = 1$, $\alpha\alpha = \alpha^2$, and so on. Thus, if we consider only two p-based oriented loops as generators, denoted by the symbols α and β respectively with the prescribed orientation and obeying no further constraints, we form a non-commutative free group in two generators, denoted by Θ_2.

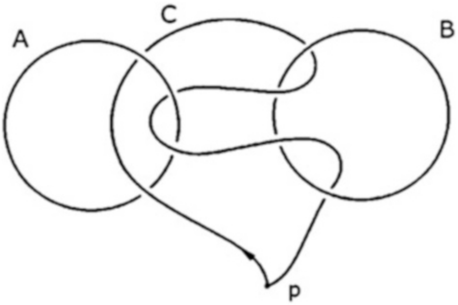
6.11 The Borromean Topological Link Semantics of the Non-commutative Free Group Θ_2

We remind that the Borromean link is characterized topologically by the property of splittability as follows: The Borromean link is a non-splittable 3-link, such that every 2-sublink of this 3-link is completely splittable. According to the defining property of the Borromean link, it is a non-splittable 3-link because not even one of the three loops, or any pair of them, can be separated from the rest without cutting. A 2-sublink is simply any sub-collection of two loops obtained by erasing the loop that does not belong to this sub-collection. Since the Borromean link is characterized by the property that if we erase any one of the three interlocking loops, then the remaining two loops become unlinked, we obtain that every 2-sublink of the non-splittable 3-link is completely splittable.

We will show that the topological information incorporated in the specification of the Borromean link can be encoded algebraically by exploiting the non-commutative group-structure of the free group Θ_2 generated by two oriented loops, which are based at the same fixed point of 3-d space. The property of irreducibility of a string of symbols in the group Θ_2 is the guiding idea for the algebraic encoding of the Borromean link in terms of the structure of Θ_2. The crucial observation is that algebraic irreducibility in Θ_2 can be used to model the topological property of non-splittability of a 3-link, where complete splittability of all 2-sublinks is encoded by the unique identity element of Θ_2. In particular, the group-theoretic commutator induced by the generators of Θ_2:

$$[\alpha, \beta^{-1}] = \alpha\beta^{-1}\alpha^{-1}\beta$$

produces an irreducible non-commutative string of symbols in Θ_2. This string represents a new based loop γ as a product loop composed by the ordered composition of the based oriented loops $\alpha \circ \beta^{-1} \circ \alpha^{-1} \circ \beta$. We call the product loop γ the Borromean loop and the formula or multiplicative string $\alpha\beta^{-1}\alpha^{-1}\beta$ in Θ_2 the Borromean loop formula.



The algebraic irreducibility of the commutator $[\alpha, \beta^{-1}]$ in the group Θ_2 encodes the topological non-splittability property of the Borromean 3-link. We notice that deletion of both α and α^{-1} (corresponding to removal of the circle A) reduces the formula to the identity 1 (and the same happens symmetrically for both β and β^{-1}). This fact models algebraically in the terms of Θ_2 that every 2-sublink of the Borromean 3-link is completely splittable. We conclude that the topological information of the Borromean 3-link can be completely encoded in terms of the algebraic structure of the non-commutative multiplicative free group in two generators Θ_2. In particular, the group-theoretic commutator $[\alpha, \beta^{-1}]$ in Θ_2, encodes algebraically the gluing condition of the based oriented loops α and β^{-1} (with respect to the circles A and B respectively in the prescribed orientation), and therefore the non-splittability of the Borromean 3-link, together with the complete splittability of all 2-sublinks of this 3-link.

We note that the Borromean topological link is characterized by threefold symmetry. In the algebraic terms of the group Θ_2 this is reflected on the fact that if we consider any two of the based loops α, β^{-1}, γ, then the third is expressed by the group commutator of the other two. The threefold symmetry of the Borromean link may be broken by reducing the free non-commutative group on two generators Θ_2 to the free nilpotent group on two generators of nilpotent class 2, which is precisely the Weyl–Heisenberg group $\mathbb{H}$. More concretely, we may choose the based loops α, β^{-1} such that $\gamma = [\alpha, \beta^{-1}] = \alpha\beta^{-1}\alpha^{-1}\beta$ and impose the relations $[\alpha, \gamma] = [\beta^{-1}, \gamma] = 1$.

6.12 Unitary Representation of the Group Θ_2: Transferring the Borromean Link to the Quantum State Space

We have shown that the topological information of the Borromean 3-link can be completely encoded in terms of the algebraic structure of the non-commutative multiplicative group in two generators Θ_2. In particular, the group Θ_2 encodes the non-splittability of the Borromean 3-link, together with the complete splittability of all 2-sublinks of this 3-link.

A natural question arising from our previous analysis is if there exists a representation of this non-commutative group Θ_2 in the Hilbert space of state vectors of a quantum system. The representation theory of the free group on two generators on an abstract Hilbert space has been first studied in detail from a pure mathematical viewpoint by Choi (1980). Here, we are going to follow a quite simplified approach, while we put the emphasis on the intended physical semantics.

It is important to stress the fact that such a representation of Θ_2 would transfer the Borromean link topology to the Hilbert space objects which carried this representation. Intuitively, the Borromean 3-link expresses the particular connectivity property of three based oriented loops, where any two of them are unlinked, which is captured algebraically by means of the structure of the group Θ_2. Topological connectivity in this context is associated with the non-splittability of this link as a 3-link. If we metaphorically think of this connectivity property as indistinguishability in a quantum theoretic context pertaining to interference of transition amplitudes, then it becomes quite natural to expect that a representation of the group Θ_2 would be feasible by means of unitary transformations. The analogy goes deeper by the fact that in a Borromean 3-link the act of cutting a based loop leads to complete splittability of the remaining 2-link. Analogously, the act of measurement does not preserve the degree of indistinguishability between states (since a measurement creates information by distinguishing among alternatives), the corresponding unitary group action breaks down, and for instance, in the double slit experiment the two interfering alternatives become completely distinguishable. The interpretational aspects of transferring the Borromean link topology in the quantum state space via the action of one-parameter unitary groups, and their concomitant representation in terms of oppositely oriented pairs of based loops, requires a more detailed presentation that will be undertaken shortly. At present, we are going to prove that a unitary representation of the group Θ_2 indeed exists, and thus the semantics of the Borromean topological link can be transferred appropriately in the quantum state space by means of one-parameter unitary groups. As a final remark, we would like to remind Feynman's saying that the complete mystery of quantum mechanics is engulfed in the double slit experiment. We would like to add in this respect that the double slit experiment may require the understanding of the Borromean-link topology in its manifestation via the Θ_2 group action on the state space by one-parameter unitary groups.

First, we need to define the notion of a unitary representation of the group Θ_2 as follows: A unitary representation of the group Θ_2 consists of a Hilbert space of states $\mathcal{H}$, together with a group homomorphism from Θ_2 to the group of unitary operators on $\mathcal{H}$.

Second, we note that the non-commutative group Θ_2 is a free multiplicative group in two generators g_1 and g_2. Given two unitary operators U_1 and U_2 in the Hilbert space of states $\mathcal{H}$, there exists a unique group homomorphism $\zeta : \Theta_2 \to B(\mathcal{H})$, where $B(\mathcal{H})$ is the algebra of bounded linear operators in $\mathcal{H}$, which sends g_1 to U_1 and g_2 to U_2, just by the universal property of free groups (Mac Lane 1998). Since ζ is a group homomorphism $\zeta(g_i) = U_i$ is unitary operator for each $j = 1, 2$. Therefore, since $\{g_1, g_2\}$ generates Θ_2 as a free group, ζ must be a unitary representation of the group Θ_2 in the Hilbert space of states $\mathcal{H}$.

Third, we know that $B(\mathcal{H})$ has the structure of a $\star$-algebra over the complexes. If we consider the free group $\star$-algebra of Θ_2, generated by finite linear combinations of elements of Θ_2 with complex coefficients, then we have the following: Given a unitary representation ζ of Θ_2 in the Hilbert space of states $\mathcal{H}$, then this representation extends by linearity to a $\star$-homomorphism of the group $\star$-algebra of Θ_2, denoted by $C^\star(\Theta_2)$, to the $\star$-algebra $B(\mathcal{H})$.

Fourth, the algebra $C^\star(\Theta_2)$ is characterized uniquely up to isomorphism by the following universal property: Given any unitary representation,

$$\zeta : \Theta_2 \to B(\mathcal{H})$$

of the group Θ_2, there exists a unique $\star$-homomorphism of the group $\star$-algebra of Θ_2, $C^\star(\Theta_2)$, to the $\star$-algebra $B(\mathcal{H})$, denoted by

$$\tilde{\zeta} : C^\star(\Theta_2) \to B(\mathcal{H})$$

that satisfies:

$$\tilde{\zeta}(\gamma_g) = \zeta(g)$$

for every $g \in \Theta_2$, where $\gamma_g \in C^\star(\Theta_2)$. Thus, if we consider the generating set of symbols $\{g_1, g_2\}$ of Θ_2 as a free group we obtain the relations:

$$\tilde{\zeta}(\gamma_{g_1}) = \zeta(g_1) = U_1$$

$$\tilde{\zeta}(\gamma_{g_2}) = \zeta(g_2) = U_2$$

where U_1 and U_2 are unitary operators in the Hilbert space of states $\mathcal{H}$.

We consider a faithful representation of the group $\star$-algebra of Θ_2, $C^\star(\Theta_2)$, in the Hilbert space of states $\mathcal{H}$, such that we identify:

$$C^\star(\Theta_2) \equiv C^\star(U_1, U_2)$$

where U_1 and U_2 are unitary operators in the Hilbert space $\mathcal{H}$, considered as universal, in the following sense: For any other pair V_1 and V_2 of unitary operators in the Hilbert space $\mathcal{H}$, the assignment $U_1 \to V_1$, $U_2 \to V_2$, extends to a $\star$-homomorphism from $C^\star(U_1, U_2)$ to $C^\star(V_1, V_2)$. Now, by utilizing the spectral theorem, we may always choose two self-adjoint operators A and B in $B(\mathcal{H})$, such that $U_1 = e^{iA}$ and $U_2 = e^{iB}$.

Next, we consider the set of all continuous functions:

$$\mathbb{T} = \{f : [0, 1] \to B(\mathcal{H})\}$$

such that $f(0)$ are scalar operators. The set $\mathbb{T}$ can be endowed with the structure of a $C^\star$-algebra, which is denoted by the same symbol. We will show that the $C^\star$-algebra $C^\star(\Theta_2) \equiv C^\star(U_1, U_2)$ can be imbedded in $\mathbb{T}$ as a $C^\star$-subalgebra.

We have already seen that if we choose two observables represented as self-adjoint operators A and B in $B(\mathcal{H})$, then we identify $U_1 = e^{iA}$ and $U_2 = e^{iB}$. Next, we define two continuous functions f_{U_1} and f_{U_2} in the $C^\star$-algebra $\mathbb{T}$ such that their image in $B(\mathcal{H})$ is unitary, as follows:

$$f_{U_1} : [0, 1] \to B(\mathcal{H}),\ f_{U_2} : [0, 1] \to B(\mathcal{H})$$

$$[0, 1] \ni t \mapsto f_{U_1}(t) := e^{itA} \in B(\mathcal{H})$$

$$[0, 1] \ni t \mapsto f_{U_2}(t) := e^{itB} \in B(\mathcal{H})$$

Then, it is clear that we may consider the $C^\star$-algebra generated by the continuous functions f_{U_1} and f_{U_2}, denoted by $C^\star(f_{U_1}, f_{U_2})$. Then, by the universality property of $C^\star(\Theta_2) \equiv C^\star(U_1, U_2)$ there exists a $\star$-homomorphism from $C^\star(\Theta_2) \equiv C^\star(U_1, U_2)$ to $C^\star(f_{U_1}, f_{U_2})$, specified precisely by the assignments $U_1 \mapsto f_{U_1}$ and $U_2 \mapsto f_{U_2}$. From the other side, we may consider the evaluation morphism $f \mapsto f(1)$, which clearly defines a $\star$-homomorphism from $C^\star(f_{U_1}, f_{U_2})$ to $C^\star(\Theta_2) \equiv C^\star(U_1, U_2)$. The above two $\star$-homomorphisms are inverse to each other, and thus induce an isomorphism:

$$C^\star(\Theta_2) \cong C^\star(f_{U_1}, f_{U_2})$$

The significance of this theorem is the following: It is obvious that the algebra $C^\star(f_{U_1}, f_{U_2})$ is a $C^\star$-subalgebra of $\mathbb{T}$. Hence, $C^\star(\Theta_2) \equiv C^\star(U_1, U_2)$ can be imbedded in $\mathbb{T}$ as a $C^\star$-subalgebra. The crucial fact is that the $C^\star$-algebra of continuous functions $\mathbb{T}$ has no nontrivial projections (Cohen 1979). This means that $C^\star(\Theta_2) \equiv C^\star(U_1, U_2)$ imbedded in $\mathbb{T}$ as a $C^\star$-subalgebra has no nontrivial projections either. This is important because it shows that the non-commutative spectrum of $C^\star(\Theta_2) \equiv C^\star(U_1, U_2)$ is highly connected. It is instructive to remind that if a nontrivial projection exists then the corresponding spectrum set of this projection is both closed and open. As a result, if we consider, for example, a Boolean algebra of projections, the spectrum of this algebra is a totally disconnected space (Johnstone 1986). In contradistinction the spectrum of $C^\star(U_1, U_2)$ is a highly non-commutative connected topological space.

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Chapter 7
Borromean Link in Quantum Gravity A Topological Approach to the "ER = EPR" Conjecture: Modelling the Correspondence Between GHZ Entanglement and Planck-Scale Wormholes via the Borromean Link

7.1 Introduction: The "ER = EPR" Conjecture and Planckian Wormholes

In the absence of an exact quantum gravity theory, the "ER = EPR" conjecture constitutes a recently introduced proposal by Maldacena and Susskind (2013), aiming to shed light on the relations among spacetime geometry, quantum field theory and quantum information theory, which is receiving significant attention currently in relation to its substantiation, proof, and groundbreaking implications. The "ER = EPR" is a short-hand that joins two ideas proposed by Einstein in 1935. One involved the quantum correlations implied by what he called "spooky action at a distance", referring to the phenomenon of *entanglement* between quantum particles (EPR entanglement, named after Einstein, Podolsky, and Rosen) (Einstein 1935). The other showed how two black holes could be connected "non-locally" via "topological handles" in space-time, known as "wormholes" (ER, for Einstein-Rosen bridges) (Einstein and Rosen 1935). If the conjecture "ER = EPR" is correct, then the ideas of quantum entanglement and wormholes are not disjoint, but they are two manifestations of the same essentially topological idea. Effectively, this underlying *connectedness* would form the foundation of quantum space-time.

More precisely, the "ER = EPR" conjecture is grounded in the context of *duality* between a gravitational theory formulated in the bulk and a quantum field theory formulated on the boundary, targeting the correspondence between ER bridges or wormholes and entanglement. In a sense, the "ER = EPR" conjecture implicates on a cosmological scale that a complex network of entangled subsystems of the universe as a whole is also a complex network of ER bridges. In particular, since ER bridges refer to the connectivity between black holes, the "ER = EPR" conjecture implies that *black holes connected by ER bridges are entangled*, and also conversely that *entangled black holes are connected by ER bridges*.

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This connective link was first recognized in the context of the *AdS/CFT correspondence*, where a wormhole between two asymptotically AdS regions is dual to two non-interacting conformal field theories in a thermally entangled state (Maldacena and Susskind 2013; Maldacena 2003), see also Van Raamsdonk (2010) and Witten (1998). It was also noticed that the area of the minimal surface cut representing the entanglement entropy and the length of the wormhole is proportional to correlations between two dual CFTs (Ryu and Takayanagi 2006). This idea was extended by the proposal that spacetime connectedness in AdS is related to quantum entanglement in the dual field theory. In this context, the "ER = EPR" conjecture is a far-reaching generalization of the above, since it postulates that entanglement is actually *equivalent* to the existence of wormholes in spacetime. This, in turn, is based on the conception of entanglement as an *interchangeable resource*, meaning that the various forms of entanglement, like vacuum entanglement or entangled particles or wormholes or even clouds of Hawking radiation, are inter-transformable into one another by means of *local unitary transformations*, which are in principle possible. In this manner, an ordinary kind of quantum entanglement, like a Bell pair, can be re-interpreted in terms of the geometric properties of wormholes, and inversely (Susskind 2016, 2014b; Maldacena and Susskind 2013).

For instance, an extended solution of the *Schwarzschild black hole* can be interpreted as two black holes in the same space located far away from each other, but connected by a wormhole. Minimal radius of the wormhole depends on the choice of the spacelike slice. Usually the constant $t = 0$ spacelike slice is considered, together with *two AdS exterior regions* connected by a wormhole. According to the AdS/CFT correspondence, the solution in AdS space referring to these two black holes corresponds to a highly entangled state defined on the left and right corresponding CFTs on the boundary. Now, one can think of the entanglement between left and right CFTs as a representation of the entanglement between the black holes themselves. At a further stage of development of these ideas, based on the duality between maximal entanglement and wormholes, the "ER = EPR" conjecture suggests of thinking about early *Hawking radiation* in terms of a black hole that is connected to the interior of the initial emitting black hole by *numerous wormholes* making them dependent (Maldacena and Susskind 2013).

However, the example of the maximally entangled GHZ state suggests that an arbitrary entangled state *cannot* be represented as a classical Einstein-Rosen bridge, thus it is necessary to think of a model of a *Planckian wormhole* going beyond the classical description of a wormhole in spacetime (Susskind 2016). This fact points to the conclusion that the relation between entanglement and wormholes is more *complex* and a concrete refined mathematical framework is necessary to establish the validity and universality of the ER = EPR conjecture.

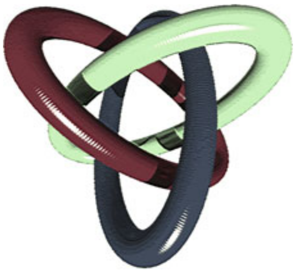
From a broad conceptual standpoint, the two perennial problems in the interface between quantum theory and general relativity, namely the *quantum state reduction* or quantum measurement problem in quantum physics and the problem of *singularities* in general relativity, may be considered as targeting precisely the issue of transition into and out of a local space-time event structure, respectively (von Müller 2015). Given that the quantum state reduction associated with the outcome of a measurement procedure is necessitated in virtue of entanglement between the quantum system and the measurement means, the latter being in this way the conceptual converse of the former, the "ER = EPR" conjecture may be refined conceptually by thinking of it in the categorial context of a universal topological mechanism by means of which the folding out of a local space-time event structure takes place. This naturally generates the question, if there exists such a *universal mechanism of a topological nature*, which would manifest appropriately these two inverse types of transition.

7.2 The Borromean Topology as the Universal Means to Qualify the "ER = EPR" Conjecture

7.2.1 GHZ Entangled State as a Borromean Link

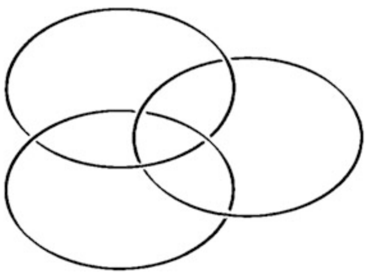
The notion of a *topological link* is based on the underlying idea of connectivity among a collection of *unknotted tame closed curves*, called simply *loops* (Kawauchi 1996). A topological N-link is a collection of N loops, where N is a natural number. Regarding the connectivity of a collection of N loops, the crucial property is the property of *splittability* of the corresponding N-link. We say that a topological N-link is splittable if it can be deformed continuously, such that part of the link lies within B and the rest of the link lies within C, where B, C denote mutually exclusive solid spheres (balls). Intuitively, the property of splittability of an N-link means that the link can come at least partly apart *without* cutting. Complete splittability means that the link can come completely apart without cutting. On the other side, non-splittability means that not even one of the involved loops, or any pair of them, or any combination of them, can be separated from the rest without cutting.

The "Borromean rings" consist of three rings, which are linked together in such a way that each of the rings lies completely over one of the other two, and completely under the other, as it is shown at the picture below:



This particular type of topological linking displayed by the "Borromean rings" is called the "Borromean link," and is characterized by the following distinguishing property: *If any one of the rings is removed from the "Borromean link," the remaining two come completely apart*. It is important to emphasize that the rings should be modelled in terms of unknotted tame closed curves and not as perfectly circular geometric circles (Cromwell et al. 1998; Debrunner 1961; Hatcher 2002; Lindström and Zetterström 1991).

From the viewpoint of the theory of topological links, the Borromean link constitutes an *interlocking family* of three loops, such that if any one of them is cut at a point and removed, then the remaining two loops become completely unlinked. In more precise terms, the Borromean link is characterized topologically by the property of splittability as follows: The Borromean link is a *non-splittable 3-link* (because it consists of three loops), such that *every 2-sublink* of this 3-link is *completely splittable*. It is clear that it is a non-splittable 3-link because not even one of the three loops, or any pair of them, can be separated from the rest without cutting. A 2-sublink is simply any sub-collection of two loops obtained by erasing the loop that does not belong to this sub-collection. Since the Borromean link is characterized by the property that if we erase any one of the three interlocking loops, then the remaining two loops become unlinked, it is clear that every 2-sublink of the non-splittable 3-link is completely splittable.



The existence of topological links, like the Borromean link, may be thought of as a form of *topological entanglement*. From the other side, one of the basic distinguishing features between classical and quantum systems is the phenomenon of *quantum entanglement*. Thus, there arises the natural question if there exists any type of *correspondence* between the forms of topological and quantum entanglement. In the context of this, Aravind proposed to investigate the correspondence between topological and quantum entanglement based on the following analogy (Aravind 1997): If the state of a simple quantum system is to be thought of as a ring (topological circle or loop), then the state of an entangled quantum system should be thought of as a topological link. Moreover, the measurement of a subsystem of an entangled system should be thought of as the process of cutting of the corresponding loop. The *caveat* of this approach is that there are many possible measurements on a subsystem of a composite entangled system, and consequently the proposed correspondence should depend on the choice of the measurement basis. We will show in the sequel how it is possible to *overcome* this issue.

It is well known that the state space of a composite quantum system is given by the tensor product of the state spaces of the component subsystems. In the case of two subsystems A and B, if the state vector of the composite system can be written as $|\psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$, where $|\psi_A\rangle$, $|\psi_B\rangle$ denote some state vector of the subsystem A and B respectively, then the state vector $|\psi\rangle$ is called separable. Otherwise, if the pure state $|\psi\rangle$ of the composite system cannot be written in the above form, it is called entangled.

In the simplest case, we may consider *qubits*, that is quantum systems whose state space is 2-d. Let us consider a basis of the 2-d state space consisting of the state vectors $|0\rangle$ and $|1\rangle$. So we may consider a composite quantum system consisting of two qubits. It is immediate to see that there are states of the composite qubit system, for example:

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

where we follow the general convention:

$$|\psi_1\rangle \otimes |\psi_2\rangle := |\psi_1\rangle|\psi_2\rangle := |\psi_1\psi_2\rangle$$

which are not separable, and thus they are entangled.

If we follow Aravind's analogy, then an *entangled state* of a composite *two qubit* system corresponds to a *non-splittable topological 2-link*, whereas a *separable state* corresponds to *splittable 2-link*.

Let us now consider the case of *three qubit* systems denoted by A, B, and C correspondingly. The composite quantum system of these three qubits is characterized by the state space given by the tensor product of the state spaces of the three component subsystems. We consider the so-called *GHZ state* (Greenberger-Horne-Zeilinger) of the composite system defined by Greenberger et al. (1990):

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$$

It is clear that the GHZ state $|\psi\rangle$ of the composite system is entangled. Moreover, the GHZ state is a *symmetric* state under permutations of the states of the three component subsystems. Thus, it can be considered as representing a *non-splittable 3-link*. Now, we may consider a measurement basis of the composite system given by the projection operators $P_0 := |0\rangle\langle 0| \otimes Id \otimes Id$ and $P_1 := |1\rangle\langle 1| \otimes Id \otimes Id$. These projections correspond to potential measurements only on qubit A. After a measurement is performed, the composite system is either in the state $|000\rangle$ or in the state $|111\rangle$. Both of these states are separable. Therefore, a measurement carried on the qubit A can be thought of as a process of *cutting* the corresponding loop. Consequently, the remaining *2-sublink* of the initial non-splittable 3-link becomes *completely splittable*. Clearly, due to the permutation symmetry of the GHZ state, one may consider a potential measurement only on qubit B or only on qubit C without affecting the argument. Hence, the entangled GHZ state of the composite 3-qubit system is analogous to a *Borromean link*. The weak point of this analogy is that it is dependent on the measurement basis.

It has been pointed out that the above problem of dependence on the measurement basis can be rectified by assuming that the process of cutting a loop is actually represented by taking the *reduced density operator* of the GHZ state with respect to the qubit corresponding to this loop. For instance, if the qubit A is traced out in the GHZ state, then the reduced density operator of the remaining system consisting of the qubits B and C is given by:

$$\rho^{BC} = tr_A \rho = \frac{1}{2}(|00\rangle\langle 00| + |11\rangle\langle 11|)$$

We note that this is a *separable mixed state* formed by the mixture of the separable pure states $|00\rangle$ and $|11\rangle$, and thus it reflects the Borromean property.

From our theoretic perspective, although this analogy is instructive for thinking about a possible bridge between the notions of topological and quantum entanglement, in particular with reference to the Borromean link, it suffers from the unjustifiable initial assumption that the state of a quantum system may be thought of in terms of a loop.

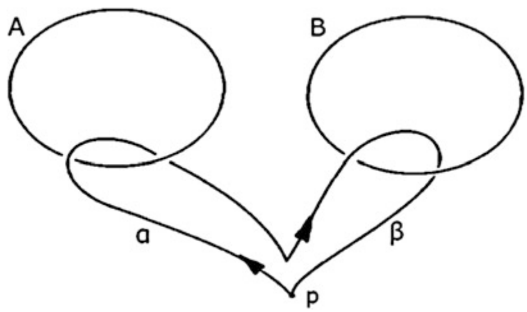
This problem has been solved by establishing an algebraic, *group-theoretic model* of the Borromean link (Zafiris 2016a), thus effectively *algebraizing* the Borromean topology, which admits a *representation* on the Hilbert space of state vectors, as it will become clear in the sequel.

7.2.2 Non-commutative Group Structure of Based Oriented Loops in 3-d Space

First, we consider a loop in three-dimensional space as an unknotted tame closed curve. Since any such closed curve can be continuously deformed to a topological circle it is enough to think of such a circle in 3-d space, denoted by A. Second, we consider a *based oriented loop in 3-d space*, which may pass through this circle A a *finite* number of times, each one with a prescribed *orientation*.

A based loop means simply that it starts and ends at a fixed *reference point* p of the 3-d space. The orientation of the loop can be thought of in terms of an observer, which is fixed at the point p, such that: If the loop passes through the circle one time with direction away from the observer, it is denoted by α^1, whereas if it passes one time with direction toward the observer, it is denoted by α^{-1}. We note that any other loop with the same properties can be continuously deformed to the loop α. Thus, the algebraic symbol α actually denotes the *equivalence class* $[\alpha]$ of all loops of kind α, passing through the circle A once with the prescribed orientation.

Taking into account the algebraic encoding of based oriented loops in relation to topological circles in 3-d space, we can define the *composition of two oriented loops* under the proviso that they are *based* on the *same point* p in 3-d space. Notice that the composition operation $\alpha \circ \beta$ of the p-based oriented loops α and β in relation to circles A and B correspondingly is *not a commutative operation*, meaning that the order of composition is not allowed to be reversed. Clearly, the rule of composition produces a based oriented loop $\alpha \circ \beta$ in 3-d space in relation to the circles A and B in the prescribed order. We think of the composition rule $\alpha \circ \beta$ as the non-commutative *multiplicative product* of the oriented loops α and β based at the same point p in 3-d space, which we may simply denote as $\alpha\beta$. It is immediate to verify that the above defined multiplication is an associative operation.



Having established the *closure* of the elements of the generic form χ under non-commutative associative multiplication as previously, we look for the existence of an *identity element*, as well as for the existence of *inverses* with respect to this operation. There is an obvious candidate for each based oriented loop α, namely the loop α^{-1}, where the orientation has been reversed. If we consider the compositions $\alpha \circ \alpha^{-1}$, $\alpha^{-1} \circ \alpha$ we obtain in both cases as a multiplication product the based loop at the same point, which does not pass through any circle at all. Thus, we name the latter loop as the multiplicative identity 1 in our algebraic structure, such that $\alpha\alpha^{-1} = \alpha^{-1}\alpha = 1$. It is also easy to verify that $1\alpha = \alpha 1 = 1$. We conclude that the set of symbols of the generic form χ representing based oriented loops in relation to circles X, endowed with the non-commutative multiplication operation of composition product of loops based at the same point, form the algebraic structure of a *non-commutative group*, denoted by Θ.

It is instructive to emphasize that the equality sign in the non-commutative group Θ is interpreted topologically as an equivalence relation of p-based oriented loops under continuous deformation. By making use of the multiplication operation in Θ we may form any permissible string of symbols in this group, which can be reduced into an *irreducible* form by using only the group-theoretic relations $\alpha\alpha^{-1} = \alpha^{-1}\alpha = 1$, $\alpha\alpha = \alpha^2$, and so on. Thus, if we consider only two p-based oriented loops as generators, denoted by the symbols α and β respectively with the prescribed orientation and obeying no further constraints, we form a *non-commutative free group in two generators*, denoted by Θ_2.

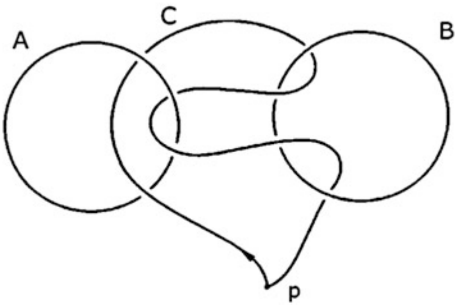
7.2.3 The Borromean Topological Link Semantics of the Non-commutative Free Group Θ_2

We remind that the Borromean link is characterized topologically by the property of splittability as follows: The Borromean link is a non-splittable 3-link, such that every 2-sublink of this 3-link is completely splittable. According to the defining property of the Borromean link, it is a non-splittable 3-link because not even one of the three loops, or any pair of them, can be separated from the rest without cutting. A 2-sublink is simply any sub-collection of two loops obtained by erasing the loop that does not belong to this sub-collection. Since the Borromean link is characterized by the property that if we erase any one of the three interlocking loops, then the remaining two loops become unlinked, we obtain that every 2-sublink of the non-splittable 3-link is completely splittable.

We will show that the topological information incorporated in the specification of the Borromean link can be *encoded algebraically* by exploiting the non-commutative group-structure of the free group Θ_2 generated by two oriented loops, which are based at the same fixed point of 3-d space. The property of irreducibility of a string of symbols in the group Θ_2 is the guiding idea for the algebraic encoding of the Borromean link in terms of the structure of Θ_2. The crucial observation is that *algebraic irreducibility* in Θ_2 can be used to model the *topological* property of *non-splittability* of a 3-link, where complete splittability of all 2-sublinks is encoded by the unique identity element of Θ_2. In particular, the *group-theoretic commutator* induced by the generators of Θ_2:

$$[\alpha, \beta^{-1}] = \alpha\beta^{-1}\alpha^{-1}\beta$$

produces an irreducible non-commutative string of symbols in Θ_2. This string represents a *new based loop* γ as a *product loop* composed by the ordered composition of the based oriented loops $\alpha \circ \beta^{-1} \circ \alpha^{-1} \circ \beta$. We call the product loop γ the *Borromean loop* and the formula or multiplicative string $\alpha\beta^{-1}\alpha^{-1}\beta$ in Θ_2 the Borromean loop formula.



The algebraic irreducibility of the commutator $[\alpha, \beta^{-1}]$ in the group Θ_2 encodes the topological non-splittability property of the Borromean 3-link. We notice that deletion of both α and α^{-1} (corresponding to removal of the circle A) reduces the formula to the identity 1 (and the same happens symmetrically for both β and β^{-1}). This fact models algebraically in the terms of Θ_2 that every 2-sublink of the Borromean 3-link is completely splittable. We conclude that the topological information of the Borromean 3-link can be completely encoded in terms of the algebraic structure of the non-commutative multiplicative free group in two generators Θ_2. In particular, the group-theoretic commutator $[\alpha, \beta^{-1}]$ in Θ_2 encodes algebraically the *gluing condition* of the based oriented loops α and β^{-1} (with respect to the circles A and B respectively in the prescribed orientation), and therefore the non-splittability of the Borromean 3-link, together with the complete splittability of all 2-sublinks of this 3-link.

We note that the Borromean topological link is characterized by *threefold symmetry*. In the algebraic terms of the group Θ_2 this is reflected on the fact that if we consider any two of the based loops α, β^{-1}, γ, then the third is expressed by the group commutator of the other two. The threefold symmetry of the Borromean link may be broken by reducing the free non-commutative group on two generators Θ_2 to the free nilpotent group on two generators of nilpotent class 2, which is precisely the *Weyl-Heisenberg group* $\mathbb{H}$. More concretely, we may choose the based loops α, β^{-1} such that $\gamma = [\alpha, \beta^{-1}] = \alpha\beta^{-1}\alpha^{-1}\beta$ and impose the relations $[\alpha, \gamma] = [\beta^{-1}, \gamma] = 1$.

7.2.4 Unitary Representation of the Group Θ_2 and Realization of the Borromean Link in the Quantum State Space

We have shown that the topological information of the Borromean 3-link can be completely encoded in terms of the algebraic structure of the non-commutative multiplicative group in two generators Θ_2. In particular, the group Θ_2 encodes the non-splittability of the Borromean 3-link, together with the complete splittability of all 2-sublinks of this 3-link.

A natural question arising from our previous analysis is if there exists a *representation of this non-commutative group* Θ_2 in the Hilbert space of state vectors of a quantum system. Such a representation would *transfer* the Borromean topology to the Hilbert space objects which carried this representation. Intuitively, the Borromean 3-link expresses the particular connectivity property of three based oriented loops, where any two of them are unlinked, which is captured algebraically by means of the structure of the group Θ_2. Topological connectivity in this context is associated with the non-splittability of this link as a 3-link.

If we metaphorically think of this connectivity property as *indistinguishability* in a quantum theoretic context, then it becomes quite natural to expect that a representation of the group Θ_2 would be feasible by means of *unitary transformations*. The analogy goes deeper by the fact that in a Borromean 3-link the act of cutting a based loop leads to complete splittability of the remaining 2-link. Analogously, the act of taking the reduced density operator does not preserve the degree of indistinguishability between states and the corresponding unitary group action breaks down.

We have shown that a unitary representation of the group Θ_2 indeed *exists*, and thus the semantics of the Borromean topological link can be transferred appropriately in the quantum state space by means of *one-parameter unitary groups* (Zafiris 2016a). In this way, the Borromean topology can be transferred in the quantum state space via the *action* of one-parameter unitary groups, and their concomitant representation in terms of *oppositely oriented pairs of based loops* under the choice of a reference state vector.

Moreover, if we consider a *faithful* representation of the group $\star$-algebra of Θ_2, $C^\star(\Theta_2)$, in the Hilbert space of states $\mathcal{H}$, such that we identify:

$$C^\star(\Theta_2) \equiv C^\star(U_1, U_2)$$

where U_1 and U_2 are *unitary operators* in the Hilbert space $\mathcal{H}$, considered as a *universal pair*, then $C^\star(U_1, U_2)$ has *no nontrivial projections*. This is important because it shows that the non-commutative spectrum of $C^\star(\Theta_2) \equiv C^\star(U_1, U_2)$ is *highly connected*.

It is instructive to remind that if a nontrivial projection exists then the corresponding spectrum set of this projection is both closed and open. As a result, if we consider, for example, a Boolean algebra of projections, the spectrum of this algebra is a totally disconnected space. In contradistinction the spectrum of $C^\star(U_1, U_2)$ is a highly non-commutative connected topological space.

7.2.5 Topological Links in Geometrodynamics

According to the paradigm of *Geometrodynamics* (Misner et al. 1970), we may *foliate* a spacetime manifold X into three-dimensional spacelike leaves Σ_t by utilizing a one-parameter family of embeddings $\varepsilon_t : \Sigma \hookrightarrow X$, such that $\varepsilon_t(\Sigma) = \Sigma_t$.

In the geometrodynamical formulation, the three-dimensional Riemannian manifold (Σ, h) is thought of as dynamically evolving, where the corresponding metric at time t, $h_t = \varepsilon_t{}^* g$, is derived by pulling back the spacetime metric g via ε_t. It is implicitly assumed that all *three-dimensional spacelike leaves* Σ_t are *mutually disjoint*, such that the Lorentzian manifold $(\mathbb{R} \times \Sigma, \varepsilon^* g)$ represents X, where the leaves of the considered foliation correspond to the *constant time hypersurfaces*.

In particular, if we consider that the Lorentzian manifold $(\mathbb{R} \times \Sigma, \varepsilon^* g)$ represents X, the *singular loci* may be localized within the three-dimensional manifold Σ. In this context, if Σ has a non-trivial topology, it is known that spacetime is *geodesically incomplete*, and thus singular (Clarke 1993; Hawking and Ellis 1973; Heller and Sasin 1995). The simplest way to implement a non-trivial topology on Σ is via the hypothesis of *non-simple connectivity* (Gannon 1975). More precisely, the existence of singular loci in Σ makes Σ a multiple-connected topological space, and thus topologically different from $\mathbb{R}^3$.

We may now consider the system of Einstein-Maxwell equations without sources for the Maxwell field. In this case, Σ is orientable and bears the *standard wormhole topology*, that is homotopically equivalent to $S^1 \times S^2 - \{\text{point}\}$, such that the magnetic flux lines thread through the wormhole. The *homology class* of all 2-spheres containing both of the wormhole mouths has zero charge, whereas the two individual wormhole mouths may be considered as having equal and opposite charges. In this context, a wormhole may be thought of in terms of a *one-dimensional homology class* in spacetime. From general results of low-dimensional geometric topology, we know that every homology class of a four-dimensional spacetime can be represented by an *embedded submanifold* (Scorpan 2005). Using the geometrodynamic foliation, we may restrict this representation to Σ. In this manner, we can instantiate a higher-order wormhole solution, for example, by considering an appropriate *two-dimensional homology class*. We argue that these higher-order wormhole solutions provide *models of Planckian wormholes* that substantiate the "ER = EPR" conjecture.

We are going to outline a general method of generating these types of solutions. For this purpose, we may consider a singular locus with boundary in $\mathbb{R}^3$ or in its compactification S^3, which is excised from $\mathbb{R}^3$ or S^3. We consider a singular locus as a singular disk cut-off from S^3, which may be visualized in terms of a cone whose apex is at infinity and whose base lies at the boundary of the singular locus. A singular disk of this form excised from S^3 gives rise to a *two-dimensional relative homology class* of S^3, which may be interpreted according to the above as a two-dimensional embedded compact submanifold. The circular boundary of this singular disk is a closed and nowhere dense subset with respect to an open set of S^3. Analogously, we may consider the excision of more than one singular disks from S^3, such that their circular boundaries collectively define a closed and nowhere dense subset of an open set of S^3. We propose to think of these *circular singular boundaries* as giving rise to *topological links*.

According to our hypothesis, a collection of circular singular boundaries defining a closed and nowhere dense subset of an open set of S^3 gives rise to a topological link in S^3. We may now replace the loop components of such a topological link by open non-intersecting *tubular neighborhoods* such that the complement of the link in S^3 can be given the structure of a three-dimensional compact and oriented manifold with boundary. Clearly, this space is *homologically equivalent* to the original one since it is just its deformation retract. Next, we may consider an *ordering* of the loops $l_1, l_2, \dots l_N$ constituting the link, or equivalently an ordering of their tubular neighborhoods $\lambda_1, \lambda_2, \dots \lambda_N$. Then, if we take λ_i, λ_j, together with their ordering, we define the relative homology class σ_{ij} that is represented by the compact oriented embedded submanifold whose two boundary components lie on the total boundary, that is the first one in $\partial\lambda_i$ and the second in $\partial\lambda_j$. The orientation is defined as being negative on the first boundary component and positive on the second, so that we have a path from λ_i to λ_j in this case.

7.2.6 The Borromean Rings as a Model of Planckian Wormholes

According to the formalism of Geometrodynamics, we consider the Lorentzian manifold $(\mathbb{R} \times \Sigma, \varepsilon^* g)$ as a representative of X, where the singularities are localized within the three-dimensional manifold Σ. We remind that if Σ is multiple-connected as a topological space, then spacetime is geodesically incomplete. According to our proposal, a collection of circular singular boundaries defining a closed and nowhere dense subset of an open set of S^3 gives rise to a topological link in S^3.

In this context, it is important to examine if there exists a *universal* way via which we can obtain the three-dimensional manifold Σ by the *information* incorporated in a topological link in S^3 representing the singular boundaries, forming collectively a closed and nowhere dense subset. This sheds more light and is guiding in our quest of exploring generalized wormhole-types of solutions based on topological links and their associated homology classes.

It turns out that a universal way to obtain Σ by using a topological link in S^3 representing the singular boundaries, according to the above, actually exists and is based on the notion of a *universal topological link*. In view of the type of solutions we are interested in, such a universal link is defined by the *Borromean rings*. In particular, using methods of geometric topology, it can be shown that any compact oriented three-dimensional manifold Σ without boundary can be obtained as the *branched covering space of the 3-sphere* S^3 with branch set the Borromean rings (Hilden et al. 1987). In this manner, the Borromean rings constitute a universal topological link.

The notion of a branched covering space is a generalization of the standard notion of a covering space, characterized as a local homeomorphism bearing the unique path lifting and homotopy lifting property (Hatcher 2002). More precisely, a branched covering space of the 3-sphere S^3 is considered as a map from Σ to S^3 such that this map is a *covering space* after we delete or exclude a locus of points, called the *branched locus*. The universality property says that Σ can be obtained in this way if the branched locus is formed by the Borromean rings, considered as a closed and nowhere dense set with respect to an open set in S^3 in our setting. We may extend this closed and nowhere dense subset to four dimensions by considering a *timelike axis* perpendicular to the Borromean rings, which plays the role of a threefold symmetry axis of rotation.

In our context, we conclude that if a triad of circular singular boundaries defining a closed and nowhere dense subset of an open set of S^3 are connected in the form of the Borromean topological link, then Σ as a compact oriented three-dimensional manifold can be obtained as the branched covering space of the 3-sphere S^3 with branch set these Borromean-linked boundaries. Based on these findings, we would like to explore their semantics in relation to the instantiation of a higher-order wormhole solution.

For this purpose, we remind that the standard wormhole solution is thought of in terms of a one-dimensional homology class in a space homotopically equivalent to $S^1 \times S^2 - \{\text{point}\}$. In our framework, we do not need to impose a particular topology on Σ ab initio, since it can now be derived universally as the branched covering space of S^3 over the branch nowhere dense subset of singular boundaries forming a Borromean link. The fact that the Borromean link is a non-splittable 3-link, such that every 2-sublink of this 3-link is completely splittable, is characterized in homology theory by a non-vanishing *triple Massey product* (Hatcher 2002), where all pairwise intersection products of one-dimensional homology classes vanish, reflecting the fact that the components of the Borromean link are not pairwise linked. If we denote the components of the Borromean link $\mathcal{B}$ by $\lambda_1, \lambda_2, \lambda_3$, the triple Massey product is expressed as a *two-dimensional cohomology class in the dense complement* of $\mathcal{B}$ in S^3, that is it defines a non-trivial class in $H^2(S^3 \backslash (\lambda_1 \sqcup \lambda_2 \sqcup \lambda_3)$. Since the Borromean link is characterized cohomologically by a higher order invariant, it provides the means to model Planckian wormholes in agreement with the "ER = EPR" conjecture.

Conclusively, the Borromean topological link can be used for modelling both, a higher-order wormhole solution, to be thought of as a Planckian wormhole, and the entanglement properties of the GHZ state, in agreement with the "ER = EPR" conjecture, such that the latter is extended beyond the domain of classical ER bridges. Thus, the Borromean topology provides the sought for universal mechanism to qualify and understand the relation between entanglement and wormholes, and thus addresses effectively the validity of the ER=EPR conjecture from a generalized conceptual and technical framework.

7.3 Delving Deeper into the "ER = EPR" Conjecture from the Borromean Topological Viewpoint

7.3.1 Homology Classes and Holographic Entanglement Entropy

According to the *AdS/CFT correspondence* a wormhole between two asymptotically AdS regions is dual to two non-interacting quantum conformal field theories in a thermally entangled state. In particular, the extended AdS-Schwarzschild black hole solution can be interpreted as two black holes in the same space located a big distance from each other, but connected by a wormhole (Maldacena and Susskind 2013; Maldacena 2003). It is standard to consider the constant $t = 0$ spacelike slice together with *two AdS exterior regions* connected by a wormhole in this manner. Thus, in the context of the above correspondence, the solution referring to these two black holes connected by a wormhole is dual to a highly entangled quantum state, called the thermofield-double state (Israel 1976), defined on the left and right corresponding quantum CFTs on the boundary and being time-reversal symmetric. Therefore, one can legitimately think of an ER bridge between two black holes as giving rise to a highly entangled quantum state between the left and right corresponding boundary quantum CFTs, and thus obtain the implication "ER $\Rightarrow$ EPR". The pertinent question is if the inverse statement, i.e. if a highly entangled quantum state of the previous form functions as a representation of the connectivity between the two black holes by an ER bridge, and thus, if the implication "EPR $\Rightarrow$ ER" is actually legitimate.

It has been shown that the entanglement entropy of the thermofield-double state is equal to the Bekenstein-Hawking entropy of either black hole, and therefore, proportional to the area of the black hole horizon. In this context, Ryu and Takayanagi proposed a generalization that allows the calculation of the entanglement entropy referring to a region of the CFT, called holographic entanglement entropy (Ryu and Takayanagi 2006). For this purpose, they consider a division of the AdS boundary time slice into two regions. This division may be extended to the time slice of the bulk spacetime in the dual gravity representation. If the regions on the boundary are denoted by A and B correspondingly, the boundary ∂A of the region A may be extended to a surface γ_A in the bulk at the depicted time slice, such that $\partial A = \partial \gamma_A$. Of course, there exists a multiplicity of possible ways that this becomes feasible, but Ryu and Takayanagi argued that there exists a unique surface having minimal area, identifying it with γ_A, such that the holographic entanglement entropy of A is proportional to the area of γ_A. In the case of a black hole the minimal surface wraps around the black hole horizon.

The problem with the validity of the Ryu-Takayanagi prescription for the calculation of the entanglement entropy referring to a region of the CFT becomes apparent when we consider two black holes that are entangled, for instance in the thermofield-double quantum state. Then, the entanglement entropy for the region of CFT containing the thermofield-double state should be augmented by the entanglement entropy of this state. Note that if there is no entanglement between the black holes, then the Ryu-Takayanagi original prescription remains in force. It has been pointed out by Susskind (2016) that the resolution of this problem comes about if we consider that the entanglement between the two black holes induces an ER bridge between them, and thus, modifies the global topology of the time slice in the bulk. In more precise mathematical terms, we propose that the entanglement between the two black holes should be thought of as giving rise to a non-trivial *one-dimensional homology class*, which can be always represented by an *embedded submanifold* of the time slice in the bulk. In this way, the Ryu-Takayanagi prescription of the minimal area surface should be modified according to this one-dimensional homology class. We notice that in view of the above modification in the calculation of the entanglement entropy referring to a region of the CFT containing the thermofield-double state, the implication "EPR $\Rightarrow$ ER" seems to be justified.

In this manner, as we have already proposed in the previous section, we can instantiate a higher-order wormhole solution, for example, by considering an appropriate *two-dimensional homology or cohomology class* and calculate their contribution to the entanglement entropy referring to a region of CFT. The rationale is that these higher-order wormhole solutions provide *models of Planckian wormholes* that substantiate the "ER = EPR" conjecture. This strategy allows the extension of the "ER = EPR" conjecture beyond the domain of classical ER bridges between two black holes. For instance, the entanglement properties of the GHZ state are precisely modelled by the Borromean topological link. Considering the circular boundaries $\lambda_1, \lambda_2, \lambda_3$ connected in the form of the Borromean link $\mathcal{B}$ and defining a closed and nowhere dense subset of an open set of S^3, we obtain a *two-dimensional cohomology class in the dense complement* of $\mathcal{B}$ in S^3, denoted by $H^2(S^3 \backslash (\lambda_1 \sqcup \lambda_2 \sqcup \lambda_3)$. This gives rise to a higher-order wormhole solution, which may be interpreted as a Planck-scale wormhole. Remarkably, due to the threefold symmetry of the Borromean topological link any of the three boundaries may be considered as the connectivity bridge for the other two. Additionally, due to the universality of the Borromean link, Σ as a compact oriented three-dimensional manifold can be generated without any ad hoc assumptions as the branched covering space of the 3-sphere S^3 with branch set these Borromean-linked boundaries.

In the sequel, we will attempt to shed more light on the universality of the Borromean topological link and the instrumental role it plays in qualifying the "ER = EPR" conjecture under the intended semantics. The guiding idea is that quantum entanglement constitutes an *interchangeable resource* (Maldacena and Susskind 2013; Susskind 2016), meaning that the various forms of entanglement, like vacuum entanglement or entangled particles or wormholes or even clouds of Hawking radiation, are inter-transformable into one another by means of *local unitary transformations* for some fixed entanglement entropy. Given that the calculation of entanglement entropy for a region of CFT depends on the existence of non-trivial homology classes, to be thought of as emerging by the particular linkage properties of boundaries, it is worth investigating if the Borromean link functions as a building block for more complex links. This is also suggested by the fact that the group-theoretic model of the Borromean link admits a representation in terms of unitary transformations that we are going to examine in more detail.

7.3.2 Realization of the Borromean Link by Means of Continuous One-Parameter Groups of Unitary Operators

Given that the Borromean link is a non-splittable 3-link, such that every 2-sublink of this 3-link is completely splittable, the topological information of its particular connectivity properties can be encoded algebraically in terms of the non-commutative multiplicative group-structure of the free group Θ_2 generated by two oriented loops, which are based at the same fixed point of 3-d space. More precisely, the group-theoretic commutator $[\alpha, \beta^{-1}]$ in Θ_2 encodes algebraically the gluing condition of the based oriented loops α and β^{-1}, and therefore, the non-splittability of the Borromean 3-link, together with the complete splittability of all 2-sublinks of this 3-link.

We are going to prove that a unitary representation of the group Θ_2 indeed exists, and thus the semantics of the Borromean topological link can be transferred appropriately in the quantum state space by means of one-parameter unitary groups, where the action of the latter type of groups in Hilbert space has been first studied by Stone (1932). Given the conception of entanglement as an interchangeable resource, the extended validity of the "ER = EPR" conjecture implicates that the class of unitary transformations making this possible come from the unitary representation of the non-commutative group Θ_2, or equivalently from the unitary realization of the Borromean link on the state space. Of course, this claim is based on the universality of the Borromean link in its function to play the role of a building block for the realization of more complex links, a topic that we will be investigated in the sequel.

First, we need to define the notion of a unitary representation of the group Θ_2 as follows: A unitary representation of the group Θ_2 consists of a Hilbert space of states $\mathcal{H}$, together with a group homomorphism from Θ_2 to the group of unitary operators on $\mathcal{H}$.

Second, we note that the non-commutative group Θ_2 is a free multiplicative group in two generators g_1 and g_2. Given two unitary operators U_1 and U_2 in the Hilbert space of states $\mathcal{H}$, there exists a unique group homomorphism $\zeta : \Theta_2 \to B(\mathcal{H})$, where $B(\mathcal{H})$ is the algebra of bounded linear operators in $\mathcal{H}$, which sends g_1 to U_1 and g_2 to U_2, just by the universal property of free groups. Since ζ is a group homomorphism $\zeta(g_i) = U_i$ is unitary operator for each $j = 1, 2$. Therefore, since $\{g_1, g_2\}$ generates Θ_2 as a free group, ζ must be a unitary representation of the group Θ_2 in the Hilbert space of states $\mathcal{H}$.

Third, we know that $B(\mathcal{H})$ has the structure of a $\star$-algebra over the complexes. If we consider the free group $\star$-algebra of Θ_2, generated by finite linear combinations of elements of Θ_2 with complex coefficients, then we have the following: Given a unitary representation ζ of Θ_2 in the Hilbert space of states $\mathcal{H}$, then this representation extends by linearity to a $\star$-homomorphism of the group $\star$-algebra of Θ_2, denoted by $C^\star(\Theta_2)$, to the $\star$-algebra $B(\mathcal{H})$.

Fourth, the algebra $C^\star(\Theta_2)$ is characterized uniquely up to isomorphism by the following universal property: Given any unitary representation,

$$\zeta : \Theta_2 \to B(\mathcal{H})$$

of the group Θ_2, there exists a unique $\star$-homomorphism of the group $\star$-algebra of Θ_2, $C^\star(\Theta_2)$, to the $\star$-algebra $B(\mathcal{H})$, denoted by

$$\tilde{\zeta} : C^\star(\Theta_2) \to B(\mathcal{H})$$

that satisfies:

$$\tilde{\zeta}(\gamma_g) = \zeta(g)$$

for every $g \in \Theta_2$, where $\gamma_g \in C^\star(\Theta_2)$. Thus, if we consider the generating set of symbols $\{g_1, g_2\}$ of Θ_2 as a free group we obtain the relations:

$$\tilde{\zeta}(\gamma_{g_1}) = \zeta(g_1) = U_1$$

$$\tilde{\zeta}(\gamma_{g_2}) = \zeta(g_2) = U_2$$

where U_1 and U_2 are unitary operators in the Hilbert space of states $\mathcal{H}$.

We consider a faithful representation of the group $\star$-algebra of Θ_2, $C^\star(\Theta_2)$, in the Hilbert space of states $\mathcal{H}$, such that we identify:

$$C^\star(\Theta_2) \equiv C^\star(U_1, U_2)$$

where U_1 and U_2 are unitary operators in the Hilbert space $\mathcal{H}$, considered as universal, in the following sense: For any other pair V_1 and V_2 of unitary operators in the Hilbert space $\mathcal{H}$, the assignment $U_1 \to V_1$, $U_2 \to V_2$, extends to a $\star$-homomorphism from $C^\star(U_1, U_2)$ to $C^\star(V_1, V_2)$. Now, by utilizing the spectral theorem, we may always choose two self-adjoint operators A and B in $B(\mathcal{H})$, such that $U_1 = e^{iA}$ and $U_2 = e^{iB}$.

Next, we consider the set of all continuous functions:

$$\mathbb{T} = \{f : [0, 1] \to B(\mathcal{H})\}$$

such that $f(0)$ are scalar operators. The set $\mathbb{T}$ can be endowed with the structure of a $C^\star$-algebra, which is denoted by the same symbol. We will show that the $C^\star$-algebra $C^\star(\Theta_2) \equiv C^\star(U_1, U_2)$ can be imbedded in $\mathbb{T}$ as a $C^\star$-subalgebra.

We have already seen that if we choose two observables represented as self-adjoint operators A and B in $B(\mathcal{H})$, then we identify $U_1 = e^{iA}$ and $U_2 = e^{iB}$. Next, we define two continuous functions f_{U_1} and f_{U_2} in the $C^\star$-algebra $\mathbb{T}$ such that their image in $B(\mathcal{H})$ is unitary, as follows:

$$f_{U_1} : [0,1] \to B(\mathcal{H}),\ f_{U_2} : [0,1] \to B(\mathcal{H})$$

$$[0,1] \ni t \mapsto f_{U_1}(t) := e^{itA} \in B(\mathcal{H})$$

$$[0,1] \ni t \mapsto f_{U_2}(t) := e^{itB} \in B(\mathcal{H})$$

Then, it is clear that we may consider the $C^\star$-algebra generated by the continuous functions f_{U_1} and f_{U_2}, denoted by $C^\star(f_{U_1}, f_{U_2})$. Then, by the universality property of $C^\star(\Theta_2) \equiv C^\star(U_1, U_2)$ there exists a $\star$-homomorphism from $C^\star(\Theta_2) \equiv C^\star(U_1, U_2)$ to $C^\star(f_{U_1}, f_{U_2})$, specified precisely by the assignments $U_1 \mapsto f_{U_1}$ and $U_2 \mapsto f_{U_2}$. From the other side, we may consider the evaluation morphism $f \mapsto f(1)$, which clearly defines a $\star$-homomorphism from $C^\star(f_{U_1}, f_{U_2})$ to $C^\star(\Theta_2) \equiv C^\star(U_1, U_2)$. The above two $\star$-homomorphisms are inverse to each other, and thus induce an isomorphism:

$$C^\star(\Theta_2) \cong C^\star(f_{U_1}, f_{U_2})$$

The significance of this theorem is the following: It is obvious that the algebra $C^\star(f_{U_1}, f_{U_2})$ is a $C^\star$-subalgebra of $\mathbb{T}$. Hence, $C^\star(\Theta_2) \equiv C^\star(U_1, U_2)$ can be imbedded in $\mathbb{T}$ as a $C^\star$-subalgebra. The crucial fact is that the $C^\star$-algebra of continuous functions $\mathbb{T}$ has no nontrivial projections. This means that $C^\star(\Theta_2) \equiv C^\star(U_1, U_2)$ imbedded in $\mathbb{T}$ as a $C^\star$-subalgebra has no nontrivial projections either. This is important because it shows that the non-commutative spectrum of $C^\star(\Theta_2) \equiv C^\star(U_1, U_2)$ is highly connected, in contradistinction to the spectrum of a Boolean algebra of projection operators resolving a complete set of commuting observables, which is a totally disconnected space.

Conclusively, the Borromean topological link is realized in the quantum state space via the combined *action* of one-parameter unitary groups, and their concomitant representation in terms of *oppositely oriented pairs of based loops* under the choice of a reference state vector giving rise to a highly connected non-commutative topological spectrum. This is particularly interesting in relation to considering complete sets of measurements in quantum mechanics, where following Susskind (2016), we think of a measurement as a process that entangles the system with the apparatus. The relation stems from Stone's theorem, which establishes a bijective correspondence between continuous one-parameter groups of unitary operators and observables. Given that $C^\star(\Theta_2) \equiv C^\star(U_1, U_2) \cong C^\star(f_{U_1}, f_{U_2})$ according to the above, the non-commutativity of quantum observables can be traced back to the non-commutative realization of the Borromean link in terms of one-parameter groups of unitary operators. Furthermore, if the threefold symmetry of the Borromean link is broken by reduction of the free non-commutative group on two generators Θ_2 to the free nilpotent group on two generators of nilpotent class 2, we obtain the *Weyl-Heisenberg group* $\mathbb{H}$ (Zafiris 2016a).

7.3.3 *The 2-Sphere and 3-Sphere Representations of the Borromean Link and Qubits*

The unit 2-sphere S^2 constitutes the space of pure states or equivalently rays of a 2-level quantum mechanical system, called usually a qubit. The unit 2-sphere may be thought of as embedded in the standard three-dimensional space $\mathbb{R}^3$. The Hilbert space of normalized unit state vectors of a qubit is the 3-sphere S^3, and thus the unit 2-sphere is considered as the base space of the Hopf fibration:

$$S^1 \hookrightarrow S^3 \twoheadrightarrow S^2$$

We note that each pair of antipodal points of S^2 corresponds to mutually orthogonal state vectors. The north and south poles are chosen to correspond to the standard orthonormal basis vectors $|0\rangle$ and $|1\rangle$ correspondingly. In the case of a spin-$\frac{1}{2}$ system, these correspond to the spin-up and spin-down states of this system.

We consider the unit sphere S^2 as the set of points of three-dimensional space $\mathbb{R}^3$ that lie at distance 1 from the origin. Then, the non-commutative group $SO(3)$ denotes the group of rotation operators on $\mathbb{R}^3$ with center at the origin, viz. linear transformations from $\mathbb{R}^3$ to $\mathbb{R}^3$ represented as 3×3 matrices with determinant one. These are called orthogonal matrices, characterized by the fact that their columns form an orthonormal basis of $\mathbb{R}^3$. Rotations around an axis going through the origin are the isometries of three-dimensional Euclidean space $\mathbb{R}^3$ leaving the origin fixed. Note that a 3×3 orthogonal transformation preserves the inner product for any pair of vectors in $\mathbb{R}^3$, and moreover it is an isometry of $\mathbb{R}^3$ that takes the unit sphere S^2 to itself.

In this context, we ask the following question: Does there exist a representation of the Borromean link via its algebraic encoding in terms of the non-commutative free group Θ_2 on the unit sphere S^2, which lifts to a unitary representation on S^3? Such a representation definitely exists if we are able to locate a subgroup of the non-commutative group $SO(3)$, which is isomorphic to Θ_2. In fact, we are able to prove the following theorem:

There are rotation operators A and B about two independent axes through the origin in $\mathbb{R}^3$ generating a non-commutative subgroup of $SO(3)$, which is isomorphic to the group Θ_2.

In other words, there exists an isomorphic copy of Θ_2 in $SO(3)$ generated by two independent rotations A and B. The term independent refers to the requirement that all rotations performed by sequences of A and B and their inverses are distinct strings in Θ_2. Actually, most pairs of rotations in $SO(3)$ are independent in the above sense, so that even by picking A and B randomly would do. For instance, one could consider two counterclockwise rotations A and B about the z-axis and the x-axis respectively of the same angle $\arccos(3/5)$. The proof is based on showing that no reduced word in the symbols A and B and their inverses collapses to the identity transformation (3×3 identity matrix). Intuitively, if we choose two counterclockwise rotations A and B about the z-axis and the x-axis of the same angle, then this specific angle needs to be an irrational number of degrees. More precisely, given an initial orientation, if the specified angle is an irrational number of degrees, then none of the distinct strings of rotations in Θ_2 performed by sequences of A and B and their inverses can give back the initial orientation. Thus, no reduced word in Θ_2 collapses to the identity transformation.

The existence of an isomorphic copy of Θ_2 in $SO(3)$ has the following consequence: Each rotation belonging to the non-commutative free subgroup Θ_2 of $SO(3)$ fixes two points in the unit sphere S^2, namely the intersection of S^2 with the axis of rotation passing through the origin. If we take the union of all these points, they form a countable set of points. Then, not only there exists an action of Θ_2 on the unit sphere S^2 (as a subgroup of $SO(3)$ generated by A and B) but this action is actually free on S^2 modulo the countable set of fixed points K.

Thus, we can partition $S^2\backslash K$ into a disjoint union of orbits for the action of Θ_2. If we choose a base point for an orbit, we may identify it with Θ_2 due to the freeness of this action. Moreover, if a countable collection K of points as above is removed from S^2 they can be restored by rotations around an axis through the origin which has zero overlap with K. In this way, the action of the group Θ_2 via rotations allows to resolve the whole unit sphere S^2. The crucial point again is that the algebraic irreducibility of the commutator $[A, B]$ of the rotations A and B generating an isomorphic copy of Θ_2 in $SO(3)$ expresses the Borromean topological non-splittability or non-separability of these three rotations belonging to the subgroup of $SO(3)$ isomorphic with Θ_2. This interpretation provides a topological justification of the fact that one cannot specify a finitely additive rotation-invariant probability measure on all subsets of the unit sphere S^2. In the terminology of von Neumann, since the group of rotation operators $SO(3)$ contains an isomorphic copy of Θ_2 it is not amenable.

An immediate consequence of the above is that the group of 2×2 complex unitary matrices with unit determinant $SU(2)$ is also not amenable, that is it also contains an isomorphic copy of Θ_2. The reason is that topologically, the simply-connected special unitary group $SU(2)$ is a covering space of the non-simply connected group of rotations $SO(3)$, and in particular it is a double cover. More concretely, there exists a two-to-one surjective homomorphism of groups:

$$\Delta : SU(2) \twoheadrightarrow SO(3)$$

whose kernel is given by $Ker\Delta = \mathbb{Z}_2 = \{+1, -1\}$.

Hence, it is clear that there exists an isomorphic copy of Θ_2 in $SU(2)$. More precisely, if A and B are rotations generating an isomorphic copy of Θ_2 in $SO(3)$, and $\Delta : SU(2) \twoheadrightarrow SO(3)$ is the covering projection, then $\overline{A}$ and $\overline{B}$ generate a free subgroup of the form Θ_2 in $SU(2)$, for any $\overline{A}$ and $\overline{B}$ with $\Delta\overline{A} = A$ and $\Delta\overline{B} = A$. Since $SU(2)$ is a double cover of $SO(3)$ there exist exactly two elements of the form $\overline{A}$, namely U and $-U$ such that $\Delta U = \Delta(-U) = A$ (the same holds for $\overline{B}$ respectively).

We conclude that there exists a representation of the group Θ_2 on the unit sphere S^2, which lifts to a unitary representation on S^3. The representation of the group Θ_2 on the unit sphere S^2 is given by the free subgroup of rotations of $SO(3)$ generated by A and B according to the above. Concomitantly, this representation lifts to a unitary representation on S^3 by the free subgroup of unitary operators of $SU(2)$ generated by $\overline{A}$ and $\overline{B}$. Thus, the Hilbert space of normalized unit state vectors of a qubit or of a spin-$\frac{1}{2}$ system carries a unitary representation of the group Θ_2. This means that the algebraic irreducibility of the commutator $[\overline{A}, \overline{B}]$ of the unitary operators $\overline{A}$ and $\overline{B}$ generating an isomorphic copy of Θ_2 in $SU(2)$ expresses the Borromean topological non-splittability or non-separability of these three unitary operators. Moreover, since the action of the group Θ_2 by strings of rotations in two generators allows to resolve S^2, such that the same lifted action resolves S^3 as well by strings of corresponding unitary operators, we conclude that the property of Borromean connectivity is transferred via these actions to the space of rays S^2 and the space of unit state vectors of a qubit S^3.

In the case of a spin-$\frac{1}{2}$ system, the unitary rotation operator U corresponding to a rotation A with axis $\mathbf{n}$ and angle θ is given explicitly by:

$$U(\theta\mathbf{n}) = e^{\frac{-i}{\hbar}\theta\mathbf{n}\cdot\mathbf{S}}$$

where $\mathbf{S}$ is the generator of the spin-$\frac{1}{2}$ unitary rotation group $SU(2)$. Since the operator $\mathbf{S}$ may be represented in matrix form in terms of the Pauli matrices σ_1, σ_2 and σ_3, as $\mathbf{S} = \frac{\hbar}{2}\sigma$, where $\sigma = (\sigma_1, \sigma_2, \sigma_3)$, we obtain:

$$U(\theta\mathbf{n}) = e^{\frac{-i}{2}\theta\mathbf{n}\cdot\sigma}$$

Now, a unitary rotation $U(\theta\mathbf{n}) = U$ corresponds to a rotation A with axis $\mathbf{n}$ and angle θ as follows: For any vector v of $\mathbb{R}^3$ such that $v \mapsto Av$, we have:

$$v \cdot \sigma \mapsto U(v \cdot \sigma)U^{\dagger} := (Av) \cdot \sigma$$

Thus, for a rotation A with axis $\mathbf{n}$ and angle θ, we obtain a unitary rotation operator U such that $\Delta U = A$. If we change the angle of rotation θ by 2π, the rotation A remains the same, whereas the unitary rotation U changes sign, that is $U \to -U$. This is consistent with the double covering projection of unitary rotations to ordinary rotations. Thus, we have that $\Delta U = \Delta(-U) = A$, that is both unitary rotations U and $-U$ correspond by the double covering map to the same rotation A.

7.3.4 Borromean Einstein-Rosen Bridge for Tripartite-Entangled Black Holes

We follow the hypothesis that the degrees of freedom of a black hole can be represented as a system of qubits (Maldacena and Susskind 2013; Susskind and Zhao 2014; Susskind 2014a, 2016). More precisely, we assume that a black hole can be described in terms of a system of K qubits, where K is of order the entropy. We consider a pair of entangled black holes, which are in the thermofield-double state, denoted by $|TFD\rangle$. In information-theoretic terms, this pair of entangled black holes can be represented by a maximally entangled state of $2K$ qubits. If we additionally choose the standard orthonormal basis of a qubit consisting of the state vectors $|0\rangle$ and $|1\rangle$, then the initial state of the black hole pair can be written as a product of K maximally entangled Bell pairs:

$$|TFD\rangle \sim (|00\rangle + |11\rangle)^{\otimes K}$$

Thus, as a starting point, we may first consider a single maximally entangled Bell pair of qubits:

$$|\mathrm{Bell}\rangle \sim (|00\rangle + |11\rangle)$$

Let us also invoke a third qubit, which acts as an apparatus qubit, in the sense that it measures a complete set of commuting observables related with one of the other two qubits. Then, the process of measurement leads to a tripartite qubit entanglement, described by the *GHZ state* (Greenberger-Horne-Zeilinger) of the composite system:

$$|\mathrm{ghz}\rangle \sim (|000\rangle + |111\rangle)$$

It is clear that $|\mathrm{ghz}\rangle$ is entangled, and moreover symmetric under permutations of the states of the three qubits. We conclude that whenever a process of measurement is carried out on one of the qubits of a maximally entangled Bell pair, then there emerges a tripartite entangled GHZ state. The GHZ-type of entanglement is characterized by the following properties:

1. If any two qubits are traced over, the density matrix of the third qubit is maximally mixed, and thus, any qubit is maximally entangled with the union of the other two;
2. If any one qubit is traced over the density matrix of the other two is separable.

For instance, we may consider a measurement basis of the composite system of qubits A, B, and C given by the projection operators $P_0 := |0\rangle\langle 0| \otimes Id \otimes Id$ and $P_1 := |1\rangle\langle 1| \otimes Id \otimes Id$, corresponding to a potential measurement carried out only on qubit A. After the measurement the composite system is either in the state $|000\rangle$ or in the state $|111\rangle$, and clearly both of these states are separable. Due to the permutation symmetry of the GHZ state, the argument remains the same if, for example, a potential measurement applies to qubit B instead of A. If we trace over the qubit A in the GHZ state, then the reduced density operator of the remaining system consisting of the qubits B and C is given by:

$$\rho^{BC} = tr_A \rho = \frac{1}{2}(|00\rangle\langle 00| + |11\rangle\langle 11|)$$

where ρ^{BC} denotes a *separable mixed state* formed by the mixture of the separable pure states $|00\rangle$ and $|11\rangle$.

It has been pointed out by Susskind (2016) that the entangled GHZ state of three qubits $|\text{ghz}\rangle \sim (|000\rangle + |111\rangle)$ can be expressed in terms of a symmetric tensor with three indices T_{ijk} corresponding to the qubits. This tensor may be transformed by the action of local unitary transformations preserving the properties of the GHZ-type of tripartite entanglement. Notwithstanding the permutation symmetry reflected in the specification of the tensor T_{ijk} as a symmetric tensor, the basic properties (1) and (2) of the GHZ-type of tripartite entanglement are not reflected in the tensor notation.

For this reason, we propose that the density operator ρ of the GHZ-entangled qubits actually forms a Borromean topological link, that is a non-splittable 3-link, such that every 2-sublink of this 3-link is completely splittable. In this manner, the process of cutting a loop from the Borromean link is actually represented by taking the reduced density operator of the GHZ state with respect to the qubit corresponding to this loop. In our previous example, if we trace over the qubit A in the GHZ state, then the reduced density operator of the remaining system consisting of the qubits B and C is given by:

$$\rho^{BC} = tr_A \rho = \frac{1}{2}(|00\rangle\langle 00| + |11\rangle\langle 11|)$$

where ρ^{BC} is a separable mixed state, thus reflecting the splittability property of the remaining 2-sublink of the Borromean 3-link.

More generally, we already have shown that the Borromean topological link, through its algebraic encoding in terms of the non-commutative group Θ_2, is realized in a quantum state space via the combined action of one-parameter unitary groups, and the concomitant representation of each one of them in terms of oppositely oriented pairs of loops based at a reference state vector, giving rise to a highly connected non-commutative topological spectrum. Given that the GHZ-type of tripartite entanglement is preserved by the action of local unitary transformations on the GHZ-state vector, we can identify a copy of each local unitary group with the cyclic orbit of its action oriented in both possible ways and based at the GHZ-state vector. Therefore, we obtain a realization of the Borromean link in terms of one-parameter unitary groups, or equivalently via their oriented orbit-loops based on the GHZ-state vector.

We come back now to a pair of entangled black holes, which is represented by a maximally entangled state of $2K$ qubits. We remind that if we choose the standard orthonormal basis of a qubit consisting of the state vectors $|0\rangle$ and $|1\rangle$, then the initial state of the black hole pair can be written as a product of K maximally entangled Bell pairs:

$$|TFD\rangle \sim (|00\rangle + |11\rangle)^{\otimes K}$$

Then, we may invoke again a third system of K qubits, which information-theoretically can also be considered as a black hole, to be thought of as an apparatus, in the sense that it measures a complete set of commuting observables related with one of the other two black holes. Then, the process of measurement leads to a tripartite black hole entanglement, described by the product of *GHZ states*:

$$|\mathrm{GHZ}\rangle = |\mathrm{ghz}\rangle^{\otimes K} \sim (|000\rangle + |111\rangle)^{\otimes K}$$

The properties of the tripartite black hole entanglement are precisely analogous to the properties of the tripartite qubit entanglement:

1. If any two black holes are traced over, the density matrix of the third black hole is maximally mixed, and thus, any black hole is maximally entangled with the union of the other two;
2. If any one black hole is traced over, the density matrix of the other two is separable, so that there is no entanglement between any two black holes.

We conclude that *the tripartite black hole entanglement follows the rules of the Borromean topological linking in analogy to the tripartite qubit case*. In view of the "ER = EPR" conjecture, the tripartite black hole entanglement gives rise to a non-classical Einstein-Rosen bridge, to be thought of as a Planck-scale one, which bears all the properties of the Borromean topological linking. This gives a precise form to what is called a GHZ-core or GHZ-brane by Susskind, which cannot be removed by any local unitary transformations since it is invariant under their action.

From the viewpoint of Susskind, since a tripartite GHZ-entangled state of three qubits is represented by a symmetric tensor, the analogous tripartite GHZ-entangled state of three black holes is represented by a tensor network corresponding to the product of the former ones. This tensor network is conceived as evolving in time and growing, where this growth represents the growth in complexity.

From the proposed perspective of the Borromean topology, which incorporates all the pertinent properties of the tripartite black hole entanglement, we obtain a Borromean topological network whose complexity grows by iteration. More precisely, in the formation of the Borromean link, if each of the three rings is substituted by a triplet of rings forming a Borromean link themselves, then all the properties of Borromean linking are preserved, thus obtaining a Borromean ring of Borromean rings. This may be iterated ad infinitum, giving rise to higher and higher orders of complexity. This is how a Borromean network grows outward from the GHZ-core.

The most important characteristic of the non-classical Einstein-Rosen bridge arising from the maximally entangled GHZ-state of the tripartite black hole entanglement is that essentially any of the three black holes serves as a wormhole only relationally, that is as a bridge in relation to the other two, in the sense that removal of any of them leaves the remaining two completely unlinked topologically.

7.3.5 The Borromean Link as the Building Block of All Higher-Order Linkings

We remind that the topological linking information of the Borromean 3-link can be completely encoded in terms of the algebraic structure of the non-commutative multiplicative free group in two generators Θ_2. In particular, the group-theoretic commutator $\gamma = [\alpha, \beta^{-1}]$ in Θ_2 encodes algebraically the *gluing condition* of the topological circles A and B by means of composing based oriented loops referring to them, and therefore, the non-splittability of the Borromean 3-link, together with the complete splittability of all 2-sublinks of this 3-link. It is precisely this gluing condition that allows, through the representability of the group Θ_2 in terms of one-parameter unitary groups, to model the entanglement properties of the tripartite GHZ state by means of a Borromean 3-link. Therefore, according to the "ER = EPR" conjecture, the tripartite black hole entanglement gives rise to a non-classical Einstein-Rosen bridge, which bears all the properties of the Borromean topological linking. As we have already pointed out the Borromean link can be iterated to obtain successively higher orders of Borromean linking, and thus build up a Borromean network of growing complexity.

A natural question arising in this setting is if the Borromean link plays the fundamental role of a building block for higher-order linkings, which extend and further qualify the "ER = EPR" conjecture in the intended semantics. The crucial observation is that *the algebraic irreducibility of the commutator* $\gamma = [\alpha, \beta^{-1}]$ *in the group* Θ_2 *encodes the topological non-splittability property of the Borromean 3-link*. We stress again that deletion of both symbols α and α^{-1} or equivalently cutting of both based oriented loops α and α^{-1} (corresponding to removal of the topological circle A) reduces the formula to the identity 1 (and the same happens symmetrically for both β and β^{-1}), modelling in the terms of Θ_2 that every 2-sublink of the Borromean 3-link is completely splittable. Therefore, at an initial stage, the idea of using the Borromean link as a building block for analogous links of a higher type means employing the group commutator iteratively as an encoding device for these higher links.

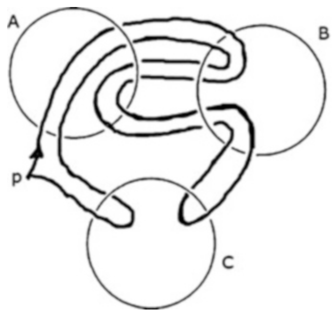
We consider the case of a total non-splittable 4-link all 3-sublinks of which are completely splittable, denoted by $\Sigma(4, 3)$, which constitutes the first direct generalization of the Borromean link as a non-splittable 3-link all 2-sublinks of which are completely splittable, denoted by $\Sigma(3, 2)$, respectively. This case involves the gluing of three topological circles A, B, and C by a higher Borromean loop. Thus, we may proceed as follows: First, we glue the circles A and B by the standard Borromean loop and then we glue analogically this product with C. Algebraically speaking, the first step is simply the commutator $\xi = [\alpha, b]$, where for simplicity we have redefined β^{-1} as b, i.e. $\beta^{-1} := b$. The first iteration of this procedure, which involves the gluing of the product ξ with γ (in relation to the topological circle C), reads simply as the commutator of ξ with γ. We conclude that a higher Borromean loop that solves the problem is given in the structural terms of the group Θ_2 simply as follows:

$$\delta = [\xi, \gamma] = [[\alpha, b], \gamma]$$

If we expand this formula, by using the definition of the group commutator as well as the group theoretic relation $(\chi\psi)^{-1} = \psi^{-1}\chi^{-1}$, where χ, ψ may stand for arbitrary strings of elements of the group Θ_2, we obtain the following unfolded expression for the higher Borromean loop formula:

$$\delta = [\xi, \gamma] = [[\alpha, b], \gamma] = \{\alpha b \alpha^{-1} b^{-1}\}\gamma\{b\alpha b^{-1}\alpha^{-1}\}\gamma^{-1}$$

Now it becomes clear how the Borromean link becomes a building block via terms of the form $\lambda\mu\lambda^{-1}\mu^{-1} = [\lambda, \mu]$ for expressing higher order links of the same type. We can also see that deletion of all incidences of any of the symbols (which involves the simultaneous deletion of the inverse symbol as well, according to our preceding explanation) reduces the formula to the identity 1 in the group Θ_2. If we decode back the obtained algebraic solution in topological terms, then the topological solution of the problem of finding a non-splittable 4-link whose all 3-sublinks are completely splittable by means of Borromean building blocks is illustrated as follows:



We recapitulate by emphasizing that the Θ_2 group commutator acts as an encoding device for these higher links of the type $\Sigma(4, 3)$ in two ways: First, the commutator provides the gluing scheme of link-formation by means of Borromean loops. Second, due to the fact that deletion of all incidences of any of the involved symbols reduces the commutator to the identity 1 in the group Θ_2, the commutator also encodes the information of complete splittability of any remaining sublink after removing any of the constituents of the total non-splittable link. In this manner, a $\Sigma(4, 3)$ link can be constructed in terms of Borromean link building blocks simply by iterating once the commutator formation.

According to the above, we may simplify the proposed algebraic algorithm of constructing a $\Sigma(4, 3)$ link within the group Θ_2, keeping in mind the corresponding topological semantics, as follows: If we start with three symbols a, b, c, we first glue a with b together by means of the commutator $[a, b]$, and then we glue their glued product $[a, b]$ with c to obtain the stacked commutator $[[a, b], c]$. This final glued product gives the required fourth symbol in the group Θ_2, which decodes topologically as a $\Sigma(4, 3)$ link. In an analogous manner, by iterating twice the commutator formation starting with four symbols a, b, c, d, we obtain a $\Sigma(5, 4)$ link. The same process can be clearly repeated inductively, so that we finally can construct any $\Sigma(N, N-1)$ link by means of Borromean building blocks, or more precisely, Borromean connectivity units, where $N \geq 3$.

We call this algorithmic procedure the operation of commutator stacking in consecutive nested levels. In this way, a $\Sigma(N, N-1)$ link is algorithmically constructed by a stacked commutator in $(N-1)$ symbols, where $N \geq 3$. For convenience, we call it a stacked commutator of order $(N-1)$. Note that the order of the stacked commutator in any link of the form $\Sigma(N, N-1)$ coincides with the number of symbols that separate if we remove any symbol from the total non-splittable N-link.

We proceed to consider the case of an arbitrary topological link of the general form $\Sigma(N, K)$. A link of the form $\Sigma(N, K)$ is defined as a link of N topological circles, such that each K-sublink is completely splittable, but each $(K+1)$-sublink, $(K+2)$-sublink, $(K+3)$-sublink, ..., $(N-1)$-sublink up to the N-link itself, is non-splittable. The natural question emerging in this context is if it is possible to express a general link $\Sigma(N, K)$ in terms of Borromean building blocks, encoded algebraically by the gluing operator of symbols, viz. by the commutator in the group Θ_2. We already know that this is feasible in case that $K = (N-1)$ solely by means of the operation of commutator stacking of order $(N-1)$. Hence, we simply call a $\Sigma(N, N-1)$ link a Borromean stack of order $(N-1)$.

Clearly a link of the general form $\Sigma(N, K)$ cannot be expressed solely as a Borromean stack. But there exists another natural algorithmic operation on Borromean building blocks, which is described by taking an appropriate product of commutators in the group Θ_2. Intuitively speaking, this natural operation should express a procedure of Borromean extension in length, or simply the formation of a Borromean chain out of Borromean links of some appropriate length. Most remarkably, these two operations can be effectively combined by means of forming Borromean chains out of arbitrary Borromean stacks. This is the crucial idea that allows the algorithmic construction of a general link $\Sigma(N, K)$ in terms of Borromean building blocks, and effectively, the formation and growth of Borromean networks of arbitrary complexity based solely on the Borromean linking property at the core.

The simplest case involves the consideration of three symbols a, b, c in the group Θ_2 under the established topological semantics. Considering these three symbols we can construct a Borromean stack of order 3, expressed by the stacked commutator formula $[[a, b], c]$, and encoding the information of a $\Sigma(4, 3)$ link, as we explained previously. Considering these three symbols a, b, c in the group Θ_2 we can also construct out of them three distinct commutators, namely $[a, b]$, $[a, c]$, and $[b, c]$. Since each of these commutators gives a new symbol in the group Θ_2, we may take their product which is also a new symbol in the group Θ_2. We notice that each of these commutators $[a, b]$, $[a, c]$, $[b, c]$ gives separately a Borromean link. Thus, their product $[a, b] \circ [a, c] \circ [b, c]$ is actually a composition of three different

Borromean links, which gives a Borromean chain out of their composition product that has length 3.

Next, we realize that the formation of this Borromean chain is the appropriate algorithmic operation on Borromean building blocks to express a $\Sigma(4, 2)$ link. We can immediately see this as follows: First, we notice that deletion of any one of the symbols a, b, c, in the Borromean chain of length 3, $[a, b] \circ [a, c] \circ [b, c]$, reduces this chain to a standard Borromean link. For instance, if we delete the symbol a, what remains is the Borromean link $[b, c]$ and analogously for the other two cases. Second, we notice that deletion of any two of the symbols a, b, c reduces this chain to unity. Hence, we conclude that the Borromean chain of length 3 provides the formula for the fourth symbol in the group Θ_2, such that the defining properties of a $\Sigma(4, 2)$ link are satisfied, and most important, this link is expressed solely in terms of Borromean building blocks. A significant thing to notice is that the length of the Borromean chain solving the problem in the $\Sigma(4, 2)$ case is given by the number of combinations of two symbols out of three, where a combination is simply the formation of the commutator of two symbols in this case.

In the same manner that we can form Borromean chains out of Borromean links, we can form Borromean chains out of Borromean stacks, thus combining effectively these two algorithmic operations on Borromean links. If Borromean stacking is conceived as a process of Borromean complexity extension in depth by nesting into consecutive ordered layers, then Borromean chain formation is conceived as a process of Borromean complexity extension in length. Using these two operations separately or in combination, which actually involves the formation of Borromean chains out of Borromean stacks, we are able to express an arbitrarily complex link $\Sigma(N, K)$ solely in terms of Borromean building blocks.

Before we investigate the general case, it is instructive to describe the formation of a $\Sigma(5, 3)$ link, which exemplifies the above rules of Borromean complexity growth. From the definition of a $\Sigma(5, 3)$ link, the crucial observation is that if we remove any of the constituent topological circles what remains is a $\Sigma(4, 3)$ link, which we already know that is expressed by means of a Borromean stack of order 3, or equivalently, by the stacked commutator formula $[[a, b], c]$ in three symbols. Thus, in order to express a $\Sigma(5, 3)$ link, if we consider four symbols a, b, c, d, we look for a formula such that deletion of any of them causes the formula to reduce to the one of a Borromean stack of order 3. This is accomplished by the algorithmic operation of forming a Borromean chain of appropriate length out of Borromean stacks. In the present case of a $\Sigma(5, 3)$ link, since we require that deletion of any of the four involved symbols a, b, c, d reduces the formula to a Borromean stack of order 3, we just need to form a Borromean chain out of Borromean stacks of order 3, where the length of the chain is given by the number of combinations of three symbols (which is the number of symbols involved in a Borromean stack of order 3) out of four symbols a, b, c, d. We immediately deduce that the sought-after formula expressing a $\Sigma(5, 3)$ link is given by the Borromean chain of length 4, composed out of Borromean stacks of order 3, and described explicitly as follows:

$$\chi = [[a, b], c] \circ [[a, b], d] \circ [[a, c], d] \circ [[b, c], d].$$

Finally, we consider an arbitrarily complex link of the general form $\Sigma(N, K)$, where $1 \leq K \leq N$, and prove that it can be constructed solely in terms of Borromean building blocks within the group Θ_2. For any K, we already know that the link $\Sigma(K + 1, K)$ is expressed by means of a Borromean stack of order K. Next, we consider $(K + 1)$ symbols in Θ_2, and we wish to construct a $\Sigma(K + 2, K)$ link. The crucial observation is that if we remove any topological circle from a $\Sigma(K + 2, K)$ link, what remains is a $\Sigma(K + 1, K)$ link. Thus, we treat this case in complete analogy to the case of a $\Sigma(5, 3)$ link, discussed previously. More precisely, we form a Borromean chain out of Borromean stacks of order K, where the length of this chain is given by the number of combinations of K symbols out of $(K + 1)$ symbols. The formula expressing this Borromean chain provides the sought-after $(K + 2)$ symbol. Now, we consider $(K + 2)$ symbols, and we wish to construct a $\Sigma(K+3, K)$ link. We just have to form a Borromean chain out of Borromean stacks of order K, where the length of this chain is given by the number of combinations of K symbols out of $(K+2)$ symbols. The formula expressing this new Borromean chain provides the sought after $(K + 3)$ symbol in Θ_2. We continue the same process of formation of new Borromean chains of appropriate combinatorial length composed by Borromean stacks of order K, stage by stage, until we reach N. This completes the proof of the proposition that an arbitrarily complex link of the general form $\Sigma(N, K)$ can be constructed solely in terms of Borromean building blocks, or equivalently, Borromean connectivity topological units. We may summarize these findings in the form of the following theorem:

An arbitrarily complex link of the general form $\Sigma(N, K)$*, where* $1 \leq K \leq N$*, can be constructed solely in terms of Borromean building blocks, by means of forming Borromean stacks and Borromean chains out of Borromean stacks of appropriate order and length, respectively.*

In view of the "ER = EPR" conjecture, and having shown that the Borromean link serves as a model of both the tripartite GHZ-type of entanglement and the corresponding non-classical Einstein-Rosen bridge, the above theorem provides the strongest evidence for the correctness of this conjecture in generalized form.

7.3.6 Criterion of Locality in the Quantum Domain

The previous considerations involving the role of the Borromean link with respect to the validity of the "ER = EPR" conjecture require a re-thinking of the foundations of quantum mechanics, especially in relation to what functions as a *criterion of locality in the quantum domain*. This is ultimately connected with the conception of entanglement as an interchangeable resource by means of local unitary transformations. The notion of local is usually employed heuristically or by appealing to the classical intuition where the notion of a system can be unambiguously defined as separated by the rest of the world. The phenomenon of quantum entanglement is then conceived as a kind of "glue" between a priori distinguishable subsystems referring to the indirect specification of their compatible observables without any violation of the standard spacetime locality. Given our conception of measurement as a process that entangles the system with the apparatus, the freedom of moving the Heisenberg cut stands as a warning that these rigid distinctions implicated by classical intuitions are not well-defined and may lead to inconsistencies when they are employed to support locality arguments in the quantum domain.

In this way, we may adopt an alternative perspective which views *quantum entanglement through the lenses of indistinguishability* between subsystems or between a system and an apparatus. This perspective is grounded on the role of unitary transformations and induces a criterion of locality that is independent of classical intuitions, based instead on the distinction between the non-commutativity of the global algebra of quantum observables and the commutativity of any subalgebra of simultaneously measurable observables. Reciprocally, since the non-commutativity can be traced back to the non-commutative realization of the Borromean link in terms of one-parameter groups of unitary transformations, the seeds of locality should be sought for in the specification and role of a one-parameter unitary group. We note that a concrete unitary group of this type is provided by a unitary representation of $\mathbb{R}$ on the Hilbert space $\mathcal{H}$ of state vectors, under the qualification of the translations in the real line as "time" translations or "space" translations in a specified direction. Stone's theorem (Stone 1932) gives a bijective correspondence between continuous one-parameter groups of unitary operators and observables, and in this manner, *complete sets of simultaneously measurable observables forming a commutative subalgebra provide a viable criterion of locality in the quantum domain*.

Intuitively, the criterion of locality is associated with what can be *spectrally distinguished, and thus localized*, by means of the orthogonal projections belonging into the simultaneous resolution of all observables forming this commutative subalgebra. Technically, the orthogonal idempotent elements (orthogonal projections) of this commutative subalgebra of observables constitute a local Boolean frame. Each local Boolean frame has the structure of a complete Boolean algebra of orthogonal projection operators obtained by the simultaneous spectral resolution of a complete set of compatible observables—represented as self-adjoint operators—with respect to a complete orthonormal basis of eigenstates. We stress that all possible observables cannot be simultaneously measurable with respect to a single universal global logical Boolean frame as is the case in all classical theories of physics. Thus, there exists a multiplicity of potential local Boolean frames, where each one of them stands for a context of co-measurable observables. In this way, each local Boolean frame provides spectrally the localization means for the probabilistic evaluation of all the observables belonging into the associated commutative algebra. Thus, the evaluation of every single observed event in the quantum domain requires taking explicitly into account the specific local Boolean frame with respect to which the corresponding observable is localized.

The remarkable fact is that each observable instantiates a Boolean algebra of orthogonal projection operators, which is utilized for the expression of a state vector as a linear superposition with respect to the associated complete orthonormal basis of eigenstates of this observable. In this way, a Boolean frame functions as a means of inducing spectral differentiations in an initially objectively indistinguishable state in terms of the resolution into orthogonal projection operators. In other words, orthogonal projections induce potential differentiations in a quantum state, which are realized only if a measurement is actually performed. Thus, observables through their spectral resolution in terms of orthogonal projectors can be thought of as potential distinguishability filters acting on a quantum state. In this way, a measurement process creates information by actualizing differentiations with reference to the associated filters, or else refines the grain of resolution associated to a quantum state.

Conclusively, the spectral resolution of an observable gives rise to a logical structural invariant characterizing a whole algebra of observables commuting with the considered one. *This logical invariant bears the structure of a complete Boolean algebra of orthogonal projection operators whose spectrum defines the means of localization in the quantum domain*. We note that the same event may be associated with more than one applicable local Boolean frames, for instance, if the same projection belongs to two different overlapping Boolean frames, or if projections in two different local Boolean frames admit a common spectral refinement (i.e., they are compatible under pulling back) in a third local Boolean frame. In this case, we should consider all pertinent local Boolean frames at once, together with their interconnecting transformations, and thus form a Boolean localization system that supplies the covariance property under homomorphisms among these local Boolean frames. From a topological viewpoint, a Boolean localization system is characterized as a sheaf-theoretic structure (Zafiris 2004a,b, 2006a,b, 2007; Epperson and Zafiris 2013).

We come now to examine the implication of Stone's theorem referring to the bijective correspondence between observables and one-parameter groups of unitary transformations, where the parameter is considered to by varying continuously on the real line. First, we note that the proof of the theorem is based on the spectral resolution of a self-adjoint operator in terms of orthogonal projections. Given the localization function of this resolution, qualified in terms of a Boolean frame, Stone's theorem characterizes group-theoretically the preservation condition of the spectral distinctions induced by this local Boolean frame. More precisely, a unitary transformation is an automorphism of the Hilbert space of state vectors preserving the inner product structure, and thus realized by means of a unitary operator. The inner product between two state vectors, interpreted as the transition amplitude from one to the other, if viewed from the perspective of the space of rays can be thought of as the degree of overlap between the corresponding rays or projection operators. Given that a projection operator functions as a distinguishability filter, the overlap provides the degree of indistinguishability between the associated states. Thus, a unitary transformation is simply a transformation which *preserves the degree of indistinguishability between states*. In this way, the real-valued continuously varying parameter in a one-parameter group of unitary operators, associated bijectively with an observable via its resolving local Boolean frame and infinitesimally generated by it, is simply a parameter indexing continuously the preservation of the degree of spectral indistinguishability between quantum states. This is the crucial aspect that distinguishes the temporal or spatial meaning of such a parameter in comparison to the classical semantics of these terms (Zafiris 2016a).

We are going to show in the sequel how the analysis of entanglement, i.e. EPR-type of correlations, fits well with the criterion of locality in the quantum domain, instantiated by means of local Boolean frames. Keeping in mind the perspective of indistinguishability with reference to the quantum states themselves, without invoking a priori artificial distinctions among subsystems, but being able to localize using Boolean frames and preserve the induced spectral distinctions using the corresponding one-parameter unitary groups, provides a viable understanding of the thesis that entanglement is an interchangeable resource.

7.3.7 Quantum Locality and Entanglement: EPR Correlations

In the Hilbert space formulation of quantum theory, and expressed in the usual terminology, *entanglement* refers to the phenomenon displayed by a *composite* quantum system according to which the behavior and properties of a composite quantum system consisting of several subsystems are *not reducible* to the properties of the individual subsystems. This is manifested in the particular type of *correlations* found in the *joint probability distributions* of events in which different of these subsystems are involved. More precisely, in the Hilbert space formulation, a single quantum system is represented by a complex Hilbert space of states, whereas a composite quantum system is represented by the *tensor product* of the Hilbert spaces of states of its subsystems over the complex numbers.

Let us briefly recall that a *state* of a quantum system is represented by a positive, Hermitian (self-adjoint) operator of trace 1 called a *density operator* (or statistical operator). The density operator is represented as a linear sum of orthogonal, one-dimensional projections operators:

$$\rho = \Sigma_i \lambda_i P_i$$

where the weight coefficients λ_i are positive numbers summing up to unity, that is $\Sigma_i \lambda_i = 1$. In case that in the above linear sum only a single coefficient λ is different from zero, the state is called a *pure state*, that is $\rho = \lambda P$. Equivalently, a pure state can be characterized by a unit state vector Ψ belonging to the one-dimensional subspace of the Hilbert space (*ray*) into which P projects. Thus, a non-pure state, called a *mixed state*, is considered as a weighted linear sum (called a convex combination) of pure states (or one-dimensional projection operators) where the total weight is one. Moreover, a mixed state can be decomposed in many different ways in the form:

$$\rho = \Sigma_i \gamma_i \rho_i$$

where the coefficients γ_i are positive weights and the ρ_i are other density operators (which may correspond to pure states).

The particular form of the density operator expressed by $\rho = \Sigma_i \lambda_i P_i$ is called the *spectral decomposition* of the density operator in terms of *one-dimensional projectors*. In the Hilbert space formulation, each Hermitian operator has associated with it a *complete Boolean algebra*, identified as a Boolean algebra of projection operators belonging to its spectral decomposition. Hence, given a complete set of observables of a quantum system, there always exists a complete Boolean algebra of projection operators, viz. a Boolean subalgebra of the global non-Boolean event algebra of a quantum system resolving spectrally all these observables, if and only if these observables are *simultaneously measurable*, and thus form a commutative subalgebra. This encapsulates the criterion of locality in the quantum domain expressed in terms of local Boolean frames that are observable-induced. In a nutshell, an observational or measurement procedure induces a local Boolean frame of orthogonal projectors, where each one of these projectors corresponds to one of the possible results of the measurement. The probability for the result ε in state ρ is given by:

$$\mu_\varepsilon = Tr(\rho P_\varepsilon)$$

where the numbers (probabilities) μ_ε are obtained by the *trace* of the operator in the parenthesis, whereas their positivity is induced by the positivity of ρ.

In view of this criterion of locality, something that is not emphasized in the standard expositions is that the form of decomposition of a density operator in terms of a *convex combination* of one-dimensional, orthogonal projection operators is *not unique* but depends on the local Boolean frame of projectors associated with some observational procedure. Thus, the weight coefficients:

$$\lambda_i = Tr(\rho P_i)$$

where P_i are one-dimensional projectors, should be interpreted as *conditional probabilities* with respect to the local Boolean frame that P_i belongs to. Notice that since P_i may belong to different overlapping local Boolean frames, the coefficients λ_i should be interpreted as conditional probabilities with respect to a Boolean localization system of compatible overlapping local Boolean frames. Thus, λ_i is the conditional probability of a whole equivalence class (Boolean germ), conditioned by the corresponding Boolean localization system (Zafiris 2006b). We conclude that the density operator of a quantum system provides a description of states in which all possible decompositions are in a well-defined sense *implicitly* present at once, albeit *potentially*.

In the Hilbert space of a composite quantum system consisting of two subsystems, $H = H_1 \otimes H_2$, there exists a special class of state vectors, called *product states*, which have the product form:

$$\Psi = \Psi^{[1]} \otimes \Psi^{[2]}$$

corresponding to the cartesian product of the state vectors of the two subsystems. But the tensor product Hilbert space $H = H_1 \otimes H_2$ contains additionally all *linear combinations* of such product states. More precisely, if we choose *orthonormal bases* $\Psi^{[1]}{}_a$ and $\Psi^{[2]}{}_b$ in H_1 and H_2 correspondingly, a general vector state of the tensor product Hilbert space $H = H_1 \otimes H_2$ is written in the following form:

$$\Psi = \sum_{ab} \eta_{ab} \Psi^{[1]}{}_a \otimes \Psi^{[2]}{}_b$$

where the state vector Ψ represents a pure state of the total system composed of subsystems [1] and [2]. We observe that in the general case a pure state of the total system is *not reduced to the product* of the state vectors of the two subsystems. Thus, it constitutes a correlated or *entangled state*, meaning that each subsystem does *not* possess a separable state within the composite system. Conceptually, this means that each one of the subsystems of the composite system does *not* have an individual, separable, and definite state independently of the state of the composite system, and most significantly, this is the case *irrespective* of the spatial distance between the subsystems.

The essential aspect of entanglement phenomena, besides the explication of a situation where the behavior of the whole is not reduced to the behavior of its parts, or else, that the whole is more than the sum of its parts, is that the parts do not assume an individuation or localization independently of the whole. Put differently, there exists a *mutually implicative bidirectional relation* between the parts and the whole, being reminiscent of a topological structure called a sheaf. To avoid a diversion into sheaf theory, it is enough to point out that the notion of a part (i.e., what is called in standard terminology a subsystem of a composite system) becomes definable only by means of localization of the whole, which is observable-induced in the quantum domain and expressed via local Boolean frames (criterion of locality).

After this brief comment, and keeping up with the usual terminology employing the notion of subsystems of a composite system, we point out the possibility of assigning a notion of *partial state* to each of the subsystems [1] and [2], although each one of them does *not* possess an individual, separable state, independently of the state of the composite system. This notion of partial state would encompass all the *statistical information* about H_1 that the density operator of the composite system ρ_{12} incorporates. In order to be able to define such a notion of partial state for each of the subsystems [1] and [2] it is necessary to consider *local actions* of each one of [1] and [2]. By a local action of subsystem [1], for example, we mean a measurement which can be performed by an observable of subsystem [1]. This is an operation which is represented by Hermitian operators of the form $A^{[1]} \otimes 1^{[2]}$, where $A^{[1]}$ is a Hermitian operator in H_1 corresponding to the chosen observable. Notice that this is equivalent to employing a *local Boolean frame* consisting of projection operators belonging to the spectral decomposition of the chosen observable. Similarly, a local action of subsystem [2] is represented by Hermitian operators of the form $1^{[1]} \otimes B^{[2]}$.

We stress the fact that none of the alleged subsystems [1] and [2] have access to all of the observables (Boolean frames) of the composite system. More concretely, the algebras of observables of the subsystems can be obtained by the operation of *restriction* or *localization* of the algebra of observables of the composite system to each one of them. Thus, the subsystems are able to see the state ρ_{12} of the composite system *only partially*. Then, we define the partial state of subsystem [1] as the *reduced density operator* $\rho_1 = Tr_{[2]}\rho_{12}$ obtained by *partial tracing* over subsystem [2] (and analogously for subsystem [2]) by the following requirements:

$$Tr\big(\rho_1 A^{[1]}\big) = Tr\big(\rho_{12}\big(A^{[1]} \otimes 1^{[2]}\big)\big)$$

$$Tr\big(\rho_2 B^{[2]}\big) = Tr\big(\rho_{12}\big(1^{[1]} \otimes B^{[2]}\big)\big)$$

Thus, for all observables $A^{[1]}$ of subsystem [1], and all observables $B^{[2]}$ of subsystem 2 the reduced density operators ρ_1 and ρ_2 correspondingly constitute simply restrictions or localizations of ρ_{12} to the respective subsystems. Reciprocally, giving priority to the criterion of locality in the quantum domain, these subsystems are precisely distinguished locally in terms of the corresponding Boolean frames resolving the above observables. We note that *many* different states ρ_{12} of the composite system may have the same restrictions on the algebras of observables of the two subsystems. Hence, from the point of view of subsystems [1] and [2] many different states of the composite system have *identical restrictions* to each one of them. According to the above requirements, the reduced density operator ρ_1, for example, reproduces the same statistical distribution for an event caused by a local action of subsystem [1] as ρ_{12} does, in the sense that we could either apply the operation $A^{[1]} \otimes 1^{[2]}$ on the composite system (thus leaving subsystem [2] unaffected) or apply the operation $A^{[1]}$ directly on subsystem [1]. Note, that the assumption of considering an event caused by a local action of subsystem [1] and leaving subsystem [2] unaffected respects the requirement of *Einstein locality* in spacetime if the two subsystems are sufficiently separated, meaning that the probability of an event at subsystem [1] is independent of subsystem [2] in a region which is *spacelike separated* with respect to this event.

However, it is important to realize that the reduced density operators ρ_1 and ρ_2 are *not* sufficient to determine the *probabilities of pairs* of correlated events between the two subsystems. These pairs of correlated events are implied by the entanglement of the states of the composite system if we consider *compatible local actions* of the subsystems, meaning measurements which can be performed by *compatible observables* of subsystems [1] and [2]. Equivalently, correlations between events of the subsystems can be observed with coincidence measurements performed between compatible local Boolean frames within some Boolean localization system of the composite system corresponding to these compatible observables. The condition of local Boolean frame compatibility between observables of the subsystems [1] and [2] means that, given the reduced density operators ρ_1 and ρ_2, they constitute *restrictions or localizations* of some pure state of the composite system only if their *eigenvalues* are identical with respect to these compatible Boolean frames. Equivalently, a vector state of the tensor product Hilbert space $H = H_1 \otimes H_2$ reflecting the condition of local Boolean frames compatibility is written in the following form:

$$\Psi = \sum_j \eta_j \Psi^{[1]}{}_j \otimes \Psi^{[2]}{}_j$$

where the state vector Ψ represents a pure state of the composite system, $\Psi^{[1]}{}_j$, $\Psi^{[2]}{}_j$ are orthonormal bases of the Hilbert spaces H_1 and H_2 of the subsystems [1] and [2] respectively, corresponding to the spectral decompositions of ρ_1 and ρ_2 with respect to compatible Boolean frames of [1] and [2] (or compatible local actions of the two subsystems) and η_j are the identical eigenvalues of [1] and [2] with respect to these bases.

In the physical state of affairs the entanglement-correlated pairs of events usually refer to some *conserved* physical quantity like charge, energy, momentum, or spin orientation of the composite system in relation to its subsystems (corresponding to some specified observable of the combined system) and persist *irrespective* of the metrical distance between the subsystems. It is important for the understanding of these entanglement correlations to emphasize the significance of the locality criterion in the quantum domain pertaining to the crucial role of compatibility between local Boolean frames (with respect to which events occur by measurement of corresponding observables) in Boolean localization systems. This is the case because entanglement correlations *cannot be reduced* to correlations between assumed pre-existing states assigned to the subsystems before the occurrence of events (with respect to their corresponding local Boolean frames).

In this manner, we realize that the criterion of locality in the quantum domain should be invoked explicitly in the analysis of quantum entanglement. More precisely, it is instructive to summarize the main points as follows:

1. The notion of a quantum subsystem becomes spectrally distinguishable, and thus localizable, only insofar a *complete Boolean frame* is designated corresponding to the measurement of some observable and followed by the registration of some observed event. In particular, the notion of a subsystem before the existence of some observed event should be thought of as a *potential locality*, which under the designation of some Boolean frame acquires the interpretation of a *probability function* (via its partial state description) for the evaluation of event-probabilities pertaining to the realization of this subsystem as a reference linkage among observed events referring to the corresponding observable;
2. The separation of a composite system into subsystems does not correspond to a partition of a system into subsystems with respect to their corresponding density operators pertaining to their partial description. The only consistent description is via the *algebraic (sheaf-theoretic) operation of restriction or localization of the algebra of observables* of the composite system into appropriate subalgebras of observables corresponding to potential localities (subsystems) which can be

realized only after the designation of local Boolean frames. Intuitively, these subalgebras contain only observables which are "visible" by the so designated subsystems, distinguished in this way only after the appearance of concrete events. Furthermore, the observable-induced localized spectral distinguishability of subsystems within a total system, for example of the subsystems [1] and [2] according to the preceding, is effectuated by considering observables of the form $A^{[1]} \otimes 1^{[2]}$ and $1^{[1]} \otimes B^{[2]}$ within the algebra of observables of the total system;

3. The observable-induced localized spectral distinguishability of subsystems within a total system allows an understanding of entanglement correlations between the subsystems under the condition of *compatibility* between their corresponding local Boolean frames within a Boolean localization system of a total system. The condition of compatibility means that given the reduced density operators ρ_1 and ρ_2 in the case of two localized subsystems, they constitute restrictions of some pure state of the composite system only if their eigenvalues are identical with respect to these compatible Boolean frames.

   Reflecting on the above, we conclude that the notion of entanglement or non-separability pertaining to the description of a composite quantum system with reference to its localized parts and conversely requires to take seriously into account the *intrinsic relativity* of this notion with respect to the depiction of certain *compatible local Boolean frames distinguishing the subsystems* and corresponding to compatible observables.

7.3.8 Generic Gravitational Properties and Forcing Conditions via the Borromean Modelling of the "ER = EPR" Conjecture at the Planck Scale

In Sects. 3.6 and 3.7 we scrutinized the EPR side of the generalized "ER = EPR" conjecture from the perspective of the formulated criterion of locality pertaining to the quantum domain. This is intrinsically associated with the function of local Boolean frames resolving spectrally complete sets of simultaneously measurable observables together with their compatibility relations. In this way, referring to a system via a measurement procedure requires a process of spectral localization of the global non-commutative algebra of observables with respect to a commutative subalgebra of co-measurable observables whose orthogonal projections constitute always a local Boolean frame. In turn, the corresponding by Stone's theorem one-parameter unitary group generated by such a local Boolean frame preserves the spectral distinctions induced by this local Boolean frame in the transition between states. Thus, a one-parameter unitary group carries the seed of Boolean frame spectral locality in the quantum domain providing the ground to qualify Susskind's thesis that entanglement is an interchangeable resource. From the inverse viewpoint, given this understanding of locality in the quantum domain, the hallmark of global non-commutativity can be traced to the non-commutative realization of the Borromean link in terms of one-parameter unitary groups, giving rise under a nilpotency condition to the Weyl-Heisenberg group.

It is precisely the role of the Borromean topological link in addressing the “ER = EPR” conjecture that invites for a study of the implications for quantum gravity in the ER side of this correspondence. According to Sects. 2.5 and 2.6 we consider the Lorentzian manifold $(\mathbb{R} \times \Sigma, \varepsilon^* g)$ as a representative of spacetime X, where the singularities are localized within the three-dimensional manifold Σ. We consider that Σ is a multiple-connected topological space, so that spacetime is geodesically incomplete. The crucial idea is that a collection of circular singular boundaries defining a closed and nowhere dense subset of an open set of S^3 gives rise to a topological link in S^3. Among all these links, the Borromean link plays a universal role, in the sense that we can construct the three-dimensional manifold Σ by the information incorporated in the Borromean link in S^3 representing the singular boundaries without any further assumptions. This is due to the fact that any compact oriented three-dimensional manifold Σ without boundary can be obtained as the branched covering space of S^3 with branch set the Borromean-linked boundaries. This proposition, together with the algebraic-topological result that the Borromean link $\mathcal{B}$ gives rise to a two-dimensional cohomology class in the dense complement of $\mathcal{B}$ in S^3, provides the strongest indication that it expresses a non-classical (Planck scale) Einstein-Rosen bridge. Clearly, we may extend the closed and nowhere dense subset of the Borromean-linked boundaries to four dimensions by considering a timelike axis perpendicular to the Borromean rings, which plays the role of a threefold symmetry axis. Concomitantly, any other type of non-classical Einstein-Rosen bridge can be constructed in terms of Borromean buildings blocks.

Given the universality of characterization of these Planck-scale Einstein-Rosen bridges in this setting, there appears the possibility that active gravitational mass/energy may emerge from these purely topological considerations taking into account the constraints imposed by Einstein’s field equations in the vacuum (Arnowitt et al. 1962). From a physical perspective, this may be interpreted in a novel way according to Wheeler’s insight referring to “mass without mass” (Misner and Wheeler 1957) as follows: *Localized configurations of topologically singular loci in open sets of X restricted to closed nowhere dense subsets and giving rise to topologically linked boundaries when restricted to Σ amount to active gravitational mass/energy in their complementary open dense subsets*. If we also employ the “positive mass theorem” in this setting, considered in the vacuum case, this gravitational mass/energy should be non-zero and strictly positive.

In turn, the above proposition implies that *in the quantum gravity regime a property can be characterized as gravitationally generic if it occurs and holds on a dense open set*. In this way, a gravitationally non-generic property should appear only on a closed nowhere dense subset. We think of a nowhere dense subset relative to an open set as the analogue of a set of measure zero in measure theory. Note that this is only an analogy to guide intuition since nowhere dense subsets relative to an open set can have non-vanishing Lebesgue measure in general. The notion of a generic property has its roots in mathematical logic and model theory. It has been introduced by Cohen in the context of generalized models of set theory using the technique of *forcing conditions defined over a partially ordered set* (Cohen 2008). We propose to use the same method to construct distinguishable extensions of the smooth spacetime manifold model of classical general relativity in the light of the qualification of the "ER = EPR" conjecture from the perspective of the Borromean topology. It is instructive to think of the notion of topological density in physical terms, viz. as an indicator of gravitational energy density caused by sources. In this context, the notion of genericity should be implemented by forcing conditions. More concretely, a condition forces a gravitational property if this property holds on a dense open set. A forcing condition forces every gravitational property either to hold or not in relation to the criterion of density, and thus, a forcing condition is generic in this sense.

The idea is to induce such forcing conditions in the bulk by considering local actions of observables on the boundary and using the "ER = EPR" correspondence in this generalized setting. More precisely, we already know that if we consider a maximally entangled pair of two parties, then a local action of an observable of any of them corresponding to an observational procedure of a complete set of commuting observables (and thus, incorporating the criterion of locality in the quantum domain) carried out by a third party leads to a GHZ-type of entanglement, which in turn corresponds to the Borromean linking property. Therefore, by applying the "ER = EPR" correspondence we can instantiate a Planck scale Einstein-Rosen bridge that links three circular singular boundaries and defining a closed and nowhere dense subset of an open set of S^3. This can be extended to the bulk, so that we obtain a closed and nowhere dense subset of an open set in the bulk bearing the property that its restriction to the boundary forms a Borromean link. Clearly the same procedure can be employed for higher order links given that all of them can be constructed in terms of Borromean building blocks. In this setting, the singular loci in the bulk form closed and nowhere dense subsets with respect to an open set in the bulk. Moreover, *local actions of observables can be partially ordered, which corresponds to an ordering of the formed link components*. The pertinent problem now is to construct *distinguishable extensions of the smooth model of the bulk entering the quantum gravity regime using the obtained partial order of forcing conditions*.

The rationale behind this approach, connected closely with the far-reaching implications of qualifying the "ER = EPR" conjecture from the viewpoint of Borromean topological networks, is the following: We know that Einstein's field equations in classical general relativity are expressed in terms of the sheaf of smooth functions defined over the spacetime manifold. Every distinguishable extension of the smooth spacetime model according to the proposed schema can be expressed in terms of a *new sheaf of coefficients incorporating the forcing conditions*. Thus, there exists the possibility that Einstein's equations may *retain their form* if formulated in terms of the new sheaf, such that the transition to the quantum gravity regime can be implemented via the "ER = EPR" correspondence in the proposed generalized setting only by means of incorporating the forcing conditions appropriately into a new sheaf of coefficients.

7.3.9 Generic Gravitational Algebras and the Transition to Quantum Gravity

It is physically reasonable to expect that an admissible algebra sheaf of coefficients in term of which an extension of the smooth spacetime manifold model may take place over singular loci, should be distribution-like (Zafiris 2016b). For example, we may think of a matter distribution confined to a submanifold of spacetime whose density is integrable over this submanifold. In the context of a linear field theory this should be naturally modelled in terms of a linear distribution. Unfortunately, this is not possible in the context of gravity, which is a non-linear theory. In other words, Schwarz's linear distributions are not suitable candidates for expressing the information of singular loci.

The unsuitability of linear distributions rests on the fact that the space $\mathbb{D}'$ they form is only a linear space, but it is not an algebra. This is characterized as the "Schwarz Impossibility," and may be formulated as follows: There is no symmetric bilinear morphism:

$$\circ : \mathbb{D}'(V) \times \mathbb{D}'(V) \ni (S, T) \to S \circ T \in \mathbb{D}'(V)$$

so that $S \circ T$ is the usual point-wise product of continuous functions, when $S, T \in \mathbb{C}^0(V)$. Equivalently, $\mathbb{D}'(V)$ is not closed under any multiplication that extends the usual multiplication of continuous functions, where V is an open subset X. Since all the involved arguments are of a local character, without loss of generality, we may simply consider V as an open subset of $\mathbb{R}^4$.

A physically natural way to bypass "Schwarz Impossibility" is to assume the existence of an embedding morphism $\mathbb{D}'(V) \hookrightarrow \mathbb{A}(V)$, which embeds the vector space of distributions $\mathbb{D}'(V)$ as a vector subspace in $\mathbb{A}(V)$, where $\mathbb{A}(V)$ is the quotient algebra:

$$\mathbb{A}(V) = \mathbb{K}(V)/\mathbb{I},$$

and $\mathbb{K}(V)$ is a subalgebra in $\mathbb{C}^\infty(V)^\Lambda$, for some index set Λ, whereas $\mathbb{I}$ is an ideal in $\mathbb{K}(V)$. This approach was proposed by Rosinger (1990) in order to express generalized solutions of non-linear partial differential equations. In our context, the crucial idea is that we can define a *partial order on this indexing set, which is identified as a partial order of forcing conditions* according to the analysis of the previous section. We may describe the generation of these algebras locally as follows:

Let $V \subseteq \mathbb{R}^4$ be an open set, and $L = (\Lambda, \leq)$ be the right directed partial order on the index set Λ generated by forcing conditions in our context of qualifying the "ER = EPR" correspondence in this generalized setting. That is, for all λ, $\mu \in \Lambda$, there exists $\nu \in \Lambda$ such that $\lambda, \mu \leq \nu$. With respect to the usual componentwise operations, $\mathbb{C}^\infty(V)^\Lambda$ is a unital and commutative algebra over the reals. We define the following ideal $\mathbb{I}_L$ in $\mathbb{C}^\infty(V)^\Lambda$ whose physical meaning will be described in the sequel:

$$\mathbb{I}_L(V) = \left\{ \phi = (\phi_\lambda)_{\lambda\in\Lambda} \,\middle|\, \begin{array}{l} \exists\ \Gamma \subset V \text{ closed nowhere dense:} \\ \forall\ x \in [V \setminus \Gamma] \text{ being dense:} \\ \exists\ \lambda \in \Lambda\text{:} \\ \forall\ \mu \in \Lambda,\ \mu \geq \lambda\text{:} \\ \quad \phi_\mu(x) = 0,\ \partial^p \phi_\mu(x) = 0 \end{array} \right\}$$

In the above definition, we think of Γ collectively as a singular locus in $\mathbb{R}^4$, characterized as a closed and nowhere dense subset relative to the open set $V \subseteq \mathbb{R}^4$, such that its complement $V \setminus \Gamma$ in V is dense. The unital and commutative algebra $\mathbb{C}^\infty(V)^\Lambda$ contains smooth functions ϕ_λ indexed by the set Λ and defined over V, to be thought of as diagrams or sequences of Λ-indexed smooth functions. The requirement of the right directed partial order on the specified index set Λ, which is denoted by $L = (\Lambda, \leq)$, is necessary in order that the above set forms actually an ideal in $\mathbb{C}^\infty(V)^\Lambda$. Now, the ideal $\mathbb{I}_L(V)$ in $\mathbb{C}^\infty(V)^\Lambda$ includes all these sequences of smooth functions ϕ_λ that vanish asymptotically outside the singular locus Γ together with all their partial derivatives. Therefore, intuitively speaking, the ideal of the form $\mathbb{I}_L(V)$ incorporates all these sequences of smooth functions indexed by Λ whose support covers the singular locus Γ, whereas they vanish outside it. In this manner, the information of the singular locus Γ is encoded in the ideal $\mathbb{I}_L(V)$ in $\mathbb{C}^\infty(V)^\Lambda$. Hence, the quotient commutative algebra $\mathbb{A}_L(V) = \mathbb{C}^\infty(V)^\Lambda / \mathbb{I}_L(V)$ is an algebra of residues of sequences of smooth functions modulo the singular information ideal $\mathbb{I}_L(V)$.

A natural question in the above context refers to the requirement that the complement $V \setminus \Gamma$ of the singular locus Γ in V should be dense. The necessity of this requirement can be understood by the fact that we wish to obtain an embedding ι of the algebra of smooth functions $\mathbb{C}^\infty(V)$ into the algebra of generalized functions $\mathbb{A}_L(V)$:

$$\iota : \mathbb{C}^\infty(V) \hookrightarrow \mathbb{A}_L(V) = \frac{\mathbb{C}^\infty(V)^\Lambda}{\mathbb{I}_L(V)}$$

such that:

$$\varphi \hookrightarrow \iota(\varphi) = \Delta(\varphi) + [\mathbb{I}_L(V)]$$

where $\Delta_\Lambda |_V$: $\mathbb{C}^\infty(V) \to \mathbb{C}^\infty(V)^\Lambda$ is the diagonal morphism with respect to Λ, defined for an open set V as follows:

$$\Delta_\Lambda(\varphi) |_V = \left\{ \Delta(\varphi) = (\varphi_\lambda)_{\lambda\in\Lambda} \mid \varphi_\lambda = \varphi,\ \forall\lambda \in \Lambda, \varphi \in \mathbb{C}^\infty(V) \right\}.$$

Hence, for every smooth function φ in $\mathbb{C}^{\infty}(V)$, the diagonal image $\Delta(\varphi)$ of φ in $\mathbb{C}^{\infty}(V)^{\Lambda}$ is a sequence of smooth functions all identical to φ, indexed by Λ. The embedding ι is feasible according to the above, if and only if the ideal $\mathbb{I}_L(V)$ satisfies the off diagonality condition:

$$\mathbb{I}_L(V) \cap \Delta_\Lambda |_V = \{0\}.$$

Therefore, it remains to show that if the complement $V \setminus \Gamma$ of the singular locus Γ in V is dense, then the ideal $\mathbb{I}_L(V)$ actually satisfies the above off diagonality condition. So we suppose that $V \setminus \Gamma$ is dense in V, and consider a smooth function χ in $\mathbb{C}^{\infty}(V)$. If $\Delta_\Lambda(\chi) |_V := \Delta(\chi)$ belongs to the ideal $\mathbb{I}_L(V)$, then the asymptotic vanishing condition implies that $\chi = 0$ in $V \setminus \Gamma$, and therefore, we must have $\chi = 0$ in V because $V \setminus \Gamma$ is dense in V by hypothesis. Thus, it follows that the ideal $\mathbb{I}_L(V)$ satisfies the off diagonality condition, as required.

Conclusively, there exists a canonical injective homomorphism of commutative algebras, or equivalently, an embedding ι of the algebra of smooth functions $\mathbb{C}^{\infty}(V)$ into the algebra of generalized functions $\mathbb{A}_L(V)$:

$$\iota : \mathbb{C}^{\infty}(V) \hookrightarrow \mathbb{A}_L(V) = \frac{\mathbb{C}^{\infty}(V)^{\Lambda}}{\mathbb{I}_L(V)}$$

Furthermore, it follows immediately that the partial differential operators:

$$\partial^p : \mathbb{C}^{\infty}(V)^{\Lambda} \ni \phi = (\phi_\lambda) \mapsto \partial^p \phi = \left(\partial^p \phi_\lambda\right) \in \mathbb{C}^{\infty}(V)^{\Lambda}$$

satisfy the inclusion:

$$\partial^p \left(\mathbb{I}_L(V)\right) \subseteq \mathbb{I}_L(V).$$

Thus, the standard partial derivative operators on $\mathbb{C}^{\infty}(V)$ extend to $\mathbb{A}_L(V)$:

$$\partial^p : \mathbb{A}_L(V) \ni [\phi + \mathbb{I}_L(V)] \mapsto [\partial^p \phi + \mathbb{I}_L(V)] \in \mathbb{A}_L(V),$$

We conclude that the above embedding of commutative algebras extends to an embedding of differential algebras. Therefore, the following diagram commutes:

$$\begin{tikzcd}
\mathbb{C}^{\infty}(V) \arrow[r, "\partial^p"] \arrow[d] & \mathbb{C}^{\infty}(V) \arrow[d] \\
\mathbb{A}_L(V) \arrow[r, "\partial^p"] & \mathbb{A}_L(V)
\end{tikzcd}$$

We emphasize that the above embedding preserves not only the algebraic structure of $\mathbb{C}^{\infty}(V)$, but also its differential structure. The off diagonality condition implies also the existence of an injective, linear morphism:

$$\mathbb{D}'(V) \hookrightarrow \mathbb{A}_L(V).$$

Therefore, the differential algebra $\mathbb{A}_L(V)$ contains the space of distributions as a linear subspace, where those algebras that admit linear embeddings of distributions are characterized in terms of such off diagonality conditions.

Finally, it is important to note that a subset of a topological space is closed and nowhere dense if and only if it satisfies this condition locally. This is the key idea used to prove that the algebras of generalized functions $\mathbb{A}_L(V)$ form actually sheaves of commutative algebras, which additionally are soft and flasque or flabby (Mallios and Rosinger 1999, 2001). The softness property of the sheaves of the form $\mathbb{A}_L$ means that any section over any closed subset can be extended to a global section. Thus, these types of sheaves characterize cohomologically the topological property of paracompactness by means of acyclicity. Equivalently, soft sheaves are acyclic over a paracompact topological space. Moreover, sheaves of the form $\mathbb{A}_L$ are not only soft, but they are flasque or flabby as well, which is a local property (Grothendieck 1957, 1958). This means that the restriction morphism of sections in the sheaf definition is an epimorphism. Hence, in this case, we can always extend any local section by zero to obtain a global section of $\mathbb{A}_L$.

We may recapitulate by pointing out that the first basic idea involved in the construction of these distribution-like algebra sheaves of coefficients $\mathbb{A}_L$ is to model a locus of singularities Γ in $\mathbb{R}^4$ as a closed and nowhere dense subset relative to an open set $V \subseteq \mathbb{R}^4$, such that its complement $V \setminus \Gamma$ in V is dense. The second basic idea is to express such a closed and nowhere dense locus as an ideal in an algebra sheaf constructed as an extension of the smooth one over a partially ordered set. This stands for a partially ordered set of forcing conditions obtained by means of qualifying the "ER = EPR" correspondence via the Borromean topological link at the Planck scale. In this manner, the ideal expressing algebraically a locus of singularities contains diagrams of locally defined smooth functions indexed by Λ whose support covers the singular locus Γ, whereas they vanish outside it. Then, it can be shown that the quotient commutative algebra sheaf $\mathbb{A}_L(V) = \mathbb{C}^{\infty}(V)^{\Lambda} / \mathbb{I}_L(V)$ is a soft and flasque algebra sheaf of residues of diagrams of smooth functions modulo the closed nowhere dense singular ideal $\mathbb{I}_L(V)$.

The possibility of obtaining distinguishable extensions of the smooth spacetime model in the transition to the quantum gravity regime, in terms of generalized algebra sheaves of coefficients, is based on the realization that the validity of the de Rham complex, in its sheaf-theoretic guise, *is not restricted exclusively to coordinatizing the tensorial physical quantities by smooth coefficients* $\mathbb{C}^{\infty}$, as it is actually the case when the $\mathbb{R}$-spectrum of the coefficients is a smooth manifold. This is a very important fact that has been established recently in the context of the theory of abstract differential geometry, which generalizes the differential-geometric framework of smooth manifolds using exclusively sheaf-theoretic means

(Mallios 1998a,b; Mallios and Zafiris 2016). The physical significance of this development is that we may construct appropriate distribution-like sheaves of coefficients satisfying the validity of the de Rham complex, and therefore, formulate Einstein's field equations in terms of these distribution-like coefficients instead of the smooth ones, generalizing in this way an old idea of Geroch (1972). More precisely, this is the case if the following sequence of $\mathbb{R}$-linear sheaf morphisms:

$$\mathbb{A} \to \Omega^1(\mathbb{A}) \to \ldots \to \Omega^n(\mathbb{A}) \to \ldots \tag{7.1}$$

is a complex of $\mathbb{R}$-vector space sheaves, identified as the sheaf-theoretic de Rham complex of $\mathbb{A}$.

In this case, if the cohomological condition expressing the Poincaré Lemma, $Ker(d^0) = \mathbb{R}$ is satisfied with respect to the algebra sheaf $\mathbb{A}$, and requiring that $\mathbb{A}$ is a soft algebra sheaf, viz. any section over any closed subset of X can be extended to a global section, we obtain that the sequence:

$$\mathbf{0} \to \mathbb{R} \to \mathbb{A} \to \Omega^1(\mathbb{A}) \to \ldots \to \Omega^n(\mathbb{A}) \to \ldots \tag{7.2}$$

is an exact sequence of $\mathbb{R}$-vector space sheaves. Thus, the sheaf-theoretic de Rham complex of the algebra sheaf $\mathbb{A}$ constitutes an acyclic resolution of the constant sheaf $\mathbb{R}$.

For instance, referring to the classical differential geometry of smooth manifolds, the de Rham complex, expressed in terms of local smooth coefficients and their differential forms of higher orders, provides such an acyclic resolution of the constant sheaf $\mathbb{R}$. What has been uncovered by the framework of abstract differential geometry is that the smooth algebra sheaf $\mathbb{C}^\infty(X)$ is not unique in this respect. More concretely, *any other soft algebra sheaf* $\mathbb{A}$ *constituting an acyclic resolution of the constant sheaf* $\mathbb{R}$ is a viable source of coefficients for the coordinatization of the tensors, maintaining at the same time all their covariance properties in terms of the new local coefficients.

It can be shown without difficulty that the distribution-like soft algebra sheaves of the form $\mathbb{A}_L$ actually constitute an acyclic resolution of the constant sheaf of the reals. Thus, we conclude that the de Rham complex can be rigorously expressed in terms of these coefficients instead of the smooth ones in the transition to the quantum gravity regime, and consequently Einstein's equations can be formulated with respect to coefficients from the algebra sheaf $\mathbb{A}_L$ instead of the smooth ones from $\mathbb{C}^\infty$. Consequently, the validity of Einstein's equations can be extended over loci of singularities in a covariant manner by utilizing coefficients from the sheaf $\mathbb{A}_L$ for expressing all involved differential geometric tensorial quantities (Zafiris 2016b).

Finally, using the criterion of gravitational genericity we will attempt to explain the structure of the algebras $\mathbb{A}_L$. In particular, is their constitution based on the notion of *gravitationally generic properties* formulated in the previous section? We focus our attention on the fact that the definition of these algebras is based on the extension of the algebra of smooth functions with respect to a partially ordered set L. The latter stands for a partially ordered set of forcing conditions in the sense of the model-theoretic method of forcing introduced by Cohen in mathematical logic (Cohen 2008). Thus, the setup of these algebras involves the extension of $\mathbb{C}^{\infty}$ to $\mathbb{C}^{\infty\Lambda}$, where Λ is the indexing set of the right directed partial order $L = (\Lambda, \leq)$. We also remind that this partial order is necessary in order that the set $\mathbb{I}_L(V)$ is qualified as an ideal in $\mathbb{A}_L$. Since the off-diagonal ideal $\mathbb{I}_L(V)$ subsumes algebraically the information of some singular locus Γ, characterized as a closed and nowhere dense subset relative to an open set V, and thus as a bearer of a gravitationally non-generic property, the quotient algebra of the form $\mathbb{A}_L(V) = \mathbb{C}^{\infty}(V)^{\Lambda}/\mathbb{I}_L(V)$ incorporates only properties defined on dense open sets. Hence, according to our definition, $\mathbb{A}_L(V)$ incorporates gravitationally generic properties and for this reason they should be called *generic gravitational algebras in the transition from classical gravity to quantum gravity*. This is legitimate since the partial order $L = (\Lambda, \leq)$ is actually a partial order of generic forcing conditions induced at the quantum level by local actions of observables in the Borromean topological schema of addressing the validity of the "ER = EPR" correspondence at the Planck scale.

We note that the initial conception of the general method of forcing has been formulated in the context of models of set theory. It demonstrates that if we start from a standard model of set theory, we can construct distinguishable extensions of this model by means of a generic set of forcing conditions, such that a proposition about a property is true in the generic extension, if and only if it is forced by some forcing condition in the generic set. Note that the generic set of forcing conditions should not be contained in the initial standard model we started with.

What we propose is that the method of forcing can be applied equally well to construct distinguishable extensions of the classical smooth model of spacetime in the transition to the quantum gravity regime according to the introduced criterion of gravitational genericity and the Borromean topological qualification of the "ER = EPR" correspondence. *The key idea is to use the generalized form of the "ER = EPR" conjecture involving the modelling of non-classical Einstein-Rosen bridges in terms of Borromean topological links and induce the sought for partial order of forcing conditions from ordering local actions of observables at the quantum level*, as it has been explained in detail in Sect. 3.8. Most important, these extensions of the classical smooth spacetime model are characterized by the property that the form of Einstein's equations in vacuum remains *invariant*, such that the transition to quantum gravity, via the Planck scale Borromean linking properties underlying the "ER = EPR" correspondence, can be implemented by incorporating a partially ordered set of forcing conditions induced at the quantum level into a generic gravitational algebra sheaf.

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