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Vladimir A. Smirnov

Analytic Tools for Feynman Integrals

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Preface

The goal of this book is to describe, from my point of view, the most powerful methods for evaluating multiloop Feynman integrals that are currently used in practice. This book supersedes my previous Springer book 'Evaluating Feynman Integrals' and its textbook version 'Feynman Integral Calculus'. After the publication of these two books, powerful new methods have appeared and existing methods have been improved in essential ways. One more qualitative change is that most of the methods and the corresponding algorithms have been implemented in computer codes which are often public. In such situations, I prefer to describe these algorithms, rather than to provide 'hand-made' solutions, as I did in my two previous books. However, I do not explain how to use the corresponding codes and just refer to Internet pages and papers where tutorials and examples can be found.

In comparison to my two previous books, three new chapters have been added: One chapter is on sector decomposition, while a second describes a new method by Lee. The third new chapter concerns the asymptotic expansions of Feynman integrals in momenta and masses, which were described in detail in my other Springer book 'Applied Asymptotic Expansions in Momenta and Masses'. In this chapter, I first present a short summary of existing strategies for obtaining an expansion of a given Feynman integral in a particular limit. Then I describe, following papers that appeared after the publication of this book, how one can reveal algorithmically the regions relevant to a given limit within the strategy of expansion by regions. The chapter on Baikov's method has been reduced, in the present book, to a section in the chapter on integration by parts. The chapters on the method of Mellin-Barnes representation and on the method of integration by parts are written in a new way, with an emphasis on the corresponding algorithms and computer codes. The chapter on the method of differential equations has a new section and a new conclusion.

Although all the necessary definitions concerning Feynman integrals are provided in the book, it would be helpful for the reader to know the basics of perturbative quantum field theory, e.g., by following the first few chapters of the well-known textbooks by Bogoliubov and Shirkov and/or Peskin and Schroeder.

I would like to express my deepest gratitude to many people. In particular, I would like to thank Michal Czakon, Lance Dixon, Claude Duhr, Bernd Jantzen, Michail Kalmykov, Roman Lee, Pierpaolo Mastrolia, Francesco Moriello, and Alexander Smirnov for all kinds of valuable assistance. I would also like to acknowledge my co-authors Pavel Baikov, Stefan Bekavac, Martin Beneke, Zvi Bern, Konstantin Chetyrkin, Michal Czakon, Andrzej Czarnecki, Andrei Davydychev, Vittorio Del Duca, Lance Dixon, James Drummond, Claude Duhr, Burkhard Eden, Nigel Glover, Andrei Grozin, Gudrun Heinrich, Johannes Henn, Paul Heslop, Tobias Huber, Bernd Jantzen, Bernd Kniehl, Gregory Korchemsky, David Kosower, Johann Kühn, Roman Lee, Sven Moch, Alexander Penin, Dirk Seidel, Adrian Signer, Alexander Smirnov, Emery Sokatchev, Matthias Steinhauser, Michail Tentyukov, and Oleg Veretin for fruitful collaboration which helped me to understand better how Feynman integrals can be evaluated. I also have a debt of gratitude to Vittorio Del Duca, Johann Kühn, and Matthias Steinhauser for continual support. Last but not least, I would like to thank my family for permanent love, sympathy, patience, and understanding.

Moscow, September 2012 Vladimir A. Smirnov

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Chapter 1
Introduction

The important mathematical problem of evaluating Feynman integrals arises quite naturally in elementary-particle physics when one treats various quantities in the framework of perturbation theory. Usually, it turns out that a given quantum-field amplitude that describes a process where particles participate cannot be completely treated in the perturbative way. However it also often turns out that the amplitude can be factorized in such a way that different factors are responsible for contributions of different scales. According to a factorization procedure a given amplitude can be represented as a product of factors some of which can be treated only non-perturbatively while others can be indeed evaluated within perturbation theory, i.e. expressed in terms of Feynman integrals over loop momenta.

A useful way to perform the factorization procedure is provided by solving the problem of asymptotic expansion of Feynman integrals in the corresponding limit of momenta and masses that is determined by the given kinematical situation. A universal way to solve this problem is based on the so-called strategy of expansion by regions [1, 21]. This strategy can be itself regarded as a (semi-analytical) method of evaluation of Feynman integrals according to which a given Feynman integral depending on several scales can be approximated, with increasing accuracy, by a finite sum of first terms of the corresponding expansion, where each term is written as a product of factors depending on different scales. The expansion by regions applicable to general limits as well as the expansion by subgraphs applicable to limits typical of Euclidean space are described in details in my other book [21] and, in a very brief way, in this book in Chap. 9. The main goal of this chapter is to present a general algorithm [9, 17] which has appeared after the publication of the book [21] and provides the possibility to find regions relevant to a given limit in a systematic way.

One needs to take into account various graphs that contribute to a given process. The number of graphs greatly increases when the number of loops gets large. For a given graph, the corresponding Feynman amplitude is represented as a Feynman integral over loop momenta, due to some Feynman rules. The Feynman integral, generally, has several Lorentz indices. The standard way to handle tensor quantities

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is to perform a tensor reduction that enables us to write the given quantity as a linear combination of tensor monomials with scalar coefficients. Therefore we will imply that we deal with scalar Feynman integrals and consider only them in examples.

A given Feynman graph therefore generates various scalar Feynman integrals that have the same structure of the integrand with various distributions of powers of propagators (indices). Let us observe that some powers can be negative, due to some initial polynomial in the numerator of the Feynman integral. A straightforward strategy is to evaluate, by some methods, every scalar Feynman integral resulting from the given graph. If the number of these integrals is small this strategy is quite reasonable. In non-trivial situations, where the number of different scalar integrals can be at the level of hundreds and thousands, this strategy looks too complicated. A well-known optimal strategy here is to derive, without calculation, and then apply some relations between the given family of Feynman integrals as *recurrence relations*. A well-known standard way to obtain such relations is provided by the method of integration by parts[1] (IBP) [7] which is based on putting to zero any integral of the form

$$\int \mathrm{d}^d k_1 \mathrm{d}^d k_2 \ldots \frac{\partial f}{\partial k_i^\mu}$$

over loop momenta $k_1, k_2, \ldots, k_i, \ldots$ within dimensional regularization with the space-time dimension $d = 4 - 2\varepsilon$ as a regularization parameter [5, 6, 8]. Here f is an integrand of a Feynman integral; it depends on the loop and external momenta. More precisely, one tries to use IBP relations in order to express a general dimensionally regularized integral from the given family as a linear combination of some irreducible integrals which are also called *master* integrals. Therefore the whole problem decomposes into two parts: solving the IBP relations and evaluating the master integrals. Observe that in such complicated situations, with the great variety of relevant scalar integrals, one really needs to know a *complete* solution of the recursion problem, i.e. to learn how an *arbitrary* integral with general integer powers of the propagators and powers of irreducible monomials in the numerator can be evaluated.

To illustrate the methods of evaluation that we are going to study in this book let us first orient ourselves at the evaluation of individual Feynman integrals, which might be master integrals, and take the simple scalar one-loop graph Γ shown in Fig. 1.1 as an example. The corresponding Feynman integral constructed with scalar propagators is written as

[1] As is explained in textbooks on integral calculus, the method of IBP is applied with the help of the relation $\int_a^b \mathrm{d}x uv' = uv|_a^b - \int_a^b \mathrm{d}x u'v$ as follows. One tries to represent the integrand as uv' with some u and v in such a way that the integral on the right-hand side, i.e. of $u'v$ will be simpler. We do not follow this idea in the case of Feynman integrals. Instead we only use the fact that an integral of the derivative of some function is zero, i.e. we always neglect the corresponding surface terms. So the name of the method looks misleading. It is however unambiguously accepted in the physics community.

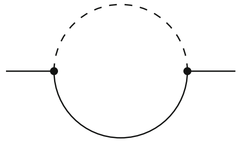

Fig. 1.1 One-loop self-energy graph. The *dashed line* denotes a massless propagator

$$F_\Gamma(q^2, m^2; d) = \int \frac{\mathrm{d}^d k}{(k^2 - m^2)(q-k)^2}, \tag{1.1}$$

where the usual $+\mathrm{i}0$ is implied in the propagators.

The same picture Fig. 1.1 can also denote the Feynman integral with general powers of the two propagators,

$$F_\Gamma(q^2, m^2; a_1, a_2; d) = \int \frac{\mathrm{d}^d k}{(k^2 - m^2)^{a_1}[(q-k)^2]^{a_2}}. \tag{1.2}$$

Suppose, one needs to evaluate the Feynman integral $F_\Gamma(q^2, m^2; 2, 1; d) \equiv F(2, 1; d)$ which is finite in four dimensions, $d = 4$. (It can also be depicted by Fig. 1.1 with a dot on the massive line.) There is a lot of ways to evaluate it. For example, a straightforward way is to take into account the fact that the given function of q is Lorentz-invariant so that it depends on the external momentum through its square, q^2. One can choose a frame $q = (q_0, \mathbf{0})$, introduce spherical coordinates for $\boldsymbol{k}$, integrate over angles, then over the radial component and, finally, over k_0. This strategy can be, however, hardly generalized to multi-loop[2] Feynman integrals.

Another way is to use a dispersion relation that expresses Feynman integrals in terms of a one-dimensional integral of the imaginary part of the given Feynman integral, from the value of the lowest threshold to infinity. This dispersion integral can be expressed by means of the well-known Cutkosky rules. We will not apply this method, which was, however, very popular in the early days of perturbative quantum field theory, and only briefly comment on it in Appendix E.

Let us now turn to the methods that will be indeed actively used in this book. To illustrate them all let me use this very example of Feynman integrals (1.2) and present main ideas of these methods, with the obligation to present the methods in great details in the rest of the book.

First, we will exploit the well-known technique of alpha or Feynman parameters. In the case of $F(2, 1; d)$, one writes down the following Feynman-parametric formula:

$$\frac{1}{(k^2 - m^2)^2(q-k)^2} = 2\int_0^1 \frac{\xi \mathrm{d}\xi}{[(k^2 - m^2)\xi + (1-\xi)(q-k)^2 + \mathrm{i}0]^3}. \tag{1.3}$$

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[2] Since the Feynman integrals are rather complicated objects the word 'multi-loop' means the number of loops greater than one ;-)

Then one can change the order of integration over ξ and k, perform integration over k with the help of the formula (10.1) (which we will derive in Chap. 3) and obtain the following representation:

$$F(2,1;d) = -\mathrm{i}\pi^{d/2}\Gamma(1+\varepsilon)\int_0^1 \frac{\mathrm{d}\xi\,\xi^{-\varepsilon}}{[m^2 - q^2(1-\xi) - \mathrm{i}0]^{1+\varepsilon}}. \tag{1.4}$$

This integral can easily be evaluated at $d = 4$ with the following result:

$$F(2,1;4) = \mathrm{i}\pi^2 \frac{\ln\left(1 - q^2/m^2\right)}{q^2}. \tag{1.5}$$

In principle, any given Feynman integral $F(a_1, a_2; d)$ with concrete numbers a_1 and a_2 can similarly be evaluated by Feynman parameters. In particular, $F(1,1;d)$ reduces to

$$F(1,1;d) = \mathrm{i}\pi^{d/2}\Gamma(\varepsilon)\int_0^1 \frac{\mathrm{d}\xi\,\xi^{-\varepsilon}}{[m^2 - q^2(1-\xi) - \mathrm{i}0]^{\varepsilon}}. \tag{1.6}$$

There is an ultraviolet (UV) divergence which manifests itself in the first pole of the function $\Gamma(\varepsilon)$, i.e. at $d = 4$. The integral can be evaluated in expansion in a Laurent series in ε, for example, up to ε^0. We obtain

$$\begin{aligned} F(1,1;d) = \mathrm{i}\pi^{d/2}\mathrm{e}^{-\gamma_{\mathrm{E}}\varepsilon}\Bigg[&\frac{1}{\varepsilon} - \ln m^2 + 2 \\ &- \left(1 - \frac{m^2}{q^2}\right)\ln\left(1 - \frac{q^2}{m^2}\right) + O(\varepsilon)\Bigg], \end{aligned} \tag{1.7}$$

where γ_{E} is the Euler's constant.

In fact, the integration in (1.6) can straightforwardly be performed at general ε with the result

$$-\mathrm{i}\pi^{d/2}m^{-2\varepsilon}\Gamma(\varepsilon-1)\,{}_2F_1\left(1,\varepsilon;2-\varepsilon;q^2/m^2\right) \tag{1.8}$$

which can then be expanded in ε. However, for sufficiently complicated Feynman integrals, this strategy of evaluating at general ε and expanding results is hardly feasible.

Alpha parameters are closely related to Feynman parameters. For usual propagators, one starts from the representation

$$\frac{1}{k^2 - m^2} = -\mathrm{i}\int_0^\infty \mathrm{d}\alpha\,\mathrm{e}^{\mathrm{i}(k^2-m^2)\alpha}, \tag{1.9}$$

changes the order of integration over alpha parameters and loop momenta and takes d-dimensional integrals over the loop momenta. For example, one obtains

$$F(1,1;d) = \mathrm{e}^{-\mathrm{i}\pi(1+d/2)/2}\pi^{d/2}\int_0^\infty\int_0^\infty \mathrm{e}^{\mathrm{i}q^2\frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2}-\mathrm{i}m^2\alpha_1}\frac{\mathrm{d}\alpha_1\mathrm{d}\alpha_2}{(\alpha_1+\alpha_2)^{d/2}}. \quad (1.10)$$

Then one can turn to Feynman parameters, i.e. to (1.6), by changing variables by $\alpha_1 = \eta\xi$, $\alpha_2 = \eta(1-\xi)$ and integrating over η.

We will study the method of Feynman and alpha parameters in Chap. 3, by deriving a lot of useful formulae and considering various examples. The next chapter also deals with parametric representations which are used there to resolve the singularity structure in ε. In contrast to examples of Chap. 3, where some subtractions are used for this purpose when analytically evaluating Feynman integrals, here the goal is to do this in an algorithmic way by introducing so-called sector decompositions which can be used either for analysis of convergence of regularized or renormalized Feynman integrals, or for numerical evaluation.

To illustrate the basic idea of sector decompositions let us turn again to the integral (1.1) which can be represented by (1.10) and reveal its UV divergence. (And let us forget that we did this in a simple way using Feynman parameters (1.6) where the divergence manifested itself as a pole of the overall gamma function.) We cannot expand the integral in ε under the integral sign because the initial term, i.e. its value at $d = 4$ is divergent. In alpha parameters, UV divergences manifest themselves as divergences at small values, so that let us consider just the integral

$$I(\varepsilon) = \int_0^1\int_0^1 \mathrm{d}\alpha_1\mathrm{d}\alpha_2(\alpha_1+\alpha_2)^{\varepsilon-2}f(\alpha_1,\alpha_2), \quad (1.11)$$

where f is regular at the origin.

To reveal the analytic structure in ε near $\varepsilon = 0$ let us decompose the integration domain into two sectors, $\alpha_1 \le \alpha_2$ and $\alpha_2 \le \alpha_1$ and represent I as $I_1 + I_2$. The two integrals are similar so that let us consider only I_1. Let us introduce sector variables by $\alpha_1 = t_1 t_2,\ \alpha_2 = t_2$. We have again an integration over the unit square:

$$I_1(\varepsilon) = \int_0^1\int_0^1 \mathrm{d}t_1\mathrm{d}t_2\ t_2^{\varepsilon-1}\ g(t_1,t_2), \quad (1.12)$$

where $g(t_1,t_2) = (1+t_1)^{\varepsilon-2}f(t_1t_2,t_2)$. Such a form of the integral easily provides the possibility of expanding under the integral sign. To reveal the pole in ε we then write down $g(t_1,t_2)$ as $g(t_1,0)$ plus $g(t_1,t_2)-g(t_1,0)$. Taking explicitly the integration over t_2 in the first term we arrive at

$$I_1(\varepsilon) = \frac{1}{\varepsilon}\int_0^1 \mathrm{d}t_1\ g(t_1,0) + \int_0^1\int_0^1 \mathrm{d}t_1\mathrm{d}t_2\ t_2^{\varepsilon-1}\ [g(t_1,t_2)-g(t_1,0)]\,. \quad (1.13)$$

We see that we have achieved our goal because the first term is just a simple pole in ε while the second term can be expanded in ε. In Chap. 4, such procedure will be extended to general Feynman integrals and various methods of sector decompositions [2–4, 10, 19] will be described.

A powerful method of evaluating Feynman integrals is based on the Mellin–Barnes (MB) representation [20, 22]. The underlying idea is to replace a sum of terms raised to some power by the product of these terms raised to certain powers, at the cost of introducing an auxiliary integration that goes from $-\mathrm{i}\infty$ to $+\mathrm{i}\infty$ in the complex plane. The most obvious way to apply this representation is to write down a massive propagator in terms of massless ones. For $F(2, 1; 4)$, we obtain

$$\frac{1}{(m^2-k^2)^2} = \frac{1}{2\pi\mathrm{i}}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}z \frac{(m^2)^z}{(-k^2)^{2+z}}\Gamma(2+z)\Gamma(-z). \tag{1.14}$$

Applying (1.14) to the first propagator in (1.2), changing the order of integration over k and z and evaluating the internal integral over k by means of the one-loop formula (10.7) (which we will derive in Chap. 3) we arrive at the following onefold MB integral representation:

$$\begin{aligned} F(2, 1; d) = & -\frac{\mathrm{i}\pi^{d/2}\Gamma(1-\varepsilon)}{(-q^2)^{1+\varepsilon}}\frac{1}{2\pi\mathrm{i}}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}z \left(\frac{m^2}{-q^2}\right)^z \\ & \times \frac{\Gamma(1+\varepsilon+z)\Gamma(-\varepsilon-z)\Gamma(-z)}{\Gamma(1-2\varepsilon-z)}. \end{aligned} \tag{1.15}$$

The contour of integration is chosen in the standard way: the poles with a $\Gamma(\cdots+z)$ dependence are to the left of the contour and the poles with a $\Gamma(\cdots-z)$ dependence are to the right of it. If $|\varepsilon|$ is small enough we can choose this contour as a straight line parallel to the imaginary axis with $-1 < \mathrm{Re}z < 0$. For $d = 4$, we obtain

$$F(2, 1; 4) = -\frac{\mathrm{i}\pi^2}{q^2}\frac{1}{2\pi\mathrm{i}}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}z \left(\frac{m^2}{-q^2}\right)^z \Gamma(z)\Gamma(-z). \tag{1.16}$$

By closing the integration contour to the right and taking a series of residues at the points $z = 0, 1, \ldots$, we reproduce (1.5). Using the same technique, any integral from the given family can similarly be evaluated.

We will study the method of MB representation in Chap. 5. This method provides the possibility to resolve singularities in ε in an easy way. We will see, through various examples, how one can analytically evaluate rather complicated Feynman integrals. Moreover, this method can be applied almost in an automatic way because various public computer codes for the application of this method are available.

Let us, however, think about a more economical strategy based on IBP relations which would enable us to evaluate any integral (1.2) as a linear combination of some master integrals. Putting to zero dimensionally regularized integrals of $\frac{\partial}{\partial k}\cdot k f(a_1, a_2)$ and $q \cdot \frac{\partial}{\partial k} f(a_1, a_2)$, where $f(a_1, a_2)$ is the integrand in (1.2), and writing down obtained relations in terms of integrals of the given family we obtain the following two IBP relations:

$$d - 2a_1 - a_2 - 2m^2 a_1 \mathbf{1}^+ - a_2 \mathbf{2}^+(\mathbf{1}^- - q^2 + m^2) = 0, \tag{1.17}$$

$$a_2 - a_1 - a_1 \mathbf{1}^+(q^2 + m^2 - \mathbf{2}^-) - a_2 \mathbf{2}^+(\mathbf{1}^- - q^2 + m^2) = 0, \tag{1.18}$$

in the sense that they are applied to the general integral $F(a_1, a_2)$. Here the standard notation for increasing and lowering operators has been used, e.g. $\mathbf{1}^+\mathbf{2}^- F(a_1, a_2) = F(a_1 + 1, a_2 - 1)$.

Let us observe that any integral with $a_1 \leq 0$ is zero because it is a massless tadpole which is naturally put to zero within dimensional regularization. Moreover, any integral with $a_2 \leq 0$ can be evaluated in terms of gamma functions for general d with the help of (10.3) (which we will derive in Chap. 3). The number a_2 can be reduced either to one or to a non-positive value using the following relation which is obtained as the difference of (1.17) multiplied by $q^2 + m^2$ and (1.18) multiplied by $2m^2$:

$$\begin{aligned}(q^2 - m^2)^2 a_2 \mathbf{2}^+ = {} & (q^2 - m^2) a_2 \mathbf{1}^- \mathbf{2}^+ \\ & - (d - 2a_1 - a_2) q^2 - (d - 3a_2) m^2 + 2m^2 a_1 \mathbf{1}^+ \mathbf{2}^- .\end{aligned} \tag{1.19}$$

Indeed, when the left-hand side of (1.19) is applied to $F(a_1, a_2)$, we obtain integrals with reduced a_2 or, due to the first term on the right-hand side, reduced a_1.

Suppose now that $a_2 = 1$. Then we can use the difference of relations (1.17) and (1.18),

$$d - a_1 - 2a_2 - a_1 \mathbf{1}^+(\mathbf{2}^- - q^2 + m^2) = 0, \tag{1.20}$$

and rewrite it down, at $a_2 = 1$, as

$$(q^2 - m^2) a_1 \mathbf{1}^+ = a_1 + 2 - d + a_1 \mathbf{1}^+ \mathbf{2}^- . \tag{1.21}$$

This relation can be used to reduce the index a_1 to one or the index a_2 to zero. We see that we can now express any integral of the given family as a linear combination of the integral $F(1, 1)$ and simple integrals with $a_2 \leq 0$ which can be evaluated for general d in terms of gamma functions. In particular, we have

$$F(2, 1) = \frac{1}{m^2 - q^2} \left[(1 - 2\varepsilon) F(1, 1) - F(2, 0) \right] . \tag{1.22}$$

At this point, we might stop our activity because we have already essentially solved the problem. However, mathematically (and aesthetically), it is natural to be more curious and wonder about the minimal number of master integrals which form a linearly independent basis in the family of integrals $F(a_1, a_2)$. We will do this in

Chap. 6. We will also consider other simple examples where IBP relations can be solved 'by hand', as in this example.

The are two news. The bad news is that solving IBP relations by hand is hardly possible at the high modern complexity level of practical calculations. The good news is that one can solve IBP relations automatically using various algorithms. The most popular one is the Laporta's algorithm [14, 15] based on solving overconstrained systems of linear equations. There are public codes where this algorithm is implemented. We will turn to this algorithm in Chap. 6 where some other algorithms will be also briefly presented.

Two powerful methods described in this book are based on equations: differential equations for one of them and difference equations for the other one. Within both of them, it is assumed that one can solve IBP relations for the family of Feynman integrals to which a given integral belongs. Practically, these methods are used to evaluate master integrals.

Let us illustrate the method of differential equations (DE) [11–13, 18] again with the help of our favourite example. To evaluate the master integral $F(1,1)$ let us observe that its derivative in m^2 is nothing but $F(2,1)$ (because $\big(\partial/(\partial m^2)\big)\big(1/(k^2-m^2)\big) = 1/(k^2-m^2)^2$) which is expressed, according to our reduction procedure, by (1.22). Therefore we arrive at the following differential equation for $f(m^2) = F(1,1)$:

$$\frac{\partial}{\partial m^2} f(m^2) = \frac{1}{m^2-q^2}\big[(1-2\varepsilon) f(m^2) - F(2,0)\big], \tag{1.23}$$

where the quantity $F(2,0)$ is a simpler object because it can be evaluated in terms of gamma functions for general ε. The general solution to this equation can easily be obtained by the method of the variation of the constant, with fixing the general solution from the boundary condition at $m = 0$. Eventually, the above result (1.7) can successfully be reproduced.

As we will see in Chap. 7, the strategy of the method of DE in much more nontrivial situations is similar: one takes derivatives of a master integral in some arguments, expresses them in terms of original Feynman integrals, by means of some variant of solution of IBP relations, and solves resulting differential equations.

The recently developed method based on difference equations [16] uses relations between Feynman integrals in shifted dimension, d. To illustrate it let us turn again to our favourite example. To evaluate the master integral $F_1(d) \equiv F(1,1;d)$ let us use its alpha representation (1.10) and consider $F_1(d-2)$. Up to simple changes of exponents in the prefactors, the most essential change is the appearance of the extra factor $(\alpha_1 + \alpha_2)$ in the integrand. Then each of these two terms can be described as a Feynman integral with a shifted index, i.e. either $F(2,1;d)$ or $F(1,2;d)$. As we will see in Chap. 6 any integral of this family can be reduced to the two master integrals, $F_1(d)$, and $F_2(d) = F(1,0;d)$. (A partial reduction, where $F(2,0;d)$ can be reduced further, to $F_2(d)$, is given by (1.22).) This is how one obtains the following dimensional recurrence relation for the master integral $F_1(d)$:

$$F_1(d-2) = 2\frac{(d-3)x}{(1-x)^2}F_1(d) - \frac{(d-2)(1+x)}{2(1-x)^2}F_2(d), \tag{1.24}$$

where we set $x = q^2/m^2$ and $m = 1$.

We will see in Chap. 8 how this and other similar equations can be systematically solved. In this example, one can arrive at a result in terms of a hypergeometric function which, after using some identity, can be reduced to (1.8). Within this method, one obtains solutions in terms of multiple series with excellent convergence. For one-scale integrals, this provides the possibility to evaluate each term in an ε expansion with a big accuracy and then obtain analytic results in very high orders of this expansion.

As promised in the beginning of this introduction, the semi-analytic method of expansions in limits of momenta and masses [1, 21] is briefly presented in Chap. 9. Let us take again the integral $F(2, 1; d)$ given by (1.2) as an example and study it in the limit $m^2/q^2 \to 0$. As explained in Chap. 9, one can proceed either by expansion by regions, or using an explicit formula for the expansion written in graph-theoretical language. In both cases, one has the sum of two contributions to the expansion. One of them is obtained by expanding the propagator $1/(k^2 - m^2)^2$ in a Taylor series in the mass m and the other one is obtained by expanding the propagator $1/(q-k)^2$ in a Taylor series in the loop momentum k. This and other typical examples are studied in Chap. 9. It will be also explained how to find regions relevant to a given limit by a geometrical algorithm [9, 17].

Before studying these methods, basic definitions are presented in Chap. 2 where tools for dealing with Feynman integrals are also introduced. Methods for evaluating individual Feynman integrals are studied in Chaps. 3–5, 7–9 and the reduction problem is studied in Chap. 6. In Appendix A, one can find a table of basic one-loop and two-loop Feynman integrals as well as some useful auxiliary formulae. Appendix B contains definitions and properties of special functions that are used in this book. A table of summation formulae for onefold series is given in Appendix C. In Appendix D, a table of onefold MB integrals is presented.

Some other methods are briefly characterized in Appendix E. These are mainly old methods whose details can be found in the literature. If I do not present some methods, this means that either I do not know about them, or I do not know physically important situations where they work not worse than the methods I present.

I will use almost the same examples in Chaps. 3–9 and Appendix E to illustrate all the methods. On the one hand, this is done in order to have the possibility to compare them. On the other hand, the methods often work together: for example, MB representation can be used in alpha or Feynman parametric integrals, the methods based on differential and difference equations require a solution of the reduction problem, boundary conditions within the method of DE can be obtained by means of the method of MB representation, etc.

Basic notational conventions are presented below. The notation is described in more detail in the List of Symbols. In the Index, one can find numbers of pages where definitions of basic notions are introduced.

1.1 Notation

We use Greek and Roman letters for four-indices and spatial indices, respectively:

$$x^\mu = (x^0, \boldsymbol{x}),$$
$$q\cdot x = q^0 x^0 - \boldsymbol{q}\cdot\boldsymbol{x} \equiv g_{\mu\nu} q^\mu x^\nu .$$

The parameter of dimensional regularization is

$$d = 4 - 2\varepsilon .$$

The d-dimensional Fourier transform and its inverse are defined as

$$\tilde{f}(q) = \int d^d x\, \mathrm{e}^{\mathrm{i}q\cdot x} f(x),$$
$$f(x) = \frac{1}{(2\pi)^d} \int \mathrm{d}^d q\, \mathrm{e}^{-\mathrm{i}x\cdot q} \tilde{f}(q).$$

In order to avoid Euler's constant γ_E in Laurent expansions in ε, we usually pull out the factor $\mathrm{e}^{-\gamma_\mathrm{E}\varepsilon}$ per loop.

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Chapter 2
Feynman Integrals: Basic Definitions and Tools

In this chapter, basic definitions for Feynman integrals are given, ultraviolet (UV), infrared (IR) and collinear divergences are characterized, and basic tools such as alpha parameters are presented. Various kinds of regularizations, in particular dimensional one, are presented and properties of dimensionally regularized Feynman integrals are formulated and discussed.

2.1 Feynman Rules and Feynman Integrals

In perturbation theory, any quantum field model is characterized by a Lagrangian, which is represented as a sum of a free-field part and an interaction part, $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\mathrm{I}}$. Amplitudes of the model, e.g. S-matrix elements and matrix elements of composite operators, are represented as power series in coupling constants. Starting from the S-matrix represented in terms of the time-ordered exponent of the interaction Lagrangian which is expanded with the application of the Wick theorem, or from Green functions written in terms of a functional integral treated in the perturbative way, one obtains that, in a fixed perturbation order, the amplitudes are written as finite sums of Feynman diagrams which are constructed according to Feynman rules: lines correspond to $\mathcal{L}_0$ and vertices are determined by $\mathcal{L}_{\mathrm{I}}$. The basic building block of the Feynman diagrams is the propagator that enters the relation

$$T\phi_i(x_1)\phi_i(x_2) =: \phi_i(x_1)\phi_i(x_2) : + D_{F,i}(x_1 - x_2). \tag{2.1}$$

Here $D_{F,i}$ is the Feynman propagator of the field of type i and the colons denote a normal product of the free fields. The Fourier transforms of the propagators have the form

$$\tilde{D}_{F,i}(p) \equiv \int d^4x\, \mathrm{e}^{\mathrm{i}p\cdot x} D_{F,i}(x) = \frac{\mathrm{i}Z_i(p)}{(p^2 - m_i^2 + \mathrm{i}0)^{a_i}}, \tag{2.2}$$

V. A. Smirnov, *Analytic Tools for Feynman Integrals*, Springer Tracts in Modern Physics 250, DOI: 10.1007/978-3-642-34886-0_2,

where m_i is the corresponding mass, Z_i is a polynomial and $a_i = 1$ or 2 (for the gluon propagator in the general covariant gauge). The powers of the propagators a_l will be also called *indices*. For the propagator of the scalar field, we have $Z = 1$, $a = 1$. This is not the most general form of the propagator. For example, in the axial or Coulomb gauge, the gluon propagator has another form. We usually omit the causal i0 for brevity. Polynomials associated with vertices of graphs can be taken into account by means of the polynomials Z_l. We also omit the factors of i and $(2\pi)^4$ that enter in the standard Feynman rules (in particular, in (2.2)); these can be included at the end of a calculation.

Eventually, we obtain, for any fixed perturbation order, a sum of Feynman amplitudes labelled by Feynman graphs[1] constructed from the given type of vertices and lines. In the commonly accepted physical slang, the graph, the corresponding Feynman amplitude and the integral are all often called the 'diagram'. A Feynman graph differs from a graph by distinguishing a subset of vertices which are called *external*. The external momenta or coordinates on which a Feynman integral depends are associated with the external vertices.

Thus quantities that can be computed perturbatively are written, in any given order of perturbation theory, through a sum over Feynman graphs. For a given graph Γ, the corresponding Feynman amplitude

$$G_\Gamma(q_1, \ldots, q_{n+1}) = (2\pi)^4 \,\mathrm{i}\, \delta\left(\sum_i q_i\right) F_\Gamma(q_1, \ldots, q_n) \tag{2.3}$$

can be written in terms of an integral over loop momenta

$$F_\Gamma(q_1, \ldots, q_n) = \int \mathrm{d}^4 k_1 \ldots \int \mathrm{d}^4 k_h \prod_{l=1}^{L} \tilde{D}_{F,l}(p_l), \tag{2.4}$$

where $\mathrm{d}^4 k_i = \mathrm{d}k_i^0\, \mathrm{d}\boldsymbol{k}_i$, and a factor with a power of 2π is omitted, as we have agreed. The Feynman integral F_Γ depends on n linearly independent external momenta $q_i = (q_i^0, \boldsymbol{q}_i)$; the corresponding integrand is a function of L internal momenta p_l, which are certain linear combinations of the external momenta and $h = L - V + 1$ chosen loop momenta k_i, where L, V and h are numbers of lines, vertices and (independent) loops, respectively, of the given graph.

One can choose the loop momenta by fixing a *tree* T of the given graph, i.e. a maximal connected subgraph without loops, and correspond a loop momentum to each line not belonging to this tree. Then we have the following explicit formula for the momenta of the lines:

[1] When dealing with graphs and Feynman integrals one usually does not bother about the mathematical definition of the graph and thinks about something that is built of lines and vertices. So, a graph is an ordered family $\{\mathcal{V}, \mathcal{L}, \pi_\pm\}$, where $\mathcal{V}$ is the set of vertices, $\mathcal{L}$ is the set of lines, and $\pi_\pm : \mathcal{L} \to \mathcal{V}$ are two mappings that correspond the initial and the final vertex of a line. By the way, mathematicians use the word 'edge', rather than 'line'.

$$p_l = \sum_{i=1}^{h} e_{il} k_i + \sum_{i=1}^{n} d_{il} q_i , \tag{2.5}$$

where $e_{il} = \pm 1$ if l belongs to the jth loop and $e_{il} = 0$ otherwise, $d_{il} = \pm 1$ if l lies in the tree T on the path with the momentum q_i and $d_{il} = 0$ otherwise. The signs in both sums are defined by orientations.

After a tensor reduction is performed one can deal only with scalar Feynman integrals. For the tensor reduction, various projectors can be applied. For example, in the case of Feynman integrals contributing to the electromagnetic formfactor (see Fig. 2.1) $\Gamma^\mu(p_1, p_2) = \gamma^\mu F_1(q^2) + \sigma^{\mu\nu} q_\nu F_2(q^2)$, where $q = p_1 - p_2$, γ^μ and $\sigma^{\mu\nu}$ are γ- and σ-matrices, respectively, the following projector can be applied to extract scalar integrals which contribute to the formfactor F_1 in the massless case (with $F_2 = 0$):

$$F_1(q^2) = \frac{\mathrm{Tr}\left[\gamma_\mu \not{p}_2 \Gamma^\mu(p_1, p_2) \not{p}_1\right]}{2(d-2)\, q^2}, \tag{2.6}$$

where $\not{p} = \gamma^\mu p_\mu$ and d is the parameter of dimensional regularization (to be discussed shortly in Sect. 2.4).

Anyway, after applying some projectors, one obtains, for a given graph, a family of Feynman integrals which have various powers of the scalar parts of the propagators, $1/(p_l^2 - m_l^2)^{a_l}$, and various monomials in the numerator. The denominators p_l^2 can be expressed linearly in terms of scalar products of the loop and external momenta. The factors in the numerator can also be chosen as quadratic polynomials of the loop and external momenta raised to some powers. It is convenient to consider both types of the quadratic polynomials on the same footing and treat the factors in the numerators as extra factors in the denominator raised to negative powers. The set of the denominators for a given graph is linearly independent. It is natural to complete this set by similar factors coming from the numerator in such a way that the whole set will be linearly independent.

Therefore we come to the following family of scalar integrals generated by the given graph:

$$F(a_1, \ldots, a_N) = \int \cdots \int \frac{\mathrm{d}^4 k_1 \ldots \mathrm{d}^4 k_h}{E_1^{a_1} \ldots E_N^{a_N}} , \tag{2.7}$$

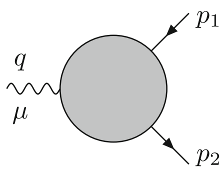


Fig. 2.1 Electromagnetic formfactor

where k_i, $i = 1, \ldots, h$, are loop momenta, a_i are integer indices, and the denominators are given by

$$E_l = \sum_{i \geq j \geq 1} A_l^{ij}\, r_i \cdot r_j - m_l^2\,, \tag{2.8}$$

with $l = 1, \ldots, N$. The momenta r_i are either the loop momenta $r_i = k_i$, $i = 1, \ldots, h$, or independent external momenta $r_{h+1} = q_1, \ldots, r_{h+n} = q_l$ of the graph.

For a usual Feynman graph, the denominators E_r determined by some matrix A are indeed quadratic. However, a more general class of Feynman integrals where the denominators are linear with respect to the loop and/or external momenta also often appears in practical calculations. Linear denominators usually appear in asymptotic expansions of Feynman integrals within the strategy of expansion by regions [1, 31]. Such expansions provide a useful link of an initial theory described by some Lagrangian with various effective theories where, indeed, the denominators of propagators can be linear with respect to the external and loop momenta. For example, one encounters the following denominators: $p \cdot k$, with an external momentum p on the light cone, $p^2 = 0$, for the Sudakov limit and with $p^2 \neq 0$ for the quark propagator of Heavy Quark Effective Theory (HQET) [17, 22, 25]. Some non-relativistic propagators appear within threshold expansion and in the effective theory called Non-Relativistic QCD (NRQCD) [4, 5, 21, 37], for example, the denominator $k_0 - \boldsymbol{k}^2/(2m)$.

2.2 Divergences

As has been known from early days of quantum field theory, Feynman integrals suffer from divergences. This word means that, taken naively, these integrals are ill-defined because the integrals over the loop momenta generally diverge. The *ultraviolet* (UV) *divergences* manifest themselves through a divergence of the Feynman integrals at large loop momenta. Consider, for example, the Feynman integral corresponding to the one-loop graph Γ of Fig. 2.2 with scalar propagators. This integral can be written as

$$F_\Gamma(q) = \int \frac{\mathrm{d}^4 k}{(k^2 - m_1^2)\left[(q-k)^2 - m_2^2\right]}, \tag{2.9}$$

where the loop momentum k is chosen as the momentum of the first line. Introducing four-dimensional (generalized) spherical coordinates $k = r\hat{k}$ in (2.9), where $\hat{k}$ is on the unit (generalized) sphere and is expressed by means of three angles, and counting powers of propagators, we obtain, in the limit of large r, the following divergent behaviour: $\int_\Lambda^\infty \mathrm{d}r\, r^{-1}$. For a general diagram, a similar power counting at large values of the loop momenta gives $4h(\Gamma) - 1$ from the Jacobian that arises when one introduces generalized spherical coordinates in the $(4 \times h)$-dimensional space of h

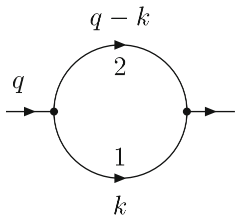


Fig. 2.2 One-loop self-energy diagram

loop four-momenta, plus a contribution from the powers of the propagators and the degrees of its polynomials, and leads to an integral $\int_\Lambda^\infty \mathrm{d}r\, r^{\omega-1}$, where

$$\omega = 4h - 2L + \sum_l n_l \tag{2.10}$$

is the (UV) *degree of divergence* of the graph. (Here n_l are the degrees of the polynomials Z_l.)

This estimate shows that the Feynman integral is UV convergent overall (no divergences arise from the region where all the loop momenta are large) if the degree of divergence is negative. We say that the Feynman integral has a logarithmic, linear, quadratic, etc. overall divergence when $\omega = 0, 1, 2, \ldots$, respectively. To ensure a complete absence of UV divergences it is necessary to check convergence in various regions where some of the loop momenta become large, i.e. to satisfy the relation $\omega(\gamma) < 0$ for all the subgraphs γ of the graph. We call a subgraph UV divergent if $\omega(\gamma) \geq 0$. In fact, it is sufficient to check these inequalities only for *one-particle-irreducible* (1PI) subgraphs (which cannot be made disconnected by cutting a line). It turns out that these rough estimates are indeed true—see some details in Sect. 4.4.

If we turn from momentum space integrals to some other representation of Feynman diagrams, the UV divergences will manifest themselves in other ways. For example, in coordinate space, the Feynman amplitude (i.e. the inverse Fourier transform of (2.3)) is expressed in terms of a product of the Fourier transforms of propagators

$$\prod_{l=1}^{L} D_{F,l}(x_{l_\mathrm{i}} - x_{l_\mathrm{f}}) \tag{2.11}$$

integrated over four-coordinates x_i corresponding to the internal vertices. Here l_i and l_f are the beginning and the end, respectively, of a line l.

The propagators in coordinate space,

$$D_{F,l}(x) = \frac{1}{(2\pi)^4} \int \mathrm{d}^4 p\, \tilde{D}_{F,l}(p) \mathrm{e}^{-\mathrm{i}x\cdot p}, \tag{2.12}$$

are singular at small values of coordinates $x = (x_0, \boldsymbol{x})$. To reveal this singularity explicitly let us write down the propagator (2.2) in terms of an integral over a so-called alpha-parameter

$$\tilde{D}_{F,l}(p) = \mathrm{i}\, Z_l \left(\frac{1}{2\mathrm{i}}\frac{\partial}{\partial u_l}\right) \mathrm{e}^{2\mathrm{i}u_l\cdot p}\Big|_{u_l=0} \frac{(-\mathrm{i})^{a_l}}{\Gamma(a_l)} \int_0^\infty \mathrm{d}\alpha_l\, \alpha_l^{a_l-1} \mathrm{e}^{\mathrm{i}(p^2-m^2)\alpha_l}. \quad (2.13)$$

which turns out to be a very useful tool both in theoretical analyses and practical calculations.

To present an explicit formula for the scalar (i.e. for $a = 1$ and $Z = 1$) propagator

$$\tilde{D}_F(p) = \int_0^\infty \mathrm{d}\alpha\, \mathrm{e}^{\mathrm{i}(p^2-m^2)\alpha} \quad (2.14)$$

in coordinate space we insert (2.14) into (2.12), change the order of integration over p and α and take the Gaussian integrations explicitly using the formula

$$\int \mathrm{d}^4k\, \mathrm{e}^{\mathrm{i}(\alpha k^2 - 2q\cdot k)} = -\mathrm{i}\pi^2\alpha^{-2}\mathrm{e}^{-\mathrm{i}q^2/\alpha}, \quad (2.15)$$

which is nothing but the product of four one-dimensional Gaussian integrals:

$$\begin{aligned} \int_{-\infty}^{\infty} \mathrm{d}k_0\, \mathrm{e}^{\mathrm{i}(\alpha k_0^2 - 2q_0k_0)} &= \sqrt{\frac{\pi}{\alpha}}\, \mathrm{e}^{-\mathrm{i}q_0^2/\alpha + \mathrm{i}\pi/4}, \\ \int_{-\infty}^{\infty} \mathrm{d}k_j\, \mathrm{e}^{-\mathrm{i}(\alpha k_j^2 - 2q_jk_j)} &= \sqrt{\frac{\pi}{\alpha}}\, \mathrm{e}^{\mathrm{i}q_j^2/\alpha - \mathrm{i}\pi/4}, \quad j = 1,2,3 \end{aligned} \quad (2.16)$$

(without summation over j in the last formula).

The final integration is then performed using [28] or in MATHEMATICA [39] with the following result:

$$\begin{aligned} D_F(x) &= -\frac{\mathrm{i}m}{4\pi^2\sqrt{-x^2+\mathrm{i}0}} K_1\left(\mathrm{i}m\sqrt{-x^2+\mathrm{i}0}\right) \\ &= -\frac{1}{4\pi^2}\frac{1}{x^2-\mathrm{i}0} + O\left(m^2\ln m^2\right) \end{aligned} \quad (2.17)$$

where K_1 is a Bessel special function [15]. The leading singularity at $x = 0$ is given by the value of the coordinate space massless propagator.

Thus, the inverse Fourier transform of the convolution integral (2.9) equals the square of the coordinate-space scalar propagator, with the singularity $(x^2 - \mathrm{i}0)^{-2}$. Power-counting shows that this singularity produces integrals that are divergent in the vicinity of the point $x = 0$, and this is the coordinate space manifestation of the UV divergence.

The divergences caused by singularities at small loop momenta are called *infrared* (IR) *divergences*. First we distinguish IR divergences that arise at general values of the external momenta. A typical example of such a divergence is given by the graph of Fig. 2.2 when one of the lines contains the second power of the corresponding propagator, so that $a_1 = 2$. If the mass of this line is zero we obtain a factor $1/(k^2)^2$ in the integrand, where k is chosen as the momentum of this line. Then, keeping in mind the introduction of generalized spherical coordinates and performing power-counting at small k (i.e. when all the components of the four-vector k are small), we again encounter a divergent behaviour $\int_0^{\Lambda} \mathrm{d}r\, r^{-1}$ but now at small values of r. There is a similarity between the properties of IR divergences of this kind and those of UV divergences. One can define, for such off-shell IR divergences, an IR degree of divergence, in a similar way to the UV case. A reasonable choice is provided by the value

$$\tilde{\omega}(\gamma) = -\omega(\Gamma/\overline{\gamma}) \equiv \omega(\overline{\gamma}) - \omega(\Gamma), \tag{2.18}$$

where $\overline{\gamma} \equiv \Gamma \backslash \gamma$ is the completion of the subgraph γ in a given graph Γ and Γ/γ denotes the reduced graph which is obtained from Γ by reducing every connectivity component of γ to a point. The absence of off-shell IR divergences is guaranteed if the IR degrees of divergence are negative for all massless subgraphs γ whose completions $\overline{\gamma}$ include all the external vertices in the same connectivity component [11, 30]. (See details in Sect. 4.4.) The off-shell IR divergences are the worst but they are in fact absent in physically meaningful theories. However, they play an important role in asymptotic expansions of Feynman diagrams—see [31] and Chap. 9.

The other kinds of IR divergences arise when the external momenta considered are on a surface where the Feynman diagram is singular: either on a mass shell or at a threshold. Consider, for example, the graph Fig. 2.2, with the indices $a_1 = 1$ and $a_2 = 2$ and the masses $m_1 = 0$ and $m_2 = m \neq 0$ on the mass shell, $q^2 = m^2$. With k as the momentum of the second line, the corresponding Feynman integral is of the form

$$F_\Gamma(q; d) = \int \frac{\mathrm{d}^4 k}{k^2 (k^2 - 2q\cdot k)^2}. \tag{2.19}$$

At small values of k, the integrand behaves like $1/[4k^2(q\cdot k)^2]$, and, with the help of power counting, we see that there is an *on-shell IR divergence* which would not be present for $q^2 \neq m^2$.

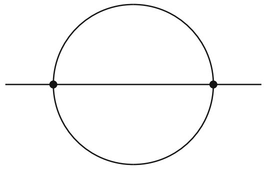
If we consider Fig. 2.2 with equal masses and indices $a_1 = a_2 = 2$ at the threshold, i.e. at $q^2 = 4m^2$, it might seem that there is a *threshold IR divergence* because, choosing the momenta of the lines as $q/2 + k$ and $q/2 - k$, we obtain the integral

$$\int \frac{\mathrm{d}^4 k}{(k^2 + q\cdot k)^2 (k^2 - q\cdot k)^2}, \tag{2.20}$$

with an integrand that behaves at small k as $1/(q \cdot k)^4$ and is formally divergent. However, the divergence is in fact absent. (The threshold singularity at $q^2 = 4m^2$ is, of course, present.) Nevertheless, threshold IR divergences do exist. For example, the sunset[2] diagram of Fig. 2.3 with general masses at threshold, $q^2 = (m_1 + m_2 + m_3)^2$,

[2] called also the sunrise diagram, or the London transport diagram.

Fig. 2.3 Sunset diagram



is divergent in this sense when the sum of the integer powers of the propagators is greater than or equal to five (see, e.g., [14]).

The IR divergences characterized above are local in momentum space, i.e. they are connected with special points of the loop integration momenta. *Collinear* divergences arise at lines parallel to certain light-like four-vectors. A typical example of a collinear divergence is provided by the massless triangle graph of Fig. 2.4. Let us take $p_1^2 = p_2^2 = 0$ and all the masses equal to zero. The corresponding Feynman integral is

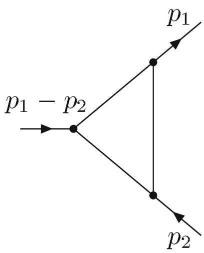
$$\int \frac{\mathrm{d}^4 k}{(k^2 - 2p_1\cdot k)(k^2 - 2p_2\cdot k)k^2}. \tag{2.21}$$

At least an on-shell IR divergence is present, because the integral is divergent when $k \to 0$ (componentwise). However, there are also divergences at non-zero values of k that are collinear with p_1 or p_2 and where $k^2 \sim 0$. This follows from the fact that the product $1/[(k^2-2p\cdot k)k^2]$, where $p^2 = 0$ and $p \neq 0$, generates collinear divergences. To see this let us take residues in the upper complex half plane when integrating this product over k_0. For example, taking the residue at $k_0 = -|\boldsymbol{k}|+\mathrm{i}0$ leads to an integral containing $1/(p\cdot k) = 1/[p^0|\boldsymbol{k}|(1-\cos\theta)]$, where θ is the angle between the spatial components $\boldsymbol{k}$ and $\boldsymbol{p}$. Thus, for small θ, we have a divergent integration over angles because of the factor $\mathrm{d}\cos\theta/(1-\cos\theta) \sim \mathrm{d}\theta/\theta$. The second residue generates a similar divergent behaviour—this can be seen by making the change $k \to p-k$.

Another way to reveal the collinear divergences is to introduce the light-cone coordinates $k_\pm = k_0 \pm k_3,\ \underline{k} = (k_1, k_2)$. If we choose p with the only non-zero component p_+, we will see a logarithmic divergence coming from the region $k_- \sim \underline{k}^2 \sim 0$ just by power counting.

These are the main types of divergences of usual Feynman integrals. Various special divergences arise in more general Feynman integrals (2.7) that can contain linear propagators and appear on the right-hand side of asymptotic expansions in momenta

Fig. 2.4 One-loop triangle diagram



and masses and in associated effective theories. For example, in the Sudakov limit, one encounters divergences that can be classified as UV collinear divergences. Another situation with various non-standard divergences is provided by threshold expansion and the corresponding effective theories, NRQCD and pNRQCD, where special power counting is needed to characterize the divergences.

2.3 Alpha Representation

A useful tool to analyze the divergences of Feynman integrals is the so-called alpha representation based on (2.13). It can be written down for any Feynman integral. For example, for (2.9), one inserts (2.13) for each of the two propagators, takes the four-dimensional Gaussian integral by means of (2.15) to obtain

$$F_\Gamma(q) = \mathrm{i}\pi^2 \int_0^\infty \int_0^\infty \mathrm{d}\alpha_1\, \mathrm{d}\alpha_2\, (\alpha_1+\alpha_2)^{-2} \times \exp\left(\mathrm{i}q^2 \frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2} - \mathrm{i}(m_1^2\alpha_1 + m_2^2\alpha_2)\right). \tag{2.22}$$

For a usual general Feynman integral, this procedure can also explicitly be implemented. Using (2.13) for each propagator of a general usual Feynman integral (i.e., with usual propagators (2.2)) one takes (see, e.g., [23]) $4h$-dimensional Gauss integrals by means of a generalization of (2.15) to the case of an arbitrary number of loop integration momenta:

$$\int \mathrm{d}^4k_1 \ldots \mathrm{d}^4k_h \exp\left[\mathrm{i}\left(\sum_{i,j} A_{ij} k_i \cdot k_j + 2\sum_i q_i \cdot k_i\right)\right] = \mathrm{i}^{-h}\pi^{2h} (\det A)^{-2} \exp\left[-\mathrm{i}\sum_{i,j} A_{ij}^{-1} q_i \cdot q_j\right]. \tag{2.23}$$

Here A is an $h \times h$ matrix and A^{-1} its inverse.[3]

The elements of the inverse matrix involved here are rewritten in graph-theoretical language (see details in [7, 23]), and the resulting alpha representation takes the form [8]

[3] In fact, the matrix A involved here equals $e\beta e^+$ with the elements of an arbitrarily chosen column and row with the same number deleted. Here e is the incidence matrix of the graph, i.e. $e_{il} = \pm 1$ if the vertex i is the beginning/end of the line l, e^+ is its transpose and β consists of the numbers $1/\alpha_l$ on the diagonal—see, e.g., [23].

$$
\begin{aligned}
F_\Gamma(q_1,\ldots,q_n;d) = & \frac{\mathrm{i}^{-a-h}\pi^{2h}}{\prod_l \Gamma(a_l)} \\
& \times \int_0^\infty \mathrm{d}\alpha_1 \ldots \int_0^\infty \mathrm{d}\alpha_L \prod_l \alpha_l^{a_l-1} \mathcal{U}^{-2} Z \mathrm{e}^{\mathrm{i}\mathcal{V}/\mathcal{U}-\mathrm{i}\sum m_l^2\alpha_l},
\end{aligned}
\tag{2.24}
$$

where $a = \sum a_l$, and $\mathcal{U}$ and $\mathcal{V}$ are the well-known functions

$$
\mathcal{U} = \sum_{T\in T^1} \prod_{l\notin T} \alpha_l, \tag{2.25}
$$

$$
\mathcal{V} = \sum_{T\in T^2} \prod_{l\notin T} \alpha_l \left(q^T\right)^2 . \tag{2.26}
$$

In (2.25), the sum runs over trees of the given graph, and, in (2.26), over *2-trees*, i.e. subgraphs that do not involve loops and consist of two connectivity components; $\pm q^T$ is the sum of the external momenta that flow into one of the connectivity components of the 2-tree T. (It does not matter which component is taken because of the conservation law for the external momenta.) The products of the alpha parameters involved are taken over the lines that do not belong to the given tree T. The functions $\mathcal{U}$ and $\mathcal{V}$ are homogeneous functions of the alpha parameters with the homogeneity degrees h and $h+1$, respectively. See also [6] for various properties of these two basic functions.

The factor Z is responsible for the non-scalar structure of the diagram:

$$
Z = \prod_l Z_l \left(\frac{1}{2\mathrm{i}}\frac{\partial}{\partial u_l}\right) \mathrm{e}^{\mathrm{i}(2B-K)/\mathcal{U}} \Big|_{u_1=\ldots u_L=0} , \tag{2.27}
$$

where (see, e.g., [30, 40])

$$
B = \sum_l u_l \sum_{T\in T_l^1} q_T \prod_{l'\notin T} \alpha_{l'}, \tag{2.28}
$$

$$
K = \sum_{T\in T^0} \prod_{l\notin T} \alpha_l \left(\sum_l \pm u_l\right)^2 . \tag{2.29}
$$

In (2.28), the sum is taken over trees T_l^1 that include a given line l, and q_T is the total external momentum that flows through the line l (in the direction of its orientation). In (2.29), the sum is taken over pseudotrees T^0 (a *pseudotree* is obtained from a tree by adding a line), and the sum in l is performed over the loop (circuit) of the pseudotree T, with a sign dependent on the coincidence of the orientations of the line l and the pseudotree T.

Let me emphasize that this terrible-looking machinery for evaluating the determinant of the matrix A that arises from Feynman integrals, as well as for evaluating the elements of the inverse matrix, together with interpreting these results from the graph-theoretical point of view, is exactly the same as that used in the problem of the solution of Kirchhoff's laws for electrical circuits, a problem typical of the nineteenth century. Recall, for example, that the parameters α_l play the role of ohmic resistances and that the expression (2.25) for the function $\mathcal{U}$ as a sum over trees is a Kirchhoff result.

In practical calculations, one often derives the alpha representation for concrete diagrams by hand, rather than deduces it from the general formulae presented above. For the derivation, one can also use the public code `UF.m` [29] which is applicable also for general quadratic and linear propagators.

2.4 Regularization

The standard way of dealing with divergent Feynman integrals is to introduce a *regularization*. This means that, instead of the original ill-defined Feynman integral, we consider a quantity which depends on a regularization parameter, λ, and formally tends to the initial, meaningless expression when this parameter takes some limiting value, $\lambda = \lambda_0$. This new, regularized, quantity turns out to be well-defined, and the divergence manifests itself as a singularity with respect to the regularization parameter. Experience tells us that this singularity can be of a power or logarithmic type, i.e. $\ln^n(\lambda - \lambda_0)/(\lambda - \lambda_0)^i$.

Although a regularization makes it possible to deal with divergent Feynman integrals, it does not actually remove UV divergences, because this operation is of an auxiliary character so that sooner or later it will be necessary to switch off the regularization. To provide finiteness of physical observables evaluated through Feynman diagrams, another operation, called *renormalization*, is used. This operation is described, at the Lagrangian level, as a redefinition of the bare parameters of a given Lagrangian by inserting counterterms. The renormalization at the diagrammatic level is called *R-operation* and removes the UV divergence from individual Feynman integrals. It is, however, beyond the scope of the present book. (See, however, some details in Sect. 14.6, where the method of IR rearrangement is briefly described.)

An obvious way of regularizing Feynman integrals is to introduce a cut-off at large values of the loop momenta. Another well-known regularization procedure is the Pauli–Villars regularization [26], which is described by the replacement

$$\frac{1}{p^2 - m^2} \to \frac{1}{p^2 - m^2} - \frac{1}{p^2 - M^2}$$

and its generalizations. For finite values of the regularization parameter M, this procedure clearly improves the UV asymptotics of the integrand. Here the limiting value of the regularization parameter is $M = \infty$.

If we replace the integer powers a_l in the propagators by general complex numbers λ_l we obtain an *analytically regularized* [32] Feynman integral where the divergences of the diagram are encoded in the poles of this regularized quantity with respect to the analytic regularization parameters λ_l. For example, power counting at large values of the loop momentum in the analytically regularized version of (2.9) leads to the divergent behaviour $\int_\Lambda^\infty \mathrm{d}r\, r^{\lambda_1+\lambda_2-3}$, which results in a pole $1/(\lambda_1+\lambda_2-2)$ at the limiting values of the regularization parameters $\lambda_l = 1$.

For example, in the case of the analytically regularized integral of Fig. 2.2, we obtain

$$F_\Gamma(q;\lambda_1,\lambda_2) = \frac{\mathrm{e}^{-\mathrm{i}\pi(\lambda_1+\lambda_2+1)/2}\pi^2}{\Gamma(\lambda_1)\Gamma(\lambda_2)} \int_0^\infty \int_0^\infty \mathrm{d}\alpha_1\,\mathrm{d}\alpha_2\, \frac{\alpha_1^{\lambda_1-1}\alpha_2^{\lambda_2-1}}{(\alpha_1+\alpha_2)^2} \times \exp\left(\mathrm{i}q^2\frac{\alpha_1\alpha_2}{(\alpha_1+\alpha_2)} - \mathrm{i}(m_1^2\alpha_1+m_2^2\alpha_2)\right). \quad (2.30)$$

After the change of variables $\eta = \alpha_1+\alpha_2,\ \xi = \alpha_1/(\alpha_1+\alpha_2)$ and explicit integration over η, we arrive at

$$F_\Gamma(q;\lambda_1,\lambda_2) = \mathrm{e}^{\mathrm{i}\pi(\lambda_1+\lambda_2)}\frac{\mathrm{i}\pi^2\Gamma(\lambda_1+\lambda_2-2)}{\Gamma(\lambda_1)\Gamma(\lambda_2)} \times \int_0^1 \mathrm{d}\xi\, \frac{\xi^{\lambda_1-1}(1-\xi)^{\lambda_2-1}}{\left[m_1^2\xi+m_2^2(1-\xi)-q^2\xi(1-\xi)-\mathrm{i}0\right]^{\lambda_1+\lambda_2-2}}. \quad (2.31)$$

Thus the UV divergence manifests itself through the first pole of the gamma function $\Gamma(\lambda_1+\lambda_2-2)$ in (2.31), which results from the integration over small values of η due to the power $\eta^{\lambda_1+\lambda_2-3}$.

The alpha representation turns out to be very useful for the introduction of *dimensional* regularization, which is a commonly accepted computational technique successfully applied in practice and which will serve as the main kind of regularization in this book. Let us imagine that the number of space–time dimensions differs from four. To be more precise, the number of space dimensions is considered to be $d-1$, rather than three. (But we still think of an integer number of dimensions.) The derivation of the alpha representation does not change much in this case. The only essential change is that, instead of (2.15), we need to apply its generalization to an arbitrary number of dimensions, d:

$$\int \mathrm{d}^d k\, \mathrm{e}^{\mathrm{i}(\alpha k^2-2q\cdot k)} = \mathrm{e}^{\mathrm{i}\pi(1-d/2)/2}\pi^{d/2}\alpha^{-d/2}\mathrm{e}^{-\mathrm{i}q^2/\alpha}. \quad (2.32)$$

So, instead of (2.22), we have the following formula in d dimensions:

$$F_\Gamma(q;d) = \mathrm{e}^{-\mathrm{i}\pi(1+d/2)/2}\pi^{d/2}\int_0^\infty\int_0^\infty \mathrm{d}\alpha_1\,\mathrm{d}\alpha_2\,(\alpha_1+\alpha_2)^{-d/2}$$
$$\times\exp\left(\mathrm{i}q^2\frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2} - \mathrm{i}(m_1^2\alpha_1+m_2^2\alpha_2)\right). \qquad (2.33)$$

The only two places where something has been changed are the exponent of the combination $(\alpha_1+\alpha_2)$ in the integrand and the exponents of the overall factors.

Now, in order to introduce dimensional regularization, we want to consider the dimension d as a complex number. So, by definition, the dimensionally regularized Feynman integral for Fig. 2.2 is given by (2.33) and is a function of q^2 as given by this integral representation. We choose $d = 4-2\varepsilon$, where the value $\varepsilon = 0$ corresponds to the physical number of the space–time dimensions. By the same change of variables as used after (2.30), we obtain

$$F_\Gamma(q;d) = \mathrm{e}^{-\mathrm{i}\pi(1+d/2)/2}\pi^{d/2}\int_0^\infty \mathrm{d}\eta\,\eta^{\varepsilon-1}$$
$$\times\int_0^1 \mathrm{d}\xi\,\exp\{\mathrm{i}q^2\xi(1-\xi)\eta - \mathrm{i}[m_1^2\xi+m_2^2(1-\xi)]\eta\}. \qquad (2.34)$$

This integral is absolutely convergent for $0 < \mathrm{Re}\,\varepsilon < \Lambda$ (where $\Lambda = \infty$ if both masses are non-zero and $\Lambda = 1$ otherwise; this follows from an IR analysis of convergence, which we omit here) and defines an analytic function of ε, which is extended from this domain to the whole complex plane as a meromorphic function.

After evaluating the integral over η, we arrive at the following result:

$$F_\Gamma(q;d) = \mathrm{i}\pi^{d/2}\Gamma(\varepsilon)\int_0^1 \frac{\mathrm{d}\xi}{\left[m_1^2\xi+m_2^2(1-\xi)-q^2\xi(1-\xi)-\mathrm{i}0\right]^\varepsilon}. \qquad (2.35)$$

The UV divergence manifests itself through the first pole of the gamma function $\Gamma(\varepsilon)$ in (2.35), which results from the integration over small values of η in (2.34).

This procedure of introducing dimensional regularization is easily generalized [8, 9, 11] to an arbitrary usual Feynman integral. Instead of (2.23), we use

$$\int \mathrm{d}^d k_1 \ldots \mathrm{d}^d k_h \exp\left[\mathrm{i}\left(\sum_{i,j} A_{ij} k_i\cdot k_j + 2\sum_i q_i\cdot k_i\right)\right]$$
$$= \mathrm{e}^{\mathrm{i}\pi h(1-d/2)/2}\pi^{hd/2}(\det A)^{-d/2}\exp\left[-\mathrm{i}\sum_{i,j} A_{ij}^{-1} q_i\cdot q_j\right], \qquad (2.36)$$

and the resulting d-dimensional alpha representation takes the form [8, 9]

$$F_\Gamma(q_1,\ldots,q_n;d) = (-1)^a \frac{e^{i\pi[a+h(1-d/2)]/2}\pi^{hd/2}}{\prod_l \Gamma(a_l)} \\ \times \int_0^\infty d\alpha_1 \ldots \int_0^\infty d\alpha_L \prod_l \alpha_l^{a_l-1} \mathcal{U}^{-d/2} Z e^{i\mathcal{V}/\mathcal{U} - i\sum m_l^2 \alpha_l}. \tag{2.37}$$

Let us now define[4] the *dimensionally regularized* Feynman integral by means of (2.37), treating the quantity d as a complex number. This is a function of kinematical invariants constructed from the external momenta and contained in the function $\mathcal{V}$. In addition to this, we have to take care of polynomials in the external momenta and the auxiliary variables u_l hidden in the factor Z. We treat these objects q_i and u_l, as well as the metric tensor $g_{\mu\nu}$, as elements of an algebra of covariants, where we have, in particular,

$$\left(\frac{\partial}{\partial u_l^\mu}\right) u_{l'}^\nu = g_\mu^\nu \delta_{l,l'}, \quad g_\mu^\mu = d.$$

This algebra also includes the γ-matrices with anticommutation relations $\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2g_{\mu\nu}$ so that $\gamma^\mu\gamma_\mu = d$, the tensor $\varepsilon_{\kappa\mu\nu\lambda}$, etc.

Thus the dimensionally regularized Feynman integrals are defined as linear combinations of tensor monomials in the external momenta and other algebraic objects with coefficients that are functions of the scalar products $q_i \cdot q_j$. However, this is not all, because we have to see that the α-integral is well-defined. Remember that it can be divergent, for various reasons.

The alpha representation is not only an important technique to evaluate Feynman integrals either analytically (as explained in the next chapter) or numerically (as explained in Sects. 4.2 and 4.3) but also a convenient tool to analyze their convergence. The adequate technique for the numerical evaluation and the analysis of convergence is the same: these are sector decompositions (appeared, for the first time, in [18]) of a given alpha-parametric integral where new variables are introduced in such a way that the integrand factorizes, i.e. takes the form of a product of some powers of the sector variables with a non-zero function. Eventually, in the new

---

[4] An alternative definition of algebraic character [19, 35, 38] (see also [13]) exists and is based on certain axioms for integration in a space with non-integer dimension. It is unclear how to perform the analysis within such a definition, for example, how to apply the operations of taking a limit, differentiation, etc. to algebraically defined Feynman integrals in d dimensions, in order to say something about the analytic properties with respect to momenta and masses and the parameter of dimensional regularization. After evaluating a Feynman integral according to the algebraic rules, one arrives at some concrete function of these parameters but, *before integration*, one is dealing with an abstract algebraic object. Let us remember, however, that, in practical calculations, one usually does not bother about precise definitions. From the purely pragmatic point of view, it is useless to think of a diagram when it is not calculated. On the other hand, from the pure theoretical and mathematical point of view, such a position is beneath criticism.

variables, the analysis of convergence reduces to power counting in one-dimensional integrals.

For Feynman integrals considered at *Euclidean* external momenta q_i, i.e. when any sum of incoming momenta is spacelike,

$$\left(\sum_{i\in I} q_i\right)^2 < 0 \tag{2.38}$$

this analysis is described in Sect. 4.4.

As a result of this analysis, any Feynman integral at *Euclidean* external momenta is defined as meromorphic function of d with series of UV and IR poles [9, 27, 30, 33, 34, 36]. Here it is also assumed that there are no massless detachable subgraphs, i.e. massless subdiagrams with zero external momenta. For example, a *tadpole*, i.e. a line with coincident end points, is a detachable subgraph. However, such diagrams are naturally put to zero in case they are massless—see a discussion below.

Observe that increasing $\text{Re}\,\varepsilon$ improves UV convergence and decreasing $\text{Re}\,\varepsilon$ improves IR convergence. If a given Feynman integral is only UV or IR divergent one can apply sector decompositions and choose an appropriate domain of ε to provide the convergence and then analytically continue the integral to the whole complex plane of ε. If there are both UV and IR divergences in a given Feynman integral so that changing ε improves one kind of convergence and spoils the other kind. As explained in Sect. 4.4 one can, however, exploit an auxiliary analytic regularization and provide an ambiguous definition[5] [11] of dimensionally regularized Feynman integrals in this situation.

There are no similar mathematical results for general Feynman integrals in cases where at least some of external momenta squared are non-negative. However, one can follow a simple recipe which is implicitly adopted at least by the authors of so-called modern sector decompositions initiated in [2, 3]. According to this recipe, various subintegrals appearing in parametric representations are considered in their own domains of ε where they are convergent. We will continue this discussion in the end of Sect. 4.4 after presenting various sector decompositions in Sects. 4.1–4.3.

Let me now emphasize that one is forced to evaluate Feynman integrals on a mass shell or a threshold because they are really needed in practice. In fact, such integrals will be mainly considered in this book as examples illustrating the methods described. However, in every concrete example considered below, we will see that every Feynman diagram is indeed an analytical function of d, both in intermediate steps of a calculation and, of course, in our results.

---

[5] Besides [11], the problem of defining UV and IR divergent Feynman integrals within dimensional regularization was studied in [10] where Mellin–Barnes integrals were applied for this purpose.

2.5 Properties of Dimensionally Regularized Feynman Integrals

We can formally write down dimensionally regularized Feynman integrals as integrals over d-dimensional vectors k_i:

$$F_\Gamma(q_1, \ldots, q_n; d) = \int \mathrm{d}^d k_1 \ldots \int \mathrm{d}^d k_h \prod_{l=1}^{L} \tilde{D}_{F,l}(p_l). \quad (2.39)$$

If a tensor reduction was already performed we deal with the corresponding scalar integrals represented by the d-dimensional version of (2.7)

$$F(q_1, \ldots, q_n; a_1, \ldots, a_N; d) = \int \cdots \int \frac{\mathrm{d}^d k_1 \ldots \mathrm{d}^d k_h}{E_1^{a_1} \ldots E_N^{a_N}}, \quad (2.40)$$

where the denominators E_i have the form (2.8). The indices a_i can be either positive or non-positive so that numerators in Feynman integrals correspond to negative indices.

In order to obtain dimensionally regularized integrals with their dimension independent of ε, a factor of $\mu^{-2\varepsilon}$ per loop, where μ is a massive parameter, is introduced. This parameter serves as a renormalization parameter for schemes based on dimensional regularization. Therefore, we obtain logarithms and other functions depending not only on ratios of given parameters, e.g. q^2/m^2, but also on q^2/μ^2 etc. However, we will usually omit this μ-dependence for brevity (i.e. set $\mu = 1$) so that you will meet sometimes quantities like $\ln q^2$ which should be understood in the sense of $\ln(q^2/\mu^2)$.

We have reasons for using the notation (2.39), because dimensionally regularized Feynman integrals as defined above possess the standard properties of integrals of the usual type in integer dimensions. In particular,

- the integral of a linear combination of integrands equals the same linear combination of the corresponding integrals;
- one may cancel the same factors in the numerator and denominator of integrands.

These properties follow directly from the above definition. A less trivial property is that

- a derivative of an integral with respect to a mass or momentum equals the corresponding integral of the derivative.

This is also a consequence (see [11, 30]) of the definition of dimensionally regularized Feynman integrals based on the alpha representation and the corresponding analysis of convergence presented in Sect. 4.4. To prove this statement, one uses standard algebraic relations between the functions entering the alpha representation [9, 23]. (We note again that these are relations quite similar to those encoded in the solutions of Kirchhoff's laws for a circuit defined by the given graph.) A corollary of the last property is the possibility of integrating by parts and always neglecting surface terms:

$$\int \mathrm{d}^d k_1 \ldots \int \mathrm{d}^d k_h \left(\frac{\partial}{\partial k_i} \cdot r_j \right) \prod_{l=1}^{L} \tilde{D}_{F,l}(p_l) = 0 \; , \quad i = 1, \ldots, h, \tag{2.41}$$

where r_j is a loop or external momentum.

This property is the basis for solving the reduction problem for Feynman integrals using IBP relations [12]—see Chap. 6.

The next property says that

- any diagram with a detachable massless subgraph is zero.

Let us consider, for example, the massless tadpole diagram, which can be reduced by means of alpha parameters to a scaleless one-dimensional integral:

$$\int \frac{\mathrm{d}^d k}{k^2} = -\mathrm{i}^{\varepsilon} \pi^{d/2} \int_0^{\infty} \mathrm{d}\alpha \, \alpha^{\varepsilon-2}. \tag{2.42}$$

We divide this integral into two pieces, from 0 to 1 and from 1 to ∞, evaluate these two integrals and find results that are equal except for opposite signs, which lead to the zero value.[6] It should be emphasized here that the two pieces that contribute to the right-hand side of (2.42) are convergent in *different* domains of the regularization parameter ε, namely, $\mathrm{Re}\,\varepsilon > -1$ and $\mathrm{Re}\,\varepsilon < -1$, with no intersection.

A massless Feynman integral with a zero external momentum can appear either in the beginning when using Feynman rules, or after some manipulations: after using partial fractions, integration by parts, etc. We can also include in this second class all such integrals that appear on the right-hand side of asymptotic expansions in momenta and masses [1, 31]—see Chap. 9. In any case, one sets such integrals to zero. In fact, in any massless Feynman integral at zero external momenta, one can reveal an internal one-dimensional integral with a pure power, similar to the above integral for the tadpole (2.42). We will come back to this point in Chap. 4.

On-shell and threshold Feynman integrals have been already mentioned many times, so that let us consider several typical one-loop examples. We must realize that, generally, an on-shell or threshold Feynman integral is *not* the value of the given Feynman integral $F_\Gamma(q^2, \ldots)$, defined as a function of q^2 and other kinematical variables, at a value of q^2 on a mass shell or at a threshold. Consider, for example, the Feynman integral corresponding to Fig. 2.2, with $m_1 = 0,\ m_2 = m,\ a_1 = 1,\ a_2 = 2$. We know an explicit result for the diagram given by (1.5). There is a logarithmic singularity at threshold, $q^2 = m^2$, so that we cannot strictly speak about the value of the integral there. Still we can certainly define the threshold Feynman integral by putting $q^2 = m^2$ *in the integrand* of the integral over the loop momentum or over the alpha parameters. And this is what was really meant and will be meant by 'on-shell' and 'threshold' integrals. In this example, we obtain an integral which can be evaluated by means of (10.13) (to be derived in Chap. 3):

[6] These arguments can be found, for example, in [20], and, ironically, even in a pure mathematical book [16].

$$\int \frac{\mathrm{d}^d k}{k^2(k^2 - 2q \cdot k)^2} = \mathrm{i}\pi^{d/2} \frac{\Gamma(\varepsilon)}{2(m^2)^{1+\varepsilon}}. \tag{2.43}$$

This integral is divergent, in contrast to the original Feynman integral defined for general q^2.

Thus on-shell or threshold dimensionally regularized Feynman integrals are defined by the alpha representation or by integrals over the loop momenta with restriction of some kinematical invariants to appropriate values in the corresponding integrands. In this sense, these regularized integrals are ‘formal’ values of general Feynman integrals at the chosen variables.

Note that the products of the free fields in the Lagrangian are not required to be normal-ordered, so that products of fields of the same sort at the same point are allowed. The formal application of the Wick theorem therefore generates values of the propagators at zero. For example, in the case of the scalar free field, with the propagator

$$D_F(x) = \frac{\mathrm{i}}{(2\pi)^4} \int \mathrm{d}^4 k \frac{\mathrm{e}^{-\mathrm{i}x\cdot k}}{k^2 - m^2}, \tag{2.44}$$

which satisfies $(\Box + m^2) D_F(x) = -\mathrm{i}\delta(x)$, we have

$$T\phi(x)\phi(x) =: \phi^2(x) : + D_F(0). \tag{2.45}$$

The value of $D_F(x)$ at $x = 0$ does not exist, because the propagator is singular at the origin according to (2.17). However, we imply the *formal* value at the origin rather than the ‘honestly’ taken value. This means that we set x to zero in some integral representation of this quantity. For example, using the inverse Fourier transformation, we can define $D_F(0)$ as the integral (2.44) with x set to zero *in the integrand*. Thus, by definition,

$$D_F(0) = \frac{\mathrm{i}}{(2\pi)^4} \int \frac{\mathrm{d}^4 k}{k^2 - m^2}. \tag{2.46}$$

This integral is, however, quadratically divergent, as Feynman integrals typically are. So, we understand $D_F(0)$ as a dimensionally regularized formal value when we put $x = 0$ in the Fourier integral and obtain, using (10.1) (which we will derive shortly),

$$\int \frac{\mathrm{d}^d k}{k^2 - m^2} = -\mathrm{i}\pi^{d/2} \Gamma(\varepsilon - 1)(m^2)^{1-\varepsilon}. \tag{2.47}$$

This Feynman integral in fact corresponds to the tadpole ϕ^4 theory graph shown in Fig. 2.5. The corresponding quadratic divergence manifests itself through an UV pole in ε—see (2.47).

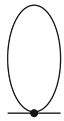


Fig. 2.5 Tadpole

Observe that one can trace the derivation of the integrals tabulated in Sect. 10.1 and see that the integrals are convergent in some non-empty domains of the complex parameters λ_l and ε and that the results are analytic functions of these parameters with UV, IR and collinear poles.

Before continuing our discussion of setting scaleless integrals to zero, let us present an analytic result for the one-loop massless triangle integral with two on-shell external momenta, $p_1^2 = p_2^2 = 0$. Using (10.28) (which we will derive in Chap. 3), we obtain

$$\int \frac{\mathrm{d}^d k}{(k^2 - 2p_1\cdot k)(k^2 - 2p_2\cdot k)k^2} = -\mathrm{i}\pi^{d/2} \frac{\Gamma(1+\varepsilon)\Gamma(-\varepsilon)^2}{\Gamma(1-2\varepsilon)(-q^2)^{1+\varepsilon}}. \tag{2.48}$$

A double pole at $\varepsilon = 0$ arises from the IR and collinear divergences.

A similar formula with a monomial in the numerator can be obtained also straightforwardly:

$$\int \frac{\mathrm{d}^d k\; k^\mu}{(k^2 - 2p_1\cdot k)(k^2 - 2p_2\cdot k)k^2} = \mathrm{i}\pi^{d/2} \frac{\Gamma(\varepsilon)\Gamma(1-\varepsilon)^2}{\Gamma(2-2\varepsilon)} \frac{p_1^\mu + p_2^\mu}{(-q^2)^{1+\varepsilon}}. \tag{2.49}$$

Now only a simple pole is present, because the factor k^μ kills the IR divergence.

Consider now a massless one-loop integral with the external momentum on the massless mass shell, $p^2 = 0$:

$$\int \frac{\mathrm{d}^d k}{(p-k)^2 k^2}. \tag{2.50}$$

If we write down the alpha representation for this integral we obtain the same expression (2.42) as for $p = 0$ because only p^2, equal to zero in both cases, is involved there. In spite of this obvious fact, there is still a qualitative difference: for $p = 0$, there are UV and IR poles which enter with opposite signs and, for $p^2 = 0$ (but with $p \neq 0$ as a d-dimensional vector), there is a similar interplay of UV and collinear poles.

Now we follow the arguments presented in [24] and write down the following identity for (2.50), with $p = p_1$:

$$\int \frac{\mathrm{d}^d k}{(k^2 - 2p_1\cdot k)k^2}$$
$$= \int \frac{\mathrm{d}^d k}{(k^2 - 2p_1\cdot k)(k^2 - 2p_2\cdot k)} - \int \frac{\mathrm{d}^d k\; 2p_2\cdot k}{(k^2 - 2p_1\cdot k)(k^2 - 2p_2\cdot k)k^2}, \tag{2.51}$$

where $p_2^2 = 0$ and $p_1 \cdot p_2 \neq 0$. We then evaluate the integrals on the right-hand side by means of (10.7) and (2.49), respectively, and obtain a zero value. This fact again exemplifies the consistency of our rules.

Thus we are going to systematically apply the properties of dimensionally regularized Feynman integrals in any situation, no matter where the external momenta are considered to be. Moreover, we will believe that these properties are also valid for more general Feynman integrals given by the dimensionally regularized version of (2.7) which can contain linear propagators.

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Chapter 3
Evaluating by Alpha and Feynman Parameters

Feynman parameters[1] are very well known and often used in practical calculations. They are closely related to alpha parameters introduced in Chap. 2. The use of both kinds of parameters enables us to transform Feynman integrals over loop momenta into parametric integrals where Lorentz invariance becomes manifest. Using alpha parameters we will first evaluate one and two-loop integrals with general complex powers of the propagators, within dimensional regularization, for which results can be written in terms of gamma functions for general values of the dimensional regularization parameter. We will show then how these formulae, together with simple algebraic manipulations, enable us to evaluate some classes of Feynman integrals.

We then turn to various characteristic one-loop examples where results cannot be written in terms of gamma functions. In such situations, we will be usually oriented at the evaluation in expansion in powers of ε up to some fixed order. We then introduce Feynman parameters and present the so-called Cheng–Wu theorem which provides a very useful trick that can greatly simplify the evaluation. Finally, we proceed at the two-loop level by presenting more complicated examples of evaluating Feynman integrals by Feynman and alpha parameters.

3.1 Simple One- and Two-Loop Formulae

A lot of one- and two-loop formulae can be derived, using alpha and Feynman parameters, for general complex indices with results expressed in terms of gamma functions. A collection of such formulae is presented in Sect. 10.1.

Let us evaluate, for example, the dimensionally regularized massive tadpole Feynman diagram of Fig. 2.5 with a general power of the propagator,

$$F_\Gamma(q;\lambda;d) = \int \frac{\mathrm{d}^d k}{(-k^2+m^2)^\lambda}. \tag{3.1}$$

[1] See, e.g. textbooks [20] and [7] and a recent review [26].

V. A. Smirnov, *Analytic Tools for Feynman Integrals*, Springer Tracts in Modern Physics 250, DOI: 10.1007/978-3-642-34886-0_3,

We apply the alpha representation of the analytically regularized scalar propagator given by (2.13) with $Z = 1$, i.e.

$$\frac{1}{(-k^2+m^2)^\lambda} = \frac{\mathrm{i}^\lambda}{\Gamma(\lambda)} \int_0^\infty \mathrm{d}\alpha\, \alpha^{\lambda-1} \mathrm{e}^{\mathrm{i}(k^2-m^2)\alpha}, \tag{3.2}$$

change the order of integration over k and α, take the Gaussian k integral by means of (2.32), again apply (3.2) written in the reverse order, i.e.

$$\int_0^\infty \mathrm{d}\alpha\, \alpha^{\lambda-1} \mathrm{e}^{-\mathrm{i}A\alpha} = \frac{\Gamma(\lambda)\, \mathrm{i}^{-\lambda}}{(A-\mathrm{i}0)^\lambda}, \tag{3.3}$$

and arrive at (10.1). In particular, this table formula gives (2.47).

Let us now turn to the dimensionally regularized Feynman diagram of Fig. 2.2 with general powers of the propagators,

$$F_\Gamma(q;\lambda_1,\lambda_2;d) = \int \frac{\mathrm{d}^d k}{(-k^2+m_1^2)^{\lambda_1}[-(q-k)^2+m_2^2]^{\lambda_2}}. \tag{3.4}$$

From now on, we will use the following convention: when powers of propagators are integers we use them with $+k^2+\mathrm{i}0$, but when they are non-integral or complex, we take the opposite sign, i.e. $-k^2-\mathrm{i}0$. The second choice is more natural if we wish to obtain a Euclidean, $-q^2$, dependence of the results (see, e.g. (3.6)). We will also prefer to use a_l for integer and λ_l for general complex indices. In the latter case, the alpha representation is obtained from (2.37) by replacing a_l by λ_l and dropping out the factor $(-1)^a$.

Starting from the alpha representation of Fig. 2.2, with the basic functions $\mathcal{U} = \alpha_1+\alpha_2$ and $\mathcal{V} = \alpha_1\alpha_2 q^2$, and using the change of variables $\alpha_1 = \xi\eta,\ \alpha_2 = \eta(1-\xi)$ we obtain the dimensionally regularized version of (2.31), i.e.

$$\begin{aligned} F_\Gamma(q;\lambda_1,\lambda_2;d) &= \mathrm{i}\pi^{d/2} \frac{\Gamma(\lambda_1+\lambda_2+\varepsilon-2)}{\Gamma(\lambda_1)\Gamma(\lambda_2)} \\ &\quad \times \int_0^1 \frac{\mathrm{d}\xi\, \xi^{\lambda_1-1}(1-\xi)^{\lambda_2-1}}{\left[m_1^2\xi + m_2^2(1-\xi) - q^2\xi(1-\xi) - \mathrm{i}0\right]^{\lambda_1+\lambda_2+\varepsilon-2}}. \end{aligned} \tag{3.5}$$

Suppose that the masses are zero. In this case the integral over ξ can be evaluated in terms of gamma functions, and we arrive at the following result:

$$\int \frac{\mathrm{d}^d k}{(-k^2)^{\lambda_1}[-(q-k)^2]^{\lambda_2}} = \mathrm{i}\pi^{d/2} \frac{G(\lambda_1,\lambda_2)}{(-q^2)^{\lambda_1+\lambda_2+\varepsilon-2}}, \tag{3.6}$$

where

$$G(\lambda_1,\lambda_2) = \frac{\Gamma(\lambda_1+\lambda_2+\varepsilon-2)\Gamma(2-\varepsilon-\lambda_1)\Gamma(2-\varepsilon-\lambda_2)}{\Gamma(\lambda_1)\Gamma(\lambda_2)\Gamma(4-\lambda_1-\lambda_2-2\varepsilon)}. \quad (3.7)$$

The one-loop formula (3.6) can graphically be described by Fig. 3.1.

In the case where the powers of propagators are equal to one, we have

$$\int \frac{\mathrm{d}^d k}{k^2(q-k)^2} = \mathrm{i}\pi^{d/2} \frac{\Gamma(\varepsilon)\Gamma(1-\varepsilon)^2}{\Gamma(2-2\varepsilon)(-q^2)^{\varepsilon}}. \quad (3.8)$$

Observe that although the indices of the diagrams are integer at the beginning, non-integer indices shifted by amounts proportional to ε appear after an intermediate integration, e.g. after the use of (3.8) inside a bigger diagram.

Another formula that can be derived from (3.5) gives a result for the integral

$$\int \frac{\mathrm{d}^d k}{(-k^2+m^2)^{\lambda_1}(-k^2)^{\lambda_2}}.$$

Indeed, we set $q=0$, $m_1=m$ and $m_2=0$, take an integral over ξ and obtain (10.4).

Consider now the following integral that arises in calculations in HQET [13, 16, 18]:

$$\int \frac{\mathrm{d}^d k}{(-k^2)^{\lambda_1}(2v\cdot k+\omega-\mathrm{i}0)^{\lambda_2}}.$$

Since the denominator of one of the propagators is not quadratic we cannot use the general formula of the alpha representation. Still we proceed by alpha parameters, i.e. apply (3.2) to the first propagator and a similar Fourier representation

$$\frac{1}{(-A-\mathrm{i}0)^{\lambda}} = \frac{\mathrm{i}^{\lambda}}{\Gamma(\lambda)}\int_0^{\infty} \mathrm{d}\alpha\, \alpha^{\lambda-1}\mathrm{e}^{\mathrm{i}A\alpha}, \quad (3.9)$$

with $A=-2v\cdot k-\omega$, to the second propagator. Changing the order of integration as above and evaluating a Gaussian integral over k we then apply (3.3) to take the integral

$$\int_0^{\infty} \alpha_1^{\lambda_1+\varepsilon-3}\mathrm{e}^{-\mathrm{i}\alpha_2^2 v^2/\alpha_1}\mathrm{d}\alpha_1$$

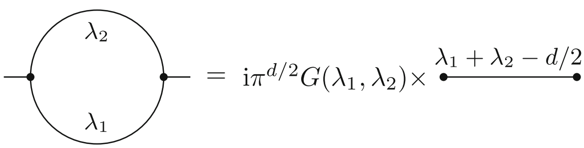

Fig. 3.1 Graphical interpretation of (3.6)

and, finally, an integral over α_2, and arrive at (10.25).

This formula can be used to calculate the integral

$$\int \frac{\mathrm{d}^d k}{(-k^2)^{\lambda_1}(-2v\cdot(q-k)-\mathrm{i}0)^{\lambda_2}}. \tag{3.10}$$

The graphical interpretation of the corresponding result is shown in Fig. 3.2, where the dashed line denotes the propagator $1/(-2v\cdot k)$ and $\bar{G}$ is the function that enters the right-hand side of (10.25).

The following one-loop integral is typical for the evaluation of the one-loop static quark potential:

$$\int \frac{\mathrm{d}^d k}{(-k^2)^{\lambda_1}[-(q-k)^2]^{\lambda_2}(-2v\cdot k-\mathrm{i}0)^{\lambda_3}}.$$

Here $v \cdot q = 0$. (Typically, one chooses $q = (0, \mathbf{q})$ and $v = (1, \mathbf{0})$.) One of the propagators is not quadratic so that we proceed by alpha parameters and represent each of the three factors as an alpha integral. After taking a Gaussian integral over k we obtain

$$\frac{\mathrm{i}^{\lambda_1+\lambda_2+\lambda_3+\varepsilon-1}\pi^{d/2}}{\prod_l \Gamma(\lambda_l)} \int_0^\infty \int_0^\infty \int_0^\infty \left(\prod_{l=1}^3 \alpha_l^{\lambda_l-1}\mathrm{d}\alpha_l\right)(\alpha_1+\alpha_2)^{\varepsilon-2}$$
$$\times \exp\left(\mathrm{i}\frac{q^2\alpha_1\alpha_2 - v^2\alpha_3^2}{\alpha_1+\alpha_2}\right).$$

Then the integral over α_3 can be evaluated by the change $\alpha_3 = \sqrt{t}$ and (3.3). After that the integration over α_1 and α_2 is taken, as before, by introducing the variables $\eta = \alpha_1 + \alpha_2$, $\xi = \alpha_1/(\alpha_1+\alpha_2)$, with the result (10.27).

Using alpha parameters one can also derive the formula (10.42) for the formal Fourier transformation within dimensional regularization. This formula provides another way to derive (3.6). In fact, the initial integral is nothing but the convolution of the two functions, $\tilde{f}_i = 1/(-k^2-\mathrm{i}0)^{\lambda_i}$, $i = 1, 2$. Then one uses the well-known mathematical formula

$$\left(\tilde{f}_1 * \tilde{f}_2\right)(q) = (2\pi)^d (f_1 \tilde{\ } f_2)$$

for the convolution of two Fourier transforms, applies (10.42) and arrives at (3.6).

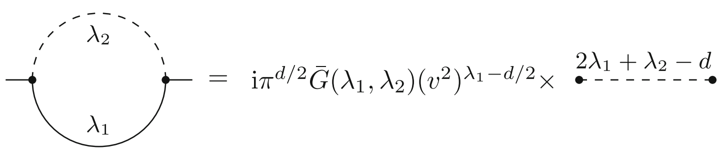

Fig. 3.2 Result for (3.10) in the graphical form

3.2 Auxiliary Tricks

3.2.1 Recursively One-Loop Feynman Integrals

Massless integrals are often evaluated with the help of successive application of the one-loop formula (3.6). In addition one can use the fact that a sequence of two lines with scalar propagators with the same mass and the indices a_1 and a_2 can be replaced by one line with index $a_1 + a_2$. Consider, for example, the two-loop diagram shown in Fig. 3.3. The internal one-loop integral can be evaluated by use of (3.8) and is effectively replaced, according to Fig. 3.1, by a line with index ε. Then the sequence of two massless lines with indices 1 and ε is replaced by one line with index $1+\varepsilon$, and the one-loop diagram so obtained, which has indices 2 and $1+\varepsilon$, is evaluated by means of the one-loop formula (3.6), with the following result expressed in terms of gamma functions: $G(1, 1)G(2, 1+\varepsilon)/(-q^2)^{1+2\varepsilon}$. The class of Feynman diagrams that can be evaluated in this way by means of (3.6) can be called *recursively one-loop*.

Another example where two tabulated one-loop integration formulae can successively be applied is given by the two-loop scalar diagram of Fig. 3.4 with general complex indices and two zero masses,

$$\int\int \frac{\mathrm{d}^d k\, \mathrm{d}^d l}{(-k^2)^{\lambda_1}[-(k+l)^2]^{\lambda_2}(m^2-l^2)^{\lambda_3}}.$$

Here one can first apply the one-loop massless integration formula (3.6), then apply (10.4) and obtain (10.39).

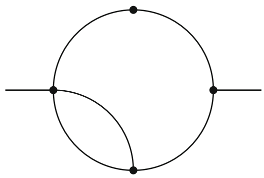

Fig. 3.3 A recursively one-loop diagram

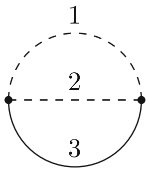

Fig. 3.4 Vacuum two-loop diagram with the masses 0, 0 and m

3.2.2 Partial Fractions

When evaluating dimensionally regularized Feynman integrals one uses their properties, in particular the possibility of manipulations based on the properties listed in Sect. 2.5. Here the following standard decomposition proves to be useful:

$$\frac{1}{(x+x_1)^{a_1}(x+x_2)^{a_2}} = \sum_{i=0}^{a_1-1} \binom{a_2-1+i}{a_2-1} \frac{(-1)^i}{(x_2-x_1)^{a_2+i}(x+x_1)^{a_1-i}}$$
$$+ \sum_{i=0}^{a_2-1} \binom{a_1-1+i}{a_1-1} \frac{(-1)^{a_1}}{(x_2-x_1)^{a_1+i}(x+x_2)^{a_2-i}}, \tag{3.11}$$

where $a_1, a_2 > 0$ and

$$\binom{n}{j} = \frac{n!}{j!(n-j)!}$$

is a binomial coefficient.

For example, the vacuum one-loop Feynman integral with two different masses,

$$\int \frac{\mathrm{d}^d k}{(k^2-m_1^2)(k^2-m_2^2)},$$

can be evaluated by (3.11) and (10.1), with the result

$$\mathrm{i}\pi^{d/2}\Gamma(\varepsilon-1)\frac{m_2^{2-2\varepsilon}-m_1^{2-2\varepsilon}}{m_1^2-m_2^2}.$$

If one of the indices, e.g. a_2 is non-positive, a similar decomposition is performed by expanding $(x+x_2)^{-a_2}$ in powers of $x+x_1$. Let us note that if one proceeds by `MATHEMATICA` [27], one can use, for given integer values of a_1 and a_2, the command `Apart` to perform partial fractions decompositions.

3.2.3 Dealing with Numerators

As we have agreed we suppose that a tensor reduction for a given class of Feynman integrals was performed so that we start with evaluating scalar integrals. Let us, however, mention that one can also evaluate integrals with Lorentz indices. A lot of one-loop Feynman integrals with numerators can be found in Sect. 10.1. One can reduce evaluating such a one-loop integral to an integral with a product $k^{\alpha_1}\dots k^{\alpha_N}$. Then one can switch to traceless monomials and back using (10.43a) and (10.43b). An integral with a traceless monomial independent of other Lorentz indices is again traceless. If it depends on one external momentum it should be proportional to its traceless monomial. This is how tabulated integrals for traceless monomials, e.g. (10.8), can be derived. Then one can turn back to usual monomials using (10.43b). (In Sect. 10.2, one can find also other useful formulae for various traceless monomials.) However, I would recommend to apply the recently developed algorithm and the corresponding public code [19] which can work successfully, for example, for four-loop propagator integrals with up to six indices in the numerator.

Another way of dealing with numerators is by shifting dimension. In the case of a general h-loop Feynman integral with standard propagators, let us observe that the function (2.27) in (2.37) can be taken into account by shifting the space–time dimension d and indices a_l of a given diagram because any factor that arises after the differentiation with respect to the auxiliary parameters u_l is a sum of products of positive integer powers of the α-parameters and negative integer powers of the function $\mathcal{U}$. In particular, the factor $1/\mathcal{U}^n$ is taken into account by the shift $d \to d+2n$. Then the shift of a power of a parameter α_l can be translated into a shift of the power of the corresponding propagator, in particular, a multiplication by α_l can be described by the operator $\mathrm{i}a_l\mathbf{l}^+$ where $\mathbf{l}^+$ increases the index a_l by one, the multiplication by α_l^2 can be described by the operator $-a_l(a_l+1)\mathbf{l}^{++}$, etc.

This observation enables us to express any given Feynman integral with numerators through a linear combination of scalar integrals with shifted indices and shifted dimensions. Systematic algorithms oriented towards implementation on a computer, with a demonstration up to two-loop level, have been constructed in [23, 24]. We will come back to this point in Chap. 6 when solving IBP recurrence relations.

At the one-loop level, this property has been used [9] to derive a general formula for the Feynman integrals

$$F^{(N)}_{\alpha_1\ldots\alpha_n}(\lambda_1,\ldots,\lambda_N,d) = \int \mathrm{d}^d k \frac{k_{\alpha_1}\ldots k_{\alpha_n}}{\prod_{i=1}^N[-(q_i-k)^2+m_i^2]^{\lambda_i}}, \tag{3.12}$$

depending on the external momenta $q_1-q_2,\ldots,q_N-q_1$ and the general masses m_i:

$$\begin{aligned} F^{(N)}_{\alpha_1\ldots\alpha_n}(\lambda_1,\ldots,\lambda_N,d) = & \sum_{r,\kappa_1,\ldots,\kappa_N:\, 2r+\sum\kappa_i=n} \frac{(-1)^r}{2^r} \\ & \times \left\{ \{[g]^r[q_1]^{\kappa_1}\ldots[q_N]^{\kappa_N}\}_{\alpha_1\ldots\alpha_n} \left(\prod_{i=1}^N(\lambda_i)_{\kappa_i}\right)\right. \\ & \left. \times F^{(N)}(\lambda_1+\kappa_1,\ldots,\lambda_N+\kappa_N,d+2(n-r))\right\}, \end{aligned} \tag{3.13}$$

where $\{[g]^r[q_1]^{\kappa_1}\ldots[q_N]^{\kappa_N}\}_{\alpha_1\ldots\alpha_n}$ is symmetric in its indices and is composed of the metric tensor and the vectors q_i. Tabulated formulae with numerators presented in Appendix A can be derived by means of (3.13).

Let us now present a simple one-loop example and illustrate the trick with turning to integrals without numerators. Consider the Feynman integral corresponding to Fig. 3.5 with a numerator

$$
\begin{aligned}
&F(q^2, m^2; a_1, a_2, a_3; n; d) \\
&\quad = \int \frac{\mathrm{d}^d k\, (l\cdot k)^n}{(k^2 - 2p_1\cdot k)^{a_1}(k^2 - 2p_2\cdot k)^{a_2}(k^2 - m^2)^{a_3}}\,,
\end{aligned}
\tag{3.14}
$$

where l is a momentum not related to p_1 and p_2. The alpha representation (2.37) takes the form

$$
\begin{aligned}
F(q^2, m^2; a_1, a_2, a_3; n; d) &= (-1)^a \frac{\mathrm{i}^{a_1+a_2+a_3+\varepsilon-1}\pi^{d/2}}{\prod_l \Gamma(a_l)} \\
&\times \int_0^\infty \int_0^\infty \int_0^\infty \mathrm{d}\alpha_1 \mathrm{d}\alpha_2 \mathrm{d}\alpha_3 \prod_l \alpha_l^{a_l-1} \mathcal{U}^{-d/2} \exp\left\{\mathrm{i}\mathcal{V}/\mathcal{U} - \mathrm{i}m^2\alpha_3\right\} \\
&\times \left(\frac{1}{2\mathrm{i}}\frac{\partial}{\partial r}\right)^n \exp\left\{\frac{\mathrm{i}[2rl\cdot(\alpha_1 p_1 + \alpha_2 p_2) + r^2 l^2]}{\alpha_1+\alpha_2+\alpha_3}\right\}\Bigg|_{r=0}\,,
\end{aligned}
\tag{3.15}
$$

where

$$
\mathcal{U} = \alpha_1 + \alpha_2 + \alpha_3, \quad \mathcal{V} = q^2 \alpha_1 \alpha_2 .
$$

Taking into account the arguments above we see, for example, that

$$
\begin{aligned}
F(a_1, a_2, a_3; 1; d) = -\frac{1}{\pi}\,&[a_1 l\cdot p_1 F(a_1+1, a_2, a_3; 0; d+2) \\
&+ a_2 l\cdot p_2 F(a_1, a_2+1, a_3; 0; d+2)]\,,
\end{aligned}
\tag{3.16}
$$

$$
\begin{aligned}
F(a_1, a_2, a_3; 2; d) = \frac{l^2}{2\pi}&F(a_1, a_2, a_3; 0; d+2) \\
&+ \frac{1}{\pi^2}\Big[a_1(a_1+1)(l\cdot p_1)^2 F(a_1+2, a_2, a_3; 0; d+4) \\
&+ 2a_1a_2(l\cdot p_1)(l\cdot p_2) F(a_1+1, a_2+1, a_3; 0; d+4) \\
&+ a_2(a_2+1)(l\cdot p_2)^2 F(a_1, a_2+2, a_3; 0; d+4)\Big]\,.
\end{aligned}
\tag{3.17}
$$

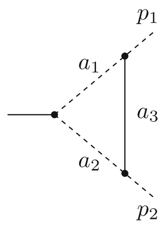

Fig. 3.5 Triangle diagram with the masses 0, 0, m, external momenta $p_1^2 = p_2^2 = 0$ and general indices

Such a reduction of numerators can be performed for any Feynman integral. The corresponding algebraic manipulations can easily be implemented on a computer.

3.3 One-Loop Examples

Let us present examples of evaluation of Feynman diagrams by means of alpha parameters with results which are not written in terms of gamma functions for general d. We first turn to the example considered in the introduction.

Example 3.1 One-loop propagator Feynman integrals (1.2) corresponding to Fig. 1.1.

We apply (3.5) to obtain

$$F(q^2, m^2; a_1, a_2; d) = \mathrm{i}\pi^{d/2}(-1)^{a_1+a_2} \frac{\Gamma(a_1 + a_2 + \varepsilon - 2)}{\Gamma(a_1)\Gamma(a_2)} \times \int_0^1 \frac{\mathrm{d}\xi \; \xi^{a_2-1}(1-\xi)^{1-a_2-\varepsilon}}{\left[m^2 - q^2\xi - \mathrm{i}0\right]^{a_1+a_2+\varepsilon-2}} . \tag{3.18}$$

For example, we have

$$F(q^2, m^2; 2, 1; d) \equiv \int \frac{\mathrm{d}^d k}{(k^2 - m^2)^2 (q-k)^2} \tag{3.19}$$

$$= -\mathrm{i}\pi^{d/2}\Gamma(1+\varepsilon) \int_0^1 \frac{(1-\xi)^{-\varepsilon}\mathrm{d}\xi}{\left[m^2 - q^2\xi - \mathrm{i}0\right]^{1+\varepsilon}} .$$

Suppose that we are interested only in the value of this (finite) integral exactly in four dimensions. The integral over ξ is then evaluated easily at $\varepsilon = 0$ with the result (1.5). Similarly, Feynman integrals corresponding to Fig. 1.1 with various integer indices a_i can be evaluated. In particular, we obtain (1.7).

Let us now evaluate

$$F(q^2, m^2; 1, 2; d) \equiv \int \frac{\mathrm{d}^d k}{(k^2 - m^2)[(q-k)^2]^2}$$

$$= -\mathrm{i}\pi^{d/2}\Gamma(1+\varepsilon) \int_0^1 \frac{\xi^{-1-\varepsilon}(1-\xi)\mathrm{d}\xi}{\left[m^2 - q^2(1-\xi) - \mathrm{i}0\right]^{1+\varepsilon}} \tag{3.20}$$

in an expansion in ε up to the finite part. This time, there is an IR pole in ε which is generated due to integration over small ξ. The standard procedure to extract the pole is to make a subtraction of the integrand, integrate the subtracted expression by expanding the integrand in ε and integrate the subtracted term explicitly. In our case, this is achieved by the following decomposition of the integral:

$$
\begin{aligned}
F(q^2, m^2; 1, 2; d) &= -\mathrm{i}\pi^{d/2}\Gamma(1+\varepsilon) \\
&\times \left[\int_0^1 \frac{\mathrm{d}\xi}{\xi^{1+\varepsilon}} \left\{ \frac{1-\xi}{[m^2 - q^2(1-\xi)]^{1+\varepsilon}} - \frac{1}{(m^2-q^2)^{1+\varepsilon}} \right\} \right. \\
&\left. + \frac{1}{\left(m^2-q^2\right)^{1+\varepsilon}} \int_0^1 \frac{\mathrm{d}\xi}{\xi^{1+\varepsilon}} \right].
\end{aligned} \tag{3.21}
$$

The last integral is

$$
\int_0^1 \frac{\mathrm{d}\xi}{\xi^{1+\varepsilon}} = -\frac{1}{\varepsilon}\, \xi^{-\varepsilon}\Big|_0^1 = -\frac{1}{\varepsilon}.
$$

When evaluating it we imply that the real part of ε is positive and then obtain a result which can be continued analytically to the whole complex plane of ε. We will later follow such prescriptions in similar situations.

The first integral is now convergent uniformly in ε and can be evaluated by expanding the integrand in a Taylor series in ε. Expanding up to ε^0 and evaluating the corresponding integral we obtain the following result:

$$
\begin{aligned}
&F(q^2, m^2; 1, 2; d) \\
&\quad = \frac{\mathrm{i}\pi^{d/2} \mathrm{e}^{-\gamma_{\mathrm{E}}\varepsilon}}{m^2 - q^2} \left[\frac{1}{\varepsilon} - \ln(m^2 - q^2) - \frac{m^2}{q^2} \ln\left(1 - \frac{q^2}{m^2}\right) \right].
\end{aligned} \tag{3.22}
$$

Here and in all the expansions in ε below we pull out the factor $\mathrm{e}^{-\gamma_{\mathrm{E}}\varepsilon}$, with Euler's constant γ_{E}, per loop in order to avoid γ_{E} in our results.

The next one-loop example is

Example 3.2 The triangle diagram of Fig. 3.5.

The Feynman integral for Fig. 3.5 with general integer indices looks like (3.14) with $n = 0$, i.e.

$$
\begin{aligned}
&F(q^2, m^2; a_1, a_2, a_3; d) \\
&\quad = \int \frac{\mathrm{d}^d k}{(k^2 - 2p_1 \cdot k)^{a_1} (k^2 - 2p_2 \cdot k)^{a_2} (k^2 - m^2)^{a_3}},
\end{aligned} \tag{3.23}
$$

where $q = p_1 - p_2$, $q^2 \equiv -Q^2 = -2p_1 \cdot p_2$. The alpha representation (2.37) takes the form (3.15) with $n = 0$.

Introducing variables $\alpha_1 = \xi_1\eta$, $\alpha_2 = \xi_2\eta$ and $\alpha_3 = (1-\xi_1-\xi_2)\eta$ and integrating over η we obtain

$$F(q^2, m^2; a_1, a_2, a_3; d) = \frac{i\pi^{d/2}(-1)^{a_1+a_2+a_3}\Gamma(a+\varepsilon-2)}{\prod_l \Gamma(a_l)}$$
$$\times \int_0^1 d\xi_1 \int_0^{1-\xi_1} d\xi_2 \frac{\xi_1^{a_1-1}\xi_2^{a_2-1}(1-\xi_1-\xi_2)^{a_3-1}}{[Q^2\xi_1\xi_2 + m^2(1-\xi_1-\xi_2)]^{a+\varepsilon-2}}. \quad (3.24)$$

This can be a reasonable starting point for the evaluation of integrals with any given indices a_i. Let us evaluate the integral with $a_1 = a_2 = a_3 = 1$ at $d = 4$. Then the integral is finite:

$$F(q^2, m^2; 1, 1, 1; 4) = -i\pi^2 \int_0^1 d\xi_1 \int_0^{1-\xi_1} \frac{d\xi_2}{Q^2\xi_1\xi_2 + m^2(1-\xi_1-\xi_2)}.$$

A straightforward integration gives the following result:

$$F(q^2, m^2; 1, 1, 1; 4)$$
$$= \frac{i\pi^2}{Q^2}\left(Li_2(x) - \frac{1}{2}\ln^2 x + \ln x \ln(1-x) - \frac{\pi^2}{3}\right), \quad (3.25)$$

where $\mathrm{Li}_2(x)$ is the dilogarithm (see (11.7)) and $x = m^2/Q^2$.

Example 3.3 The massless on-shell box diagram of Fig. 3.6, i.e. with $p_i^2 = 0, i = 1, 2, 3, 4$.

With the loop momentum chosen as the momentum of line 1, the Feynman integral takes the form

$$F(s, t; a_1, a_2, a_3, a_4; d)$$
$$= \int \frac{d^d k}{(k^2)^{a_1}[(k+p_1)^2]^{a_2}[(k+p_1+p_2)^2]^{a_3}[(k-p_3)^2]^{a_4}}, \quad (3.26)$$

where $s = (p_1+p_2)^2$ and $t = (p_1+p_3)^2$ are Mandelstam variables.

The trees and 2-trees relevant to the functions $\mathcal{U}$ and $\mathcal{V}$ are shown in Figs. 3.7 and 3.8. Four more existing 2-trees, for example the 2-tree with the component consisting of the lines 1 and 2 and the component consisting of the isolated vertex with the external momentum p_4, do not contribute to the function $\mathcal{V}$ because the product $\alpha_3\alpha_4$ is multiplied by the corresponding external momentum squared which is zero.

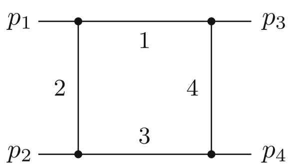

Fig. 3.6 Box diagram

Fig. 3.7 Trees contributing to the function $\mathcal{U}$ for the box diagram

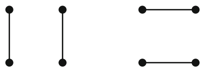

Fig. 3.8 2-trees contributing to the function $\mathcal{V}$ for the massless on-shell box diagram

We have (2.37) with

$$\mathcal{U} = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, \quad \mathcal{V} = t\alpha_1\alpha_3 + s\alpha_2\alpha_4. \tag{3.27}$$

Introducing new variables by $\alpha_1 = \eta_1\xi_1, \alpha_2 = \eta_1(1-\xi_1), \alpha_3 = \eta_2\xi_2, \alpha_4 = \eta_2(1-\xi_2)$, with the Jacobian $\eta_1\eta_2$, and evaluating an integral over η_2 due to the delta function and an integral over η_1 in terms of gamma functions we obtain

$$\begin{aligned} &F(s,t;a_1,a_2,a_3,a_4;d) \\ &\quad = (-1)^a \mathrm{i}\pi^{d/2} \frac{\Gamma(a+\varepsilon-2)\Gamma(2-\varepsilon-a_1-a_2)\Gamma(2-\varepsilon-a_3-a_4)}{\Gamma(4-2\varepsilon-a)\prod\Gamma(a_l)} \\ &\quad\quad \times \int_0^1\int_0^1 \mathrm{d}\xi_1\mathrm{d}\xi_2 \frac{\xi_1^{a_1-1}(1-\xi_1)^{a_2-1}\xi_2^{a_3-1}(1-\xi_2)^{a_4-1}}{[-s\xi_1\xi_2 - t(1-\xi_1)(1-\xi_2) - \mathrm{i}0]^{a+\varepsilon-2}}, \end{aligned} \tag{3.28}$$

where $a = a_1 + a_2 + a_3 + a_4$.

Consider, for example, integral with all the indices equal to one. We have

$$\begin{aligned} F(s,t;d) = \mathrm{i}\pi^{d/2} &\frac{\Gamma(2+\varepsilon)\Gamma(-\varepsilon)^2}{\Gamma(-2\varepsilon)} \\ &\times \int_0^1\int_0^1 \frac{\mathrm{d}\xi_1\mathrm{d}\xi_2}{[-t\xi_1\xi_2 - s(1-\xi_1)(1-\xi_2) - \mathrm{i}0]^{2+\varepsilon}}. \end{aligned} \tag{3.29}$$

Then the integration over ξ_2 results in

$$\begin{aligned} F(s,t;d) &= -\mathrm{i}\pi^{d/2} \frac{\Gamma(1+\varepsilon)\Gamma(-\varepsilon)^2}{\Gamma(-2\varepsilon)} \\ &\times \int_0^1 \frac{\mathrm{d}\xi}{s-(s+t)\xi}\left[(-t)^{-1-\varepsilon}\xi^{-1-\varepsilon} - (-s)^{-1-\varepsilon}(1-\xi)^{-1-\varepsilon}\right]. \end{aligned} \tag{3.30}$$

The singularity at $s - (s+t)\xi = 0$ is absent because the rest of the integrand is zero at this point. To calculate this integral in expansion in ε one needs, however, to separate the two terms in the square brackets. In order not to run into divergence due to the denominator one can perform an auxiliary subtraction at $s - (s+t)\xi = 0$. We obtain

$$F(s,t;d) = -\mathrm{i}\pi^{d/2} \frac{\Gamma(1+\varepsilon)\Gamma(-\varepsilon)^2}{\Gamma(-2\varepsilon)} \left[f(s,t;\varepsilon) + f(t,s;\varepsilon) \right], \tag{3.31}$$

where

$$f(s,t;\varepsilon) = (-t)^{-1-\varepsilon} \int_0^1 \frac{\mathrm{d}\xi}{s-(s+t)\xi} \left[\xi^{-1-\varepsilon} - \left(\frac{s}{s+t} \right)^{-1-\varepsilon} \right]. \tag{3.32}$$

To expand the function f in a Laurent series in ε one needs to perform another subtraction, at $\xi = 0$, which we make by the replacement

$$\frac{1}{s-(s+t)\xi} \to \frac{(s+t)\xi}{s(s-(s+t)\xi)} + \frac{1}{s}. \tag{3.33}$$

Then the integral with the first term can be evaluated by expanding the integrand in ε while the second term is integrated explicitly. Eventually, we arrive at the following result:

$$F(s,t;d) = \frac{\mathrm{i}\pi^{d/2} \mathrm{e}^{-\gamma_{\mathrm{E}}\varepsilon}}{st} \left(\frac{4}{\varepsilon^2} - \left[\ln(-s) + \ln(-t) \right] \frac{2}{\varepsilon} \right.$$
$$\left. + 2\ln(-s)\ln(-t) - \frac{4\pi^2}{3} \right) + O(\varepsilon). \tag{3.34}$$

Although we are oriented at calculations in expansion in ε, let us, for completeness, present a simple result for general ε [17] which can straightforwardly be obtained from (3.31):

$$F(s,t;d) = -\frac{\mathrm{i}\pi^{d/2}\Gamma(-\varepsilon)^2\Gamma(\varepsilon)}{st\Gamma(-2\varepsilon)} \left[(-t)^{-\varepsilon} {}_2F_1\left(1,-\varepsilon;1-\varepsilon;1+\frac{t}{s}\right) \right.$$
$$\left. + (-s)^{-\varepsilon} {}_2F_1\left(1,-\varepsilon;1-\varepsilon;1+\frac{s}{t}\right) \right], \tag{3.35}$$

where ${}_2F_1$ is the Gauss hypergeometric function (see (11.1)).

3.4 Feynman Parameters

Let us now turn to the alpha representation of scalar dimensionally regularized integrals (2.40) with general denominators E_i which are quadratic or linear with respect to the loop momenta. It has the same form (2.37) with $Z = 1$, i.e.

$$F(q_1,\ldots,q_n;a_1,\ldots,a_N;d) = (-1)^a \frac{\mathrm{e}^{\mathrm{i}\pi[a+h(1-d/2)]/2}\pi^{hd/2}}{\prod_l \Gamma(a_l)}$$
$$\times \int_0^\infty \mathrm{d}\alpha_1 \ldots \int_0^\infty \mathrm{d}\alpha_N \prod_l \alpha_l^{a_l-1} \mathcal{U}^{-d/2} e^{-\mathrm{i}\mathcal{W}/\mathcal{U}}. \tag{3.36}$$

For Feynman integrals (2.40) with standard propagators $1/(p^2-m^2+i0)^a$ associated with the lines of some graph. We have

$$\mathcal{W} = -\mathcal{V} + \mathcal{U}\sum m_l^2 \alpha_l, \tag{3.37}$$

with the functions $\mathcal{U}$ and $\mathcal{V}$ given by (2.25) and (2.26).

In the case of general denominators E_i, the functions $\mathcal{U}$ and $\mathcal{W}$ in (3.36) can be obtained easily by the public code `UF.m` [22]. If all the indices are integer and some of them are negative, i.e. correspond to numerators of a given integral one can use (3.36) and take the limit where some indices a_i tend to negative integers. After this, the integration is taken only over the parameters corresponding to positive indices and one obtains in the integrand a polynomial due to the differentiation in the parameters corresponding to negative indices and setting these parameters to zero.

Let us now present the alpha representation of scalar dimensionally regularized integrals in a modified form by making the change of variables $\alpha_l = \eta\alpha_l'$, where $\sum \alpha_l' = 1$. Starting from (3.36), performing the integration over η from 0 to ∞ explicitly and omitting primes from the new variables, we obtain

$$F_\Gamma(q_1,\ldots,q_n;d) = (-1)^a \frac{\left(\mathrm{i}\pi^{d/2}\right)^h \Gamma(a-hd/2)}{\prod_l \Gamma(a_l)}$$
$$\times \int_0^\infty \mathrm{d}\alpha_1 \ldots \int_0^\infty \mathrm{d}\alpha_L\, \delta\left(\sum \alpha_l - 1\right) \frac{\mathcal{U}^{a-(h+1)d/2} \prod_l \alpha_l^{a_l-1}}{\mathcal{W}^{a-hd/2}}. \tag{3.38}$$

The folklore Cheng–Wu theorem [5] (see also [2]) says that the same formula (3.38) holds with the delta function

$$\delta\left(\sum_{l\in\nu} \alpha_l - 1\right), \tag{3.39}$$

where ν is an arbitrary subset of $\{1,\ldots,N\}$, when the integration over the rest of the α-variables, i.e. for $l \overline{\in} \nu$, is extended to the integration from zero to infinity. Observe that the integration over α_l for $l \in \nu$ is bounded at least by 1 from above, as in the case where all the α-variables are involved in the sum in the argument of the delta function.

One can prove this theorem straightforwardly by changing variables and calculating the corresponding Jacobian. But a simpler way to prove it[2] is to start from the alpha

[2] Thanks to A.G. Grozin for pointing out this possibility!

representation (2.37), introduce new variables by $\alpha_l = \eta\alpha_l'$ for all $l = 1, 2, \ldots, L$, where $\eta = \sum_{l\in\nu}\alpha_l$, and immediately arrive at (3.38) with the delta function (3.39). Let me emphasize that this theorem holds not only for (3.38) corresponding to Feynman diagrams with standard propagators but also for the alpha representation derived for Feynman diagrams with various linear propagators.

As we will see below in multiple examples, an adequate choice of the delta function in (3.38) can greatly simplify the evaluation. Note that one can use various homogeneous substitutions which keep the form of the delta function in (3.38)—see Sect. 3.1 of [10] and references therein.

In addition to alpha parameters, the closely related Feynman parameters are often used. For a product of two propagators, one writes down the following relation:

$$\frac{1}{(m_1^2 - p_1^2)^{\lambda_1}(m_2^2 - p_2^2)^{\lambda_2}} = \frac{\Gamma(\lambda_1+\lambda_2)}{\Gamma(\lambda_1)\Gamma(\lambda_2)}\int_0^1 \frac{\mathrm{d}\xi\, \xi^{\lambda_1-1}(1-\xi)^{\lambda_2-1}}{\left[(m_1^2 - p_1^2)\xi + (m_2^2 - p_2^2)(1-\xi)\right]^{\lambda_1+\lambda_2}}. \tag{3.40}$$

This relation is usually applied to a pair of appropriately chosen propagators if an explicit integration over a loop momentum then becomes possible. Then new Feynman parameters can be introduced for other factors in the integral, etc. In fact, any choice of the Feynman parameters can be achieved by starting from the alpha representation (3.38) and making certain changes of variables. However, the possibility of an intermediate explicit loop integration of the kind mentioned above can be hidden in the alpha integral.

The generalization of (3.40) to an arbitrary number of propagators is of the form

$$\frac{1}{\prod A_l^{\lambda_l}} = \frac{\Gamma(\sum\lambda_l)}{\prod\Gamma(\lambda_l)}\int_0^1 \mathrm{d}\xi_1 \ldots \int_0^1 \mathrm{d}\xi_L \prod_l \xi_l^{\lambda_l-1}\frac{\delta\left(\sum\xi_l - 1\right)}{\left(\sum A_l\xi_l\right)^{\sum\lambda_l}}, \tag{3.41}$$

where $A_l = m_l^2 - p_l^2$.

For the evaluation of diagrams with a small number of loops, the choice of applying either alpha or Feynman parameters is usually just a matter of taste. In particular, if we apply (3.41) to a two-loop diagram and then integrate over two loop momenta, with the help of (10.1) and its generalizations to integrals with numerators, we obtain the same result as that obtained starting from (3.38).

For completeness, here is a one more parametric representation which is related to Feynman parameters and is often used in practice:

$$\frac{1}{A^{\lambda_1}B^{\lambda_2}} = \frac{\Gamma(\lambda_1+\lambda_2)}{\Gamma(\lambda_1)\Gamma(\lambda_2)}\int_0^\infty \frac{x^{\lambda_2-1}\,\mathrm{d}x}{(A+Bx)^{\lambda_1+\lambda_2}}. \tag{3.42}$$

3.5 Two-Loop Examples

At the two-loop level, we first consider the

Example 3.4 Two-loop vacuum diagram of Fig. 3.9 with the masses $m, 0, m$ and general complex powers of the propagators.

The Feynman integral is written as

$$F(m^2; \lambda_1, \lambda_2, \lambda_3; d)$$
$$= \int\int \frac{\mathrm{d}^d k \, \mathrm{d}^d l}{(-k^2+m^2)^{\lambda_1}[-(k+l)^2]^{\lambda_2}(-l^2+m^2)^{\lambda_3}}. \tag{3.43}$$

The two basic functions in the alpha representation are $\mathcal{U} = \alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1$ and $\mathcal{V} = 0$. We apply (3.38) to obtain

$$F(m^2; \lambda_1, \lambda_2, \lambda_3; d) = \left(\mathrm{i}\pi^{d/2}\right)^2 \frac{\Gamma(\lambda+2\varepsilon-4)}{\prod \Gamma(\lambda_l)(m^2)^{\lambda+2\varepsilon-4}}$$
$$\times \int_0^\infty \int_0^\infty \int_0^\infty \delta\left(\sum_l \alpha_l - 1\right) \frac{\left(\prod_{l=1}^3 \alpha_l^{\lambda_l-1} \mathrm{d}\alpha_l\right)(\alpha_1+\alpha_3)^{4-\lambda-2\varepsilon}}{(\alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_3\alpha_1)^{2-\varepsilon}}.$$

Now we exploit the freedom provided by the Cheng–Wu theorem and choose the argument of the delta function as $\alpha_1 + \alpha_3 - 1$. The integration over α_2 is performed from 0 to ∞. Resulting integrals are evaluated in terms of gamma functions for general ε and we arrive at the table formula (10.38).

Consider now

Example 3.5 Two-loop massless propagator diagram of Fig. 3.10 with arbitrary integer powers of the propagators,

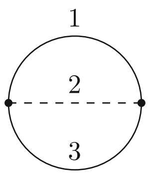

Fig. 3.9 Vacuum two-loop diagram with the masses $m, 0, m$

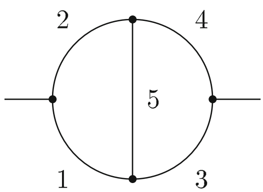

Fig. 3.10 Two-loop propagator diagram

Fig. 3.11 Trees contributing to the function $\mathcal{U}$ for Fig. 3.10

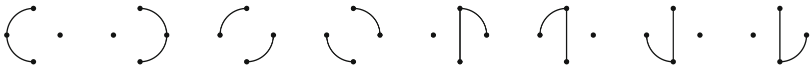

Fig. 3.12 2-trees contributing to the function $\mathcal{V}$ for Fig. 3.10

$$
\begin{aligned}
&F(q^2; a_1, a_2, a_3, a_4, a_5; d) \\
&\quad = \int\int \frac{\mathrm{d}^d k\, \mathrm{d}^d l}{(k^2)^{a_1}[(q-k)^2]^{a_2}(l^2)^{a_3}[(q-l)^2]^{a_4}[(k-l)^2]^{a_5}}. \qquad (3.44)
\end{aligned}
$$

The sets of trees and 2-trees relevant to the two basic functions in the alpha representation are shown in Figs. 3.11 and 3.12

We have, correspondingly,

$$
\mathcal{U} = (\alpha_1+\alpha_2+\alpha_3+\alpha_4)\alpha_5 + (\alpha_1+\alpha_2)(\alpha_3+\alpha_4), \qquad (3.45)
$$

$$
\begin{aligned}
\mathcal{V} &= [(\alpha_1+\alpha_2)\alpha_3\alpha_4 + \alpha_1\alpha_2(\alpha_3+\alpha_4) + (\alpha_1+\alpha_3)(\alpha_2+\alpha_4)\alpha_5]q^2 \\
&\equiv \overline{\mathcal{V}} q^2. \qquad (3.46)
\end{aligned}
$$

As we will see in Sect. 6.1, any diagram of this class can be evaluated for general ε in terms of gamma functions. This is however hardly seen from its alpha representation. In spite of the fact that the evaluation by alpha parameters is not an optimal method for this class of integrals, let us evaluate, for the sake of illustration, this diagram for all powers of the propagators equal to one, using its alpha representation. It is finite at $d = 4$, both in the UV and IR sense. Representation (3.38) takes the form

$$
F(q^2; 1, 1, 1, 1, 1; 4) = \frac{(\mathrm{i}\pi^2)^2}{q^2} \int_0^\infty \mathrm{d}\alpha_1 \ldots \int_0^\infty \mathrm{d}\alpha_5 \frac{\delta\left(\sum \alpha_l - 1\right)}{\mathcal{U}\overline{\mathcal{V}}}. \qquad (3.47)
$$

We exploit the Cheng–Wu theorem by choosing the delta function $\delta(\alpha_5 - 1)$, with the integration over the rest of the four variables from zero to infinity. Then one can delegate the integration procedure to MATHEMATICA [27] and obtain the well-known result[3]:

$$
F(q^2; 1, 1, 1, 1, 1; 4) = \frac{\left(\mathrm{i}\pi^2\right)^2}{q^2} 6\zeta(3), \qquad (3.48)
$$

where $\zeta(z)$ is the Riemann zeta function.

---

[3] This result was first obtained in [21] by means of expansion in Chebyshev polynomials in momentum space. In [6], it was reproduced using Gegenbauer polynomials in coordinate space.

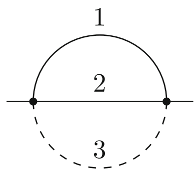

Fig. 3.13 Sunset diagram with the masses m, m, 0

In the rest of this chapter, we will consider just two more examples which are, however, more complicated than the previous ones.

Example 3.6 Two classes of two-loop integrals[4] with integer powers of the propagators:

$$F_{\pm}(q^2; a_1, a_2, a_3) = \int\int \frac{\mathrm{d}^d k\, \mathrm{d}^d l}{(k^2 + q\cdot k)^{a_1} (l^2 + q\cdot l)^{a_2} [(k \pm l)^2]^{a_3}}. \quad (3.49)$$

It turns out that the F_- is simple. Indeed we rewrite the first denominator $k^2 + q\cdot k$ as $(k + q/2)^2 - q^2/4$ and similarly the second denominator, make the change of variables $k = k' - q/2$, $l = l' - q/2$ and recognize F_- as a two-loop vacuum diagram with the mass $m^2 = q^2/4$ shown in Fig. 3.9 which was evaluated in Example3.4—see (10.38).

The integrals F_+ are, however, not so simple. Using the same manipulation as above we see that they are graphically recognized as sunset diagrams of Fig. 3.13 at threshold, i.e. $q^2 = 4m^2$. We start from the alpha representation (2.37) with $Z = 1$. The two basic functions are

$$\mathcal{U} = \alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1, \quad \mathcal{V} = \alpha_1\alpha_2\alpha_3 q^2. \quad (3.50)$$

After using the threshold condition $m^2 = q^2/4$ we obtain

$$F_+(q^2; a_1, a_2, a_3) = \frac{(-1)^a \mathrm{i}^{a+2\varepsilon-2}}{\prod \Gamma(a_l)} \times \int_0^\infty \int_0^\infty \int_0^\infty \left(\prod_{l=1}^3 \alpha_l^{a_l - 1} \mathrm{d}\alpha_l \right) \mathcal{U}^{\varepsilon-2} \exp\left\{ -\mathrm{i} \frac{q^2 \mathcal{W}}{4\mathcal{U}} \right\}, \quad (3.51)$$

where

$$\mathcal{W} = (\alpha_1 + \alpha_2)\alpha_1\alpha_2 + \alpha_3(\alpha_1 - \alpha_2)^2. \quad (3.52)$$

[4] They were involved, in particular, in the calculation [1, 8] of two-loop matching coefficients of the vector current in QCD and Non-Relativistic QCD (NRQCD) [3, 4, 15, 25].

Proceeding as with the general alpha representation we come to

$$F_{+}(q^2; a_1, a_2, a_3) = \frac{(-1)^a \left(\mathrm{i}\pi^{d/2}\right)^2}{(q^2/4)^{a+2\varepsilon-4}} \frac{\Gamma(a+2\varepsilon-4)}{\prod \Gamma(a_l)}$$
$$\times \int_0^\infty \int_0^\infty \int_0^\infty \delta\left(\sum \alpha_l - 1\right) \left(\prod_{l=1}^3 \alpha_l^{a_l-1} \mathrm{d}\alpha_l\right) \frac{\mathcal{U}^{a+3\varepsilon-6}}{\mathcal{W}^{a+2\varepsilon-4}}. \quad (3.53)$$

We continue to exploit the Cheng–Wu theorem in an appropriate way. We choose the delta function in (3.53) as $\delta(\alpha_1 + \alpha_2 - 1)$ and obtain an integral over $\xi = \alpha_1$ from 0 to 1, with $\alpha_2 = 1 - \xi$, and an integral over $t = \alpha_3$ from 0 to ∞:

$$F_{+}(q^2; a_1, a_2, a_3) = \frac{(-1)^a \left(\mathrm{i}\pi^{d/2}\right)^2}{(q^2/4)^{a+2\varepsilon-4}} \frac{\Gamma(a+2\varepsilon-4)}{\prod \Gamma(a_l)}$$
$$\times \int_0^1 \mathrm{d}\xi \, \xi^{a_1-1}(1-\xi)^{a_2-1} \int_0^\infty \mathrm{d}t \, \frac{t^{a_3-1}[t+\xi(1-\xi)]^{a+3\varepsilon-6}}{[t(1-2\xi)^2+\xi(1-\xi)]^{a+2\varepsilon-4}}. \quad (3.54)$$

This two-parametric integral representation can be used for the evaluation of any diagram of the given class in expansion in ε. Let us show how the integral with all the indices equal to one can be evaluated in expansion in ε up to the finite part. We start with (3.54) which gives

$$F_{+}(q^2; 1, 1, 1) = -\frac{\left(\mathrm{i}\pi^{d/2}\right)^2 \Gamma(2\varepsilon-1)}{(q^2/4)^{2\varepsilon-1}}$$
$$\times \int_0^1 \mathrm{d}\xi \int_0^\infty \mathrm{d}t \, \frac{[t+\xi(1-\xi)]^{3\varepsilon-3}}{[t(1-2\xi)^2+\xi(1-\xi)]^{2\varepsilon-1}}. \quad (3.55)$$

Observe that the integrand is invariant under the transformation $\xi \to 1-\xi$. We write the integral as twice the integral from 0 to $1/2$ over ξ, change the variable ξ by

$$\xi = \frac{1-\sqrt{1-x}}{2}, \quad (3.56)$$

with the Jacobian $1/(4\sqrt{1-x})$, and rescale $t \to t/4$ to obtain

$$F_{+}(q^2; 1, 1, 1) = -\left(\mathrm{i}\pi^{d/2}\right)^2 \Gamma(2\varepsilon-1)(q^2/2)^{1-2\varepsilon}$$
$$\times \int_0^1 \frac{\mathrm{d}x}{\sqrt{1-x}} \int_0^\infty \mathrm{d}t \, \frac{[t(1-x)+x]^{1-2\varepsilon}}{(t+x)^{3-3\varepsilon}}. \quad (3.57)$$

Remember that our integral is UV divergent. The overall divergence is quadratic since the UV degree of divergence is $\omega = 2$, and there are three one-loop logarithmically divergent subgraphs, so that, presumably, there should be poles up to the second order in ε. One source of the poles is the overall gamma function $\Gamma(2\varepsilon - 1)$. Another power of $1/\varepsilon$ comes from the integration over t and x in (3.57), namely from the region of small t and x. To have the possibility to perform an expansion in ε we have to reveal the singularity at $\varepsilon = 0$. Similarly to what we did in Example 3.3, let us perform a subtraction according to the identity

$$[t(1-x)+x]^{1-2\varepsilon} = \left\{[t(1-x)+x]^{1-2\varepsilon} - (t+x)^{1-2\varepsilon}\right\} + (t+x)^{1-2\varepsilon}.$$

Now, the integral with the expression in braces can be evaluated by expanding the integrand in a Laurent series in ε, while the last term can be integrated by hand with a result expressed in terms of gamma functions which can be, of course, expanded in ε after the evaluation:

$$\int_0^1 \frac{\mathrm{d}x}{\sqrt{1-x}} \int_0^\infty \frac{\mathrm{d}t}{(t+x)^{2-\varepsilon}} = \frac{\sqrt{\pi}\,\Gamma(\varepsilon)}{(1-\varepsilon)\Gamma(\varepsilon+1/2)}.$$

The integration of the subtracted part up to order ε^0 can straightforwardly be done by `MATHEMATICA` [27]. Finally, we obtain the following result:

$$F_+(q^2;1,1,1) = \left(\mathrm{i}\pi^{d/2}\mathrm{e}^{-\gamma_{\mathrm{E}}\varepsilon}\right)^2 \left(\frac{q^2}{4}\right)^{1-2\varepsilon} \times \left[\frac{1}{\varepsilon^2} + \frac{2}{\varepsilon} + \frac{11\pi^2}{12} - \frac{1}{2} + O(\varepsilon)\right]. \tag{3.58}$$

Consider now

Example 3.7 Non-planar two-loop massless vertex diagram of Fig. 3.14 with $p_1^2 = p_2^2 = 0$.

The Feynman integral can be written as

$$F(Q^2;a_1,\ldots,a_6;d) = \int\int \frac{\mathrm{d}^d k\,\mathrm{d}^d l}{[(k+l)^2 - 2p_1\cdot(k+l)]^{a_1}} \times \frac{1}{[(k+l)^2-2p_2\cdot(k+l)]^{a_2}(k^2-2p_1\cdot k)^{a_3}(l^2-2p_2\cdot l)^{a_4}(k^2)^{a_5}(l^2)^{a_6}}, \tag{3.59}$$

where $Q^2 = -(p_1-p_2)^2 = 2p_1\cdot p_2$, and the loop momenta are chosen as the momenta flowing through lines 5 and 6.

Let us proceed by Feynman parameters following [12] where some integrals of this class were calculated. (They were also evaluated in [14] and [17].) We write down Feynman parametric formula (3.40) for the pairs of the propagators (3, 5) and (4, 6):

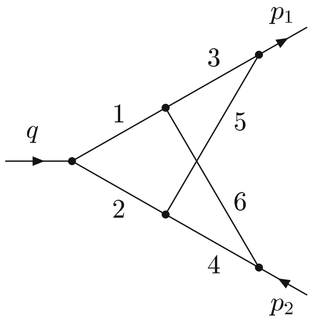

Fig. 3.14 Non-popular vertex diagram

$$\frac{1}{(k^2-2p_1\cdot k)^{a_3}(k^2)^{a_5}} = \frac{(-1)^{a_3+a_5}\Gamma(a_3+a_5)}{\Gamma(a_3)\Gamma(a_5)} \times \int_0^1 \frac{\mathrm{d}\xi_1\, \xi_1^{a_3-1}(1-\xi_1)^{a_5-1}}{[-(k-\xi_1 p_1)^2 - \mathrm{i}0]^{a_3+a_5}} \quad (3.60)$$

and, similarly, for the second pair, with the replacements

$$\xi_1 \to \xi_2,\ p_1 \to p_2,\ k \to l,\ a_3 \to a_4,\ a_5 \to a_6.$$

Then we change the integration variable $l \to r = k + l$ and integrate over k by means of our one-loop tabulated formula (3.6):

$$\int \frac{\mathrm{d}k}{[-(k-\xi_1 p_1)^2]^{a_3+a_5}[-(r-\xi_2 p_2 - k)^2]^{a_4+a_6}} = \mathrm{i}\pi^{d/2}\frac{G(a_3+a_5, a_4+a_6)}{[-(r-\xi_1 p_1 - \xi_2 p_2)^2]^{a_3+a_4+a_5+a_6+\varepsilon-2}}. \quad (3.61)$$

Then we apply Feynman parametric formula (3.41) to the propagators 1 and 2 and the propagator resulting from the right-hand side of (3.61), with a resulting integral over r evaluated by (10.1):

$$\int \frac{\mathrm{d}^d r}{[-(r^2 - Q^2 A(\xi_1,\xi_2,\xi_3,\xi_4))]^{a+\varepsilon-2}} = \mathrm{i}\pi^{d/2}\frac{\Gamma(a+2\varepsilon-4)}{\Gamma(a+\varepsilon-2)}\frac{1}{(Q^2)^{a+2\varepsilon-4}A(\xi_1,\xi_2,\xi_3,\xi_4)^{a+2\varepsilon-4}}, \quad (3.62)$$

where $a = a_1 + \cdots + a_6$ and

$$A(\xi_1,\xi_2,\xi_3,\xi_4) = \xi_3\xi_4 + (1-\xi_3-\xi_4)[\xi_2\xi_3(1-\xi_1)+\xi_1\xi_4(1-\xi_2)].$$

Thus we arrive at the following intermediate result valid for general powers of the propagators:

$$F(Q^2; a_1, \dots, a_6; d) = \frac{(-1)^a \left(\mathrm{i}\pi^{d/2}\right)^2}{(Q^2)^{a+2\varepsilon-4}} \frac{\Gamma(2-\varepsilon-a_{35})\Gamma(2-\varepsilon-a_{46})}{\prod \Gamma(a_l)\Gamma(4-2\varepsilon-a_{3456})}$$
$$\times \Gamma(a+2\varepsilon-4) \int_0^1 \mathrm{d}\xi_1 \dots \int_0^1 \mathrm{d}\xi_4\, \xi_1^{a_3-1}(1-\xi_1)^{a_5-1}\xi_2^{a_4-1}(1-\xi_2)^{a_6-1}$$
$$\times \xi_3^{a_1-1}\xi_4^{a_2-1}(1-\xi_3-\xi_4)_+^{a_{3456}+\varepsilon-3} A(\xi_1,\xi_2,\xi_3,\xi_4)^{4-2\varepsilon-a}. \quad (3.63)$$

We use the shorthand notation $a_{35} = a_3 + a_5$, $a_{3456} = a_3 + a_4 + a_5 + a_6$. As usually, $X_+ = X$ for $X > 0$ and $X_+ = 0$ otherwise.

This four-parametric integral representation can be used for the evaluation of Feynman integrals of this class with various indices. Let us use it in the case $a_1 = \dots = a_6 = 1$ and evaluate the corresponding Feynman integral in expansion in ε up to the finite part. We have

$$F(Q^2; 1, \dots, 1; d) = \frac{\left(\mathrm{i}\pi^{d/2}\right)^2}{(Q^2)^{2+2\varepsilon}} \frac{\Gamma(2+2\varepsilon)\Gamma(-\varepsilon)^2}{\Gamma(-2\varepsilon)}$$
$$\times \int_0^1 \mathrm{d}\xi_1 \dots \int_0^1 \mathrm{d}\xi_4 \frac{(1-\xi_3-\xi_4)_+^{1+\varepsilon}}{A(\xi_1,\xi_2,\xi_3,\xi_4)^{2+2\varepsilon}}. \quad (3.64)$$

We introduce new variables by $\xi_3 = \xi\eta$, $\xi_4 = (1-\xi)\eta$ and integrate over ξ_2 to obtain

$$F(Q^2; 1, \dots, 1; d) = -\frac{\left(\mathrm{i}\pi^{d/2}\right)^2}{(Q^2)^{2+2\varepsilon}} \frac{\Gamma(1+2\varepsilon)\Gamma(-\varepsilon)^2}{\Gamma(-2\varepsilon)} \int_0^1 \mathrm{d}\eta\, \eta^{-1-2\varepsilon}(1-\eta)^{\varepsilon}$$
$$\times \int_0^1 \int_0^1 \frac{\mathrm{d}\xi \mathrm{d}\xi_1}{\xi - \xi_1} \left\{ \xi^{-1-2\varepsilon}[(1-\xi)\eta + (1-\eta)(1-\xi_1)]^{-1-2\varepsilon} \right.$$
$$\left. - (1-\xi)^{-1-2\varepsilon}[\xi\eta + (1-\eta)\xi_1]^{-1-2\varepsilon} \right\}. \quad (3.65)$$

The singularity of the denominator at $\xi = \xi_1$ is spurious because the numerator is zero at this point. We notice that, due to the symmetry of the integrand, the integral over ξ and ξ_1 equals twice the integral over the domain $0 \le \xi_1 \le \xi \le 1$. Following [12] again, we turn to the variable z by $\xi_1 = z\xi$, make the changes $\eta \to 1-\eta$, $z \to 1-z$ and come to

$$F(Q^2; 1, \dots, 1; d) = -2\frac{\left(\mathrm{i}\pi^{d/2}\right)^2}{(Q^2)^{2+2\varepsilon}} \frac{\Gamma(1+2\varepsilon)\Gamma(-\varepsilon)^2}{\Gamma(-2\varepsilon)} f(\varepsilon), \quad (3.66)$$

where

$$f(\varepsilon) = \int_0^1 \mathrm{d}\eta\, \eta^{\varepsilon}(1-\eta)^{-1-2\varepsilon} \int_0^1 \mathrm{d}\xi\, \xi^{-1-2\varepsilon}$$
$$\times \int_0^1 \frac{\mathrm{d}z}{z} \left\{ [1-\xi(1-\eta z)]^{-1-2\varepsilon} - (1-\xi)^{-1-2\varepsilon}(1-\eta z)^{-1-2\varepsilon} \right\}. \qquad (3.67)$$

At this point it is claimed in [12] that, in principle, it is possible to evaluate this integral, in expansion in ε up to the finite part, performing appropriate subtractions of the integrand. Still another way was chosen: to expand various quantities of the type $(1-X)^{\lambda}$ in a binomial series, with subsequent integration and summing up resulting multiple series. (This procedure can be qualified as another method of evaluation.) Let us, however, realize the possibility of making subtractions. Indeed, the situation is complicated because we are dealing with a three-parametric integral so that several subtractions that would reveal the singularities that generate poles in ε are necessary.

Since the prefactor in (3.66) involves a simple pole in ε we have to evaluate the function $f(\varepsilon)$ given by (3.67) up to order ε^1. There are several sources of the poles: the points $\xi = 0$, $\xi = 1$, $\eta = 0$, $\eta = 1$, and $z = 1$. The following strategy of subtractions is suitable for the calculation. Let us first decompose f into the sum $f_1 + f_2$ according to the subtraction of the braces in (3.67) at $\eta = 0$, i.e.

$$\left[(1-\xi(1-\eta z))^{-1-2\varepsilon} - (1-\xi)^{-1-2\varepsilon} \right]$$
$$+ (1-\xi)^{-1-2\varepsilon} \left[1 - (1-\eta z)^{-1-2\varepsilon} \right]. \qquad (3.68)$$

Let us start with f_1. We perform subtraction of the integrand at $\eta = 1$ according to the decomposition of the first part of (3.68) into

$$\left[(1-\xi(1-z))^{-1-2\varepsilon} - (1-\xi)^{-1-2\varepsilon} \right]$$
$$+ \left[(1-\xi(1-\eta z))^{-1-2\varepsilon} - (1-\xi(1-z))^{-1-2\varepsilon} \right]. \qquad (3.69)$$

The first term in (3.69) does not depend on η so that the corresponding integration over η is performed in terms of gamma functions. Then the integral

$$\int_0^1 \mathrm{d}\xi\, \xi^{-1-2\varepsilon} \int_0^1 \frac{\mathrm{d}z}{z} \left\{ [1-\xi(1-z)]^{-1-2\varepsilon} - (1-\xi)^{-1-2\varepsilon} \right\}$$

appears. We need a subtraction at $\xi = 1$ here because when $\xi \to 1$ the factor $z^{-1-2\varepsilon}$ generating a pole in ε arises. So we replace $\xi^{-1-2\varepsilon}$ by $1 + \left(\xi^{-1-2\varepsilon} - 1\right)$. The first term corresponding to unity, after integration over ξ, gives the following integral evaluated in terms of gamma functions

$$\int_0^1 \frac{\mathrm{d}z}{1-z}\left(1 - z^{-1-2\varepsilon}\right) = \psi(-2\varepsilon) + \gamma_{\mathrm{E}},$$

where $\psi(z)$ is the logarithmical derivative of the gamma function, i.e. $\psi(z) = \Gamma'(z)/\Gamma(z)$. Thus we obtain the following contribution to our result:

$$\begin{aligned} f_{11} &= -\frac{\Gamma(1+\varepsilon)\Gamma(-2\varepsilon)}{2\varepsilon\Gamma(1-\varepsilon)} \\ &= \frac{1}{8\varepsilon^3} - \frac{\pi^2}{24\varepsilon} - \frac{3\zeta(3)}{4} - \frac{3\pi^4}{80}\varepsilon + O(\varepsilon^2). \end{aligned} \tag{3.70}$$

Starting from the second term we obtain an integral which can be evaluated by expanding the integrand in ε and performing the integration, e.g. in `MATHEMATICA` [27], with the following contribution:

$$f_{12} = \frac{\pi^2}{12\varepsilon} + 5\zeta(3) + \frac{43\pi^4}{180}\varepsilon + O(\varepsilon^2). \tag{3.71}$$

In the second part of (3.69), we make the same replacement (with the same motivation) as before, i.e. $\xi^{-1-2\varepsilon} \to 1 + \left(\xi^{-1-2\varepsilon} - 1\right)$. The second part here again produces an integral which can be evaluated by expanding the integrand in ε, with the following contribution:

$$f_{13} = \zeta(3) + \frac{11\pi^4}{120}\varepsilon + O(\varepsilon^2). \tag{3.72}$$

The unity gives a part where the integration over ξ is explicitly taken. The corresponding result is proportional to the sum of these two two-parametric integrals:

$$\begin{aligned} &\int_0^1\int_0^1 \mathrm{d}\eta\mathrm{d}z\eta^{\varepsilon}(1-\eta)^{-1-2\varepsilon}\left(1-\eta^{-1-2\varepsilon}\right) \\ &\quad + \int_0^1\int_0^1 \mathrm{d}\eta\mathrm{d}z\eta^{\varepsilon}(1-\eta)^{-1-2\varepsilon}\left[\frac{1-(\eta z)^{-2\varepsilon}}{1-\eta z} - \frac{1-z^{-2\varepsilon}}{1-z}\right]. \end{aligned} \tag{3.73}$$

The first integral can be evaluated in terms of gamma functions, with the following contribution:

$$\begin{aligned} f_{14} &= \frac{\Gamma(-2\varepsilon)}{4\varepsilon^2}\left[\frac{\Gamma(1+\varepsilon)}{\Gamma(1-\varepsilon)} - \frac{\Gamma(1-\varepsilon)}{\Gamma(1-3\varepsilon)}\right] \\ &= -\frac{\pi^2}{12\varepsilon} - \zeta(3) - \frac{\pi^4}{36}\varepsilon + O(\varepsilon^2). \end{aligned} \tag{3.74}$$

In the second integral, one can expand the integrand in ε. Here is the corresponding contribution:

$$f_{15} = -\zeta(3) - \frac{\pi^4}{72}\varepsilon + O(\varepsilon^2). \tag{3.75}$$

Let us now deal with f_2 defined by the second part of (3.68). The integration over ξ is performed explicitly, and the following integral over z arises:

$$\int_0^1 \frac{\mathrm{d}z}{z}\left[(1-\eta z)^{-1-2\varepsilon} - 1\right].$$

When $z \to 1$ a factor $(1-\eta)^{-1-2\varepsilon}$ appears so that we need a subtraction at $z = 1$. We make the replacement $1/z \to 1 + (1-z)/z$. The unity generates a part which is integrated explicitly over z and then over η. The resulting contribution is then

$$\begin{aligned} f_{21} &= -\frac{\Gamma(-2\varepsilon)^2\Gamma(\varepsilon)}{\Gamma(-4\varepsilon)}\left[\frac{1}{2\varepsilon}\left(\frac{\Gamma(-4\varepsilon)}{\Gamma(-3\varepsilon)} - \frac{\Gamma(-2\varepsilon)}{\Gamma(-\varepsilon)}\right) + \frac{\Gamma(-2\varepsilon)}{\Gamma(-\varepsilon)}\right] \\ &= \frac{1}{8\varepsilon^3} + \frac{1}{2\varepsilon^2} + \frac{\pi^2}{12\varepsilon} - \frac{\pi^2}{6} + 2\zeta(3) + \left(\frac{29\pi^4}{360} - 7\zeta(3)\right)\varepsilon + O(\varepsilon^2). \end{aligned} \tag{3.76}$$

Starting from the second term and performing one more subtraction we obtain the following integral

$$\begin{aligned} &\int_0^1\int_0^1 \mathrm{d}\eta \mathrm{d}z \eta^{\varepsilon}(1-\eta)^{-1-2\varepsilon}\frac{1-z}{z} \\ &\quad \times \left\{\left[(1-\eta z)^{-1-2\varepsilon} - (1-z)^{-1-2\varepsilon}\right] + \left[(1-z)^{-1-2\varepsilon} - 1\right]\right\}. \end{aligned} \tag{3.77}$$

For the part corresponding to the second square brackets, one can explicitly integrate over η and then expand the integrand in ε and integrate over z with the following resulting contribution:

$$\begin{aligned} f_{22} &= -\frac{\Gamma(-2\varepsilon)^3\Gamma(1+\varepsilon)}{\Gamma(-4\varepsilon)\Gamma(1-\varepsilon)}\left[\frac{1}{2\varepsilon} + 1 - \psi(-2\varepsilon) - \gamma_{\mathrm{E}}\right] \\ &= -\frac{1}{2\varepsilon^2} - \frac{\pi^2}{6\varepsilon} + \frac{\pi^2}{6} - 2\zeta(3) + \left(\frac{\pi^4}{90} + 7\zeta(3)\right)\varepsilon + O(\varepsilon^2). \end{aligned} \tag{3.78}$$

For the part corresponding to the first square brackets in (3.77), one can expand the integrand in ε and integrate over z and η with the following resulting contribution:

$$f_{23} = -\frac{\pi^2}{6\varepsilon} - 9\zeta(3) + \frac{19\pi^4}{45}\varepsilon + O(\varepsilon^2). \tag{3.79}$$

Collecting all the eight contributions obtained and taking into account the prefactor in (3.66) we arrive at the well-known analytical result[5] [12]

$$F(Q^2; 1, \ldots, 1; d) = \frac{\left(\mathrm{i}\pi^{d/2}\mathrm{e}^{-\gamma_{\mathrm{E}}\varepsilon}\right)^2}{(Q^2)^{2+2\varepsilon}} \times \left(\frac{1}{\varepsilon^4} - \frac{\pi^2}{\varepsilon^2} - \frac{83\zeta(3)}{3\varepsilon} - \frac{59\pi^4}{120}\right) + O(\varepsilon). \quad (3.80)$$

In [12], a similar algorithm based on Feynman parameters has been developed for the evaluation of planar massless two-loop vertex diagrams. It has turned out that the evaluation, by Feynman parameters, in the planar case is more complicated. As we will see in Sect. 6.1, there is, however, a better choice of an appropriate method in this situation and the planar vertex diagrams of this class are in fact much simpler than the non-planar ones.

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Chapter 4
Sector Decompositions

In this chapter, various sector decompositions are described. They are used both for theoretical and practical purposes: for an analysis of convergence of Feynman integrals, to prove theorems on renormalization and asymptotic expansion, to evaluate Feynman integrals numerically in a Laurent expansion in $\varepsilon = (4-d)/2)$ etc.

The well-known way to analyze convergence of Feynman integrals is to decompose an initial integration domain of alpha parameters into appropriate subdomains (*sectors*) and introduce, in each sector, new variables in such a way that the integrand properly factorizes, i.e. becomes equal to a product of monomials in the new variables times a non-singular function. The sectors developed by Hepp [21] and Speer [43] were successfully used starting from the sixties for proving mathematical theorems on analytically or/and dimensionally regularized and renormalized Feynman integrals [6–8, 17, 21, 25, 29, 30, 34, 39–43, 48]. These sectors are described in Sect. 4.1.

It turns out that the Hepp and Speer sectors are generally not applicable if the external momenta q_i are not Euclidean, i.e. if $\left(\sum_{i\in\nu} q_i\right)^2 \geq 0$ for some subset ν. This means that they do not provide a proper factorization of the integrand. To deal with Feynman integrals without restricting to Euclidean external momenta, Binoth and Heinrich introduced sector decompositions of a new kind [11, 12] and provided a powerful method of evaluating Feynman integrals numerically in situations with severe UV, IR and collinear divergences. In contrast to Hepp and Speer sectors, the sectors of [11, 12] are introduced recursively, according to so-called *sector decomposition strategies*. Various recursive sector decompositions and the corresponding public computer algorithms to evaluate Feynman integrals numerically are characterized in Sect. 4.2. Furthermore, Hepp and Speer sectors are also described as recursive strategies. In Sect. 4.3, a recently developed sector decomposition based on geometrical ideas [22] is presented. Finally, in Sect. 4.4, we discuss the analysis of convergence of Feynman integrals and present an ambiguous definition of dimensionally regularized Feynman integrals in situations where both UV and IR divergence are present.

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4.1 Hepp and Speer Sectors

In fact, the problem of resolving singularities in Feynman parametric integrals and revealing poles in ε is very close to the mathematical problem [20] on analytical properties of the distribution (generalized function) P_+^λ with respect to λ. Here P is a polynomial of $x_1, \ldots, x_n$ and λ a complex parameter. The action of the functional P_+^λ on a test function ϕ is given by the integral

$$\int_{P\geq 0} P(x_1, \ldots, x_n)^\lambda \phi(x_1, \ldots, x_n) \mathrm{d}x_1 \ldots, \mathrm{d}x_n.$$

According to the conjecture of Gel'fand, this functional is a meromorphic functions of λ. This conjecture was proven in [4, 9]. However, as it usually happens, proofs of mathematical theorems can hardly be applied in practice.

Let us start from the alpha representation (3.38) and introduce what Binoth and Heinrich called *primary sectors* [11, 12]. The set of primary sectors corresponds to the set of the lines of a given graph. In the sector Δ_l defined by $\alpha_i \leq \alpha_l,\ i \neq l$, the sector variables are introduced by $\alpha_i = t_i \alpha_l,\ i \neq l$. The integration over α_l is then taken due to the delta function in the integrand.

Alternatively, one can start directly from (2.24) and introduce primary sectors $\alpha_i \leq \alpha_l,\ i \neq l$ there with the new variables, $\alpha_i' = \alpha_i/\alpha_l$ belonging to a unit hypercube. For example, in the case of $l = L$, using the homogeneity properties of the functions in the representation, explicitly integrating over α_L and omitting primes at α_i we obtain the contribution of Δ_L as

$$F^{(L)} = (-1)^L \frac{\left(\mathrm{i}\pi^{d/2}\right)^h \Gamma(a - hd/2)}{\prod_l \Gamma(a_l)} \int_0^1 \ldots \int_0^1 \prod_l^{L-1} \alpha_l^{a_l - 1}$$
$$\times\, \hat{\mathcal{U}}_\Gamma^{a-(h+1)d/2} \mathcal{W}^{hd/2-a} \mathrm{d}\alpha_1 \ldots \mathrm{d}\alpha_{L-1}, \tag{4.1}$$

where

$$\mathcal{W}_\Gamma = -\hat{\mathcal{V}}_\Gamma + \hat{\mathcal{U}}_\Gamma \left(\sum_{l=1}^{L-1} m_l^2 \prod_{l=l'}^{L-1} \alpha_{l'} + m_L^2 \right), \tag{4.2}$$

$$\hat{\mathcal{U}}_\Gamma = \mathcal{U}(\alpha_1, \ldots, \alpha_{L-1}, 1), \hat{\mathcal{V}}_\Gamma = \mathcal{V}_\Gamma(\alpha_1, \ldots, \alpha_{L-1}, 1) \tag{4.3}$$

and the functions $\mathcal{U}$ and $\mathcal{V}$ are given by (2.25) and (2.26). Without loss of generality let us consider only this primary sector.

The primary sectors are not sufficient for a proper factorization so that they are further decomposed into smaller sectors. First we will study well-known Hepp and Speer sectors. The Hepp sectors which are

$$\alpha_1 \leq \ldots \leq \alpha_L \tag{4.4}$$

and $(L-1)!-1$ other sectors obtained from this one by permutations. The sector variables are introduced by

$$\alpha_l = t_l \ldots t_L, \tag{4.5}$$

so that $t_l = \alpha_l/\alpha_{l+1}, i = 1, \ldots, L-1$ and $\alpha_L = t_L$.

We will not, however, study Hepp sectors because the Speer sectors are more economical, in the sense that they are larger, so that their number is smaller, and they are quite sufficient to provide a proper factorization in the case of massless Feynman integrals at Euclidean external momenta. Therefore, the factorization formulae for both functions in the parametric representation in the case of Hepp sectors can be obtained from the corresponding formulae for Speer sectors.

Let us now show that the function $\mathcal{U}$ is proper factorized in Speer sectors. We can consider $\mathcal{U}$ dependent on all L α-variables and set $\alpha_L = 1$ in the end. Thus, we will deal with $\mathcal{U}$ rather than with $\hat{\mathcal{U}}$.

Let us imply that the graph Γ is a *connected* graph, i.e. any two vertices of Γ can be connected by a path in Γ. However, we are going to consider various subgraphs of the graph and they can be disconnected, i.e. consist of several connectivity components. A subgraph γ of Γ is determined by a subset of lines $\mathcal{L}(\gamma)$ and includes all the vertices incident to these lines. (Sometimes isolated vertices are added to a subgraph. For example, `Mathematica` [47] produces isolated vertices as bi-connected components—see below.) The number of loops of a subgraph is

$$h(\gamma) = L(\gamma) - V(\gamma) + c(\gamma), \tag{4.6}$$

where $V(\gamma)$ and $c(\gamma)$ are, respectively the numbers of the vertices and connectivity components.

We need extra graph-theoretical definitions. *An articulation vertex* of a graph Γ is a vertex whose deletion disconnects Γ. Any graph with no articulation vertices is said to be *bi-connected* (or, *one-vertex-irreducible* (1VI)). Otherwise, it is called *one-vertex-reducible* (1VR). In other words, in a 1VR graph, one can distinguish two subsets of its lines and a vertex (an articulation vertex) such that any path between vertices from these two subsets goes through this vertex. From now on let us suppose that we are dealing with a 1VI graph. It is natural to treat a single line as a 1VI graph since we cannot decompose it into two parts.

Any subgraph can be represented as the union of its 1VI components, i.e. maximal 1VI subgraphs. Consider, for example, the two-loop self-energy graph of Fig. 3.10. The subgraphs $\{1, 2, 5\}$ and $\{1, 2, 3, 4\}$ are 1VI. The subgraph $\{1, 2, 3, 5\}$ is 1VR and its 1VI components are $\{1, 2, 5\}$ and $\{3\}$. The subgraph $\{1, 2, 3\}$ is 1VR and its 1VI components are $\{1\}$, $\{2\}$ and $\{3\}$.

A set f of 1VI subgraphs is called an *ultraviolet (UV) forest* if the following conditions hold:

(i) for any pair $\gamma, \gamma' \in f$, we have either $\gamma \subset \gamma'$, $\gamma' \subset \gamma$ or $\mathcal{L}(\gamma \cap \gamma') = \emptyset$;
(ii) if $\gamma^1, \ldots, \gamma^n \in f$ and $\mathcal{L}(\gamma^i \cap \gamma^j) = \emptyset$ for any pair from this family, the subgraph $\cup_i \gamma^i$ is 1VR.

In other words, the number of loops in $\cup_i \gamma^i$ (where γ^i are disjoint with respect to lines and belong to a UV forest) is equal to the sum of the numbers of loops of γ^i. The term 'UV' is used because the UV divergences are due to the integration over small values of α_l where the exponent in (2.24) is irrelevant and they are generated by the singularities of the factor $\mathcal{U}_\Gamma^{-d/2}$. We are going to show that the resolution of the UV singularities can be performed by the use of sectors associated with 1VI subgraphs.

For example, the set $\{1\}$, $\{2\}$, $\{3\}$ of subgraphs of Fig. 3.10 is a UV forest and $\{1, 2, 5\}$, $\{3\}$ is also a UV forest but the set $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$ is not a UV forest because the condition (ii) breaks down.

Let $\mathcal{F}$ be a *maximal* UV forest (i.e. there are no UV forests that include $\mathcal{F}$) of a given graph Γ. An element $\gamma \in \mathcal{F}$ is called *trivial* if it consists of a single line and is not a loop line. Any maximal UV forest has h non-trivial and $L - h$ trivial elements.

Let us define the mapping σ: $\mathcal{F} \to \mathcal{L}$ such that $\sigma(\gamma) \in \mathcal{L}(\gamma)$ and $\sigma(\gamma) \notin \mathcal{L}(\gamma')$ for any $\gamma' \subset \gamma, \gamma' \in \mathcal{F}$. The inverse mapping σ^{-1}: $\mathcal{L} \to \mathcal{F}$ exists and can be defined as follows: $\sigma^{-1}(l)$ is the minimal element of the UV forest $\mathcal{F}$ that contains the line l. Let us denote by γ_+ the minimal element of $\mathcal{F}$ that strictly includes the given element γ.

For a given maximal UV forest $\mathcal{F}$, let us define the corresponding sector (f-sector) as

$$\mathcal{D}_\mathcal{F} = \left\{ \underline{\alpha} | \alpha_l \leq \alpha_{\sigma(\gamma)},\ l \in \gamma \in \mathcal{F} \right\} . \tag{4.7}$$

The intersection of two different f-sectors is of measure zero; the union of all the sectors gives the whole integration domain of the alpha parameters. For a given f-sector, let us introduce new variables labelled by the elements of $\mathcal{F}$,

$$\alpha_l = \prod_{\gamma \in \mathcal{F}:\, l \in \gamma} t_\gamma , \tag{4.8}$$

where the corresponding Jacobian is $\prod_\gamma t_\gamma^{L(\gamma)-1}$. The inverse formula is

$$t_\gamma = \begin{cases} \alpha_{\sigma(\gamma)}/\alpha_{\sigma(\gamma_+)} & \text{if } \gamma \text{ is not maximal} \\ \alpha_{\sigma(\gamma)} & \text{if } \gamma \text{ is maximal} \end{cases} . \tag{4.9}$$

Consider, for example, the following maximal UV forest $\mathcal{F}$ of Fig. 3.10 consisting of $\gamma^1 = \{1\},\ \gamma^2 = \{2\},\ \gamma^3 = \{3\},\ \gamma^4 = \{1, 2, 5\},\ \gamma^5 = \Gamma$. The mapping σ is $\sigma(\gamma^1) = 1,\ \sigma(\gamma^2) = 2,\ \sigma(\gamma^3) = 3,\ \sigma(\gamma^4) = 5,\ \sigma(\gamma^5) = 4$. The sector associated with this maximal UV forest is given by $\mathcal{D}_\mathcal{F} = \{\alpha_{1,2} \leq \alpha_5 \leq \alpha_4,\ \alpha_3 \leq \alpha_4\}$ and the sector variables are $t_{\gamma^1} = \alpha_1/\alpha_5,\ t_{\gamma^2} = \alpha_2/\alpha_5,\ t_{\gamma^3} = \alpha_3/\alpha_4,\ t_{\gamma^4} = \alpha_5/\alpha_4,\ t_{\gamma^5} = \alpha_4$.

All the maximal UV forests of the given graph can be constructed at least in two ways.

Way 1. Let us imply that the lines are enumerated. Let us consider the sequence of subgraphs γ_l consisting of lines $\{1, 2, \ldots, l\}$, respectively, with $l = 1, \ldots, L$. For each l, let us take the 1VI component of γ_l that includes the line l. The set of all these components is a maximal UV forest. Then we construct in a similar way the UV forests for other $L! - 1$ enumerations of the set of lines. After this we leave only distinct maximal UV forests.

Way 2. Since we consider a 1VI graph we include it into any maximal forest. Let us delete a line from it. The resulting graph is decomposed as the union of its 1VI components which we include into the maximal UV forest. Then we continue this process by deleting a line from some 1VI component which is not a single line, etc.

In the sector corresponding to a given maximal UV forest f, the function $\mathcal{U}_\Gamma$ takes the form

$$\mathcal{U}_\Gamma = \prod_{\gamma \in f} t_\gamma^{h(\gamma)} \left[1 + P_f\right], \tag{4.10}$$

where P_f is a non-negative polynomial and the product is over elements of the given maximal UV forest f. As we agreed we call such a factorization *proper*.

To prove this factorization formula we use the relation

$$\prod_{l \notin T} \alpha_l = \prod_{\gamma \in f} t_\gamma^{L(\gamma \setminus T)}, \tag{4.11}$$

where T is a tree or a 2-tree. (We will need the latter case later.) Applying (4.6) for γ and $\gamma \cap T$, i.e. $h(\gamma \cap T) = L(\gamma \cap T) - V(\gamma \cap T) + c(\gamma \cap T) = 0$ and taking the difference $h(\gamma \cap T) - h(\gamma)$ we obtain

$$L(\gamma \setminus T) = h(\gamma) + c(\gamma \cap T) - c(\gamma) \equiv h(\gamma \cup T). \tag{4.12}$$

Therefore, the factorization formula (4.10) will follow from the factorization of the polynomial

$$\sum_T \prod_{\gamma \in f} t_\gamma^{c(\gamma \cap T) - c(\gamma)},$$

and the problem reduces to constructing a tree that provides the minimal value of the non-negative quantity $c(\gamma \cap T) - c(\gamma)$. Let T_0 be the tree composed of all trivial elements of the given maximal UV-forest F. In other words, this tree can be constructed as follows. One uses an order of lines which was used within Way 1 for the construction of the given maximal UV forest f and includes the given line in the tree if a loop is not generated. One can observe that this tree T_0 provides the zero value of $c(\gamma \cap T_0) - c(\gamma)$ for all the elements of the given maximal forest.

We also need to factorize the second function (4.1), i.e. $\mathcal{W}_\Gamma$. Let us first consider the pure massless case where $\mathcal{W}_\Gamma = -\mathcal{V}_\Gamma$. As we could see in the previous section, the use of f-sectors provides a proper factorization (4.10) of the function $\mathcal{U}$ so that the factor $\mathcal{U}_\Gamma^{a-(h+1)d/2}$ in (4.1) is properly factorized. However, these sectors generally do not provide a factorization of $\mathcal{V}_\Gamma$. This can be seen using our example of Fig. 3.10.

We are going to use smaller sectors which are in fact obtained from the f-sectors generated by the graph Γ by a further decomposition. As for $\mathcal{U}$, we will deal with $\mathcal{V}$ depending on all the L variables α_l.

Let Γ^∞ be the graph obtained from Γ by adding a new vertex v^∞ and connecting it with all the external $n+1$ vertices by additional lines. These lines are only auxiliary and no propagators correspond to them. When writing down the function $\mathcal{U}$ for Γ^∞, let us include, by definition, these additional lines into any tree. In particular, then in the case of two external vertices (i.e. for $n = 1$) we have

$$\mathcal{V}_\Gamma = \mathcal{U}_{\Gamma^\infty} q^2$$

where q is the only external momentum.

Let us define sectors in a way similar to the previous case of $\mathcal{U}$ but using, instead of 1VI subgraphs, another set of subgraphs which we call *s-irreducible*. If a subgraph γ does not have all the external vertices in the same connectivity component and if it is 1VI let us call it s-irreducible as well. If a subgraph γ has all the external vertices in the same connectivity component let us call it s-irreducible if the graph γ^∞ is 1VI. Let us call an s-irreducible subgraph trivial if it is a single line which is not a loop line and which does not connect the external vertices.

The maximal forests consisting of s-irreducible subgraphs can be constructed again by Way 1 or Way 2.

Let us define sectors[1] in a way similar to the sectors discussed in the previous section. Let us introduce sector variables by the same formula (4.8) as above. The factorization of the function $\mathcal{V}$ follows from its definition (2.26) and the auxiliary relations (4.11) and (4.12). The 2-tree that provides the minimal value of the non-negative quantity $c(\gamma \cap T) - c(\gamma)$ can be constructed by a procedure similar to the procedure used for the function U: one considers the lines in the order used for the construction of the given f-forest by Way 1 and includes the given line into the 2-tree if a loop is not generated *and* if this is not the line whose inclusion would connect all the external vertices.

By construction, for such a 2-tree T_0, we obtain $c(\gamma \cap T_0) - c(\gamma) = \theta(\gamma)$ where $\theta(\gamma) = 1$ if the external vertices are connected in γ and $\theta(\gamma) = 0$ otherwise. Hence we obtain a proper factorization

$$-\mathcal{V}_\Gamma = \prod_{\gamma \in f} t_\gamma^{h(\gamma)+\theta(\gamma)} \left(-q_{T_0}^2 + P_\mathcal{V}\right), \tag{4.13}$$

where $q_{T_0}^2 < 0$ is the square of the external momentum flowing into one of the connectivity components of T_0, $P_\mathcal{V}$ is a non-negative polynomial (at Euclidean external momenta).

Obviously, the Speer sectors can be obtained from those associated with the graph Γ by a further decomposition, so that the factorization of the function $\mathcal{U}_\Gamma$ in the

[1] We call them *Speer sectors* although they are not exactly the same as in [43]. See, however, a recent paper [5], where the authors use 'genuine' Speer sectors.

corresponding variables also holds and has the form similar to (4.10) with the same exponents.

Let us turn to the case of non-zero masses. Two subcases are simple. If for a given primary sector, $m_L \neq 0$ then the term m_L^2 in (4.1) is dominating so that it is sufficient to achieve a factorization of the function $\mathcal{U}$ using f-sectors. In another partial case where all the masses are non-zero and the given Feynman integral is vacuum, the function $\mathcal{W}$ is determined only by the massive term which takes a proper factorized form immediately after the introduction of the primary sectors.

In the general case, s-sectors are not sufficient to proper factorize $\mathcal{W}$ although they are sufficient both for $\mathcal{W}$ and the massive term. This can be seen in simple examples. It turns out that we can achieve a factorization of the whole function $\mathcal{W}$ by using more advanced sectors. In fact, it suffices just to change the definition of the s-irreducibility which we used to construct s-sectors applied to factorize the function $\mathcal{V}$.

As before, if a subgraph γ does not have all the external vertices in the same connectivity component and if it is 1VI let us call it s-irreducible as well. Let γ^m be the subgraph consisting of all the massive lines. If a subgraph γ has all the external vertices in the same connectivity component let us call it s-irreducible if the graph $\gamma^\infty/(\gamma \cap \gamma^m)$ is 1VI.

The s-sectors constructed with this definition of s-irreducibility provide a proper factorization of the function $\mathcal{W}$. To see this, let reorganize the integrand in (4.1):

$$F^{(L)} = (-1)^L \frac{\left(\mathrm{i}\pi^{d/2}\right)^h \Gamma(a-hd/2)}{\prod_l \Gamma(a_l)} \int_0^1 \dots \int_0^1 \prod_l^{L-1} \alpha_l^{a_l-1}$$

$$\times\ \hat{\mathcal{U}}_\Gamma^{-d/2} \left[-\frac{\mathcal{V}_\Gamma}{\mathcal{U}_\Gamma} + \sum_{l=1}^{L-1} m_l^2 \prod_{l=l'}^{L-1} \alpha_{l'} \right]^{hd/2-a} \mathrm{d}\alpha_1 \dots \mathrm{d}\alpha_{L-1}. \qquad (4.14)$$

(We observed above that the case with $m_L \neq 0$ is simple so that we imply that $m_L = 0$.)

In each of resulting contributions of the s-sectors, the first term in the square brackets can be written as the product $t_{\gamma_0} t_{\gamma_{0+}} \dots$ times a positive polynomial, where γ_0 is the minimal element of a given s-forest including all the external vertices in the same connectivity component, γ_{0+} is the minimal element that contains γ_0 inside etc. Let us consider various massive terms. If $m_l \neq 0$ and $l \in \gamma_0$ then $m_l^2 \alpha_l = m_l^2 \prod_{\gamma \in \mathcal{F}:\, l \in \gamma} t_\gamma$ is divisible by $t_{\gamma_0} t_{\gamma_{0+}} \dots$ so that the first term determines the factorization. Let us consider the nest of the elements which include the external vertices in the same connectivity component

$$\gamma_0 \subset \gamma_{0+} \subset \dots \gamma^* \subset \gamma_+^* \subset \dots$$

Here γ^* is the minimal element of the given s-forest including all the massive lines. It turns out that the term $m_l^2 \alpha^l$ with $l = \sigma(\gamma^*)$ is dominating in the whole function $\mathcal{W}/\mathcal{U}$, in the sense that it includes the minimal powers of the sector variables, $t_{\gamma^*} t_{\gamma_+^*} \dots$.

Therefore, we arrive at the following factorization formula:

$$\frac{\mathcal{W}_\Gamma}{\mathcal{U}_\Gamma} = \prod_{\gamma\in\mathcal{F}} t_\gamma^{h(\gamma)+\theta^*(\gamma)} H_{\mathcal{W}}, \tag{4.15}$$

where $\theta^*(\gamma) = 1$ if the external vertices are connected in γ and all the massive lines belong to γ. Otherwise, $\theta^*(\gamma) = 0$. Here $H_{\mathcal{W}}$ is a function analytic in a vicinity of the point $t_\gamma = 0,\ \gamma \in \mathcal{F}$.

Therefore we see that any Feynman integral at Euclidean external momenta can be decomposed into contributions of Speer sectors, with properly factorized integrands. Such form can be used both for numerical calculations and for proving mathematical results. We will describe in the next section how numerical calculations with these sectors as well as with other various sectors are organized. We will perform the analysis of convergence of Feynman integrals in Sect. 4.4 and come back to the discussion of the definition of dimensionally regularized Feynman integrals with both UV and IR divergences.

4.2 Recursive Sector Decompositions

Let us now think of applying sector decompositions to the numerical evaluation of Feynman integrals. Indeed, Speer sectors seem quite optimal for this purpose, and the corresponding procedure has been implemented in a computer code—see below. The bad news is that, although Hepp and Speer sectors can successfully be used for proving theorems on renormalization [21, 39, 48] and on asymptotic expansions in limits of momenta and masses typical of Euclidean space (see [29, 30] and Appendix B of [34]), they are not sufficient for resolving the singularities of the integrand in the case of Feynman integrals on a mass shell or at a threshold. Let us consider again the massless on-shell box diagram,[2] i.e. Example 3.3 of Sect. 3.3, with the basic functions $\mathcal{U}$ and $\mathcal{V}$ given by (3.27), and try to apply the Hepp sectors to resolve the singularities. (Hepp sectors have more chances to provide a proper factorization because they are obtained from Speer sectors by an extra decomposition.)

Let us consider the primary sector Δ_4 given by $\alpha_i \le \alpha_4$, with $i = 1, 2, 3$. We obtain an integral over α_1, α_2 and α_3 from 0 to 1. The two basic functions take the form

$$\hat{\mathcal{U}} = 1 + \alpha_1 + \alpha_2 + \alpha_3, \quad \hat{\mathcal{V}} = s\alpha_1\alpha_3 + t\alpha_2.$$

The first function has already a proper factorized form. However, the second function is not proper factorized in some of the Hepp sectors, in particular, in $\alpha_2 \le \alpha_1 \le \alpha_2$. Indeed, if we introduce the corresponding sector variables by $\alpha_2 = t_1t_2t_3$, $\alpha_1 =$

[2] Let me emphasize that according to the terminology accepted in the book (and used starting from the sixties and seventies) the external momenta of this diagram *are not* Euclidean because all the end-points are on the light cone. One can consider Euclidean values of the Mandelstam variables, i.e. $s < 0$ and $t < 0$ but the configuration of the four external momenta is not Euclidean.

$t_2 t_3$, $\alpha_3 = t_3$, we obtain $\hat{\mathcal{V}} = t_1 t_3(st_3 + t\ t_1 t_2)$ so that this expression is not of the desired form and a further sector decomposition is desirable.

In a pioneer paper [11, 12] Binoth and Heinrich invented recursive sector decompositions which work at least in the case where all the kinematic invariants in the second function (4.2) in the alpha representation have the same sign.[3]

Binoth and Heinrich suggested to introduce sectors recursively, step by step. Let us consider only the contribution of the primary sector Δ_L given by (4.1) because other primary sectors are quite similar. Let us choose a subset $I = \{i_1, \dots i_k\}$ of $\{1, \dots, n\}$, with $n \equiv L-1$. The unit hypercube in (4.1) $\{(t_1, \dots, t_n)|0 \le t_i \le 1 \forall i \in (1, \dots, n)\}$ is then decomposed into k sectors

$$S_l = \{(t_1, \dots, t_n)|t_i \le t_{i_l} \forall i \in I\},$$

for $l = 1, \dots, k$, and the new variables are introduced as follows:

$$\begin{aligned} t_i &= t_i' \ \forall i \notin I \\ t_{i_l} &= t_{i_l}' \\ t_{i_r} &= t_{i_l}' t_{i_r}' \ \forall i_r \in I, r \ne l \end{aligned}$$

It is easy to verify that the integration region in the new variables t_i' is again a unit hypercube. Then for each of the k resulting sectors subsets of the indices are chosen and new sectors are introduced in a similar way. The rules according to which these subsets are chosen form a sector decomposition strategy. The goal of any such strategy is to arrive, at some step, at a proper factorization of the functions $\hat{\mathcal{U}}$ and $\mathcal{W}$. At this step, the procedure terminates and the sum of integrals over resulting final sectors can be used for numerical evaluation.

Indeed, at this point, the contribution of each of the final sectors takes a proper factorized form

$$\int_0^1 \dots \int_0^1 \prod_{i=1}^n t_i^{a_i + b_i \varepsilon} f(t_1, \dots, t_n; \varepsilon) \tag{4.16}$$

where a function f is a product of a polynomial of the form $1 + P_U$ with positive coefficients in P_U and a polynomial of the form $s + P_W$ with positive coefficients in P_W and a positive linear function s of kinematic invariants which should be set to concrete values before the procedure of numerical integration. To make the poles in ε manifest, the integrations over the final sector variables t_i are analyzed one by one. Each of the integrals of the variables t_i has the form

$$G(\varepsilon) = \int_0^1 t^{a+b\varepsilon} g(t) \tag{4.17}$$

[3] Typically, the kinematic invariants are considered positive. The transition to the negative sign is trivial and reduces to taking into account a phase factor and replacing $-i0$ by $+i0$.

where a is integer, $g(t)$ is an infinitely differentiable function of a given sector variable $t = t_i$ as well as other variables $t_{i'}$ and ε.

If $a \geq 0$, no poles in ε arise so that G can be expanded in ε in a Taylor series under the integral sign. If $a < 0$, poles do arise. To reveal them one subtracts first terms of the Taylor series of $g(t)$ in t at the origin up to order $-1-a$ and obtains, after explicitly integrating the subtracted terms,

$$G = \sum_{k=0}^{-1-a} \frac{g^{(k)}(0,\varepsilon)}{k!(a+k+b\varepsilon+1)} + \int_0^1 t^{a+b\varepsilon} \left[g(t) - \sum_{k=0}^{-1-a} \frac{g^{(k)}(0,\varepsilon)}{k!} t^k \right]. \quad (4.18)$$

In fact, only the last term, i.e. at $k = -1-a$, in the first sum here produces a pole in ε. On the other hand, the integral over the remainder can be expanded in ε under the integral sign.

One analyzes the integrations one by one and one obtains, as a result, the possibility to evaluate numerically each term of the Laurent expansion in ε. Details and a discussion of problems of numerical integration can be found in [11, 12] and in later papers which we will discuss shortly.

When looking for an optimal sector decomposition strategy it is, of course, reasonable to achieve a minimal number of the final sectors. The first strategy of [11, 12] was already quite optimal in this respect. According to this strategy one looks, at each sector decomposition step, for a minimal set $I = \{i_1, \ldots i_k\}$ such that at least one of the functions $\hat{\mathcal{U}}$ and $\mathcal{W}$ vanishes for $t_{i_1}, \ldots t_{i_k} = 0$. In the time when there were no public codes for sector decompositions, subsequent numerous practical calculations have shown that the code developed by Binoth and Heinrich and based on this strategy successfully works for complicated Feynman integrals with multiple IR and collinear divergences. For example, analytical results for double and triple boxes [3, 31–33, 35–37, 45] were numerically confirmed by means of this code.

However, a sector decomposition strategy, in particular the strategy of [11, 12] does not always terminate. Closed loops can appear within such an algorithm. Bogner and Weinzierl [13, 14] were first to present strategies which were guaranteed to terminate. They also developed a first public code for sector decompositions. Within this code, four strategies were implemented three of which were guaranteed to terminate. Strategy A [49] is conceptually the simplest one but it results in too many sectors; Strategy B has been described in [44], Strategy C is an improved version of strategy B. Finally, Strategy X is a variant of the strategy of Binoth and Heinrich [11, 12]. It is not guaranteed to terminate but produces less sectors than Strategies A–C.

In the second public code `FIESTA` [28] a new strategy (Strategy S) was implemented. It is also guaranteed to terminate. Since then it was successfully applied in various calculations. In the corresponding algorithm, one uses more general sectors. For any n-dimensional vector $(v_1, \ldots, v_n)$ (where $n = L-1$) in the positive quadrant with at least two non-zero coordinates one considers the set $I = \{i | v_i \neq 0\}$ and separates the unit hypercube into k parts by

$$S_l = \{(t_1, \ldots, t_n) | t_i^{d_i} \leq t_{i_l}^{d_{i_l}} \forall i \in I\}, \quad l = 1, \ldots, k$$

where $\{i_1, \dots i_k\} = I$, and the exponents d_i are defined by

$$\begin{pmatrix} d_{i_1} \\ d_{i_2} \\ d_{i_3} \\ \vdots \\ d_{i_k} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & \dots & 1 \\ 1 & 1 & 0 & \dots & 1 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & 1 & \dots & 0 \end{pmatrix}^{-1} \begin{pmatrix} v_{i_1} \\ v_{i_2} \\ v_{i_3} \\ \vdots \\ v_{i_k} \end{pmatrix} \tag{4.19}$$

The new sector variables in S_l are defined by

$$\begin{aligned} t_i &= t'_i \; \forall i \notin I \\ t_{i_l} &= (t')_{i_l}^{v_{i_l}} \\ t_{i_r} &= (t')_{i_l}^{v_{i_r}} t'_{i_r} \; \forall i \in I, r \neq l. \end{aligned}$$

One can check that the integration region in the variables t' is still the unit hypercube.

To choose the vector v, one considers the set of weights W of the polynomial P defined as the set of all possible $(w_1, \dots, w_n)$ where $ct_1^{w_1} \dots t_n^{w_n}$ is one of the monomials of P. One says that a weight is higher than another one if their difference is a set of non-negative numbers. If P had a unique lowest weight, a monomial could be factored out and we would represent P in the required form. Hence it becomes reasonable to try to minimize the number of lowest weights of P. One considers the convex hull of W and choose one of its facets visible from the origin. Now v is chosen to be the normal vector to this facet. This vector is chosen in such a way because the vectors formed by the weights of the vectors t'_{i_r} for $r \neq l$ are orthogonal to v and therefore belong to the facet F. Hence there is a good chance that after a single sector decomposition step only one of the vertices of considered facet is left to be a lowest weight.

It turns out that the Hepp and Speer sectors can also be defined recursively [26], in the style of the modern sector decompositions. With the Hepp sectors, the situation is obvious: they are reproduced when we choose, as subsets I, *maximal* subsets of lines at each step, i.e. with one line less than before this. To reproduce the choice of Speer sectors within a sector decomposition let me remind the Way 2 (see Sect. 4.3) to construct sectors. One has just to consider only subsets of the indices $I = \{i_1, \dots, i_k\}$ that correspond to s-irreducible subgraphs.

One more conclusion made in [26] is that the final sectors obtained within Strategy S (for massless Feynman integrals at Euclidean external momenta) exactly coincide with the Speer sectors—see a proof in the Appendix of that paper. This is a rather nontrivial relation because Speer sectors are defined in the graph-theoretical language while Strategy S is based on the geometrical language and reduces to finding convex hulls of polytopes. We will consider one more geometrical strategy in the next section.

An updated version `SecDec` [18] of the code by Binoth and Heinrich was also made public. Recently a new version `SecDec 2.0` [15, 16] of this code was published. In contrast to all the previously existing codes of sectors decompositions, it can work at general configurations of kinematic invariants, i.e. not only when kinematic invariants in the function $\mathcal{W}$ has the same sign but also in physical regions.[4] The code is based on contour deformations in parametric integrals. This idea was first suggested in [38] and was later further developed and successfully applied in numerous one-loop calculations. At the two-loop level, it was applied in [1, 2, 23]. More explicitly (see [10, 24, 38]), the integration over every parameter α_i in an integral of the type (4.1) over a hypercube is shifted in the complex plane by $\alpha_i \to \alpha_i - i\tau_i$ where τ_i depends on all the integration variables. It is chosen to be proportional to $\alpha_i(1-\alpha_i)$ so that the two end-points of each of the segments are not shifted.

4.3 Geometrical Sector Decompositions

A new method of sector decompositions was recently suggested [22]. It is not recursive so that the corresponding sectors are similar, in their character, to Speer sectors, rather than to the previously developed modern sector decompositions. Like Strategy S described in the previous section, this method is based on geometrical ideas. Let us illustrate it using the simple example of the triangle massless diagram shown in Fig. 4.1. Let us restrict ourselves to the value of this Feynman integral at the symmetrical Euclidean point $p_1^2 = p_2^2 = p_3^2 = -\mu^2$.

The basic functions are

$$\mathcal{U} = \alpha_1 + \alpha_2 + \alpha_3, \tag{4.20}$$

$$-\mathcal{V} = (\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3)\,\mu^2. \tag{4.21}$$

The three primary sectors are symmetric so that let us consider only the primary sector $\alpha_1, \alpha_2 \leq \alpha_3$. So, the function $-\mathcal{V}$ gives $P(\alpha_1, \alpha_2) = \alpha_1 + \alpha_2 + \alpha_1\alpha_2$. Since the function $\mathcal{U}$ takes the form $1 + \alpha_1 + \alpha_2$ and turns out to be factorized from the beginning the problem is reduced to the factorization of the function $\mathcal{V}$ only. Let us, therefore, ignore the function $\mathcal{U}$ and study the toy example of the integral

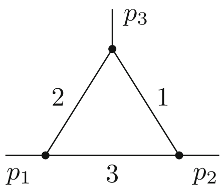

Fig. 4.1 A triangle diagram

[4] FIESTA 2 [27] can also work for Feynman integrals at threshold where some individual terms in the function $\mathcal{W}$ are negative. A typical situation is with the term $-2\alpha_1\alpha_2$ accompanied by $\alpha_1^2 + \alpha_2^2$ so that the whole combination is non-negative. Such combinations can be found by FIESTA 2 in an automatic way.

$$I = \int_0^1 \int_0^1 (P(\alpha_1, \alpha_2))^\lambda \, d\alpha_1 d\alpha_2, \tag{4.22}$$

where λ a complex regularization parameter. (In fact, $\lambda = -1 - \varepsilon$.)

Let us decompose the integration domain into three regions where one of the three terms of the polynomial P is dominating. So, these are the regions with $\alpha_2, \alpha_1\alpha_2 \leq \alpha_1$, $\alpha_1, \alpha_1\alpha_2 \leq \alpha_2$, and $\alpha_1, \alpha_2 \leq \alpha_1\alpha_2$. Since the third region has zero measure we are left with the two regions, $\alpha_2 \leq \alpha_1$ and $\alpha_1 \leq \alpha_2$ which give equal contributions. We can turn to sector variables in the first region by $\alpha_2 = t_1 t_2, \alpha_1 = t_2$ to obtain a desirable factorization:

$$I_1 = \int_0^1 \int_0^1 t_2^{1+\lambda} (1 + t_1 + t_1 t_2)^\lambda \, dt_1 dt_2. \tag{4.23}$$

In fact, all the sector decomposition strategies provide this very solution of the problem.

Instead of this simple way to achieve a proper factorization in our example, we will follow a lengthy way which, however, admits a generalization for *any* Feynman integral considered in a kinematics with all the terms in the function (4.2) of the same sign. Let us introduce new variables by $\alpha_l = e^{-y_l}$. We have

$$\begin{aligned} I_1 &= \int_0^1 \int_0^{\alpha_1} (P(\alpha_1, \alpha_2))^\lambda \, d\alpha_1 d\alpha_2 \\ &= \int_{\Delta_1} e^{-y_1 - y_2 - \lambda y_1} \left(1 + e^{y_1 - y_2} + e^{-y_2}\right)^\lambda dy_1 dy_2, \end{aligned} \tag{4.24}$$

where $\Delta_1 = \{y \in \mathbb{R}_+^2 \, | y_1 \leq y_2\}$ and $y = (y_1, y_2)$.

For any point in Δ_1 one can introduce barycentric coordinates $y_l \in \mathbb{R}_+^2$ by

$$y_l = \sum_{l'=1}^{2} (v_{l'})_l u_{l'} \tag{4.25}$$

where $v_1 = (1, 1)$ and $v_2 = (0, 1)$ are vectors with integer coordinates. They correspond to the edges of the region Δ_1. So, we have

$$y_1 = u_1, \quad y_2 = u_1 + u_2. \tag{4.26}$$

The contribution of the region Δ_1 takes the form

$$I_1 = \int_{\mathbb{R}_+^2} e^{-2u_1 - u_2 - \lambda u_1} \left(1 + e^{-u_2} + e^{-u_1 - u_2}\right)^\lambda du_1 du_2, \tag{4.27}$$

Changing variables one again (the last time) by $t_l = e^{-u_l}$ we arrive at the factorization (4.23).

The importance of this very lengthy way (which looks too complicated in our example) is explained by the fact that it can be used for general Feynman integrals (at least with all the kinematic invariants of the same sign).

One starts from a contribution of a given primary sector (4.1) given by an integral over a unit hypercube. Let us first consider a simplified situation with only one function in the integrand, for example, $\mathcal{W}$. So, we are dealing with an integral

$$I = \int_0^1 \ldots \int_0^1 P^{\lambda} \mathrm{d}\alpha_1 \ldots \mathrm{d}\alpha_n \tag{4.28}$$

where $n = L - 1$ and $\lambda = hd/2 - a$ and $P = \mathcal{W}$ is a polynomial of n variables α_l. All the kinematic invariants present in it should be set to concrete values. Let us make one more assumption: suppose that all the monomials enter with coefficients equal to one. (In fact, this condition is always satisfied for $P = \mathcal{U}$.) Then P is a sum of N monomials $\alpha_1^{w_1} \ldots \alpha_n^{w_n}$ characterized by vectors $w = (w_1, \ldots, w_N)$ composed of the w_l, i.e. $P = \sum_{i=1} \prod_{l=1}^n \alpha_l^{(w_i)_l}$.

Then one decomposes (4.28) into N regions Δ_i where one of the N monomials is dominating, i.e. $\prod_{l=1}^n \alpha_l^{(w_j)_l} \leq \prod_{l=1}^n \alpha_l^{(w_i)_l}$ for all $j \neq i$. As in our simple example, one turns to the new variables $\alpha_l = \mathrm{e}^{-y_l}$ and the contribution of the region Δ_i is written as

$$\int_{D_i} \mathrm{e}^{-y_1 - \ldots - y_n - \lambda(w_i, y)} \left(1 + \sum_{j=1:j\neq i}^{N} \mathrm{e}^{-(w_j - w_i, y)} \right)^{\lambda} \mathrm{d}y_1 \ldots \mathrm{d}y_n, \tag{4.29}$$

where $D_i = \{y \in \mathbb{R}_+^n \,\big|((w_j - w_i), y) \geq 0 \; \forall j \neq i\}$, $y = (y_1, \ldots, y_n)$, and (w, y) is the scalar product of vectors in n-dimensional Euclidean space.

Geometrically, the region D_i can be described as $C(Z_i)^* \cap \mathbb{R}_+^n$ with $Z_i = (w_1 - w_i, \ldots, w_n - w_i)$. Moreover, the convex polyhedral cone of a finite S is defined as

$$C(S) = \left\{ \sum_{\nu \in S} r_\nu \nu \,| r_\nu \geq 0, \nu \in S \right\}.$$

The corresponding dual cone is defined as

$$C(S)^* = \{y \in \mathbb{R}^n \,|(\nu, y) \geq 0 \; \forall \nu \in C(S)\}.$$

If a given region D_i is a simplicial cone, i.e. a convex polyhedral cone with n edges in n-dimensional space (like in our example) one can turn to barycentric coordinates $u_l \in R_+^n$

$$y_l = \sum_{l'=1}^{n} (v_{l'})_l u_{l'}, \tag{4.30}$$

where v_l are integer vectors corresponding to edges of D_i, then change the variables to $t_l = \mathrm{e}^{-u_l}$ and obtain, up to the determinant of the linear transformation (4.30), the following proper factorized representation for the contribution of the region Δ_i:

$$\int_0^1 \cdots \int_0^1 \prod_{m=1}^{n} t_m^{\sum_{l=1}^{n}(v_m)_l(1+\lambda(w_i)_l)-1}$$
$$\times \left(1 + \sum_{j=1:j\neq i}^{N} \prod_{m=1}^{n} t_m^{\sum_{l=1}^{n}((w_j)_l-(w_i)_l)(v_m)_l}\right)^{\lambda} \mathrm{d}t_1 \ldots \mathrm{d}t_n. \tag{4.31}$$

If the region D_i is not a simplicial cone, one needs to perform its triangulation, i.e. to decompose this convex polyhedral cone into simplicial cones. One can apply various algorithms for the triangulation, in particular, the algorithm developed by the authors of [22].

One more complication which was not seen in our example is that there are two basic functions to factorize. One way [22] to proceed is to consider the intersection of the dual cones corresponding to these functions. Another way is to study the product of the two basic functions and the resulting dual cone.

Conceptually, this geometrical algorithm [22] is oriented at the minimal number of resulting sectors. Multiple checks confirm these feature. In [46], a `FORM` implementation of this algorithm was described. It stays private at the moment. However, an implementation of this algorithm is present in the second version [27] of the public code `FIESTA` [28].

Let me formulate some recommendations on which method of sector decompositions to apply in practice. For massless Feynman integrals at Euclidean external momenta, Speer sectors look preferable. Since the geometrical approach provides less sectors it is reasonable to try to use it. It works indeed very well at least up to three loops. The most problematic step of this method is finding convex hulls of a given set. This procedure is applied many times during the process of triangulation so that, in four loops, this process can take too much time and this obstacle can prevent to go further to numerical integration where the advantage of this method (the minimal number of sectors) manifests itself. To evaluate numerically Feynman integrals in four loops and beyond one can try `FIESTA` or `SecDec`.

4.4 Analysis of Convergence and a Definition of Dimensionally Regularized UV and IR Divergent Feynman Integrals

Let us now apply Speer sectors to the analysis of convergence of Feynman integrals. Let us introduce analytic regularization by substituting the powers of propagators a_l by $a_l + \lambda_l$ with general complex numbers λ_l. For simplicity, let us assume that the powers of propagators are equal to one. (If $a_l > 1$, one can represent such a line by a sequence of a_l lines.) Let us use factorizations (4.10) and (4.15). A given s-forest $\mathcal{F}$ can be represented as $\mathcal{F}_1 \cup \mathcal{F}_2 \cup \Gamma$, where $\mathcal{F}_2 \cup \Gamma$ is the nest of elements which have all the external vertices in the same connectivity component and include all the massive lines. We can represent the contribution of the corresponding sector as

$$G^{\mathcal{F}} = \int_0^1 \dots \int_0^1 \prod_{\gamma \in \mathcal{F}_1 \cup \mathcal{F}_2} \mathrm{d}t_\gamma \left(\prod_{\gamma \in \mathcal{F}_1} t_\gamma^{\lambda(\gamma)+h(\gamma)\varepsilon-\omega(\gamma)/2-1} \right)$$
$$\times \left(\prod_{\gamma \in \mathcal{F}_2} t_\gamma^{-\lambda(\bar{\gamma})-h(\Gamma/\gamma)\varepsilon+(\omega(\Gamma)-\omega(\gamma))/2-1} \right) H(\underline{t}) \tag{4.32}$$

where H is a function analytic in a vicinity of the point $\underline{t} = 0$, $\underline{t}$ denotes the set of all the sector variables, $\bar{\gamma} = \Gamma \backslash \gamma$ and

$$\lambda(\gamma) = \sum_{l \in \gamma} \lambda_l. \tag{4.33}$$

Remember that there is no integration over t_Γ which dropped out when turning to the primary sectors.

The domain of the regularization parameters λ_l and ε where these sector integrals are convergent is determined by the inequalities

$$\mathrm{Re}\,\lambda(\gamma) + h(\gamma)\,\mathrm{Re}\,\varepsilon > \omega(\gamma)/2, \tag{4.34a}$$
$$\mathrm{Re}\,\lambda(\bar{\gamma}) + h(\Gamma/\gamma)\,\mathrm{Re}\,\varepsilon < (\omega(\Gamma) - \omega(\gamma))/2, \tag{4.34b}$$

which correspond, respectively, to $\gamma \in \mathcal{F}_1$ and $\gamma \in \mathcal{F}_2$. In fact, the first inequalities express UV convergence while the second inequalities express IR convergence. This can be seen with another scenario of analyzing divergences, with a primary decomposition of each α integral over $[0, 1]$ and $[1, \infty)$, where the first part is responsible for the UV divergences and the second part for IR divergences. Then, using appropriate sectors of Speer type, one can arrive at (4.34a), (4.34b), where the first/second inequalities correspond to UV/IR divergences.

Speer has proven [43] the following statement.

Theorem The domain (4.34a), (4.34b) is non-empty for any graph without massless detachable subgraphs.

Proof Let us define, following Speer, the analytic regularization parameters

$$\lambda_l^{(0)} = (2 - \varepsilon)\left(1 + \delta - \frac{|T_l^1|}{|T^1|}\right) - 1, \tag{4.35}$$

where T_l^1 is the set of trees containing the line l (As before, $|\dots|$ is the number of elements in the corresponding finite set.) and let us prove that these parameters satisfy (4.34a) and (4.34b) for sufficiently small $\delta > 0$. Since the imaginary part of the regularization parameters is irrelevant we will assume that they all are real. Let us also assume that ε is in a vicinity of the origin. At least, let $|\varepsilon| < 2$ so that it will be allowed to divide inequalities by $2 - \varepsilon$.

Let us start with (4.34a). Substituting (4.35) into (4.34a) and using the identity

$$\sum_{l \in \gamma} |T_l^1| = \sum_{T \in T_1} L(\gamma \cap T)$$

we obtain, by dividing by $2 - \varepsilon$,

$$\frac{1}{|T^1|} \sum_{T \in T_1} L(\gamma \cap T) < (1 + \delta) L(\gamma) - h(\gamma). \tag{4.36}$$

Then using (4.12) we obtain the following relations

$$\begin{aligned} L(\gamma \cap T) = L(\gamma) - L(\gamma \backslash T) &= L(\gamma) - h(\gamma) - (c(\gamma \cap T) - c(\gamma)) \\ &\leq L(\gamma) - h(\gamma) \end{aligned} \tag{4.37}$$

from which the validity of (4.36) follows for any $\delta > 0$.

Substituting (4.35) into (4.34b), using the relation $L(\bar{\gamma} \cap T) = L(\Gamma) - h(\Gamma) - L(\gamma \cap T)$ and dividing by $2 - \varepsilon$ we arrive at the inequality

$$\frac{1}{|T^1|} \sum_{T \in T_1} L(\gamma \cap T) < L(\gamma) - h(\gamma) - \delta L(\gamma). \tag{4.38}$$

Let us first drop the term with δ and take into account the arguments used after (4.36). Then we observe that in the obtained relation the equality sign can take place if and only if $c(\gamma \cap T) - c(\bar{\gamma}) = 0$ for any tree T. Such situation is possible only if γ and $\bar{\gamma}$ are bi-components of Γ. Remember that the inequalities (4.34b) are written for subgraphs that involve all the masses and external vertices, so that $\bar{\gamma}$ turns out to be a massless detachable subgraph. Therefore, if such subgraphs are absent, we have the strict inequality (4.38) at $\delta = 0$. This means that with the δ term we can choose δ sufficiently small so that the inequality (4.38) is still satisfied.

Let us illustrate the choice of parameters (4.35) on the simple massless diagram shown in Fig. 4.2. The basic functions are

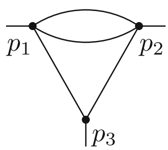

Fig. 4.2 A two-loop massless diagram

$$\mathcal{U} = (\alpha_1 + \alpha_2)\alpha_3 + (\alpha_1 + \alpha_2)\alpha_4 + \alpha_1\alpha_2, \tag{4.39}$$

$$\mathcal{V} = \alpha_1\alpha_2\alpha_3 p_1^2 + \alpha_1\alpha_2\alpha_4 p_2^2 + (\alpha_1 + \alpha_2)\alpha_3\alpha_4 p_3^2. \tag{4.40}$$

According to (4.35) we have

$$\lambda_1^{(0)} = \lambda_2^{(0)} = (2-\varepsilon)\left(\frac{3}{5} + \delta\right) - 1, \tag{4.41a}$$

$$\lambda_1^{(0)} = \lambda_2^{(0)} = (2-\varepsilon)\left(\frac{2}{5} + \delta\right) - 1, \tag{4.41b}$$

where we used the values $|T_1^1| = |T_2^1| = 2$ and $|T_3^1| = |T_4^1| = 3$ with $|T^1| = 5$.

In this section, we assumed the scalar case. Modifications for the general Feynman integrals are straightforward: one adds $n_l/2$ to the right-hand side of (4.35), where n_l is the degree of the polynomial in the numerator of the lth propagator. In addition, $\omega(\gamma)/2$ should be replaced by $[\omega(\gamma)/2]$ and $(\omega(\Gamma) - \omega(\gamma))/2$ by $[(\omega(\Gamma) - \omega(\gamma) + 1)/2]$ where the square brackets denote the integer part of a number.

Writing down the inequalities (4.34a) and (4.34b) when the regularization is switched off we obtain the well-known conditions of the UV and IR finiteness of a given Feynman integral:

$$\omega(\gamma) < 0, \tag{4.42a}$$

$$\omega(\gamma) - \omega(\Gamma) < 0. \tag{4.42b}$$

The UV conditions (4.42a) are written for 1PI subgraphs and the IR conditions (4.42b) are written for subgraphs which contain all the external momenta in the same connectivity component and include all the massive lines. Therefore, the IR condition can be formulated as the negativity of the IR degree of divergence (2.18).

From the convergence domain (4.34a) and (4.34b), a given Feynman integral (considered as the sum over Speer sectors) can be continued analytically as a function of $\lambda_l,\ l \in \Gamma$ and ε to the whole complex space C^{L+1} of the regularization parameters. To do this one applies presubtractions in the sector integrals, similar to what was done in Sect. 4.3 when revealing poles in ε.

Here one applies as many subtractions as necessary:

$$\int_0^1 \mathrm{d}t\, t^\lambda \phi(t) = \int_0^1 \mathrm{d}t\, t^\lambda \left[\phi(t) - \sum_{j=0}^{n} \frac{\phi^{(j)}(0)}{j!} t^j\right] + \sum_{j=0}^{n} \frac{\phi^{(j)}(0)}{j!(\lambda + j + 1)}, \tag{4.43}$$

in order to analytically continue such an integral to the whole complex plane. (Here λ is a linear combination of λ_l and ε and t is a sector variable.)

Now, we can use Speer theorem to unambiguous define dimensionally regularized Feynman integrals at Euclidean external momenta. Suppose that there are both UV and IR divergences so that for any ε a given Feynman integral is formally divergent in some sense. Let us introduce an auxiliary analytic regularization as described in this section. The theorem shows the existence of a non-empty domain of the regularization parameters $(\lambda_1, \ldots, \lambda_L; \varepsilon)$. Let us now define, following [19], this Feynman integral within dimensional regularization as the analytic continuation of the sum of the contributions of Speer sectors to the point $(0, \ldots, 0; \varepsilon)$.

What about Feynman integrals without restriction to Euclidean external momenta? As a first natural extension, let us consider the situation where all the kinematic invariants in the second function (4.2) in the alpha representation have the same sign. Thanks to the modern sector decompositions, we do have the possibility to introduce sectors which provide a proper factorization of the sector contributions. Remember that some of the sector decomposition strategies are guaranteed to terminate and that we even have a non-recursive geometrical strategy described in the previous section. However, we do not have a generalization of the Speer theorem to this more general situation so that we cannot immediately present a similar definition of dimensionally regularized Feynman integral using the trick with the auxiliary analytic regularization.

Let us realize, however, that practically oriented people and even the authors of the various sector decomposition strategies do not bother that some parametric subintegrals in final sectors are convergent in a certain domain of ε, for example, at $\mathrm{Re}\, \varepsilon > 0$ while other subintegrals are convergent in its completion in the complex plane, i.e. at $\mathrm{Re}\, \varepsilon < 0$.

Consider, for example, the Feynman integral corresponding to the graph depicted in Fig. 4.2 with the indices $a_1 = a_2 = a_3 = 1$, $a_4 = 2$. There is a UV divergence in the upper loop for which $\omega(\gamma) = 0$ and there is an IR divergence in the fourth line because of the index $a_4 = 2$. Formally, this is in agreement with (4.42a) where $\gamma = \{1, 2, 3\}$ with $\omega(\gamma) = \omega(\Gamma) = -2$. Both divergences manifest themselves at least in the Speer sector $\alpha_1 \leq \alpha_2 \leq \alpha_3$ which is a part of the primary sector $\alpha_1, \alpha_2, \alpha_3 \leq \alpha_4$. (In fact, this is a Hepp sector.) Factorizing the functions (4.39) and (4.40) (and finding agreement with (4.32)) we reveal the following product of the sector variables in the integrand:

$$t_2^{\varepsilon-1} t_3^{-\varepsilon-1},$$

where the sector variables t_i correspond to subgraphs $\{1\}$, $\{1, 2\}$ and $\{1, 2, 3\}$. Indeed, we see that either the integral over t_2 or the integral over t_3 is divergent at a given value of ε.

Since this is an integral at Euclidean external momenta we know that we can introduce an analytic regularization and, starting from a domain of the regularization parameters where the integral is convergent, and analytically continue the integral by switching off the analytic regularization. After introducing the analytic regularization by inserting $\prod_l \alpha_{\lambda_l}$ into the initial Feynman integral the product of powers of the sector variables becomes

$$t_1^{\lambda_1} t_2^{\lambda_1+\lambda_2+\varepsilon-1} t_3^{-\lambda_4-\varepsilon-1}.$$

At this point, we can apply Speer theorem. Although it is formulated for the indices $a_l = 1$, we can also apply it in our situation with $a_4 = 2$ by switching to $a_4 = 1$, using parameters (4.41a) and (4.41b) with ε in a small vicinity of the origin and analytically continuing to $\lambda_1 = \lambda_2 = \lambda_3 = 0$ and $\lambda_4 = 1$. (For $a_4 = 1$, in the product above only the power of t_3 changes to $-\lambda_4 - \varepsilon$.) This can be done by the standard procedure typical when using practical sector decompositions: one subtracts first terms of a Taylor series of the function which is in addition to the product of the sector variables and explicitly evaluates integrals with these first terms. In our example, the poles of the type $1/(\lambda_1+\lambda_2+\varepsilon)$ and $1/(1-\lambda_4-\varepsilon)$ are made manifest this way. Then, in these terms and in the integral with remainders of such Taylor series, one can let $\lambda_l \to 0, l = 1, 2, 3$ and $\lambda_4 \to 1$. From such pole terms we obtain ε-poles $1/\varepsilon$ and $-1/\varepsilon$.

Of course, for this sector contribution we can choose λ_1 and λ_2 sufficiently positive and λ_4 sufficiently negative to provide convergence and then perform an analytic continuation from this domain. Let me emphasize that there are also other sector contributions and the power of Speer theorem is that it provides the values of analytic regularization parameters simultaneously for *all* the sectors.

I think, it is a good problem to prove a kind of the Speer theorem at least for some classes of Feynman integrals without restriction to Euclidean external momenta. Still even without such an extension, we can live with dimensionally regularized Feynman integrals defined via their sector decompositions and applying the recipe which is usually implied in practice:

Consider integration over every sector variable in its own convergence domain of ε and continue analytically the integral to the whole complex plane by Taylor presubtractions.

At least I do not know examples where this recipe breaks down, i.e. leads to erroneous results.

Observe that setting to zero diagrams with detachable massless subgraph can be considered as a corollary of this recipe. Indeed, in any such diagram, one can reveal, introducing an overall integration variable for alpha parameters associated with the detachable massless subgraph, an integral without scale, i.e. an integral of a pure power of this variable from zero to infinity. Then one can decompose it into two pieces, as described in Chap. 2, and treat each piece in its own convergence domain of ε, with the zero total result.

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Chapter 5
Evaluating by MB Representation

One often uses Mellin integrals[1] when dealing with Feynman integrals. These are integrals over contours in a complex plane along the imaginary axis of products of gamma functions in the numerator and denominator. In particular, the inverse Mellin transform is given by such an integral. We will, however, deal with a very specific technique in this field. The key ingredient of the method presented in this chapter is the Mellin–Barnes (MB) representation used to replace a sum of two terms raised to some power by the product of these terms raised to some powers. Our goal is to use such a factorization in order to achieve the possibility to perform integrations in terms of gamma functions, at the cost of introducing extra Mellin integrations. Then one obtains a multiple Mellin integral with gamma functions. The next step is the resolution of the singularities in ε by means of shifting contours and taking residues. It turns out that multiple MB integrals are very convenient for this purpose. The final step is to perform at least some of the Mellin integrations explicitly, by means of the first and the second Barnes lemmas and their corollaries and/or evaluate these integrals by closing the integration contours in the complex plane and summing up corresponding series.

In Sect. 5.1 we start with simple one-loop examples which illustrate the two basic strategies of resolving singularities in ε. In Sect. 5.2 we discuss general properties of multiple MB integrals we are going to deal with and outline basic steps of the method. Section 5.3 contains various examples where MB representation are derived and checked. Then in Sect. 5.4 the two basic strategies are described in details. In Sects. 5.5 and 5.6 they are illustrated through examples. In Sects. 5.7 and 5.8 MB integrals are used in two different ways to study asymptotic behaviour of Feynman integrals. Finally, in Sect. 5.9 public codes related to the method of MB representation are listed and the status of the method is briefly characterized.

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[1] First examples of application of Mellin integrals to Feynman integrals can be found in [8, 65].

V. A. Smirnov, *Analytic Tools for Feynman Integrals*, Springer Tracts in Modern Physics 250, DOI: 10.1007/978-3-642-34886-0_5,

5.1 A One-Loop Example

Our basic tool is the following simple formula:

$$\frac{1}{(X+Y)^{\lambda}} = \frac{1}{\Gamma(\lambda)} \frac{1}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}z \, \Gamma(\lambda+z)\Gamma(-z) \frac{Y^z}{X^{\lambda+z}}. \tag{5.1}$$

Here the contour of integration is chosen in the standard way: the poles with a $\Gamma(\cdots+z)$ dependence (let us call them *left* poles, for brevity) are to the left of the contour and the poles with a $\Gamma(\cdots-z)$ dependence (*right* poles) are to the right of it. See Fig. 5.1, where a possible contour C is shown in the case of $\lambda = -1/4 - \mathrm{i}/2$. (This terminology is useful and, although it often happens that the first right pole is to the left of the first left pole of a given integrand, this, hopefully, will not cause misunderstanding.)

We will use decompositions $X+Y$ of various functions in integrals over Feynman and alpha parameters. But the simplest way[2] to apply this representation is to write down a massive propagator as a continuous superposition of massless ones: Mellin–Barnes (MB) representation

$$\frac{1}{(m^2-k^2)^{\lambda}} = \frac{1}{\Gamma(\lambda)} \frac{1}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}z \frac{(m^2)^z}{(-k^2)^{\lambda+z}} \Gamma(\lambda+z)\Gamma(-z). \tag{5.2}$$

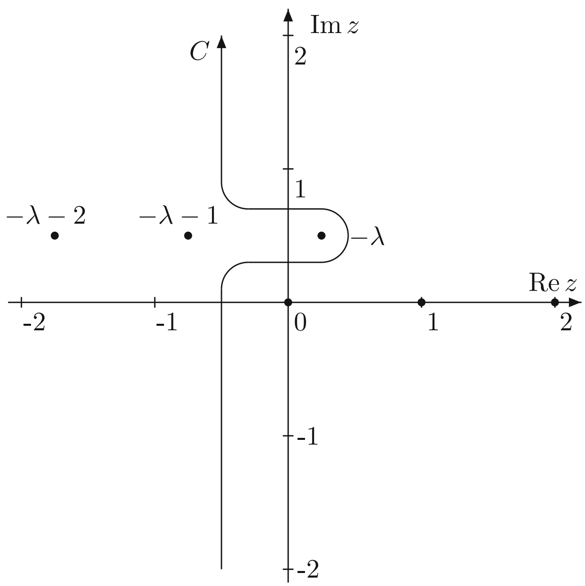


Fig. 5.1 A possible integration contour in (5.1) for $\lambda = -1/4 - \mathrm{i}/2$

[2] Historically, it was first advocated and applied in [16].

Our first example is the same as Example 3.1:

Example 5.1 One-loop propagator Feynman integrals (1.2) corresponding to Fig. 1.1.

We insert (5.2) with $\lambda = a_1$ into (1.2), apply (3.6) and obtain the following result:

$$F(q^2, m^2; a_1, a_2; d) = \frac{\mathrm{i}\pi^{d/2}(-1)^{a_1+a_2}\Gamma(2-\varepsilon-a_2)}{\Gamma(a_1)\Gamma(a_2)(-q^2)^{a_1+a_2+\varepsilon-2}}$$
$$\times \frac{1}{2\pi\mathrm{i}}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}z \left(\frac{m^2}{-q^2}\right)^z \Gamma(a_1+a_2+\varepsilon-2+z)$$
$$\times \frac{\Gamma(2-\varepsilon-a_1-z)\Gamma(-z)}{\Gamma(4-2\varepsilon-a_1-a_2-z)}. \tag{5.3}$$

The rules for choosing an integration contour that goes from $-\mathrm{i}\infty$ to $+\mathrm{i}\infty$ in the complex z-plane are the same as before: the right poles (in $\Gamma(\cdots - z)$) are to the right of the contour and the left poles (in $\Gamma(\cdots + z)$) are to left.

This representation can be used to evaluate any integral of this family in a Laurent expansion in ε. In particular, for $F(q^2, m^2; 2, 1; d)$, we obtain (1.15) and, at $d = 4$ come to

$$F(2, 1; 4) = \frac{\mathrm{i}\pi^2}{q^2}\frac{1}{2\pi\mathrm{i}}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}z \left(\frac{m^2}{-q^2}\right)^z \frac{\Gamma(1+z)\Gamma(-z)^2}{\Gamma(1-z)} \tag{5.4}$$

with an integration contour at $-1 < \mathrm{Re}z < 0$. Using properties of the gamma function we obtain (1.16).

Here is a subtle point: if we look at (1.16) we observe that there is a product $\Gamma(z)\Gamma(-z)$ which would be bad if it was present from the beginning because we could not satisfy our agreement about choosing the integration contours. Indeed, here the right and left poles at $\varepsilon = 0$ glue together and there is no space between them. However, the situation is unambiguous because we have fixed an integration contour with $-1 < \mathrm{Re}z < 0$ and we are free to perform identical transformations of the integrand after that. A moral of this discussion is the recipe to derive the MB representation for *general* powers of the propagators a_l and fix appropriate integration contours at this point. Then, for concrete integer indices a_l, we are allowed to make transformations like $\Gamma(1+z)\Gamma(-z) = -\Gamma(z)\Gamma(1-z)$, but it is necessary to remember about the choice of the contours made before this.

The integral (1.16) can be evaluated, according to the Cauchy theorem, by closing the integration contour to the right and taking a series of residues (with the minus sign, of course) at the points $z = 0, 1, 2, \ldots$. The residue at $z = 0$ gives $\mathrm{i}\pi^2 \ln\left(-q^2/m^2\right)/q^2$ and the residues at $z = 1, 2, \ldots$ give the series

$$-\frac{\mathrm{i}\pi^2}{q^2}\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{m^2}{q^2}\right)^n .$$

As a result, we reproduce (1.5).

In the case of the indices equal to one we use (5.3) to obtain

$$F(q^2, m^2; 1, 1; d) = \frac{\mathrm{i}\pi^2 \Gamma(1-\varepsilon)}{(-q^2)^\varepsilon} \frac{1}{2\pi \mathrm{i}} \int_C \mathrm{d}z f(z, \varepsilon), \tag{5.5}$$

with

$$f(z, \varepsilon) = \left(\frac{m^2}{-q^2}\right)^z \frac{\Gamma(\varepsilon + z)\Gamma(-z)\Gamma(1-\varepsilon-z)}{\Gamma(2-2\varepsilon-z)}. \tag{5.6}$$

Our goal is to evaluate (5.5) in a Laurent expansion in ε. Possible integration contours C in (5.5) in the cases $\mathrm{Re}\,\varepsilon > 0$ and $\mathrm{Re}\,\varepsilon < 0$ are shown in Figs. 5.2 and 5.3, respectively. In the former case, a contour can be chosen as a straight line parallel to the imaginary axis, while in the latter case, there is no such choice.

Let us now introduce two basic strategies for resolving singularities in ε in MB integrals. We will call them Strategy A and Strategy B. They both have computer implementations as described in the next section. Strategy A [51] is a modified variant of the strategy suggested in [53] (which will be described in Sect. 5.6) and Strategy B was suggested in [64].

We know in advance that the given integral has a pole in ε because the diagram is UV-divergent. There are no explicit functions with singularities in ε so that the pole is generated by the MB integration. So, if we just set $\varepsilon = 0$ (or take more terms of

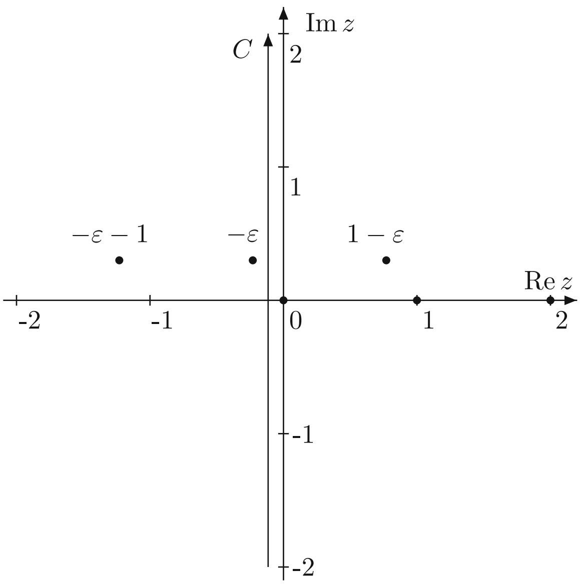


Fig. 5.2 A possible integration contour in (5.5) in the case $\mathrm{Re}\,\varepsilon > 0$

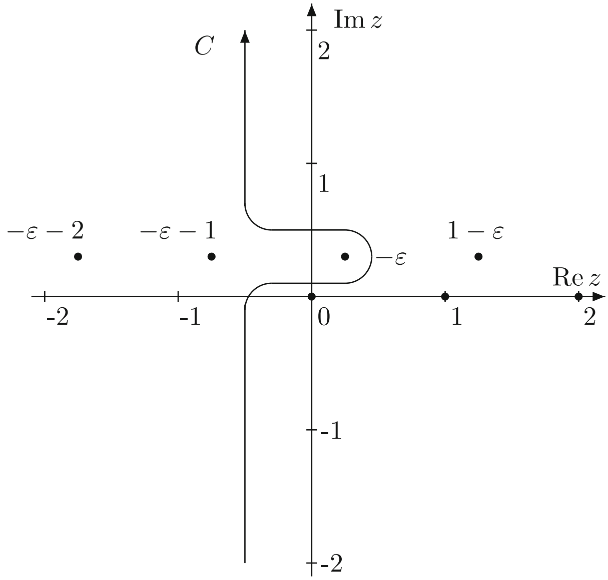


Fig. 5.3 A possible integration contour in (5.5) in the case $\mathrm{Re}\,\varepsilon < 0$

the ε-expansion) we will obtain a wrong result which does not have a pole at $\varepsilon = 0$. Moreover, it is clear that we cannot do this because the integration contour in (5.5) should go between the point $z = 0$ and $z = -\varepsilon$ so that, at $\varepsilon = 0$, there is no place for the contour. Still it is clear that such a 'naive part' of the ε-expansion has to be present in the right result which is, presumably, obtained from it by some extra terms.

Let us fix this naive part by choosing some straight contour for it. If we set $\varepsilon = 0$ we obtain the product of three gamma functions

$$\Gamma(z)\Gamma(-z)\Gamma(1-z).$$

For any straight contour, the real part of the argument of at least one of these gamma function turns out to be negative so that the standard prescription of the positivity of the argument of any gamma function when crossing the real axis has to be violated. It is not reasonable to choose a contour at $1 < \mathrm{Re}\,z < 2$ or $2 < \mathrm{Re}\,z < 3$ etc. because we would violate this prescription for two gamma function. So, we are left with the choices $0 < \mathrm{Re}\,z < 1$, $-1 < \mathrm{Re}\,z < 0$ etc. For example, let us fix it at $\mathrm{Re}\,z = -1/4$ so that we spoil the gamma function $\Gamma(\varepsilon + z)$, in the sense that we violate our prescription for it. Let us formally denote this transition by $\Gamma(\varepsilon+z) \to \Gamma^{(1)}(\varepsilon+z)$ where $\Gamma^{(1)}(\varepsilon + z)$ means that the rule $\mathrm{Re}(\varepsilon + z) > 0$ when crossing the real axis is changed to $-1 < \mathrm{Re}(\varepsilon + z) < 0$.

It is natural to violate the prescriptions in a minimal way, i.e. we do not need to spoil this gamma function more, e.g., by switching to $\Gamma^{(2)}(\varepsilon+z)$ with the prescription $-2 < \mathrm{Re}(\varepsilon + z) < -1$. We can write down (5.6) as

$$
\begin{aligned}
\frac{1}{2\pi \mathrm{i}}\int_C f(z,\varepsilon)\mathrm{d}z &= \frac{1}{2\pi \mathrm{i}}\int_{C_0} f(z,\varepsilon)\mathrm{d}z \\
&\quad + \left(\frac{1}{2\pi \mathrm{i}}\int_C f(z,\varepsilon)\mathrm{d}z - \frac{1}{2\pi \mathrm{i}}\int_{C_0} f(z,\varepsilon)\mathrm{d}z\right) \\
&= \frac{1}{2\pi \mathrm{i}}\int_{C_0} f(z,\varepsilon)\mathrm{d}z + \mathrm{res}_{z=\varepsilon} f(z,\varepsilon) \\
&= \frac{1}{2\pi \mathrm{i}}\int_{C_0} f(z,\varepsilon)\mathrm{d}z + \left(\frac{m^2}{-q^2}\right)^{-\varepsilon} \frac{\Gamma(\varepsilon)}{\Gamma(2-\varepsilon)},
\end{aligned}
$$

where C_0 is the straight contour with $\mathrm{Re}\, z = -1/4$, and we consider ε with $|\varepsilon| < 1/4$.

The way how the singularity in ε was resolved corresponds to Strategy A which will be described in the next section in the general situation. The crucial point is that we can safely expand the integrand in a Laurent series in ε in the integral over C_0. (In this particular example, this is just a Taylor series.) Its value at $\varepsilon = 0$ gives the following contribution to (5.5):

$$
\mathrm{i}\pi^2 \frac{1}{2\pi \mathrm{i}}\int_{C'} \mathrm{d}z \left(\frac{m^2}{-q^2}\right)^z \frac{\Gamma(z)\Gamma(-z)}{1-z}.
$$

This MB integral can be evaluated by closing the integration contour to the right in the complex z-plane, as in the previous example. Combining the corresponding result with the residue calculated above we arrive at (1.7).

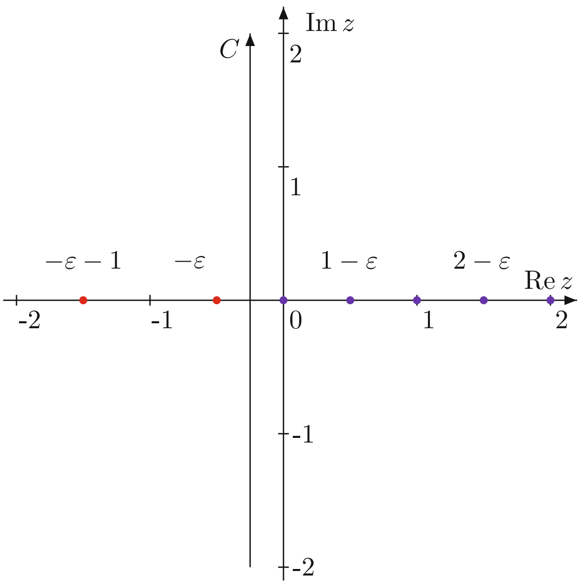


Fig. 5.4 A choice of ε and an integration contour within Strategy B

Let us turn to Strategy B [64]. Without loss of generality, we may consider ε real. Let us choose ε and a straight contour such that the arguments of all the gamma functions in the numerator of (5.6) will be positive when crossing the real axis. For example, we can take $\varepsilon = 1/2$, $\mathrm{Re}z = -1/4$—see Fig. 5.4. Let us then keep the contour fixed and tend ε to zero. When the first pole of $\Gamma(\varepsilon + z)$ is crossed we add a residue and tend ε to zero. As a result we arrive at the same intermediate result (5.7).

5.2 Evaluating Multiple MB Integrals

The first step of the method is to derive an appropriate MB representation. Of course, it is advantageous to have a minimal number of MB integrations. In every case, we will derive MB representations for *general* powers of the propagators. This is useful and important for several reasons. First, if we obtain a MB representation for general indices which we might imagine as complex we will certainly have unambiguous prescriptions for choosing integration contours. Second, such general formulae can be checked using various partial simple cases. Finally, starting from a general formula we can derive a lot of formulae by setting some indices to zero and thereby turning to graphs where the corresponding lines are contracted to a point. We will illustrate all these features through multiple examples in Sect. 5.3.

Multiple MB integrals which arise in the evaluation of Feynman integrals have the following general form:

$$\frac{1}{(2\pi \mathrm{i})^n} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \ldots \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \frac{\prod_i \Gamma\left(a_i + b_i \varepsilon + \sum_j c_{ij} z_j\right)}{\prod_{i'} \Gamma\left(a'_{i'} + b'_{i'} \varepsilon + \sum_j c'_{i'j} z_j\right)} \prod_k x_k^{d_k} \prod_{l=1}^{n} \mathrm{d}z_l \,, \tag{5.7}$$

where $a_i, \ldots, c'_{i'j}$ are rational numbers, x_k are ratios of kinematic invariants and/or masses, and their exponents, d_k, are linear combinations of ε and z-variables. Typically, $c_{ij} = \pm 1$.

In the second step, one resolves the singularity structure of integrals (5.7) in ε, taking residues and shifting contours, with the goal to obtain a sum of integrals where one can expand integrands in Laurent series in ε. For this, we will apply Strategy A and Strategy B introduced in the previous section in the one-loop case. They were suggested in [51] and [64], respectively, and they both have public computer implementations described in [51] and [17]. In the next section, we will describe these strategies as well as the 'the old Strategy A' [53] which we call in such a way because the strategy of [51] was motivated by it. For completeness, we present in Sect. 5.6 two examples considered with the old Strategy A.

The third step of the method is to evaluate integrals expanded in ε after the second step. Here one can use corollaries of the first and the second Barnes lemmas (13.1) and (13.47). A table of these formulae is presented in Appendix D. If Barnes lemmas do not work at this point one can shift contours to the right (or left), replace a given

MB integral by a series of residues and transform it into a multiple series with the hope to sum it up. In the case of onefold series, one can use summation formulae of Appendix C.

In fact, we are going to be pragmatic and not to bother whether the change of the order of integration over MB variables is legitimate. The analysis of the validity of the manipulations with MB integrals that we use is certainly possible in every example—see, e.g., the proof [64] when deriving an MB representation for the non-planar double box diagram. In fact, the crucial point is not the convergence of the integral in the basic identity (5.1), but the interchange of the order of integrations between the Mellin–Barnes integral and the parameter integrals.

Usually, at least at large values in the complex plane, the convergence of MB integrals is perfect[3] because gamma functions have exponential decrease in both imaginary directions. This property can be used for numerical checks. Moreover, in complicated situations, one can decompose a given integrand into pieces and choose an order of integration for every piece in a special way, with the possibility to integrate explicitly, using table formulae of Appendix D.

We will apply some standard properties of integration for multiple MB integrals. We will use changes of variables of the type $z \to \pm z + z_0$. When doing this we will, of course, trace how the nature of various poles is transformed. Note that, after such a change, $z \to -z$, right poles become left poles.

The integration by parts (IBP) is also possible in multiple MB integrals, although it is reasonable to apply it in rare situations. Still sometimes it is useful. For example, tabulated formulae of Appendix D with the factor $1/z^2$ were derived using the IBP identity

$$\int_C \mathrm{d}z \frac{f(z)}{z^2} = \int_C \mathrm{d}z \frac{f'(z)}{z} . \tag{5.8}$$

5.3 How to Derive and Check MB Representations

In simplest situations, it is sufficient to apply (5.2). Consider Example 3.2 where we can apply it to the only massive propagator in (3.23) and then evaluate the resulting massless triangle integral by (10.28) to obtain a onefold MB representation.

To present other possibilities to derive MB Representations let us consider again Example 3.3:

[3] In some situations, e.g. in a MB integral for the Gauss hypergeometric function, the asymptotic exponents of gamma functions cancel each other so that the convergence is defined by the value of the argument x which is present in the MB integral as x^z. Depending on whether $|x| < 1$ or $|x| > 1$, one has to close the integration contour to the right or to the left. Closing the contours to the different sides corresponds to an analytical continuation with respect to the argument x. However, there are certainly problems with the convergence in physical regions of kinematic variables, where factors of the type x^z, with $x < 0$, are present—see [17].

Example 5.2 The massless on-shell box diagram of Fig. 3.6, i.e. with $p_i^2 = 0,\ i = 1, 2, 3, 4$.

Let us start with (3.28). The natural idea here is to apply (5.1) to the denominator of the integrand. We do this with $X = -s\xi_1\xi_2$. After that we change the order of integration over z and the parameters ξ_1 and ξ_2 and evaluate the parametric integrals in terms of gamma functions:

$$\begin{aligned} F(s, t; a_1, a_2, a_3, a_4; d) = {} & \frac{(-1)^a \mathrm{i}\pi^{d/2}}{\Gamma(4 - 2\varepsilon - a) \prod \Gamma(a_l)(-s)^{a+\varepsilon-2}} \\ & \times \frac{1}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}z \left(\frac{t}{s}\right)^z \Gamma(a + \varepsilon - 2 + z)\Gamma(a_2 + z)\Gamma(a_4 + z)\Gamma(-z) \\ & \times \Gamma(2 - a_1 - a_2 - a_4 - \varepsilon - z)\Gamma(2 - a_2 - a_3 - a_4 - \varepsilon - z), \end{aligned} \tag{5.9}$$

where $a = a_1 + a_2 + a_3 + a_4$.

We now turn to a class of one-loop Feynman integrals with two more parameters.

Example 5.3 The massless box diagram of Fig. 3.6 with two legs on shell, $p_3^2 = p_4^2 = 0$, and two legs off shell, $p_1^2, p_2^2 \neq 0$.

We proceed like in the pure on-shell case, using alpha parameters, and obtain

$$\begin{aligned} F(s, t, p_1^2, p_2^2; a_1, \ldots, a_4; d) = {} & \mathrm{i}\pi^{d/2}(-1)^a \frac{\Gamma(a + \varepsilon - 2)}{\prod \Gamma(a_l)} \\ & \times \int_0^\infty \cdots \int_0^\infty \left(\prod_{l=1}^4 \alpha_l^{a_l - 1} \mathrm{d}\alpha_l\right) \delta\left(\sum_{l=1}^4 \alpha_l - 1\right) \\ & \times (-s\alpha_1\alpha_3 - t\alpha_2\alpha_4 - p_1^2\alpha_1\alpha_2 - p_2^2\alpha_2\alpha_3 - \mathrm{i}0)^{2-a-\varepsilon}. \end{aligned} \tag{5.10}$$

We have chosen the delta function of the sum of all the α-variables so that the factor with a power of the function $\mathcal{U}$ is equal to one.

Now we need a generalization of (5.1) to the case of several terms which is easily obtained by induction:

$$\begin{aligned} \frac{1}{(X_1 + \cdots + X_n)^\lambda} = {} & \frac{1}{\Gamma(\lambda)} \frac{1}{(2\pi \mathrm{i})^{n-1}} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \cdots \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}z_2 \ldots \mathrm{d}z_n \prod_{i=2}^n X_i^{z_i} \\ & \times X_1^{-\lambda - z_2 - \cdots - z_n} \Gamma(\lambda + z_2 + \cdots + z_n) \prod_{i=2}^n \Gamma(-z_i). \end{aligned} \tag{5.11}$$

We use (5.11) to replace the last factor in (5.10) by a product of four factors thus separating terms with t, p_1^2 and p_2^2 from s. After that we introduce new variables by $\alpha_1 = \eta_1\xi_1,\ \alpha_2 = \eta_1(1 - \xi_1),\ \alpha_3 = \eta_2\xi_2,\ \alpha_4 = \eta_2(1 - \xi_2)$ and arrive at a product of three parametric integrals evaluated in terms of gamma functions. Eventually we obtain the following threefold MB representation of the general Feynman integral of the given class:

$$
\begin{aligned}
F(s,t,p_1^2,p_2^2;a_1,\ldots,a_4;d) &= \frac{\mathrm{i}\pi^{d/2}(-1)^a}{\Gamma(4-2\varepsilon-a)\prod\Gamma(a_l)(-s)^{a+\varepsilon-2}} \\
&\times \frac{1}{(2\pi\mathrm{i})^3}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty}\mathrm{d}z_2\mathrm{d}z_3\mathrm{d}z_4\frac{(-p_1^2)^{z_2}(-p_2^2)^{z_3}(-t)^{z_4}}{(-s)^{z_2+z_3+z_4}} \\
&\times \Gamma(a+\varepsilon-2+z_2+z_3+z_4)\Gamma(a_2+z_2+z_3+z_4)\Gamma(a_4+z_4) \\
&\times \Gamma(2-\varepsilon-a_{234}-z_3-z_4)\Gamma(2-\varepsilon-a_{124}-z_2-z_4) \\
&\times \Gamma(-z_2)\Gamma(-z_3)\Gamma(-z_4).
\end{aligned}
\tag{5.12}
$$

In this chapter, we continue to use our notation: $a_{124} = a_1 + a_2 + a_4$, etc. with $a = a_{1234}$. This representation can be, of course, used for evaluating these Feynman integrals. However, we will use it below only as an auxiliary result when deriving an MB representation for the massless on-shell double box diagrams.

One of the advantages of general formulae is that they provide a lot of partial cases. For example (5.12) immediately gives a twofold MB representation for

Example 5.4 The massless box diagram of Fig. 3.6 with three legs on shell, $p_2^2 = p_3^2 = p_4^2 = 0$, and one leg off shell, $p_1^2 \neq 0$.

Indeed we put p_2^2 to zero in the 'naive' sense, i.e. in the integrand of the corresponding Feynman integral or in some parametric representation. This is equivalent to setting p_2^2 to zero in the sense of the leading term of the hard part of the asymptotic expansion in the limit $p_2^2 \to 0$ (see details in [57]), which corresponds to taking residues (with the minus sign) of the poles of $\Gamma(-z_3)$. So we just take minus residue of the integrand at $z_3 = 0$. Thus we obtain

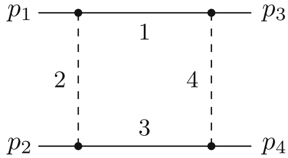
$$
\begin{aligned}
F(s,t,p_1^2;a_1,\ldots,a_4;d) &= \frac{\mathrm{i}\pi^{d/2}(-1)^a}{\Gamma(4-2\varepsilon-a)\prod\Gamma(a_l)(-s)^{a+\varepsilon-2}} \\
&\times \frac{1}{(2\pi\mathrm{i})^2}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty}\mathrm{d}z_2\mathrm{d}z_4\frac{(-p_1^2)^{z_2}(-t)^{z_4}}{(-s)^{z_2+z_4}}\Gamma(a+\varepsilon-2+z_2+z_4) \\
&\times \Gamma(a_2+z_2+z_4)\Gamma(a_4+z_4)\Gamma(2-\varepsilon-a_{234}-z_4) \\
&\times \Gamma(2-\varepsilon-a_{124}-z_2-z_4)\Gamma(-z_2)\Gamma(-z_4).
\end{aligned}
\tag{5.13}
$$

Let us now turn to massive diagrams.

Example 5.5 The on-shell box with two massive and two massless lines shown in Fig. 5.5, with $p_1^2 = \cdots = p_4^2 = m^2$.

The derivation of the corresponding MB representation is quite straightforward. The combination that is involved in the corresponding integral over alpha or Feynman parameters has now an additional piece as compared with the massless case:

Fig. 5.5 On-shell box with two massive and two massless lines. The *solid lines* denote massive, the *dashed lines* massless particles



$$\mathcal{V} - \mathcal{U} \sum m_l^2 \alpha_l = s\alpha_1\alpha_3 + t\alpha_2\alpha_4 - m^2(\alpha_1 + \alpha_3)^2.$$

This term can be separated from the rest terms at the cost of introducing one more MB integration according to (5.11). This time, let us introduce new parametric variables in a slightly different way, $\alpha_1 = \eta_1\xi_1,\ \alpha_2 = \eta_2\xi_2,\ \alpha_3 = \eta_1(1-\xi_1),\ \alpha_4 = \eta_2(1-\xi_2)$, in order to make $(\alpha_1 + \alpha_3)^2$ simpler. Evaluating the parametric integrals we arrive at the following massive generalization of (5.9):

$$\begin{aligned} F(s,t,m^2;a_1,a_2,a_3,a_4;d) &= \frac{(-1)^a \mathrm{i}\pi^{d/2}}{\Gamma(4-2\varepsilon-a)\prod \Gamma(a_l)(-s)^{a+\varepsilon-2}} \\ &\times \frac{1}{(2\pi \mathrm{i})^2} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}z_1 \mathrm{d}z_2 \frac{(-t)^{z_1}(m^2)^{z_2}}{(-s)^{z_1+z_2}} \Gamma(a+\varepsilon-2+z_1+z_2) \\ &\times \Gamma(a_2+z_1)\Gamma(a_4+z_1)\Gamma(-z_1)\Gamma(-z_2)\Gamma(2-a_{124}-\varepsilon-z_1-z_2) \\ &\times \Gamma(2-a_{234}-\varepsilon-z_1-z_2)\frac{\Gamma(4-a_{122344}-2\varepsilon-2z_1)}{\Gamma(4-a_{122344}-2\varepsilon-2z_1-2z_2)}, \end{aligned} \tag{5.14}$$

where $a_{122344} = a_1 + 2a_2 + a_3 + 2a_4$, etc. Observe that the onefold representation (5.9) in the massless case follows from (5.14) when we put m to zero. As it was discussed above we do this by taking the limit $m \to 0$ in the sense of the leading term of the hard part of the expansion. Here this means that we just take minus residue at $z_2 = 0$ with respect to the variable z_2 which enters the integrand as the exponent of m^2.

The general MB representation (5.14) can be used to derive an MB representation for the triangle diagram shown in Fig. 5.6. This class of Feynman integrals is obtained from the corresponding box integrals if we set $a_4 = 0$. If we do this blindly in (5.14) we obtain a zero result due to $\Gamma(a_4)$ in the denominator. This is, of course, not true. Let us think of a_4 as a complex number and analyze the behaviour in the limit $a_4 \to 0$ similarly to what we do when analyzing how singularities in ε are generated. We identify the product $\Gamma(a_4+z_1)\Gamma(-z_1)$ responsible for the generation of the singularity when $a_4 \to 0$. To reveal this singularity we can take minus residue at the point $z_1 = 0$ and shift the integration contour over z_1. The contribution of the new integral is indeed zero because of the factor $1/\Gamma(a_4)$. The contribution of the residue produces $\Gamma(a_4)$ which cancels this factor in the denominator, and we put a_4 to zero after that. Changing the numbering $2 \leftrightarrow 3$, for convenience,

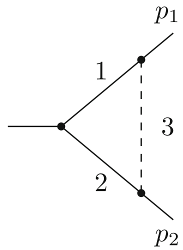


Fig. 5.6 *Triangle* diagram with the masses $m, m, 0$ and external momenta on-shell, $p_1^2 = p_2^2 = m^2$. The *dashed line* denotes a massless propagator

we obtain the following onefold MB representation[4] for integrals corresponding to Fig. 5.6:

$$\frac{(-1)^a \mathrm{i}\pi^{d/2} \Gamma(4-2\varepsilon-a_1-a_2-2a_3)}{\Gamma(4-2\varepsilon-a_1-a_2-a_3)\Gamma(a_1)\Gamma(a_2)(-s)^{a+\varepsilon-2}}$$
$$\times \frac{1}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}z \left(\frac{m^2}{-s}\right)^z \Gamma(a+\varepsilon-2+z)\Gamma(-z)$$
$$\times \frac{\Gamma(2-a_1-a_3-\varepsilon-z)\Gamma(2-a_2-a_3-\varepsilon-z)}{\Gamma(4-2\varepsilon-a_1-a_2-2a_3-2z)}. \quad (5.15)$$

Observe that if we want to have a representation for massive propagator-type diagrams by setting $a_3 = 0$ we will not reduce the number of integrations: there is no $\Gamma(a_3)$ in the denominator and, on the other hand, no singularities in the limit $a_3 \to 0$ are generated. So, one can simply apply (5.15) with $a_3 = 0$ for this class of diagrams.

The general MB representation (5.14) provides in a very similar way a MB representation for another triangle diagram obtained from Fig. 5.5. We shrink the line 3 to a point and obtain Fig. 5.7. The corresponding onefold MB representation takes the form

$$\frac{(-1)^a \mathrm{i}\pi^{d/2}}{\Gamma(4-2\varepsilon-a)\Gamma(a_1)\Gamma(a_2)\Gamma(a_4)(m^2)^{a+\varepsilon-2}}$$
$$\times \frac{1}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}z \left(\frac{-t}{m^2}\right)^z \Gamma(a+\varepsilon-2+z)\Gamma(-z)$$
$$\times \Gamma(a_2+z)\Gamma(a_4+z)\Gamma(4-2\varepsilon-a_1-2a_2-2a_4-2z), \quad (5.16)$$

where $t = (p_1 + p_3)^2$.

Among other partial cases of the massive on-shell boxes let us mention the case where $a_1 = a_2 = 0$. Then we obtain a massless one-loop propagator-type diagram which is evaluated by (3.6). On the other hand, one can see that to perform the limit $a_1, a_2 \to 0$ it is necessary to take two residues in the integrand and somehow compensate the corresponding gamma functions in the denominator. Eventually one

[4] In [26], it was demonstrated that this Feynman integral reduces, for any values of the three indices, to a two-point function in the shifted dimension $d - 2a_3$.

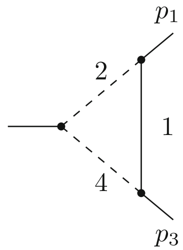


Fig. 5.7 *Triangle* diagram with the masses $m, 0, 0$ and external momenta on-shell, $p_1^2 = p_3^2 = m^2$, obtained from the box of Fig. 5.5

arrives at the known result. This procedure is just an additional check for the initial MB representation (5.14).

The representation (5.14) can straightforwardly be generalized to various off-shell cases, similarly to how we obtained the generalizations (5.12) and (5.13). For the box of Fig. 5.5 with two massive and two massless lines, two legs on shell, $p_3^2 = p_4^2 = m^2$, and two legs off shell we obtain the following fourfold MB representation:

$$
\begin{aligned}
&\frac{(-1)^a \mathrm{i}\pi^{d/2}(-s)^{2-a-\varepsilon}}{\Gamma(4-2\varepsilon-a)\prod\Gamma(a_l)}\frac{1}{(2\pi\mathrm{i})^4}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty}\dots\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty}\left(\prod_{j=1}^{4}\mathrm{d}z_j\,\Gamma(-z_j)\right)\\
&\times\frac{(m^2-p_1^2)^{z_1}(m^2-p_2^2)^{z_2}(-t)^{z_3}(m^2)^{z_4}}{(-s)^{z_1+z_2+z_3+z_4}}\Gamma(a_2+z_1+z_2+z_3)\Gamma(a_4+z_3)\\
&\times\Gamma(2-a_{124}-\varepsilon-z_1-z_3-z_4)\Gamma(2-a_{234}-\varepsilon-z_2-z_3-z_4)\\
&\times\frac{\Gamma(4-a_{122344}-2\varepsilon-z_1-z_2-2z_3)}{\Gamma(4-a_{122344}-2\varepsilon-z_1-z_2-2z_3-2z_4)}\\
&\times\Gamma(a+\varepsilon-2+z_1+z_2+z_3+z_4).
\end{aligned}
\tag{5.17}
$$

For the box of Fig. 5.5 with two legs on shell, $p_2^2 = p_4^2 = m^2$, and two legs off shell, we obtain:

$$
\begin{aligned}
&\frac{(-1)^a \mathrm{i}\pi^{d/2}(-s)^{2-a-\varepsilon}}{\Gamma(4-2\varepsilon-a)\prod\Gamma(a_l)}\frac{1}{(2\pi\mathrm{i})^4}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty}\dots\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty}\left(\prod_{j=1}^{4}\mathrm{d}z_j\,\Gamma(-z_j)\right)\\
&\times\frac{(m^2-p_1^2)^{z_1}(m^2-p_3^2)^{z_2}(-t)^{z_3}(m^2)^{z_4}}{(-s)^{z_1+z_2+z_3+z_4}}\Gamma(a_2+z_1+z_3)\Gamma(a_4+z_2+z_3)\\
&\times\Gamma(2-a_{124}-\varepsilon-z_1-z_2-z_3-z_4)\Gamma(2-a_{234}-\varepsilon-z_3-z_4)\\
&\times\frac{\Gamma(4-a_{122344}-2\varepsilon-z_1-z_2-2z_3)}{\Gamma(4-a_{122344}-2\varepsilon-z_1-z_2-2z_3-2z_4)}\\
&\times\Gamma(a+\varepsilon-2+z_1+z_2+z_3+z_4).
\end{aligned}
\tag{5.18}
$$

Finally, for the box of Fig. 5.5 with two legs on shell, $p_1^2 = p_4^2 = m^2$, and two legs off shell, we obtain:

$$
\begin{aligned}
&\frac{(-1)^a \mathrm{i}\pi^{d/2}(-s)^{2-a-\varepsilon}}{\Gamma(4-2\varepsilon-a)\prod \Gamma(a_l)} \frac{1}{(2\pi \mathrm{i})^4} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \dots \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \left(\prod_{j=1}^{4} \mathrm{d}z_j \, \Gamma(-z_j) \right) \\
&\times \frac{(m^2-p_3^2)^{z_1}(m^2-p_2^2)^{z_2}(-t)^{z_3}(m^2)^{z_4}}{(-s)^{z_1+z_2+z_3+z_4}} \Gamma(a_2+z_2+z_3)\Gamma(a_4+z_1+z_3) \\
&\times \Gamma(2-a_{124}-\varepsilon-z_1-z_3-z_4)\Gamma(2-a_{234}-\varepsilon-z_2-z_3-z_4) \\
&\times \frac{\Gamma(4-a_{122344}-2\varepsilon-z_1-z_2-2z_3)}{\Gamma(4-a_{122344}-2\varepsilon-z_1-z_2-2z_3-2z_4)} \\
&\times \Gamma(a+\varepsilon-2+z_1+z_2+z_3+z_4). \qquad (5.19)
\end{aligned}
$$

In two loops, our first example is the same as Example 3.7:

Example 5.6 Non-planar two-loop massless vertex diagram of Fig. 3.14 with $p_1^2 = p_2^2 = 0$.

We are again dealing with two-loop vertex Feynman integrals (3.59). We start with the four-parametric representation (3.63) obtained within the method of Feynman parameters in Chap. 3. Let us turn to the variables $\xi_3 = \xi\eta, \ \xi_4 = (1-\xi)\eta$ and apply (5.1) to the resulting denominator in the integrand:

$$
\begin{aligned}
&\frac{\Gamma(a+2\varepsilon-4)}{[\eta\xi(1-\xi)+(1-\eta)(\xi\xi_2(1-\xi_1)+(1-\xi)\xi_1(1-\xi_2))]^{a+2\varepsilon-4}} \\
&\quad = \frac{1}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \frac{\mathrm{d}z_1 \, \Gamma(-z_1)\eta^{z_1}\xi^{z_1}(1-\xi)^{z_1}}{(1-\eta)^{a+2\varepsilon-4+z_1}} \\
&\quad \times \frac{\Gamma(a+2\varepsilon-4+z_1)}{[\xi\xi_2(1-\xi_1)+(1-\xi)\xi_1(1-\xi_2)]^{a+2\varepsilon-4+z_1}}. \qquad (5.20)
\end{aligned}
$$

Then we again apply (5.1) to transform the last line of (5.20) into

$$
\frac{1}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \frac{\mathrm{d}z_2 \, \Gamma(a+2\varepsilon-4+z_1+z_2)\Gamma(-z_2)\xi^{z_2}\xi_2^{z_2}(1-\xi_1)^{z_2}}{(1-\xi)^{a+2\varepsilon-4+z_1+z_2}\xi_1^{a+2\varepsilon-4+z_1+z_2}(1-\xi_2)^{a+2\varepsilon-4+z_1+z_2}}.
$$

After that all the integrals over the parameters ξ_1, ξ_2, ξ, η can be evaluated in terms of gamma functions, and we come to the following twofold MB representation of (3.59) with general powers of the propagators:

$$
\begin{aligned}
&F(Q^2; a_1, \dots, a_6; d) = \frac{(-1)^a \left(\mathrm{i}\pi^{d/2}\right)^2 \Gamma(2-\varepsilon-a_{35})}{(Q^2)^{a+2\varepsilon-4}\Gamma(6-3\varepsilon-a)\prod \Gamma(a_l)} \\
&\times \frac{\Gamma(2-\varepsilon-a_{46})}{\Gamma(4-2\varepsilon-a_{3456})} \frac{1}{(2\pi \mathrm{i})^2} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}z_1 \mathrm{d}z_2 \Gamma(a+2\varepsilon-4+z_1+z_2)
\end{aligned}
$$

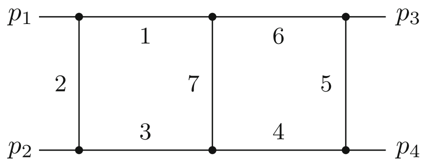


Fig. 5.8 Double box

$$
\begin{aligned}
&\times \Gamma(-z_1)\Gamma(-z_2)\Gamma(a_4+z_2)\Gamma(a_5+z_2)\Gamma(a_1+z_1+z_2) \\
&\times \frac{\Gamma(2-\varepsilon-a_{12}-z_1)\Gamma(4-2\varepsilon+a_2-a-z_2)}{\Gamma(4-2\varepsilon-a_{1235}-z_1)\Gamma(4-2\varepsilon-a_{1246}-z_1)} \\
&\times \Gamma(4-2\varepsilon+a_3-a-z_1-z_2)\Gamma(4-2\varepsilon+a_6-a-z_1-z_2). \qquad (5.21)
\end{aligned}
$$

Let us now consider

Example 5.7 Massless on-shell planar double box diagram of Fig. 5.8.

As in Example 5.2 we have $p_i^2 = 0,\ i = 1, 2, 3, 4$. Let us consider double boxes with the irreducible numerator $(k + p_1 + p_2 + p_4)^2$ and the routing of the external momenta as in [3]. Then the general double box Feynman integral takes the form

$$
\begin{aligned}
K(s,t;a_1,\ldots,a_8,\varepsilon) &= \int\int \frac{d^dk\,d^dl}{(k^2)^{a_1}[(k+p_1)^2]^{a_2}[(k+p_1+p_2)^2]^{a_3}} \\
&\times \frac{[(k+p_1+p_2+p_4)^2]^{-a_8}}{[(l+p_1+p_2)^2]^{a_4}[(l+p_1+p_2+p_4)^2]^{a_5}(l^2)^{a_6}[(k-l)^2]^{a_7}}\,, \qquad (5.22)
\end{aligned}
$$

As usual, we consider the factor corresponding to the irreducible numerator as an extra propagator but, really, we are interested only in non-positive integer values of a_8. In fact, there are two possible independent irreducible numerators but the derivation of the MB representation is simple only when we take one of them into account.

In order to derive a MB representation for (5.22) it is possible to start from the alpha representation and then apply (5.1) to the corresponding functions $\mathcal{U}$ and $\mathcal{V}$. This is not, however, an optimal way. In particular, this was done in the first calculation of the master double box [53] but a resulting MB representation turned out to be fivefold, with essential complications in the calculations. However, one can proceed using a fourfold MB representation. Let us also mention that in the case of non-planar on-shell double boxes it was possible to achieve [64] the minimal number of integrations equal to four starting from the global alpha representation.

To arrive at a fourfold MB representation let us use the 'loop by loop' derivation suggested in [66]. According to this procedure one starts from a one-loop subintegral, derives an MB representation for it using alpha parameters, then inserts this result into the given integral and obtains an MB integral for a Feynman integral with one loop less and indices depending on MB integration variables. Then one selects a next one-loop subintegral etc. One can use the public code AMBRE [33–35] for this.

The loop by loop procedure was applied to the massless double box in [3]. Let us do this by observing that (5.22) can be represented as

$$K(s,t;a_1,\ldots,a_8,\varepsilon) = \int \frac{d^dk\,[(k+p_1+p_2+p_4)^2]^{-a_8}}{(k^2)^{a_1}[(k+p_1)^2]^{a_2}[(k+p_1+p_2)^2]^{a_3}} \times F(s,(k+p_1+p_2+p_4)^2,k^2,(k+p_1+p_2)^2;a_6,a_7,a_4,a_5;d), \quad (5.23)$$

where the integral of four propagators dependent on l has been recognized as the box with two legs off shell. Then we can use (5.12). After inserting it into (5.23) we obtain the massless on-shell box with the indices $a_1 - z_2, a_2, a_3, a_8 - z_4$ for which we apply our representation (5.9). After these straightforward manipulations, we change the variables $z_2 \to z_2 - z_4, z_3 \to z_3 - z_4, z_4 \to z_1 + z_4$, and arrive at the following fourfold MB representation of (5.22) (see also [3]):

$$\begin{aligned} K(s,t;a_1,\ldots,a_8,\varepsilon) &= \frac{\left(\mathrm{i}\pi^{d/2}\right)^2(-1)^a}{\prod_{l=2,4,5,6,7}\Gamma(a_l)\Gamma(4-a_{4567}-2\varepsilon)(-s)^{a-4+2\varepsilon}} \\ &\times \frac{1}{(2\pi\mathrm{i})^4}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty}\cdots\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty}\left(\prod_{j=1}^{4}\mathrm{d}z_j\right)\left(\frac{t}{s}\right)^{z_1}\Gamma(a_2+z_1)\Gamma(-z_1) \\ &\times \frac{\Gamma(z_2+z_4)\Gamma(z_3+z_4)\Gamma(a_{1238}-2+\varepsilon+z_4)\Gamma(a_7+z_1-z_4)}{\Gamma(a_1+z_3+z_4)\Gamma(a_3+z_2+z_4)\Gamma(4-a_{1238}-2\varepsilon+z_1-z_4)} \\ &\times \frac{\Gamma(a_8-z_2-z_3-z_4)\Gamma(a_5+z_1+z_2+z_3+z_4)\Gamma(-z_1-z_2-z_3-z_4)}{\Gamma(a_8-z_1-z_2-z_3-z_4)} \\ &\times \Gamma(a_{4567}-2+\varepsilon+z_1-z_4)\Gamma(2-a_{128}-\varepsilon+z_2)\Gamma(2-a_{238}-\varepsilon+z_3) \\ &\times \Gamma(2-a_{567}-\varepsilon-z_1-z_2)\Gamma(2-a_{457}-\varepsilon-z_1-z_3). \end{aligned} \quad (5.24)$$

Let us check (5.24). Let us observe that, if we start from (5.24), we have to obtain, in the limit $a_{1,3,4,6} \to 0$ with $a_8 = 0$, the massless sunset diagram with the indices a_2, a_5, a_7. This corresponds to shrinking the horizontal lines—see Fig. 5.9. Indeed, we can start from (5.25) and perform the limit $a_3 \to 0$ by taking minus the residue at $z_1 = 4 - a_{1257} - 2\varepsilon$ in order to take into account the singularity of the integral of $\Gamma(a-4+2\varepsilon+z_1)\Gamma(4-a_{1257}-2\varepsilon-z_1)$. Then we can set $a_1 = 0$ and reproduce a known result.

Let us think of shrinking the vertical lines. We should obtain the product of two one-loop massless propagator-type integrals with the indices (a_1, a_3) and (a_4, a_6) in the limit $a_{2,5,7} \to 0$ with $a_8 = 0$—see Fig. 5.9. Yes, we do this by a similar analysis and similar manipulations: take minus residue at $z_1 = 0$ and set $a_2 = 0$, then take minus residue at $z_4 = -z_2 - z_3$ and set $a_5 = 0$, then take residues at $z_2 = 0$ and $z_3 = 0$ and set $a_7 = 0$.

The general fourfold representation (5.24) contains a lot of information. In particular, it is very easy to derive MB representations for the two classes of Feynman integrals corresponding to the graphs shown in Fig. 5.10. The integrals for the box

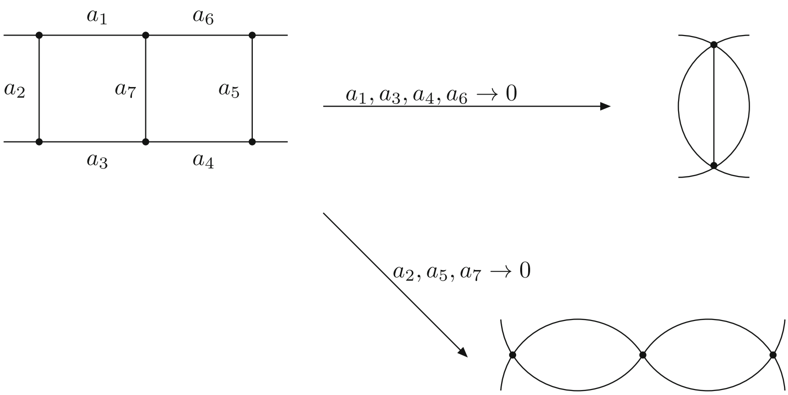


Fig. 5.9 *Horizontal* and *vertical* checks for the double box

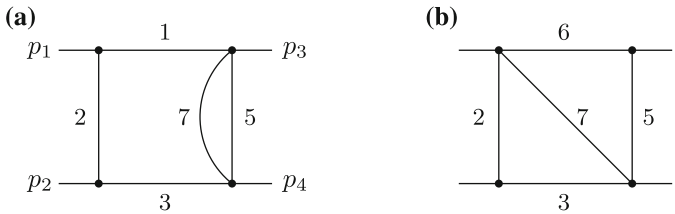


Fig. 5.10 Boxes with a one-loop insertion (**a**) and boxes with a diagonal (**b**) obtained from Fig. 5.8

with a one-loop insertion are obtained from the double box integrals at $a_4 = a_6 = 0$. (For simplicity, we consider the case $a_8 = 0$.) There are $\Gamma(a_4)$ and $\Gamma(a_6)$ in the denominator of (5.24) but, of course, the limit $a_4, a_6 \to 0$ is not zero see Fig. 5.9. Indeed, we can distinguish the product

$$\Gamma(a_{4567} - 2 + \varepsilon + z_1 - z_4)\Gamma(2 - a_{567} - \varepsilon - z_1 - z_2)\Gamma(z_2 + z_4)$$

which generates, due to integration over z_2 and z_4, the singularity of the type $\Gamma(a_4)$—remember our discussion in Sect. 5.2. So, to perform this limit we take a residue at $z_4 = -z_2$ and minus residue at $z_2 = 2 - a_{567} - \varepsilon - z_1$ and then set $a_4 = 0$. We still have $\Gamma(a_6)$ in the denominator, but there is also the product $\Gamma(a_{567} - 2 + \varepsilon + z_1 + z_3)\Gamma(2 - a_{57} - \varepsilon - z_1 - z_3)$ which generates the singularity of the type $\Gamma(a_6)$. Therefore, we take minus residue at $z_3 = 2 - a_{57} - \varepsilon - z_1$, then set $a_6 = 0$ and arrive at the following onefold MB representation:

$$K(a_1, a_2, a_3, 0, a_5, 0, a_7, 0) = \frac{\left(\mathrm{i}\pi^{d/2}\right)^2 (-1)^a \Gamma(2 - a_5 - \varepsilon)\Gamma(2 - a_7 - \varepsilon)}{\prod \Gamma(a_l)\Gamma(4 - a_{57} - 2\varepsilon)\Gamma(6 - a - 3\varepsilon)}$$

$$\times \frac{1}{(-s)^{a-4+2\varepsilon}} \frac{1}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}z \left(\frac{t}{s}\right)^z \Gamma(a-4+2\varepsilon+z)\Gamma(a_{57}-2+\varepsilon+z)$$
$$\times \Gamma(a_2+z)\Gamma(4-a_{1257}-2\varepsilon-z)\Gamma(4-a_{2357}-2\varepsilon-z)\Gamma(-z). \quad (5.25)$$

The integrals for the box with a diagonal are obtained from the double box integrals at $a_1 = a_4 = 0$. We start from the limit $a_4 \to 0$ as in the previous case. Then we observe that there is no $\Gamma(a_1)$ in the denominator and no gluing of right and left poles when $a_1 \to 0$. So, we just set $a_1 = 0$. After that the integration over z_3 involves only four gamma functions

$$\Gamma(2-a_{23}-\varepsilon+z_3)\Gamma(a_5+z_1+z_3)\Gamma(2-a_{57}-\varepsilon-z_1-z_3)\Gamma(-z_3).$$

The integral is evaluated by the first Barnes lemma (13.1), and we obtain

$$K(0,a_2,a_3,0,a_5,a_6,a_7,0) = \frac{\left(\mathrm{i}\pi^{d/2}\right)^2 \Gamma(2-a_{23}-\varepsilon)\Gamma(2-a_{56}-\varepsilon)}{\prod \Gamma(a_l)\Gamma(4-a_{237}-2\varepsilon)\Gamma(4-a_{567}-2\varepsilon)}$$
$$\times \frac{(-1)^a \Gamma(2-a_7-\varepsilon)}{\Gamma(6-a-3\varepsilon)(-s)^{a-4+2\varepsilon}} \frac{1}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}z \left(\frac{t}{s}\right)^z \Gamma(a-4+2\varepsilon+z)$$
$$\times \Gamma(a_2+z)\Gamma(a_5+z)\Gamma(-z)$$
$$\times \Gamma(4-a_{2357}-2\varepsilon-z)\Gamma(4-a_{2567}-2\varepsilon-z). \quad (5.26)$$

So, these two classes of integrals are rather simple because they are given only by onefold MB representations. See—[63] where an algorithmic procedure of their evaluation was described.

5.4 Two Basic Strategies of Resolving Singularities in ε

According to the old strategy A [53], one performs an analysis of the integrand to understand how poles in ε arise. The guiding principle is that the product $\Gamma(a+z)$ $\Gamma(b-z)$, where a and b can depend on the rest of the integration variables, generates, due to the integration over z, the singularity of the type $\Gamma(a+b)$. Indeed, if we shift an initial contour of integration over z across the point $z = -a$ we obtain an integral over a new contour which is not singular at $a+b = 0$, while the corresponding residue involves an explicit factor $\Gamma(a+b)$. This observation shows that any contour of one of the following integrations over the rest of the MB variables should be chosen according to this dependence, $\Gamma(a+b)$. Hence one thinks of integrations in various orders and then identifies some 'key' gamma functions which are crucial for the generation of poles in ε. Then one takes residues and shifts contours, starting from the first poles of these key gamma functions. The same analysis and procedure is then applied to the contributions of the residues.

To present (the new) Strategy A [51] let us explicitly formulate what was implied in the old Strategy A [53]. When we take care of one of the key gamma functions we shift a contour and take a residue. Let $\Gamma(A_i)$ with $A_i = a_i + b_i\varepsilon + \sum_j c_{ij} z_j$ be one of the key gamma functions in (5.7). Without loss of generality, we can consider ε real. Then changing the nature of the first pole of this gamma function means changing the rule for an admissible contour, i.e. that, instead of the condition $\mathrm{Re} A_i > 0$ when crossing the real axis in the process of the integration, we have the condition $-1 < \mathrm{Re}\, A_i < 0$. As in our first example in Sect. 5.1 let us denote this transition by replacing $\Gamma(A_i)$ by $\Gamma^{(1)}(A_i)$. The initial rule for the contour can be changed again and then we have the condition $-n < \mathrm{Re}\, A_i < -n+1$ for $n = 2, 3, \ldots$ with the notation $\Gamma^{(n)}(A_i)$.

In contrast to Strategy B where one has straight contours in the beginning, Strategy A is oriented at straight contours in the end. Apparently, it is desirable to achieve a minimal number of terms after the resolution of the singularities in ε. To do this, let us try to look for contours which are going to have in the end of this procedure and for which the gamma functions in the numerator are changed, in the above sense, in a minimal way. To formalize this requirement, let us introduce the function $\sigma(x) = [(1-x)_+]$ where $[\ldots]$ is the integer part of a number and $x_+ = x$ for $x > 0$ and 0, otherwise. In other words, if $-n < x < -n+1$ then $\sigma(x) = n$ for $n > 0$ and $\sigma(x) = 0$ for $n \leq 0$.

So, let us set $\varepsilon = 0$ and look for contours, i.e. $\mathrm{Re}\, z_i$, for which the sum

$$\sum_i \sigma\left(\mathrm{Re}\, A_i|_{\varepsilon=0}\right) \equiv \sum_i \sigma\left(a_i + \sum_j c_{ij} \mathrm{Re}\, z_j\right)$$

is minimal.

After such a choice is done we identify gamma functions which should be changed, in the above sense, in order to arrive at a final integral where a Laurent expansion in ε is possible. In fact, this step replaces the first step in the old Strategy A where one identified such key gamma functions after the analysis characterized above.

Then the second step in Strategy A is the same as in the old version: we take care of the distinguished gamma functions, i.e. take a residue and replace Γ by $\Gamma^{(1)}(A_i)$ (and, possibly, $\Gamma^{(1)}(A_i)$ by $\Gamma^{(2)}(A_i)$ etc.) We proceed iteratively, as in the previous strategy: every residue is considered from scratch, i.e. treated in the same way as the initial MB integral.

Let me emphasize that although this strategy aims at minimizing the number of resulting terms, we cannot exclude that there is another way of resolving the singularities in ε that is the best one in this sense. For example, it can happen, in rather complicated examples, that different orders of changing the key gamma functions lead to different numbers of resulting terms. Still I believe that such a difference is negligible and that the Strategy A provides a resolution of the singularities in ε at least very close to the theoretically best one.

The difference of the new and the old Strategies A is minor. In multiple applications of the old Strategy A one can see that resulting contours were straight indeed. This difference can still be seen in the following simple example of the integral

$$\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \Gamma(1+\varepsilon+z)\Gamma(-1/2+\varepsilon+z)\Gamma(3/2-\varepsilon-z)\Gamma(-z)\,dz$$

which can be evaluated at general ε by the first Barnes lemma (13.1). Within the old Strategy A, one can observe that there is no gluing of poles so that one can expand the integrand in ε. However, a resulting contour cannot be chosen as a straight line. Rather, within the new Strategy A, we have to choose a gamma function to be modified and, as a result, we obtain a residue and an integral over a straight line which both can be expanded in ε.

Let us now turn to Strategy B [64]. First, one chooses a domain of the regularization parameter ε and values of the real parts of the integration variables, z_i, $w, \ldots$ in such a way that *all* the integrations over the MB variables can be performed over straight lines parallel to imaginary axis. In fact this is not always possible. However, in these situations, one can introduce an auxiliary analytic regularization to provide the existence of such straight contours. Then one tends ε to zero, and whenever a pole of some gamma function is crossed one takes into account the corresponding residue. (If the auxiliary analytic regularization was introduced, one first performs, in a similar way, the analytic continuation to zero values of the corresponding analytic parameters.) It is simple to organize this procedure in such a way that no more than one pole is crossed at the same time. For every resulting residue, which involves one integration less, a similar procedure is applied, and so on.

Strategy B [64] is algorithmic in its character and, indeed, two algorithmic descriptions were formulated in [2, 17]. The public code called `MB.m` was presented in [17]. Strategy B was successfully applied both before and after `MB.m` was developed—see, e.g., [3, 5, 9, 10, 12, 18, 20–22, 24, 25, 32, 39].

The additional code `MBresolve.m` where Strategy A is implemented was presented in [51]. It works together with `MB.m`. A package which includes `MB.m` and `MBresolve.m` can be downloaded from [41]. As an example of a recent application of this package let me mention [52]. So, if one needs to resolve the singularity structure of a MB integral there is no reason to proceed by hand. With the use of `MB.m` or `MBresolve.m` one solves this problem automatically. Moreover, these and some additional codes provide the possibility to evaluate automatically resulting MB integrals (which are coefficients at powers of ε). This point will be illustrated through examples in the next section.

5.5 Two-Loop Examples

As in Chap. 3 let us turn to Example 3.7 and evaluate the integral (3.59) given by the MB representation (5.21) with all indices equal to one. We have

$$F(Q^2; 1, \ldots, 1; d) = \frac{\left(i\pi^{d/2}\right)^2}{(Q^2)^{2+2\varepsilon}} f(\varepsilon), \qquad (5.27)$$

with

$$f(\varepsilon) = \frac{\Gamma(-\varepsilon)^2}{\Gamma(-3\varepsilon)\Gamma(-2\varepsilon)} V(\varepsilon)$$

and

$$\begin{aligned} V(\varepsilon) = \frac{1}{(2\pi \mathrm{i})^2} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} & \mathrm{d}z_1 \mathrm{d}z_2 \Gamma(2+2\varepsilon+z_1+z_2)\Gamma(1+z_1+z_2) \\ \times \; & \Gamma(1+z_2)^2 \Gamma(-z_1)\Gamma(-z_2) \frac{\Gamma(-\varepsilon-z_1)}{\Gamma(-2\varepsilon-z_1)^2} \\ \times \; & \Gamma(-1-2\varepsilon-z_2)\Gamma(-1-2\varepsilon-z_1-z_2)^2. \end{aligned} \tag{5.28}$$

After the useful change of variables $z_1 \to -1 - z_1 - z_2$, we obtain

$$\begin{aligned} V(\varepsilon) = \frac{1}{(2\pi \mathrm{i})^2} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} & \mathrm{d}z_1 \mathrm{d}z_2 \frac{\Gamma(1+z_1+z_2)\Gamma(1-\varepsilon+z_1+z_2)}{\Gamma(1-2\varepsilon+z_1+z_2)^2} \\ \times \; & \Gamma(-2\varepsilon+z_1)^2 \Gamma(-z_1)\Gamma(1+2\varepsilon-z_1) \\ \times \; & \Gamma(1+z_2)^2 \Gamma(-1-2\varepsilon-z_2)\Gamma(-z_2). \end{aligned} \tag{5.29}$$

For this simple example, the codes `MB.m` and `MBresolve.m` produce identical intermediate results. These are MB integrals which are then expanded in a Laurent series in ε. We do this up to ε^0. There are three contributions. One of them is without integration:

$$C_1 = \frac{1}{\varepsilon^4} - \frac{\pi^2}{2\varepsilon^2} - \frac{110\zeta(3)}{3\varepsilon} - \frac{41\pi^4}{40}. \tag{5.30}$$

The second contribution is given by a onefold MB integral:

$$\begin{aligned} C_2 = 3\frac{1}{2\pi \mathrm{i}} \int_{r-\mathrm{i}\infty}^{r+\mathrm{i}\infty} & \mathrm{d}z \Gamma(1+z)\Gamma(z)\Gamma(-z)^2 \Bigg(\frac{1}{\epsilon^2} + \frac{3\psi(-z) - 2\psi(z) + \gamma_{\mathrm{E}}}{\epsilon} \\ & + \frac{1}{12} \Big(6\psi(-z)^2 - 24\psi(z)\psi(-z) + 48\psi(z+1)\psi(-z) + 36\gamma_{\mathrm{E}}\psi(-z) \\ & + 12\psi(z)^2 - 12\psi(z+1)^2 - 24\gamma_{\mathrm{E}}\psi(z) - 24\psi(z)\psi(z+1) \\ & - 66\psi'(-z) + 12\psi'(z) - 12\psi'(z+1) - 7\pi^2 + 6\gamma_{\mathrm{E}}^2 \Big) \Bigg), \end{aligned} \tag{5.31}$$

where the integration is over the straight contour with $r = -1/2$ or some other number between -1 and 0.

The integral of the $1/\varepsilon^2$ and $1/\varepsilon$ terms can be evaluated by means of corollaries of the first Barnes lemma (13.1). In `MB.m` the evaluation of any integral of the form

$$\frac{1}{2\pi \mathrm{i}} \int_{r-\mathrm{i}\infty}^{r+\mathrm{i}\infty} \mathrm{d}z \, \Gamma(a_1+z)\Gamma(a_2+z)\Gamma(a_3-z)\Gamma(a_4-z). \tag{5.32}$$

and similar integrals with a derivative of one of the gamma functions involved is implemented. To do this one uses the command `Barnes1` and specifies an integration variable. Similarly, the evaluation of integrals of the form

$$\frac{1}{2\pi \mathrm{i}}\int_{r-\mathrm{i}\infty}^{r+\mathrm{i}\infty} \mathrm{d}z\, \frac{\Gamma(a_1+z)\Gamma(a_2+z)\Gamma(a_3+z)\Gamma(a_4-z)\Gamma(a_5-z)}{\Gamma(a_6+z)}, \tag{5.33}$$

with $a_6 = a_1+a_2+a_3+a_4+a_5$, using the second Barnes lemma (13.47) is implemented. However, the integration of the ε^0 term cannot be done by corollaries of Barnes lemmas. Here one can apply the `PSLQ` algorithm using as input a numerical result for the ε^0 part of (5.31) with a sufficiently high accuracy—see a brief description of this algorithm in Sect. 14.8. This gives the value $7\pi^4/10$ for the ε^0 part of (5.31) and the result

$$C_2 = -\frac{\pi^2}{2\varepsilon^2} + \frac{9\zeta(3)}{\varepsilon} + \frac{7\pi^4}{10}. \tag{5.34}$$

The third contribution is

$$\begin{aligned} &6\frac{1}{(2\pi \mathrm{i})^2}\int_{r1-\mathrm{i}\infty}^{r1+\mathrm{i}\infty} \mathrm{d}z_1 \int_{r2-\mathrm{i}\infty}^{r2+\mathrm{i}\infty} \mathrm{d}z_2 \Gamma(1+z_1+z_2)\Gamma(2+z_1+z_2)\Gamma(1+z_2)^2 \\ &\quad\times \Gamma(-1-z_1-z_2)^2\Gamma(-1-z_2)\Gamma(-z_2), \end{aligned} \tag{5.35}$$

where $r_1 = r_2 = -1/4$. This integral can straightforwardly be evaluated by a successive application of the first Barnes lemma with the result $-\pi^4/6$. Summing up the three parts of the result we reproduce the result (3.80) obtained in [36].

Let us now apply (5.24) to the evaluation, in expansion in ε up to the finite part, of the double box without numerator and with all powers of the propagators equal to one. We know in advance that it has poles up to the fourth order in ε, due to IR and collinear divergences. Representation (5.24) gives

$$K(s,t;1,\ldots,1,0,\varepsilon) = -\frac{\left(\mathrm{i}\pi^{d/2}\right)^2}{(-s)^{3+2\varepsilon}} F(x,\varepsilon), \tag{5.36}$$

where $x = t/s$ and

$$\begin{aligned} F(x,\varepsilon) = &\frac{1}{\Gamma(-2\varepsilon)}\frac{1}{(2\pi \mathrm{i})^4}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty}\ldots\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty}\left(\prod_{j=1}^{4}\mathrm{d}z_j\right)x^{z_1} \\ &\times \frac{\Gamma(1+z_1)\Gamma(-z_1)\Gamma(-1-\varepsilon-z_1-z_2)\Gamma(-1-\varepsilon-z_1-z_3)}{\Gamma(1+z_2+z_4)\Gamma(1+z_3+z_4)\Gamma(1-2\varepsilon+z_1-z_4)} \\ &\times \Gamma(2+\varepsilon+z_1-z_4)\Gamma(1+z_1+z_2+z_3+z_4)\Gamma(1+z_1-z_4) \end{aligned}$$

$$\times \Gamma(z_2+z_4)\Gamma(z_3+z_4)\Gamma(-\varepsilon+z_2)\Gamma(-\varepsilon+z_3)$$
$$\times \Gamma(1+\varepsilon+z_4)\Gamma(-z_2-z_3-z_4)\,. \tag{5.37}$$

We use the integrand of (5.37) as an input for the code `MB.m`. After using commands for resolving singularities in ε, we obtain a linear combination of 28 MB integrals which are at most twofold. Some integrations can now be taken by Barnes lemmas. One can look for such possibilities himself (herself) but it is more reasonable at this point to use an additional public code `barnesroutines.m` by Kosower [45]. The command `DoAllBarnes` looks for possible applications of Barnes lemmas, in some order and with some priority, and applies the corresponding commands `Barnes1` and `Barnes2` of `MB.m`.

After this we obtain a linear combination of five onefold MB integrals and a term without integration. These are MB integrals without dependence on $x=t/s$, either with four gamma functions or with five gamma functions in the numerator and one gamma function in the denominator, times a linear combination of the ψ-function and its derivatives, and MB integrals including x^z, with six gamma functions and a ψ-function. All these integrals can be evaluated by closing the contour to the right, taking residues at the points $z=0,1,2,\ldots$ and summing up resulting series with the help of the table of formulae [31] presented in Appendix C. Alternatively, one can apply the public codes `SUMMER` [68] and `XSummer` [47] written in `FORM` [67] for the summation of these series.

Collecting all the contributions we reproduce the result of [53]:

$$K(s,t;1,\ldots,1,0,\varepsilon) = -\frac{\left(\mathrm{i}\pi^{d/2}\mathrm{e}^{-\gamma_{\mathrm{E}}\varepsilon}\right)^2}{(-s)^{2+2\varepsilon}t}\, f\left(\frac{t}{s};\varepsilon\right), \tag{5.38}$$

where

$$\begin{aligned} f(x,\varepsilon) = &-\frac{4}{\varepsilon^4}+\frac{5\ln x}{\varepsilon^3}-\left(2\ln^2 x-\frac{5}{2}\pi^2\right)\frac{1}{\varepsilon^2} \\ &-\left(\frac{2}{3}\ln^3 x+\frac{11}{2}\pi^2\ln x-\frac{65}{3}\zeta(3)\right)\frac{1}{\varepsilon} \\ &+\frac{4}{3}\ln^4 x+6\pi^2\ln^2 x-\frac{88}{3}\zeta(3)\ln x+\frac{29}{30}\pi^4 \\ &-\left[2\,\mathrm{Li}_3(-x)-2\ln x\,\mathrm{Li}_2(-x)-(\ln^2 x+\pi^2)\ln(1+x)\right]\frac{2}{\varepsilon} \\ &-4\left[S_{2,2}(-x)-\ln x\,S_{1,2}(-x)\right]+44\,\mathrm{Li}_4(-x) \\ &-4\left[\ln(1+x)+6\ln x\right]\mathrm{Li}_3(-x) \\ &+2\left(\ln^2 x+2\ln x\ln(1+x)+\frac{10}{3}\pi^2\right)\mathrm{Li}_2(-x) \end{aligned}$$

$$
\begin{aligned}
&+\left(\ln^2 x+\pi^2\right)\ln^2(1+x)\\
&-\frac{2}{3}[4\ln^3 x+5\pi^2\ln x-6\zeta(3)]\ln(1+x)+O(\varepsilon),
\end{aligned}
\tag{5.39}
$$

where $S_{k,n}(x)$ are generalized (Nielsen) polylogarithms [30, 43, 44]—see (11.8). Generalized polylogarithms are partial cases of multiple polylogarithms—see (11.43) and (11.46) in Appendix B.

This result is in agreement with the leading behaviour in the (Regge) limit $t/s \to 0$ obtained in [63] by use of the strategy of expansion by regions [6, 57, 60, 61] described in Chap. 9. Keeping the two leading powers of x we have

$$
\begin{aligned}
f(x,\varepsilon) = &-\frac{4}{\varepsilon^4}+\frac{5\ln x}{\varepsilon^3}-\left(2\ln^2 x-\frac{5}{2}\pi^2\right)\frac{1}{\varepsilon^2}\\
&-\left(\frac{2}{3}\ln^3 x+\frac{11}{2}\pi^2\ln x-\frac{65}{3}\zeta(3)\right)\frac{1}{\varepsilon}\\
&+\frac{4}{3}\ln^4 x+6\pi^2\ln^2 x-\frac{88}{3}\zeta(3)\ln x+\frac{29}{30}\pi^4\\
&+2x\left(\frac{1}{\varepsilon}(\ln^2 x-2\ln x+\pi^2+2)\right.\\
&\left.-\frac{1}{3}\{4\ln^3 x+3\ln^2 x+(5\pi^2-36)\ln x+2[33+5\pi^2-3\zeta(3)]\}\right)\\
&+O(x^2\ln^3 x,\varepsilon).
\end{aligned}
\tag{5.40}
$$

A remarkable feature of (5.39) is that the result has uniform transcendentality, i.e. every term has transcendentality weight four. (We assume that the weight of $\mathrm{Li}_n(-x)$, $S_{k,n-k}(-x)$ and $\zeta(n)$ is n and the weight of a product is the sum of the weights of the corresponding factors.

5.6 Examples with the Old Strategy A

Although the old Strategy A does not have an implementation in a computer code let us now consider examples where it was successfully applied because it might happen that somebody provides such an implementation which could be better than the new Strategy A. In fact, the resolution of singularities in ε in the first example is rather simple and results in a few terms. On the other hand, this example will give the possibility to discuss some additional techniques and limitations of the method of MB representation. Then we will just outline calculations for the second example (massless triple box) and discuss mainly the end of the calculation. One more reason for not forgetting about the old Strategy A is that it proved useful in a series of nontrivial calculations.

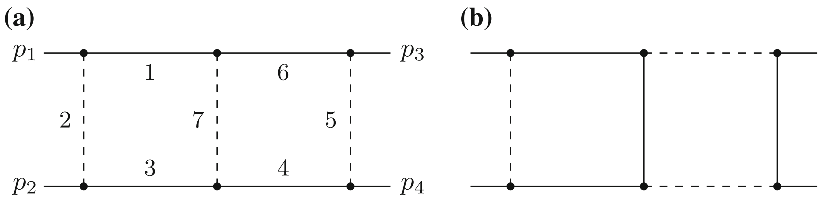
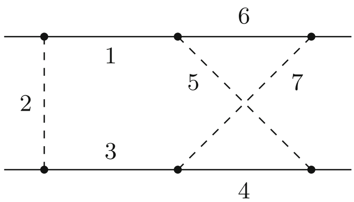


Fig. 5.11 Planar massive on-shell double boxes: **a** first type, **b** second type. The *solid lines* denote massive, the *dashed lines* massless particles

Fig. 5.12 Non-planar massive on-shell double box



Let us turn to

Example 5.8 Massive on-shell double box diagrams shown in Figs. 5.11 and 5.12.

This is an important class of Feynman integrals with one more parameter, with respect to the massless on-shell double boxes. In particular, it is relevant to Bhabha scattering.

The general double box Feynman integral of the first type (see Fig. 5.11a) takes the form

$$
\begin{aligned}
B_{\mathrm{PL},1}(s,t,m^2;a_1,\ldots,a_8,\varepsilon) &= \int\int \frac{d^dk\,d^dl}{(k^2-m^2)^{a_1}[(k+p_1)^2]^{a_2}} \\
&\times \frac{[(k+p_1+p_2+p_4)^2]^{-a_8}}{[(k+p_1+p_2)^2-m^2]^{a_3}[(l+p_1+p_2)^2-m^2]^{a_4}[(l+p_1+p_2+p_4)^2]^{a_5}} \\
&\times \frac{1}{(l^2-m^2)^{a_6}[(k-l)^2]^{a_7}}, \qquad (5.41)
\end{aligned}
$$

where $p_i^2 = m^2$, $i = 1,2,3,4$, and we consider a (non-negative) power $-a_8$ of the factor $(k+p_1+p_2+p_4)^2$ in the numerator as in the massless case.

To derive an appropriate MB representation for (5.41) we proceed loop by loop, similarly to the massless case, i.e. recognize the internal integral over l as a massive box with two legs off-shell for which we use representation (5.17). After that the integral over k can be recognized as the massive on-shell box represented by (5.14), and we obtain the following sixfold MB representation [56]:

$$
\begin{aligned}
B_{\mathrm{PL},1}(s,t,m^2;a_1,\ldots,a_8,\varepsilon) &= \frac{\left(\mathrm{i}\pi^{d/2}\right)^2(-1)^a(-s)^{4-a-2\varepsilon}}{\prod_{j=2,4,5,6,7}\Gamma(a_j)\Gamma(4-a_{4567}-2\varepsilon)} \\
&\times \frac{1}{(2\pi\mathrm{i})^6}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty}\cdots\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} dw \prod_{j=1}^{5}\mathrm{d}z_j \left(\frac{m^2}{-s}\right)^{z_1+z_5}\left(\frac{t}{s}\right)^{w}\Gamma(a_2+w)\Gamma(-w) \\
&\times \frac{\Gamma(z_2+z_4)\Gamma(z_3+z_4)\Gamma(4-a_{13}-2a_{28}-2\varepsilon+z_2+z_3)\Gamma(a_7+w-z_4)}{\Gamma(a_1+z_3+z_4)\Gamma(a_3+z_2+z_4)} \\
&\times \frac{\Gamma(a_{1238}-2+\varepsilon+z_4+z_5)\Gamma(a_{4567}-2+\varepsilon+w+z_1-z_4)}{\Gamma(4-a_{46}-2a_{57}-2\varepsilon-2w-2z_1-z_2-z_3)} \\
&\times \frac{\Gamma(a_8-z_2-z_3-z_4)\Gamma(-w-z_2-z_3-z_4)\Gamma(2-a_{238}-\varepsilon+z_3-z_5)}{\Gamma(4-a_{1238}-2\varepsilon+w-z_4)\Gamma(a_8-w-z_2-z_3-z_4)} \\
&\times \frac{\Gamma(a_5+w+z_2+z_3+z_4)\Gamma(2-a_{567}-\varepsilon-w-z_1-z_2)}{\Gamma(4-a_{13}-2a_{28}-2\varepsilon+z_2+z_3-2z_5)} \\
&\times \Gamma(2-a_{457}-\varepsilon-w-z_1-z_3)\Gamma(2-a_{128}-\varepsilon+z_2-z_5) \\
&\times \Gamma(4-a_{46}-2a_{57}-2\varepsilon-2w-z_2-z_3)\Gamma(-z_1)\Gamma(-z_5). \qquad (5.42)
\end{aligned}
$$

This general formula can be used to evaluate various Feynman integrals of the given family. Let us consider the example of the Feynman integral without numerator and $a_i = 1$ for $i = 1, 2, \ldots, 7$. Then (5.42) takes the form

$$
\begin{aligned}
B^{(0)}(s,t,m^2,\varepsilon) &\equiv B_{\mathrm{PL},1}(s,t,m^2;1,\ldots,1,0,\varepsilon) = -\frac{\left(\mathrm{i}\pi^{d/2}\right)^2}{\Gamma(-2\varepsilon)(-s)^{3+2\varepsilon}} \\
&\times \frac{1}{(2\pi\mathrm{i})^6}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty}\cdots\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} dw \prod_{j=1}^{5}\mathrm{d}z_j \left(\frac{m^2}{-s}\right)^{z_1+z_5}\left(\frac{t}{s}\right)^{w} \\
&\times \frac{\Gamma(1+w)\Gamma(-w)\Gamma(2+\varepsilon+w+z_1-z_4)\Gamma(-1-\varepsilon-w-z_1-z_2)}{\Gamma(1-2\varepsilon+w-z_4)\Gamma(1+z_2+z_4)\Gamma(1+z_3+z_4)} \\
&\times \frac{\Gamma(-1-\varepsilon-w-z_1-z_3)\Gamma(-z_1)\Gamma(-\varepsilon+z_2-z_5)\Gamma(-\varepsilon+z_3-z_5)}{\Gamma(-2\varepsilon+z_2+z_3-2z_5)\Gamma(-2-2\varepsilon-2w-2z_1-z_2-z_3)} \\
&\times \Gamma(1+\varepsilon+z_4+z_5)\Gamma(-z_5)\Gamma(-2\varepsilon+z_2+z_3)\Gamma(1+w-z_4) \\
&\times \Gamma(1+w+z_2+z_3+z_4)\Gamma(-2-2\varepsilon-2w-z_2-z_3) \\
&\times \Gamma(z_2+z_4)\Gamma(z_3+z_4)\Gamma(-z_2-z_3-z_4). \qquad (5.43)
\end{aligned}
$$

Observe that, because of the presence of the factor $\Gamma(-2\varepsilon)$ in the denominator, we are forced to take some residue in order to arrive at a non-zero result at $\varepsilon = 0$, so that the integral is effectively fivefold.

Let us apply our strategy of shifting contours and taking residues, with the goal to decompose (5.43) into pieces where the Laurent expansion ε of the integrand becomes possible. We will evaluate this integral in expansion in ε up to a finite part. We know in advance that the poles in ε are now only of the second order because collinear divergences are absent. This is how such procedure can be performed in this case [56]:

1. Take minus residue at $z_3 = -2-2\varepsilon-2w-z_2$, then minus residue at $w = -1-2\varepsilon$, then a residue at $z_4 = 0$, then a residue at $z_2 = 0$, expand in a Laurent series in ε up to a finite part. Let us denote the resulting integral over z_1 and z_5 by B_1.
2. Take minus residue at $z_3 = -2-2\varepsilon-2w-z_2$, then minus residue at $w = -1-2\varepsilon$, then a residue at $z_4 = 0$, and change the nature of the first pole of $\Gamma(z_2)$ (choose a contour from the opposite side, i.e. the pole z_2 will be now right), then expand in ε. Denote this integral over z_1, z_2 and z_5 by B_2.
3. Take minus residue at $z_3 = -2-2\varepsilon-2w-z_2$, then minus residue at $w = -1-2\varepsilon$, then change the nature of the first pole of $\Gamma(z_4)$, then take a residue at $z_2 = -z_4$, then take a residue at $z_4 = -\varepsilon$ and expand in ε. This resulting integral over z_1 and z_5 is denoted by B_3.
4. Take minus residue at $z_3 = -2-2\varepsilon-2w-z_2$, then minus residue at $w = -1-2\varepsilon$, then change the nature of the first pole of $\Gamma(z_4)$, then take a residue at $z_2 = -z_4$, then change the nature of the first pole of $\Gamma(2(\varepsilon + z_4))$ and expand in ε. The resulting integral over z_1, z_4 and z_5 is denoted by B_4.
5. Take minus residue at $z_3 = -2-2\varepsilon-2w-z_2$, then minus residue at $w = -1-2\varepsilon$, then change the nature of the first pole of $\Gamma(z_4)$, then change the nature of the first pole of $\Gamma(z_2+z_4)$ and expand in ε. The resulting integral over z_1, z_2, z_4 and z_5 is denoted by B_5.
6. Take minus residue at $z_3 = -2 - 2\varepsilon - 2w - z_2$, then change the nature of the first pole of $\Gamma(-2(1 + 2\varepsilon + w))$, then take minus residue at $z_4 = 1 + w$, then minus residue at $z_2 = -1 - 2\varepsilon - w$ and expand in ε. The resulting integral over w, z_1 and z_5 is denoted by B_6.
7. Change the nature of the first pole of $\Gamma(-2-2\varepsilon-2w-z_2-z_3)$, then take minus residue at $z_4 = -z_2 - z_3$, then a residue at $z_3 = 2\varepsilon - z_2$, then take a residue at $z_2 = 2\varepsilon$ and expand in ε. The resulting integral over w, z_1 and z_5 is denoted by B_7.

One can see that all the other contributions vanish at $\varepsilon = 0$. By a suitable change of variables, one can observe that $B_7 = B_6$. In fact, the dependence of the first five contributions on the Mandelstam variable t is trivial: they are just proportional to $1/t$.

The two-dimensional integrals B_1 and B_3 are products of one-dimensional integrals which can be evaluated by closing the contour to the left and summing up resulting series with the help of formulae [26] of Appendix C.

To evaluate the three-parametric integral B_4 it is reasonable to observe that the integrand only changes its sign after the transformation $\{z_4 \to -z_4, z_1 \to z_5, z_5 \to z_1\}$. If we take into account that the change of variables $z_4 \to -z_4$ implies that the initial integration contour $-1 < \mathrm{Re} z_4 < 0$ becomes $0 < \mathrm{Re} z_4 < 1$ we will obtain a simple equation for B_4 and conclude that the value of the integral equals $1/2$ times the residue at $z_4 = 0$. The latter quantity turns out to be a factorized integral over z_1 and z_5 which is evaluated like B_1 and B_3.

The three-dimensional integral B_2 is evaluated by closing the integration contours over z_1 and z_5 to the left, summing up resulting series and applying a similar procedure to a final integral in z_2. The corresponding result is naturally expressed in terms of polylogarithms, up to Li_3, depending on s and m^2 in terms of the variable

$$v = \left[\frac{\sqrt{4m^2 - s} + \sqrt{-s}}{\sqrt{4m^2 - s} - \sqrt{-s}} \right]^2 .$$

The form of this result provides a hint about a possible functional dependence of the result for the four-dimensional integral B_5, and a heuristic procedure which was explicitly formulated in [31] turns out to be successfully applicable here. First, all the contributions, in particular B_4, are analytic functions of s in a vicinity of the origin. One can observe that any given term of the Taylor expansion can be evaluated straightforwardly because the corresponding integrals over z_2 and z_4 are taken recursively. It is, therefore, possible to evaluate enough first terms (say, 30) of this Taylor expansion. Then one takes into account the type of the functional dependence mentioned above, turns to a new Taylor series in terms of the variable $v - 1$ and assumes that the nth term of this Taylor series is a linear combination, with unknown coefficients, of the following quantities of weights 1, 2, 3, and 4, respectively:

$$\frac{1}{n}, \tag{5.44}$$

$$\frac{1}{n^2}, \frac{S_1(n)}{n}, \tag{5.45}$$

$$\frac{1}{n^3}, \frac{S_1(n)}{n^2}, \frac{S_2(n)}{n}, \frac{S_1(n)^2}{n}, \tag{5.46}$$

$$\begin{aligned} &\frac{1}{n^4}, \frac{S_1(n)}{n^3}, \frac{S_2(n)}{n^2}, \frac{S_1(n)^2}{n^2}, \\ &\quad \frac{S_3(n)}{n}, \frac{S_{12}(n)}{n}, \frac{S_1(n)S_2(n)}{n}, \frac{S_1(n)^3}{n}. \end{aligned} \tag{5.47}$$

where $S_k(n) = \sum_{j=1}^{n} j^{-k}$, etc. are nested sums (see Appendix C). Using the information about the first terms of the Taylor series one solves a system of linear equations, finds those unknown coefficients and checks this solution with the help of the next Taylor coefficients.

This experimental mathematics has turned out to be quite successful for the evaluation of B_5. Finally, the contribution B_6 is a product of a one-dimensional integral over z_1, which is easily evaluated, and a two-dimensional integral over w and z_5 which involves a non-trivial dependence on t and is evaluated by closing the integration contour in z_5 to the left, summing up a resulting series in terms of Gauss hypergeometric function for which one can apply the parametric representation (11.5). After that the internal integral over w is taken by the same procedure and, finally, one takes the parametric integral.

The final result takes the following form [56]:

$$B^{(0)}(s,t,m^2;\varepsilon) = -\frac{\left(\mathrm{i}\pi^{d/2}\mathrm{e}^{-\gamma_{\mathrm{E}}\varepsilon}\right)^2 x^2}{s^2(-t)^{1+2\varepsilon}} \times \left[\frac{b_2(x)}{\varepsilon^2} + \frac{b_1(x)}{\varepsilon} + b_{01}(x) + b_{02}(x,y) + O(\varepsilon)\right], \quad (5.48)$$

where $x = 1/\sqrt{1-4m^2/s}$, $y = 1/\sqrt{1-4m^2/t}$, and

$$b_2(x) = 2(m_x - p_x)^2, \quad (5.49)$$

$$\begin{aligned} b_1(x) = &-8\left[\mathrm{Li}_3\left(\frac{1-x}{2}\right) + \mathrm{Li}_3\left(\frac{1+x}{2}\right) + \mathrm{Li}_3\left(\frac{-2x}{1-x}\right)\right. \\ &\left.+ \mathrm{Li}_3\left(\frac{2x}{1+x}\right)\right] + 4(m_x - p_x)\left[\mathrm{Li}_2\left(\frac{1-x}{2}\right) - \mathrm{Li}_2\left(\frac{-2x}{1-x}\right)\right] \\ &- (4/3)m_x^3 + 4m_x^2 p_x - 6m_x p_x^2 + (2/3)p_x^3 + 4l_2(m_x p_x + p_x^2) \\ &- 2l_2^2(m_x + 3p_x) - (\pi^2/3)(4l_2 - m_x - 3p_x) + (8/3)l_2^3 + 14\zeta(3), \end{aligned} \quad (5.50)$$

$$\begin{aligned} b_{01}(x) = &-8(m_x - p_x)\left[\mathrm{Li}_3(x) - \mathrm{Li}_3(-x) - \mathrm{Li}_3\left(\frac{1+x}{2}\right)\right. \\ &\left.+ \mathrm{Li}_3\left(\frac{1-x}{2}\right) - \mathrm{Li}_3\left(\frac{2x}{1+x}\right) + \mathrm{Li}_3\left(\frac{-2x}{1-x}\right)\right] \\ &+ 16\mathrm{Li}_2\left(\frac{1-x}{2}\right)(\mathrm{Li}_2(x) - \mathrm{Li}_2(-x)) \\ &+ 4\left[\mathrm{Li}_2(x)^2 + \mathrm{Li}_2(-x)^2 + 4\mathrm{Li}_2\left(\frac{1-x}{2}\right)^2\right] - 8\mathrm{Li}_2(x)\,\mathrm{Li}_2(-x) \\ &- (8/3)[\pi^2 - 6l_2^2 + 6l_x p_x - 6m_x(l_x + p_x - 2l_2)]\mathrm{Li}_2\left(\frac{1-x}{2}\right) \\ &- (4/3)[\pi^2 - 6l_2^2 + 3m_x^2 + 6m_x(2l_2 - 2l_x - p_x) + 12l_x p_x - 3p_x^2] \\ &\times (\mathrm{Li}_2(x) - \mathrm{Li}_2(-x)) + 8(m_x - p_x)\left[(p_x - m_x + 2l_2)\mathrm{Li}_2\left(\frac{2x}{1+x}\right)\right. \\ &\left.+ 2(l_x - m_x + l_2)\mathrm{Li}_2\left(\frac{-2x}{1-x}\right)\right] - 8(m_x - p_x)(2l_x - p_x - 5m_x + 4l_2) \\ &\times (-m_x p_x + l_2(m_x + p_x) - l_2^2 + \pi^2/6) \\ &- (20/3)m_x^4 + (164/3)m_x^3 p_x - 40m_x^2 p_x^2 - (4/3)m_x p_x^3 - (8/3)p_x^4 \end{aligned}$$

$$
\begin{aligned}
&+ 8m_x l_x (m_x^2 - 3m_x p_x + 2p_x^2) \\
&- 4l_2(7m_x^3 + 21m_x^2 p_x - 4m_x l_x p_x - 23m_x p_x^2 + 4l_x p_x^2 - p_x^3) \\
&- \pi^2((17/3)m_x^2 - (4/3)m_x l_x - 2m_x p_x + (4/3)l_x p_x - (7/3)p_x^2) \\
&+ l_2^2(84m_x^2 - 8m_x l_x - 16m_x p_x + 8l_x p_x - 44p_x^2) \\
&- (8/3)l_2(6l_2^2 - \pi^2)(3m_x - 2p_x) - (4/3)\pi^2 l_2^2 + 4l_2^4 + \pi^4/9. \qquad (5.51)
\end{aligned}
$$

The last piece of the finite part comes from B_6 and B_7:

$$
\begin{aligned}
b_{02}(x, y) = {}& 2(p_x - m_x)\Bigg\{4\left[\mathrm{Li}_3\left(\frac{1-x}{2}\right) - \mathrm{Li}_3\left(\frac{1+x}{2}\right)\right. \\
&+ \mathrm{Li}_3\left(\frac{(1-x)y}{1-xy}\right) - \mathrm{Li}_3\left(\frac{-(1+x)y}{1-xy}\right) + \mathrm{Li}_3\left(\frac{-(1-x)y}{1+xy}\right) \\
&\left. - \mathrm{Li}_3\left(\frac{(1+x)y}{1+xy}\right)\right] + 2\left[\mathrm{Li}_3\left(\frac{(1+x)(1-y)}{2(1-xy)}\right) - \mathrm{Li}_3\left(\frac{(1-x)(1+y)}{2(1-xy)}\right)\right. \\
&\left. - \mathrm{Li}_3\left(\frac{(1-x)(1-y)}{2(1+xy)}\right) + \mathrm{Li}_3\left(\frac{(1+x)(1+y)}{2(1+xy)}\right)\right] \\
&+ 2(m_y + p_y - m_{xy} - p_{xy}) \\
&\times \left[2\mathrm{Li}_2\,(x) - 2\mathrm{Li}_2\,(-x) + \mathrm{Li}_2\left(\frac{-2x}{1-x}\right) - \mathrm{Li}_2\left(\frac{2x}{1+x}\right)\right] \\
&+ 4(m_{xy} - p_{xy})(\mathrm{Li}_2\,(-y) - \mathrm{Li}_2\,(y)) - 4(m_x + p_x - 2l_2)\mathrm{Li}_2\left(\frac{1-x}{2}\right) \\
&- 4(m_{xy} - p_{xy})\mathrm{Li}_2\left(\frac{1-y}{2}\right) - 4(m_x + l_y - m_{xy})\mathrm{Li}_2\left(\frac{(1-x)y}{1-xy}\right) \\
&+ 4(p_x + l_y - m_{xy})\mathrm{Li}_2\left(\frac{-(1+x)y}{1-xy}\right) \\
&- 4(m_x + l_y - p_{xy})\mathrm{Li}_2\left(\frac{-(1-x)y}{1+xy}\right) \\
&+ 4(p_x + l_y - p_{xy})\mathrm{Li}_2\left(\frac{(1+x)y}{1+xy}\right) \\
&+ 2(m_x + p_x + m_y + p_y - 2m_{xy} - 2l_2)\mathrm{Li}_2\left(\frac{(1-x)(1+y)}{2(1-xy)}\right) \\
&+ 2(m_x + p_x + m_y + p_y - 2p_{xy} - 2l_2)\mathrm{Li}_2\left(\frac{(1-x)(1-y)}{2(1+xy)}\right) \\
&+ 2p_x^2(m_y + p_y - m_{xy} - p_{xy}) + 2p_x(2(m_y l_y + m_y p_y + l_y p_y) \\
&+ m_{xy}(-m_y - 2l_y - 3p_y + 3m_{xy}) + p_{xy}(-3m_y - 2l_y - p_y + 3p_{xy})) \\
&+ 2m_x(2p_x + m_y - 2l_y + p_y)(m_y + p_y - m_{xy} - p_{xy}) - p_y^2(m_{xy} + p_{xy}) \\
&+ 2p_y(2m_{xy}^2 + p_{xy}^2) + m_y^2(2p_y - m_{xy} - p_{xy})
\end{aligned}
$$

$$
\begin{aligned}
&+ 2m_y(p_y^2 + m_{xy}^2 + 2p_{xy}^2 - p_y(3m_{xy} + p_{xy})) - 2(m_{xy}^3 + p_{xy}^3) \\
&+ 2l_2((4m_y + 4p_y - 3m_{xy})m_{xy} + (2m_y + 2p_y - 3p_{xy})p_{xy} \\
&- 2(p_x + 2m_x)(m_y + p_y - m_{xy} - p_{xy}) - m_y^2 - 4m_y p_y - p_y^2) \\
&+ 2l_2^2(3(m_y + p_y) - 2(2m_{xy} + p_{xy})) \\
&- (\pi^2/3)(m_y + p_y - 8m_{xy} + 6p_{xy})\Big\}. \qquad (5.52)
\end{aligned}
$$

The following abbreviations are used here: $l_z = \ln z$ for $z = x, y, 2$, $p_z = \ln(1+z)$ and $m_z = \ln(1-z)$ for $z = x, y, xy$.

This result is presented in such a way that it is manifestly real at small negative values of s and t. From this Euclidean domain, it can easily be continued analytically to any other domain.

The result (5.48)–(5.52) is in agreement with the leading power behaviour in the (Sudakov) limit of the fixed-angle scattering, $m^2 \ll |s|, |t|$ which can be alternatively obtained [56] by use of the strategy of expansion by regions [6, 57]:

$$
\begin{aligned}
B^{(0)}(s,t,m^2;\varepsilon) = &-\frac{\left(\mathrm{i}\pi^{d/2}\mathrm{e}^{-\gamma_{\mathrm{E}}\varepsilon}\right)^2}{s^2(-t)^{1+2\varepsilon}} \\
&\times \left\{2\frac{L^2}{\varepsilon^2} - \left[(2/3)L^3 + (\pi^2/3)L + 2\zeta(3)\right]\frac{1}{\varepsilon}\right. \\
&- (2/3)L^4 + 2\ln(t/s)L^3 - 2(\ln^2(t/s) + 4\pi^2/3)L^2 \\
&+ \Big[4\mathrm{Li}_3(-t/s) - 4\ln(t/s)\mathrm{Li}_2(-t/s) + (2/3)\ln^3(t/s) \\
&- 2\ln(1+t/s)\ln^2(t/s) + (8\pi^2/3)\ln(t/s) - 2\pi^2\ln(1+t/s) + 10\zeta(3)\Big]L \\
&\left. + \pi^4/36\right\} + O(m^2L^3, \varepsilon), \qquad (5.53)
\end{aligned}
$$

where $L = \ln(-m^2/s)$. This asymptotic behaviour is reproduced when one starts from the result (5.48)–(5.52).

Starting from (5.42), one can try to evaluate integrals with other indices. For example, the integral $B_{\mathrm{PL},1}(s,t,m^2;1,\ldots,1,-1,\varepsilon)$ was evaluated in [40]. There is the same problem as in the massless case [3] connected with spurious singularities in MB integrals. It can also be cured by introducing an auxiliary analytic regularization, e.g. with $a_8 = -1+\lambda$. The singularities in the corresponding MB integral are first resolved with respect to λ and then with respect to ε when λ and ε tend to zero. (This possibility to introduce an auxiliary analytic regularization is present in the codes `MB.m` and `MBresolve.m`.) In the result [40], one meets not only usual polylogarithms but also the harmonic polylogarithm (HPL) [50] (see Appendix B), $H_{-1,0,0,1}(-(1-x)/(1+x))$ with x defined after (5.48). In fact, HPLs are partial cases of multiple polylogarithms—see (11.43) and (11.45) in Appendix B.

The situation with the analytical evaluation of the other two types of massive on-shell double boxes shown in Figs. 5.11b and 5.12 is not so satisfactory as with the

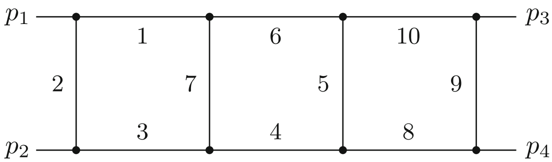


Fig. 5.13 Triple box

planar double box of the first type. For the planar double box of the second type with the indices equal to one and without numerator the pole part was evaluated in [40]. Some preliminary results for the similar non-planar double box can be found in [40].

The corresponding master integrals for the three types of the massive on-shell double boxes were characterized in [20–22], where the authors tried to evaluate them both by the method of MB representation and by differential equations. An essential problem within the method of differential equations is that one encounters differential equations of third order and higher. So, analytic results are available at the moment only for a restricted subset of the set of the master integrals for massive on-shell double boxes. However, since for applications to Bhabha scattering, one can turn to the limit of the small electron mass, the values for the master integrals can be substituted by first terms of the corresponding asymptotic expansion [23].

Let us turn to

Example 5.9 The massless on-shell triple box diagram of Fig. 5.13.

The general planar triple box Feynman integral without numerator takes the form

$$
\begin{aligned}
T(s,t;a_1,\ldots,a_{10},\varepsilon) = \int\int\int & \frac{\mathrm{d}^d k\, \mathrm{d}^d l\, \mathrm{d}^d r}{(k^2)^{a_1}[(k+p_2)^2]^{a_2}[(k+p_1+p_2)^2]^{a_3}} \\
& \times \frac{1}{[(l+p_1+p_2)^2]^{a_4}[(r-l)^2]^{a_5}(l^2)^{a_6}[(k-l)^2]^{a_7}} \\
& \times \frac{1}{[(r+p_1+p_2)^2]^{a_8}[(r+p_1+p_2+p_4)^2]^{a_9}(r^2)^{a_{10}}} .
\end{aligned}
\tag{5.54}
$$

To derive a suitable MB representation for (5.54) we follow the loop by loop procedure like in the derivation of (5.24). We recognize the internal integral over the loop momentum r as a box with two legs off-shell given by (5.12). After inserting it into (5.54) we obtain an MB integral of the on-shell double box with certain indices dependent on MB integration variables. These straightforward manipulations lead [59] to the following sevenfold MB representation of (5.54):

$$
T(s,t;a_1,\ldots,a_{10},\varepsilon) = \frac{\left(\mathrm{i}\pi^{d/2}\right)^3 (-1)^a (-s)^{6-a-3\varepsilon}}{\prod_{j=2,5,7,8,9,10} \Gamma(a_j)\Gamma(4-a_{589(10)}-2\varepsilon)}
$$
$$
\times \frac{1}{(2\pi \mathrm{i})^7} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \cdots \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \prod_{j=1}^{7} \mathrm{d}z_j \left(\frac{t}{s}\right)^{z_1} \frac{\Gamma(a_2+z_1)\Gamma(-z_1)\Gamma(z_2+z_4)}{\Gamma(a_1+z_3+z_4)\Gamma(a_3+z_2+z_4)}
$$

$$
\begin{aligned}
&\times \frac{\Gamma(2-a_{12}-\varepsilon+z_2)\Gamma(2-a_{23}-\varepsilon+z_3)\Gamma(a_7+z_1-z_4)\Gamma(-z_5)\Gamma(-z_6)}{\Gamma(4-a_{467}-2\varepsilon+z_5+z_6+z_7)\Gamma(4-a_{123}-2\varepsilon+z_1-z_4)} \\
&\times \frac{\Gamma(z_3+z_4)\Gamma(a_{123}-2+\varepsilon+z_4)\Gamma(z_1+z_2+z_3+z_4-z_7)}{\Gamma(a_6-z_5)\Gamma(a_4-z_6)} \\
&\times \Gamma(2-a_{59(10)}-\varepsilon-z_5-z_7)\Gamma(2-a_{589}-\varepsilon-z_6-z_7)\Gamma(a_9+z_7) \\
&\times \Gamma(a_{467}-2+\varepsilon+z_1-z_4-z_5-z_6-z_7)\Gamma(a_5+z_5+z_6+z_7) \\
&\times \Gamma(a_{589(10)}-2+\varepsilon+z_5+z_6+z_7)\Gamma(2-a_{67}-\varepsilon-z_1-z_2+z_5+z_7) \\
&\times \Gamma(2-a_{47}-\varepsilon-z_1-z_3+z_6+z_7)\Gamma(-z_2-z_3-z_4), \qquad (5.55)
\end{aligned}
$$

where $a = \sum_{i=1}^{10} a_i$, $a_{589(10)} = a_5 + a_8 + a_9 + a_{10}$, etc.

Let us consider the triple box with all the indices $a_i = 1$:

$$
\begin{aligned}
&T^{(0)}(s,t,\varepsilon) \equiv T(1,\ldots,1;s,t,\varepsilon) \\
&= \frac{\left(\mathrm{i}\pi^{d/2}\right)^3}{\Gamma(-2\varepsilon)(-s)^{4+3\varepsilon}} \frac{1}{(2\pi\mathrm{i})^7} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \cdots \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \prod_{j=1}^{7} \mathrm{d}z_j \left(\frac{t}{s}\right)^{z_1} \Gamma(1+z_1) \\
&\times \frac{\Gamma(-z_1)\Gamma(-\varepsilon+z_2)\Gamma(-\varepsilon+z_3)\Gamma(1+z_1-z_4)\Gamma(-z_2-z_3-z_4)}{\Gamma(1+z_2+z_4)\Gamma(1+z_3+z_4)\Gamma(1-2\varepsilon+z_1-z_4)} \\
&\times \frac{\Gamma(z_2+z_4)\Gamma(z_3+z_4)\Gamma(-z_5)\Gamma(-z_6)\Gamma(z_1+z_2+z_3+z_4-z_7)}{\Gamma(1-z_5)\Gamma(1-z_6)\Gamma(1-2\varepsilon+z_5+z_6+z_7)} \\
&\times \Gamma(2+\varepsilon+z_5+z_6+z_7)\Gamma(-1-\varepsilon-z_5-z_7)\Gamma(-1-\varepsilon-z_6-z_7) \\
&\times \Gamma(1+z_7)\Gamma(1+\varepsilon+z_1-z_4-z_5-z_6-z_7)\Gamma(-\varepsilon-z_1-z_2+z_5+z_7) \\
&\times \Gamma(1+\varepsilon+z_4)\Gamma(-\varepsilon-z_1-z_3+z_6+z_7)\Gamma(1+z_5+z_6+z_7). \qquad (5.56)
\end{aligned}
$$

Starting from (5.56), the resolution of singularities in ε was performed in [59] within the old Strategy A. An intermediate result, in an expansion in ε up to the finite part, was given by MB integrals with up to five integrations. Taking some of these integrations with the help of the table of formulae presented in Appendix D, it was possible to reduce all the integrals to no more than twofold MB integrals of gamma functions and their derivatives. A similar expression can, of course, be now obtained using the codes `MB.m` (or `MBresolve.m`) and `barnesroutines.m`.

In the twofold MB integrals where one more integration (over a variable different from z_1) can hardly be performed in terms of gamma functions, it was possible [59] to perform it with z_1 in a vicinity of an integer point $z_1 = n = 0, 1, 2, \ldots$, in expansion in $z = z_1 - n$, with a sufficient accuracy. Then one obtained power series where, in addition to nested sums with one index, various nested sums (see Appendix C) appeared. These series were summed up in terms of HPLs. Of course, such twofold series can be now handled with `SUMMER` [68] and `XSummer` [47].

Eventually, the following result was obtained [59]:

$$T^{(0)}(s,t;\varepsilon) = -\frac{\left(\mathrm{i}\pi^{d/2}\mathrm{e}^{-\gamma_{\mathrm{E}}\varepsilon}\right)^3}{s^3(-t)^{1+3\varepsilon}} \sum_{j=0}^{6} \frac{c_j(x,L)}{\varepsilon^j}, \tag{5.57}$$

where $x = -t/s$, $L = \ln(s/t)$, and

$$c_6 = \frac{16}{9}, \quad c_5 = -\frac{5}{3}L, \quad c_4 = -\frac{3}{2}\pi^2, \tag{5.58}$$

$$\begin{aligned} c_3 = {} & 3(H_{0,0,1}(x) + LH_{0,1}(x)) + \frac{3}{2}(L^2+\pi^2)H_1(x) \\ & - \frac{11}{12}\pi^2 L - \frac{131}{9}\zeta(3), \end{aligned} \tag{5.59}$$

$$\begin{aligned} c_2 = {} & -3\left(17H_{0,0,0,1}(x) + H_{0,0,1,1}(x) + H_{0,1,0,1}(x) + H_{1,0,0,1}(x)\right) \\ & - L\left(37H_{0,0,1}(x) + 3H_{0,1,1}(x) + 3H_{1,0,1}(x)\right) - \frac{3}{2}(L^2+\pi^2)H_{1,1}(x) \\ & - \left(\frac{23}{2}L^2 + 8\pi^2\right)H_{0,1}(x) - \left(\frac{3}{2}L^3 + \pi^2 L - 3\zeta(3)\right)H_1(x) \\ & + \frac{49}{3}\zeta(3)L - \frac{1411}{1080}\pi^4, \end{aligned} \tag{5.60}$$

$$\begin{aligned} c_1 = {} & 3\left(81H_{0,0,0,0,1}(x) + 41H_{0,0,0,1,1}(x) + 37H_{0,0,1,0,1}(x) + H_{0,0,1,1,1}(x)\right. \\ & + 33H_{0,1,0,0,1}(x) + H_{0,1,0,1,1}(x) + H_{0,1,1,0,1}(x) + 29H_{1,0,0,0,1}(x) \\ & \left. + H_{1,0,0,1,1}(x) + H_{1,0,1,0,1}(x) + H_{1,1,0,0,1}(x)\right) + L\left(177H_{0,0,0,1}(x)\right. \\ & + 85H_{0,0,1,1}(x) + 73H_{0,1,0,1}(x) + 3H_{0,1,1,1}(x) + 61H_{1,0,0,1}(x) \\ & \left. + 3H_{1,0,1,1}(x) + 3H_{1,1,0,1}(x)\right) \\ & + \left(\frac{119}{2}L^2 + \frac{139}{12}\pi^2\right)H_{0,0,1}(x) + \left(\frac{47}{2}L^2 + 20\pi^2\right)H_{0,1,1}(x) \\ & + \left(\frac{35}{2}L^2 + 14\pi^2\right)H_{1,0,1}(x) + \frac{3}{2}\left(L^2+\pi^2\right)H_{1,1,1}(x) \\ & + \left(\frac{23}{2}L^3 + \frac{83}{12}\pi^2 L - 96\zeta(3)\right)H_{0,1}(x) \\ & + \left(\frac{3}{2}L^3 + \pi^2 L - 3\zeta(3)\right)H_{1,1}(x) \\ & + \left(\frac{9}{8}L^4 + \frac{25}{8}\pi^2 L^2 - 58\zeta(3)L + \frac{13}{8}\pi^4\right)H_1(x) \\ & - \frac{503}{1440}\pi^4 L + \frac{73}{4}\pi^2\zeta(3) - \frac{301}{15}\zeta(5), \end{aligned} \tag{5.61}$$

$$\begin{aligned} c_0 = {} & -\left(951H_{0,0,0,0,0,1}(x) + 819H_{0,0,0,0,1,1}(x) + 699H_{0,0,0,1,0,1}(x)\right. \\ & + 195H_{0,0,0,1,1,1}(x) + 547H_{0,0,1,0,0,1}(x) + 231H_{0,0,1,0,1,1}(x) \\ & + 159H_{0,0,1,1,0,1}(x) + 3H_{0,0,1,1,1,1}(x) + 363H_{0,1,0,0,0,1}(x) \\ & + 267H_{0,1,0,0,1,1}(x) + 195H_{0,1,0,1,0,1}(x) + 3H_{0,1,0,1,1,1}(x) \end{aligned}$$

$$
\begin{aligned}
&+ 123H_{0,1,1,0,0,1}(x) + 3H_{0,1,1,0,1,1}(x) + 3H_{0,1,1,1,0,1}(x) \\
&+ 147H_{1,0,0,0,0,1}(x) + 303H_{1,0,0,0,1,1}(x) + 231H_{1,0,0,1,0,1}(x) \\
&+ 3H_{1,0,0,1,1,1}(x) + 159H_{1,0,1,0,0,1}(x) + 3H_{1,0,1,0,1,1}(x) \\
&+ 3H_{1,0,1,1,0,1}(x) + 87H_{1,1,0,0,0,1}(x) + 3H_{1,1,0,0,1,1}(x) \\
&+ 3H_{1,1,0,1,0,1}(x) + 3H_{1,1,1,0,0,1}(x)\big) \\
&- L\big(729H_{0,0,0,0,1}(x) + 537H_{0,0,0,1,1}(x) + 445H_{0,0,1,0,1}(x) \\
&+ 133H_{0,0,1,1,1}(x) + 321H_{0,1,0,0,1}(x) + 169H_{0,1,0,1,1}(x) \\
&+ 97H_{0,1,1,0,1}(x) + 3H_{0,1,1,1,1}(x) + 165H_{1,0,0,0,1}(x) \\
&+ 205H_{1,0,0,1,1}(x) + 133H_{1,0,1,0,1}(x) + 3H_{1,0,1,1,1}(x) \\
&+61H_{1,1,0,0,1}(x) + 3H_{1,1,0,1,1}(x) + 3H_{1,1,1,0,1}(x)\big) \\
&- \left(\frac{531}{2}L^2 + \frac{89}{4}\pi^2\right) H_{0,0,0,1}(x) - \left(\frac{311}{2}L^2 + \frac{619}{12}\pi^2\right) H_{0,0,1,1}(x) \\
&- \left(\frac{247}{2}L^2 + \frac{307}{12}\pi^2\right) H_{0,1,0,1}(x) - \left(\frac{71}{2}L^2 + 32\pi^2\right) H_{0,1,1,1}(x) \\
&- \left(\frac{151}{2}L^2 - \frac{197}{12}\pi^2\right) H_{1,0,0,1}(x) - \left(\frac{107}{2}L^2 + 50\pi^2\right) H_{1,0,1,1}(x) \\
&- \left(\frac{35}{2}L^2 + 14\pi^2\right) H_{1,1,0,1}(x) - \frac{3}{2}\left(L^2 + \pi^2\right) H_{1,1,1,1}(x) \\
&- \left(\frac{119}{2}L^3 + \frac{317}{12}\pi^2 L - 455\zeta(3)\right) H_{0,0,1}(x) \\
&- \left(\frac{47}{2}L^3 + \frac{179}{12}\pi^2 L - 120\zeta(3)\right) H_{0,1,1}(x) \\
&- \left(\frac{35}{2}L^3 + \frac{35}{12}\pi^2 L - 156\zeta(3)\right) H_{1,0,1}(x) \\
&- \left(\frac{3}{2}L^3 + \pi^2 L - 3\zeta(3)\right) H_{1,1,1}(x) \\
&- \left(\frac{69}{8}L^4 + \frac{101}{8}\pi^2 L^2 - 291\zeta(3)L + \frac{559}{90}\pi^4\right) H_{0,1}(x) \\
&- \left(\frac{9}{8}L^4 + \frac{25}{8}\pi^2 L^2 - 58\zeta(3)L + \frac{13}{8}\pi^4\right) H_{1,1}(x) - \left(\frac{27}{40}L^5 + \frac{25}{8}\pi^2 L^3\right. \\
&\left. - \frac{183}{2}\zeta(3)L^2 + \frac{131}{60}\pi^4 L - \frac{37}{12}\pi^2\zeta(3) + 57\zeta(5)\right) H_1(x) \\
&+ \left(\frac{223}{12}\pi^2\zeta(3) + 149\zeta(5)\right) L + \frac{167}{9}\zeta(3)^2 - \frac{624607}{544320}\pi^6.
\end{aligned}
\tag{5.62}
$$

The above result was confirmed with the help of numerical integration using sector decompositions [14, 15] described in Chap. 4. Another natural check of the result is its agreement with the leading power Regge asymptotic behaviour [58] which was

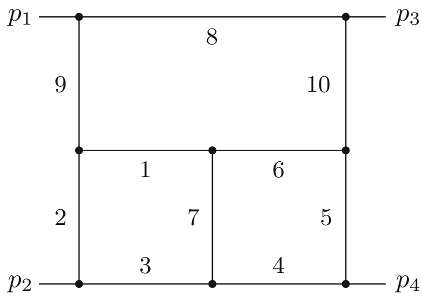


Fig. 5.14 Three-loop tennis court graph

evaluated by an independent method based on the strategy of expansion by regions [6, 57] described in Chap. 9. As in the case of the double box the result (5.62) has uniform transcendentality: every term has transcendentality weight six.

The same procedure was used [11] to evaluate a little bit more complicated four-point three-loop diagram:

Example 5.10 The massless on-shell tennis court[5] diagram of Fig. 5.14.

An appropriate MB representation was derived straightforwardly in the loop by loop procedure. As a result, an eightfold MB representation was derived for the general diagram $W(s,t;a_1,\ldots,a_{11},\varepsilon)$ of Fig. 5.14 with the eleventh index corresponding to the numerator $[(l_1+l_3)^2]^{-a_{11}}$, where $l_{1,3}$ are the momenta flowing through lines 1 and 3 in the same direction. This general MB representation satisfies horizontal and vertical checks described in Sect. 5.3. In the particular case of the first ten indices equal to one and the numerator $[l_1+l_3)^2$, i.e. with $a_{11}=-1$

$$
\begin{aligned}
W(s,t;1,\ldots,1,-1,\varepsilon) &= -\frac{\left(\mathrm{i}\pi^{d/2}\right)^3}{\Gamma(-2\varepsilon)(-s)^{1+3\varepsilon}t^2} \\
&\times \frac{1}{(2\pi\mathrm{i})^8}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty}\ldots\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} dw\,\mathrm{d}z_1 \prod_{j=2}^{7}\mathrm{d}z_j\,\Gamma(-z_j)\left(\frac{t}{s}\right)^w \Gamma(1+3\varepsilon+w) \\
&\times \frac{\Gamma(-3\varepsilon-w)\Gamma(1+z_1+z_2+z_3)\Gamma(-1-\varepsilon-z_1-z_3)\Gamma(1+z_1+z_4)}{\Gamma(1-z_2)\Gamma(1-z_3)\Gamma(1-z_6)\Gamma(1-2\varepsilon+z_1+z_2+z_3)} \\
&\times \frac{\Gamma(-1-\varepsilon-z_1-z_2-z_4)\Gamma(2+\varepsilon+z_1+z_2+z_3+z_4)}{\Gamma(-1-4\varepsilon-z_5)\Gamma(1-z_4-z_7)\Gamma(2+2\varepsilon+z_4+z_5+z_6+z_7)} \\
&\times \Gamma(-\varepsilon+z_1+z_3-z_5)\Gamma(2-w+z_5)\Gamma(-1+w-z_5-z_6) \\
&\times \Gamma(z_5+z_7-z_1)\Gamma(1+z_5+z_6)\Gamma(-1+w-z_4-z_5-z_7) \\
&\times \Gamma(-\varepsilon+z_1+z_2-z_5-z_6-z_7)\Gamma(1-\varepsilon-w+z_4+z_5+z_6+z_7) \\
&\times \Gamma(1+\varepsilon-z_1-z_2-z_3+z_5+z_6+z_7). \qquad (5.63)
\end{aligned}
$$

---

[5] Well, this is only one half of the court for singles.

For the resolution of the singularities and evaluation of resulting MB integrals, the above comments for the triple box also take place. In particular, for such calculation, it is highly recommended to use public codes `MB.m`, `MBresolve.m`, `barnesroutines.m`, `SUMMER.m` and `XSummer.m`, rather than proceed by hand and mouse like in [11]. The result obtained in [11] has the same structure as (5.62). It is also uniformly transcendental.

The analytic results for triple box and the tennis court were crucial to arrive at an Ansatz [11] (previously formulated in two loops [1]) that determines the analytic dependence of maximally helicity violating amplitudes with n external gluons in the $N = 4$ supersymmetric Yang–Mills theory for $n = 4$ and $n = 5$ and provides a kind of a background for $n \geq 6$, where the amplitude is written in an exponential form with this Ansatz plus a remainder with respect to it.

5.7 Applying MB Representation to Expand Feynman Integrals in Momenta and Masses

To expand a given Feynman integral in some limit, where certain masses and/or kinematical invariants are large with respect to the rest of these parameters, one can successfully apply expansion by regions [6, 60, 61], as explained in details in the book [57] and briefly in Chap. 9. An alternative technique for solving the problem of asymptotic expansion is provided by multiple MB representations. Let us see how it works using our simple examples. One can proceed either at a general value of ε, or after the resolution of the singularities and expansion in ε. In the case of Example 4.1, let us proceed after the expansion in ε and use the MB representation (5.3) to expand such Feynman integrals in the two different limits, $m^2/q^2 \to 0$ and $q^2/m^2 \to 0$. Consider, for example, $F(2, 1; 4)$ represented by (5.4).

This is an integral over the variable z, with the ratio m^2/q^2 present in the form $(m^2/q^2)^z$. The initial integration contour is at $-1 < \mathrm{Re}z < 0$. Let us observe that if we follow the procedure used to evaluate this integral, i.e. close the integration contour to the right and pick up (minus) residues at $z = 0, 1, 2, \ldots, n, \ldots$ we will obtain terms of the asymptotic expansion in the limit $m^2/q^2 \to 0$. Indeed, one can prove that the remainder of this expansion determined by picking up the $(n + 1)$-st residue is of order $(m^2)^{n+1}$. Thus we obtain

$$F(2, 1; 4) = \frac{\mathrm{i}\pi^2}{q^2}\left[\ln\frac{-q^2}{m^2} - \frac{m^2}{q^2} - \frac{m^4}{2(q^2)^2} - \cdots\right]. \tag{5.64}$$

If we are interested in the opposite limit, $q^2/m^2 \to 0$, the natural idea is to close the integration contour to the left and take residues at the points $z = -1, -2, \ldots$ to obtain

$$F(2, 1; 4) = -\frac{\mathrm{i}\pi^2}{m^2}\left[1 + \frac{q^2}{2m^2} + \frac{(q^2)^2}{3m^4} + \cdots\right]. \tag{5.65}$$

Consider now Example 5.2, where IR and collinear divergences are present. Let us proceed at a general value of ε. We can use MB representation (5.9) for expanding Feynman integrals with various indices in the two different limits, $t/s \to 0$ and $s/t \to 0$. There is again the typical dependence of the ratio of t and s on z of the form $(t/s)^z$. The procedure of using (5.9) to obtain an asymptotic expansion in the limit $t/s \to 0$ is standard: to shift the integration contour to the right. For the integral with given indices a_l, the points where it is necessary to take (minus) residues are given by the right poles of the gamma functions, in our terminology: at $z = 0, 1, 2, \ldots$ and at $z = 2 - \max\{a_1, a_3\} - a_2 - a_4 - \varepsilon + n$ with $n = 0, 1, 2, \ldots$. For example, for $F(s, t; d)$ represented by (5.9) at the indices equal to one, these are the two series of residues at $z = 0, 1, 2, \ldots$ and $z = -1 - \varepsilon, -\varepsilon, 1 - \varepsilon, \ldots$ which reproduce the hard and collinear contributions, respectively, to the asymptotic expansion within expansion by regions—see Example 9.4 in Chap. 9. We obtain

$$\begin{aligned}
F(s,t;d) = {} & \frac{\mathrm{i}\pi^{d/2}}{\Gamma(-2\varepsilon)} \left\{ \frac{\Gamma(1+\varepsilon)\Gamma(-\varepsilon)^2}{s(-t)^{1+\varepsilon}} \left[\ln\frac{t}{s} + 2\psi(-\varepsilon) - \psi(1+\varepsilon) + \gamma_{\mathrm{E}} \right] \right. \\
& - \frac{\Gamma(\varepsilon)\Gamma(1-\varepsilon)^2}{s^2(-t)^{\varepsilon}} \left[\ln\frac{t}{s} + 2\psi(1-\varepsilon) - \psi(\varepsilon) - 1 + \gamma_{\mathrm{E}} \right] \\
& + \frac{\Gamma(2+\varepsilon)\Gamma(-1-\varepsilon)^2}{(-s)^{2+\varepsilon}} \\
& + \frac{\Gamma(\varepsilon-1)\Gamma(2-\varepsilon)^2(-t)^{1-\varepsilon}}{2s^3} \left[\ln\frac{t}{s} + 2\psi(2-\varepsilon) - \psi(\varepsilon-1) - \frac{3}{2} + \gamma_{\mathrm{E}} \right] \\
& \left. + \frac{\Gamma(3+\varepsilon)\Gamma(-2-\varepsilon)^2 t}{(-s)^{3+\varepsilon}} \right\} + \cdots . \qquad (5.66)
\end{aligned}$$

To obtain the asymptotic expansion in the opposite limit, $s/t \to 0$, one shifts the integration contour to the left and takes residues at the left poles at $z = 2 - \min\{a_2, a_4\} - n$ and at $z = 2 - a - \varepsilon - n$ with $n = 0, 1, 2, \ldots$. For $F(s, t; d)$, these are the two series of residues at $z = -1, -2, \ldots$ and $z = -2-\varepsilon, -3-\varepsilon, -4-\varepsilon, \ldots$. One can check that the resulting expansion is nothing but (5.66) with the interchange $s \to t, t \to s$—this should be the case because of the symmetry of the initial integral.

In these two examples, terms of asymptotic expansions were obtained as residues in onefold MB integrals. If a Feynman integrals is represented by a multiple MB integral one can choose the power of the small parameter of an expansion, x, as one the integration variables so that there is the factor x^z in the integrand. If there are gamma functions with the $-z$ dependence and without dependence on the rest of the integration variables one can similarly shift the z-integration contour to the right and obtain a series given by MB integrals with one integration less. If the argument of a gamma function in the integrand equals $-z$ plus a linear combination other z-variables one can follow the old Strategy A and analyze how the poles in z are generated due to the integration over the other variables. In this procedure, z plays the role of ε.

However, it is much better to proceed 'after' the expansion in ε because this procedure can be automated. Suppose that we have to analyze the asymptotic expansion of a given multiple MB integral in the limit $x \to 0$ and x appears in the integrand as x^z with z one of the MB integration variables. After resolving singularities either by `MB.m` or `MBresolve.m` one can perform a Laurent expansion in ε and obtain a linear combination of MB integrals over straight contours in MB variables, in particular in z. Then one can apply a simple procedure of shifting the contour in the variable z to the right. Whenever a pole of some gamma function depending on z is crossed, one takes a residue (with the minus sign) and obtains an MB integral with one integration less. As a result, one obtains an asymptotic expansion in the limit $x \to 0$ with terms given by MB integrals. One can then apply to these integrals commands of the `MB.m` and `barnesroutines.m` packages.

This procedure was implemented in `Mathematica` [69] as the public code `MBasymptotics.m` [19]. For example, one can consider the massless on-shell double box as an example, start from (5.37), apply `MB.m` or `MBresolve.m`, then the command `MBexpand` of `MB.m` to perform an expansion in ε, then apply `MBasymptotics.m` to obtain at most onefold MB integrals. Then one can convert these integrals into series and either apply summation formulae of Appendix C or `SUMMER` and `XSummer` and eventually reproduce (5.40). Similarly, the result (5.53) for the leading behaviour of the massive on-shell double box in the limit of the small mass can be reproduced starting from (5.43).

It is not clear in advance which way is simpler: expanding by MB representation, or, by regions. My experience tells me that, usually, expanding by regions is certainly preferable, but sometimes, it looks more convenient to derive an appropriate MB representation and proceed as described in this section. But sometimes, this is just a matter of taste. In complicated situations, the two strategies can successfully be combined. In particular, extracting the leading asymptotic behaviour from a general MB representation can show what kind of contributions one gets and will help detecting all regions which contribute. For example, the calculation [11] of the tennis court diagram of Fig. 5.14 provided a hint for finding a non-trivial contribution within expansion by regions which was, in turn, used to check the result. See also [42], where both strategies to expand Feynman integrals in the Sudakov limit were combined.

The asymptotic behaviour in various limits was evaluated with the help of MB representation in many papers—see, e.g., [4, 13, 37, 38] and [23]. In the second paper, the very code `MBasymptotics.m` was successfully applied.

5.8 Combining MB Representation with Sector Decompositions to Expand Feynman Integrals

One more way [48] to use MB representation to expand Feynman integrals in momenta and masses is to combine it with modern sector decomposition described in Chap. 4. In fact, this idea was exploited many years ago. For example, in [7, 49] the asymptotic expansion of Feynman integrals in various limits of momenta and masses was studied using Mellin transform and Hepp or Speer sectors. One more example of applying sector decompositions can be found in [27–29] (see also references therein) where leading and subleading logarithms in asymptotic expansions of Feynman integrals in the high-energy limit were studied. This approach was successfully applied up to two-loop level.

Let me present, following [62], an algorithm that provides an expansion in powers and logarithms of t/s with numerical coefficients. Let us suppose that we are studying a limit where kinematic invariants and masses are decomposed into two groups. At the level of the contribution of the primary sectors, we have a decomposition of the function (3.37) in (4.1) into two parts,

$$\mathcal{W} = \mathcal{W}_1 + \mathcal{W}_2, \tag{5.67}$$

and that those in the term W_1 are much smaller than in W_2. We introduce the parameter of expansion, λ by multiplying by it the terms of the first group. Let us then separate the two groups of terms by introducing a onefold MB integral,

$$\frac{\Gamma(a-hd/2)}{(\lambda\mathcal{W}_1+\mathcal{W}_2)^{a-hd/2}} = \frac{1}{2\pi \mathrm{i}}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}z\, \lambda^z \frac{\Gamma(a-hd/2+z)\Gamma(-z)}{\mathcal{W}_1^{-z}\mathcal{W}_2^{a-hd/2+z}}, \tag{5.68}$$

so that we obtain

$$\mathcal{F}^{(L)} = \frac{\left(\mathrm{i}\pi^{d/2}\right)^h}{\prod_l \Gamma(a_l)} \frac{1}{2\pi \mathrm{i}}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}z\, \Gamma(a-hd/2+z)\Gamma(-z)\lambda^z$$
$$\times \int_0^1 \dots \int_0^1 \hat{\mathcal{U}}^{a-(h+1)d/2}\, \mathcal{W}_1^z\, \mathcal{W}_2^{-a+hd/2-z} \prod_l^{L-1} \left(\alpha_l^{a_l-1}\alpha_l\right). \tag{5.69}$$

The idea of using MB representation is to reduce the problem of expansion to the analysis of poles in the variable z of the integrand. To pick up terms of expansion in the limit $\lambda \to 0$ one closes the integration contour to the right and takes residues in z. (The residues are taken with the minus sign according to the Cauchy theorem.) In addition to the poles of one of the two explicitly present gamma functions $\Gamma(-z)$ at $z = 0, 1, 2, \dots$, we have poles coming from the parametric integration. In fact, we have to distinguish poles which are of the same character as poles of gamma functions with $-z$ dependence.

The algorithm to evaluate numerically first terms of the asymptotic expansion at $\lambda \to 0$ implemented within of `FIESTA 2` [62] consists of the following steps.

Step 1. The resolution of singularities of the integral over $\alpha_1, \dots, \alpha_{L-1}$ by a sector decomposition. Instead of the two functions in the integrand of (4.1), we have the three functions $\hat{\mathcal{U}}$, $\mathcal{W}_1$ and $\mathcal{W}_2$ raised to certain powers depending not only on ε but also (for $\mathcal{W}_{1,2}$) on the MB integration variable z. As a result of this procedure we obtain a sum of parametric integrals where all these three functions are proper factorized, i.e. represented as powers of sector variables times positive functions. Therefore each of resulting parametric integrals is represented as an integral over $t_1, \ldots, t_{L-1}$ from 0 to 1 of $\prod t_i^{r_i-1}$ times a product of positive functions raised to some powers. Here exponents r_i have the form $b_i\varepsilon + c_i z + n_i$ where b_i, c_i, n_i are rational numbers.

Step 2. Let us reveal singularities in ε generated by the MB integration over z. The integral of $t_i^{b_i\varepsilon+c_i z+n_i-1}$ generates a z-dependence of the type $\Gamma(b_i\varepsilon + c_i z + n_i)$. We are concentrating on sector integrals with $c_i < 0$ because they are relevant to our limit.

Using Taylor subtractions of sufficient order for the rest of the integrand we decompose every integral over such t_i into the corresponding integral with a remainder and an integral of the subtracted terms which is evaluated analytically. The remainder in such subtractions contributes to the remainder of the whole expansion. When increasing the order of expansion it tends to zero with a given power of λ. The explicit integration of every Taylor subtracted term provides a singular behavior in z as a rational function. We take residues in z at these points, similarly to the residues of the explicit $\Gamma(-z)$.

Step 3. Every resulting residue is a sector integral where a proper factorization due to the sector decomposition has been achieved. It is treated numerically within `FIESTA 2`.

This algorithm within `FIESTA 2` can successfully be applied at least at the two-loop level. For example, the expansion of the massless on-shell double box (5.40) can be numerically confirmed.

5.9 Conclusion

The method of MB representation is a powerful method which has good chances to be developed and optimized further. Let me list the public codes which can be applied within this method:

- To derive MB representations for planar graphs one can use `AMBRE` [33–35]. One can easily check MB representations derived at general indices, for example, with `Mathematica` [69].
- To resolve the singularities in ε in multiple MB integrals one can apply `MB.m` and `MBresolve.m`.
- After the resolution of the singularities in ε and expansion in ε one can apply various commands from `barnesroutines.m`. One can evaluate MB integrals numerically within `MB.m`.
- When the integration in multiple MB integrals is hardly performed explicitly, one can convert them into multiple series and apply such packages as `SUMMER` [68] and `XSummer` [47] for summation.
- One can expand MB integrals in various limits of momenta and masses using `MBasymptotics` and `FIESTA 2` [62].

All the codes listed above (apart from `AMBRE` and `FIESTA 2`) can be found at [41].

The technique of multiple MB representations is not always optimal. This holds at least for non-planar double boxes with one leg off-shell. Although first analytical results were obtained with its help [54, 55] the adequate technique here turned out to be the method of differential equations which will be studied in Chap. 7. On the other hand, massive on-shell double box diagrams considered in this chapter provide an example of a situation where it is natural to combine these two methods. Still for the moment, the problem of the evaluation of this class of integrals is not solved.

As an open problem within the method of MB representation let me mention the automatic derivation of MB representations for non-planar graphs. `AMBRE` is hardly applicable here. Indeed, it turns out that for MB representations derived within the loop by loop procedure for nonplanar diagrams, there can be another source of poles,[6] in addition to the poles revealed by `MB.m`. This feature can be illustrated through Example 5.6. In the corresponding MB representation derived within the loop by loop strategy one meets, in particular, the following onefold MB integral

$$\frac{1}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \frac{\Gamma(1+2\varepsilon+z)\Gamma(-z)}{1+\varepsilon+z} \mathrm{e}^{-\mathrm{i}\pi z} \mathrm{d}z .$$

There is no gluing of poles so that a pole in ε cannot be generated by the integration over finite regions. Still a pole is generated and this can be seen by an explicit evaluation of this integral by closing the integration contour to the right and summing up the resulting series. This can be seen also by analyzing the asymptotic behaviour of the integrand at infinity. Let us set $z = x + iy$ and use the following formula of the asymptotic behaviour of the gamma function at large arguments in the complex plane:

$$\Gamma(x \pm iy) \sim \sqrt{2\pi} \mathrm{e}^{\pm i \frac{\pi}{4}(2x-1)} \mathrm{e}^{\pm iy(\ln y - 1)} \mathrm{e}^{-\frac{\pi}{2} y} y^{x-1/2} \tag{5.70}$$

where $y \to +\infty$.

We can conclude that the leading asymptotic behaviour when $y \to +\infty$ is $1/y^{1-2\varepsilon}$ which explains the appearance of the pole. Let me still mention that, for this concrete diagram, there is a better way to proceed with the 'good' MB representation (5.21).

A safe way to derive MB representation for non-planar graphs is to start from alpha parameters and introduce MB integrations when separating terms contributing to the basic functions. Hopefully, there exists a natural algorithm for this procedure with a minimal number of resulting MB integrations.

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Chapter 6
IBP and Reduction to Master Integrals

The next method[1] in our list is based on integration by parts[2] (IBP) [18] within dimensional regularization, i.e. property (2.41). The idea is to write down various equations (2.41) for integrals of derivatives with respect to loop momenta and use this set of relations between Feynman integrals in order to solve the reduction problem, i.e. to find out how a general Feynman integral of the given class can be expressed linearly in terms of some basic (*master*) integrals. In contrast to the evaluation of the master integrals, which is performed, at a sufficiently high level of complexity, in a Laurent expansion in ε, the reduction problem is usually solved at *general* d, and the expansion in ε does not provide simplifications here.

The reduction to master integrals can be performed in the two different ways: one can stop the reduction when one arrives at integrals which can be expressed in terms of gamma functions at general d or to try to reduce any given integral to true master integrals. The latter variant is the reduction problem in the ultimate mathematical sense, i.e. the reduction to irreducible integrals which cannot be reduced further.

For many years IBP relations were solved by hand. There is a lot of example of such solutions in the literature. To illustrate this procedure we consider in Sect. 6.1 various simple examples. There are several public codes for certain classes of Feynman integrals, with the IBP reduction done by hand. Typically, this reduction is in the first way, i.e. to Feynman integrals expressed in terms of gamma functions at general d. In the rest of this chapter, we will turn to algorithmic ways to solve IBP relations, in particular to the well-known Laporta's algorithm [43, 44]. For algorithmic IBP reductions, the second way is typical, i.e. a reduction of any given integral to true master integrals.

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[1] A recent alternative review on the method of IBP can be found in [33].

[2] For one loop, IBP was used in [36]. The crucial step—an appropriate modification of the *integrand* before differentiation, with an application at the two-loop level (to massless propagator diagrams)—was taken in [18] and, in a coordinate-space approach, in [71]. The case of three-loop massless propagators was treated in [18].

V. A. Smirnov, *Analytic Tools for Feynman Integrals*, Springer Tracts in Modern Physics 250, DOI: 10.1007/978-3-642-34886-0_6,

6.1 Solving IBP Relations by Hand

The first example is very simple:

Example 6.1 One-loop vacuum massive Feynman integrals

$$F(a) = \int \frac{\mathrm{d}^d k}{(k^2 - m^2)^a}. \tag{6.1}$$

In this chapter, we are concentrating on the dependence of Feynman integrals on the powers of the propagators so that we will usually omit dependence on dimension, masses and external momenta. Let us forget that we know the explicit result (10.1) and try to exploit information following from IBP. Let us use the IBP identity

$$\int \mathrm{d}^d k \frac{\partial}{\partial k} \cdot k \frac{1}{(k^2 - m^2)^a} = 0, \tag{6.2}$$

with $(\partial/(\partial k)) \cdot k = (\partial/(\partial k_\mu))k_\mu$. To write down resulting quantities in terms of integrals (6.1) we replace k^2 by $(k^2 - m^2) + m^2$. We obtain

$$(d - 2a)F(a) - 2am^2 F(a+1) = 0. \tag{6.3}$$

This gives the following recurrence relation:

$$F(a) = \frac{d - 2a + 2}{2(a-1)m^2} F(a-1). \tag{6.4}$$

We see that any Feynman integral with integer $a > 1$ can be expressed recursively in terms of one integral $F(1) \equiv I_1$ which we therefore consider as a master integral. (Observe that all the integrals with non-positive integer indices are zero since they are massless tadpoles.) In this example, we have explicitly:

$$F(a) = \frac{(-1)^a\,(1 - d/2)_{a-1}}{(a-1)!(m^2)^{a-1}} I_1, \tag{6.5}$$

where $(x)_a$ is the Pochhammer symbol. So, the only master integral is

$$I_1 = -\mathrm{i}\pi^{d/2}\Gamma(1 - d/2)(m^2)^{d/2-1}. \tag{6.6}$$

As in Chap. 3 let us consider

Example 6.2 Massless one-loop propagator Feynman integrals

$$F(a_1, a_2) = \int \frac{\mathrm{d}^d k}{(k^2)^{a_1}[(q-k)^2]^{a_2}}. \tag{6.7}$$

(As we have agreed, the dependence on q^2 and d is omitted.) For integer powers of the propagators, these integrals are zero whenever one of the indices is non-positive. Let us forget the explicit result (3.6) and try to apply the IBP identity

$$\int \mathrm{d}^d k \frac{\partial}{\partial k} \cdot k \frac{1}{(k^2)^{a_1}[(q-k)^2]^{a_2}} = 0. \tag{6.8}$$

We recognize different terms resulting from the differentiation as integrals (6.7) and obtain the following relation

$$d - 2a_1 - a_2 - a_2 \mathbf{2}^+(\mathbf{1}^- - q^2) = 0 \tag{6.9}$$

which is understood as applied to the general integral $F(a_1, a_2)$ with the standard notation for increasing and lowering operators, e.g. $\mathbf{2}^+\mathbf{1}^- F(a_1, a_2) = F(a_1 - 1, a_2 + 1)$. We rewrite it as

$$a_2 q^2 \mathbf{2}^+ = a_2 \mathbf{1}^- \mathbf{2}^+ + 2a_1 + a_2 - d \tag{6.10}$$

and obtain the possibility to reduce the sum of the indices $a_1 + a_2$. Explicitly, applying (6.10) to the general integral and shifting the index a_2, we have

$$\begin{aligned} F(a_1, a_2) = -\frac{1}{(a_2 - 1)q^2}&[(d - 2a_1 - a_2 + 1)F(a_1, a_2 - 1) \\ &- (a_2 - 1)F(a_1 - 1, a_2)]. \end{aligned} \tag{6.11}$$

Indeed, $a_1 + a_2$ on the right-hand side is less by one than on the left-hand side. This relation can be applied, however, only when $a_2 > 1$. Suppose now that $a_2 = 1$. Then we use the symmetry property $F(a_1, a_2) = F(a_2, a_1)$ and apply (6.11) interchanging a_1 and a_2 and setting $a_2 = 1$:

$$F(a_1, 1) = -\frac{d - a_1 - 1}{(a_1 - 1)q^2} F(a_1 - 1, 1). \tag{6.12}$$

This relation enables us to reduce the index a_1 to one and we see that the two relations (6.11) and (6.12) provide the possibility to express any integral of the given family in terms of the only master integral $I_1 = F(1, 1)$ given by (3.8), i.e. $F(a_1, a_2) = c(a_1, a_2)I_1$, and the corresponding coefficient function $c(a_1, a_2)$ is constructed as a rational function of d.

Let us now complete the analysis for the example considered in the introduction, i.e. once again consider our favourite example:

Example 6.3 One-loop propagator Feynman integrals (1.2) corresponding to Fig. 1.1.

We stopped in Chap. 1 at the point where we were able to express any integral (1.2) in terms of the master integral $I_1 = F(1, 1)$ and integrals with $a_2 \leq 0$ which can be evaluated for general d in terms of gamma functions by means of (10.3).

So, for any given indices a_1, a_2, we obtain, as a result of the reduction,

$$F(a_1, a_2) = c_1(a_1, a_2) I_1 + \sum_{i_1 > 0, i_2 \leq 0} c'(i_1, i_2) F(i_1, i_2), \tag{6.13}$$

where the sum is finite and $c'(i_1, i_2)$ are rational functions of q^2, m^2 and d. Let us now try to understand what the true master integrals are. We want to have really irreducible integrals, i.e. that cannot be expressed linearly in terms of other integrals.

Suppose that $a_2 \leq 0$, Then we can apply (1.17) to reduce a_1 to one. In the case $a_1 = 1$, we use relation (1.17) multiplied by $\mathbf{2}^-$ to express the term $2m^2 a_1 \mathbf{1}^+ \mathbf{2}^-$ in (1.19). Thus, we obtain the following relation

$$(d - a_2 - 1)\mathbf{2}^- = (q^2 - m^2)^2 a_2 \mathbf{2}^+ + (q^2 + m^2)(d - 2a_2 - 1) \tag{6.14}$$

that can be used to increase the index a_2 to zero or one starting from negative values. We come to the conclusion that there are two irreducible integrals $I_1 = F(1, 1)$ given by (1.7) and $I_2 = F(1, 0)$ which equals the right-hand side of (6.6), and any integral from our family can linearly be expressed in terms of them. This reduction procedure to I_1 and I_2 can easily be implemented on a computer. Observe that the integrals I_1 and I_2 cannot be linearly expressed through each other, with a coefficient which is a rational function of d, because, at general d, I_1 is a non-trivial function of q^2 and m^2 while I_2 is independent of q^2. Explicitly, instead of relation (6.13), we now have

$$F(a_1, a_2) = c_1(a_1, a_2) I_1 + c_2(a_1, a_2) I_2. \tag{6.15}$$

For example, we obtain

$$\begin{aligned} F(3, 2) = &-\frac{(d-5)(d-3)(-4m^2 + dm^2 - 8q^2 + dq^2)}{2(m^2 - q^2)^4} I_1 \\ &+ \frac{(d-2)((96 - 39d + 4d^2)m^4 - 2(3d - 14)m^2 q^2 + (d-4)(q^2)^2}{8m^4(m^2 - q^2)^4} I_2. \end{aligned}$$

Let us now come back to the point where our reduction was incomplete, in the mathematical sense, and we had (6.13). Suppose that we made the observation that all the integrals with nonpositive a_2 are proportional to $F(1, 0)$ with coefficients that are rational functions of d. Then we can write down Eq. (6.15) immediately and say that the coefficient function at I_2 is obtained as

$$c_2(a_1, a_2) = \frac{1}{F(1, 0)} \sum_{i_1 > 0, i_2 \leq 0} c'(i_1, i_2) F(i_1, i_2), \tag{6.16}$$

where the integrals on the right-hand side are evaluated by (10.3). This ratio can be simplified. If we proceed with `Mathematica` [76] we can try to apply the command `FullSimplify` at least for smaller values of the indices. Alternatively, one can recursively apply identities for gamma functions of the form $\Gamma(a + r\varepsilon)$ reducing them to $\Gamma(1 + r\varepsilon)$. So, it looks like we can have the desired reduction (6.15) without the second part of the described procedure. Let me however emphasize that this result was obtained with the help of *analytical* information on the integrals involved, while the second part of the reduction was done using only *algebraical* IBP equations, without such additional analytical information.

Let us now derive, using IBP, a simple rule which can be useful in many situations. Let us consider the triangle diagram of Fig. 6.1 with general indices, $m_3 = 0$ and general masses m_1 and m_2. The general Feynman integral for this graph is

$$F(a_1, a_2, a_3) = \int \frac{\mathrm{d}^d k}{[(k + p_1)^2 - m_1^2]^{a_1} [(k + p_2)^2 - m_2^2]^{a_2} (k^2)^{a_3}} . \tag{6.17}$$

Let us write down the IBP identity with the operator $(\partial/\partial k)\cdot k$ acting on the integrand of (6.17). Then we obtain the following 'triangle' rule:

$$\begin{aligned} 1 = {} & \frac{1}{d - a_1 - a_2 - 2a_3} \\ & \times [a_1 \mathbf{1}^+ (\mathbf{3}^- - (p_1^2 - m_1^2)) + a_2 \mathbf{2}^+ (\mathbf{3}^- - (p_2^2 - m_2^2))]. \end{aligned} \tag{6.18}$$

This identity can be applied to a triangle as a subgraph in a bigger graph. Suppose that the external upper right line in Fig. 6.1 has the mass m_1 and the external lower right line has the mass m_2 but these are internal lines for the bigger graph. Then the factors $(p_1^2 - m_1^2)$ and $(p_2^2 - m_2^2)$ effectively reduce the indices of the corresponding lines (with the momenta p_1 and p_2) by one.

For example, the triangle rule alone can provide an IBP reduction, in the first way, of the massless Feynman integrals (3.44) corresponding to Fig. 3.10. We have already considered these diagrams in Example 3.5 in Chap. 3. Let us first observe that if $a_5 = 0$ the integrals over k and l decouple and can be evaluated in terms of gamma functions by use of (3.6):

$$\begin{aligned} F(a_1, a_2, a_3, a_4, 0) = {} & (-1)^{a_1+a_2+a_3+a_4} \left(\mathrm{i}\pi^{d/2}\right)^2 \\ & \times \frac{G(a_1, a_2) G(a_3, a_4)}{(-q^2)^{a_1+a_2+a_3+a_4+2\varepsilon-4}} . \end{aligned} \tag{6.19}$$

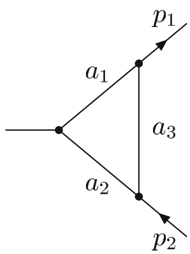

Fig. 6.1 Triangle diagram with general integer indices

Moreover, if some other index a_l is zero, the integral becomes recursively one-loop (see Sect. 3.2.1), i.e. it can be evaluated in terms of gamma functions by successively applying the same one-loop formula, for example,

$$F(a_1, a_2, a_3, 0, a_5) = (-1)^{a_1+a_2+a_3+a_5} \left(\mathrm{i}\pi^{d/2}\right)^2 \times \frac{G(a_3, a_5)G(a_2, a_1+a_3+\varepsilon-2)}{(-q^2)^{a_1+a_2+a_3+a_5+2\varepsilon-4}}. \tag{6.20}$$

If all the five indices are positive we can apply the triangle rule (in the massless case) to the left triangle in Fig. 3.10 and reduce the sum $a_3 + a_4 + a_5$ by one, so that proceeding recursively we can eventually reduce one of the indices a_3, a_4, a_5 to zero and obtain an integral expressed in terms of gamma functions at general d.

Therefore, any Feynman integral of this family can be evaluated at general d in terms of gamma functions. For example, the following result for the integral with all the indices equal to one can be obtained:

$$\begin{aligned} F(1,1,1,1,1) &= \frac{1}{\varepsilon}\left[F(2,1,0,1,1) - F(2,1,1,1,0)\right] \\ &= \frac{1}{\varepsilon} G(1,1)\left[G(2,1) - G(2,1+\varepsilon)\right] \frac{\left(\mathrm{i}\pi^{d/2}\right)^2}{(-q^2)^{1+2\varepsilon}} \\ &= -\frac{(\mathrm{i}\pi^{d/2}\mathrm{e}^{-\gamma_{\mathrm{E}}\varepsilon})^2}{(-q^2)^{1+2\varepsilon}} \left[6\zeta(3) + \left(\frac{\pi^4}{10} + 12\zeta(3)\right)\varepsilon \right. \\ &\quad \left. + \left(\frac{\pi^4}{5} + (24-\pi^2)\zeta(3) + 42\zeta(5)\right)\varepsilon^2\right] + \cdots, \end{aligned} \tag{6.21}$$

so that the well-known result [17, 52] at order ε^0 is again (as in Sect. 3.5) reproduced.

Similarly, planar two-loop massless vertex diagrams of Fig. 6.2 with $p_1^2 = p_2^2 = 0$ and general integer powers of the propagators can be reduced by IBP relations in the first way using the triangle rule so that any such integral can be evaluated at general d in terms of gamma functions. (As it was mentioned in Chap. 3, the evaluation of such

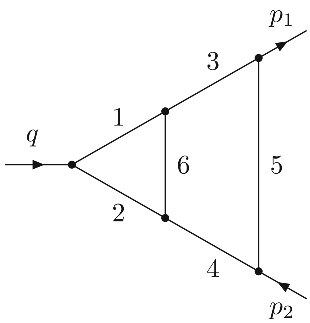

Fig. 6.2 Planar vertex diagram

Feynman integrals by Feynman parameters is rather cumbersome.) After applying the triangle rule one arrives at the possibility to apply (10.7) and (10.28).

For example, the well-known result [27, 70] for Fig. 6.2 at all the six indices equal to one was reproduced in this way in [42]:

$$
\begin{aligned}
&\frac{(\mathrm{i}\pi^{d/2})^2}{(Q^2)^{2+2\varepsilon}} \frac{1}{\varepsilon} \left[\frac{1}{2\varepsilon} G_2(2,2) G_3(2+\varepsilon, 1, 1) \right. \\
&\qquad \left. - G_2(2,1) \left(\frac{1}{\varepsilon} G_3(2,1,1+\varepsilon) + G_3(1,1,1) \right) \right] \\
&\qquad = \frac{(\mathrm{i}\pi^{d/2} \mathrm{e}^{-\gamma_{\mathrm{E}}\varepsilon})^2}{(Q^2)^{2+2\varepsilon}} \left(\frac{1}{4\varepsilon^4} + \frac{5\pi^2}{24\varepsilon^2} + \frac{29\zeta(3)}{6\varepsilon} + \frac{3\pi^4}{32} + O(\varepsilon) \right). \qquad (6.22)
\end{aligned}
$$

Historically, IBP relations were first successfully applied in [18] to three-loop massless propagators diagrams shown in Fig. 6.3. The corresponding algorithm [28, 29] called MINCER was implemented in FORM [72]. This is a hand-made reduction in the first way, i.e. to integrals expressed in terms of gamma functions and expanded up to a desirable order in ε. Since this a reduction based on explicit recursive relations it is very fast and quite competitive with respect to modern algorithmic reductions considered in the next sections of this chapter. MINCER has been successfully applied in numerous calculations.

In [13, 19, 21, 22, 30, 31], the problem of reduction for two-loop on-shell diagrams was solved: in [30, 31], relevant recurrence relations were derived and used to find all necessary integrals, and, in [13], a general algorithm implemented in the REDUCE [35] package Recursor was constructed. The reduction in the three-loop case was developed in [44, 45] and, completely, in [49, 50] with an implementation in FORM [72] (although no details of the reduction procedure were presented, as in many other cases).

The reduction of two-loop bubble integrals with different masses was solved in [20]. Three-loop vacuum diagrams with one mass were considered in [2, 13, 61]. The corresponding computer package MATAD was developed in [61]. The reduction of two- and three-loop propagator diagrams in HQET was solved in [14] and implemented in the code Grinder [32, 77].

Let me emphasize that, after public computer codes of solving IBP relations automatically have been appeared, there is no need to solve IBP relations by hand because this procedure is time consuming and success is not guaranteed, while one can easily apply these computer codes which are universal. The success depends

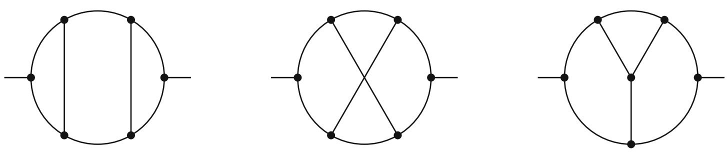

Fig. 6.3 Three-loop massless planar, non-planar and Mercedez–Benz propagator diagrams

only on the complexity of a given problem, in particular, the number of indices for a given family of Feynman integrals and the number of kinematic invariants.

6.2 General Setup for IBP Reduction

Suppose that we have a family of scalar Feynman integrals associated with a given graph,

$$F(q_1, \ldots, q_n; a_1, \ldots, a_N; d) = \int \cdots \int \prod_{i=1}^{h} \mathrm{d}^d k_i \frac{1}{\prod_{j=1}^{N} E_j^{a_j}}, \tag{6.23}$$

with integer indices a_j, where the denominators E_j are given by

$$E_r = \sum_{i \geq j \geq 1}^{h} A_r^{ij}\, k_i \cdot k_j + \sum_{i=1}^{h} B_r^i \cdot k_i + D_r, \tag{6.24}$$

i.e are quadratic or linear functions of the external momenta q_i and the loop momenta k_i. Our goal is to exploit IBP relations (2.41) in order to develop and algorithm for a reduction of any Feynman integral (6.23) to master integrals. Explicitly, let us use the IBP relations

$$\int \cdots \int \prod_{i'=1}^{h} \mathrm{d}^d k_{i'} \frac{\partial}{\partial k_i} \left(p_j \prod_{j'=1}^{N} E_j^{-a_{j'}} \right) = 0 \tag{6.25}$$

written for $i = 1, \ldots, h$ and $j = 1, \ldots, N$ with $p_j = k_j$ for $j = 1, \ldots, h$ and $p_{h+1} = q_1, \ldots, p_{h+n} = q_n$ with $N = h + n$.

After differentiating, the scalar products $k_i \cdot k_j$ and $k_i \cdot q_j$ are expressed linearly in terms of the factors E_i of the denominator, and one obtains the IBP relations in the following form:

$$\sum \alpha_i F(a_1 + b_{i,1}, \ldots, a_N + b_{i,N}) = 0. \tag{6.26}$$

Now one can substitute all possible $(a_1, \ldots, a_N)$ on the left-hand sides of (6.26) and obtain an infinite set of relations between integrals (6.23).

In fact, in addition to IBP relations, one uses also symmetries of the given family of integrals. Typically they have the following form:

$$F(a_1, \ldots, a_N) = (-1)^{\sum d_i a_i} F(a_{\pi(1)}, \ldots, a_{\pi(N)}),$$

where d_i are fixed and are equal to either one or zero, and π is a permutation. One more type of additional relations corresponds to parity conditions. For example, Feynman integrals can be zero if the sum of some subset of indices is odd and each index is nonpositive.

One also takes into account boundary conditions which have the form

$$F(a_1, \ldots, a_N) = 0 \text{ when } a_{i_1} \leq 0, \ldots a_{i_k} \leq 0 \tag{6.27}$$

for some subset of indices. These conditions are connected with zero values of integrals without scale.

Feynman integrals (6.23) can be considered as elements of the field of functions $\mathcal{F}$ of N integer arguments $a_1, \ldots, a_N$. This is an infinitely dimensional linear space. The IBP relations as well as additional relations mentioned above can be considered as elements of the adjoint vector space $\mathcal{F}^*$, i.e. the linear functionals on $\mathcal{F}$, so that, for any $r \in \mathcal{F}^*$, there is a corresponding value $\langle r, f \rangle$ for any given $f \in \mathcal{F}$. The simplest basis of this space is the set of elements $H^*_{a_1,\ldots,a_N}$ which are defined as follows:

$$\langle H^*_{a_1,\ldots,a_N}, f \rangle = f(a_1, \ldots, a_N).$$

After having fixed the set of IBP relations and additional relations we can generate by them an infinitely dimensional vector subspace $\mathcal{R} \subset \mathcal{F}^*$. Now one considers the set of solutions of all those relations, that is the intersection of the kernels of all functionals $r \in \mathcal{R}$. This is a vector subspace of $\mathcal{F}$, that will be denoted with $\mathcal{S}$. A Feynman integral considered as a function of the integer variables $a_1, \ldots, a_n$ is an element of the space $\mathcal{S}$ for it satisfies the IBP relations and other relations mentioned above. Formally,

$$\mathcal{S} = \{f \in \mathcal{F} : \langle r, f \rangle = 0 \; \forall \, r \in \mathcal{R}\}.$$

As it was proven in [56] the dimension of $\mathcal{S}$ is finite, i.e. the number of master integrals is always finite. We will give more comments on this theorem below.

When talking about expressing one Feynman integral by another, it is usually assumed that we consider the consequences of relations $\mathcal{R}$. Let us say that an integral $F(a_1, \ldots, a_N)$ can be expressed via some other integrals $F(a^1_1, \ldots, a^1_N)$, …, $F(a^k_1, \ldots, a^k_N)$ if there exists $r \in \mathcal{R}$ such that

$$\begin{aligned}\langle r, F \rangle = F(a_1, \ldots, a_N) \\ + \sum k_{a'_1,\ldots,a'_N} F(a'_1, \ldots, a'_N).\end{aligned} \tag{6.28}$$

Let us turn to the notion of irreducibility of Feynman integrals. Suppose we have two integrals $F(a_1, \ldots, a_N)$ and $F(a'_1, \ldots, a'_N)$ that can be expressed one by another, for example, due to a symmetry of the diagram. Of course, it is reasonable to choose only one of them as a master integral. However there seems to be nothing natural in this choice, for they are equivalent. So, even having fixed a set of relations, we do not have enough information to define master integrals. The only thing we know that their number is equal to the dimension of $\mathcal{S}$.

Therefore, to define master (or, irreducible) integrals, we need to choose a certain priority between the points $(a_1, \ldots, a_N)$, formally, to introduce a complete ordering on them (that will be denoted with the symbol $\prec$ and named as *lower*). There are different ways to do that but at least it looks natural to have simpler integrals corresponding to the minimal elements in this ordering.

We will introduce an ordering in two steps. First of all, let us realize that the Feynman integrals are simpler, from the analytic point of view, if they have more non-positive indices. In fact, in numerous examples of solving of IBP relations by hand, in particular, in the examples of Sect. 6.1, the natural goal was to reduce indices to zero or negative values. The big experience reflected in many calculations has led to the natural idea to decompose the whole region of the integer indices into so-called *sectors*.[3] This decomposition is standard in algorithmic approaches of solving IBP relations (on which the corresponding computer codes are based), in particular in the so-called Laporta's algorithm [43, 44] which is characterized in the next section.

The sectors are 2^N regions labeled by subsets $\nu \subseteq \{1, \ldots, N\}$, where $\sigma_\nu = \{(a_1, \ldots, a_N) : a_i > 0 \;\text{ if }\; i \in \nu,\;\; a_i \leq 0 \;\text{ if }\; i \notin \nu\}$. A sector σ_ν is said to be *lower* than a sector σ_μ if $\nu \subset \mu$. Furthermore, $F(a_1, \ldots, a_N) \succ F(a'_1, \ldots, a'_N)$ if the sector of $(a'_1, \ldots, a'_N)$ is lower than the sector of $(a_1, \ldots, a_N)$. To define an ordering completely one has to introduce it in some way inside the sectors (this will be discussed below).

Equivalently, we can start from the set of elements $\{d_1, \ldots, d_n\}$ called *directions*, where all d_i are equal to 1 or -1. For any given direction $\nu = \{d_1, \ldots, d_N\}$, we consider the sector $\sigma_\nu = \{(a_1, \ldots, a_N) : (a_i - 1/2)d_i > 0\}$. In other words, in a given sector, the indices corresponding to ± 1 are positive (non-positive). It is natural to assume that the 'corner point' $((d_1 + 1)/2, \ldots, (d_N + 1)/2)$ (those numbers are either ones or zeros) of the sector $\sigma_{\{d_1,\ldots,d_N\}}$ is lower than all other points of this sector.

To define an ordering completely we introduce it in some way inside the sectors. After this, we can define what a master integral is. It is such an integral $F(a_1, \ldots, a_N)$ that there is no element $r \in \mathcal{R}$ acting on F according to relation (6.28) such that all the points $(a'_1, \ldots, a'_N)$ are lower than $(a_1, \ldots, a_N)$.

Well, this definition is a little bit tautological. Still it is quite natural and practical and it is present in all the automatic solutions of IBP relations.

6.3 Laporta's Algorithm

The idea of the Laporta's algorithm [43, 44] is to solve systems of equations for individual Feynman integrals. Let $\mathcal{F}_M$ be the set the subspace of $\mathcal{F}$ generated by $H_{a_1,\ldots,a_n}$ where $\sum_i |a_i| \leq M$ and $\mathcal{R}_M$ be the intersection of $\mathcal{R}$ with the subset of $\mathcal{F}^*$ generated by $H^*_{a_1,\ldots,a_n}$ where $\sum_i |a_i| \leq M$. The limit of the difference between the dimensions of $\mathcal{F}_M$ and $\mathcal{R}_M$ when M tends to infinity is the dimension of $\mathcal{S}$ which is finite

[3] So, the word 'sector' is used here to denote a domain of integer variables, while in Chap. 4 it is used for domains of continuous variables (alpha or Feynman parameters).

according to [56], so that there is a certain M such that $\mathcal{R}_M$ has 'enough' relations to express any given integral $F(a_1, \ldots, a_N)$ with $\sum_i |a_i| \leq M$ as a linear combination of master integrals with the use of those relations. So, the method is based on finding such an M and solving a system of linear equations which might be huge.

Of course, at any M, this system has a solution, i.e. any integral of the given family is linearly expressed in terms of some integrals. The point is that starting from some minimal M the situation stabilizes, i.e. the set of the integrals on the right hand side of solutions stays the same, so these are master integrals.

A breakthrough in the implementation of this algorithm came due to the following two publications: the first practical successful implementation was achieved for the reduction of massless double box diagrams with one leg off-shell [24], and detailed prescriptions for the implementation of this method in a general situation were presented in [43].

To see how the Laporta's algorithm works let us turn to our favourite Example 6.3 for which a manual solution was presented in Sect. 6.1. Let us define the left-hand sides of the IBP relations (1.17) and (1.18) written at the level of individual integrals:

$$\begin{aligned} L_1(a_1, a_2) = (d - 2a_1 - a_2)F(a_1, a_2) - 2m^2 a_1 F(a_1 + 1, a_2) \\ - a_2(F(a_1 - 1, a_2 + 1) + (m^2 - q^2)F(a_1, a_2 + 1)), \end{aligned} \tag{6.29}$$

$$\begin{aligned} L_2(a_1, a_2) = -a_1((q^2 + m^2)F(a_1 + 1, a_2) - F(a_1 + 1, a_2 - 1)) \\ (a_2 - a_1)F(a_1, a_2) - a_2(F(a_1 - 1, a_2 + 1) + (m^2 - q^2)F(a_1, a_2 + 1)). \end{aligned} \tag{6.30}$$

Let us consider the sector $a_1 > 0, a_2 \leq 0$, use IBP relations at various (a_1, a_2) with $a_1 + |a_2| \leq M$ and solve the corresponding linear systems of equations with respect to the integrals $F(a_1, a_2)$ involved. At $M = 1$, we solve the system

$$L_1(1, 0) = 0, \quad L_2(1, 0) = 0$$

and obtain

$$\begin{aligned} F(2, -1) &= \frac{d(m^2 + q^2) - 2q^2}{2m^2} F(1, 0), \\ F(2, 0) &= \frac{(d - 2)}{2m^2} F(1, 0). \end{aligned}$$

At $M = 2$, we solve the system of the six equations

$$L_i(1, 0) = 0, \quad L_i(2, 0) = 0, \quad L_i(1, -1) = 0$$

with $i = 1, 2$ and obtain

$$F(2, -2) = \frac{(2 + d)m^4 + 2(2 + d)m^2 q^2 + (d - 2)(q^2)^2}{2m^2} F(1, 0),$$

$$
\begin{aligned}
F(3,-1) &= \frac{(d-2)(-4q^2+d(m^2+q^2))}{8m^4}F(1,0),\\
F(3,0) &= \frac{(d-4)(d-2)}{8m^4}F(1,0),\\
F(1,-1) &= (m^2+q^2)F(1,0),\\
F(2,-1) &= \frac{-2q^2+d(m^2+q^2)}{2m^2}F(1,0),\\
F(2,0) &= \frac{d-2}{2m^2}F(1,0).
\end{aligned}
$$

Here the situation is stable from the very beginning, i.e. we reveal just one master integral $I_2 = F(1,0)$ in the considered sector. In complicated problems, such a stabilization can take place for large M. For example, in the case of the double boxes with one leg off-shell, it was necessary [24] to solve linear systems of dozens of thousands of equations for dozens of thousands of variables.

At the moment, there are three public codes based on Laporta's algorithm: `AIR` [1] written in `MAPLE`, `FIRE` written in `MATHEMATICA`[4] and `Reduze` [62, 73, 74] written in `C++`, and a lot of private codes.[5]

6.4 Algebraic Structure of IBP Relations

Lee has observed [46] that the IBP relations have the structure of a Lie algebra. Indeed, they are generated by the operators

$$
O_{ij} = \frac{\partial}{\partial k_i}\cdot p_j \tag{6.31}
$$

in (6.25) which can be considered as generators of linear transformation of variables in the integrals (6.23), with the commutation relations

$$
\left[O_{ik}, O_{i'j'}\right] = \delta_{ij'}O_{i'j} - \delta_{i'j}O_{ij'}. \tag{6.32}
$$

This observation was useful in various respects. *First*, it turns out that the IBP relations are not linearly independent, so that it is sufficient to consider a subset of them when performing an IBP reduction, for example, with the Laporta's algorithm. This is a variant of choosing a basic set of the IBP relations [46] generated by the following operators:

[4] A `C++` version of `FIRE` is private at the moment.

[5] By T. Gehrmann and E. Remiddi, S. Laporta, M. Czakon, Y. Schröder, A. Pak, C. Sturm, P. Marquard and D. Seidel, V. Velizhanin, ...

$$\begin{aligned}
&\frac{\partial}{\partial k_i} \cdot k_{i+1}, \quad i = 1, \ldots, h, \quad k_{h+1} \equiv k_1; \\
&\frac{\partial}{\partial k_1} \cdot p_j, \quad j = 1, \ldots, n; \\
&\sum_{i=1}^{h} \frac{\partial}{\partial k_i} \cdot k_i .
\end{aligned} \tag{6.33}$$

Second, the algebraic structure of IBP relations provides the possibility to show [46] that so-called Lorentz-invariance (LI) identities [23] follow from the IBP relations. The LI identities express the fact that Feynman integrals are Lorentz scalars. For several years, they were used within Laporta algorithm, together with the IBP relations (e.g., in [23]), and it was not clear whether they provide new information. They have the form

$$q_i^{\mu} q_j^{\nu} \left(\sum_r q_{r[\nu} \frac{\partial}{\partial q_r^{\mu]}} \right) F(a_1, \ldots, a_N) = 0. \tag{6.34}$$

To prove that the LI identities follow from the IBP relations one can observe [46] that the operator

$$\sum_{i=1}^{h+n} p_{i[\nu} \frac{\partial}{\partial p_i^{\mu]}} = \sum_{i=1}^{h} k_{i[\nu} \frac{\partial}{\partial k_i^{\mu]}} + \sum_{i=1}^{n} q_{i[\nu} \frac{\partial}{\partial q_i^{\mu]}}, \tag{6.35}$$

gives zero when acting on the integrand of a Feynman integral because this is a scalar quantity. Therefore the operator in (6.34) acting on the integrand can be represented as follows:

$$\begin{aligned}
q_i^{\mu} q_j^{\nu} \sum_{r=1}^{h} q_{r[\mu} \frac{\partial}{\partial q_r^{\nu]}} F &= q_i^{\mu} q_j^{\nu} \sum_{r=1}^{h+n} p_{r[\nu} \frac{\partial}{\partial p_r^{\mu]}} - q_i^{\mu} q_j^{\nu} \sum_{r=1}^{h} k_{r[\nu} \frac{\partial}{\partial k_r^{\mu]}} \\
&= -q_i^{\mu} q_j^{\nu} \sum_{r=1}^{h} k_{r[\nu} \frac{\partial}{\partial k_r^{\mu]}} \\
&= \sum_{r=1}^{h} \left[(q_i \cdot k_r)\, q_j \cdot \frac{\partial}{\partial k_r} - \left(q_j \cdot k_r\right) q_i \cdot \frac{\partial}{\partial k_r} \right] \\
&= \sum_{r=1}^{h} \left[\frac{\partial}{\partial k_r} \cdot q_j \,(q_i \cdot k_r) - \frac{\partial}{\partial k_r} \cdot q_i \left(q_j \cdot k_r\right) \right].
\end{aligned}$$

Since the factors $(q_i \cdot k_r)$ and $(q_j \cdot k_r)$ can linearly be expressed in terms of the ‘denominators’ E_j in (6.23) the last equation shows that the LI identities are expressed in terms of the IBP relations.

Third, the structure of a Lie algebra helped to formulate [46] a simple criterion that shows whether integrals of a given sector are all identically zero (because they are scaleless integrals) or not. According to this criterion it is sufficient to check whether the corner integral of the sector is zero in virtue of the IBP relations.

Fourth, it was shown in [46] how to develop an algorithm which looks for solutions of the IBP relations in a given sector, similarly to looking for such solution 'by hand'. At present, a private code based on this algorithm works successfully for some rather complicated cases [47]. As in the situations with solutions found by hand, the corresponding solutions are practically very fast. (Well, if they exist.)

Fifth, the use of the structure of a Lie algebra for IBP relations was the starting point in the proof of the theorem [56] on the finite number of the master integrals.

6.5 Baikov's Method

One more systematic method to solve IBP relations (6.23) [3, 4, 10, 11, 60] is from the beginning oriented at an IBP reduction to a minimal number of master integrals,

$$F(\underline{a}) = \sum_i c_i(\underline{a}) I_i\,, \tag{6.36}$$

where underlined letters denote collections of variables, i.e. $\underline{a} = (a_1, \ldots, a_N)$, etc. The coefficient functions satisfy the natural normalization conditions

$$c_i(I_j) = \delta_{ij} \tag{6.37}$$

which simply mean that any master integral cannot be expressed in terms of other master integrals. In fact, the master integrals are integrals of the given family, $I_i = F(\underline{a}_i)$, where $\underline{a}_i = (a_{i1}, \ldots, a_{iN})$ are some concrete sets of the indices, and, by definition, $c_i(I_j) = c_i(a_{i1}, \ldots, a_{iN})$. Typically, a_{ir} are equal to 1, 0, −1, 2.

Consider, first, the case of vacuum Feynman integrals which are functions of some masses and are defined by (6.23) with

$$E_r = \sum_{h \geq i \geq j \geq 1} A_r^{ij}\, k_i \cdot k_j - m_r^2\,, \tag{6.38}$$

with $r = 1, \ldots, N = h(h+1)/2$.

The IBP relations in the vacuum case originate from the following N equations:

$$\int \cdots \int \mathrm{d}^d k_1 \ldots \mathrm{d}^d k_h \frac{\partial}{\partial k_i} \cdot \left(\frac{k_j}{E_1^{a_1} \ldots E_N^{a_N}} \right) = 0\,, \quad i \geq j\,. \tag{6.39}$$

After differentiating, resulting scalar products $k_i \cdot k_j$ are expressed in terms of the denominators E_r. When we invert the relations (6.38) we obtain a matrix which is inverse, in some sense, to the matrix A_r^{ij}. So, we write down the IBP relations in the following form:

$$\sum_{r,r',i'} \bar{A}_r^{i'i} \tilde{A}_{r'}^{ji'} \left(\mathbf{r}'^- + m_{r'}^2 \right) a_r \mathbf{r}^+ = (d-h-1)\delta_{ij}/2, \tag{6.40}$$

where the operators $\mathbf{r}^+$ and $\mathbf{r}^-$ increase and lower indices. (They were used in the examples of Sect. 6.1 for various concrete values of r.)

Moreover, $\bar{A}_r^{ij} = A_r^{ij}$ for $i = j$, $A_r^{ij}/2$ for $i > j$ and $A_r^{ji}/2$ for $i < j$. The matrix $\tilde{A}$ is defined as follows. Take the quadratic $N \times N$ matrix A, where the first index is labelled by pairs (i, j) with $i \geq j$, and the second index is r. The corresponding inverse matrix $(A^{-1})_r^{ij}$ (with $i \geq j$) satisfies

$$\sum_{r=1}^{N} A_r^{ij} \, (A^{-1})_r^{i'j'} = \delta_{ii'}\delta_{jj'}. \tag{6.41}$$

Then $\tilde{A}_r^{ij}$ is the symmetrical extension of $(A^{-1})_r^{ij}$ to all values i, j.

To construct the coefficient functions $c_i(\underline{a})$ in the vacuum case, the following basic representation [3, 4] can be applied:

$$\int \ldots \int \frac{\mathrm{d}x_1 \ldots \mathrm{d}x_N}{x_1^{a_1} \ldots x_N^{a_N}} \left[P(\underline{x'}) \right]^{(d-h-1)/2}, \tag{6.42}$$

where the polynomial P will be defined shortly and the parameters $\underline{x'} = (x'_1, \ldots, x'_N)$ are obtained from $\underline{x} = (x_1, \ldots, x_N)$ by the shift $x'_i = x_i + m_i^2$.

Integration over the parameters x_i is understood in some way, with the requirement that the IBP in this parametric integral is valid. In this case, such objects satisfy the initial IBP relations (6.40). This property can be verified straightforwardly if we take into account that the operator $a_r \mathbf{r}^+$ is transformed into the differential operator $\partial/\partial x_r$ and the operator $\mathbf{r}^-$ is transformed into the multiplication by x_r.

Now, the basic polynomial P of $\underline{x}$ which enters (6.42) is [3, 4]

$$P(\underline{x}) = \det_{ij} \left(\sum_{r=1}^{N} \tilde{A}_r^{ij} \, x_r \right). \tag{6.43}$$

Here are simple practical prescriptions for evaluating the basic polynomials:

1. Solve the system

$$\sum_{i \geq j \geq 1} A_r^{ij} \, k_i \cdot k_j = E_r, \quad r = 1, \ldots, N$$

   with respect to $k_i \cdot k_j$, $i \geq j$.

2. Replace E_r by x_r on the right-hand side of this solution.
3. Extend this expression to all values of i and j in the symmetrical way.
4. Take the determinant of this matrix to obtain P.

In fact, the basic polynomial is defined up to a normalization factor independent of the variables x_j. This will be clear when constructing the coefficient functions which will be themselves normalized at some point.

For general Feynman integrals, the problem can be reduced to the vacuum case [3, 4, 10]. If there is one external momentum, q, so that we are dealing with a family of propagator-type integrals, one involves into the game coefficients of the Taylor expansion of $F(\underline{a})$ in q^2,

$$F(q^2; a_1, \ldots, a_N) \sim \sum_{a_{N+1}=1}^{\infty} (q^2 - m_{N+1}^2)^{a_{N+1}-1} F(a_1, \ldots, a_N, a_{N+1}). \quad (6.44)$$

It turns out [3, 4, 10] that the so defined objects $F(a_1, \ldots, a_N, a_{N+1})$ satisfy vacuum IBP relations.

To formulate a prescription for corresponding basic polynomials in the non-vacuum case, we need first to present a preliminary discussion of constructing master integrals. To identify candidates for master integrals in a first approximation, we will analyze integrals where the indices corresponding to irreducible numerators are set to zero and other indices are either zero or one. Let $F(\underline{a_i})$ with $a_{ij} = 1$ or 0 be a candidate to be considered as a master integral.

Let us remember the examples of this chapter where according to the setup formulated in Sect. 6.2 the reduction always goes down so that a master integral $I_i = F(\underline{a_i}) = F(a_{i1}, \ldots, a_{ir}, \ldots, a_{iN})$ never appears in the decomposition of a given Feynman integral in terms of master integrals

$$F(\underline{a}) = \cdots + c_i(a_1, \ldots, a_r, \ldots, a_N) I_i + \cdots$$

if $a_r \leq 0$ and $a_{ir} > 0$. Therefore, we come to the natural condition for the coefficient function $c_i(\underline{a})$ of $F(\underline{a_i})$: if $a_{ir} = 1$ then $c_i(a_1, \ldots, a_r, \ldots, a_N) = 0$ for $a_r \leq 0$.

This condition can be realized easily [3, 4] in an automatic way by treating the integration over x_j as a Cauchy integral around the origin in the complex x_j-plane,

$$\frac{1}{2\pi i} \oint \frac{\mathrm{d}x_j}{x_j^{a_j}} \int \ldots \left[P(\underline{x})\right]^{(d-h-1)/2}. \quad (6.45)$$

According to the Cauchy theorem, this expression reduces to the Taylor expansion of order $a_j - 1$ of the integrand in x_j so that it becomes a linear combination of terms

$$\int \ldots \int \left[P_i(\underline{x})\right]^{z-n_d} \prod_{j:a_{ij} \leq 0} \frac{\mathrm{d}x_j}{x_j^{n_j}}, \quad (6.46)$$

where $z = (d - h - 1)/2$, and $P_i(\underline{x})$ is obtained from $P(\underline{x})$ by setting to zero all the variables x_j with j such that $a_{ij} = 1$. We will use n_j instead of a_j for powers of x_j in auxiliary parametric integrals. Observe that the parameter n_d in such integrals plays the role of the shift of the dimension.

Suppose that we are not interested in higher terms of the Taylor expansion in powers of $(q^2 - m_{N+1}^2)$ in (6.44), i.e. we need just the value at $q^2 = m_{N+1}^2$, i.e. the term with $a_{N+1} = 1$. Then the integration over x_{N+1} should be understood in the sense of Cauchy integration so that, effectively, x_{N+1} is set to zero. So, if $\hat{P}(x_1, \ldots, x_N, x_{N+1})$ is the basic polynomial for the corresponding vacuum problem, then the basic polynomial for the initial propagator-type problem is obtained as

$$P(\underline{x}) \equiv P(x_1, \ldots, x_N) = \hat{P}(x_1, \ldots, x_N, 0). \tag{6.47}$$

In the case of n independent external momenta $q_1, \ldots, q_n$, one includes into the procedure all the terms of the formal Taylor expansions in the scalar products $q_i \cdot q_j$. One is usually interested only in the value at some $q_i \cdot q_j$ and not in the derivatives at these points. (Otherwise, it would be necessary to deal with a generalization of (6.44), where the initial Feynman integrals are rescaled by the Gram determinant $\det(p_i \cdot p_j)$ which is raised to the power $(h+n+1-d)/2$—see [3, 4, 10].) Then the transition to the vacuum problem, which effectively increases the number of loops, $h \to h + n$, can be performed as follows:

1. Introduce a complete set of invariants by considering, in addition to $k_i \cdot k_j$, $i \geq j$ and $k_i \cdot q_j$, also invariants generated by the external momenta, i.e. the scalar products $q_i \cdot q_j$, $i \geq j$. Let $p_i = k_i, i = 1, \ldots, h$ and $p_i = q_i, i = h + 1, \ldots, h + n$ so that the total number of the kinematical invariants becomes $\hat{N} = (h+n)(h+n+1)/2$.
2. Introduce, in some way, the corresponding new propagators.
3. Solve the system
$$\sum_{i \geq j \geq 1} A_r^{ij}\, p_i \cdot p_j = E_r, \quad r = 1, \ldots, \hat{N}$$
with respect to $p_i \cdot p_j$.
4. Evaluate the basic polynomial $\hat{P}$ for such a vacuum problem.
5. Obtain $P(\underline{x}) \equiv P(x_1, \ldots, x_N) = \hat{P}(x_1, \ldots, x_N, 0, \ldots, 0)$.

Let me emphasize that this strategy is applicable not only to usual Feynman integrals with quadratic denominators but also for more general Feynman integrals with the denominators (6.24): one treats additional vectors, like the quark velocity, on the same footing as the true external momenta and considers Feynman integrals as functions of various scalar products.

Now, we want to apply the basic parametric representation for two closely related purposes:

- identifying master integrals,
- constructing the corresponding coefficient functions.

According to the discussion above, let us consider integrals where the indices corresponding to irreducible numerators are set to zero and other indices are either zero or one. Let $I_i = F(\underline{a}_i) = F(a_{i1}, \ldots, a_{ir}, \ldots, a_{iN})$. For indices equal to one, we understand the corresponding integration over x_j in the basic parametric representation (6.42) in the Cauchy sense. This leads to a Taylor expansion of order $a_j - 1$ of the integrand in x_j and gives a linear combination of (6.46).

Let us try to understand whether a given candidate can be considered as a master integral. Suppose that $P_i = 0$. Then there is no other way as to consider the coefficient function equal to zero. Therefore, this integral cannot be a master integral and has to be recognized as a reducible integral within the reduction problem.

Let us assume a weaker condition: the parametric integral involves an integral without scale which we set, by definition, to zero. Then, again, we cannot construct the coefficient function in a non-trivial way so that the corresponding integral is considered reducible.

After such analysis, we obtain a preliminary list of master integrals. Sometimes one has to consider master integrals which differ from $F(\underline{a}_i)$ by some indices $a_{ij} < 0$. The number of such additional master integrals is connected with the degree of the polynomial P_i with respect to some of the parameters x_j.

Before discussing further the general recipes to construct coefficient functions let us turn once again to our favourite example which is labelled in this chapter as Example 6.3. The transition to the corresponding vacuum problem reduces to adding a new propagator, $1/(q^2 - s)^{a_3}$. We again consider these integrals at general q^2 and are not interested in derivatives so that, effectively, the corresponding index will be $a_3 = 1$ and the corresponding variable x_3 is set to zero. The resulting basic polynomial is

$$P(x_1, x_2) = -(x_1 - x_2 + m^2)^2 - q^2(q^2 - 2m^2 - 2(x_1 + x_2)). \tag{6.48}$$

There are two master integrals $F(1, 1) = I_1$ given by (1.5) and $F(1, 0) = I_2$ given by the right-hand side of (6.6). We want to construct the corresponding coefficient function with the normalization conditions (6.37), i.e.

$$c_1(1, 1) = 1, \quad c_1(1, 0) = 0, \quad c_2(1, 1) = 0, \quad c_2(1, 0) = 1.$$

The coefficient function of I_1 is simply obtained:

$$c_1(a_1, a_2) = \frac{\left(q^2 - m^2\right)^{(d-3)}}{(a_1 - 1)!(a_2 - 1)!} \times \left(\frac{\partial}{\partial x_1}\right)^{a_1 - 1} \left(\frac{\partial}{\partial x_2}\right)^{a_2 - 1} [P(x_1, x_2)]^{(d-3)/2} \Bigg|_{x_i = 0}. \tag{6.49}$$

For the coefficient function $c_2(a_1, a_2)$ of I_2, we obtain linear combinations of one-parametric integrals

$$f(n_1, n_2) = \int \frac{dx}{x^{n_1}} [P_2(x)]^{(d-3)/2-n_2}, \tag{6.50}$$

where

$$P_2(x) = P(x_1, x)|_{x_1=0} = \alpha x^2 + \beta x + \gamma \tag{6.51}$$

with $\alpha = -1$, $\beta = 2(m^2 + q^2)$, $\gamma = -(m^2 - q^2)^2$.

Consider first the case $a_2 \leq 0$. Then n_1 is always non-positive here, and $f(n_1, n_2)$ can be understood as an integral between the roots

$$x^{(1,2)} = \left(m \mp \sqrt{q^2}\right)^2$$

of the quadratic polynomial $P_2(x)$, using the following explicit formula:

$$\begin{aligned}&\int_{x_1}^{x_2} dx\, x^k (x - x_1)^{\alpha_1} (x_2 - x)^{\alpha_2} \\ &\quad = \sum_{r=0}^{k} x_1^{k-r} (x_2 - x_1)^{\alpha_1+\alpha_2+r+1} \frac{k!}{(k-r)!r!} \frac{\Gamma(1+\alpha_2)\Gamma(1+\alpha_1+r)}{\Gamma(\alpha_1+\alpha_2+r+2)}.\end{aligned} \tag{6.52}$$

The evaluation at $a_1 = 1$ and $a_2 = 0$ provides a normalization factor to satisfy the normalization condition $c_2(1, 0) = 1$, and we obtain the following expression for $c_2(a_1, a_2)$ at $a_2 \leq 0$:

$$\begin{aligned}c_2^0(a_1, a_2) = c_2^0(a_1, a_2) \equiv &\frac{\Gamma(d-1)}{4^{d-2}(m^2q^2)^{(d-2)/2}\Gamma((d-1)/2)^2} \\ &\times \frac{1}{(a_1-1)!} \int_{x^{(1)}}^{x^{(2)}} \frac{dx}{x^{a_2}} \left(\frac{\partial}{\partial x_1}\right)^{a_1-1} [P(x_1, x)]^{(d-3)/2} \bigg|_{x_1=0}.\end{aligned} \tag{6.53}$$

In the case $a_2 > 0$, the integrals $f(n_1, n_2)$ appear also with $n_1 > 0$. When taken seriously they can be evaluated in terms of a Gauss hypergeometric function. Instead of doing this, let us apply IBP to our parametric integrals $f(n_1, n_2)$. This gives the relation

$$\begin{aligned}f(n_1, n_2) = &\frac{(d-3)/2 - n_2}{n_1 - 1} \\ &\times (2\alpha f(n_1 - 2, n_2 + 1) + \beta f(n_1 - 1, n_2 + 1))\end{aligned} \tag{6.54}$$

which can be used to reduce n_1 to one or zero. Moreover, the identity

$$P_2^{(d-3)/2-n_2} = P_2^{(d-3)/2-n_2-1} P_2$$

leads to the relation

$$f(1, n_2) = \frac{1}{\gamma} \left(f(1, n_2 - 1) - \alpha f(-1, n_2) - \beta f(0, n_2) \right) \tag{6.55}$$

which can be used to reduce n_2 to zero.

This means that we can express any $f(n_1, n_2)$ as a linear combination of an *auxiliary* master integral $f(1, 0)$ and integrals $f(n_1, n_2)$ with $n_1 \leq 0$ which can be evaluated in terms of gamma functions. We believe that the coefficient functions are rational functions of everything. The only chance to satisfy this property here is to construct $c_2(a_1, a_2)$ as a linear combination of $c_2^0(a_1, a_2)$ and the first coefficient function $c_1(a_1, a_2)$:

$$c_2(a_1, a_2) = c_2^0(a_1, a_2) + A c_1(a_1, a_2). \tag{6.56}$$

The constant A is determined by the normalization condition $c_2(1, 1) = 0$:

$$A = -c_2^0(1, 1). \tag{6.57}$$

After this, the dependence on $f(1, 0)$ drops out and $c_2(a_1, a_2)$ indeed turns out to be a rational function.

Observe that integrating over some real domain, in particular between the roots of a quadratic polynomial when constructing coefficient functions, with a subsequent normalization, is in fact equivalent to solving IBP relations for our auxiliary parametric integrals. If there is such a possibility to understand a given parametric integral it is reasonable to use it. If there is no such possibility, e.g. one meets a polynomial of the third degree, or, an integration over one of the x-variables leads to inconvenient integrals over the rest variables, then there is no other way as to treat the auxiliary parametric integrals in a pure algebraic way by solving the corresponding IBP relations. In the above example, the situation with $a_2 \leq 0$ could be treated algebraically, by IBP in the initial two-parametric integral, but integrating over x_2 has simplified the situation.

Let us extend what was done in the above example to the general situation. After a preliminary analysis, with the help of (6.42), we obtain a preliminary list of candidates for the master integrals. Let us define a relation of partial ordering of the master integrals as follows:

$$F(\underline{a}_1) < F(\underline{a}_2) \text{ if } a_{1j} \leq a_{2j} \quad \text{for all} \quad j,$$

and the strict inequality holds at least for one index.

The master integrals can be grouped into families characterized by their maximal integrals. Let us start from the master integrals which have most non-negative indices. Usually, the corresponding parametric integral for the coefficient function can be understood in such a way that it results in integrations in terms of gamma functions.

Consider now a situation with two master integrals with $F(\underline{a}_2) < F(\underline{a}_1)$, and suppose that we already know c_1. If $a_{2i} = 1$ we have also $a_{1i} = 1$. To construct an algorithm for the coefficient function $c_2(\underline{a})$ we start with the case of negative indices a_j for those indices j where $a_{1j} = 1$ since in this case we have $c_1(\underline{a}) = 0$. Experience shows that the integrations for $c_2(\underline{a})$ result in ratios of gamma functions which in particular can be used to satisfy the normalization $c_2(\underline{a}_2) = 1$.

In a next step one considers the case $a_j > 0$. Then the corresponding parametric representation usually leads to integrals which cannot be evaluated in terms of gamma functions. (See the above example.) Thus, at first sight it looks hopeless to achieve that the coefficient functions have to be rational functions of d. The way out is to look for an expression for the coefficient function $c_2(\underline{a})$ which is a linear combination of $c_1(\underline{a})$ and the basic parametric representation for $c_2(\underline{a})$ denoted by $c_2^0(\underline{a})$

$$c_2(\underline{a}) = c_2^0(\underline{a}) + A c_1(\underline{a}). \tag{6.58}$$

The constant A is determined by the normalization condition $c_2(\underline{a}_1) = 0$ which gives

$$A = -c_2^0(\underline{a}_1). \tag{6.59}$$

Then IBP relations are applied to the parametric integrals and the corresponding relations are used to express any given parametric integral in terms of auxiliary (parametric) master integrals and expressions which are straightforwardly evaluated in terms of gamma functions. The dependence on the new auxiliary master integrals has to drop out[6] in order to provide a rational dependence of the coefficient functions on d.

In fact, this strategy can be generalized to the case of several master integrals with more complicated hierarchies.

The most complicated example of applying the strategy formulated in this section is the evaluation of two-loop Feynman integrals for the heavy quark static potential [60]. Well, let us realize that this is a simple example, from the modern point of view. I do not have doubts that the modern public codes based on Laporta's algorithm can provide easily an IBP reduction in this case because the corresponding two families of Feynman integrals have only seven indices. In higher loop orders, this strategy was never applied. In particular, the reduction to master integrals of Feynman integrals for the heavy quark static potential in the three-loop approximation was done in [59] with the public code `FIRE` [55, 75]. These are reduction problems with twelve indices. Still I believe that this strategy can be optimized and automated in order

[6] This cancellation serves as a good check of the algorithm, similarly to cancellations of spurious poles in ε on the right-hand side of various asymptotic expansions in momenta and/or masses—see Chap. 9.

to solve complicated reduction problems. For example, it looks reasonable to apply Laporta's algorithm when solving IBP relations for auxiliary integrals appearing in the evaluation of coefficient functions.

As to the author of the considered method, he has turned to another strategy within his method based on an expansion in inverse power of d and applied to one-scale integrals. The idea is to evaluate the coefficient functions in this expansion using the method of stationary phase applied to the Ansatz (6.42). Practically, a coefficient function at a given master integral is expanded in powers of $1/d$ up to sufficiently high order and then the assumption that the coefficient functions are rational functions of d is used. So, the coefficient function is written as a ratio of two polynomials of some degree in d, with unknown coefficients, and it is matched in the expansion in $1/d$. The coefficients of the two polynomials are then found after solving these matching relations.

This strategy turned out to be quite powerful. It was successfully used for the reduction of the four-loop massless propagator integrals, where the number of the indices equals fourteen, with multiple applications to various physical problems—see [7–9] and references therein.

6.6 Shifting Dimension and Gröbner Bases

Both techniques implied in the title of this section were initiated by Tarasov. As it was pointed out in Sect. 3.2.3 one can rid of numerators and negative indices at the cost of shifting dimension [63, 64]. Then one can solve IBP reduction working with positive indices and sifted dimension which turns out to be one more parameter. Some prescriptions of this technique were presented in [66, 67]. Another example of its applications [53, 54] is provided by the calculation of Feynman integrals relevant to the two-loop quark potential (considered within Baikov's method in [60] as it was pointed out in the previous section). It was also used to solve the reduction problem for two-loop propagator integrals with arbitrary masses [65], with a public `Mathematica` implementation in [51], and obtain new results for the two-loop sunset diagram with equal masses [69].

One more approach to solve reduction problems for Feynman integrals is based on the theory of Gröbner bases [15] that have arisen naturally when characterizing the structure of ideals of polynomial rings. The first attempt to apply this theory to Feynman integrals was made in [66, 68], where IBP relations were reduced to differential equations. To do this, one assumes that there is a non-zero mass for each line. The typical combination $a_i\mathbf{i}^+$, where $\mathbf{i}^+$ is a shift operator, is naturally transformed into the operator $\frac{\partial}{\partial m_i^2}$ of differentiation in the corresponding mass squared. Then one can apply some standard algorithms for constructing corresponding Gröbner bases for differential equations. Another attempt was made in [25] where Janet bases were used.

As an application of the method of [66], the solution of the reduction problem for two-loop self-energy diagrams with five general masses was obtained in [68], with an agreement with an earlier solution [65]. Moreover, the solution of the reduction problem for massless two-loop off-shell vertex diagrams (which was first obtained in [12] within Laporta's algorithm) was reproduced in [37, 38].

One more approach based on Gröbner-type bases, called the s-bases, has been developed in [57, 58]. Here is its very brief description.

Suppose first that we are interested in expressing any integral in the positive sector $\sigma_{\{1,\ldots,n\}}$ as a linear combination of a finite number of integrals in it. The left-hand sides of IBP relations (2.41) can be expressed in terms of operators of multiplication A_i and shift operators $Y_i = \mathbf{i}^+$, $Y_i^- = \mathbf{i}^-$, where

$$(A \cdot F)(a_1, \ldots, a_n) = a_i F(a_1, \ldots, a_n) \text{ and} \tag{6.60}$$

$$(Y_i^{\pm} \cdot F)(a_1, \ldots, a_n) = F(a_1, \ldots, a_i \pm 1, \ldots, a_n). \tag{6.61}$$

Let $\mathcal{A}_{1,\ldots,n}$ be the algebra generated by shift operators Y_i^+ and multiplication operators A_i. It acts on the field of functions $\mathcal{F}$ of n integer variables. One can choose certain elements f_i corresponding to IBP relations and write

$$(f_i \cdot F)(a_1, \ldots, a_n) = 0. \tag{6.62}$$

The choice of elements f_i is not unique, we will choose them so that they do not depend on Y_i^-. Consider the left ideal $\mathcal{I} \subset \mathcal{A}_{1,\ldots,n}$ generated by the elements f_i. This ideal is named as the ideal of IBP relations. For any element $X \in \mathcal{I}$ we have

$$(XF)(1, 1, \ldots, 1) = 0. \tag{6.63}$$

Also we have

$$F(a_1, \ldots, a_n) = (Y_1^{a_1-1} \ldots Y_n^{a_n-1} F)(1, 1, \ldots, 1). \tag{6.64}$$

The idea of the algorithm is to reduce the operator on the right-hand side of (6.64) using the elements of the ideal $\mathcal{I}$. Suppose we are interested in $F(a_1, a_2, \ldots, a_n)$. The reduction problem becomes equivalent to reducing the monomial $Y_1^{a_1-1} \ldots Y_n^{a_n-1}$ modulo the ideal of the IBP relations. After obtaining an expression like

$$Y_1^{a_1-1} \ldots Y_n^{a_n-1} = \sum r_i f_i + \sum c_{i_1,\ldots,i_n} Y_1^{i_1-1} \ldots Y_n^{i_n-1} \tag{6.65}$$

it is left to apply (6.65) to F at $a_1 = 1, \ldots, a_n = 1$ and obtain the following expression:

$$F(a_1, \ldots, a_n) = \sum c_{i_1,\ldots,i_n} F(i_1, \ldots, i_n), \tag{6.66}$$

where integrals on the right-hand side are master integrals. (Here the formulae (6.62) and (6.64) are used.)

To do the reduction we need an ordering of monomials of operators Y_i or, similarly, an ordering of points $(a_1, \ldots, a_n)$ in the sector: (for two monomials $M_1 = Y_1^{i_1-1} \ldots Y_n^{i_n-1}$ and $M_2 = Y_1^{j_1-1} \ldots Y_n^{j_n-1}$ we have $(M_1 \cdot F)(1, \ldots, 1) \succ (M_2 \cdot F)(1, \ldots, 1)$ if and only if $M_1 \succ M_2$). Then the reduction procedure becomes similar to the division of polynomials. But one needs to introduce a proper ordering.

A monomial is defined by its degree, i.e. a set of n non-negative integers ($\mathbb{N}^n$). Thus defining an ordering on monomials is equivalent to defining an ordering on $\mathbb{N}^n$. We require the following properties:

(i) for any $a \in \mathbb{N}^n$ not equal to $(0, \ldots 0)$ one has $(0, \ldots 0) \prec a$;
(ii) for any $a, b, c \in \mathbb{N}^n$ one has $a \prec b$ if and only if $a + c \prec b + c$.

For example, the lexicographical ordering is defined the following way: a set $(i_1, \ldots, i_n)$ is said to be higher than a set $(j_1, \ldots, j_n)$ (denoted by $(i_1, \ldots, i_n) \succ (j_1, \ldots, j_n)$) if there is $l \leq n$ such that $i_1 = j_1, i_2 = j_2, \ldots, i_{l-1} = j_{l-1}$ and $i_l > j_l$. The degree-lexicographical ordering is $(i_1, \ldots, i_n) \succ (j_1, \ldots, j_n)$ if $\sum i_k > \sum j_k$, or $\sum i_k = \sum j_k$ and $(i_1, \ldots, i_n) \succ (j_1, \ldots, j_n)$ in the sense of the lexicographical ordering.

An ordering can be defined by a non-degenerate $n \times n$ matrix $(a_{k,l})$: for two sets of numbers $(i_1, \ldots, i_n)$ and $(j_1, \ldots, j_n)$ one first compares $\sum_l i_l a_{1,l}$ and $\sum_l j_l a_{1,l}$. If the first number is greater, then the first degree is greater; if the first number is smaller, then the first degree is smaller; and if those numbers are equal we compare $\sum_l i_l a_{2,l}$ and $\sum_l j_l a_{2,l}$ and so on. For example, the following matrices correspond to a lexicographical, a degree-lexicographical and a reverse degree-lexicographical ordering for $n = 5$:

$$\begin{pmatrix} 1&0&0&0&0 \\ 0&1&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \end{pmatrix}, \quad \begin{pmatrix} 1&1&1&1&1 \\ 1&0&0&0&0 \\ 0&1&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \end{pmatrix}, \quad \begin{pmatrix} 1&1&1&1&1 \\ 1&1&1&1&0 \\ 1&1&1&0&0 \\ 1&1&0&0&0 \\ 1&0&0&0&0 \end{pmatrix}$$

Such an approach encounters the following problem: the reduction does not always lead to a reasonable number of irreducible integrals, so one has to build a special basis of the ideal first. Obviously having elements with lowest possible degrees is equivalent to obtaining master integrals with minimal possible degrees. Therefore one needs to build special bases. This can be done by an algorithm based on the Buchberger algorithm with the use of S-polynomials and reductions [15].

Moreover, one has to keep in mind that we are interested in integrals not only in the positive sector. The algorithm of [57, 58] aims to build a set of special bases of the ideal of IBP relations (s-bases). The idea is to consider the algebra $\mathcal{A}_\nu$ generated by operators A_i and operators Y_i^+ for $i \in \nu$ and Y_i^- for $i \notin \nu$. Then for σ_ν one again considers the ideal of IBP relations in $\mathcal{A}_\nu$. Now one has to construct *sector bases*

(s-bases), rather than Gröbner bases for all sectors, where an s-basis for a sector σ_ν is a set of elements of $\mathcal{I}$ which provides the possibility of a reduction to master integrals in σ_ν and integrals whose indices lie in lower sectors, i.e. $\sigma_{\nu'}$ for $\nu' \subset \nu$.

This leads to considering many sectors—seemingly the problem becomes harder. But the important simplification is that one is not trying to solve the reduction problem in each sector separately but allows to reduce the integrals in a given sector to lower sectors—similarly to the 'by hand' solutions. It is also worth noting that it is most complicated to construct s-bases for minimal sectors.

The construction of s-bases is close to the Buchberger algorithm but it can be terminated when the 'current' basis already provides the needed reduction. The basic operations are the same, i.e. calculating S-polynomials and reducing them modulo current basis, with a chosen ordering. After constructing s-bases for all non-trivial sectors one obtains a recursive (with respect to the sectors) procedure to evaluate $F(a_1, \ldots, a_n)$ at any point and thereby reduce a given integral to master integrals.

This algorithm was applied in practice to the reduction of a family of three-loop Feynman integrals necessary for the analysis of decoupling of c-quark loops in b-quark HQET [34]. The main difficulty in applying the algorithm in more complicated situations (starting from ten indices) is that the construction of an s-basis for a given sector is not guaranteed. It turns out that it is very difficult to find an optimal ordering for this procedure. This problems is still open at the moment. There is however a way to avoid this problem: to combine this approach with the Laporta's algorithm, i.e. to construct s-bases in sectors where this construction is easy and non time and memory consuming and turn to Laporta's algorithm in the rest of the sectors. This option exists in the public code `FIRE` [55, 75].

6.7 Obtaining Additional Relations

If we deal with one family of Feynman integrals we can use a criterion [5, 6] based on Baikov's method to prove that the master integrals we have revealed are true master integrals, in particular, their number is minimal. If we deal with several families of Feynman integrals (for example, relevant to a given physical problem) we take the union of the sets of the master integrals corresponding to the individual families. (It often happens that some master integrals belong to different families.) Then we could try to find relations in this united family. In this section, two examples of finding such relations are presented.

The first example is given by the diagram of Fig. 6.4a. This is an integral with the numerator $k \cdot p$. It is denoted by I_{11} in [44] and belongs to the set of the master integrals contributing to the three-loop $g-2$ factor. It was present, in addition to the corresponding master integral I_{10} without numerator. Indeed, if one runs an IBP reduction for such integral with numerators one obtains two master integrals in the upper sector, i.e. with positive four indices associated with the propagators. Later it was observed in [45] that I_{11} is a linear combination of the integrals I_{14} and I_{18} (or, J_{14} and J_{18} in the notation of [45]). This linear connection was present in [45]

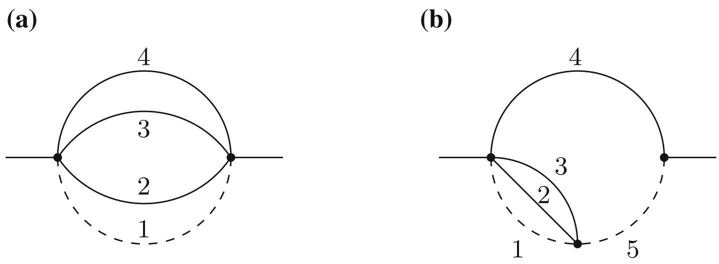

Fig. 6.4 The integral I_{11} (**a**) and the auxiliary diagram (**b**) used for its reduction

with the coefficient at I_{14} expanded in ε up to a certain power, rather than exactly at general d. In [48] this relation was presented at general d:

$$I_{11} = \frac{2d-5}{2(d-2)} I_{14} - \frac{1}{4} I_{18}. \tag{6.67}$$

In the notation of [48], we have $I_{14} = G_{4,4}$ and $I_{18} = G_3$.

The relation (6.67) can be derived using a symmetry of Feynman integrals. In the case of Feynman integrals connected with I_{11} no symmetry can help to reduce the number of the master integrals and we have two master integrals in the highest sector. However, if our goal is to reduce the number of master integrals for several families of Feynman integrals considered together we can profit from a symmetry. To reduce I_{11} it is enough to consider the family of Feynman integrals corresponding to the graph Fig. 6.4b. In particular, we have $F(1, 2, 1, 1, 1) - F(1, 1, 2, 1, 1) = 0$. However this relation is automatically satisfied after applying an IBP reduction. It turns out that a missing relation can be revealed at the next level of indices: if we reduce $F(1, 2, 1, 2, 1) - F(1, 1, 2, 2, 1) = 0$ to the master integrals we indeed obtain an equation which leads to (6.67).

Let us turn to one more way to obtain extra relations between master integrals. using the diagram of Fig. 6.5a considered at $p^2 = m^2$ as an example. In [39] it was shown that a linear combination of the three master integral in the highest sector of

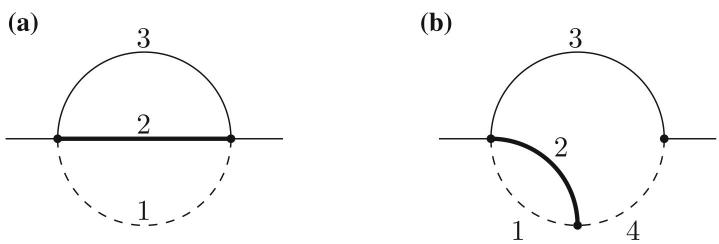

Fig. 6.5 A two-loop diagram with the masses 0, M, m (**a**) and the auxiliary diagram (**b**) used for its reduction

the family of integrals associated with Fig. 6.5a is a function which is expressed in terms of gamma functions at general d. It was observed that this function is given by a two-loop vacuum diagram with two zero masses. This analysis was based on explicit representations of Feynman integrals for Fig. 6.5a in terms of hypergeometric functions and recurrence relation between hypergeometric functions. Later such relation was derived in [41] using a trick with an introduction of an auxiliary mass and, in [40] using recurrence relations between MB integrals representing Fig. 6.5a.

It turns out that this extra relation can be derived using IBP reduction and differentiation with respect to M. Similarly to the previous example, let us consider a diagram with one more propagator depicted in Fig. 6.5b. Using the numbering in this figure let us consider $F(1, 1, 1, 2, 0)$. First, we reduce it to master integrals. Second, we use the fact that $F(1, 1, 1, 2, 0) = -\frac{\partial}{\partial M^2} F(1, 1, 1, 1, 0)$. So, we reduce $F(1, 1, 1, 1, 0)$ to master integrals and then take a derivative in M^2. Then we take the difference of the two results and straightforwardly arrive at the relation

$$
\begin{aligned}
&(3d-8)F(1,1,1,0,0) + 4m^2 F(1,1,2,0,0)\\
&+ 2M^2 F(1,2,1,0,0) + (2-d)F(1,1,0,1,0) = 0
\end{aligned}
\tag{6.68}
$$

which is noting but the additional relation of [39].

6.8 Conclusion

As a result of an IBP reduction one obtains, for any given integral of a family under consideration, a result of the form (6.36), with coefficient functions c_i which are rational functions of everything. For example, irreducible quadratic polynomials in d in the denominator are forbidden because they would show the presence of poles with a non-zero imaginary part. However, the analysis of convergence of Feynman integrals performed in Sect. 4.4 shows that the poles in ε (and d) can be present only at real points. Therefore, if one obtains a quadratic polynomial in d in the denominator of a coefficient at a master integral, this means that there are some relations between the master integrals revealed before this. After finding such relations, for example, using some symmetries (see the previous section), one has to obtain a reduction with rational coefficients.

It is usually desirable to turn to a set of master integrals whose coefficient functions are not singular at $\varepsilon = 0$—see [16] for a discussion of this idea and its practical implementations. Another natural requirement oriented at numerical calculations is to use, together with the IBP relations, some additional relations in order to build a basis in the situation where all integrals are evaluated up to order $O(\varepsilon)$. See [26], where this idea was algorithmically implemented for planar integrals at two loops, using additional relations with appropriately chosen Gram determinants.

The power of public and private codes of IBP reduction is permanently increasing. The complexity of problems of IBP reduction can be measured in terms of the number of indices and the number of kinematical invariants. For example, the IBP reduction for four-loop propagator integrals, i.e. with fourteen indices, looked hardly feasible ten years ago. Now, it is a relatively simple problem, not only for an implementation of the Baikov's method applied in [7–9] but also for some other private codes. At the moment, some codes can work with eighteen indices (for example, massless vertex integrals with two end-points on the light cone) and even with twenty indices (massless five-loop propagators). Well, at least, up to some level of complexity of integrals of corresponding families.

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Chapter 7
Evaluation by Differential Equations

In contrast to the method of alpha parametric representation, the method of MB representation and many other methods of evaluating individual Feynman integrals, the two methods presented in this and subsequent chapter are oriented at the evaluation of master integrals. This means that we have a solution of the IBP relations [21] for a given family of Feynman integrals, using some technique described in the previous chapter.

The method of differential equations (DE) was suggested in [31–35] and developed in [46] and later papers (see references below). The idea of the method is to take some derivatives of a given master integral with respect to kinematical invariants and masses. Then the result of this differentiation is written in terms of Feynman integrals of the given family and, according to the known reduction, in terms of the master integrals. Therefore, one obtains a system of differential equations for the master integrals which can be solved with appropriate boundary conditions.

We will consider typical one-loop examples in Sect. 7.1, a two-loop example in Sect. 7.2 and a three-loop example in Sect. 7.3. The status of the method, i.e. its perspectives and open problems will be discussed in Sect. 7.4, together with a brief review of its applications.

7.1 One-Loop Examples

Of course, we start with our favourite example.

Example 7.1 One-loop propagator diagram corresponding to Fig. 1.1.

After solving the corresponding reduction problem in Chap. 6, we know that there are two master integrals, $F(1, 1) = I_1$ and $F(1, 0) = I_2$. The second one is a simple one-scale integral given by the right-hand side of (6.6). We have started to evaluate I_1 in Chap. 1, by differentiating in m^2 and arrived at the Eq. (1.23) for $f(m^2) = F(1, 1)$. To be very pedantic, let us rewrite it in terms of our true master integrals,

V. A. Smirnov, *Analytic Tools for Feynman Integrals*, Springer Tracts in Modern Physics 250, DOI: 10.1007/978-3-642-34886-0_7,

$$\frac{\partial}{\partial m^2} f(m^2) = \frac{1}{m^2-q^2}\left[(1-2\varepsilon) f(m^2) \frac{1-\varepsilon}{m^2} I_2\right], \tag{7.1}$$

although this does not make an essential difference here.

Let us turn to the new function by $f(m^2) = \mathrm{i}\pi^{d/2}(m^2)^{-\varepsilon} y(m^2)$. We obtain the following differential equation for it:

$$y' - \frac{m^2(1-\varepsilon)-\varepsilon q^2}{m^2(m^2-q^2)} y = -\frac{\Gamma(\varepsilon)}{m^2-q^2}. \tag{7.2}$$

It can be solved by the method of the variation of the constant. The general solution to the corresponding homogeneous equation, with a zero on the right-hand side of (7.2), is

$$y(m^2) = C(m^2-q^2)^{1-2\varepsilon}(m^2)^{-\varepsilon}. \tag{7.3}$$

Then we make $C = C(m^2)$ dependent on m^2, solve this equation and obtain

$$f(m^2) = \mathrm{i}\pi^{d/2}(m^2-q^2)^{1-2\varepsilon}\left[-\Gamma(\varepsilon)\int_0^{m^2} \frac{\mathrm{d}x\; x^{-\varepsilon}}{(x-q^2)^{2-2\varepsilon}} + C_1\right], \tag{7.4}$$

where the constant C_1 can be determined from the boundary value $f(0)$ which is a massless one-loop diagram evaluated by means of (10.7). This gives

$$\begin{aligned} f(m^2) = &-\mathrm{i}\pi^{d/2}(m^2-q^2)^{1-2\varepsilon}\Gamma(\varepsilon) \\ &\times\left[\int_0^{m^2} \frac{\mathrm{d}x\; x^{-\varepsilon}}{(x-q^2)^{2-2\varepsilon}} - \frac{\Gamma(1-\varepsilon)^2}{\Gamma(2-2\varepsilon)(-q^2)^{1-\varepsilon}}\right]. \end{aligned} \tag{7.5}$$

If we turn to expansion in ε and take terms up to ε^0 into account we will reproduce (1.7).

The next example is also an old one.

Example 7.2 The triangle diagram of Fig. 3.5.

The only master integral that is not expressed in terms of gamma functions for general d is $F(1,1,1) = I_1 = f(m^2)$. We have already calculated it in Example 3.2. Let us now do this by DE. As in the previous example, we take the derivative $\frac{\partial}{\partial m^2} f(m^2)$ and obtain $F(1,1,2)$ which can be reduced to the master integrals:

$$\begin{aligned} F(1,1,2) = \frac{1}{m^2(m^2-Q^2)}\Big[&\frac{1}{2}(d-4)(2m^2-Q^2)I_1 \\ &+(d-3)I_2 + \frac{2-d}{2m^2} I_3\Big]. \end{aligned} \tag{7.6}$$

Let us again, as above, confine ourselves to the evaluation up to the finite part in ε. Then the first term on the right-hand side of (7.6) is irrelevant because it is proportional to ε. So, we obtain, at $\varepsilon = 0$,

$$\frac{\partial}{\partial m^2} f(m^2) = \mathrm{i}\pi^2 \frac{\ln(m^2/Q^2)}{m^2(m^2 - Q^2)}. \tag{7.7}$$

Thus, the evaluation of I_1 at $d = 4$ reduces to taking an integral of the right-hand side of (7.7). The boundary condition is simple: this function vanishes in the large mass limit. This can be seen, for example, by examining this behaviour Consequently, the known result (3.25) is once again reproduced.

If one needs to evaluate I_1 at general ε, or obtain higher terms of expansion in ε by DE, one can start from (7.6) and solve the so-obtained differential equation, applying the method of the variation of the constant quite similarly to Example 7.1.

Let us now turn, following [12], to

Example 7.3 The on-shell box diagram with two massive and two massless lines shown in Fig. 7.1, with $p_1^2 = \ldots = p_4^2 = m^2$.

These are functions of the three variables s, t and m^2. The following combinations arise naturally in the problem:

$$x = \frac{\sqrt{4m^2 - s} - \sqrt{-s}}{\sqrt{4m^2 - s} + \sqrt{-s}}, \quad y = \frac{\sqrt{4m^2 - t} - \sqrt{-t}}{\sqrt{4m^2 - t} + \sqrt{-t}}. \tag{7.8}$$

We again assume that we know a solution of the corresponding reduction problem. The reduction based on the Laporta algorithm [26, 27, 37, 39, 40] leads [12] to the family of the master integrals shown in Fig. 7.2: $I_1 = F(1, 1, 1, 1)$, $I_2 = F(1, 0, 1, 1) = F(0, 1, 1, 1)$, $I_{31} = F(1, 1, 0, 0)$, $I_{32} = F(0, 0, 1, 1)$ and $I_4 = F(1, 0, 0, 0) = F(0, 1, 0, 0)$, where I_2 and I_4 are present in two copies.

Suppose that we want to evaluate I_1 by DE. Therefore, we assume that all the master integrals with the number of lines less than four are already known. The integrals I_4 and I_{32} are given by (2.47) and (3.8). The value of the master integral $I_{31} = F(1, 1, 0, 0)$ is very well-known and can be obtained by various methods. To be self-consistent, let us observe that one can apply MB representation (5.15), set $a_1 = a_2 = 1, a_3 = 0$ and evaluate this integral by closing the integration contour

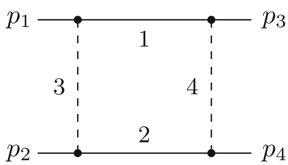

Fig. 7.1 On-shell box with two massive and two massless lines. The *solid lines* denote massive, the *dashed lines* massless particles

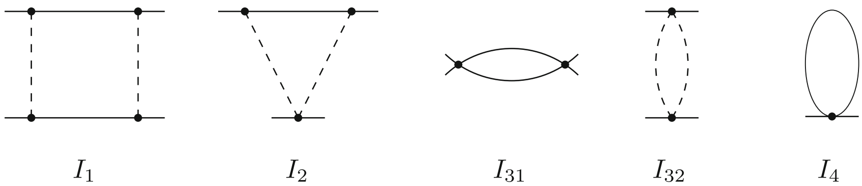

Fig. 7.2 Master integrals for Fig. 7.1

and summing up the resulting series. Within the method of DE, it is important to present this and later results in terms of the variables (7.8):

$$I_{31} = \frac{\mathrm{i}\pi^{d/2} \mathrm{e}^{-\gamma_{\mathrm{E}}\varepsilon}}{(m^2)^\varepsilon} \left[\frac{1}{\varepsilon} + 2 - 2\left(\frac{1}{2} - \frac{1}{1-x}\right) H_0(x) \right] + O(\varepsilon). \tag{7.9}$$

Here and in subsequent formulae, usual logarithms and polylogarithms are written in terms of HPLs [48]—see Appendix B. Moreover, it is necessary to rewrite the quantity q^2 in (3.8) in terms of these variables, i.e. make the substitution $q^2 \to t \to -(1-y)^2/(m^2 y)$ in the factor $(-q^2)^\varepsilon$ and then expand it in ε.

Finally, we need I_2 which can be obtained using (5.16) at $a_1 = a_2 = a_4 = 1$ and evaluating this integral by closing the integration contour to the right. In [12], this result was obtained by DE. It is also naturally written in terms of the variables (7.8):

$$I_2 = \frac{\mathrm{i}\pi^2}{2m^2} \left[\frac{1}{1+y} - \frac{1}{1-y} \right] \left[\frac{2}{3}\pi^2 + H_{0,0}(y) + 2H_{0,1}(y) \right] + O(\varepsilon). \tag{7.10}$$

Observe that higher terms of this and other expansions in ε can be found in [12].

The starting point is to take derivatives in s or t and write them down as linear combinations of integrals of the given class. In order to do this, one observes that taking derivatives in the external momenta reduces to taking derivatives in s and t:

$$p_i \cdot \frac{\partial}{\partial p_j} = \sum_{r=1}^{6} p_i \cdot \frac{\partial s_r}{\partial p_j} \frac{\partial}{\partial s_r}, \tag{7.11}$$

where $s_i = p_i^2$, $i = 1, 2, 3, 4$, are invariants with the on-shell condition, $s_i = m^2$, and $s_5 = s$, $s_6 = t$. This linear system of six equations can easily be solved, i.e. the derivatives $\partial/\partial s_r$ can be expressed linearly in terms of the derivatives $p_i \cdot \partial/\partial p_j$ with $i, j = 1, 2, 3$—see [12].

One can use here the following expressions [22] which are equivalent to that of [12] due to the on-shell conditions:

$$s\frac{\partial}{\partial s} = \frac{1}{2} \left[p_1 + p_2 - \frac{s}{4m^2 - s - t}(p_2 + p_3) \right] \cdot \frac{\partial}{\partial p_2}, \tag{7.12}$$

$$t\frac{\partial}{\partial t} = \frac{1}{2}\left[p_1 + p_3 - \frac{t}{4m^2 - s - t}(p_2 + p_3)\right]\cdot\frac{\partial}{\partial p_3}. \tag{7.13}$$

So, we take partial derivatives of $I_1 = f(s,t)$ with respect to s and t, using (7.12) and (7.13), and obtain, on the right-hand side, a linear combination of integrals corresponding to Fig. 7.1. Every integral can be written in terms of the master integrals, according to the reduction procedure, and we obtain

$$\frac{\partial f}{\partial s} = -\frac{1}{2}\left(\frac{1}{s} + \frac{d-5}{4m^2 - s} - \frac{d-4}{4m^2 - s - t}\right) f + g_1, \tag{7.14}$$

$$\frac{\partial f}{\partial t} = \frac{1}{2}\left(\frac{d-6}{t} + \frac{d-4}{4m^2 - s - t}\right) f + g_2, \tag{7.15}$$

where

$$\begin{aligned} g_1 = &-(d-4)\left[\frac{1}{4m^2 s} - \frac{4m^2 - t}{4m^2 t(4m^2 - s)} + \frac{1}{t(4m^2 - s - t)}\right] I_2 \\ &+ \frac{2(d-3)}{t}\left[\frac{1}{(4m^2 - s)^2} + \frac{1}{t(4m^2 - s)} - \frac{1}{t(4m^2 - s - t)}\right] I_{31} \\ &- \frac{d-3}{2m^2 - t}\left[\frac{1}{s} + \frac{1}{4m^2 - s}\right] I_{32} \\ &+ \frac{d-2}{m^2 t}\left[\frac{1}{(4m^2 - s)^2} + \frac{1}{t(4m^2 - s)} - \frac{1}{t(4m^2 - s - t)}\right] I_4, \end{aligned} \tag{7.16}$$

$$\begin{aligned} g_2 = &-\frac{d-4}{4m^2 - s}\left[\frac{1}{t} + \frac{1}{4m^2 - s - t}\right] I_2 \\ &- \frac{2(d-3)}{(4m^2 - s)^2}\left[\frac{1}{t} + \frac{1}{4m^2 - s - t}\right] I_{31} \\ &- \frac{d-2}{m^2(4m^2 - s)^2}\left[\frac{1}{t} + \frac{1}{4m^2 - s - t}\right] I_4. \end{aligned} \tag{7.17}$$

It is sufficient to use one of the two equations to evaluate $f(s,t)$. Let it be (7.14). Then (7.15) can be used for a non-trivial check. One needs also a boundary condition when solving (7.14): it can be obtained using the fact that the function $f(s,t)$ is regular at $s = 0$. Multiplying (7.14) by s and taking the limit $s \to 0$ one obtains

$$f(0,t) = -\frac{d-4}{2m^2} I_2 + \frac{d-3}{m^2 t} I_{32}. \tag{7.18}$$

Equation (7.14) can be solved in a Laurent expansion in ε,

$$f(s,t) = \sum_{j=-1} f_j(s,t)\varepsilon^j. \tag{7.19}$$

As a result, one obtains a set of nested differential equations from (7.14),

$$\frac{\mathrm{d}f_j}{\mathrm{d}s} = -\frac{1}{2}\left(\frac{1}{s} + \frac{1}{4m^2 - s}\right) f_j + h_j, \tag{7.20}$$

where the functions h_j involve, in addition to the corresponding term of the expansion of the function g_1, a piece coming from f_{j-1}. These equations can be solved by the method of the variation of the constant.

The homogeneous equation corresponding to (7.20), which is the same for all f_j, takes the following form in the new variable x given by (7.8):

$$\left(\frac{\mathrm{d}}{\mathrm{d}x} - \frac{1}{x} + \frac{1}{1+x} - \frac{1}{1-x}\right) f^{(0)}(x) = 0, \tag{7.21}$$

with the solution

$$f^{(0)}(x) = \frac{x}{(1-x)(1+x)}. \tag{7.22}$$

Then the solution of the jth differential equation in (7.20) can be written as

$$f_j(x, y) = f^{(0)}(x)\left[A_j + \int \mathrm{d}x \frac{h_j(x, y)}{f^{(0)}(x)}\right], \tag{7.23}$$

where A_j is a constant which can be fixed by imposing the boundary condition (7.18) expanded in ε.

Observe that the combinations of the kinematical invariants involved on the right-hand side of (7.14) and (7.16) and, therefore, present in h_j can be represented as

$$4m^2 - s = m^2\frac{(1+x)^2}{x}, \quad 4m^2 - s - t = m^2\frac{(x+y)(1+xy)}{xy}. \tag{7.24}$$

After that the integration in (7.23), order by order in ε, becomes straightforward. All the quantities are prepared in such a form that the integration is taken in terms of HPLs of the next weight, also of the arguments x and y. So, one arrives at

$$I_1 = \frac{\mathrm{i}\pi^{d/2}\mathrm{e}^{-\gamma_{\mathrm{E}}\varepsilon}}{(m^2)^{2+\varepsilon}}\left[\frac{1}{1+x} - \frac{1}{1-x}\right]\left[\frac{1}{1-y} - \frac{1}{(1-y)^2}\right] H_0(x)$$
$$\times\left[\frac{1}{\varepsilon} + H_0(y) + 2H_1(y)\right] + O(\varepsilon). \tag{7.25}$$

Further terms of this expansion in ε can be found in [12].

7.2 Two-Loop Example

We turn again to Feynman integrals appeared in Example 3.6.

Example 7.4 Sunset diagram of Fig. 3.13 with one zero mass and two equal non-zero masses at a general value of the external momentum squared.

The general Feynman integral of this family is given by

$$F(a_1, a_2, a_3, a_4, a_5) = \int\int \frac{\mathrm{d}^d k \mathrm{d}^d l \; (2q\cdot k)^{-a_3}(2q\cdot l)^{-a_4}}{(k^2-m^2)^{a_1}(l^2-m^2)^{a_2}[(q-k-l)^2]^{a_5}}, \tag{7.26}$$

so that there are two irreducible numerators in the problem. According to a solution of the IBP reduction problem there are three master integrals, $I_1 = F(1,1,0,0,1)$, $\bar{I}_1 = F(1,1,-1,0,1)$ and $I_2 = F(1,1,0,0,0)$. The last of them is the square of the massive tadpole given by the right-hand side of (2.47). Let us now evaluate I_1 and $\bar{I}_1$ by DE. For convenience, let us use, instead of $\bar{I}_1$, the integral with $a_1 = a_2 = a_5 = 1$ and the numerator equal to the product of the momenta (flowing in the same direction) of the massless and one of the massive lines,

$$\tilde{I}_1 = \frac{1}{2}\left(q^2 I_1 - \bar{I}_1 - I_2\right). \tag{7.27}$$

We start with taking derivatives. We use the homogeneity of the integrals I_1 and $\tilde{I}_1$ with respect to q^2 and m^2, with the help of Euler's theorem, set $q^2 = s$ and obtain

$$s f'(s) = (1-2\varepsilon) f(s) - \frac{\partial}{\partial m^2} f(s), \tag{7.28}$$

$$s \tilde{f}'(s) = 2(1-\varepsilon)\tilde{f}'(s) - \frac{\partial}{\partial m^2}\tilde{f}(s), \tag{7.29}$$

where $f(s) = I_1$ and $\tilde{f}(s) = \tilde{I}_1$, and we have already put $m^2 = 1$ after differentiating with respect to the mass which results in indices equal to 2 instead of 1 on one of the massive lines. The IBP reduction gives

$$F(2,1,0,0,1) = \frac{1}{m^2(4m^2-q^2)}\Big[\big((d-3)m^2 - (d-2)q^2\big) I_1 + \frac{3}{2}(d-2)\bar{I}_1 + \frac{1}{2}(d-2) I_2\Big], \tag{7.30}$$

$$F(2,1,-1,0,1) = \frac{2}{4m^2-q^2}\Big[-\big(2(d-3)m^2 + (d-1)q^2\big) I_1 + 3(d-2)\bar{I}_1 + (d-2) I_2\Big]. \tag{7.31}$$

$$F(2,1,0,-1,1) = \frac{1}{m^2(4m^2-q^2)}\Big[\big(4(d-3)m^4-(d-2)(q^2)^2\big)I_1 \\ +\frac{3}{2}(d-2)q^2\bar{I}_1-(d-2)(2m^2-q^2)I_2\Big]. \tag{7.32}$$

As a result we arrive at the following differential equations for the functions $f(s)$ and $\tilde{f}(s)$:

$$sf'(s) = \frac{1}{s-4}\Big[\big(3s-2-4\varepsilon(s-1)\big)f(s) \\ +4(\varepsilon-1)(h(s)+3\tilde{f}(s))\Big], \tag{7.33}$$

$$s\tilde{f}'(s) = \frac{1}{2}(\varepsilon-1)\Big[h(s)-sf(s)+2\tilde{f}(s)\Big], \tag{7.34}$$

where h originates from I_2.

As in the previous example, it is convenient to turn to the new variable x given by (7.8), or, vice versa,

$$s = -\frac{(1-x)^2}{x}. \tag{7.35}$$

Then we obtain the following equations:

$$f'(x) = \frac{1}{x(x^2-1)}\Big[\big(3-4x+3x^2-4\varepsilon(1-x+x^2)\big)f(x) \\ -4(\varepsilon-1)x(h(x)+3\tilde{f}(x))\Big], \tag{7.36}$$

$$\tilde{f}'(x) = \frac{1}{2x^2(x-1)}(\varepsilon-1)(1+x) \\ \times\Big[(x-1)^2f(x)+x(h(x)+2\tilde{f}(x))\Big]. \tag{7.37}$$

The second function $\tilde{f}(x)$ can be eliminated from this system in order to obtain a separate equation for the first one:

$$f''(x)+\frac{(3\varepsilon(x-1)^2+6x-2)}{x(x^2-1)}f'(x) \\ +\frac{(2\varepsilon-1)(2x+\varepsilon(1-4x+x^2))}{x^2(x-1)^2}f(x)+\frac{2(\varepsilon-1)^2}{x(x-1)^2}h(x)=0. \tag{7.38}$$

Then we turn to solving this equation in expansion in ε, as in the previous examples,

$$f(x) = \frac{f_{-2}(x)}{\varepsilon^2}+\frac{f_{-1}(x)}{\varepsilon}+f_0(x)+\cdots. \tag{7.39}$$

As usual, we need a general solution of the corresponding homogeneous equation at $\varepsilon = 0$:

$$f''(x) + \frac{2(3x-1)}{x(x^2-1)} f'(x) - \frac{2}{x(x-1)^2} f(x) = 0. \tag{7.40}$$

Two independent solutions are

$$\phi_1(x) = \frac{1-x+x^2}{(x-1)^2}, \tag{7.41}$$

$$\phi_2(x) = \frac{4x(1-x+x^2)H_0(x) - 1 + 7x - 3x^2 - x^3 + x^4}{x(x-1)^2}, \tag{7.42}$$

with the Wronskian

$$w(x) = \frac{(x+1)^4}{x^2(x-1)^2}. \tag{7.43}$$

The solutions are presented in a form similar to the previous example, in terms of HPLs.

The equation for f_{-2} has the inhomogeneous term

$$r_{-2}(x) = -\frac{2}{x(x-1)^2}. \tag{7.44}$$

Its solution is written as

$$\begin{aligned} f_{-2}(x) = &\left[c_1 - \int \mathrm{d}x \, \frac{\phi_2(x) r_{-2}(x)}{w(x)}\right] \phi_1(x) \\ &+ \left[c_2 + \int \mathrm{d}x \, \frac{\phi_1(x) r_{-2}(x)}{w(x)}\right] \phi_2(x), \end{aligned} \tag{7.45}$$

where c_1 and c_2 are integration constants. We obtain

$$\begin{aligned} f_{-2}(x) = &\frac{1}{x(x-1)^2} \Big[x(c_1(1-x+x^2) - x) - c_2(1 - 7x + 3x^2 + x^3 - x^4) \\ &+ 4c_2 x(1-x+x^2)H_0(x)\Big]. \end{aligned} \tag{7.46}$$

The integration constants are evaluated from the regular behaviour of the solution at $x \to 0$ so that $1/x$ and $\sqrt{x}$ in the asymptotic expansion of (7.46) are forbidden. This gives the values $c_1 = 1$ and $c_2 = 0$, with

$$f_{-2}(x) = 1. \tag{7.47}$$

The inhomogeneous term for $f_1(x)$ is

$$r_{-1}(x) = \frac{1 - 8x + x^2}{x^2(x-1)^2}. \tag{7.48}$$

Proceeding in a similar way we obtain the following solution:

$$f_{-1}(x) = \frac{1}{2x(x-1)^2}\Big[1 - 6x - x^2 - 2x^3 + 2c_1 x(1 - x + x^2) \\ -2c_2(1 - 7x + 3x^2 + x^3 - x^4) + 2(4c_2 - 1)x(1 - x + x^2)H_0(x)\Big]. \tag{7.49}$$

The regularity condition at $x = 0$ gives $c_1 = 13/4$ and $c_2 = 1/4$, with

$$f_{-1}(x) = \frac{1 + 10x + x^2}{4x}. \tag{7.50}$$

Finally, for f_0, we have the inhomogeneous term

$$r_0(x) = -\frac{3 - 9x + 2(48 + \pi^2)x^2 - 9x^3 + 3x^4}{6x^3(x-1)^2}. \tag{7.51}$$

Similarly, we obtain the following solution:

$$f_0(x) = \frac{1}{24x(x-1)^2}\Big[(x-1)^2(39 + 66x + 4\pi^2 x + 39x^2) \\ +12(1 - 4x + 4x^3 - x^4)H_0(x) - 48x(1 - x + x^2)H_{0,0}(x)\Big]. \tag{7.52}$$

The second function

$$\tilde{f} = \frac{\tilde{f}_{-2}(x)}{\varepsilon^2} + \frac{\tilde{f}_{-1}(x)}{\varepsilon} + \tilde{f}_0(x) + \cdots . \tag{7.53}$$

can be now obtained in a pure algebraic way, with the following results:

$$\begin{aligned}
\tilde{f}_{-2}(x) &= -\frac{1 + x^2}{4x}, \\
\tilde{f}_{-1}(x) &= -\frac{1 + 11x + 11x^3 + x^4}{24x^2}, \\
\tilde{f}_0(x) &= \frac{1}{48x^2(x-1)^2}\Big[-(x-1)^2\Big((2\pi^2 - 11)x(1 + x^2) \\
&\quad + 13(1 + x^4) + 44x^2\Big) - 4\Big(1 - 9x(1 - x^2)(1 - x + x^2) - x^6\Big)H_0(x) \\
&\quad + 24x(1 - 2x + 4x^2 - 2x^3 + x^4)H_{0,0}(x)\Big].
\end{aligned} \tag{7.54}$$

The corresponding result for the master integral $\bar{I}_1$ can be obtained easily from (7.47), (7.49), (7.52) and (7.54), using (7.27). These results are in agreement with [23, 24], where another choice of the master integrals was used (with higher powers of the propagators, instead of integrals with numerators).

7.3 Three-Loop Example

At least in higher loops, evaluating Feynman integrals by DE can hardly be performed without computer. There are no public codes for the method of DE and there is a lot of private codes. In fact, there can be no universal code for this method because every problem has its own peculiarities. Moreover, manipulations and results turn out to be cumbersome. Therefore I would like to describe only very schematically an example of a three-loop calculation, following [9], where all the twenty seven master integrals for three-loop on-shell QCD Feynman integrals with two masses were evaluated. These are propagators integrals with masses M, m and 0 and external on-shell momentum, $q^2 = M^2$.

The method of MB representation and the method of DE often participate in a competition. It turns out that sometimes one of these two methods works essentially better than the other method. In this calculation, the method of DE works very well for almost all the master integrals depending on the two non-zero masses, i.e. which are nontrivial functions of the ratio $x = m/M$. However, for the master integrals of Fig. 7.3 (integrals with the indices of lines equal to one as well as integrals with a dot on a line with the mass M) the method of DE works successfully up to order $1/\varepsilon$. When evaluating the finite part of the integrals by DE, one meets not only singularities of the form $1/x$ and $1/(1 \pm x)$, but also the singularities $1/(1 \pm 2x)$ (for $M_{5.2}$) and $1/(1 \pm x/2)$ (for $M_{5.3}$).

Thus, it was not possible to integrate the differential equations in terms of the usual HPLs with three weight functions[1] so that the finite part was evaluated by the method of MB representation.

In particular, the master integrals with the (maximal) number of lines equal to six shown in Fig. 7.4 were successfully evaluated by DE. Differential equations which

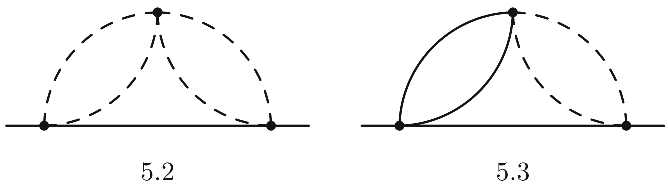

Fig. 7.3 Master integrals which pose difficulties within the method of DE. *Solid lines* denote propagators with the mass *M* and *dashed lines* propagators with the mass *m*

[1] Presumably, this can be done using multiple polylogarithms (11.43).

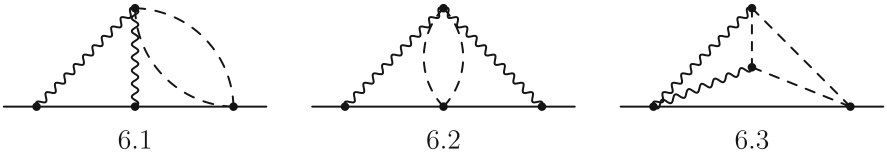

Fig. 7.4 Master integrals with six lines. *Wavy lines* denote massless propagators

arise in the problem are relatively simple. They are obtained by differentiating in the mass m which results in putting a dot on one of the corresponding lines. For example, for $M_{6.2}$, one obtains, after an IBP reduction, the following master integrals on the right-hand side of the differential equations: $M_{3.1}$, $M_{3.2}$, $M_{4.3}$, $M_{4.8}$, $M_{4.3a}$, $M_{4.8b}$. The first four of them are shown in Figs. 7.5 and 7.6. The integrals of Fig. 7.5 can be expressed explicitly in terms of gamma functions at general ε, using (10.1). The vacuum integral $M_{4.3a}$ differs from $M_{4.3}$ by a dot on a line with the mass M. (In fact, in the evaluation of [9], an integral with two dots was chosen.) The integral $M_{4.8b}$ differs from $M_{4.8}$ also by a dot on the line with the mass M. The integrals $M_{4.3}$, $M_{4.8}$, $M_{4.3a}$, $M_{4.8b}$ were evaluated also by DE—see [9]. To obtain the integration constants in differential equations, boundary conditions at $x = 1$ were used. For example, for $M_{6.2}$, the corresponding value could be taken from [39, 40, 43, 44].

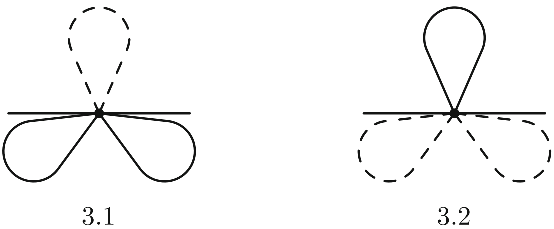

Fig. 7.5 Master integrals *lower* than $M_{6.2}$ with three lines

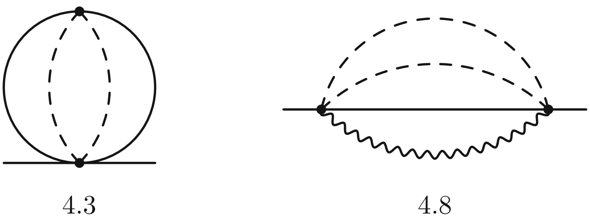

Fig. 7.6 Master integrals *lower* than $M_{6.2}$ with four lines

One naturally obtains a result in terms of HPLs

$$
\begin{aligned}
M_{6.2} = {} & \left(\mathrm{i}\pi^{d/2}\right)^3 M^{-6\varepsilon}\Gamma^3(1+\varepsilon)\left[\frac{1}{3\varepsilon^3}+\frac{7}{3\varepsilon^2}+\frac{31}{3\varepsilon}\right. \\
& + \frac{4}{3}\pi^2 x^2 H_0(x) + 16x^2 H_{-3,0}(x) - 8x^2 H_{-2,0}(x) + 4x^2 H_{-1,0}(x) \\
& - \frac{8}{3}\pi^2 x^2 H_{0,0}(x) - 8x^2 H_{0,0}(x) - 4x^2 H_{1,0}(x) + 8x^2 H_{2,0}(x) \\
& - 16x^2 H_{3,0}(x) - 16x^2 H_{-2,0,0}(x) + 8x^2 H_{-1,0,0}(x) - 8x^2 H_{1,0,0}(x) \\
& + 16x^2 H_{2,0,0}(x) - 4H_{-1,0}(x) + 4H_{1,0}(x) - 8H_{-1,0,0}(x) + 8H_{1,0,0}(x) \\
& \left. - \frac{4\pi^4 x^2}{15} - \frac{2\pi^2 x^2}{3} + \frac{8\zeta(3)}{3} + \frac{2\pi^2}{3} + \frac{103}{3} + O(\varepsilon)\right].
\end{aligned}
$$

Here one more term in ε as compared with [9] is presented. The ε^1 term as well as results for all the other master integrals can be found on the web page cited in [9].

7.4 Conclusion

At first sight, the method of DE cannot be applied to integrals dependent on one scale since the dependence on the only scale parameter is trivial and can be obtained immediately by power counting. However, one can introduce, for a one-scale integral, an additional scale parameter, apply the corresponding differential equation, get the boundary condition at a different, more suitable point and then return to the single scale value. An example of this strategy can be found in [8].

I admit that it might seem, from the previous examples,[2] that the method of DE is not optimal. In particular, the results for Example 7.4 can be, probably, derived by MB representation in a simpler way. However, the method of DE is very powerful indeed and, in some situations, the very best one. An important feature of the strategy outlined above is that it can straightforwardly be generalized to more complicated classes of multiloop Feynman integrals, with a computer implementation of all the steps. The method of DE, coupled with solving the IBP reduction problem by means of algorithms described in the previous chapter, in particular by Laporta's algorithm [26, 27, 37, 39, 40], has become a powerful industry for obtaining results for various phenomenologically important classes of Feynman integrals—see, e.g. [2–5, 7, 10–19, 25, 38, 41, 42, 47, 49, 50].

I think, the first impressive example of this technique was evaluating master integrals by DE for the massless double boxes with one leg off-shell, $p_1^2 \neq 0,\ p_2^2 = p_3^2 = p_4^2 = 0$, performed in [26, 27]. To express results obtained by DE for this family of integrals, the authors introduced two-dimensional HPLs (2dHPLs) [26, 27] which are natural generalizations of HPLs to the case of functions of two variables.

[2] Simple instructive examples can be found also in the reviews [1, 6, 36].

To define them [26, 27] one uses, instead of the functions (11.10), the following set of functions of the two variables x and y labelled by the four indices 0, -1, $-y$ and $-1/y$:

$$g(0; x) = \frac{1}{x}, \quad g(-1; x) = \frac{1}{1+x}, \quad g(-y; x) = \frac{1}{x+y}, \tag{7.55}$$

$$g(-1/y; x) = \frac{1}{x+1/y}. \tag{7.56}$$

Then 2dHPLs are defined as the set of functions generated by repeated integrations with these functions similarly to (11.9).

The 2dHPLs, as well as the HPLs themselves, turned out to be partial cases of multiple polylogarithms which is clear from the definition (11.43) presented in Appendix B. Still the introduction of 2dHPLs as new functions of mathematical physics, without referring to multiple polylogarithms, was quite useful. Indeed, in [28–30], one discovered, independently of mathematicians, basic properties of 2dHPLs and, moreover, developed packages for their numerical evaluation [28–30].

Here are two more examples where new functions were introduced: generalized HPLs in [4, 5] which were necessary to evaluate some two-loop massive Feynman diagrams and some generalized 2dHPLs [11] which were necessary to evaluate two-loop massless diagrams with three off-shell legs. These functions were not studied from the mathematical point of view, so that their introduction was used just to parameterize the results obtained. It turns out, however, that they are again partial cases of the well studied multiple polylogarithms. Indeed, in the first of these two cases, generalized HPLs were constructed recursively similar to HPLs with some new building functions, in addition to (11.10), in particular

$$g(\pm 4, x) = \frac{1}{4 \mp x} \quad \text{and} \quad g(\pm r, x) = \frac{1}{\sqrt{x(4 \mp x)}}.$$

So, such generalized HPLs have indices not only 0, ± 1 as the usual HPLs do, but also ± 4, $\pm r$ etc.

In particular, one has [45] the following expression in terms of multiple polylogarithms

$$\begin{aligned}
H_{0,0,-4,-r}(x) = \frac{1}{72}\mathrm{sgn}(y-1)\Big[&-108\zeta(3)G(0;y) + 216\zeta(3)G(1;y) \\
&+ 12\pi^2 G(0,0;y) - 24\pi^2 G(0,1;y) - 24\pi^2 G(1,0;y) + 48\pi^2 G(1,1;y) \\
&+ 144G(0,0,-1,0;y) - 72G(0,0,0,0;y) - 288G(0,1,-1,0;y) \\
&+ 144G(0,1,0,0;y) - 288G(1,0,-1,0;y) + 144G(1,0,0,0;y) \\
&+ 576G(1,1,-1,0;y) - 288G(1,1,0,0;y) + 72\pi^2 \ln 2\; G(0;y) \\
&- 144\pi^2 \ln 2\; G(1;y) + 576\mathrm{Li}_4\left(\frac{1}{2}\right) + \pi^4 + 24\ln^4 2 + 48\pi^2 \ln^2 2\Big)
\end{aligned}$$

with $x = (1-y)^2/y$. In the second of these two examples, generalized 2dHPLs [11] were introduced using building functions $1/\lambda$, $1/(x\lambda)$, with $\lambda \equiv \lambda(x, y) = \sqrt{(1-x-y)^2 - 4xy}$, and some other functions. However, all these generalized 2dHPLs can be expressed again in terms of multiple polylogarithms—see [20].

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Chapter 8
Evaluating Master Integrals by Dimensional Recurrence and Analyticity

In this chapter, the method called 'dimensional recurrence and analyticity' (DRA) is described following recent papers by Lee [24–26] and his papers with coauthors [27–32, 34]. It is based on so-called dimensional recurrence relations (DRR) which express a given master integral considered in dimension $d-2$ or $d+2$ as a linear combination of Feynman integrals in dimension d with shifted indices. In the next section, two kinds of such relations are described. As in the case of the method of differential equations it is assumed that one can perform an IBP reduction [12] for a given family of Feynman integrals. Using a solution of IBP relations with the help of the algorithms described in Chap. 6, the linear combinations on the right-hand sides of the DRR can be represented as linear combinations of master integrals so that we obtain a difference equation (or, a system of difference equations) with respect to the variable d. Then this equation is solved by finding its solution in the form of a series and fixing then the arbitrariness encoded in the solution of the corresponding homogeneous equation with the help of information about properties of the given Feynman integral as an analytic function of d.

The general procedure in the case of one master integral in a given sector is formulated in Sect. 8.2, following [24]. Section 8.3 contains various multiloop examples. (More examples can be found in [24–31, 34].) In Sect. 8.4, we consider, following [32], the case of several master integrals in a given sector, where matrix difference equations have to be solved. Finally, in Sect. 8.5, we speculate about possible irrational constants present in ε-expansions of one-scale Feynman integrals.

8.1 Dimensional Recurrence Relations

One can use two types of DRR [24, 44, 45] for master integrals. Let us suppose that a given master integral $F(d)$ has the form

V. A. Smirnov, *Analytic Tools for Feynman Integrals*, Springer Tracts in Modern Physics 250, DOI: 10.1007/978-3-642-34886-0_8,

$$F(a_1, \ldots, a_N; d) = \frac{1}{\pi^{hd/2}} \int \cdots \int \frac{\mathrm{d}^d k_1 \ldots \mathrm{d}^d k_h}{E_1^{a_1} \ldots E_N^{a_N}}, \tag{8.1}$$

where the dependence on masses and external momenta is suppressed and indices a_i are positive. Typically, one has $a_i = 1, 2, 3$. The overall factor is introduced in order to get rid of powers of π as coefficients in DRR. Let also suppose that the integral corresponds to a graph and that propagators are usual quadratic propagators. We will usually indicate only d as the argument of F and suppress a dependence on masses and kinematic invariants because d is the main variable in this approach.

To express the integral $F(d-2)$, i.e. in dimension shifted by 2, let us start from the alpha representation (3.36). The shift $d \to d-2$ in the overall coefficient results in the factor $\mathrm{e}^{-ih/2}/\pi$. As it was pointed out in Sect. 3.2.3 this shift in the integrand results in the additional factor $\mathcal{U}$ [44, 45]. Every monomial present in $\mathcal{U}$ is a product of some parameters α_l which can be taken into account by the corresponding product of factors $(-ia_l)\mathbf{l}^+$ involving shifting operators. Since the number of alpha parameters in every monomial equals h we obtain one more factor $\mathrm{e}^{-\mathrm{i}h/2}/\pi$. Using the explicit representation (2.25) of the function $\mathcal{U}$ and taking into account the normalization in (8.1) we arrive at the following formula

$$F(d-2) = (-1)^h \sum_T \prod_{l \notin T} a_l \mathbf{l}^+ F(d) \tag{8.2}$$

which can be qualified as the raising relation because the integrals on the right hand side are in dimension greater by two. Observe that, for Feynman integrals in Euclidean space, the factor $(-1)^h$ is absent.

A more general way to derive raising relations valid also for negative indices can be found in [25].

The lowering DRR can be obtained using manipulations which were used within Baikov's approach [1, 2] (see Sect. 6.5) where integrals over loop momenta are transformed into integrals over scalar products. We consider now the general master integral (2.40) where some indices can be negative. (Typically, one has $a_i = -1, -2$.) We suppose that the 'denominators' E_i form a complete basis in the linear space of scalar products $k_i \cdot k_j$ and $k_i \cdot p_j$ where k_i and $p_1, \ldots, p_n$ are the loop and external momenta, correspondingly. An h-loop Feynman integral with an integrand I is transformed as follows:

$$\frac{1}{\pi^{hd/2}} \int \cdots \int \mathrm{d}k_1 \ldots \mathrm{d}k_h \, I = \frac{(-1)^h \pi^{-hn/2 - h(h-1)/4}}{\Gamma\left((d-n-h+1)/2\right) \ldots \Gamma\left((d-n)/2\right)}$$
$$\times \int \left(\prod_{i=1}^{h} \prod_{j=i}^{h+n} \mathrm{d}s_{ij} \right) \frac{[V(k_2, \ldots k_h, p_1, \ldots, p_n)]^{(d-n-h-1)/2}}{[V(p_1, \ldots, p_n)]^{(d-n-1)/2}} I, \tag{8.3}$$

where

$$V\left(q_1,\ldots,q_M\right) = \begin{vmatrix} q_1^2 & \ldots & q_1\cdot q_M \\ \vdots & \ddots & \vdots \\ q_1\cdot q_M & \ldots & q_M^2 \end{vmatrix} \tag{8.4}$$

is the Gram determinant constructed with the vectors $q_1,\ldots,q_M$ and I is a function of $N = k(k+1)/2 + kn$ scalar products s_{ij}, and, in Euclidean space, the factor $(-1)^h$ is absent.

We choose $I = 1/(E_1^{a_1}\ldots E_N^{a_N})$, i.e the integrand in (2.40). Since the functions E_i form a basis, the function $V\left(l_2,\ldots l_L, p_1,\ldots,p_n\right)$ has the form of some polynomial of degree $h+n$ of $E_1,\ldots,E_N$:

$$V\left(k_2,\ldots k_L, p_1,\ldots,p_n\right) = P\left(E_1, E_2,\ldots,E_N\right). \tag{8.5}$$

Replacing $d \to d+2$ in (8.3), we obtain the lowering DRR

$$F\left(d+2\right)\left(a_1,\ldots,a_N\right) = \frac{(2\mu)^h\left[V\left(p_1,\ldots,p_n\right)\right]^{-1}}{(d-n-h+1)_h} \times \left(P\left(\mathbf{1}^-,\ldots,\mathbf{N}^-\right)F\left(d\right)\right)\left(a_1,\ldots,a_N\right), \tag{8.6}$$

where $\mathbf{i}^-$ is used again for the lowering operators.

The raising relation (8.2) looks preferable since the right-hand side of it contains integrals with indices shifted by at most the number of loops h, while the right-hand side of the lowering relation contains integrals with indices shifted by $h+n$. This difference can be important in complicated calculations because an IBP reduction of integrals appearing on the right-hand side can be too complicated in the case of the lowering relation. However, the lowering relation can be directly used also for master integrals with negative indices. On the other hand, the two kinds of the relations are equivalent so that one might use both of them for checks.

8.2 General Prescriptions

Suppose, we have to evaluate a master integral $F(d)$ which is the only master integral in a given sector. The procedure is recursive so that let us also suppose that master integrals in *lower* sectors (in the sense of the definition in Sect. 6.2) are already known. To evaluate $F(d)$ let us use the corresponding raising or lowering relation. To be specific, let us chose the raising relation. The next step is to perform an IBP reduction of integrals present on the right-hand side. As a result, we express $F(d-2)$ as a linear combination of master integrals considered in d dimensions, in particular, $F(d)$ itself. The case where no other master integrals are present is trivial and the given master integral can be evaluated in terms of gamma functions at general d. So, we obtain a difference equation of the following form

$$F\,(d-2) = C\,(d)\,F\,(d) + R\,(d)\,, \tag{8.7}$$

where $R\,(d)$ is the inhomogeneous part which is a linear combination of lower master integrals, i.e. in lower sectors. Coefficients at master integrals are rational functions of everything, i.e. of d, masses and kinematic invariants. Experience tells us that the coefficient at $F(d)$ itself can be represented as

$$C\,(d) = c\frac{\prod_{i=1}^{n}\,(b_i - d/2)}{\prod_{j=1}^{m}\left(b'_j - d/2\right)}. \tag{8.8}$$

where b_i and b'_i are numbers and a possible dependence on masses and kinematic invariants is hidden in the overall coefficient c.

As in the case of differential equations, it is reasonable, first, to solve, the corresponding homogeneous equation $F\,(d-2) = C\,(d)\,F\,(d)$. Its specific solution is represented as $1/\Sigma\,(d)$ where, similarly to an integrating factor in the theory of differential equations, the function $\Sigma\,(d)$ is called *a summing factor*. To construct solutions of the homogeneous equation one replaces linear factors in (8.8) by some gamma functions. One can check easily that a possible solution is

$$\Sigma^{-1}\,(d) = \,c^{-d/2}\frac{\prod_{i=1}^{n}\Gamma\,(b_i - d/2)}{\prod_{j=1}^{m}\Gamma\left(b'_j - d/2\right)}. \tag{8.9}$$

One more solution is

$$\Sigma^{-1}\,(d) = \left[(-1)^{n+m}\,c\right]^{-d/2}\frac{\prod_{j=1}^{m}\Gamma\left(d/2+1-b'_j\right)}{\prod_{i=1}^{n}\Gamma\,(d/2+1-b_i)}. \tag{8.10}$$

Starting from one summing factor we can obtain another one but multiplying the first one by any periodic function of d so that a general solution of the homogeneous equation has the form

$$F_0\,(d) = \omega\,(d)\,\,/\Sigma\,(d)\,, \tag{8.11}$$

where $\omega\,(d) = \omega\,(d+2)$ is an arbitrary periodic function of d and $\Sigma^{-1}\,(d)$ is a fixed non-zero solution.

Let us now consider the inhomogeneous Equation (8.7) and turn to a new unknown function by the substitution

$$F\,(d) = \Sigma^{-1}\,(d)\,g\,(d)\,. \tag{8.12}$$

After multiplying both parts by $\Sigma\,(d-2)$, we obtain a much simpler equation for $g(d)$:

$$g\,(d-2) = g\,(d) + r\,(d)\,, \tag{8.13}$$

where $r\,(d) = R\,(d)\,\Sigma\,(d-2)$.

We have

$$\begin{aligned} g(d) &= g(d-2) - r(d) = g(d-4) - r(d) - r(d-2) = \cdots \\ &= g(-\infty) - \sum_{k=0}^{\infty} r(d-2k) \qquad (8.14) \\ &= g(d+2) + r(d+2) = g(d+4) + h(d+2) + r(d+4) = \cdots \\ &= g(+\infty) + \sum_{k=1}^{\infty} r(d+2k). \qquad (8.15) \end{aligned}$$

We can ignore formal quantities $g(\pm\infty)$ because each of the series involved provides a solution of (8.13) *if* the series is convergent.

Remember that the function $r(d)$ is a sum over master integrals lower than the given master integral. Experience shows that if there is just one lower master integral then only one series, in (8.14) or (8.15), is convergent. In cases with two and more lower master integrals, it can happen that the contribution of one master integral gives a series convergent at $d \to \infty$ while a second master integral gives a series convergent at $d \to -\infty$—see an example below. However, this property does not always hold.

In the general situation, the idea suggested in [24] is to decompose $r\,(d)$ into two terms, $r_+\,(d)$ and $r_-\,(d)$, decreasing fast enough at $\pm\infty$, respectively and providing series convergent like geometrical progressions:

$$r\,(d) = r_+\,(d-2) + r_-\,(d)\,, \quad r_\pm\,(d \pm 2k) \overset{k\to\infty}{<} a^k, \quad |a| < 1. \tag{8.16}$$

We have, correspondingly,

$$R\,(d) = R_+\,(d-2) + R_-\,(d)\ \ , \tag{8.17}$$

with

$$\left|\lim_{d\to+\infty} \frac{C\,(d+2)\,R_+\,(d+2)}{R_+\,(d)}\right| < 1, \quad \left|\lim_{d\to-\infty} \frac{C^{-1}\,(d-2)\,R_-\,(d-2)}{R_-\,(d)}\right| < 1. \tag{8.18}$$

We arrive at the following general solution of the difference Equation (8.7):

$$F\,(d) = \Sigma^{-1}\,(d)\,\omega\,(d) + \sum_{k=0}^{\infty} s_+\,(d,k) - \sum_{k=0}^{\infty} s_-\,(d,k)\,, \tag{8.19}$$

$$s_{+}(d,k) = \frac{r_{+}(d+2k)}{\Sigma(d)} = \left[(-1)^{n+m}c\right]^{k} \frac{\prod_{i=1}^{n}\left(\frac{d}{2}+1-b_i\right)_k}{\prod_{j=1}^{m}\left(\frac{d}{2}+1-b'_j\right)_k} R_{+}(d+2k)\,,$$

$$s_{-}(d,k) = \frac{r_{-}(d-2k)}{\Sigma(d)} = c^{-1-k} \frac{\prod_{j=1}^{m}\left(b'_j-\frac{d}{2}\right)_{k+1}}{\prod_{i=1}^{n}\left(b_i-\frac{d}{2}\right)_{k+1}} R_{-}(d-2k)\,. \qquad (8.20)$$

Once the summing factor $\Sigma(d)$ (with $1/\Sigma(d)$ satisfying the homogeneous equation) is fixed, an arbitrariness of the solution of the inhomogeneous equation (8.7) is encoded in the periodic function $\omega(d)$ of period 2. To take into account the periodic condition, we can turn to the complex variable $z = e^{i\pi d}$ with $\omega(d) = \Omega(z)$.

To fix the function $\omega(d)$ one can use analytical properties of the integral $F(d)$ which we are evaluating. It is sufficient to reveal these properties in an arbitrary stripe of width 2, $S = \{d|\, d_0 < \mathrm{Re} d \leqslant d_0 + 2\}$, which is called the *basic stripe*. In fact, a proper choice of the summing factor $\Sigma(d)$ can greatly simplify this analysis as will be seen in examples below.

The basic properties of a given Feynman integral $F(d)$ as a function of d are the position of poles (which can be only on the real axis) and the behaviour in the limits $\mathrm{Im}\, d \to \pm\infty$. The position of poles can be revealed automatically, using the code `FIESTA` [42, 43] of sector decompositions described in Chap. 4. Well, sometimes poles at some values of d are spurious in the sense that they appear in different sectors and cancel in the sum. However, one can check whether a pole is present by running `FIESTA` and evaluating the pole part numerically.

The behaviour at large values of $\mathrm{Im}\, d$ can be analyzed using parametric representations. Indeed, using (3.38) we can observe that in the limits $\mathrm{Im}\, d \to \pm\infty$ the parametric integral itself does not contribute so that this behaviour is determined by the overall gamma function. Using the formula (5.70) we arrive at the estimate

$$|F(d)| \leq C' \mathrm{e}^{-\frac{\pi}{4} h \mathrm{Im} d} (\mathrm{Im}\, d)^{a-h\mathrm{Re}\, d-1/2} \qquad (8.21)$$

where C' is a constant and a the sum of the indices, as before.

When one takes into account the position of poles at some values $d = d_i$ in a given basic stripe one writes down a linear combination of terms $b_i/(z - \mathrm{e}^{\mathrm{i}\pi d_i})$ (or, higher poles). Then it is convenient to use the trigonometric identity

$$\cot\left(\frac{\pi}{2}(d-d_i)\right) = i\,\frac{z+\mathrm{e}^{\mathrm{i}\pi d_i}}{z-\mathrm{e}^{\mathrm{i}\pi d_i}} \qquad (8.22)$$

and use as an Ansatz a linear combination of terms $\cot\left(\frac{\pi}{2}(d-d_i)\right)$. The revealed analytic properties of $F(d)$ can be insufficient to fix completely the periodic function $\omega(d)$. Typically, some coefficients in such an Ansatz can be undetermined. To fix them one can evaluate some pole parts of the given integral $F(d)$ at $d = d_i$. In particular, one can do this by the method of MB representation and accompanying public codes because the method is flexible with respect to the shift of dimension.

After fixing the function $\omega(d)$, one arrives at a result in terms of a multiple series with very good properties: these series are convergent uniformly with respect to d and summands have a factorized structure. For one-scale integrals, one can then perform a Laurent expansion in ε under the sum sign and evaluate coefficients at powers of ε with a very high accuracy: several hundreds of digits, or even thousands. Then one can successfully apply the PSLQ algorithm [15, 16] briefly described in Sect. 14.8 and thereby arrive at analytical results. For integrals with two and more scales, one can use the series obtained in order to arrive at analytical results at general d, for example, in terms of hypergeometric functions.

This general scenario will be illustrated through examples in the next section and generalized in Sect. 8.4, to the case of two and more master integrals in a given sector.

8.3 Multiloop Examples

Let us now turn again to our favourite example and continue the analysis started in the introduction.

Example 8.1 One-loop propagator Feynman integral (1.1) corresponding to Fig. 1.1.

The IBP reduction constructed explicitly in Example 6.3 shows that there are two master integrals, $F(d) = F(1, 1; d)$, and $F_2(d) = F(1, 0; d)$ The raising DRR for $F(d)$ following from (8.2) takes the form (1.24). Let us choose the summing factor

$$\Sigma\,(d) = 2^d (1-x)^{-d} x^{d/2} \sin^2 \frac{\pi d}{2} \Gamma\left(\frac{d-1}{2}\right). \tag{8.23}$$

Turning to a new function by (8.12) we obtain a simpler equation (8.13) with

$$r\,(d) = 2^{d-3}(d-2)(x+1)\frac{x^{d/2-1}}{(1-x)^d} \sin^2 \frac{\pi d}{2} \Gamma\left(1-\frac{d}{2}\right) \Gamma\left(\frac{d-3}{2}\right). \tag{8.24}$$

If we assume that x is in a small vicinity of the origin, then only r_+ in (8.16) is non-zero. Analyzing the behaviour of $\Sigma(d)$ and $\Sigma(d)F(d)$ (using (8.21) at $\mathrm{Im}\, d \to \pm\infty$) we see that these functions increase slower than $\mathrm{e}^{|\pi d|}$. Moreover, they do not have poles in the stripe $(2, 4]$. Therefore, the function ω in (8.7) is a constant. At $d = 4$, both $\Sigma(d)F(d)$ and $\Sigma(d)r_+(d)$ take zero values. This means that the function $\omega(d)$ in (8.7) should be set to zero and we are left with the specific solution of the inhomogeneous equation determined by r_+. We obtain

$$F(d) = -\frac{1}{1+x}\left(\Gamma(1-d/2)\right.$$
$$\left. -\Gamma(-d/2)\frac{2x}{(1-x)^2}\,{}_2F_1\left(1, \frac{d-1}{2}; \frac{d+2}{2}; -\frac{4x}{(x-1)^2}\right)\right). \quad (8.25)$$

Using identities for the hypergeometric function one can observe that this result coincides with (1.8).

Consider next

Example 8.2 Three-loop vacuum diagram of Fig. 8.1.

The Feynman integral has the form

$$F(d) = \frac{1}{\pi^{3d/2}}\int\int\int \frac{\mathrm{d}^d k\,\mathrm{d}^d l \mathrm{d}^d r}{\left(k^2+m^2\right)\left(l^2+m^2\right)\left(r^2+m^2\right)\left((k+l+r)^2+m^2\right)}, \quad (8.26)$$

where we imply Euclidean space. Let us set $m = 1$.

There is one master integral in lower sectors—see Fig. 8.2. It is given by the third power of the massive tadpole:

$$F_1(d) = \Gamma^3(1-d/2)\,. \quad (8.27)$$

Using the parametric representation

$$F(d) = \Gamma(4-3d/2)\int \frac{\mathrm{d}\alpha_1\mathrm{d}\alpha_2\mathrm{d}\alpha_3\mathrm{d}\alpha_4\delta\left(\sum\alpha_i - 1\right)}{\left[\alpha_1\alpha_2\alpha_3+\alpha_1\alpha_2\alpha_4+\alpha_1\alpha_3\alpha_4+\alpha_2\alpha_3\alpha_4\right]^{d/2}} \quad (8.28)$$

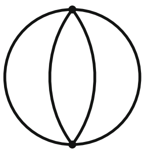

Fig. 8.1 Three-loop vacuum diagram

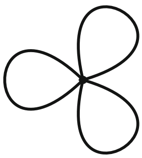

Fig. 8.2 The lower master integral for Fig. 8.1

one can conclude that the integral $F(d)$ is an analytic function in the stripe $S = \{d|\mathrm{Re}\, d \in [-2, 0)\}$. Moreover, it is clear that any Euclidean Feynman integral with all massive lines is an analytic function in the whole half-plane Re $d < 0$.

Using (8.2) and an IBP reduction we obtain the following DRR

$$F(d-2) = -\frac{(3d-10)_3\,(d-2)}{128(d-4)}F(d) - \frac{(11d-38)(d-2)^3}{64(d-4)}F_2(d) \tag{8.29}$$

where $(x)_n = x\,(x+1)\ldots(x+n-1)$ is the Pochhammer symbol.

It is convenient to choose the summing factor $\Sigma(d)$ with

$$\Sigma^{-1}\,(d) = 4^d\frac{\Gamma\,(3/2-d/2)\,\Gamma\,(3-3d/2)}{\Gamma\,(2-d/2)}. \tag{8.30}$$

It has neither poles nor zeros in S. Introducing a new function by (8.12) we obtain (8.13) with

$$r\,(d) = -\frac{(11d-38)\,\Gamma^4\,(2-d/2)}{4^d\,\Gamma\,(5/2-d/2)\,\Gamma\,(6-3d/2)}. \tag{8.31}$$

The function $r\,(d-2k)$ decreases as $(16/27)^k$ at $k \to \infty$, so that we can use (8.14) and obtain a general solution of (8.29) in the form (8.20) with $s_+ = 0$:

$$\begin{aligned}F(d) &= 4^d\frac{\Gamma\,(3/2-d/2, 3-3d/2)}{\Gamma\,(2-d/2)}\omega(z) \\ &+ \frac{1}{16\Gamma\,(2-d/2)}\sum_{k=1}^{\infty}\frac{(11d-16-22k)\,\Gamma^4\,(1+k-d/2)}{(3/2-d/2)_k\,(3-3d/2)_{3k}}16^k.\end{aligned} \tag{8.32}$$

The integral $F(d)$ is an analytic function in S. According to (8.21) it behaves like

$$|F(d)| \lesssim |\Gamma\,(4-3d/2)| \sim e^{-\frac{3\pi|\mathrm{Im}\,d|}{4}}\,|\mathrm{Im}\,d|^{7/2-3\mathrm{Re}\,d/2} \tag{8.33}$$

when $\mathrm{Im}\,d \to \pm\infty$. The summing factor is also an analytic function in S with the behaviour

$$|\Sigma\,(d)| \lesssim e^{\frac{3\pi|\mathrm{Im}\,d|}{4}}\,|\mathrm{Im}\,d|^{-2+3\mathrm{Re}\,d/2}\,. \tag{8.34}$$

Finally, the specific solution of the inhomogeneous equation given by the second line of (8.32) is also an analytic function in S with the behaviour $\mathrm{e}^{-\frac{3\pi|\mathrm{Im}\,d|}{4}}\,|\mathrm{Im}\,d|^{\sigma}$ with some σ.

Observe that the limits $\mathrm{Im}\,d \to \pm\infty$ correspond to the limits $z \to 0, \infty$. So, from (8.32), (8.33) and (8.34) we conclude that

$$\omega(z) \overset{z\to 0,\infty}{\sim} |\mathrm{Im}\,d|^{\nu}\,, \tag{8.35}$$

where ν is a real number. Taking into account that

$$\lim_{z\to\infty}\frac{|\mathrm{Im}\, d|^{\nu}}{|z|^{\alpha}}=\lim_{z\to 0}\frac{|\mathrm{Im}\, d|^{\nu}}{|z|^{-\alpha}}=0 \tag{8.36}$$

for any ν and any $\alpha > 0$, we conclude that $\omega(z)$ is an analytic function in the extended complex plane of z, except, maybe $z = 0$ and $z = \infty$ and growing slower than any positive(negative) power of $|z|$ when z tends to infinity (zero). This is sufficient to claim that $\omega(z)$ is an analytic function in the extended complex plane of z, so that this is a constant.

We can fix this constant by the condition $F(0) = 1$ which, e.g., follows from the parametric representation. We finally obtain

$$\begin{aligned}F(d) &= 4^{d}\frac{\Gamma(3/2-d/2,3-3d/2)}{\Gamma(2-d/2)}\frac{1}{\sqrt{\pi}}\sum_{k=0}^{\infty}\frac{(1+11k/8)\,(k!)^{4}}{(3/2)_k\,(3)_{3k}}16^{k}\\ &+\frac{1}{16\Gamma(2-d/2)}\sum_{k=1}^{\infty}\frac{(11d-16-22k)\,\Gamma^{4}[k+1-d/2]}{(3/2-d/2)_k\,(3-3d/2)_{3k}}16^{k}.\end{aligned} \tag{8.37}$$

Both series here are very well convergent, with the behaviour $(16/27)^k$. Moreover, this convergence is uniform with respect to d and this is a very important feature of results obtained with DRA. So, one can safely expand series under the sum sign. Since the convergence of series which appear as coefficients at powers of ε is excellent one can achieve a very high accuracy and then successfully apply the PSLQ algorithm [15, 16]. As a result one obtains

$$\begin{aligned}F(d) &= \Gamma(1+\epsilon)^3\Bigg\{\frac{2}{\epsilon^3}+\frac{23}{3\epsilon^2}+\frac{35}{2\epsilon}+\frac{275}{12}+\epsilon\left(-\frac{189}{8}+\frac{112\zeta_3}{3}\right)\\
&-\epsilon^2\left(\frac{14917}{48}-280\zeta_3+\frac{136\pi^4}{45}+\frac{32}{3}\pi^2\ln^2 2-\frac{32\ln^4 2}{3}-256a_4\right)\\
&-\epsilon^3\left(\frac{48005}{32}-\frac{4060\zeta_3}{3}+\frac{68\pi^4}{3}+80\pi^2\ln^2 2-80\ln^4 2-1920a_4-\frac{272}{15}\pi^4\ln 2\right.\\
&\left.-\frac{64}{3}\pi^2\ln^3 2+\frac{64\ln^5 2}{5}-1536a_5+1240\zeta_5\right)-\epsilon^4\left(\frac{1108525}{192}-5390\zeta_3\right.\\
&+\frac{986\pi^4}{9}+\frac{1160}{3}\pi^2\ln^2 2-\frac{1160\ln^4 2}{3}-9280a_4-136\pi^4\ln 2-160\pi^2\ln^3 2\\
&+96\ln^5 2-11520a_5+9300\zeta_5+\frac{32\pi^6}{5}+\frac{272}{5}\pi^4\ln^2 2+32\pi^2\ln^4 2-\frac{64\ln^6 2}{5}\\
&\left.-9216a_6-3840s_6+\frac{4880\zeta_3^2}{3}\right)-\epsilon^5\left(\frac{2570029}{128}-\frac{57967\zeta_3}{3}+\frac{1309\pi^4}{3}\right.\\
&+1540\pi^2\ln^2 2-1540\ln^4 2-36960a_4-\frac{1972}{3}\pi^4\ln 2-\frac{2320}{3}\pi^2\ln^3 2\end{aligned}$$

$$
\begin{aligned}
&+ 464 \ln^5 2 \; - 55680 a_5 + 44950 \zeta_5 + 48\pi^6 + 408\pi^4 \ln^2 2 \; + 240\pi^2 \ln^4 2 \; - 96 \ln^6 2 \\
&- 69120 a_6 - 28800 s_6 + 12200 \zeta_3^2 - \frac{3824}{135}\pi^6 \ln 2 \; - \frac{544}{5}\pi^4 \ln^3 2 \; - \frac{192}{5}\pi^2 \ln^5 2 \\
&+ \frac{384 \ln^7 2}{35} - 55296 a_7 + \frac{74240}{7} \ln 2\, s_6 - \frac{720\pi^4 \zeta_3}{7} - \frac{92800}{7} \ln 2\, \zeta_3^2 \\
&- \frac{130360\pi^2 \zeta_5}{21} - 22320 \ln^2 2\, \zeta_5 + \frac{772868 \zeta_7}{7} - \frac{74240 s_{7a}}{7} + \frac{87040 s_{7b}}{7} \Bigg) \\
&+ O\left(\epsilon^6\right) \Bigg\}, \qquad (8.38)
\end{aligned}
$$

in agreement with [8]. Here

$$
\begin{aligned}
a_n &= \mathrm{Li}_n\,(1/2)\,, \\
s_6 &= \zeta_{-5,-1} + \zeta_6, \\
s_{7a} &= \zeta_{-5,1,1} + \zeta_{-6,1} + \zeta_{-5,2} + \zeta_{-7}, \\
s_{7b} &= \zeta_7 + \zeta_{5,2} + \zeta_{-6,-1} + \zeta_{5,-1,-1}, \qquad (8.39)
\end{aligned}
$$

and $\zeta_{\ldots} \equiv \zeta(\ldots)$ are multiple zeta values (MZV) defined by (12.4). In fact, this master integral belongs to the set of three-loop master integrals relevant to the $g-2$ factor which were evaluated up to transcendentality weight six in [22, 23, 35, 36]. The evaluation of these master integrals up to transcendentality weight seven was preformed in [31] by DRA.

The next example is the same as Examples 3.7 and 5.6:

Example 8.3 Non-planar two-loop massless vertex diagram of Fig. 3.14 with $p_1^2 = p_2^2 = 0$.

The Feynman integral (considered in Minkowski space) is given by (3.59) where we now include the factor $1/\pi^d$. There are two lower master integrals corresponding to graphs shown in Fig. 8.3. They can be evaluated easily in terms of gamma functions using one-loop integration formulae (10.7) and (10.28):

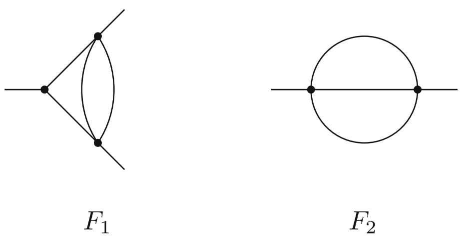

Fig. 8.3 Two lower master integrals for Fig. 3.14

$$F_1(d) = -\frac{\Gamma(4-d)\Gamma\left(2-\frac{d}{2}\right)\Gamma\left(\frac{d}{2}-1\right)^2\Gamma(d-3)^2}{\Gamma(d-2)\Gamma\left(\frac{3d}{2}-4\right)}, \tag{8.40}$$

$$F_2(d) = \frac{\Gamma(3-d)\Gamma\left(\frac{d}{2}-1\right)^3}{\Gamma\left(\frac{3d}{2}-3\right)}. \tag{8.41}$$

The integral has on-shell IR and collinear divergences at $d = 4$ and an UV divergence at $d = 6$. Let us choose the basic stripe as $S = \{d|\mathrm{Re}d \in (4, 6]\}$. FIESTA reports that $F(d)$ is finite at $\mathrm{Re}d \in (4, 6)$. It also shows that at $d = 6$ the integral has a simple pole.

Using the lowering recurrence relation and an IBP reduction of the corresponding integrals we obtain the following DRR:

$$\begin{aligned} F(d) &= -\frac{(2d-7)_4}{(d-4)_2}F(d+2) + R_+(d) + R_-(d+2), \\ R_+(d) &= -\frac{2\left(43d^4 - 478d^3 + 1963d^2 - 3530d + 2352\right)}{(d-3)(d-4)^3}F_2(d), \\ R_-(d+2) &= -\frac{2\left(37d^3 - 313d^2 + 858d - 752\right)}{(3d-8)(d-4)^2}F_1(d). \end{aligned} \tag{8.42}$$

It turns out that $R_\pm(d)$ satisfy the conditions (8.18), i.e. this distribution of the inhomogeneous term corresponds exactly to the two lower master integrals.

It is reasonable to choose the summing factor as

$$\Sigma(d) = 4^d(4-d)\sin\left(\frac{\pi}{2}(d-5)\right)\sin^2\left(\frac{\pi}{2}(d-4)\right)\Gamma\left(d-\frac{7}{2}\right). \tag{8.43}$$

Since $\Sigma(d) \sim (d-6)^2$ near $d = 6$ we have

$$\lim_{d\to 6}\Sigma(d)F(d) = 0 \tag{8.44}$$

and, therefore, the function $\Sigma(d)F(d)$ is regular at $d = 6$.

According to (8.20), the general solution of (8.42) takes the form

$$\begin{aligned} F(d) &= \Sigma^{-1}(d)\left[\omega(d) + \sum_{k=0}^{\infty} r_+(d+2k) - \sum_{k=0}^{\infty} r_-(d-2k)\right], \\ r_+(d) &= \frac{\sqrt{\pi}2^d\sin(\pi d)\Gamma\left(\frac{3}{2}-\frac{d}{2}\right)\Gamma\left(\frac{d}{2}-2\right)^2\Gamma\left(d-\frac{7}{2}\right)}{(d-3)\Gamma\left(\frac{3d}{2}-3\right)} \\ &\quad\times\left(43d^4 - 478d^3 + 1963d^2 - 3530d + 2352\right), \end{aligned} \tag{8.45}$$

$$r_-(d) = -\frac{\pi^2 2^{2d-6}\Gamma\left(\frac{d}{2}-3\right)\Gamma\left(d-\frac{11}{2}\right)}{(5-d)\Gamma\left(\frac{3d}{2}-6\right)}\left(37d^3-535d^2+2554d-4016\right).$$

Due to the choice of the summing factor, the function $r_+(d)$ does not have any singularities in the region $\operatorname{Re} d > 4$. The function $r_-(d)$ has simple poles at

$$\begin{aligned} d &= d_1(k), d_2(k), 5, 6 \\ d_1(k) &= 5\tfrac{1}{2} - 2k, \quad d_2(k) = 4\tfrac{1}{2} - 2k, \quad k = 0, 1, \ldots \end{aligned} \tag{8.46}$$

Taking into account that $\Sigma(d)\,F(d)$ is analytic in S, we obtain

$$\begin{aligned} \omega(d) &= b + b_1 \cot\tfrac{\pi}{2}\left(d - 5\tfrac{1}{2}\right) + b_2 \cot\tfrac{\pi}{2}\left(d - 4\tfrac{1}{2}\right) \\ &\quad + b_3 \cot\tfrac{\pi}{2}(d-5) + b_4 \cot\tfrac{\pi}{2}(d-6), \\ b_1 &= \frac{\pi}{2}\sum_{k=0}^{\infty}\operatorname*{Res}_{d=d_1(k)} r_-(d-2k), \quad b_2 = \frac{\pi}{2}\sum_{k=0}^{\infty}\operatorname*{Res}_{d=d_2(k)} r_-(d-2k), \\ b_3 &= \frac{\pi}{2}\operatorname*{Res}_{d=5} r_-(d) = 256\pi^{7/2}, \quad b_4 = \frac{\pi}{2}\operatorname*{Res}_{d=6} r_-(d) = 1280\pi^{7/2}. \end{aligned} \tag{8.47}$$

Taking into account the values of $b_{3,4}$ one can make a natural guess $b_{1,2} = f_{1,2}\pi^{7/2}$ and then check with the accuracy of thousand digits that

$$b_1 = b_2 = -256\pi^{7/2}. \tag{8.48}$$

The constant b can be fixed using the condition (8.44) and the properties of $r_\pm$:

$$r_+(6+2k) = 0, \quad r_-(6-2\epsilon) - \frac{2560\pi^{5/2}}{-2\epsilon} = O(\epsilon).$$

We obtain

$$b = \sum_{k=0}^{\infty} r_-(4-2k) = 0. \tag{8.49}$$

Therefore, relations (8.45), (8.43), (8.47), (8.48), and (8.49) lead to a result for $F(d)$ in terms of a well convergent series. Then one can perform an ε-expansion under the sum sign, evaluate coefficients at powers of ε with a high accuracy, apply PSLQ and reproduce (3.80) [19] as well as results for higher terms obtained in [18].

Let us now consider

Example 8.4 Two-loop vacuum diagram of Fig. 8.4 with two different masses.

The Feynman integral has the form

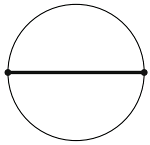

Fig. 8.4 Two-loop vacuum diagram with the masses m, M, m

$$F(d) = \frac{1}{\pi^d} \int \frac{\mathrm{d}^d k \, \mathrm{d}^d l}{\left(k^2 + m^2\right)\left(l^2 + m^2\right)\left((k+l)^2 + M^2\right)}, \tag{8.50}$$

where we imply Euclidean space. Let us set $m = 1$.

Master integrals for lower sectors are products of one-loop tadpoles and are represented in terms of gamma functions at general d:

$$F_1(d) = \Gamma^2 (1 - d/2),$$
$$F_2(d) = \Gamma^2 (1 - d/2) M^{d-4}.$$

The basic stripe can be chosen as $S = \{d | \mathrm{Re} d \in [0, 2)\}$. The DRR takes the form

$$F(d) = \frac{M^2 \left(4 - M^2\right)}{(d-2)(d-3)} F(d-2) + \frac{\left(M^2 - 2 - 2M^{d-2}\right)}{(d-2)(d-3)} \Gamma^2 (2 - d/2). \tag{8.51}$$

Let us suppose that $M < 2$. The result for $M > 2$ can be obtained by an analytical continuation. Let us choose the summing factor as

$$\Sigma(d) = \frac{M^{-d} \left(4 - m^2\right)^{-d/2}}{\Gamma(2-d)}. \tag{8.52}$$

Using (8.12) we obtain a simpler equation (8.13) with

$$r(d) = \frac{\left(M^2 - 2 - 2M^{d-2}\right) \Gamma^2 (2 - d/2)}{M^d \left(4 - M^2\right)^{d/2} \Gamma(4-d)}. \tag{8.53}$$

The limit

$$\lim_{d \to -\infty} \frac{r(d+2)}{r(d)} = \left(1 - \frac{M^2}{4}\right) \max\left(1, M^2\right) \leqslant 1 \tag{8.54}$$

shows that $r(d) = r_-(d)$ and allows us to represent the general solution in the form

$$F(d) = \Sigma^{-1}(d)\,\omega(z) + H(d),$$
$$H(d) = \frac{\Gamma(1 - d/2)\,\Gamma(2 - d/2)}{2(3-d)}$$

$$\times \sum_{k=0}^{\infty} \left(1 - \frac{M^2}{4}\right)^k \left(\left(M^2 - 2\right) M^{2k} - 2M^{d-2}\right) \frac{(2-d/2)_k}{(5/2-d/2)_k}$$
$$= \frac{\Gamma(1-d/2)\,\Gamma(2-d/2)}{2(3-d)} \left(\left(M^2-2\right) {}_2F_1\left(\left.{1,2-d/2 \atop 5/2-d/2}\right| M^2 - \frac{M^4}{4}\right)\right.$$
$$\left. -2M^{d-2}\, {}_2F_1\left(\left.{1,2-d/2 \atop 5/2-d/2}\right| 1 - \frac{M^2}{4}\right)\right).$$

The functions $\Sigma(d)\,F(d)$ and $\Sigma(d)\,H(d)$ are analytic in S, They are also bounded at $\mathrm{Im} d \to \pm\infty$, so $\omega(z)$ is a constant. Fixing the constant by the condition $F(0) = M^{-2}$ we obtain

$$\omega(z) = M^{-2} - H(0) = \frac{4\pi\theta\left(2 - M^2\right)}{M^3\left(4 - M^2\right)^{3/2}}. \tag{8.55}$$

One can check that $\theta\left(2 - M^2\right)$ in this formula cancels the discontinuity of the function $\left(M^2 - 2\right) {}_2F_1\left(\left.{1,2-d/2 \atop 5/2-d/2}\right| M^2 - \frac{M^4}{4}\right)$ in $H(t)$. Using properties of the hypergeometric function one can represent the result as

$$F(d) = \frac{\Gamma\left(1-\frac{d}{2}\right)\Gamma\left(2-\frac{d}{2}\right)}{d-3}$$
$$\times \left({}_2F_1\left(\left.{1,3-d \atop 5/2-d/2}\right| \frac{M^2}{4}\right) + M^{d-2}\, {}_2F_1\left[\left.{1,2-d/2 \atop 3/2}\right| \frac{M^2}{4}\right]\right),$$

in agreement with [14].

Let us now briefly describe, following [27, 31], how the evaluation of a set of master integrals can be organized.

Example 8.5 Master integrals for three-loop massless form factors shown in Fig. 8.5. Two external momenta flowing to the right are on the light cone, $p_1^2 = p_2^2 = 0$.

These Feynman integrals were evaluated analytically in an ε-expansion up to weight six in [4, 17, 20, 21], with the exception of the finite parts of $A_{9,2}$ and $A_{9,4}$ which were later evaluated by DRA in [27]. Then the evaluation of all these master integrals up to weight eight was done by DRA in [31]. This is how the evaluation was organized.

It turns out that there is at most one master integral in every sector. This means that one can choose integrals with indices 1 and 0 as master integrals. So, there are no dots on lines in Fig. 8.5 and no numerators in the master integrals are assumed.

Master integrals naturally form a partially ordered set in the sense of the definitions of Sect. 6.2. In other words, a master integral is *lower* than another master integral if the Feynman graph for the former can be obtained by contracting some internal lines from the Feynman graph of the latter. This ordering enables us to introduce

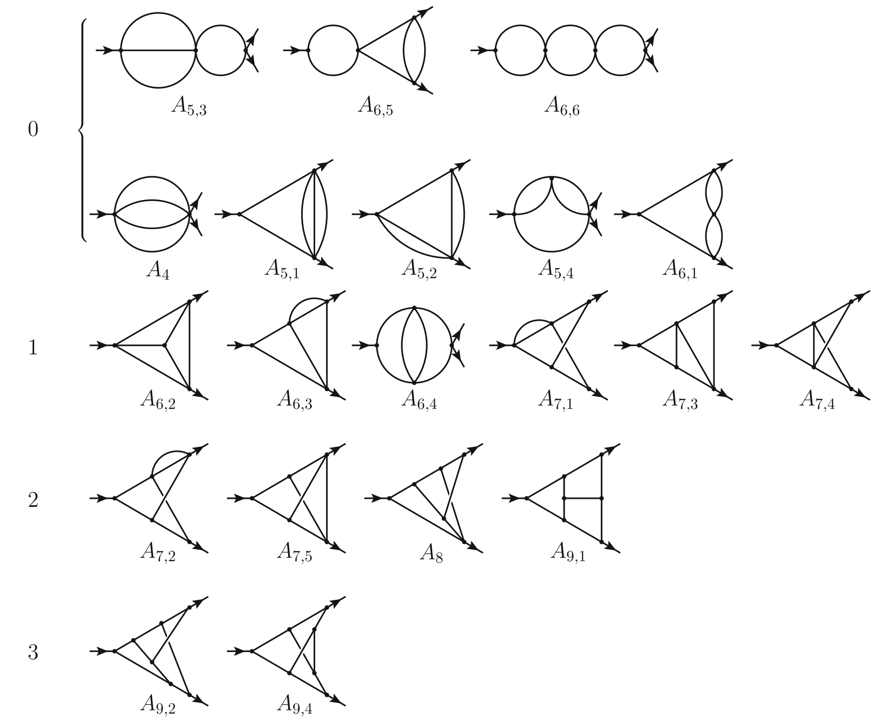

Fig. 8.5 Master integrals for three-loop massless form factors. The complexity level is indicated to the *left*

the notion of *complexity level* (cl) of a given master integral which is the maximal number of nested lower master integrals. According to this definition, the master integrals with zero complexity level have no lower master integrals. The DRR for such integral is obviously homogeneous and its explicit solution is expressed in terms of Γ-functions. Moreover, it turns out that for three-loop on-shell massless vertex master integrals any integral expressed in terms of Γ-functions has zero complexity level. It might happen that this situation is general.

In Fig. 8.5, there are four rows of diagrams corresponding to complexity levels 0, 1, 2 and 3. Therefore, one starts the calculation from the complexity level 1, then turns to the complexity level 2 and, finally, calculates the two master integrals of complexity level 3, i.e. $A_{9,2}$ and $A_{9,4}$. Let us, for example, consider $A_{6,3}(d)$. The raising DRR takes the form

$$\begin{aligned}
F(1,1,1,1,1,1;d-2) &= F(1,1,1,2,2,2;d) + F(1,1,2,1,2,2;d) \\
&+ F(1,1,2,2,1,2;d) + F(1,2,1,2,1,2;d) + F(1,2,1,2,2,1;d) \\
&+ F(1,2,2,1,1,2;d) + F(1,2,2,1,2,1;d) + F(1,2,2,2,1,1;d) \\
&+ F(2,1,1,2,1,2;d) + F(2,1,1,2,2,1;d) + F(2,1,2,1,1,2;d) \\
&+ F(2,1,2,1,2,1;d) + F(2,1,2,2,1,1;d).
\end{aligned}$$

After performing an IBP reduction of the integrals on the right-hand side we obtain

$$\begin{aligned} A_{6,3}(d-2) = {} & \frac{8(-3+d)(-9+2d)(-7+2d)(-10+3d)}{-16+3d} A_{6,3}(d) \\ & + \frac{32(-3+d)(-9+2d)(-7+2d)(-5+2d)(-10+3d)}{(-5+d)(-4+d)^2(-16+3d)} \\ & \times (-8+3d)(-32+7d)\ A_4(d) \end{aligned}$$

where

$$A_4(d) = \frac{\Gamma(4-3d/2)\Gamma(d/2-1)^4}{\Gamma(2d-4)}.$$

Here and in what follows, we omit, for brevity, a power-like dependence of the master integrals on $q^2 + i0$ which can easily be restored by power counting.

Let us choose the summing factor

$$\begin{aligned} \Sigma(d) = {} & \frac{1}{\sqrt{\pi}} 32^{4d-\frac{5}{2}} \left(\frac{d}{2} - \frac{5}{3}\right) \sin\left(\frac{1}{2}\pi(d-5)\right) \sin\left(\frac{1}{2}\pi\left(d - \frac{14}{3}\right)\right) \\ & \times \sin^2\left(\frac{\pi d}{2}\right) \Gamma\left(\frac{d}{2} - \frac{5}{4}\right) \Gamma\left(\frac{d}{2} - \frac{3}{4}\right) \Gamma\left(\frac{d}{2} - \frac{1}{2}\right). \end{aligned}$$

Using (8.12) we obtain the equation (8.13) with

$$\begin{aligned} r(d) = {} & \frac{1}{3(d-5)} 1024\pi(32-7d) \sin\left(\frac{\pi d}{2}\right) \sin(\pi d) \cos\left(\frac{1}{6}(3\pi d + \pi)\right) \\ & \times \Gamma\left(7 - \frac{3d}{2}\right) \Gamma\left(\frac{d}{2} - 2\right)^3. \end{aligned}$$

Here only the function $r_+(d)$ is non-zero in (8.16) so that we have

$$A_{6,3}(d) = (\omega(d) + r_+(d))/\Sigma(d). \tag{8.56}$$

For $A_{6,3}$, FIESTA says that in the stripe $(3, 5]$ there can be simple poles at $d = 10/3, 4, 14/3, 5$. Taking into account this information and analyzing the behaviour at infinity ($\mathrm{Im} d \to \pm\infty$) with the help of the (8.21) provides the following Ansatz for $\Omega(z)$:

$$a_0 + \frac{a_1}{z - e^{-2i\pi/3}} + \frac{a_2}{z-1} \tag{8.57}$$

which corresponds to the following Ansatz for $\omega(d)$:

$$b_0 + b_1 \cot\left(\frac{1}{2}\pi(d-4)\right) + b_2 \cot\left(\frac{1}{2}\pi\left(d-\frac{10}{3}\right)\right).$$

The constants b_i can be fixed by using a onefold MB representation for $A_{6,3}$ which can be obtained from the general representation (5.21):

$$\omega(d) = 512\pi^4 3^{-1/2}\left(\cot\left(\frac{1}{2}\pi\left(d-\frac{10}{3}\right)\right) - \cot\left(\frac{1}{2}\pi(d-4)\right)\right).$$

We obtain the following result:

$$\begin{aligned} A_{6,3}(d) = \frac{\omega(d)}{\Sigma(d)} - \frac{1}{\Sigma(d)} \sum_{k=0}^{\infty} \frac{2048\pi^2(-1)^k(7d+14k-32)}{3(d+2k-5)} \\ \times \frac{\sin^2\left(\frac{\pi d}{2}\right)\cos\left(\pi\left(\frac{d}{2}+\frac{1}{6}\right)\right)\cos\left(\frac{\pi d}{2}\right)\csc\left(\frac{3\pi d}{2}\right)\Gamma\left(\frac{d}{2}+k-2\right)^3}{\Gamma\left(\frac{3d}{2}+3k-6\right)}. \end{aligned} \tag{8.58}$$

As an example of a master integral with complexity level two, let us consider $A_{7,2}$. There are four lower master integrals, A_4, $A_{5,1}$, $A_{5,2}$, and $A_{6,3}$. Three of them are expressed in terms of Γ-functions and $A_{6,3}$ is given by (8.58).

The lowering DRR has the form

$$\begin{aligned} A_{7,2}(d+2) &= c_{7,2}(d)A_{7,2}(d) \\ &\quad + c_{6,3}(d)A_{6,3}(d) + c_{5,2}(d)A_{5,2}(d) + c_{5,1}(d)A_{5,1}(d) + c_4(d)A_4(d). \end{aligned}$$

where

$$c_{7,2} = -\frac{8(d-4)^2(d-3)}{5(d-2)(d-1)(5d-18)(5d-16)(5d-14)(5d-12)}, \tag{8.59}$$

and other coefficients are rational functions of d (presented in [27]).

Using (8.59) we choose the summing factor as

$$\Sigma(d) = \frac{(d-3)\cos\left(\frac{\pi d}{2}\right)\cos\left(\frac{\pi}{6}-\frac{\pi d}{2}\right)\cos\left(\frac{\pi d}{2}+\frac{\pi}{6}\right)\Gamma\left(\frac{5d}{2}-9\right)}{\Gamma\left(\frac{d}{2}-2\right)^2}. \tag{8.60}$$

Turning to the new function by (8.12) we obtain (8.13) with $g = \tilde{A}_{7,2}(d)$ in the following form:

$$\tilde{A}_{7,2}(d+2) = \tilde{A}_{7,2}(d) + \tilde{A}_{6,3}(d) + \tilde{A}_{5,2}(d) + \tilde{A}_{5,1}(d) + \tilde{A}_4(d), \tag{8.61}$$

where $\tilde{A}_n(d) = \Sigma(d+2)c_n(d)A_n(d)$. The general solution can easily be constructed using the explicit form of the integrals A_4, $A_{5,1}$, $A_{5,2}$, and $A_{6,3}$:

$$\tilde{A}_{7,2}(d) = \sum_{l=0}^{\infty}\left[\tilde{A}_{5,2}(d-2-2l) + \tilde{A}_{5,1}(d-2-2l) + \tilde{A}^{2}_{6,3}(d-2-2l)\right] \\ - \sum_{l=0}^{\infty}\tilde{A}^{1,1}_{6,3}(d+2l)\sum_{k=0}^{\infty}A^{1,2}_{6,3}(d+2l+2k) - \sum_{l=0}^{\infty}\tilde{A}_4(d+2l) + \omega(d). \tag{8.62}$$

Applying FIESTA to $A_{7,2}$ we see that the integral has simple poles at $d = 14/3, 5, 16/3, 6$. The function $\Sigma(d)$ has simple zeros at $d = 14/3, 5, 16/3$, therefore, $\tilde{A}_{7,2}(d)$ is regular everywhere in S except the point $d = 6$, where it has a simple pole. Besides, from the explicit form of the summing factor and from the parametric representation of $A_{7,2}(d)$ it is clear that $\tilde{A}_{7,2}(d)$ grows slower than any positive (negative) power of $|z|$ when $\mathrm{Im}\, d \to -\infty$ ($\mathrm{Im}\ d \to +\infty$). This fixes $\omega(d)$ up to a function

$$a_1 + a_2 \cot\left(\frac{\pi}{2}(d-6)\right). \tag{8.63}$$

In order to fix the two remaining constants, one can use data obtained from the Mellin–Barnes representation of $A_{7,2}(d)$ which can easily be obtained from the general Mellin–Barnes representation for the non-planar on-shell vertex diagram given by (5.21):

$$A_{7,2}(d) = \frac{\Gamma\left(\frac{d}{2}-2\right)\Gamma\left(\frac{d}{2}-1\right)^2\Gamma(d-3)}{\Gamma(d-2)\Gamma\left(\frac{3d}{2}-5\right)\Gamma(2d-7)}\frac{1}{(2\pi \mathrm{i})^2}\int\int\frac{\Gamma(-z_1)\Gamma(-z_2)}{\Gamma(d-z_1-4)} \\ \times\frac{\Gamma(z_2+1)^2}{\Gamma\left(\frac{3d}{2}-z_1-5\right)}\Gamma\left(\frac{d}{2}-z_1-2\right)\Gamma\left(\frac{3d}{2}-z_2-6\right)\Gamma(z_1+z_2+1) \\ \times\Gamma(d-z_1-z_2-5)\Gamma\left(\frac{3d}{2}-z_1-z_2-6\right)\Gamma\left(-\frac{3d}{2}+z_1+z_2+7\right)\mathrm{d}z_1\,\mathrm{d}z_2.$$

Using the codes of [13] and [39] (described in Chap. 5) at $d = 6-2\varepsilon$ and $d = 5-2\varepsilon$ one can straightforwardly obtain

$$A_{7,2}(6-2\varepsilon) = -\frac{41}{15552\varepsilon} + O(\varepsilon^0), \\ A_{7,2}(5-2\varepsilon) = -\frac{\pi^{5/2}}{24\varepsilon} + O(\varepsilon^0). \tag{8.64}$$

Using these two values and also taking into account the fact that the singularities of the inhomogeneous part should be cancelled, one obtains

$$\omega(d) = \frac{\pi^3}{20\sqrt{5}} \tan\left(\frac{\pi}{10} - \frac{\pi d}{2}\right) - \frac{\pi^3}{36} \tan\left(\frac{\pi}{6} - \frac{\pi d}{2}\right)$$
$$- \frac{\pi^3}{20\sqrt{5}} \tan\left(\frac{\pi d}{2} + \frac{\pi}{10}\right) + \frac{\pi^3}{36} \tan\left(\frac{\pi d}{2} + \frac{\pi}{6}\right) + \frac{\pi^3}{60} \cot^3\left(\frac{\pi d}{2}\right)$$
$$+ \frac{13\pi^3}{180} \cot\left(\frac{\pi d}{2}\right) + \frac{\pi^3}{20\sqrt{5}} \cot\left(\frac{\pi}{5} - \frac{\pi d}{2}\right) - \frac{\pi^3}{20\sqrt{5}} \cot\left(\frac{\pi d}{2} + \frac{\pi}{5}\right).$$

Equations (8.62), (8.65), and (8.60) determine a result for $A_{7,2}(d)$.

Let us realize that there is a double sum in (8.62). Making a shift $k \to k - l$, we obtain the following triangle sum with the factorized summand:

$$\sum_{l=0}^{\infty} \tilde{A}_{6,3}^{1,1}(d + 2l) \sum_{k=l}^{\infty} A_{6,3}^{1,2}(d + 2k). \tag{8.65}$$

The factorized form of the summand essentially simplifies the numerical calculation of the sum, making it possible to organize the calculations without nested do-loops. Let me emphasize that this is a general feature of multiple series which arise within the presented method.

This procedure was applied to all the integrals shown in Fig. 8.5. For example, the result for $A_{9,4}$ up to weight eight is [27, 31]

$$A_{9,4}(d) = e^{-3\gamma_E\epsilon}\left\{-\frac{1}{9\epsilon^6} - \frac{8}{9\epsilon^5} + \epsilon^{-4}\left(1 + \frac{43\pi^2}{108}\right)\right.$$
$$+ \epsilon^{-3}\left(\frac{14}{9} + \frac{53\pi^2}{27} + \frac{109\zeta_3}{9}\right) + \epsilon^{-2}\left(-17 - \frac{311\pi^2}{108} + \frac{608\zeta_3}{9} - \frac{481\pi^4}{12960}\right)$$
$$- \epsilon^{-1}\left(-84 - \frac{11\pi^2}{18} + \frac{949\zeta_3}{9} - \frac{85\pi^4}{108} + \frac{2975\pi^2\zeta_3}{108} - \frac{3463\zeta_5}{45}\right) - \left(339\right.$$
$$\left.- \frac{77\pi^2}{4} - \frac{434\zeta_3}{9} + \frac{2539\pi^4}{2592} + \frac{299\pi^2\zeta_3}{3} - \frac{7868\zeta_5}{15} + \frac{247613\pi^6}{466560} + \frac{3115\zeta_3^2}{6}\right)$$
$$- \epsilon\left(-1242 + 112\pi^2 - 589\zeta_3 + \frac{487\pi^4}{432} - \frac{19499\pi^2\zeta_3}{108} + \frac{30067\zeta_5}{45}\right.$$
$$\left.+ \frac{25567\pi^6}{30240} + \frac{18512\zeta_3^2}{9} + \frac{38903\pi^4\zeta_3}{2592} + \frac{113629\pi^2\zeta_5}{540} - \frac{8564\zeta_7}{63}\right)$$
$$- \epsilon^2\left(4293 - \frac{1887\pi^2}{4} + 3756\zeta_3 - \frac{491\pi^4}{32} + \frac{4019\pi^2\zeta_3}{18} + \frac{7874\zeta_5}{15}\right.$$
$$- \frac{9901847\pi^6}{3265920} - \frac{26291\zeta_3^2}{6} + \frac{9037\pi^4\zeta_3}{135} + \frac{35728\pi^2\zeta_5}{45} - \frac{72537\zeta_7}{14}$$
$$\left.\left.+ \frac{30535087\pi^8}{31352832} - \frac{152299}{216}\pi^2\zeta_3^2 + \frac{730841\zeta_3\zeta_5}{135} - \frac{76288}{81}\zeta_{-6,-2}\right) + O\left(\epsilon^3\right)\right\}.$$

8.4 Evaluating Multicomponent Master Integrals

Suppose now that we have to evaluate two or more master integrals in a given sector, $F_1, \ldots, F_k$, with $k \geq 2$. This is how one can proceed in this situation, according to [32] where details can be found. Let us denote by $\mathbf{F}$ the column-vector composed of F_i and call it multicomponent master integral (MMI). For each of the components, one can construct the lowering or raising DRR which has the form (8.7) in the former case and (8.7) with $d - 2 \to d + 2$ in the latter case. For convenience, let us turn to the variable $\nu = d/2$ in this section. Now, on the right-hand side, we have a linear combination of the components F_i so that, in particular, in the raising case, we obtain the following matrix difference equation:

$$\mathbf{F}(\nu + 1) = \mathbb{C}(\nu)\, \mathbf{F}(\nu) + \mathbf{R}(\nu)\,, \tag{8.66}$$

where $\mathbb{C}$ is a $k \times k$ matrix with elements rational in ν and the vector $\mathbf{R}$ involves master integrals only in lower sectors.

As in the case $k = 1$, we need, first, to construct a solution of the corresponding homogeneous equation, i.e. without $\mathbf{R}(\nu)$ on the right-hand side. More precisely, we have to find k solutions. To do this, let us involve into the game so-called cut-integrals[1] which are obtained by replacing propagators $1/(E_j + i0)^{a_j}$ by the differences $1/(E_j + i0)^{a_j} - 1/(E_j - i0)^{a_j}$ which are proportional to $\delta^{(a_j - 1)}\left(E_j\right)$ and are zero at $a_j \leq 0$. Let us replace all the propagators according to this rule, denote the result of the replacement by $\Delta F(d)$ and call it 'maximal cut'.

It is easy to conclude that both IBP and dimensional recurrence relations are not sensitive to the change of the sign of i0 in any subset of the propagators. Therefore, Eq. (8.66) holds for any cut integral obtained from F_i present in and, in particular, for ΔF_i. Since cutting a line, in the above sense, gives a zero result if an index is a non-positive integer, we see that the inhomogeneous term drops out and, therefore, the maximally cut MMI satisfies [32] the homogeneous equation corresponding to (8.66):

$$\Delta\mathbf{F}(\nu + 1) = \mathbb{C}(\nu)\, \Delta\mathbf{F}(\nu)\,. \tag{8.67}$$

Let us see how this property can be used to find solutions of homogeneous equations associated with DRR, using the example of the two master integrals shown in Fig. 8.6, where dashed lines denote static propagators $1/(v \cdot k + \mathrm{i}0)$ and the condition $v \cdot q = 0$ is implied for the scalar product of v and the external momentum q. One chooses $v = (1, \mathbf{0})$ and $q = (0, \mathbf{q})$. We use the following normalization:

---

[1] The cut integrals are also the basic tool of the powerful generalized unitarity technique [5, 6] which provides the possibility to construct scattering amplitudes. In fact, the strategy of writing an Ansatz as a linear combination of some basic scalar integrals and constructing the corresponding coefficient functions is very similar to the strategy of solving IBP relations, especially within Baikov's method outlined in Sect. 6.5.

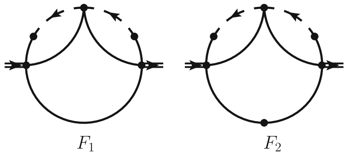

Fig. 8.6 Master integrals F_1 and F_2

$$F_a(d) = \frac{(-1)^a}{\left(\mathrm{i}\pi^{d/2}\right)^3} \int\int\int \frac{\mathrm{d}^d k \, \mathrm{d}^d l \, \mathrm{d}^d r}{k^2 r^2 ((l+q)^2)^a (k-l)^2 (l-r)^2 (v \cdot k)(v \cdot r)}, \quad (8.68)$$

where $a = 1$ and 2 and $+\mathrm{i}0$ is implied in all the propagators.

The lower master integrals are shown in Fig. 8.7.

Here the labelling of the master integrals is taken from the future paper [33]. Moreover, in this labeling, $F_1 = P_{71}$ and $F_1 = P_{72}$ but the notation F_i is more convenient within this section. The lowering DRR reads:

$$\mathbf{F}(\nu+1) = \mathbb{C}(\nu)\,\mathbf{F}(\nu) + \mathbf{R}(\nu), \quad (8.69)$$

where $\mathbf{F}(\nu) = \begin{pmatrix} F_1(\nu) \\ F_2(\nu) \end{pmatrix}$. The inhomogeneous term $\mathbf{R}(\nu) = \begin{pmatrix} R_1(\nu) \\ R_2(\nu) \end{pmatrix}$ contains only the lower master integrals, and $\mathbb{C}(\nu) = \begin{pmatrix} C_{11}(\nu) & C_{12}(\nu) \\ C_{21}(\nu) & C_{22}(\nu) \end{pmatrix}$ is a matrix with rational elements. (See explicit expressions for the functions $C_{ij}(\nu)$ and $\mathbf{R}(\nu)$ in [32].) Observe that although $C_{ij}(\nu)$ are quite cumbersome, the determinant of the matrix $\mathbb{C}(\nu)$ has a simple factorized form:

$$\det \mathbb{C}(\nu) = -\frac{(\nu-2)(4\nu-7)^2(4\nu-5)^2}{16(\nu-1)^5(2\nu-3)^2(8\nu-13)(8\nu-11)(8\nu-9)(8\nu-7)}. \quad (8.70)$$

This seems to be a general situation.

Let us now find two solutions $\mathbf{F}_h^i(\nu)$, $i = 1, 2$ of the homogeneous equation

$$\mathbf{F}(\nu+1) = \mathbb{C}(\nu)\,\mathbf{F}(\nu). \quad (8.71)$$

Solving this equation is equivalent to solving the second-order difference equation for

$$F(\nu+2) + C_1(\nu)\,F(\nu+1) + C_2(\nu)\,F(\nu) = 0, \quad (8.72)$$

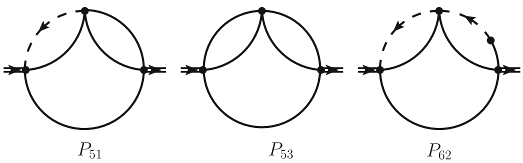

Fig. 8.7 Lower masters integrals P_{51}, P_{53}, P_{62}

where C_1 and C_2 can be expressed in terms of the matrix elements of $\mathbb{C}(\nu)$.

The maximally cut MMI $\varDelta\mathbf{F}$ satisfies 8.71. Observe that contracting the lower line of F_1 in Fig. 8.6 we obtain a scaleless integral which is zero so that it is not necessary to cut this line. Let us omit the factors $-2\pi i$ from each cut and let us take F_1 and perform the replacements $1/(k^2+i0) \to \delta(k^2)$ and $1/(v\cdot k+i0) \to \delta(v\cdot k) = \delta(k_0)$ for all the propagators apart from $1/(l+q)^2$.

Let us, first, integrate over the loop momenta of the two identical one-loop subdiagrams consisting of one static and two usual propagators

$$J(l) = \int \frac{\mathrm{d}^d k}{\pi^{d/2}} \delta(k_0)\delta(k^2)\delta(l^2 - 2l\cdot k), \tag{8.73}$$

where $\delta(k_0)$ comes from $1/(v\cdot k+i0) = 1/(k_0+i0)$. Here is a subtle point because in Minkowskian metrics we might conclude that this integral is zero due to the kinematical restrictions. Indeed, in Minkowskian space the first two δ-functions result in $k=0$, which is incompatible with the last δ-function. Let us instead use the metric signature $(1, 1, -1, -1, \ldots)$, so that $k^2 = k_0^2 + k_1^2 - k_2^2 - \cdots - k_d^2 = k_0^2 + k_1^2 - \mathbf{k}^2$. Then a straightforward integration gives

$$J(l) = \frac{2^{2-d}\varOmega(d-2)}{\pi^{d/2}} \frac{(-l^2)^{d-4}}{(\mathbf{l}^2)^{(d-3)/2}}, \tag{8.74}$$

where $\mathbf{l}^2 = -l_1^2 + \mathbf{l}^2$, and $\varOmega(d) = 2\pi^{d/2}/\varGamma(d/2)$ is the volume of the unit hypersphere in Euclidean d-dimensional space.

To take the final integral

$$\varDelta F_1(\nu) = \frac{1}{\mathrm{i}^6} \int \frac{\mathrm{d}^d l}{\pi^{d/2}} \frac{J(l)^2}{(-(l+q)^2)} \tag{8.75}$$

we turn to Euclidean space and separate the two terms in the denominator of $1/(l_0^2 + (\mathbf{l}+\mathbf{q})^2)$ introducing a onefold MB representation. The factor $\frac{1}{\mathrm{i}^6}$ corresponds to six 'time-like' integration variables, two per each loop momenta.

Then the internal integration is taken straightforwardly and we arrive at the following result:

$$\begin{aligned}\varDelta F_1(\nu) = {} & \frac{2^{4-4\nu}\varGamma(6-3\nu)}{\varGamma(\nu-1)^2\,\varGamma\left(8-4\nu, 4\nu-\frac{13}{2}\right)} \frac{1}{2\pi\mathrm{i}} \int \mathrm{d}z \frac{\varGamma(-z)\varGamma\left(z+\frac{1}{2}\right)}{\varGamma(z+5-2\nu)} \\ & \times \varGamma\left(3\nu - \frac{11}{2} - z\right) \varGamma(z-4\nu+8)\,\varGamma(z+\nu-1)\,.\end{aligned} \tag{8.76}$$

It is easy to convert this representation into a linear combination of ${}_3F_2$ hypergeometric functions.

There are two series of poles from the right of the integration contour and three series of poles from the left:

$$
z_1 = n, \quad z_2 = 3\nu - \frac{11}{2} + n,
$$
$$
z_3 = -\frac{1}{2} - n, \quad z_4 = 4\nu - 8 - n, \quad z_5 = 1 - \nu - n,
$$

where $n = 0, 1, \ldots$. It turns out that the sum of the residues taken at each of these five series of poles gives a solution of 8.72. This can be checked either numerically, or using the Zeilberger's method of creative telescoping [46, 47]. We assume, of course, that the corresponding sums are defined in some region of ν where they converge and then analytically continued to the whole ν complex plane. As two independent solutions let us choose the contribution of the series of residues at z_1 and z_4. The solutions have the form

$$
F^1_{1,h}(\nu) = \frac{\sqrt{\pi} 2^{4-4\nu} \Gamma(6-3\nu) \Gamma\left(3\nu - \frac{11}{2}\right)}{\Gamma(5-2\nu)\Gamma(\nu-1)\Gamma\left(4\nu - \frac{13}{2}\right)} \, {}_3F_2\left(\left. \begin{array}{c} 8-4\nu, \frac{1}{2}, \nu-1 \\ 5-2\nu, \frac{13}{2}-3\nu \end{array} \right| 1 \right),
$$
$$
F^2_{1,h}(\nu) = \frac{32\Gamma(6-3\nu)\Gamma(5\nu-9)\Gamma\left(\frac{5}{2}-\nu\right)}{2^{4\nu}(8\nu-15)\Gamma(\nu-1)^2\Gamma(2\nu-3)}
$$
$$
\times {}_3F_2\left(\left. \begin{array}{c} 8-4\nu, \frac{5}{2}-\nu, 4-2\nu \\ 10-5\nu, \frac{17}{2}-4\nu \end{array} \right| 1 \right).
$$

Analytical properties of these solutions can be found from the above representation. The series for the hypergeometric functions ${}_3F_2$ here converge at $\operatorname{Re}\nu < 5/2$. In order to determine the analytical properties of $F^1_{1,h}(\nu)$ and $F^2_{1,h}(\nu)$ in the region $\operatorname{Re}\nu \geqslant 5/2$, one can use the recurrence relation (8.72). It would be more convenient to use the representation in terms of series converging uniformly in d. Luckily, both hypergeometric functions appear to be nearly-poised, and it is possible to transform them to Saalschutzian ${}_4F_3$ whose series converge uniformly in ν. Explicit expressions of $F^1_{1,h}(\nu)$ and $F^2_{1,h}(\nu)$ in terms of Saalschutzian ${}_4F_3$ can be found in [32].

Therefore, the matrix of fundamental solutions in (8.71) has the form $\mathbb{F}_h(\nu) = \begin{pmatrix} F^1_{1,h}(\nu) & F^2_{1,h}(\nu) \\ F^1_{2,h}(\nu) & F^2_{2,h}(\nu) \end{pmatrix}$, where $F^1_{2,h}(\nu)$ and $F^2_{2,h}(\nu)$ are obtained form the first equation of the system (8.71):

$$
F^1_{2,h}(\nu) = \frac{F^1_{1,h}(\nu+1) - C_{11}(\nu) F^1_{1,h}(\nu)}{C_{12}(\nu)}, \tag{8.77a}
$$

$$
F^2_{2,h}(\nu) = \frac{F^2_{1,h}(\nu+1) - C_{11}(\nu) F^2_{1,h}(\nu)}{C_{12}(\nu)}. \tag{8.77b}
$$

As in the case of one master integral in a given sector, we are now going to construct a summing factor which is inverse to a solution of the homogeneous equation and satisfies

$$\mathbb{S}(\nu) = \mathbb{S}(\nu + 1)\,\mathbb{C}(\nu)\,. \tag{8.78}$$

According to the recipe formulated in Sect. 5 of [34], the summing factor can be constructed from the fundamental solution as

$$\mathbb{S}(\nu) = \mathbb{W}(\nu)\, S(\nu) \begin{pmatrix} F^2_{2,h}(\nu) & -F^2_{1,h}(\nu) \\ -F^1_{2,h}(\nu) & F^1_{1,h}(\nu) \end{pmatrix}, \tag{8.79}$$

where

$$S(\nu) = \frac{2^{2\nu}(\nu-2)\Gamma(2\nu-3)^2\Gamma\left(4\nu-\frac{13}{2}\right)}{\Gamma\left(2\nu-\frac{7}{2}\right)^2\Gamma(2-\nu)^2\sin(\pi\nu)}$$

is a solution of the equation $S(\nu) = S(\nu+1)\det\mathbb{C}(\nu)$ and $\mathbb{W}(\nu)$ is an arbitrary periodic matrix. Using (8.69) and (8.79) we obtain the relation

$$(\mathbb{S}\mathbf{F})(\nu-1) = (\mathbb{S}\mathbf{F})(\nu) + \mathbb{S}(\nu-1)\,\mathbf{R}(\nu) \tag{8.80}$$

which implies

$$(\mathbb{S}\mathbf{F})(\nu) = \mathbf{W}(\nu) + \sum_{+\infty}\mathbb{S}(\nu-1)\,\mathbf{R}(\nu)\,, \tag{8.81}$$

where $\mathbf{W}(\nu)$ is an arbitrary periodic column-vector and the notation $\sum_{\pm\infty} f(\nu)$ introduced in [37] means

$$\sum_{+\infty} f(\nu) = -\sum_{n=0}^{\infty} f(\nu+n)\,,$$
$$\sum_{-\infty} f(\nu) = \sum_{n=1}^{\infty} f(\nu-n)\,. \tag{8.82}$$

Now we need to determine $\mathbf{W}(\nu)$ from the analytical properties of $(\mathbb{S}\mathbf{F})(\nu)$ which depend on our choice of $\mathbb{W}(\nu)$. In particular, if we choose $\mathbb{W}(\nu) = 1$, the function $(\mathbb{S}\mathbf{F})$ has singularities at $\nu = 2, 2\frac{1}{6}, 2\frac{1}{5}, 2\frac{1}{3}, 2\frac{2}{5}, 2\frac{1}{2}, 2\frac{3}{5}, 2\frac{2}{3}, 2\frac{4}{5}, 2\frac{5}{6}$ in the stripe $\mathrm{Re}\,\nu \in [2, 3)$. In order to cancel these singularities, we can choose $\mathbb{W}(\nu)$ to be properly degenerate matrix at the points of the singularities but we should also try not to spoil the behavior of $(\mathbb{S}\mathbf{F})$ at $\nu \to \pm i\infty$. Therefore, it is very useful to eliminate also the explicit and hidden zeros of $\mathbb{S}$, which, at $\mathbb{W}(\nu) = 1$, are located at the points $\nu = 2\frac{1}{8}, 2\frac{3}{8}, 2\frac{5}{8}, 2\frac{7}{8}, \pm i\infty$. We finally choose

$$\begin{aligned}\mathbb{W}(\nu) = {} & \frac{(1+c)(1+2c)}{c^2} \\ & \times \begin{pmatrix} 2^5(1-c)\left(1-2c-4c^2\right) & 2^5\frac{1+c}{2c^2-1}(1-2c)^2 \\ -\frac{c}{\sqrt{2}}\left(1-2c-4c^2\right) & \frac{c(1-2c)}{\sqrt{2}(2c^2-1)}\left(1-2c-4c^2\right) \end{pmatrix},\end{aligned} \tag{8.83}$$

where $c = \cos(2\pi\nu)$. With this choice of the summing factor, $(\mathbb{S}\mathbf{F})$ is holomorphic in the stripe $\operatorname{Re}\nu \in [2, 3)$ and grows at $\nu \to \pm i\infty$ slower than $\exp(2\pi|\nu|)$. Taking into account the singularities of $\sum_{+\infty} \mathbb{S}(\nu-1)\mathbf{R}(\nu)$, we obtain

$$\mathbf{W}(\nu) = \frac{4\pi^2}{\sin^2(\pi\nu)}\left(\pi - 2\arctan\left(4\sqrt{5}\right)\cos^2(\pi\nu)\right)\begin{pmatrix} -64 \\ \sqrt{2} \end{pmatrix}. \tag{8.84}$$

Multiplying (8.81) by $\mathbb{S}^{-1}(\nu)$ we obtain

$$\mathbf{F}(\nu) = \mathbb{S}^{-1}(\nu)\mathbf{W}(\nu) + \mathbb{S}^{-1}(\nu)\sum_{+\infty}\mathbb{S}(\nu-1)\mathbf{R}(\nu). \tag{8.85}$$

With $\mathbb{S}(\nu)$, $\mathbf{W}(\nu)$, $\mathbf{R}(\nu)$ determined by (8.79), (8.83), (8.84), and the explicit expression for the right-hand side of (8.69) presented in [32] the above representation (8.85) gives a result for the MMI $\mathbf{F}(\nu) = \begin{pmatrix} F_1(\nu) \\ F_2(\nu) \end{pmatrix}$ in terms of a series.

Let us make two remarks about the two terms in this representation of $F_{1,2}(\nu)$. The second term, in fact, does not depend on the explicit form of the summing factor $\mathbb{S}(\nu)$ because

$$\mathbb{S}^{-1}(\nu)\mathbb{S}(\nu+n) = \begin{cases} \prod_{k=1}^{n}\mathbb{C}(\nu+k), & n \geqslant 0 \\ \prod_{k=0}^{-n-1}\mathbb{C}^{-1}(\nu-k), & n < 0 \end{cases}$$

is always a finite product of rational matrices. This product can be evaluated recursively, so that one can organize a numerical evaluation without nested loops. The first term can explicitly be written as a combination of the fundamental solutions $\mathbf{F}_h^1$ and $\mathbf{F}_h^2$:

$$\mathbb{S}^{-1}(\nu)\mathbf{W}(\nu) = \begin{pmatrix} F_{1,h}(\nu) \\ F_{2,h}(\nu) \end{pmatrix},$$

$$\begin{aligned} F_{1,h} &= \frac{2^5\pi^{5/2}\left(\pi - (c+1)\arctan\left(4\sqrt{5}\right)\right)}{(1-c)c^2(2c+1)} \\ &\quad\times\left[\frac{(2c-1)\left(4c^3-2c+1\right)}{2c^2-1}F_{1,h}^1 - \left(8c^3-4c+1\right)F_{1,h}^2\right], \end{aligned} \tag{8.86}$$

$$F_{2,h} = \frac{F_{1,h}(\nu+1) - C_{11}(\nu)F_{1,h}(\nu)}{C_{12}(\nu)}. \tag{8.87}$$

Now, since the evaluation of all the nested sums appearing in representation (8.85) can be organized in one loop, it is easy to calculate $\mathbf{F}(\nu)$ with a high precision and apply the PSLQ algorithm. Then we obtain [32]

$$
\begin{aligned}
F_1\,(2-\epsilon) &= \frac{28\pi^4}{135\epsilon} + \frac{116\pi^2\zeta(3)}{9} + \pi^4\left(\frac{224}{135} - 4\ln(2)\right) + \frac{226\zeta(5)}{3} \\
&+ \Bigg(-192s_6 + \frac{1808\zeta(5)}{3} - \frac{8\zeta(3)^2}{3} + \frac{928\pi^2\zeta(3)}{9} + 64\pi^2\mathrm{Li}_4\left(\frac{1}{2}\right) + \frac{8}{3}\pi^2\ln^4(2) \\
&- \frac{20}{3}\pi^4\ln^2(2) - 32\pi^4\ln(2) - \frac{428\pi^6}{2835} + \frac{1792\pi^4}{135}\Bigg)\epsilon \\
&+ \Bigg(-768\mathrm{Li}_4\left(\frac{1}{2}\right)\zeta(3) - 128\pi^2\mathrm{Li}_5\left(\frac{1}{2}\right) + 512\pi^2\mathrm{Li}_4\left(\frac{1}{2}\right) - 1536s_6 \\
&+ \frac{384}{7}s_6\ln(2) - \frac{384s_{7a}}{7} - \frac{3072s_{7b}}{7} + \frac{4960\zeta(7)}{21} + \frac{35519\pi^2\zeta(5)}{42} \\
&+ \frac{14464\zeta(5)}{3} - \frac{64\zeta(3)^2}{3} - \frac{31457\pi^4\zeta(3)}{945} + \frac{7424\pi^2\zeta(3)}{9} - 32\zeta(3)\ln^4(2) \\
&+ 372\zeta(5)\ln^2(2) + 32\pi^2\zeta(3)\ln^2(2) - \frac{480}{7}\zeta(3)^2\ln(2) - \frac{3424\pi^6}{2835} + \frac{14336\pi^4}{135} \\
&+ \frac{16}{15}\pi^2\ln^5(2) + \frac{64}{3}\pi^2\ln^4(2) - \frac{40}{9}\pi^4\ln^3(2) - \frac{160}{3}\pi^4\ln^2(2) - \frac{3079}{315}\pi^6\ln(2) \\
&- 256\pi^4\ln(2)\Bigg)\epsilon^2 + O(\epsilon^3),
\end{aligned}
$$

$$
\begin{aligned}
F_2\,(2-\epsilon) &= -\frac{\pi^4}{\epsilon} - 93\zeta(5) - 14\pi^2\zeta(3) - 2\pi^4\ln(2) + \Bigg(-96s_6 + 120\zeta(3)^2 \\
&+32\pi^2\mathrm{Li}_4\left(\frac{1}{2}\right) + \frac{4}{3}\pi^2\ln^4(2) - \frac{10}{3}\pi^4\ln^2(2) - \frac{989\pi^6}{420}\Bigg)\epsilon \\
&+ \Bigg(-384\mathrm{Li}_4\left(\frac{1}{2}\right)\zeta(3) - 64\pi^2\mathrm{Li}_5\left(\frac{1}{2}\right) + \frac{192}{7}s_6\ln(2) - \frac{192s_{7a}}{7} \\
&- \frac{1536s_{7b}}{7} - \frac{32666\zeta(7)}{7} - \frac{40585\pi^2\zeta(5)}{84} + \frac{35047\pi^4\zeta(3)}{630} - 16\zeta(3)\ln^4(2) \\
&+ 186\zeta(5)\ln^2(2) + 16\pi^2\zeta(3)\ln^2(2) - \frac{240}{7}\zeta(3)^2\ln(2) + \frac{8}{15}\pi^2\ln^5(2) \\
&- \frac{20}{9}\pi^4\ln^3(2) - \frac{3079}{630}\pi^6\ln(2)\Bigg)\epsilon^2 + O(\epsilon^3),
\end{aligned}
$$

where $\zeta_{\ldots}$ are multiple zeta values (C.4) and the constants s_6 and s_{7i} are defined by (8.39). The terms up to ϵ^1 are in agreement with the previous results [40, 41].

8.5 What Numbers Can Appear in Epsilon Expansions?

Let me emphasize that if we apply the DRA method and obtain a result in terms of a multiple well convergent series, going to higher powers of ϵ is an easy procedure, in contrast to other methods, in particular, to the method of MB representation, where the complexity of the evaluation rapidly increases for higher powers. This feature provides the possibility to check hypotheses about types of irrational constants present in ε-expansions.

It looks like massless propagator diagrams form the simplest class of one-scale Feynman integrals. Calculational experience shows that only MZV appear in their ε-expansions. In one loop, this is an obvious statement following from the one-loop formula (3.6). In two loops, this property follows already from an IBP reduction to integrals expressed in terms of gamma functions as described in Sect. 6.1. It was also proven in [7] for more general massless propagator diagrams with indices linear dependent on ε. Brown has proven [10] that this property holds for convergent scalar massless planar propagator diagrams with the degree of divergence $\omega = -2$ up to five loops.

Brown also has proven that for three four-loop non-planar diagrams, $M_{4,5}$ in Fig. 8.8 and two more convergent diagrams, every coefficient in a Taylor expansion in ϵ is a rational linear combination of MZV and multiple polylogarithms (see (11.43)) with sixth roots of unity as arguments. In fact, those two 'additional' diagrams *are not* master integrals and, consequently, they do not appear in the list of the twenty eight master integrals shown in Fig. 8.8 which are *all* four-loop massless propagator master integrals [3]. So, they can be reduced to other master integrals [29] which turn our to be planar and should have only MZV in their ε-expansions according to the above mentioned result of [10].

The ε-expansion of $M_{4,5}$ was evaluated in [29] up to transcendentality weight twelve and only MZV were observed there. Moreover, this was also done in [30] with DRA for *all* the four-loop massless propagator master integrals, i.e. diagrams of Fig. 8.8 (where the complexity levels of the master integrals are also shown), with the same qualitative conclusion that only MZV are present there. (For the IBP reduction of integrals in DRR, the `C` version of `FIRE` [38] was used.) Since any other four-loop massless propagator integral, with any integer powers of numerators and propagators, can be represented, due to an IBP reduction, as a linear combination of the master integrals, with coefficients which are rational functions of d, we come to the conclusion that any four-loop massless propagator integral has only MZV in its epsilon expansion up to transcendentality weight twelve. This means that if we want to find something beyond MZV in four-loop massless propagator diagrams we have to go to higher transcendentality weights. This is certainly possible within DRA method.

Taking these results into account one obtains more motivations to try to prove that there are only MZV in massless propagator diagrams. One more confirmation of this hypothesis, for an infinite series of diagrams, is the proof [11] of the conjecture of [9] about so-called zig-zag propagators graphs expressed in terms of odd zeta

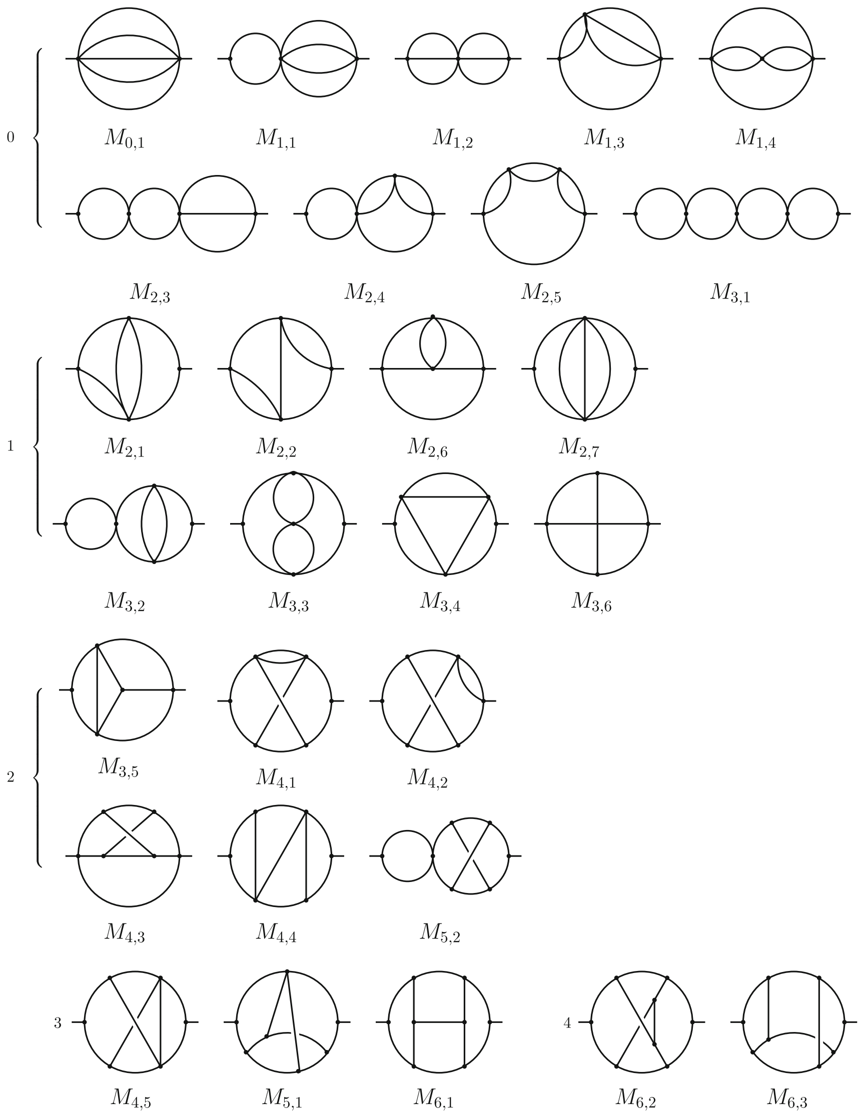

Fig. 8.8 Master integrals for massless four-loop propagator diagrams. The complexity level is indicated to the left

values $\zeta(2n-1)$. Another alternative is to continue to look for unusual constants in higher loops. Keeping in mind the dramatic progress of the last years in the field of evaluating Feynman integrals, this also looks to be a possible scenario.

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Chapter 9
Asymptotic Expansions in Momenta and Masses

If a given Feynman integral depends on kinematic invariants and masses which essentially differ in scale, a natural idea is to expand it in ratios of small and large parameters. As a result, the integral is written as a series of simpler quantities than the original integral itself and it can be substituted by a sufficiently large number of terms of such an expansion. For limits typical of Euclidean space (for example, the off-shell large-momentum limit or the large-mass limit), one can write down the corresponding asymptotic expansion in terms of a sum over certain subgraphs of a given graph [3–7, 14, 15, 17]. This prescription of expansion by subgraphs has been mathematically proven (see [14] and Appendix B.2 of [17]).

For limits typical of Minkowski space (i.e. which cannot be formulated in Euclidean space) the universal strategy of expansion by regions [2, 16–19] is available. The two strategies are explained in details in my book [17]. The goal of this chapter is to present important additional developments which have appeared after its publication [9, 10]. It will be explained how to reveal algorithmically regions relevant to a given limit.

In the first two sections, the two strategies are formulated and illustrated through simple one- and two-loop examples. In Sect. 9.3, the expansion by regions is formulated in terms of parametric representations of Feynman integrals. Then, in Sects. 9.4–9.6, a geometrical algorithm [9, 10] to reveal regions is described, following [10] and [9], correspondingly. Finally, in Sect. 9.7 the mathematical status of expansion by regions is discussed.

9.1 Expansion by Subgraphs

Let us consider a Feynman integral F_Γ, corresponding to a graph Γ, in an off-shell large-momentum limit. So, F_Γ depends on the large Euclidean external momenta $Q_1, \ldots, Q_{n_1}$ and the small external momenta $q_1, \ldots, q_{n_2}$. All the masses are supposed to be small. Its asymptotic expansion in this limit is described by the following

V. A. Smirnov, *Analytic Tools for Feynman Integrals*, Springer Tracts in Modern Physics 250, DOI: 10.1007/978-3-642-34886-0_9,

short formula [3–7, 14, 15, 17]

$$F_\Gamma \sim \sum_\gamma F_{\Gamma/\gamma} \circ \mathcal{M}_\gamma F_\gamma, \tag{9.1}$$

which needs some explanations.

The sum (9.1) runs over *asymptotically irreducible (AI)* subgraphs. To define this class of subgraphs let us denote by $\hat{\gamma}$ the graph that is obtained from a given subgraph γ by identifying[1] all the external vertices associated with the large external momenta. All the graphs of this form are subgraphs of the graph $\hat{\Gamma}$. So, a subgraph γ is AI if

(i) it contains all the vertices with the large external momenta and
(ii) $\hat{\gamma}$ is one-particle-irreducible (1PI).

Furthermore, F_γ and $F_{\Gamma/\gamma}$ are Feynman integrals for the graphs γ and Γ/γ, correspondingly. The operator $\mathcal{M}_\gamma$ corresponding to an AI subgraph γ is the Taylor expansion operator with respect to its masses and small external momenta:

$$\mathcal{M}_\gamma F_\Gamma = \int \mathrm{d}k_1 \ldots \mathrm{d}k_h \, \Pi_{\Gamma/\gamma} \mathcal{T}_{q_1,\ldots,m_1,\ldots,k_{h(\gamma)+1},\ldots,k_h} \Pi_\gamma, \tag{9.2}$$

where the integrand is naturally subdivided into the factors Π_γ and $\Pi_{\Gamma/\gamma}$ corresponding to γ and Γ/γ. It is implied that the large external momenta flow through γ, so that there is no dependence on them in $\Pi_{\Gamma/\gamma}$. The loop momenta $k_{h(\gamma)+1}, \ldots, k_h$ are external for γ and, by definition, they are considered small.

For any fixed order of expansion, $\mathcal{M}_\gamma F_\gamma$ is a polynomial in the external momenta and the loop momenta of Γ/γ. The symbol $\circ$ denotes the insertion of the polynomial which stands to the right of it into the reduced vertex of the graph Γ/γ, i.e. to the vertex to which the subgraph γ was reduced.

As explained in [17], the remainder of expansion (9.1) can be described by the forest formula based on operators $\mathcal{M}_\gamma$ and, in this formula, the sum runs over *nests* of AI subgraphs, i.e. families which can be ordered with respect to the inclusion, $\gamma^1 \subset \gamma^2 \subset \ldots$.

Let me illustrate this general prescription using a couple of examples, in particular, our favourite one.

Example 9.1 One-loop propagator Feynman integral (1.2) corresponding to Fig. 1.1 with the indices 2 and 1:

$$F_\Gamma(q^2, m^2; d) = \int \frac{\mathrm{d}^d k}{(k^2 - m^2)^2 (q-k)^2}. \tag{9.3}$$

There are contributions of two subgraphs: the graph Γ itself and the subgraph γ consisting of the massless line. The contribution from Γ is obtained by the expansion of the massive propagator in a Taylor series in the mass m. The contribution from

[1] Another possibility is to introduce a new vertex and connect it with each external vertex by a new line.

γ is obtained by expanding the propagator $1/(q-k)^2$ in a Taylor series in the loop momentum k which is external for the subgraph:

$$\frac{1}{(q-k)^2} = \frac{1}{q^2} + \frac{2q \cdot k - k^2}{(q^2)^2} + \frac{(2q \cdot k - k^2)^2}{(q^2)^3} + \cdots . \tag{9.4}$$

There is no contribution from the subgraph consisting of the massive line because the corresponding integral is a zero scaleless integral.

Adding the two contributions we arrive at the following expansion:

$$\begin{aligned} F_\Gamma(q^2, m^2; d) \sim & \int \frac{\mathrm{d}^d k}{(k^2)^2 (q-k)^2} - 2m^2 \int \frac{\mathrm{d}^d k}{(k^2)^3 (q-k)^2} + \cdots \\ & + \frac{1}{q^2} \int \frac{\mathrm{d}^d k}{(k^2 - m^2)^2} + \frac{1}{(q^2)^2} \int \frac{(2q \cdot k - k^2)\, \mathrm{d}^d k}{(k^2 - m^2)^2} + \cdots . \end{aligned} \tag{9.5}$$

Evaluating the integrals on the right-hand side by use of (10.6) and (10.7), we obtain the following result:

$$\begin{aligned} F_\Gamma(q^2, m^2; d) \sim & \frac{\mathrm{i}\pi^{d/2}}{(-q^2)^{1+\varepsilon}} \frac{\Gamma(1-\varepsilon)^2 \Gamma(\varepsilon)}{\Gamma(1-2\varepsilon)} \left(1 + 2\varepsilon \frac{m^2}{q^2} + \cdots \right) \\ & + \frac{\mathrm{i}\pi^{d/2}}{q^2 (m^2)^\varepsilon} \Gamma(\varepsilon) \left(1 + \frac{\varepsilon}{1+\varepsilon} \frac{m^2}{q^2} + \cdots \right). \end{aligned} \tag{9.6}$$

The pole in ε in the contribution from Γ is of IR nature, while the pole in the contribution from γ is UV. In fact, poles are present only in the leading-order terms. We observe that the poles are cancelled to produce a result finite at $\varepsilon = 0$:

$$F_\Gamma(q^2, m^2; 4) \sim \frac{\mathrm{i}\pi^2}{q^2} \left[\ln \left(\frac{-q^2}{m^2} \right) - \frac{m^2}{q^2} + \cdots \right]. \tag{9.7}$$

An arbitrary, nth, term of the expansion can easily be evaluated, with a subsequent summation to obtain (1.5).

If both masses are non-zero in such an example we have three contributions. Let us consider

Example 9.2 The one-loop self-energy diagram of Fig. 2.2 given by (2.9) with equal non-zero masses:

$$F_\Gamma(q^2, m^2; d) = \int \frac{\mathrm{d}^d k}{(k^2 - m^2)[(q-k)^2 - m^2]}. \tag{9.8}$$

The contribution from the subgraph consisting of the line for $1/[(q-k)^2 - m^2]$ changes to

$$\frac{1}{(q-k)^2-m^2} = \frac{1}{q^2} + \frac{2q\cdot k - k^2 + m^2}{(q^2)^2} + \frac{(2q\cdot k - k^2 + m^2)^2}{(q^2)^3} + \cdots \quad (9.9)$$

and we obtain a non-zero contribution from the subgraph consisting of the line for $1/(k^2 - m^2)$ which is, of course, equal to the previous one. Then the calculations are similar to the Example 9.1. Eventually, one can evaluate general terms of expansion for all the contributions and sum them up, reproducing the results (1.7) or (at general ε) (1.8).

Let us now consider

Example 9.3 The two-loop propagator diagram of Fig. 3.10 with arbitrary integer powers of the propagators and general masses which are small with respect to the external momentum.

The Feynman integral has the form

$$F_\Gamma(a_1, \ldots, a_5; m_1, \ldots, m_5) = \int \frac{\mathrm{d}^d k}{(k^2 - m_1^2)^{a_1}[(q-k)^2 - m_2^2]^{a_2}} \times \int \frac{\mathrm{d}^d l}{(l^2 - m_3^2)^{a_3}[(q-l)^2 - m_4^2]^{a_4}[(k-l)^2 - m_5^2]^{a_5}} . \quad (9.10)$$

The corresponding set of AI subgraphs is shown in Fig. 9.1. The other subgraphs are not AI and do not contribute to (9.1). For example, the subgraph {1, 2, 5} does not have a path between external vertices, and the subgraph {1, 3, 5} is one-particle reducible even after identifying the two external vertices. In the ‘physical’ language, the flow of the external momentum cannot be distributed through line 5.

Let us now choose a specific case, where $m_1 = \ldots = m_4 = m$ and $m_5 = 0$ and all the indices a_i are equal to one. The contribution of Type 1, i.e. when $\gamma = \Gamma$, is obtained by Taylor-expanding the propagators in the masses. The resulting massless two-loop self-energy diagrams given by (3.44) can easily be evaluated by means of the triangle rule following from IBP as it was explained in Sect. 6.1.

All four contributions of Type 2 (see Fig. 9.1) are equal to each other because of the symmetry of the diagram. The subgraph {2, 3, 4, 5} generates the following contribution:

$$\int \frac{\mathrm{d}^d k}{k^2 - m^2} \int \mathrm{d}^d l \, \mathcal{T}_{k,m} \frac{1}{(l^2 - m^2)[(q-k)^2 - m^2](k-l)^2[(q-l)^2 - m^2]}$$
$$= \frac{1}{q^2} \int \frac{\mathrm{d}^d k}{k^2 - m^2} \int \frac{\mathrm{d}^d l}{(l^2)^2 (q-l)^2} + \cdots$$
$$= -\left(\mathrm{i}\pi^{d/2}\right)^2 \frac{\Gamma(\varepsilon - 1) G(2,1)}{(-q^2)^{2+\varepsilon}(m^2)^{\varepsilon - 1}} + \cdots , \quad (9.11)$$

where the function G is given by (3.7). All the terms in the Type 2 contributions are products of massless one-loop integrals and massive vacuum integrals with numerators and can analytically be evaluated by means of (10.2) and (10.10), respectively.

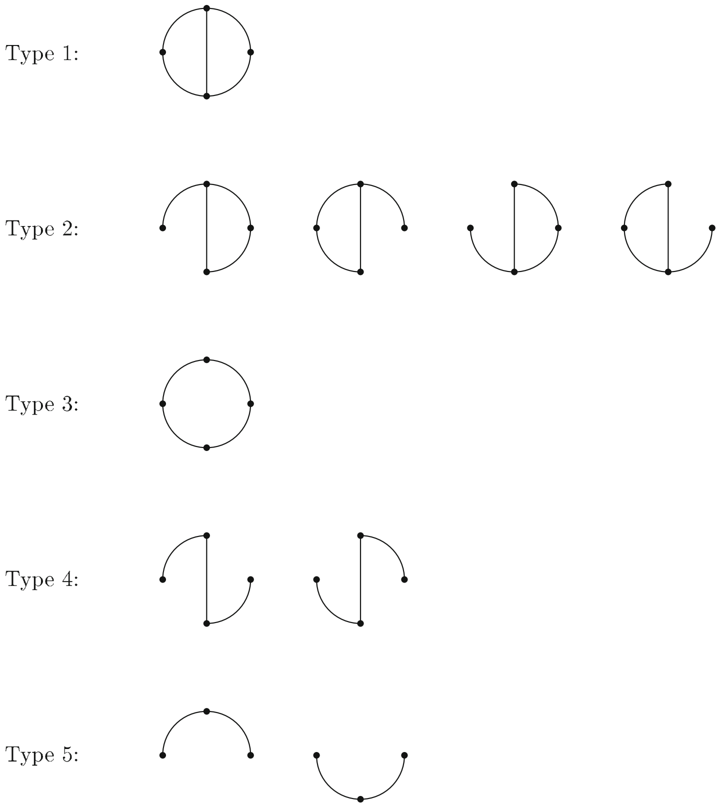

Fig. 9.1 The subgraphs contributing to the large-momentum expansion of Fig. 3.10

In our simple case, the Type 3 contribution is zero because the fifth line is massless. It would be non-zero in the case $m_5 \neq 0$ when the corresponding series in the result would have the same structure as that of the Type 2 contribution and would be evaluated by means of the same formulae.

For Type 4, we have two equal contributions, from $\gamma = \{1, 4, 5\}$ and $\{2, 3, 5\}$. Let $\gamma = \{1, 4, 5\}$. According to our prescriptions, we choose the loop momenta in a different way and let the external momentum flow through all three lines of the subgraphs. We obtain the following contribution:

$$
\begin{aligned}
&\int \frac{\mathrm{d}^d k}{k^2 - m^2} \int \frac{\mathrm{d}^d l}{l^2 - m^2} \mathcal{T}_{k,l,m} \frac{1}{[(q-l)^2 - m^2](q-k-l)^2[(q-k)^2 - m^2]} \\
&\quad = \frac{1}{(q^2)^3} \int \frac{\mathrm{d}^d k}{k^2 - m^2} \int \frac{\mathrm{d}^d l}{l^2 - m^2} + \cdots \\
&\quad = \left(\mathrm{i}\pi^{d/2}\right)^2 \frac{\Gamma(\varepsilon - 1)^2}{(q^2)^3 (m^2)^{2\varepsilon - 2}} + \cdots , \qquad (9.12)
\end{aligned}
$$

where all the terms on the right-hand side are products of tadpoles with numerators evaluated by means of (10.2).

There are two subgraphs of Type 5, {1, 3} and {2, 4}, with equal contributions. The subgraph {2, 4} gives

$$
\begin{aligned}
&\int\int \frac{\mathrm{d}^d k\, \mathrm{d}^d l}{(k^2-m^2)(l^2-m^2)(k-l)^2} \mathcal{T}_{k,l,m} \frac{1}{[(q-k)^2-m^2][(q-l)^2-m^2]} \\
&\quad = \frac{1}{(q^2)^2} \int\int \frac{\mathrm{d}^d k\, \mathrm{d}^d l}{(k^2-m^2)(l^2-m^2)(k-l)^2} + \cdots \\
&\quad = \left(\mathrm{i}\pi^{d/2}\right)^2 \frac{\Gamma(\varepsilon)^2}{(1-\varepsilon)(1-2\varepsilon)(q^2)^2(m^2)^{2\varepsilon-1}} + \cdots, \qquad (9.13)
\end{aligned}
$$

where all the terms on the right-hand side can be evaluated by means of (10.38) and its generalization to the case with numerators.

In the leading order (LO) of the expansion of our diagram, only Type 1 contributes. In the next-to-leading order (NLO) m^2, we also have contributions of Types 2 and 5 because Type 3 gives zero and Type 4 starts from the order m^4. Although the original diagram is finite, there are poles up to the second order in the individual contributions: IR poles in Type 1, products of UV and IR poles in Types 2–4 and UV poles in Type 5. Collecting the LO and NLO contributions we observe that the poles in ε are cancelled and we obtain the following result:

$$
\begin{aligned}
&F_\Gamma(1,\ldots,1; m,m,m,m,0) \\
&\quad \sim \left(\mathrm{i}\pi^2\right)^2 \left(\frac{6\zeta(3)}{q^2} + \frac{2m^2}{(q^2)^2}\left[\ln^2\left(-q^2/m^2\right) + 4\ln\left(-q^2/m^2\right) + 6\right]\right) + \cdots. \qquad (9.14)
\end{aligned}
$$

The large mass expansion as well as expansions in other limits typical of Euclidean space are described by the same formula (9.1) with the corresponding changes in the definition of AI subgraphs—see [15, 17].

9.2 Expansion by Regions

The strategy of expansion by regions [2, 16–19] consists of the following prescriptions:

- Divide the space of the loop momenta into various regions and, in every region, expand the integrand in a Taylor series with respect to the parameters that are considered small there.
- Integrate the integrand, expanded in the appropriate way in every region, over the *whole integration domain* of the loop momenta.
- Set to zero any scaleless integral.

For our favourite example of the integral (9.3), we have two regions that determine the asymptotic behaviour: $k \sim m$ and $k \sim q$. For the first of them, the massive propagator is not expanded and the massless propagator is expanded in k. For the second of them, the massive propagator is expanded in m and the massless propagator is not expanded. Thus, the region where $k \sim m$ corresponds to the contribution from the subgraph γ consisting of the massless line and the region where $k \sim q$ corresponds to the contribution from the graph Γ within expansion by subgraphs.

In fact, for limits typical of Euclidean space, where we have two scales, for example Q and q, there is a simple equivalence of the two strategies. For a given loop momentum we define regions where this momentum is large or small:

$$\text{large}, \;\; k \sim Q, \tag{9.15a}$$

$$\text{small}, \;\; k \sim q. \tag{9.15b}$$

Let us then define the set of regions labelled by 1PI subgraphs of the given graph:

$$\begin{aligned} k_i &\sim Q, \quad \text{if } \; k_i \text{ is a loop momentum of } \gamma, \\ k_i &\sim q, \quad \text{if } \; k_i \text{ is not a loop momentum of } \gamma. \end{aligned} \tag{9.16}$$

In the contribution from the region corresponding to a given γ, we can expand every propagator from γ not only in its masses and the small external momenta flowing through it but also in the rest of the loop momenta of the whole graph (which actually correspond to the reduced graph Γ/γ). We thus obtain nothing but the contribution from the subgraph γ within the method of expansion by subgraphs. So we reproduce, within expansion by regions, the general prescriptions (9.1).

However, expansion by regions works for much more general limits than expansion by subgraphs. Here two limits typical of Minkowski space are considered. (A lot of other examples can be found in [17].) Let us turn again to the diagram of Example 3.3, i.e.

Example 9.4 The massless on-shell box diagram of Fig. 3.6 with $p_i^2 = 0, \; i = 1, 2, 3, 4$ and all the indices equal to one

$$F(s, t; d) = \int \frac{\mathrm{d}^d k}{(k^2 + 2p_1 \cdot k)(k^2 - 2p_2 \cdot k)k^2(k + r)^2}, \tag{9.17}$$

where $r = p_1 + p_3$.

Let us expand it in the Regge limit, $|t| \ll |s|$ (which corresponds to the scattering at small angles). Let us choose the external momenta as follows:

$$p_{1,2} = (\mp Q/2, 0, 0, Q/2), \;\; r = (T/Q, 0, \sqrt{T + T^2/Q^2}, 0), \tag{9.18}$$

where $s = -Q^2$ and $t = -T$. The regions that are typical in this limit are *hard* and *collinear*. We have, in particular, the following region:

$$1-collinear(1c), \quad k_+ \sim T/Q, \ k_- \sim Q\,, \ \underline{k} \sim \sqrt{T}, \tag{9.19}$$

where $\underline{k} = (k_1, k_2)$ and $k_\pm$ are the light-cone coordinates $k_\pm = k_0 \pm k_3$ with $2p_{1,2} \cdot k = Qk_\pm$. The (2c) region is defined by the permutation of k_+ and k_- (i.e. p_1 and p_2).

The hard region generates a Taylor expansion of the integrand in t but, owing to the kinematics, this is a Taylor expansion in the four-vector $r = p_1 + p_3$. The leading hard term contributes in the next-to-leading-order, $1/s^2$. This term is given by the forward-scattering box, with $p_3 = -p_1$ and $p_4 = -p_2$, and can be evaluated by means of alpha parameters, with the following result:

$$F^{(\mathrm{h}),\,\mathrm{LO}} = \mathrm{i}\pi^{d/2} \frac{\Gamma(-\varepsilon)^2 \Gamma(1+\varepsilon)}{(1+\varepsilon)\Gamma(-2\varepsilon)(-s)^{2+\varepsilon}}. \tag{9.20}$$

In the (2c) region, k^2 and $p_2 \cdot k$ are of order T, while $p_1 \cdot k$ is of order Q^2. Moreover, $(k+r)^2 \equiv k^2 + 2k \cdot r - T \sim (k+\tilde{r})^2$, where

$$\tilde{r} = (T/(2Q), 0, \sqrt{T}, -T/(2Q)), \tag{9.21}$$

with $2p_1 \cdot \tilde{r} = 0$ and $-2p_2 \cdot \tilde{r} = \tilde{r}^2 = -T$.

Thus the (2c) contribution is obtained by expanding the propagator $1/(k^2+2p_1 \cdot k)$ in a Taylor series in k^2, and by expansion also in a Taylor series in $2p_1 \cdot r$. (Observe that we are dealing with a function of three kinematical variables, $2p_1 \cdot r$, $2p_2 \cdot r$ and r^2. So we expand the integrand (e.g. in the alpha representation) in $2p_1 \cdot r$ and then set $2p_1 \cdot r = T$.) Only the leading term in the Taylor expansion in k^2 is non-zero, because, starting from the next order, the factor k^2 cancels the propagator $1/k^2$ and we obtain a zero scaleless integral.

It turns out that the individual collinear contributions are not regularized dimensionally, so that we introduce an auxiliary analytic regularization by considering the powers of the first two propagators to be $1+\lambda_1$ and $1+\lambda_2$, with $\lambda_1 \neq \lambda_2$, evaluate both contributions and then take the limit $\lambda_2 \to \lambda_1 \to 0$ in the sum of the two contributions. The leading analytically regularized (2c) contribution is then easily evaluated by means of alpha parameters, with the following result:

$$F^{(2\mathrm{c}),\,\mathrm{LO}} = \mathrm{i}\pi^{d/2} \frac{\Gamma(\lambda_2-\lambda_1)\Gamma(1+\lambda_2+\varepsilon)\Gamma(-\lambda_2-\varepsilon)^2}{\Gamma(1+\lambda_2)\Gamma(-\lambda_1-\lambda_2-2\varepsilon)(Q^2)^{1+\lambda_1} T^{1+\lambda_2+\varepsilon}}. \tag{9.22}$$

The (1c) contribution is obtained by the change $\lambda_1 \leftrightarrow \lambda_2$. Summing the two collinear contributions and evaluating the next-to-leading contribution in a similar way, we obtain, in the limit $\lambda_{1,2} \to 0$,

$$\begin{aligned} F^{(\mathrm{c})} = \mathrm{i}\pi^{d/2} \frac{\Gamma(-\varepsilon)^2 \Gamma(1+\varepsilon)}{\Gamma(-2\varepsilon) s (-t)^{1+\varepsilon}} \Bigg[& \ln(t/s) + 2\psi(-\varepsilon) - \psi(1+\varepsilon) + \gamma_{\mathrm{E}} \\ & + \varepsilon \frac{t}{s} \left(\ln \frac{t}{s} + 2\psi(-\varepsilon) - \psi(1+\varepsilon) + \gamma_{\mathrm{E}} - \frac{1}{\varepsilon} - 1 \right) + \cdots \Bigg]. \end{aligned} \tag{9.23}$$

Combining (9.20) and (9.23) in the limit $\varepsilon \to 0$ we see that, up to the finite part in ε, the hard and collinear terms at next-to-leading order cancel each other. In fact, this phenomenon takes place in an arbitrary order of the expansion starting from the NLO, so that we are left with only the leading collinear contribution which reproduces the result obtained with Feynman parameters and given by (3.34).

As a second example of a limit typical of Minkowski space let us take the threshold limit. Let us consider

Example 9.5 The one-loop self-energy diagram of Fig. 2.2 given by (2.9) with equal non-zero masses in the threshold limit $q^2 \to 4m^2$.

Let us choose the loop momentum in another way (see Fig. 9.2) to obtain, instead of (9.8) (used in Example 9.2), the following integral:

$$F(q^2, y; d) = \int \frac{\mathrm{d}^d k}{(k^2 + q \cdot k - y)(k^2 - q \cdot k - y)}, \tag{9.24}$$

in order to make explicit the dependence of the propagators on the expansion parameter, $y = m^2 - q^2/4$, of the problem. This is a usual convention: in the case of two equal non-zero masses in the threshold, to let half of the external momentum flow through one of the massive lines and the other half of it through the other massive line. So we have transformed to the new variables $(q^2, m^2) \to (q^2, y)$.

Let us perform expansion by regions. The primary task is to identify all relevant regions in the problem. The hard region always contributes to the expansion in any limit. In this example, the hard region generates a 'naive' Taylor expansion of the integrand in y, where the integrals can be evaluated by means of partial fractions and the tabulated formula (10.13):

$$\begin{aligned} F^{(\mathrm{h})} &= \int \mathrm{d}^d k \, \mathcal{T}_y \frac{1}{(k^2 + q \cdot k - y)(k^2 - q \cdot k - y)} \\ &= \int \frac{\mathrm{d}^d k}{(k^2 + q \cdot k)(k^2 - q \cdot k)} + \cdots \\ &= \mathrm{i}\pi^{d/2} \left(\frac{4}{q^2}\right)^{\varepsilon} \sum_{n=0}^{\infty} \frac{\Gamma(n+\varepsilon)}{n!(1 - 2\varepsilon - 2n)} \left(\frac{-4y}{q^2}\right)^n . \end{aligned} \tag{9.25}$$

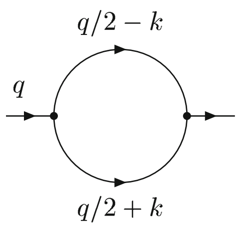

Fig. 9.2 One-loop propagator diagram with two non-zero masses in the threshold

Let us look for other regions. The soft region , $k \sim \sqrt{y}$, gives

$$
\begin{aligned}
F^{(\mathrm{s})} &= \int \frac{\mathrm{d}^d k}{(q \cdot k + \mathrm{i}0)(-q \cdot k + \mathrm{i}0)} + \cdots \\
&= -\int \frac{\mathrm{d}^d k}{(q \cdot k + \mathrm{i}0)(q \cdot k - \mathrm{i}0)} + \cdots = 0,
\end{aligned} \tag{9.26}
$$

because these are integrals without scale. Although the product in the integrand is, strictly speaking, ill-defined because of pinching singularities, the presence of a scaleless integral in the components of k additional to the linear combination $q \cdot k$ shows that the integral is zero, according to our prescriptions.

The ultrasoft region, $k \sim y/\sqrt{q^2}$, also generates (for the same reason) a zero contribution:

$$
\int \frac{\mathrm{d}^d k}{(q \cdot k - y + i0)(q \cdot k + y - i0)} + \cdots = 0. \tag{9.27}
$$

In order to find the missing contribution, let us choose the frame $q = (q_0, \mathbf{0})$ (keeping in mind the non-relativistic flavour of the problem). We have

$$
F = \int \frac{\mathrm{d}k_0 \, \mathrm{d}^{d-1}\boldsymbol{k}}{(\boldsymbol{k}^2 - k_0^2 + q_0 k_0 + y - \mathrm{i}0)(\boldsymbol{k}^2 - k_0^2 - q_0 k_0 + y - \mathrm{i}0)}. \tag{9.28}
$$

In any region other than the hard one, we have to suppose that some component of k is small. It is easy to observe that we have no chances of arriving at a non-zero contribution if we do not suppose that k_0 is small. When k_0 is small, i.e. at least $|k_0| \leq \sqrt{y}$, we can neglect k_0^2 in comparison with $q_0 k_0$. Thus both propagators are expanded in k_0^2, and we obtain

$$
\int \frac{\mathrm{d}k_0 \, \mathrm{d}^{d-1}\boldsymbol{k}}{(\boldsymbol{k}^2 + q_0 k_0 + y - \mathrm{i}0)(\boldsymbol{k}^2 - q_0 k_0 + y - \mathrm{i}0)} + \cdots . \tag{9.29}
$$

This series is already composed of quantities that are homogeneous with respect to the expansion parameter, y. Each term can be evaluated by, first, integrating over k_0 using Cauchy's theorem. To be consistent we have to decide that, for any term arising from the Taylor expansion in k_0^2, we will close the integration contour in the same half-plane. Let this be the upper half-plane, for definiteness. Observe that in this example the results for the contributions of this type do not depend on this choice. Observe also that, starting from some order of the expansion, the integrand does not vanish when $k_0 \to \infty$. Nevertheless, we do not pay attention to this fact and all the resulting integrals are by definition obtained by use of Cauchy's theorem. In fact, such manipulations can be proven by introducing an auxiliary analytic regularization—see Appendix A.2.3 of [8], where a similar example of a triangle diagram is studied in detail.

It turns out, however, that in this example, only the leading term of the contribution (9.29) survives because, for any of the subsequent terms, the resulting integrals in the vector component, $\boldsymbol{k}$, are integrals without scale. The leading term gives

$$\frac{\pi \mathrm{i}}{\sqrt{q^2}} \int \frac{\mathrm{d}^{d-1}\boldsymbol{k}}{\boldsymbol{k}^2 + y} = \mathrm{i}\pi^{d/2} \Gamma(\varepsilon - 1/2) \sqrt{\frac{\pi y}{q^2}} y^{-\varepsilon}, \tag{9.30}$$

where the spatial integral has been evaluated by means of the $(d-1)$-dimensional variant (with the replacement $\varepsilon \to \varepsilon + 1/2$) of (10.1).

Let us now come back to (9.29) and remember that we supposed that we had started from the region with small k_0. We have to say something about $\boldsymbol{k}$. We do not want the combination $\boldsymbol{k}^2 + q_0 k_0 + y$ to be expanded further because, otherwise, we will arrive at zero scaleless integrals. This requirement fixes absolutely the order of all the quantities involved, and we arrive at the following characterization of this new region [2]:

$$\textit{potential}(\mathrm{p}), \quad k_0 \sim y/\sqrt{q^2}, \quad \boldsymbol{k} \sim \sqrt{y}. \tag{9.31}$$

It is called 'potential' because it is connected with the Coulomb potential.

Thus we have contributions from two regions, and the whole expansion of the given diagram near threshold consists of (9.25) and

$$F^{(\mathrm{p})} = \mathrm{i}\pi^{d/2} \Gamma(\varepsilon - 1/2) \sqrt{\frac{\pi y}{q^2}} y^{-\varepsilon}. \tag{9.32}$$

The sum of these two contributions successfully reproduces the known analytic result for the given diagram,

$$F(q^2, y; d) = \mathrm{i}\pi^{d/2} \Gamma(\varepsilon) y^{-\varepsilon} \, {}_2F_1\left(\frac{1}{2}, \varepsilon; \frac{3}{2}; -\frac{q^2}{4y}\right), \tag{9.33}$$

which can be obtained (as well as an equivalent result (1.8)) by means of Feynman parameters. By use of (11.3), the right-hand side of (9.33) can be rewritten as a sum of two terms, exactly corresponding to the hard and potential contributions (9.25) and (9.32).

The general prescriptions for expanding Feynman integrals at threshold look as follows [2]:

- Choose the canonical routing for the flow of the external momenta for the given threshold. In particular, when there are two equal non-zero masses in the threshold, let one half of the external momentum flow through one of the massive lines and the other half of it through the other massive line. In the general situation with masses $m_i \equiv \xi_i m$ with $\sum \xi_i = 1$, let a portion $\xi_i q$ flow through line i.
- Choose the frame $q = (q_0, \mathbf{0})$.
- Consider the various regions where any loop momentum can be of one of the following four types:

$$
\begin{aligned}
\text{(hard)},\;\; & k_0 \sim \sqrt{q^2}\,,\;\; \boldsymbol{k} \sim \sqrt{q^2}\,, \\
\text{(soft)},\;\; & k_0 \sim \sqrt{y}\,,\;\; \boldsymbol{k} \sim \sqrt{y}\,, \\
\text{(potential)},\;\; & k_0 \sim y/\sqrt{q^2}\,,\;\; \boldsymbol{k} \sim \sqrt{y}\,, \\
\text{(ultrasoft)},\;\; & k_0 \sim y/\sqrt{q^2}\,,\;\; \boldsymbol{k} \sim y/\sqrt{q^2}.
\end{aligned}
$$

- Try various choices of the loop momenta (and at the same time avoid double counting).
- In accordance with the general strategy of expansion by regions formulated in the beginning of this section, extend the integration to the whole space and set scaleless integrals to zero.

9.3 Expansion by Regions in Parametric Representations

In fact, it is a non-trivial task to reveal the typical regions for a given limit. Usually, one starts from considering one-loop examples, checks the results against known analytical results, then proceeds in two loops etc. One can also use the second version [12] of the code `FIESTA` [13] to obtain numerically several first terms of a given asymptotic expansion, as explained in Sect. 5.8.

It turns out that it is reasonable to switch to expansion by regions for parametric representations of Feynman integrals [16] in order to arrive at an efficient algorithm for revealing regions. So, let us start from the parametric representation (3.36) or, equivalently, from (3.38), i.e.

$$
F_\Gamma(q_1,\ldots,q_n;d) = (-1)^a \frac{\left(\mathrm{i}\pi^{d/2}\right)^h \Gamma(a-hd/2)}{\prod_l \Gamma(a_l)}
\times \int_0^\infty \mathrm{d}\alpha_1 \ldots \int_0^\infty \mathrm{d}\alpha_L\, \delta\left(\sum \alpha_l - 1\right) \frac{\mathcal{U}^{a-(h+1)d/2} \prod_l \alpha_l^{a_l-1}}{(\mathcal{W}(s_1,s_2,\ldots))^{a-hd/2}}, \tag{9.34}
$$

where, in addition to (3.38), a dependence on kinematic invariants and/or masses in the function $\mathcal{W}$ is indicated. This collection of variables is denoted by $s_i\,,\;\; i = 1, 2, \ldots$.

Let us suppose that we have to study the asymptotic behaviour in a one-scale limit, i.e. every mass and kinematic invariant has a certain scaling expressed in powers of the small parameter of the problem, $s_i \to s_i' = \rho^{\kappa_i} s_i$, $i = 1, 2, \ldots$. The strategy of expansion by regions formulated in terms of parametric integrals [16, 17] states that the asymptotic expansion in this limit is given by a sum over regions in the space of alpha parameters. Let us realize that we are not starting from an 'honest' decomposition of an initial alpha integral over some regions. Rather, by regions we imply various relations between the integration variables, i.e. scalings of the alpha parameters in terms of the small parameter, ρ. So a region where $\alpha_1 \sim \rho^{r_1}, \ldots, \alpha_N \sim \rho^{r_N}$ is labelled by a vector $r = (r_1, \ldots, r_N)$ composed of the weights r_l.

The contribution of the region r is obtained by scaling the masses and kinematic invariants according to the given limit as well as by substituting $\alpha_l \to \alpha'_l = \rho^{r_l}\alpha_l,\ l = 1, \ldots, N$ in the integrand of (2.37) and expanding it in powers of ρ. Here one treats the product of the differentials as belonging to the integrand so that this gives the factor $\rho^{\sum r_l}$.

Let $I(\alpha_1, \ldots, \alpha_N; s_1, s_2, \ldots)$ be the integrand in (9.34), excluding the delta function. Explicitly, the contribution of the region r is given by the prefactor in (9.34) times $\rho^{\sum r_l}$ times the integral

$$\int_0^\infty \mathrm{d}\alpha_1 \ldots \int_0^\infty \mathrm{d}\alpha_N\, \delta\left(\sum \alpha_l - 1\right) I(\alpha'_1, \ldots, \alpha'_N; s'_1, s'_2, \ldots) \tag{9.35}$$

with the integrand expanded in powers of ρ.

Let us write down the LO contribution of a given region in a more explicit way. For the two basic functions in (3.38) we have

$$\begin{aligned} \mathcal{U}(\alpha'_1, \ldots, \alpha'_N) &= \sum_{j=n_{\min}(r;\mathcal{U})}^{n_{\max}(r;\mathcal{U})} \rho^j \mathcal{U}_j(\alpha_1, \ldots, \alpha_N), \\ \mathcal{W}(\alpha'_1, \ldots, \alpha'_N; s'_1, s'_2, \ldots) &= \sum_{j=n_{\min}(r;\mathcal{W})}^{n_{\max}(r;\mathcal{W})} \rho^j \mathcal{W}_j(\alpha_1, \ldots, \alpha_N; s_1, s_2, \ldots), \end{aligned} \tag{9.36}$$

where $\mathcal{U}_j$ and $\mathcal{W}_j$ are polynomials. According to the prescription formulated above the LO contribution of a region r is represented as

$$\rho^{\sum_l r_l a_l + n_{\min}(r;\mathcal{U})(a-(h+1)d/2) - n_{\min}(r;\mathcal{W})(a-hd/2)} \tag{9.37}$$

times

$$\begin{aligned} &(-1)^a \frac{\left(\mathrm{i}\pi^{d/2}\right)^h \Gamma(a - hd/2)}{\prod_l \Gamma(a_l)} \\ &\times \int_0^\infty \mathrm{d}\alpha_1 \ldots \int_0^\infty \mathrm{d}\alpha_L\, \delta\left(\sum \alpha_l - 1\right) \frac{\mathcal{U}_{n_{\min}(r;\mathcal{U})}^{a-(h+1)d/2} \prod_l \alpha_l^{a_l-1}}{\left(\mathcal{W}_{n_{\min}(r;\mathcal{W})}(s_1, s_2, \ldots)\right)^{a-hd/2}}. \end{aligned}$$

A region is determined up to adding a vector $(c, \ldots, c)$ with an arbitrary real number c because the corresponding contribution stays the same. In particular, the leading power behaviour determined by (9.37) is independent of such c.

The terms of the expansion come from various regions and can be ordered according to accompanying powers of ρ. After keeping some first terms of the expansion one can set $\rho = 1$ and write down the given Feynman integral as these selected first terms plus a remainder which has a sufficiently fast decrease in the given limit.

It turns out that only a finite number of regions contribute to the expansion because for the rest of the regions one obtains integrals without scale which are put to zero. In the next section, we will consider a geometrical algorithm that provides the possibility to find the relevant regions for a given limit.

9.4 How to Reveal Relevant Regions

Our goal is to find regions which have non-zero contributions. In fact, a contribution is zero if this is a scaleless integral. A sufficient condition for this is the homogeneity of the integrand with respect to a strict subset of the variables α_l, i.e the homogeneity of the functions $\mathcal{U}_{n_{\min}(r;\mathcal{U})}$ and $\mathcal{W}_{n_{\min}(r;\mathcal{W})}$ determined by (9.37) and involved in the LO contribution (9.38). Let ν be a strict subset of the set $\{1,\ldots,N\}$. So, if

$$\begin{aligned}\mathcal{U}_{n_{\min}(r;\mathcal{U})}\big|_{\alpha_l\to\kappa\alpha_l,l\in\nu} &= \kappa^{d_\mathcal{U}}\mathcal{U}_{n_{\min}(r;\mathcal{U})},\\ \mathcal{W}_{n_{\min}(r;\mathcal{W})}\big|_{\alpha_l\to\kappa\alpha_l,l\in\nu} &= \kappa^{d_\mathcal{W}}\mathcal{W}_{n_{\min}(r;\mathcal{W})},\end{aligned} \tag{9.38}$$

for some $d_\mathcal{U}$ and $d_\mathcal{W}$ then the LO contribution is zero because it is a scaleless integral. In this situation, all the next orders of the expansion are also zero because they differ from the LO contribution by changing powers of the functions $\mathcal{U}_{n_{\min}(r;\mathcal{U})}$ and $\mathcal{W}_{n_{\min}(r;\mathcal{W})}$ and inserting some polynomials. Instead of checking the homogeneity of the two functions we can consider their product $\mathcal{U}\mathcal{W}$.

Let us consider the function $\mathcal{W}(\alpha_1,\ldots,\alpha_N;s'_1,s'_2,\ldots)$, i.e. the function $\mathcal{W}$ where the scaling of the kinematic invariants according to a considered limit is introduced. This is a polynomial of α_l, s_i and ρ, i.e. each term considered at fixed numerical values of the kinematic invariants takes the form $\alpha_1^{w_1}\ldots\alpha_N^{w_N}\rho^{w_0}$, so that each monomial is labelled by an $(N+1)$-dimensional vector $w=(w_1,\ldots,w_N,w_0)$. So, the polynomial $\mathcal{W}$ generates, for a given limit, a finite set of points $P[\mathcal{W}]$ in the $(N+1)$-dimensional vector space. Since $\mathcal{W}$ is homogeneous in the variables α_l, all these points belong to the N-dimensional hyperplane $w_1+\cdots+w_N=h+1$. Similarly, the points of the set $P[\mathcal{U}]$ belong to the $(N-1)$-dimensional hyperplane $w_1+\cdots+w_N=h$, $w_0=0$.

Let us now introduce the scaling corresponding to a given region $r=(r_1,\ldots,r_N)$, i.e. $\alpha_l\to\alpha'_l=\rho^{r_l}\alpha_l$, $l=1,\ldots,N$. Then a given monomial $\alpha_1^{w_1}\ldots\alpha_N^{w_N}\rho^{w_0}$ transforms into $\alpha_1^{w_1}\ldots\alpha_N^{w_N}\rho^{r_1w_1+\cdots r_Nw_N+w_0}$ so that its dependence on ρ becomes $\rho^{w\cdot\bar r}$ where $\bar r=(r,1)\equiv(r_1,\ldots,r_N,1)$ and $w\cdot\bar r$ is the scalar product in $\mathbb{R}^{N+1}$.

The contribution of r is obtained by expanding in ρ when one keeps the minimal powers of ρ and obtains $P[\mathcal{W}_{n_{\min}(r;\mathcal{W})}]$ and $P[\mathcal{U}_{n_{\min}(r;\mathcal{U})}]$ defined by (9.36) as subsets of all the points $P[\mathcal{W}(\alpha'_1,\ldots,\alpha'_N;s'_1,s'_2,\ldots)]$ and $P[\mathcal{U}(\alpha'_1,\ldots,\alpha'_N)]$, correspondingly.

Before looking for relevant regions in the general case, let us turn to Example 9.2, i.e. the Feynman integral (9.8) in the large-momentum limit so that the corresponding scaling is $m^2 \to m'^2 = \rho m^2$. We have $\mathcal{U} = \alpha_1 + \alpha_2$ and $\mathcal{W}(m'^2) = \rho m^2 \alpha_1^2 + \rho m^2 \alpha_2^2 + (2\rho m^2 - q^2)\alpha_1\alpha_2$. To reveal relevant regions we consider the product of the two functions $\mathcal{U}\mathcal{W}$. The points for the polynomials involved are shown in Fig. 9.3. The three points $P[\mathcal{W}]$ are $A = (0, 2, 1)$, $B = (2, 0, 1)$, $C = (1, 1, 0)$. The two points of $P[\mathcal{U}]$ are $D = (0, 1, 0)$, $E = (1, 0, 0)$. The six points for $P[\mathcal{U}\mathcal{W}]$ are $A + D, A + E, B + D, B + E, C + D, C + E$.

As it was pointed out after (9.8), there are three contributions to the expansion. In the alpha-parametric language, they correspond to the following vectors in the three-dimensional space and the corresponding faces of the polyhedron of weights for $P[\mathcal{U}\mathcal{W}]$:

$$
\begin{aligned}
\bar{r} &= (0, 0, 1), \quad P[\mathcal{W}_{n_{\min}(r;\mathcal{W})}\mathcal{U}_{n_{\min}(r;\mathcal{U})}] = \{C + D, C + E\},\\
\bar{r} &= (1, 0, 1), \quad P[\mathcal{W}_{n_{\min}(r;\mathcal{W})}\mathcal{U}_{n_{\min}(r;\mathcal{U})}] = \{A + D, C + D\},\\
\bar{r} &= (0, 1, 1), \quad P[\mathcal{W}_{n_{\min}(r;\mathcal{W})}\mathcal{U}_{n_{\min}(r;\mathcal{U})}] = \{B + E, C + E\}.
\end{aligned}
$$

Let the function $\mathcal{W}$ be positive. It turns out [10] that if a region $(r_1, \ldots, r_N)$ gives a non-zero contribution then the vector $\bar{r} = (r_1, \ldots, r_N, 1)$ is orthogonal to one of the facets (i.e. faces of maximal dimension) of the envelope (or, the convex hull) of the set $P[\mathcal{W}(\alpha'_1, \ldots, \alpha'_N; s'_1, s'_2, \ldots)\mathcal{U}(\alpha'_1, \ldots, \alpha'_N)]$ and this vector is directed inside this set. For example, the fourth facet in Fig. 9.3 is irrelevant because the corresponding normal vector (where the third component is always equal to 1) is directed outside the set of weights.

To prove this statement let us consider the convex hull $H[\mathcal{W}\mathcal{U}]$ of the set $P[\mathcal{W}(\alpha'_1, \ldots, \alpha'_N; s'_1, s'_2, \ldots)\mathcal{U}(\alpha'_1, \ldots, \alpha'_N)]$. Observe that $P[\mathcal{W}\mathcal{U}]$ is not of full dimension $N+1$ because the points in it satisfy the condition $w_1 + \ldots + w_N = 2h+1$

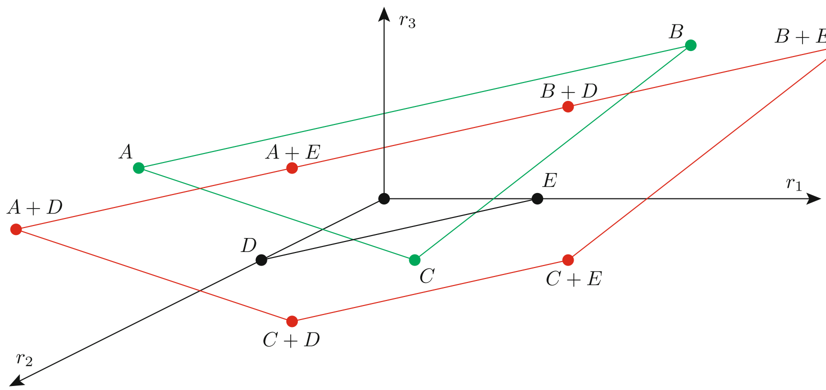

Fig. 9.3 Points corresponding to $P[\mathcal{U}]$, $P[\mathcal{W}(m'^2)]$ and $P[\mathcal{U}\mathcal{W}(m'^2)]$ for the one- loop propagator diagram

following from the homogeneity of the functions $\mathcal{W}$ and $\mathcal{U}$. Let us take a tangent hyperplane, orthogonal to $\bar{r}$. Its intersection with $H[\mathcal{W}\mathcal{U}]$ is a face G of $H[\mathcal{W}\mathcal{U}]$. Let us suppose that this face is not a facet. Then its dimension is no more than the dimension of the whole space of weights minus three, i.e. $N-2$. Hence the projection of G along the $(N+1)$-st axis also has at most dimension $N-2$. Therefore there are at least two linear independent vectors orthogonal to G, and we can choose its non-zero linear combination $(u_1, \ldots, u_N, 0)$ such that at least one of the u_i is equal to zero. However, the existence of such a vector means that the function $\mathcal{U}\mathcal{W}$ is homogeneous with respect to some rescaling of a strict subset of integration variables, i.e. relations of the type (9.38) are satisfied. Therefore, the integral under consideration is zero because it is scaleless.

Thus, regions with non-zero contributions correspond only to vectors, orthogonal to facets of $H[\mathcal{W}\mathcal{U}]$. The points corresponding to other terms should be 'above' the chosen hyperplane. Hence the regions with non-zero contributions correspond to the 'bottom side' of $H[\mathcal{W}\mathcal{U}]$.

An algorithm to find all the relevant regions [10] is based on the property proven above. So, the problem to find the regions reduces to a known problem of computation geometry—finding a convex hull of a given set of points in a finite-dimensional space. Here the public algorithm `quickhull` [1] can be applied. The corresponding algorithm of finding regions is called `asy.m` and can be downloaded from [11]. Let us point out that we meet the problem of finding a convex hull already for third time in this book: it appeared in Sects. 4.2 and 4.3 when studying various algorithms of sector decompositions.

Let us consider Examples 9.2 and 9.4 using `asy.m`. For the Feynman integral (9.8) in the large-momentum limit the code `asy.m` reports about the regions (0, 0), (1, 0) and (0, 1).

For the box in the Regge limit, it gives the regions (0, 0, 0, 0), (1, 0, 0, 0) and (0, 0, 1, 0). The contribution of the region (0, 0, 0, 0) is nothing but the hard contribution which is obtained by expanding the integrand in a Taylor series in t. In particular, the leading term can be evaluated in gamma functions in agreement with the result (9.20) obtained within expansion by regions in momentum space.

According to the prescriptions formulated in the previous section, the region (1, 0, 0, 0) gives, at leading order,

$$\begin{aligned} \mathrm{i}\pi^{d/2}\Gamma(2+\varepsilon) \int_0^\infty \ldots \int_0^\infty \mathrm{d}\alpha_1 \ldots \mathrm{d}\alpha_4 \delta\left(\sum \alpha_l - 1\right) \\ \times (\alpha_2+\alpha_3+\alpha_4)^{2\varepsilon} \left(-s\alpha_1\alpha_3 - t\alpha_2\alpha_4\right)^{-2-\varepsilon} . \end{aligned}$$

However, this contribution is not regularized dimensionally, so let us introduce an auxiliary analytic regularization, like in the example considered in the momentum-space picture. Introducing such a regularization in the exponents of α_1 and α_3, we obtain the integral

$$\mathrm{i}\pi^{d/2}\frac{\Gamma(2+\varepsilon+\lambda_1+\lambda_2)}{\Gamma(1+\lambda_1)\Gamma(1+\lambda_2)}\int_0^\infty\ldots\int_0^\infty \mathrm{d}\alpha_1\ldots\mathrm{d}\alpha_4\delta\left(\sum\alpha_l-1\right)\alpha_1^{\lambda_1}\alpha_3^{\lambda_2}$$
$$\times\,(\alpha_2+\alpha_3+\alpha_4)^{2\varepsilon+\lambda_1+\lambda_2}\,(-s\alpha_1\alpha_3-t\alpha_2\alpha_4)^{-2-\varepsilon-\lambda_1-\lambda_2}$$

which can easily be evaluated recursively in terms of gamma functions. We obtain nothing but the 2-collinear contribution (9.22). Similarly, the region (0, 0, 1, 0) gives the 1-collinear contribution so that we reproduce the result obtained in momentum space.

9.5 Revealing Potential Regions

The first version of the code `asy.m` was not capable of revealing potential regions. Let us consider again Example 9.5 where `asy.m` reported only about the hard region. The reason for this can be seen in the corresponding parametric representation,

$$F(q^2,y)=\mathrm{i}\pi^{d/2}\,\Gamma(\varepsilon)$$
$$\times\int_0^\infty\int_0^\infty\frac{(\alpha_1+\alpha_2)^{2\varepsilon-2}\,\delta\left(\alpha_1+\alpha_2-1\right)\;\mathrm{d}\alpha_1\mathrm{d}\alpha_2}{\left[\frac{q^2}{4}(\alpha_1-\alpha_2)^2+y(\alpha_1+\alpha_2)^2-\mathrm{i}0\right]^\varepsilon}, \tag{9.39}$$

where the parameters α_i are integrated from 0 to ∞ (restricted by the delta function). As it was pointed out in [10], it is the region where $\alpha_1 \approx \alpha_2$ (more precisely $\alpha_1-\alpha_2\sim y^{1/2}$) which causes problems. In other words, the polynomial in the square brackets in (9.39) (considered at positive q^2 and y) has terms of different sign, such that cancellations occur because of the presence of the negative term $-q^2\alpha_1\alpha_2/2$.

To reveal the missing potential contribution, one can perform a simple trick [9]. We decompose the integration domain into two subdomains, $\alpha_1\le\alpha_2$ and $\alpha_2\le\alpha_1$. The two resulting integrals are equal to each other, but such an equality will not generally take place for any integral. In the first domain we turn to new variables by $\alpha_1=\alpha_1'/2,\;\;\alpha_2=\alpha_2'+\alpha_1'/2$, remove the primes at α_i and obtain the integral (again from 0 to ∞ with the usual restrictions via the delta function)

$$\mathrm{i}I\pi^{d/2}\,\frac{\Gamma(\varepsilon)}{2}\int_0^\infty\int_0^\infty\frac{(\alpha_1+\alpha_2)^{2\varepsilon-2}\,\delta\left(\alpha_1+\alpha_2-1\right)\;\mathrm{d}\alpha_1\mathrm{d}\alpha_2}{\left[\frac{q^2}{4}\alpha_2^2+y(\alpha_1+\alpha_2)^2-\mathrm{i}0\right]^\varepsilon}. \tag{9.40}$$

The goal of this trick was to make the line $\alpha_1=\alpha_2$ (in the old variables) the border of an integration domain which turned out to be (in the new variables) $\alpha_2=0$. The Mathematica command for treating such parametrical integrals as well as other commands of `asy2.m` are described in details in [9]. This command reports about the regions (0, 0) and (0, 1/2). The first of them obviously corresponds to the hard region in momentum-space picture and the second of them provides the potential contribution.

According to the prescriptions for writing down contributions of regions formulated in Sect. 9.3, the contribution of the k-th order expansion of (9.40) in the potential region reads

$$\mathrm{i}\pi^{d/2} \frac{\Gamma(\varepsilon)}{2\,k!} \int_0^\infty \int_0^\infty \mathrm{d}\alpha_1 \mathrm{d}\alpha_2\, \alpha_2^k \left(\frac{\partial}{\partial \alpha_1}\right)^k \frac{\alpha_1^{2\varepsilon-2}\,\delta\left(\alpha_1 - 1\right)}{\left(\frac{q^2}{4}\alpha_2^2 + y\,\alpha_1^2\right)^\varepsilon}. \tag{9.41}$$

Only the leading order ($k = 0$),

$$\mathrm{i}\pi^{d/2} \frac{\Gamma(\varepsilon)}{2} \int_0^\infty \frac{\mathrm{d}\alpha_2}{\left(\frac{q^2}{4}\alpha_2^2 + y\right)^\varepsilon}, \tag{9.42}$$

yields a non-vanishing contribution which is evaluated in terms of gamma functions at general ε. Taking into account that we have two identical integrals after our decomposition, we reproduce the result (9.30).

In fact, such a trick of making manifest squares of some linear combination of the integration parameters was already used in the algorithm of the code `FIESTA` [12] (described in Sect. 4.2) in order to evaluate numerically Feynman integrals at a threshold. Using the implementation of this procedure in `FIESTA` it turned out to be possible to automate the above trick for a general Feynman integral.

The trick applied in the previous example can be generalized to look for potential regions for any diagram. The corresponding 'preresolution' algorithm implemented in `asy2.m` tries to eliminate factorized combinations of terms in the function $\mathcal{W}$ which potentially cancel each other, like $(\alpha_1 - \alpha_2)^2$ in the example above. It checks all pairs of variables (say, x and y) which are part of monomials with opposite sign. For all those pairs the code tries to build a linear combination z of x and y such that in the variables x and z or y and z this monomial disappears. The code checks whether in the new variables the number of monomials with opposite sign decreases. For all such pairs the code recursively repeats the initial procedure in the new variables. As a result it creates a tree of possible bisections and corresponding replacements of variables. A leaf of this tree is a set of sectors and functions such that one cannot decrease the number of monomials with opposite sign any longer. Ideally it means that all monomials now have the same sign. The code analyzes all leafs and chooses one of those with the minimal number of opposite-sign monomials (or the minimal number of sectors if the numbers of monomials with opposite sign coincide). After finishing with the preresolution, the code performs the replacements and looks for regions in all those sectors, using the algorithm of the original code `asy.m` described in [10].

It was checked [9] that the updated version `asy2.m` works in various examples of the threshold expansion (considered in [2, 17]): a triangle, a box, the two-loop propagator diagram (with the masses $m, m, m, m, 0$), a two-loop vertex diagram. Because of the decomposition of a given integration domain into subdomains, the number of resulting integrals for various regions increases a little bit. For example, the (hard-hard) region for the two-loop propagator diagram is described by six integrals, the (potential-ultrasoft) region is also described by six integrals, etc. However, the (potential-hard) region is described by four integrals with some regions (with scalings composed of powers 1, 1/2 and 0), and four more integrals with a set of regions of a different type (composed of 1 and 0).

9.6 Revealing Glauber Regions

It turns out that the previous version of the code `asy.m` failed not only for potential but also for so-called Glauber regions. Let us consider

Example 9.6 The one-loop five-point integral in Fig. 9.4.

Here two initial-state partons both perform a collinear splitting into two partons each with momenta p_1, p_2 and q_1, q_2, respectively. While two partons, one of each pair, collide with a large centre-of-mass energy $Q = \sqrt{(p_2 + q_2)^2}$, the two remaining partons exchange a particle with the small mass m. We will use the simplified kinematics $p_1 = p_2 = p$ and $q_1 = q_2 = q$ with $p^2 = q^2 = 0$ and $(p + q)^2 = 2p \cdot q = Q^2$ in the limit $m^2/Q^2 \to 0$:

$$F(Q^2, m^2) = \int \frac{\mathrm{d}^d k}{(k^2 - m^2)(k^2 - 2p \cdot k)(k^2 + 2p \cdot k)} \times \frac{1}{(k^2 - 2q \cdot k)(k^2 + 2q \cdot k)}. \tag{9.43}$$

This five-point integral is similar to the Sudakov form factor example treated in Sect. 6 of [8] and can be expanded in loop-momentum space employing the following regions:

- a hard region where $k \sim Q$,
- a 1-collinear region where $k^2 \sim p \cdot k \sim m^2$ and $q \cdot k \sim Q^2$,
- a 2-collinear region where $k^2 \sim q \cdot k \sim m^2$ and $p \cdot k \sim Q^2$,

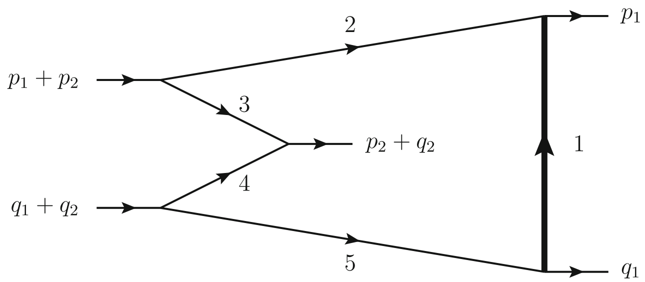

Fig. 9.4 One-loop five-point diagram exhibiting a Glauber contribution

- a Glauber region where $p \cdot k \sim q \cdot k \sim m^2$, and the components of k perpendicular to the plane spanned by p, q scale as $k_\perp \sim m$.

The Glauber region provides the leading contribution scaling as $(m^2)^{-2-\varepsilon}$, whereas the collinear contributions start with $(m^2)^{-1-\varepsilon}$ and the hard contribution starts with $(m^2)^0$.

The alpha representation for (9.43) takes the form

$$F(Q^2, m^2) = -\mathrm{i}\pi^{d/2}\, \Gamma(3+\varepsilon) \int_0^\infty \ldots \int_0^\infty \mathrm{d}\alpha_1 \cdots \mathrm{d}\alpha_5$$
$$\times \frac{\delta\left(\sum_i \alpha_i - 1\right)\ (\alpha_1 + \cdots + \alpha_5)^{1+2\varepsilon}}{\left[\alpha_1(\alpha_1 + \cdots + \alpha_5)m^2 + (\alpha_2 - \alpha_3)(\alpha_4 - \alpha_5)Q^2 - \mathrm{i}0\right]^{3+\varepsilon}}. \tag{9.44}$$

The code `asy.m` reports on the following three regions: (0, 0, 0, 0, 0), (0, 0, 0, 1, 1) and (0, 1, 1, 0, 0). The first region is hard; its contribution starts with $(m^2)^0$. The second and third regions start with order $(m^2)^{-1-\varepsilon}$. They correspond to the two collinear regions stated for the momentum-space expansion above. But `asy.m` does not find anything corresponding to the Glauber region; in particular, none of the regions found by `asy.m` provides the leading $(m^2)^{-2-\varepsilon}$ contribution.

Let us observe that, as in the previous section about potential contributions, the polynomial in the square brackets of (9.44) has terms of different sign. The missing Glauber contribution stems from the parameter region where either $(\alpha_2 - \alpha_3) \sim (m^2)^1$ or $(\alpha_4 - \alpha_5) \sim (m^2)^1$. So let us decompose the parametric integral into four parts corresponding to the domains where the two factors $(\alpha_2 - \alpha_3)$ and $(\alpha_4 - \alpha_5)$ are either positive or negative and then introduce new variables in such a way that this product takes the form $\pm\alpha_2'\alpha_4'$. For example, in the domain $\alpha_2 \le \alpha_3$, $\alpha_5 \le \alpha_4$ we change the variables by $\alpha_2 = \alpha_3'/2,\ \ \alpha_3 = \alpha_2' + \alpha_3'/2$ and by $\alpha_4 = \alpha_4' + \alpha_5'/2,\ \ \alpha_5 = \alpha_5'/2$, similarly to the example in the previous section. However, in the threshold expansion the cancelling terms appeared in squared form such that a transformation between one pair of variables was sufficient. Here two separate factors involve cancellations, which requires a twofold change of variables.

Removing the primes from the variables α_i, the parametric integral reads

$$F(Q^2, m^2) = 2(I_+ + I_-) \tag{9.45}$$

with

$$I_\pm = -\,\mathrm{i}\pi^{d/2}\, \frac{\Gamma(3+\varepsilon)}{4} \int_0^\infty \ldots \int_0^\infty \mathrm{d}\alpha_1 \cdots \mathrm{d}\alpha_5$$
$$\times \frac{\delta(\alpha_1 - 1)\ (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5)^{1+2\varepsilon}}{\left[\alpha_1(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5)m^2 \pm \alpha_2\alpha_4 Q^2 - \mathrm{i}0\right]^{3+\varepsilon}}, \tag{9.46}$$

where we have chosen the argument of the delta function as $\alpha_1 - 1$, according to the explanations presented after (3.38) in Sect. 3.4. So we may also write

$$I_{\pm} = -\mathrm{i}\pi^{d/2} \frac{\Gamma(3+\varepsilon)}{4} \int_0^\infty \cdots \int_0^\infty \mathrm{d}\alpha_2 \cdots \mathrm{d}\alpha_5$$
$$\times \frac{(1+\alpha_2+\alpha_3+\alpha_4+\alpha_5)^{1+2\varepsilon}}{\left[(1+\alpha_2+\alpha_3+\alpha_4+\alpha_5)m^2 \pm \alpha_2\alpha_4 Q^2 - \mathrm{i}0\right]^{3+\varepsilon}}. \quad (9.47)$$

It is sufficient to consider the expansion of I_+ and obtain a result for I_- by analytically continuing $Q^2 \to -Q^2 - i0$, taking into account that the dependence on Q^2 is power-like.

The code `asy2.m` applied to the integral I_+ again reveals three regions: (0, 0, 0, 0, 0), (0, 1, 0, 0, 0 and (0, 0, 0, 1, 0).

The evaluation of the contributions to each region is straightforward. The first region is the hard one. The contributions of the second and third regions are not individually regularized by dimensional regularization, as it often happens for Sudakov-type limits. We use an auxiliary analytic regularization by introducing additional powers $\alpha_2^{\delta_2}\alpha_3^{\delta_3}\alpha_4^{\delta_4}\alpha_5^{\delta_5}$ of the new variables into the integrand of (9.47), taking the limit $\delta_2, \delta_3, \delta_4, \delta_5 \to 0$ in the end. The LO contribution of the second and third regions to the integral $F(Q^2, m^2)$ reads

$$-\, i\pi^{d/2} \frac{i\pi\Gamma(\varepsilon)}{2Q^2(m^2)^{2+\varepsilon}}. \quad (9.48)$$

This agrees with the leading contribution of the Glauber region in the momentum-space expansion.

We have found the leading Glauber contribution of order $(m^2)^{-2-\varepsilon}$. But we seem to have lost the two collinear regions with the scalings (0, 0, 0, 1, 1) and (0, 1, 1, 0, 0) found before the change of variables. In fact, we can evaluate the contributions from these two regions by expanding the integral (9.47). The resulting integrals are scaleless and regularized by the parameters δ_3, δ_5, so they vanish, and `asy2.m` is right in omitting these two regions.

To have an additional check we can represent the integral (9.47) including the auxiliary analytic regularization factor $\alpha_2^{\delta_2}\alpha_3^{\delta_3}\alpha_4^{\delta_4}\alpha_5^{\delta_5}$ in terms of a onefold MB representation:

$$I_{\pm} = -\mathrm{i}\pi^{d/2} \frac{\Gamma(1+\delta_3)\Gamma(1+\delta_5)}{4} \frac{1}{2\pi\mathrm{i}} \int \mathrm{d}z\, (m^2)^z (\pm Q^2 - i0)^{-3-\varepsilon-z}$$
$$\times \Gamma(-z)\Gamma(-2-\varepsilon+\delta_2-z)\Gamma(-2-\varepsilon+\delta_4-z)$$
$$\times \frac{\Gamma(1-\delta_2-\delta_3-\delta_4-\delta_5+z)\Gamma(3+\varepsilon+z)}{\Gamma(-1-2\varepsilon-z)}. \quad (9.49)$$

The asymptotic expansion of $I_\pm$ in the limit $m^2/Q^2 \to 0$ is obtained by taking the residues of the poles of the functions $\Gamma(\ldots - z)$. The poles of $\Gamma(-z)$ correspond to the hard region, while the poles of the two functions $\Gamma(-2-\varepsilon+\delta_{2,4}-z)$ provide the contributions of the second and third regions. So `asy2.m` has found all contributing regions.

In the MB integral (9.49) we can safely take the limit $\delta_2,\delta_3,\delta_4,\delta_5 \to 0$, add up I_+ and I_-, and arrive at the MB representation

$$F(Q^2, m^2) = \mathrm{i}\pi^{d/2} \frac{\mathrm{i}}{2} \frac{1}{2\pi \mathrm{i}} \int \mathrm{d}z \, (m^2)^z (Q^2)^{-3-\varepsilon-z} \mathrm{e}^{\mathrm{i}\pi(\varepsilon+z)/2}$$
$$\times \Gamma(-z)\Gamma(-2-\varepsilon-z)\Gamma\left(\tfrac{-1-\varepsilon-z}{2}\right) \frac{\Gamma(1+z)\Gamma\left(\tfrac{3+\varepsilon+z}{2}\right)}{\Gamma(-1-2\varepsilon-z)}. \tag{9.50}$$

The LO contribution to $F(Q^2, m^2)$ is obtained from the residue of the single pole at $z = -2-\varepsilon$, in agreement with (9.48). The NLO contribution stems from the residue of the double pole at $z = -1-\varepsilon$ and reads

$$\mathrm{i}\pi^{d/2} \frac{\Gamma(1+\varepsilon)}{(Q^2)^2 (m^2)^{1+\varepsilon}} \left(\mathrm{i}\frac{\pi}{2} + 2\psi(-\varepsilon) - \psi(1+\varepsilon) + \gamma_{\mathrm{E}} - \ln \frac{Q^2}{m^2} - 1 \right). \tag{9.51}$$

This agrees with the NLO contributions of the second and third regions.

At next-to-next-to-leading order (NNLO) there is a contribution from the residue of the single pole at $z = -\varepsilon$ which reads

$$-\,\mathrm{i}\pi^{d/2} \frac{\mathrm{i}\pi\Gamma(2+\varepsilon)}{4(Q^2)^3 (m^2)^{\varepsilon}} \tag{9.52}$$

and agrees with the NNLO contributions of the second and third regions. The second NNLO contribution comes from the residue of the single pole at $z = 0$. It is given by

$$-\,\mathrm{i}\pi^{d/2} \frac{\mathrm{i}\,\mathrm{e}^{\mathrm{i}\pi\varepsilon/2}\, \Gamma(-2-\varepsilon)\Gamma\left(\tfrac{1+\varepsilon}{2}\right)\Gamma\left(\tfrac{1-\varepsilon}{2}\right)}{2(Q^2)^{3+\varepsilon}\, \Gamma(-1-2\varepsilon)} \tag{9.53}$$

and agrees with the LO contribution of the hard region. So indeed all contributions to the five-point integral up to NNLO are correctly reproduced by the contributions of the three regions found by `asy2.m` after the decomposition of the integral and the change of variables.

The trick applied in this example was generalized to an arbitrary diagram [9] by modifying the preresolution algorithm. When revealing Glauber regions for a general diagram, the preresolution algorithm of `asy2.m` tries to eliminate monomials with opposite sign in the polynomial $\mathcal{W}$ by automatically separating the integration into domains and performing changes of variables. If the preresolution is enabled, the code warns the user once the elimination of monomials with opposite sign has not been successful, such that possibly not all regions are revealed. This is the case if some monomials of opposite sign remain in the polynomial $\mathcal{W}$ after tries to eliminate them, or if symbols with unknown signs are present in the polynomial. Eventually, the code either reveals all relevant regions or issues a warning.

More examples and details, in particular instructions for using `asy2.m` can be found in [9].

9.7 Mathematical Status of Expansion by Regions

The strategy of expansion by regions still has the status of experimental mathematics. One way to try to prove it is to start from an 'honest' decomposition of a given integral into non-intersecting regions. At the level of a one-parametric integral which is similar to a Feynman integral in the large-momentum limit, expansion by regions was illustrated in [17]. This point of view was taken in [8] where it was shown explicitly and illustrated using various one-loop examples that one can start from a decomposition into non-intersecting regions and arrive at an expansion by regions in the sense of this chapter. The formalism presented in [8] is formulated for general integrals, including one-loop and multi-loop integrals as well as parametric representations, phase-space integrals and integrals unrelated to Feynman diagrams. Up to now, however, it has mainly been tested with one-loop examples, so its applicability to multi-loop integrals is not yet clear.

It is the alpha parametric picture that looks to be suitable for proving expansion by regions. Let me emphasize that to prove expansion by regions at least for some specific limit typical of Minkowski space is a natural mathematical problem. It looks reasonable to start from situations where the function $\mathcal{W}$ is positive. Perhaps, this problem is not specifically related to Feynman integrals. Let us present an example [9] of a one-dimensional parametric integral, without any relevance to Feynman integrals, and show that expansion by regions works successfully. To do this, we will use `asy2.m`.

Let us consider the integral

$$F(t) = \int_0^\infty (t + \alpha + \alpha^2)^\lambda \mathrm{d}\alpha, \tag{9.54}$$

with λ a complex parameter, in the limit $t \to 0$. We assume that λ is in the domain $\mathrm{Re}\lambda < -1/2$ in order to have an absolute convergence of the integral which then can be continued analytically to the whole complex plane as an analytic function of λ. Running the code `asy2.m` we obtain the two regions (1) and (0).

The LO terms from each region can be evaluated analytically in terms of gamma functions at general λ, with the results

$$\frac{t^{\lambda+1}\Gamma(-\lambda-1)}{\Gamma(-\lambda)} \frac{\Gamma(-2\lambda-1)\Gamma(\lambda+1)}{\Gamma(-\lambda)} .$$

They can be checked easily by deriving the onefold MB representation

$$F(t) = \frac{1}{2\pi \mathrm{i}} \frac{1}{\Gamma(-\lambda)} \int \Gamma(-z)\Gamma(\lambda - z + 1)\Gamma(-2\lambda + 2z - 1) t^z \mathrm{d}z$$

and evaluating the first terms of the asymptotic expansion in the limit $t \to 0$ by shifting the contour to the right and taking residues at the poles of the two gamma functions in the integrand.

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Chapter 10
Appendix A: Tables

10.1 Table of Integrals

Each Feynman integral presented here can be evaluated straightforwardly by use of alpha or Feynman parameters. Results are presented for the 'Euclidean' dependence, $-k^2$, of the denominators, which is more natural when the powers of propagators are general complex numbers. As usual, $-k^2$ is understood in the sense of $-k^2 - \mathrm{i}0$, etc. Moreover, denominators with a linear dependence on k are also understood in this sense, e.g. $2p \cdot k \to 2p \cdot k - \mathrm{i}0$, although sometimes this i0 dependence is explicitly indicated to avoid misunderstanding.

$$\int \frac{\mathrm{d}^d k}{(-k^2 + m^2)^\lambda} = \mathrm{i}\pi^{d/2} \frac{\Gamma(\lambda + \varepsilon - 2)}{\Gamma(\lambda)} \frac{1}{(m^2)^{\lambda+\varepsilon-2}} . \tag{10.1}$$

$$\int \mathrm{d}^d k \frac{k^{\alpha_1} \ldots k^{\alpha_{2n}}}{(-k^2 + m^2)^\lambda} = \mathrm{i}\pi^{d/2} \frac{\Gamma(\lambda - n + \varepsilon - 2)}{2^n \Gamma(\lambda)} \frac{(-1)^n g_{\mathrm{s}}^{\alpha_1 \ldots \alpha_{2n}}}{(m^2)^{\lambda-n+\varepsilon-2}} , \tag{10.2}$$

where $g_{\mathrm{s}}^{\alpha_1 \ldots \alpha_{2n}} = g^{\alpha_1 \alpha_2} \ldots g^{\alpha_{2n-1}\alpha_{2n}} + \cdots$ (with $(2n-1)!!$ terms in the sum) is a combination symmetrical with respect to the permutation of any pair of indices. If the number of monomials in the numerator is odd, the corresponding integral is zero.

$$\int \mathrm{d}^d k \frac{(2l \cdot k)^{2n}}{(-k^2 + m^2)^\lambda}$$
$$= \mathrm{i}\pi^{d/2} (-2)^n (2n-1)!! \frac{\Gamma(\lambda - n + \varepsilon - 2)}{\Gamma(\lambda)} \frac{(l^2)^n}{(m^2)^{\lambda-n+\varepsilon-2}} . \tag{10.3}$$

$$\int \frac{\mathrm{d}^d k}{(-k^2 + m^2)^{\lambda_1} (-k^2)^{\lambda_2}}$$
$$= \mathrm{i}\pi^{d/2} \frac{\Gamma(\lambda_1 + \lambda_2 + \varepsilon - 2)\Gamma(-\lambda_2 - \varepsilon + 2)}{\Gamma(\lambda_1)\Gamma(2 - \varepsilon)} \frac{1}{(m^2)^{\lambda_1+\lambda_2+\varepsilon-2}} . \tag{10.4}$$

V. A. Smirnov, *Analytic Tools for Feynman Integrals*, Springer Tracts in Modern Physics 250, DOI: 10.1007/978-3-642-34886-0_10,

$$\int \mathrm{d}^d k \frac{k^{\alpha_1} \dots k^{\alpha_{2n}}}{(-k^2+m^2)^{\lambda_1}(-k^2)^{\lambda_2}}$$
$$= \mathrm{i}\pi^{d/2} \frac{(-1)^n}{2^n} g_{\mathrm{s}}^{\alpha_1 \dots \alpha_{2n}} \frac{\Gamma(\lambda_1+\lambda_2-n+\varepsilon-2)\Gamma(n-\lambda_2-\varepsilon+2)}{\Gamma(\lambda_1)\Gamma(n-\varepsilon+2)(m^2)^{\lambda_1+\lambda_2-n+\varepsilon-2}}. \quad (10.5)$$

$$\int \mathrm{d}^d k \frac{(2l \cdot k)^{2n}}{(-k^2+m^2)^{\lambda_1}(-k^2)^{\lambda_2}} = \mathrm{i}\pi^{d/2}(-2)^n(2n-1)!!$$
$$\times \frac{\Gamma(\lambda_1+\lambda_2-n+\varepsilon-2)\Gamma(n-\lambda_2-\varepsilon+2)(l^2)^n}{\Gamma(\lambda_1)\Gamma(n-\varepsilon+2)(m^2)^{\lambda_1+\lambda_2-n+\varepsilon-2}}. \quad (10.6)$$

$$\int \frac{\mathrm{d}^d k}{(-k^2)^{\lambda_1}[-(q-k)^2]^{\lambda_2}}$$
$$= \mathrm{i}\pi^{d/2} \frac{\Gamma(2-\varepsilon-\lambda_1)\Gamma(2-\varepsilon-\lambda_2)}{\Gamma(\lambda_1)\Gamma(\lambda_2)\Gamma(4-\lambda_1-\lambda_2-2\varepsilon)} \frac{\Gamma(\lambda_1+\lambda_2+\varepsilon-2)}{(-q^2)^{\lambda_1+\lambda_2+\varepsilon-2}}. \quad (10.7)$$

Let $k^{(\alpha_1 \dots \alpha_n)} = k^{\alpha_1} \dots k^{\alpha_n} + \cdots$ be traceless with respect to any pair of indices, i.e. $g_{\alpha_i \alpha_j} k^{(\alpha_1 \dots \alpha_n)} = 0$—see (10.43b) below. Then

$$\int \mathrm{d}^d k \frac{k^{(\alpha_1 \dots \alpha_n)}}{(-k^2)^{\lambda_1}[-(q-k)^2]^{\lambda_2}} = \mathrm{i}\pi^{d/2} \frac{A_{\mathrm{T}}(\lambda_1, \lambda_2; n) q^{(\alpha_1 \dots \alpha_n)}}{(-q^2)^{\lambda_1+\lambda_2+\varepsilon-2}}, \quad (10.8)$$

where

$$A_{\mathrm{T}}(\lambda_1, \lambda_2; n) = \frac{\Gamma(\lambda_1+\lambda_2+\varepsilon-2)\Gamma(n+2-\varepsilon-\lambda_1)\Gamma(2-\varepsilon-\lambda_2)}{\Gamma(\lambda_1)\Gamma(\lambda_2)\Gamma(4+n-\lambda_1-\lambda_2-2\varepsilon)}. \quad (10.9)$$

For pure monomials, the corresponding formula has one more finite summation:

$$\int \mathrm{d}^d k \frac{k^{\alpha_1} \dots k^{\alpha_n}}{(-k^2)^{\lambda_1}[-(q-k)^2]^{\lambda_2}}$$
$$= \frac{\mathrm{i}\pi^{d/2}}{(-q^2)^{\lambda_1+\lambda_2+\varepsilon-2}} \sum_{r=0}^{[n/2]} A_{\mathrm{NT}}(\lambda_1, \lambda_2; r, n) \frac{1}{2^r} (q^2)^r \{[g]^r [q]^{n-2r}\}^{\alpha_1 \dots \alpha_n}, \quad (10.10)$$

where

$$A_{\mathrm{NT}}(\lambda_1, \lambda_2; r, n)$$
$$= \frac{\Gamma(\lambda_1+\lambda_2+\varepsilon-2-r)\Gamma(n+2-\varepsilon-\lambda_1-r)\Gamma(2-\varepsilon-\lambda_2+r)}{\Gamma(\lambda_1)\Gamma(\lambda_2)\Gamma(4+n-\lambda_1-\lambda_2-2\varepsilon)}, \quad (10.11)$$

and $\{[g]^r[q]^{n-2r}\}^{\alpha_1\ldots\alpha_n}$ is symmetric in its indices and is composed of the metric tensor and the vector q.

$$\int \mathrm{d}^d k \frac{(2l\cdot k)^n}{(-k^2)^{\lambda_1}[-(q-k)^2]^{\lambda_2}} = \frac{\mathrm{i}\pi^{d/2}}{(-q^2)^{\lambda_1+\lambda_2+\varepsilon-2}} \times \sum_{r=0}^{[n/2]} A_{\mathrm{NT}}(\lambda_1,\lambda_2;r,n)\frac{n!}{r!(n-2r)!}(q^2)^r(l^2)^r(2q\cdot l)^{n-2r}, \tag{10.12}$$

$$\int \frac{\mathrm{d}^d k}{(-k^2)^{\lambda_1}(-k^2+2p\cdot k)^{\lambda_2}} = \mathrm{i}\pi^{d/2}\frac{\Gamma(\lambda_1+\lambda_2+\varepsilon-2)\Gamma(-2\lambda_1-\lambda_2-2\varepsilon+4)}{\Gamma(\lambda_2)\Gamma(-\lambda_1-\lambda_2-2\varepsilon+4)}\frac{1}{(p^2)^{\lambda_1+\lambda_2+\varepsilon-2}}. \tag{10.13}$$

$$\int \mathrm{d}^d k \frac{k^{(\alpha_1\ldots\alpha_n)}}{(-k^2)^{\lambda_1}(-k^2+2p\cdot k)^{\lambda_2}} = \mathrm{i}\pi^{d/2}B_{\mathrm{T}}(\lambda_1,\lambda_2;n)\frac{p^{(\alpha_1\ldots\alpha_n)}}{(p^2)^{\lambda_1+\lambda_2+\varepsilon-2}}, \tag{10.14}$$

where

$$B_{\mathrm{T}}(\lambda_1,\lambda_2;n) = \frac{\Gamma(\lambda_1+\lambda_2+\varepsilon-2)\Gamma(-2\lambda_1-\lambda_2+n-2\varepsilon+4)}{\Gamma(\lambda_2)\Gamma(-\lambda_1-\lambda_2+n-2\varepsilon+4)}. \tag{10.15}$$

$$\int \mathrm{d}^d k \frac{k^{\alpha_1}\ldots k^{\alpha_n}}{(-k^2)^{\lambda_1}(-k^2+2p\cdot k)^{\lambda_2}} = \frac{\mathrm{i}\pi^{d/2}}{(p^2)^{\lambda_1+\lambda_2+\varepsilon-2}} \times \sum_{r=0}^{[n/2]} B_{\mathrm{NT}}(\lambda_1,\lambda_2;r,n)\frac{(-1)^r}{2^r}(p^2)^r\{[g]^r[p]^{n-2r}\}^{\alpha_1\ldots\alpha_n}, \tag{10.16}$$

where

$$B_{\mathrm{NT}}(\lambda_1,\lambda_2;r,n) = \frac{\Gamma(\lambda_1+\lambda_2+\varepsilon-2-r)\Gamma(-2\lambda_1-\lambda_2+n-2\varepsilon+4)}{\Gamma(\lambda_2)\Gamma(-\lambda_1-\lambda_2+n-2\varepsilon+4)}. \tag{10.17}$$

$$\int \mathrm{d}^d k \frac{(2l\cdot k)^n}{(-k^2)^{\lambda_1}(-k^2+2p\cdot k)^{\lambda_2}} = \frac{\mathrm{i}\pi^{d/2}}{(q^2)^{\lambda_1+\lambda_2+\varepsilon-2}} \times \sum_{r=0}^{[n/2]} B_{\mathrm{NT}}(\lambda_1,\lambda_2;r,n)(-1)^r\frac{n!}{r!(n-2r)!}(p^2)^r(l^2)^r(2p\cdot l)^{n-2r}. \tag{10.18}$$

Let $p\cdot q = 0$. Then

$$\int \mathrm{d}^d k \frac{(p\cdot k)^{b_1}(q\cdot k)^{b_2}}{(-k^2)^{\lambda_1}[-(l-k)^2]^{\lambda_2}}$$
$$= \frac{\mathrm{i}\pi^{d/2}}{(-l^2)^{\lambda_1+\lambda_2+\varepsilon-2}} \sum_{r=0}^{[(b_1+b_2)/2]} A_{\mathrm{NT}}(\lambda_1,\lambda_2;r,b_1+b_2)\frac{b_1!b_2!}{4^r}(l^2)^r$$
$$\times \sum_{r_1=\max\{0,r-[b_2/2]\}}^{\min\{r,[b_1/2]\}} \frac{(p\cdot l)^{b_1-2r_1}(q\cdot l)^{b_2-2r+2r_1}(p^2)^{r_1}(q^2)^{r-r_1}}{r_1!(r-r_1)!(b_1-2r_1)!(b_2-2r+2r_1)!}, \qquad (10.19)$$

and

$$\int \mathrm{d}^d k \frac{(p\cdot k)^{b_1}(q\cdot k)^{b_2}}{(-k^2)^{\lambda_1}(-k^2+2q\cdot k)^{\lambda_2}}$$
$$= \mathrm{i}\pi^{d/2}\frac{(p^2)^{b_1/2}}{(q^2)^{\lambda_1+\lambda_2+\varepsilon-2-b_1/2-b_2}} B_{pq}(\lambda_1,\lambda_2;b_1,b_2), \qquad (10.20)$$

for even b_1 (and are equal to zero for odd b_1), where

$$B_{pq}(\lambda_1,\lambda_2;b_1,b_2)$$
$$= \sum_{r=b_1/2}^{b_1/2+[b_2/2]} \frac{(-1)^r}{4^r}\frac{b_1!b_2!}{(b_1/2)!(r-b_1/2)!} B_{\mathrm{NT}}(\lambda_1,\lambda_2;r,b_1+b_2). \qquad (10.21)$$

$$\int \frac{\mathrm{d}^d k}{(-k^2+m^2)^{\lambda_1}(2p\cdot k)^{\lambda_2}}$$
$$= \frac{\mathrm{i}\pi^{d/2}}{(p^2)^{\lambda_2/2}(m^2)^{\lambda_1+\lambda_2/2+\varepsilon-2}} \frac{\Gamma(\lambda_2/2)\Gamma(\lambda_1+\lambda_2/2+\varepsilon-2)}{2\Gamma(\lambda_1)\Gamma(\lambda_2)}. \qquad (10.22)$$

$$\int \mathrm{d}^d k \frac{k^{(\alpha_1,\ldots,\alpha_n)}}{(-k^2+m^2)^{\lambda_1}(2p\cdot k)^{\lambda_2}}$$
$$= \mathrm{i}\pi^{d/2}\frac{\Gamma((\lambda_2+n)/2)}{2\Gamma(\lambda_1)\Gamma(\lambda_2)} \frac{\Gamma(\lambda_1+(\lambda_2-n)/2+\varepsilon-2)}{(m^2)^{\lambda_1+(\lambda_2-n)/2+\varepsilon-2}} \frac{p^{(\alpha_1,\ldots,\alpha_n)}}{(p^2)^{(\lambda_2+n)/2}}. \qquad (10.23)$$

$$\int \frac{\mathrm{d}^d k}{(-k^2+2p\cdot k)^{\lambda_1}(2p\cdot k)^{\lambda_2}}$$
$$= \frac{\mathrm{i}\pi^{d/2}}{(p^2)^{\lambda_1+\lambda_2+\varepsilon-2}} \frac{\Gamma(\lambda_1+\lambda_2+\varepsilon-2)\Gamma(2\lambda_1+\lambda_2+2\varepsilon-4)}{\Gamma(\lambda_1)\Gamma(2\lambda_1+2\lambda_2+2\varepsilon-4)}. \qquad (10.24)$$

$$\int \frac{d^d k}{(-k^2)^{\lambda_1}(2v\cdot k+\omega-i0)^{\lambda_2}}$$
$$= i\pi^{d/2}\frac{\Gamma(2-\lambda_1-\varepsilon)\Gamma(2\lambda_1+\lambda_2+2\varepsilon-4)}{\Gamma(\lambda_1)\Gamma(\lambda_2)}(v^2)^{\lambda_1+\varepsilon-2}\omega^{-2\lambda_1-\lambda_2-2\varepsilon+4}. \quad (10.25)$$

$$\int d^d k \frac{k^{(\alpha_1,\ldots,\alpha_n)}}{(-k^2)^{\lambda_1}(2v\cdot k+\omega-i0)^{\lambda_2}} = i\pi^{d/2}\omega^{-2\lambda_1-\lambda_2+n-2\varepsilon+4}$$
$$\times \frac{v^{(\alpha_1,\ldots,\alpha_n)}}{(v^2)^{-\lambda_1+n-\varepsilon+2}}\frac{\Gamma(2-\lambda_1+n-\varepsilon)\Gamma(2\lambda_1+\lambda_2-n+2\varepsilon-4)}{\Gamma(\lambda_1)\Gamma(\lambda_2)}. \quad (10.26)$$

Let $v\cdot q=0$. Then

$$\int \frac{d^d k}{(-k^2)^{\lambda_1}[-(q-k)^2]^{\lambda_2}(-2v\cdot k-i0)^{\lambda_3}}$$
$$= i\pi^{d/2}\frac{\Gamma(-\lambda_1-\lambda_3/2-\varepsilon+2)\Gamma(-\lambda_2-\lambda_3/2-\varepsilon+2)}{\Gamma(-\lambda_1-\lambda_2-\lambda_3-2\varepsilon+4)}$$
$$\times \frac{\Gamma(\lambda_1+\lambda_2+\lambda_3/2+\varepsilon-2)\Gamma(\lambda_3/2)}{2\Gamma(\lambda_1)\Gamma(\lambda_2)\Gamma(\lambda_3)(-q^2)^{\lambda_1+\lambda_2+\lambda_3/2+\varepsilon-2}(v^2)^{\lambda_3/2}}. \quad (10.27)$$

Let $p_1^2=p_2^2=0,\ q=p_1-p_2$. Then

$$\int \frac{d^d k}{(-k^2+2p_1\cdot k)^{\lambda_1}(-k^2+2p_2\cdot k)^{\lambda_2}(-k^2)^{\lambda_3}}$$
$$= i\pi^{d/2}\frac{\Gamma(-\lambda_1-\lambda_3-\varepsilon+2)\Gamma(-\lambda_2-\lambda_3-\varepsilon+2)}{\Gamma(\lambda_1)\Gamma(\lambda_2)\Gamma(-\lambda_1-\lambda_2-\lambda_3-2\varepsilon+4)}$$
$$\times \frac{\Gamma(\lambda_1+\lambda_2+\lambda_3+\varepsilon-2)}{(-q^2)^{\lambda_1+\lambda_2+\lambda_3+\varepsilon-2}}, \quad (10.28)$$

$$\int \frac{d^d k}{(-k^2+2p_1\cdot k)^{\lambda_1}(-k^2+2p_2\cdot k)^{\lambda_2}(2p_2\cdot k)^{\lambda_3}} = i\pi^{d/2}\frac{\Gamma(-\lambda_1-\varepsilon+2)}{\Gamma(\lambda_1)\Gamma(\lambda_2)}$$
$$\times \frac{\Gamma(\lambda_1+\lambda_2+\varepsilon-2)\Gamma(-\lambda_2-\lambda_3-\varepsilon+2)}{\Gamma(-\lambda_1-\lambda_2-\lambda_3-2\varepsilon+4)(-q^2)^{\lambda_1+\lambda_2+\lambda_3+\varepsilon-2}}, \quad (10.29)$$

$$\int \frac{d^d k}{(2p_1\cdot k)^{\lambda_1}(-k^2+2p_2\cdot k)^{\lambda_2}(-k^2+m^2)^{\lambda_3}}$$
$$= i\pi^{d/2}\frac{\Gamma(\lambda_2-\lambda_1)\Gamma(\lambda_2+\lambda_3+\varepsilon-2)\Gamma(-\lambda_2-\varepsilon+2)}{\Gamma(\lambda_2)\Gamma(\lambda_3)\Gamma(-\lambda_1-\varepsilon+2)(-q^2)^{\lambda_1}(m^2)^{\lambda_2+\lambda_3+\varepsilon-2}}, \quad (10.30)$$

$$\int \frac{d^d k}{(2p_1\cdot k)^{\lambda_1}(-k^2+2p_2\cdot k)^{\lambda_2}(-k^2+m^2)^{\lambda_3}(Q^2-2p_1\cdot k)^{\lambda_4}}$$
$$= i\pi^{d/2}\frac{\Gamma(\lambda_2-\lambda_1)\Gamma(\lambda_2+\lambda_3+\varepsilon-2)\Gamma(-\lambda_2-\lambda_4-\varepsilon+2)}{\Gamma(\lambda_2)\Gamma(\lambda_3)\Gamma(-\lambda_1-\lambda_4-\varepsilon+2)}$$
$$\times \frac{1}{(Q^2)^{\lambda_1+\lambda_4}(m^2)^{\lambda_2+\lambda_3+\varepsilon-2}}, \tag{10.31}$$

$$\int \frac{d^d k}{(2p_1\cdot k+m^2)^{\lambda_1}(2p_2\cdot k+m^2)^{\lambda_2}(-k^2)^{\lambda_3}}$$
$$= i\pi^{d/2}\frac{\Gamma(\lambda_1+\lambda_3+\varepsilon-2)\Gamma(\lambda_2+\lambda_3+\varepsilon-2)\Gamma(-\lambda_3-\varepsilon+2)}{\Gamma(\lambda_1)\Gamma(\lambda_2)\Gamma(\lambda_3)(-q^2)^{-\lambda_3-\varepsilon+2}(m^2)^{\lambda_1+\lambda_2+2\lambda_3+2\varepsilon-4}}. \tag{10.32}$$

Let $p_1^2=0,\ p_2^2=-m^2,\ q=p_1-p_2$. Then

$$\int \frac{d^d k}{(2p_1\cdot k)^{\lambda_1}(-k^2+2p_2\cdot k+m^2)^{\lambda_2}(-k^2)^{\lambda_3}} = i\pi^{d/2}\frac{\Gamma(\lambda_2+\lambda_3+\varepsilon-2)}{(m^2)^{\lambda_2+\lambda_3+\varepsilon-2}}$$
$$\times \frac{\Gamma(-\lambda_1-\lambda_3-\varepsilon+2)\Gamma(-\lambda_2-\varepsilon+2)}{\Gamma(\lambda_2)\Gamma(\lambda_3)\Gamma(-\lambda_1-\lambda_2-\lambda_3-2\varepsilon+4)(-q^2)^{\lambda_1}}, \tag{10.33}$$

$$\int \frac{d^d k}{(2p_1\cdot k)^{\lambda_1}(-k^2+2p_2\cdot k-m^2)^{\lambda_2}(-k^2)^{\lambda_3}(-q^2-2p_1\cdot k)^{\lambda_4}}$$
$$= i\pi^{d/2}\frac{\Gamma(\lambda_2+\lambda_3+\varepsilon-2)}{(m^2)^{\lambda_2+\lambda_3+\varepsilon-2}}$$
$$\times \frac{\Gamma(-\lambda_1-\lambda_3-\varepsilon+2)\Gamma(-\lambda_2-\lambda_4-\varepsilon+2)}{\Gamma(\lambda_2)\Gamma(\lambda_3)\Gamma(-\lambda_1-\lambda_2-\lambda_3-\lambda_4-2\varepsilon+4)(-q^2)^{\lambda_1+\lambda_4}}. \tag{10.34}$$

Let $P^2=M^2,\ p^2=0,\ (P-p)^2=0$. Then

$$\int \frac{d^d k}{(-k^2+2P\cdot k)^{\lambda_1}(-k^2+2p\cdot k)^{\lambda_2}(-k^2)^{\lambda_3}}$$
$$= i\pi^{d/2}\frac{\Gamma(-\lambda_1-\lambda_2-2\lambda_3-2\varepsilon+4)\Gamma(\lambda_1+\lambda_2+\lambda_3+\varepsilon-2)}{\Gamma(\lambda_1)\Gamma(-\lambda_1-\lambda_2-\lambda_3-2\varepsilon+4)}$$
$$\times \frac{\Gamma(-\lambda_2-\lambda_3-\varepsilon+2)}{\Gamma(-\lambda_3-\varepsilon+2)(M^2)^{\lambda_1+\lambda_2+\lambda_3+\varepsilon-2}}. \tag{10.35}$$

Let $p_1^2 = 0,\ p_2^2 = m^2,\ Q^2 = 2p_1\cdot p_2$. Then

$$\int \frac{\mathrm{d}^d k}{(2p_1\cdot k)^{\lambda_1}(-k^2+2p_2\cdot k)^{\lambda_2}(-k^2)^{\lambda_3}(Q^2-2p_1\cdot k)^{\lambda_4}}$$
$$= \mathrm{i}\pi^{d/2}\frac{\Gamma(\lambda_3-\lambda_4)\Gamma(-\lambda_1-\lambda_2-2\lambda_3-2\varepsilon+4)}{\Gamma(\lambda_2)\Gamma(\lambda_3)\Gamma(-\lambda_1-\lambda_2-\lambda_3-\lambda_4-2\varepsilon+4)}$$
$$\times\frac{\Gamma(\lambda_2+\lambda_3+\varepsilon-2)}{(Q^2)^{\lambda_1+\lambda_4}(m^2)^{\lambda_2+\lambda_3+\varepsilon-2}}, \tag{10.36}$$

$$\int \frac{\mathrm{d}^d k}{(2p_1\cdot k)^{\lambda_1}(-k^2+2p_2\cdot k)^{\lambda_2}(-k^2)^{\lambda_3}} = \frac{\mathrm{i}\pi^{d/2}}{(Q^2)^{\lambda_1}(m^2)^{\lambda_2+\lambda_3+\varepsilon-2}}$$
$$\times\frac{\Gamma(\lambda_2+\lambda_3+\varepsilon-2)\Gamma(-\lambda_1-\lambda_2-2\lambda_3-2\varepsilon+4)}{\Gamma(\lambda_2)\Gamma(-\lambda_1-\lambda_2-\lambda_3-2\varepsilon+4)}. \tag{10.37}$$

The following integrals are related to two-loop diagrams:

$$\int\int \frac{\mathrm{d}^d k\,\mathrm{d}^d l}{(-k^2+m^2)^{\lambda_1}[-(k+l)^2]^{\lambda_2}(-l^2+m^2)^{\lambda_3}}$$
$$= \left(\mathrm{i}\pi^{d/2}\right)^2 \frac{\Gamma(\lambda_1+\lambda_2+\varepsilon-2)\Gamma(\lambda_2+\lambda_3+\varepsilon-2)\Gamma(2-\varepsilon-\lambda_2)}{\Gamma(\lambda_1)\Gamma(\lambda_3)}$$
$$\times\frac{\Gamma(\lambda_1+\lambda_2+\lambda_3+2\varepsilon-4)}{\Gamma(\lambda_1+2\lambda_2+\lambda_3+2\varepsilon-4)\Gamma(2-\varepsilon)(m^2)^{\lambda_1+\lambda_2+\lambda_3+2\varepsilon-4}}, \tag{10.38}$$

$$\int\int \frac{\mathrm{d}^d k\,\mathrm{d}^d l}{(-k^2)^{\lambda_1}[-(k+l)^2]^{\lambda_2}(m^2-l^2)^{\lambda_3}}$$
$$= \left(\mathrm{i}\pi^{d/2}\right)^2 \frac{\Gamma(\lambda_1+\lambda_2+\lambda_3+2\varepsilon-4)}{(m^2)^{\lambda_1+\lambda_2+\lambda_3+2\varepsilon-4}}$$
$$\times\frac{\Gamma(\lambda_1+\lambda_2+\varepsilon-2)\Gamma(2-\varepsilon-\lambda_1)\Gamma(2-\varepsilon-\lambda_2)}{\Gamma(\lambda_1)\Gamma(\lambda_2)\Gamma(\lambda_3)\Gamma(2-\varepsilon)}, \tag{10.39}$$

$$\int\int \frac{\mathrm{d}^d k\,\mathrm{d}^d l}{[-2v\cdot(k+l)]^{\lambda_1}(-k^2+m^2)^{\lambda_2}(-l^2+m^2)^{\lambda_3}}$$
$$= \left(\mathrm{i}\pi^{d/2}\right)^2 \frac{\Gamma(\lambda_1/2+\lambda_2+\varepsilon-2)\Gamma(\lambda_1/2+\lambda_3+\varepsilon-2)}{\Gamma(\lambda_1+\lambda_2+\lambda_3+2\varepsilon-4)}$$
$$\times\frac{\Gamma(\lambda_1/2)\Gamma(\lambda_1/2+\lambda_2+\lambda_3+2\varepsilon-4)}{2\Gamma(\lambda_1)\Gamma(\lambda_2)\Gamma(\lambda_3)(m^2)^{\lambda_1/2+\lambda_2+\lambda_3+2\varepsilon-4}(v^2)^{\lambda_1/2}}, \tag{10.40}$$

$$\int\int \frac{\mathrm{d}^d k\, \mathrm{d}^d l}{[-2v\cdot(k+l)]^{\lambda_1}[-(k+l)^2]^{\lambda_2}(-k^2+m^2)^{\lambda_3}(-l^2+m^2)^{\lambda_4}}$$

$$= \frac{\left(\mathrm{i}\pi^{d/2}\right)^2 \Gamma(\lambda_1/2+\lambda_2+\lambda_3+\varepsilon-2)\Gamma(\lambda_1/2+\lambda_2+\lambda_4+\varepsilon-2)}{2\Gamma(\lambda_1)\Gamma(\lambda_3)\Gamma(\lambda_4)\Gamma(\lambda_1+2\lambda_2+\lambda_3+\lambda_4+2\varepsilon-4)}$$

$$\times \frac{\Gamma(\lambda_1/2)\Gamma(\lambda_1/2+\lambda_2+\lambda_3+\lambda_4+2\varepsilon-4)\Gamma(2-\lambda_1/2-\lambda_2-\varepsilon)}{\Gamma(2-\lambda_1/2-\varepsilon)(m^2)^{\lambda_1/2+\lambda_2+\lambda_3+\lambda_4+2\varepsilon-4}(v^2)^{\lambda_1/2}}. \tag{10.41}$$

This is the (inverse) Fourier transformation of $(-q^2 - i0)^{-\lambda}$ in d dimensions:

$$\frac{1}{(2\pi)^d}\int \mathrm{d}^d q \frac{\mathrm{e}^{-\mathrm{i}x\cdot q}}{(-q^2-i0)^\lambda} = \frac{\mathrm{i}\Gamma(d/2-\lambda)}{4^\lambda \pi^{d/2}\Gamma(\lambda)} \frac{1}{(-x^2+i0)^{d/2-\lambda}}. \tag{10.42}$$

10.2 Some Useful Formulae

To traceless expressions and back:

$$k^{\alpha_1}\dots k^{\alpha_N} = \frac{1}{N!}\sum_{r=0}^{[N/2]} \frac{1}{2^r(d/2+N-2r)_r}(k^2)^r\{[g]^r[k]^{(N-2r)}\}^{\alpha_1\dots\alpha_N}, \tag{10.43a}$$

$$k^{(\alpha_1\dots\alpha_N)} = \frac{1}{N!}\sum_{r=0}^{[N/2]} \frac{1}{2^r(2-N-d/2)_r}(k^2)^r\{[g]^r[k]^{N-2r}\}^{\alpha_1\dots\alpha_N}, \tag{10.43b}$$

where $\{[g]^r[k]^{N-2r}\}^{\alpha_1\dots\alpha_N}$ is defined after (10.11) and $(a)_n$ is the Pochhammer symbol (11.2).

Furthermore,

$$(k\cdot p)^N = \sum_{r=0}^{[N/2]} a_{N,r}(k^2)^r(p^2)^r(k\cdot p)^{(N-2r)}, \tag{10.44}$$

$$(k\cdot p)^{(N)} = \sum_{r=0}^{[N/2]} b_{N,r}(k^2)^r(p^2)^r(k\cdot p)^{N-2r}, \tag{10.45}$$

$$k_{(\alpha_1\dots\alpha_N)}k^{(\alpha_1\dots\alpha_N)} = \frac{(d-2)_N}{2^N\,((d-2)/2)_N}(k^2)^N, \tag{10.46}$$

where $(k\cdot p)^{(N)} = k_{(\alpha_1\dots\alpha_N)}p^{(\alpha_1\dots\alpha_N)}$ and

$$a_{N,r} = \frac{N!}{4^r r!(N-2r)!(d/2+N-2r)_r}, \tag{10.47}$$

$$b_{N,r} = \frac{1}{4^r r!(N-2r)!(2-N-d/2)_r}. \tag{10.48}$$

Summation formulae:

$$\begin{aligned}
&[(k_1)^m (k_2)^n * g_\mathrm{s}] \equiv k_1^{\alpha_1} \ldots k_1^{\alpha_m} k_2^{\alpha_{m+1}} \ldots k_2^{\alpha_{m+n}} g_{\mathrm{s},\, \alpha_1 \ldots \alpha_{m+n}} \\
&\quad = \sum_{j \geq 0,\ j+\min\{m,n\}\ \text{even}}^{\min\{m,n\}} \frac{m!n!}{2^{(m+n)/2-j}((m-j)/2)!((n-j)/2)!j!} \\
&\quad \times (k_1^2)^{(m-j)/2} (k_2^2)^{(n-j)/2} (k_1 \cdot k_2)^j,
\end{aligned} \tag{10.49}$$

$$\begin{aligned}
&[(k_1)^m (k_2)^n * \{[g]^r [k_3]^{m+n-2r}\}] \\
&\quad = \sum_{r_1=\max\{0,2r-n\}}^{\min\{2r,m\}} \sum_{j \geq 0,\ j+r_1\ \text{even}}^{\min\{r_1, 2r-r_1\}} \frac{1}{(m-r_1)!(n-2r+r_1)!} \\
&\quad \times \frac{m!n!}{2^{r-j}((r_1-j)/2)!(r-(r_1+j)/2)!j!} (k_1^2)^{(r_1-j)/2} (k_2^2)^{r-(r_1+j)/2} \\
&\quad \times (k_1 \cdot k_2)^j (k_1 \cdot k_3)^{m-r_1} (k_2 \cdot k_3)^{n-2r+r_1}.
\end{aligned} \tag{10.50}$$

In particular,

$$\begin{aligned}
&[(k_1)^m (k_2)^n * \{[g]^r [k_3]^{N-2r}\}] \\
&\quad = \binom{n}{N-2r} (k_2 \cdot k_3)^{N-2r} [(k_1)^m (k_2)^{n-N+2r} * g_\mathrm{s}],
\end{aligned} \tag{10.51}$$

where $k_1 \cdot k_3 = 0$, $N = m + n$, and

$$\begin{aligned}
&[p^{b_1} q^{b_2} * \{[g]^r [l]^{n-2r}\}] \\
&\quad = \frac{b_1! b_2!}{2^r} \sum_{r_1=\max\{0,r-[b_2/2]\}}^{\min\{r,[b_1/2]\}} \frac{(p \cdot l)^{b_1-2r_1} (q \cdot l)^{b_2-2r+2r_1} (p^2)^{r_1} (q^2)^{r-r_1}}{r_1!(r-r_1)!(b_1-2r_1)!(b_2-2r+2r_1)!},
\end{aligned} \tag{10.52}$$

where $p \cdot q = 0$ and $n = b_1 + b_2$.

$$\begin{aligned}
&[(k_1)^m (k_2)^n (k_3)^{l-m-n} * g_\mathrm{s}] \\
&\quad = \sum_{j_1 \geq 0,\ j_1+m\ \text{even}} \ \sum_{j_2 \geq 0,\ j_2+n\ \text{even}} \ \sum_{j_3 \geq 0,\ j_3+l-m-n\ \text{even}} a(l, m, n, j_1, j_2, j_3) \\
&\quad \times (k_1^2)^{(m-j_1)/2} (k_2^2)^{(n-j_2)/2} (k_3^2)^{(l-m-n-j_3)/2} \\
&\quad \times (k_1 \cdot k_2)^{(j_1+j_2-j_3)/2} (k_1 \cdot k_3)^{(j_1-j_2+j_3)/2} (k_2 \cdot k_3)^{(-j_1+j_2+j_3)/2},
\end{aligned}$$

$$a(l, m, n, j_1, j_2, j_3) = \frac{2^{(j_1+j_2+j_3-l)/2} m! n! (l-m-n)!}{((m-j_1)/2)!((n-j_2)/2)!((l-m-n-j_3)/2)!} \times \frac{\theta(j_1+j_2-j_3)\theta(j_1-j_2+j_3)\theta(-j_1+j_2+j_3)}{((j_1+j_2-j_3)/2)!((j_1-j_2+j_3)/2)!((-j_1+j_2+j_3)/2)!}, \tag{10.53}$$

where $\theta(n) = 1$ for $n \geq 0$ and $\theta(n) = 0$ otherwise.

Chapter 11
Appendix B: Some Special Functions

The Gauss hypergeometric function [2] is defined by the series

$$ {}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n, \tag{11.1}$$

where

$$(x)_n = \Gamma(x+n)/\Gamma(x) \tag{11.2}$$

is the Pochhammer symbol. This power series has the radius of convergence equal to one. It is analytically continued to the whole complex plane, with a cut, usually chosen as $[1, \infty)$. The analytic continuation to values of z where $|z| > 1$ is given by

$$\begin{aligned} {}_2F_1(a,b;c;z) = & \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-z)^{-a} \, {}_2F_1\left(a, 1-c+a; 1-b+a; \frac{1}{z}\right) \\ & + \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(-z)^{-b} \, {}_2F_1\left(b, 1-c+b; 1-a+b; \frac{1}{z}\right). \end{aligned} \tag{11.3}$$

Another formula for the analytic continuation is

$$ {}_2F_1(a,b;c;z) = (1-z)^{-a} \, {}_2F_1\left(a, c-b; c; \frac{z}{z-1}\right). \tag{11.4}$$

This is a useful parametric representation:

$$ {}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 \mathrm{d}x \; x^{b-1}(1-x)^{c-b-1}(1-zx)^{-a}. \tag{11.5}$$

MB representations for the Gauss hypergeometric function can be found in Sect. 13.3.

V. A. Smirnov, *Analytic Tools for Feynman Integrals*, Springer Tracts in Modern Physics 250, DOI: 10.1007/978-3-642-34886-0_11,

The polylogarithms [7] and generalized (Nielsen) polylogarithms [1, 5, 6] are defined by

$$\mathrm{Li}_a(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^a} \tag{11.6}$$

$$= \frac{(-1)^a}{(a-1)!} \int_0^1 \frac{\ln^{a-1} t}{t - 1/z}\, \mathrm{d}t \tag{11.7}$$

and

$$S_{a,b}(z) = \frac{(-1)^{a+b-1}}{(a-1)!b!} \int_0^1 \frac{\ln^{a-1} t \ln^b(1-zt)}{t}\, \mathrm{d}t, \tag{11.8}$$

where a and b are positive integers.

The harmonic polylogarithms [9] $H_{a_1,a_2,\ldots,a_n}(x)$ (or $H(a_1, a_2, \ldots, a_n; x)$) (HPL), with $a_i = 1, 0, -1$, are defined recursively by

$$H_{a_1,a_2,\ldots,a_n}(x) = \int_0^x f_{a_1}(t) H_{a_2,\ldots,a_n}(t)\, \mathrm{d}t, \tag{11.9}$$

where

$$f_{\pm 1}(x) = \frac{1}{1 \mp x}, \quad f_0(x) = \frac{1}{x}, \tag{11.10}$$

$$H_{\pm 1}(x) = \mp \ln(1 \mp x), \quad H_0(x) = \ln x, \tag{11.11}$$

and at least one of the indices a_i is non-zero. For all $a_i = 0$, one has

$$H_{0,0,\ldots,0}(x) = \frac{1}{n!} \ln^n x. \tag{11.12}$$

The number n in (11.9) (as well as in (11.43) for multiple polylogarithms) is called *weight*. Up to weight 4, HPLs with the indices 0 and 1 can be expressed in terms of usual polylogarithms [9]:

$$H_0(x) = \ln x, \tag{11.13}$$

$$H_1(x) = -\ln(1-x), \tag{11.14}$$

$$H_{0,0}(x) = \frac{1}{2!} \ln^2 x, \tag{11.15}$$

$$H_{0,1}(x) = \mathrm{Li}_2(x), \tag{11.16}$$

$$H_{1,0}(x) = -\ln x \ln(1-x) - \mathrm{Li}_2(x), \tag{11.17}$$

$$H_{1,1}(x) = \frac{1}{2!} \ln^2(1-x), \tag{11.18}$$

$$H_{0,0,0}(x) = \frac{1}{3!}\ln^3 x, \tag{11.19}$$

$$H_{0,0,1}(x) = \mathrm{Li}_3(x), \tag{11.20}$$

$$H_{0,1,0}(x) = -2\mathrm{Li}_3(x) + \ln x\,\mathrm{Li}_2(x), \tag{11.21}$$

$$H_{0,1,1}(x) = S_{1,2}(x), \tag{11.22}$$

$$H_{1,0,0}(x) = -\frac{1}{2}\ln(1-x)\ln^2 x - \ln x\,\mathrm{Li}_2(x) + \mathrm{Li}_3(x), \tag{11.23}$$

$$H_{1,0,1}(x) = -2S_{1,2}(x) - \ln(1-x)\mathrm{Li}_2(x), \tag{11.24}$$

$$H_{1,1,0}(x) = S_{1,2}(x) + \ln(1-x)\,\mathrm{Li}_2(x) + \frac{1}{2}\ln x\;\ln^2(1-x), \tag{11.25}$$

$$H_{1,1,1}(x) = -\frac{1}{3!}\ln^3(1-x), \tag{11.26}$$

$$H_{0,0,0,0}(x) = \frac{1}{4!}\ln^4 x, \tag{11.27}$$

$$H_{0,0,0,1}(x) = \mathrm{Li}_4(x), \tag{11.28}$$

$$H_{0,0,1,0}(x) = \ln x\,\mathrm{Li}_3(x) - 3\mathrm{Li}_4(x), \tag{11.29}$$

$$H_{0,0,1,1}(x) = S_{2,2}(x), \tag{11.30}$$

$$H_{0,1,0,0}(x) = \frac{1}{2}\ln^2 x\,\mathrm{Li}_2(x) - 2\ln x\,\mathrm{Li}_3(x) + 3\mathrm{Li}_4(x), \tag{11.31}$$

$$H_{0,1,0,1}(x) = -2S_{2,2}(x) + \frac{1}{2}\mathrm{Li}_2(x)^2, \tag{11.32}$$

$$H_{0,1,1,0}(x) = \ln x\;S_{1,2}(x) - \frac{1}{2}\mathrm{Li}_2(x)^2, \tag{11.33}$$

$$H_{0,1,1,1}(x) = S_{1,3}(x), \tag{11.34}$$

$$\begin{aligned} H_{1,0,0,0}(x) = {} & -\frac{1}{6}\ln^3 x\;\ln(1-x) - \frac{1}{2}\ln^2 x\,\mathrm{Li}_2(x) \\ & + \ln x\,\mathrm{Li}_3(x) - \mathrm{Li}_4(x), \end{aligned} \tag{11.35}$$

$$H_{1,0,0,1}(x) = -\frac{1}{2}\mathrm{Li}_2(x)^2 - \ln(1-x)\mathrm{Li}_3(x), \tag{11.36}$$

$$\begin{aligned} H_{1,0,1,0}(x) = {} & 2\ln(1-x)\mathrm{Li}_3(x) - \ln x\;\ln(1-x)\mathrm{Li}_2(x) - 2\ln x\;S_{1,2}(x) \\ & + \frac{1}{2}\mathrm{Li}_2(x)^2 + 2S_{2,2}(x), \end{aligned} \tag{11.37}$$

$$H_{1,0,1,1}(x) = -\ln(1-x)S_{1,2}(x) - 3S_{1,3}(x), \tag{11.38}$$

$$\begin{aligned} H_{1,1,0,0}(x) = {} & \frac{1}{4}\ln^2 x\;\ln^2(1-x) - \ln(1-x)\mathrm{Li}_3(x) \\ & + \ln x\;\ln(1-x)\mathrm{Li}_2(x) + \ln x\;S_{1,2}(x) - S_{2,2}(x), \end{aligned} \tag{11.39}$$

$$H_{1,1,0,1}(x) = \frac{1}{2}\ln^2(1-x)\mathrm{Li}_2(x) + 2\ln(1-x)S_{1,2}(x) + 3S_{1,3}(x), \tag{11.40}$$

$$H_{1,1,1,0}(x) = -\frac{1}{6}\ln x \ \ln^3(1-x) - \frac{1}{2}\ln^2(1-x)\,\mathrm{Li}_2\,(x) - \ln(1-x)S_{1,2}(x) - S_{1,3}(x), \tag{11.41}$$

$$H_{1,1,1,1}(x) = \frac{1}{4!}\ln^4(1-x). \tag{11.42}$$

Analytic properties of HPLs (and 2dHPLs) which allow to continue them to any domain are described in [3]. A `Mathematica` package dealing with HPLs is presented in [8].

The HPLs are partial cases of multiple polylogarithms introduced by mathematicians and defined recursively by [4]

$$G(a_1,\ldots,a_n;z) = \int_0^z \frac{\mathrm{d}t}{t-a_1} G(a_2,\ldots,a_n;t), \tag{11.43}$$

with $G(z) = 1$ and where $a_i, z \in \mathbb{C}$. In the special case where $a_i = 0$ for all i one has by definition

$$G(0,\ldots,0;x) = \frac{1}{n!}\ln^n x. \tag{11.44}$$

So, the HPLs correspond to $a_i \in \{-1, 0, 1\}$:

$$H(a_1,\ldots,a_n;x) = (-1)^k\,G(a_1,\ldots,a_n;x), \quad a_i \in \{-1,0,1\}, \tag{11.45}$$

where k is the number of elements a_i equal to $+1$. Here are other partial cases where multiple polylogarithms reduce to polylogarithms and generalized polylogarithms:

$$G(\mathbf{0}_{n-1}, a; x) = -\mathrm{Li}_n\left(\frac{x}{a}\right), \qquad G(\mathbf{0}_n, \mathbf{a}_p; x) = (-1)^p\,S_{n,p}\left(\frac{x}{a}\right), \tag{11.46}$$

where $\mathbf{a} = (a_1,\ldots,a_n)$, $\mathbf{a}_n = (\underbrace{a,\ldots,a}_{n})$ and $a \neq 0$. The properties of the multiple polylogarithms have been studied very well. For example, they form a shuffle algebra.

The multiple polylogarithms can also be represented [4] as multiple nested sums

$$\mathrm{Li}_{m_1,\ldots,m_k}(x_1,\ldots,x_k) = \sum_{n_1<n_2<\cdots<n_k} \frac{x_1^{n_1}x_2^{n_2}\cdots x_k^{n_k}}{n_1^{m_1}n_2^{m_2}\cdots n_k^{m_k}} \tag{11.47}$$

$$= \sum_{n_k=1}^{\infty}\frac{x_k^{n_k}}{n_k^{m_k}}\sum_{n_{k-1}=1}^{n_k-1}\cdots\sum_{n_1=1}^{n_2-1}\frac{x_1^{n_1}}{n_1^{m_1}}. \tag{11.48}$$

where the series converges at least at $|x_i| < 1$. (Some authors use the reverse summation convention $n_1 > \cdots > n_k$.) The two definitions are related by

$$\mathrm{Li}_{m_1,\ldots,m_k}(x_1, \ldots, x_k) = (-1)^k \, G_{m_k,\ldots,m_1}\left(\frac{1}{x_k}, \ldots, \frac{1}{x_1 \ldots x_k}\right), \tag{11.49}$$

where

$$G_{m_1,\ldots,m_k}(t_1, \ldots, t_k) = G(\underbrace{0, \ldots, 0}_{m_1-1}, t_1, \ldots, \underbrace{0, \ldots, 0}_{m_k-1}, t_k; 1). \tag{11.50}$$

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Chapter 12
Appendix C: Summation Formulae

In this appendix, summation formulae for series involving nested sums and binomial coefficients are presented. Some of them are used in this book. These series have a canonical form, in the sense that the argument of a nested sum S_k is $n-1$, there are powers of $1/n^j$ etc. If the arguments are somehow shifted one can reduce the summation problem to the tables below by some manipulations with a given series. But I would recommend very much, in case of series with nested sums and inverse powers of n, to apply the packages `SUMMER` [13] and `XSummer` [10] written in `FORM` [12].

Nested sums are defined as follows [13]:

$$S_i(n) = \sum_{j=1}^{n} \frac{1}{j^i}, \quad S_{ik}(n) = \sum_{j=1}^{n} \frac{S_k(j)}{j^i}, \tag{12.1}$$

$$S_{ikl}(n) = \sum_{j=1}^{n} \frac{S_{kl}(j)}{j^i}, \quad S_{iklm}(n) = \sum_{j=1}^{n} \frac{S_{klm}(j)}{j^i}, \tag{12.2}$$

etc. Properties and algorithms for the nested sums (also for negative indices which are defined with $(-1)^j$) are presented in [13]. In particular, for positive indices, we have

$$S_{j,k}(n) + S_{k,j}(n) = S_j(n)S_k(n) + S_{j+k}(n). \tag{12.3}$$

The nested sums are closely connected with multiple zeta values

$$\zeta(m_1, \ldots, m_k) = \sum_{i_1=1}^{\infty} \sum_{i_2=1}^{i_1-1} \cdots \sum_{i_k=1}^{i_{k-1}-1} \prod_{j=1}^{k} \frac{\mathrm{sgn}(m_j)^{i_j}}{i_j^{|m_j|}}. \tag{12.4}$$

see, e.g., [1, 3–5, 11, 15] and the reviews [2, 8, 16]. The number $m_1 + \cdots + m_k$ is called *weight*.

V. A. Smirnov, *Analytic Tools for Feynman Integrals*, Springer Tracts in Modern Physics 250, DOI: 10.1007/978-3-642-34886-0_12,

The sums with one index are connected with the ψ function (the logarithmical derivative of the gamma function) as

$$\psi(n) = S_1(n-1) - \gamma_{\mathrm{E}}, \tag{12.5}$$

$$\psi^{(k)}(n) = (-1)^k k!\,\left(S_{k+1}(n-1) - \zeta(k+1)\right), \quad k = 1, 2, \ldots, \tag{12.6}$$

where $\zeta(z)$ is the Riemann zeta function

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}. \tag{12.7}$$

12.1 Some Number Series

These are series up to weight 6 with at least $1/n^2$ dependence:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}, \tag{12.8}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^3} = \zeta(3), \tag{12.9}$$

$$\sum_{n=1}^{\infty} S_1(n-1)\frac{1}{n^2} = \zeta(3), \tag{12.10}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}, \tag{12.11}$$

$$\sum_{n=1}^{\infty} S_1(n-1)\frac{1}{n^3} = \frac{\pi^4}{360}, \tag{12.12}$$

$$\sum_{n=1}^{\infty} S_1(n-1)^2\frac{1}{n^2} = \frac{11\pi^4}{360}, \tag{12.13}$$

$$\sum_{n=1}^{\infty} S_2(n-1)\frac{1}{n^2} = \frac{\pi^4}{120}, \tag{12.14}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^5} = \zeta(5), \tag{12.15}$$

$$\sum_{n=1}^{\infty} S_1(n-1)\frac{1}{n^4} = 2\zeta(5) - \frac{\pi^2\zeta(3)}{6}, \tag{12.16}$$

$$\sum_{n=1}^{\infty} S_2(n-1)\frac{1}{n^3} = \frac{\pi^2\zeta(3)}{2} - \frac{11\zeta(5)}{2}, \tag{12.17}$$

$$\sum_{n=1}^{\infty} S_1(n-1)^2\frac{1}{n^3} = \frac{\pi^2\zeta(3)}{6} - \frac{3\zeta(5)}{2}, \tag{12.18}$$

$$\sum_{n=1}^{\infty} S_3(n-1)\frac{1}{n^2} = \frac{9\zeta(5)}{2} - \frac{\pi^2\zeta(3)}{3}, \tag{12.19}$$

$$\sum_{n=1}^{\infty} S_1(n-1)^3\frac{1}{n^2} = \frac{\pi^2\zeta(3)}{6} + \frac{15\zeta(5)}{2}, \tag{12.20}$$

$$\sum_{n=1}^{\infty} S_1(n-1)S_2(n-1)\frac{1}{n^2} = \frac{7\zeta(5)}{2} - \frac{\pi^2\zeta(3)}{6}, \tag{12.21}$$

$$\sum_{n=1}^{\infty} S_{12}(n-1)\frac{1}{n^2} = 9\zeta(5) - \frac{2\pi^2\zeta(3)}{3}, \tag{12.22}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}, \tag{12.23}$$

$$\sum_{n=1}^{\infty} S_1(n-1)\frac{1}{n^5} = \frac{\pi^6}{1260} - \frac{\zeta(3)^2}{2}, \tag{12.24}$$

$$\sum_{n=1}^{\infty} S_2(n-1)\frac{1}{n^4} = -4\frac{\pi^6}{2835} + \zeta(3)^2, \tag{12.25}$$

$$\sum_{n=1}^{\infty} S_1(n-1)^2\frac{1}{n^4} = \frac{37\pi^6}{22680} - \zeta(3)^2, \tag{12.26}$$

$$\sum_{n=1}^{\infty} S_3(n-1)\frac{1}{n^3} = -\frac{\pi^6}{1890} + \frac{\zeta(3)^2}{2}, \tag{12.27}$$

$$\sum_{n=1}^{\infty} S_4(n-1)\frac{1}{n^2} = \frac{5\pi^6}{2268} - \zeta(3)^2, \tag{12.28}$$

$$\sum_{n=1}^{\infty} S_{13}(n-1)\frac{1}{n^2} = \frac{61\pi^6}{45360}, \tag{12.29}$$

$$\sum_{n=1}^{\infty} S_2(n-1)^2\frac{1}{n^2} = \frac{59\pi^6}{22680} - \zeta(3)^2, \tag{12.30}$$

$$\sum_{n=1}^{\infty} S_1(n-1)^3\frac{1}{n^3} = -\frac{11\pi^6}{5040} + 2\zeta(3)^2, \tag{12.31}$$

$$\sum_{n=1}^{\infty} S_1(n-1)S_2(n-1)\frac{1}{n^3} = -\frac{121\pi^6}{45360} + 2\zeta(3)^2, \tag{12.32}$$

$$\sum_{n=1}^{\infty} S_{12}(n-1)\frac{1}{n^3} = \frac{41\pi^6}{22680} - \zeta(3)^2, \tag{12.33}$$

$$\sum_{n=1}^{\infty} S_1(n-1)S_3(n-1)\frac{1}{n^2} = \frac{167\pi^6}{45360} - \frac{3\zeta(3)^2}{2}, \tag{12.34}$$

$$\sum_{n=1}^{\infty} S_1(n-1)^2 S_2(n-1)\frac{1}{n^2} = \frac{23\pi^6}{3780} - \zeta(3)^2, \tag{12.35}$$

$$\sum_{n=1}^{\infty} S_1(n-1)^4\frac{1}{n^2} = \frac{859\pi^6}{22680} + 3\zeta(3)^2, \tag{12.36}$$

$$\sum_{n=1}^{\infty} S_{112}(n-1)\frac{1}{n^2} = \frac{17\pi^6}{4536} - \zeta(3)^2, \tag{12.37}$$

$$\sum_{n=1}^{\infty} S_1(n-1)S_{12}(n-1)\frac{1}{n^2} = \frac{313\pi^6}{45360} - 2\zeta(3)^2. \tag{12.38}$$

Series up to weight 6 with the factor $1/n$ where the convergence is provided by other factors:

$$\sum_{n=1}^{\infty} \psi'(n+1)\frac{1}{n} = \zeta(3), \tag{12.39}$$

$$\sum_{n=1}^{\infty} \psi'(n+1)S_1(n)\frac{1}{n} = \frac{7\pi^4}{360}, \tag{12.40}$$

$$\sum_{n=1}^{\infty} \psi''(n+1)\frac{1}{n} = -\frac{\pi^4}{180}, \tag{12.41}$$

$$\sum_{n=1}^{\infty} \psi'(n+1)S_1(n)^2\frac{1}{n} = \frac{\pi^2\zeta(3)}{3}, \tag{12.42}$$

$$\sum_{n=1}^{\infty} \psi'(n+1)^2\frac{1}{n} = \frac{5\pi^2\zeta(3)}{6} - 9\zeta(5), \tag{12.43}$$

$$\sum_{n=1}^{\infty} \psi''(n+1)S_1(n)\frac{1}{n} = -\frac{2\pi^2\zeta(3)}{3} + 7\zeta(5), \tag{12.44}$$

$$\sum_{n=1}^{\infty} \psi'''(n+1)\frac{1}{n} = -\pi^2\zeta(3) + 12\zeta(5), \tag{12.45}$$

$$\sum_{n=1}^{\infty} \psi''''(n+1)\frac{1}{n} = -\frac{2\pi^6}{105} + 12\zeta(3)^2, \tag{12.46}$$

$$\sum_{n=1}^{\infty} \psi'''(n+1) S_1(n) \frac{1}{n} = \frac{\pi^6}{1512}, \tag{12.47}$$

$$\sum_{n=1}^{\infty} \psi''(n+1) S_1(n)^2 \frac{1}{n} = \frac{\pi^6}{90} - 8\zeta(3)^2, \tag{12.48}$$

$$\sum_{n=1}^{\infty} \psi'(n+1)^2 S_1(n) \frac{1}{n} = -\frac{\pi^6}{432} + 2\zeta(3)^2, \tag{12.49}$$

$$\sum_{n=1}^{\infty} \psi'(n+1) S_1(n)^3 \frac{1}{n} = \frac{269\pi^6}{22680}, \tag{12.50}$$

$$\sum_{n=1}^{\infty} \psi'(n+1)\psi''(n+1) \frac{1}{n} = \frac{61\pi^6}{22680} - 2\zeta(3)^2. \tag{12.51}$$

Series of weight 7 with at least $1/n^2$ dependence:

$$\sum_{n=1}^{\infty} \frac{1}{n^7} = \zeta(7), \tag{12.52}$$

$$\sum_{n=1}^{\infty} S_1(n-1) \frac{1}{n^6} = 3\zeta(7) - \frac{\pi^2\zeta(5)}{6} - \frac{\pi^4\zeta(3)}{90}, \tag{12.53}$$

$$\sum_{n=1}^{\infty} S_2(n-1) \frac{1}{n^5} = -11\zeta(7) + \frac{5\pi^2\zeta(5)}{6} + \frac{\pi^4\zeta(3)}{45}, \tag{12.54}$$

$$\sum_{n=1}^{\infty} S_1(n-1)^2 \frac{1}{n^5} = -\zeta(7) + \frac{\pi^2\zeta(5)}{6} - \frac{\pi^4\zeta(3)}{180}, \tag{12.55}$$

$$\sum_{n=1}^{\infty} S_3(n-1) \frac{1}{n^4} = 17\zeta(7) - \frac{5\pi^2\zeta(5)}{3}, \tag{12.56}$$

$$\sum_{n=1}^{\infty} S_1(n-1)^3 \frac{1}{n^4} = \frac{119\zeta(7)}{16} + \frac{\pi^2\zeta(5)}{3} - \frac{11\pi^4\zeta(3)}{120}, \tag{12.57}$$

$$\sum_{n=1}^{\infty} S_1(n-1) S_2(n-1) \frac{1}{n^4} = \frac{61\zeta(7)}{16} - \frac{\pi^2\zeta(5)}{3} + \frac{\pi^4\zeta(3)}{}, \tag{12.58}$$

$$\sum_{n=1}^{\infty} S_{12}(n-1) \frac{1}{n^4} = \frac{141\zeta(7)}{8} - \frac{5\pi^2\zeta(5)}{4} - \frac{\pi^4\zeta(3)}{24}, \tag{12.59}$$

$$\sum_{n=1}^{\infty} S_4(n-1) \frac{1}{n^3} = -18\zeta(7) + \frac{5\pi^2\zeta(5)}{3} + \frac{\pi^4\zeta(3)}{90}, \tag{12.60}$$

$$\sum_{n=1}^{\infty} S_{13}(n-1) \frac{1}{n^3} = -\frac{73\zeta(7)}{4} + \frac{5\pi^2\zeta(5)}{3} + \frac{\pi^4\zeta(3)}{72}, \tag{12.61}$$

$$\sum_{n=1}^{\infty} S_1(n-1)S_3(n-1)\frac{1}{n^3} = -\frac{85\zeta(7)}{8} + \frac{11\pi^2\zeta(5)}{12} + \frac{\pi^4\zeta(3)}{72}, \tag{12.62}$$

$$\sum_{n=1}^{\infty} S_2(n-1)^2\frac{1}{n^3} = \frac{13\zeta(7)}{8} - \frac{5\pi^2\zeta(5)}{6} + \frac{11\pi^4\zeta(3)}{180}, \tag{12.63}$$

$$\sum_{n=1}^{\infty} S_1(n-1)S_{12}(n-1)\frac{1}{n^3} = -\frac{113\zeta(7)}{16} + \frac{7\pi^2\zeta(5)}{12} + \frac{\pi^4\zeta(3)}{72}, \tag{12.64}$$

$$\sum_{n=1}^{\infty} S_1(n-1)^2S_2(n-1)\frac{1}{n^3} = -\frac{77\zeta(7)}{8} - \frac{\pi^2\zeta(5)}{3} + \frac{7\pi^4\zeta(3)}{60}, \tag{12.65}$$

$$\sum_{n=1}^{\infty} S_1(n-1)^4\frac{1}{n^3} = -\frac{109\zeta(7)}{8} - \frac{5\pi^2\zeta(5)}{6} + \frac{37\pi^4\zeta(3)}{180}, \tag{12.66}$$

$$\sum_{n=1}^{\infty} S_{112}(n-1)\frac{1}{n^3} = -\frac{61\zeta(7)}{4} + \frac{5\pi^2\zeta(5)}{4} + \frac{\pi^4\zeta(3)}{40}, \tag{12.67}$$

$$\sum_{n=1}^{\infty} S_1(n-1)S_4(n-1)\frac{1}{n^2} = \frac{173\zeta(7)}{16} - \frac{3\pi^2\zeta(5)}{4} - \frac{\pi^4\zeta(3)}{60}, \tag{12.68}$$

$$\sum_{n=1}^{\infty} S_1(n-1)S_{13}(n-1)\frac{1}{n^2} = \frac{61\zeta(7)}{4} - \frac{3\pi^2\zeta(5)}{2} + \frac{\pi^4\zeta(3)}{36}, \tag{12.69}$$

$$\sum_{n=1}^{\infty} S_1(n-1)^2S_3(n-1)\frac{1}{n^2} = \frac{301\zeta(7)}{16} - \frac{3\pi^2\zeta(5)}{4} - \frac{\pi^4\zeta(3)}{15}, \tag{12.70}$$

$$\sum_{n=1}^{\infty} S_1(n-1)S_2(n-1)^2\frac{1}{n^2} = -\frac{77\zeta(7)}{16} + \frac{13\pi^2\zeta(5)}{12} - \frac{\pi^4\zeta(3)}{30}, \tag{12.71}$$

$$\sum_{n=1}^{\infty} S_1(n-1)^2S_{12}(n-1)\frac{1}{n^2} = \frac{423\zeta(7)}{16} - \frac{\pi^2\zeta(5)}{6} - \frac{37\pi^4\zeta(3)}{360}, \tag{12.72}$$

$$\sum_{n=1}^{\infty} S_1(n-1)^3S_2(n-1)\frac{1}{n^2} = \frac{307\zeta(7)}{16} + \frac{5\pi^2\zeta(5)}{12} - \frac{13\pi^4\zeta(3)}{180}, \tag{12.73}$$

$$\sum_{n=1}^{\infty} S_1(n-1)^5\frac{1}{n^2} = \frac{1855\zeta(7)}{16} + \frac{19\pi^2\zeta(5)}{4} + \frac{11\pi^4\zeta(3)}{30}, \tag{12.74}$$

$$\sum_{n=1}^{\infty} S_1(n-1)S_{112}(n-1)\frac{1}{n^2} = \frac{73\zeta(7)}{4} - \frac{3\pi^2\zeta(5)}{4} - \frac{\pi^4\zeta(3)}{30}, \tag{12.75}$$

$$\sum_{n=1}^{\infty} S_5(n-1)\frac{1}{n^2} = 10\zeta(7) - \frac{2\pi^2\zeta(5)}{3} - \frac{\pi^4\zeta(3)}{45}, \tag{12.76}$$

$$\sum_{n=1}^{\infty} S_{14}(n-1)\frac{1}{n^2} = \frac{141\zeta(7)}{8} - \frac{19\pi^2\zeta(5)}{12} - \frac{\pi^4\zeta(3)}{360}, \tag{12.77}$$

$$\sum_{n=1}^{\infty} S_2(n-1)S_3(n-1)\frac{1}{n^2} = \frac{19\zeta(7)}{16} + \frac{5\pi^2\zeta(5)}{12} - \frac{7\pi^4\zeta(3)}{180}, \tag{12.78}$$

$$\sum_{n=1}^{\infty} S_{23}(n-1)\frac{1}{n^2} = -\frac{131\zeta(7)}{16} + \frac{4\pi^2\zeta(5)}{3} - \frac{7\pi^4\zeta(3)}{180}, \tag{12.79}$$

$$\sum_{n=1}^{\infty} S_2(n-1)S_{12}(n-1)\frac{1}{n^2} = -\frac{141\zeta(7)}{16} + \frac{5\pi^2\zeta(5)}{3} - \frac{19\pi^4\zeta(3)}{360}, \tag{12.80}$$

$$\sum_{n=1}^{\infty} S_{113}(n-1)\frac{1}{n^2} = \frac{113\zeta(7)}{16} - \frac{\pi^2\zeta(5)}{2}, \tag{12.81}$$

$$\sum_{n=1}^{\infty} S_{212}(n-1)\frac{1}{n^2} = \frac{169\zeta(7)}{16} - \frac{\pi^2\zeta(5)}{2} - \frac{7\pi^4\zeta(3)}{180}, \tag{12.82}$$

$$\sum_{n=1}^{\infty} S_{1112}(n-1)\frac{1}{n^2} = \frac{141\zeta(7)}{8} - \pi^2\zeta(5) - \frac{7\pi^4\zeta(3)}{180}. \tag{12.83}$$

12.2 Power Series of Weights 3 and 4 in Terms of Polylogarithms

The formulae of this section can be found in [7].

$$\sum_{n=1}^{\infty} S_2(n-1)\frac{z^n}{n} = -2S_{1,2}(z) - \ln(1-z)\mathrm{Li}_2(z), \tag{12.84}$$

$$\sum_{n=1}^{\infty} S_1(n-1)^2\frac{z^n}{n} = -2S_{1,2}(z) - \ln(1-z)\mathrm{Li}_2(z) - \frac{1}{3}\ln^3(1-z), \tag{12.85}$$

$$\sum_{n=1}^{\infty} S_1(n-1)\frac{z^n}{n^2} = S_{1,2}(z), \tag{12.86}$$

$$\sum_{n=1}^{\infty} \frac{z^n}{n^3} = \mathrm{Li}_3(z), \tag{12.87}$$

$$\sum_{n=1}^{\infty} S_3(n-1)\frac{z^n}{n} = -\frac{1}{2}\mathrm{Li}_2(z)^2 - \ln(1-z)\mathrm{Li}_3(z), \tag{12.88}$$

$$\sum_{n=1}^{\infty} S_{12}(n-1)\frac{z^n}{n} = 3S_{1,3}(z) - \ln(1-z)\mathrm{Li}_3\,(z) - \frac{1}{2}\mathrm{Li}_2\,(z)^2$$
$$+\ \frac{1}{2}\ln^2(1-z)\mathrm{Li}_2\,(z) + 2\ln(1-z)S_{1,2}(z), \tag{12.89}$$

$$\sum_{n=1}^{\infty} S_1(n-1)S_2(n-1)\frac{z^n}{n} = -\frac{1}{2}\mathrm{Li}_2\,(z)^2 + \ln(1-z)(S_{1,2}(z) - \mathrm{Li}_3\,(z))$$
$$+\ \frac{1}{2}\ln^2(1-z)\mathrm{Li}_2\,(z)\,, \tag{12.90}$$

$$\sum_{n=1}^{\infty} S_1(n-1)^3\frac{z^n}{n} = -\frac{1}{2}\mathrm{Li}_2\,(z)^2 + \frac{3}{2}\ln^2(1-z)\mathrm{Li}_2\,(z)$$
$$+\ln(1-z)(3S_{1,2}(z) - \mathrm{Li}_3\,(z)) + \frac{1}{4}\ln^4(1-z), \tag{12.91}$$

$$\sum_{n=1}^{\infty} S_2(n-1)\frac{z^n}{n^2} = -2S_{2,2}(z) + \frac{1}{2}\mathrm{Li}_2\,(z)^2\,, \tag{12.92}$$

$$\sum_{n=1}^{\infty} S_1(n-1)^2\frac{z^n}{n^2} = 2S_{1,3}(z) - 2S_{2,2}(z) + \frac{1}{2}\mathrm{Li}_2\,(z)^2\,, \tag{12.93}$$

$$\sum_{n=1}^{\infty} S_1(n-1)\frac{z^n}{n^3} = S_{2,2}(z), \tag{12.94}$$

$$\sum_{n=1}^{\infty} \frac{z^n}{n^4} = \mathrm{Li}_4\,(z)\,. \tag{12.95}$$

12.3 Inverse Binomial Power Series up to Weight 4

The formulae of this section (as well as other similar formulae) can be found in [6]. See a table of formulae for the corresponding number series in [9]. Let

$$y = \frac{\sqrt{4-z} - \sqrt{-z}}{\sqrt{4-z} + \sqrt{-z}}.$$

Then

$$\sum_{n=1}^{\infty} \frac{1}{\binom{2n}{n}}\frac{z^n}{n} = \frac{1-y}{1+y}\ln y, \tag{12.96}$$

$$\sum_{n=1}^{\infty} \frac{1}{\binom{2n}{n}}\frac{z^n}{n^2} = -\tfrac{1}{2}\ln^2 y, \tag{12.97}$$

$$\sum_{n=1}^{\infty}\frac{1}{\binom{2n}{n}}\frac{z^n}{n^3} = 2\mathrm{Li}_3(y) - 2\ln y\,\mathrm{Li}_2(y) - \ln^2 y\ln(1-y) + \tfrac{1}{6}\ln^3 y - 2\zeta(3), \tag{12.98}$$

$$\sum_{n=1}^{\infty}\frac{1}{\binom{2n}{n}}\frac{z^n}{n^4} = 4S_{2,2}(y) - 4\mathrm{Li}_4(y) - 4S_{1,2}(y)\ln y + 4\mathrm{Li}_3(y)\ln(1-y) + 2\mathrm{Li}_3(y)\ln y - 4\mathrm{Li}_2(y)\ln y\ln(1-y) - \ln^2 y\ln^2(1-y) + \tfrac{1}{3}\ln^3 y\ln(1-y) - \tfrac{1}{24}\ln^4 y - 4\ln(1-y)\zeta(3) + 2\ln y\,\zeta(3) + 3\zeta(4), \tag{12.99}$$

$$\sum_{n=1}^{\infty}\frac{1}{\binom{2n}{n}}\frac{z^n}{n}S_1(n-1) = \frac{1-y}{1+y} \times\left[-2\mathrm{Li}_2(-y) - 2\ln y\ln(1+y) + \tfrac{1}{2}\ln^2 y - \zeta(2)\right], \tag{12.100}$$

$$\sum_{n=1}^{\infty}\frac{1}{\binom{2n}{n}}\frac{z^n}{n}S_1(n-1)^2 = \frac{1-y}{1+y}\Big[8S_{1,2}(-y) - 4\mathrm{Li}_3(-y) + 8\mathrm{Li}_2(-y)\ln(1+y) + 4\ln^2(1+y)\ln y - 2\ln(1+y)\ln^2 y + \tfrac{1}{6}\ln^3 y + 4\zeta(2)\ln(1+y) - 2\zeta(2)\ln y - 4\zeta(3)\Big], \tag{12.101}$$

$$\sum_{n=1}^{\infty}\frac{1}{\binom{2n}{n}}\frac{z^n}{n}S_2(n-1) = -\frac{1-y}{6(1+y)}\ \ln^3 y, \tag{12.102}$$

$$\sum_{n=1}^{\infty}\frac{1}{\binom{2n}{n}}\frac{z^n}{n^2}S_2(n-1) = \tfrac{1}{24}\ln^4 y, \tag{12.103}$$

$$\sum_{n=1}^{\infty}\frac{1}{\binom{2n}{n}}\frac{z^n}{n}S_3(n-1) = \frac{1-y}{1+y}\Big[\tfrac{1}{24}\ln^4 y + 6\mathrm{Li}_4(y) + \ln^2 y\,\mathrm{Li}_2(y) - 2\zeta(3)\ln y - 4\ln y\,\mathrm{Li}_3(y) - 6\zeta(4)\Big],$$

$$\sum_{n=1}^{\infty}\frac{1}{\binom{2n}{n}}\frac{z^n}{n}S_1(n-1)S_2(n-1) = \frac{1-y}{1+y}\Big[\tfrac{1}{3}\ln^3 y\ln(1+y) - \tfrac{1}{24}\ln^4 y + \tfrac{1}{2}\zeta(2)\ln^2 y + \ln^2 y\,\mathrm{Li}_2(-y) + \ln^2 y\,\mathrm{Li}_2(y) + \zeta(3)\ln y - 4\ln y\,\mathrm{Li}_3(-y) - 4\ln y\,\mathrm{Li}_3(y) + \zeta(4) + 8\mathrm{Li}_4(-y) + 6\mathrm{Li}_4(y)\Big], \tag{12.104}$$

$$\sum_{n=1}^{\infty} \frac{1}{\binom{2n}{n}} \frac{z^n}{n^2} S_1(n-1) = 4\mathrm{Li}_3(-y) - 2\mathrm{Li}_2(-y)\ln y - \tfrac{1}{6}\ln^3 y + 3\zeta(3) + \zeta(2)\ln y, \tag{12.105}$$

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{\binom{2n}{n}} \frac{z^n}{n^2} S_1(n-1)^2 &= -8S_{1,2}(-y)\ln y + 4\mathrm{Li}_3(-y)\ln y \\ &\quad - 2\mathrm{Li}_2(-y)\ln^2 y + 4\mathrm{Li}_2(-y)^2 - \tfrac{1}{24}\ln^4 y + 4\zeta(2)\mathrm{Li}_2(-y) \\ &\quad + \zeta(2)\ln^2 y + 4\zeta(3)\ln y + \tfrac{5}{2}\zeta(4),\end{aligned} \tag{12.106}$$

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{\binom{2n}{n}} \frac{z^n}{n^3} S_1(n-1) &= 4H_{-1,0,0,1}(-y) + S_{2,2}\left(y^2\right) \\ &\quad - 4S_{2,2}(y) - 4S_{2,2}(-y) - 6\mathrm{Li}_4(-y) - 2\mathrm{Li}_4(y) + 4S_{1,2}(-y)\ln y \\ &\quad + 4S_{1,2}(y)\ln y - 2S_{1,2}\left(y^2\right)\ln y + 4\mathrm{Li}_3(-y)\ln(1-y) \\ &\quad + 2\mathrm{Li}_3(-y)\ln y + 2\mathrm{Li}_3(y)\ln y - \mathrm{Li}_2(y)\ln^2 y \\ &\quad - 4\mathrm{Li}_2(-y)\ln y\ln(1-y) - \tfrac{1}{3}\ln^3 y\ln(1-y) + \tfrac{1}{24}\ln^4 y \\ &\quad + 2\zeta(2)\mathrm{Li}_2(y) - \tfrac{1}{2}\zeta(2)\ln^2 y + 2\zeta(2)\ln y\ln(1-y) \\ &\quad + 6\zeta(3)\ln(1-y) - 3\zeta(3)\ln y - 4\zeta(4),\end{aligned} \tag{12.107}$$

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{\binom{2n}{n}} \frac{z^n}{n} S_1(n-1)^3 &= \frac{1-y}{1+y}\Big[-48S_{1,2}(-y)\ln(1+y) - 48S_{1,3}(-y) \\ &\quad + 24S_{2,2}(-y) - 12\zeta(2)\ln^2(1+y) - 24\ln^2(1+y)\mathrm{Li}_2(-y) \\ &\quad + 24\zeta(3)\ln(1+y) + 24\ln(1+y)\mathrm{Li}_3(-y) - 8\ln y\ln^3(1+y) \\ &\quad + 12\zeta(2)\ln y\ln(1+y) + 6\ln^2 y\ln^2(1+y) - \ln^3 y\ln(1+y) \\ &\quad + \tfrac{1}{24}\ln^4 y - \tfrac{3}{2}\zeta(2)\ln^2 y + 3\ln^2 y\,\mathrm{Li}_2(-y) \\ &\quad + \ln^2 y\,\mathrm{Li}_2(y) - 5\zeta(3)\ln y - 12\ln y\,\mathrm{Li}_3(-y) - 4\ln y\,\mathrm{Li}_3(y) \\ &\quad + \tfrac{3}{2}\zeta(4) + 12\mathrm{Li}_4(-y) + 6\mathrm{Li}_4(y)\Big].\end{aligned} \tag{12.108}$$

12.4 Power Series of Weights 5 and 6 in Terms of HPLs

$$\sum_{n=1}^{\infty} \frac{z^n}{n^5} = H_{0,0,0,0,1}(z), \tag{12.109}$$

$$\sum_{n=1}^{\infty} S_1(n-1)\frac{z^n}{n^4} = H_{0,0,0,1,1}(z), \tag{12.110}$$

$$\sum_{n=1}^{\infty} S_2(n-1)\,\frac{z^n}{n^3} = H_{0,0,1,0,1}(z), \tag{12.111}$$

$$\sum_{n=1}^{\infty} S_1(n-1)^2\,\frac{z^n}{n^3} = H_{0,0,1,0,1}(z) + 2H_{0,0,1,1,1}(z), \tag{12.112}$$

$$\sum_{n=1}^{\infty} S_3(n-1)\,\frac{z^n}{n^2} = H_{0,1,0,0,1}(z), \tag{12.113}$$

$$\begin{aligned}\sum_{n=1}^{\infty} S_1(n-1)^3\,\frac{z^n}{n^2} &= H_{0,1,0,0,1}(z) + 3H_{0,1,0,1,1}(z)\\ &\quad + 3H_{0,1,1,0,1}(z) + 6H_{0,1,1,1,1}(z),\end{aligned} \tag{12.114}$$

$$\begin{aligned}\sum_{n=1}^{\infty} S_1(n-1)S_2(n-1)\,\frac{z^n}{n^2} &= H_{0,1,0,0,1}(z) + H_{0,1,0,1,1}(z)\\ &\quad + H_{0,1,1,0,1}(z),\end{aligned} \tag{12.115}$$

$$\sum_{n=1}^{\infty} S_{12}(n-1)\,\frac{z^n}{n^2} = H_{0,1,0,0,1}(z) + H_{0,1,1,0,1}(z), \tag{12.116}$$

$$\sum_{n=1}^{\infty} S_4(n-1)\,\frac{z^n}{n} = H_{1,0,0,0,1}(z), \tag{12.117}$$

$$\sum_{n=1}^{\infty} S_{13}(n-1)\,\frac{z^n}{n} = H_{1,0,0,0,1}(z) + H_{1,1,0,0,1}(z), \tag{12.118}$$

$$\begin{aligned}\sum_{n=1}^{\infty} S_1(n-1)S_3(n-1)\,\frac{z^n}{n} &= H_{1,0,0,0,1}(z) + H_{1,0,0,1,1}(z)\\ &\quad + H_{1,1,0,0,1}(z),\end{aligned} \tag{12.119}$$

$$\sum_{n=1}^{\infty} S_2(n-1)^2\,\frac{z^n}{n} = H_{1,0,0,0,1}(z) + 2H_{1,0,1,0,1}(z), \tag{12.120}$$

$$\begin{aligned}&\sum_{n=1}^{\infty} S_1(n-1)S_{12}(n-1)\,\frac{z^n}{n} = H_{1,0,0,0,1}(z) + H_{1,0,0,1,1}(z)\\ &\quad + H_{1,0,1,0,1}(z) + 2H_{1,1,0,0,1}(z) + H_{1,1,0,1,1}(z) + 2H_{1,1,1,0,1}(z),\end{aligned} \tag{12.121}$$

$$\begin{aligned}\sum_{n=1}^{\infty} S_1(n-1)^2S_2(n-1)\,\frac{z^n}{n} &= H_{1,0,0,0,1}(z) + 2H_{1,0,0,1,1}(z)\\ &\quad + 2H_{1,0,1,0,1}(z) + 2H_{1,0,1,1,1}(z) + 2H_{1,1,0,0,1}(z)\\ &\quad + 2H_{1,1,0,1,1}(z) + 2H_{1,1,1,0,1}(z),\end{aligned} \tag{12.122}$$

$$\sum_{n=1}^{\infty} S_1(n-1)^4 \frac{z^n}{n} = H_{1,0,0,0,1}(z) + 4H_{1,0,0,1,1}(z) + 6H_{1,0,1,0,1}(z) + 12H_{1,0,1,1,1}(z) + 4H_{1,1,0,0,1}(z) + 12H_{1,1,0,1,1}(z) + 12H_{1,1,1,0,1}(z) + 24H_{1,1,1,1,1}(z), \tag{12.123}$$

$$\sum_{n=1}^{\infty} S_{112}(n-1) \frac{z^n}{n} = H_{1,0,0,0,1}(z) + H_{1,0,1,0,1}(z) + H_{1,1,0,0,1}(z) + H_{1,1,1,0,1}(z), \tag{12.124}$$

$$\sum_{n=1}^{\infty} \frac{z^n}{n^6} = H_{0,0,0,0,0,1}(z), \tag{12.125}$$

$$\sum_{n=1}^{\infty} S_1(n-1) \frac{z^n}{n^5} = H_{0,0,0,0,1,1}(z), \tag{12.126}$$

$$\sum_{n=1}^{\infty} S_2(n-1) \frac{z^n}{n^4} = H_{0,0,0,1,0,1}(z), \tag{12.127}$$

$$\sum_{n=1}^{\infty} S_1(n-1)^2 \frac{z^n}{n^4} = H_{0,0,0,1,0,1}(z) + 2H_{0,0,0,1,1,1}(z), \tag{12.128}$$

$$\sum_{n=1}^{\infty} S_3(n-1) \frac{z^n}{n^3} = H_{0,0,1,0,0,1}(z), \tag{12.129}$$

$$\sum_{n=1}^{\infty} S_1(n-1)^3 \frac{z^n}{n^3} = H_{0,0,1,0,0,1}(z) + 3H_{0,0,1,0,1,1}(z) + 3H_{0,0,1,1,0,1}(z) + 6H_{0,0,1,1,1,1}(z), \tag{12.130}$$

$$\sum_{n=1}^{\infty} S_1(n-1)S_2(n-1) \frac{z^n}{n^3} = H_{0,0,1,0,0,1}(z) + H_{0,0,1,0,1,1}(z) + H_{0,0,1,1,0,1}(z), \tag{12.131}$$

$$\sum_{n=1}^{\infty} S_{12}(n-1) \frac{z^n}{n^3} = H_{0,0,1,0,0,1}(z) + H_{0,0,1,1,0,1}(z), \tag{12.132}$$

$$\sum_{n=1}^{\infty} S_4(n-1) \frac{z^n}{n^2} = H_{0,1,0,0,0,1}(z), \tag{12.133}$$

$$\sum_{n=1}^{\infty} S_{13}(n-1) \frac{z^n}{n^2} = H_{0,1,0,0,0,1}(z) + H_{0,1,1,0,0,1}(z), \tag{12.134}$$

$$\sum_{n=1}^{\infty} S_1(n-1)S_3(n-1) \frac{z^n}{n^2} = H_{0,1,0,0,0,1}(z) + H_{0,1,0,0,1,1}(z) + H_{0,1,1,0,0,1}(z), \tag{12.135}$$

$$\sum_{n=1}^{\infty} S_2(n-1)^2 \frac{z^n}{n^2} = H_{0,1,0,0,0,1}(z) + 2H_{0,1,0,1,0,1}(z), \tag{12.136}$$

$$\sum_{n=1}^{\infty} S_1(n-1)S_{12}(n-1) \frac{z^n}{n^2} = H_{0,1,0,0,0,1}(z) + H_{0,1,0,0,1,1}(z) + H_{0,1,0,1,0,1}(z) + 2H_{0,1,1,0,0,1}(z) + H_{0,1,1,0,1,1}(z) + 2H_{0,1,1,1,0,1}(z), \tag{12.137}$$

$$\sum_{n=1}^{\infty} S_1(n-1)^2 S_2(n-1) \frac{z^n}{n^2} = H_{0,1,0,0,0,1}(z) + 2H_{0,1,0,0,1,1}(z) + 2H_{0,1,0,1,0,1}(z) + 2H_{0,1,0,1,1,1}(z) + 2H_{0,1,1,0,0,1}(z) + 2H_{0,1,1,0,1,1}(z) + 2H_{0,1,1,1,0,1}(z), \tag{12.138}$$

$$\sum_{n=1}^{\infty} S_1(n-1)^4 \frac{z^n}{n^2} = H_{0,1,0,0,0,1}(z) + 4H_{0,1,0,0,1,1}(z) + 6H_{0,1,0,1,0,1}(z) + 12H_{0,1,0,1,1,1}(z) + 4H_{0,1,1,0,0,1}(z) + 12H_{0,1,1,0,1,1}(z) + 12H_{0,1,1,1,0,1}(z) + 24H_{0,1,1,1,1,1}(z), \tag{12.139}$$

$$\sum_{n=1}^{\infty} S_{112}(n-1) \frac{z^n}{n^2} = H_{0,1,0,0,0,1}(z) + H_{0,1,0,1,0,1}(z) + H_{0,1,1,0,0,1}(z) + H_{0,1,1,1,0,1}(z), \tag{12.140}$$

$$\sum_{n=1}^{\infty} S_1(n-1)S_4(n-1) \frac{z^n}{n} = H_{1,0,0,0,0,1}(z) + H_{1,0,0,0,1,1}(z) + H_{1,1,0,0,0,1}(z), \tag{12.141}$$

$$\sum_{n=1}^{\infty} S_1(n-1)S_{13}(n-1) \frac{z^n}{n} = H_{1,0,0,0,0,1}(z) + H_{1,0,0,0,1,1}(z) + H_{1,0,1,0,0,1}(z) + 2H_{1,1,0,0,0,1}(z) + H_{1,1,0,0,1,1}(z) + 2H_{1,1,1,0,0,1}(z), \tag{12.142}$$

$$\sum_{n=1}^{\infty} S_1(n-1)^2 S_3(n-1) \frac{z^n}{n} = H_{1,0,0,0,0,1}(z) + 2H_{1,0,0,0,1,1}(z) + H_{1,0,0,1,0,1}(z) + 2H_{1,0,0,1,1,1}(z) + H_{1,0,1,0,0,1}(z) + 2H_{1,1,0,0,0,1}(z) + 2H_{1,1,0,0,1,1}(z) + 2H_{1,1,1,0,0,1}(z), \tag{12.143}$$

$$\sum_{n=1}^{\infty} S_1(n-1)S_2(n-1)^2 \frac{z^n}{n} = H_{1,0,0,0,0,1}(z) + H_{1,0,0,0,1,1}(z) + 2H_{1,0,0,1,0,1}(z) + 2H_{1,0,1,0,0,1}(z) + 2H_{1,0,1,0,1,1}(z) + 2H_{1,0,1,1,0,1}(z) + H_{1,1,0,0,0,1}(z) + 2H_{1,1,0,1,0,1}(z), \tag{12.144}$$

$$\sum_{n=1}^{\infty} S_1(n-1)^2 S_{12}(n-1)\,\frac{z^n}{n} = H_{1,0,0,0,0,1}(z) + 2H_{1,0,0,0,1,1}(z)$$
$$+ 2H_{1,0,0,1,0,1}(z) + 2H_{1,0,0,1,1,1}(z) + 3H_{1,0,1,0,0,1}(z)$$
$$+ 2H_{1,0,1,0,1,1}(z) + 3H_{1,0,1,1,0,1}(z) + 3H_{1,1,0,0,0,1}(z)$$
$$+ 4H_{1,1,0,0,1,1}(z) + 4H_{1,1,0,1,0,1}(z) + 2H_{1,1,0,1,1,1}(z)$$
$$+ 6H_{1,1,1,0,0,1}(z) + 4H_{1,1,1,0,1,1}(z) + 6H_{1,1,1,1,0,1}(z), \tag{12.145}$$

$$\sum_{n=1}^{\infty} S_1(n-1)^3 S_2(n-1)\,\frac{z^n}{n} = H_{1,0,0,0,0,1}(z) + 3H_{1,0,0,0,1,1}(z)$$
$$+ 4H_{1,0,0,1,0,1}(z) + 6H_{1,0,0,1,1,1}(z) + 4H_{1,0,1,0,0,1}(z)$$
$$+ 6H_{1,0,1,0,1,1}(z) + 6H_{1,0,1,1,0,1}(z) + 6H_{1,0,1,1,1,1}(z)$$
$$+ 3H_{1,1,0,0,0,1}(z) + 6H_{1,1,0,0,1,1}(z) + 6H_{1,1,0,1,0,1}(z)$$
$$+ 6H_{1,1,0,1,1,1}(z) + 6H_{1,1,1,0,0,1}(z)$$
$$+ 6H_{1,1,1,0,1,1}(z) + 6H_{1,1,1,1,0,1}(z), \tag{12.146}$$

$$\sum_{n=1}^{\infty} S_1(n-1)^5\,\frac{z^n}{n} = H_{1,0,0,0,0,1}(z) + 5H_{1,0,0,0,1,1}(z) + 10H_{1,0,0,1,0,1}(z)$$
$$+ 20H_{1,0,0,1,1,1}(z) + 10H_{1,0,1,0,0,1}(z) + 30H_{1,0,1,0,1,1}(z)$$
$$+ 30H_{1,0,1,1,0,1}(z) + 60H_{1,0,1,1,1,1}(z) + 5H_{1,1,0,0,0,1}(z)$$
$$+ 20H_{1,1,0,0,1,1}(z) + 30H_{1,1,0,1,0,1}(z) + 60H_{1,1,0,1,1,1}(z)$$
$$+ 20H_{1,1,1,0,0,1}(z) + 60H_{1,1,1,0,1,1}(z)$$
$$+ 60H_{1,1,1,1,0,1}(z) + 120H_{1,1,1,1,1,1}(z), \tag{12.147}$$

$$\sum_{n=1}^{\infty} S_1(n-1) S_{112}(n-1)\,\frac{z^n}{n} = H_{1,0,0,0,0,1}(z) + H_{1,0,0,0,1,1}(z)$$
$$+ H_{1,0,0,1,0,1}(z) + 2H_{1,0,1,0,0,1}(z) + H_{1,0,1,0,1,1}(z)$$
$$+ 2H_{1,0,1,1,0,1}(z) + 2H_{1,1,0,0,0,1}(z) + H_{1,1,0,0,1,1}(z)$$
$$+ 2H_{1,1,0,1,0,1}(z) + 3H_{1,1,1,0,0,1}(z)$$
$$+ H_{1,1,1,0,1,1}(z) + 3H_{1,1,1,1,0,1}(z), \tag{12.148}$$

$$\sum_{n=1}^{\infty} S_5(n-1)\,\frac{z^n}{n} = H_{1,0,0,0,0,1}(z), \tag{12.149}$$

$$\sum_{n=1}^{\infty} S_{14}(n-1)\,\frac{z^n}{n} = H_{1,0,0,0,0,1}(z) + H_{1,1,0,0,0,1}(z), \tag{12.150}$$

$$\sum_{n=1}^{\infty} S_2(n-1) S_3(n-1)\,\frac{z^n}{n} = H_{1,0,0,0,0,1}(z) + H_{1,0,0,1,0,1}(z)$$
$$+ H_{1,0,1,0,0,1}(z), \tag{12.151}$$

$$\sum_{n=1}^{\infty} S_{23}(n-1)\,\frac{z^n}{n} = H_{1,0,0,0,0,1}(z) + H_{1,0,1,0,0,1}(z), \tag{12.152}$$

$$\begin{aligned}\sum_{n=1}^{\infty} S_{12}(n-1)S_2(n-1)\,\frac{z^n}{n} = {} & H_{1,0,0,0,0,1}(z) + 2H_{1,0,0,1,0,1}(z)\\ & + H_{1,0,1,0,0,1}(z) + H_{1,0,1,1,0,1}(z)\\ & + H_{1,1,0,0,0,1}(z) + 2H_{1,1,0,1,0,1}(z),\end{aligned} \tag{12.153}$$

$$\begin{aligned}\sum_{n=1}^{\infty} S_{113}(n-1)\,\frac{z^n}{n} = {} & H_{1,0,0,0,0,1}(z) + H_{1,0,1,0,0,1}(z)\\ & + H_{1,1,0,0,0,1}(z) + H_{1,1,1,0,0,1}(z),\end{aligned} \tag{12.154}$$

$$\begin{aligned}\sum_{n=1}^{\infty} S_{212}(n-1)\,\frac{z^n}{n} = {} & H_{1,0,0,0,0,1}(z) + H_{1,0,0,1,0,1}(z)\\ & + H_{1,0,1,0,0,1}(z) + H_{1,0,1,1,0,1}(z),\end{aligned} \tag{12.155}$$

$$\begin{aligned}\sum_{n=1}^{\infty} S_{1112}(n-1)\,\frac{z^n}{n} = {} & H_{1,0,0,0,0,1}(z) + H_{1,0,0,1,0,1}(z) + H_{1,0,1,0,0,1}(z)\\ & + H_{1,0,1,1,0,1}(z) + H_{1,1,0,0,0,1}(z) + H_{1,1,0,1,0,1}(z)\\ & + H_{1,1,1,0,0,1}(z) + H_{1,1,1,1,0,1}(z).\end{aligned} \tag{12.156}$$

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Chapter 13
Appendix D: Table of MB Integrals

13.1 MB Integrals with Four Gamma Functions

This is the first Barnes lemma:

$$\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dz\, \Gamma(\lambda_1+z)\Gamma(\lambda_2+z)\Gamma(\lambda_3-z)\Gamma(\lambda_4-z)$$
$$= \frac{\Gamma(\lambda_1+\lambda_3)\Gamma(\lambda_1+\lambda_4)\Gamma(\lambda_2+\lambda_3)\Gamma(\lambda_2+\lambda_4)}{\Gamma(\lambda_1+\lambda_2+\lambda_3+\lambda_4)}. \tag{13.1}$$

Results for integrals with $\psi(\lambda_1+z), \ldots$ are obtained from (13.1) by differentiating with respect to $\lambda_1, \ldots$. Second derivatives give, in a similar way, results for integrals with products of two different functions $\psi(\lambda_i \pm z)$ and with the combinations $\psi'(\lambda_i \pm z) + \psi(\lambda_i \pm z)^2$.

Various corollaries can be derived from (13.1). For example,

$$\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dz\, \Gamma(\lambda_1+z)\Gamma^*(\lambda_2+z)\Gamma(-\lambda_2-z)\Gamma(\lambda_3-z)$$
$$= \Gamma(\lambda_1-\lambda_2)\Gamma(\lambda_2+\lambda_3)\big[\psi(\lambda_1-\lambda_2)-\psi(\lambda_1+\lambda_3)\big], \tag{13.2}$$

$$\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dz\, \Gamma(\lambda_1+z)\Gamma(\lambda_2+z)\Gamma^*(-\lambda_2-z)\Gamma(\lambda_3-z)$$
$$= \Gamma(\lambda_1-\lambda_2)\Gamma(\lambda_2+\lambda_3)\big[\psi(\lambda_2+\lambda_3)-\psi(\lambda_1+\lambda_3)\big]. \tag{13.3}$$

The asterisk is used to indicate that the first pole of the corresponding gamma function is of the opposite nature, i.e. the first pole of $\Gamma(\lambda_2+z)$ in (13.2) is considered right and the first pole of $\Gamma(-\lambda_2-z)$ in (13.3) is considered left.

V. A. Smirnov, *Analytic Tools for Feynman Integrals*, Springer Tracts in Modern Physics 250, DOI: 10.1007/978-3-642-34886-0_13,

These are four formulae with the psi function with the same condition as in (13.2):

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dz\, \Gamma(\lambda_1+z)\Gamma^*(\lambda_2+z)\Gamma(-\lambda_2-z)\Gamma(\lambda_3-z)\psi(\lambda_1+z) \\
&\quad = \Gamma(\lambda_1-\lambda_2)\Gamma(\lambda_2+\lambda_3)\big[\psi(\lambda_1-\lambda_2)^2-\psi(\lambda_1-\lambda_2)\psi(\lambda_1+\lambda_3) \\
&\qquad + \psi'(\lambda_1-\lambda_2)-\psi'(\lambda_1+\lambda_3)\big], \qquad (13.4)
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dz\, \Gamma(\lambda_1+z)\Gamma^*(\lambda_2+z)\Gamma(-\lambda_2-z)\Gamma(\lambda_3-z)\psi(\lambda_2+z) \\
&\quad = -\frac{1}{2}\Gamma(\lambda_1-\lambda_2)\Gamma(\lambda_2+\lambda_3)\big[\psi(\lambda_1-\lambda_2)^2-\psi(\lambda_1+\lambda_3)^2 \\
&\qquad + 2\psi(\lambda_1-\lambda_2)(\gamma_E-\psi(\lambda_2+\lambda_3))-2\psi(\lambda_1+\lambda_3)(\gamma_E-\psi(\lambda_2+\lambda_3)) \\
&\qquad + \psi'(\lambda_1-\lambda_2)+\psi'(\lambda_1+\lambda_3)\big], \qquad (13.5)
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dz\, \Gamma(\lambda_1+z)\Gamma^*(\lambda_2+z)\Gamma(-\lambda_2-z)\Gamma(\lambda_3-z)\psi(-\lambda_2-z) \\
&\quad = \frac{1}{2}\Gamma(\lambda_1-\lambda_2)\Gamma(\lambda_2+\lambda_3)\big[\psi(\lambda_1-\lambda_2)^2+2\gamma_E\psi(\lambda_1+\lambda_3) \\
&\qquad + \psi(\lambda_1+\lambda_3)^2-2\psi(\lambda_1-\lambda_2)(\gamma_E+\psi(\lambda_1+\lambda_3)) \\
&\qquad + \psi'(\lambda_1-\lambda_2)-\psi'(\lambda_1+\lambda_3)\big], \qquad (13.6)
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dz\, \Gamma(\lambda_1+z)\Gamma^*(\lambda_2+z)\Gamma(-\lambda_2-z)\Gamma(\lambda_3-z)\psi(\lambda_3-z) \\
&\quad = \Gamma(\lambda_1-\lambda_2)\Gamma(\lambda_2+\lambda_3)\big[\psi(\lambda_1-\lambda_2)\psi(\lambda_2+\lambda_3) \\
&\qquad - \psi(\lambda_1+\lambda_3)\psi(\lambda_2+\lambda_3)-\psi'(\lambda_1+\lambda_3)\big]. \qquad (13.7)
\end{aligned}$$

These are four formulae with the psi function with the same condition as in (13.3):

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dz\, \Gamma(\lambda_1+z)\Gamma(\lambda_2+z)\Gamma^*(-\lambda_2-z)\Gamma(\lambda_3-z)\psi(\lambda_1+z) \\
&\quad = -\Gamma(\lambda_1-\lambda_2)\Gamma(\lambda_2+\lambda_3) \\
&\qquad \times \left[\psi(\lambda_1-\lambda_2)(\psi(\lambda_1+\lambda_3)-\psi(\lambda_2+\lambda_3))+\psi'(\lambda_1+\lambda_3)\right], \qquad (13.8)
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dz\, \Gamma(\lambda_1+z)\Gamma(\lambda_2+z)\Gamma^*(-\lambda_2-z)\Gamma(\lambda_3-z)\psi(\lambda_2+z) \\
&\quad = \frac{1}{2}\Gamma(\lambda_1-\lambda_2)\Gamma(\lambda_2+\lambda_3)\big[(\psi(\lambda_1+\lambda_3)-\psi(\lambda_2+\lambda_3))^2 \\
&\qquad + 2\gamma_E(\psi(\lambda_1+\lambda_3)-\psi(\lambda_2+\lambda_3))-\psi'(\lambda_1+\lambda_3)+\psi'(\lambda_2+\lambda_3)\big], \qquad (13.9)
\end{aligned}$$

$$\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dz\,\Gamma(\lambda_1+z)\Gamma(\lambda_2+z)\Gamma^*(-\lambda_2-z)\Gamma(\lambda_3-z)\psi(-\lambda_2-z)$$
$$= \frac{1}{2}\Gamma(\lambda_1-\lambda_2)\Gamma(\lambda_2+\lambda_3)$$
$$\times\big[2(\psi(\lambda_1-\lambda_2)-\gamma_E)(\psi(\lambda_2+\lambda_3)-\psi(\lambda_1+\lambda_3))$$
$$+\psi(\lambda_1+\lambda_3)^2-\psi(\lambda_2+\lambda_3)^2-\psi'(\lambda_1+\lambda_3)-\psi'(\lambda_2+\lambda_3)\big], \tag{13.10}$$

$$\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dz\,\Gamma(\lambda_1+z)\Gamma(\lambda_2+z)\Gamma^*(-\lambda_2-z)\Gamma(\lambda_3-z)\psi(\lambda_3-z)$$
$$=\Gamma(\lambda_1-\lambda_2)\Gamma(\lambda_2+\lambda_3)\big[\psi(\lambda_2+\lambda_3)^2-\psi(\lambda_1+\lambda_3)\psi(\lambda_2+\lambda_3)$$
$$-\psi'(\lambda_1+\lambda_3)+\psi'(\lambda_2+\lambda_3)\big], \tag{13.11}$$

This is an example with the gluing of two poles:

$$\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dz\,\Gamma(\lambda_1+z)\Gamma(\lambda_2+z)\Gamma^{**}(-1-\lambda_2-z)\Gamma(\lambda_3-z)$$
$$=\Gamma(\lambda_1-\lambda_2-1)\Gamma(\lambda_2+\lambda_3)\big[1-\lambda_1+\lambda_2$$
$$+(\lambda_1+\lambda_3-1)(\psi(\lambda_1+\lambda_3-1)-\psi(\lambda_2+\lambda_3))\big], \tag{13.12}$$

where the first two poles of $\Gamma(-1-\lambda_2-z)$, i.e. $z=-\lambda_2$ and $z=-\lambda_2-1$, are considered left, with the corresponding change in notation. Here it is implied that $\lambda_1+\lambda_3\neq 1$.

In the case $\lambda_1+\lambda_3=1$, we have

$$\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dz\,\Gamma(1-\lambda_1+z)\Gamma(\lambda_2+z)\Gamma^{**}(-1-\lambda_2-z)\Gamma(\lambda_1-z)$$
$$=(\lambda_1+\lambda_2-1)\Gamma(\lambda_1+\lambda_2)\Gamma(-\lambda_1-\lambda_2). \tag{13.13}$$

Here is one more example of such an integral:

$$\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dz\,\Gamma(1-\lambda_1+z)\Gamma^*(\lambda_2+z)\Gamma^*(-1-\lambda_2-z)\Gamma(\lambda_1-z)$$
$$=\Gamma(\lambda_1+\lambda_2)\Gamma(-\lambda_1-\lambda_2)$$
$$\times\big[(\lambda_1+\lambda_2)(\psi(-\lambda_1-\lambda_2)-\psi(1+\lambda_1+\lambda_2))-1\big]. \tag{13.14}$$

Furthermore, we have

$$\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dz\,\Gamma^*(\lambda_1+z)\Gamma^*(\lambda_2+z)\Gamma(-\lambda_2-z)\Gamma(-\lambda_1-z)$$
$$=\Gamma(\lambda_1-\lambda_2)\Gamma(\lambda_2-\lambda_1)\left[2\gamma_E+\psi(\lambda_1-\lambda_2)+\psi(\lambda_2-\lambda_1)\right], \tag{13.15}$$

where the poles $z = -\lambda_1$ and $z = -\lambda_2$ are right. These are four more formulae with these conditions:

$$\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dz\, \Gamma^*(\lambda_1+z)\Gamma^*(\lambda_2+z)\Gamma(-\lambda_2-z)\Gamma(-\lambda_1-z)\psi(\lambda_1+z)$$
$$= -\frac{1}{4}\Gamma(\lambda_1-\lambda_2)\Gamma(\lambda_2-\lambda_1)\Big[2\gamma_E^2+\pi^2-4\psi(\lambda_1-\lambda_2)\psi(\lambda_2-\lambda_1)$$
$$+4\gamma_E(\psi(\lambda_2-\lambda_1)-2\psi(\lambda_1-\lambda_2))-4\psi(\lambda_1-\lambda_2)^2-4\psi'(\lambda_1-\lambda_2)$$
$$+2\psi(\lambda_2-\lambda_1)^2+2\psi'(\lambda_2-\lambda_1)\Big], \tag{13.16}$$

$$\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dz\, \Gamma^*(\lambda_1+z)\Gamma^*(\lambda_2+z)\Gamma(-\lambda_2-z)\Gamma(-\lambda_1-z)\psi(\lambda_2+z)$$
$$= -\frac{1}{4}\Gamma(\lambda_1-\lambda_2)\Gamma(\lambda_2-\lambda_1)\Big[2\gamma_E^2+\pi^2+2\psi(\lambda_1-\lambda_2)^2$$
$$+4\psi(\lambda_1-\lambda_2)(\gamma_E-\psi(\lambda_2-\lambda_1))-8\gamma_E\psi(\lambda_2-\lambda_1)-4\psi(\lambda_2-\lambda_1)^2$$
$$+2\psi'(\lambda_1-\lambda_2)-4\psi'(\lambda_2-\lambda_1)\Big], \tag{13.17}$$

$$\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dz\, \Gamma^*(\lambda_1+z)\Gamma^*(\lambda_2+z)\Gamma(-\lambda_2-z)\Gamma(-\lambda_1-z)\psi(-\lambda_2-z)$$
$$= -\frac{1}{4}\Gamma(\lambda_1-\lambda_2)\Gamma(\lambda_2-\lambda_1)\big[2\gamma_E^2+\pi^2-2\psi(\lambda_1-\lambda_2)^2$$
$$-4\psi(\lambda_1-\lambda_2)(\gamma_E+\psi(\lambda_2-\lambda_1))-2\psi'(\lambda_1-\lambda_2)\big], \tag{13.18}$$

$$\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dz\, \Gamma^*(\lambda_1+z)\Gamma^*(\lambda_2+z)\Gamma(-\lambda_2-z)\Gamma(-\lambda_1-z)\psi(-\lambda_1-z)$$
$$= -\frac{1}{4}\Gamma(\lambda_1-\lambda_2)\Gamma(\lambda_2-\lambda_1)\big[2\gamma_E^2+\pi^2-2\psi(\lambda_2-\lambda_1)^2$$
$$-4(\gamma_E+\psi(\lambda_1-\lambda_2))\psi(\lambda_2-\lambda_1)-2\psi'(\lambda_2-\lambda_1)\big]. \tag{13.19}$$

There are similar formulae with different understanding of the nature of the poles:

$$\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dz\, \Gamma(\lambda_1+z)\Gamma^*(\lambda_2+z)\Gamma(-\lambda_2-z)\Gamma^*(-\lambda_1-z)$$
$$= 2\Gamma(\lambda_1-\lambda_2)\Gamma(\lambda_2-\lambda_1)\left[\gamma_E+\psi(\lambda_1-\lambda_2)\right], \tag{13.20}$$

where the pole $z = -\lambda_1$ is left and the pole and $z = -\lambda_2$ is right, and

$$\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dz\, \Gamma^*(\lambda_1+z)\Gamma(\lambda_2+z)\Gamma^*(-\lambda_2-z)\Gamma(-\lambda_1-z)$$
$$= 2\Gamma(\lambda_1-\lambda_2)\Gamma(\lambda_2-\lambda_1)\left[\gamma_E+\psi(\lambda_2-\lambda_1)\right], \tag{13.21}$$

where the pole $z = -\lambda_1$ is right and the pole and $z = -\lambda_2$ is left. These are four more formulae with these conditions:

$$\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dz\, \Gamma^*(\lambda_1+z)\Gamma(\lambda_2+z)\Gamma^*(-\lambda_2-z)\Gamma(-\lambda_1-z)\psi(\lambda_1+z)$$
$$= -\frac{1}{4}\Gamma(\lambda_1-\lambda_2)\Gamma(\lambda_2-\lambda_1)\big[2\gamma_E^2+\pi^2+4\gamma_E\psi(\lambda_2-\lambda_1)+2\psi(\lambda_2-\lambda_1)^2$$
$$-8\psi(\lambda_1-\lambda_2)(\gamma_E+\psi(\lambda_2-\lambda_1))+2\psi'(\lambda_2-\lambda_1)\big], \tag{13.22}$$

$$\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dz\, \Gamma^*(\lambda_1+z)\Gamma(\lambda_2+z)\Gamma^*(-\lambda_2-z)\Gamma(-\lambda_1-z)\psi(\lambda_2+z)$$
$$= -\frac{1}{4}\Gamma(\lambda_1-\lambda_2)\Gamma(\lambda_2-\lambda_1)\big[2\gamma_E^2+\pi^2-4\gamma_E\psi(\lambda_2-\lambda_1)$$
$$-6\psi(\lambda_2-\lambda_1)^2-6\psi'(\lambda_2-\lambda_1)\big], \tag{13.23}$$

$$\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dz\, \Gamma^*(\lambda_1+z)\Gamma(\lambda_2+z)\Gamma^*(-\lambda_2-z)\Gamma(-\lambda_1-z)\psi(-\lambda_2-z)$$
$$= -\frac{1}{4}\Gamma(\lambda_1-\lambda_2)\Gamma(\lambda_2-\lambda_1)\big[2\gamma_E^2+\pi^2+4\gamma_E\psi(\lambda_2-\lambda_1)+2\psi(\lambda_2-\lambda_1)^2$$
$$-8\psi(\lambda_1-\lambda_2)(\gamma_E+\psi(\lambda_2-\lambda_1))+2\psi'(\lambda_2-\lambda_1)\big], \tag{13.24}$$

$$\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dz\, \Gamma^*(\lambda_1+z)\Gamma(\lambda_2+z)\Gamma^*(-\lambda_2-z)\Gamma(-\lambda_1-z)\psi(-\lambda_1-z)$$
$$= -\frac{1}{4}\Gamma(\lambda_1-\lambda_2)\Gamma(\lambda_2-\lambda_1)\big[2\gamma_E^2+\pi^2-4\gamma_E\psi(\lambda_2-\lambda_1)$$
$$-6\psi(\lambda_2-\lambda_1)^2-6\psi'(\lambda_2-\lambda_1)\big]. \tag{13.25}$$

Furthermore, we have

$$\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dz\, \Gamma(\lambda_1+z)\Gamma(\lambda_2+z)\Gamma^*(-\lambda_2-z)\Gamma^*(-\lambda_1-z)$$
$$= \Gamma(\lambda_1-\lambda_2)\Gamma(\lambda_2-\lambda_1)\left[2\gamma_E+\psi(\lambda_1-\lambda_2)+\psi(\lambda_2-\lambda_1)\right], \tag{13.26}$$

where the poles $z = -\lambda_1$ and $z = -\lambda_2$ are left. These are four more formulae with these conditions:

$$\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dz\, \Gamma(\lambda_1+z)\Gamma(\lambda_2+z)\Gamma^*(-\lambda_2-z)\Gamma^*(-\lambda_1-z)\psi(\lambda_1+z)$$
$$= -\frac{1}{4}\Gamma(\lambda_1-\lambda_2)\Gamma(\lambda_2-\lambda_1)\big[2\gamma_E^2+\pi^2-2\psi(\lambda_1-\lambda_2)^2$$
$$-4\psi(\lambda_1-\lambda_2)(\gamma_E+\psi(\lambda_2-\lambda_1))-2\psi'(\lambda_1-\lambda_2)\big], \tag{13.27}$$

$$\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dz\, \Gamma(\lambda_1+z)\Gamma(\lambda_2+z)\Gamma^*(-\lambda_2-z)\Gamma^*(-\lambda_1-z)\psi(\lambda_2+z)$$
$$= -\frac{1}{4}\Gamma(\lambda_1-\lambda_2)\Gamma(\lambda_2-\lambda_1)\big[2\gamma_E^2+\pi^2-4(\gamma_E+\psi(\lambda_1-\lambda_2))\psi(\lambda_2-\lambda_1)$$
$$-2\psi(\lambda_2-\lambda_1)^2-2\psi'(\lambda_2-\lambda_1)\big], \tag{13.28}$$

$$\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dz\, \Gamma(\lambda_1+z)\Gamma(\lambda_2+z)\Gamma^*(-\lambda_2-z)\Gamma^*(-\lambda_1-z)\psi(-\lambda_2-z)$$
$$= -\frac{1}{4}\Gamma(\lambda_1-\lambda_2)\Gamma(\lambda_2-\lambda_1)\big[2\gamma_E^2+\pi^2-4\psi(\lambda_1-\lambda_2)^2$$
$$+4\gamma_E\psi(\lambda_2-\lambda_1)+2\psi(\lambda_2-\lambda_1)^2-4\psi(\lambda_1-\lambda_2)(2\gamma_E+\psi(\lambda_2-\lambda_1))$$
$$-4\psi'(\lambda_1-\lambda_2)+2\psi'(\lambda_2-\lambda_1)\big], \tag{13.29}$$

$$\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dz\, \Gamma(\lambda_1+z)\Gamma(\lambda_2+z)\Gamma^*(-\lambda_2-z)\Gamma^*(-\lambda_1-z)\psi(-\lambda_1-z)$$
$$= -\frac{1}{4}\Gamma(\lambda_1-\lambda_2)\Gamma(\lambda_2-\lambda_1)\big[2\gamma_E^2+\pi^2+2\psi(\lambda_1-\lambda_2)^2$$
$$+4\psi(\lambda_1-\lambda_2)(\gamma_E-\psi(\lambda_2-\lambda_1))-8\gamma_E\psi(\lambda_2-\lambda_1)$$
$$-4\psi(\lambda_2-\lambda_1)^2+2\psi'(\lambda_1-\lambda_2)-4\psi'(\lambda_2-\lambda_1)\big]. \tag{13.30}$$

We also have

$$\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dz\, \Gamma(\lambda_1+z)\Gamma^*(\lambda_2+z)\Gamma(-\lambda_2-z)^2$$
$$= -\Gamma(\lambda_1-\lambda_2)\psi'(\lambda_1-\lambda_2), \tag{13.31}$$

where the pole $z=-\lambda_2$ is right. These are three more formulae with this condition:

$$\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dz\, \Gamma(\lambda_1+z)\Gamma^*(\lambda_2+z)\Gamma(-\lambda_2-z)^2\psi(\lambda_1+z)$$
$$= -\Gamma(\lambda_1-\lambda_2)\left[\psi(\lambda_1-\lambda_2)\psi'(\lambda_1-\lambda_2)+\psi''(\lambda_1-\lambda_2)\right], \tag{13.32}$$
$$\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dz\, \Gamma(\lambda_1+z)\Gamma^*(\lambda_2+z)\Gamma(-\lambda_2-z)^2\psi(\lambda_2+z)$$
$$= \Gamma(\lambda_1-\lambda_2)\psi'(\lambda_1-\lambda_2)\left[2\gamma_E+\psi(\lambda_1-\lambda_2)\right], \tag{13.33}$$
$$\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dz\, \Gamma(\lambda_1+z)\Gamma^*(\lambda_2+z)\Gamma(-\lambda_2-z)^2\psi(-\lambda_2-z)$$
$$= \frac{1}{2}\Gamma(\lambda_1-\lambda_2)\left[2\gamma_E\psi'(\lambda_1-\lambda_2)-\psi''(\lambda_1-\lambda_2)\right]. \tag{13.34}$$

We also have

$$\frac{1}{2\pi \mathrm{i}}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}z\, \Gamma(\lambda_1+z)\Gamma(\lambda_2+z)\Gamma^*(-\lambda_2-z)^2$$
$$= \frac{1}{4}\Gamma(\lambda_1-\lambda_2)\left[\pi^2+2(\gamma_{\mathrm{E}}+\psi(\lambda_1-\lambda_2))^2-2\psi'(\lambda_1-\lambda_2)\right], \tag{13.35}$$

where the pole $z=-\lambda_2$ is left,

$$\frac{1}{2\pi \mathrm{i}}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}z\, \Gamma(\lambda_1+z)^2\Gamma^*(-\lambda_1-z)\Gamma(\lambda_2-z)$$
$$= -\Gamma(\lambda_1+\lambda_2)\psi'(\lambda_1+\lambda_2), \tag{13.36}$$

where the pole $z=-\lambda_1$ is left, and

$$\frac{1}{2\pi \mathrm{i}}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}z\, \Gamma^*(\lambda_1+z)^2\Gamma(-\lambda_1-z)\Gamma(\lambda_2-z)$$
$$= \frac{1}{4}\Gamma(\lambda_1+\lambda_2)\left[2(\gamma_{\mathrm{E}}+\psi(\lambda_1+\lambda_2))^2+\pi^2-2\psi'(\lambda_1+\lambda_2)\right], \tag{13.37}$$

where the pole $z=-\lambda_1$ is right. These are three more formulae with this condition:

$$\frac{1}{2\pi \mathrm{i}}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}z\, \Gamma^*(\lambda_1+z)^2\Gamma(-\lambda_1-z)\Gamma(\lambda_2-z)\psi(\lambda_1+z)$$
$$= \frac{1}{6}\Gamma(\lambda_1+\lambda_2)[\psi(\lambda_1+\lambda_2)^3+3\psi(\lambda_1+\lambda_2)\left(\psi'(\lambda_1+\lambda_2)-\gamma_{\mathrm{E}}^2+\frac{\pi^2}{6}\right)$$
$$-2\gamma_{\mathrm{E}}^3-\gamma_{\mathrm{E}}\pi^2+6\gamma_{\mathrm{E}}\psi'(\lambda_1+\lambda_2)-4\zeta(3)-2\psi''(\lambda_1+\lambda_2)], \tag{13.38}$$

$$\frac{1}{2\pi \mathrm{i}}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}z\, \Gamma^*(\lambda_1+z)^2\Gamma(-\lambda_1-z)\Gamma(\lambda_2-z)\psi(-\lambda_1-z)$$
$$= -\frac{1}{12}\Gamma(\lambda_1+\lambda_2)[12\gamma_{\mathrm{E}}\psi(\lambda_1+\lambda_2)^2+2\psi(\lambda_1+\lambda_2)^3$$
$$+3\psi(\lambda_1+\lambda_2)\left(6\gamma_{\mathrm{E}}^2+\frac{\pi^2}{3}-2\psi'(\lambda_1+\lambda_2)\right)$$
$$+2(4\gamma_{\mathrm{E}}^3+2\gamma_{\mathrm{E}}\pi^2-6\gamma_{\mathrm{E}}\psi'(\lambda_1+\lambda_2)+8\zeta(3)+\psi''(\lambda_1+\lambda_2))], \tag{13.39}$$

$$\frac{1}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}z\, \Gamma^*(\lambda_1+z)^2 \Gamma(-\lambda_1-z)\Gamma(\lambda_2-z)\psi(\lambda_2-z)$$
$$= \frac{1}{4}\Gamma(\lambda_1+\lambda_2)\Big[4\gamma_{\mathrm{E}}\psi(\lambda_1+\lambda_2)^2 + 2\psi(\lambda_1+\lambda_2)^3 + 4\gamma_{\mathrm{E}}\psi'(\lambda_1+\lambda_2)$$
$$+\psi(\lambda_1+\lambda_2)(2\gamma_{\mathrm{E}}^2+\pi^2+2\psi'(\lambda_1+\lambda_2)) - 2\psi''(\lambda_1+\lambda_2)\Big]. \tag{13.40}$$

In some situations, it is possible to evaluate MB integrals with higher derivatives of the ψ function. Here are some examples:

$$\frac{1}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}z\, \Gamma(\lambda_1+z)^2 \Gamma(\lambda_2-z)^2\psi(\lambda_1+z)$$
$$= \frac{\Gamma(\lambda_1+\lambda_2)^4}{\Gamma(2(\lambda_1+\lambda_2))}\big[2\psi(\lambda_1+\lambda_2) - \psi(2(\lambda_1+\lambda_2))\big], \tag{13.41}$$

$$\frac{1}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}z\, \Gamma(\lambda_1+z)^2 \Gamma(\lambda_2-z)^2\psi(\lambda_1+z)^2$$
$$= \frac{\Gamma(\lambda_1+\lambda_2)^4}{\Gamma(2(\lambda_1+\lambda_2))}\Big[4\psi(\lambda_1+\lambda_2)^2 - 4\psi(\lambda_1+\lambda_2)\psi(2(\lambda_1+\lambda_2))$$
$$+\psi(2(\lambda_1+\lambda_2))^2 - \psi'(2(\lambda_1+\lambda_2))\Big], \tag{13.42}$$

$$\frac{1}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}z\, \Gamma(\lambda_1+z)^2 \Gamma(\lambda_2-z)^2\psi'(\lambda_1+z)$$
$$= 2\frac{\Gamma(\lambda_1+\lambda_2)^4}{\Gamma(2(\lambda_1+\lambda_2))}\psi'(\lambda_1+\lambda_2), \tag{13.43}$$

$$\frac{1}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}z\, \Gamma(\lambda_1+z)^2 \Gamma(\lambda_2-z)^2\psi(\lambda_1+z)\psi(\lambda_2-z)$$
$$= \frac{\Gamma(\lambda_1+\lambda_2)^4}{\Gamma(2(\lambda_1+\lambda_2))}\Big[4\psi(\lambda_1+\lambda_2)^2 - 4\psi(\lambda_1+\lambda_2)\psi(2(\lambda_1+\lambda_2))$$
$$+\psi(2(\lambda_1+\lambda_2))^2 + \psi'(\lambda_1+\lambda_2) - \psi'(2(\lambda_1+\lambda_2))\Big], \tag{13.44}$$

$$\frac{1}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}z\, \Gamma(\lambda_1+z)^2 \Gamma(\lambda_2-z)^2\psi(\lambda_1+z)^2\psi(\lambda_2-z)$$
$$= \frac{\Gamma(\lambda_1+\lambda_2)^4}{\Gamma(2(\lambda_1+\lambda_2))}\big[8\psi(\lambda_1+\lambda_2)^3 - 12\psi(\lambda_1+\lambda_2)^2\psi(2(\lambda_1+\lambda_2))$$
$$+2\psi(\lambda_1+\lambda_2)(3\psi(2(\lambda_1+\lambda_2))^2 + 2\psi'(\lambda_1+\lambda_2) - 3\psi'(2(\lambda_1+\lambda_2)))$$
$$+\psi(2(\lambda_1+\lambda_2))(3\psi'(2(\lambda_1+\lambda_2)) - 2\psi'(\lambda_1+\lambda_2))$$
$$-\psi(2(\lambda_1+\lambda_2))^3 - \psi''(2(\lambda_1+\lambda_2))\big], \tag{13.45}$$

$$\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dz\, \Gamma(\lambda_1+z)^2\Gamma(\lambda_2-z)^2\psi'(\lambda_1+z)\psi(\lambda_2-z)$$
$$= \frac{\Gamma(\lambda_1+\lambda_2)^4}{\Gamma(2(\lambda_1+\lambda_2))}\big[4\psi(\lambda_1+\lambda_2)\psi'(\lambda_1+\lambda_2)$$
$$-2\psi(2(\lambda_1+\lambda_2))\psi'(\lambda_1+\lambda_2)+\psi''(\lambda_1+\lambda_2)\big], \qquad (13.46)$$

13.2 MB Integrals with Six Gamma Functions

This is the second Barnes lemma:

$$\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dz\, \frac{\Gamma(\lambda_1+z)\Gamma(\lambda_2+z)\Gamma(\lambda_3+z)\Gamma(\lambda_4-z)\Gamma(\lambda_5-z)}{\Gamma(\lambda_6+z)}$$
$$= \frac{\Gamma(\lambda_1+\lambda_4)\Gamma(\lambda_2+\lambda_4)\Gamma(\lambda_3+\lambda_4)\Gamma(\lambda_1+\lambda_5)}{\Gamma(\lambda_1+\lambda_2+\lambda_4+\lambda_5)\Gamma(\lambda_1+\lambda_3+\lambda_4+\lambda_5)}$$
$$\times\frac{\Gamma(\lambda_2+\lambda_5)\Gamma(\lambda_3+\lambda_5)}{\Gamma(\lambda_2+\lambda_3+\lambda_4+\lambda_5)}, \qquad (13.47)$$

where $\lambda_6 = \lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_5$.

Here is a collection of its corollaries:

$$\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dz\, \frac{\Gamma(\lambda_1+z)\Gamma(\lambda_2+z)\Gamma(\lambda_3+z)\Gamma^*(-\lambda_3-z)\Gamma(\lambda_4-z)}{\Gamma(\lambda_5+z)}$$
$$= \frac{\Gamma(\lambda_1-\lambda_3)\Gamma(\lambda_2-\lambda_3)\Gamma(\lambda_3+\lambda_4)}{\Gamma(\lambda_1+\lambda_2-\lambda_3+\lambda_4)}\big[\psi(\lambda_1+\lambda_2-\lambda_3+\lambda_4)$$
$$+\psi(\lambda_3+\lambda_4)-\psi(\lambda_1+\lambda_4)-\psi(\lambda_2+\lambda_4)\big], \qquad (13.48)$$

where $\lambda_5 = \lambda_1+\lambda_2+\lambda_4$ and the pole $z=-\lambda_3$ is considered left,

$$\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dz\, \frac{\Gamma(\lambda_1+z)\Gamma(\lambda_2+z)\Gamma^*(\lambda_3+z)\Gamma(-\lambda_3-z)\Gamma(\lambda_4-z)}{\Gamma(\lambda_5+z)}$$
$$= \frac{\Gamma(\lambda_1-\lambda_3)\Gamma(\lambda_2-\lambda_3)\Gamma(\lambda_3+\lambda_4)}{\Gamma(\lambda_1+\lambda_2-\lambda_3+\lambda_4)}\big[\psi(\lambda_1-\lambda_3)+\psi(\lambda_2-\lambda_3)$$
$$-\psi(\lambda_1+\lambda_4)-\psi(\lambda_2+\lambda_4)\big], \qquad (13.49)$$

where $\lambda_5 = \lambda_1+\lambda_2+\lambda_4$ and the pole $z=-\lambda_3$ is considered right,

$$\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dz\, \frac{\Gamma(\lambda_1+z)\Gamma(\lambda_2+z)\Gamma^*(-\lambda_3+z)\Gamma(\lambda_3-z)^2}{\Gamma(\lambda_4+z)}$$
$$= -\frac{\Gamma(\lambda_1+\lambda_3)\Gamma(\lambda_2+\lambda_3)}{\Gamma(\lambda_1+\lambda_2+2\lambda_3)}\big[\psi'(\lambda_1+\lambda_3)+\psi'(\lambda_2+\lambda_3)\big], \qquad (13.50)$$

where $\lambda_4 = \lambda_1 + \lambda_2 + \lambda_3$ and the pole $z = \lambda_3$ is considered right,

$$\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dz \, \frac{\Gamma(\lambda_1+z)\Gamma^*(\lambda_2+z)^2\Gamma(-\lambda_2-z)\Gamma(\lambda_3-z)}{\Gamma(\lambda_4+z)}$$
$$= \frac{\Gamma(\lambda_1-\lambda_2)\Gamma(\lambda_2+\lambda_3)}{2\Gamma(\lambda_1+\lambda_3)}\Big[\frac{\pi^2}{2} + (\gamma_E - \psi(\lambda_1-\lambda_2) + \psi(\lambda_1+\lambda_3)$$
$$+ \psi(\lambda_2+\lambda_3))^2 + \psi'(\lambda_1-\lambda_2) + \psi'(\lambda_1+\lambda_3) - \psi'(\lambda_2+\lambda_3)\Big], \tag{13.51}$$

where $\lambda_4 = \lambda_1 + \lambda_2 + \lambda_3$ and the pole $z = -\lambda_2$ is considered right,

$$\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dz \, \frac{\Gamma(\lambda_1+z)\Gamma(\lambda_2+z)^2\Gamma^*(-\lambda_2-z)\Gamma(\lambda_3-z)}{\Gamma(\lambda_4+z)}$$
$$= \frac{\Gamma(\lambda_1-\lambda_2)\Gamma(\lambda_2+\lambda_3)}{\Gamma(\lambda_1+\lambda_3)}\left[\psi'(\lambda_1+\lambda_3) - \psi'(\lambda_2+\lambda_3)\right], \tag{13.52}$$

where $\lambda_4 = \lambda_1 + \lambda_2 + \lambda_3$ and the pole $z = -\lambda_2$ is considered left.

The integrals (13.47) can be evaluated recursively in the case where the difference $\lambda_6 - (\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5)$ is a positive integer. In particular, we have

$$\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dz \, \frac{\Gamma(\lambda_1+z)\Gamma(\lambda_2+z)\Gamma(\lambda_3+z)\Gamma(\lambda_4-z)\Gamma(-z)}{\Gamma(\lambda_5+z)}$$
$$= \frac{(\Gamma(1+\lambda_2+\lambda_3+\lambda_4))^{-1}\Gamma(\lambda_1)\Gamma(\lambda_3)\Gamma(\lambda_2+\lambda_4)}{\Gamma(1-\lambda_1-\lambda_3-\lambda_4)\Gamma(1+\lambda_1+\lambda_2+\lambda_4)\Gamma(\lambda_1+\lambda_3+\lambda_4)}$$
$$\times \big[\Gamma(1+\lambda_2)\Gamma(1-\lambda_1-\lambda_3-\lambda_4)\Gamma(\lambda_1+\lambda_4)\Gamma(\lambda_3+\lambda_4)$$
$$- \Gamma(\lambda_2)\Gamma(-\lambda_1-\lambda_3-\lambda_4)\Gamma(1+\lambda_1+\lambda_4)\Gamma(1+\lambda_3+\lambda_4)\big], \tag{13.53}$$

where $\lambda_5 = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + 1$, and

$$\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dz \, \frac{\Gamma(\lambda_1+z)\Gamma(\lambda_2+z)\Gamma(\lambda_3+z)\Gamma(\lambda_4-z)\Gamma(-z)}{\Gamma(\lambda_5+z)}$$
$$= \frac{(\Gamma(2+\lambda_2+\lambda_3+\lambda_4))^{-1}\Gamma(\lambda_1)\Gamma(\lambda_3)\Gamma(\lambda_2+\lambda_4)}{\Gamma(1-\lambda_1-\lambda_3-\lambda_4)\Gamma(2+\lambda_1+\lambda_2+\lambda_4)\Gamma(\lambda_1+\lambda_3+\lambda_4)}$$
$$\times \big[\Gamma(2+\lambda_2)\Gamma(1-\lambda_1-\lambda_3-\lambda_4)\Gamma(\lambda_1+\lambda_4)\Gamma(\lambda_3+\lambda_4)$$
$$- 2\Gamma(1+\lambda_2)\Gamma(-\lambda_1-\lambda_3-\lambda_4)\Gamma(1+\lambda_1+\lambda_4)\Gamma(1+\lambda_3+\lambda_4)$$
$$+ \Gamma(\lambda_2)\Gamma(-1-\lambda_1-\lambda_3-\lambda_4)\Gamma(2+\lambda_1+\lambda_4)\Gamma(2+\lambda_3+\lambda_4)\big], \tag{13.54}$$

where $\lambda_5 = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + 2$.

Here are more corollaries of the second Barnes lemma:

$$\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\frac{dz}{z}\,\Gamma(\lambda_1+z)\Gamma(\lambda_2+z)\Gamma(\lambda_3-z)\Gamma(\lambda_4-z)$$
$$=\frac{\Gamma(2-\lambda_1-\lambda_3)\Gamma(1-\lambda_2-\lambda_3)\Gamma(\lambda_1+\lambda_3-1)\Gamma(\lambda_2+\lambda_3)}{\Gamma(1-\lambda_1)\Gamma(1-\lambda_2)}$$
$$\times\left[\Gamma(1-\lambda_1)\Gamma(1-\lambda_2)-\Gamma(2-\lambda_1-\lambda_2-\lambda_3)\Gamma(\lambda_3)\right], \quad (13.55)$$

where $\lambda_1+\lambda_2+\lambda_3+\lambda_4=2$, and the pole at $z=0$ is considered left,

$$\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\frac{dz}{z}\,\Gamma(\lambda_1+z)\Gamma(\lambda_2+z)\Gamma(\lambda_3-z)\Gamma(\lambda_4-z)$$
$$=-\Gamma(\lambda_1)\Gamma(\lambda_2)\Gamma(2-\lambda_1-\lambda_2-\lambda_3)\Gamma(\lambda_3)$$
$$+\frac{\Gamma(2-\lambda_1-\lambda_3)\Gamma(1-\lambda_2-\lambda_3)\Gamma(\lambda_1+\lambda_3-1)\Gamma(\lambda_2+\lambda_3)}{\Gamma(1-\lambda_1)\Gamma(1-\lambda_2)}$$
$$\times\left[\Gamma(1-\lambda_1)\Gamma(1-\lambda_2)-\Gamma(2-\lambda_1-\lambda_2-\lambda_3)\Gamma(\lambda_3)\right], \quad (13.56)$$

where $\lambda_1+\lambda_2+\lambda_3+\lambda_4=2$, and the pole at $z=0$ is considered right,

$$\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}dz\,\frac{\Gamma^*(\lambda+z)^2\Gamma^*(z)\Gamma(-z)\Gamma(-\lambda-z)}{\Gamma(\lambda+1+z)}$$
$$=-\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\frac{dz}{z}\,\Gamma(\lambda+z)\Gamma(z)\Gamma^*(-z)\Gamma^*(-\lambda-z)$$
$$=\frac{1}{6\lambda}\Gamma(\lambda)\Gamma(-\lambda)\big[12(\gamma_E+\psi(\lambda))+2\lambda\pi^2$$
$$+3\lambda((\psi(\lambda)-\psi(-\lambda))^2-\psi'(\lambda)+\psi'(-\lambda))\big], \quad (13.57)$$

where the nature of the poles at $z=0$ and $z=-\lambda$ is indicated by asterisks, according to our conventions,

$$\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}dz\,\frac{\Gamma(\lambda+z)^2\Gamma(z)\Gamma^*(-z)\Gamma^*(-\lambda-z)}{\Gamma(\lambda+1+z)}$$
$$=-\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\frac{dz}{z}\,\Gamma^*(\lambda+z)\Gamma^*(z)\Gamma(-z)\Gamma(-\lambda-z)=\frac{1}{\lambda^2}\Gamma(\lambda)\Gamma(-\lambda)$$
$$\times\left[1+\lambda(\psi(\lambda)+\psi(-\lambda)+2\gamma_E)-\lambda^2\left(\psi'(\lambda)-\frac{\pi^2}{6}\right)\right], \quad (13.58)$$

$$\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dz\, \frac{\Gamma(\lambda+z)^2\Gamma^*(z)\Gamma(-z)\Gamma^*(-\lambda-z)}{\Gamma(\lambda+1+z)}$$
$$= -\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \frac{dz}{z}\, \Gamma(\lambda+z)\Gamma^*(z)\Gamma(-z)\Gamma^*(-\lambda-z)$$
$$= \frac{1}{\lambda}\Gamma(\lambda)\Gamma(-\lambda)\left[2(\gamma_E+\psi(\lambda)) - \lambda\left(\psi'(\lambda) - \frac{\pi^2}{6}\right)\right]. \qquad (13.59)$$

We also have

$$\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \frac{dz}{z^2}\, \Gamma(\lambda_1+z)\Gamma(\lambda_2+z)\Gamma(\lambda_3-z)\Gamma(\lambda_4-z)$$
$$= \frac{\Gamma(2-\lambda_1-\lambda_3)\Gamma(1-\lambda_2-\lambda_3)\Gamma(2-\lambda_1-\lambda_2-\lambda_3)\Gamma(\lambda_3)}{\Gamma(2-\lambda_1)\Gamma(1-\lambda_2)}$$
$$\times\, \Gamma(\lambda_1+\lambda_3-1)\Gamma(\lambda_2+\lambda_3)\big[1+(\lambda_1-1)(\psi(2-\lambda_1)+\psi(1-\lambda_2)$$
$$-\,\psi(2-\lambda_1-\lambda_2-\lambda_3)-\psi(\lambda_3))\big], \qquad (13.60)$$

where $\lambda_1+\lambda_2+\lambda_3+\lambda_4 = 2$, and the pole at $z=0$ is considered left,

$$\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \frac{dz}{z^2}\, \Gamma(\lambda_1+z)\Gamma(\lambda_2+z)\Gamma(\lambda_3-z)\Gamma(\lambda_4-z)$$
$$= \Gamma(2-\lambda_1-\lambda_2-\lambda_3)\Gamma(\lambda_3)\big[-\Gamma(\lambda_1)\Gamma(\lambda_2)(\psi(\lambda_1)+\psi(\lambda_2)$$
$$-\,\psi(2-\lambda_1-\lambda_2-\lambda_3)-\psi(\lambda_3))$$
$$+\, \frac{\Gamma(2-\lambda_1-\lambda_3)\Gamma(1-\lambda_2-\lambda_3)\Gamma(\lambda_1+\lambda_3-1)\Gamma(\lambda_2+\lambda_3)}{\Gamma(2-\lambda_1)\Gamma(1-\lambda_2)}$$
$$\times\, \big[1+(\lambda_1-1)(\psi(2-\lambda_1)+\psi(1-\lambda_2)$$
$$-\,\psi(2-\lambda_1-\lambda_2-\lambda_3)-\psi(\lambda_3))\big]\big], \qquad (13.61)$$

where $\lambda_1+\lambda_2+\lambda_3+\lambda_4 = 2$, and the pole at $z=0$ is considered right,

$$\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \frac{dz}{z}\, \Gamma(\lambda_1+z)\Gamma^*(\lambda_2+z)\Gamma(-\lambda_2-z)\Gamma^*(-\lambda_1-z)$$
$$= -\frac{1}{\lambda_1^2\lambda_2}\Gamma(\lambda_1-\lambda_2)\Gamma(\lambda_2-\lambda_1)\big[2\lambda_1-\lambda_2$$
$$+\,\lambda_1(\lambda_1+\lambda_2)(\gamma_E+\psi(\lambda_1-\lambda_2)) - \lambda_1(\lambda_1-\lambda_2)$$
$$\times\, (\psi(-\lambda_1)-\psi(-\lambda_2)+\psi(\lambda_2-\lambda_1)-\psi(1-\lambda_1+\lambda_2))\big], \qquad (13.62)$$

where the pole at $z=0$ is left and the nature of the first poles of the gamma functions is shown by asterisks,

$$\frac{1}{2\pi \mathrm{i}}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty}\frac{\mathrm{d}z}{z}\,\Gamma(\lambda_1+z)\Gamma(\lambda_2+z)\Gamma^*(-\lambda_2-z)\Gamma^*(-\lambda_1-z)$$
$$=\frac{1}{\lambda_1^2\lambda_2^2}\Gamma(\lambda_1-\lambda_2)\Gamma(\lambda_2-\lambda_1)\big[\lambda_1^2-\lambda_1\lambda_2+\lambda_2^2$$
$$-\lambda_1\lambda_2(\lambda_1+\lambda_2)\gamma_{\mathrm{E}}+\lambda_1(\lambda_1-\lambda_2)\lambda_2(\psi(-\lambda_1)-\psi(-\lambda_2))$$
$$-\lambda_1\lambda_2(\lambda_2\psi(\lambda_1-\lambda_2)+\lambda_1\psi(\lambda_2-\lambda_1))\big], \tag{13.63}$$

where the pole at $z=0$ is left,

$$\frac{1}{2\pi \mathrm{i}}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty}\frac{\mathrm{d}z}{z^2}\,\Gamma(\lambda_1+z)\Gamma^*(\lambda_2+z)\Gamma(-\lambda_2-z)\Gamma^*(-\lambda_1-z)$$
$$=\frac{1}{\lambda_1^3\lambda_2^2}\Gamma(\lambda_1-\lambda_2)\Gamma(\lambda_2-\lambda_1)\big[2(\lambda_1^2+\lambda_1\lambda_2-\lambda_2^2)$$
$$+\lambda_1(\lambda_1^2+\lambda_2^2)(\psi(\lambda_1-\lambda_2)+\gamma_{\mathrm{E}})$$
$$-\lambda_1(\lambda_1^2-\lambda_2^2)(\psi(-\lambda_1)-\psi(-\lambda_2)+\psi(-\lambda_1+\lambda_2)-\psi(1-\lambda_1+\lambda_2))$$
$$-\lambda_1^2\lambda_2(\lambda_1-\lambda_2)(\psi'(-\lambda_1)-\psi'(-\lambda_2))\big], \tag{13.64}$$

where the pole at $z=0$ is left,

$$\frac{1}{2\pi \mathrm{i}}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty}\frac{\mathrm{d}z}{z^2}\,\Gamma(\lambda_1+z)\Gamma(\lambda_2+z)\Gamma^*(-\lambda_2-z)\Gamma^*(-\lambda_1-z)$$
$$=-\frac{1}{\lambda_1^3\lambda_2^3}\Gamma(\lambda_1-\lambda_2)\Gamma(\lambda_2-\lambda_1)\Big[(\lambda_1+\lambda_2)(2\lambda_1^2-3\lambda_1\lambda_2+2\lambda_2^2)$$
$$-\lambda_1\lambda_2(\lambda_1^2+\lambda_2^2)\gamma_{\mathrm{E}}+\lambda_1\lambda_2(\lambda_1^2-\lambda_2^2)\psi(-\lambda_1)$$
$$-\lambda_1\lambda_2^3(\psi(\lambda_1-\lambda_2)-\psi(-\lambda_2))-\lambda_1^3\lambda_2(\psi(-\lambda_2)+\psi(\lambda_2-\lambda_1))$$
$$+\lambda_1^3\lambda_2^2(\psi'(-\lambda_1)-\psi'(-\lambda_2))-\lambda_1^2\lambda_2^3(\psi'(-\lambda_1)-\psi'(-\lambda_2))\Big], \tag{13.65}$$

where the pole at $z=0$ is left,

$$\frac{1}{2\pi \mathrm{i}}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty}\frac{\mathrm{d}z}{z^2}\,\Gamma(\lambda+z)\Gamma(z)\Gamma^*(-z)\Gamma^*(-\lambda-z)$$
$$=-\frac{1}{6\lambda^3}\Gamma(\lambda)\Gamma(-\lambda)\big[12-6\lambda(2\gamma_{\mathrm{E}}+\psi(-\lambda)+\psi(\lambda))$$
$$+\lambda^2(\pi^2-6\psi'(-\lambda))-3\lambda^3(\psi''(-\lambda)+2\zeta(3))\big], \tag{13.66}$$

where the pole at $z=0$ is left,

$$\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\frac{dz}{z^2}\,\Gamma(\lambda+z)\Gamma^*(z)\Gamma(-z)\Gamma^*(-\lambda-z)$$
$$=\frac{1}{6\lambda^3}\Gamma(\lambda)\Gamma(-\lambda)\big[-12+6\lambda(2\gamma_E+\psi(-\lambda)+\psi(\lambda))-\lambda^2(\pi^2-6\psi'(-\lambda))$$
$$-\lambda^3(\pi^2(\psi(-\lambda)-\psi(\lambda))+(\psi(-\lambda)-\psi(\lambda))^3-2\psi''(-\lambda)-\psi''(\lambda)$$
$$+3(\psi(-\lambda)-\psi(\lambda))(\psi'(-\lambda)+\psi'(\lambda))-6\zeta(3))\big], \tag{13.67}$$

where the pole at $z=0$ is right,

$$\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\frac{dz}{z}\,\Gamma(\lambda+z)^2\Gamma^*(-\lambda-z)^2$$
$$=-\frac{1}{6\lambda^4}\left[6+\lambda^2(\pi^2-6\psi'(-\lambda))+12\lambda^3\zeta(3)\right], \tag{13.68}$$

where the pole at $z=0$ is left,

$$\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\frac{dz}{z^2}\,\Gamma(\lambda+z)^2\Gamma^*(-\lambda-z)^2$$
$$=\frac{1}{3\lambda^5}\left[12+\lambda^2(\pi^2-6\psi'(-\lambda))-3\lambda^3(\psi''(-\lambda)-2\zeta(3))\right], \tag{13.69}$$

where the pole at $z=0$ is left,

$$\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\frac{dz}{z^2}\,\Gamma(\lambda+1+z)^2\Gamma(-\lambda-z)^2$$
$$=2\Gamma(1+\lambda)^2\Gamma(-\lambda)^2(\psi(-\lambda)-\psi(1+\lambda))-\psi''(-\lambda), \tag{13.70}$$

where the pole at $z=0$ is right.

13.3 The Gauss Hypergeometric Function and MB Integrals

The Gauss hypergeometric function can be defined in terms of MB integrals:

$${}_2F_1(a,b;c;x)$$
$$=\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\frac{\Gamma(a+z)\Gamma(b+z)\Gamma(-z)}{\Gamma(c+z)}(-x)^z dz \tag{13.71}$$

$$= \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c-a)\Gamma(c-b)}$$
$$\times \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \Gamma(a+z)\Gamma(b+z)\,\Gamma(c-a-b-z)\Gamma(-z)(1-x)^z \mathrm{d}z. \quad (13.72)$$

Combining these two formulae with (11.4) gives the following useful formula:

$$\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \Gamma(a+z)\Gamma(b+z)\,\Gamma(c-z)\Gamma(-z)\,x^z \mathrm{d}z$$
$$= \Gamma(a+c)\Gamma(b+c)\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \frac{\Gamma(a+z)\Gamma(b+z)\Gamma(-z)}{\Gamma(a+b+c+z)}(x-1)^z \mathrm{d}z. \quad (13.73)$$

Chapter 14
Appendix E: A Brief Review of Some Other Methods

In this appendix, some methods which were not considered in Chaps. 3–9 are briefly reviewed. The method based on dispersion relations was successfully used from the early days of quantum field theory. The Gegenbauer Polynomial x-Space Technique [21], the method of gluing [23] and the method based on star-triangle uniqueness relations [34, 53, 54, 72] are methods for evaluating massless diagrams. The method of IR rearrangement [74], also in a generalized version based on the R^*-operation [22, 69], is a method oriented at renormalization-group calculations.

One more method (in addition to the method presented in Chap. 8) based on difference equations [60] is briefly described. It also provides the possibility to obtain numerical results with a high precision. Some methods which could be characterized as based on experimental mathematics are discussed. In particular, this is the integer relation algorithm called PSLQ [36, 37] which provides the possibility to obtain a result for a given one-scale Feynman integral, when we strongly suspect that it is a linear combination of some transcendental numbers with rational coefficients, provided we know the result numerically with a high accuracy. Then a list of references to papers where Feynman integrals are evaluated by summing up series and expanding hypergeometric functions at (half-)integer parameters is presented. Finally, a new method based on the notion of symbols is advertised.

14.1 Dispersion Integrals

A given propagator scalar Feynman integral can be written as

$$F(q^2) = \frac{1}{2\pi \mathrm{i}} \int_{s_0}^{\infty} \mathrm{d}s \, \frac{\Delta F(s)}{s - q^2 - \mathrm{i}0}, \tag{14.1}$$

where the discontinuity $\Delta F(s) = 2\mathrm{i}\,\mathrm{Im}(F(s + \mathrm{i}0))$ is given, according to Cutkosky rules, by a sum over cuts in a given channel of integrals, where the propagators $\mathrm{i}/(k^2 - m^2 + \mathrm{i}0)$ in the cut are replaced by $2\pi\mathrm{i}\,\theta(k_0)\delta(k^2 - m^2)$, while the propagators

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to the left of the cut stay the same, and the propagators to the right of the cut change the causal prescription and become $-\mathrm{i}/(k^2 - m^2 - \mathrm{i}0)$.

Let us again consider our favourite example of Fig. 1.1, with the indices equal to one. This time, let us include all the necessary factors of i from each propagator and the factor $-\mathrm{i}$ corresponding to the definition of the Feynman integral with i on the right-hand side of (2.3). We have

$$\begin{aligned}\Delta F(q^2) &= 4\pi^2 \int \mathrm{d}^d k\, \theta(k_0)\delta(k^2 - m^2)\theta(q_0 - k_0)\delta\big[(q-k)^2\big] \\ &= \frac{2\pi^2}{q_0}\Omega_{d-1}\int_0^{q_0} \mathrm{d}r\, r^{d-2}\,\delta\left[\left(\frac{q_0^2 - m^2}{2q_0}\right)^2 - r^2\right] \\ &= \frac{2^{4-d}\pi^{(d+3)/2}}{\Gamma((d-1)/2)}\frac{(q^2-m^2)_+^{d-3}}{(q^2)^{(d-2)/2}}, \end{aligned} \tag{14.2}$$

where $X_+ = X$ for $X > 0$ and $X_+ = 0$ otherwise, as usual. We have chosen $q = (q_0, \mathbf{0})$ and introduced $(d-1)$-dimensional spherical coordinates with the surface of the unit sphere in d dimensions equal to

$$\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}. \tag{14.3}$$

For $d = 4$, this gives

$$\Delta F(s) = \frac{2\pi^3 (q^2 - m^2)_+}{q^2}. \tag{14.4}$$

Integrating from the threshold $s_0 = m^2$ in the dispersion integral (14.1) (where a subtraction is needed) leads to the finite part of (1.7) (where the factors of i mentioned above were dropped) up to a renormalization constant.

In this calculation, a phase-space integral corresponding to a two-particle cut with the masses m and 0 was evaluated. The evaluation of three- and four-particle phase-space integrals is much more complicated. Although we have less integrations in integrals corresponding to cuts, because of the δ-functions, resulting integrals are still rather nasty so that the evaluation of Feynman integrals via their imaginary part by means of Cutkosky rules (see [71] for a typical example) was successful only up to some complexity level. On the other hand, the phase-space integrals are needed for the calculation of the real radiation. It has turned out that the development of methods of evaluating Feynman integrals resulted in similar techniques for the phase-space integrals. Now, one applies, for the evaluation of the phase-space integrals, the strategy of the reduction to master integrals, using IBP, and DE applied for the evaluation of the master integrals—see, e.g., [3, 4]. Moreover, the technique of the sector decompositions described in Chap. 4 is also applicable here and was successfully applied in NNLO calculations—see references in Chap. 4.

It turns out that Feynman integrals with propagators replaced by delta functions naturally arise also within the generalized unitarity technique [10, 11] which is a powerful modern method of constructing scattering amplitudes.

14.2 Gegenbauer Polynomial x-Space Technique

The Gegenbauer polynomial x-space technique (GPXT) [21] is based on the $SO(d)$ symmetry of Euclidean Feynman integrals. According to (10.42), the dimensionally regularized scalar massless propagator in coordinate space is

$$D_F(x_1 - x_2) = \frac{1}{(2\pi)^d} \int \mathrm{d}^d q \frac{\mathrm{e}^{-\mathrm{i}x\cdot q}}{q^2} = \frac{\Gamma(1-\varepsilon)}{4\pi^{d/2}\left[(x_1-x_2)^2\right]^{1-\varepsilon}}, \tag{14.5}$$

where $x^2 = x_0^2 + \boldsymbol{x}^2$. It can be expanded in Gegenbauer polynomials [35] as

$$\begin{aligned} \frac{1}{[(x_1-x_2)^2]^\lambda} = &\frac{1}{(\max\{|x_1|, |x_2|\})^{2\lambda}} \\ &\times \sum_{n=0}^{\infty} C_n^\lambda\left(\hat{x}_1 \cdot \hat{x}_2\right) \left(\frac{\min\{|x_1|, |x_2|\}}{\max\{|x_1|, |x_2|\}}\right)^{n/2}, \end{aligned} \tag{14.6}$$

where $|x| = \sqrt{x^2}$, $\lambda = 1 - \varepsilon$ and $\hat{x} = x/|x|$. The polynomials C_n^λ are orthogonal on the unit sphere [35]:

$$\int \mathrm{d}\hat{x}_2 C_n^\lambda\left(\hat{x}_1 \cdot \hat{x}_2\right) C_m^\lambda\left(\hat{x}_2 \cdot \hat{x}_3\right) = \frac{\lambda}{n+\lambda}\delta_{n,m} C_n^\lambda\left(\hat{x}_1 \cdot \hat{x}_3\right). \tag{14.7}$$

The normalization is such that $\int \mathrm{d}\hat{x} = 1$. So, the strategy of GPXT is to turn to coordinate space, represent each propagator by (14.6), evaluate integrals over angles by (14.7) and sum up resulting multiple series.

First results for non-trivial multiloop diagrams within dimensional regularization were obtained by GPXT: for example, the value of the non-planar diagram (see the second diagram of Fig. 6.3 with all the powers of the propagators equal to one), with the famous result proportional to $20\zeta(5)$ [21].

The GPXT as well as the method of gluing (see below) were crucial in many important analytical calculations, for example, of the three-loop ratio $R(s)$ in QCD [21] and the five-loop β-function in the ϕ^4 theory [19]. More details on the GPXT can be found in the review [56]. See also [9] where the application of GPXT is reduced systematically to the evaluation of nested sums (defined in Appendix C).

14.3 Gluing

The dependence of an h-loop dimensionally regularized scalar propagator massless Feynman integral corresponding to a graph Γ on the external momentum can easily be found by power counting:

$$F_\Gamma(q;d) = \left(\mathrm{i}\pi^{d/2}\right)^h C_\Gamma(\varepsilon)(q^2)^{\omega/2-h\varepsilon}, \tag{14.8}$$

where ω is the degree of divergence given by (2.10) and $C_\Gamma(\varepsilon)$ is a meromorphic function which is finite at $\varepsilon = 0$ if the integral is convergent, both in the UV and IR sense. (Of course, there are no collinear divergences in propagator integrals.)

It turns out that the values $C_\Gamma(0)$ are the same for graphs connected by some transformations based on gluing. The gluing can be of two types: by vertices and by lines. Let Γ be a graph with two external vertices. Let us denote by $\hat{\Gamma}$ the graph obtained from it by identifying these vertices, and by $\bar{\Gamma}$ the graph obtained from it by adding a new line which connects them. Then the following properties hold [23]:

- *Gluing by vertices.* Let us suppose that two UV- and IR-convergent graphs, Γ_1 and Γ_2, have degrees of divergence $\omega_1 = \omega_2 = -4$ and that $\hat{\Gamma}_1$ and $\hat{\Gamma}_2$ are the same. Then $C_{\Gamma_1}(0) = C_{\Gamma_2}(0)$.
- *Gluing by lines.* Let us suppose that two UV- and IR-convergent graphs, Γ_1 and Γ_2, have degrees of divergence $\omega_1 = \omega_2 = -2$ and that $\bar{\Gamma}_1$ and $\bar{\Gamma}_2$ are the same. Then $C_{\Gamma_1}(0) = C_{\Gamma_2}(0)$.

For example, the first and the second diagrams in Fig. 6.3 with all the indices equal to one produce the same graph after the gluing the external vertices. It is shown in Fig. 14.1. Therefore, one could obtain the value of the more complicated non-planar diagram (proportional to $20\zeta(5)$) from a simpler planar diagram [23].

The method of gluing was successfully applied in the combination with GPXT—see the references above. As a recent application of the method of gluing let us refer to the evaluation of the master integrals for massless four-loop propagator diagrams. The evaluation of all the 28 master integrals (shown in Fig. 8.8) was done in [5] in an algebraic way using various gluing relations.

One more recent development of the method of gluing is its application in coordinate space in order to evaluate non-planar four-loop master integrals. This was one of the steps of the evaluation [33] of the five-loop correction to the anomalous dimen-

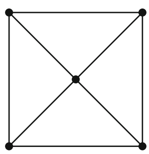

Fig. 14.1 The graph $\hat{\Gamma}$ obtained by gluing of vertices

sion of the Konishi operator in the $N = 4$ super Yang–Mills theory. In fact, this was a more general gluing, with the help of an auxiliary analytic regularization which was introduced on the 'background' of dimensional regularization—see details in [33].

14.4 Star-Triangle Relations

The method based on star-triangle uniqueness relations can be applied to massless diagrams. As in the case of GPXT, the coordinate space language is used, where the propagators have the form $1/(x^2)^\lambda$ up to a coefficient depending on ε—see, e.g., (14.5).

The basic uniqueness relation [34, 72] connects diagrams with different numbers of loops. It is graphically shown in Fig. 14.2, where $\lambda_i' = d/2 - \lambda_i$ and

$$v(\lambda_1, \lambda_2, \lambda_3) = \pi^{d/2} \prod_i \frac{\Gamma(d/2 - \lambda_i)}{\Gamma(\lambda_i)}. \tag{14.9}$$

This equation holds when the vertex on the left-hand side is *unique*, i.e. $\lambda_1 + \lambda_2 + \lambda_3 = d$. The triangle on the right-hand side, with $\lambda_1' + \lambda_2' + \lambda_3' = d/2$, is also called unique. Remember that, in coordinate space, the triangle diagram does not involve integration and is just a product of the three propagators,

$$\left[(x_1 - x_2)^2\right]^{-\lambda_3} \left[(x_2 - x_3)^2\right]^{-\lambda_1} \left[(x_3 - x_1)^2\right]^{-\lambda_2},$$

while the star diagram is an integral over the coordinate corresponding to the central vertex.

The relation (14.9) can be used to simplify a given diagram. *Almost unique* relations introduced in [70], with $\lambda_1+\lambda_2+\lambda_3 = d-1$, can be also useful. Sometimes one introduces an auxiliary analytic regularization, to satisfy (almost) unique relations, which can be switched off in the end of the calculation. For example, using (almost) unique relations, the general ladder massless scalar propagator diagram with an arbitrary number of loops, h, with all the indices a_i equal to one (see the first diagram

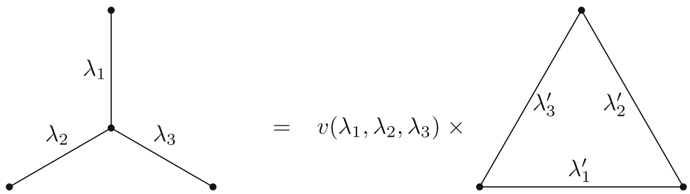

Fig. 14.2 Uniqueness equation

of Fig. 6.3 and imagine a general number of rungs), was evaluated [8] with a result proportional to $\zeta(2h-1)$.

Another example of applications of the uniqueness relations is the evaluation of the diagram $M_{4,4}$ which is one of the twenty eight master integrals shown in Fig. 8.8 for the four-loop massless propagator diagrams. To evaluate this diagram at $d=4$, uniqueness relations were used together with functional equations in [53, 54]. In this calculation, the initial problem was reduced to the problem of expansion of the propagator diagram of Fig. 3.10 with the indices $a_1 = \ldots = a_4 = 1$, $a_5 = 1+\lambda$ in a Taylor series in λ up to λ^4. This diagram, at various indices, was investigated in many papers starting from the old result for all indices equal to one [67] which was later reproduced [21] by GPXT, an analytical result for this diagram with general values of the indices a_1 and a_2 and other integer indices [21], an analysis of this diagram from the group-theoretical point of view [15, 16], an extension of the previous results with the help of GPXT [55], etc. See also a recent paper [44] where the history of evaluating the diagram of Fig. 3.10 is presented in details, with extensive references.

Eventually, this diagram was evaluated at $d=4$ with the result

$$\left(\mathrm{i}\pi^2\right)^4 (441/8)\zeta(7)/q^2$$

which is of transcendentality weight seven. Modern methods provide the possibility to go systematically to higher orders of expansion in ε. Using the DRA method presented in Chap. 8 it was possible to evaluate the ε-expansion of this and all other diagrams of Fig. 8.8 up to weight twelve [62].

14.5 Duality for Planar Graphs

Although we already used the word ‘planar’ many times let me present some definitions. A *planar* graph is a graph that can be drawn on a plane without crossing lines. Let Γ be a Feynman graph, i.e. the set of its vertices is decomposed into two subsets of internal and external vertices. We denote by Γ^∞ the graph obtained from Γ by connecting each external vertex with one additional vertex. A Feynman graph is planar if Γ^∞ is planar.

There are some additional relations for Feynman integrals corresponding to planar Feynman graphs. For a planar Feynman graph Γ, one constructs the corresponding *dual* graph $\tilde{\Gamma}$ [66] as follows. We draw a point in each of the h domains of the plane corresponding to loops of Γ. We divide the external domain of Γ by extending each of n external lines of Γ to infinity and choose a point inside each of the resulting subdomains. Then we connect by lines each pair of the points belonging to neighbouring domains. We consider as external vertices all the exterior points and denote this by drawing external lines. For example, in Fig. 14.3, a three-loop Feynman graph and the corresponding three-loop dual graph marked with dashed lines are shown. We have $\tilde{\tilde{\Gamma}} = \Gamma$.

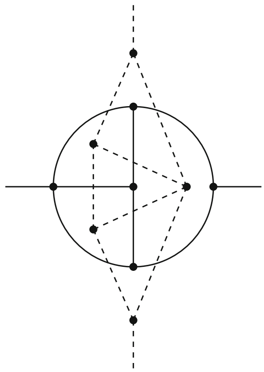

Fig. 14.3 A planar three-loop graph and the corresponding dual graph

It turns out that, for planar Feynman graphs, the corresponding Feynman integrals in coordinate space

$$F(x_1, \ldots, x_n) \int \ldots \int \prod_l D_l \left(\sum_i e_{il} x_i \right) \mathrm{d}^d x_{n+1} \ldots \mathrm{d}^d x_V \tag{14.10}$$

can be written as integrals over loop momenta. (As before, V is the number of vertices, n is the number of external vertices and e is the incidence matrix, i.e. $e_{il} = \pm 1$ if i is the beginning (end) of l.) We naturally enumerate the vertices of Γ starting from its external vertices. Let us introduce momenta for the dual graph by $p_i = x_i - x_{i+1}$, $i = 1, 2, \ldots, V-1$. The momenta $p_1, \ldots, p_{n-1}$ will be independent external momenta and the momenta $p_n, \ldots, p_{V-1}$ the loop momenta. Then the argument of the lth propagator, $\sum_i e_{il} x_i$, can also be associated with the line of the dual graph which crosses l.

Let me emphasize that this transition to the momentum-space picture is done just by a linear change of variables, rather than by the standard way, via Fourier transform. Moreover, this correspondence holds at general d. Let us take the coordinate-space Feynman integral corresponding to the graph of Fig. 14.3 drawn with solid lines. Then it can be represented as the momentum-space Feynman integral corresponding to the graph of Fig. 14.3 drawn with dashed lines. A much more non-trivial application of this duality relation can be found in [33] where 22 of the 24 coordinate-space master integrals correspond to planar graph so that results for them could be obtained by identifying them as momentum-space integrals of Fig. 8.8 and using the results of [5].

There is also a duality relation in the sense of the Fourier transform. As it was explicitly proven in [24], the Fourier transform of a Feynman integral for a planar graph can be written as the Feynman integral for Γ, with propagators which are Fourier transformed original propagators. Observe that if one starts from usual propagators in momentum space one obtains integrals with propagators raised to powers depending on ε—see (14.5).

Duality relations were often applied together with star-triangle relations of Sect. 14.4.

14.6 IR Rearrangement and *R**

The method of IR rearrangement is a special method for the evaluation of UV counterterms which are necessary to perform renormalization. The counterterms are introduced into the Lagrangian, i.e. the dependence of the bare parameters (coupling constants, masses, etc.) of a given theory on a regularization parameter (e.g., d within dimensional regularization) is adjusted in such a way that the renormalized physical quantities become finite when the regularization is removed. The renormalization can be described at the diagrammatic level, i.e. the renormalized Feynman integrals can be obtained by applying the so-called R*-operation* which removes the UV divergence from individual Feynman integrals. Thus, for any R-operation, the quantity RF_Γ is UV finite at $d = 4$.

As is well known, the requirement for the R-operation to be implemented by inserting counterterms into the Lagrangian leads to the following structure [14]:

$$RF_\Gamma = \sum_{\gamma_1,\ldots,\gamma_j} \Delta(\gamma_1)\ldots\Delta(\gamma_j)F_\Gamma \equiv R'\,F_\Gamma + \Delta(\Gamma)\,F_\Gamma, \tag{14.11}$$

where $\Delta(\gamma)$ is the corresponding counterterm operation, and the sum is over all sets $\{\gamma_1,\ldots,\gamma_j\}$ of disjoint UV-divergent 1PI subgraphs, with $\Delta(\emptyset) = 1$. The 'incomplete' R-operation R', by definition, includes all the counterterms except the overall counterterm $\Delta(\Gamma)$. For example, if a graph is primitively divergent, i.e. does not have divergent subgraphs, the R-operation is of the form $RF_\Gamma = [1 + \Delta(\Gamma)]\,F_\Gamma$.

The action of the counterterm operations is described by

$$\Delta(\gamma)\,F_\Gamma = F_{\Gamma/\gamma} \circ P_\gamma, \tag{14.12}$$

where $F_{\Gamma/\gamma}$ is the Feynman integral corresponding to the reduced graph Γ/γ, and the right-hand side of (14.12) denotes the Feynman integral that differs from $F_{\Gamma/\gamma}$ by insertion of the polynomial P_γ in the external momenta and internal masses of γ into the vertex v_γ to which the subgraph γ was reduced. The degree of each P_γ equals the degree of divergence $\omega(\gamma)$. It is implied that a UV regularization is present in (14.11) and (14.12) because these quantities are UV-divergent. The coefficients of the polynomial P_γ are connected in a straightforward manner with the counterterms of the Lagrangian.

A specific choice of the counterterm operations for the set of the graphs of a given theory defines a *renormalization scheme*. In the framework of dimensional renormalization, i.e. renormalization schemes based on dimensional regularization, the polynomials P_γ have coefficients that are linear combinations of pure poles in

$\varepsilon = (4-d)/2$. In the minimal subtraction (MS) scheme [46], these polynomials are defined recursively by equations of the form

$$P_\gamma \equiv \Delta(\gamma)\, F_\gamma = -\hat{K}_\varepsilon R'\, F_\gamma \tag{14.13}$$

for the graphs γ of the given theory. Here $\hat{K}_\varepsilon$ is the operator that picks up the pole part of the Laurent series in ε. The modified MS scheme [7] ($\overline{\mathrm{MS}}$) is obtained from the MS scheme by the replacement $\mu^2 \to \mu^2 \mathrm{e}^{\gamma_{\mathrm{E}}}/(4\pi)$ for the massive parameter of dimensional regularization that enters through the factors of $\mu^{2\varepsilon}$ per loop.

If Γ is a logarithmically divergent diagram the corresponding counterterm is just a constant. To simplify its calculation it is tempting to put to zero the masses and external momenta. This is, however, a dangerous procedure because it can generate IR divergences. Consider, for example, the two-loop graph of Fig. 14.4a. It contributes to the mass renormalization in the ϕ^4 theory. To evaluate the corresponding counterterm it is necessary to compute $R'\, F_\gamma$, according to (14.13). Here $R' = 1 + \Delta_1$, where Δ_1 is the counterterm operation for the logarithmically divergent subgraph of Fig. 14.4a. We consider each of the two resulting terms separately. The last term is simple. The first one is just the pole part of the given diagram. If we put the mass to zero we will obtain an IR divergence. There is another option which is safe: we put the mass to zero and let the external momentum q flow in another way through the graph: from the bottom vertex, rather than from the right vertex—see Fig. 14.4b. Then the resulting Feynman integral is IR-convergent and, at the same time, much simpler because it is now recursively one-loop and can be evaluated in terms of gamma functions.

This is a simple example of the trick called *IR rearrangement* and invented in [74]. In a general situation, one tries to put as many masses and external momenta to zero as possible and, probably, let the external momentum flow through the graph in such a way that the resulting diagram is IR-convergent and simple for calculation. Consider now the three-loop graph of Fig. 14.4c contributing to the β-function in the ϕ^4 theory. It is also logarithmically divergent. When calculating its counterterm, it is dangerous to put the masses to zero and let the external momentum flow from the bottom to the top vertex, because we run into IR divergences either due to the left or the right pair of the lines. Still there is a possibility not to generate IR divergences: to put the masses of the central loop and the external momentum to zero. The resulting three-loop

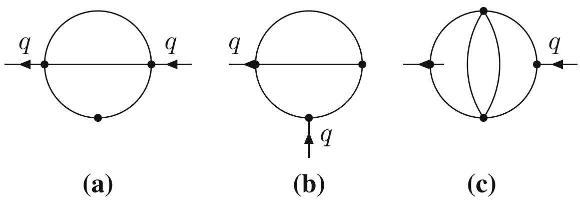

Fig. 14.4 **a** A two-loop graph contributing to the mass renormalization. **b** A possible IR rearrangement. **c** A three-loop graph contributing to the β-function

Feynman integral is evaluated in terms of gamma functions, first, by integrating the massless subintegral by (10.7) and then by (10.38).

Quite recently the method of IR rearrangement was applied to four-loop diagrams in [33] in coordinate space, where instead of setting an external momentum to zero we integrate over a coordinate.

At a sufficiently high level, such a safe IR rearrangement is not always possible. However, there is a way to put as many masses and momenta to zero and still have control on IR divergences. Formally, we have

$$P_\gamma = -\hat{K}_\varepsilon R'^* \, F_\gamma(q), \tag{14.14}$$

where it is implied that all the masses are put to zero, and one external momentum is chosen to flow through the diagram in an appropriate way. (Another version is to put all the external momenta to zero and leave one non-zero mass.)

The operation R^* removes not only UV but also (off-shell) IR divergences in a similar way [22], i.e. by a formula which generalizes (14.11). Now, it includes IR counterterms $\tilde{\Delta}(\gamma)$ which are defined in a full analogy to the UV counterterms $\Delta(\gamma)$. They are defined for subgraphs irreducible in the IR sense, with the IR degree of divergence given by (2.18). Now, they are local in momentum space. For example, the IR counterterm corresponding to the logarithmically divergent (in the IR sense, i.e. with the IR degree of divergence $\tilde{\omega}(\gamma) = 0$) factor $1/(k^2)^2$ for the two lower lines in Fig. 14.4a (when they are massless) is proportional to $\delta^{(d)}(k)/\varepsilon$. More details on the R^*-operation can be found in [69]. So, according to (14.14), one can safely put to zero all the momenta and masses but one, in a way which is the simplest for the calculation, at the cost of generating IR divergences which should be removed with the help of IR counterterms. Finally, the problem of the evaluation of the UV counterterms for graphs with positive degrees of divergence can be reduced, by differentiating in momenta and masses, to the case $\omega = 0$.

The R^*-operation was successfully applied in renormalization group calculations—see, e.g., [19].

14.7 Difference Equations

Basic prescriptions of a method based on difference equations can be found in [60] and an informal introduction in [61]. It is analytical in nature but is used to obtain numerical results with a high precision. The starting point of this approach is to choose a propagator, in an arbitrary way, treat its power, n, as the basic integer variable and fix other powers of the propagators (typically, equal to one). Then the general Feynman integral of a given family is written as

$$F(n) = \int \cdots \int \mathrm{d}^d k_1 \ldots \mathrm{d}^d k_h \frac{H}{E_1^n E_2 \ldots E_N}, \tag{14.15}$$

where H is a numerator. After combining various IBP relations, one can obtain a difference equation for $F(n)$:

$$c_0(n)F(n) + c_1(n)F(n+1) + \cdots + c_r(n)F(n+r) = G(n), \tag{14.16}$$

where the right-hand side contains Feynman integrals F_1, F_2, ... which have one or more denominators E_2, E_3, ... less with respect to (14.15). These integrals are treated in a similar way, by means of equations of the type (14.16) so that one obtains a triangular system of difference equations. This system is solved, starting from the simplest integrals that have the minimum number of denominators, with the help of an Ansatz in the form of a factorial series,

$$\mu^n \sum_{l=0}^{\infty} \frac{b_l\, n!}{\Gamma(n-K+l+1)}, \tag{14.17}$$

where the values of parameters μ, b_l and K are obtained from these values for the factorial series corresponding to the right-hand side of (14.16).

This method was successfully applied, with a precision of several dozens up to hundreds of digits, to the calculation of various multiloop Feynman integrals [57–60].

Observe that, although this method is numerical, it requires serious mathematical efforts. The same feature holds for any modern method of numerical evaluation. One can say that the boarder between analytical and numerical methods becomes rather vague at the moment.

Sometimes it is claimed that sooner or later we will achieve the limit in the process of analytical evaluation of Feynman integrals so that we will be forced to proceed only numerically. However, the dramatic progress in the field of analytical evaluation of Feynman integrals shows that we have not yet exhausted our abilities. So, the natural strategy is to combine available analytical and numerical methods in an appropriate way.

14.8 Experimental Mathematics and PSLQ

When evaluating Feynman integrals, various tricks are used. One usually does not bother about mathematical proofs of the tricks, partially, because of the pragmatical orientation and strong competition and, partially, because, now, there are a lot of possibilities to check obtained results, both in the physical and mathematical way.

An example of such ‘experimental mathematics’ suggested in [40] was described in Sect. 5.5, where it was supposed that the nth coefficient of the Taylor series c_n of a piece of the result for the master massive double box is a linear combination of the 15 functions (5.44)–(5.47) of the variable n. Then the possibility to evaluate the first 15 coefficients $c_1, c_2, \ldots, c_{15}$ was used and the corresponding linear system for unknown coefficients in the given linear combination was solved. At this point, a pure mathematician could say that there is no mathematical proof of this procedure and its validity is not guaranteed at all even after we (successfully) check it by calculating more terms of the Taylor expansion, starting from the 16th and comparing it with what we have from the obtained solution. Still I believe that this pure mathematician will believe in the result when he/she looks at some details of the calculation. Indeed, suppose that we forget about just one of the functions in (5.44)–(5.47) and follow our procedure. Then we indeed obtain a different solution of our system of 14 equations but it blows up and looks so ugly, in terms of rational numbers with hundreds of digits in the numerator and denominator, that this pure mathematician will say that our previous solution, with nice rational numbers, is true and there is no need for mathematical proofs.

There are a lot of other elements of experimental mathematics in dealing with Feynman integrals. Indeed, we never hesitate to change the order of integration over alpha and Feynman parameters and over MB parameters, it is not known in advance which IBP equations within the algorithm formulated in [60] are really independent, etc. One more example of experimental mathematics[1] is provided by the so-called[2] PSLQ algorithm [36, 37]. It can be applied when we evaluate a one-scale Feynman integral in a Laurent expansion in ε. Let us suppose that, in a given order of expansion in ε, we understand which transcendental numbers can appear in the result and that we can obtain the result numerically with a high accuracy. For example, in the finite part of the ε-expansion in two loops we can expect at least $x_{i-1} = \zeta(i)$ with $i = 2, 3, 4$ or, equivalently, $x_1 = \pi^2$, $x_2 = \zeta(3)$ and $x_3 = \pi^4$. Then the PSLQ algorithm could be of use. In this particular example, it gives the possibility to estimate whether or not a given number, x can be expressed linearly as $x = c_1x_1 + c_2x_2 + c_3x_3$ with rational coefficients c_i.

The PSLQ is an example of an 'integer relation algorithm'. If $x_1, x_2, \ldots, x_n$ are some real numbers, it gives the possibility to find the n integers c_i such that $c_1x_1 + c_2x_2 + \cdots + c_nx_n = 0$ or provide bounds within which this relation is impossible. (In the above situation, we consider our numerical result as x_4, in addition to the x_i, $i = 1, 2, 3$.) More formally, suppose that x_i are given with the precision of ν decimal digits. Then we have an integer relation with the norm bound N if

$$|c_1x_1 + \cdots + c_nx_n| < \varepsilon, \tag{14.18}$$

provided that $\max|c_i| < N$, where $\varepsilon > 0$ is a small number of order $10^{-\nu}$. With a given accuracy ν, a detection threshold ε and a norm bound N as an input, the PSLQ algorithm enables us to find out whether the relation (14.18) exists or not at some confidence level (see details in [36, 37]).

The PSLQ algorithm has been successfully applied in the evaluation of various single-scale Feynman integrals—see, e.g., earlier applications in [6, 17, 38, 39, 52]. For example, almost all the results mentioned in Chap. 8 were obtained using PSLQ.

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[1] The very term 'experimental mathematics' can be found on the web page where, in particular, the PSLQ algorithm is described [75].

[2] The name 'PSLQ' comes from the words 'partial sum' and 'lower-diagonal' used in the algorithm.

The experience obtained in the calculations shows that one needs at least seven digits for each independent transcendental number.

In addition to various private implementations of PSLQ, there is now an implementation within `Mathematica` [76].

14.9 Evaluating Feynman Integrals by Summing up Series

Any Feynman integral can be converted into a multiple series. To do this, one can start from the derivation of an MB representation, either in the loop-by-loop strategy in the case of planar graphs, or using the alpha representation. Then the (multiple) MB integral obtained can be transformed straightforwardly into a sum of (multiple) series, by closing the integration contour(s) in the complex plane to the right or to the left. This can be done both at general ε or after the resolution of the singularities in ε. Then one can try to sum up the series obtained by available summation formulae or computer codes, for example, by tables of summation formulae presented in Appendix C and the codes `SUMMER` [73] and `XSummer` [64]. Some other summation formulae can be found in [1, 45]. Algorithms for summing up various series were presented in [1, 50]. See also [13] and references therein. Most corresponding computer codes are private, with some exceptions, for example `Sigma` [68].

In some situations, one arrives at a onefold series which can be recognized as a hypergeometric series ${}_pF_q$ with parameters depending on ε. Then one can apply existing formulae for the expansion of ${}_pF_q$ near integer and half-integer parameters. A `Mathematica` package for doing this in the case of ${}_2F_1$ and ${}_3F_2$ was developed in [47]. Here are papers on other various cases: [25–27, 45, 48, 49, 51, 52, 65].

14.10 Symbols

A new tool for evaluating Feynman integrals is the so-called symbol map which associates an element of a tensor algebra with a given transcendental function. Our experience tells us that analytical results for Feynman integrals in a Laurent expansion in ε can be expressed in terms of multiple polylogarithms [41] (see (11.43)) and, in particular, HPLs and usual polylogarithms. However, these functions can depend on masses and kinematical invariants in a rather complicated way, via some rational and irrational combinations. Moreover, there are various identities between them, for example, corresponding to a shuffle algebra. In particular, for the polylogarithms, various functional identities can be found in [63]. The goal of using symbols is to map functional identities among multiple polylogarithms onto algebraic relations among the corresponding symbols.

Let $F : \mathbb{C}^n \to \mathbb{C}$ be a complex valued function depending on n complex variables x_k, $1 \le k \le n$ and let the total differential of F be expressed in the form

$$\mathrm{d}F = \sum_i F_i \, \mathrm{d} \log R_i \,, \tag{14.19}$$

where F_i and R_i are functions of the variables x_k, and R_i are rational functions. Then the symbol of F is defined recursively by [32, 43]

$$\mathcal{S}(F) = \sum_i \mathcal{S}(F_i) \otimes R_i \,. \tag{14.20}$$

In the case where F is a multiple polylogarithm the differential of F can be represented in an explicit form. For example, in the special case where all the arguments of the multiple polylogarithm are generic (i.e., they are mutually different and do not take particular values), one obtains [42]

$$\mathrm{d}G(a_{n-1}, \ldots, a_1; a_n) = \sum_{i=1}^{n-1} G(a_{n-1}, \ldots, \hat{a}_i, \ldots, a_1; a_n) \, \mathrm{d} \ln \left(\frac{a_i - a_{i+1}}{a_i - a_{i-1}} \right) . \tag{14.21}$$

The symbol of $G(a_1, \ldots, a_{n-1}; a_n)$ is then defined in the form

$$\begin{aligned} &\mathcal{S}(G(a_{n-1}, \ldots, a_1; a_n)) \\ &\qquad = \sum_{i=1}^{n-1} \mathcal{S}(G(a_{n-1}, \ldots, \hat{a}_i, \ldots, a_1; a_n)) \otimes \left(\frac{a_i - a_{i+1}}{a_i - a_{i-1}} \right) . \end{aligned} \tag{14.22}$$

The symbol satisfies the following identities:

$$R_1 \cdots \otimes (R_a R_b) \otimes \cdots R_k = R_1 \cdots \otimes R_a \otimes \cdots R_k + R_1 \cdots \otimes R_b \otimes \cdots R_k , \tag{14.23}$$

$$R_1 \cdots \otimes (c R_a) \otimes \cdots R_k = R_1 \cdots \otimes R_a \otimes \cdots R_k , \tag{14.24}$$

$$R_1 \cdots \otimes (\pm 1) \otimes \cdots R_k = 0 \tag{14.25}$$

for any constant c and rational functions R_i. The first of these properties is similar to the basic property of the usual logarithm.

For example, the symbols of the classical polylogarithms and the ordinary logarithms are given by

$$\mathcal{S}(\mathrm{Li}_n(z)) = -(1-z) \otimes \underbrace{z \otimes \ldots \otimes z}_{(n-1)\ \text{times}}, \tag{14.26}$$

$$\mathcal{S}\left(\frac{1}{n!}\ln^n z\right) = \underbrace{z \otimes \ldots \otimes z}_{n\ \text{times}}. \tag{14.27}$$

Here are two examples which show how symbols work to prove functional identities between dilogarithms. Let us, first, prove the very well known identity

$$\mathrm{Li}_2(1-x) + \ln(1-x)\ln x = -\mathrm{Li}_2(x) + \frac{\pi^2}{6}. \tag{14.28}$$

The symbols of the functions involved are

$$\mathcal{S}[\mathrm{Li}_2(1-x)] = -x \otimes (1-x), \tag{14.29}$$

$$\mathcal{S}[\ln(1-x)\ln x] = (1-x) \otimes x + x \otimes (1-x), \tag{14.30}$$

$$\mathcal{S}[\mathrm{Li}_2(x)] = -(1-x) \otimes x. \tag{14.31}$$

We then get

$$\begin{aligned}
&\mathcal{S}\big[\mathrm{Li}_2(1-x) + \ln(1-x)\ln x\big] \\
&\quad = -x \otimes (1-x) + \big[(1-x) \otimes x + x \otimes (1-x)\big] \\
&\quad = (1-x) \otimes x = \mathcal{S}\big[-\mathrm{Li}_2(x)\big].
\end{aligned} \tag{14.32}$$

So $\mathrm{Li}_2(1-x) + \ln(1-x)\ln x$ is equal to $-\mathrm{Li}_2(x)$ up to terms whose symbols vanish. Putting $x = 1$, we see that

$$\big[\mathrm{Li}_2(1-x) + \ln(1-x)\ln x\big]_{|x=1} = 0, \tag{14.33}$$

$$-\mathrm{Li}_2(1) = -\frac{\pi^2}{6}, \tag{14.34}$$

and arrive at (14.28).

Let us now turn to

$$\mathrm{Li}_2\left(1-\frac{1}{x}\right) + \frac{1}{2}\ln^2 x = -\mathrm{Li}_2(1-x). \tag{14.35}$$

We have (14.29) and

$$\mathcal{S}\left[\mathrm{Li}_2\left(1-\frac{1}{x}\right)\right] = -\frac{1}{x} \otimes \left(1-\frac{1}{x}\right) = x \otimes (1-x) - x \otimes x, \tag{14.36}$$

$$\mathcal{S}[\ln^2 x] = 2\,x \otimes x. \tag{14.37}$$

We then get

$$\mathcal{S}\left[\mathrm{Li}_2\left(1-\frac{1}{x}\right)+\frac{1}{2}\ln^2 x\right]=\left[x\otimes(1-x)-x\otimes x\right]+x\otimes x \tag{14.38}$$

$$=x\otimes(1-x) \tag{14.39}$$

$$=\mathcal{S}\left[-\mathrm{Li}_2\,(1-x)\right]. \tag{14.40}$$

So $\mathrm{Li}_2\,(1-1/x)-\frac{1}{2}\ln^2 x$ is equal to $-\mathrm{Li}_2(1-x)$ up to terms whose symbols vanish. Putting $x=1$, we see that both functions vanish and arrive at (14.35).

The first impressive application of the symbol technology in perturbative calculations was presented in [43] for the two-loop hexagon Wilson loop which, according to the conjecture in [2], is related to a six-point amplitude in $N=4$ supersymmetric Yang–Mills theory. The previously obtained result [28, 29] which took seventeen pages in a journal was rewritten in two lines. The result of [28, 29] was written in terms of multiple polylogarithms with a cumbersome dependence on three conformal variables. So the authors of [43] started from this result, calculated its symbol and then adjusted a much simpler function having the same symbol. Since the knowledge of the symbol determines a transcendental function up to lower weight functions multiplied by numerical constants it was necessary to fix this ambiguity. This was done by matching the result of [28, 29] in appropriate limits.

First direct applications of the symbol technology to Feynman integrals can be found in [30, 31], where the massless hexagon integral and one-mass hexagon integral in six dimensions were evaluated. As a recent application, let me mention the evaluation of three-point two-loop massless Feynman integrals [18], with much simpler results than in [12]. I am confident very much that symbols will help a lot in the nearest future to evaluate important classes of Feynman integrals which could not be evaluated up to now by existing tools.

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List of Symbols

a_l	Power of a propagator (index)
$\tilde{D}_F$	Propagator in coordinate space
$D_F, D_{F,i}$	Propagator in momentum space
d	Space–time dimension
E_i	Denominator of propagator
F	Feynman integral
${}_2F_1(a,b;c;z)$	Gauss hypergeometric function
$G(\lambda_1,\lambda_2)$	Function in one-loop massless integration formula
$G(a_1,\ldots,a_n;z)$	Multiple polylogarithm
$g_{\mu\nu}$	Metric tensor
$H_{a_1,a_2,\ldots,a_n}(x)$	Harmonic polylogarithm (HPL)
h	Number of loops
k	Loop momentum
L	Number of lines
$\mathrm{Li}_a(z)$	Polylogarithm
l	Loop momentum
m	Mass
p	External or internal momentum
$Q^2=-q^2$	Euclidean external momentum squared
q	External momentum
$S_{a,b}(z)$	Generalized polylogarithm
S_j, S_{jk},...	Nested sums
$s=(p_1+p_2)^2$	Mandelstam variable
T	Tree, 2-tree, pseudotree
$t=(p_1+p_3)^2$	Mandelstam variable
t_l	Sector variable
$\mathcal{U}$	Function of alpha parameters
$u=(p_1+p_4)^2$	Mandelstam variable

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u_l	Auxiliary parameter
V	Number of vertices
$\mathcal{V}$	Function of alpha parameters
$\mathcal{W}$	Function of alpha parameters
w	Variable in MB integrals
x	Coordinate
x_i	Variable in the basic parametric representation
z, z_i	Variable in MB integrals
α_l	Alpha parameter
$\beta_l = 1/\alpha_l$	Inverse alpha parameter
Γ	Graph
$\Gamma(x)$	Gamma function (first Euler integral)
γ	Subgraph
$\gamma_E = 0.577216\ldots$	Euler's constant
$\delta(x)$	Delta function
$\varepsilon = (4-d)/2$	Parameter of dimensional regularization
$\zeta(z)$	Riemann zeta function
$\zeta(m_1, \ldots, m_k)$	Multiple zeta value
λ_l	Parameter of analytic regularization
ξ, ξ_i	Feynman parameter
τ_l	Sector variable
$\psi(x) =' (z)/(z)$	Logarithmical derivative of the gamma function
ω	Degree of UV divergence

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