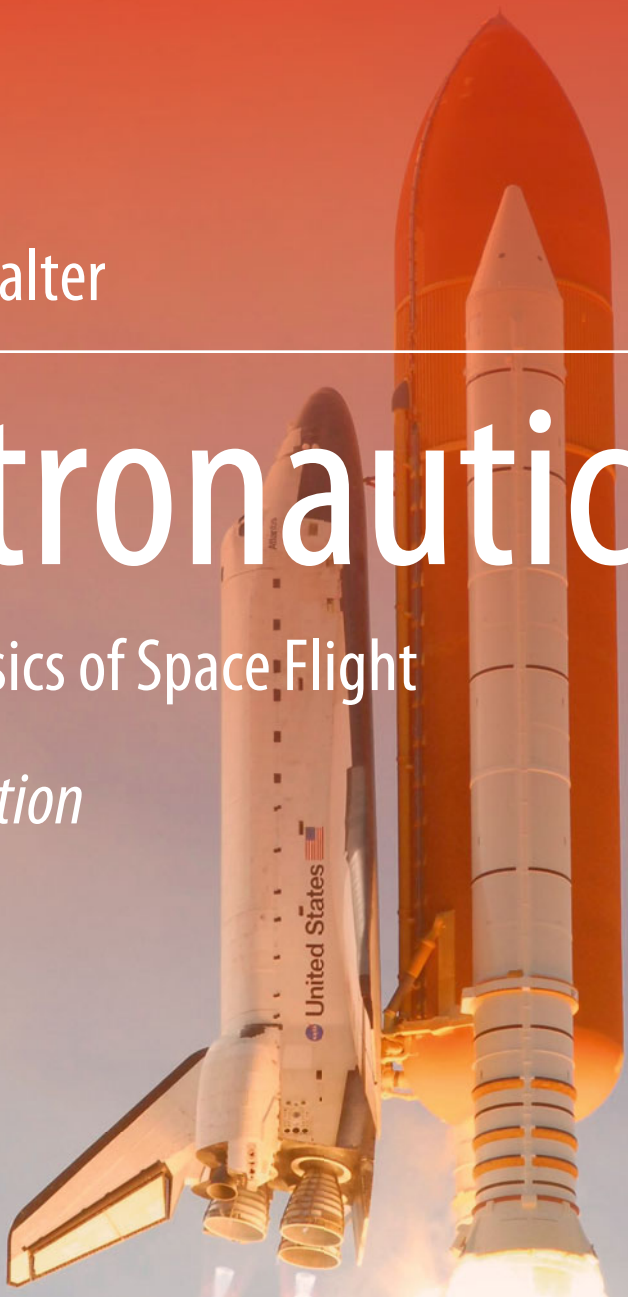
Ulrich Walter

Astronautics

The Physics of Space Flight

Third Edition



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Ulrich Walter
Institute of Astronautics
Technical University of Munich
Garching
Germany

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Cover: The Space Shuttle *Atlantis* launched on February 7, 2008, to ferry on its 29th flight the European science laboratory *Columbus* to the International Space Station. (Used with permission of NASA)

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This book is dedicated to the astronauts and cosmonauts, who lost their lives in the pursuit of space exploration

Preface to the Third Edition

This textbook is about all basic physical aspects of spaceflight. Not all have been covered in the past editions. So, what is new in this third edition? First, there are new sections covering new topics, such as

- Sections 1.2 and 1.3 dealing with the physics of a jet engine and general rocket performance have been widely extended to more sophisticated effects.
- Sections 7.4.5 and 7.4.6 describe two general solutions to Newton's gravitational equation of motion.
- Section 7.7 studies stellar orbits, which are not subject to the standard but more general types of gravitational potentials.
- Hypersonic flow theory for reentry vehicles is expounded in Sect. 6.2 as a basis to understand how lift and drag come about and in particular how both depend on the angle of attack, the most important control parameter to guide a winged body through the flight corridor (see Fig. 10.22).
- Accordingly, the reentry of a Space Shuttle, which in this book even more serves as a case study, is explained in Sect. 10.7 in greater detail and in terms of NASA terminology.
- In Sect. 8.1, the different basic types of orbit maneuvers are discussed and exemplified.
- A new form of solution of Lambert's problem is derived in Sect. 8.2.3, which is visualized in Fig. 8.8.
- Section 8.4.3 discusses modern super-synchronous transfer orbits to GEO.
- Relative motion in near-circular orbits is examined in Sect. 8.5.4.
- The virial theorem for bounded and unbounded n-body systems is derived in Sect. 11.1.2 and used to discuss the stability of an n-body system.
- Section 12.3 (*Gravitational Perturbation Effects*) has been revised and greatly extended including other and higher order perturbation terms.
- Chapter 14 has been radically revised: There is a new Sect. 14.1 on orbit geometric issues (eclipse duration and access area) and a fully revised Sect. 14.2 on orbit determination.

– There is a whole new Chap. 16 dedicated to thermal radiation physics and modeling. It serves the same purpose as Chap. 15 *Spacecraft Attitude Dynamics*, namely to provide insight into some basic and important physics of a spacecraft in space.

Some sections have been substantially revised and there a hundreds more or less significant extensions of established topics of space fight as already covered in the 2nd version of this textbook.

I put a lot of effort into introducing and using a proper terminology, or establishing one if not existent. An example of the former is the distinction between orbital velocity v, angular velocity ω, angular frequency ω_i, and orbital frequency n, which are sometimes confused. Orbital velocity v is the speed of motion of a body on an orbit. Angular velocity ω is the instantaneous speed of angular motion, while angular frequency ω_i is the number of revolutions in a given time. Finally, orbital frequency n (a.k.a. mean motion) is the time average of the angular velocity over one orbital period T (see Eq. (7.4.10)). Thus, $n = 2\pi/T$; it therefore can be considered both as a mean angular velocity (i.e., mean angular motion) and as a frequency, the orbital frequency. Because proper terminology is essential, the conventional "symbols used" table on the following pages also serves the purpose of enabling one to look up the proper terminology for a physical quantity.

Because physics is independent of the choice of the reference system, the third version consequently uses a reference system-free vector notation (except auxiliary corotating reference systems in Sects. 6.3 and 7.3). All reference systems, the transformations between them, and the vector representations in the different common reference systems are summed up in Sect. 13.1.

Finally, I feel the need to a very personal comment on textbooks in general. When I was a student, I bought some expensive but basic physics textbooks, which are still in my office shelf and serve as my reference books, because true physics is eternal. Compare buying a textbook with a marriage. You do not just buy it. It must have a kind of visual—a tactile sensuality: You open it with joyful anticipation. Your fingers glide over the pages, and they slowly turn one page after the other. You like the layout, the way the book talks to you, and how it explains the world from a point of view you have never considered before. You just love it, and thus it will become part of your daily scientific work. You may forget little physical details, but you will always remember that the one you are looking for is on top of the left-hand page somewhere in the middle of the book. You will never forget that visual detail, and therefore you will always find the answer to your question quite swiftly. I have about a handful of such key textbooks, which I would not sell in my lifetime. I sense that these books were written for guiding me through my scientific life. For me, writing this book was for giving back to other people what many scientists before had given to me. We all are standing on the shoulders of giants. May this textbook keep and pass the body of basic knowledge to you and future generations.

Garching, Germany Ulrich Walter

Preface to the Second Edition

Textbooks are subject to continuous and critical scrutiny of students. So is this one. Having received many questions to the book in my lectures and by e-mail, I constantly improved and updated the content such that already after three years it was time to have also the reader benefit from this. You will therefore find the textbook quite revised as for instance rocket staging (Chapter 3), engine design (Section 4.4), radial orbits (Section 7.5), or the circular restricted thee-body problem (Section 11.4). But there are also new topics, namely Lambert transfer (Section 8.2), relative orbits (Section 8.5), and orbital rendezvous (Section 8.6), higher orbit perturbations including frozen orbits (Sections 12.3.6 and 12.3.7), resonant perturbations and resonant orbits (Section 12.4), and relativistic perturbations (Section 12.6.2). Along with this also the structure of the content has changed slightly. Therefore the section and equation numbers are not always identical to the first edition.

Nevertheless the overall structure still serves the same intention: It is set up for a two semester course on astronautics. Chapter 1–7 (except Sect. 1.4), Section 8.1, and Chapters 9–10 is the basic subject matter an aerospace student should know or have been exposed to at least once. The sequence of the chapters is first rocket basics (Chapter 1–5), thereafter a flight into space "once around", starting with ascent flight (Chapter 6), then space orbits (Chapter 7) and basic orbital maneuvers (Section 8.1), interplanetary flight (Chapter 9), and finally reentry (Chapter 10). The second part of the textbook is more advanced material, which I lecture together with satellite technology in an advanced course for true rocket scientists and space engineers.

The careful reader might have noticed that the book now comes with a subtitle: *The Physics of Space Flight*. This was decided to provide a quick comprehension of the nature of this textbook. In addition, because the Space Shuttle and the ISS are running examples in this textbook, a picture of the launching Space Shuttle Atlantis was chosen as a new frontispiece. Unfortunately, I couldn't find an equally attractive picture of my Space Shuttle Columbia.

Preface to the First Edition

There is no substitute for true understanding

Kai Lai Chung

If you want to cope with science, you have to understand it – truly understand it. This holds in particular for astronautics. "To understand" means that you have a network of relationships in your mind, which permits you to deduce an unknown fact from well-known facts. The evolution of a human being from birth to adulthood and beyond consists of building up a comprehensive knowledge network of the world, which makes it possible to cope with it. That you are intelligent just means that you are able to do that – sometimes you can do it better, and sometimes worse.

True understanding is the basis of everything. There is nothing that would be able to substitute true understanding. Computers do not understand – they merely carry out programmed deterministic orders. They do not have any understanding of the world. This is why even a large language computer will always render a false translation of the phrase: "He fed her cat food." Our world experience intuitively tells us that "He fed a woman's cat some food." But a computer does not have world experience, and thus does not generally know that cat food is nasty for people. Most probably, and according to the syntax, it would translate it as: "He fed a woman some food that was intended for cats.", what the Google translator actually does when translating this phase into other languages. No computer program in the world is able to substitute understanding. You have to understand yourself. Only when you understand are you able to solve problems by designing excellent computer programs. Nowadays, real problems are only solved on computers – written by bright engineers and scientists.

The goal of this book is to build up a network of astronautic relationships in the mind of the reader. If you don't understand something while reading this book, I made a mistake. The problem of a relational network, though, is that the underlying logic can be very complex, and sometimes it seems that our brains are not suitable for even the simplest logic. If I asked you, "You are not stupid, are you?", you would normally answer, "No!" From a logical point of view, a double negation of

an attribute is the attribute itself. So your "No!" means that you consider yourself stupid. You, and also we scientists and engineers, do not want this embarrassing mistake to happen time and time again, and so we use mathematics. Mathematical logic is the guardrail of human thinking. Physics, on the other hand, is the art of applying this logic consistently to nature in order to be able to understand how it works. So it comes as no surprise to find a huge amount of formulas and a lot of physics in this book.

Some might think this is sheer horror. But now comes the good news. Most of the formulas are just intermediate steps of our elaborations. To understand astronautics, you only need to engage in the formulas shaded gray and to remember those bordered black. There you should pause and try to understand their meaning because they will tell you the essential story and lift the secrets of nature. Though you don't need to remember all the other formulas, as a student you should be able to derive these stepping stones for yourself. Thereby you will always be able to link nodes in your relational network whenever you deem it necessary. To treat formulas requires knowing a lot of tricks. You will learn them only by watching others doing such "manipulation" and, most importantly, by doing it yourself. Sometimes you will see the word "exercise" in brackets. This indicates that the said calculation would be a good exercise for you to prove to yourself that you know the tricks. Sometimes it might denote that there is not the space to fully lay out the needed calculation because it is too lengthy or quite tricky. So, you have to guess for yourself whether or not you should do the exercise. Nonetheless, only very few of you will have to derive formulas professionally later. For the rest of you: just try to follow the story and understand how consistent and wonderful nature is. Those who succeed will understand the words of Richard Feynman, the great physicist, who once expressed his joy about this by saying: "The pleasure of finding things out."

Take the pleasure to find out about astronautics.

Acknowledgements

Key parts of the new Chap. 16 are the two sections of thermal modeling. Thermal modeling of vehicles in space requires not only high skill, but also a lot of expert knowledge gathered in daily work. I am happy and very thankful to Philipp Hager (Thermal Engineer in the Thermal Control Section of the European Space Agency at ESTEC) and to Markus Czupalla (Full Professor at the Department of Aerospace Engineering, University of Applied Sciences Aachen, Germany) that they agreed to contribute these important sections.

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My special thanks go to Julia Bruder for her tedious work of translating the original German manuscript into English. Many expounding passages of this book would not be in place without the bright questions of my students, who reminded me of the fact that a lot of implicit meanings that scientists have become used to are not that trivial as they seem to be.

Many figures in this book were drawn by the interactive plotting program *gnuplot v4.0*. My sincere thanks to its authors Thomas Williams, Colin Kelley, Hans-Bernhard Bröker, and many others for establishing and maintaining this versatile and very useful tool for free public use. The author is grateful to the GeoForschungsZentrum Potsdam, Germany's National Research Centre for Geosciences for providing the geoid views and the visualization of the spherical harmonics in the color tables on pages 566, 568, and 569.

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Abbreviations

AOA	Angle of attack
AU	Astronomical unit
CFPAR	Constant flight path angle rate
CM	Center of mass
CPR	Constant pitch rate
CR3BP	Circular restricted three-body problem
EGM96	Earth gravitational model 1996
EL_1	Sun-Earth libration point L_1
EoM	Equation of motion
ET	External tank
EQW	Equinoctial coordinate system EQW (see Sect. 7.3.5)
FPA	Flight path angle
GEO	Geostationary orbit
GEODSS	Ground-based electro-optical deep space surveillance
GG	Gravity gradient
GMT	Greenwich mean time
GMST	Greenwich mean sidereal time
GSO	Geosynchronous orbit
GTO	Geostationary transfer orbit
GVE	Gaussian variational equation
IAU	International Astronomical Union
ICRF	International Celestial Reference Frame
IJK	Cartesian equatorial coordinate system (see Sect. 13.1.4)
ITRF	International Terrestrial Reference Frame
ISS	International Space Station
JD	Julian date
LEO	Low earth orbit, 100 km $< h <$ 2000 km
LL_1	Earth-Moon (Lunar) libration point L_1
LPE	Lagrange's planetary equations
LVLH	Local vertical, local horizontal (reference frame)

MECO	Main engines cut-off
MEO	Medium earth orbit, 2000 km $< h <$ *GEO*
MJD	Modified julian date
NTW	Corotating Cartesian topocentric satellite coordinate system *NTW* (see Sect. 13.1.4)
OMS	Orbital maneuvering system
PQW	Cartesian geocentric perifocal coordinate system *PQW* (see Sect. 13.1.4)
R&D	Rendezvous and docking
R3BP	Restricted three-body problem
RAAN	Right ascension of ascending node
RSW	Corotating Cartesian topocentric satellite coordinate system *RSW* (see Sect. 13.1.4)
RTG	Radioisotope thermoelectric generator
S/C	Spacecraft
SOI	Sphere of influence
SRB	Solid rocket booster
SSME	Space shuttle main engine
SSO	Sun-synchronous orbit
SSTO	Super-synchronous transfer orbit
TAEM	Terminal area energy management
TDRS	Tracking and data relay satellite
TDRSS	Tracking and data relay satellite system
TTPR	Thrust-to-power ratio
UT	Universal time
VDF	Velocity distribution function
WSB	Weak stability boundary

Symbols Used and Terminology

$x,\ a,\ b(.),\ f(\cdot)$	Scalars/scalar-valued functions
$\boldsymbol{x}, \boldsymbol{a}, \boldsymbol{b}(.), \boldsymbol{f}(\cdot)$	Vectors/vector-valued functions
$\boldsymbol{X},\ \boldsymbol{\Omega},\ \boldsymbol{J}(\cdot), \boldsymbol{M}(\cdot)$	Matrices/matrix-valued functions

Superscripts

T	Transpose of a vector or matrix

Subscripts

0	At the beginning (zero); or osculating (momentary)
a	With respect to the atmosphere
air	Atmosphere
apo	Apoapsis
B	Body system
c	Combustion; or commensurate
CM	Center of mass (a.k.a. barycenter)
col	Collision
crit	Critical (maximal deceleration)
D	Aerodynamic drag
div	(Jet) divergence
e	At exit, *or* ejection; or at entry interface
eff	Effective
esc	Escape (velocity)
ex	Exhaust
ext	External
EQW	Equinoctial coordinate system *EQW* (see Sect. 7.3.5)

f	Final (mass); or frozen orbit
F	Force
G	Gravitation
GEO	Geostationary
GG	Gravity gradient
h	Horizontal
H	Hohmann
i	Initial (mass)
id	Ideal engine
I	Inertial reference frame
IJK	Cartesian equatorial coordinate system (see Sect. 13.1.4)
IR	Infrared
ion	Ionic
in	Initial, at entry, incoming
int	Internal
jet	Propellant exhaust jet
kin	Kinetic (energy)
L	Aerodynamic lift; or payload; or libration point
LVLH	Local vertical, local horizontal (reference frame)
max	Maximum
min	Minimum
micro	Microscopic
n	Nozzle; or normal (vertically to …)
NTW	Corotating Cartesian topocentric satellite coordinate system *NTW* (see Sect. 13.1.4)
opt	Optimal (value)
out	Final, at exit, outgoing
p	Propellant; or planet; or perturbation; or periapsis (only in the case of epoch t_p)
P	Principal axes system; or orbital period
per	Periapsis; or periodic
PQW	Cartesian geocentric perifocal coordinate system *PQW* (see Sect. 13.1.4)
pot	Potential (energy)

r	Radial; or reflection; or radiation
rms	Root-mean-square (a.k.a. quadratic mean)
RSW	Corotating Cartesian topocentric satellite coordinate system *RSW* (see Sect. 13.1.4)
s	Structural
S/C	Spacecraft
sec	Secular
sk	Station keeping
sol	Solar
SOI	Sphere of influence
syn	Synodic
t	Tangential; or throat (of thruster)
T	Transfer orbit
tot	Total
trans	Translation; or transition
TTPR	Thrust-to-power ratio
v	Vertical
VDF	Velocity distribution function
vib	Vibration
θ	Vertically to radial
ω	Rotation (or centrifugal); or argument of periapsis (apsidal line)
Ω	Relating to the ascending node (draconitic)
∞	External, at infinity
$*$	Effective (thrust), total
$\oplus$	Earth
$\odot$	Sun
∇	Spacecraft
$\bullet$	Inner (orbit); or black body
$\bigcirc$	Outer (orbit); or in orbit plane
$\varnothing$	Diameter; or cross section
$\parallel$	Parallel to …
$\perp$	Vertical to … ($A_\perp$ = effectively wetted surface area)
$\times$	At orbit crossing

Latin Symbols

a	Semi-major axis (of a Keplerian orbit); or speed of sound; or acceleration
A	Area
A_p	Daily global index of geomagnetic activity, $0 \leq A_p \leq 400$
b	Semi-minor axis (of a Keplerian orbit); or $b := L \cdot \tan \gamma_e / (2D)$
B	Ballistic coefficient (without index: for drag), (see Eq. (6.2.19))
c	$c := L \cdot \cot \gamma_e / (2D)$; or speed of light
c^*	Characteristic velocity, $c^* := p_0 A_t / m_p$
c_p	Specific heat capacity at constant pressure
c_V	Specific heat capacity at constant volume
C	Jacobi constant
C_3	Characteristic energy, $C_3 := v_\infty^2$
C_∞	Infinite-expansion coefficient
C_D	Drag coefficient
C_f	Thrust coefficient; or skin friction drag coefficient
C_L	Lift coefficient
C_n	Nozzle coefficient (a.k.a. nozzle efficiency)
C_n^m	Multipole coefficient of the cosine term
δx	Variation (small changes) of x
$\delta v_{\|\|}$	Differential increase in orbital velocity due to kick-burn in flight direction
$\delta v_{\perp \mathrm{O}}$	Differential increase in orbital velocity due to kick-burn vertical to flight direction, within orbital plane, outbound
$\delta v_{\perp\perp}$	Differential increase in orbital velocity due to kick-burn vertical to flight direction and vertical to orbital plane, parallel to angular momentum
Δv	Delta-v budget
$\boldsymbol{D}$	Drag force
D	Aerodynamic drag, $D = \|\boldsymbol{D}\|$; or diameter
$diag(\ldots)$	Diagonal matrix with elements $(\ldots)$
e	Eccentricity; or electrical charge unit; or Eulerian number, e = 2.718281828 …
E	Energy; or (elliptic) eccentric anomaly
$Ei(x)$	Exponential integral (see Eq. (10.4.5))
f^x	f function (see definition Eq. (10.4.6))
F	Force (without index: gravitational force); or hyperbolic anomaly

$\boldsymbol{F}_*$	Thrust force (total), (a.k.a. propellant force)
F_*	Thrust (total)
F_e	Ejection thrust
F_{ex}	Momentum thrust
F_p	Pressure thrust
$F_{10.7}$	Daily solar flux index at wavelength of 10.7 cm in units 10^{-22} W m^2 Hz^{-1} (= 1 *Jansky* = 1 *solar flux unit*)
$\boldsymbol{g}(\boldsymbol{r})$	(Earth's) gravitational field
g	(Earth's) mean gravitational acceleration, $g = GM_\oplus/r^2$ (see Sect. 7.1.2)
g_o	(Earth's) mean gravitational acceleration at its surface, $g_0 := g(R_\oplus) = GM_\oplus/R_\oplus^2 = \mu_\oplus/R_\oplus^2 = 9.7982876\ \mathrm{m\,s^{-2}}$ (see Sect. 7.1.2)
G	Gravitational constant, $G = 6.67259 \times 10^{-11}$ m^3 kg^{-1} s^{-2}; or generic anomaly
h	(Mass-specific) angular momentum (i.e., per mass unit); or molar enthalpy; or height (above sea level); altitude
H	Enthalpy; or scale height
i	Inclination
$\boldsymbol{I}$	Inertia tensor
$I_x,\ I_y,\ I_z$	Principal moments of inertia
I_{sp}	(Weight-)specific impulse
j	Charge flow density (a.k.a. charge flux)
j_n	Reduced harmonic coefficient of order n
J_{nm}	Harmonic coefficient
k_B	Boltzmann constant, $k_B = 1.380650 \times 10^{23}$ J K^{-1}
$\boldsymbol{L}$	Lift force
L	Aerodynamic lift, $L = \lvert\boldsymbol{L}\rvert$; or angular momentum
L^*	Characteristic length of a combustion chamber
m	Body mass (without index: of a spacecraft)
m_i	In the collinear configuration: masses ordered according to their index, $m_3 < m_2 < m_1$
m_i'	In the collinear configuration: masses with m_2' located between m_1' and m_3', and $m_1' \geq m_3'$
$\dot{m}$	Mass flow rate (without index: of a spacecraft)
M	Central mass (central body); or total mass of a system of bodies; or mean anomaly; or molar mass
Ma	Mach number $Ma := v/a$

n	Rocket stage number; or mean motion (a.k.a orbital frequency); or mean number of excited degrees of freedom of gas molecules; or particle density
N	Particle number
p	Pressure; or propellant; or linear momentum $p = mv$; or semi-latus rectum, $p := h^2/\mu = a(1 - e^2)$; or $p := H/(\varepsilon_e R)$
P	Power
$P_n(x)$	Legendre polynomials of degree n
$P_n^m(x)$	Unnormalized associated Legendre polynomials of degree n and order m
q	Electrical charge density; or $q := H \cot^2 \gamma_e / (\varepsilon_e R)$
$\dot{q}$	Heat flux (*a.k.a.* heat flow density), $\dot{q} := \dot{Q}/A$
$\dot{Q}$	Heat flow rate
r	Orbit radius; or ratio
$\boldsymbol{r}$	Radial vector, a.k.a. position vector
R	Radius (of a celestial body, in particular Earth's radius); or residual perturbational potential; or universal gas constant, $R = 8.314\ \mathrm{J\,K^{-1}\ mol^{-1}}$
$\boldsymbol{R}$	Rotation (matrix)
$R_\oplus$	Equatorial scale factor of the Earth gravitational model EGM96, $R_\oplus = 6378.1363$ km, equaling roughly Earth's mean equatorial radius
R_s	Specific gas constant of standard atmosphere, $R_s = 286.91\ \mathrm{J\,K^{-1}\ kg^{-1}}$
Re	Reynolds number
S_n^m	Multipole coefficient of the sine term
$\mathrm{sgn}(x)$	Sign function (sign of x): $\mathrm{sgn}(x) = x/\lvert x\rvert$
St	Stanton number, $St \approx 0.1\%$
t	Time
t_p	Time at passage through periapsis, a.k.a. epoch (see end of Sect. 7.3.1)
T	Temperature; or orbital period; or torque
u	Argument of latitude
$\boldsymbol{u}$	Unit vector as a basis of a reference system (example: $\boldsymbol{u}_r \equiv \hat{\boldsymbol{r}} = \boldsymbol{r}/r$)

U	Internal energy of a gas; or total electrical voltage; or potential (for which holds $\boldsymbol{F} = -m \cdot dU/d\boldsymbol{r}$) (without index: gravitational potential)
v	Velocity (orbital v of the spacecraft, or drift v of propellant gas)
$\boldsymbol{v}$	Velocity vector
$v_{\triangleright}$	First cosmic velocity, $v_{\triangleright} = \sqrt{g_0 R_{\oplus}} = 7.905\ \mathrm{km\,s^{-1}}$ (see Eq. (7.4.4))
$v_{\triangleright\triangleright}$	Second cosmic velocity, $v_{\triangleright\triangleright} = \sqrt{2 g_0 R_{\oplus}} = 11.180\ \mathrm{km\,s^{-1}}$ (see Sect. 7.4.4)
v_*	Effective exhaust velocity
v_e	Ejection velocity
$\bar{v}_e$	Mean ejection velocity, $\bar{v}_e := \langle v_e \rangle_\mu$
v_{ex}	Exhaust velocity, $v_{ex} = \eta_{div} \bar{v}_e$
v_h	$v_h := \mu/h$
V	Volume; or electric potential
yr	Year(s)

Greek Symbols

α	Thrust angle; or angle of attack (AOA); or proper acceleration; or mass-specific power output of an electrical plant; or absorptivity (*a.k.a.* absorption coefficient)
β	$\beta := v/c$or geocentric latitude; or (orbit) beta angle
δ	Deflection angle
$\delta(x)$	Dirac delta function
δ_{nm}	Kronecker delta
δx	A finite (not differential) but small variation of x; or small variation of an orbital element over one orbital revolution; or small error of x
δv	An impulsive maneuver (kick-burn) varying the orbital velocity by δv
Δ	Impact parameter
Δ_i	Distance of the libration point i normalized to the distance between the two primaries in the R3BP (see Sect. 11.3.1)

Δx	The amount of change of parameter x
ε	Structural ratio; or specific orbital energy (*a.k.a.* specific mechanical energy); or expansion ration; or $\varepsilon := v^2/v_c^2$ (see Eq. (10.2.11)); or emissivity (*a.k.a.* emission coefficient emittance)
ε_0	Vacuum permittivity
γ	Flight path angle; or $\gamma := 1\Big/\sqrt{1-\beta^2} = 1\Big/\sqrt{1-v^2/c^2}$
κ	Heat capacity ratio; or reduced drag; κ_D, or lift, κ_L, coefficient
λ	Payload ratio; or dimensionless altitude variable (see definition Eq. (10.2.12)); or geographic longitude
λ_{nm}	Equilibrium longitude
ζ_d	Discharge correction factor
ζ_v	Velocity correction factor
η	Efficiency (a.k.a. loss factor), *in particular* thermal efficiency
η_{div}	Nozzle-divergence loss factor
η_{ec}	Energy conversion efficiency
ρ	(Atmospheric) mass density; or normalized position vector, $\rho = (\xi, \eta, \zeta) = \boldsymbol{r}/d$; or surface reflectivity; or inverse radius, $\rho := 1/r$
μ	Standard gravitational parameter: $\mu := GM$. For Earth: $\mu_\oplus = GM_\oplus = 3.9860044105 \times 10^5$ km^3 s^{-2} (as of EGM96); or reduced mass, $\mu := m_2/(m_1 + m_2)$ with $m_2 < m_1$; or mass ratio (see Eq. (1.3.1)); or mass flux (*a.k.a.* mass flow density), $\mu := \dot{m}_p/A$ (see Sect. 4.1.3)
$\mu(\theta)$	Angular mass flow distribution function; or bank angle (roll angle)
μ_i	Mass ratio of the ith partial rocket (see Eq. (3.1.7))
σ	Stefan–Boltzmann constant, $\sigma = 5.6704 \times 10^{-8}\, W\, m^{-2}\, K^{-4}$; or proper speed, $\sigma = c \cdot \operatorname{arctanh}\beta$
τ	Proper time (*a.k.a.* eigentime); or dimensionless time
θ	True anomaly, (*a.k.a.* orbit angle); or pitch angle, $\theta := \alpha + \gamma$
χ	Collinear configuration parameter (see Sect. 11.2.2)
$\chi(\lambda, \varepsilon_e)$	See definition Eq. (10.4.7)

$\boldsymbol{\omega}$	Angular velocity vector
ω	Angular velocity, $\omega = d\theta/dt = \lvert\boldsymbol{\omega}\rvert$; or argument of periapsis
ω_i	Angular frequency (*a.k.a.* circular frequency)
Ω	Right ascension of ascending node (RAAN); or effective potential in the CR3BP; or solid angle

Diacritics

$\dot{x}$	First-order time derivative of a quantity x, $\dot{x} = dx/dt$
$\ddot{x}$	Second-order time derivative of a quantity x, $\ddot{x} = d^2x/dt^2$
x'	First-order derivative with respect to a specified variable
x''	Second-order derivative with respect to a specified variable
$\bar{x}$	Geometric mean
$\underline{\text{ex}}\text{pression}$	The underlined letters of an arbitrary expression will be used as subscript for an upcoming variable to indicate its special meaning. Example: $\underline{\text{ex}}$haust velocity $\boldsymbol{v}_{ex}$
$\hat{\boldsymbol{r}}$	Unit vector along direction $\boldsymbol{r}$, $\hat{\boldsymbol{r}} = \boldsymbol{r}/r$

Others

:= or =:	Definition equation. The symbol on the colon's side is defined by the expression on the other side of the equation
$= const$	The expression preceding the equation sign is constant (invariant) with respect to a given variable
$\Rightarrow$	From this follows …
@	The condition following this symbol applies to the equation preceding it
$O(\varepsilon^n)$	Landau notation (a.k.a. Big O notation): $O(\varepsilon^n)$ is the magnitude ($\underline{\text{o}}$rder) of the residual power (here ε^n) of a power series expansion. $O(\varepsilon^n)$ means: The residual is of order ε^n
$n!$	Factorial of the nonnegative integer n, $n! = \prod_{k=1}^{n} k$, $0! = 1$
$(2n-1)!!$	Double factorial of the odd positive integer $i = 2k-1$, $(2n-1)!! = \prod_{k=1}^{n} (2k-1)$
[…]	Square brakes denote the units of a given physical quantity
[a, b]	Closed interval between numbers a and b
$\angle(\boldsymbol{a}, \boldsymbol{b})$	Angle between vector **a** and vector **b**
$\langle \boldsymbol{y} \rangle_x$	Average of y with respect to x over interval [a, b], $\langle y \rangle_x := \frac{1}{b-a} \int_a^b y \cdot dx$
$\langle y \rangle$	Time average, $\langle y \rangle \equiv \langle y \rangle_t$

$\bullet$	Inner orbit (relative to another given orbit)
$\bigcirc$	Outer orbit (relative to another given orbit)
$\blacksquare$	What was to be shown (quod erat demonstrandum)
Υ	First point of Aries, a.k.a. vernal point (see Sect. 13.1)
$\uparrow$	Increasing
$\uparrow\uparrow$	Strongly increasing
$\downarrow$	Decreasing
$\downarrow\downarrow$	Strongly decreasing

Chapter 1
Rocket Fundamentals

1.1 Rocket Principles

1.1.1 Repulsion Principle

Many people have had and still have misconceptions about the basic principle of rocket propulsion. Here is a comment of an unknown editorial writer of the renowned *New York Times* from January 13, 1920, about the pioneer of US astronautics, Robert Goddard, who at that time was carrying out the first experiments with liquid propulsion engines:

> Professor Goddard … does not know the relation of action to reaction, and of the need to have something better than a vacuum against which to react – to say that would be absurd. Of course he only seems to lack the knowledge ladled out daily in high schools.

The publisher's doubt whether rocket propulsion in the vacuum could work is based on our daily experience that you can only move forward by pushing backward against an object or medium. Rowing is based on the same principle. You use the blades of the oars to push against the water. But this example already shows that the medium you push against, which is water, does not have to be at rest, it may move backward. So basically it would suffice to fill a blade with water and push against it by very quickly guiding the water backward with the movement of the oars. Of course, the forward thrust of the boat gained hereby is much lower compared with rowing with the oars in the water, as the large displacement resistance in the water means that you push against a far bigger mass of water. But the principle is the same. Instead of pushing water backward with a blade, you could also use a pile of stones in the rear of your boat, and hurl them backward as fast as possible. With this, you would push ahead against the accelerating stone. And this is the basis of the propulsion principle of a rocket: it pushes against the gases it ejects backward with full brunt. So, with the propellant, the rocket carries the mass, against which it pushes to move forwards, and this is why it also works in vacuum. This repulsion principle is called the "rocket principle" in astronautics and is utilized in the classical rocket engine.

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The repulsion principle is based on the physical principle of conservation of momentum. It states that the total (linear) momentum of a closed system remains constant with time. So, if at initial time t_0 the boat (rocket) with mass m_1 plus stone (propellant) with mass m_2 had velocity $\boldsymbol{v}_0$, implying that the initial total momentum was $\boldsymbol{p}(t_0) = (m_1 + m_2)\boldsymbol{v}_0$, this must remain the same at some time $t_+ > t_0$ when the stone is hurled away with velocity $\boldsymbol{v}_2$, the boat has velocity $\boldsymbol{v}_1$ (neglecting water friction), and the total momentum is $\boldsymbol{p}(t_+) = m_1\boldsymbol{v}_1 + m_2\boldsymbol{v}_2$. That is,

$$\boldsymbol{p}(t_0) = \boldsymbol{p}(t_+) \quad \textbf{principle of the conservation of (linear) momentum}$$

from which follows

$$(m_1 + m_2) \cdot \boldsymbol{v}_0 = m_1\boldsymbol{v}_1 + m_2\boldsymbol{v}_2$$

Note *The principle of the conservation of momentum is valid only for the vectorial form of the momentum equation, which is quite often ignored. A bomb that is ignited generates a huge amount of momentum out of nothing, which apparently would invalidate an absolute value form of the momentum equation. But if you add up the vectorial momentums of the bomb's fragments, it becomes obvious that the vectorial momentum has been conserved.*

Given m_1, m_2, $\boldsymbol{v}_0$ and velocity $\boldsymbol{v}_2$ of the stone (propellant) expelled, one is able to calculate from this equation the increased boat (rocket) velocity $\boldsymbol{v}_1$. Doing so, this equation affirms our daily experience that hurling the stone backward increases the speed of the boat, while doing it forward decreases its speed.

With a rocket, the situation is a bit more complicated, as it does not eject one stone after the other, but it emits a continuous stream of tiny mass particles (typically molecules). In order to describe the gain of rocket speed by the continuous mass ejection stream adequately in mathematical and physical terms, we have to consider the ejected mass and time steps as infinitesimally small and in an external rest frame, a so-called inertial (unaccelerated, see Sect. 13.1) reference frame. This is depicted in Fig. 1.1 where in an inertial reference frame with its origin at the center of the Earth a rocket with mass m in space experiences no external forces.

At a given time t, the rocket may have velocity $\boldsymbol{v}$ and momentum $\boldsymbol{p}(t) = m\boldsymbol{v}$. By ejecting the propellant mass $dm_p > 0$ with exhaust velocity $\boldsymbol{v}_{ex}$ and hence with momentum $\boldsymbol{p}_p(t + dt) = (\boldsymbol{v} + \boldsymbol{v}_{ex}) \cdot dm_p$, it will lose part of its mass $dm = -dm_p < 0$ and hence gain rocket speed $d\boldsymbol{v}$ by acquiring momentum $\boldsymbol{p}_r(t + dt) = (m + dm)(\boldsymbol{v} + d\boldsymbol{v})$.

Note *In literature, $dm > 0$ often denotes the positive mass flow rate of the propellant, and m the mass of the rocket. This is inconsistent, and leads to an erroneous mathematical description of the relationships, because if m is the mass of the rocket, logically dm has to be the mass change of the rocket, and thus it has to be negative. This is why in this book, we will always discriminate between rocket mass m and propulsion mass m_p using the consistent description $dm = -dm_p < 0$ implying $\dot{m} = -\dot{m}_p < 0$ for their flows.*

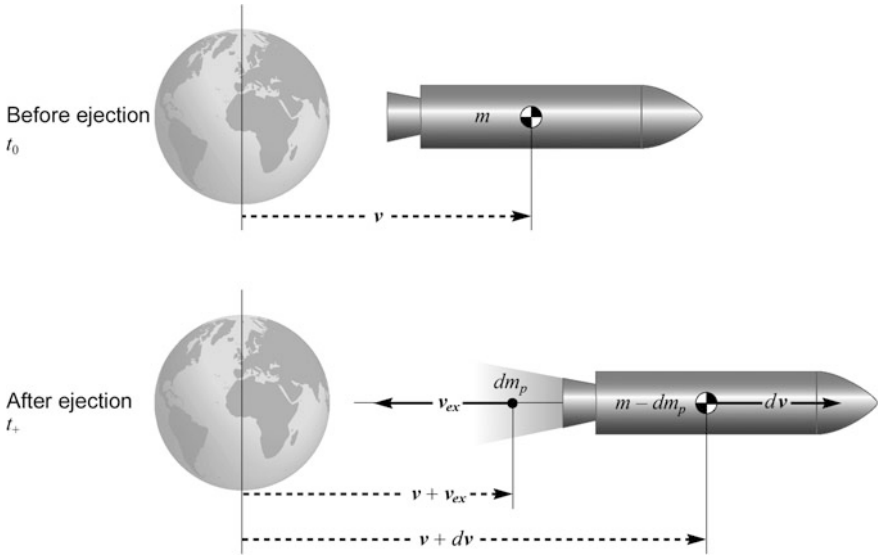


Fig. 1.1 A rocket in force-free space before (above) and after (below) it ejected a mass dm_p with exhaust velocity $\boldsymbol{v}_{ex}$, thereby gaining speed $d\boldsymbol{v}$. Velocities relative to the external inertial reference frame (Earth) are dashed and those with regard to the rocket are solid

For this line of events, we can apply the principle of conservation of momentum as follows

$$\boldsymbol{p}(t) = \boldsymbol{p}(t+dt) = \boldsymbol{p}_p(t+dt) + \boldsymbol{p}_r(t+dt)$$

From this follows,

$$m\boldsymbol{v} = -dm(\boldsymbol{v}+\boldsymbol{v}_{ex}) + (m+dm)(\boldsymbol{v}+d\boldsymbol{v}) = m\boldsymbol{v} - dm\cdot\boldsymbol{v}_{ex} + m\cdot d\boldsymbol{v} + dm\cdot d\boldsymbol{v}$$

As the double differential $dm \cdot d\boldsymbol{v}$ mathematically vanishes with respect to the single differentials dm and $d\boldsymbol{v}$, we get with division by dt

$$m\dot{\boldsymbol{v}} = \dot{m}\boldsymbol{v}_{ex}$$

According to Newton's second law (Eq. (7.1.12)), $\boldsymbol{F} = m\dot{\boldsymbol{v}}$, the term on the left side corresponds to a force, called *momentum thrust force*, due to the repulsion of the propellant, which we correspondingly indicate by

$$\boldsymbol{F}_{ex} = \dot{m}\boldsymbol{v}_{ex} \tag{1.1.1}$$

Remark *This equation can alternatively be derived from the fact that the momentum of the expelled propellant mass is* $d\boldsymbol{p}_p = dm_p\boldsymbol{v}_{ex}$. *The equivalent force according to Newton's second law (see Eq. (7.1.12)) is* $\boldsymbol{F}_p = d\boldsymbol{p}_p/dt = \dot{m}_p\boldsymbol{v}_{ex}$. *This in turn causes a reaction force (Newton's third law Eq. (7.1.11)) on the rocket of* $\boldsymbol{F}_{ex} = -\boldsymbol{F}_p = -\dot{m}_p\boldsymbol{v}_{ex} = \dot{m}\boldsymbol{v}_{ex}$. *Although this derivation is more elegant, we retain the conservation of linear momentum approach in the main text because it nicely expounds the physics behind the propulsion—the repulsion principle.*

This means that the thrust of a rocket is determined by the product of propellant mass flow rate and exhaust velocity. Observe that due to $\dot{m} = -\dot{m}_p < 0$, $\boldsymbol{F}_{ex}$ is exactly in opposite direction to $\boldsymbol{v}_{ex}$ (but depending on the steering angle of the engine, $\boldsymbol{v}_{ex}$ and hence $\boldsymbol{F}_{ex}$ do not necessarily have to be in line with the flight direction $\boldsymbol{v}$). Therefore, with regard to absolute values, we can write

$$F_{ex} = -\dot{m}v_{ex} = \dot{m}_p v_{ex} \quad \textbf{momentum thrust} \tag{1.1.2}$$

The term *momentum thrust* is well chosen, because if the expression $\dot{m}_p v_{ex}$ is integrated with regard to time, one obtains the momentum $m_p v_{ex}$, which is merely the recoil momentum of the ejected propellant.

1.1.2 Total Thrust

In the above, we have considered a simple propulsion mechanism, namely, repulsion from expelled mass. As we will see later there exist other physical effects, such as gas pressure for jet engines (Sect. 1.3) or relativistic effects close to the speed of light (Sect. 1.4), which contribute to the thrust. We take all these into account by a corresponding additional thrust term F_+ und thus obtaining the *total thrust* of a reaction engine

$$F_* := F_{ex} + F_+$$

In reference to Eq. (1.1.1) we can formally write for the total thrust force

$$\boldsymbol{F}_* = \dot{m}\boldsymbol{v}_* \quad \textbf{thrust force (total)} \ (\text{a.k.a. propellant force}) \tag{1.1.3}$$

and for its absolute values, i.e. for the total thrust

$$\boxed{F_* = -\dot{m}v_* = \dot{m}_p v_*} \quad \textbf{thrust (total)} \tag{1.1.4}$$

By doing so we have defined the *effective exhaust velocity*

$$v_* := v_{ex} + \frac{F_+}{\dot{m}_p} \quad \textbf{effective exhaust velocity} \tag{1.1.5}$$

We thus can interpret the total thrust F_* as caused by expelling mass at rate $\dot{m}_p$ at an effective exhaust velocity v_*. From this point of view,

> The **effective exhaust velocity** v_* is an effective conversion factor of all physical effects that converts the employed propellant flow $\dot{m}_p$ into total thrust F_*.

As we will see in Sect. 1.3.3, the effective exhaust velocity v_* is identical to and therefore can also be understood as "the achievable total impulse of an engine with respect to a given exhausted propellant mass m_p", called the mass-specific impulse: $v_* = I_{sp}$.

In essence, one can state that for each type of engine, one has to investigate what the thrust-generating mechanisms are, how they act, and, by writing its total thrust in the form $F_* = \dot{m}_p v_*$, determine what the effective exhaust velocity of that engine is.

Equations (1.1.3) or (1.1.4), respectively, is of vital importance for astronautics, as it describes basic physical facts, just like every other physical relationship, relating just three parameters, such as $W = F \cdot s$ or $U = R \cdot I$. This is its statement: thrust is the product of effective exhaust velocity times mass flow rate. Only both properties together make up a powerful thruster. The crux of the propellant is not its "energy content" (actually, the energy to accelerate the propellant might be provided externally, which is the case with ion propulsions), but the fact that it possesses mass, which is ejected backward and thus accelerates the rocket forward by means of conservation of momentum. The higher the mass flow rate, the larger the thrust. If "a lot of thrust" is an issue, for instance, during launch, when the thrust has to overcome the pull of the Earth's gravity, and since the exhaust speed of engines is limited, you need thrusters with a huge mass flow rate. The more the better. Each of the five first stage engines of a Saturn V rocket had a mass flow rate of about 2.5 metric tons per second, in total 12.5 tons per second, to achieve the required thrust of 33,000 kN (corresponds to 3400 tons of thrust). This tremendous mass flow rate is exactly why, for launch, chemical thrusters are matchless up to now, and they will certainly continue to be so for quite some time.

1.1.3 Equation of Rocket Motion

Knowing the thrust of the rocket, we now wonder what the trajectory of a powered rocket looks like. To determine it, we have to account not only for the thrust but also for all possible external forces. They are typically summarized to one external force F_{ext}:

$$\boldsymbol{F}_{ext} := \boldsymbol{F}_G + \boldsymbol{F}_D + \boldsymbol{F}_L + \ldots \tag{1.1.6}$$

with

$\boldsymbol{F}_G$ Gravitational force
$\boldsymbol{F}_D$ Aerodynamic drag
$\boldsymbol{F}_L$ Aerodynamic lift

For each of these external forces a virtual point within the rocket can be assumed the external force effectively acts on (Fig. 1.2). This point has a unique location with regard to the geometry of the rocket, and it is in general different for every type of forces. For instance, the masses of the rocket can be treated as lumped together in the center of mass where the gravitational force applies; the aerodynamic drag and lift forces effectively impact the spacecraft at the so-called center of pressure; and possible magnetic fields have still another imaginary point of impact. If the latter do not coincide with the center of mass, which in general is the case, the distance in between results in torques due to the inertial forces acting effectively at the center of mass. In this textbook, we disregard the resulting complex rotational movements, and we just assume that all the points of impact coincide with the center of mass or, alternatively, that the torques are compensated by thrusters.

Newton's second law, Eq. (7.1.12), gives us an answer to the question of how the rocket will move under the influence of all non-inertial forces $\boldsymbol{F}_i$, including the rocket's thrust $\boldsymbol{F}_*$:

$$m\dot{\boldsymbol{v}} = \sum_{all\ i} \boldsymbol{F}_i$$

We therefore find the following equation of motion for the rocket

$$m\dot{\boldsymbol{v}} = \boldsymbol{F}_* + \boldsymbol{F}_{ext}$$

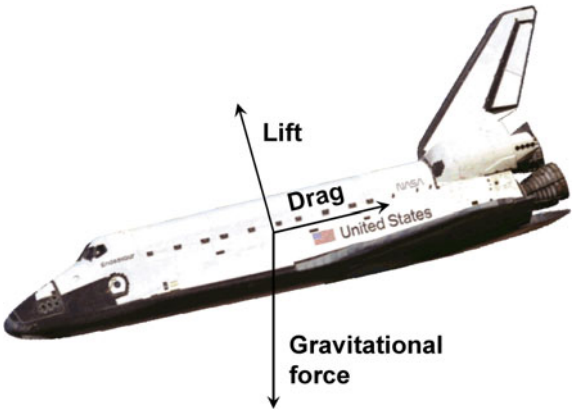


Fig. 1.2 External forces acting on a Space Shuttle upon reentry

and with Eq. (1.1.4), we finally obtain

$$\boxed{m\dot{\boldsymbol{v}} = \dot{m}\boldsymbol{v}_* + \boldsymbol{F}_{ext}} \quad \textbf{equation of rocket motion} \tag{1.1.7}$$

This is the key differential equation for the motion of the rocket. In principle, the speed can be obtained by a single integration step and its position by a double integration. Note that this equation not only applies to rockets but also to any type of spacecraft during ascent flight, reentry, or when flying in space with or without propulsion.

1.2 Jet Engine

Any propulsion system that acts according to the repulsion principle provides thrust by expelling reaction mass is called a *reaction engine*. This definition includes not only the classical jet engines, such as the thermal jet engine (see Chap. 4), resistojets, or arcjets working with neutral gases, but also engines that work with "ion gases", i.e., plasma, such as ion thrusters (see Chap. 5) or Hall effect thrusters, where the ions interact via the Coulomb interaction and therefore also create pressure. A *rocket engine* is a reaction engine that merely stores all the propellant in the rocket.

A jet engine is a reaction engine that generates thrust by discharging a gas jet at high speed. Gas is a loose accumulation of molecules, which at high speed move around and collide with each other and with the volume boundary thus generating pressure. The gas pressure in the jet is highly specific for jet engines and determines their performance, because on one hand it creates an additional thrust component called *pressure thrust*, but on the other hand also causes thrust losses owing to nozzle divergence. Both effects are considered in this section.

1.2.1 Nozzle Divergence

An exhaust jet that is everywhere parallel to the average thrust direction, as assumed in Sect. 1.1.1, is an ideal situation. In practice there is jet spraying, i.e., depending on the nozzle shape and on internal gas dynamics we have diverging components of the otherwise axisymmetric gas flow. We account for that by expressing the differential gas mass as a conical outflow shell with width $d\theta$ as

$$d\dot{m}_p = \dot{m}_p \mu(\theta) \sin\,\theta \cdot d\theta$$

where $\mu(\theta)$ is the dimensionless and axisymmetric **angular mass-flow distribution function**, and θ is the cone shell half angle, which is half of the aperture angle of the conical shell, as measured against its centerline. Any engine property x that depends on jet spraying then needs to be evaluated in terms of this mass flow angle distribution

$$\langle x \rangle_\mu := \frac{\int_0^{\pi/2} x(\theta) \cdot \mu(\theta) \sin\,\theta \cdot d\theta}{\int_0^{\pi/2} \mu(\theta) \sin\,\theta \cdot d\theta} \tag{1.2.1}$$

Take the momentum thrust as an example. Let $v_e(\theta)$ be the **ejection velocity** (a.k.a. velocity distribution function, VDF) of the jet mass flow at ejection angle θ. The contribution of this mass flow to the momentum thrust is the **ejection thrust** $F_e(\theta) = \dot{m}_p v_e(\theta)$ projected onto the centerline, $F_{e\|}(\theta) = \dot{m}_p v_e(\theta)\cos(\theta)$. For the total momentum thrust it then follows from Eq. (1.2.1)

$$\begin{aligned} F_{ex} &= \dot{m}_p \langle v_e \cos\,\theta \rangle_\mu \\ &=: \eta_{div} \dot{m}_p \bar{v}_e =: \eta_{div} \bar{F}_e \end{aligned} \quad \textbf{momentum thrust} \tag{1.2.2}$$

with

$$\bar{v}_e := \langle v_e \rangle_\mu \quad \textbf{mean ejection velocity} \tag{1.2.3}$$

$$\bar{F}_e = \dot{m}_p \bar{v}_e \quad \textbf{mean ejection thrust} \tag{1.2.4}$$

$$\eta_{div} = \frac{\langle v_e \cos\,\theta \rangle_\mu}{\langle v_e \rangle_\mu} \leq 1 \quad \textbf{nozzle-divergence loss factor} \tag{1.2.5}$$

From this and Eq. (1.1.2), $F_{ex} = \dot{m}_p v_{ex}$, follows that

$$v_{ex} = \eta_{div} \bar{v}_e \quad \textbf{exhaust velocity} \tag{1.2.6}$$

From this we see that the nozzle-divergence loss factor is an important figure of merit of a jet engine because it affects the engine's efficiency via a reduced exhaust velocity.

Example

What is the nozzle-divergence loss factor for a common conical nozzle with cone half angle α and $v_e(\theta) = const = v_e \equiv \bar{v}_e$?

Having

$$\mu(\theta) = \begin{cases} 1 & @ \quad \theta < \alpha \\ 0 & @ \quad \theta \geq \alpha \end{cases}$$

we find according to Eq. (1.2.1)

$$\langle v_e(\theta) \cos(\theta) \rangle_\mu = v_e \frac{\int_0^\alpha \cos\theta \, \sin\,\theta \cdot d\theta}{\int_0^\alpha \sin\,\theta \cdot d\theta} = \frac{v_e}{2} \frac{1 - \cos^2\alpha}{1 - \cos\alpha}$$

We hence obtain with Eq. (1.2.5) and $v_e = \langle v_e \rangle_\mu$

$$\eta_{div} = \frac{1 + \cos\alpha}{2} \quad @ \text{ conical nozzle} \tag{1.2.7}$$

Typical divergence cone half angles in use are $\alpha = 12°–18°$, with corresponding loss factors of $\eta_{div} = 0.989–0.975$. So, these losses amount to 1.1–2.5%.

In practice an ideal contoured Rao nozzle (see Sect. 4.4.2) is the most common. Its conical exit wall angle is typically $\theta = 7°–12°$. The corresponding conical exhaust plume therefore causes a loss of $\eta_{div} = 0.989–0.996$. So, exhaust losses due to the Rao nozzle contour do not exceed 1% and are therefore a small contribution to the total thrust losses of typically 2–8% mainly due to shock formation in the nozzle and boundary layer losses owing to friction with the nozzle wall.

1.2.2 Pressure Thrust

The working fluid of a jet engine is gas. While the jet engine has an internal gas pressure, there might also exist an external gas pressure from the gas molecules of the atmosphere surrounding a rocket during ascent. In order to understand the impact of the propellant gas pressure and external ambient pressure on the engine's thrust, let us first have a look at the general pressure and flow conditions in a typical jet engine depicted in Fig. 1.3 by a thrust chamber of a rocket engine.

Continuity Equation

Let us have a look at the general propellant gas flow in a rocket engine. A propellant mass dm_p perfuses a given engine cross section of area A with velocity v (see Fig. 1.4). During the time interval dt, it will have passed through the volume $dV = A \cdot ds = Av \cdot dt$. Therefore,

$$dm_p = \rho \cdot dV = \rho A v \cdot dt$$

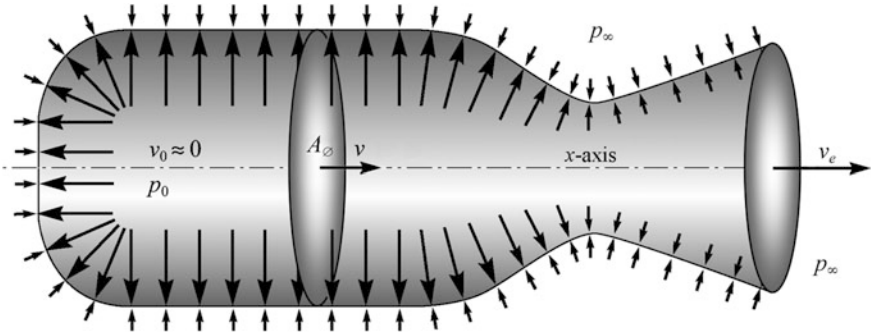


Fig. 1.3 Pressure and velocity conditions inside and outside a thrust chamber

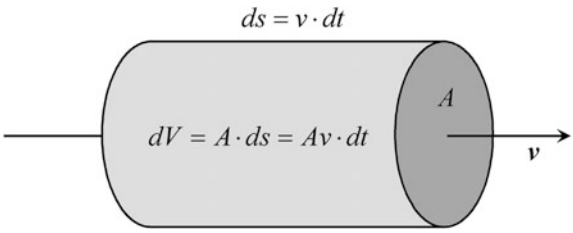


Fig. 1.4 The volume dV that a mass flow with velocity v passes in time dt

where ρ is the mass density. As the number of molecules that enter and exit this volume, is preserved, we derive for the mass flow rate the equation

$$\boxed{\dot{m}_p = \rho v A = const} \quad \textbf{continuity equation} \tag{1.2.8}$$

The continuity equation is a direct outcome of the transport of mass particles as a conserved quantity, which expresses the fact that the number of mass particles cannot increase or decrease, but can only move from place to place. This is exactly what the word "continuity" means.

Pressure Thrust

Jet thrust is generated by gas pressure. To see how, we denote by p the varying engine pressure acting from inside on the wall and exerting the force $d\boldsymbol{F} = p \cdot d\boldsymbol{A}$ on a wall segment $d\boldsymbol{A}$, which points outward. In the area surrounding the chamber (here "chamber" is the abbreviation for thrust chamber, i.e. the engine's casing including the nozzle, if existent) we assume a constant external pressure p_∞. Quite generally, the total propellant force $\boldsymbol{F}_*$ generated by the chamber must be the sum of all effective forces acting on the entire chamber wall having surface S_C:

$$\boldsymbol{F}_* = \iint_{S_C} d\boldsymbol{F} = \iint_{S_C} (p - p_\infty) \cdot d\boldsymbol{A}$$

Of course, no force is acting at the imaginary exit surface. Therefore S_C does not include the flat exit area A_e. By denoting the closed surface $S_O = S_C \cup A_e$ we can rewrite the above equation as

$$\boldsymbol{F}_* = \oiint_{S_O} (p - p_\infty) \cdot d\boldsymbol{A} - (p_e - p_\infty)\boldsymbol{A}_e \tag{1.2.9}$$

The second term is called *pressure thrust force* $\boldsymbol{F}_p$.

$$\boldsymbol{F}_p = -(p_e - p_\infty)\boldsymbol{A}_e \quad \textbf{pressure thrust force} \tag{1.2.10}$$

We will discuss its properties in a moment after having evaluated the first term in Eq. (1.2.9).

Emergence of Momentum Thrust

We now assume an axisymmetric thrust chamber, which generally is the case, with an according axially symmetric gas flow along the chamber axis. This implies that we can treat the hydrodynamics along the chamber axis, which we denote by $\boldsymbol{u}_x$ and which points from the front of the chamber to the exit, as one-dimensional. The propellant gas enters the chamber at the front side with velocity $v_0 \approx 0$ and by arbitrary means is accelerated along the chamber axis to v_e at the exit; therefore $\boldsymbol{u}_x = \hat{\boldsymbol{v}} = \hat{\boldsymbol{A}}_e =: \boldsymbol{u}_e$. Overall, the gas is accelerated, although there may be times when it is decelerated along its path through the chamber. The axial symmetry reduces the first term in Eq. (1.2.9) to a force on the effective front side and a reversed force on the effective and imaginary closed rear side, i.e.,

$$\oiint_{S_O} (p - p_\infty) \cdot d\boldsymbol{A} = -[(p_0 - p_\infty) - (p_e - p_\infty)]A_\phi \boldsymbol{u}_x = (p_e - p_0)A_\phi \boldsymbol{u}_x \quad (1.2.11)$$

where A_ϕ is the chamber cross-section. Note that at the front side where $p = p_0$ we have $d\hat{\boldsymbol{A}} = -\boldsymbol{u}_x$, which causes the negative sign. According to hydrodynamics, an accelerated flow is intimately connected via the mass density ρ to a pressure gradient as $\nabla p = -\rho \cdot d\boldsymbol{v}/dt$. This is the so-called Euler equation, which reads in our one-dimensional case

$$\frac{dp}{dx} = -\rho \cdot \frac{dv_x}{dt} \quad \textbf{Euler equation}$$

So, the gas pressure decreases with increasing gas velocity. This seemingly paradoxical effect is called the *Bernoulli effect*. After separating the variables dp and dx and then integrating this equation we have with the continuity equation $\dot{m}_p = const$ (Eq. (1.2.8)) and because $v_0 \approx 0$

$$\int_{p_0}^{p} dp = p - p_0 = -\int_0^x \frac{dm_p}{A_\phi dx}\frac{dv_x}{dt}dx = -\frac{1}{A_\phi}\int_{v_0}^{v}\frac{dm_p}{dt}dv_x = -\frac{\dot{m}_p}{A_\phi}v$$

Therefore, at any point along the chamber axis we have

$$(p - p_0)A_\phi = -\dot{m}_p v$$

Applying this result to the exit and inserting it into Eq. (1.2.11) and taking into account also nozzle divergence (see Sect. 1.2.1) we obtain the thrust component

$$\boldsymbol{F}_{ex} = \oiint_{S_O} (p - p_\infty) \cdot d\boldsymbol{A} = -\dot{m}_p \boldsymbol{v}_{ex} \quad \textbf{momentum thrust force} \quad (1.2.12)$$

which is the momentum thrust. It is remarkable that we have recovered Eq. (1.1.1) on hydrodynamic grounds rather than on first principles.

Total Thrust
Inserting the above results into Eq. (1.2.9) we obtain for the total thrust

$$\boldsymbol{F}_* = -\dot{m}_p \boldsymbol{v}_{ex} - (p_e - p_\infty) A_e \cdot \boldsymbol{u}_e \tag{1.2.13}$$

where $\boldsymbol{u}_e$ is the unit vector of the exit surface in the direction of the exhaust jet. Because the thrust is antiparallel to the exit flow we finally have for the absolute value of the thrust

$$\boxed{F_* = \dot{m}_p v_{ex} + (p_e - p_\infty) A_e} \tag{1.2.14}$$

Hence, the total thrust is the sum of the momentum thrust and the pressure thrust. The wording "pressure thrust", on one hand, is conclusive because it originates from the very special fact that the rocket engine works with gases that produce pressure. On the other hand, and as according to Eq. (1.2.12), the exhaust and momentum thrust is also generated by a pressure on the chamber because of its internal pressure gradient. In the end, it is solely pressure that accelerates the gas engine, and with it the rocket.

Effective Exhaust Velocity
If we apply Eq. (1.2.14) to the definition of the effective exhaust velocity as given by Eq. (1.1.5) with $F_+ = F_p = (p_e - p_\infty) A_e$ and keeping in mind Eq. (1.2.6), we get

$$v_* = v_{ex} + (p_e - p_\infty) \frac{A_e}{\dot{m}_p} \tag{1.2.15}$$

The expression "effective exhaust velocity" makes it clear that thrust is essentially caused by the exhaust velocity $v_{ex} = \eta_{div} \bar{v}_e$ modified by a pressure-thrust-equivalence exhaust velocity term. Indeed, as we will see from Eq. (1.2.18) the pressure thrust for a real engine chamber is only a small contribution. For an ideally adapted nozzle with $p_e = p_\infty$ (see Sect. 4.2.1) it even vanishes.

1.2.3 Momentum versus Pressure Thrust

Ultimately, if it is only pressure that drives a rocket engine, how does this fit together with the rocket principle discussed in Sect. 1.1.1, which was based on repulsion and not on pressure? And what is the physical meaning of pressure thrust? You often find the statement that pressure thrust occurs when the pressure at the exit (be it nozzle exit or combustion chamber exit) hits the external pressure. The pressure difference at this point times the surface is supposed to be the pressure thrust. Though the result is right, the explanation is not. First, the exit pressure does

not abruptly meet the external pressure. When the exhaust gas hits and merges with the ambient atmospheric gas there is rather a smooth pressure transition from the exit pressure to the external pressure covering in principle an infinite volume behind the engine. In the course of this process the pressure difference and with it the abstract force $(p_e - p_\infty)A_\phi$ is irreversibly lost. Second, even if such a pressure difference could be traced back mathematically to a specific surface, this would not cause a thrust, because, as we will see later, the gas in the nozzle expands backward with supersonic speed, and such a gas cannot have a causal effect on the engine to exert a thrust on it.

Momentum Thrust

For a true explanation, let us imagine for a moment and purely hypothetically, a fully closed and idealized rectangular thrust chamber (see Fig. 1.5) with the same pressure and flow conditions as in the real thrust chamber. The surface force on the front side would be $F_{front} = (p_0 - p_\infty)A_\phi$, and $F_{rear} = (p_e - p_\infty)A_\phi$ on the rear side. Hence, the net forward thrust would be $F_{ex} = F_{front} - F_{rear} = (p_0 - p_e) \cdot A_\phi$. Owing to the Bernoulli effect, this translates into $F_{ex} = \dot{m}_p v_{ex}$. Therefore, we can say the following,

> The **momentum thrust** F_{ex} physically results from the fact that, in a hypothetically closed thrust chamber, the Bernoulli effect of the mass flow causes a bigger chamber pressure on the front side than on the back side, with the net pressure force $F_{ex} = \dot{m}_p v_{ex}$.

The momentum thrust can also be described in a different mathematical form. If we apply the continuity equation (1.2.8) to the exit of the engine we obtain $\dot{m}_p = \rho_e \bar{v}_e A_e$. Inserting this into $F_{ex} = \dot{m}_p v_{ex}$ yields

$$F_{ex} = \dot{m}_p v_{ex} = \eta_{div} \rho_e A_e \bar{v}_e^2 \tag{1.2.16}$$

This equation begs the question whether the momentum thrust of a rocket engine is linearly or quadratically dependent on $\bar{v}_e$. The answer depends on the engine in question. Depending on the engine type (e.g., electric or chemical engine), a change of its design in general will vary all parameters v_e and $\dot{m}_p, \rho_e, A_e$ in a specific way.

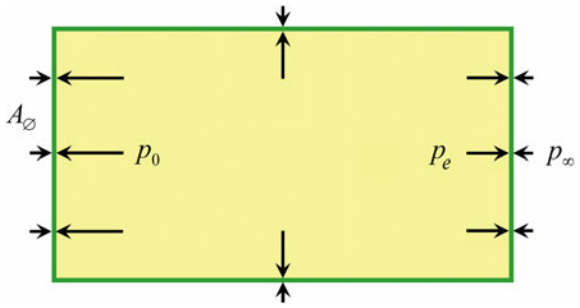


Fig. 1.5 Pressure conditions of the idealized rectangle thrust chamber if it would be, hypothetically, fully closed

This is why the demanding goal of engine design is to tune all engine parameters, including $\bar{v}_e$, such that the total thrust is maximized. Hence, it is not only $\bar{v}_e$ alone, which is decisive for the momentum thrust of a rocket engine but also it is necessary to adjust all relevant engine parameters in a coordinated way.

Pressure Thrust and Its Significance

In order to have the hypothetical gas flow indeed flowing, we need to make a hole with area A_e into the rear side of the hypothetically closed thrust chamber of Fig. 1.5. Once this is done, the counterthrust at the rear side decreases by $\Delta F_{rear} = -(p_e - p_\infty)A_e$, which in turn increases the total thrust by the same amount. This contribution is the pressure thrust. Therefore,

> The **pressure thrust** F_p is the additional thrust that originates from the absence of the counter pressure force at the exit opening of the engine.

If the exit pressure happens to be equal to the external pressure, then the external pressure behaves like a wall, the pressure thrust vanishes, and we have an ideally adapted nozzle (see Sect. 4.2.1).

Given Eq. (1.2.16) we are able to qualitatively derive the significance of pressure thrust. We do so by rating it against the momentum thrust. Because $F_p = (p_e - p_\infty)A_e \leq p_e A_e = F_p\big|_\infty$ we have

$$\frac{F_p}{F_{ex}} \approx \frac{F_p}{F_e} \leq \frac{F_p}{F_e}\bigg|_\infty = \frac{p_e A_e}{\rho_e A_e \bar{v}_e^2} = \frac{\rho_e R T_e / M_p}{\rho_e \bar{v}_e^2} \tag{1.2.17}$$

For jet engines one can make use of various results of Sect. 4.1, namely Eq. (4.1.6); the ideal gas law Eq. (4.1.1) with the universal gas constant R, molar propellant mass M_p, average number of excited degrees of freedom of the gas molecules, $n \approx 8$; and Eq. (4.1.3), which defines the thermal efficiency at the exit η_e. This leads to

$$\frac{F_p}{F_e} \leq \frac{R}{2c_p M_p}\frac{1-\eta_e}{\eta_e} = \frac{1}{n+2}\left(\frac{1}{\eta_e} - 1\right)$$

With $n \approx 8$ and $\eta_e = 0.5 - 1.0$ for jet engines we finally have

$$\frac{F_p}{F_{ex}} \leq \frac{1}{10} \cdot \left(\frac{1}{\eta_e} - 1\right) \approx 0 - 10\% \quad @ \text{ jet engines} \tag{1.2.18}$$

For a so-*called ideally adapted nozzle* where $p_e = p_\infty$ (see Sect. 4.2.1) then of course $F_p = 0$. Because the exhaust temperature T_e generally decreases with increasing v_e (cf. Eq. (4.1.6) for their general relation) we see that the pressure thrust becomes rapidly less important with increasing ejection velocity. This will be particularly important for ion thrusters with exhaust velocity 10 times larger than for thermal engines.

1.3 Rocket Performance

In this section, we define and examine those parameters that characterize a rocket, in particular its figures of merit.

1.3.1 Payload Considerations

When looking at Eq. (2.2.3), at first glance one might think that the burnout final mass is identical to the payload mass, $m = m_f = m_L$. That would mean that if you only choose the launch mass big enough one would be able to get a payload of any size into space. However, this thought discards the structural mass m_s of the rocket, which includes the mass of the outer and inner mechanical structure of the rocket in particular the tank mass, the mass of the propulsion engines including propellant supply (pumps), avionics incl. cable harness, energy support systems, emergency systems, and so on. Structural mass trades directly with payload mass, and hence

$$m_f = m_s + m_L$$

In practice, structural mass limits the payload mass to such a severe extent that later on we will have to look for alternative propulsion concepts, the so-called staging concepts, to reduce m_s. For further considerations, in particular for the later stage optimization, we define the following mass ratios:

$$\mu := \frac{m_f}{m_0} = \frac{m_s + m_L}{m_0} \qquad \textbf{mass ratio} \tag{1.3.1}$$

$$\varepsilon := \frac{m_s}{m_0 - m_L} = \frac{m_s}{m_s + m_p} \qquad \textbf{structural ratio} \tag{1.3.2}$$

$$\lambda := \frac{m_L}{m_0 - m_L} = \frac{m_L}{m_s + m_p} \qquad \textbf{payload ratio} \tag{1.3.3}$$

Observe that for the last two ratios the structural mass and the payload mass are not taken relative to the total mass, but to the total mass reduced by the payload mass. This is done in view of consistency with the equivalent, more general definitions for the upcoming rocket staging (see Eqs. (3.1.3)–(3.1.5)). From the above definitions it follows that

$$\mu(1+\lambda) = \frac{m_s + m_L}{m_0} \frac{m_s + m_p + m_L}{m_s + m_p} = \frac{m_s + m_L}{m_s + m_p} = \varepsilon + \lambda$$

hence

$$\mu = \frac{m_f}{m_0} = \frac{\varepsilon + \lambda}{1 + \lambda} \tag{1.3.4}$$

So the rocket Eq. (2.2.2) can be written as

$$\frac{\Delta v}{v_*} = -\ln \frac{\varepsilon + \lambda}{1 + \lambda}$$

or

$$\lambda = \frac{e^{-\Delta v / v_*} - \varepsilon}{1 - e^{-\Delta v / v_*}} \tag{1.3.5}$$

This equation is represented in Fig. 1.6. It directly relates the payload ratio to the achievable propulsion demand at a given structural ratio of the rocket and effective exhaust velocity of the engine. So, because the structural mass is not negligibly small, it is not possible to achieve any propulsion demands you like. In numbers this says that:

> As $\varepsilon = 0.05$ represents the lower limit of the structural mass of a rocket, the obtainable propulsion demand is limited to $\Delta v < 2.5 \cdot v_*$ at $\lambda = 3\ \%$.

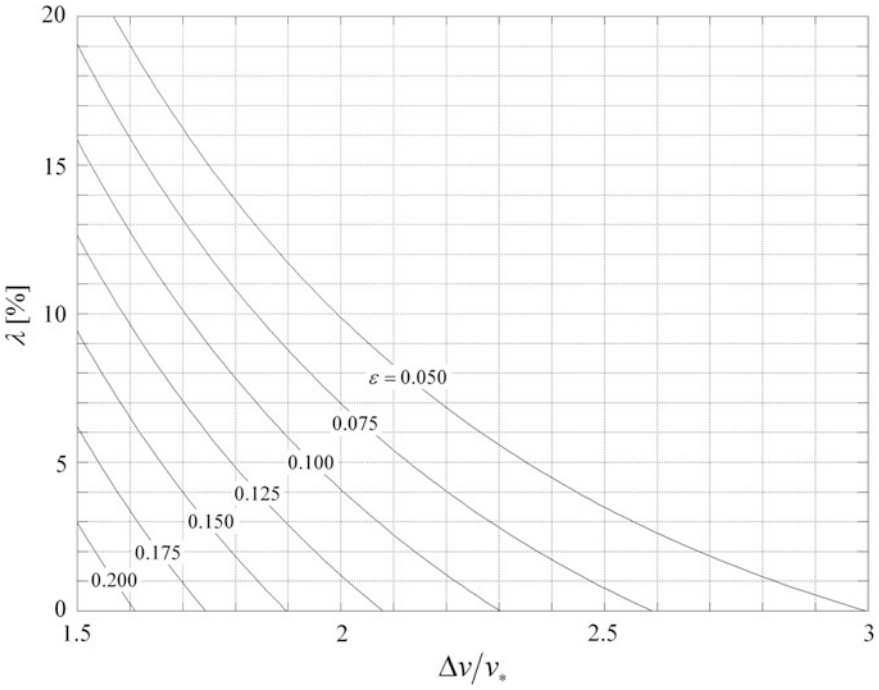


Fig. 1.6 Obtainable payload ratios at a given propulsion demand for different structural ratios

The effective velocity of chemical rockets ascending through Earth's atmosphere is limited to $v_* \leq 4$ km/s, limiting the available propulsion demand to $\Delta v < 10$ km/s. If, for instance, the goal is to get in a single stage with $\varepsilon = 0.1$ into low Earth orbit (so-called SSTO) for which in practice $\Delta v = 9$ km/s is required (see Sect. 6.4.7), then even with an optimal $v_* = 4$ km/s for a LOX/LH2 engine the achievable payload ratio is a mere $\lambda = 0.6\%$. Even if the structural ratio would be a smashing $\varepsilon = 0.075$ we would arrive at only $\lambda = 3.4\%$. So, in principle, a SSTO rocket is possible, but only at the expense of an unacceptable low payload mass. This is why there is no way around a staged rocket to which we come in Chap. 3 (cf. considerations following Eq. (3.3.5)).

1.3.2 Rocket Efficiency

The principle of rocket propulsion is that a certain amount of energy is utilized to accelerate propulsion mass in order to gain rocket speed via repulsion and hence rocket kinetic energy. Of course, it is the goal to design a rocket that from a given amount of spent energy extracts as much kinetic energy as possible. The quantity to measure this is the *total rocket efficiency* η_{tot}. It is defined as

$$\begin{aligned} \eta_{tot} &:= \frac{\textit{gained rocket kinetic energy}}{\textit{utilized energy}} \\ &= \frac{E_{kin}(v_0 + \Delta v) - E_{kin}(v_0)}{E_0} \end{aligned} \qquad \textbf{total rocket efficiency} \qquad (1.3.6)$$

The utilized energy is converted into rocket kinetic energy in two steps. First the engine converts the utilized energy into thrust with internal efficiency η_{int}

$$\eta_{\text{int}} := \frac{\textit{generated thrust energy}}{\textit{utilized energy}} = \frac{\frac{1}{2} m_p v_*^2}{E_0} \qquad \textbf{internal efficiency} \qquad (1.3.7)$$

The internal efficiency (a.k.a. total engine efficiency η_{tot}, see also Sect. 1.3.3) is independent from the motion state of the rocket. It is therefore characteristic for an engine and has to be evaluated separately for different kinds of engines (see for instance Eq. (4.2.7)). In a second (propulsion) step the thrust energy is converted into kinetic energy of the rocket based on the conservation of momentum. The efficiency of this second conversion step is called *external efficiency*—a.k.a. *integral* or *mechanical efficiency*—of a rocket. It is defined as

$$\begin{aligned} \eta_{ext} &:= \frac{\textit{gained rocket kinetic energy}}{\textit{generated thrust energy}} \\ &= \frac{\frac{1}{2} m_f (v_0 + \Delta v)^2 - \frac{1}{2} m_0 v_0^2}{\frac{1}{2} m_p v_*^2} \end{aligned} \qquad \textbf{external efficiency} \qquad (1.3.8)$$

In total, we have

$$\eta_{tot} = \eta_{ext} \cdot \eta_{\text{int}} \tag{1.3.9}$$

Let us have a closer look at the external efficiency. The key point is that velocity is a property relative to a reference frame. Velocity and hence kinetic energy changes when a different reference frame is assumed. Although never mentioned in literature explicitly, the reference frame assumed here is the one in which the rocket had zero velocity at the beginning of the propulsion phase, $v_0 = 0$. Applying this condition and Eq. (2.0.1) to Eq. (1.3.8) yields

$$\eta_{ext} = \frac{m_f \Delta v^2}{m_p v_*^2} = \frac{m_f}{m_0 - m_f} \frac{\Delta v^2}{v_*^2} = \frac{\left(\Delta v / v_*\right)^2}{m_0 / m_f - 1}$$

With Eq. (2.2.3) we finally obtain

$$\eta_{ext} = \frac{\left(\Delta v / v_*\right)^2}{\exp(\Delta v / v_*) - 1} \tag{1.3.10}$$

This function is displayed in Fig. 1.7. It has a maximum at $\Delta v / \Delta v = 1.59362\ldots$, which, according to Eq. (2.2.3), corresponds to $m_f / m_0 = 0.203188\ldots$. From this it is sometimes inferred that the optimal operating point is around this maximum and an acceptable economic limit usually is reached at about $\Delta v \approx 3 v_*$, when the payload portion is only 5.0%. It is therefore argued that a rocket can be operated efficiently only for $\Delta v < 3 v_*$.

Some words of caution are in place. The argument of external efficiency is pointless for practical considerations. First, because it depends on a reference frame that can be chosen arbitrarily. Second, the objective of a thrust maneuver is to achieve a given delta-v. The kinetic energy gained by the maneuver is irrelevant in

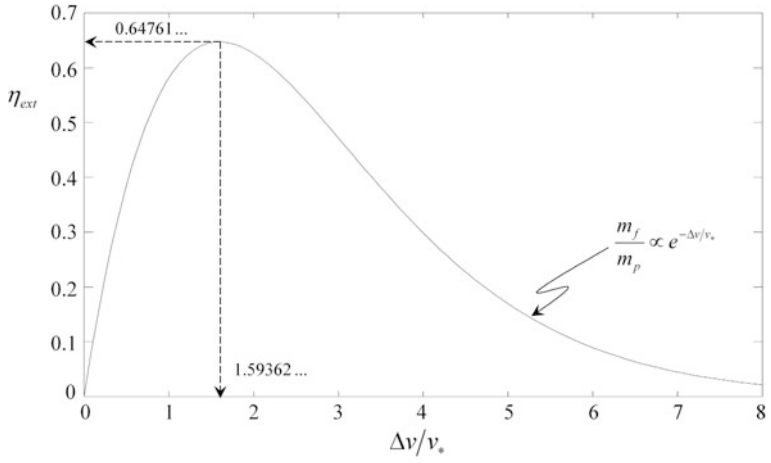


Fig. 1.7 External efficiency of a rocket as a function of the propulsion demand

contrast. The only thing that matters is this: How much propellant is to be expended to achieve a given Δv? The answer is provided by the rocket Eq. (2.2.2) or Eq. (2.2.4), respectively. Only the rocket equation is able to tell whether an impulsive maneuver is efficient or not—apart from the fact that efficiency is a discretionary notion (we could instead define a momentum efficiency). The external energy efficiency therefore is of no practical relevance, which is why it is rarely used. In contrast the internal efficiency is a valuable figure of merit of an engine because it determines $v_* = \sqrt{2E_0\eta_{\text{int}}/m_p}$: the better η_{int} the higher v_* (cf. Sect. 4.2.4) and all the more Δv is achieved.

Transmitted Spacecraft Power
The power transmitted to the spacecraft with velocity v is simply calculated according to classic physics with the product of force times velocity, i.e.,

$$P_{S/C} = F_* \cdot v \qquad \textbf{transmitted spacecraft power} \tag{1.3.11}$$

Note that forces, such as F_*, are independent of the chosen reference system. Therefore, the transmitted spacecraft power is valid both in the rocket system and the external inertial reference frame in which v is measured. However, observe that v depends on the chosen external reference frame.

1.3.3 Performance Parameters

In this section we summarize all those engine figures, which characterize the performance of any rocket engine.

Total and Specific Impulse
The so-called total impulse I_{tot} of an engine is the integral product of total thrust and propulsion duration

$$I_{tot} := \int_0^t F_* dt = v_* \int_0^t \dot{m}_p \cdot dt = m_p v_* \quad @ \quad v_* = const \quad \textbf{total impulse} \tag{1.3.12}$$

The latter is only valid as long as the effective exhaust velocity is constant. This is, in its strict sense, not the case during launch where the external pressure and hence the effective exhaust velocity varies due to the pressure thrust.

The total impulse can be used to define the very important (weight-)specific impulse, which is defined as "*the achievable total impulse of an engine with respect to a given exhausted propellant weight* $m_p g_0$", i.e., with Eq. (1.3.12)

$$\boxed{I_{sp} := \frac{I_{tot}}{m_p g_0} = \frac{v_*}{g_0}} \; @ \; v_* = const \quad (\textbf{weight}-)\textbf{specific impulse} \tag{1.3.13}$$

By this definition the specific impulse has the curious, but simple, dimension "second". Typical values are 300–400 s for chemical propulsion, 300–1500 s for

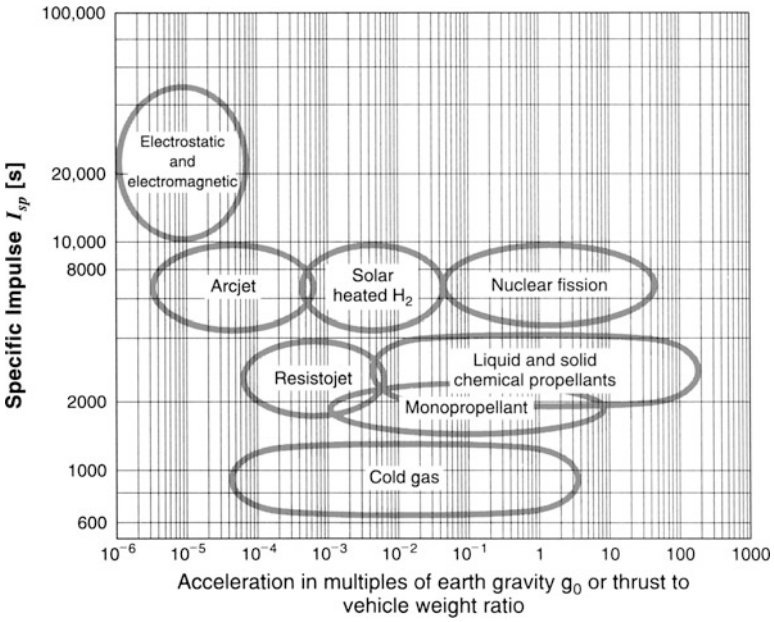


Fig. 1.8 Specific impulse and specific thrust of different propulsion systems. *Credit* Sutton (2001)

electrothermal propulsion (Resistojet, Arcjet), and approximately 2000–6000 s for electrostatic (ion thrusters) and electromagnetic engines (see Fig. 1.8). The specific impulse characterizes the general performance and besides the thrust is therefore a figure of merit of an engine.

In Europe, in particular at ESA, the **mass-specific impulse** with definition "I_{sp} *is the achievable total impulse of an engine with respect to a given exhausted propellant mass* m_p" is more common. This leads to the simple identity $I_{sp} = v_*$. However the definition "I_{sp} = weight-specific impulse" is worldwide more established, which is why we also will use it throughout this book. In either case you should keep in mind that quite generally:

> The specific impulse is an important figure of merit of an engine, and is in essence the effective exhaust velocity.

Jet Power

The mechanical power of an exhaust jet, the so called jet power, is defined as the change of the kinetic energy of the ejected gas (jet energy) per time unit. In other words, P_{jet} describes the time rate of expenditure of the jet energy. With this definition, and averaging over the mass-flow distribution (see Sect. 1.1.2), and from Eq. (1.2.2) we get

$$P_{jet} := \left\langle \frac{dE_{jet}}{dt} \right\rangle_{\mu} = \left\langle \frac{d}{dt}\left(\frac{1}{2} m_p v_e^2\right) \right\rangle_{\mu} \quad \textbf{jet power} \tag{1.3.14}$$
$$= \frac{1}{2}\dot{m}_p \left\langle v_e^2 \right\rangle_{\mu} = \frac{1}{2}\frac{\bar{F}_e \bar{v}_e}{\eta_{VDF}} \geq \frac{1}{2}\bar{F}_e \bar{v}_e$$

Here we have defined the so-called

$$\eta_{VDF} := \frac{\langle v_e \rangle_{\mu}^2}{\left\langle v_e^2 \right\rangle_{\mu}} \leq 1 \quad \textbf{VDF loss factor} \tag{1.3.15}$$

where VDF is the velocity distribution function $v_e(\theta)$ as of Sect. 1.2.1. The VDF loss factor takes account of the power loss owing to the flow velocity in the nozzle to be weighted with the angular distribution of the mass flow. Note that, though the angular mass-flow distribution function $\mu(\theta)$ determines the amount of loss, it is the VDF that causes the loss. Since if $v_e(\theta) = const$ we have $\langle v_e \rangle_{\mu}^2 = \left\langle v_e^2 \right\rangle_{\mu}$. For the proof of $\eta_{VDF} \leq 1$, and hence that η_{VDF} is a true loss factor and not just a correction factor, see Problem 1.3. Note that this loss is due to comparing thrust with jet power. In the first case the flow velocity, in the second case the square of the flow velocity, has to be averaged. This has nothing to do with divergence losses, which are solely accounted for by η_{div}.

Note that forces, such as $\bar{F}_e$, are independent of the chosen reference system, whereas the velocity $\bar{v}_e$ is defined with respect to the rocket. So jet power is a property with respect to the rocket.

In the case where we have uniform ejection velocities, then $\bar{v}_e = v_e$ and $\eta_{VDF} = 1$, and

$$P_{jet} = \frac{1}{2}\dot{m}_p v_e^2 = \frac{1}{2} F_e v_e \quad @\ v_e(\theta) = const = v_e \tag{1.3.16}$$

The latter holds because of Eq. (1.2.2) with $\bar{v}_e = v_e$.

Total Engine Efficiency

With the jet power as defined in Eq. (1.3.14) we can express the dimensionless *total engine efficiency* (a.k.a. *internal efficiency*, see Eq. (1.3.7)) of a real rocket as

$$\eta_{tot} := \frac{E_{jet}}{E_{in}} = \frac{P_{jet}}{P_{in}} = \frac{1}{2}\frac{\dot{m}_p \left\langle v_e^2 \right\rangle_{\mu}}{P_{in}} = \frac{1}{2}\frac{\bar{F}_e \bar{v}_e}{\eta_{VDF} P_{in}} \tag{1.3.17}$$

We now factorize the total efficiency by taking real versus ideal rocket engines (for the definition of an ideal rocket engine see box in Sect. 4.1.1) into consideration and accounting in the following equation for the losses consecutively from right to left starting from the power into the propellant to the ideal jet power

$$\eta_{tot} = \frac{\bar{F}_e \bar{v}_e}{2\eta_{VDF} P_{in}} = \frac{\dot{m}_p \langle v_e \rangle_\mu^2}{2\eta_{VDF} P_{in}} = \frac{\dot{m}_p \langle v_e \rangle_\mu^2}{2\eta_{VDF} P_{jet}} \frac{P_{jet}}{P_{jet,id}} \frac{P_{jet,id}}{P_{in}}$$

where

$$P_{jet} = \frac{1}{2} \dot{m}_p \left\langle v_e^2 \right\rangle_\mu \quad \text{real jet power}$$

$$P_{jet,id} = \frac{1}{2} \dot{m}_{p,id} \left\langle v_{e,id}^2 \right\rangle_\mu \quad \text{ideal jet power}$$

and $\dot{m}_p$ is the total exit mass-flow, while $v_e(\theta)$ is the exit flow velocity into the conical shell with cone half-angle θ for any given real rocket engine. On the other hand $\dot{m}_{p,id}$ and $v_{e,id}(\theta)$ are those for an ideal rocket engine as expounded in Chap. 4 for thermal engines and Chap. 5 for ion engines. We hence obtain

$$\eta_{tot} = \frac{1}{\eta_{VDF}} \frac{\langle v_e \rangle_\mu^2}{\left\langle v_e^2 \right\rangle_\mu} \frac{\dot{m}_p}{\dot{m}_{p,id}} \frac{\left\langle v_e^2 \right\rangle_\mu}{\left\langle v_{e,id}^2 \right\rangle_\mu} \frac{P_{jet,id}}{P_{in}} = \frac{\dot{m}_p}{\dot{m}_{p,id}} \frac{\left\langle v_e^2 \right\rangle_\mu}{\left\langle v_{e,id}^2 \right\rangle_\mu} \frac{P_{jet,id}}{P_{in}}$$

This factorization gives rise to the definition of the following three correction and efficiency factors

$$\zeta_d := \frac{\dot{m}_p}{\dot{m}_{p,id}} = \frac{\dot{m}_p a_0 \sqrt{n}}{A_t C_\infty p_0} = 0.98 - 1.15 \quad \textbf{discharge correction factor} \tag{1.3.18}$$

where the third term follows from Eq. (4.3.2) for a thermal engine. The discharge correction factor relates the total mass flow in the real rocket to that of the ideal rocket. In the real rocket the gas flow experiences friction with the chamber and nozzle walls, which decelerates the flow. On the other hand, chemical reactions in the flow leading to a higher molecular weight, liquid and solid particles in the combustion products, and a lower gas density owing to heat loss and hence cooling of the gas, all these three effects increase the mass flow. In total all these processes usually make up $\zeta_d > 1$.

$$\zeta_v := \sqrt{\frac{\left\langle v_e^2 \right\rangle_\mu}{\left\langle v_{e,id}^2 \right\rangle_\mu}} = 0.85 - 0.98 \quad \textbf{velocity correction factor} \tag{1.3.19}$$

This factor relates the real mean to the ideal mean ejection velocity. Decline in real velocity is due to the friction of the gas with the walls leading to a boundary layer with reduced velocity, which is transmitted to neighboring gas layers due to gas viscosity. But also velocity increases are possible, for instance if post-combustion occurs in the nozzle.

$$\eta_{ec} := \frac{P_{jet,id}}{P_{in}} = \frac{E_{jet,id}}{E_{in}} \leq 1 \quad \textbf{energy conversion efficiency} \tag{1.3.20}$$

η_{ec} accounts for combustion losses (≈1%) in the chamber (see Fig. 4.6), but also for heat losses to the chamber and nozzle walls (≈3%). A very huge contribution is the energy loss of about 25–30% into internal excitation of the gas molecules and gas liquid and solid particles. So, most of the combustion enthalpy as the total input power into a chemical engine is spent on this drain. From this it is evident that efficient combustion is at the heart of any thrust-to-power optimization.

In summary we get the following expression for the total engine efficiency

$$\eta_{tot} = \zeta_d \zeta_v^2 \eta_{ec} \quad \textbf{total engine efficiency} \tag{1.3.21}$$

As we will see in Sect. 4.2.4 we have $\eta_{tot} \leq 0.735$ for today's chemical thrusters.

Thrust Correction Factor

From Eq. (1.2.2) the important ratio of real total thrust of a rocket engine with an ideally adapted nozzle (see Sect. 4.2.1: $F_p = 0$) to the ideal momentum thrust is derived to be

$$\frac{F_*}{\bar{F}_{e,id}} = \frac{\eta_{div}\bar{F}_e}{\bar{F}_{e,id}} = \eta_{div}\zeta_d \frac{\bar{v}_e}{\bar{v}_{e,id}} = \eta_{div}\zeta_d \frac{\langle v_e \rangle_\mu}{\langle v_{e,id} \rangle_\mu}$$

Since according to Eq. (1.3.15) $\langle v_e \rangle_\mu = \sqrt{\eta_{VDF} \langle v_e^2 \rangle_\mu}$ we obtain with Eq. (1.3.19) for $p_e = p_\infty$

$$\eta_T = \frac{F_*}{F_{e,id}} = \eta_{div}\zeta_d\zeta_v = \eta_{div}(0.92 - 1.00) \quad \textbf{thrust correction factor} \tag{1.3.22}$$

Thrust-to-Power Ratio

Finally, we define the important *thrust-to-total-power ratio*, which describes the thrust received from the total electrical power.

$$r_{TTPR} := \frac{F_*}{P_{in}}$$

For a rocket engine with an ideally adapted nozzle ($p_e = p_\infty$, see Sect. 4.2.1) we have $F_* = F_{ex}$ and therefore

$$r_{TTPR} = \frac{F_{ex}}{P_{in}} \quad @ \quad p_e = p_\infty$$

Upon determining its value we circumvent the problem that the thrust-to-power ratio is not a dimensionless ratio by expressing P_{in} through the total engine efficiency Eq. (1.3.17) and rewrite the thrust-to-power ratio with Eq. (1.2.2) as

$$r_{TTPR} = \frac{F_{ex}}{P_{in}} = \eta_{div}\bar{F}_e \frac{2\eta_{VDF}}{\bar{F}_e \bar{v}_e}\eta_{tot} = \frac{2}{\bar{v}_e}\eta_{div}\eta_{VDF}\eta_{tot}$$

Hence with Eq. (1.3.21) we get

$$r_{TTPR} = \frac{F_*}{P_{in}} = \frac{2}{\bar{v}_e}\eta_{div}\eta_{VDF}\zeta_d\zeta_v^2\eta_{ec} \quad @\ p_e = p_\infty \quad \textbf{thrust-to-power ratio} \quad (1.3.23)$$

We have seen from Sect. 1.2.1 that for today's Rao-optimized bell nozzles we have

$$\eta_{div} = 0.989 - 0.996$$

Thus, we only have to deal with the dimensionless total efficiency and are always able to easily shift back to thrust-to-power ratio.

1.4 Relativistic Rocket[1]

All that has been said up to here was based on Newton's classical mechanics. It holds as long as the speed of the rocket v is well below the speed of light c. We know from the theory of special relativity, which Einstein developed at the beginning of the last century, that physics behaves differently if $v \approx c$. May rockets eventually fly close to the speed of light? In order to find out we need to know what is needed to get it close to the speed of light and how it performs there. But note that the need to apply relativistic physics depends on the precision that is needed to describe a given situation. A satellite navigation system in Earth orbit, for instance, needs a high precision time keeping system onboard with a stability of less than $\Delta t/t \approx 10^{-12}$ that allows determining a position on Earth with roughly 10 cm accuracy. At an orbital speed of 3.9 km/s relativity contributes to the time deviation with $\Delta t/t = v^2/2c^2 \approx 8.5 \times 10^{-11}$ that is not negligible. Therefore, at much lower speeds, relativity must also be taken into account if the accuracy of the description is high.

Our goal here is to understand how relativity works for a spacecraft close to the speed of light and how this relates to classical mechanics at lower speeds. We start out by assuming a one-dimensional motion of the rocket, thrust direction and hence acceleration along the x-axis. The main conclusions of relativity will not be touched by this restriction. This implies that the position of a rocket in time can be appropriately described by the 2-vector (x, t). We define two reference frames: the

[1]Section 1.4 is partly adapted from Walter (2006) with contributions from Westmoreland (2010).

"primed" reference frame of an external inertial observer $O'(x', t')$ and the "unprimed" reference frame of the rocket under consideration $R\ (x, t)$, which is supposed to have an instantaneous velocity v relative to O'.

1.4.1 Space Flight Dynamics

For relativistic physics, it is important to note that among all existing reference frames, there is one preferred frame: the rest frame. This is the frame of the object under consideration in which it is at rest. Any other external observer having velocity v relative to this rest frame observes the properties of the object such as length, time, speed, and acceleration differently as the object itself. Since there may be an infinite number of observers and, therefore, many different views of the object properties, relativistic physics holds that only one has a proper view of the object: the object itself. In this sense, relativity is an absolute concept.

Relativistic physics, therefore, introduces the notion of "proper". In general, a "proper" measure of a quantity is that taken in the relevant instantaneous rest frame, thus also called proper reference frame. So "proper" is everything an astronaut experiences in his rocket. This is why we will not put a prime on such quantities and those as observed from outside will carry a prime. In general the observed values depend on the reference frame with of course one exception: $v = v'$. Adopting this notion, what is of relevance first is how the proper measures relate to the measures of external observers.

Proper time (also called *eigentime*) τ is the time that the watch of an astronaut in a rocket shows. Special relativity holds that τ is related to the time t' of the external observer O' by

$$d\tau \equiv dt = \sqrt{1 - \beta^2} \cdot dt' \tag{1.4.1}$$

where we adopt the convenient relativistic notations $\beta := v/c$ and $\gamma := 1\Big/\sqrt{1 - \beta^2}$. We will sometimes denote dt by $d\tau$ in order to point out that the proper time is meant. It should be noted that in special relativity, Eq. (1.4.1) holds for any condition of the rest frame even if it is accelerated, because, and contrary to common misjudgment, special relativity is not restricted to constant relative velocities or inertial reference frames.

Einstein pointed out that acceleration is an absolute concept: an astronaut does not experience rocket velocity in his rest frame, but he does so for acceleration. Let us assume that the astronaut experiences an acceleration α. Then, special relativity tells us that this is related to the acceleration a' as seen by an external observer through

$$\alpha = \gamma^3 a' \qquad \textbf{proper acceleration} \tag{1.4.2}$$

Because acceleration is an absolute concept, we are apt to define

$$d\sigma := \alpha(t) \cdot dt \tag{1.4.3}$$

$d\sigma$ is an increase in speed as measured in the instantaneous rest frame. We integrate to get

$$\sigma(\tau) = \int_0^\tau \alpha(t) \cdot dt \tag{1.4.4}$$

This equation tells us that σ is the integral of the acceleration as experienced in the proper reference frame and hence is the speed as experienced by an astronaut, who sees the outer world going by. Since this is the true meaning of proper, σ is a proper speed. In order to find the relation of this proper speed to the relative speed v we apply Eqs. (1.4.1) and (1.4.2) to Eq. (1.4.4) and get

$$\sigma = \int_0^{t'} \gamma^2 a' \cdot dt'$$

As $a' = dv/dt' = c \cdot d\beta/dt'$, we find

$$\sigma = \int_0^v \frac{dv}{1 - v^2/c^2} = c \int_0^\beta \frac{d\xi}{1 - \xi^2} = c \cdot \operatorname{arctanh} \beta \qquad \textbf{proper speed}$$

or

$$\beta = \frac{v}{c} = \tanh\left(\frac{\sigma}{c}\right) \tag{1.4.5}$$

It is now shown that the proper speed is proper in a more general sense. Let us consider a second rocket or any other object in space having the known speed u relative to the astronaut's R system. We want to know what its speed u' is as measured by O'. Special relativity tells us that

$$u' = \frac{u + v}{1 + uv/c^2} = c \frac{\beta_u + \beta_v}{1 + \beta_u \beta_v} \tag{1.4.6}$$

The problem with this transformation equation is that it is not linear as in classical physics where the Galileo transformation $u' = u + v$ holds. In addition, Eq. (1.4.6) limits u' to the range $0 \le u' \le c$ if v starts out from below c. This can be seen

immediately if one inserts even limiting velocities $u = c$. This is Einstein's famous law that nothing goes faster than the speed of light. It is exactly this non-linearity and limited range of values that cause problems when treating special relativity mathematically. We now apply Eq. (1.4.5) to Eq. (1.4.6) and find

$$\tanh\left(\sigma'_u/c\right) = \tanh(\sigma_u/c + \sigma_v/c)$$

where we have used the algebraic equation for any two values x and y:

$$\frac{\tanh x + \tanh y}{1 + \tanh x \cdot \tanh y} = \tanh(x + y)$$

As this must hold for any proper speed values; we find

$$\sigma'_u = \sigma_u + \sigma_v \tag{1.4.7}$$

i.e., proper speed recovers the linearity of speed transformation in special relativity.

According to Eq. (1.4.5) the proper speed goes to infinity if the externally observed speed goes to the speed of light. This is to say that from an astronaut's point of view there is no speed limit. His subjective impression is that he can actually travel much faster than the speed of light. But of course he cannot travel faster than infinitely fast. This is the reason why the observer also sees a speed limit: the speed of light. So the ultimate reason why nothing can ever go faster than the speed of light is that no proper space traveler can ever go faster than infinitely fast. Note that from this point of view photons always travel infinitely fast. They experience that any distance in the universe is zero: for them the universe is one point. Because their proper time is zero one might say they don not even exist. But this would be wrong. They come into existence at one point in our universe, they transfer energy, momentum, angular momentum, and information to any other point in proper zero time, thereby causally linking any two parts in our universe and at the instance their work is done they are gone. This is why causality is the basic conservation law and hence the cement of our universe, and not the speed of light. The speed of light c may vary throughout our universe, but the fact that the proper time at $v = c$ is always zero and cannot become negative—implying that no inverse causality is possible—is firm.

In order to show that the concept of proper speed has relevance to the concept of classical speed, we finally show that for small speeds, the proper speed turns over into the classical concept of speed v for $v \to 0$

$$\sigma = c \cdot \operatorname{arctanh}\beta = c\left(\beta + 1/3\beta^3 + \ldots\right) \approx c\beta = v \quad @ \quad v \to 0 \tag{1.4.8}$$

We summarize by noting that the proper speed exhibits four important properties: it is proper, it transforms linearly, its takes on real numbers and it turns over into the classical concept of velocity at low speeds. This implies that it is a natural extension of the classical speed into special relativity and is mathematically integrable.

1.4.2 Relativistic Rocket Equation

With the concept of proper speed at hand, we start out to derive the relativistic rocket equation. We want to do this in its most general form. The two physically distinct rocket propulsion systems are mass propulsion and photon propulsion. We take both into account and assume that upon combustion a portion ε of the propellant mass will be converted into energy with a certain efficiency η and that a portion δ of it expels the exhaust mass with velocity v_e, while the other portion $(1-\delta)$ is expelled as exhaust photons, and the rest is lost. Therefore, the overall energy scheme looks like Fig. 1.9. In the rest frame R of the rocket, momentum conservation holds. Taking the momentums of both exhaust components and that of the rocket into account, we can write

$$(1-\varepsilon)dm \cdot \gamma_e v_e + (1-\delta)\eta\varepsilon \cdot dm \cdot c + (m+dm)dv = 0$$

where the factor $\gamma_e = 1\Big/\sqrt{1-v_e^2/c^2}$ takes into account the Lorentz factor of the ejected exhaust mass, and again we count dm negatively since m is the mass of the rocket. From the above equation, we find

$$dv = -v_* \frac{dm}{m} \tag{1.4.9}$$

whereby we have defined the effective exhaust velocity

$$v_* := (1-\varepsilon)\gamma_e v_e + (1-\delta)\eta\varepsilon\, c$$

Note that all terms in Eq. (1.4.9) are unprimed and are therefore terms measured in the proper reference frame including dv. (observe that in this equation, the effective exhaust velocity is operationally defined as explicated in Sect. 1.1.2).

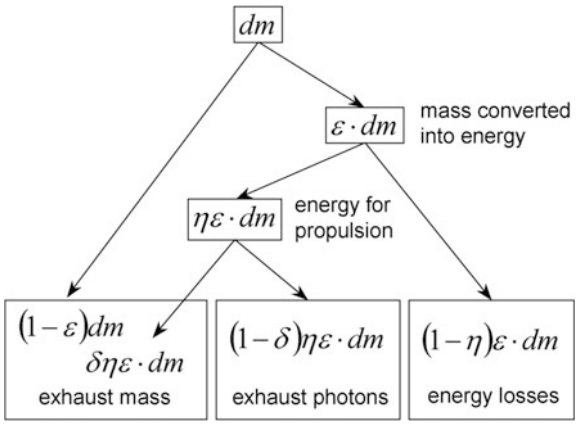


Fig. 1.9 Energy scheme for a relativistic rocket with energy losses and expelled propulsion mass and photons

Now, in classical physics, the relation $dv = dv'$ holds and hence the equation can be readily integrated to yield the classical rocket equation $\Delta v = v_* \ln(m_0/m)$, see Eq. (2.2.2). But $dv = dv'$ is no longer valid for relativistic speeds. However, if we identify $dv = d\sigma$ we can again directly integrate to obtain

$$\boxed{\Delta\sigma = v_* \ln\frac{m_0}{m}} \quad \textbf{relativistic rocket equation} \tag{1.4.10}$$

So the relativistic rocket equation is to the utmost extent complementary to the classical rocket Eq. (2.2.2). In order to show that Eq. (1.4.10) is in accordance with today's more convenient form of the relativistic rocket equation we apply Eq. (1.4.5) and the algebraic equation for the free variable x,

$$\operatorname{arctanh} x = \ln\sqrt{\frac{1+x}{1-x}}$$

to Eq. (1.4.10) and, with Eq. (1.4.6), we derive the common form of the relativistic rocket equation

$$\frac{m_0}{m} = \left(\frac{1+\Delta\beta}{1-\Delta\beta}\right)^{\frac{1}{2\beta_*}} \tag{1.4.11}$$

with

$$\beta_* = \frac{v_*}{c}.$$

From Eqs. (1.4.3) and (1.4.9), we can also derive the thrust F_* of the relativistic rocket in its rest frame

$$F_* = m\alpha = m\frac{d\sigma}{d\tau} = -v_*\frac{dm}{d\tau} = -\dot{m}v_* \quad \textbf{relativistic rocket thrust} \tag{1.4.12}$$

which is identical to the classical Eq. (1.1.4).

1.4.3 Exhaust Considerations

Because a portion of the converted energy propels the exhaust mass, the energy obtained from the propellant $dE_m = \delta\eta\varepsilon \cdot dm \cdot c^2$ has to equal the relativistic energy of the propelled mass $dm_e = (1-\varepsilon)dm$, i.e.,

$$dE_m = \delta\eta\varepsilon \cdot dm \cdot c^2 = \gamma_e dm_e \cdot c^2 - dm_e \cdot c^2 = (\gamma_e - 1)(1-\varepsilon)dm \cdot c^2 \quad (1.4.13)$$

This implies that for a given δ, η, the two terms γ_e (or β_e) and ε are interrelated, namely,

$$\varepsilon = \frac{\gamma_e - 1}{\delta\eta + \gamma_e - 1} \text{ or } 1 - \varepsilon = \frac{\delta\eta}{\delta\eta + \gamma_e - 1} \quad (1.4.14)$$

or

$$\beta_e = \sqrt{1 - \left[\frac{1-\varepsilon}{1-\varepsilon(1-\delta\eta)}\right]^2} \quad (1.4.15)$$

and

$$\gamma_e = \frac{1}{\sqrt{1-\beta_e^2}} = \frac{1-\varepsilon(1-\delta\eta)}{1-\varepsilon} \quad (1.4.16)$$

the other way around. We summarize by saying that internal energy considerations determine the relativistic exhaust mass velocity.

If we insert these results into the effective exhaust velocity from Eq. (1.4.9), we obtain

$$\begin{aligned}\beta_* &:= (1-\varepsilon)\gamma_e\beta_e + (1-\delta)\eta\varepsilon \\ &= \sqrt{\delta\eta\varepsilon(2 - 2\varepsilon + \delta\eta\varepsilon)} + (1-\delta)\eta\varepsilon\end{aligned} \quad \textbf{effective exhaust velocity} \quad (1.4.17)$$

For a rocket that exhausts just mass, $\delta = 1$, we find

$$\beta_* = \sqrt{\eta\varepsilon(2 - 2\varepsilon + \eta\varepsilon)} \quad @ \quad \delta = 1 \quad (1.4.18)$$

In case the rocket has 100% efficiency, $\eta = 1$, we find the expression

$$\beta_* = \sqrt{2\varepsilon - \varepsilon^2} \quad @ \quad \delta, \eta = 1$$

For a photon rocket, $\varepsilon = 1$ and $\delta = 0$, we get

$$\beta_* = \eta \quad (1.4.19)$$

Matter–Antimatter Annihilation Thruster

As an example let us assume a matter–antimatter annihilation thruster as shown in Fig. 1.10). We assume that our rocket annihilates H_2 and anti-H_2 ($\overline{H}_2$) molecules

stored as solid pellets in a storage tank below 14 K, the freezing temperature of hydrogen and hence also anti-hydrogen, typically at 1–2 K to avoid sublimation. In order to confine the neutral antimatter, either their diamagnetism would hold them together in a strong external magnetic field or they would be electrically charged and suspended in an array of electrostatic traps. Otherwise, we neglect all the technical obstacles that come along with such storing devices. Upon annihilation of an H and an $\overline{H}$ atom, each having a total rest mass of 938.8 meV, 22.30% of them are converted into charged pions, 14.38% into neutral pions, and the electron and positron into two γ-rays. The charged pions can be deflected backward by a magnetic field to provide propulsion force. Let us assume that this can be done with 100% efficiency. The neutral pions are lost because after a 0.06 μm travel distance, they decay into 709.1 meV γ-rays, which has to be considered as a major hazard to the crew. As long as the γ-rays cannot be directed backward as well (there seems to be no practical way of doing that) thus adding to the thrust via photonic propulsion this thruster is a purely mass-exhaust one, hence $\delta = 1$.

So, effectively, we have 418.8 meV of pion rest mass as propulsion mass, while the rest is converted into energy, i.e., $\varepsilon = 1 - 418.8/(2 \times 938.8) = 0.7769$. About 748.6 meV of the energy goes into the kinetic energy of the pions, $\varepsilon\eta = 748.6/(2 \times 938.8)$ and therefore $\eta = 0.5132$, and the rest is lost. From Eq. (1.4.18), we then find with $\delta = 1$ an *ultimate effective exhaust velocity* of

$$\beta_* = 0.5804 \quad @ \quad H - \overline{H}\,\text{annihilation} \tag{1.4.20}$$

For a given total rocket mass at a given time this, can be used to calculate the travel speed at this instance from rocket Eq. (1.4.10) or (1.4.11).

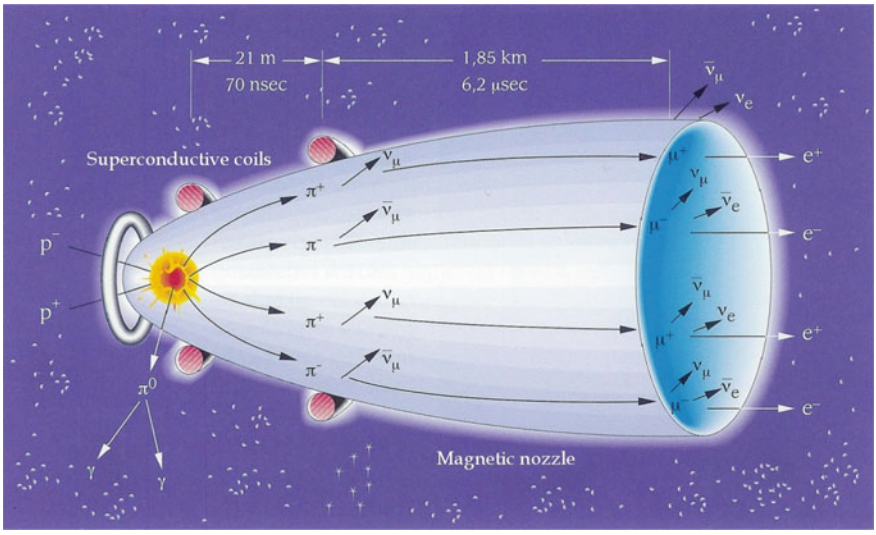


Fig. 1.10 Working scheme of a matter–antimatter annihilation thruster. *Copyright* U. Walter

1.4.4 External Efficiency

As for a classical rocket in Sect. 1.3.2, we want to derive the external rocket efficiency η_{ext} of a relativistic rocket that was defined by

$$\eta_{ext} := \frac{rocket\ kinetic\ \ energy\ \ at\ burnout}{generated\ thrust\ energy} =: \frac{E_{kin}}{E_*}$$

From Eq. (1.4.13) plus the photon energy, we have

$$E_* = (\gamma_e - 1)(1-\varepsilon)m_p c^2 + (1-\delta)\eta\varepsilon m_p c^2 = [(\gamma_e - 1)(1-\varepsilon) + (1-\delta)\eta\varepsilon]\left(\frac{m_i}{m} - 1\right)mc^2$$
$$= \frac{\eta(\gamma_e - 1)}{\delta\eta + \gamma_e - 1}\left(e^{\sigma/v_e} - 1\right)mc^2 = \eta\varepsilon\left(e^{\sigma/v_e} - 1\right)mc^2$$

In the second line, we have applied Eq. (1.4.14) and the relativistic rocket Eq. (1.4.10). And trivially,

$$E_{kin} = \gamma mc^2 - mc^2 = (\gamma - 1)mc^2$$

Employing the definition, we derive

$$\eta_{ext} = \frac{\gamma - 1}{\gamma_e - 1}\frac{\delta\eta + \gamma_e - 1}{\eta(e^{\sigma/v_*} - 1)} = \frac{1}{\eta\varepsilon}\frac{\gamma - 1}{e^{\sigma/v_*} - 1} \qquad \textbf{relativistic external efficiency} \tag{1.4.21}$$

For non-relativistic speeds, i.e., $\gamma \to 1 - v^2/2c^2$, $\sigma \to v$, and for $\delta = 1$, we recover the classical external rocket efficiency (see Eq. (1.3.10)):

$$\eta_{ext} = \frac{(v/v_*)^2}{e^{v/v_*} - 1}$$

Note that while the external efficiency in the relativistic regime depends on the internal efficiency η, this does not hold for classical speeds.

1.4.5 Space–Time Transformations

It is an important and well-known feature of special relativity that observed values for space and time intervals depend on the reference frame of the external observer. This is what the word "relativity" actually refers to. With the concept of proper speed, it is easy to derive the space–time transformation equations between the proper (absolute) reference frame spacecraft and that of an external observer, which we now denote (σ, τ) and (x', v', t'), respectively. From Eqs. (1.4.1) and (1.4.5) and denoting $\varphi := \sigma/c$, the so-called *rapidity*, we find

$$c \cdot dt' = \cosh\phi \cdot c d\tau$$

and because

$$dx' = v \cdot dt' = c \cdot \tanh\phi \cdot dt' = c \cdot \tanh\phi \cdot \cosh\phi \cdot d\tau = \sinh\phi \cdot c \cdot d\tau$$

we can write in short-hand vector notation:

$$\begin{bmatrix} dx' \\ c \cdot dt' \end{bmatrix} = \begin{bmatrix} \sinh\phi \\ \cosh\phi \end{bmatrix} \cdot c \cdot d\tau \tag{1.4.22}$$

Note that because $dx = 0$ for the rocket in the rest frame we do not have a transformation matrix as in the general case. In order to derive the space–time transformations for any rocket–observer relation we have to determine the rapidity (proper speed) and then solve the two differential Eqs. (1.4.22). This will be done now for the two most simple cases.

Cruising Rocket

For a cruising (non-accelerated) rocket, $\cosh\phi = \cosh(\sigma/c) = \gamma = const$ and Eq. (1.4.22) can easily be integrated to give the well-known space–time transformation between inertial observers:

$$t' = \frac{1}{\gamma}\tau \tag{1.4.23a}$$

$$x' = \frac{v}{\gamma}\tau \tag{1.4.23b}$$

$$\frac{v}{c} = \tanh\frac{\sigma}{c} \tag{1.4.23c}$$

As an example: A rocket that travels 90% the speed of light as seen from an external observer or $\sigma = 1.47 \cdot c$ of proper speed would cross our Milky Way with diameter d = 100,000 ly within $t' = d/v$ = 111,000 yr or τ = 48 000 yr in proper time.

Constant-Acceleration Rocket

If the acceleration α is constant, $\sigma = c\phi = \alpha\tau$. By integrating Eq. (1.4.22), we find

$$\frac{\alpha t'}{c} = \sinh\frac{\alpha\tau}{c} = \sinh\frac{\sigma}{c} \tag{1.4.24a}$$

Also, with Eq. (1.4.24a) and $\cosh x = \sqrt{1 + \sinh^2 x}$, we have

$$\frac{\alpha x'}{c^2} = \cosh\frac{\alpha\tau}{c} - 1 = \sqrt{1 + \left(\frac{\alpha t'}{c}\right)^2} - 1 \tag{1.4.24b}$$

Applying Eq. (1.4.24a) with $\tanh x = \sinh x \Big/ \sqrt{1+\sinh^2 x}$ to Eq. (1.4.5) yields

$$\frac{v}{c} = \tanh\frac{\sigma}{c} = \frac{\alpha t'}{\sqrt{c^2 + (\alpha t')^2}} \tag{1.4.24c}$$

From Eq. (1.4.24b) we find

$$\left(x' + \frac{c^2}{\alpha}\right)^2 - (ct')^2 = \left(\frac{c^2}{\alpha}\right)^2$$

This denotes that the space–time trajectory of a rocket with constant acceleration is a hyperbola.

Let us reconsider the case of an ultimate manned $H - \overline{H}$ annihilation rocket with $\beta_* = 0.5804$, which we assume to cross the Milky Way ($x'_f = 100\,000\ ly$ 100,000 ly) with a comfortable acceleration of $\alpha = 1g$. According to Eq. (1.4.24b), this would take only $\tau_f = 11.9\ yr$ in proper time of an astronaut! His final proper speed would be $\sigma_f = \alpha\tau_f = 12.2\ c$ and the rocket's mass ratio can be calculated from the rocket Eq. (1.3.10) to be $m_i/m_f = 1.35 \cdot 10^9$. If the final spacecraft mass is, say, 100 metric tons (Space Shuttle), then the launch mass is $m_i = 1.35 \times 10^{14}$ kg. Moreover, if we assume that the H, $\overline{H}$ fuel is stored in liquid form with density 70 kg m^{-3}, the two tanks together must have dimensions of $14 \times 14 \times 14$ km^3! Not to say anything about the engines that would have to propel such a gigantic space ship at 1 g.

1.5 Problems

Problem 1.1 *Balloon Propulsion*
Consider a balloon that is propelled by exhausting its air with density $\rho = 1.29$ g dm^{-3}. The balloon has a volume of 2 dm^3, the exit (throat) diameter is $A_t = 0.5$ cm^2. Let us assume the balloon exhausts the gas with constant mass flow rate within 2 s. Show that the momentum thrust $F_e = 0.026$ N and the pressure thrust $F_p = 0.013$ N and hence the momentum thrust is roughly twice as big as the pressure thrust.

Hint: Observe that the ejection velocity at the throat does not reach the speed of sound. Make use of the Bernoulli's equation $p + \frac{1}{2}\rho v^2 = const.$

Problem 1.2 *Nozzle Exit Area of an SSME*
The thrust of a Space Shuttle main engine (SSME) was at 100% power level 1.817×10^6 N at sea level, and 2.278×10^6 N in vacuum. By using only this information, derive that the nozzle exit area is $A_e = 4.55\,\mathrm{m}^2$.

Problem 1.3 *Proof of* $\eta_{VDF} \leq 1$
With the definition of $\langle x \rangle_\mu$ in Eq. (1.2.1) prove that $\langle v_e \rangle_\mu^2 \leq \langle v_e^2 \rangle_\mu$ and therefore $\eta_{VDF} \leq 1$.

Hint: Start out with Jensen's inequality

$$f\left(\sum_{i=1}^{n} \lambda_i x_i\right) \leq \sum_{i=1}^{n} \lambda_i f(x_i) \quad @ \; f \text{ convex}, \sum_{i=1}^{n} \lambda_i = 1, \lambda_i \geq 0$$

Show that from this follows

$$\left(\sum_{i=1}^{n} c_i x_i\right)^2 \leq \left(\sum_{i=1}^{n} c_i x_i^2\right) \cdot \left(\sum_{i=1}^{n} c_i\right) \quad @ \; c_i \geq 0$$

By making the transition from a finite sum to an integral, show that this may be written as

$$\left(\int_0^{\pi/2} x\mu \, \sin\theta \cdot d\theta\right)^2 \leq \left(\int_0^{\pi/2} x^2 \mu \, \sin\theta \cdot d\theta\right) \cdot \left(\int_0^{\pi/2} \mu \, \sin\theta \cdot d\theta\right) \quad @ \; \mu \geq 0$$

Chapter 2
Rocket Flight

We now want to tackle the problem of solving the equation of rocket motion Eq. (1.1.7). As will be seen in Sect. 2.1, even for many simple cases it can be solved only by numerical methods. Since this is not the objective of this book, we will treat only those important cases that can be analyzed analytically. This will give rise to an important characteristic quantity the so-called "delta-v budget". Its relevance will be explored in Sect. 2.4.1.

Before turning to this, we will introduce some essential rocket mass definitions we will use in this chapter

m	instantaneous total mass of the rocket
m_0	total launch mass of the rocket
m_f	mass of the rocket at burnout(final mass)
m_p	propellant mass of the rocket before launch or before a maneuver
m_s	structural mass of the rocket
m_L	payload mass

From this it is obvious that

$$\begin{aligned} m_0 &= m_p + m_s + m_L \\ m_f &= m_0 - m_p = m_s + m_L \end{aligned} \tag{2.0.1}$$

In the following calculations, the variable parameter m can often be interpreted as the instantaneous mass or the mass at burnout, so in most cases $m = m_f$ is valid.

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2.1 General Considerations

Before we come to the few very important cases that can be examined analytically, let us have a look on the general solution of a flight in an external field. This may be any relevant external field, but it is almost always the gravitational field. To do this, we separate the variables on the left side of the motion Eq. (1.1.7), and we get

$$d\boldsymbol{v} = \frac{\boldsymbol{F}_* + \boldsymbol{F}_{ext}}{m}\,dt \tag{2.1.1}$$

$\boldsymbol{F}_*$ and $\boldsymbol{F}_{ext}$ are generally dependent on time. For example, during ascent in the atmosphere, the pressure thrust continually varies according to Eq. (1.2.14) because of the changing atmospheric pressure and/or the mass flow rate, which in particular holds for solid propellant rockets. In addition, the thrust direction changes because of the so-called gimbaling, i.e., the steering of the nozzle to change flight direction. In all these cases, with a given $\boldsymbol{F}_*(t)$, $\boldsymbol{F}_{ext}(t)$, $\dot{m}_p(t)$, or

$$m(t) = m_0 - \int_0^t \dot{m}_p(t')dt' \tag{2.1.2}$$

respectively, one can calculate the resultant velocity change by explicit integration:

$$\Delta\boldsymbol{v}(t) = \int_0^t \frac{\boldsymbol{F}_* + \boldsymbol{F}_{ext}}{m}\,dt' = \int_{m_0}^{m} \frac{\boldsymbol{v}_*}{m}\,dm + \int_0^t \frac{\boldsymbol{F}_{ext}}{m}\,dt' \tag{2.1.3}$$

where we have assumed $t_0 = 0$ for the sake of simplicity. We have written the left side of the equation in terms of the new and characteristic quantity "delta-v budget"

$$\Delta\boldsymbol{v}(t) := \int_{\boldsymbol{v}_0}^{\boldsymbol{v}} d\boldsymbol{v} = \boldsymbol{v}(t) - \boldsymbol{v}_0 \quad \textbf{delta-}\boldsymbol{v}\textbf{ budget} \tag{2.1.4}$$

or just "delta-v" for short, which will turn out to be quite handy to describe spacecraft (S/C) maneuvers in space (see Sect. 2.4.1). It describes the total change of the rocket's velocity vector due to all forces acting on the S/C over the time t.

In order to determine the position of the S/C as a function of time, Eq. (2.1.3) needs to be integrated once more. For nearly every practical case these integrations need to be done by numerical methods. There is only one important case where both integrations can be performed fully analytically for an external force: the continuous tangential thrust maneuver under a gravitational force. This important case will be covered separately in Sect. 8.4.5. We now consider some other important specific limiting cases.

2.2 Rocket in Free Space

A limiting case occurs in free space when there are no external forces, $\boldsymbol{F}_{ext} = 0$. In addition, in free space thrust maneuvers typically take place with a $\boldsymbol{v}_*$, which is constant both in absolute value and in direction. This special but most common situation simplifies Eq. (2.1.3) considerably to

$$\Delta \boldsymbol{v} = \boldsymbol{v}_* \int_{m_0}^{m} \frac{dm}{m}$$

so that it can be integrated straight away

$$\Delta \boldsymbol{v} = -\boldsymbol{v}_* \cdot \ln \frac{m_0}{m} \quad @ \quad \boldsymbol{v}_* = const \tag{2.2.1}$$

with $\Delta \boldsymbol{v} = \boldsymbol{v} - \boldsymbol{v}_0$, (v_0, m_0) the initial, and (v, m) the final state of the S/C. Note that the velocity change is independent of thrust level, of the duration, or any time dependence of the thrust, and hence via Eq. (1.1.5) on any variation of the mass flow rate. So, in free space, for a delta-v any specific engine characteristics does not matter, or whether a velocity-change boost is carried out over a short or a long time period. All what matters is initial and final mass and a constant effective exhaust velocity. Note that during ascent through an atmosphere v_* varies somewhat along with the pressure thrust due to the decreasing ambient atmospheric pressure. Therefore, the condition $v_* = const$ is not strictly fulfilled in this case.

Keeping in mind that $\Delta \boldsymbol{v}$ is always strictly antiparallel to $\boldsymbol{v}_*$ (see Fig. 2.1), Eq. (2.2.1) can be rewritten as an absolute value equation

$$\boxed{\Delta v = v_* \ln \frac{m_0}{m}} \quad @ \quad v_* = const \quad \textbf{rocket equation} \,(\text{single stage}) \tag{2.2.2}$$

which is also known as **Tsiolkovsky rocket equation** or *ideal rocket equation*. Observe the close relationship to the relativistic rocket equation Eq. (1.4.10). Next

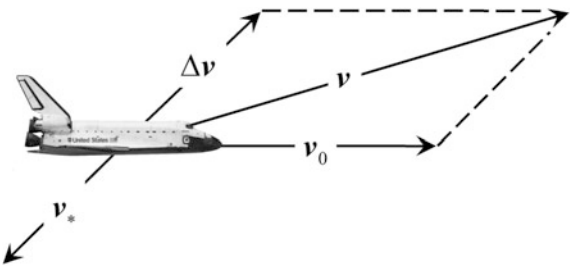


Fig. 2.1 Direction of delta-v for a maneuver in free space

to Eq. (1.1.5) this rocket equation is the most important equation in rocket flight. It can be written alternatively as

$$\frac{m}{m_0} = e^{-\Delta v/v_*} \tag{2.2.3}$$

or applying $m = m_f = m_0 - m_p$ at burnout

$$\frac{m_p}{m_0} = 1 - e^{-\Delta v/v_*} \quad @ \quad v_* = const \tag{2.2.4}$$

2.3 Rocket in a Gravitational Field

In this section we apply the equation of rocket motion (1.1.7) to evaluate the impact of an external field on the flight of a powered spacecraft. In outer space the external field is nearly always a gravitational field.

Only during ascent will a rocket also be subject to drag and sometimes also lift. The consideration of drag and lift complicates the treatment which is why we treat ascent motion only qualitatively in Chap. 6. Upon reentry there is also drag and lift, while the spacecraft is powerless. So, while the considerations in this section do not apply for reentry, the absence of thrust permits treating the motion in more detail, as done in Chap. 10.

Because the equation of rocket motion (1.1.7) is universal it also applies for a spacecraft without propulsion in a gravitational field as for powerless planet orbiting satellites or deep space probes. Although this is a frequent and hence extremely important case, it is somewhat difficult to treat in full detail. This is why we explore it separately in Chap. 7.

2.3.1 Impulsive Maneuvers

We first study the case when a spacecraft undergoes an impulsive maneuver in the presence of a gravitational field exerting the force F_G. Formally, "impulsive maneuver" means that for $t \to 0$ the total impulse $\lim_{t\to 0} \int_0^t F_* \, dt' = \Delta p > 0$ of the maneuver takes on a finite value. But since $\lim_{t\to 0} \int_0^t F_G \, dt' = 0$, this indirectly implies

$$F_* \gg F_G,$$

i.e., a minute but powerful thrust maneuver. We therefore find

For an impulsive maneuver external fields can be neglected and the so-called "delta-v" is determined solely by the thrust characteristics.

Note that, if the propellant mass expelled in an impulsive maneuver is not negligibly small compared to the total rocket mass, the delta-v of the maneuver is not

$$\Delta v = \int_0^t \frac{F_*}{m} dt' \neq \frac{F_*}{m} \int_0^t dt' = \frac{F_* t}{m} = \frac{\dot{m} v_* t}{m} = \frac{m_p v_* t}{tm} = v_* \frac{m_p}{m} = v_* \left(\frac{m_0}{m} - 1 \right)$$

as assumed quite frequently. Rather for a constant thrust (equals constant mass flow rate at $v_* = const$)

$$\Delta v = \int_0^t \frac{F_*}{m} dt' = \int_{m_0}^{m_f} \frac{v_*}{m} dm = v_* \int_{m_0}^{m_f} \frac{dm}{m} = v_* \cdot \ln \frac{m_0}{m}$$

i.e., the rocket Eq. (2.2.2). Impulsive maneuvers are of high relevance for orbit transfers. This is why we will investigate their effects on orbits in more detail in Chap. 8.

2.3.2 Brief Thrust

If the thrust $\boldsymbol{F}_* = -\dot{m}_p \boldsymbol{v}_*$ is constant, i.e., $\dot{m}_p = const$ and $\boldsymbol{v}_* = const$, but if $F_* \gg F_G$ does not apply, we have to take into account the effect of the gravitational field during the maneuver. From Eq. (2.1.3) we find

$$\Delta \boldsymbol{v}(t) = \boldsymbol{v}_* \int_{m_0}^{m} \frac{dm}{m} + \int_{m_0}^{m} \frac{\boldsymbol{F}_G}{\dot{m}} \frac{dm}{m}$$

Often the thrust maneuver is short compared to any variation of the gravitational field, which implies $\boldsymbol{F}_G$=$const$. In this case

$$\Delta \boldsymbol{v} = -\left(\boldsymbol{v}_* - \frac{\boldsymbol{F}_G}{\dot{m}_p} \right) \cdot \ln \frac{m_0}{m_f} = \frac{\boldsymbol{F}_* + \boldsymbol{F}_G}{\dot{m}_p} \cdot \ln \frac{m_0}{m_f} \quad @ \ \boldsymbol{v}_*, \boldsymbol{F}_G = const \qquad (2.3.1)$$

This is quite an interesting result. It claims that for a brief thrust maneuver we only need to substitute $\boldsymbol{v}_* \rightarrow \boldsymbol{v}_* - \boldsymbol{F}_G / \dot{m}_p$ in the familiar rocket Eq. (2.2.1). In closing, we mention that the impulsive maneuver considered above is just a limiting case of Eq. (2.3.1) for $\boldsymbol{F}_* = -\dot{m}_p \boldsymbol{v}_* \gg \boldsymbol{F}_G$.

2.3.3 Gravitational Loss

We now consider the ascent of a rocket from a celestial body (cf. Sect. 6.4) with the gravitational force F_G as the only external force acting on the rocket. From Eq. (2.1.1), we find for the instantaneous speed gain

$$d\boldsymbol{v} = \frac{\boldsymbol{v}_*}{m} dm + \frac{\boldsymbol{F}_G}{m} dt$$

We assume that in course of the ascent the gravitational field does not change significantly and therefore $\boldsymbol{F}_G = m\boldsymbol{g}_0 \approx const$. To find the absolute value of the instantaneous speed gain we multiply this equation with the instantaneous unit velocity vector $\hat{\boldsymbol{v}}(t)$, finding

$$dv = \frac{\hat{\boldsymbol{v}}\boldsymbol{v}_*}{m} dm + g_0 \cos(\gamma + 90°) \cdot dt = \frac{\hat{\boldsymbol{v}}\boldsymbol{v}_*}{m} dm - g_0 \sin\gamma \cdot dt$$

where $\gamma = \angle(\boldsymbol{v}, \boldsymbol{F}_G) - 90°$ is the so-called *flight path angle*, which is the angle between the flight path and the local horizon (see Figs. 6.8 and 7.9). For a steering-free ascent $\hat{\boldsymbol{v}}\boldsymbol{v}_* = -v_* \approx const$ holds. In general $\hat{\boldsymbol{v}}\boldsymbol{v}_*/m$ is the mass-specific speed gain in a gravitational-free environment, the integral of which is not of specific interest for us here, so we will denote it quite generally as Δv_0. Therefore, we find for the velocity after time t

$$\Delta v = \Delta v_0 \underbrace{- g_0 \int_0^t \sin\gamma \cdot dt'}_{gravitational\ loss} \tag{2.3.2}$$

So, in contrast to a rocket in free space, the achieved velocity for ascent depends on the flight direction relative to the local horizon and the time t to engine shutdown. For ascent $\gamma > 0°$ and therefore the integral term in Eq. (2.3.2) is positive. (For a reentering S/C with $\gamma < 0°$ it would be negative.) Therefore, an ascending rocket does not achieve the same velocity increase as for a propulsion maneuver in free space, which is why the entire integral term is called **gravitational loss** term. Gravitational loss is particularly striking for vertical ascent, when $\gamma = 90°$. In this case

$$\Delta v = \Delta v_0 - g_0 t \quad @\ \text{vertical ascent}$$

In the extreme case when the thrust at ascent just balances the gravitational force, then $g_0 t = \Delta v_0$ and therefore $\Delta v = v_0 = 0$: the rocket barely hovers above the

launch pad until the fuel is used up. We, therefore, derive the following general rule:

> The longer the ascent time and the larger the angle between the flight path and the gravitational force, the smaller is the final speed at engine shutdown.

Therefore, for vertical ascent, the specific impulse $I_{sp} \propto v_*$ is not the only figure of merit of an engine but also a high thrust, which reduces ascent time and thus, gravitational losses. Consequently, powerful but admittedly low-efficient chemical boosters are regularly used during vertical ascent; while for the upper stages when the rocket turns horizontally higher-efficient but lower-power liquid hydrogen/oxygen thrusters take over.

Though we have found a clue how to get into orbit efficiently we are still far from answering the question: What is the optimal ascent trajectory? We will investigate this problem in more detail in Sect. 6.4.

2.4 Delta-v Budget and Fuel Demand

2.4.1 Delta-v Budget

The figure $\Delta \boldsymbol{v}$ appeared in the above equations for the first time. It has a special, double relevance in astronautics. On the one hand, it represents the mass-specific momentum change of a rocket: $\Delta \boldsymbol{v} = \Delta \boldsymbol{p}/m$. Momentum changes are necessary to change from a given Keplerian orbit to another Keplerian orbit, or from a Hohmann transfer orbit into a planetary orbit, or vice versa. We know from conservation laws that momentum is a basic physical parameter. Another important basic parameter is energy. To track energy changes is very important, as the initial increase in kinetic energy $\Delta E = m\boldsymbol{v} \cdot \Delta \boldsymbol{v}$ generated by a small $\Delta \boldsymbol{v}$ may transfer into different forms of energy by means of the energy conservation law, e.g., into potential energy, and in lower Earth orbits unfortunately also into frictional energy. This is why it should come as no surprise that a rocket, which formally gains velocity through a kick-burn Δv, may actually decrease its total velocity v when for instance a rocket fires in a gravitational potential. Then more kinetic energy is transferred into potential energy than kinetic energy is produced by the kick-burn. Overall, due to a higher final orbit and in accordance with $v = \sqrt{\mu(2/r - 1/a)}$ (see Eq. (7.3.14)) the orbital velocity is paradoxically reduced, even though the spacecraft initially received a velocity increase Δv.

The nice feature with $\Delta \boldsymbol{v}$ is that it measures all the possible energy demands of a mission. Since it also measures momentum and angular momentum demand, it is a perfect measure for the total thrust demand for an entire mission.

Remark *Changes of angular momentum caused by thrust maneuvers are related to the Δv demand in a quite complicated way, as because of $h = |\boldsymbol{r} \times \boldsymbol{p}| = mrv \cdot \sin(\angle(\boldsymbol{r}, \boldsymbol{v})) = mrv \cdot \cos\gamma$, every change of angular momentum depends strongly on the thrust direction and flight path angle γ. For a circular orbit and for a small tangential thrust $\Delta \boldsymbol{v}_{||}$, from the above it is easy to show that $\Delta h = mr \cdot \Delta v_{||}$.*

Even better, according to Eq. (2.2.4), the all important propellant demand m_p has a one-to-one correspondence to Δv. This is a very handy relationship: Although the required propellant is actually the determining factor in space flight, later on, it does not show up any more in orbit calculations. Rather it is the Δv measure that comes from astrodynamical considerations, such as from the vis-viva Eq. (7.2.15). Equation (2.2.4) now links the two parameters in a very convenient way. Even more conveniently, the propellant demand of two successive Δv maneuvers corresponds to the sum $\Delta v_1 + \Delta v_2$ of the individual maneuvers according to

$$\Delta v = v_* \ln \frac{m_0}{m_2} = v_* \ln \frac{m_0 m_1}{m_1 m_2} = v_* \ln \frac{m_0}{m_1} + v_* \ln \frac{m_1}{m_2} = \Delta v_1 + \Delta v_2$$

Since the propulsion effort is independent on the sign of Δv, we derive for the effort

$$\boxed{\Delta v = \sum_{\text{all maneuvers}} |\Delta v_i|} \tag{2.4.1}$$

This is called *delta-v* budget (a.k.a. *propulsion demand*) or just *delta-v* for short. So we can make the following comment:

> Δv (delta-v budget) is a perfect quantity to determine mission effort, because it relates astrodynamical properties (momentum, angular momentum, and total energy changes), which may be derived from mission considerations, in a convenient additive way to the propellant demand of a mission.

2.4.2 Fuel Demand—Star Trek Plugged

Keep in mind that, according to Eq. (2.2.4), the propellant demand and the propulsion demand have a non-linear relationship: propellant demand strongly grows with increasing propulsion demand. For practical purposes, the relative change of the launching mass of a rocket $\Delta m_0/m_0$ for a given change in propulsion demand $\Delta(\Delta v)$ is a very interesting relation. This relation of relative change can be shown (exercise, Problem 1.2) to be

$$\frac{\Delta m_0}{m_0} = \exp\left[\frac{\Delta(\Delta v)}{v_*}\right] - 1 \approx \frac{\Delta(\Delta v)}{v_*} \quad @ \quad \Delta(\Delta v) \ll v_* \tag{2.4.2}$$

Example

An interplanetary probe is to be accelerated to escape velocity to leave the solar system. A flyby maneuver via Jupiter requires $\Delta v_2 = 6.33$ km/s. Direct escape from the Earth orbit requires $\Delta v_1 = 8.82$ km/s. Question: How much more

launching mass do you need for direct escape, if the chemical propulsions have an I_{sp} of 306 s? Answer:

$$v_* = g_0 \cdot I_{sp} = 9.80 \cdot 306 \text{ m/s} = 3.00 \text{ km/s}$$

$$\frac{\Delta m_0}{m_0} = \exp\left[\frac{2.49}{3.00}\right] - 1 = 1.29$$

The additional propulsion demand for direct escape is only 39%, the increase of the launch mass due to the additional propellant demand, however, is 129%!

Let us have a closer look at Eq. (2.2.4). It refutes what many science fiction fans believe: that good classical propulsion just needs a lot of energy.

Remark *With "good classical propulsion" we refer to classical recoil propulsion not to exotic propulsion like warp propulsion. When you see on a cinema screen a spacecraft accelerating with a thundering roar (which of course does not make sense at all in a vacuum as outer space) during a spacecraft battle, this obviously is recoil propulsion.*

The truth, however, is this: What a flight maneuver needs more than anything else is propellant mass. A lot of it. As for large maneuvers, such as an inversion of the flight direction, Δv gets very large, the exponent tends to zero, and the used propellant mass tends to 100% of the spacecraft mass, which is an extremely uncomfortable perspective for the passengers. You could object arguing that "Star Trek" et al. have engines providing unlimited exhaust velocity v_*, which would reduce the propellant demand in line with Eq. (2.2.4). But that is not possible. Because, from Einstein, we know that the maximum possible exhaust velocity is the velocity of light c. Assuming that Star Trek's relativistic rocket engines (of course) have $v_* = c$, one can prove (cf. Sect. 1.4) that, for big maneuvers, Eq. (2.2.4) takes on the form

$$\frac{m_p}{m_0} = 1 - \frac{1}{2}\sqrt{1 - \left(\frac{\Delta v}{c}\right)^2} \quad @ \quad v_* = c, \quad \Delta v \to c$$

If Capt. Kirk now wants to carry out a reversion maneuver, then he has to decelerate first, i.e., $\Delta v = v \approx c$, then he has to accelerate again in the opposite direction, i.e., again $\Delta v = v \approx c$. This relativistic equation has to be used for both maneuvers one after the other, leading to

$$\frac{m_p}{m_0} = 1 - \frac{1}{4}\left[1 - \left(\frac{v}{c}\right)^2\right] \quad @ \quad v_* = c, \quad \Delta v = 2v \to c$$

In other words, if, in a galactic fight with an enemy, Kirk only flew with 90% velocity of light (which would indeed be below his dignity), he would need 95.25% of the spacecraft's mass as propellant for one single reversion maneuver. If he flew with 99% velocity of light, he would already need 99.5% of the spacecraft's mass.

It is quite strange that you never see any of the necessary huge propellant tanks in the movies.

But propellant shortage would be Kirk's smallest problem. The energy required for the reversion maneuver would be more of a problem. A relativistic calculation for the total energy demand of a full slam on the brakes would be

$$E = \frac{2m_0c^2}{\sqrt{1 - v^2/c^2}}$$

If the spacecraft does a reversion maneuver using up the double amount of energy, and let us assume the ideal case that Kirk gets his energy from a 100% efficient annihilation of matter and antimatter—the very best that could be imagined—then he would need the energy mass-equivalent

$$m = \frac{4m_0}{\sqrt{1 - v^2/c^2}}$$

of matter and antimatter, half the amount of each. So, if Kirk flies with 90% velocity of light, he would need the mass equivalent of 9.18, and with 99% velocity of light he would need 28.4 spacecraft masses. But from a logical point of view, this is not possible at all, as the spacecraft only has one spacecraft mass.

2.5 Problems

Problem 2.1 *Launch Mass Changes*
Prove Eq. (2.4.2) by deriving the relative change of the launch mass of a rocket $\Delta m_0/m_0$ for a relative change of the propulsion demand $\Delta(\Delta v)/(\Delta v)$. Find the approximation for small $\Delta(\Delta v)$.

Chapter 3
Rocket Staging

In Sect. 1.3.1, we found out that there are limits to the obtainable payload mass because of the finite structural mass. This limit is crucial for the construction of a rocket. If a rocket is to reach the low Earth orbit, a propulsion demand of about 9 km s^{-1} has to be taken into account (see end of Sect. 6.4.7), which is at the limit of feasibility for today's chemical propulsions. If a higher payload ratio beyond 3% is required, or the S/C needs to leave the gravitational field of the Earth requiring a higher propulsion demand, one has to take measures to increase the rocket efficiency.

The best method to do this is rocket staging. "Staging" means to construct a rocket such that some tanks and/or engines are integrated into one stage, which can be jettisoned after use thereby reducing the mass to be further accelerated. Figure 3.1 depicts four types of rocket staging, where parallel staging and serial staging are the most common.

3.1 Serial Staging

3.1.1 Definitions

Serial staging (a.k.a. multistaging, multistepping, tandem staging) means that several rockets (n stages) sit on top of each other. One stage after the other is fired during operation, and the burnt-out stages are jettisoned. The advantages are:

1. The engines can be adapted to the changing environment upon ascent: the lowest stage can be chosen for a high thrust to quickly escape the Earth's gravitational potential, whereas the upper stage(s) in (almost) free space can be dimensioned for best efficiency (I_{sp}).
2. Jettisoning the lower engines, which are no longer necessary, and the tanks, decreases structural mass and hence increases payload mass.

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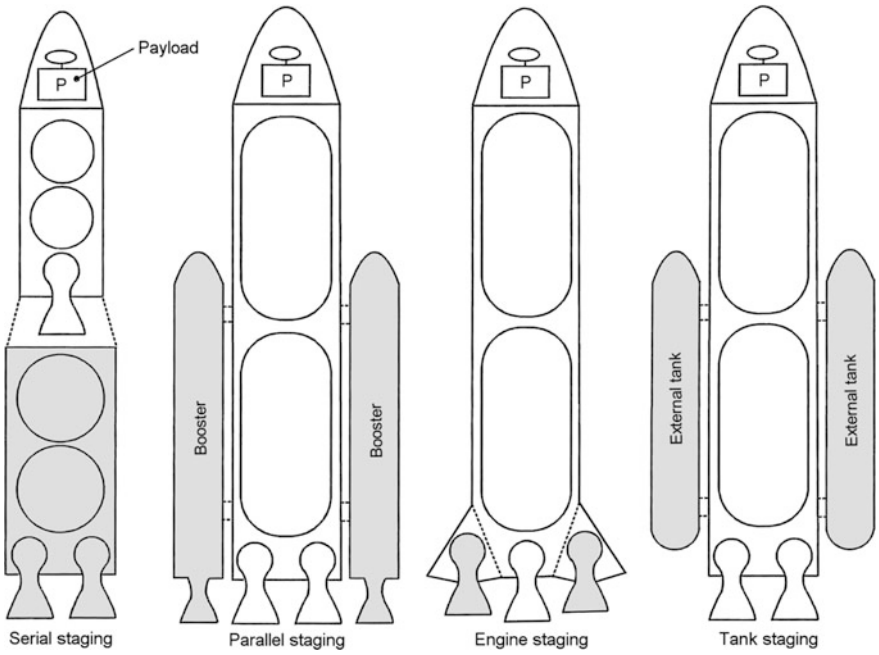


Fig. 3.1 Four types of rocket staging. Stages that are jettisoned during ascent are marked in gray

The concept of serial staging dates back to the first military applications in 1529. Figure 3.2 depicts the sketch of the first staged rocket as introduced by the Austrian military technician Conrad Haas in his *Kunstbuch* (German, meaning *Art Book*). The concept of staging then passed via a publication of the German rocket pioneer Johannes Schmidlap to the Polish military engineer Kazimierz Siemienowicz, whose book *Artis Magnae Artilleriae pars prima* (*Great Art of Artillery, the First Part*) published in 1650 was translated into many languages in Europe, and became for two centuries the basic artillery manual. So rocket staging became a well-known method in artillery rocketry and passed from there also to the rocket-hype decades 1920–1940.

Already in those days the question was raised, whether ignition of the next stage should wait until the rocket's speed had reduced to zero, i.e., make use of the full impetus or to ignite it immediately. The answer is given by Eq. (2.3.2), in saying: Any delay of ignition will lift the fuel of the next stage to higher altitudes thus converting kinetic energy of the rocket into potential energy of the fuel which is a waste of effort (see Problem 3.1). So, the gravitational loss demands for a preferably short ascent and hence a preferably short ignition sequence.

For a mathematical analysis of serial staging, we introduce the concept of a **partial rocket** *i*. Let us assume we have four separate propulsion units (see Fig. 3.3), which are set up in four stages on top of each other. The first partial

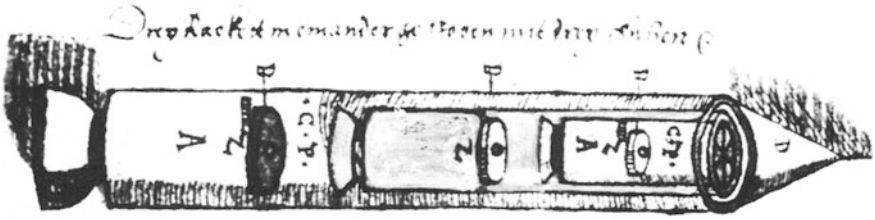


Fig. 3.2 A three-stage rocket of Conrad Haas from 1529. The picture inscription (German) reads: "Three nested rockets with three shots". Note that each stage has already a bell-shaped nozzle. By courtesy of Barth (2005)

rocket refers to the sum of all the four propulsion units plus the true payload. The second partial rocket refers to the sum of the three stages 2–4 arranged on top of that, plus the true payload. The third partial rocket to the sum of the third and fourth stage plus true payload. Finally, last the fourth partial rocket to the upper unit plus the true payload.

We define the payload of a partial rocket as the mass of the next partial rocket, i.e.,

$$m_{L,i} := m_{0,i+1} \tag{3.1.1}$$

The **true payload** is defined as the payload of the last partial rocket. In line with the single-stage rocket $m_{0,i}$ and $m_{f,i}$ will be the initial and the final mass of the ith partial stage; therefore

$$\begin{aligned} m_{0,i} &= m_{p,i} + m_{s,i} + m_{0,i+1} \\ m_{f,i} &= m_{0,i} - m_{p,i} = m_{s,i} + m_{0,i+1} \end{aligned} \tag{3.1.2}$$

We also define the following ratios (cf. Eqs. (1.3.1)–(1.3.3)) for each partial rocket i:

$$\mu_i := \frac{m_{f,i}}{m_{0,i}} = \frac{m_{0,i+1} + m_{s,i}}{m_{0,i}} = \frac{m_{L,i} + m_{s,i}}{m_{0,i}} < 1 \quad \textbf{mass ratio} \tag{3.1.3}$$

$$\varepsilon_i := \frac{m_{s,i}}{m_{0,i} - m_{0,i+1}} = \frac{m_{s,i}}{m_{s,i} + m_{p,i}} < 1 \quad \textbf{structural ratio} \tag{3.1.4}$$

$$\lambda_i := \frac{m_{L,i}}{m_{0,i} - m_{0,i+1}} = \frac{m_{L,i}}{m_{s,i} + m_{p,i}} = \frac{m_{0,i+1}}{m_{s,i} + m_{p,i}} < \frac{m_{L,i}}{m_{s,i}} \quad \textbf{payload ratio} \tag{3.1.5}$$

The inverse value of the mass ratio is sometimes called the *growth factor of the stage*. Observe that the structural mass and the payload mass of a partial rocket are

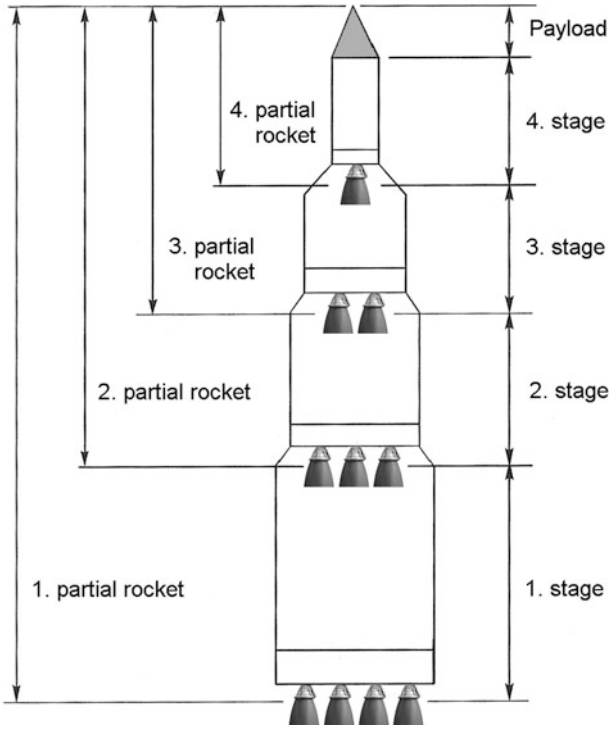


Fig. 3.3 Definitions for a staged rocket

taken relative to the mass of its lower stage and not its total mass. This is reasonable because it will facilitate the optimization of partial rockets, which will be the main objective of this chapter. Because

$$1+\lambda_i = \frac{m_{s,i}+m_{p,i}+m_{0,i+1}}{m_{s,i}+m_{p,i}} = \frac{m_{0,i}}{m_{s,i}+m_{p,i}}$$

it follows that

$$\frac{\lambda_i}{1+\lambda_i} = \frac{m_{0,i+1}}{m_{s,i}+m_{p,i}}\frac{m_{p,i}+m_{s,i}}{m_{0,i}} = \frac{m_{0,i+1}}{m_{0,i}}$$

and

$$\frac{\varepsilon_i}{1+\lambda_i} = \frac{m_{s,i}}{m_{p,i}+m_{s,i}}\frac{m_{p,i}+m_{s,i}}{m_{0,i}} = \frac{m_{s,i}}{m_{0,i}}.$$

For the so-called total *payload ratio* (Ratio of the true payload to the total launch mass. Its inverse value is sometimes called *total growth factor*):

$$\lambda_* := \frac{m_{L,n}}{m_{0,1}} = \frac{m_{L,n}}{m_{0,i}} \frac{m_{0,i}}{m_{0,i-1}} \cdots \frac{m_{0,3}}{m_{0,2}} \frac{m_{0,2}}{m_{0,1}} \quad \textbf{total payload ratio}$$

we therefore obtain

$$\lambda_* = \prod_{i=1}^{n} \frac{\lambda_i}{1+\lambda_i} \tag{3.1.6}$$

and

$$\mu_i = \frac{m_{L,i} + m_{s,i}}{m_{0,i}} = \frac{\varepsilon_i + \lambda_i}{1+\lambda_i} \tag{3.1.7}$$

Remark *In order to be consistent to Eq. (3.1.5) λ_* should be defined as $\lambda_* := m_{L,n}/(m_{0,1} - m_{L,n})$. In this case in all the following equations $\lambda_* \to \lambda_*/(1+\lambda_*)$ should be replaced. In Eq. (3.2.2) the replacement should read $\lambda_* \to \lambda_*(1+\lambda_*)^2$. Because the inconsistent definition is used throughout the literature and because it is insignificantly different to the consistent form we will adopt it here also.*

3.1.2 Rocket Equation

We are now looking for a serial-staged rocket equation equivalent to Eq. (2.2.2) for the single-staged rocket. According to Eq. (2.2.2) for each partial rocket i,

$$\Delta v_i = v_{*,i} \ln \frac{m_{0,i}}{m_{f,i}} = -v_{*,i} \ln \mu_i$$

holds. Serial staging with instant firing of the following stage means that the terminal velocity of one partial rocket is the initial velocity of the following partial rocket, i.e., $v_{f,i} = v_{0,i+1}$. So the following is valid:

$$\Delta v = v_{f,n} - v_{0,1} = \left(v_{f,n} - v_{0,n}\right) + \left(v_{f,n-1} - v_{0,n-1}\right) + \cdots + \left(v_{f,1} - v_{0,1}\right) = \sum_{i=1}^{n} \Delta v_i$$

Therefore we get for the total velocity increase (propulsion demand):

$$\Delta v = -\sum_{i=1}^{n} v_{*,i} \ln \mu_i = \sum_{i=1}^{n} v_{*,i} \ln \frac{1+\lambda_i}{\varepsilon_i + \lambda_i} \quad \textbf{serial-stage rocket equation} \tag{3.1.8}$$

3.2 Serial-Stage Optimization

3.2.1 Road to Stage Optimization

Our goal now is to optimize the serial stages such that we get a maximum payload into orbit or to achieve a maximum velocity gain. In order to know how to perform the optimization let us state the problem explicitly. For the optimal construction of a serial-staged rocket, the following quantities must be considered:

- **Technically given quantities**
  - exhaust velocities $v_{*,i}$
  - structural ratios ε_i
  - total launch mass $m_{0,1}$ of the rocket
- **Technically variable parameters**
  - number of stages n
  - payload ratios λ_i
- **Target quantities**
  - total payload ratio λ_*
  - propulsion demand Δv

The latter are determined by the technical parameters through Eqs. (3.1.6) and (3.1.8).

The objective of a stage optimization now is to first specify one target quantity and then to maximize the other by variation of n and λ_i. However, n is not a true variable: first because n is an integer; and, second, because, with every additional stage, the rocket becomes more efficient (see Fig. 3.6) and therefore an optimal rocket would have infinitive many stages. So, optimizing n cannot be the true objective. Rather the following holds. Because every stage adds to the propulsion demand by a summand (see Eq. (3.1.8)), in the following optimization procedure n is the smallest stage number, for which at a given propulsion demand the optimized payload ratios can be just determined. We will see at the end of Sect. 3.2.2 precisely what this means.

Therefore, the payload ratios λ_i remain as the only variables, which have to be optimized by the following two different optimization approaches:

1. max λ_*: maximize the total payload ratio λ_* at a given Δv
2. max Δv: maximize the obtainable Δv at a given λ_*

The first approach is taken for instance by Ruppe (1966) and the second for instance by Griffin and French (2004). So, with max λ_* Eq. (3.1.6) is the target function to maximize, and Eq. (3.1.8) is the secondary condition; and for max Δv it is the other way round. As both proceedings are described in literature, and also used in practice, we also want to explore both, and will see that in principle they are equivalent.

3.2.2 General Optimization

Now λ_* and Δv are to be maximized with respect to λ_i under secondary conditions. This can be achieved with the so-called Lagrangian Multiplier Method, whereby a secondary condition can be taken into account by adding it to the partial derivatives (to be set to zero) via a so-called Lagrangian Multiplier γ.

Remark *For a comprehensible description of the Lagrangian multiplier method and why it works see for instance: Reif (1965, Appendix A10), or* http://mathworld.wolfram.com/LagrangeMultiplier.html.

By this method we find for both max cases and for each partial rocket $j = 1, \ldots, n$ the condition equations

$$\frac{\partial \lambda_*}{\partial \lambda_j} + \gamma \frac{\partial(\Delta v)}{\partial \lambda_j} = 0 \quad @ \quad \max \lambda_* \tag{3.2.1a}$$

and

$$\frac{\partial(\Delta v)}{\partial \lambda_j} + \gamma' \frac{\partial \lambda_*}{\partial \lambda_j} = 0 \quad @ \quad \max \Delta v \tag{3.2.1b}$$

respectively. Obviously these two equations actually set the same mathematical problem if one identifies $\gamma = 1/\gamma'$. Applying the partial derivations

$$\frac{\partial(\Delta v)}{\partial \lambda_j} = v_{*,j} \frac{\varepsilon_j + \lambda_j}{1 + \lambda_j} \frac{\partial}{\partial \lambda_j} \left(\frac{1 + \lambda_j}{\varepsilon_j + \lambda_j} \right) = -v_{*,j} \frac{1 - \varepsilon_j}{1 + \lambda_j} \frac{1}{\varepsilon_j + \lambda_j}$$
$$\frac{\partial \lambda_*}{\partial \lambda_j} = \frac{\lambda_*}{\lambda_j / (1 + \lambda_j)} \frac{\partial}{\partial \lambda_j} \left(\frac{\lambda_j}{1 + \lambda_j} \right) = \frac{\lambda_*}{\lambda_j (1 + \lambda_j)}$$

to Eq. (3.2.1a) yields

$$\frac{\lambda_*}{\lambda_j (1 + \lambda_j)} = \gamma v_{*,j} \frac{1 - \varepsilon_j}{1 + \lambda_j} \frac{1}{\varepsilon_j + \lambda_j}$$

From this it follows that

$$v_{*,j} \lambda_j \frac{1 - \varepsilon_j}{\varepsilon_j + \lambda_j} = \frac{\lambda_*}{\gamma} =: \alpha = const \quad @ \quad j = 1, \ldots, n \tag{3.2.2}$$

This is a key equation: first, because it claims that if $v_{*,i} = v_* = const$ and $\varepsilon_i = const$ then all payload ratios are equal; and, second, because, once the constant α is known, all optimized payload ratios follow from it immediately by solving for λ_j:

$$\lambda_{j,opt} = \frac{\alpha\varepsilon_j}{(1-\varepsilon_j)v_{*,j} - \alpha} \quad @ \quad j = 1, \ldots, n \tag{3.2.3}$$

Inserting this equation into Eqs. (3.1.8) and (3.1.6) one arrives at

$$\Delta v = \sum_{i=1}^{n} v_{*,i} \ln \frac{v_{*,i} - \alpha}{\varepsilon_i v_{*,i}} \tag{3.2.4}$$

and

$$\lambda_* = \prod_{i=1}^{n} \frac{\alpha\varepsilon_i}{(1-\varepsilon_i)(v_{*,i} - \alpha)} \tag{3.2.5}$$

The two optimization methods now differ merely in that in the

- $\max\lambda_*$ case, at a given Δv the constant α is numerically determined from secondary condition Eq. (3.2.4), which inserted into Eq. (3.2.5) yields the maximized λ_*,
- $\max\Delta v$ case, at a given λ_* the constant α is numerically determined from secondary condition Eq. (3.2.5), which inserted into Eq. (3.2.4) yields the maximized Δv.

Quite generally α can be determined from the above equations only numerically. If, for instance, Newton's method is applied, the equation for the recursive iteration for solving Eq. (3.2.4) reads

$$\alpha_{i+1} = \alpha_i - \frac{\Delta v - \sum_{i=1}^{n} v_{*,i} \ln \frac{v_{*,i} - \alpha_i}{\varepsilon_i v_{*,i}}}{\sum_{i=1}^{n} \frac{v_{*,i}}{v_{*,i} - \alpha_i}} \tag{3.2.6}$$

and that for Eq. (3.2.5)

$$\alpha_{i+1} = \alpha_i \cdot \left\{ 1 + \frac{\ln\left[\lambda_* \cdot \prod_{i=1}^{n} \frac{1-\varepsilon_i}{\varepsilon_i}\left(\frac{v_{*,i}}{\alpha_i} - 1\right)\right]}{\sum_{i=1}^{n} \frac{v_{*,i}}{v_{*,i} - \alpha_i}} \right\} \tag{3.2.7}$$

Note that due to either of the constraints $\lambda_{i,opt} > 0$, $\Delta v_i > 0$, or $\lambda_i/(1+\lambda_i) < 1$ we have from Eqs. (3.2.3), (3.2.4), or (3.2.5) that α is restricted to

$$0 < \alpha < \min_{i=1,n}\left[v_{*,i}(1-\varepsilon_i)\right] \tag{3.2.8}$$

Newton's iteration as given by Eqs. (3.2.6) and (3.2.7) does not take this restriction into account, so it must be addressed as a constraint in the iteration process.

If we insert α so derived into Eq. (3.2.3), we also obtain the optimal payload ratios. With this we have finally achieved the stage optimization goal.

Remark *One might argue that the procedure laid out here will not work in the max λ_* case because according to Eq. (3.2.2), the constant α depends on λ_*, which in itself is a variable to be determined. Rather one would have to set up a system of $n+1$ equations from the corresponding secondary condition (3.2.4) or (3.2.5) plus the n Eqs. (3.2.3) to find from this the $n+1$ quantities (λ_i, α) self-consistently. The point is that λ_* is not a variable to be optimized (which are the λ_i) but a target quantity, which is mathematically a (yet to be determined) constant. With the introduction of α in Eq. (3.2.2) we merely redefine the constant Lagrange multiplier γ with the help of the quasi-constant λ_*.*

Example

Let us consider the Saturn V rocket from the Apollo era. Its characteristic partial rocket parameters are given for Apollo 11 in Fig. 3.4.

If we optimize the staging with the above procedure, we arrive at the dependencies as given in Fig. 3.5. Saturn V provided a $\Delta v = 12.4$ km s^{-1} to the Moon. Given this, we find as the optimal result $\lambda_* = 0.0175$, which is only marginally better than the actual Saturn V total payload ratio of $\lambda_* = 0.0162$, corresponding to 47 tons of payload (Command Module, Service Module, and Lunar Module) into translunar trajectory. Note that a two-stage version of Saturn V with the same staging parameters would have either permitted to transfer only half of the payload mass to the Moon or would have been twice as heavy on the launch pad at the given payload mass.

Choosing the Number of Stages

When α needs to be determined from Eq. (3.2.4) or Eq. (3.2.5) the number of stages (partial rockets) n, which occurs in both equations, deserves a treatment: What should n be chosen? In general, it can be said that, as long as Eq. (3.2.8) is obeyed, any additional stage adds to λ_* or Δv, respectively. So the answer is: "The more stages the better!" (cf. Fig. 3.6 for a rocket with uniform stages). However, since every stage adds to the complexity and hence to structural weight and to the cost of a rocket, the smallest number of stages might be pursued. Is there a lowest n that can be attained? In Eq. (3.2.5) $n \geq 1$ can be chosen arbitrarily, since we see from Eq. (3.1.6) that for every given $\lambda_* < 0.5$ and stage number n one can find a λ_i and

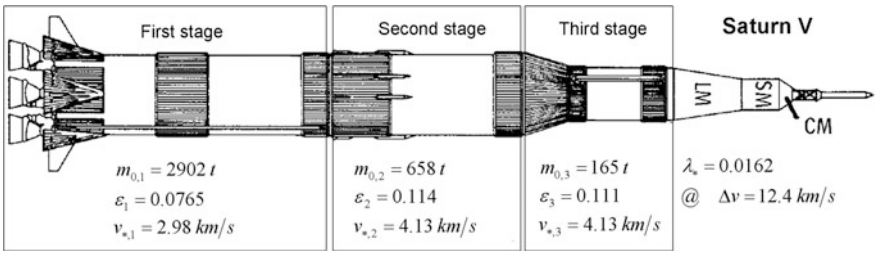


Fig. 3.4 Characteristic stage parameters of Saturn V (Apollo 11)

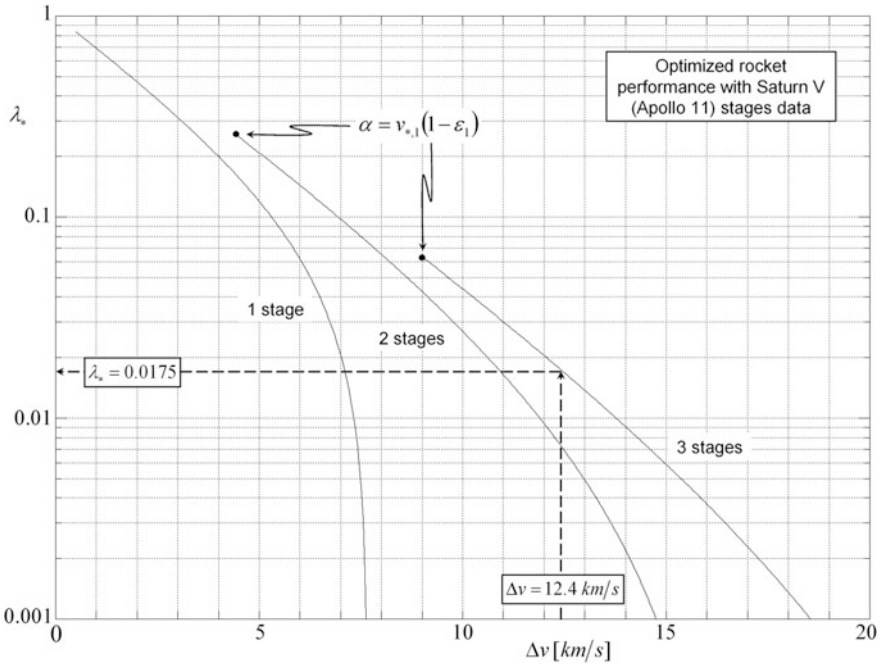


Fig. 3.5 Optimized rocket performance with Saturn V (Apollo 11) stage parameters as given in Fig. 3.4

hence an α. However, in Eq. (3.2.4) there may be a lower bound $n_{\min} > 1$. In the max λ_* case this happens because n has to be raised such that for a given Δv there exists an α. By the same token, in the max Δv case n has to be raised such that for an α derived from Eq. (3.2.5) a required propulsion demand is achieved.

But most importantly staging is governed by mission design rather than by optimization considerations. Take Saturn V as an example. The staging was chosen essentially such that the first two stages carried the third partial rocket into a low earth parking orbit (LEO), while the third stage injected the payload cluster into a translunar trajectory. Since the same will be true for future missions to Mars, the staging will have to be adapted to the mission sequence: launch pad → LEO parking and assembly orbit → low Mars parking orbit → descent to Mars surface → ascent to low Mars parking orbit → Earth reentry. For that reason the number of stages is determined by the sequence steps, while the partial rocket mass for each step is determined from the given delta-v budget for that step from the one stage rocket equation and working in the sequence backward beginning at the final payload mass (reentry capsule). The above stage optimization procedure has to be

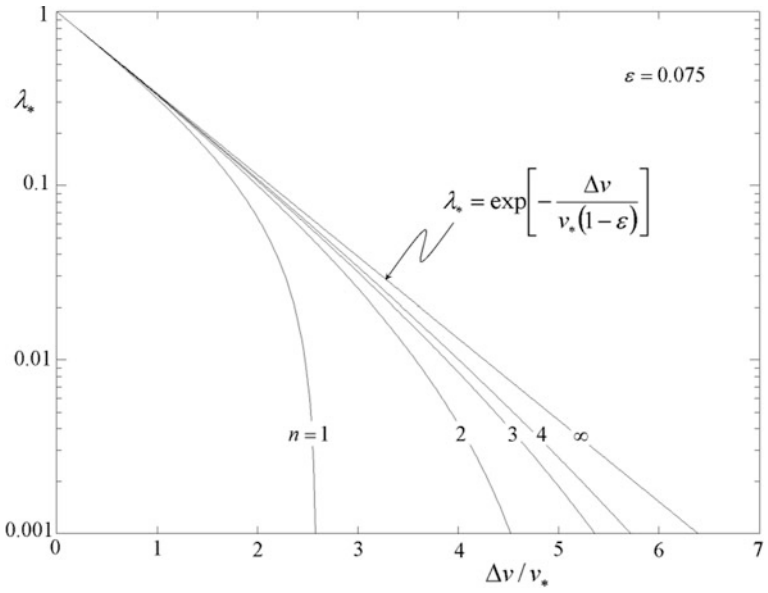


Fig. 3.6 Total payload ratio λ_* as a function of the normalized terminal velocity $\Delta v/v_*$

considered only to tune the overall mission. For the Apollo missions to the Moon, for instance, the LEO apogee kick-burn was not performed by the second stage but already by the third stage, which two and a half hours later reignited to inject the payload into a translunar trajectory. This trade-off shifted the payload ratios between the two partial rockets.

In summary we have the following guidelines for choosing the number of stages:

Guidelines for choosing *n*

1. Adapt the stages and hence also their number n to the propulsion needs of a mission sequence.
2. There is a minimum number of stages $n_{\min}$ such that only for $n \geq n_{\min}$ there exists an α and hence a self-consistent solution $\Delta v(\lambda_*)$ or $\lambda_*(\Delta v)$ from Eqs. (3.2.4) and (3.2.5).
3. The performance of a rocket increases with its number of stages—provided that $\alpha \leq v_{*,i}(1 - \varepsilon_i)$ holds for all of them—at the expense of rocket complexity.

3.3 Analytical Solutions

We are now seeking for analytical solutions to Eqs. (3.2.4) and (3.2.5). We start out with the most simple case of uniform staging and move toward more general cases. Whatsoever, the solution equation will be identical for both maximization cases, because mathematically we just merge the two equations by eliminating the constant α. We always obtain *one* equation, which relates Δv and λ_* via ε_i and $v_{*,i}$. This is why we will disregard in the following the index *max* of the variable to be maximized.

3.3.1 Uniform Staging

The most simple case is the uniform staging where the structural ratios and effective exhaust velocities are all the same:

$$v_{*,i} = v_* = const$$

and

$$\varepsilon_i = \varepsilon = const.$$

Via Eq. (3.2.3) this implies that all payload ratios must be identical, i.e., $\lambda_i = \lambda = const$. The single optimal payload ratio for all stages can derived from Eq. (3.1.6) as

$$\frac{1}{\lambda_{opt}} = \frac{1}{\lambda_*^{1/n}} - 1 \tag{3.3.1}$$

The maximized velocity gain of an n-staged rocket is finally obtained by inserting Eq. (3.3.1) into the serial-stage rocket Eq. (3.1.8)

$$\Delta v = -v_* \sum_{i=1}^{n} \ln\left[\lambda_*^{1/n}(1-\varepsilon)+\varepsilon\right] = -nv_* \ln\left[\lambda_*^{1/n}(1-\varepsilon)+\varepsilon\right] \tag{3.3.2}$$

From this follows

$$\lambda_*^{1/n} = \frac{e^{-\Delta v/nv_*} - \varepsilon}{1-\varepsilon} \tag{3.3.3}$$

This dependence is depictured in Fig. 3.6 for different stage numbers. Obviously the rocket performance increases steadily with increasing stage number.

In the max Δv case the maximized Δv is calculated with a given λ_* from Eq. (3.3.2), and with Eq. (3.3.3) it is just the other way around. In any case, the

smaller the total payload ratio λ_*, the bigger is Δv. The absolute maximum Δv is achieved when λ_* vanishes, in which case we find

$$\Delta v_{\max} = -nv_* \ln \varepsilon \quad @\ \lambda_* = 0 \tag{3.3.4}$$

This sets an upper limit to what can be achieved. If one inserts Eq. (3.3.3) into Eq. (3.3.1) we derive for the optimized uniform payload ratio

$$\lambda_{opt} = \frac{\lambda_*^{1/n}}{1 - \lambda_*^{1/n}} = \frac{e^{-\Delta v/nv_*} - \varepsilon}{1 - e^{-\Delta v/nv_*}} \tag{3.3.5}$$

Having found an optimal solution to the serial-staged rocket we might ask: How does it compare with the single-stage rocket? In terms of achievable total payload ratio, Eq. (3.3.3) compares with Eq. (1.3.4) for the single stage. For a single stage rocket (SSTO, $\Delta v = 9\ \mathrm{km\,s^{-1}}$) with 7.5% structural ratio and $v_* = 4\ \mathrm{km\,s^{-1}}$ we are able to lift only 3.4% total payload ratio into LEO. With an optimized two-stage rocket we lift 7.3%, and hence twice as much under the same conditions; and with a three-stage rocket the total payload ratio we obtain is only a little more, viz. 7.9%. Obviously a two-stage rocket is a perfect vehicle to LEO. This is why nearly all rockets into LEO have two stages.

This simple case of uniform staging enables us to calculate the ultimate rocket with an infinite number of stages. The transition to infinite stages can be performed in Eq. (3.3.2). The result is (exercise, Problem 3.2):

$$\Delta v = -v_*(1 - \varepsilon) \ln \lambda_* \quad @\ n \to \infty$$

From this follows

$$\lambda_* = \exp\left[-\frac{\Delta v}{v_*(1 - \varepsilon)}\right] \quad @\ n \to \infty \tag{3.3.6}$$

So, even in this ideal case the total payload ratio of an optimized rocket decreases exponentially with Δv budget. Figure 3.6 displays the functional dependencies of Eqs. (3.3.3) and (3.3.6). So, with a given structural ratio Δv and stage number n the achievable total payload ratio λ_* can be easily derived and vice versa. The result shows that already with a few stages it is possible to considerably increase the fraction of the payload mass. However, it also demonstrates that with more than three or four stages, the benefit hardly outweighs the additional complexity of the rocket. Obviously the straight line Eq. (3.3.6) is an asymptote for the achievable λ_* and Δv, respectively. And, whatever the number of stages, the total payload ratio always decreases exponentially with an increasing terminal velocity.

3.3.2 Uniform Exhaust Velocities

We now relax the conditions to

$$\begin{aligned} v_{*,i} &= v_* = const \\ \varepsilon_i &\quad \text{arbitrary} \end{aligned}$$

From Eq. (3.2.4) it follows that

$$e^{\Delta v/v_*} = \prod_{i=1}^{n} \frac{v_* - \alpha}{\varepsilon_i v_*} = \prod_{i=1}^{n} \frac{1}{\varepsilon_i} \prod_{i=1}^{n} \left(1 - \frac{\alpha}{v_*}\right) = \frac{1}{\bar{\varepsilon}^n} \left(1 - \frac{\alpha}{v_*}\right)^n$$

with $\bar{x} = \left(\prod_{i=1}^{n} x_i\right)^{1/n}$ geometric mean. Solving for α/v_* yields

$$\frac{\alpha}{v_*} = 1 - \bar{\varepsilon} \cdot e^{\Delta v/n v_*} \tag{3.3.7}$$

Inserting this into Eq. (3.2.5)

$$\lambda_* = \prod_{i=1}^{n} \frac{\alpha \varepsilon_i}{(1 - \varepsilon_i)(v_* - \alpha)} = \frac{\bar{\varepsilon}^n}{\overline{(1 - \varepsilon)}^n} \cdot \frac{1}{(v_*/\alpha - 1)^n}$$

delivers

$$\lambda_*^{1/n} = \frac{e^{-\Delta v/n v_*} - \bar{\varepsilon}}{\overline{1 - \varepsilon}} \tag{3.3.8}$$

From this follows

$$\Delta v = -n v_* \ln\left[\lambda_*^{1/n}\left(\overline{1 - \varepsilon}\right) + \bar{\varepsilon}\right] \tag{3.3.9}$$

depending on which is the target quantity to be maximized. Inserting Eq. (3.3.7) into Eq. (3.2.3) one obtains for the optimized payload ratios

$$\lambda_{i,opt} = \frac{e^{-\Delta v/n v_*} - \bar{\varepsilon}}{\frac{\bar{\varepsilon}}{\varepsilon_i} - e^{-\Delta v/n v_*}} = \frac{\lambda_*^{1/n}}{\frac{\bar{\varepsilon}}{\varepsilon_i}\frac{1-\varepsilon_i}{\overline{1-\varepsilon}} - \lambda_*^{1/n}} \tag{3.3.10}$$

These are the analytical results for a rocket with uniform exhaust velocities $v_{*,i} = v_*$.

3.3.3 Uneven Staging

We finally examine the most general case where both the exhaust velocities and the structural ratios are uneven

$$\begin{array}{ll} v_{*,i} & \text{arbitrary} \\ \varepsilon_i & \text{arbitrary} \end{array}$$

Due to the complexity of Eqs. (3.2.4) and (3.2.5) no exact analytical solutions can be given. But in the following we will provide approximate solutions, which for a draft rocket design are sufficiently exact. Without loss of generality we first define

$$v_{*,i} = v_* + \Delta v_{*,i} \quad \text{and} \quad \varepsilon_i = \varepsilon + \Delta\varepsilon_i$$

where

$$v_* = \frac{1}{n}\sum_{i=1}^{n} v_{*,i} \quad \text{and} \quad \varepsilon = \frac{1}{n}\sum_{i=1}^{n} \varepsilon_i$$

are the arithmetic means of the exhaust velocities and structural ratios of all n stages. Assuming $\Delta v_{*,i} \ll v_*$, $\Delta\varepsilon_i \ll \varepsilon$, and according to Appendix B it is possible to derive the following approximate solutions

$$\lambda_*^{1/n} = \frac{e^{-\Delta v/nv_* - C} - \bar{\varepsilon}}{\overline{1-\varepsilon}} \quad @ \quad \exp\left(-\frac{\Delta v}{nv_*}\right) \ll \min_{i=1,n}\left(\frac{\bar{\varepsilon}}{\delta_i}\right) \tag{3.3.11}$$

with

$$C = \frac{1}{n}\sum_i \frac{\Delta v_{*,i}}{v_*}\left\{\frac{\Delta\varepsilon_i}{\varepsilon_i} + \frac{1}{2}\left(1 - \frac{1}{\bar{\varepsilon}}e^{-\frac{\Delta v}{nv_*}}\right)^2 \frac{\Delta v_{*,i}}{v_*} + O\left[\left(\frac{\Delta\varepsilon_i}{\varepsilon}\right)^2\right] + O\left[\left(\frac{\Delta v_{*,i}}{v_*}\right)^2\right]\right\}$$

From this follows

$$\Delta v = -nv_*\left\{\ln\left[\lambda_*^{1/n}\left(\overline{1-\varepsilon}\right) + \bar{\varepsilon}\right] + C'\right\} \tag{3.3.12}$$

with

$$C' = \frac{1}{n}\sum_i \frac{\Delta v_{*,i}}{v_*}\left\{\frac{\Delta\varepsilon_i}{\varepsilon_i} + \frac{\lambda_*^{2/n}}{2}\left(\frac{\overline{1-\varepsilon}}{\bar{\varepsilon}}\right)^2 \frac{\Delta v_{*,i}}{v_*} + O\left[\left(\frac{\Delta\varepsilon_i}{\varepsilon}\right)^2\right] + O\left[\left(\frac{\Delta v_{*,i}}{v_*}\right)^2\right]\right\}$$

and

$$\lambda_{i,opt} = \frac{e^{-\Delta v/nv_* - C'} - \bar{\varepsilon}}{\frac{\bar{\varepsilon}}{\varepsilon_i} - e^{-\Delta v/nv_* - C'}} = \frac{\lambda_*^{1/n} + K}{\frac{\bar{\varepsilon}}{\varepsilon_i}\frac{1-\varepsilon_i}{1-\bar{\varepsilon}} - \lambda_*^{1/n} - K} \tag{3.3.13}$$

with $K = \left(\lambda_*^{1/n} + \frac{\bar{\varepsilon}}{1-\varepsilon}\right) C'$

Note that even for quite uneven exhaust velocities the solutions are acceptable.

Example

Saturn V for instance had three stages with $v_{*,1} = 304$ s and $v_{*,2} = v_{*,3} = 421$ s at vacuum, therefore $v_* = 382$ s. A calculation of C via the above equations introduces a relative error of only

$$\frac{\Delta C}{C} \approx \sum_{i=1}^{3} \left(\frac{\Delta v_{*,i}}{v_*}\right)^3 = 0.64\%$$

Nevertheless, in specific cases Eqs. (3.2.4) and (3.2.5) should rather be solved numerically.

3.4 Parallel Staging

Parallel staging means that several stages (here: k stages) are mounted, and also activated in parallel (see Fig. 3.1). Let us determine the corresponding rocket equation. The total thrust of a parallel-staged rocket is the sum of the generally different thrusts of all stage engines

$$F_* = \sum_{i=1}^{k} F_{*,i} = \sum_{i=1}^{k} \dot{m}_{p,i} v_{*,i} \tag{3.4.1}$$

Also the total mass flow rate is the sum of all stage engines

$$\dot{m}_p = \sum_{i=1}^{k} \dot{m}_{p,i} \tag{3.4.2}$$

Analogous to Eq. (1.1.5) we can set up a thrust equation

$$F_* = \dot{m}_p v_* \tag{3.4.3}$$

whereby we have indirectly defined a effective exhaust velocity

$$v_* := \frac{F_*}{\dot{m}_p} = \frac{\sum_{i=1}^{k} \dot{m}_{p,i} v_{*,i}}{\sum_{i=1}^{k} \dot{m}_{p,i}} \quad \textbf{effective exhaust velocity} \tag{3.4.4}$$

Mathematically speaking this is an arithmetic mean exhaust velocity. In words, we have the following:

> For parallel staging the effective exhaust velocity is calculated from the mean of exhaust velocities weighted by the respective mass flow rates of the stage thrusters.

If the exhaust velocities are all the same, $v_{*i} = v_{*j}$, Eq. (3.4.4) results in $v_* = v_{*i}$, as expected. With this definition the rocket equation of a parallel-staged rocket is identical to that of a single-stage rocket Eq. (2.2.2)

$$\boxed{\Delta v = v_* \cdot \ln \frac{m_0}{m}} \quad \textbf{parallel-staging rocket equation} \tag{3.4.5}$$

with v_* given by Eq. (3.4.4). Note that owing to this identity any combination of serial staging and parallel staging can be easily calculated.

Advantages and Disadvantages of Parallel Staging Compared with Serial Staging

Advantages

+ The total engine weight is fully used during propulsion time. It does not have to be carried as "dead" payload of the following stage as in serial staging.
+ Thereby in near-Earth space one achieves a higher acceleration and hence a lower gravitational loss (see Sect. 2.3.3).
+ Attaching additional boosters easily adapts a launcher to larger payload masses.
+ Empty tank mass can be minimized by propellant transfer with collecting pipes.
+ Development costs are minimized by standardizing structure and engines.
+ A smaller overall length of a rocket reduces bending moments and longitudinal and lateral oscillations.

Disadvantages

– Structural load is higher after launch.
– This implies higher dynamic drag losses.
– Due to the relatively long combustion time the boosters have to be dimensioned to operate over a higher altitude range. Therefore, the nozzle cross sections cannot be optimally dimensioned for all altitudes implying less thrust. With serial staging the engines can be far better adapted to the respective operative ranges to achieve higher effective exhaust velocities.

After Saturn V, which was a true serial three-stage rocket, NASA quickly passed over to two-stage rockets with parallel staging of the first stage by reflecting on the archetypal Russian R-7 rocket. This mixed type of staging is a good trade-off between the pros and cons of serial and parallel staging. Engines that are just flanged on to the first stage are called *strap-on boosters* or just *boosters*. Today virtually any launcher is build like that: Delta, Soyuz, Ariane, and Shuttle. Parallel

staging of only the first stage has the advantage that by using two, four or six boosters the thrust can be easily adapted to varying payloads without changing the rocket design.

3.5 Other Types of Staging

Instead of integrating a tank plus engine in one stage, it is also possible (see Fig. 3.1) to stage only the tank (**tank staging**) or the engines (**engine staging**). These staging types can be combined with serial staging and parallel staging. Tank staging is an interesting alternative, for instance, when a propellant component has a very low density (such as hydrogen), leading to a large and hence heavy empty tank at the end of the launching phase.

A quite interesting option of tank staging has been discussed since the 1970s: Only one type of engine for two different propellants (dual propellant propulsion). Initially, a propellant with a higher density (e.g., kerosene and liquid oxygen) is burnt because of the higher thrust. At burnout, one merely switches to a propellant with lower density, but higher specific impulse, (e.g., liquid hydrogen and oxygen). This staging type would be very effective (just one engine mass) and reliable (no full-stage separation) if the empty propellant tanks, would be attached external to the rocket to be jettisoned.

3.6 Problems

Problem 3.1 *Ignition Sequencing*
Assume a serial two-stage sounding rocket, which ascends vertically and provides a Δv_1 by its first stage and a Δv_2 by its second stage. Now consider the two cases: The sounding rocket ignites its second stage

1. immediately after burn-out of the first stage.
2. after the second partial rocket has reached $v = 0$.

Assuming $g = const$ and the ideal case where all thrust phases have duration $t \to 0$, show that the payload in the first case climbs by $\Delta h = \Delta v_1 \cdot \Delta v_2 / g$ higher than in the second case.

Problem 3.2 *Infinite-Stages Rocket*
Starting from Eq. (3.3.2) show that for infinite many stages equation $\Delta v = -v_*(1-\varepsilon)\ln\lambda_*$ holds (see Eq. (3.3.6)).

Hint: Define the auxiliary variable $x := 1 - \varepsilon$ and make use of $\lim_{n\to\infty}\left[n\left(\lambda_*^{1/n} - 1\right)\right] = \ln\lambda_*$.

Problem 3.3 *Space Shuttle Propulsion*
Consider a Space Shuttle with a total launch mass of 2017 t (1 t = 1000 kg); orbiter mass of 111 t; external tank (ET) of total mass 738 t, 705 t of which is H_2/O_2 fuel. There are two external Solid Rocket Boosters (SRB) of total mass 584 t each, of which is 500 t solid fuel, firing at $I_{sp} = 300$ s. The three Space Shuttle Main Engines (SSME) had a $v_* = 4.3$ km s^{-1} and a fuel flow rate of 500 kg s^{-1} each. Finally the orbiter incorporated two thrusters called Orbital Maneuvering Systems (OMS) with $v_* = 3.0$ km s^{-1} and 11 t of UDMH/N_2O_4 fuel.

There are three propulsion phases: during the first 120 s of ascent the SRBs burned in parallel with the three SSMEs. Then the SRBs were jettisoned and the orbiter with the ET continued ascent during this second phase until the ET was empty and the SSMEs were cut off. In the third phase, which included the apogee boost into the target orbit, orbit maintenance maneuvers and deorbit burn, the orbiter fired its OMSs.

Assuming that the mass flow rate is constant during all these phases show that the maximum Δv achievable with this Shuttle system adds up to $\Delta v_{\max} = 9.63$ km s^{-1}.

Chapter 4
Thermal Propulsion

The thrust from thermal propulsion engines, such as jet engines, results from the exhaust of propellant gases, which is achieved by the rapid expansion of the heated gas. The heat usually comes from the combustion of chemical propellants—which we will assume in the following without loss of generality—or from the supply of external heat, or from both. A chemical propellant, therefore, serves two different purposes at the same time: it is a provider of mass for the required mass flow rate and a provider of energy to accelerate itself to ejection velocity.

We now want to know how an engine converts combustion heat into thrust, that is, how the expansion of propellant gases can be described in terms of thermodynamics, and how, with a given amount of energy in the combustion chamber, we can determine and maximize the thrust of the propulsion with an adequate combustion chamber and nozzle design. In doing so, we will first assume a propulsion engine with an arbitrarily formed combustion chamber and nozzle. It is not important for us how the propellant actually gets into the chamber, but we merely assume that it somehow appears there with a given mass rate $\dot{m}_p$, and that it carries a certain amount of energy per mole, which, upon combustion. heats it. The energy might even be supplied externally. The total received thermal energy per mole of propellant mass is the molar enthalpy, which we label h_0. The conditions mentioned above generally apply to mono-, and bipropellant, hybrid, dual mode, and thermoelectric thrusters. Even cold gas propulsion is applicable if one considers the product of pressure times molar volume in the combustion chamber as a molar enthalpy. Actually, we have the following (see Fig. 4.1):

Essential engine requirements A propellant gas with a known pressure p_0, a received molar enthalpy h_0, and excited internal degrees of freedom n (see Eq. (4.1.2)) flows with mass flow rate $\dot{m}_p$ and with sound velocity (see Eq. (4.1.13)) through a narrow throat having cross section A_t, and escapes through a widening nozzle by means of controlled expansion. This constrictional geometry is called a *Laval nozzle.*

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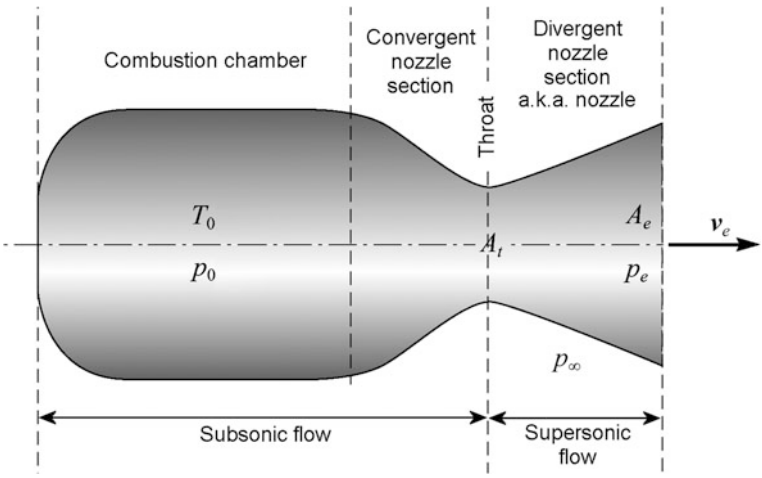


Fig. 4.1 Thrust chamber with Laval nozzle: sections and parameters

This chapter deals with understanding the behavior of the propellant gas in such an engine, and its optimal design for maximum thrust. Section 4.1 will follow the gas flow from the combustion chamber (index 0), through the throat of the combustion chamber (index t, throat) along the nozzle to the nozzle exit (index e, exit), and outside (index ∞) (see Fig. 4.1). In Sects. 4.2 and 4.4, we will analyze the optimal engine design after having determined the thrust of a thermal engine in Sect. 4.3.

4.1 Engine Thermodynamics

4.1.1 Physics of Propellant Gases

The properties of a thermal thruster are essentially determined by the properties of the propellant gas while flowing through the engine, from the combustion chamber right down to the nozzle exit. In order to understand the thrust characteristics, we therefore need to understand the basic behavior of a gas. Let us have a look at the physical and chemical characteristics of a propellant gas, before we apply this knowledge to calculate the thrust and optimize the design of the engine. The theory we will lay out below is based on the assumption that we have an *ideal engine*.

An **ideal thermal engine** is characterized by:

- The propellant is an ideal gas. In particular no viscous effects occur.
- Any change of gas state is adiabatic (heat losses to the walls are negligible).
- No frictional losses, no boundary layer losses.
- The gas is spatially homogeneous in the chamber.
- The gas flow is one-dimensional, i.e. the flow is axisymmetric.
- Gas properties (in particular the drift velocity) are constant across any plane normal to the flow.
- Steady gas flow at the exit of the combustion chamber and beyond (i.e. no shock waves).
- The gas composition does not change in space and time after combustion.
- No multi-phase flow (i.e., no liquid drops or solid particles in the gas).
- The acceleration of the rocket is negligible compared to acceleration of the gas.

In addition we assume (if not stated otherwise) that there is no beam divergence at the nozzle exit. In terms of Sect. 1.3.3 this all means that

$$\overline{v}_e = v_e$$
$$\eta_{div} = \eta_{VDF} = \zeta_d = \zeta_v = 1$$

Due to the last assumption our theory is a one-dimensional theory of gas flow, which allows us to describe spatial conditions in the engine by only one variable, i.e., the distance traveled along the engine axis.

Ideal Gas Thermodynamics

With these assumptions we are now going to exploit the behavior of the gas along the engine axis. Because the gas is ideal, the so-called *intensive thermodynamic variables* pressure p, gas density $\rho = m_p/V$, and gas temperature T (variables not depending on the amount of gas, in contrast to those which do depend on the amount of gas such as gas mass and volume, the so-called *extensive thermodynamic variables*) are interdependent as described by the ideal gas law:

$$p = \rho \frac{R}{M_p} T \quad \textbf{ideal gas law} \tag{4.1.1}$$

with

R **universal gas constant** with value $R = 8.314\ \mathrm{J\,K^{-1}mol^{-1}}$

M_p **molar mass** (mass per mole = mass per 6.022×10^{23} molecules) of the propellant

When the gas pressure forms due to combustion in the chamber, and when the gas thermally expands along the nozzle axis, internal energy, pressure, density, and temperature are constantly changing. We assume that the corresponding conversion

processes are reversible and adiabatic. Physically speaking, such processes are called isentropic—they preserve entropy. The thermodynamic potential that adequately describes an isentropic process is the so-called enthalpy. It encompasses the internal energy U (thermal energy) of the gas, which corresponds to the microscopic, statistical motion of the molecules and internal molecular energy, plus the macroscopic displacement work pV of the gas, but not its macroscopic flow energy (kinetic energy). The enthalpy H and the molar enthalpy h of an ideal gas with mass m_p are given by

$$H := U + pV = m_p c_p T \tag{4.1.2}$$

and

$$h = \frac{\kappa}{\kappa - 1} RT$$

with

$$c_p = \frac{\kappa}{\kappa - 1} \frac{R}{M_p} \quad \textit{specific thermal heat capacity} \text{ at constant pressure}$$

$$\kappa = \frac{c_p}{c_V} = 1 + \frac{2}{n} \approx 1 + \frac{2}{8} = 1.25 \quad \textbf{heat capacity ratio}$$

c_V *specific thermal heat capacity* at constant volume
n average number of **excited degrees of freedom** of the gas molecules

Molecular Degrees of Freedom

The number of excited degrees of freedom n of the gas particles is an important characteristic of the gas, since it determines in how many micromechanical forms thermal energy is stored in the gas. So, n, as well as the heat capacity ratio, may be considered as a indicator of the energy storage capacity of the gas molecules. As will be seen in a moment, among other things, this determines the temperature of the gas. The value of n depends on the specific type and composition of the gas. A propellant gas is usually composed of different types of molecules. Each gas component can move in all three directions in space, so it always has three translatory degrees of freedom $n_{trans} = 3$. If the gas component is monatomic, the atom is not able to take up any more internal energies (gas ionization can be neglected in propellant engines), and $n = n_{trans} = 3$. For diatomic molecules there are two additional rotational degrees of freedom $n_{rot} = 2$(two rotational axes perpendicular to the molecular axis—the rotation around the molecular axis does not count, as quantum-mechanically there exists no corresponding moment of inertia), and one vibrational degree of freedom along the molecular axis, $n_{vib} = 1$, so in total $n = 6$. Polyatomic molecules mostly have a three-dimensional configuration, and thus three rotational axes and three vibrational degrees of freedom, so $n = 9$. Two important exceptions, however, are: the linear CO_2 molecule with $n_{rot} = 2$ and $n_{vib} = 1$, therefore $n = 6$, and the important planar H_2O

with $n_{rot} = 3$ and $n_{vib} = 2$, implying $n = 8$. For a propellant gas mixture the actual number of degrees of freedom is a stoichiometric average over all gas components, so in general it is not integer.

If you heat up a gas, the heat energy is distributed via collisions evenly between the molecules to cause translational, rotational, and vibrational motion. According to the equipartition theorem of statistical mechanics each degree of freedom of a particle, be it translatory, rotational, or vibrational, has the same average energy in thermal equilibrium, namely $\frac{1}{2}k_BT$, no matter how massive the particle. But only translational motion determines the temperature of the gas. Physically speaking, the temperature of a gas is the kinetic energy of the average microscopic, translational motion, the so-called root-mean-square velocity v_{rms}, i.e., of the velocity of its molecules, only. The corresponding relation is

$$\frac{n_{trans}}{2}k_BT = \frac{1}{2}m_p v_{rms}^2$$

where k_B is the Boltzmann constant. If one were able to limit the motion of the gas to a line or a surface, it would therefore have the translational energy $\frac{1}{2}k_BT$ and $\frac{2}{2}k_BT$, respectively. Generally, gas can move freely in all three dimensions, in case of which it possesses $\frac{3}{2}k_BT$ translational energy. Although rotational and vibrational motions physically also constitute energy, they do not contribute to temperature. Nevertheless, they altogether make up the internal energy U of the gas. Because there is no lower limit to translational energy and since the quantum energy of molecule rotation is very low, translational and rotational modes are always excited. Quantum vibrational energies are much higher and start to get excited at around 800 K. Therefore at intermediate temperatures the number of excited degrees of freedom increases while κ monotonously decreases with increasing temperatures. Dissociation of molecules starts at about 2500 K and finally at extremely high temperatures, beyond 9000 K, which do not occur in thrusters, one also would have to consider ionizing degrees of freedom, i.e., gas plasma.

The number of theoretically accessible degrees of freedom of the molecules is between 3 for monatomic noble gases and 9 for three-dimensional polyatomic molecules, or equivalently $1.22 \leq \kappa \leq 1.67$. We therefore arrive at the following rule of thumb:

> Due to the high combustion chamber temperatures, almost all degrees of freedom of the mostly polyatomic molecules are excited, and as the gas is also a mixture of different components, $n \approx 8$ is a good average value for any rocket propellant with the corresponding heat capacity ratio $\kappa \approx 1.25$.

Two examples are: $n(O_2/H_2) = 7.41$, $n(O_2/N_2H_4) = 8.70$, where X/Y denotes all reaction components of the oxidant X and the propellant Y. In the first case, the

planarity of the resulting water molecule is obviously responsible for a relatively low number of degrees of freedom.

Example

What is the heat capacity ratio of dry air at standard conditions?

The molar composition of dry air is 78% N_2, 21% O_2 and 1% Ar, which is 99% diatomic molecules with excited degrees of freedom $n = 5$ (at ambient temperature the one vibration mode is not excited at room temperature) and 1% atoms with $n = 3$. So we find $\kappa_{air} = 0.99 \cdot 1.400 + 0.01 \cdot 1.667 = 1.403$. This is exactly the value as given in relevant tables of thermodynamics.

Thermal Efficiency

On the path from the combustion chamber along the nozzle to the exit, the energy of the gas is continuously converted: Internal energy (heat) is converted into gas expansion work pV and macroscopic flow energy, and the other way round. Thereby the intensive thermodynamic variables T, p, and ρ change in line with the gas Eq. (4.1.1) and the macroscopic drift velocity v changes as well. In physics the so-called *thermal efficiency* η (a.k.a. *thermodynamic* or *ideal cycle efficiency*) describes how efficiently internal energy (heat) is converted into work and gas flow in course of these energy changes. As we assume only adiabatic (isentropic) processes the following thermodynamic relations hold:

$$\eta = 1 - \frac{T}{T_0} = 1 - \left(\frac{p}{p_0}\right)^{\frac{2}{n+2}} = 1 - \left(\frac{\rho}{\rho_0}\right)^{\frac{2}{n}} \leq 1 \quad \textbf{thermal efficiency} \tag{4.1.3}$$

From this follows

$$\rho = \rho_0 (1 - \eta)^{n/2} \tag{4.1.4}$$

In the following, it would be rather tedious to express the thermodynamic equations either by p (which is appropriate in most circumstances), or by T, or by ρ. We want to free ourselves from this ambivalence by the following convention

> We use the thermal efficiency η as a substitute for p, T and ρ, which permits us to always and immediately shift to any of these intensive thermodynamic variables by the application of Eq. (4.1.3).

In this way η is like a convenient exchange currency. This representation also has the very practical benefit of considerably simplifying the equations, though the equations derived will look quite different from those in the literature. Therefore for the most important equations, we will also cite them in their familiar notation.

4.1.2 Flow Velocity

Let v be the gas flow velocity (drift velocity) and T its temperature at any point along the engine axis. According to the general law of the conservation of energy the total energy, i.e., the kinetic energy (macroscopic flow energy) plus the enthalpy of the gas, expressed by the state variables v, T, has to be the same as in the combustion chamber (v_0, T_0)

$$\frac{1}{2}m_p v_0^2 + m_p c_p T_0 = \frac{1}{2}m_p v^2 + m_p c_p T \tag{4.1.5}$$

Assuming that the gas flow velocities in the combustion chamber at the location of heat generation are negligibly small, $v_0 \approx 0$, we derive from this the gas flow velocity at any position in the engine

$$v^2 = 2c_p T_0 \left(1 - \frac{T}{T_0}\right) = 2c_p T_0 \eta \tag{4.1.6}$$

By making use of Eq. (4.1.2) this equation leads to the important relationship

$$\boxed{v = a_0\sqrt{n\,\eta}} \tag{4.1.7}$$

Remark 1 Equation (4.1.7) *is known in literature as St. Venant–Wantzel equation*

$$v = \sqrt{\frac{2\kappa}{\kappa - 1}\frac{R}{M_p}T_0\left[1 - (p/p_0)^{\frac{\kappa-1}{\kappa}}\right]}$$

Remark 2 *Although* v_0 *is not exactly zero, the neglected term* $\frac{1}{2}m_p v_0^2$ *in* Eq. (4.1.6) *formally can be ascribed to the chamber temperature* via

$$T_0 \to T_0' = T_0 + \frac{1}{2}\frac{v_0^2}{c_p}$$

The expression $m_p c_p T_0'$ *is then the total combustion enthalpy.*

In the above we have made use of the expression for the sound velocity

$$a = \sqrt{\frac{\kappa p}{\rho}} = \sqrt{\frac{\kappa RT}{M_p}} = \sqrt{\frac{\kappa RT_0}{M_p}}\sqrt{\frac{T}{T_0}} =: a_0\sqrt{1-\eta} \quad \textbf{speed of sound} \tag{4.1.8}$$

as given by physics. The latter is a result of Eq. (4.1.3). Note that Eq. (4.1.7) is valid for the gas velocity at any point along the flow right down to the nozzle exit. Also note that a decreasing pressure or temperature leads to an increase of thermal

efficiency and also to an increasing flow velocity. We will have a closer look at this counter-intuitive behavior later on.

4.1.3 Flow at the Throat

We recall from Eq. (1.2.8) that the following continuity equation holds for the mass flow rate

$$\boxed{\dot{m}_p = \rho\, vA = const} \quad \textbf{continuity equation} \tag{4.1.9}$$

Since it is a result of mass conservation, the mass flow rate $\dot{m}_p$ must be constant over any cross section along the flow track.

Remark *Strictly speaking, we presume* $\dot{m}_p = const$ *from which an areal cross section is defined* via *the continuity equation, on which* ρ *und* v *are constant. If the gas jet diverges, i.e., is no longer axial, such as in the nozzle, the area is no longer flat, but in lowest approximation a sphere segment. The flow vector is normal to the surface of this segment. The center of the sphere is the imaginary point where the gas flow lines converge. Depending on whether the exiting jet is under- or over-expanding (see* Fig. 4.4 *and according text) the exit area is a convex or concave sphere segment, respectively.*

We now define the important parameter *mass flux* (a.k.a. *mass flow density)* $\mu := \dot{m}_p/V$. For μ we find from Eqs. (4.1.4), (4.1.7), and (4.1.8):

$$\mu := \frac{\dot{m}_p}{A} = a_0\rho_0\sqrt{n\eta(1-\eta)^n} = a\rho\sqrt{\frac{n\eta}{1-\eta}} \tag{4.1.10}$$

Because $\dot{m}_p = const$, μ is not constant. As mass flow is constant and the nozzle cross section has its minimum at the throat, the mass flux μ at a constant a_0 and ρ_0 has a maximum at the throat with

$$\eta_t = \frac{1}{n+1} \tag{4.1.11}$$

Equation (4.1.11) is easily derived by zeroing the first derivation of Eq. (4.1.10). By applying Eqs. (4.1.11) to (4.1.10) we find for the mass flux at the throat

$$\mu_t = \frac{\dot{m}_p}{A_t} = a_0\rho_0\sqrt{\left(\frac{n}{n+1}\right)^{n+1}} = a_t\rho_t \tag{4.1.12}$$

where the last term results from the last term of Eq. (4.1.10) due to $n\eta_t/(1-\eta_t) = 1$. In addition we get from Eqs. (4.1.9), (4.1.12), and (4.1.8) the flow velocity at the throat

$$v_t = \frac{\dot{m}_p}{\rho_t A_t} = \frac{\rho_t a_t A_t}{\rho_t A_t} = a_t = a_0\sqrt{\frac{n}{n+1}} \tag{4.1.13}$$

In words this is equivalent to the following:

> The flow velocity just reaches sound velocity at the throat. This is an essential property of thermal propulsion engines.

Note *Sound velocity at the throat here is not the conventional* $a_{air} \approx 343.4\ \mathrm{m\,s^{-1}}$ *at standard atmosphere, but due to* Eq. (4.1.8) $a_t = a_{air}\cdot\sqrt{T_t/T_{air}} \approx 1200\ \mathrm{m\,s^{-1}}$. *So,* a_t *is much bigger and in addition dependent on* T_t *and hence on the temperature and pressure conditions in the pressure chamber.*

Let us pause for a moment to ponder about what we have achieved so far, and what lies still ahead of us. Strictly speaking, the above maximum mass flux at the throat, and with it this whole chapter, describes a key property of a thermal thruster. In view of the maximization principle, the physical principle of mass conservation as expressed in Eqs. (4.1.9) and (4.1.10), and the conversation of energy (whose result led to Eq. (4.1.7)), all considerations following now are just more or less clever applications, new definitions and analytical conversions of Eqs. (4.1.7), (4.1.9), (4.1.10), and (4.1.12).

4.1.4 Flow in the Nozzle

Behind the throat, the gas expands into the widening nozzle (this constrictional geometry from the chamber through a narrow throat to a nozzle with an hourglass-shape that widens in the flow direction, is called a *Laval nozzle*, see Fig. 4.3) and exits the nozzle at the exit cross section A_e with velocity v_e. Obviously, the exit gas pressure p_e and v_e will depend on the expansion ratio A_e/A_t. Applying Eq. (4.1.10) to the nozzle exit and from Eq. (4.1.12) we find for this expansion ratio

$$\varepsilon := \frac{A_e}{A_t} = \frac{\dot{m}_p}{A_t}\frac{A_e}{\dot{m}_p} = \frac{a_0\rho_0\sqrt{\left(\frac{n}{n+1}\right)^{n+1}}}{a_0\rho_0\sqrt{n\,\eta_e(1-\eta_e)^n}}$$

So

$$\varepsilon := \frac{A_e}{A_t} = \frac{C_\infty}{n+2}\frac{1}{\sqrt{(1-\eta_e)^n \eta_e}} \qquad \textbf{expansion ratio} \tag{4.1.14}$$

with

$$C_\infty := (n+2)\sqrt{\frac{n^n}{(n+1)^{n+1}}} \approx 0.624\frac{n+2}{\sqrt{n+1}} \qquad \textbf{infinite-expansion coefficient}$$

where the latter follows for $n \approx 8$ from Problem 4.2. We call C_∞ the *infinite-expansion coefficient* for reasons, which will become clear in Sect. 4.3.2. For our standard $n = 8$ ($\kappa = 1.25$) it has the value $C_\infty = 2.0810$.

Remark 1 In literature Eq. (4.1.14) reads

$$\varepsilon = \frac{A_e}{A_t} = \sqrt{\frac{\kappa-1}{2}\left(\frac{2}{\kappa+1}\right)^{\frac{\kappa+1}{\kappa-1}}} \cdot \left(\frac{p_0}{p_e}\right)^{1/\kappa}\left[1-\left(\frac{p_e}{p_0}\right)^{1-1/\kappa}\right]^{-1/2}$$

Remark 2 *In the literature the quantity* $\Gamma = C_\infty/\sqrt{n+2}$ *is sometimes called Vandenkerckhove Function.*

From Eq. (4.1.14) we see that there is a one-to-one correspondence between ε and η_e and due to Eq. (4.1.3) also between ε and all intensive thermodynamic variables p_e, T_e, ρ_e. We could therefore invert Eq. (4.1.14) to derive for a given ε the quantity $\eta_e(\varepsilon)$ and hence also $p_e(\varepsilon)$, $T_e(\varepsilon)$, $\rho_e(\varepsilon)$ via

$$\left(\frac{p_e(\varepsilon)}{p_0}\right)^{\frac{2}{n+2}} = \frac{T_e(\varepsilon)}{T_0} = \left(\frac{\rho_e(\varepsilon)}{\rho_0}\right)^{\frac{2}{n}} = 1-\eta_e(\varepsilon)$$

However, $\eta_e(\varepsilon)$ can be derived only numerically since the relation between η_e and ε is too convoluted. In Fig. 4.2, we therefore provide a graphical solution for η_e and p_e. It illustrates that η_e monotonously increases and hence p_e monotonously decreases with an increasing ε. Analogous diagrams can of course be drawn also for T_e and ρ_e. Because any cross section along the nozzle axis can be considered as a momentaneous exit, all these dependencies also holds along the nozzle axis. We can thus depicture the dependency of all intensive variables along a nozzle, which is done in Fig. 4.3.
From a practical point of view it is interesting to know how the expansion ratio changes if the chamber pressure is varied at $p_e = const$. From all the above we derive, after some minor algebra, that for small variations

$$\frac{\Delta\varepsilon}{\varepsilon} = \left[1-\frac{p_e}{2(p_0-p_e)}\right]\frac{\Delta p_0}{p_0} \approx \frac{\Delta p_0}{p_0}$$

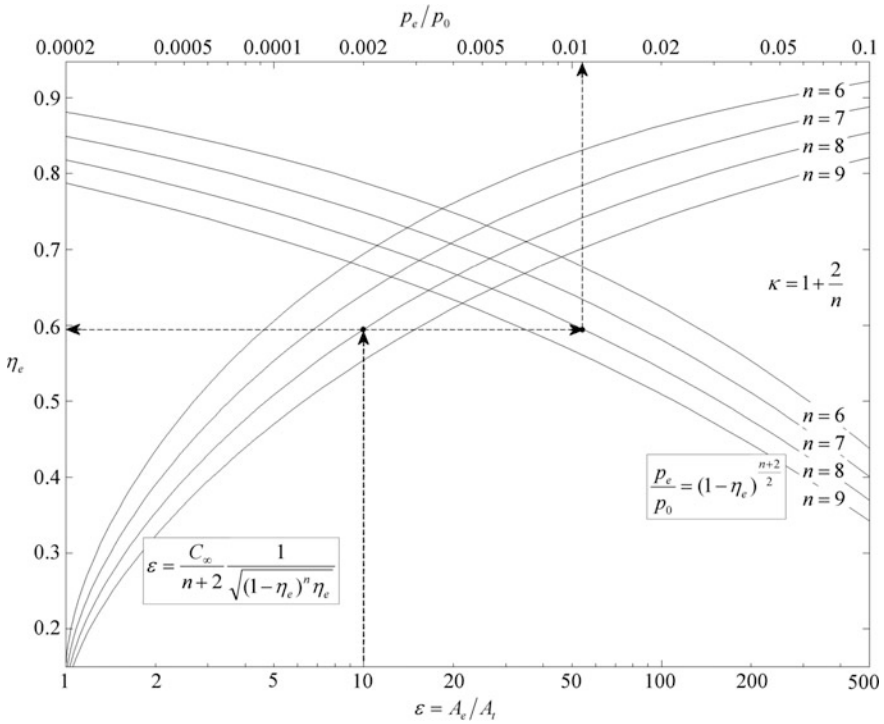


Fig. 4.2 Thermal efficiency η_e and pressure p_e at nozzle exit versus the expansion ratio ε for different excited degrees of freedom n of the gas molecules. In the figure shown is the example for $\varepsilon = 10$ and $n = 8$ leading to $\eta_e \approx 0.6$ and $p_e/p_0 \approx 0.01$

Flow velocity

According to Eqs. (4.1.3) and (4.1.7) the flow velocity increases with increasing expansion ratio (see Fig. 4.3). This is exactly what a nozzle is designed for. How big is this increase? The relation between the ejection velocity v_e and v_t can be derived from Eqs. (4.1.7) and (4.1.13) to be

$$v_e = a_0\sqrt{n\eta_e} = v_t\sqrt{n+1}\sqrt{\eta_e} \tag{4.1.15}$$

In other words:

> The velocity gain factor of a nozzle is $\sqrt{n+1}\sqrt{\eta_e} \approx 3\sqrt{\eta_e}$, which in vacuum tends to the value of 3. So a nozzle increases momentum thrust by 200%, but because the expansion at the same time reduces the pressure thrust, the gain in total thrust is less than 67% (see Sect. 4.3.3).

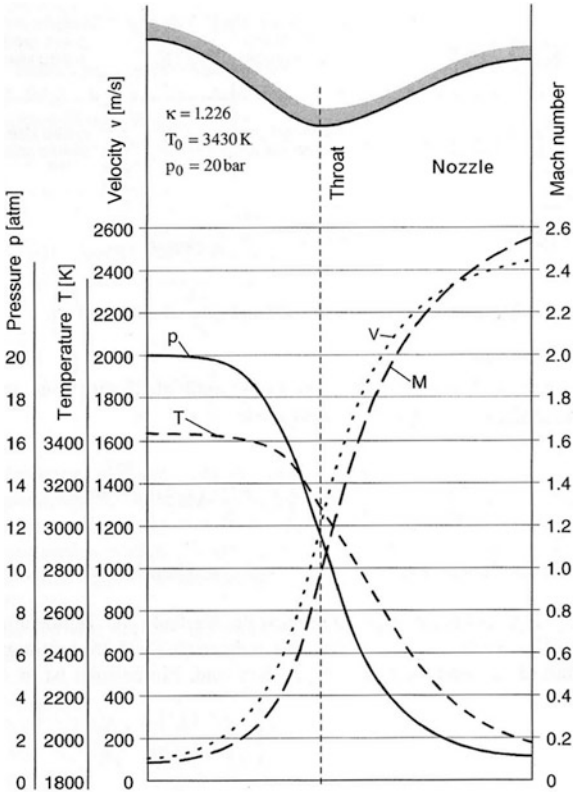


Fig. 4.3 Course of thermodynamic variables along a Laval nozzle. *Credit* Messerschmid, Fasoulas (2000)

It might be surprising to learn that the flow velocity increases with decreasing flow pressure. Intuitively, one would expect the contrary. Let us see why this seemingly paradoxical behavior occurs.

Hydrodynamics

This weird behavior is due to the hydrodynamic nature of the gas flow. The continuity Eq. (4.1.9)

$$\rho v A = const$$

hereby plays a key role. It states that, if the cross section A of the nozzle increases, the product ρv has to decrease. It now crucially depends on the dependence $\rho \propto 1/v^{\alpha}$, or equivalently its differential $d\rho/\rho = -\alpha \cdot dv/v$, and hence on the exponent α how v behaves: If $\alpha > 1$, then due to $\rho \cdot v \propto 1/v^{\alpha} \cdot v = v^{1-\alpha}$ v increases with increasing A, otherwise v decreases, or vice versa.

So we have to determine α to solve the paradox. We start out by examining the ideal gas Eq. (4.1.1), $p = \rho RT/M_p$, and the equation of energy conservation (4.1.6), $v^2 = 2c_p(T_0 - T)$. To arrive at one-to-one relations between the intensive thermodynamic variables p, ρ, T, v, which we are looking for, we need an additional relation between any two of them. This is furnished by the fact that according to thermodynamics for adiabatic processes $pV^\kappa = const$ holds. Because in general $\rho V = const$, we find $p \propto \rho^\kappa$ and hence

$$\frac{dp}{p} = \kappa \frac{d\rho}{\rho} \tag{4.1.16}$$

So p and ρ change in the same way: increasing p implies increasing ρ and vice versa (which we write in short as $p \uparrow \leftrightarrow \rho \uparrow$). In order to apply this differential equation, we have to differentiate $p = \rho RT/M_p$ and $v^2 = 2c_p(T_0 - T)$, which yields

$$\frac{dp}{p} = \frac{d\rho}{\rho} + \frac{dT}{T}$$

and

$$v\, dv = -c_p dT$$

With Eq. (4.1.16) we find the one-to-one relations

$$\frac{d\rho}{\rho}(\kappa - 1) = \frac{dT}{T} = -\frac{v^2}{c_p T}\frac{dv}{v} \tag{4.1.17}$$

Because $\kappa > 1$, they state the relationship $\rho \uparrow \leftrightarrow T \uparrow \leftrightarrow v \downarrow$. The latter inverse behavior, which seems quite strange, is due to the law of energy conservation, to which we will turn later. Because of Eqs. (4.1.2) and (4.1.8) it follows that $c_p(\kappa - 1)T = \kappa RT/M_p = a^2$ and hence

$$\frac{d\rho}{\rho} = -\frac{v^2}{a^2}\frac{dv}{v} = -(Ma)^2\,\frac{dv}{v} \tag{4.1.18}$$

where $Ma := v/a$ is the so-called **Mach number**. It is dimensionless by relating the flow velocity to the instantaneous speed of sound. With Eq. (4.1.18) we have identified $\alpha = (Ma)^2$. To find the explicit dependencies $v \leftrightarrow A$ and $\rho \leftrightarrow A$, we differentiate $\rho vA = const$ and get

$$\frac{d\rho}{\rho} + \frac{dv}{v} + \frac{dA}{A} = 0$$

Inserting Eq. (4.1.18) into this, we finally obtain

$$\frac{dv}{v} = \frac{1}{(Ma)^2 - 1}\frac{dA}{A} \quad \textbf{area-velocity equation} \tag{4.1.19}$$

or

$$\frac{d\rho}{\rho} = -\frac{(Ma)^2}{(Ma)^2 - 1}\frac{dA}{A}$$

From this we can read off the change of flow velocity as a function of change of cross section. We have to discern two cases:

1. subsonic case $(Ma < 1)$: $A \uparrow \rightarrow \rho \uparrow \ \& \ v \downarrow \leftrightarrow \ T \uparrow$
2. supersonic case $(Ma > 1)$: $A \uparrow \rightarrow \rho \downarrow \ \& \ v \uparrow \leftrightarrow \ T \downarrow$

In the subsonic case the flow velocity declines along the nozzle and its density increases, while for the supersonic case things are reverse.

> **Note** *This implies that, if the flow would not reach sound velocity at the throat, the flow in the nozzle would stay subsonic and even decrease. The condition $v = a$ at the throat therefore is a critical condition for a thermal propulsion engine.*

Physical Interpretation

This leaves open the question, why the flow behaves so differently at subsonic and supersonic speeds. This is due to the kinetic energy of the flow. Let us take a look at the law of energy conservation of the gas as given by Eq. (4.1.5)

$$m_p c_p T + \frac{1}{2} m_p v^2 = const$$

It shows that the kinetic energy increases quadratically with flow velocity. So any change in gas temperature implies decreasing flow speed changes for increasing flow speeds. This is expressed explicitly in Eq. (4.1.17): relative density and temperature changes are rigidly coupled, while the coupling between temperature and speed changes is quadratically in v. Therefore, for a given $\Delta\rho/\rho$ and with increasing v the absolute value of $\Delta v/v$ decreases.

So, overall the following happens. The space available for a given amount of gas enlarges with increasing cross section along the nozzle, $A \uparrow$. Thereby the density and the speed of the gas, flowing into the enlarging space ahead of it, change such that the amount of gas remains constant (conservation of mass, $\rho v A = const$). If the density would increase, $\rho \uparrow$, then due to $\rho v A = const$, the flow speed v would have to strongly decrease, $v \downarrow\downarrow$. But at supersonic speeds this is not possible, because due to the gas equation $\rho \uparrow \leftrightarrow \ T \uparrow$ and because of the equation of energy conservation v decreases only little. Only at subsonic flow speeds the flow speed reduction would be big enough to compensate. Therefore at supersonic speeds the density has to drop, $\rho \downarrow$, causing the gas temperature via the gas equation to drop as well, which in turn raises the flow speed only little, $v \uparrow$. But this is all what is necessary according to $\rho v A = const$ because the density reduction already counteracts the increase of cross section to a large amount.

From the above equation of energy conservation with implication $T \uparrow \leftrightarrow v \downarrow$ one can read off another important feature: the first term is enthalpy, which is the internal microscopic kinetic energy plus displacement work pV, and the second term is the kinetic flow energy of the gas. But because temperature is a direct measure of the translational microscopic motion of the gas molecules, as seen early on in this chapter, the inverse dependency can be interpreted in the following way:

> Supersonic expansion along the nozzle converts part of the microscopically disordered motion of the molecules (microscopic translations) into an increasingly macroscopically directed motion (flow velocity) of the gas.

In view of the second law of thermodynamics, which always demands that entropy increases and hence implies a decreasing motional order, one may ask how an increasingly directed motion can form from random motion. The answer is that the second law of thermodynamics holds for a gas in equilibrium. In a diverging nozzle just behind the throat, where gas molecules move at and above the speed of sound, the gas is in a highly non-equilibrium phase. Physics (see e.g. Reimann (2002)) has shown that a non-equilibrium gas in an asymmetric structure, such as a bell nozzle that causes an asymmetric potential in which the molecules move, can approach equilibrium by rectifying random motion into orderly directed molecular flow along the asymmetry line. This effect is known as *fluctuation-driven transport*.

Upon temperature reduction, fewer rotational and vibrational modes of the gas molecules are excited. By means of impact processes they decay into microscopic translations thus adding slightly to the flow velocity of the gas. However, this additional conversion effect, which lowers the excited degrees of freedom n and adds to the nozzle efficiency (see Sect. 4.3.3), is not considered in our derivations. We assume that n or κ is constant with temperature.

4.2 Ideally Adapted Nozzle

4.2.1 Ideal-Adaptation Criterion

A nozzle is designed to increase ejection velocity. However, our goal is not to maximize ejection velocity, but thrust. In order to do so, let us have a look at the expression for total thrust. If the exit surface normal $\boldsymbol{u}_e$ and $\boldsymbol{v}_e$ go into the same directionamount of total thrust of a real engine according to Eq. (1.2.14) (for the sake of simplicity we drop the average sign throughout this section)

$$F_* = \eta_{div}\dot{m}_p v_e + (p_e - p_\infty)A_e$$

Expanding the gas through the nozzle not only increases v_e but also reduces p_e. The first term on the right-hand side tells us that this implies an increase of the momentum thrust, but the second term leads to a reduction of the pressure thrust. So, what we really want to know is how to choose p_e or v_e to maximize the thrust as the combined force of momentum thrust and pressure thrust.

Our optimization problem is even a bit more complicated. According to the above equation, thrust depends on three parameters, p_e, v_e, and A_e, if the mass flow rate is considered to be constant. The latter assumption is generally admissible, as supersonic speed in the nozzle has the positive effect that erratic flow variations cannot expand backward into the combustion chamber. The Laval nozzle acts like a barrier, and the mass flow rate of the propulsion is only determined by the pressure in the combustion chamber and the diameter of the exit (see Eq. (4.3.2)). It is our long term goal to find that optimal combination of $p_{e,opt}$, $v_{e,opt}$, and $A_{e,opt}$ for which thrust is maximum.

Mathematically speaking optimization means that we are looking for that combination where any variations δA_e, δv_e, and δp_e do not lead to any further increase of F_*, that is, $\delta F_* = 0$. If we consider infinitesimally small variations, this can be expressed mathematically by the total differential as follows:

$$dF_* = dF_A + dF_e + dF_p = \frac{\partial F_*}{\partial A_e} dA_e + \frac{\partial F_*}{\partial v_e} dv_e + \frac{\partial F_*}{\partial p_e} dp_e = 0 \qquad (4.2.1)$$

Deriving the partial derivatives from Eq. (1.2.14) we find

$$dF_* = \left(p_{e,opt} - p_\infty\right) dA_e + \eta_{div} \dot{m}_p dv_e + A_{e,opt} dp_e = 0$$

We now consider the fact that the thermodynamic variables v_e and p_e are directly dependent on each other via Eqs. (4.1.15) and (4.1.3). With these relationships and with application of Eq. (4.1.10) we can calculate (exercise, Problem 4.1) the derivation dv/dp straightforwardly. Alternatively, Eq. (4.1.16) can be inserted into Eqs. (4.1.17) and (4.1.9) applied. In both cases one obtains

$$dv = -\frac{A}{\dot{m}_p} dp$$

This relation describes the change of flow velocity with pressure at any cross section A along the nozzle while the mass flow rate $\dot{m}_p$ is constant. Interestingly, it states that the flow velocity increases at decreasing gas pressure (see discussion of this effect in Sect. 4.1.4). Applying this equation to the nozzle exit and inserting it into the above equation yields

$$dF_* = \left(p_{e,opt} - p_\infty\right) dA_e + (1 - \eta_{div}) A_{e,opt} dp_e = 0$$

This brings us to the required condition for maximum thrust:

$$\frac{dA_e}{dp_e} = -\frac{(1-\eta_{div})A_{e,opt}}{p_{e,opt}-p_\infty}$$

From Eq. (4.1.14) we derive that

$$\frac{dA_e}{dp_e} = -\frac{A_{e,opt}}{p_0}\frac{1}{n+2}\frac{n-(1-\eta_e)}{\eta_e(1-\eta_e)^{n/2+1}}$$

and therefore we obtain for the optimal exit pressure

$$\begin{aligned}\frac{p_{e,opt}}{p_\infty} &= 1-\frac{p_0}{p_\infty}(1-\eta_{div})(n+2)\frac{\eta_e(1-\eta_e)^{n/2+1}}{n-(1-\eta_e)}\\ &= 1-(1-\eta_{div})\frac{(n+2)\eta_e}{n-(1-\eta_e)}\\ &\approx 1-(1-\eta_{div})\underbrace{\frac{10\eta_e}{7+\eta_e}}_{0.79-1.25}\end{aligned}$$

The second equation follows from Eq. (4.1.3) and is exact. The last holds because $n \approx 8$, and because $\eta_{div} \approx 1$ we have $p_{e,opt} \approx p_\infty = 0-1\ b$. From Eq. (4.1.3) and typically $p_0 \approx 100\ b$ hence it follows that $\eta_e = 1.0-0.6$. We therefore find approximately

$$p_{e,opt} \approx \eta_{div} p_\infty \tag{4.2.2}$$

and the maximal thrust is

$$F_{*,\max} = \eta_{div}\dot{m}_p v_{e,opt} - p_\infty(1-\eta_{div})A_{e,opt}$$

and

$$\eta_{e,opt} = 1-\left(\eta_{div}\frac{p_\infty}{p_0}\right)^{\frac{2}{n+2}}$$

For an ideal engine with $\eta_{div} = 1$ we thus can state that

> The thrust (propellant force) for an ideal thermal engine achieves its maximum for $p_e = p_\infty$, i.e., when the pressure thrust vanishes.

Remark *We actually only showed that F_* is optimal at $p_e = p_\infty$. For the proof that it really is maximal see* Fig. 4.7 *in* Sect. 4.3.2.

4.2.2 Ideal Nozzle Design

For $p_e = p_\infty$ the corresponding *optimal* ejection *velocity* can be derived from Eq. (4.1.15) as

$$v_{e,opt} = a_0\sqrt{n\eta_{e,opt}} \approx a_0\sqrt{n\eta_\infty} \tag{4.2.3}$$

By the same token, we derive from $p_e = p_\infty$ and hence $\eta_e \to \eta_\infty$ the optimal expansion ratio ε_{opt} from Eq. (4.1.14) to be

$$\varepsilon_{opt} = \frac{A_{e,opt}}{A_t} = \frac{C_\infty}{n+2}\frac{1}{\sqrt{\left(1-\eta_{e,opt}\right)^n \eta_{e,opt}}} \approx \frac{C_\infty}{n+2}\frac{1}{\sqrt{(1-\eta_\infty)^n\eta_\infty}} \quad \textbf{optimal expansion ratio} \tag{4.2.4}$$

Because we rather are pursuing $A_{e,opt}$ the question remains: Given Eq. (4.2.4) what is the design value of the throat cross section A_t to obtain $A_{e,opt}$? We will answer this question in Sect. 4.4.1. Thus, we have finally determined the optimal parameters $p_{e,opt}$, $v_{e,opt}$, and $A_{e,opt}$.

Flow Expansion

A nozzle that achieves $p_e = p_\infty$ is called an **ideally adapted nozzle**. In a nozzle where the optimum is not achieved, be it $p_e < p_\infty$, which is called *over-expansion* because the jet is over expanded within the nozzle (see Fig. 4.4), or $p_e > p_\infty$, called *under-expansion* because the jet was not able to properly expand within the nozzle, divergences arise behind the nozzle exit. The jet direction then is not longer parallel to the engine axis. This leads to a loss of thrust, as there are thrust components perpendicular to the engine axis, which are then irreversibly lost. Depending on the altitude, the I_{sp} losses due to non-adapted nozzles are typically in the range 0–15%. As pressure adiabatically lowers with increasing volume, thrust reductions due to an over- or under-expanding jet can be counteracted by an increase or decrease of the exit surface, which is equivalent to extending or reducing the length of the Laval nozzle.

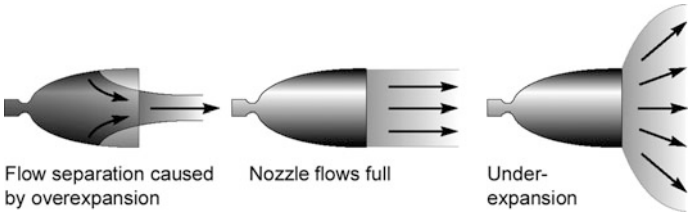


Fig. 4.4 Flow conditions of an over-expanding (left), ideally expanding (middle), and under-expanding (right) nozzle

4.2.3 Shock Attenuation and Pogos

For an ideally adapted nozzle $dF_* = dF_e + dF_p = 0$. Therefore $dF_p = -dF_e$ must hold. In other words, every tiny pressure thrust variation is counterbalanced by a momentum thrust variation. This is quite a remarkable effect, as pressure variations occur within every combustion chamber in the form of shock waves, which travel from the combustion chamber right beyond the nozzle exit. The reverse interdependency $dF_p = -dF_e$ ensures that none of these shock waves has any effect on the total thrust.

The situation is different with pogos, which are the nightmare of thruster manufacturers—and astronauts. Pogos are thrust oscillations along the engine axis, which may put the rocket under enormous stress even beyond its structural limits. As astronaut Michael Collins stated, "The first stage … vibrated longitudinally so that someone riding on it would be bounced up and down as if on a pogo stick." Pogos result from a negative feedback between thrust and mass flow rate. Assume, for instance, that at a given time a an unsteady combustion causes a temporary larger heat release and therefore an increase of chamber pressure. This partially impedes the propellant flow into the chamber, which in turn results in local and even overall pressure decrease in the chamber. The transient pressure build up in the propellant supply lines in combination with the pressure drop inside the chamber yields an increase in propellant mass flow rate (note that due to differences in propellant densities and velocities there will be a phase shift between the two mass flow rates). The frequency of this pogo resonance usually is of the order of 10 Hertz. The partial blocking of the flow rate yields a local extinction of the flame. The influx of unburnt propellant into the hot environment quite often results in a sudden reaction (explosion) with an appropriate heat release and local pressure build-up which starts the cycle again.

Pogos are not taken care of in our above considerations, as we assumed a constant mass flow rate. A small accumulator connected to the fuel line, as for instance the pogo suppressor with about the size of a basketball charged with hot gaseous oxygen in the Shuttle orbiter, as shown in the Fig. 4.5 usually damps them out, because it gives and takes additional propellant depending on the acceleration on the propellant and hence on the fuel pressure in the fuel line.

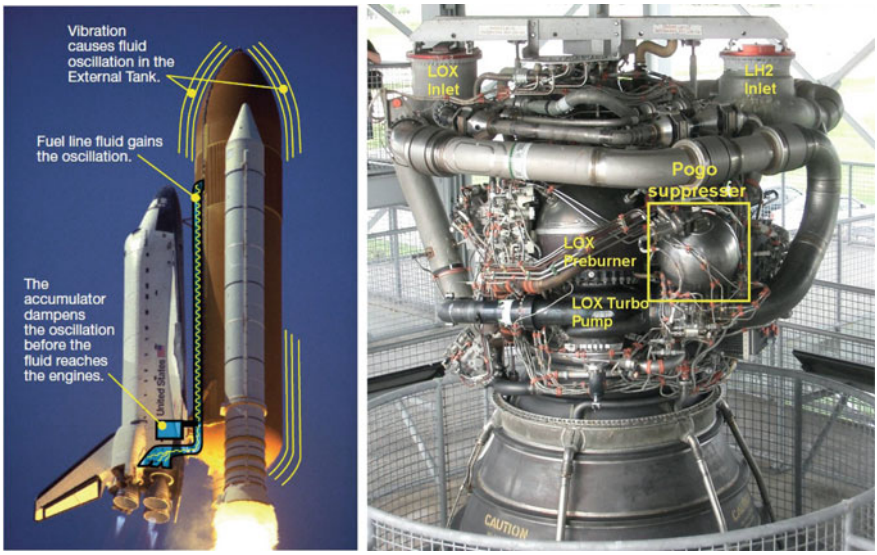


Fig. 4.5 Pogo resonance (left) and pogo suppressor system (right) in the SSME of the Space Shuttle

4.2.4 Ideal Engine Performance

According to Sect. 1.3.3 engine performance and hence rocket performance is measured very generally by the engine's figure of merit: the specific impulse $I_{sp} = v_*/g_0$. Now that we have derived an expression for the ejection velocity of thermal engines, we are ready to determine their I_{sp} from thermodynamic values. Usually I_{sp} is cited in literature for an ideally adapted nozzle in vacuum. At these conditions we get a *maximum obtainable specific impulse*, which can be calculated from Eq. (4.1.15) for $p_e = p_\infty = 0$. This yields with Eq. (4.1.8)

$$g_0 I_{sp,\max} = v_{e,\max} = a_0\sqrt{n} = \sqrt{(n+2)\frac{RT_0}{M_p}} = \sqrt{\frac{2h_0}{M_p}} \quad @ \text{ vacuum} \tag{4.2.5}$$

where $h_0 = M_p H_0/m_p$ is the molar form of the available enthalpy H_0, which comprises the combustion enthalpy H_p (heat of reaction) of the propellant with **combustion efficiency** η_c, that is, $\eta_c H_p$, and the externally supplied energy E_{ext}

$$H_0 = \eta_c H_p + E_{ext} \tag{4.2.6}$$

η_c is determined by the heat losses of the engine, which are quite considerable (see Fig. 4.6). From Eqs. (4.2.5) and (4.2.6) we get the remarkably simple result

For an ideally adapted engine in vacuum, its maximum specific impulse does not depend on the design of the engine or the chamber pressure, but it depends exclusively on the propellant properties and the combustion efficiency of the engine. The best propellant is a propellant with the highest combustion molar enthalpy h_p and lowest molar mass M_p.

Because H_2/O_2 displays one of the highest molar enthalpies and hydrogen has the lowest molar mass, it fuels chemical thrusters with the highest efficiency available. Apart from employing a better propellant, the specific impulse can only be further increased at a given mass flow rate by utilizing the combustion enthalpy as much as possible in the combustion chamber (e.g., by a more efficient precombustion), or by injecting additional external energy (e.g., external nuclear–thermal energy), or by reducing heat losses.

Total Engine Efficiency

The total engine efficiency of a propulsion engine is a measure for the effectiveness of converting the energy released into the combustion chamber E_{in} into exhaust energy (kinetic energy of the exhaust jet). We recall from Eq. (1.3.7) that for $v_e(\theta) \approx const = v_e$ it is given as $\eta_{tot} = \frac{1}{2} m_p v_e^2 / E_{in}$. Without any supply of external energy, one then obtains

$$\eta_{tot} = \frac{m_p v_e^2}{2\eta_c H_p} = \left(\frac{v_e}{v_{e,\max}} \right)^2 \quad @\ E_{ext} = 0 \tag{4.2.7}$$

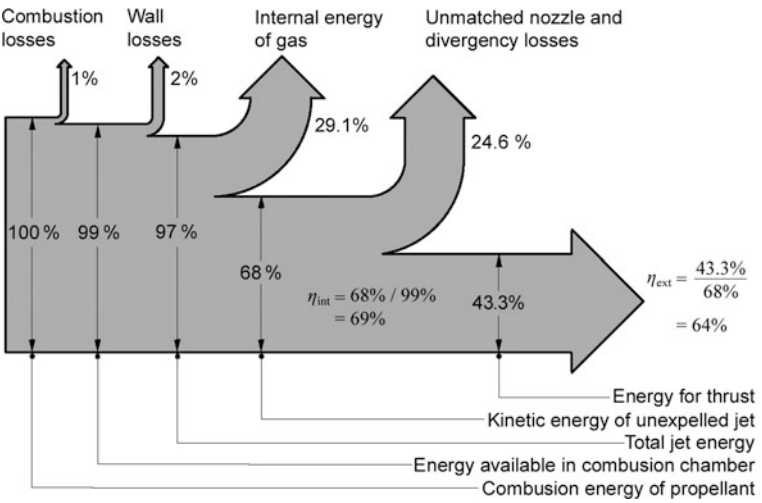


Fig. 4.6 Energy flow of the third stage engine of Ariane 1

Example

The SSME cryogenic engine of the Space Shuttle burned H_2/O_2 with $\eta_c H_p/m_p = 13.4\,\mathrm{MJ/kg}$, i.e., $v_{e,\max} = 5.18\,\mathrm{km/s}$, and exhausted the propellant with $v_e = 4.44\,km/s$. The internal efficiency therefore was $\eta_{tot} = 0.735$, which, equaling the Russian RD-0120 engine. was, and still is the best value for chemical engines.

4.3 Engine Thrust

We saw above that the figure of merit of an engine, $I_{sp,\max}$, does not depend on the engine design, the chamber pressure in particular. The all-important thrust on the other hand does. How does the thrust depend on the engine design? We recall from Eq. (1.2.14) that the thrust of an ideal engine having $v_{ex} = v_e$ is given by

$$F_* = \dot{m}_p v_e + (p_e - p_\infty)A_e$$

Inserting $v_e = a_0\sqrt{n\eta_e}$ from Eq. (4.1.15) we get for the thrust

$$F_* = \dot{m}_p a_0\sqrt{n\eta_e} + (p_e - p_\infty)A_e \tag{4.3.1}$$

How is the mass flow rate $\dot{m}_p$ related to chamber pressure p_0? To figure this out, we apply Eq. (4.1.12)

$$\dot{m}_p a_0\sqrt{n} = a_0^2\rho_0 A_t n\sqrt{\frac{n^n}{(n+1)^{n+1}}}$$

Since $a_0^2\rho_0 = \kappa p_0$ (see Eq. (4.1.8)) and because $\kappa n = n+2$ we get the important relation

$$\dot{m}_p = \frac{A_t C_\infty}{a_0\sqrt{n}}p_0 \tag{4.3.2}$$

where C_∞ is the infinite-expansion coefficient defined in Eq. (4.1.14) . Because of Eq. (4.3.2) we now can rewrite thrust more conveniently for engineering purposes as

$$\boxed{F_* = p_0 A_t C_\infty\sqrt{\eta_e} + (p_e - p_\infty)A_e} \tag{4.3.3}$$

For an ideally adapted nozzle with $p_e = p_\infty$ we have $\eta_e = \eta_\infty$ and thus we find for the thrust

$$F_* = p_0 A_t C_\infty\sqrt{\eta_\infty} \quad @\ p_e = p_\infty \tag{4.3.4}$$

In vacuum the thrust of an ideally adapted engine achieves its maximum value

$$F_* = p_0 A_t C_\infty \quad @\ p_e = p_\infty = 0 \tag{4.3.5}$$

This is merely a theoretical value, as the nozzle exit cross section A_e would then grow infinitely big, as we see from Eq. (4.1.14).

4.3.1 Engine Performance Parameters

Characteristic Velocity c^*

Chamber pressure, propellant mass flow rate, and cross section of the throat need to be in balance to each other. With a larger cross section of the throat, the mass flow rate has to rise, in order to maintain the chamber pressure. Equation (4.3.2) describes the interplay between these three parameters. Their ratio determines the so-called *characteristic velocity* (commonly pronounced "cee-star")

$$c^* := \frac{p_0 A_t}{\dot{m}_p} = \frac{a_0\sqrt{n}}{C_\infty} = \frac{1}{C_\infty}\sqrt{\frac{2h_0}{M_p}} = \frac{g_0 I_{sp,\max}}{C_\infty} = const \tag{4.3.6}$$

characteristic velocity

Because both C_∞ and $I_{sp,\max}$ depend only on propellant properties, c^* is constant and an alternative figure of merit for the engine. From Eq. (4.3.6) and because from Eq. (4.2.6) $h_0 = \eta_c h_p$ it follows that c^* can be considered as a parameter to rate the propellant combustion performance, i.e. the energy level of the propellant, the burn efficiency of the injector, and the heat loss efficiency of the chamber. In practice, the value of c^* for a given propellant and thrust chamber design is guessed from existing experience and refined during development testing.

Given Eq. (4.3.6) the ejection velocity from Eq. (4.1.15) can be written as

$$v_e = c^* C_\infty \sqrt{\eta_e} = g_0 I_{sp,\max}\sqrt{\eta_e} \tag{4.3.7}$$

Thrust Coefficient C_f

Equation (4.3.3) gives rise to the definition of the so-called *thrust coefficient* C_f, which is of high practical importance for engine design:

$$C_f := \frac{F_*}{p_0 A_t} \tag{4.3.8}$$

The rationale is that according to Eq. (4.3.6) the product $p_0 A_t = \dot{m}_p c^* = const$ for $\dot{m}_p = const$. Therefore, when optimizing an engine design at a given fuel

consumption only C_f matters. Comparing Eqs. (4.3.3) with (4.3.8) and applying Eq. (4.3.6), we derive the following expression for C_f

$$C_f = \frac{v_*}{c^*} = C_\infty \sqrt{\eta_e} + \frac{p_e - p_\infty}{p_0}\frac{A_e}{A_t} \quad \textbf{thrust coefficient} \tag{4.3.9}$$

The thrust coefficient is a dimensionless parameter used to measure the gas expansion performance through the nozzle, i.e. the design quality of the nozzle. According to Sect. 4.1.4, both η_e and p_e/p_∞ are a function of $\varepsilon = A_e/A_t$. alone. Therefore, $C_f(\varepsilon, p_0/p_\infty)$.

One might assume that $p_0 A_t$ might be the thrust of just the chamber and C_f is an efficiency factor that accounts for the nozzle contribution. As we will see in Sect. 4.3.3, C_f is not the nozzle efficiency factor but is directly related to it. So C_f just has engineering and no physical relevance.

Characteristic velocity and thrust coefficient are two key engine parameters for design engineers. So it is worth investigating what the actually achievable values for these two parameters are. For $p_0 A_t = set$ we therefore find from Eqs. (4.3.6) and (4.3.8) and the results of Sect. 1.3.3, the correction factors

$$\eta_{c^*} := \frac{\text{actual } c^*}{\text{ideal } c^*} = 1/\frac{\text{actual } \dot{m}_p}{\text{ideal } \dot{m}_p} = \frac{1}{\zeta_d} = 0.87 - 1.02$$

$$\eta_f := \frac{\text{actual } C_f}{\text{ideal } C_f} = \frac{\text{actual } F_*}{\text{ideal } F_*} = \eta_T = \eta_{div} = 0.92 - 1.00$$

4.3.2 Thrust Performance

Let us examine the thrust at a given fuel consumption, i.e., for $\dot{m}_p = const$. We rewrite Eq. (4.3.3) with Eqs. (4.3.8) and (4.3.6) as

$$\boxed{F_* = c^* \dot{m}_p C_f(\varepsilon, p_0/p_\infty)} \tag{4.3.10}$$

In order to determine the thrust performance we thus only need to determine $C_f(\varepsilon, p_0/p_\infty)$. We do so by solving Eq. (4.1.14) for $\eta_e(\varepsilon)$ (cf. Fig. 4.2), and plug the result into Eq. (4.3.9). However, this can be done only numerically, the result of which is depicted in Fig. 4.7. Recall from Sect. 4.1.4 that $\Delta\varepsilon/\varepsilon = [1 - O(p_e/p_0)]\,\Delta p_0/p_0 \approx \Delta p_0/p_0$. The nearly horizontal curves display the change of the thrust coefficient with the expansion ratio ε at a given chamber pressure ratio p_0/p_∞. The curves confirm our theoretical conclusion from Sect. 4.2.1 that maximal thrust occurs at $p_e = p_\infty$ indicated by the line crossing all others. The region around the thrust maximum, in particular for high combustion chamber pressures, is so flat that a slightly suboptimally adapted nozzle does not gravely reduce thrust.

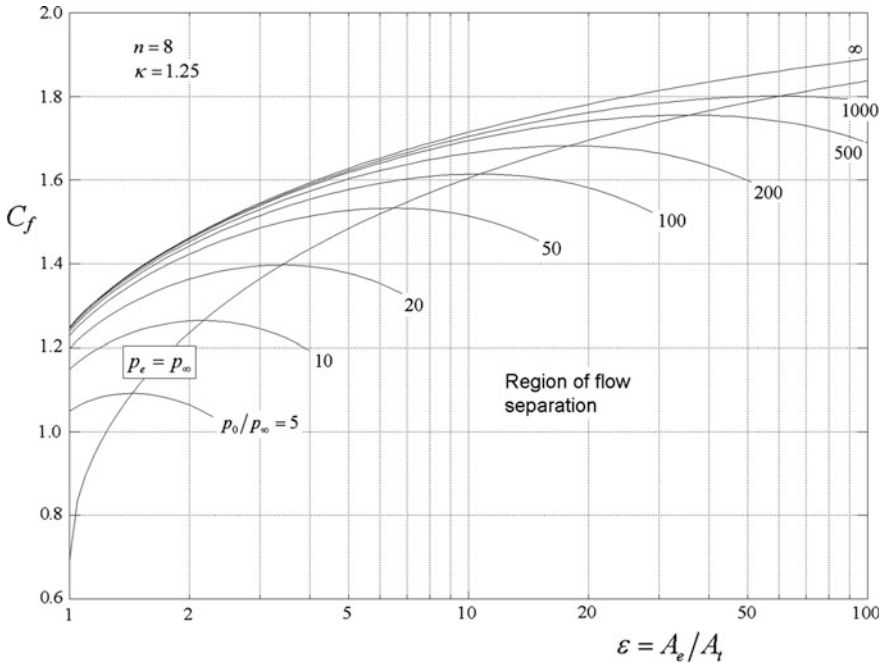


Fig. 4.7 The dependence of the thrust coefficient C_f on the expansion ratio for $n = 8$ ($\kappa = 1.25$)

Nevertheless, in course of the ascent through the atmosphere into space, over-expansion and under-expansion losses on average are about 10%.

Figure 4.7 shows that the thrust coefficient C_f steadily increases until for $A_e/A_t \to \infty$, i.e., in vacuum ($\eta_e \to \eta_\infty = 1$), $C_f = C_\infty = 2.0810$. In this case the infinitely sized nozzle allows the exhaust jet to expand to zero ambient pressure, which is why we call C_∞ infinite-expansion coefficient. Likewise the thrust tends to its limiting value of Eq. (4.3.5).

Thrust Sensitivity Analysis I—Dependence on Chamber Pressure

We now want to study the dependence of the thrust from key chamber design parameters.

We first analyze the thrust variations of an ideally adapted engine for varying chamber pressure at a given $\dot{m}_p = const$. To have the engine remain ideally adapted, the pressure thrust must remain zero and therefore $p_e = p_\infty = const$. Because ε is to be adjusted such that $p_e = p_\infty$ the only variable according to Eq. (4.3.10), is p_0, which from Eq. (4.3.6) corresponds to a variable A_t, namely $dp_0/p_0 = -dA_t/A_t$. Due to $F_* = c^* \dot{m}_p C_f = c^* \dot{m}_p C_\infty \sqrt{\eta_\infty}$ we then have

$$\frac{dF_*}{F_*} = \frac{dC_f}{C_f} = \frac{1}{2}\frac{d\eta_\infty}{\eta_\infty} = \left(\frac{p_0}{2\eta_\infty}\frac{d\eta_\infty}{dp_0}\right)\frac{dp_0}{p_0}$$

From $\eta_\infty = 1 - (p_\infty/p_0)^{\frac{2}{n+2}}$ follows that

$$\frac{p_0}{2\eta_\infty}\frac{d\eta_\infty}{dp_0} = \frac{1}{n+2}\frac{1-\eta_\infty}{\eta_\infty} = \frac{1}{n+2}\frac{1}{(p_0/p_\infty)^{\frac{2}{n+2}}-1}$$

and therefore the relative change of the thrust is

$$\begin{aligned}\frac{\Delta F_*}{F_*} &= \frac{1}{n+2}\frac{1-\eta_\infty}{\eta_\infty}\frac{\Delta p_0}{p_0} \\ &= -\frac{1}{n+2}\frac{1}{(p_0/p_\infty)^{2/(n+2)}-1}\frac{\Delta A_t}{A_t}\end{aligned} \quad @\ \begin{matrix} p_e = p_\infty \\ \dot{m}_p = const \end{matrix} \tag{4.3.11}$$

So, the increase in efficiency is notably effective for low chamber pressures. We find for $n = 8$ and for a typical pressure ratio $p_0/p_\infty \approx 75/0.5 = 150$, $\eta_\infty \approx 0.63$ and hence

$$\frac{\Delta F_*}{F_*} \approx 0.06\,\frac{\Delta p_0}{p_0} = -0.06\,\frac{\Delta A_t}{A_t} \quad @\ \begin{matrix} p_e = p_\infty \\ \dot{m}_p = const \end{matrix}$$

Thrust Sensitivity Analysis II—Dependence on Mass Flow Rate
An interesting case in point is when the thrust of an engine changes just due to a variation in mass flow rate (propellant injection rate). We then have $\varepsilon = const \rightarrow \eta_e = const \rightarrow v_e = const$. From $\eta_e = 1 - (p_e/p_0)^{2/(n+2)}$ then follows

$$p_e = p_0(1-\eta_e)^{\frac{n+2}{2}} = \dot{m}_p c^*(1-\eta_e)^{\frac{n+2}{2}}\varepsilon.$$

Therefore

$$F_* = \dot{m}_p v_e + (p_e - p_\infty)A_e = \dot{m}_p\left[v_e + c^*(1-\eta_e)^{\frac{n+2}{2}}\varepsilon\right] - p_\infty A_e$$

and with Eqs. (4.3.2) and (4.3.7)

$$\begin{aligned}\frac{dF_*}{d\dot{m}_p} &= v_e + c^*(1-\eta_e)^{\frac{n+2}{2}} \\ &= c^*\left(C_\infty\sqrt{\eta_e} + \frac{p_e}{p_0}\varepsilon\right) = c^*\left(C_f + \frac{p_\infty}{p_0}\varepsilon\right)\end{aligned} \quad @\ \varepsilon = const$$

With $F_* = \dot{m}_p v_* = \dot{m}_p c^* C_f$ and with Eq. (4.3.9) the relative thrust change as a function of a relative propellant injection rate change therefore is

$$\frac{\Delta F_*}{F_*} = \left(1 + \frac{p_\infty \varepsilon}{C_f p_0}\right) \frac{\Delta \dot{m}_p}{\dot{m}_p} \quad @\ \varepsilon = const$$

For $p_e = p_\infty$, $\eta_\infty \approx 0.63$ (see above), and with Eq. (4.2.4)

$$\frac{p_\infty \varepsilon}{C_f p_0} = \frac{(1-\eta_\infty)^{(n+2)/2}}{C_\infty \sqrt{\eta_\infty}} \varepsilon_{opt} = \frac{1}{n+2} \frac{1-\eta_\infty}{\eta_\infty} \approx 0.06 \quad @\ p_e = p_\infty$$

we alternatively have

$$\frac{\Delta F_*}{F_*} = \left(1 + \frac{1}{n+2} \frac{1-\eta_\infty}{\eta_\infty}\right) \frac{\Delta \dot{m}_p}{\dot{m}_p} \approx 1.06 \frac{\Delta \dot{m}_p}{\dot{m}_p} \quad @\ \begin{matrix} p_e = p_\infty \\ \varepsilon = const \end{matrix} \tag{4.3.12}$$

So, mainly the momentum thrust contributes while the pressure thrust (0.06) contributes only little owing to deviations from the ideally adapted nozzle induced by the change in the mass flow rate.

4.3.3 Nozzle Efficiency

The thrust coefficient might be of engineering interest, but it does not provide an answer to the ultimate question: What is the thrust a nozzle adds to the engine performance? We therefore define the nozzle coefficient (nozzle efficiency) as follows:

$$C_n := \frac{thrust\ of\ an\ engine\ with\ nozzle}{thrust\ of\ an\ engine\ without\ nozzle} = \frac{F_*}{F_{*,plain}} \tag{4.3.13}$$

by assuming no divergence losses in both cases, i.e. $\eta_{div} = 1$. According to Eq. (1.2.14), an engine without nozzle would provide the thrust

$$F_{*,plain} = \dot{m}_p v_t + (p_t - p_\infty) A_t$$

From Eqs. (4.1.11), (4.1.3), and (4.3.2) we find for the pressure at the throat

$$p_t = p_0 \left(\frac{n}{n+1}\right)^{\frac{n+2}{2}} = p_0 \frac{n}{\sqrt{n+1}} \frac{C_\infty}{n+2} \approx 0.55 \cdot p_0$$

With the relation $v_t = a_0 \sqrt{n} / \sqrt{n+1}$ from Eq. (4.1.13) and with $\dot{m}_p a_0 \sqrt{n} = p_0 A_t C_\infty$ from Eq. (4.3.2) we can rewrite it to

$$\begin{aligned} F_{*,plain} &= p_0 A_t C_\infty \left(\frac{1}{\sqrt{n+1}} + \frac{n}{\sqrt{n+1}} \frac{1}{n+2} - \frac{p_\infty}{C_\infty p_0} \right) \\ &= p_0 A_t \left(\frac{2C_\infty \sqrt{n+1}}{n+2} - \frac{p_\infty}{p_0} \right) \end{aligned} \tag{4.3.14}$$

As shown in Problem 4.2 at any practical rate

$$2C_\infty \frac{\sqrt{n+1}}{n+2} = 1.25$$

Therefore we arrive at the interesting result that an engine without nozzle provides the thrust

$$F_{*,plain} = p_0 A_t \left(1.25 - \frac{p_\infty}{p_0} \right) \approx 1.2 \cdot p_0 A_t \tag{4.3.15}$$

The latter is due to $p_\infty / p_0 \approx 0.01$. For the thrust of an engine with nozzle, Eq. (4.3.8) states that $F_* = p_0 A_t C_f$. If we insert this equation and Eq. (4.3.14) into Eq. (4.3.13) we obtain

$$C_n = \frac{C_f}{\frac{2C_\infty \sqrt{n+1}}{n+2} - \frac{p_\infty}{p_0}} \approx \frac{C_f}{1.25 - \frac{p_\infty}{p_0}} \tag{4.3.16}$$

with

$$C_f = \frac{v_*}{c^*} = C_\infty \sqrt{\eta_e} + \frac{p_e - p_\infty}{p_0} \frac{A_e}{A_t}$$

We therefore arrive at the approximation adequate at any practical rate

$$\boxed{C_n \approx 0.81 \cdot C_f} \quad \textbf{nozzle coefficient} \tag{4.3.17}$$

The nozzle efficiency thus can be estimated to be 81% of the thrust coefficient. Because $F_* = C_n \cdot F_{*,plain}$ and in comparison to $F_* = C_f \cdot p_0 A_t$ the nozzle coefficient C_n is a more vivid substitute for C_f. The two are virtually in a direct relation to each other.

If the nozzle is ideally adapted, then $C_f = C_\infty \sqrt{\eta_\infty}$, and we get

$$C_n = \frac{C_\infty \sqrt{\eta_\infty}}{1.25 - p_\infty / p_0} \quad @\ p_e = p_\infty \tag{4.3.18}$$

For a nozzle ideally adapted to outer space one gets from Eq. (4.3.16) with $C_f = C_\infty$

$$C_n = \frac{n+2}{2\sqrt{n+1}} \approx \frac{5}{3} = 1.667 \quad @\ p_e = p_\infty = 0 \tag{4.3.19}$$

Thus, in space a nozzle ideally adapted to vacuum always increases the thrust by approximately 67% independent of the pressure in its combustion chamber. However, this is an ideal case because a nozzle adapted to vacuum in space would have an infinite size. Because $p_e = p_\infty = 0$ is a limiting condition, therefore, the nozzle efficiency in general is always lower than 67%.

Example

The Space Shuttle SSME LH2/LOX engine had $I_{sp}(vacuum) = 455\ s$, $I_{sp}(1\ bar) = 363\ s$, and $c^* = 2330\,m/s$. Therefore, during ascent through Earth's atmosphere, one gets $1.53 \leq C_f \leq 1.91$ and thus $1.23 \leq C_n \leq 1.53$. So, a SSME nozzle provides at sea level 23% and in space 53% more engine thrust.

4.4 Engine Design

For engine design, aspects of maximum performance, engine reliability, cost, and, last but not least, system constraints need to be considered. An engine design for a reusable launch system for instance has to be a compromise between efficiency and durability. An optimum engine design may even contradict an optimum system design. As an example consider the mixture ratio of LOX/LH2 engines. While the engine designer would favor a mixture ratio around 5, which corresponds to maximum specific impulse and slightly lower hot gas temperatures, stage or launcher architects favor higher mixture ratios, which result in in significantly lower LH2 tank volumes, however at the expense of a lower specific impulse and a higher hot gas temperature. Therefore, every engine usually is unique in that it is tailored to the specific requirements of a rocket for a mission.

Despite these many design aspects this section deals only with the layout of an engine to achieve maximum total thrust. We already know two engine parameters that govern engine geometry: the cross-sectional area of the combustion chamber throat, A_t, and the cross-sectional area of the nozzle exit, A_e. In order to optimize the engine layout we need to design both, combustion chamber and nozzle. Let us go into details.

4.4.1 Combustion Chamber

Equation (4.3.6) determines the design of the combustion chamber: On the left-hand side of the equation we have the technically variable parameters, on the right-hand side we only have propellant-specific parameters. For instance, with a given (turbo pumps, combustion rate) mass flow rate $\dot{m}_p$, and a maximum

admissible pressure of the combustion chamber p_0, we can determine the necessary cross section A_t of the throat

$$A_t = \frac{\dot{m}_p}{p_0 C_\infty}\sqrt{\frac{2h_0}{M_p}} = \frac{F_*}{p_0 C_\infty \sqrt{\eta_\infty}} \tag{4.4.1}$$

The chamber volume V_c remains undefined by these considerations. It only comes into play when internal combustion processes are analyzed. Then the so-called *characteristic length* (commonly pronounced "L-star")

$$L^* = \frac{V_c}{A_t} \quad \textbf{characteristic length} \tag{4.4.2}$$

plays an important role. From the continuation Eq. (4.1.9) we derive for the chamber volume $V_c = \dot{m}_p t_s/\rho$, where t_s is the propellant residence time in the chamber. Hence, essentially $L^* \propto t_s$. Experimentally, the characteristic velocity c^* increases monotonically with L^* to an asymptotic maximum, due to better mixing and burning on its path through the chamber. On the other hand, longer thrust chambers result in higher weight and more chamber surface area to be cooled (typically by fuel) and more friction losses. Yet, in expander cycle engines a minimum surface is needed in order to extract sufficient heat from the chamber to drive the turbines. If the chamber is too long resonances may occur, which cause standing pressure waves, so-called *thrust oscillations*, a problem which became widely known in early 2008 in the course of the solid-fuel boosters design phase of NASA's Ares I rocket. Therefore, the choice of the right L^* is subject to quite contradictory requirements.

Without going into details of a chamber stability analysis, we may mention that typically $L^* = 0.76 - 1.02$ m for LOX/LH2 engines, $L^* = 1.02 - 1.27$ m for LOX/RP-1 engines, and $L^* = 0.76 - 0.89$ m for hydrazine-based N_2O_4 engines (see Humble et al. (1995)). Optimum L^* relies on past experience and evaluation of actual firings of experimental thrust chambers. If we assume a common cylindrical combustion chamber with cross section A_c and length L_c, i.e., $V_0 = A_c L_c$, and follow the rough relationship for chamber contraction ratio

$$\frac{A_c}{A_t} = 7.44\left(A_t/1\,\mathrm{cm}^2\right)^{-0.3} + 1.25 \tag{4.4.3}$$

suggested by Huzel and Huang (see Humble et al. (1995, p. 222)), which is based on engine test data and simple gas-dynamic considerations, the chamber length should be scaled as

$$L_c = \frac{V_c}{A_c} = L^*\frac{A_t}{A_c} = \frac{L^*}{7.44(A_t/1\,\mathrm{cm}^2)^{-0.3} + 1.25} \tag{4.4.4}$$

In general, it can be said that in solid-fuel boosters with their very high characteristic length longitudinal waves leading to thrust oscillations are a standing problem. In liquid thrusters these modes are dampened out by the throat (no pressure reflection) at the back-side and by the injection system and propellant distribution devices and even by the injected droplets (droplet drag) itself acting as active acoustic elements on the front side of the chamber. Therefore transversal, rather than longitudinal, waves play a major role here.

In closing we note that although computer codes today sufficiently simulate selected combustion processes such as propellant vaporization, mixing, and combustion, there currently exists no end-to-end approach to determine the optimal fuel injection mechanism and chamber configuration from scratch. Good engine design still relies on a substantial amount of experimental data that makes it a laborious undertaking. This is why one often falls back on existing reliable engines such as for the Ares I and Ares V rockets, which were slated for the canceled US Constellation program.

4.4.2 Nozzles

One essential result of our engine design considerations for an ideal rocket (ideal rocket criteria see Sect. 4.1.1) is that according to Eq. (4.3.10), thrust depends (see also Fig. 4.7) just on the expansion ratio ε via the parameter p_0. So, the nozzle design is determined solely by the areal ratio of its end faces, and not by its exact shape in between. In fact, from a thermodynamic point of view, the shape of the nozzle casing and how the gas expands in it is irrelevant as long as the gas flow is one-dimensional and expands steadily and adiabatically. So any smooth contour that precludes shock formation will do. This purports that the nozzle should not expand too rapidly behind the throat, implying that the angle of the nozzle casing against the nozzle axis should not be too large. From an engineering point of view, a well-designed shape can reduce the mechanical strain of the nozzle along its axis.

For a real rocket any nozzle design has to respect that the gas exhibits a two-dimensional flow pattern (radial and axial dependence), that the shape of the divergent supersonic part dictates shock formation and hence performance gain/loss, and that there are boundary layer losses due to friction with the nozzle wall. For instance, accounting for the latter a shorter nozzle may reduce the thermodynamic thrust gain, but increases total thrust gain due to less boundary losses in addition to less weight and less side loads at engine start-up (see bell nozzle below).

We do not want to go into details about these implications for the nozzle shape, but mention the four principal nozzle designs.

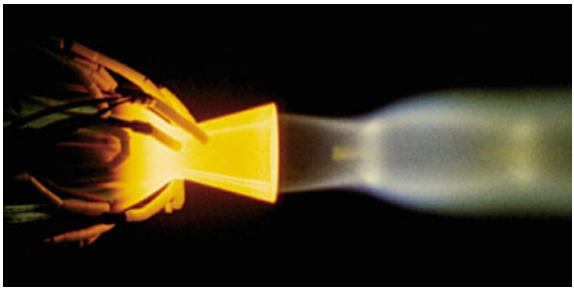


Fig. 4.8 A small chemical thruster with conical nozzle. *Credit* DLR

Conical Nozzle

If mechanical strain is not an issue, for instance for engines with little thrust, simple conical nozzles are a practical solution (see Fig. 4.8). They are simple to design and manufacture. However, they lead to some thrust losses due to oblique shocks at the discontinuous transition from the throat to the cone 12°–18°. The conical nozzle with a 15° divergent half-angle has become almost a standard because it is a good compromise on the basis of weight, length, and performance.

Bell Nozzle

The most common contoured nozzle is the bell-shaped nozzle (see Fig. 4.9). The shape, which usually is a parabolic-geometry approximation, is ideal because the gases quickly expand conically (angle of expansion typically 30°–60°) behind the throat. The expansion waves emanating from this quick expansion diminish the compression effects caused by the flow reorientation, thereby leading to relatively little nozzle losses. The longer the gases run along the nozzle, the less divergence occurs because of the bell shape, and at the exit the gases are expelled almost parallel to the nozzle axis. A near optimal thrust bell nozzle contour uses the parabolic approximation procedures suggested by Rao (1958), in particular



Fig. 4.9 Bell nozzle of the Aestus upper stage engine from Ariane 5 with CC-NE I/F. *Credit* ESA/Arianespace

the so-called thrust-optimized parabolic (TOP) nozzles, used for instance for the Vulcain, J-2S, and SSME engines.

Combustion and Start-Up Instabilities

During engine start-up significant lateral off-axis loads, so-called side loads, as shown in Fig. 4.10 occur in a thrust-optimized contour nozzle, which are of quite a complex nature. They are caused by

1. combustion waves of hydrogen-rich jet afterburning in the nozzle, the transversal component of which hits the wall, thereafter by
2. the oscillation of the supersonic jets of the Mach disk flow and subsequent axial oscillation of the Mach disk itself, then by
3. asymmetric flow separation patterns of types FSS (free-shock separation), RSS (restricted-shock separation), and FSS-to-RSS transitions caused on one hand by over-expansion (recall that at launch elevation the nozzle is overexpanding) and on the other by the Coanda effect, which draws the supersonic core jet to the wall, and finally by
4. asymmetric lip lambda shock oscillation.

In outer space, where the thrust reaches its maximum, the thrust force may lead to axial buckling of the nozzle (Fig. 4.10). Axial buckling of the, at that time, new Vulcain 2 engine was the reason for the total failure of Ariane 5 ECA on December 11, 2002.

During engine start-up 10% of the thrust level is due to side loads. They can be reduced by ideal contoured nozzles, which are usually less sensitive towards flow separation during start-up. In addition, the danger of nozzle damage due to side-load deformations and buckling deformations increases with decreasing nozzle size and hence with decreasing distance to the throat. So, shorter nozzles do less damage.

Owing to this load situation, the cross over from the heavy reinforced combustion chamber (CC), to the lighter nozzle extension (NE), the so-called CC-NE I/F, is not

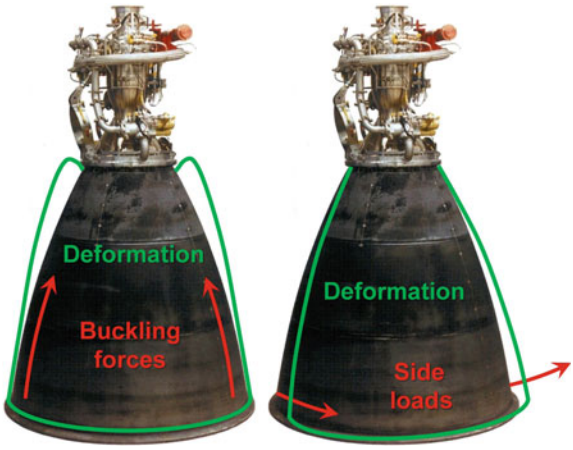


Fig. 4.10 Deformations of a Bell nozzle caused by buckling forces at high altitudes (left) and side loads at engine start-up (right)

at the throat, but is located farther downstream (see Fig. 4.8). Its actual location is always a compromise between the weight of the thruster and its tolerance for deformation loads.

An equivalent 15° half-angle conical nozzle is commonly used as a standard to specify bell nozzles. For instance, the length of an 80% bell nozzle (distance between throat and exit plane) is 80% of that of a 15° half-angle conical nozzle having the same throat area, radius below the throat, and area expansion ratio.

Both conical and Bell nozzles belong to the class of so-called convergent-divergent nozzles (CD nozzle, a.k.a. Laval nozzle), where the gas flow is convergent before the throat and divergent after the throat.

Circular Nozzles

In circular nozzles (a.k.a. annular nozzles) combustion occurs along a ring, or annulus, around the base of the nozzle. There are two basic types of circular nozzles:

1. *Radial outflow nozzles* where the exhaust is expanded radially outward such as in the expansion-deflection (E-D) (Fig. 4.11), reverse-flow (R-F), and horizontal-flow (H-F) nozzles.
2. *Radial inflow nozzles* where the exhaust is expanded radially inward, such as the spike nozzle (a.k.a. *(full-length radial) plug nozzle*) (Fig. 4.12) and aerospike nozzle (a.k.a. *truncated (radial) plug nozzle*).

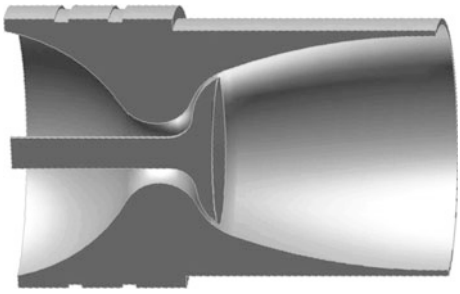


Fig. 4.11 Cutaway view of an expansion–deflection nozzle with a pintle deflecting the flow outward toward the wall. *Credit* RedHotIceCube, Creative Commons

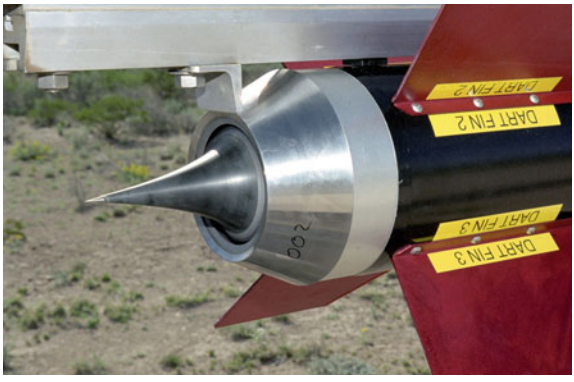


Fig. 4.12 A spike nozzle having a plug at the center as used in NASA's Dryden Aerospike Rocket Test in March 2004. *Credit* NASA

The radial outflow nozzle has an inner free-jet boundary with the outer nozzle wall contour (see Fig. 4.8). Conversely, radial inflow nozzles have an inner but no outer wall, the latter is just the outer surface of the annular flow as a free-jet boundary, which is self-adjusted by ambient pressure. Compared to a bell-shaped nozzle, annular nozzles are more complex to design but operate more efficiently across a wide range of altitudes.

For a **plug nozzle** (spike nozzle) the word "plug" refers to the center body that blocks the flow from what would be the center portion of a traditional nozzle (see Fig. 4.12).

An **aerospike nozzle** is a plug nozzle with a truncated plug. The term "aerospike" follows from the fact that the virtual bell is formed on the inner side by the spike and on the other by the air. Since this property holds also for the plug nozzle, both terms are often used interchangeably. However, aerospike was originally used only for a truncated plug nozzle. As a result of the truncation, the turbulent wake, which at high altitudes would form aft of the base, results in a high base drag and reduced efficiency. This drawback can greatly be alleviated by a "base bleed" into the region aft of the base forming an "air spike".

Circular nozzles suffer from large surfaces which have to be cooled and problems with non-symmetric combustion owing to uneven annular distribution of the exhaust gases. This results in thrust vector diversion and local shock waves leading to local heat load peaks and hence temperature peaks. To prevent thruster damage the entire plug area needs to be overly cooled.

Linear Nozzles

Plug nozzles and aerospike nozzles also come in the form of linear nozzles, where the combustion occurs in a line.

In a **linear aerospike engine**, such as the hailed Rocketdyne RS-2200 (Fig. 4.13), many small combustion chambers are placed in a line along two sides of the truncated straight spike nozzle. Throttling of either side therefore allows



Fig. 4.13 Test run of the linear aerospike engine RS-2200 at Rocketdyne. *Credit* NASA-MSFC

steering of the engine. The disadvantages of the linear aerospike are severe heat losses and thrust divergences and hence thrust losses at the two open sides of the engine, which could be prevented by winglets, but those would again need to be cooled.

4.4.3 Design Guidelines

The essential of this section is summarized by the following coarse engine design guidelines:

1. Choose fuel and oxidizer (h_0, n, M_p) from efficiency and practical considerations. From this follows

$$C_\infty := (n+2)\sqrt{\frac{n^n}{(n+1)^{n+1}}}$$

2. Choose the altitude and hence the external pressure p_∞ to which the engine should be ideally adapted: $p_e = p_\infty$
3. Determine the required total engine thrust F_* (at the altitude to which the engine is to be adapted).
4. Choose the chamber pressure p_0 from thruster efficiency (see Fig. 4.7) and chamber material considerations (such as strength and thermal conductivity), and that the chamber volume scales with (see Eqs. (4.4.2) with (4.3.6))

$$V_c = L^* A_t = \frac{L^* c^* F_*}{v_* p_0} \propto \frac{F_*}{p_0}$$

5. From this and from Eqs. (4.3.4) and (4.2.4) follows

$$\eta_\infty = 1 - \left(\frac{p_\infty}{p_0}\right)^{2/(n+2)}$$

$$A_t = \frac{F_*}{p_0 C_\infty \sqrt{\eta_\infty}}$$

$$A_e = \frac{C_\infty}{n+2} \frac{A_t}{\sqrt{(1-\eta_\infty)^n \eta_\infty}}$$

6. Finally and as described in Sect. 4.4.1 choose with the chosen fuel and oxidizer the characteristic length of the chamber L^*. With this the chamber length L_c and its cylindrical diameter D_c should be

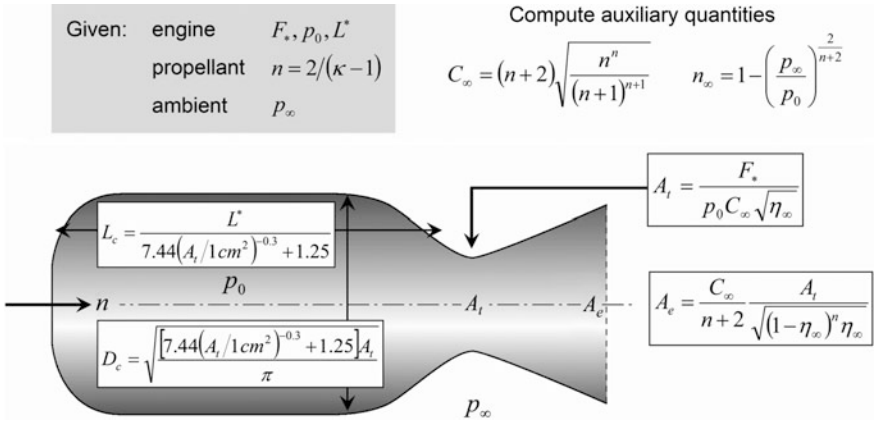


Fig. 4.14 Design flow to determine the optimal engine design

$$L_c = \frac{L^*}{7.44(A_t/1\ \mathrm{cm}^2)^{-0.3} + 1.25}$$

$$D_c = 2\sqrt{\frac{\left[7.44(A_t/1\ \mathrm{cm}^2)^{-0.3} + 1.25\right]A_t}{\pi}}$$

Obviously, the choice of the fuel and oxidizer, a properly chosen external pressure p_∞, chamber pressure p_0, and characteristic length L^*, i.e., finding the detailed dimensions of the pressure chamber, is of high importance and requires a lot of expertise.

In Fig. 4.14, the design guidelines are outlined as a workflow to determine the optimal engine design.

4.5 Problems

Problem 4.1 *Gas Velocity-Pressure Relation in a Nozzle*
Prove explicitly with Eqs. (4.1.7) and (4.1.3) and by applying Eq. (4.1.10) that for the gas velocity–pressure dependence in a nozzle holds

$$dv = -\frac{A}{\dot{m}_p} dp$$

Problem 4.2 *Approximation of the Infinite-Expansion Coefficient*
Prove that for $n \approx 8$ at any practical rate

$$C_\infty = \frac{36}{35\sqrt{e}} \frac{n+2}{\sqrt{n+1}} \left(1 - \frac{n-8}{315} + O\left(n^2\right)\right) \approx 0.624 \frac{n+2}{\sqrt{n+1}}$$

Hint: Consider the expression

$$\frac{n+2}{C_\infty \sqrt{n+1}} = \sqrt{\left(\frac{n+1}{n}\right)^n}$$

Chapter 5
Electric Propulsion

5.1 Overview

Electric propulsion engines differ from thermal engines in that, among other things, the propellant does not serve as an energy source to heat and accelerate the propellant mass in the combustion chamber. Rather acceleration is achieved by accelerating ions in an electric field, the energy of which needs to be provided externally by an electric current source. This is both an advantage and a disadvantage at the same time. The advantage is that, theoretically, any amount of energy can be applied to the propellant mass which would in principle permit unlimited exhaust speeds, hence unlimited specific impulse, and therefore unlimited efficiency of the engine. The disadvantage is that the structural mass of the rocket stage increases due to the additional mass of the electric generator, which directly trades with payload mass. Massive generators are required especially for high-I_{sp} engines, so their additional mass may outweigh propellant savings. Therefore, comparisons between different propulsion systems always need to consider the total propulsion system mass: propulsion system, consumed propellant, plus energy supply system.

Another disadvantage of electric propulsion is that ions repel each other, permitting only very low particle densities in the engine chamber, which in turn leads to mass flow densities many orders of magnitude lower than those of chemical engines. This results in very small thrusts. For this reason electric propulsions will not replace launch thrusters in the long run, as thrust is a key figure for launch. This is apart from the problem that their exit pressure is much lower than ambient pressure, which by itself rules out their employment for launch. On the other hand, once outer space has been reached, and especially with interplanetary flights with long flight times, continuous operation with a highly effective ion or Hall-effect thruster often pays off in comparison with a two-impulse transfer with low-efficient chemical propulsions. This is shown in Table 5.1 with the example of a Mars mission.

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Table 5.1 Comparison of all-chemical and ion-solar engines for the example of a Mars mission

Parameter	Spacecraft description	
	All-chemical voyager (190 days transit)	Ion-solar voyager (190 days transit)
Injected weight (not including launch vehicle)	3530 kg	4350 kg
Power level	All chemical	23 kW
Approach velocity	4.3 km s^{-1}	1.8 km s^{-1}
Weight at approach	2400 kg	2330 kg
Weight in orbit (excluding retro insert weight)	840 kg	1630 kg
Orbit spacecraft fraction	0.35	0.70
Lander weight	1040 kg	1040 kg
Scientific payload	210 kg	810 kg
Percent scientific payload weight at approach	8.9%	34.9%

5.2 Ion Thruster

Let us have a closer look at ion propulsion as it becomes more and more of practical importance. It is based on the acceleration of cold plasma in a high electric field (see Fig. 5.1). The inflowing propellant atoms are ionized in a relatively voluminous reaction chamber by hitting them with circulating electrons, which knock outer-shell electrons out off the atoms. "Cold" here means that, during the ionization process no internal states of the atom are excited. This is achieved by using noble gas atoms, which occur only as single atoms and hence quite naturally lack any rotational or vibrational modes to be excited. These singly charged ions then

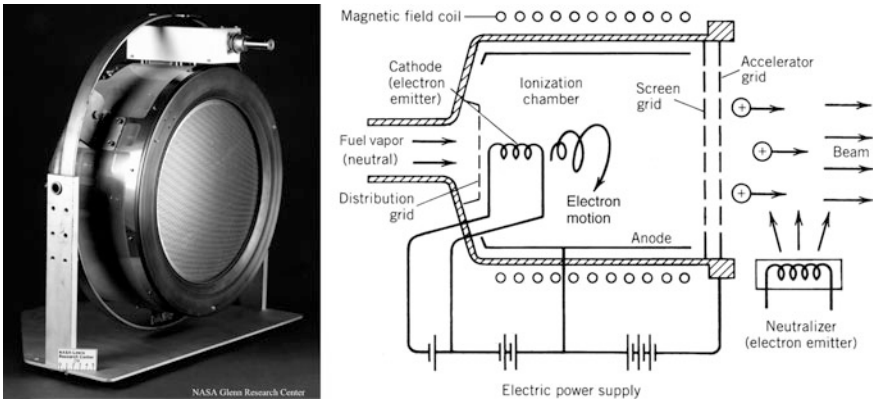


Fig. 5.1 The xenon-fueled NSTAR ion thruster of NASA's Glenn Research Center (left). *Credit* NASA/GRC. Schematic of an ion thruster (right). The electrostatic zone is between the screen grid (anode) and accelerator grid (cathode). *Credit* Sutton (2001), U. Walter

enter a very narrow electrostatic zone where they are accelerated in a high electric field. After emerging from this zone at an extremely high speed, typically 40 km s^{-1}, they are neutralized with the electrons separated earlier in the reaction chamber.

To begin with the physics of the acceleration mechanism, we define some essential quantities. Let Q be the total charge, q the charge density, N the number, n the number density, and ρ the mass density of the ions with unit charge e and mass m_{ion} in the thruster. Between these quantities and the ion beam current I_b the following basic relations hold:

$$\begin{array}{ll} Q = Ne & q = ne \\ \dot{Q} = \dot{N}e = I_b & \dot{q} = \dot{n}e \\ m_i = Nm_{ion} = m_{ion} \cdot Q/e & \rho = nm_{ion} = m_{ion} \cdot q/e \\ \dot{m}_i = \dot{N}m_{ion} = m_{ion} \cdot I_b/e & \dot{\rho} = \dot{n}m_{ion} = m_{ion} \cdot \dot{q}/e \end{array} \tag{5.2.1}$$

where m_i is the total ion mass. Now we turn to the key question: What is the thrust $F_* = \dot{m}_i v_*$ of such an ion thruster?

5.2.1 Ion Acceleration and Flow

The crucial component of the engine for thrust generation is the acceleration zone, which determines the required parameters $\dot{m}_p, v_e$. In order to derive them, we need to understand the charge distribution in the electrostatic zone between the two grids in detail (see Fig. 5.2). We assume the engine axis as the x-axis along which the ions are moving. They enter the zone through bores in the screen grid (anode) at $x = 0$. Then they are accelerated by an electric potential $V(x)$ (the form of which

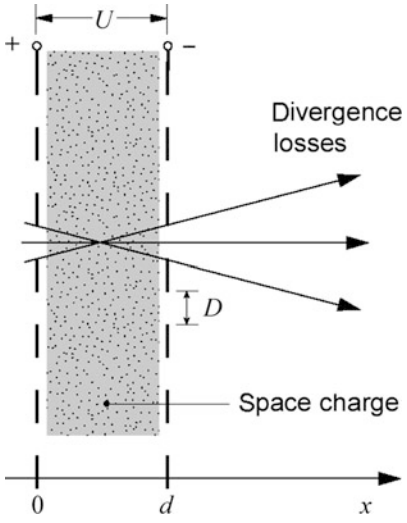


Fig. 5.2 Geometrical and electric relations in the electrostatic zone between the two grids

still has to be derived) between the screen grid (anode) and the accelerator grid (cathode) with separation d. The acceleration voltage across the grids is $U := V(d)$. Finally, the accelerated ions exit the zone through bores in the accelerator grid at $x = d$.

The charges flowing through this zone with velocity v generate a space charge with density q, which reduces the electric potential $V(x)$ so that the inflow of the charges is slightly reduced. The balance between charge and electric potential is physically described by the Poisson equation (cf. Eq. (7.1.1))

$$\frac{d^2V}{dx^2} = \frac{q(x)}{\varepsilon_0} \tag{5.2.2}$$

with ε_0 being the vacuum permittivity. Because of charge conservation along the whole acceleration distance $0 \leq x \leq d$, the resulting charge flow density $j = qv$ must be constant

$$j = qv = const \quad \textbf{charge continuity equation} \tag{5.2.3}$$

This is the equivalent equation to the mass continuity Eq. (1.2.8) for mass conservation. The particle velocity v at any place within the acceleration zone is determined by the balance between kinetic energy and electric potential energy, i.e.,

$$\frac{1}{2}m_{ion}v^2 = eV \tag{5.2.4}$$

with m_{ion} the mass of an ion, and e the charge of the singly charged ion, which is the elementary charge. At the end of the acceleration distance, the ejection velocity is determined from Eq. (5.2.4) and $U := V(d)$ as

$$v_e = \sqrt{\frac{2eU}{m_{ion}}} \tag{5.2.5}$$

For typical ion thrusters, we have $v_e = 30 - 80$ km s^{-1}. Therefore, ion thrusters are about one order of magnitude more efficient than chemical thrusters. Ion engines, however, share an important property with thermal engines (see Eq. (4.2.5)):

The efficiency of an ion thruster increases with decreasing molar mass of the propellant according to $I_{sp} \propto v_e \propto 1/\sqrt{M_p}$.

With Eq. (5.2.4), the charge density at any position in the zone can be determined through Eq. (5.2.3) as

$$q = j\sqrt{\frac{m_{ion}}{2eV}} \tag{5.2.6}$$

Inserting this expression into the master Eq. (5.2.2) for the electric potential V we obtain

$$\sqrt{V}\frac{d^2V}{dx^2} = \frac{j}{\varepsilon_0}\sqrt{\frac{m_{ion}}{2e}} =: c = const$$

To solve this differential equation we multiply both sides with $V' = dV/dx$ and get

$$V'V'' = cV'V^{-1/2}$$

Direct integration of both sides and taking the square root delivers $V' = 2\sqrt{c}V^{1/4}$ or

$$\frac{dV}{V^{1/4}} = 2\sqrt{c}\cdot dx$$

Integration of both sides and taking into account that $V(x = 0) = 0$ yields

$$V^{3/2} = \frac{9}{4}\frac{j}{\varepsilon_0}\sqrt{\frac{m_{ion}}{2e}}x^2 \tag{5.2.7}$$

This is the wanted electric potential function $V(x) \propto x^{4/3}$.

Remark *In case we would not have any charges in the electrostatic zone, Eq. (5.2.2) would read* $d^2V/dx^2 = 0$ *and hence the familiar* $V(x) \propto x$ *in vacuum would result. The weak modification of the linear electric potential function to a* $x^{4/3}$ *behavior is obviously caused by the space charge of the transiting ions.*

Charge and Mass Flow Densities

At location $x = d$ the acceleration potential is $V = U$ and we derive from Eq. (5.2.7) for the charge flow density at the exit

$$j_e = \frac{4\varepsilon_0}{9}\sqrt{\frac{2e}{m_{ion}}}\frac{U^{3/2}}{d^2} \quad \textbf{Child-Langmuir law} \tag{5.2.8}$$

Inserting this into Eq. (5.2.6), the charge density of singly charged ions at the exit is calculated with for $x = d$ and $V = U$ as

$$q_e = \frac{4}{9}\frac{\varepsilon_0 U}{d^2} \tag{5.2.9}$$

It is instructive to know the particle density at the exit n_e. This is easily derived from the relation $q = ne$ to be

$$n_e = \frac{4}{9}\frac{\varepsilon_0 U}{ed^2} \tag{5.2.10}$$

For typical values $U = 2\,000$ V and $d = 2$ mm we find $n_e = 1 \times 10^{10}$ cm^{-3}. For comparison, standard dry air has a particle density of $n_{air} = 5 \times 10^{19}$ cm^{-3} which equals about that at the exit of thermal engines at launch. The exhausted gas therefore has a density of about or more than nine orders of magnitude lower than thermal engines. The reason obviously is the strong charge repulsion of the ions in the acceleration zone. At such extremely low gas densities, the gas exit pressure is far below any atmospheric pressure. Operating it in an atmosphere would thus imply that the atmospheric gas would flow from outside into the ion chamber bringing the ionization to a stall. Therefore, ion thrusters only work in the vacuum of space.

The extremely low exhaust ion gas density also implies that the mass flow rate of an ion thruster is equally low. From Eq. (5.2.1) and from the continuity Eq. (1.2.8) applied to the exit we determine the ion mass flow rate $\dot{m}_i$ as

$$\dot{m}_i = \frac{m_{ion}}{e} I_b = \rho_e v_e A_e = \frac{4}{9}\sqrt{\frac{2m_{ion}}{e}}\frac{\varepsilon_0 A_e}{d^2} U^{3/2} \tag{5.2.11}$$

where I_b is the ion beam current. A typical value for today's ion thrusters with xenon fuel is $\dot{m}_i \approx 10^{-6}$ kg s^{-1}. Xenon is the most common propellant for ion thrusters, because it is naturally occurring (87ppb in atmosphere) and gaseous at ambient temperature. In addition it has a very low chemical reactivity, a low first ionization potential, high storage density, and a high atomic mass. The high atomic mass yields a better thrust-to-power ratio (see Eq. (5.2.22)).

5.2.2 *Ideal Engine Thrust*

With these results we are now able to determine the thrust of an ideal ion engine having $\bar{v}_e = v_e$, $\eta_{div} = 1$. To evaluate the relevance of the pressure thrust F_p we recall from Eq. (1.2.17) that $F_p/F_e \propto 1/v_e^2$. Because the ejection velocity for ion thrusters is with $v_e = 30 - 80$ km s^{-1} about one order of magnitude larger than for chemical engines, we establish that the pressure thrust is about two orders of

magnitude smaller than for chemical engines and hence negligible. The engine thrust in practical terms therefore is given solely by the momentum thrust as given by the general expression for jet engines in Eq. (1.2.16), $F_e = F_{ex} = \rho_e A_e v_e^2$, where ρ_e is the mass density at the exit of the engine with cross section A_e. Because from Eq. (5.2.1) $\rho = m_{ion} q/e$ and with Eq. (5.2.5) we can write for the momentum thrust

$$F_e = \rho_e A_e v_e^2 = 2 q_e A_e U \tag{5.2.12}$$

We now make use of Eq. (5.2.11) and insert Eq. (5.2.9) into Eq. (5.2.12) yielding

$$\begin{aligned} F_e &= \dot{m}_i \, v_e = \sqrt{\frac{2 m_{ion} U}{e}} I_b \\ &= 2 q_e A_e U = \tfrac{8}{9} \varepsilon_0 \left(\tfrac{U}{d}\right)^2 A_e \end{aligned} \tag{5.2.13}$$

Here U is the voltage applied across the acceleration grid and I_b the ion beam current. The exit surface A_e is the sum of all n bores with the diameter D in the cathode plate, i.e., $A_e = n\pi D^2/4$. Hence, thrust increases quadratically with the applied acceleration voltage and inversely proportionally to the square of the distance between the electrodes. So, the perfect ion thruster has the highest possible voltage with the smallest possible acceleration distance. A practical limit is reached when electrical flash-overs happen at about 10 kV mm^{-1}. For a typical xenon propellant we have $\sqrt{2m_{Xe}/e} = 1.65$ mN A^{-1} V$^{-1/2}$ or $\sqrt{2e/m_{Xe}} = 1.21$ km s^{-1} V$^{-1/2}$. For common ion thruster parameters with $I_b = 1$ A, and $U = 1500$ V we typically get $F_e = 62$ mN and $v_e = 45$ km s^{-1}.

Comparison with Other Thrusters

With 10–200 mN the total thrust $F_* = \dot{m}_i v_*$ of an ion thruster is by several orders of magnitude smaller than that of chemical engines of comparable size. This alone wreckages its use as a launch engine. The reason is clear: Although the ejection velocity of ion thrusters is about ten times larger, the lack of momentum thrust is due to an extremely low mass flow rate, which, as already mentioned, in turn is due to the strong charge repulsion of the ions in the acceleration zone. So, taking into account the different particle masses for ion and chemical propellant, the mass flow rate for a chemical engine is about seven orders of magnitude larger than for an ion thruster at the same exit cross section (engine size). This finally results in a chemical engine with roughly 100,000 times more thrust than an ion thruster of similar size. These characteristics are summarized for the three most important engines for space propulsion in Table 5.2.

Table 5.2 Assessment of the figures of merit of three engine types in space propulsion

Engine type	$\dot{m}_p$	v_e (efficiency)	$F_* = \dot{m}_p v_e$ (thrust)
Chemical	++	O	++
Ion/Hall	– – –	+	– –
Nuclear	+	+	++

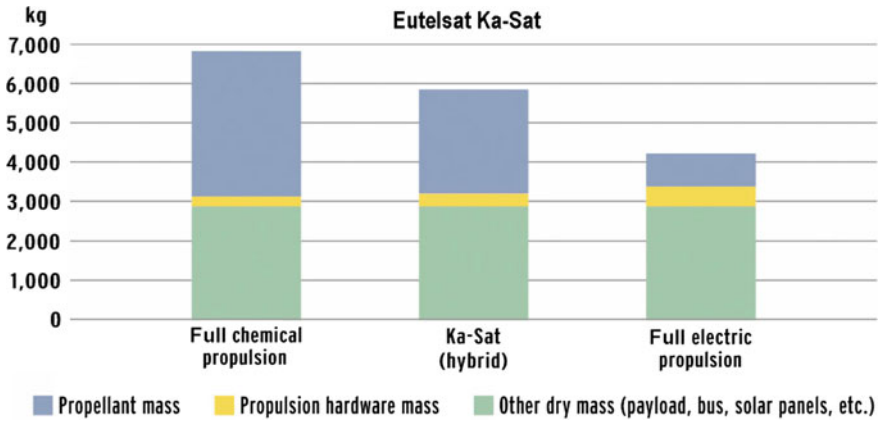


Fig. 5.3 The weight of Eutelsat's Ka-Sat in GEO at different propulsion designs. *Credit* Space News 2012

So, the nuclear (fission or even better fusion) engine would be an all-round engine if nuclear energy in space would not be banned by many societies today. This leaves us with only two distinct options: use massive chemical thrusters for launch and efficient ion thrusters for continuous in-flight orbit maneuvers or recurring station-keeping and attitude maneuvers. The promise of ion thrusters in the recurring maneuvers is depicted in Fig. 5.3 for the case of Eutelsat's Ka-Sat broadband communication satellite launched in December 2010 into GEO. With chemical thrusters for orbit-raising and ion thrusters for station keeping, i.e., with a hybrid design, it weighed 5900 kg. If it had used all chemical thrusters it would have weighed 6900 kg, too heavy for most commercial launchers. Using electric propulsion for orbit raising as well, Ka-Sat would have weighed no more than 4100 kg at launch, according to Eutelsat. At 2010's launch prices, going from a full-chemical to a all-electric propulsion design would have saved up to $60 million.

Nevertheless, for remote sensing satellites in LEO where propulsion mass for attitude-only maneuvers is not a key issue, small inexpensive chemical thrusters will still be more common than ion thrusters.

5.2.3 *Thruster Performance*

Thrust losses

Concerning thrust losses we have to take into account that ion thrusters exhibits jet divergences of typically $\alpha = 11 - 15^\circ$ yielding a divergence loss factor (see Sect. 1.2.1) of $\eta_{div} = (1 + \cos\alpha)/2 = 0.983 - 0.990$. In addition, it can be shown that the existence of multiply charged ions caused by multiple electron collisions in the discharge process causes thrust loss as expressed by a loss factor of typically $\eta_{++} = 0.98 - 0.99$. So, both make up an *exhaust loss factor* of

$$\eta_{ex} := \eta_{div}\eta_{++} = 0.96 - 0.98 \tag{5.2.14}$$

So by the same token as in Sect. 1.2.1. we have for the total thrust

$$\begin{aligned} v_* &= \eta_{ex}\, v_e \\ F_* &= \dot{m}_i v_* = \eta_{ex}\dot{m}_i\, v_e = \eta_{ex}F_e \end{aligned} \tag{5.2.15}$$

Note that at any time not all atoms in the discharge chamber are ionized. So, due to the chamber gas pressure some of the neutral atoms are leaking through the grids and are irreversibly lost. This neutral mass flow also contributes to the total propellant flow $\dot{m}_p$ but not to thrust.

Energy Conversion Loss

We recall from Sect. 1.3.3 that the *total engine efficiency*, which for an ion engine is the ratio of the jet power P_{jet} of the ion beam to the total electrical power into the thruster P_{in} is given as

$$\eta_{tot} = \zeta_d\zeta_v^2\eta_{ec} \tag{5.2.16}$$

where in our case the energy conversion efficiency η_{ec} is

$$\eta_{ec} := \frac{P_{jet,id}}{P_{in}} = \frac{I_b U}{P_{in}} \tag{5.2.17}$$

The energy conversion efficiency tells us how much of the external electric power is transferred into the power of the beam. It therefore is a figure of merit for the quality of the engine.

For ion engines there are two prevailing energy conversion losses: discharge losses, i.e. power losses due to discharging electrons into the chamber that do not ionize atoms but hit the chamber anode unimpeded, and ionization losses due to the ionization of the ions. To see how these two come about, let E be energy, in particular E^+ the single ionization energy of an ion, P power, and N the number of ions with elementary charge e. Hence according to Eq. (1.3.14) and Eq. (5.2.1)

$$\begin{aligned} E_{jet} &= \frac{1}{2}m_i v_e^2 && \textbf{jet energy} \\ P_{jet,id} &= \frac{dE_{jet,id}}{dt} = \frac{1}{2}\dot{m}_i v_e^2 = I_b U && \textbf{jet power} \\ E_{jet,id} &= \frac{1}{2}m_i v_e^2 + NE^+ && \textbf{ionized-jet energy} \\ P^+_{jet,id} &= \frac{dE^+_{jet,id}}{dt} = \frac{1}{2}\dot{m}_i v_e^2 + \dot{N}E^+ = I_b(U + E^+/e\ ) && \textbf{ionized-jet power} \end{aligned}$$

We consider these contributions by factorizing the energy conversion efficiency correspondingly

$$\eta_{ec} := \frac{P_{jet,id}}{P_{in}} = \frac{P_{jet,id}}{P^+_{jet,id}} \frac{P^+_{jet,id}}{P_{dis}} \cdot \ldots := \eta_+ \eta_{dis} \cdot \ldots = 0.55 - 0.75 \tag{5.2.18}$$

The figures given are derived from ion engines as given in literature. Discharge losses contribute (typically) by

$$\eta_{dis} \approx 0.86 \quad \textbf{discharge loss factor} \tag{5.2.19}$$

while ionization losses factor in as

$$\eta_+ := \frac{P_{jet,id}}{P^+_{jet,id}} = \frac{I_b U}{I_b (U + E^+/e)} = \frac{1}{1 + E^+/eU} \quad \textbf{ionization loss factor} \tag{5.2.20}$$

For xenon atoms with $E^+ = 12$ eV and typically $U = 1500$ V we find

$$\eta_+ = \frac{1}{1 + 0.008} = 0.99$$

Therefore while ionization losses are negligible, energy conversion is mostly limited by discharge losses.

Thrust-to-Power Ratio

With the above definitions we get from Eq. (5.2.16)

$$\eta_{tot} = \zeta_d \zeta_v^2 \eta_+ \eta_{dis} \cdot \ldots$$

In ion plasma there is no viscosity and no multiphase flow and hence $\zeta_d = 1$. Though there is velocity spread due to multiply charged ions, the correction factor ζ_v is negligible. Therefore

$$\eta_{tot} = \zeta_v^2 \eta_{ec} \approx \eta_{ec} = \eta_+ \eta_{dis} \cdot \ldots \tag{5.2.21}$$

Typical values for ion thrusters are $\eta_{tot} \approx \eta_{ec} = 0.55 - 0.75$.

We finally have from Eq. (1.3.23) and Eq. (5.2.5) for the important *thrust-to-total-power ratio*, which describes the thrust received from the total electrical power into the ion thruster,

$$\begin{aligned} r_{TTPR} &= \frac{F_*}{P_{in}} = \frac{2}{\bar{v}_e} \eta_{div}\, \eta_{VDF} \zeta_v^2 \eta_{ec} \\ &\approx \eta_{div}\, \eta_{VDF}\, \eta_{ec} \sqrt{\frac{2 m_{ion}}{eU}} \end{aligned} \quad \textbf{thrust-to-power ratio} \tag{5.2.22}$$

with

$$\eta_{div} \eta_{VDF} \eta_{ec} = 0.50 - 0.70$$

From this we learn that, though the ion thrust, $F_* \propto U^2$, increases with increasing acceleration voltage, the thrust-to-power ratio $r_{TTPR} \propto 1/\sqrt{U}$ decreases at the same time, yet at a much slower rate.

5.3 Electric Propulsion Optimization

Electric propulsion is special in that the ejection velocity and thus the specific impulse depends on the acceleration voltage and hence is variable. Due to the rocket Eq. (2.2.4) the propellant demand decreases exponentially with $v_* = g_0 I_{sp}$ (see Fig. 5.4). Would it then be feasible to get a steadily increasing payload ratio with an increasing acceleration voltage? This, unfortunately, is not the case, because with an increasing voltage also the mass of the power supply system increases which trades directly with payload mass. So, there exists an optimum I_{sp} where the total engine plus propellant mass of a spacecraft becomes minimal (see Fig. 5.4). We summarize this important fact by stating:

> The best ion engine for a mission with a given delta-v is NOT the one with the highest $I_{sp,}$ but the one with that I_{sp} that minimizes the total engine mass.

In the following we want to derive means to determine an optimal S/C system layout. The following contributions add to the total mass of a S/C with an electric propulsion system

$$m_0 = m_p + m_s + m_L + m_g = m_p + m_f \tag{5.3.1}$$

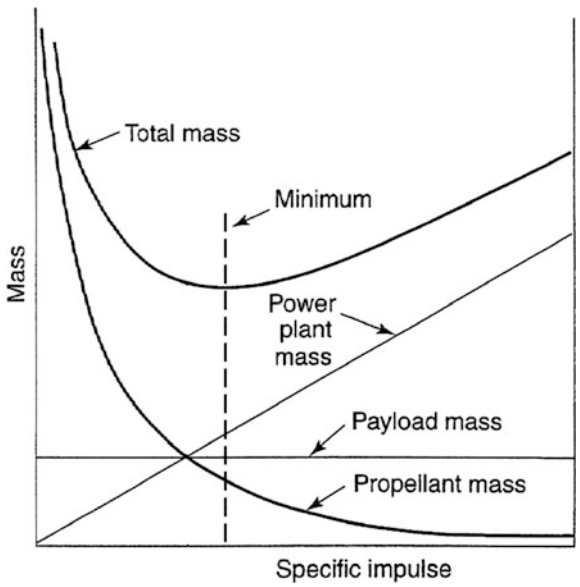


Fig. 5.4 The mass of a S/C with electrical engines has a minimum, as with increasing specific impulse the required propellant exponentially decreases, but the mass of the electric generator linearly increases

with m_s the structural mass, m_L the payload mass, and m_g the mass of the power plant. The latter can be, for instance, an RTG (radioisotope thermoelectric generator) or solar cells. If P_g is the electric power provided, then the so-called

$$\alpha := \frac{P_g}{m_g} \quad \textbf{specific power} \tag{5.3.2}$$

describes the mass-specific power output of the plant. Current plants are of order 100−200 W kg^{-1}. The supplied power is converted into exhaust jet energy $\frac{1}{2}\dot{m}_p v_e^2$ (see Eq. (1.3.16)), at a given efficiency of the thruster η_t, i.e.

$$\frac{1}{2}\dot{m}_p v_e^2 = \frac{m_p}{2t_p} v_e^2 = \eta_t \alpha m_g \tag{5.3.3}$$

Here we have assumed a continuous mass flow over the total combustion time t_p. If we define the so-called

$$v_c := \sqrt{2\alpha t_p \eta_t} \quad \textbf{characteristic velocity} \tag{5.3.4}$$

we get the following relation between propellant mass and power plant masswe get the following relation between propellant mass and power plant mass

$$m_p = \frac{m_g}{\gamma^2} \tag{5.3.5}$$

with

$$\gamma := \frac{v_e}{v_c}$$

If we define

$$\mu := \frac{m_s + m_L}{m_0} \quad \textbf{payload ratio}$$

we find from Eqs. (5.3.1) and (5.3.5)

$$\mu = 1 - \frac{m_p}{m_0} - \frac{m_g}{m_0} = 1 - \frac{m_p}{m_0}\left(1 + \gamma^2\right)$$

Because of Eq. (2.2.4)

$$e^{-\Delta v/v_e} = \frac{m_f}{m_0} = 1 - \frac{m_p}{m_0} \tag{5.3.6}$$

we get with the definition $\lambda := \Delta v / v_c$

$$\mu = e^{-\lambda/\gamma}\left(1 + \gamma^2\right) - \gamma^2 \tag{5.3.7}$$

We now want to find the dependence of the propulsion demand (here λ) provided by the electrical engine on the variable ejection velocity and hence on γ at a given payload ratio μ. For this purpose we solve Eq. (5.3.7) with regard to λ and finally get

$$\lambda = \gamma \ln\left(\frac{1+\gamma^2}{\mu+\gamma^2}\right) \tag{5.3.8}$$

The curves $\lambda = \lambda(\gamma)$ with parameter μ are shown in Fig. 5.5. It is now our goal to find the maximum Δv provided by the engine at a given μ, where v_e is our variable. To find this maximum, we have to differentiate Eq. (5.3.8) and find its root. This leads to the following conditional equation for γ:

$$\ln\frac{1+\gamma^2}{\mu+\gamma^2} = \frac{2\gamma^2(1-\mu)}{(\mu+\gamma^2)(1+\gamma^2)} \tag{5.3.9}$$

Solving for γ gives the optimized $v_e/v_c = \gamma$ and via Eq. (5.3.8) the maximized $\Delta v/v_c = \lambda$ as a function of μ as shown in Fig. 5.6.

Having found the optimal parameters, the following calculation scheme can be given to optimize an electric propulsion system.

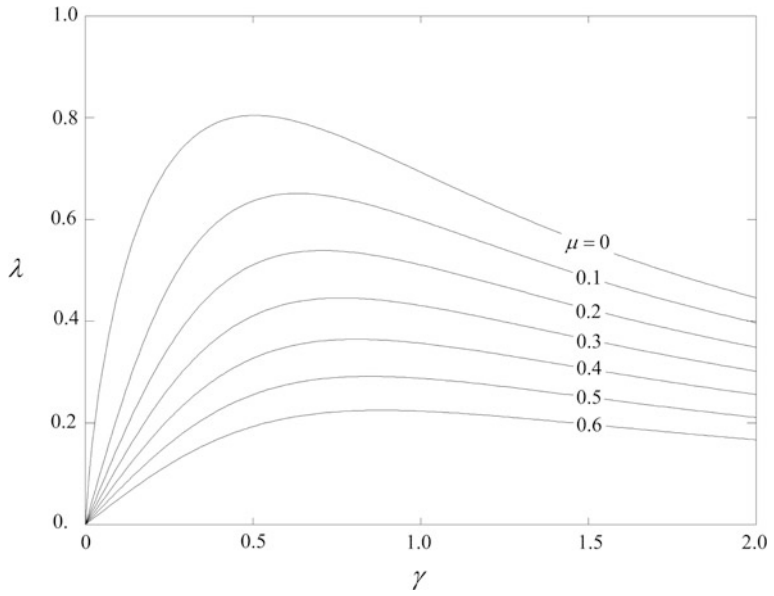


Fig. 5.5 The available normalized propulsion demand of an electric propulsion as a function of the normalized ejection velocity

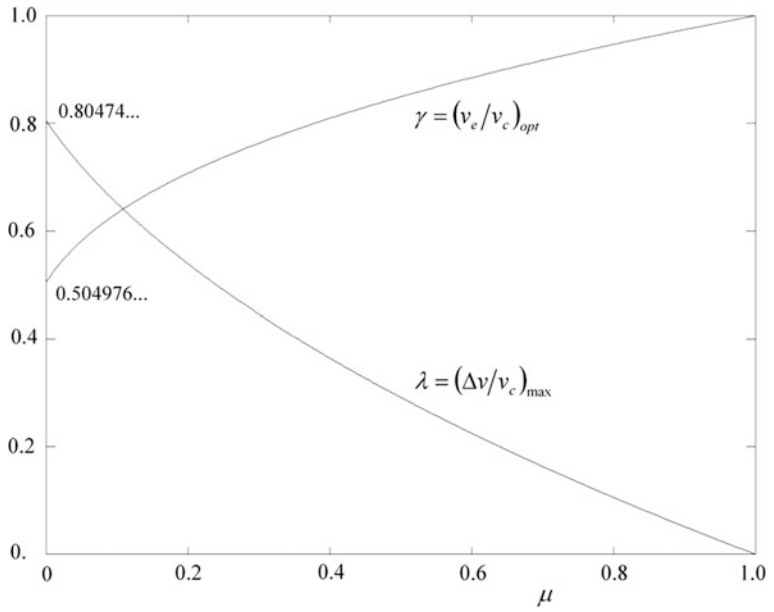


Fig. 5.6 The normalized optimal ejection velocity and the maximal available normalized propulsion demand of an electrical engine as a function of the payload μ

Calculation scheme

1. Determine from Eq. (5.3.9) or from Figure 5.6 for a given μ the optimal $\gamma = v_e/v_c$ and from Eq. (5.3.8) the corresponding maximized $\lambda = \Delta v/v_c$
2. Determine the propellant mass m_p from Eq. (5.3.6) and the power plant mass m_g from Eq. (5.3.5)
3. Determine at a given Δv the optimal v_e, or vice versa, through $v_e = \gamma \cdot \Delta v/\lambda$
4. Determine $v_c = v_e/\gamma$
5. For a given α and η_t determine the optimal burn time from $t_p = v_c^2/(2\alpha\eta_t)$.

As the payload mass is typically only a small percentage, approximate solutions can be provided for the limiting case $\mu \to 0$ (exercise, Problem 5.1)

$$\begin{aligned} \gamma &= 0.5050 \cdot (1 + 4.145\mu) \\ \lambda &= 0.8047 \cdot (1 - 2.461\mu) \\ m_p &= 0.7968 m_0 \, (1 - 2.685\mu) \\ m_g &= 0.2032 m_0 \, (1 + 5.606\mu) \\ v_e &= 0.6275 \cdot (1 + 6.606\mu)\Delta v \\ t_p &= 0.7721 \cdot (1 + 4.922\mu)\Delta v^2/\alpha\eta_t \end{aligned} \qquad @\ \mu < 0.04 \qquad (5.3.10)$$

Note *These are approximations in first order of* μ. *Therefore Eqs. (5.3.10) apply only as long as* $\mu \ll 1$. *It can be shown that for* $\mu < 0.04$ *the error* $\delta\gamma/\gamma$ *and with it also that of the other quantities remain below 5%.*

5.4 Problem

Problem 5.1 *Electric Engine Optimization (laborious)*
Prove that the linearized solution of Eq. (5.3.9) for $\mu \to 0$ are Eq. (5.3.10) by first showing with Newton's method that

$$\gamma = \gamma_0 \cdot \left[1 + \varepsilon\mu + O\left(\varepsilon^2\right)\right]$$
$$\gamma_0 = 0.504\,976\ldots$$

and

$$\varepsilon = \frac{\left(1+\gamma_0^2\right)^2}{2\gamma_0^2\left(1-\gamma_0^2\right)} = 4.145344\ldots$$

Chapter 6
Atmospheric and Ascent Flight

Now that we know the technical and physical properties of a rocket and the general equation of motion, which governs its flight, we are ready for a mission to the planets in our solar system. Before we investigate the rocket's motion in interplanetary space, it first has to ascent in Earth's gravitational field through the atmosphere. As will be shown later, ascent and reentry are subject to identical physical laws treated by the science called *flight mechanics*. The difference between the two is that reentry is powerless and the initial conditions of both mission phases are drastically different. This is why the problems we have to deal with are much different, and we therefore devote a chapter of its own (Chap. 10) to reentry after we consider orbital motion (Chap. 7), orbital maneuvering (Chap. 8), and come back from an interplanetary journey (Chap. 9). For ascent and for reentry, the properties of Earth's atmosphere are crucial. This is why we will first examine here the atmosphere's condition (Sect. 6.1) and the general laws of motion through the atmosphere (Sect. 6.2). Only after that shall we go (Sects. 6.3 and 6.4) into the specifics of how to optimize an ascent into space.

6.1 Earth's Atmosphere

From a space flight point of view, the atmosphere plays an important role during ascent, Earth orbiting, and reentry. First, because of the aerodynamics at lower altitudes and, second, because it also impacts low Earth orbits due to the residual atmospheric drag at high altitudes. To determine these influences quantitatively, we have to derive expressions that describe the density distribution in the atmosphere as a function of altitude.

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6.1.1 Density Master Equation

It is well known that atmospheric pressure, starting from sea level, decreases with increasing altitudes. To describe its quantitative dependency in mathematical terms, we first imagine that the atmosphere is a stack of infinitesimally thin layers with thickness ds (see Fig. 6.1). Without loss of generality, we assume a layer of having a surface A of arbitrary size. The volume of the layer is then $A \cdot ds$ and the mass of air within it is $dm = \rho A \cdot ds$, where ρ is the atmospheric mass density. The additional infinitesimal pressure it generates onto the ones below is the weight force per square unit

$$dp = d\left(\frac{mg}{A}\right) = \frac{g}{A} dm = \frac{g}{A} d(\rho A s) = \rho g \cdot ds$$

Here s measures the height against increasing pressure, which is in the same direction as the increase in gravitational force. But atmospheric pressure is usually given as a function of altitude h relative to sea level, i.e., in the opposite direction. So $ds = -dh$. Therefore we find for the pressure change of the atmospheric pressure

$$\frac{dp}{dh} = -\rho g(h) \tag{6.1.1}$$

where we have taken into account that with the gravitational acceleration g decreases with altitude according to

$$g(h) = g_0 \frac{R_\oplus^2}{r^2} = g_0 \frac{R_\oplus^2}{(R_\oplus + h)^2}$$

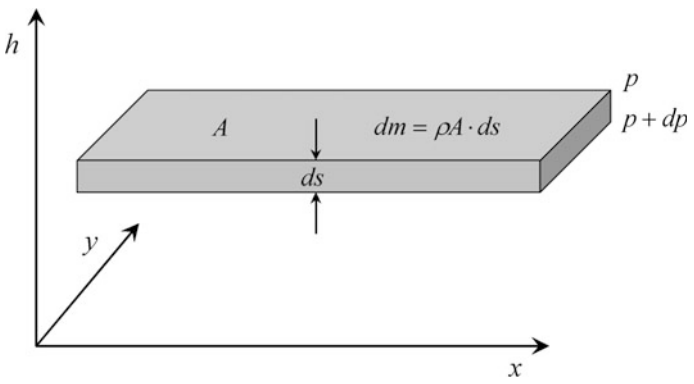


Fig. 6.1 Characteristics of an infinitesimally thin atmospheric layer

with $g_0 = 9.798\ \mathrm{m\ s^{-2}}$ and $R_\oplus = 6378.136$ km the Earth's radius. We now need to know how a gas behaves under external pressure. Earth's atmosphere can be described in a very good approximation by the ideal gas law (cf. Eq. (4.1.1))

$$p = \rho R_s T \quad \textbf{ideal gas law} \tag{6.1.2}$$

where T is the layer's temperature and

$$R_s = \frac{R}{M_{air}} = 286.91\ \frac{\mathrm{J}}{\mathrm{K \cdot kg}} \quad \begin{matrix}\textbf{specific gas constant} \\ \textbf{of standard atmosphere}\end{matrix}$$

Strictly speaking, R_s holds only for the standard atmosphere, i.e., for the standard molecular composition. As we will see, only particular parts of the atmosphere fulfill this requirement. We now differentiate Eq. (6.1.2) with regard to the altitude h. Because ρ and T depend on h, this yields with Eq. (6.1.1)

$$\frac{dp}{dh} = \frac{d\rho}{dh} R_s T + \rho R_s \frac{dT}{dh} = -\rho g(h)$$

From this follows that

$$\frac{d\rho}{\rho} + \frac{dT}{T} = -\frac{g(h) \cdot dh}{R_s T(h)} \tag{6.1.3}$$

This equation is the master equation to calculate the density function $\rho(h)$ for a given $T(h)$. By applying Eq. (6.1.2) $p(h)$ may then be derived. So all we need to know is the temperature profile $T(h)$. This is depicted in Fig. 6.2.

6.1.2 *Atmospheric Structure*

In terms of modeling, the atmosphere can be divided in two quite different zones:

1. The so-called **homosphere** extends to an altitude of roughly 120 km and includes the troposphere, stratosphere, mesosphere, and part of the thermosphere. The name derives from the fact that it is constituted by a homogeneous mixture of the standard atmospheric components, and therefore $R_s = const$. Although overall the homosphere shows only modest variations in temperature, each layer behaves meteorologically differently due to their different temperature gradients (see Fig. 6.2).
2. The so-called **heterosphere** extends above about 120 km and includes most of the thermosphere and exosphere (see Fig. 6.2). As its name indicates, the molecular constituents are heterogeneous with height and become partly ionized, all of which results in $R_s \neq const$. Its temperature deviates significantly

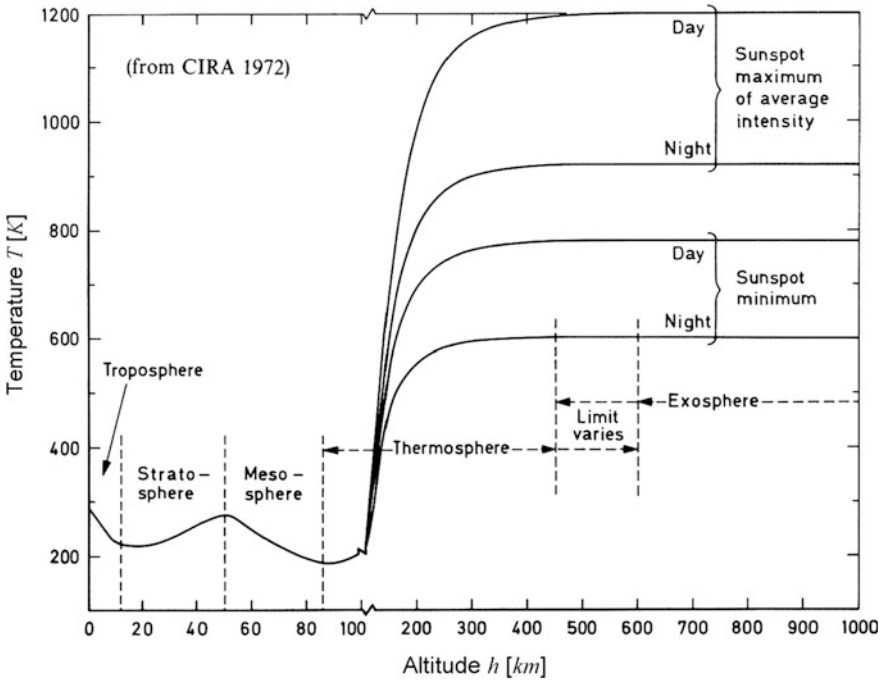


Fig. 6.2 The temperature distribution in Earth's atmosphere. *Credit* King-Hele 1987

from that in the homosphere in that it displays much higher levels of, and tremendous variations in, temperatures, which are even time-dependent.

The detailed $T(h)$ profile in the homosphere and in the heterosphere is not analytical, so an exact solution to Eq. (6.1.3) can be obtained only by numerical integration. Because this is too intricate for practical applications we have to look for ways to find approximate solutions.

Homosphere (Barometric Formula)
For general purposes the temperature within the homosphere can be considered roughly as constant with a mean value of $T = T_0 \approx 230$ K. For a constant temperature, $dT = 0$, reducing Eq. (6.1.3) to

$$\frac{d\rho}{\rho} = -\frac{R_\oplus^2}{(R_\oplus + h)^2}\frac{dh}{H}$$

Here we have introduced the so-called *scale height* as

$$H := \frac{T_0 R_s}{g_0} = T_0 \cdot 29.28 \left[\text{m K}^{-1}\right] \quad \textbf{scale height}$$

The solution is found by direct integration of both sides yielding

$$\rho = \rho_0 \exp\left[-\frac{R_\oplus h}{(R_\oplus + h)H}\right] \quad @ \quad 0 \le h \le 120 \text{ km} \tag{6.1.4}$$

where h is considered relative to sea level and ρ_0 is the density at sea level. Compared to our assumption $T = const$ we can also safely assume $h \ll R_\oplus$ which leads to the well-known barometric formula

$$\boxed{\rho = \rho_0 \exp\left(-\frac{h}{H}\right)} \quad @ \quad 0 \le h \le 120 \text{ km} \quad \textbf{barometric formula} \tag{6.1.5}$$

If one fits this formula to the actual atmospheric data in the range $0 \le h \le 120$ km one obtains the following mean values for ρ_0, H with errors $\Delta\rho/\rho < 50\%$:

$$\begin{array}{l} \rho_0 = 1.752 \text{ kg m}^{-3} \\ H = 6.7 \text{ km} \end{array} \quad @ \quad 0 \le h \le 120 \text{ km} \tag{6.1.6}$$

Equations (6.1.5) and (6.1.6) are the most convenient and hence the most common form to describe the density distribution in the homosphere. They will be used throughout this book for ascent and reentry of a S/C with sometimes adaptations of the scale height to the actual altitude conditions.

For aerodynamic purposes in later sections we note that in the homosphere of a standard atmosphere with density $\rho(h)$, the Reynolds number of a flight vehicle with speed v is given as

$$Re = 36 \times 10^6 \cdot \exp\left(-\frac{h}{H}\right) \cdot Ma \cdot \frac{l}{L} \propto \rho v$$

with

h altitude
$H = T(h) \cdot 29.28 \left[\text{m K}^{-1}\right] \approx 7.1 \text{ km}$
Ma Mac number
l characteristic length of the vehicle
$L =$ 1m.

For a vehicle reentering from space $Re \approx 10^6$.

Heterosphere

Spacecraft orbit Earth at altitudes $h > 100$ km where the atmospheric drag slowly brakes their speed and thus drags them down into lower and lower orbits. To find out their orbit life-times (see Sect. 12.7.4) the detailed density profile at those elevated altitudes needs to be known. Since in the heterosphere temperature varies strongly with height and time and R_s is not constant, a barometric formula like Eq. (6.1.5) does not hold. More appropriate atmospheric models need to be provided which by nature are, however, considerably more complex. Today's quasi-standard is the MSIS-86 model (a.k.a. CIRA-86), which is COSPAR's International Reference Atmosphere, and its newer extension MSIS-E-90. But also

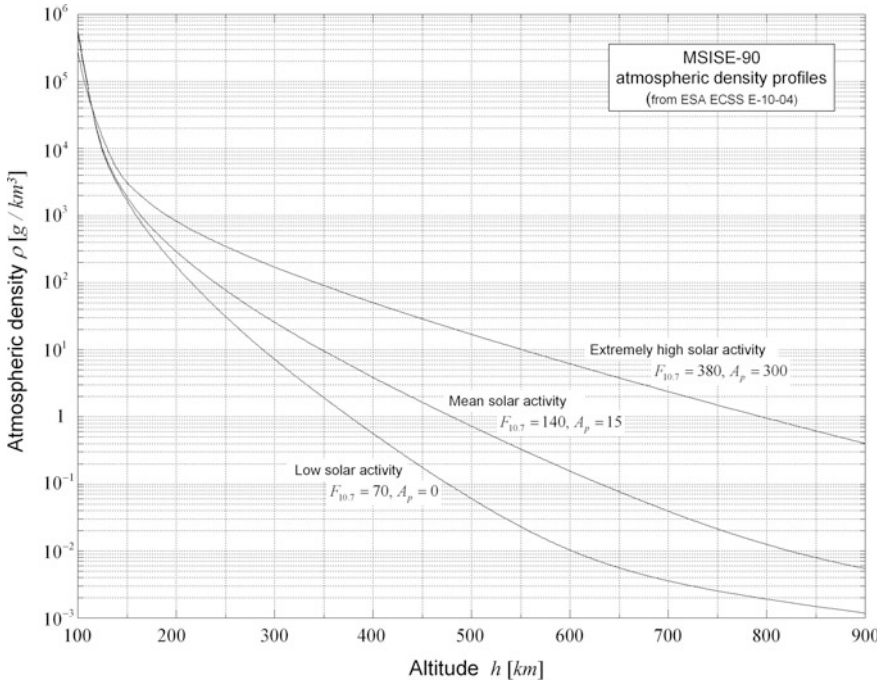


Fig. 6.3 Mean atmospheric density in the heterosphere as derived from the MSIS-E-90 model

the Jacchia 1977 (J77) model and its older variants and the Harris–Priester model from 1962 (see, e.g., Montenbruck and Gill 2000) are still frequently used. They are all based on piecewise analytical expressions for different altitudes, whose coefficients have been adapted to measured values. So there are no closed analytical expressions and the densities have to be derived numerically. Figure 6.3 depicts the mean atmospheric density profiles as derived from the MSIS-E-90 model above 100 km.

As an illustrative example we present the relatively simple Harris–Priester model. It is based on the data of the upper atmosphere derived from a solution of the heat conduction equation. It takes into account the daily, but not the yearly, temperature variations in the atmosphere. The upper atmosphere expands because of daily insolation and runs about 2 h behind, which corresponds to 30° of longitude toward the east. The density distributions of the corresponding density peaks (maxima, M) and valleys (minima, m) are described by the functions $\rho_M(h)$ and $\rho_m(h)$ by means of piecewise exponential interpolation between interpolation altitudes h_i:

$$\begin{aligned} \rho_m(h) &= \rho_m(h_i)\exp\left(\frac{h_i - h}{H_m}\right) \\ \rho_M(h) &= \rho_M(h_i)\exp\left(\frac{h_i - h}{H_M}\right) \end{aligned} \quad @ \quad h_i < h < h_{i+1} \tag{6.1.7}$$

where h is the altitude above the Earth's reference ellipsoid (see Sect. 12.2.1). The scale heights H_m and H_M are given as

$$H_m(h) = \frac{h_i - h_{i+1}}{\ln \rho_m(h_{i+1})/\ln \rho_m(h_i)}$$
$$H_M(h) = \frac{h_i - h_{i+1}}{\ln \rho_M(h_{i+1})/\ln \rho_M(h_i)}$$

The daily density variation due to insolation is modeled as a cosine variation

$$\rho(h) = \rho_m(h) + [\rho_M(h) - \rho_m(h)] \cdot \cos^n\left(\frac{\Psi}{2}\right)$$

where Ψ is the angle between the position vector of the orbiting S/C and the vector to the density peak. Density variations in geographical latitude are represented by a declinational dependence of Ψ and the exponent n: for a small inclination $n = 2$, and $n = 6$ for polar orbits. Table shows the density coefficients $\rho_M(h)$ and $\rho_m(h)$ at the different interpolation points. Note that above 300 km the atmospheric density differs by a factor of more than two between day-time $\rho_M(h)$ and night-time $\rho_m(h)$ values. This is caused by the large temperature differences in these altitudes between day and night (see Fig. 6.2).

Outer Space

The atmosphere at around 100 km is special in two aspects: First, at this altitude the atmospheric composition changes markedly. Second, above 100 km the heterosphere is so rarefied that it can no longer support aerodynamic flight. To be more specific, the limiting altitude where the flying speed of an airplane would need to reach orbital velocity in order to provide lift that would counteract the gravitational force is called the *Kármán line* and was calculated by Theodore von Kármán to be around 100 km.

For the above two reasons the Fédération Aéronautique Internationale (FAI), which is an international standard setting and record-keeping body for aeronautics and astronautics, defined the border to **outer space** at exactly 100 km. So ***space travelers*** are individuals who have been above 100 km altitude in a vehicle. As opposed to this, in the 1960s the United States adopted the definition of an astronaut as someone having been above at least 50 statute miles (equaling 80 km) altitude. Thus, outer space was and still is indirectly defined by the U.S. as space above 80 km altitude. However, only the FAI's definition is internationally accepted today. It should be noted that from a re-entry point of view the transition from astronautics to aeronautics formally takes place at the so-called entry interface (see Sect. 10.1.2) at 400,000 ft equaling 122 km. In NASA and RKA terms an ***astronaut*** is a crew member of a spacecraft, while other people, who travel aboard spacecraft, are termed ***space flight participants***.

Note that opposed to the above US definition of an astronaut (US astronauts are awarded the *astronaut badge* also called *astronaut wings*) the Association of Space Explorers (ASE, the association of all flown astronauts) defines ***astronauts*** as any "*individuals who have completed at least one orbit of the Earth in a spacecraft*".

Table 6.1 Atmospheric density coefficients of the Harris–Priester model valid for a mean solar activity

h (km)	ρ_m (g km^{-3})	ρ_M (g km^{-3})	h (km)	ρ_m (g km^{-3})	ρ_M (g km^{-3})
100	497400.0	497400.0	420	1.558	5.684
120	24900.0	244900.0	440	1.091	4.355
130	8377.0	8710.0	460	0.7701	3.362
140	3899.0	4059.0	480	0.5474	2.612
150	2122.0	2215.0	500	0.3916	2.042
160	1263.0	1344.0	520	0.2819	1.605
170	800.8	875.8	540	0.2042	1.267
180	528.3	601.0	560	0.1488	1.005
190	361.7	429.7	580	0.1092	0.7997
200	255.7	316.2	600	0.08070	0.6390
210	183.9	239.6	620	0.06012	0.5123
220	134.1	185.3	640	0.04519	0.4121
230	99.49	145.5	660	0.03430	0.3325
240	74.88	115.7	680	0.02632	0.2691
250	57.09	93.08	700	0.02043	0.2185
260	44.03	75.55	720	0.01607	0.1779
270	34.30	61.82	740	0.01281	0.1452
280	26.97	50.95	760	0.01036	0.1190
290	21.39	42.26	780	0.008496	0.09776
300	17.08	35.26	800	0.007069	0.08059
320	10.99	25.11	840	0.004680	0.05741
340	7.214	18.19	880	0.003200	0.04210
360	4.824	13.37	920	0.002210	0.03130
380	3.274	9.955	960	0.001560	0.02360
400	2.249	7.492	1000	0.001150	0.01810

This is why Allan Shepard, the first American in space, internationally is not considered to be America's first astronaut (who is John Glenn) because his space flight in 1961 was just suborbital. Only later Shepard became an astronaut owing to his participation in the Apollo 14 mission. All "Apollo moon flyers" are accepted by ASE as astronauts, because they all have completed at least one Earth orbit before translunar injection. In line with US definition, eight X-15 pilots were awarded astronaut wings in the 1960s because they reached altitudes above 80 km. But only one of them, Joseph A. Walker, flew above 100 km in 1963 (even twice) and therefore is an space traveler by FAI standards, but not an astronaut by ASE standards.

Note that beyond what is referenced here there exists no legal definition of astronaut, neither in the Outer Space Treaties, nor in the International Space Station Legal Framework, nor in the US Commercial Space Launch Act of 1984.

6.1.3 Piecewise-Exponential Model

In later chapters we will study the ascent and reentry of spacecraft through the atmosphere and the orbit life-time of satellites in low Earth orbits. For these studies the barometric formula for the homosphere is too inaccurate, while for the heterosphere the common atmospheric models are too complex to handle analytically. By examining the functional dependency of the atmospheric density in Fig. 6.3 in logarithmic representation, one recognizes that the density can be expressed quite well by piecewise straight lines corresponding to piecewise exponential functions of the form

$$\rho(h) = \rho_i \exp\left(-\frac{h - h_i}{H_i}\right) \quad @ \quad h_i < h < h_{i+1} \tag{6.1.8}$$

where h is the altitude above sea level and h_i are the base altitudes above sea level for a given altitude interval, ρ_i the corresponding nominal base density, and H_i the scale height holding for an entire interval.

They are given for the different altitude intervals in Table 6.2. Equation (6.1.8) is the density model, which we will use in the following for our general analytical studies related to the atmosphere.

Table 6.2 Altitude intervals and corresponding atmospheric coefficients for the piecewise exponential model based on the CIRA-72 atmospheric model

Altitude h (km)	Base altitude h_0 (km)	Nominal density ρ_0 (kg m^{-3})	Scale height H (km)	Altitude h (km)	Base altitude h_0 (km)	Nominal density ρ_0 (kg m^{-3})	Scale height H (km)
0–25	0	1.225	7.249	150–180	150	2.070×10^{-9}	22.523
25–30	25	3.899×10^{-2}	6.349	180–200	180	5.464×10^{-10}	29.740
30–40	30	1.774×10^{-2}	6.682	200–250	200	2.789×10^{-10}	37.105
40–50	40	3.972×10^{-3}	7.554	250–300	250	7.248×10^{-11}	45.546
50–60	50	1.057×10^{-3}	8.382	300–350	300	2.418×10^{-11}	53.628
60–70	60	3.206×10^{-4}	7.714	350–400	350	9.518×10^{-12}	53.298
70–80	70	8.770×10^{-5}	6.549	400–450	400	3.725×10^{-12}	58.515
80–90	80	1.905×10^{-5}	5.799	450–500	450	1.585×10^{-12}	60.828
90–100	90	3.396×10^{-6}	5.382	500–600	500	6.967×10^{-13}	63.822
100–110	100	5.297×10^{-7}	5.877	600–700	600	1.454×10^{-13}	71.835
110–120	110	9.661×10^{-8}	7.263	700–800	700	3.614×10^{-14}	88.667
120–130	120	2.438×10^{-8}	9.473	800–900	800	1.170×10^{-14}	124.64
130–140	130	8.484×10^{-9}	12.636	900–1000	900	5.245×10^{-15}	181.05
140–150	140	3.845×10^{-9}	16.149	1000–	1000	3.019×10^{-15}	268.00

Adopted from Vallado (2007)

6.2 Hypersonic Flow Theory

Imagine a spacecraft that experiences drag in a low Earth orbit or in ascents or reentries through the atmosphere at hypersonic speeds, i.e. at more than about Mach 5. What is the gas flow around the spacecraft and how does it generate the aerodynamic forces on the vehicle? The answer critically depends on whether the atmosphere can be treated as an assembly of free molecules or as an self-interacting gas. Physically we have a gas if the interaction between the molecules is faster than the interaction with the spacecraft. The key parameter is the *mean free path* λ, which is the distance a molecule travels between two successive collisions. The mean free path, of course, depends inversely on the density and hence on the temperature and pressure of the gas $\lambda \propto 1/n \propto T/p$. At sea level the distance between collisions is only $\lambda = 0.066$ μm. At 10 km altitude $\lambda \approx 0.2$ μm, at 50 km $\lambda \approx 80$ μm, at 100 km $\lambda \approx 2$ cm, and 200 km $\lambda \approx 250$ m. For physical effects on the spacecraft the so-called *Knudsen number*, $Kn = \lambda/L$, is relevant, where L is the length scale of the spacecraft. Obviously, if $\lambda \geq L \approx 10$ m, equaling altitudes $h \geq 150$ km, we can treat the atmospheric gas as an assembly of individual non-interacting molecules and hence $Kn \geq 1$. This is called *free molecular flow regime*. For $Kn < 0.03$, i.e. at altitudes $h \leq 90$ km, the atmosphere can be modelled as a continuum, and for $0.03 < Kn < 1$, i.e. 90 km $< h <$ 150 km, we are in a *transition regime*. Since a spacecraft upon ascent or reentry passes all three regimes we need to study them all.

Free molecular flow regime	$Kn \geq 1$	$h \geq 150$ km
Transition regime	$0.03 < Kn < 1$	90 km $< h <$ 150 km
Continuum regime	$Kn < 0.03$	$h \leq 90$ km

In the following general study of the aerodynamic forces of a vehicle, namely lift and drag, let ρ be the atmospheric mass density at a given altitude, m the mass of the vehicle, $A_\perp$ its cross-sectional area (wetted surface area) with regard to the flight direction, and v its velocity relative to the atmosphere, which to a high accuracy is identical to the speed of the S/C measured relative to the ground.

6.2.1 Free Molecular Flow

We first study the effects of free molecular flow, valid for $Kn \geq 1$ equaling altitudes $h \geq 150$ km. In this case we can treat the gas as an assembly of non-interacting molecules that behave fully ballistically. We therefore can apply statistical mechanics, from which the aerodynamic forces are simply determined by the linear

momentum transfer of the atmospheric particles per unit time. In the scattering process we define

Δp as the momentum transfer to the surface,

$\Delta p_{||}$ the proportion of momentum that is transferred in the flow direction $\hat{\boldsymbol{p}}$, *and*

$\Delta p_{\perp}$ the portion that is transferred into the direction $\hat{\boldsymbol{p}}_{\perp}$, lying in the reflection plane normal to $\hat{\boldsymbol{p}}$, i.e., $\hat{\boldsymbol{p}}\,\hat{\boldsymbol{p}}_{\perp} = 0$

All these momentum transfers critically depend on the details of the ballistic reflection process from the surface. Quite formally, if an impacting particle initially possess linear momentum p, the amount of momentum that is transferred to the surface depends on one hand on the incidence angle θ measured relative to the surface normal, and on the other hand on the way the particle is scattered off the surface and hence on the details of the scattering process (denoted by the symbol $\times$), which depends for instance on the surface (e.g., surface roughness) and boundary layer properties. These scattering details are comprised by the dimensionless momentum transfer functions $f_{||,\times}(\theta)$ and $f_{\perp,\times}(\theta)$ defined by

$$\begin{aligned} p \cdot f_{||,\times}(\theta) &:= \Delta p_{||} = \Delta \boldsymbol{p} \cdot \hat{\boldsymbol{p}} \\ p \cdot f_{\perp,\times}(\theta) &:= \Delta p_{\perp} = \Delta \boldsymbol{p} \cdot \hat{\boldsymbol{p}}_{\perp} \end{aligned} \tag{6.2.1}$$

$f_{||,\times}(\theta)$ and $f_{\perp,\times}(\theta)$ are known theoretically only for some elementary surface shapes (see e.g. Sect. 6.2.4). So, in general they are determined experimentally.

Based on the moment transfer, the aerodynamic drag and lift (which are the absolute values of the corresponding force vectors) on $A_{\perp}$ are given by Newton's second law (see Eq. 7.1.12 and Remark thereafter) as

$$\begin{aligned} D &= \frac{d\left(\Delta p_{||}\right)}{dt} = f_{||,\times}\frac{dp}{dt} = f_{||,\times}\frac{d(mv)}{dt} = f_{||,\times} v\,\dot{m} = f_{||,\times}\rho v^2 A_{\perp} \\ L &= \frac{d(\Delta p_{\perp})}{dt} = f_{||,\times}\frac{dp}{dt} = f_{\perp,\times}\frac{d(mv)}{dt} = f_{\perp,\times} v\,\dot{m} = f_{\perp,\times}\rho v^2 A_{\perp} \end{aligned}$$

where the latter follows from the continuity equation (see Eq. (1.2.8)) of the air flow, $\dot{m} = \rho v A_{\perp}$. From this derivation we recognize that the quadratic velocity dependence on one hand is due to the momentum transfer and on the other to the number of particles hitting the surface per unit time, both of which are proportional to the impinging velocity. To calculate the aerodynamic drag and lift of the entire S/C, the drag and lift of an infinitesimal surface area $dA_{\perp}$

$$\begin{aligned} dD &= f_{||,\times}\rho v^2 \cdot dA_{\perp} = f_{||,\times}\cos\theta \cdot \rho v^2 \cdot dA \\ dL &= f_{\perp,\times}\rho v^2 \cdot dA_{\perp} = f_{\perp,\times}\cos\theta \cdot \rho v^2 \cdot dA \end{aligned}$$

with different θ and/or surface properties have to be integrated over the total wetted area

$$D = \rho v^2 \iint_A f_{\parallel,\times}(\theta) \cos\theta \cdot dA \tag{6.2.2a}$$

$$L = \rho v^2 \iint_A f_{\perp,\times}(\theta) \cos\theta \cdot dA \tag{6.2.2b}$$

Specular and Diffuse Reflection

For specular and diffuse reflection the momentum transfer functions can easily be evaluated. From Fig. 6.4 we deduce that for specular reflection the momentum transfer to the surface is

$$\Delta \boldsymbol{p} = -2p \cos\theta \cdot \boldsymbol{n} \quad @ \quad \text{specular reflection} \tag{6.2.3}$$

where

$$\boldsymbol{n} = -(\cos\theta\, \hat{\boldsymbol{p}} + \sin\theta\, \hat{\boldsymbol{p}}_\perp)$$

is the surface normal vector. For diffuse reflection we must average the momentum transfer over all reflection angles. One generally assumes that the reflected intensity is proportional to the angle of the reflected molecule from $\boldsymbol{n}$. This leads to

$$\Delta \boldsymbol{p} = -p \cos\theta \cdot \left(\frac{2}{3}\boldsymbol{n} - \hat{\boldsymbol{p}}\right) \quad @ \quad \text{diffuse reflection} \tag{6.2.4}$$

Applying these results to Eqs. (6.2.1) we find the longitudinal and transverse transfer functions

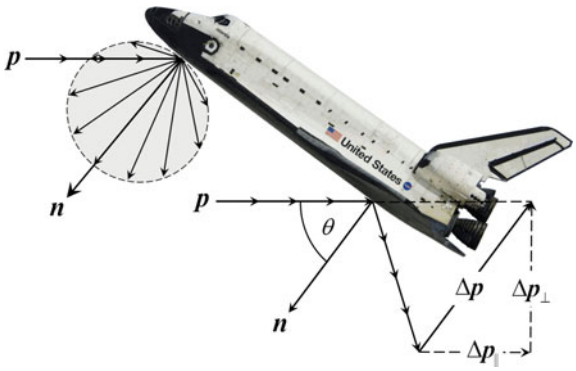


Fig. 6.4 Schematics of diffuse (top) and specular (bottom) reflection

$$\begin{aligned} f_{||,\,spec} &= 2\cos^2\theta \\ f_{\perp,\,spec} &= 2\sin\theta\,\cos\theta \end{aligned} \quad @\ \text{specular reflection} \tag{6.2.5}$$

and

$$\begin{aligned} f_{||,\,diff} &= \cos\theta\left(\tfrac{2}{3}\cos\theta + 1\right) \\ f_{\perp,\,diff} &= \tfrac{2}{3}\sin\theta\,\cos\theta \end{aligned} \quad @\ \text{diffuse reflection} \tag{6.2.6}$$

For the specular and diffuse drag and lift we therefore obtain with Eq. (6.2.2a) and (6.2.2b) for a flat plate with surface A and hence wetted area $A_\perp = A\,\cos\theta$

$$\begin{aligned} D_{spec} &= 2\rho v^2 A\,\cos^3\theta \\ L_{spec} &= 2\rho v^2 A\,\sin\theta\,\cos^2\theta \end{aligned} \quad @\ \text{flat plate specular reflection} \tag{6.2.7}$$

and

$$\begin{aligned} D_{diff} &= \rho v^2 A\,\cos^2\theta\left(\tfrac{2}{3}\cos\theta + 1\right) \\ L_{diff} &= \tfrac{2}{3}\rho v^2 A\,\sin\theta\,\cos^2\theta \end{aligned} \quad @\ \text{flat plate diffuse reflection} \tag{6.2.8}$$

Assuming only specular reflection from a flat plate we find for the so-called lift-to-drag ratio

$$\frac{L_{spec}}{D_{spec}} = \tan\theta \tag{6.2.9}$$

This is a celebrated result because it generally holds for the three regimes: inviscid supersonic, hypersonic, and free molecular flow over a flat plate.

Depending on the surface roughness one has a mix of specular and diffuse scattering for a concrete S/C surface in the free molecular flow regime.

6.2.2 *Newtonian Flow Theory*

We now turn to the hypersonic flow for $Kn < 1$, i.e. in the continuum regime with $Kn < 0.03$ and, at lower altitudes, in the transition regime with $0.03 < Kn < 1$. Here scattering between the gas molecules and hence gas dynamics needs to be accounted for.

A key feature of hypersonic flow is that the Mach angle is so small that the incoming stream lines are deflected nearly parallel and close to the surface making up a thin shock layer, as depicted in Fig. 6.5. Consequently,

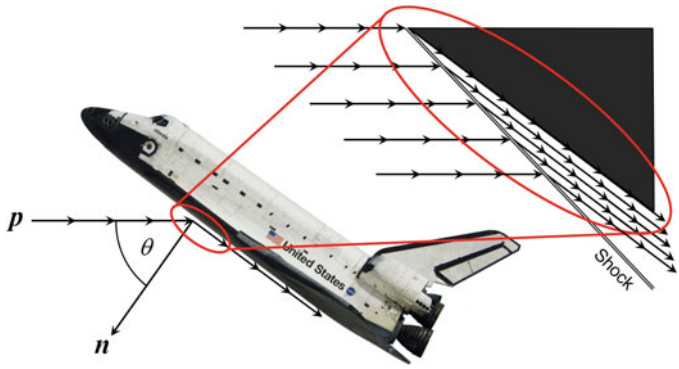


Fig. 6.5 Streamlines of hypersonic flow in Newtonian flow theory

1. the flow around a hypersonic body and the corresponding aerodynamics loses importance
2. the strong interaction, i.e. friction, between the surface layer and the body surface causes extreme gas temperatures leading to molecular vibrations and even dissociation, which causes alterations from the ideal gas concept.

Because this behavior resembles Newton's gas flow theory that he established at the end of the 17th century for wind flow around buildings, the corresponding flow theory is frequently called Newtonian flow theory, or Newtonian theory for short.

Gas Dynamics at Hypersonic Speeds

While for the main atmospheric components dissociation $O_2 \rightarrow O + O$ and $N_2 \rightarrow N + N$ commences at $v \geq 2\,\mathrm{km/s}$ and $v \geq 5\,\mathrm{km/s}$, they ionize only at $v \geq 11\,\mathrm{km/s}$. Therefore, in the case when a spacecraft such as a Space Shuttle or capsule reentries from Earth orbits with $v \approx 7.9\,\mathrm{km/s}$, ionization effects only occur partitially at stagnation points and hence are negligible, while dissociation effects in principle are relevant. Yet, dissociation has only a very mild, (and hence for our considerations also negligible), effect on lift and drag.

> **Remark** *For the Shuttle, dissociation varied considerably along the centerline and thus had a considerable effect on the pitching moment of the Shuttle. To be specific, due to these chemical effects the body flap deflection for pitch trim turned out to be twice as big as initially predicted and designed for the Shuttle. Therefore, for all Shuttle reentries body flap deflection was always at its limit and therefore under NASA's close observation.*

Therefore, we limit our analysis to ideal gas dynamics. This essentially implies that we will not consider chemical effects such as dissociation or ionization of atmospheric components at hypersonic velocities.

Hypersonic Flow Dynamics

We do not want to go into the details of hypersonic flow dynamics. For this we refer to classical textbooks (see e.g. Anderson 2006) and summarize here only those results, which are significant to determine the aerodynamic forces on a hypersonic vehicle.

As stated above the characteristics of hypersonic flow is that after impact the stream lines are nearly parallel to the surface. In microscopic terms this means that the molecules lose their momentum normal to the surface but preserve their tangential momentum as shown in Fig. 6.5. In effect, we have the following momentum transfer laws (cf. Eq. (6.2.3))

$$\Delta \boldsymbol{p} = p \cos\theta \cdot \boldsymbol{n} \tag{6.2.10}$$

which due to $f_{\|,\times} \propto \Delta\hat{\boldsymbol{p}} \cdot \hat{\boldsymbol{p}}$, $f_{\perp,\times} \propto \Delta\hat{\boldsymbol{p}} \cdot \hat{\boldsymbol{p}}_{\perp}$ leads to

$$\begin{array}{l} f_{\|,sld} = \cos^2\theta \\ f_{\perp,sld} = \sin\theta\cos\theta \end{array} \quad @ \text{ surface layer deflection} \tag{6.2.11}$$

that is, their values are only half of that in the free flow regime. Accordingly

> Although for hypersonic flow the values of the drag and lift coefficients, and hence also for drag and lift, attain only half of the values for free molecular flow, the $L/D = \tan\theta$ law holds in both flow regimes.

Since this results applies to any infinitesimally small flat surface it holds for any body shape.

Skin Friction Drag

However, apart from a modified scattering process the shear forces of the skin flow exert the following skin friction drag on the reentry vehicle surface

$$D_f = \frac{1}{2}\rho v^2 C_f A \tag{6.2.12}$$

Here C_f is the *skin friction drag coefficient*, which for turbulent boundary flow depends only weakly on the Reynolds number in the following way

$$C_f = 0.074 \cdot \mathrm{Re}^{-1/5} = C_{f0}\left(\frac{\rho_0 v_0}{\rho v}\right)^{1/5} \quad \begin{array}{l}\textbf{skin friction} \\ \textbf{drag coefficient}\end{array} \tag{6.2.13}$$

$$\rho_0 v_0 = 1.2041 \ \text{kg m}^{-3} Ma$$

where ρv is the mass flux on the surface and C_{f0} the skin friction drag coefficient, both at standard atmosphere and $Ma = 1$.

Transition regime
In the transition regime gas dynamics changes from non-slip to slip conditions and thus gas pressure changes in the boundary layer. Nevertheless, the *L/D* ratio, which was derived simply from geometrical considerations, remains unaffected. We therefore are saved from extending the result $L/D = \tan\theta$ of the Newtonian limit into the transition regime as long as the gas interaction is sufficient to create a boundary layer with a gas pressure.

6.2.3 Drag and Lift Coefficients

Owing to the specific microscopic unevenness of actual S/C surfaces (surface roughness) drag and lift obviously cannot be evaluated analytically, and often not even numerically from the integrals Eqs. (6.2.2a) and (6.2.2b). One therefore and even quite generally resorts to defining a dimensionless "effective force conversion coefficient"

$$C := \frac{\text{force}}{(\text{dynamic pressure}) \times A_\perp} \tag{6.2.14}$$

where the dynamic (air) pressure is defined as $p_a := \frac{1}{2}\rho v^2$. Observe that through *Bernoulli's principle* $p_a A_\perp = \frac{1}{2}\rho v^2 A_\perp \propto \frac{1}{2}mv^2 = E_{kin}$ the denominator is closely related to the kinetic energy of the impacting particles. With the above definition we obtain from Eq. (6.2.2a)

$$D = \frac{1}{2}\rho v^2 C_D A_\perp \tag{6.2.15a}$$

with the "effective along-force conversion coefficient", better known as the *drag coefficient,*

$$C_D = \frac{2}{A_\perp}\iint_{A_\perp} f_{\|,\times}(\theta)\cdot dA_\perp = \frac{2}{A_\perp}\iint_{A} f_{\|,\times}(\theta)\cos\theta\cdot dA \quad \textbf{drag coefficient} \tag{6.2.15b}$$

Owing to the normalization to $A_\perp$, this dimensionless drag coefficient only depends on the shape of the body (and not on its size) and on the detailed interaction of the impacting particles with the surface. By the same token, the lateral lift can be determined from Eq. (6.2.2b) as

$$L = \frac{1}{2}\rho v^2 C_L A_\perp \tag{6.2.16a}$$

with the dimensionless lift coefficient

$$C_L = \frac{2}{A_\perp} \iint\limits_{A_\perp} f_{\perp,\times}(\theta) \cdot dA_\perp = \frac{2}{A_\perp} \iint\limits_{A} f_{\perp,\times}(\theta) \cos\theta \cdot dA \quad \textbf{lift coefficient} \tag{6.2.16b}$$

Note *Note that the seemingly odd factor 2 is well-established by reference to the kinetic energy of the impacting particles. Despite this, the use of a drag coefficient defined without the factor 2 is not uncommon in the literature. Therefore, caution is always in order.*

Reduced Coefficients
For later practical purposes we also define the *reduced drag coefficient* and *reduced lift coefficient*

$$\kappa_D := \frac{C_D}{2} \frac{A_\perp H}{m} \rho_0 = \frac{1}{2} \rho_0 BH \quad \textbf{reduced drag coefficient} \tag{6.2.17}$$

$$\kappa_L := \frac{C_L}{2} \frac{A_\perp H}{m} \rho_0 = \kappa_D \frac{C_L}{C_D} = \kappa_D \frac{L}{D} \quad \textbf{reduced lift coefficient} \tag{6.2.18}$$

For hypersonic speeds we typically have $\kappa_D \approx 25$.

Ballistic Coefficient
Besides drag and lift coefficients another frequently used characteristic parameter of a S/C with mass m is

$$B := C_D \frac{A_\perp}{m} \quad \textbf{ballistic coefficient} \tag{6.2.19}$$

Consider the ballistic coefficient as a single characteristic aerodynamic parameter where all the unknowns of a S/C are lumped together. It can best be determined experimentally from the decay data of a S/C's low Earth orbit as provided by NORAD TLE and from Eq. (12.7.20) in Sect. 12.7.4. Note that in literature the ballistic coefficient is often defined inversely, i.e., $B = m/(C_D A_\perp)$, which can be recognized by its inverse dimensions.

6.2.4 Drag in Free Molecular Flow

In this section we consider simple specific geometric bodies in free molecular flow. The geometries are assumed to be either symmetric relating to the flight direction or that the bodies are tumbling so that their average shape is symmetric. In both cases the bodies do not experience any lift. We therefore study now only their drag coefficients.

Cone

We start with a cone having an aperture angle 2α. If its symmetry axis points along the direction of motion, any gas molecule strikes the surface at the same angle $\theta = 90° - \alpha$ and therefore we have according to Eqs. (6.2.5) and (6.2.6)

$$C_{D,spec} = 2f_{||,\,spec} = 4\sin^2\alpha \leq 4$$
$$C_{D,diff} = 2f_{||,\,diff} = 2\sin\alpha\left(\frac{2}{3}\sin\alpha + 1\right) \leq 3.333$$

Thus, a cone aerodynamically behaves like a plate with its drag depending only on its slenderness. The more slender the cone the less its drag. From these results we derive the following relationships for specular and diffuse drag of a cone

$$C_{D,spec} = C_{D,diff} = 2.25 \quad @ \quad \alpha = \arcsin 3/4 = 48.6°$$
$$C_{D,spec} \geq C_{D,diff} \quad @ \quad \alpha \geq 48.6°$$

Sphere

For a sphere we find for the drag coefficients evaluated over a wetted half sphere

$$C_{D,spec} = \frac{2}{A_\perp}\frac{A}{2\pi}\int_{2\pi} f_{||,spec}(\theta)\cos\theta \cdot d\Omega = \frac{2}{\pi r^2}\frac{2\pi r^2}{2\pi}\int_0^{2\pi}\int_0^{\pi/2} 2\cos^3\theta \cdot \sin\theta \cdot d\theta \cdot d\varphi$$
$$= 8\int_0^{\pi/2}\cos^3\theta\,\sin\theta \cdot d\theta = 8\left(-\frac{1}{4}\cos^3\theta\right)\Big|_0^{\pi/2} = 2$$

$$C_{D,diff} = \frac{2}{A_\perp}\frac{A}{2\pi}\int_{2\pi} f_{||,diff}(\theta)\cos\theta \cdot d\Omega = \frac{2}{\pi r^2}\frac{2\pi r^2}{2\pi}\int_0^{2\pi}\int_0^{\pi/2}\cos^2\theta\left(\frac{2}{3}\cos\theta + 1\right)\cdot\sin\theta \cdot d\theta \cdot d\varphi$$
$$= 4\int_0^{\pi/2}\left(\frac{2}{3}\cos^3\theta\sin\theta + \cos^2\theta\sin\theta\right)\cdot d\theta = 4\left(-\frac{2}{12}\cos^4\theta - \frac{1}{3}\cos^3\theta\right)\Big|_0^{\pi/2} = 2$$

Tumbling Cuboid

We finally assume a tumbling cuboid, which might come closest to the shape of a general satellite. We define our Cartesian reference frame such that the x-axis points along the flight direction of the body and the y- and z-axes normal to it. Accordingly, the surface areas of the cuboid with normal vectors pointing in the three axis directions are termed A_x, A_y, A_z. We introduce standard spherical coordinates with the polar angle θ relative to the z-axes and the azimuthal angle φ relative to the x-axes. We thus have for the three wetted areas

$$\begin{aligned} A_{\perp x} &= A_x \cos\varphi \, \sin\theta \\ A_{\perp y} &= A_y \sin\varphi \, \sin\theta \\ A_{\perp z} &= A_z \cos\theta \end{aligned} \tag{6.2.20}$$

To simplify calculations we make the assumption that the cuboid tumbles along a rotational axis, which is lateral to the direction of motion, i.e. $\varphi = 0$. In this case $A_{\perp x} = A_x \sin\theta$, $A_{\perp y} = 0$, $A_{\perp z} = A_z \cos\theta$ and we have to average the following expressions

$$C_{D,spec} = 2\frac{2\cos^2(90^\circ - \theta)\cdot A_{\perp x} + 2\cos^2\theta \cdot A_{\perp z}}{A_{\perp x} + A_{\perp z}} = 4\frac{\cos^3\theta + \lambda \, \sin^3\theta}{\cos\theta + \lambda \, \sin\theta}$$

$$\begin{aligned} C_{D,diff} &= 2\frac{\left[\cos\theta\left(\frac{2}{3}\cos\theta + 1\right)\right]\cdot A_z\cos\theta + \left[\sin\theta\left(\frac{2}{3}\sin\theta + 1\right)\right]\cdot A_x\sin\theta}{A_z\cos\theta + A_x\sin\theta} \\ &= 2\left[\frac{2}{3}\frac{\cos^3\theta + \lambda \, \sin^3\theta}{\cos\theta + \lambda \, \sin\theta} + \frac{\cos^2\theta + \lambda \, \sin^2\theta}{\cos\theta + \lambda \, \sin\theta}\right] \end{aligned}$$

where

$$\lambda := \frac{A_x}{A_z}$$

over a quarter rotation. With the use of a symbolic integrator we find

$$\left\langle \frac{\cos^3\theta + \lambda \, \sin^3\theta}{\cos\theta + \lambda \, \sin\theta} \right\rangle_\theta = \frac{2}{\pi\left(1+\lambda^2\right)}\left[\pi\left\{\frac{1}{4}\left(1+\lambda^2\right) + \frac{\lambda^2}{1+\lambda^2}\right\} - \lambda - \lambda\frac{1-\lambda^2}{1+\lambda^2}\ln\lambda\right]$$

$$\left\langle \frac{\cos^2\theta + \lambda \, \sin^2\theta}{\cos\theta + \lambda \, \sin\theta} \right\rangle_\theta = \frac{2}{\pi\left(1+\lambda^2\right)}\left[(1-\lambda)^2 + 2\frac{\lambda(1+\lambda)}{\sqrt{1+\lambda^2}}\cdot \operatorname{artanh}\left(\frac{\sqrt{1+\lambda^2}}{1+\lambda}\right)\right]$$

For $0 \le \lambda \le 1$ both functions are strictly monotonically increasing and vary by less than 18% from an mean value at $\lambda = 0.1$.

Tumbling Cube

To arrive at concrete numbers we now assume that we have a cube, $A \equiv A_x = A_y = A_z$, i.e. $\lambda = 1$. In this case we find

$$\left\langle C_{D,spec}\right\rangle_\theta = 4\left(1 - \frac{1}{\pi}\right) = 2.73 \quad @ \text{ tumbling cube}$$

and

$$\left\langle C_{D,diff}\right\rangle_\theta = 2\left[\frac{2}{3}\left(1 - \frac{1}{\pi}\right) + \frac{2\sqrt{2}}{\pi}\cdot \operatorname{artanh}\left(\frac{1}{\sqrt{2}}\right)\right] = 2.50 \quad @ \text{ tumbling cube}$$

So, the average drag owing to diffuse reflection is little less than that owing to specular reflection.

Standard Drag and Lift Coefficient Approximation
In conclusion of these examples we can state that

> For a body with arbitrary shape in free molecular flow the specular and diffuse reflection for both lift and drag is about the same; the drag coefficient amounts to about little more than 2.0 (sphere). For tumbling or irregular shapes the lift coefficient vanishes.

In face of the fact that the surfaces of spacecraft are highly irregular and therefore the drag coefficient generally is only roughly known, it is not worth discerning these two types of reflections in the free molecular flow regime. We will therefore consider only single average drag and single average lift coefficients

$$\begin{aligned} C_D &\equiv \langle C_D \rangle = \langle C_{D,spec} \rangle \approx \langle C_{D,diff} \rangle \\ C_L &\equiv \langle C_L \rangle = \langle C_{L,spec} \rangle \approx \langle C_{L,diff} \rangle \end{aligned} \tag{6.2.21}$$

for further analysis.

As a matter of fact, the more irregular the shape of a body is and if we go from 2-dimensional to 3-dimensional shapes and averages for tumbling bodies, the better the following empirical and frequently used rough value for the drag coefficient becomes

$$\boxed{C_D = 2.2 \pm 0.2} \quad @ \quad 150\ \text{km} < \text{h} < 600\ \text{km} \tag{6.2.22}$$

This value is comprehensible because it falls between the value for the shape of a sphere and a tumbling cube (see results above). For a laterally tumbling cuboid $C_D \approx \left(\langle C_{D,spec} \rangle + \langle C_{D,diff} \rangle\right)/2 = 2.2$ holds if $A_z/A_x = \lambda = 0.046$.

The above empirical value is quite crude. Actually, the drag coefficient depends on the chemical composition of the atmosphere, because both the energy transfer and the molecular speed ratio decrease as the molecular weight of the atmosphere decreases with changing molecular constituents of the atmosphere. The drag coefficient with respect to atomic oxygen (prevailing at 200–600 km altitude) is approximately 2.2, while with respect to helium (prevailing at 600–1500 km) it approximates 2.8. However, when solar activity becomes low (F10.7 < 80), helium becomes dominant at altitudes as low as 500 km. Above 1500 km, when hydrogen becomes the dominant species, the drag coefficient value is greater than 4.0. This matches the reported drag coefficient for spheres, which increases slightly from 2.2 at 250 km to 2.4 at 500 km altitude. Therefore, the drag coefficient and with it the ballistic coefficient (see below) can vary by as much as 80% over a wide range of altitudes.

Geometric Bodies in Newtonian Flow
We have seen in Sect. 6.2.2. that for Newtonian flow the drag and lift coefficient attain values only half of those for free molecular flow. Therefore the above results

also apply to Newtonian flow if the obtained results are divided by 2. More generally it can be said that

> In free molecular flow the values of drag and lift coefficients are of order 2, while in Newtonian flow they are of order 1.

6.2.5 *Aerodynamic Forces*

With all these definitions the drag in Eq. (6.2.15a) can be written with the barometric formula Eq. (6.1.5) as

$$D = \frac{1}{2}\rho v^2 C_D A_\perp = m v^2 \frac{\kappa_D}{H} \mathrm{e}^{-\frac{h}{H}} = \frac{1}{2} m B \rho v^2 \quad \textbf{drag} \tag{6.2.23}$$

Example

What is the mean atmospheric drag of the International Space Station ($h = 350$ km, $m = 419$ t, $A_\perp \approx 1050\ \mathrm{m}^2$, $C_D \approx 2.2$)?

The ballistic coefficient according to Eq. (6.2.19) is $B \approx 0.0055\ \mathrm{m}^2\,\mathrm{kg}^{-1}$. At an altitude of 350 km the average atmospheric mass density is $\langle\rho\rangle = 9.518\ \mathrm{g\,km}^{-3}$. With this we calculate the drag and the deceleration to be $a_D = D/m = 1.5 \times 10^{-6}\ \mathrm{m\,s}^{-2} = 0.15\ \mu g_0$, where $g_0 = 9.798\ \mathrm{m\,s}^{-2}$ is Earth's mean gravitational acceleration at its surface.

Remark *This residual atmospheric acceleration force is not the primary contribution to the so-called* µg condition *onboard the scientific laboratories of the ISS. By far the biggest share is the tidal forces (see Sect. 8.5.2) with a* $\approx 1\ \mu g_0$ *per meter distance from the center of mass of the ISS. This is where the expression "microgravity research" for scientific research in space comes from.*

Of course the drag force always points antiparallel to the velocity vector $\boldsymbol{D} = -D\hat{\boldsymbol{v}} = -D\boldsymbol{u}_t$ (see Figs. 6.6 and 6.7).

Remark *In line with the literature we discriminate between the notation "drag/lift" and "drag/lift forces". A drag/lift force is the force vector caused by the interaction of the air flow with the vehicle, while drag/lift is the amount of this vector.*

On the other hand and according to the definition, the lift vector $\boldsymbol{L} \perp \boldsymbol{D}$. This leaves a degree of freedom for $\boldsymbol{L}$ in the plane vertical to the flight direction (see Fig. 6.7). As S/C in low Earth orbits or at hypervelocity speeds are moving in a plane (see Sects. 6.3 and 7.2.2), usually only the in-plane component of the lift is relevant (see Fig. 6.7). In line with the literature we therefore define $L_v := \boldsymbol{L}\boldsymbol{u}_n$, i.e., L is the "outward"-component of $\boldsymbol{L}$ in the plane. Because the plane usually includes Earth's center of mass, this lift component is also known as (upward) vertical lift. This implies that L_v actually depends on the orientation of $\boldsymbol{L}$ as given by the so-called **bank angle** μ. The lift therefore is written as

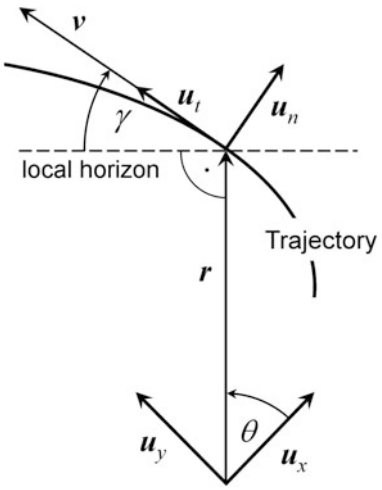


Fig. 6.6 The co-moving S/C reference system $(\boldsymbol{u}_t, \boldsymbol{u}_n)$ and the inertial Earth reference system $(\boldsymbol{u}_x, \boldsymbol{u}_y)$

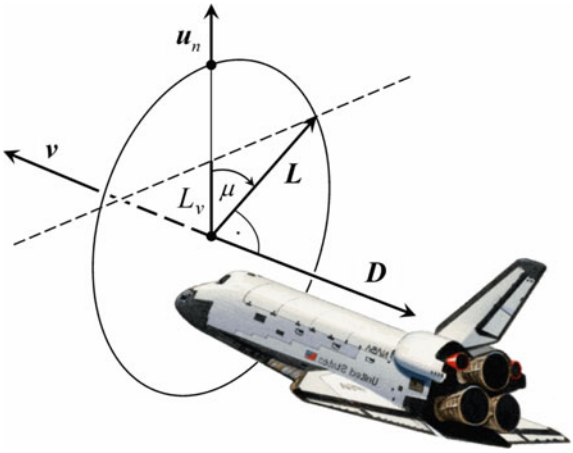


Fig. 6.7 The orientation of drag and lift forces in the co-moving S/C system

$$L = \frac{1}{2}\rho v^2 C_L A_\perp = mv^2 \frac{\kappa_L}{H} e^{-\frac{h}{H}} \quad \textbf{lift} \tag{6.2.24}$$

and

$$L_v = L\cos\mu \quad \textbf{vertical lift}$$
$$L_h = L\sin\mu \quad \textbf{horizontal lift}$$

By turning $\boldsymbol{L}$ sideways the horizontal lift provides a cross-range capability of the space vehicle (see, e.g., Space Shuttle reentry, Sect. 10.7.1). By turning the vehicle

down, $\mu = 180°$, one can even achieve a negative vertical lift, $L_v < 0$, as utilized, for instance, for Apollo reentries (see Sect. 10.5.2).

***L/D* Ratio**

From Eqs. (6.2.23) and (6.2.24) we find for the lift-to-drag ratio, L/D, which is a key parameter for reentry flight mechanics (see Sect. 10.2.1),

$$\frac{L}{D} = \frac{C_L}{C_D} \tag{6.2.25}$$

L/D actually is not constant, but largely depends on the angle of attack (AOA) α, which is the angle between the body's reference line and the oncoming flow, and somewhat also on the atmospheric density and relative speed. For hypersonic speeds, we typically have $L/D = C_L/C_D = 0.2 - 2.5$.

The lesser values apply for capsule-shaped blunt bodies. For the Apollo 4 capsule reentry the following measured relation was found in the transition and continuum regime

$$\frac{L}{D} = 0.0143 \cdot \alpha\,[^\circ] \quad @\ \text{Apollo capsule}, \alpha = 0^\circ - 40^\circ \tag{6.2.26}$$

Here α is the angle in units of degrees between the flight vector and the body center line in flight direction. On average $C_D \approx 1.22$, $C_L \approx 0.45$, $L/D = 0.369$ was measured during entry.

For winged bodies, such as the Space Shuttle, the wings and the body as a rough approximation, constitute a flat plate for the hypervelocity flow. So, the angle of attack α is related to the air flow incidence angle θ through $\theta = 90° - \alpha$. Assuming that specular and diffuse scattering are about the same (see Eq. (6.2.21)), we derive from Newtonian flow theory in the continuum regime, i.e. Eq. (6.2.11) inserted into Eq. (6.2.2a) and (6.2.2b) and with Eqs. (6.2.12) and (6.2.13)

$$\begin{aligned} D &= \rho v^2 A \sin^3\alpha + \frac{1}{2}\rho v^2 C_f A \\ L &= \rho v^2 A \sin^2\alpha \cos\alpha \end{aligned} \tag{6.2.27}$$

The drag on the spacecraft therefore monotonically increases with AOA and becomes maximal for $\alpha = 90°$. Lift also increases with increasing AOA, but becomes maximal for $\alpha = 65.5°$, and decreases thereafter. This is what one intuitively expects and what resembles the general behavior of a spacecraft at hypersonic speeds. So, in general the expression

$$\frac{L}{D} = \frac{\sin\alpha \sin 2\alpha}{2\sin^3\alpha + C_f} \quad @\ \text{winged body}, v \geq 5\ Ma \tag{6.2.28}$$

$$C_f = C_{f0}\left(\frac{\rho_0 v_0}{\rho v}\right)^{1/5}, \quad \rho_0 v_0 = 1.2041 \ \text{kg m}^{-3} Ma$$

describes the AOA-dependent lift-to-drag ratio of a hypervelocity vehicle quite well. Observe that owing to the skin friction L/D does not diverge at $\alpha = 0$, but displays a maximum of $L/D_{\max} = 1 - 2.5$ at typically $\alpha \approx 20°$. From extensive trajectory data of the first Space Shuttle mission STS-1 it can be derived that

$$C_{f0} = 0.0210 \pm 0.0005 \quad @ \ \text{Space Shuttle}, v \geq 3 \ Ma \tag{6.2.29}$$

6.3 Equations of Motion

It is our goal to derive the equations of motion of a S/C flight in the atmosphere of a celestial body. We start out with the general equation of rocket motion (1.1.7)

$$m\dot{\boldsymbol{v}} = \boldsymbol{F}_* + \boldsymbol{F}_{ext}$$

where $\boldsymbol{F}_* = \dot{m}\boldsymbol{v}_*$ is the thrust of the rocket, and $\boldsymbol{F}_{ext}$ comprises all external forces, in particular the aerodynamic forces, which are distinctive for this situation. Figure 6.8 shows the flight path of a rocket in the atmosphere with flight direction $\boldsymbol{v}$ and all relevant forces at a given point in flight. Given these forces, the equation of motion can be explicitly written as

$$m\frac{d\ddot{\boldsymbol{r}}}{dt} = \boldsymbol{F}_*(t) + m\boldsymbol{g}(\boldsymbol{r}) + \boldsymbol{D}(\boldsymbol{v}, \boldsymbol{r}) + \boldsymbol{L}(\boldsymbol{v}, \boldsymbol{r}) \tag{6.3.1}$$

For a given S/C the altitude- and velocity-dependent drag $\boldsymbol{D}$, lift $\boldsymbol{L}$, and the time-dependent thrust $\boldsymbol{F}_*$ is known, and they can be used to solve Eq. (6.3.1) numerically. The solution is the wanted $\boldsymbol{r}(t)$ and $\boldsymbol{v}(t) = d\boldsymbol{r}/dt$. For real missions, this indeed is the only possibility to determine the solution with adequate accuracy.

This would already bring us to the end of this chapter, were it not for the need to gain a physical understanding of the processes and typical flight stages. For that we first introduce appropriate reference systems, which is always an essential step when exploring physical processes. Figure 6.6 shows a reference system $(\boldsymbol{u}_t, \boldsymbol{u}_n)$ co-moving with the S/C (tangential component points into the direction of motion), on which our investigations are based.

We assume that all thrust and lift forces lie in the $(\boldsymbol{u}_t, \boldsymbol{u}_n)$-plane. In this case the S/C moves only in this plane and our treatment is reduced to a two-dimensional case.

By assuming the Earth *reference system* to be inertial, we neglect the Earth's rotation, which leads to three errors to be considered in practice:

1. The transition to a rotating Earth reference system changes the coordinates of the S/C trajectory relative to an observer on ground.
2. The atmosphere moves with the Earth's surface, which leads to cross-components of drag and lift. They are, however, negligible compared to wind forces. Even those we do not take into account here.

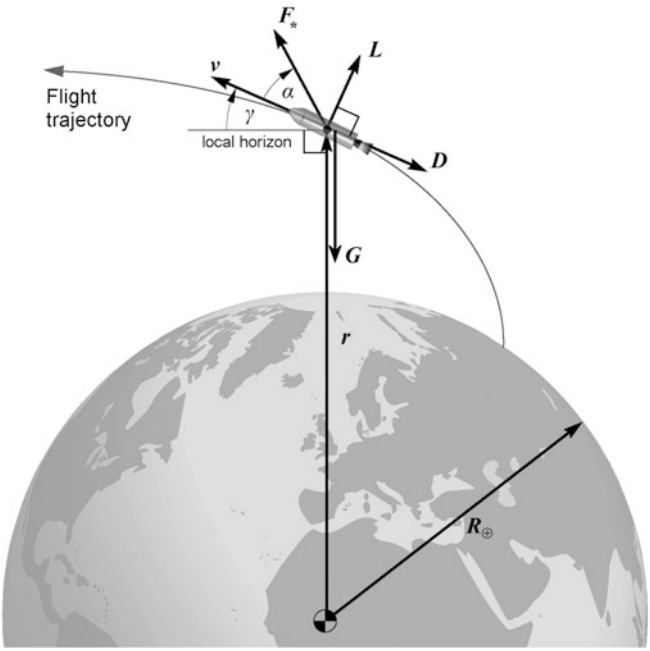


Fig. 6.8 Ascent flight of a S/C with flight path angle γ through the atmosphere subject to the effective forces: thrust F_* with thrust angle (steering angle) α, lift L, drag D, and gravitational force $G = mg$

3. At launch Earth's rotation causes a tangential velocity, which adds to the S/C velocity for a launch in the eastern direction. So placing a launch pad somewhere near the equator and launching the vehicle in an eastern direction saves propellant or alternatively enables a bigger payload. For example, the new Soyuz-2 rocket will deliver 8.5 tons of payload into LEO from Baikonur located at 45.9°N, but about 9.1 tons from Kourou at 5.1°N!

Analysis in the Co-Moving S/C System

Because lift and drag forces are conveniently defined in the $(\boldsymbol{u}_t, \boldsymbol{u}_n)$ reference system as depicted in Fig. 6.7, we will analyze the differential motion in this co-moving S/C system. The $(\boldsymbol{u}_t, \boldsymbol{u}_n)$-plane is part of the cylindrical reference system $(\boldsymbol{u}_t, \boldsymbol{u}_n, \boldsymbol{u}_h)$ with the following instantaneous unit vectors: $\boldsymbol{u}_t$ tangential to the trajectory, $\boldsymbol{u}_n$ normal to the trajectory in the motion plane (see Fig. 6.6), and $\boldsymbol{u}_h = \hat{\boldsymbol{h}}$ the vector normal to the plane equaling the angular momentum unit vector (see Sect. 7.2.2). In this reference system we have per definition

$$\boldsymbol{v} = v \cdot \boldsymbol{u}_t$$

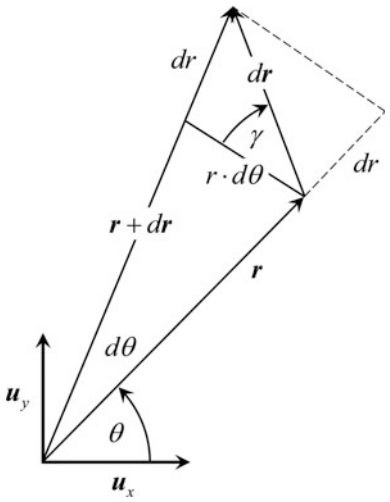


Fig. 6.9 The components of the radial vector in the inertial Earth reference system

According to Fig. 6.6 and due to a rotating $\boldsymbol{r}$ and $\boldsymbol{v}$, $(\boldsymbol{u}_t, \boldsymbol{u}_n, \boldsymbol{u}_h)$ rotates with angular velocity

$$\boldsymbol{\omega} = \left(\dot{\theta} - \dot{\gamma}\right) \cdot \boldsymbol{u}_h$$

where γ is the so-called **flight path angle**, which is the key angle in the $(\boldsymbol{u}_t, \boldsymbol{u}_n)$-system. To evaluate $\dot{\theta}$ in the $(\boldsymbol{u}_t, \boldsymbol{u}_n)$-system we derive from Fig. 6.9 $r \cdot d\theta = |d\boldsymbol{r}| \cdot \cos\gamma$. This yields (cf. Eq. (7.3.15c))

$$\dot{\theta} = \frac{v}{r}\cos\gamma \tag{6.3.2}$$

> **Note** *dr describes the change of the absolute value of the radial distance and therefore is only the radial part of the position change vector* $d\boldsymbol{r}$*, and not its value,* $dr \neq |d\boldsymbol{r}|$*, that includes the change of direction. Therefore* $v = |d\boldsymbol{r}/dt| = |\boldsymbol{v}|$ *is the absolute value of the velocity, while* $v_r = dr/dt$ *is only the radial velocity component (cf. Note in Sect. 7.2.2).*

For the acceleration vector we obtain by differentiation of $\boldsymbol{v} = v \cdot \boldsymbol{u}_t$ (see also Sect. 7.2.1)

$$\dot{\boldsymbol{v}} = \dot{v} \cdot \boldsymbol{u}_t + v \cdot \dot{\boldsymbol{u}}_t = \dot{v} \cdot \boldsymbol{u}_t + \boldsymbol{\omega} \times \boldsymbol{v}$$

With $\boldsymbol{\omega} = \left(\dot{\theta} - \dot{\gamma}\right) \cdot \boldsymbol{u}_h$, $\boldsymbol{v} = v \cdot \boldsymbol{u}_t$, and Eq. (6.3.2) we have

$$\boldsymbol{\omega} \times \boldsymbol{v} = -\left(\dot{\theta} - \dot{\gamma}\right) v \cdot \boldsymbol{u}_n = \left(\dot{\gamma} v - \frac{v^2}{r} \cos\gamma\right) \cdot \boldsymbol{u}_n \tag{6.3.3}$$

This brings us to the following form of the equation of motion

$$m\left[\dot{v} \cdot \boldsymbol{u}_t + \left(\dot{\gamma} v - \frac{v^2}{r} \cos\gamma\right) \cdot \boldsymbol{u}_n\right] = \boldsymbol{F}_* + m\boldsymbol{g} + \boldsymbol{D} + \boldsymbol{L} \tag{6.3.4}$$

Finally, we decompose the vectors on the right hand side into $\boldsymbol{u}_t, \boldsymbol{u}_n$ and obtain (cf. Fig. 6.7)

$$\begin{aligned} \boldsymbol{F}_* &= F_* \cos\alpha \cdot \boldsymbol{u}_t + F_* \sin\alpha \cdot \boldsymbol{u}_n \\ \boldsymbol{g} &= -g \sin\gamma \cdot \boldsymbol{u}_t - g \cos\gamma \cdot \boldsymbol{u}_n \\ \boldsymbol{D} &= -D \cdot \boldsymbol{u}_t \\ \boldsymbol{L} &= L \cdot \boldsymbol{u}_n \end{aligned} \tag{6.3.5}$$

where $\alpha = \angle(\boldsymbol{v}, \boldsymbol{v}_*)$ is the **thrust angle** (a.k.a. *thrust angle of attack,* which is about the **steering angle**, see Fig. 6.8), i.e., the angle between the thrust vector and the tangent to the trajectory (velocity vector). The convenience of the first and last two equations for thrust, lift, and drag forces was the reason for considering the motion in the co-moving $(\boldsymbol{u}_t, \boldsymbol{u}_n)$-system.

Scalar Equations of Motion

When inserting Eq. (6.3.5) into the vectorial equation of motion (6.3.4) the equation must hold for each component $\boldsymbol{u}_t, \boldsymbol{u}_n$ separately. In this way we derive two scalar equations of motion for a trajectory through the atmosphere:

$$\dot{v} = \frac{F_*}{m} \cos\alpha - \frac{D}{m} - g \sin\gamma \tag{6.3.6}$$

$$v\dot{\gamma} = \frac{F_*}{m} \sin\alpha + \frac{L}{m} - \left(g - \frac{v^2}{r}\right) \cos\gamma \tag{6.3.7}$$

In addition we introduce the altitude $h = r - R$ and downrange distance x, for which according to Fig. 6.9 the following holds in the Earth reference system

$$\begin{aligned} \dot{h} &= v \sin\gamma \\ \dot{x} &= v \cos\gamma \end{aligned} \tag{6.3.8}$$

We do this because this enables us to derive the time-dependent solutions $h(t)$, $x(t)$ and hence the ascent and reentry trajectory $h = h(x)$. We recall from Eqs. (6.2.23), (6.2.24), and (6.1.5) that

$$L = \frac{1}{2}\rho(r)v^2 C_L A_\perp \tag{6.3.9}$$

$$D = \frac{1}{2}\rho(r)v^2 C_D A_\perp \tag{6.3.10}$$

$$g(r) = g_0 \frac{R^2}{r^2} \tag{6.3.11}$$

$$\rho(r) = \rho_0 \exp\left(-\frac{h}{H}\right) \tag{6.3.12}$$

Here R is the radius of the celestial body in question, and $\rho_0 = 1.752\ \mathrm{kg/m^3}$ and $H = 6.7$ km are the mean barometric values for Earth as given by Eq. (6.1.6). Observe that both lift and centrifugal force (as given by the term $v^2/r \cdot \cos\gamma$) act just normal to the trajectory (see Eqs. (6.3.2) and (6.3.5)) and therefore do not show up in Eq. (6.3.6), but only in Eq. (6.3.7).

Note *Strictly speaking, the barometric formula Eq.* (6.3.12) *holds for altitudes above sea level, hence* $R =$ *sea level. Since the sea level is given by Earth's geoid (see Sect.* 12.2.1*), which roughly follows the spheriodal shape of Earth, we have* $R = R_\oplus\left(1 - 0.003353 \cdot \sin^2\beta\right)$*, where* $R_\oplus = 6378.136$ *km and* β *is the latitude.*

It is important to mention that Eqs. (6.3.6–6.3.8) implicitly require that the thrust axis coincides with the aerodynamic axis of the S/C, relative to which the thrust angle is defined, and that the center of mass lies on this axis. If this is not the case, e.g., for the Space Shuttle, then the equations of motion are far more complex, and can no longer be treated analytically. In that case one is left to solve the full equation of motion with six degrees of freedom numerically.

Only with the above form of the equation of motion it is now possible to understand flight mechanics, the science of ascent and reentry.

Normalized Equations of Motion

For studying an ascent or reentry flight in detail, the method of choice is to solve the equations of motion numerically in their time-dependent representation. Equations (6.3.6–6.3.8) are, however, not suited for that. Rather the most natural and hence optimal choice to describe the physics of nature are dimensions with unit 1. It is therefore advantageous to treat the equations of motion with dimensionless variables. In addition, such a choice makes the problem of choosing optimal spatial, time, and mass units dispensable. By defining

$$v_\triangleright = \sqrt{g_0 R}$$

and the following dimensionless variables, expressed by greek symbols

$$\mu = \frac{v}{v_\triangleright}, \quad \eta = \frac{h}{H}, \quad \chi = \frac{x}{H}, \quad \Phi = \frac{F_*}{mg_0}$$
$$\tau = \frac{g_0}{v_\triangleright} t, \quad (\ldots)' = \frac{d}{d\tau}$$

it is easy to show that the equations of motion (6.3.6), (6.3.7), and (6.3.8) can be transformed into the following normalized equations of motion

$$\begin{aligned}
\mu' &= \Phi \cos\alpha - \mu^2 \kappa_D \tfrac{R}{H} \mathrm{e}^{-\eta} - \sin\gamma \\
\mu\gamma' &= \Phi \sin\alpha + \mu^2 \kappa_D \tfrac{R}{H}\tfrac{L}{D} \mathrm{e}^{-\eta} - (1-\mu^2)\cos\gamma \\
\eta' &= \mu \tfrac{R}{H} \sin\gamma \\
\chi' &= \mu \tfrac{R}{H} \cos\gamma
\end{aligned}$$

normalized EoM (6.3.13)

The first three equations are coupled. This was not obvious at first glance from Eqs. (6.3.6) to (6.3.8), as both lift and drag exponentially depend on the altitude via the atmospheric density so that the ratio L/D is a constant. This is why we use L/D in the following as a convenient constant to characterize the S/C. Observe that upon ascent the thrusters are burning fuel at a high rate, which is usually about constant, $\dot{m}_p \approx const$. Therefore the S/C mass decreases almost linearly $m(t) = m_0 - \dot{m}_p t \geq m_L + m_s$, and hence

$$m(\tau) = m_0 - m'_p \tau \geq m_L + m_s \tag{6.3.14}$$

Therefore the thrust per S/C weight, $\Phi = F_*/(m(\tau)g_0)$, increases monotonically upon ascent.

The equations of motion in form of Eq. (6.3.13) are optimally adapted to be coded and solved numerically such as by a Runge-Kutta method. For specific problems more elaborate equations without the approximations made here are used. The relatively simple equations above however capture the general flight behavior, so we will limit ourselves to them.

6.4 Ascent Flight

The flight mechanics of ascent flights deals with the question of how to steer a rocket from a launch pad optimally (in terms of propulsion demand) to a predetermined target orbit. This is a very delicate task, which will lead, as we will see, to quite complex ascent strategies.

6.4.1 Ascent Phases

If you approach the problem naively from an orbit-mechanical point of view, you might consider the trajectory of the rocket as a transfer orbit between two Keplerian orbits (see Sect. 8.3) where the launch pad is a point on the initial Keplerian orbit and the target orbit is the final orbit. But of course this is not correct, as the rocket is not in a Keplerian orbit at lift-off. Nevertheless, this orbit-mechanical approach is still quite sensible. This is because, if the rocket were in a Keplerian orbit at lift-off, and if the initial aerodynamic drag at low altitudes did not occur, the solution of the problem would instantly be obvious: according to orbit mechanics, the optimal transfer orbit regarding propulsion demand is a Hohmann orbit. A Hohmann orbit is a two-impulse transfer. The first impulse carries the S/C into an elliptic transfer orbit, while the second impulse at apogee of the transfer orbit kicks it into the target orbit. Now, if we interpret the powered ascent phase as an "extended impulse maneuver" from zero velocity at lift-off to the entry of the powerless transfer orbit, we have found an important partial optimization of the ascent trajectory problem. So, we divide our optimal ascent into three phases (see Fig. 6.10):

1. **Thrust phase**—beginning with lift-off, in the thrust phase the launch vehicle traverses the atmosphere to the transition point, where thrust is terminated by the so-called MECO (main engines cut-off).
2. **Coasting phase**—starting at the transition point this phase succeeds the thrust phase. It is a powerless, weightless flight without aerodynamic drag on an elliptic transfer orbit to the target orbit.
3. **Apogee boost**—this maneuver transfers the S/C into the target orbit.

Our optimization problem has now been reduced to determining the Hohmann transfer orbital elements and the optimal trajectory in the thrust phase. Note that we hereby have presumed that the S/C is able to perform a final impulse maneuver. If this is not the case, for instance if the last stage is not reignitable, then we have a continuous thrust ascent all the way up to the target orbit. For this continuous thrust ascent problem we defer the reader to the relevant literature.

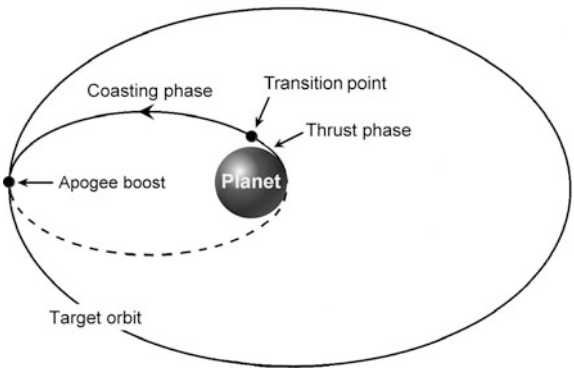


Fig. 6.10 The three ascent phases: thrust phase, coasting phase, and apogee boost

Hohmann-Transfer Orbital Elements
The orbital elements of the elliptic transfer orbit are determined on one hand by the requirement to touch the target orbit at apogee, i.e., to meet the boundary condition $r_{apo} = a_T(1 + e_T)$ (see Eq. (7.4.7)). On the other hand the transfer ellipse must touch the Earth's surface at the launch pad, i.e., $r_{per} = a_T(1 - e_T)$ (see Eq. (7.4.6)). From this follows for the two orbital elements

$$a_T = \frac{r_{apo} + r_{per}}{2}$$

$$e_T = \frac{r_{apo} - r_{per}}{r_{apo} + r_{per}}$$

Example

Let us take a Shuttle launch as an example to describe how these three phases are traversed. The launch pad at Kennedy Space Center is at sea level, so $r_{per} = 6378$ km. The target orbit typically is at an altitude of 300 km, i.e., $r_{apo} = 6678$ km. So, according to the above equations the transfer orbit has the orbital elements $a_T = 6528$ km and $e_T = 0.02298$. If the Shuttle ascended without aerodynamic drag on this ideal transfer orbit, the ascending time until reaching the target orbit would be $t_T = 43.5$ min according to Eq. (8.3.3).

Now, let us have a look at the ascent in reality. The thrust phase lasts 8.5 min, and at the transition point at an altitude of 110 km it took the Shuttle seamlessly into the transfer orbit. If the transition were not perfect, the Shuttle would, typically after 1.5 min, adjust its coasting trajectory by a so-called OMS-1 burn. Thereafter the Shuttle was in a powerless flight for 31.5 min on the elliptic transfer orbit until apogee at an altitude of 300 km. Here a so-called OMS-2 burn (apogee boost) for 2.5 min brought it into the circular target orbit. The total flight time was 42.5 min, which is just little less than the Hohmann transfer time.

Remark *Ballistic missiles after their thrust phase follow ballistic, i.e., elliptic, trajectories, which do not have their perigee at the launch pad and therefore have an additional degree of freedom, which is used to adjust the trajectory to the target distance.*

6.4.2 Optimization Problem

Having determined the transfer orbit, we are left with the final problem: Which is the optimal trajectory during the thrust phase, which smoothly transits into the

coasting phase and which consumes the least propellant? This ascent optimization problem is a typical problem of *optimal control theory* (see, e.g., Rau (2010)). Optimal control problems are inherently so complex that in general they can be solved only numerically. Here we will derive the problem setting and outline the different ascent strategies an optimal ascent is based on.

An ascent trajectory in general is determined by the equations of motion (6.3.6–6.3.8) plus the initial condition that the rocket at ignition rests on the launch-pad.

$$\begin{aligned} v(0) &= 0 \\ \gamma(0) &= 90^\circ \qquad \text{initial conditions} \\ h(0) &= h_0 \approx 0 \end{aligned} \tag{6.4.1}$$

and by the final condition that the rocket has to meet the transition point to the transfer orbit at engine cutoff. According to Eq. (10.1.4) the final condition is expressed as

$$\begin{aligned} v(t_f) &= v_f = \sqrt{\frac{\mu}{a_T\left(1-e_T^2\right)}}\sqrt{1+2e_T\cos\theta_T+e_T^2} \\ \cos\gamma(t_f) &= \cos\gamma_f = \frac{1+e_T\cos\theta_T}{\sqrt{1+2e_T\cos\theta_T+e_T^2}} \qquad \text{final conditions} \\ h(t_f) &= h_f = r_f - R = \frac{a_T\left(1-e_T^2\right)}{1+e_T\cos\theta_T} - R \end{aligned} \tag{6.4.2}$$

with t_f the time and θ_T the orbit angle where the thrust phase transfers at engine cutoff into the elliptic coasting phase. Since a_T, e_T are already determined by the transfer orbit, we only have the variables $F_*(t)$, $m(t)$, $\alpha(t)$, θ_T to find an ascent trajectory, which meets the final conditions (6.4.2). However, we also have to take into account that

$$\begin{aligned} F_* &= set \\ m(t) &= m_0 - \dot{m}_p \cdot t = set \end{aligned} \tag{6.4.3}$$

are set by thruster characteristics and the requirement that the ascent time should be as short as possible due to the gravitational loss (see Sect. 2.3.3 and below). This implies maximum thrust throughout ascent and therefore all in all thrust F_* and rocket mass m are predetermined functions of time (except temporary throttling for the Space Shuttle due to excessive drag). Therefore we only have the two variables $\alpha(t)$, θ_T to attain the optimal ascent trajectory. An optimal ascent trajectory implies that its so-called *performance index* (a.k.a. *cost functional*) *J,* which is the fuel demand, is lowest. Since the fuel demand is monotonously increasing in time and due to the gravitational loss we find the following problem setting for optimal ascent.

Formulation of the Optimal-Ascent Problem
Determine θ_T and the functional relationship $\alpha(t)$ that minimize $J = m_p(t_f)$ subject to the

1. Differential equations of ascent motion (6.3.6–6.3.8),
2. Path constraints Eq. (6.4.3), and
3. Initial and final conditions (6.4.1) and (6.4.2).

Note that the variable θ_T to be optimized can be substituted via the final condition Eqs. (6.4.2) by either v_f, γ_f or h_f.

Optimization Guidelines
Without knowing the exact solutions for $\alpha(t)$ and θ_T from numerical methods it is important to know that there are general design guidelines. To derive them we formally write $D = D(h),\ L = L(h)$, by which we indirectly include the solution of Eq. (6.3.8), so we need only to focus on the optimization treatment of the first two equations. We start this treatment by formally integrating the equation of motion (6.3.6)

$$\begin{aligned} v &= \int_0^{t_f} \frac{F_* \cos\alpha}{m} dt - \int_0^{t_f} \frac{D}{m} dt - \int_0^{t_f} g \sin\gamma \cdot dt \\ &= F_* \int_0^{t_f} \frac{dt}{m} - F_* \int_0^{t_f} \frac{1-\cos\alpha}{m} dt - \int_0^{t_f} \frac{D}{m} dt - \int_0^{t_f} g \cdot \sin\gamma \cdot dt \end{aligned}$$

from which we derive with Eq. (6.4.3), with $F_* = \dot{m}_p v_*$ (see Eq. (1.1.4)), and with Eq. (6.2.23)

$$v_f = \underbrace{v_* \ln\frac{m_0}{m_f}}_{\text{rocket equation}} \underbrace{- 2F_* \int_0^{t_f} \frac{\sin^2(\alpha/2)}{m} dt}_{\text{steering losses}} \underbrace{- \frac{1}{H} \int_0^{t_f} \kappa_D v^2 e^{-\frac{h}{H}} dt}_{\text{drag losses}} \underbrace{- \int_0^{t_f} g \sin\gamma \cdot dt}_{\text{gravitational losses}} \quad (6.4.4)$$

where κ_D is the reduced drag coefficient (see Eq. (6.2.17)). We find that there are three contributions, which reduce the velocity gain: steering losses, drag losses, and gravitational losses. Since v_f is a given final condition of the optimal-ascent problem, this equation states that reducing the target quantity m_f implies a reduction of these three velocity contributions. This is the key optimization guideline. If we were on a celestial body without any atmosphere there would not be any drag losses, so a steerless horizontal launch with $\gamma(0) = 0°$ would result in a lossless ascent. As we have an atmosphere on Earth and because rocket structures do not

support horizontal launches but only vertical ones, the initial condition is $\gamma(0) = 90°$. As we will see in Sect. 6.4.7 the loss shares of a typical ascent profile in Earth's atmosphere are: steering losses $\approx 3\%$, drag losses $\approx 27\%$, gravitational losses $\approx 70\%$. So optimal steering today is the art of reducing gravitational losses either by higher accelerations thus reducing t_f or by smaller flight path angles (early horizontal flight) at the limit of aerodynamic structural loads.

To investigate the options of affecting the flight path angle for an optimal trajectory we formally integrate Eq. (6.3.7) and find with Eq. (6.2.24)

$$\gamma_f = \frac{\pi}{2} + \underbrace{F_* \int_0^{t_f} \frac{\sin\alpha}{mv} dt}_{steering} + \underbrace{\frac{1}{H} \int_0^{t_f} \kappa_L v^2 \mathrm{e}^{-\frac{h}{H}} \cdot dt}_{lift} - \underbrace{\int_0^{t_f} \frac{g}{v} \cos\gamma \cdot dt}_{gravitation} + \underbrace{\int_0^{t_f} \frac{v}{r} \cos\gamma \cdot dt}_{centrifugal\ force} \tag{6.4.5}$$

where κ_L is the reduced lift coefficient (see Eq. (6.2.18)). Note that these terms do not imply losses. They just change the flight path angle. From this follows the secondary optimization guideline to make use of steering, lift, gravitation, and centrifugal to shape the flight path such that the steering, drag, and gravitational losses are minimized.

Optimal-Ascent Considerations

With Eqs. (6.4.4) and (6.4.5) at hand we are able to ponder about an optimum-ascent trajectory. Taking all the contributions adequately into account is quite an engineering feat, and we are not able to discuss it extensively in this book. However, we want to capture at least the essential aspects of optimization.

During the thrust phase the spacecraft changes its state vector from a vertical launch direction, $\gamma_i = \pi/2$, with zero initial velocity to a nearly horizontal flight direction, $\gamma_f = 0$, and maximum velocity at engine shutdown at the transition point. We first want to investigate the losses due to velocity direction changes. Two direction changes have to be taken into consideration: the turn into the desired target-orbit inclination, and the turn from the vertical launch direction into the incline of the transfer orbit.

As long as the S/C ascents vertically the turn into the so-called launch azimuth (angle between geographic north and the orbit trajectory, see Sect. 8.6.1) is just a tilt of the axially symmetric rocket into this direction or a roll of an asymmetrical S/C along its vertical axis into the launch azimuth. Let us take the Space Shuttle as an example for the latter. Because the launch pad at Kennedy Space Center is oriented roughly southward, which is a remnant of the Apollo era, the Shuttle first had to roll by 120° around its longitudinal axis to match its body symmetry plane (x-z plane) with the orbital plane of the International Space Station. This was the famous 120° roll maneuver. The losses due to such tilt or roll maneuvers of course can be neglected.

Concerning the turn from the vertical launch direction into the incline of the transfer orbit there and according to Eq. (6.4.5) there are four effects which contribute to a changing flight path angle: steering, lift, gravitation, and centrifugal force. The positive sign of the centrifugal term in Eq. (6.4.5) indicates that this a countering effect we have to live with. Of course, flight angle reduction can be achieved by a negative steering angle $\alpha < 0$. However, any steering brings about steering losses, which need to be kept as small as possible. If the body of the S/C is not axially symmetric, such as the winged body of the Space Shuttle, it generates lift, which can be harnessed to reduce the flight path angle more quickly by turning the Shuttle upside down ($\kappa_L \rightarrow -\kappa_L$), which actually was done. So lift can be looked at as a kind of additional steering option, which we will neglect in the following because most rockets have no lift because they are built symmetrically.

What is the least fuel consumption trajectory into space? Equation (6.4.4) with (6.4.5) claim that flight path angle reduction is most effective if high steering angles are applied at low speeds, i.e., just after lift-off. A small γ early on would also reduce the gravitational loss term $g \sin\gamma$. But you do not want to turn too early, because drag is very high at low altitudes (see exponential contribution in drag term in Eq. (6.4.4)). Because steering losses contribute with only about 3% to the total ascent losses, the optimal-ascent problem hence seems to be a matter of trading drag losses against gravitational losses and that there is a wide range of possible trajectories into space due to these contradictory requirements. Yet this is not the case because there is gravity turn.

6.4.3 *Gravity Turn*

Why wasting propellant to steer the rocket into horizontal flight when gravity does it for you?

According to Eq. (6.4.5) gravity reduces the flight path angle without any steering. It is like throwing a stone forward and upward. Gravity bends its trajectory until it flies horizontally at its apex. Now, accelerate the stone in flight, which will shift the apex into space and the increasing centrifugal force will prevent it from falling back. Thus you have the gravity turn of an ascending rocket. Since the required thrust angle is $\alpha = 0$, there are no steering losses for the velocity gain (second term in Eq. (6.4.4)). Mathematically the gravity turn maneuver can be described by setting $\alpha = 0$ in Eq. (6.3.7) and neglecting lift. We then get for the flight path angle rate

$$\dot{\gamma} = -\left(\frac{g}{v} - \frac{v}{r}\right)\cos\gamma \tag{6.4.6}$$

We see that the initial rate and hence the gravity turn is big at low speeds (but zero for vertical ascent). With rising altitudes, velocity increases, so gravity turn diminishes while centrifugal forces become stronger until $g/v - v/r = 0 \rightarrow v \approx$



Fig. 6.11 Ascent trajectory with gravity turn until main engines cut-off (MECO) of an ascending Delta II rocket from Vandenberg Air Force Base on July 2, 2014. *Credit* Rick Baldridge

$\sqrt{gR} = 7.92$ km s^{-1}. When this happens $\dot{\gamma} \approx 0$, and if the gravity turn was initiated just right, also $\gamma \approx 0°$, i.e., the trajectory is nearly circular at the transit point. This can somehow be gleaned from the image of an ascending rocket at night in Fig. 6.11. The gravity-turn ascent is equivalent to the opposite case of a ballistic entry as discussed in the Sect. 10.3.2.

You might think that gravity turn is the philosopher's stone for ascent. This is not the case, because a gravity-turn-only ascent just eliminates steering losses. But, as we will see in Sect. 6.4.7, drag and gravity losses are one order of magnitude more significant than steering losses, which are not minimal for such an ascent. So, although an optimized ascent is close to a gravity-turn ascent it needs some additional ingredients. Moreover, with vertical lift-off a gravity turn does no happen all by itself. It needs to be kicked off.

6.4.4 Pitch Maneuver

For structural reasons, the S/C is in a vertical position at lift-off. So just after lift-off the flight path angle is $\gamma = 90°$, $\alpha = 0$ and $L = D = 0$. From Eq. (6.4.6), we get $\dot{\gamma} = 0$: the S/C will ascent vertically. In order to subject it to a gravity turn, we need a so-called *initial kick angle* (a.k.a. *pitch angle*, i.e., the angle between flight direction and the vertical), which may be small, but not zero. This pitch angle is brought about by the so-called *pitch maneuver* or *pitch program*, and it amounts to approximately 3–5° (see Fig. 6.12). Only after receiving the kick angle the pitch will increase further due to the gravitational force according to Eq. (6.4.6) until it acquires about 20–30° at an altitude of 10–15 km. Note that for small celestial bodies without any atmosphere a timely pitch maneuver plus a gravity turn together

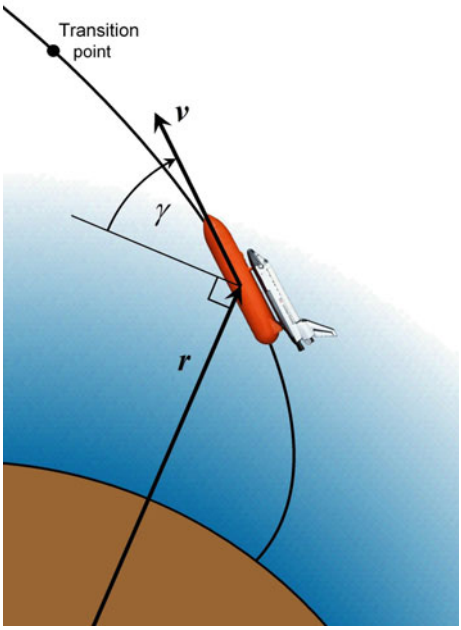


Fig. 6.12 Pitch maneuver and constant pitch-rate maneuver just after lift-off

make up an optimal ascent. So the ascent of the Apollo landing module from the Moon was virtually an ideal pitch and gravity turn.

6.4.5 Constant-Pitch-Rate Maneuver

How are drag losses minimized? According to Eq. (6.3.10) drag is small, if at low altitudes, despite a high atmospheric density, the velocity is very low. This is always the case after lift-off. But flight velocity increases rapidly, so the maximum aerodynamic pressure, so-called $q_{\max}$ (a.k.a. *max q*), is achieved at medium altitudes, and it may become quite big. At increasing altitudes, aerodynamic pressure decreases due to the exponential decline of the atmospheric density with altitude. Apart from substantial drag losses, aerodynamic pressure also puts on high dynamic loads. This is why at max q the Space Shuttle temporarily throttled down its three liquid propulsion engines to about 70% thrust (see Fig. 6.1). So from the sole perspective of drag losses, the ascent should be as slow as possible and with the shortest trajectory through the atmosphere (Fig. 6.13).

To account for this requirement the so-called constant pitch rate (CPR) maneuver is frequently used rather than the gravity turn. The **pitch angle** is defined as $\theta := \alpha + \gamma$. A CPR would therefore imply $\dot{\theta} = \dot{\alpha} + \dot{\gamma} = const < 0$. We want to know, how the vehicle needs to be steered, i.e., how $\alpha(t)$ looks like just after lift-off, to achieve a constant pitch rate. This problem is closely related to the problem, what

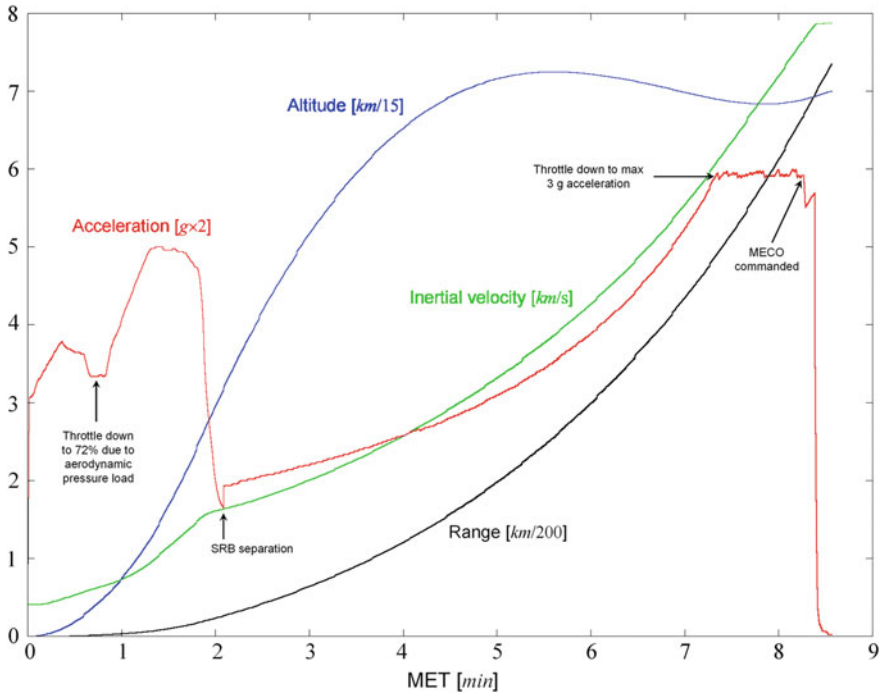


Fig. 6.13 A typical Space Shuttle ascent profile (here STS-122). MET (mission elapsed time) = time from lift-off, SRB = solid rocket booster, MECO = main engine cut-off. Observe that due to Earth's rotational velocity the inertial velocity of the Shuttle was not zero at lift-off. An overshoot in altitude is typical for any spacecraft ascent, which is due to the necessity of a longer vertical flight to reduce aerodynamic resistance rather than to obey an optimal gravity turn

the steering after lift-off is for a constant flight path angle rate (CFPAR), $\dot{\gamma} = const < 0$. We first investigate the latter problem before we come to the CPR problem.

Constant Flight Path Angle Rate

After lift-off we are at low speed, so we can neglect drag, lift, and centrifugal forces. In this case the flight path angle would decrease according to $\gamma = 90^\circ + \dot{\gamma} t$ with $\dot{\gamma} = const < 0$. We get from Eq. (6.3.7)

$$\begin{aligned} v\dot{\gamma} &= \frac{F_*}{m}\sin\alpha - g\,\cos\gamma = a_*\sin\alpha - g\,\cos\gamma \\ F_* &=: ma_* \end{aligned} \tag{6.4.7}$$

where we have defined the thrust acceleration a_*. In order to derive $\alpha(t)$ and with $\gamma(t) = set$ we need to find an expression for v. Considering Eq. (6.3.6), we find for $D \ll F_*$ and because thrust angles are always small

$$\dot{v} = a_* \cos\alpha - g\,\sin\gamma \approx a_* - g\,\sin\gamma$$

We integrate both sides with regard to time. With $m(t) = m_0 + \dot{m}t$ from Eq. (6.4.3) and with $v(t=0) = v_0$ and the condition $\dot{\gamma} = const$, we find

$$\begin{aligned} v - v_0 &= \int_0^t \frac{F_*}{m_0 + \dot{m}t'}dt' - g\int_{90^\circ}^{\gamma} \frac{\sin\gamma'}{\dot{\gamma}}\cdot d\gamma' = a_{*0}\int_0^t \frac{dt'}{1 + \dot{m}t'/m_0} - \frac{g}{\dot{\gamma}}\int_{90^\circ}^{\gamma} \sin\gamma'\cdot d\gamma' \\ &= a_*\frac{m}{\dot{m}}\ln\left(1 + \frac{\dot{m}}{m_0}t\right) + \frac{g}{\dot{\gamma}}\cos\gamma \end{aligned}$$

Inserting this into Eq. (6.4.7) yields

$$\dot{\gamma}\left[a_*\frac{m}{\dot{m}}\ln\left(1 + \frac{\dot{m}}{m_0}t\right) + v_0\right] + g\,\cos\gamma = a_*\sin\alpha - g\,\cos\gamma$$

As $\gamma = 90^\circ + \dot{\gamma}t$, $\cos\gamma = -\sin\dot{\gamma}t$, so we finally obtain for the thrust angle to steer a constant FPA rate

$$\sin\alpha(t) = \dot{\gamma}\left[\frac{m}{\dot{m}}\ln\left(1 + \frac{\dot{m}}{m_0}t\right) + \frac{mv_0}{F_*}\right] - 2\frac{mg}{F_*}\sin\dot{\gamma}t \qquad \textbf{CFPAR steering law} \qquad (6.4.8)$$

This law applies whenever $D, L \ll F_* = const$, $\dot{\gamma} = const$, and v small. After lift-off, when $\dot{\gamma}\,t \ll 1$ and $\dot{m}t \ll m_0$, that is, $m \approx m_0$, we derive the approximate result

$$\sin\alpha(t) \approx \alpha(t) \approx \dot{\gamma}\left[t\left(1 - 2\frac{m_0 g}{F_*}\right) + \frac{m_0 v_0}{F_*}\right] \qquad (6.4.9)$$

So the thrust angle increases linearly with time.

Note *In the special case $F_* = 2m_0 g$, that is, if the thrust is twice as much as the weight of the vehicle, the thrust angle to steer $\dot{\gamma} = const$ becomes constant, and zero for $v_0 = 0$. This means that in a drag-free and lift-free initial stage of ascent where $\dot{m}t \ll 2m_0$ and $\dot{\gamma}^2 t^2 \ll 6$ the gravity turn alone does the job.*

Constant Pitch Rate

We now turn back to the CPR problem. Since the flight path angle rate depends on the steering angle, $\dot{\gamma}(\alpha)$ (see Eq. (6.3.7)), we have with $\dot{\alpha} + \dot{\gamma}(\alpha) = \dot{\theta} = const$. This is a differential equation for $\alpha(t)$, which however is too complex to be solved analytically. But we can simplify this problem by making the special choice $\dot{\gamma} = const$, i.e., applying the CFPAR steering law. From $\dot{\alpha} = \dot{\theta} - \dot{\gamma} = const$ follows that

$$\alpha(t) = \left(\dot{\theta} - \dot{\gamma}\right) t + \alpha_0$$

Solving Eq. (6.4.9) for $\dot{\gamma}$ and inserting it yields

$$\alpha \frac{2 - 2g/a_{*0}}{1 - 2g/a_{*0}} = \dot{\theta}\, t + \frac{v_0 \dot{\gamma}/a_{*0}}{1 - 2g/a_{*0}} + \alpha_0$$

from which we finally obtain

$$\alpha(t) = \frac{a_*/2 - g}{a_* - g}\left(\dot{\theta} t + \alpha_0\right) + \frac{1}{2}\frac{v_0 \dot{\gamma}}{a_* - g} \qquad \begin{array}{l}\textbf{linear CPR steering law} \\ @\ \dot{\gamma} = const\end{array} \tag{6.4.10}$$

Here, as well, we find that, if $a_* = 2g$ then $\alpha(t) = const$ and no steering is needed. For $v_0 = 0$ there are even no steering losses.

6.4.6 Terminal State Control

The terminal state control problem considers the case where the ascending vehicle initially is in the state $(\boldsymbol{r}(t_i), \boldsymbol{v}(t_i)) =: (\boldsymbol{r}_i, \boldsymbol{v}_i)$ and targets the final state $\left(\boldsymbol{r}(t_f), \boldsymbol{v}(t_f)\right) =: \left(\boldsymbol{r}_f, \boldsymbol{v}_f\right)$, which for instance is given by Eq. (6.4.2). So, this is exactly the control we need to solve the optimization problem as described in Sect. 6.4.2. However, it can be shown that there exists no analytical solution if aerodynamic forces act on the vehicle. We therefore turn to the simpler case at high altitudes where we have no aerodynamic forces. In this case the master Eq. (6.3.1) reduces to the equations of motion

$$\begin{aligned} \dot{\boldsymbol{r}} &= \boldsymbol{v} = \begin{pmatrix} v_x \\ v_y \end{pmatrix}_{RSW} \\ \dot{\boldsymbol{v}} &= \boldsymbol{a}_* + \boldsymbol{g} = \begin{pmatrix} a_* \cos(\alpha + \gamma) \\ a_* \sin(\alpha + \gamma) - g \end{pmatrix}_{RSW} \end{aligned} \qquad @\ \text{no aerodynamic forces}$$

where

$$\boldsymbol{a}_* = \frac{\boldsymbol{F}_*}{m} = a_* \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}_{NTW} = a_* \begin{pmatrix} \cos(\alpha + \gamma) \\ \sin(\alpha + \gamma) \end{pmatrix}_{RSW}$$

is the thrust acceleration expressed in the *NTW* and *RSW* Cartesian topocentric satellite coordinate systems co-moving with the vehicle (see Sect. 13.1.4).

Optimal Performance Control

Let us assume that by means of an optimization technique we have found a $\gamma(t)$ and t_f that maximizes the performance index J, that is the total vehicle mass at the target point. Owing to $m(t_f) = m_0 - \dot{m}_p \cdot t_f = m_0 - m_p(t_f)$ this is equivalent to

minimizing the expended propellant or to minimizing t_f. Then it can be shown by variational calculus (see e.g. Battin 1987, p. 562ff) that in the case of absent aerodynamic forces one gets, in the most general case, a bilinear tangent steering law for $\alpha(t)$

$$\tan[\alpha(t)+\gamma(t)] = \frac{c_1(t_f - t) + c_2}{c_1'(t_f - t) + c_2'} \quad \textbf{bilinear tangent steering law}$$

where the constants c_1, c_2, c_1', c_2' are given by the constraint that the initial $(\boldsymbol{r}_i, \boldsymbol{v}_i)$ and final $(\boldsymbol{r}_f, \boldsymbol{v}_f)$ state satisfy the corresponding boundary conditions. Even more generally it was shown that the bilinear tangent steering law is independent of the optimization function.

Furthermore, if the terminal downrange distance is unconstrained, which is the case when one just needs to get into an Earth orbit, as assumed above in Eq. (6.4.2), then it can be shown that $c_1' = 0$ and the bilinear tangent steering law reduces to the linear tangent steering law

$$\tan[\alpha(t)+\gamma(t)] = a(t_f - t) + b \quad \textbf{linear tangent steering law}$$

Employing the initial and terminal boundary conditions leads to

$$\tan[\alpha(t)+\gamma(t)] = \tan(\alpha_i + \gamma_i) - \left[\tan(\alpha_i + \gamma_i) - \tan(\gamma_f)\right]\frac{t}{t_f}$$

If Earth is assumed to be flat, the *RSW* coordinate system is a Cartesian system, where *x,y* span the flight plane and *y* points up vertically. The terminal velocity vector then can be written (see Eq. (6.4.2)) as

$$\boldsymbol{v}_f = \begin{pmatrix} v_x(t_f) \\ v_y(t_f) \end{pmatrix}_{x,y} = v_f \begin{pmatrix} \cos\gamma_f \\ \sin\gamma_f \end{pmatrix}_{x,y} = \sqrt{\frac{\mu}{a_T(1-e_T^2)}} \begin{pmatrix} 1 + e_T\cos\theta_T \\ e_T\sin\theta_T \end{pmatrix}_{x,y}$$

With this, the following linear tangent steering law can be shown (Battin 1987, p. 565) to be optimal

$$\tan[\alpha(t)+\gamma(t)] = \frac{g\cdot(t_f - t) + v_y(t_f)}{v_x(t_f)} \tag{6.4.11}$$

Note that at the transition point, when $t = t_f$, we have $\tan(\alpha+\gamma) = v_y(t_f)/v_x(t_f) = \tan\gamma$ and hence the steering angle becomes $\alpha(t_f) = 0$.

It must be emphasized that a tangent steering law does not optimize the performance index *J*. Rather, a numerical optimization method, not discussed here, that obeys the initial and final boundary conditions and the qualitative characteristics as discussed in Sects. 6.4.2–6.4.5 delivers $\gamma(t)$ and the duration t_f for the optimal

trajectory of the powered flight. A linear or bilinear tangent steering law is just the means to provide the corresponding steering $\alpha(t)$ for this optimal trajectory.

A key point of terminal state control is that the resulting linear tangent steering law is linear in time. The other is that the steering at any point in time is independent of the history of the thrust acceleration in the thrust phase.

It can be shown that the linear tangent steering is continuous even when the thrust magnitude becomes zero or switches instantaneously between zero and maximum thrust magnitude. This is a perfect criterion for today's thermal thrusters, which most effectively work in a pulsed rather than in a throttling mode. Steering control that makes use of this pulsed acceleration is called *bang-bang control*. Therefore bang-bang profiles are often used for linear tangent steering.

All these advantages enables the linear tangent steering law to be applied as a single steering law through all rocket stages regardless of their variations of thrust and propellant flow rates. It is this universality and simplicity why it is commonly used as an approximate steering law in realistic ascent problems, in particular at high altitudes where the aerodynamic forces become negligible.

6.4.7 Optimal Ascent Trajectory

In summary, the following qualitative picture of an optimal ascent can be given: Immediately after vertical lift-off the S/C is rolled if required (Space Shuttle) into the target orbit inclination. It is then subjected to a pitch and constant pitch rate maneuver which requires only a low propulsion demand at these low speeds. This brings the S/C in a relatively steep trajectory to altitudes where drag has reduced to a level that a loss-free gravity turn bends the trajectory more and more horizontally. The cross-over from constant pitch rate with $\alpha \neq 0$ to gravity turn with $\alpha = 0$ of course is steady. Detailed investigations have shown that a good approximation to the ideal thrust phase trajectory is the following approach:

1. At altitudes below roughly 25 km a piecewise constant thrust angle rate profile of the empirical form

$$\dot{\alpha} = \dot{\gamma} \cdot \mathrm{e}^{-\kappa t} \tag{6.4.12}$$

   is chosen with form factors $\dot{\gamma}$, κ to be determined as part of the overall optimization. In this region of high dynamic pressure near zero steering angles are flown to reduce bending loads caused by wind shear.
2. At altitudes above 25 km a closed-loop guidance based on bilinear or linear tangent steering law is usually initiated, in which real-time integration of the dynamic equations of motion is performed to determine the value of the guidance parameters that meet the desired terminal conditions.

Since we have seen that for bilinear and linear tangent steering $\alpha(t_f) = 0$, at the end of the thrust phase the ascent trajectory passes smoothly into the elliptic transfer orbit, which finally touches and transits into the target orbit.

For such an optimized ascent trajectory the extra delta-v for an ascent into a low Earth orbit are typically:

- Steering $\Delta v \approx 0.05$ km s^{-1}
- Drag $\Delta v \approx 0.4$ km s^{-1}
- Gravitation $\Delta v \approx 1.0$ km s^{-1}
- Earth's rotation $\Delta v = -0.464 \cdot \cos\lambda$ [km s^{-1}].

Earth's rotation Δv is the surface speed of the launch pad at latitude λ due to the rotation of the Earth, which directly adds to the total delta-v as a negative (for a prograde orbit) contribution. In total, the delta-v budget for a typical 250 km parking orbit is

$$\begin{aligned}\Delta v_{tot} &= 7.75 + 0.05 + 0.4 + 1.0 - 0.464 \cdot \cos\lambda \left[\text{km s}^{-1}\right] \\ &= 9.2 - 0.464 \cdot \cos\lambda \left[\text{km s}^{-1}\right]\end{aligned}$$

So, as a rule of thumb the delta-v required to get into LEO from a low latitude launch pad (such as Kennedy Space Center) including a typical margin of about 0.3 km/s is $\Delta v_{tot} \approx 9$ km/s.

To determine an optimal-ascent trajectory with such optimized losses, for instance by determining the form factors $\dot{\gamma}$, κ, is a brilliant feat, in particular when also staging, variations in thrust, aerodynamic properties of the vehicle and winds are taken into account. In the end, a good ascent optimization is based on sophisticated software, on the knowledge of the basic ascent maneuvers, but also a lot on the skills of experienced flight mechanics engineers and on trial and error.

Chapter 7
Orbits

7.1 Fundamental Physics

After ascent, we are now in outer space. How does a spacecraft move under the influence of the gravitational forces of the Sun, planets, and moons? This is the question we will deal with in this chapter, and we are pursuing general answers to it. Let us face reality from the start: The details of motions are usually very complicated and can be determined sufficiently accurately only numerically on a computer. This is exactly how real missions are planned. But the goal for us is not numerical accuracy, but to understand the basic behavior of a spacecraft. To achieve this, it suffices to study some crucial cases. The easiest and by far the most important case is the mutual motion of two point-like (a.k.a. ideal) bodies in the gravitational field of each other, the so-called *(ideal) two-body problem* (2BP), which we study in this chapter, such as the Moon in the gravitational field of the Earth. More complicated cases can often be traced back to the two-body problem by minor simplifications.

Before we derive the corresponding equation of motion, solve it, and thus describe the motion of orbiting bodies, we want to gain insight into the basic principles of gravitation and show that even Newton's laws are the outcome of these.

7.1.1 Gravitational Potential

The existence of forces seems to be so self-evident that we deem them to be the foundation of nature. But appearances can be deceptive, and also Newton succumbed to this in the late seventeenth century. It is not forces that are fundamental, but so-called potentials that cause such forces. This was shown by Laplace one century later. The gravitational potential U is a property of space induced by the

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mass of a body and its surrounding. Like a force, you would not see it by itself. Only if you insert a test mass into this space does the potential act on it and generates an attractive force.

Poisson's Equation

The basic mutual interaction between masses and space, in which they are embedded, is described by the famous Einstein field equations of his theory of general relativity

$$G_{ik} = \frac{8\pi G}{c^4} T_{ik} \qquad \textbf{Einstein field equations}$$

Remark *The Einstein field equations are the components of a tensor equation, a system of 10 coupled partial differential equations of second order in the coordinates to determine G_{ik} from the given T_{ik}. Here the cosmological constant, which recently turned out to be significant on a cosmological scale, has been neglected. You do not really have to understand this equation and the meaning of its terms. We start our considerations with Einstein's equations to show that the origin of Newton's gravitational field is the theory of general relativity.*

G_{ik} is the so-called Einstein tensor. It describes the basic structure of space, its spacetime curvature; T_{ik} is the so-called stress-energy tensor that describes the energy and the inertial moment distribution of matter or fields in space. It corresponds to the classic energy and mass density ρ; G is the *gravitational constant*, and c is the velocity of light. The Einstein field equations tell us that matter and energy of the universe on one hand, and the curvature of space on the other, determine each other. To put it in a different way: Masses tell space how to curve, and space tells the masses how to move. In contradiction to classical Newtonian mechanics, space without any ingredient (Newton's absolute space) cannot exist.

If the curvature of space is weak and the planetary motions are far below relativistic speeds, and if the pressure in the state equation of the local matter/energy distribution is small, it is possible to show that the Einstein field equations turn into the classic potential equation called *Poisson's equation* (a.k.a. *Newtonian field equation*):

$$\nabla^2 U(\boldsymbol{r}) = 4\pi G \cdot \rho(\boldsymbol{r}) \qquad \textbf{Poisson's equation} \tag{7.1.1}$$

where

$$\begin{aligned}\nabla^2 &= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \\ &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin\vartheta}\frac{\partial}{\partial \vartheta}\left(\sin\vartheta \frac{\partial}{\partial \vartheta}\right) + \frac{1}{r^2 \sin^2\vartheta}\frac{\partial^2}{\partial \varphi^2}\end{aligned}$$

is the so-called *Laplace operator*, expressed here in cartesian and spherical coordinates.

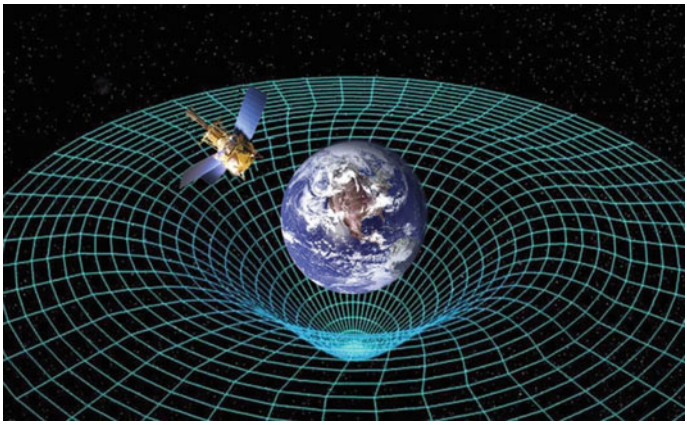


Fig. 7.1 According to the theory of general relativity mass, such as Earth, curves space, and the curvature of space is equivalent to the gravitational potential of this mass that attracts other bodies such as an orbiting satellite. Actually, space is curved (distorted) towards Earth in all three dimensions, which is hard to depict. *Credit* NASA

Equation (7.1.1) is a differential equation of second order. Reading it from right to left, it states that a given mass density $\rho(\boldsymbol{r})$ generates a gravitational potential $U(\boldsymbol{r}) \equiv U_G(\boldsymbol{r})$, which reflects the curvature of space. So, the potential can be conceived as the mass-induced curvature of space (see Fig. 7.1). As we will see in Sect. 7.1.2, the gradient of space curvature, or of the gravitational potential, respectively, in turn acts as a force on other bodies. Mathematically, the gradient of a potential is a force field. It is exactly this force field that we commonly interpret as the cause of gravitational attraction. Thus, the key statement of general relativity reads: Mass-induced gravitation and curvature of space around that mass are the same, they are just different in appearance.

Gravitational Potential of an Isotropic Mass

We now want to determine from Poisson's equation the gravitational potential of a celestial body that exhibits an isotropic mass distribution, $\rho(\boldsymbol{r}) = \rho(r)$. That is, the body is spherical with given radius R. (Gravitational anisotropies of a celestial body will be treated in Sects. 7.7.1 and 12.3.) The isotropic mass sphere of the body under consideration has total mass

$$M = 4\pi \int_0^R \rho(r) r^2 dr$$

and is generally centered at the origin O of our reference system (Fig. 7.2). This is why M is usually called the *central body*. The vector $\boldsymbol{r}$ is the radial vector, a.k.a. *position vector*, from O to any position outside the mass sphere.

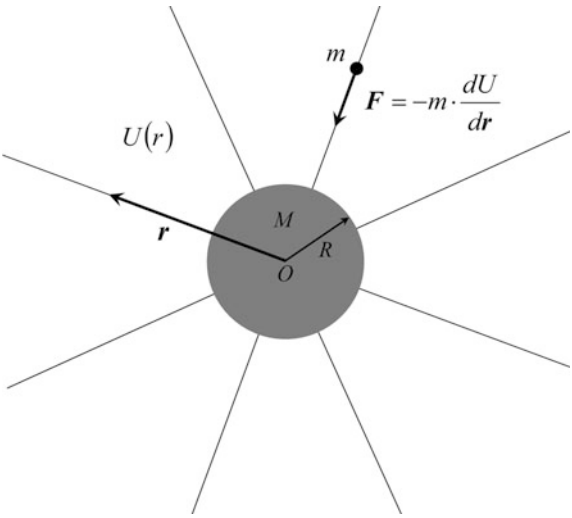


Fig. 7.2 The gravitational potential of a central body with radius R and mass M and in its surroundings with the resulting force on a test mass m. The potential is isotropic in all 3 dimensions

To determine the gravitational potential of such a body, we make use of the fact that the potential must also be isotropic. By adapting spherical coordinates r, ϑ, φ we hence have $\partial U(\boldsymbol{r})/\partial\vartheta = \partial U(\boldsymbol{r})/\partial\varphi = 0$ and $U(\boldsymbol{r}) = U(r)$. Thus, Eq. (7.1.1) reduces to

$$\frac{d}{dr}\left(r^2 \frac{dU}{dr}\right) = 4\pi G\rho r^2 \tag{7.1.2}$$

We now perform the radial integral over the sphere, which yields

$$r^2 \frac{dU}{dr} = 4\pi G \int_0^R \rho(r') r'^2 dr' = GM \quad @\ r \geq R$$

This is a telling interim result. It shows that the potential outside the body does not depend on the specifics of the radial density distribution, but just on the total mass of the central body. Note that this implies that the potential not only is independent of the actual radius of the sphere, but that a central body, such as the Sun, does not need to have a well-defined radius at all. We thus arrive at the famous result, already established by Newton, that

> Isotropic bodies of different size (including a point mass), but same total mass,generate the same gravitational potential outside their body.

An indefinite integration of the above equation finally provides the wanted gravitational potential

$$U(r) = -\frac{GM}{r} + U_0 \quad @\ r \geq R$$

We are free to choose the integration constant, $U_0 = U(r \to \infty)$, which is the zero reference. For instance, the potential energy of a body near Earth's surface is usually measured in terms of its altitude $h = r - R$ above Earth's surface at radius R, i.e. $U(R) = 0 \to U_0 = GM/R$, and hence $U(h) = -GM/(R+h) + U_0 \approx GMh$. In astrophysics, though, one always chooses $U_0 = 0$. So, for $r \to \infty$ the potential energy is defined to be zero, that is

$$U(r) = -\frac{\mu}{r} \quad @\ r \geq R \quad \textbf{gravitational potential} \tag{7.1.3}$$

with

$$\mu := GM \qquad \textbf{standard gravitational parameter}$$

Remark *The standard gravitational parameters of the Sun and Earth are called the* heliocentric *and* geocentric gravitational constants*, respectively. The square root of the heliocentric gravitational constant as determined from Kepler's third law* (see Eq. (7.4.12)) with $T = 365.256363$ *days and* $a = 1\ AU = 1.49597870 \times 10^{11}\ m$ *is called the* Gaussian gravitational constant *and deviates marginally from the heliocentric gravitational constant* (see Sect. 7.1.5).

Potential Energy
A body with small mass m placed into this gravitational potential, by definition, acquires the potential energy

$$E_{pot} := mU(r) = -\frac{\mu m}{r} \tag{7.1.4}$$

Motivated by this relation, one could also consider the gravitational potential as potential energy per mass.

Note *Here, M and m characterize the gravitational property of the masses, in contrast to their inertial property, which they also bear and to which we come in a moment.*

7.1.2 Gravitational Force

Because the potential energy varies from point to point in space and since a body tries to minimize its potential energy, a test mass m with zero initial velocity will

move along the steepest descent, the gradient of the potential energy. We therefore interpret the gradient of the potential energy as a force, the gravitational force, which mathematically is described as

$$\boldsymbol{F}(\boldsymbol{r}) = -\frac{d}{d\boldsymbol{r}} E_{pot} = -m\frac{d}{d\boldsymbol{r}} U(r) \tag{7.1.5}$$

where

$$\frac{d}{d\boldsymbol{r}} \equiv grad \equiv \nabla := \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

The negative sign occurs because the gravitational force $\boldsymbol{F} \equiv \boldsymbol{F}_G$ points in the direction of decreasing energy E_{pot}. As an illustrative example for the calculation of a gradient, let us calculate the gradient of the radial distance

$$\begin{aligned} r &= \sqrt{x^2 + y^2 + z^2} \\ \frac{dr}{d\boldsymbol{r}} &= \left(\frac{\partial r}{\partial x}, \frac{\partial r}{\partial y}, \frac{\partial r}{\partial z}\right) = \left(\frac{1}{2r} 2x,\ \frac{1}{2r} 2y,\ \frac{1}{2r} 2z\right) = \frac{\boldsymbol{r}}{r} = \hat{\boldsymbol{r}} \end{aligned} \tag{7.1.6}$$

where $\hat{\boldsymbol{r}}$ is the unit vector in the direction of $\boldsymbol{r}$. Applying $U(r)$ from Eq. (7.1.3) to (7.1.5) and because of Eq. (7.1.6), we get for the gravitational force at any point in space (which we can also interpret as a gravitational force field)

$$\boldsymbol{F}(\boldsymbol{r}) = \mu m \frac{d}{d\boldsymbol{r}} \frac{1}{r} = -\frac{\mu m}{r^2} \frac{dr}{d\boldsymbol{r}}$$

yielding with Eq. (7.1.6) *Newton's law of gravitation* for the gravitational force

$$\boxed{\boldsymbol{F}(\boldsymbol{r}) = -\frac{\mu m}{r^2} \hat{\boldsymbol{r}}} \qquad \textbf{gravitational force} \tag{7.1.7}$$

It states that the gravitational force decreases with the square of the distance from the mass at the origin. We define

$$\boldsymbol{F}(\boldsymbol{r}) =: m\boldsymbol{g}(\boldsymbol{r}) = -mg(r)\hat{\boldsymbol{r}}$$

and call $\boldsymbol{g}(\boldsymbol{r})$ the **gravitational field**, which is formally the gravitational force field per unit test mass.

Central Force Properties

Any force that like the gravitational force exhibits the factorial property $\boldsymbol{F}(\boldsymbol{r}) = -F(r)\,\hat{\boldsymbol{r}}$ and hence

$$\boldsymbol{F}(\boldsymbol{r}) = -cf(r)\,\hat{\boldsymbol{r}} \quad \textbf{central force}$$

where c is the force-specific coupling constant, is called a *central force*. If $c > 0$, i.e. if it is attractive, it points to the origin. The vector field formed by the central forces at any point in space is called a central force field. Gravitational force and Coulomb force are two familiar examples with $f(r) \propto 1/r^2$.

Let us assume $\boldsymbol{F}$ to be an arbitrary central force with continuous function $f(r)$, and $U(r)$ is its antiderivative (primitive integral or potential function), i.e. $f(r) = dU/dr$. We then have

$$\boldsymbol{F}(\boldsymbol{r}) = -cf(r)\hat{\boldsymbol{r}} = -c\frac{dU}{dr}\hat{\boldsymbol{r}} = -c\frac{dU}{dr}\frac{dr}{d\boldsymbol{r}} = -c\frac{dU}{d\boldsymbol{r}}$$

So, we have shown that

> Any central force can be written as the gradient of a central potential function.

We now show that the work done by a central force field on a body that moves between two points depends only on these points and not on the path followed. To prove this, we express the work W done by the central force field as an integral over an arbitrary path between two points A and B

$$W = \int_A^B \boldsymbol{F} \cdot d\boldsymbol{r} = -c\int_A^B \frac{dU}{d\boldsymbol{r}} \cdot d\boldsymbol{r} = -c\int_{U_A}^{U_B} dU = E_{pot,A} - E_{pot,B}$$

So, the work done depends only on the potential energy of the field at the terminal points.

Forces that obey the above definition are called **conservative forces**. Therefore, any central force is a conservative force (but not necessarily vice versa; see the following box).

Conservative Force

According to the above, one possible definition of a conservative force is:

The work done by a conservative force field on a body that moves between two points depends only on these points and not on the path followed.

Note that from this it follows that:

The work done by a conservative force field on a body that moves on a round trip is zero.

This is an alternative and more common definition of a conservative force. In addition, because the total energy, equaling the kinetic energy plus potential energy, must be conserved (see Sect. 7.1.3) on a round trip, we can characterize a conservative force also by:

After a round trip in a conservative force field a body regains its initial kinetic energy.

In turn, *a **non-conservative force** (a.k.a. dissipative force) can be characterized as one in which a body after a round trip has lost kinetic energy.*

Observe that the above equations for the work done also work backward if for total derivative $c \cdot dU = \boldsymbol{F}(\boldsymbol{r}) \cdot d\boldsymbol{r}$ holds, i.e., if U is a function of the body position only. Then the force is $\boldsymbol{F}(\boldsymbol{r}) \equiv -c \cdot dU/d\boldsymbol{r}$ and also depends solely on position. Therefore, all solely position-dependent forces are conservative forces.

If a force is velocity-dependent, time-dependent, or dependent on any other variable, it is usually not conservative. A textbook example for a non-conservative force is friction. Here $\boldsymbol{F}(\boldsymbol{v}) = -k\boldsymbol{v}$, with $k > 0$, and hence $dW/dt = \boldsymbol{F} \cdot d\boldsymbol{r}/dt = -kv^2 < 0$, the system constantly dissipates energy.

A special case is the electromagnetic field. Here $U = \phi - \boldsymbol{v}\boldsymbol{A}$, where $\phi(\boldsymbol{r}, t)$ and $\boldsymbol{A}(\boldsymbol{r}, t)$ are scalar and vector potentials, respectively. It can be shown that for the electromagnetic Lorentz force $\boldsymbol{F} = -q(\partial U/\partial \boldsymbol{r} - d\boldsymbol{A}/dt) = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$ holds, with q the electric charge (coupling constant) of the affected particle, and $\boldsymbol{E} = -\boldsymbol{\nabla}\phi - \partial \boldsymbol{A}/\partial t$ the electric field and $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$ the magnetic field. So, even though $U(\boldsymbol{r}, \boldsymbol{v}, t)$, we obtain $\boldsymbol{F} = -q \cdot \partial U/\partial \boldsymbol{r}$ if $\boldsymbol{A}$ and hence $\boldsymbol{B}$ (and $\boldsymbol{E}$, which both go hand in hand) does not vary in time. In this case the electromagnetic field is also a conservative force field. If $\boldsymbol{B}$ (and $\boldsymbol{E}$) varies in time, the electromagnetic field is not conservative.

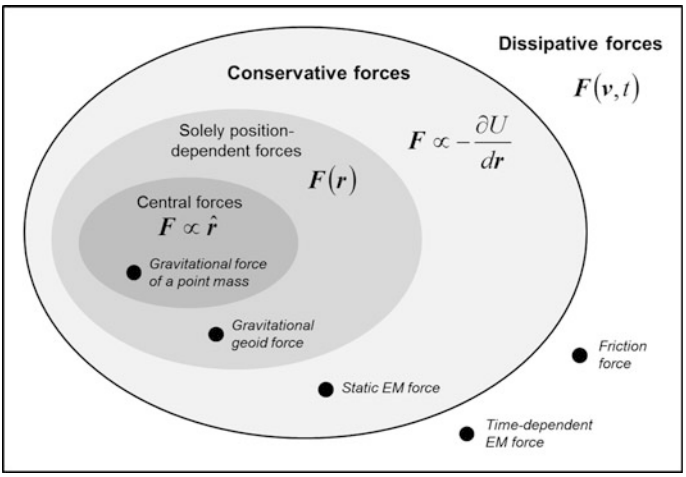


Fig. 7.3 Conservative versus dissipative forces and the different types of conservative forces with some typical examples (black dots)

We summarize all the above in Fig. 7.3 and by the following relations

$$\boldsymbol{F}(\boldsymbol{r}) = -cf(r)\hat{\boldsymbol{r}}\,(\text{central force})$$
$$\Downarrow$$
$$\boldsymbol{F}(\boldsymbol{r}) \text{ and } U(\boldsymbol{r}) \text{ are solely position-dependent}$$
$$\Downarrow$$
$$\boxed{\boldsymbol{F} = -c \cdot \partial U/\partial \boldsymbol{r} \quad \Leftrightarrow \quad \text{conservative force}}$$

A good example of a conservative force field that is even solely position-dependent, but yet not a central force, is Earth's true gravitational field (actually, the non-spherical part of it) as examined in Sect. 12.2.2.

In the above considerations we have assumed that the force in question is the only one that acts on the body. If more than one force act, the effects are separately attributable to each such force. For example, apart from the conservative gravitational force, we often have to consider external non-conservative forces such as atmospheric drag, solar radiation, or eddy fields, the last of which arise from time-dependent external magnetic fields. They change the kinetic energy of the body and hence are also called *dissipative forces*. As we will see from Sect. 7.2.1, external forces also imply that the angular momentum is no longer preserved because the system is no longer a closed system.

Earth's Gravitational Field

In Earth's environment $\boldsymbol{g}(\boldsymbol{r}) \equiv \boldsymbol{g}_{\oplus}(\boldsymbol{r})$ is Earth's gravitational field. Its absolute value is

$$g(r) = \frac{GM_{\oplus}}{r^2} = g_0 \frac{R_{\oplus}^2}{r^2} \qquad \textbf{Earth's mean gravitational acceleration}$$

Note *This definition contrasts with the one in physical geodesy, where g is* Earth's gravity*, which is the distinct acceleration at any point on Earth's surface. Earth's gravity g results jointly from its gravitational force plus the centrifugal force due to the rotation of the Earth and therefore is dependent on the local altitude and the local geographical latitude. Both forces in turn are derived from the (effective) geopotential, which comprises Earth's gravitational potential and Earth's centrifugal potential* (see Sect. 7.2.4). *The difference in these two definitions is reflected by the use of the discriminative words "gravitation" versus "gravity", and "Earth's gravitational potential" versus "geopotential".*

In particular, and for practical purposes, in the following we define

$$g_0 := \frac{GM_{\oplus}}{R_{\oplus}^2} = 9.7982876\ \mathrm{m\,s^{-2}} \qquad \textbf{Earth's mean gravitational acceleration at its surface}$$

where the numerical value follows from $\mu_\oplus = GM_\oplus = g_0 R_\oplus^2 = 3.98600442 \times 10^{14}\,\mathrm{m^3\,s^{-2}}$ (see Appendix A) and $R_\oplus = 63780.1363$ km, which is the equatorial scale factor of the Earth gravitational model EGM96 (see Sect. 12.2.2) equaling Earth's mean equatorial radius at any practical rate.

Note *Because Earth's mass is not perfectly homogeneous and spherically symmetrical $U_\oplus(r, \beta, \lambda)$ varies slightly with geocentric latitude β and geographic longitude λ* (see Eq. 12.2.3). *Therefore the gravitational acceleration at Earth's surface locally deviates from the mean value g_0, and in Earth's orbits $g(r, \beta, \gamma)$ locally deviates from Earth's mean gravitational acceleration $GM_\oplus/r^2$.*

While the gravitational attraction of masses hinge on this gravitational property their motion in a gravitational field is additionally determined by the inertial properties of the body. We will now show that Newton's laws, which are closely related to the inertia of a body, are also based on very fundamental properties of our universe.

7.1.3 Conservation Laws

In the literature it is common to assume Newton's laws and Newton's equation of motion as given and to apply them to gravitation and to derive the conservation of angular momentum and energy from them. This might be correct on mathematical grounds, but it does not mean that the conservation laws result from Newton's laws. It rather shows that the conservation laws also hold for motion under a gravitational force. Could that mean that they would not be valid in other cases? The conservation laws actually are very first principles in nature: In systems not affected by non-conservative interactions conservation laws are always valid. This property stems from very basic features of our universe, namely that time t and space $x,\ y,\ z$ are homogeneous, and the direction φ in space is isotropic.

Remark *According to Einstein's equations, space and time are homogeneous and isotropic because the masses are distributed evenly in the universe on a cosmic scale. All masses in the universe have to be considered here, because only in their entirety do they determine the gross spatial structure of the cosmos.*

The so-called *Noether's theorem* (Emmy Noether, 1918) of physics tells us that these basic features result in the following conservation laws.

Law of Conservation of Energy
Homogeneity of time, that is, the invariance of the physical action integral against continuous time shifts $t \to t + \delta t$, results in the conservation of energy

$$\boxed{\sum_{\text{all energy forms } j \text{ of all bodies } i} E_{ji} = const} \tag{7.1.8}$$

Law of Conservation of Linear Momentum
Homogeneity of space, that is, the invariance of the action integral against continuous spatial shifts $\boldsymbol{r} \to \boldsymbol{r} + \delta\boldsymbol{r}$, results in the conservation of linear momentum

$$\boxed{\sum_{\text{all bodies } i} \boldsymbol{p}_i = const} \tag{7.1.9}$$

Law of Conservation of Angular Momentum
Isotropy of the direction in space, that is, the invariance of the action integral against continuous spatial rotations $\varphi \to \varphi + \delta\varphi$, results in the conservation of angular momentum

$$\boxed{\sum_{\text{all bodies } i} \boldsymbol{L}_i = const} \tag{7.1.10}$$

Remark *You do not need to understand why symmetries correspond to conservation laws. Here is just a short summary. The pair of variables (energy, time), (linear momentum, location), and (angular momentum, rotation angle) are so-called "canonically conjugated parameters", generally written as* (p_i, q_i)*, for every particle i. If one takes the difference between kinetic and potential energies for all particles under consideration, which is called the Lagrangian L, then from the energy minimization principle Euler's equation is:* $d(\partial L/\partial \dot{q}_i)/dt - \partial L/\partial q_i = 0$ *with* $p_i \equiv \partial L/\partial \dot{q}_i$*. The invariance of the universe and hence of its Lagrangian L with regard to the shifts* $q_i \to q_i + \delta q_i$ *implies* $d\left(\sum_i \partial L/\partial q_i\right)/dt = 0$*, which in turn implies* $\sum_i \partial L/\partial \dot{q}_i = \sum_i p_i = const$*. These are the said conservation laws.*

7.1.4 Newton's Laws of Motion

We are now set to derive the equation of motion in a gravitational field. First, it is important to note that Eq. (7.1.5) generally describes the relation between any type of energy and the force derived from it. So when taking the gradient of the energy conservation Eq. (7.1.8) and employing Eq. (7.1.5) we get for our test mass m ($i = 1$)

$$0 = \sum_j -\frac{dE_j}{d\boldsymbol{r}} = \sum_j \boldsymbol{F}_j \quad \textbf{Newton's third law} \tag{7.1.11}$$

The running index j indicates all relevant energies and consequent forces. This equation states that the sum of all forces that a mass is subject to vanishes. This is a generalization of Newton's third law that Newton established for only one acting force causing a reacting force: action equals reaction. The energies relevant to our point mass are: potential energy in the gravitational field, E_{pot}, and kinetic energy, E_{kin}; there are possibly also other energies from electric, magnetic, or chemical

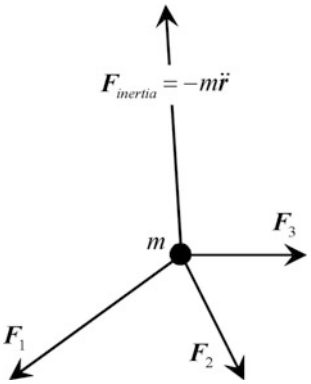


Fig. 7.4 According to Newton's second law, a mass m moves such that the inertial force caused by this accelerated motion counterbalances all other external forces acting on the mass

potentials that we will, however, neglect for our further considerations. The gravitational force derived from the potential energy is already described in Eq. (7.1.7). What is still missing though is the force derived as a gradient from the kinetic energy

$$\frac{dE_{kin}}{d\boldsymbol{r}} = \frac{1}{2}m\frac{d\boldsymbol{v}^2}{d\boldsymbol{r}} = m\boldsymbol{v}\frac{d\boldsymbol{v}}{d\boldsymbol{r}} = m\frac{d\boldsymbol{r}}{dt}\frac{d\boldsymbol{v}}{d\boldsymbol{r}} = m\frac{d\boldsymbol{v}}{dt} = m\ddot{\boldsymbol{r}}$$

According to Eq. (7.1.11) this is the so-called *inertial force*

$$\boldsymbol{F}_{inert} = -\frac{dE_{kin}}{d\boldsymbol{r}} = -m\ddot{\boldsymbol{r}} \qquad \textbf{inertial force}$$

Observe that the inertial force is antiparallel to the direction of acceleration: When you are push-starting a car, you accelerate it forward, but its inertial force pushes backwards against your palms. This derivation shows that

The inertial force is the force field of the kinetic energy.

Note *Here, m now characterizes the inertial property of the mass.*

Because external forces are a given, while a body can acquaint any state of motion, we see that a proper acceleration and hence an adjustable inertial force is the elegant means by which nature constantly complies with Newton's third law (see Fig. 7.4). Inserting this result into Eq. (7.1.11), one gets the Newton's well-known second law

$$\boxed{m\ddot{\boldsymbol{r}} = \sum_j \boldsymbol{F}_j} \qquad \textbf{Newton's second law} \tag{7.1.12}$$

where the summation is over all external forces.

Remark *To be precise, Newton's second law states that* $\boldsymbol{F} = d\boldsymbol{p}/dt$. *But since* $\boldsymbol{p} = m\boldsymbol{v}$, *this together with* Eq. (7.1.11) *is equivalent to Eq.* (7.1.12).

If the external forces vanish, Eq. (7.1.12) reduces to $\ddot{\boldsymbol{r}} = 0$ with the solution

$$\boldsymbol{r} = \boldsymbol{v}_0 t + \boldsymbol{r}_0 \qquad \textbf{Newton's first law}$$

where $\boldsymbol{v}_0$ and $\boldsymbol{r}_0$ are the initial values of our mass m. This equation states that

Every body persists in a state of rest or of uniform motion in a straight line unless it is compelled to change that state by forces impressed on it.

These are the words of Newton, by which he described his first law.
In conclusion, we have shown that

Classical Newtonian physics, in particular the equation of motion in a gravitational field, is an outcome of the theory of general relativity by taking into account the homogeneity and isotropy properties of space and time in our universe.

Significance of Newton's Second Law
At first glance, Newton's second law (Eq. (7.1.12)) seems inconspicuous. Yet, it is the general starting point for the description of the classical motion of a body. It therefore deserves a second glance. It tells us two things. First, it ensures that whatever the external forces acting on a body may be, the inertial force $\boldsymbol{F}_{inertial} = -m\ddot{\boldsymbol{r}}$ will be such that it cancels them all out. Inertia therefore is nature's wild card to achieve this. The second point is that inertial force is available only at the expense of accelerated motion of the body. So,

Accelerated motion, and with it inertial force, is the natural response of a body to external forces. Its trajectory $\boldsymbol{r}(t)$ will generally be determined by solving the differential equation of motion as given by Newton's second law.

This is why Newton's second law is so valuable for classical physics: Any determination of the motion of a body starts out at Eq. (7.1.12).

However, there is even more to Eq. (7.1.12). It gives us an immediate answer to the question why there is weightlessness all over the place in space: If the forces on a body are canceled out by inertial behavior, there will be no residual gravitational force causing weight on a scale. It tells us that this happens whenever a body is free to move. Therefore, when jumping from a tower into a pool of water, we experience the same weightlessness during the free fall as an astronaut does during his uniform free fall around a planet, called circular orbit. In the latter case, the circular motion causes a constant inertial centrifugal force counterbalancing the gravitational force. But why is an astronaut even weightless when he is flying in a spacecraft outbound on a straight line from Earth, say to the Moon, as happened nearly so during the Apollo flights? This seems a much trickier question, but the answer is found again in Newton's second law: The gravitational force pulls the astronaut back and hence

decelerates his outward motion, which in turn entails a breaking force exactly in the same manner as when you slam on the breaks in your car. This deceleration force, which now points tangentially along the trajectory, counterbalances the gravitational force in the same way as the perpendicular centrifugal force in a circular orbit.

Equation of Motion
Assuming an ideal two-body problem with an inertial point mass M centered at O and by applying Newton's second law, we finally get for the motion of a body m under the gravitational force of the central mass M, as described in Eq. (7.1.7), Newton's gravitational equation of motion:

$$\ddot{\boldsymbol{r}} = -\frac{\mu}{r^3}\boldsymbol{r} = -\frac{\mu}{r^2}\hat{\boldsymbol{r}} \tag{7.1.13}$$

with

$$\mu = GM$$

Note *Observe that the body's mass m no longer appears in this equation! The trajectory of a body is thus independent of its mass.*

Remark *In order that the masses cancel out in* Eq. (7.1.13), *we have to assume that gravitational mass and inertial mass are identical. Newton's theory is not able to explain why the gravitational and inertial mass of a body should be identical. They could just as well be different. Only the theory of general relativity provides us with a seamless explanation: acceleration forces (inertial forces) and gravitational forces are two sides of the same coin, the curvature of space. So, a body must react on acceleration and gravitation in exactly the same way: inertial force = weight force. Let us illustrate this with an example due to Einstein: If you would be standing in an elevator at an unknown place in outer space, you could not tell whether your weight is due to external gravitation or due to an acceleration of the elevator.*

7.1.5 General Two-Body Problem

The assumption that the central body M is fixed at O and the body m moves within its potential—which implies that the body m is negligibly small with respect to the central body M, $m \ll$ M—is a constraint that can easily be eliminated. Let us have a look at two bodies with unrestricted mass m_1 and m_2, moving around each other under the influence of their mutual gravitational potential. Now that we have two bodies on an equal footing, there is no exceptional point for the origin O of our reference system. We can place it wherever we want. Let $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ be the position vectors from an arbitrary O to m_1 and m_2, and $\boldsymbol{r} := \boldsymbol{r}_2 - \boldsymbol{r}_1$ the connecting vector

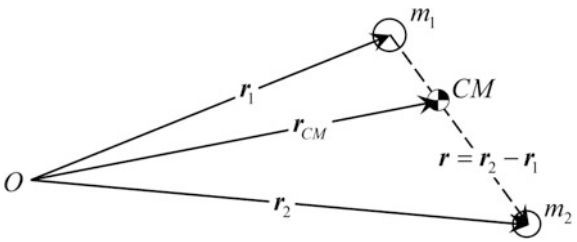


Fig. 7.5 Relevant vectors in the general two-body system

(see Fig. 7.5). According to Eq. (7.1.13), the vectorial equation of motion for each of the two bodies then is

$$\begin{aligned} m_2\ddot{\boldsymbol{r}}_2 &= -\frac{Gm_1m_2}{r^3}(\boldsymbol{r}_2 - \boldsymbol{r}_1) = -\frac{Gm_1m_2}{r^3}\boldsymbol{r} \\ m_1\ddot{\boldsymbol{r}}_1 &= -\frac{Gm_1m_2}{r^3}(\boldsymbol{r}_1 - \boldsymbol{r}_2) = +\frac{Gm_1m_2}{r^3}\boldsymbol{r} \end{aligned} \tag{7.1.14}$$

It is possible to trace bak these equation to that of the ideal two-body problem. To do so, one cancels m_1 from the first equation and m_2 from the other and then subtracts both equations from each other. This yields

$$\ddot{\boldsymbol{r}} = -\frac{\mu}{r^3}\boldsymbol{r} \tag{7.1.15}$$

with

$$\mu := G(m_1 + m_2), \quad \boldsymbol{r} = \boldsymbol{r}_2 - \boldsymbol{r}_1 \tag{7.1.16}$$

This is Newton's gravitational EoM for the connecting vector $\boldsymbol{r}$ between m_1 and m_2 describing the joint synchronous motion of two masses about each other.

Motion of the Center of Mass

The vector $\boldsymbol{r}_{CM}$ to the center of mass (CM, a.k.a. *barycenter*) per definition is the mass-weighted average of the position vectors to both bodies:

$$\boldsymbol{r}_{CM} := \frac{m_1\boldsymbol{r}_1 + m_2\boldsymbol{r}_2}{m_1 + m_2} \tag{7.1.17}$$

Because of Eq. (7.1.14) $m_1\ddot{\boldsymbol{r}}_1 + m_2\ddot{\boldsymbol{r}}_2 = 0$. This implies $\ddot{\boldsymbol{r}}_{CM} = 0$ and hence

$$\boldsymbol{r}_{CM} = \boldsymbol{v}_0 t + \boldsymbol{r}_0 \tag{7.1.18}$$

with the initial conditions $\boldsymbol{v}_0, \boldsymbol{r}_0$. So, with no external forces acting, the CM moves along a straight line in space. This is Newton's third law applied to the CM. As this happens without any acceleration, the CM is an inertial system. According to Eq. (7.1.15) both bodies move synchronously around their common CM.

Motion in the CM System

Physically, one rather would like to describe the motion of each mass in an inertial reference system. Having found with the CM a natural point in space that exhibits inertial properties it lends itself to place the origin O into it: $\boldsymbol{r}_{CM} = 0$. Then $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ are the relative vectors with regard to the CM, and Eq. (7.1.17) results in

$$\boldsymbol{r}_{CM} = \frac{m_1\boldsymbol{r}_1 + m_2\boldsymbol{r}_2}{m_1 + m_2} = 0$$

From this, it follows that

$$m_1\boldsymbol{r}_1 = -m_2\boldsymbol{r}_2 \quad \text{and} \quad m_1 r_1 = m_2 r_2$$

In addition, we have for the absolute value of the connecting vector $r = r_1 + r_2$. With this and from Eqs. (7.1.14) we get after some simple modifications

$$\ddot{\boldsymbol{r}}_1 = -\frac{\mu_1}{r_1^3}\boldsymbol{r}_1, \quad \ddot{\boldsymbol{r}}_2 = -\frac{\mu_2}{r_2^3}\boldsymbol{r}_2$$

where

$$\mu_1 = \frac{Gm_2}{(1 + m_1/m_2)^2}, \quad \mu_2 = \frac{Gm_1}{(1 + m_2/m_1)^2}$$

These are the equations of motion of each of the two masses in the CM reference system.

For all following relevant considerations and hence for the remainder of this book let m be the mass under consideration and M be the other mass. Then the equation of motion for m with position vector $\boldsymbol{r}$ in the CM system reads

$$\boxed{\ddot{\boldsymbol{r}} = -\frac{\mu}{r^2}\hat{\boldsymbol{r}} = -\frac{\mu}{r^3}\boldsymbol{r}} \quad \textbf{Newton's gravitational equation of motion} \tag{7.1.19}$$

with

$$\boxed{\mu = \frac{GM}{(1 + m/M)^2}}$$

Small Mass Approximation

To account for a small mass m moving about a large mass M (a.k.a. central body, e.g., small moon orbiting a planet), we linearly approximate Eq. (7.1.19) in m/M, which yields

$$\ddot{\boldsymbol{r}} = -\frac{\mu}{r^2}\hat{\boldsymbol{r}} \tag{7.1.20}$$

with

$$\mu := G(M - 2m) \quad @\ m \ll M$$

This is the Newtonian equation of motion relevant for all planets in the solar system. The factor μ differs from the one in Eq. (7.1.15) by $3mG$, which, in the case of the Moon circling the Earth, amounts to a non-negligible 3.7%. The μ values for all planets are given in Appendix A.

Of course for $m \to 0$ (e.g., spacecraft orbiting a planet) Eq. (7.1.20) crosses over into our primordial Eq. (7.1.13).

7.2 General Principles of Motion

Having derived the equation of motion, the next step would be to solve it in order to determine the precise motion of a body in a gravitational field. However, before we do that we will first study the general characteristics of a body's motion.

7.2.1 Vector Derivatives

We assume the most general situation, namely that a body moves in an arbitrary way (including rotation about a reference point) in space and attach an arbitrary co-moving reference system to this body, which we denote by $(\boldsymbol{u}_x, \boldsymbol{u}_y, \boldsymbol{u}_z)$. Examples are the $(\boldsymbol{u}_t, \boldsymbol{u}_n, \boldsymbol{u}_h)$-system in Fig. 6.5, the $(\boldsymbol{u}_r, \boldsymbol{u}_\theta, \boldsymbol{u}_h)$-system in Fig. 7.6,

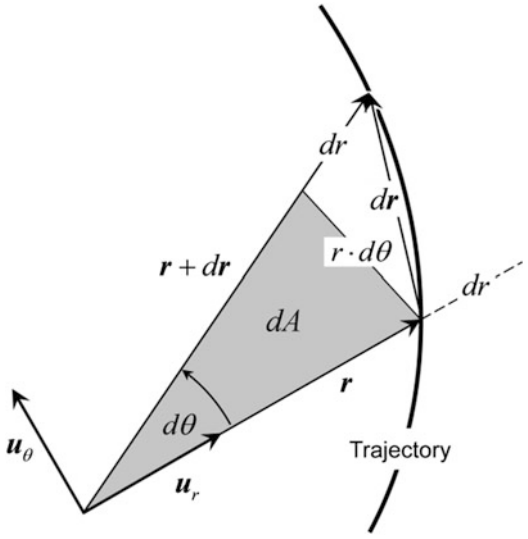


Fig. 7.6 Decomposition of the differential position vector d$\boldsymbol{r}$ in the co-moving reference system

or the $(\boldsymbol{u}_x, \boldsymbol{u}_y, \boldsymbol{u}_z)$-system in Fig. 8.20. Any vector $\boldsymbol{a}$ at a given instance t can therefore be expressed in terms of this reference frame

$$\boldsymbol{a}(t) = a_x \boldsymbol{u}_x + a_y \boldsymbol{u}_y + a_z \boldsymbol{u}_z$$

If we denote the unit vector of $\boldsymbol{a}$ by $\boldsymbol{u}_a(t)$, $\boldsymbol{a}(t)$ can also be written as

$$\boldsymbol{a}(t) = a(t) \cdot \boldsymbol{u}_a(t)$$

To derive the components of any differential vector, first have a look at Fig. 7.6 where the differential position vector

$$d\boldsymbol{r} = dr \cdot \boldsymbol{u}_r + d\boldsymbol{\theta} \times \boldsymbol{r} = dr \cdot \boldsymbol{u}_r + r \cdot d\theta \cdot \boldsymbol{u}_\theta$$

is pictured. From this we recognize that $dr \cdot \boldsymbol{u}_r$ is the radial, while $d\boldsymbol{\theta} \times \boldsymbol{r}$ is its lateral component of $d\boldsymbol{r}$. If we denote the angular velocity of $\boldsymbol{a}(t)$ as $\boldsymbol{\omega}(t) := d\boldsymbol{\theta}/dt$, its lateral unit motion therefore is

$$\dot{\boldsymbol{u}}_a = \boldsymbol{\omega} \times \boldsymbol{u}_a$$

We now aim to express the derivatives of this arbitrary $\boldsymbol{a}$ in the co-moving reference frame at the given instant t. By differentiating $\boldsymbol{a} = a \cdot \boldsymbol{u}_a$ twice, we obtain

$$\dot{\boldsymbol{a}} = \dot{a} \cdot \boldsymbol{u}_a + a\dot{\boldsymbol{u}}_a = \dot{\boldsymbol{a}}_r + \boldsymbol{\omega} \times \boldsymbol{a} \tag{7.2.1}$$

$$\ddot{\boldsymbol{a}} = \ddot{\boldsymbol{a}}_r + \underbrace{2\boldsymbol{\omega} \times \dot{\boldsymbol{a}}_r}_{\text{Coriolis force}} + \underbrace{\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{a})}_{\text{centrifugal force}} + \dot{\boldsymbol{\omega}} \times \boldsymbol{a} \tag{7.2.2}$$

where we have defined

$$\dot{\boldsymbol{a}}_r := \dot{a} \cdot \boldsymbol{u}_a = \dot{a}_x \boldsymbol{u}_x + \dot{a}_y \boldsymbol{u}_y + \dot{a}_z \boldsymbol{u}_z$$
$$\ddot{\boldsymbol{a}}_r := \ddot{a} \cdot \boldsymbol{u}_a = \ddot{a}_x \boldsymbol{u}_x + \ddot{a}_y \boldsymbol{u}_y + \ddot{a}_z \boldsymbol{u}_z$$

The annotated terms in the acceleration vector correspond to the said well-known forces in physics.

7.2.2 Motion in a Central Force Field

Let us consider the real-world case that a body m moves in a gravitational field, or more generally in a central force field, with position vector $\boldsymbol{r}$ and velocity vector $\boldsymbol{v}$. Under these conditions, its motion is determined by the equation of motion, in

particular Eq. (7.1.19) for motion in a gravitational field. Before solving this differential vector equation we first want to derive some general properties of the motion.

Conservation of Angular Momentum

We have seen in Sect. 7.1.3 that, if the body m and the central body is a closed system, i.e. if the body m is not subject to any external interaction, its angular momentum L is a conserved quantity that is given by physics as

$$\boxed{\boldsymbol{h} := \frac{\boldsymbol{L}}{m} = \boldsymbol{r} \times \boldsymbol{v} = const} \quad \textbf{mass-specific angular momentum} \text{ (invariant)} \tag{7.2.3}$$

where h is the *mass-specific angular momentum*. We will denote it *angular momentum* for short. Vector representations of $\boldsymbol{h}$ in various reference systems are found in Sect. 13.1.5.

To verify that for this motion the angular momentum is indeed conserved, we take the time-derivative of $\boldsymbol{L}$

$$\dot{\boldsymbol{L}} = m\boldsymbol{v} \times \boldsymbol{v} + \boldsymbol{r} \times m\ddot{\boldsymbol{r}} = \boldsymbol{r} \times \boldsymbol{F}$$

where the latter follows from Newton's second law, Eq. (7.1.12). We now see that the conservation of angular momentum, i.e. $\dot{\boldsymbol{L}} = 0$, hinges on the fact that the gravitational force is a central force $\boldsymbol{F} = -m \cdot g(r) \cdot \hat{\boldsymbol{r}}$ (see Sect. 7.1.2). Note that $\boldsymbol{r} \times \boldsymbol{F}$ is the torque that the force exerts on a moving body. A central force therefore is also characterized in that it affects a body without any torque.

The Orbital Plane

Let us assume that the body m has the initial velocity $\boldsymbol{v}_0$ at the initial position $\boldsymbol{r}_0$. $\boldsymbol{r}_0$ and $\boldsymbol{v}_0$ span a plane. Because of Eq. (7.2.3), the initial angular momentum $\boldsymbol{h}_0$ is vertical on $\boldsymbol{r}_0$ and $\boldsymbol{v}_0$, and also at later times $\boldsymbol{h} \cdot \boldsymbol{r} = \boldsymbol{h} \cdot \boldsymbol{v} = 0$ holds. So, r as well as v is always vertical to h. In other words, because $\boldsymbol{h} = const$, the body m always maintains its motion in the plane, that was spanned by the initial $\boldsymbol{r}_0, \boldsymbol{v}_0$.

> **Note** *Strictly speaking, the motion in a plane with* $\boldsymbol{r}, \boldsymbol{v} \perp \boldsymbol{h}$ *is valid only for* $h \neq 0$. *For* $h = 0$, *the motion is on a line* (see Sect. 7.5).

Therefore, the plane spanned by $\boldsymbol{r}, \boldsymbol{v}$ does not change with time. As will be shown later, the pair $(\boldsymbol{r}, \boldsymbol{v})$ at any point in the orbit also determines the shape of an orbit. Hence, $(\boldsymbol{r}, \boldsymbol{v})$ determines both the orientation and the shape, i.e., the full state, of an orbit, which is why it is rightly called state vector. We conclude that

> The motion of a body m always takes place in a constant plane, the orbital plane, through the center of mass common with M, perpendicular to the angular momentum $\boldsymbol{h}$, and spanned by $\boldsymbol{r}$ and $\boldsymbol{v}$.

As conservation of the angular momentum is a very general property, independent of the details of gravitational force or its potential, it is even true for spaces with dimension other than three. We will come back to this peculiarity in Sect. 7.6.

General Radial and Lateral Motion in a Plane

To obtain the general properties of the planar motion we apply the results of Sect. 7.2.1 to For convenience we choose the co-moving cylindrical reference system $(\boldsymbol{u}_r, \boldsymbol{u}_\theta, \boldsymbol{u}_h)$ with the instantaneous radial unit vector $\boldsymbol{u}_r = \hat{\boldsymbol{r}}$, $\boldsymbol{u}_\theta$ perpendicular to it in the motion plane, and $\boldsymbol{u}_r = \hat{\boldsymbol{h}}$ as the basis vectors (see Fig. 7.6). In this system any other vector in the plane is described as $\boldsymbol{a} = (a, \theta, 0)$. We also define the angular velocity

$$\omega := \dot{\theta} \qquad \textbf{angular velocity} \tag{7.2.4}$$

It is now easy to show that for $\boldsymbol{a} \equiv \boldsymbol{r} = r \cdot \boldsymbol{u}_r$ we get with $\boldsymbol{\omega} = \omega \cdot \boldsymbol{u}_h$ from Eqs. (7.2.1) and (7.2.2)

$$\dot{\boldsymbol{r}} = \boldsymbol{v} = \dot{r} \cdot \boldsymbol{u}_r + r\omega \cdot \boldsymbol{u}_\theta \tag{7.2.5}$$

$$\ddot{\boldsymbol{r}} = \left(\ddot{r} - \omega^2 r\right) \cdot \boldsymbol{u}_r + (2\omega\dot{r} + \dot{\omega} r) \cdot \boldsymbol{u}_\theta \tag{7.2.6}$$

So, while $v_r = \dot{r}$ is the radial component $v_\theta = \omega r$ is the lateral component of the velocity (see Fig. 7.5 for the differential analysis).

> **Note** *$\dot{r}$ is only the radial part of the velocity vector $\dot{\boldsymbol{r}}$, i.e. $\dot{r} = \hat{\boldsymbol{r}}\dot{\boldsymbol{r}}$, and not its value, $\dot{r} \neq |\dot{\boldsymbol{r}}|$. To avoid confusion, we will therefore from now on always write $\boldsymbol{v}$ rather than $\dot{\boldsymbol{r}}$* (cf. Note in Sect. 6.3).

With Eq. (7.2.5) we obtain for the angular momentum $\boldsymbol{h} = \boldsymbol{r} \times \dot{\boldsymbol{r}} = r \cdot r\omega \cdot \boldsymbol{u}_h$, and hence for its absolute value

$$\boxed{h = \omega\, r^2 = \dot{\theta} r^2 = const} \tag{7.2.7}$$

> **Note** *It is the angular momentum h and its conservation in time that makes a body to orbit (ω!) steadily around a central mass and prevents the masses in our universe to instantly collapse* (cf. Sect. 7.5 for trajectories with $h = 0$).

This equation is of notable significance. It states that the further a body departs on its orbit from the origin the less its angular velocity becomes, and vice versa.

From Eq. (7.2.5) finally follows with Eq. (7.2.7)

$$v^2 = \dot{r}^2 + \omega^2 r^2 = \dot{r}^2 + \frac{h^2}{r^2} \tag{7.2.8}$$

Kepler's Second Law

The conservation of angular momentum has an immediate and important geometrical implication. The infinitesimal area dA that the position vector r sweeps by advancing through $d\theta$ is determined according to Fig. 7.6, Eqs. (7.2.7) and (7.2.4) by

$$dA = \frac{1}{2} r\,(r \cdot d\theta) = \frac{1}{2} r^2 \cdot d\theta = \frac{1}{2} r^2 \omega \cdot dt = \frac{h}{2} dt$$

That is, $h = 2\dot{A} = const$. So if we consider a constant time interval Δt, we find

$$\Delta A = \int \frac{h}{2} dt = \frac{h}{2} \int dt = \frac{1}{2} h \cdot \Delta t = const \qquad \textbf{Kepler's second law} \qquad (7.2.9)$$

The angular momentum can be interpreted as a constant areal velocity of the body: The area that the position vector sweeps in equal time intervals is constant.

Kepler's second law is valid not only for bound orbits (ellipses, circles) in a gravitational field, as Kepler postulated, but for any motion of a body in any central force field. This is because the above derivation rest solely on the conservation of angular momentum, which implies a central force as we saw in Sect. 7.2.1.

Equation of Radial and Lateral Motion under a Gravitational Force
From Eq. (7.2.6) we get with Newton's gravitational equation of motion (7.1.19), $\ddot{\boldsymbol{r}} = -\mu/r^2 \cdot \boldsymbol{u}_r$ for the radial and lateral component separately

$$\ddot{r} - \omega^2 r = -\frac{\mu}{r^2}$$
$$2\omega\dot{r} + \dot{\omega} r = 0$$

By multiplying the second equation by r and by direct integration we recover $r^2\omega = h$, which is Eq. (7.2.7). Inserting this result into the first equation, one obtains

$$\ddot{r} = -\frac{\mu}{r^2} + \frac{h^2}{r^3} \qquad \textbf{Leibniz's equation} \qquad (7.2.10)$$

This is an equation of radial motion (cf. Eqs. (7.1.19) and (7.6.3)), which, derived by Leibniz, historically is of high relevance. Multiplication by $\dot{r}$ and direct integration delivers the equation of radial energy

$$\frac{1}{2}\dot{r}^2 = \frac{\mu}{r} - \frac{h^2}{2r^2} + \varepsilon$$

As we will see later in Eq. (7.3.19), the energy integration constant can be written as $\varepsilon = -\mu/2a$ where a is geometrically the semi-major axis of the orbit (i.e., $a > 0$ for ellipses and $a < 0$ for hyperbolas). This explicitly proves that the gravitational

field as a conservative force conserves both the angular momentum and the total energy of the moving body. In summary we have

$$\dot{r}^2 = \frac{2\mu}{r} - \frac{h^2}{r^2} - \frac{\mu}{a} \quad \textbf{gravitational equation of radial motion} \tag{7.2.11}$$

$$\dot{\theta} = \frac{h}{r^2} \quad \textbf{equation of lateral motion} \tag{7.2.12}$$

which are two scalar differential equations that describe the motion of the body in polar coordinates in the orbital plane. As we will see later from Eq. (7.3.7), $h^2 = \mu a(1 - e^2)$ and the second differential equation can be decoupled from the first via Eq. (7.3.5). In principle one could solve both equations of motion to derive a body's motion in the plane. However, as it turns out, rather than $r(t),\ \theta(t)$ they deliver the inverse solutions $t(r),\ t(\theta)$, which are useless for practical purposes. We therefore have to retreat to Kepler's solution by introducing an easy to treat auxiliary variable $E(t)$ (Kepler transformation, see Sect. 7.4.3), from which $r(E),\ \theta(E)$ follows.

What is the condition that Eqs. (7.2.11) and (7.2.12) are valid? The steadiness of the orbital plane in an inertial reference frame (and thus that the orbit can be described by a polar coordinate system in this plane) and $h = const$ is equivalent to the conservation of angular momentum. This in turn hinges on the fact, as shown in Sect. 7.2.1, that the potential U_G causes a central force field $\boldsymbol{F}_G = -m \cdot g(r) \cdot \hat{\boldsymbol{r}}$. However, this central force field condition generally does not hold, for instance for Earth's non-spherically symmetric gravitational potential (see Sect. 12.3.1, General Considerations) or for external perturbations on Earth orbits such as lunisolar perturbations (see Sect. 12.7.1). All non-central forces are able to tilt an orbital plane, and h may even vary over one orbital revolution, staying constant only on average over one orbital period. For this reason only Newton's second law (7.1.12) serves as a general differential equation to describe the motion of a body under the influence of any kind of forces.

7.2.3 *Vis-Viva Equation*

For the motion of a body in a gravitational field, the general law of conservation of energy Eq. (7.1.8) reduces to

$$\varepsilon_{kin} + \varepsilon_{pot} =: \varepsilon = const \tag{7.2.13}$$

where $\varepsilon_{pot} := E_{pot}/m = U(r)$ is the *specific potential energy*, $\varepsilon_{kin} = E_{kin}/m = \frac{1}{2}v^2$ is the specific *kinetic energy* and $\varepsilon := E_{tot}/m$ is the so-called *specific orbital energy* (a.k.a. *specific mechanical energy*). This results in the following important equations:

$$\frac{1}{2}v^2 = \frac{\mu}{r} + \varepsilon \tag{7.2.14}$$

$$\boxed{v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right)} \quad \textbf{vis-viva equation} \tag{7.2.15}$$

with $a := -\mu/2\varepsilon$. As we will see later in Eq. (7.3.19), a is geometrically the semi-major axis of the orbit (i.e., $a > 0$ for ellipses and $a < 0$ for hyperbolas). Observe that the gravitational equation of radial motion (7.2.11) can be also obtained by inserting Eq. (7.2.8) into Eq. (7.2.15).

Bear in mind that the vis-viva equation is crucial for any orbital motion and will be used frequently throughout this book, because at any orbit position it directly relates the orbit velocity and orbit radius. This is why it bears this very historic name.

Historic Remark

The naming *vis-viva equation*, earlier also called *equatio elegantissima*, originates with the German mathematician Leibniz (1646–1716). He coined the term "vis viva" (Latin, meaning "living force") for mv^2, which he considered as the real measure of force, to separate it from Newton's gravitational force, which people called just "vis". Later when "vis viva" was recognized as an energy, "vis" was also called by the Leibnizians "vis morte" (a.k.a. "vis mortua", meaning "dead force") to distinguish the concept "energy" from "force". You can imagine that this caused a serious fight between Newtonians and Lebnizians on whether their forces were dead or alive. Actually, this all only shows that at that time people did not really understand the difference between force, linear momentum, and energy and the relation between each other: $\boldsymbol{F} = d\boldsymbol{p}/dt = -dE/d\boldsymbol{r}$.

We now show explicitly that the energy of a body moving in a conservative field and hence in any gravitational field is indeed conserved and therefore the vis-viva equation with $a = -\mu/2\varepsilon = const$ holds. We first generalize Eq. (7.2.14) for an arbitrary potential U

$$\varepsilon = \frac{1}{2}v^2 + U$$

From this we obtain with $dv^2/dt = d\dot{\boldsymbol{r}}^2/dt = 2\dot{\boldsymbol{r}}\ddot{\boldsymbol{r}}$

$$\frac{d\varepsilon}{dt} = \dot{\boldsymbol{r}}\ddot{\boldsymbol{r}} + \frac{dU}{dt}$$

Since for the potential of a conservative field (see Sect. 7.1.2) $dU/dt = dU/d\boldsymbol{r} \cdot \dot{\boldsymbol{r}} = -\boldsymbol{f}(\boldsymbol{r})\dot{\boldsymbol{r}}$ holds, we find with Newton's second law (7.1.12), $\ddot{\boldsymbol{r}} = \boldsymbol{f}(\boldsymbol{r})$,

$$\frac{d\varepsilon}{dt} = \dot{\boldsymbol{r}}\ddot{\boldsymbol{r}} - \dot{\boldsymbol{r}}\ddot{\boldsymbol{r}} = 0$$

and therefore $\varepsilon = const$.

7.2.4 *Effective Radial Motion*

Rotational Potential

On pure mathematical grounds the expression $h^2/2r^2$ in Eq. (7.2.11) like the preceding term μ/r could be considered a potential: a radial angular momentum potential, the so-called *rotational potential,* a.k.a. *centrifugal potential*

$$U_\omega(r) := \frac{h^2}{2r^2} = \frac{1}{2}\omega^2 r^2 \tag{7.2.16}$$

The latter follows from Eq. (7.2.7). This also physically makes sense, because according to Eqs. (7.1.5), (7.2.7), and (7.1.6) the corresponding centrifugal force would be

$$\boldsymbol{F}_\omega = m\ddot{\boldsymbol{r}}_\omega = -m\frac{d}{d\boldsymbol{r}}\left(\frac{h^2}{2r^2}\right) = m\frac{h^2}{r^3}\frac{dr}{d\boldsymbol{r}} = m\frac{h^2}{r^3}\frac{\boldsymbol{r}}{r} = m\omega^2\boldsymbol{r} \tag{7.2.17}$$

This is the well-known formula for the centrifugal force in physics. It pushes the orbiting body toward the outside (positive sign). For example, for a circular orbit $r = const$, or $\dot{r} = 0$, it follows from Eq. (7.2.11) that $\mu/r = \mu/a = -h^2/2r^2$, meaning that the centrifugal force compensates the gravitational force at any point of the orbit. Generalizing Eq. (7.1.4), the corresponding rotational energy would be

$$E_\omega = mU_\omega(r) = \frac{1}{2}m\omega^2 r^2 \tag{7.2.18}$$

which is also a quite familiar expression in physics. From this equation the physical meaning of the rotational potential becomes clear: The constancy of angular momentum $h = \omega\, r^2 = const$ implies that the lateral velocity $v_\theta = \omega\, r = h/r$ increases with decreasing orbit radius. lateral kinetic energy $E_\omega = mU_\omega = \frac{1}{2}mv_\theta^2 = \frac{1}{2}m\omega^2 r^2$ increases correspondingly. The rotational potential $U_\omega(r) = \frac{1}{2}\omega^2 r^2 = h^2/2r^2$ is just a virtual potential of this effect. In summary, the closer a body comes to the central mass the more energy is required for its revolving motion. At its closest point of approach all available kinetic energy is rotational energy.

Effective Potential

It is convenient to lump together the gravitational and the rotational potential into a pseudopotential the so-called

$$U_{eff}(r) := U_G + U_\omega = -\frac{\mu}{r} + \frac{h^2}{2r^2} \qquad \textbf{effective potential} \qquad (7.2.19)$$

With this it follows from Eq. (7.2.11) that

$$\frac{1}{2}\dot{r}^2 + U_{eff}(r) = -\frac{\mu}{2a} \qquad (7.2.20)$$

The term $\dot{r}^2/2$ describes the specific radial kinetic energy. In Fig. 7.7 the effective potential for an orbiting body with a given specific angular momentum h is plotted. We recognize that if the body acquires a negative total specific energy, $\varepsilon = -\mu/2a < 0$, its motion is limited to a radial interval bounded by an inner point A and an outer point B. For $\varepsilon > 0$ the orbit is unbounded at the outer end. We determine the outer and inner bound radii from Eq. (7.2.20) and Eq. (7.3.20) with $\dot{r} = 0$ as

$$r_{A,B} = a\left(1 \pm \sqrt{1 - \frac{h^2}{\mu a}}\right) = a(1 \pm e)$$

which agrees with the result of Eq. (7.4.6) in Sect. 7.4.2. Note that the lower bound exists due to the rotational potential, i.e., due to fact that the body has to maintain its

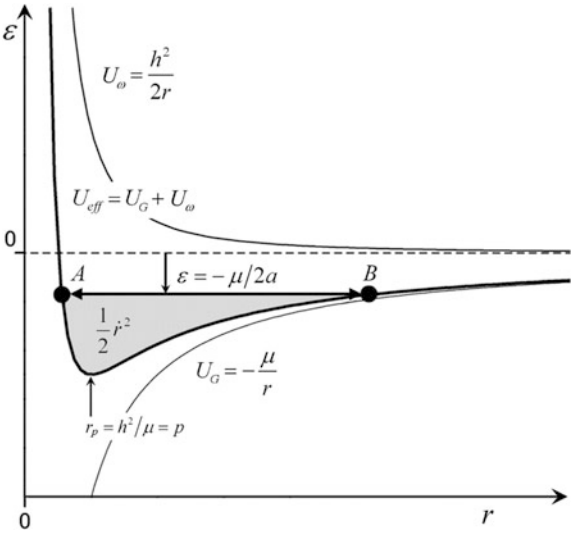


Fig. 7.7 The shape of the effective potential U_{eff} limits the motion of an orbiting body with a given h and specific total energy $\varepsilon < 0$ to the radial end points A and B. The area marked gray is the contribution of the specific radial kinetic energy

angular momentum. At the end points the radial velocity vanishes, while at the radial position $r_p = h^2/\mu = p$, where the effective potential a minimum value, it attains the maximum value $\dot{r}_{\max} = e/h = e\sqrt{\mu/p}$, as derived from Eq. (7.2.11). As we will see later in Eq. (7.3.6), this particular point is called *semi-latus rectum*. The detailed behavior of the different types of orbits will be discussed in Sect. 7.4.

We close this section by summarizing the motion principles brought about by the laws of conservation.

For an object in a gravitational field:

- The law of conservation of angular momentum restricts its motion to a plane and imposes a simple relation between its angular velocity and radial distance, $\omega(r) = h/r^2$, which gives rise to Kepler's second law.
- The law of conservation of energy requires a one-to-one relation between its velocity and radial distance as specified by the vis-viva Eq. (7.2.15).
- The law of conservation of angular momentum jointly with the law of conservation of energy impose limits on its range of motion as given by $r_{A,B} = a\left(1 \pm \sqrt{1 - h^2/\mu a}\right) = a(1 \pm e)$ for $e < 1$.

7.3 Motion in a Gravitational Field

7.3.1 Orbit Equation

So far by applying the equation of motion and general conservation laws we were able to determine general features of the motion without knowing the explicit solution. To obtain further details of the orbit, we have to solve the equation of motion (7.1.19). However, it is in general not possible to find the desired $\boldsymbol{r}(t)$ because when explicitly written for each of the coordinate components we have three equations of motion that are coupled via the $1/r^3 = 1/(x^2 + y^2 + z^2)^{3/2}$ term. Yet, it is possible to find a general expression for the orbital shape the so-called *orbit equation*. To find that we apply an elegant method that Pierre-Simon Laplace (1749–1827) is credited for. However, it dates back to an article of Jakob Hermann (1668–1733, a pupil of Bernoulli and friend of Leibniz) in 1710 in the journal *Giornale de Letterati d'Italia*, Vol. 2, pp. 447–467, which in fact provides the first solution of Newton's gravitational equation of motion at all (see Volk 1976).

In preparation to this method, we note that according to Eq. (7.2.5)

$$\boldsymbol{r} \cdot \boldsymbol{v} = \boldsymbol{r} \cdot (\dot{r} \cdot \boldsymbol{u}_r + r\omega \cdot \boldsymbol{u}_\theta) = r\dot{r}$$

which we will make use in Hermann's smart approach: For $h \neq 0$ (see Note following Eq. (7.3.7)) take the cross product of $\boldsymbol{h}$ with Eq. (7.1.19), which yields

$$\begin{aligned}\boldsymbol{h}\times\ddot{\boldsymbol{r}} &= -\frac{\mu}{r^3}(\boldsymbol{h}\times\boldsymbol{r}) = -\frac{\mu}{r^3}[(\boldsymbol{r}\times\boldsymbol{v})\times\boldsymbol{r}] = -\frac{\mu}{r^3}[\boldsymbol{v}(\boldsymbol{r}\cdot\boldsymbol{r}) - \boldsymbol{r}(\boldsymbol{r}\cdot\boldsymbol{v})] \\ &= -\frac{\mu}{r^3}(r^2\cdot\boldsymbol{v} - r\dot{r}\cdot\boldsymbol{r}) = -\mu\left(\frac{1}{r}\boldsymbol{v} - \frac{\dot{r}}{r^2}\boldsymbol{r}\right) = -\mu\frac{d}{dt}\left(\frac{\boldsymbol{r}}{r}\right) = -\mu\frac{d\hat{\boldsymbol{r}}}{dt}\end{aligned} \tag{7.3.1}$$

with $\hat{\boldsymbol{r}} = \boldsymbol{r}/r$ the unit vector in $\boldsymbol{r}$ direction. This approach is smart because owing to $\boldsymbol{h} = const$, this equation can be integrated directly to give

$$\boldsymbol{h}\times\boldsymbol{v} = -\mu\hat{\boldsymbol{r}} - \boldsymbol{A} = -\mu(\hat{\boldsymbol{r}} + \boldsymbol{e}) \tag{7.3.2}$$

with $\boldsymbol{A} = \mu\boldsymbol{e}$ the integration constant, determined by the initial conditions. Apart from $\boldsymbol{h}$, ε, and $\boldsymbol{e}$ (or $\boldsymbol{A}$, respectively) are also invariants of the system. $\boldsymbol{e}$ is called the *eccentricity vector* and $\boldsymbol{A}$ is called the *Laplace–Runge–Lenz vector* (a.k.a. *Runge-Lenz vector* or *Laplace vector*). From Eq. (7.3.2), we get

$$\boldsymbol{e} = \frac{1}{\mu}\boldsymbol{v}\times\boldsymbol{h} - \hat{\boldsymbol{r}} = \left(\frac{1}{r} - \frac{1}{a}\right)\boldsymbol{r} - \frac{1}{\mu}(\boldsymbol{r}\boldsymbol{v})\boldsymbol{v} \quad \textbf{eccentricity vector}\ (\text{invariant}) \tag{7.3.3}$$

where the latter follows from Eqs. (7.2.3) and (7.2.15). Because $\boldsymbol{v}\times\boldsymbol{h}$ and $\hat{\boldsymbol{r}}$ lie in the motion plane, so must $\boldsymbol{e}$ (and also $\boldsymbol{A}$), that is $\boldsymbol{h}\cdot\boldsymbol{e} = 0$. Vector representations of $\boldsymbol{e}$ in various reference systems are given in Sect. 13.1.5.

To directly derive the equation for the trajectory of the orbit, we multiply Eq. (7.3.2) with r and with $\boldsymbol{r}(\boldsymbol{h}\times\boldsymbol{v}) = -\boldsymbol{h}(\boldsymbol{r}\times\boldsymbol{v}) = -\boldsymbol{h}^2$ get the orbit equation in vectorial form (i.e., independent of a reference system):

$$r + \boldsymbol{e}\cdot\boldsymbol{r} = \frac{h^2}{\mu} =: p \tag{7.3.4}$$

Introducing polar coordinates (r, θ) with $\cos\theta := \hat{\boldsymbol{e}}\hat{\boldsymbol{r}}$, which are most suited and common for orbital trajectories, we obtain

$$\boxed{r = \frac{p}{1 + e\cdot\cos\theta}} \quad @\ h \neq 0 \quad \textbf{orbit equation} \tag{7.3.5}$$

with

$$p := \frac{h^2}{\mu} =: a\left(1 - e^2\right) > 0 \qquad \textbf{semi-latus rectum} \tag{7.3.6}$$

$$h^2 = \mu a\left(1 - e^2\right) \tag{7.3.7}$$

Note *The orbit equation is valid only as long as* $h \neq 0$. *If the body m takes on* $h = 0$, *then according to* Eq. (7.2.7) $\omega = 0$ *and therefore* $\theta = const$. *Thus, the body falls toward the central body M or moves away from it radially on a straight line* (see Sect. 7.5 *for details and compare Note following* Eq. (7.2.7)).

Historic Remark
The fight between Newtonians and Lebnizians as indicated by the Historic Remark in Sect. 7.2.3 was even more profound. Newtons derivation of trajectories under gravitational action was based on his three laws as outlined in his *Principia*. Having derived a vector of the gravitational forces he derived a vectorial equation of motion Eq. (7.1.19), which fully defines the two-dimensional motion in a plane. Leibniz, on the other hand, based his physics solely on his vis viva, mv^2, from which he derived the gravitational equation of radial motion Eq. (7.2.11) and from this by differentiation Leibniz's Eq. (7.2.10). This, however, is just one equation, from which the full description of two-dimensional motion cannot be derived. This weakness of his physics made him resort to his *best of all possible worlds* (most efficient orbit) *principle*, which today is known as the *principle of least action*. Only later in 1833 did William Hamilton show that from the principle of least action the laws of motion indeed follow. Newtonians, not aware of this equivalence, blamed Leibniz for this metaphysics and therefore believed that Newton's physics was superior. If Leibniz would only have known about conservation of angular momentum, he would have derived the equation of lateral motion Eq. (7.2.12). It is now quite easy to show (see Problem 7.4) that the solution of Leibniz's equation with the equation of lateral motion also delivers the orbit Eq. (7.3.5). Therefore, Newton's and Leibniz's physics stand on equal footing.

Analysis of Trajectories
A geometrical analysis of the polar Eq. (7.3.5) shows that it describes four types of trajectories (see Fig. 7.8): circle $(e = 0)$, ellipse $(0 < e < 1,\ 0 < a < \infty)$, parabola $(e = 1,\ a = \infty)$, and hyperbola $(e > 1,\ -\infty < a < 0)$. These trajectories turn out to be conic sections, a.k.a. Keplerian orbits, which will be discussed in detail in Sect. 7.4. Common to all Keplerian orbits is the symmetry line, the so-called *line of apsides*. The geometric interpretation of the elements a, e, p is (see Fig. 7.8b):

- a as defined in Eq. (7.3.6) is the semi-major axis of a Keplerian orbit and as such is a direct measure of the orbital energy ε (see Eq. (7.3.19)).
- The eccentricity e determines the type of orbit and its shape.
- The semi-latus rectum p (a.k.a. *orbital parameter*) is a direct measure of the orbital angular momentum via $p = h^2/\mu$ and as such is an important parameter

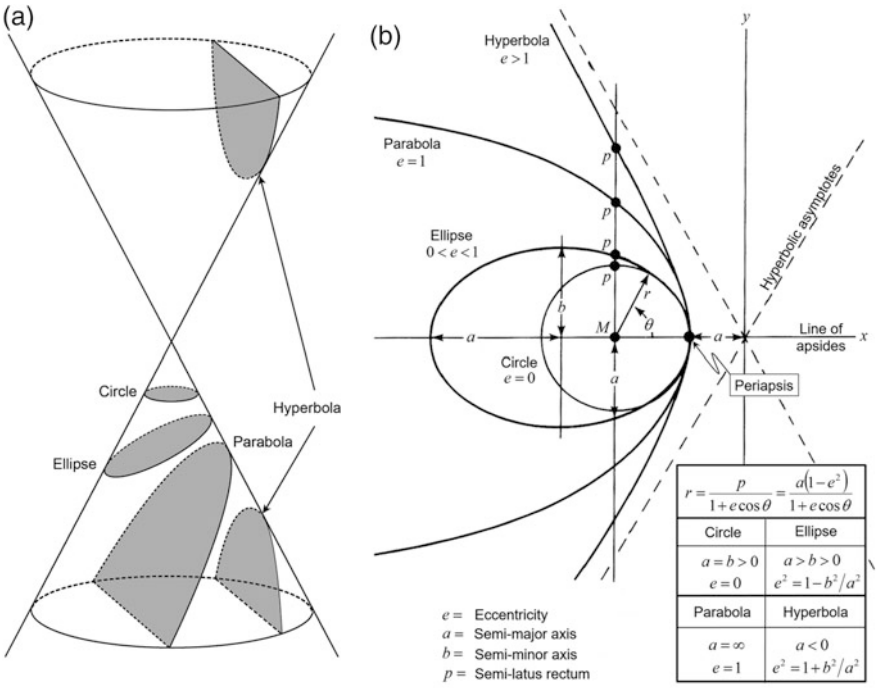


Fig. 7.8 (**a**) Keplerian orbits are conic sections. (**b**) Geometrical presentation of the parameters a, e, b, p for the four conic sections

to characterize any Keplerian orbit. Because $r(\theta = 90°) = p$, it geometrically is the distance from the focal point to the intersection of the line normal to the line of apsides with the trajectory. At this intersection point the body also achieves its maximum radial velocity (see Fig. 7.7). For any Keplerian orbit, p is also the orbit's curvature radius at periapsis (Exercise).

Having achieved geometrical interpretations of a and e, we see that the vis-viva Eq. (7.2.15)

$$v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right)$$

and Eq. (7.2.7) with (7.3.7)

$$h = \omega\, r^2 = \sqrt{\mu a(1 - e^2)} = \sqrt{\mu p}$$

are important, simple, and useful equations that link physical state properties, namely h, r, v, ω, to geometric properties of the orbit, a, e, p.

The Periapsis
Common to all Keplerian orbits is that they have a point of closest approach to the focal point. This point of closest approach is called the *periapsis*. The time of the passage of the body through the periapsis is defined as t_p. Sometimes t_p is called the (astronomical) epoch. This epoch is to be distinguished from the notion "standard epoch J2000" (= January 1, 2000, 11:58:55.816 UTC; see Sect. 13.2). So, the general meaning of "epoch" is "a reference point in time".

Significance of Eccentricity Vector
We read from Eq. (7.3.5) that r becomes minimal when $\cos\theta = \hat{\boldsymbol{e}}\hat{\boldsymbol{r}} = 1$, i.e., if $\boldsymbol{r}$ points along $\boldsymbol{e}$. Thus, the eccentricity vector $\boldsymbol{e}$ points from the focal point (central body) to the periapsis (see Fig. 7.10). Its absolute value is the orbit eccentricity e, which describes the elongation of the orbit. Because both the focal point and the periapsis lie on the line of apsides, the eccentricity vector defines the line of apsides.

Because $\hat{\boldsymbol{e}}\hat{\boldsymbol{r}} = \cos\theta$, θ measures the angle between the radial vector to the periapsis and to the current position of the body and therefore $\theta(t_p) = 0$. Hence, we can determine the exact position on the orbit with θ. This important parameter $\boldsymbol{\theta}$ is called the **true anomaly** (a.k.a. *orbit angle*). So, the orbit equation provides the orbital radius r as a function of the true anomaly θ (Figs. 7.8 and 7.10).

> **Remark** *The weird term "anomaly" for θ and later in this book for the angles M, E, F, and G dates back to the Ptolemaic astronomical system. In ancient times, any angle that could not be traced back to a true circular motion, appeared to be wrong or "anomalous".*

In summary, we figured out that

> The eccentricity vector $\boldsymbol{e}$ is the base vector relative to which the position on the orbit, the true anomaly θ, is measured: $\hat{\boldsymbol{e}}\hat{\boldsymbol{r}} = \cos\theta$. It points from the orbit's focal point (central body) to the smallest approach distance, the periapsis (see Figs. 7.8b and 7.10). Its absolute value describes the elongation of the orbit.

7.3.2 Position on the Orbit

How did we succeed in solving the apparently difficult vectorial equation of motion so swiftly? We made use of our previous knowledge that the momentum is a constant of motion and of its relation to $\boldsymbol{r}$ and $\boldsymbol{v}$, namely $\boldsymbol{h} = \boldsymbol{r} \times \boldsymbol{v}$. Therefore, we just had to integrate only once to find, besides $\boldsymbol{h}$, the second integral of motion $\boldsymbol{e}$. However, note that with Eq. (7.3.5), we actually did not achieve our goal to find $\boldsymbol{r}(t)$, we rather only found $r(\theta)$. So we still have to derive $\theta(t)$, a problem we will address now.

Because the true anomaly $\theta(t)$ evolves with time, there must exist a differential equation describing this evolution. We have found this already, it is the equation of lateral motion (7.2.12). In conjunction with the orbit Eq. (7.3.5) and $h = \sqrt{p\mu}$ it reads

$$\dot{\theta} = \frac{h}{r^2} = \sqrt{\frac{\mu}{p^3}}(1 + e \cdot \cos\theta)^2 \quad \textbf{equation of lateral motion} \tag{7.3.8}$$

Separating the variables and integrating on the right-hand side with respect to time results in

$$\int_0^{\theta} \frac{d\theta'}{(1 + e \cdot \cos\theta')^2} = \sqrt{\frac{\mu}{p^3}}(t - t_p) \tag{7.3.9}$$

Keplerian Problem

Equation (7.3.9) is to be read as follows: Given t_p, e, $p = a(1 - e^2)$. To find the true anomaly at any time t, the integral on the left hand side needs to be solved. Except for a circular and a parabolic orbit, this is too complicated to do by regular means. Even if we were able to solve, the solutions are very complicated (see Eqs. (7.4.17) and (7.4.33)). The crucial problem, however, is that these solutions display the time dependency not explicitly, that is, $\theta = \theta(t)$, but only implicitly, that is, $t = t(\theta)$. One could, though, for any given point in time solve $t = t(\theta)$ for θ numerically. But this is quite an effort. In the face of this problem, Kepler at the beginning of the seventeenth century proposed a method, which shifts the problem analytically to a simpler one, which can be solved with less effort, though still numerically. We will follow Kepler's elegant method in the next chapter. In fact, as we will see later, his method provides an easy way to solve Eq. (7.3.9) analytically also for elliptic and hyperbolic orbits.

The so-called "Keplerian problem" is historically the problem of finding the orbit position at a given time, if it was known at an earlier time. The background is the astronomical problem, even today, to find a celestial body back if it was observed at earlier times and to determine its orbital elements. Since the orbital path is constant in time, Kepler's problem lies in the difficulty of determining $\theta = \theta(t)$.

In Sect. 7.4 we will examine the specific properties of each type of orbit and the solutions to the Keplerian problem, separately.

Mean Lateral Motion

The lateral motion $\omega = \dot{\theta}(t)$ may be quite uneven. We can facilitate further calculations a lot by introducing an artificial constant lateral motion, the so-called *mean motion n* (a.k.a. *orbital frequency*). We define it as the average lateral motion $\langle\omega\rangle_T$, which of course is one orbit 2π over the orbital period T . Therefore,

$$n := \langle\omega\rangle_T = \frac{2\pi}{T} = \begin{cases} \sqrt{\mu/a^3} & @ \text{ elliptic orbits} \\ \sqrt{-\mu/a^3} & @ \text{ hyperbolic orbits} \end{cases} \quad \textbf{mean motion} \tag{7.3.10}$$

where the latter follows from Eqs. (7.4.12) and (7.4.31). Observe that while $\omega = d\theta/dt$ is the (instantaneous) angular velocity the mean motion n is formally a frequency. Only for circular orbits $n \equiv \omega$. Though the definition of n as based on a closed periodic orbit, it can be extended also to hyperbolic orbits by definition $n = \sqrt{\mu/(-a^3)}$ (see Sect. 7.4.3).

This idealized motion gives rise to the further definition to the *mean anomaly*

$$\boxed{M := n \cdot (t - t_p)} \qquad \textbf{mean anomaly} \tag{7.3.11}$$

Because both n and M can be applied to elliptic and hyperbolic orbits they are highly useful. The particular usefullness of M is due to the fact that it serves two different purposes in one parameter. First, by definition M is an angle that by intention advances linearly with time. Just as for the true anomaly θ the mean anomaly M is measured relative to the periapsis. But, in contrast to θ, M is not cyclically limited to the interval [0°, 360°]. As an unambiguous angle M perfectly serves as an orbital element (see Sect. 7.3.5). Second, because M is strictly linear with time and in addition is dimensionless, it is a perfect substitution for time and hence a orbit position sequencer.

Because the mean anomaly M is an unambiguous angle, even for multiple revolutions, and in addition advances linearly in time, M is an ideal substitute for the true anomaly θ (cf. Sect. 7.3.5) *and* the time variable t. Therefore, it is used in celestial mechanics as a standard orbital element. In particular, because $dM \propto dt$ the mean anomaly M is regularly used as the integrating variable for time averages (see, e.g., Sect. 12.3).

For later purposes we derive from Eqs. (7.3.8) and (7.3.9) the following useful relation

$$\frac{dM}{d\theta} = \frac{(1-e^2)^{3/2}}{(1+e\cdot\cos\theta)^2} = \left(\frac{r}{a}\right)^2 \frac{1}{\sqrt{1-e^2}} \tag{7.3.12}$$

7.3.3 Orbital Velocity

We now want to determine the orbital velocity $\boldsymbol{v}$ at any point on the orbit (see Fig. 7.9). With the identity $\hat{\boldsymbol{h}} \times \left(\boldsymbol{v} \times \hat{\boldsymbol{h}}\right) = \boldsymbol{v}$ we derive from Eq. (7.3.2)

$$\boxed{\boldsymbol{v} = \frac{\mu}{h}\hat{\boldsymbol{h}} \times (\boldsymbol{e} + \hat{\boldsymbol{r}})} \qquad \textbf{orbital velocity} \tag{7.3.13}$$

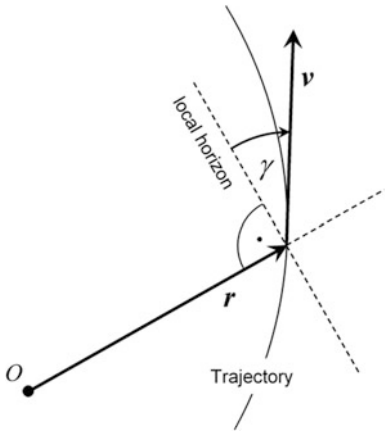


Fig. 7.9 Flight path angle γ measured relative to the local horizon in the co-moving reference system

For its absolute value we find

$$v = \frac{\mu}{h}\sqrt{\left[\hat{\boldsymbol{h}} \times (\boldsymbol{e} + \hat{\boldsymbol{r}})\right]\left[\hat{\boldsymbol{h}} \times (\boldsymbol{e} + \hat{\boldsymbol{r}})\right]}$$
$$= \frac{\mu}{h}\sqrt{\hat{\boldsymbol{h}}^2(\boldsymbol{e} + \hat{\boldsymbol{r}})^2 - \left[\hat{\boldsymbol{h}}(\boldsymbol{e} + \hat{\boldsymbol{r}})\right]^2}$$

Because $\boldsymbol{h}$ is orthogonal on both $\boldsymbol{e}$ and $\boldsymbol{r}$, the second term in the radicand vanishes and we have with $\boldsymbol{e}\hat{\boldsymbol{r}} = e\cos\theta$

$$v = \frac{\mu}{h}|\boldsymbol{e} + \hat{\boldsymbol{r}}| = \frac{\mu}{h}\sqrt{1 + 2e\cos\theta + e^2} = \sqrt{\mu\left(\frac{2}{r} - \frac{1}{a}\right)} \tag{7.3.14}$$

where the latter follows from Eq. (7.2.15). For the radial part of the velocity we find

$$\dot{r} = \boldsymbol{v}\hat{\boldsymbol{r}} = \frac{\mu}{h}\left[\hat{\boldsymbol{h}} \times (\boldsymbol{e} + \hat{\boldsymbol{r}})\right]\hat{\boldsymbol{r}} = \frac{\mu}{h}\hat{\boldsymbol{h}}[(\boldsymbol{e} + \hat{\boldsymbol{r}}) \times \hat{\boldsymbol{r}}] = \frac{\mu}{h}\hat{\boldsymbol{h}}(\boldsymbol{e} \times \hat{\boldsymbol{r}})$$

and because $\boldsymbol{e} \times \hat{\boldsymbol{r}} = e\sin\theta \cdot \hat{\boldsymbol{h}}$ we finally have

$$\dot{r} = \frac{e\mu}{h}\sin\theta \tag{7.3.15a}$$

Vector representations of $\boldsymbol{v}$ in various reference frames are given in Sect. 13.1.5.

Flight Path Angle

We now define the much used flight path angle γ. This is the angle that the velocity vector $\boldsymbol{v}$ makes with the *local horizon*, the vertical on the radial vector. According to Fig. 6.9 we have

$$\dot{r} = \boldsymbol{v}\,\hat{\boldsymbol{r}} = v\,\sin\gamma \tag{7.3.15b}$$

$$r\dot{\theta} = \boldsymbol{v}(\boldsymbol{h}\times\hat{\boldsymbol{r}}) = v\,\cos\gamma \tag{7.3.15c}$$

and therefore with Eqs. (7.3.15a), (7.3.14) and (7.2.8)

$$\begin{aligned}\sin\gamma &= \frac{e\,\sin\theta}{\sqrt{1+2e\,\cos\theta+e^2}} = \sqrt{1-\frac{\mu p}{r^2v^2}} = \sqrt{1-\frac{p}{r(2-r/a)}}\\ \cos\gamma &= \frac{1+e\,\cos\theta}{\sqrt{1+2e\,\cos\theta+e^2}} = \frac{\sqrt{\mu p}}{rv} = \sqrt{\frac{p}{r(2-r/a)}}\end{aligned} \tag{7.3.16}$$

where we have also made use of relation $r\omega = \sqrt{\mu p}/r = h/r$.

We thus can express the orbital velocity in the orbit plane also as (cf. Eq. (13.1.10))

$$\boldsymbol{v} = \frac{\mu}{h}\begin{pmatrix} e\,\sin\theta \\ 1+e\,\cos\theta\end{pmatrix}_{RSW} = v\begin{pmatrix}\sin\gamma\\ \cos\gamma\end{pmatrix}_{RSW} \tag{7.3.17}$$

7.3.4 Orbital Energy

We intuitively know that the orbital energy must somehow depend on its shape or size or on both. To derive the relationship, we square Eq. (7.3.2) on both sides

$$\mu^2(\boldsymbol{e}+\hat{\boldsymbol{r}})^2 = \mu^2\left(e^2+2\boldsymbol{e}\hat{\boldsymbol{r}}+1\right) \;=\; (\boldsymbol{h}\times\boldsymbol{v})^2 = h^2\cdot v^2$$

The latter holds because of $\boldsymbol{h}\perp\boldsymbol{v}$. From Eq. (7.3.4) it follows that $\boldsymbol{e}\hat{\boldsymbol{r}} = h^2/\mu r - 1$. Equation (7.2.14) states $v^2 = 2\mu/r + 2\varepsilon$. This applied to the above equation leads to

$$2h^2\varepsilon = \mu^2\left(e^2-1\right) \tag{7.3.18}$$

From this we derive:

> The specific orbital energy ε is negative for ellipses with $e<1$, zero for parabolas with $e=1$, or positive for hyperbolas with $e>1$.

With Eqs. (7.3.7) and (7.3.18) can be transformed into the simple expression for the specific orbital energy

$$\boxed{\varepsilon = -\frac{\mu}{2a}} \tag{7.3.19}$$

Equation (7.3.19) proves that the parameter a (semi-major axis) defined in Eq. (7.3.6) is identical to the one in the vis-viva Eqs. (7.2.11) and (7.2.15), and thus the geometrical interpretation of the semi-major axis formerly announced is correct.

Equation (7.3.19) not only is remarkably simple, but is also remarkable because as the vis-viva Eq. (7.2.15), it directly relates an important physical quantity, here the "total orbital energy", to the geometrical size of the orbit, the semi-major axis. Thereby,

> The orbital energy can be directly read from the orbital size: The larger the orbit, the larger (negative sign!) the orbital energy.

This interrelation can be easily viewed from Fig. 7.7.

Does also the angular momentum h directly relate to a geometrical property of the orbit? Absolutely, Eq. (7.3.6) is the one. It relates h to the semi-latus rectum p: $h^2 = \mu p$. However, p is not frequently used to characterize an orbit, except for a parabola. Rather a and e are used more often because they are orbital elements (see Sect. 7.3.5). In fact, and according to Eq. (7.3.18), the eccentricity can be employed to gauge the angular momentum at a given orbital energy: If $e = 1$ (radial orbit, see Sect. 7.5), then the angular momentum vanishes, and for $e = 0$ (circular orbit) the angular momentum becomes maximum, $h = \mu\sqrt{a}$. At a given orbital energy, the circular orbit is hence that bounded orbit with the biggest angular momentum.

Virial Theorem

A body on its trajectory continuously changes position and velocity, and along with this its potential and kinetic energy. Since the orbital energy is constant, kinetic energy is transformed into potential energy and the other way round, i.e., $\dot{\varepsilon}_{kin} \propto -\dot{\varepsilon}_{pot}$. Only on a circular orbit, ε_{kin} and ε_{pot} remain constant according to (see Eq. (7.4.5))

$$\varepsilon_{pot} = -2\varepsilon_{kin} = const \quad @ \text{ circular orbit}$$

However, it is possible to show that for any periodic orbit in a gravitationally bound many-body system if $\langle\varepsilon_{kin}\rangle$ and $\langle\varepsilon_{pot}\rangle$ are the kinetic and potential energies time-averaged over an orbital period, then the following holds

$$\langle\varepsilon_{pot}\rangle = -2\langle\varepsilon_{kin}\rangle = const \quad \textbf{virial theorem} \tag{7.3.20}$$

7.3.5 *Orbital Elements*

An orbital element is any quantity that specifies an orbit under consideration. We have already derived orbital elements that remain constant even under orbital motion: angular momentum $\boldsymbol{h}$, eccentricity vector $\boldsymbol{e}$, and orbital energy ε. Constant orbital elements are also called *integrals of motion* (a.k.a. *invariants of motion*) and, evidently, they are of particular interest to characterize an orbit. Although the vectors $\boldsymbol{h}$, $\boldsymbol{e}$ are of high analytical relevance, we need to embed them into an inertial frame of reference, in which they constitute descriptive values.

Invariant Angular Elements

The classical way of doing this is using the geocentric equatorial coordinate system *IJK* (see Sect. 13.1.4), in which the Cartesian coordinate system $(\boldsymbol{u}_I,\ \boldsymbol{u}_J,\ \boldsymbol{u}_K)$ has the following orientation (see Fig. 13.2): The unit vector $\boldsymbol{u}_I$ points to the vernal point, $\boldsymbol{u}_K$ to Earth's north pole, and $\boldsymbol{u}_J = \boldsymbol{u}_K \times \boldsymbol{u}_I$. With this we establish the unit vector

$$\boldsymbol{u}_n = \frac{\boldsymbol{u}_K \times \hat{\boldsymbol{h}}}{\left|\boldsymbol{u}_K \times \hat{\boldsymbol{h}}\right|} \quad \text{unit vector to the ascending node}$$

These unit vectors determine the following angular orbital elements (see Fig. 7.10)

$$i = \arccos\left(\boldsymbol{u}_K \hat{\boldsymbol{h}}\right) \qquad 0 \le i < 180^\circ \qquad (7.3.21)$$

$$\Omega = \begin{cases} \arccos(\boldsymbol{u}_I \boldsymbol{u}_n) & @\ \ \boldsymbol{u}_J \boldsymbol{u}_n \ge 0 \\ 2\pi - \arccos(\boldsymbol{u}_I \boldsymbol{u}_n) & @\ \ \boldsymbol{u}_J \boldsymbol{u}_n < 0 \end{cases} \qquad 0 \le \Omega < 360^\circ \qquad (7.3.22)$$

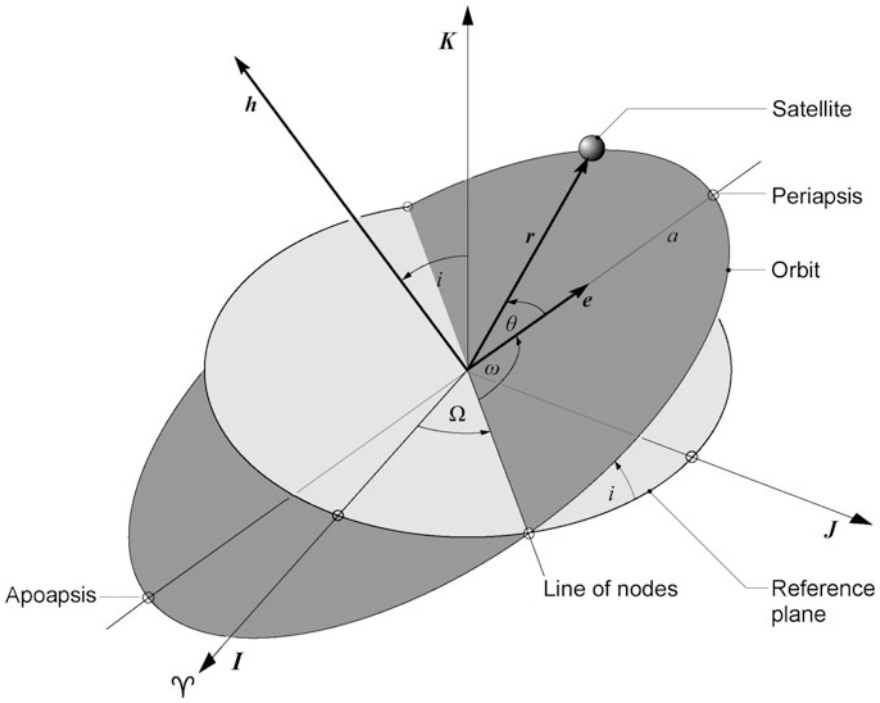


Fig. 7.10 The graphical representation of orbital elements of an elliptic satellite orbit in a geocentric equatorial coordinate system *IJK* (see Sect. 13.1.3), where the plane spanned by *I* and *J* with the *I*-axis oriented toward the vernal point ϒ is the reference plane. The orbital plane and reference plane intersect in the so-called line of nodes. The line of nodes in turn intersects the orbit in the so-called ascending node (where the satellite moves toward the upper side with respect to $\boldsymbol{h}$ of the reference plane) and in the descending node (vice versa)

$$\omega = \begin{cases} \arccos(\boldsymbol{u}_n \hat{\boldsymbol{e}}) & @ \ \boldsymbol{u}_K \hat{\boldsymbol{e}} \geq 0 \\ 2\pi - \arccos(\boldsymbol{u}_n \hat{\boldsymbol{e}}) & @ \ \boldsymbol{u}_K \hat{\boldsymbol{e}} < 0 \end{cases} \qquad 0 \leq \omega < 360^\circ \tag{7.3.23}$$

The following classification of orbits according to their inclination is common:

$i = 0^\circ$	*Equatorial Orbit*. Prominent examples are the geostationary orbits.
$0^\circ < i < 90^\circ$	*Prograde Orbit*. This orbit is used by most satellites in Low Earth Orbits.
$i \approx 90^\circ$	*Polar Orbit*. Orbits used for sun-synchronous Earth observation satellites.
$i > 90^\circ$	*Retrograde Orbit*. Very few satellites have been put into this type of orbit.

Invariant Orbit Vectors

In terms of these angular elements, the eccentricity vector can be represented in the *IJK* system as follows (see also Sect. 13.1.5)

$$\boldsymbol{e} = e \begin{pmatrix} \cos\Omega \cos\omega - \sin\Omega \sin\omega \cos i \\ \sin\Omega \cos\omega + \cos\Omega \sin\omega \cos i \\ \sin\omega \sin i \end{pmatrix}_{IJK} \quad \textbf{eccentricity vector} \tag{7.3.24}$$

Equivalently, the angular momentum unity vector $\hat{\boldsymbol{h}}$, also known as the *three-dimensional inclination vector* (a.k.a. *orbit pole vector*), given as (see Eq. (13.1.5)) as

$$\boldsymbol{I} \equiv \hat{\boldsymbol{h}} = \begin{pmatrix} \sin\Omega \sin i \\ -\cos\Omega \sin i \\ \cos i \end{pmatrix}_{IJK} \quad \textbf{three-dimensional inclination vector} \tag{7.3.25}$$

For instance, in Fig. 12.28 this 3D-inclination vector depicts the evolution of i, Ω under lunisolar perturbations. Its projection onto the *IJ* reference plane is known as

$$\boldsymbol{i} = \sin i \begin{pmatrix} \sin\Omega \\ -\cos\Omega \end{pmatrix}_{IJ} \quad \text{(two-dimensional) } \textbf{inclination vector} \tag{7.3.26}$$

Note that for $i \to 0$ conveniently

$$\boldsymbol{i} = i \begin{pmatrix} \sin\Omega \\ -\cos\Omega \end{pmatrix}_{IJ} \quad @ \ i \to 0$$

We now rotate the inclination vector in the *IJ* reference plane by 90° counter-clockwise into the direction of the ascending node and thus get the *ascending node vector*

$$\boldsymbol{n} = \sin i \begin{pmatrix} \cos\Omega \\ \sin\Omega \end{pmatrix}_{IJ} = \sin i \begin{pmatrix} \cos\Omega \\ \sin\Omega \end{pmatrix}_{EQ} \qquad \textbf{ascending node vector} \qquad (7.3.27)$$

All the above invariant quantities are orbital elements. This gives rise to the question of whether there is a minimum number of orbital elements that uniquely specify an orbit and how many of them are constant.

State Vector

We investigate the above question by going back to the basics of orbital motion. Depending on the initial conditions, orbital motion takes place in 6-dimensional phase space $\boldsymbol{r}$, $\boldsymbol{v}$, a space in which all possible states of a system are represented, with each possible state corresponding to one unique point in the phase space. Once an initial state $(\boldsymbol{r}_0, \boldsymbol{v}_0)$ is given, the two first-order differential vector equations

$$\dot{\boldsymbol{r}} = \boldsymbol{v}, \quad \dot{\boldsymbol{v}} = -\frac{\mu}{r^3}\boldsymbol{r}$$

derived from Newton's gravitation equation of motion (7.1.19) determine the future motion in phase space. Obviously, phase space has 6 dimensions and therefore the 6 elements of an initial state vector are the minimum number of orbital elements plus the independent parameter t. This is the significance of the state vector $(\boldsymbol{r}, \boldsymbol{v})$ as introduced in Sect. 7.2.2. It also has the advantage that it is the direct outcome of ranging and range rate measurements as part of orbit determination procedures (see Sect. 14.2.1). On the other hand it has the major drawback that $\boldsymbol{r}$ and $\boldsymbol{v}$ continuously change with time.

Number of Constant Orbital Elements

So, what is the maximum number of constant orbital elements? We have seen in Sect. 7.1.3 that linear momentum, angular momentum, and total energy always need to be conserved. Because the motion of a 2-body system in a central force field, such as the gravitational field and as shown in Sect. 7.2.2, always takes place in just 2 dimensions, we expect that any orbital motion exhibits $2+2+1=5$ constant (integrals) of motion and hence 5 constant orbital elements. This leaves one degree of freedom to the orbital motion, which is the 1-dimensional line of orbit. We therefore expect one additional orbital element to determine the moving position of the body on its orbit. Hence, in total there must be 5 constant orbital elements and 1 time-dependent orbital element. Obviously $\boldsymbol{h}$, $\boldsymbol{e}$ and $\theta(t)$ lend themselves as such orbital elements. Although they seem to make up $3+3+1=7$ elements, one has to consider the constraint $\boldsymbol{h} \cdot \boldsymbol{e} = 0$, and therefore the set $(\boldsymbol{h}, \boldsymbol{e}, \theta(t))$ actually represents the set of 6 independent elements and hence a minimum complete set of orbital elements.

We have seen that we may choose many different sets of 5 constant orbital elements, and any set is equivalent to any other. Next we to explore some common sets of orbital elements.

Common Sets of Orbital Elements

The most vivid orbital elements are the angular orbital elements i, Ω, ω (see Fig. 7.10) as given in Eq. (7.3.21), (7.3.22), and (7.3.23). In addition and from Eq. (7.3.7) we have from $\boldsymbol{h}$, $\boldsymbol{e}$

$$e = |\boldsymbol{e}|$$
$$a = \frac{\boldsymbol{h}^2}{\mu(1 - e^2)}$$

We thus obtain the highly important and so-called six *classical orbital elements* (see Fig. 7.10)

Classical Orbital Elements $\left(a, e, i, \Omega, \omega \,/\, t - t_p \leftrightarrow M\right)$

Two elements, so-called *metric elements*, describe the dimension of the orbit:

- $\boldsymbol{a}$ **Semi-major axis** axis, $-\infty < a < \infty$.
  It specifies the size of the orbit. Hyperbolic orbits have negative values.
- $\boldsymbol{e}$ **Eccentricity**, $0 \leq e < \infty$.
  It specifies the shape and jointly with a the type of the orbit.

The next three elements are called *angular elements*. Two elements describe the orientation of the orbital plane:

- $\boldsymbol{i}$ **Inclination**, $0 \leq i < 180°$
  Angle between the angular momentum vector h and the z-direction of the reference frame = angle between the corresponding planes perpendicular to these vectors.
- Ω **Right ascension of ascending node** (abbreviated: RAAN), $0 \leq \Omega < 360°$
  Longitude of the ascending node = angle between the line from the origin O of the reference frame to the vernal point and from O to the ascending node.

One element *determines* the orientation of the orbit in the orbital plane:

- $\boldsymbol{\omega}$ **Argument of periapsis**, $0 \leq \omega < 360°$
  Angle in the orbital plane between line of nodes and the periapsis measured in the direction of the motion.

The sixth parameter is the orbital state sequencer, it determines the location on the orbit. The following two sequencers can be used alternatively:

- $\boldsymbol{t - t_p}$ **Time after periapsis passage**
  $\boldsymbol{t_p}$ is the time, when the satellite was at periapsis (a.k.a. *epoch*).

or

M **Mean anomaly**, $M > 0$
$M := n \cdot (t - t_p)$, dimensionless (see Sect. 7.3.2).

Although M formally is an angle (however not having an geometrical interpretation) with dimension [rad], it commonly is treated as dimensionless and hence in modern orbital mechanics is a more convenient variant of time. Both sequencers have the disadvantage that their relationship to r and θ (see Sect. 7.4.2) is transcendental.

There is an alternative and more traditional set of orbital elements, called *Keplerian elements*

Keplerian Elements $(a, e, i, \Omega, \omega \;/\; \theta \leftrightarrow E, F, G)$
The Keplerian elements share the metric and angular classical orbital elements. The following two sequencers can be used alternatively:

θ **True anomaly** (a.k.a. *orbit angle*), $0 \le \theta < 360°$
Angle between the direction to the periapsis and the current position vector $\boldsymbol{r}$.

or

E **Eccentric anomaly** for elliptic orbits (see Sect. 7.4.2), $0 \le E < 2\pi$
F **Hyperbolic anomaly** for hyperbolic orbits (see Sect. 7.4.3), $0 \le F < \infty$
G **Universal anomaly** for all orbits (see Sect. 7.4.6), $0 \le G < \infty$

The advantage of this set is that the determination of $r(\theta)$, or equivalently $r(E, F, G)$, is algebraic and hence straightforward. The disadvantage is that θ is not a convenient orbital state sequencer, but time t. The inclusion of θ, which Leonhard Euler called "angle of elements", into the set of elements dates back to Johann Bernoulli.

Sometimes the compound angle

$$u = \omega + \theta \qquad \textbf{argument of latitude}$$

i.e., the position of the orbit measured relative to the line of nodes, is a more valuable parameter for orbit analysis. Note that, though $\boldsymbol{h}$, $\boldsymbol{e}$, $\theta(t)$ usually are defined in the geocentric equatorial coordinate system, of course any inertial reference system will do and only the angular elements will change by a different choice of reference system.

Degenerate Orbits
There exist two particular cases where some of the angular elements become undefined:

1. ω and θ (or M) will become undefined for $e = 0$.
2. Ω and ω will become undefined for $i = 0$.

Both happens, for example, for satellites in geostationary orbit. These degenerate cases, however, pose no problem.

In the first case, $e = 0$, where ω becomes irrelevant the argument of latitude $u = \theta$ is used as a substitute to determine the orbital position, i.e. the true anomaly is measured relative the line of nodes.

In the second case, the compound angles

$\varpi = \Omega + \omega$	**longitude of periapsis**	@ $i = 0$
$L = \Omega + \omega + \theta = \varpi + \theta = \Omega + u$	**orbital longitude**	@ $i, e = 0$
$l = \Omega + \omega + M = \varpi + M$	**mean longitude**	@ $i, e = 0$

measured relative to the vernal point Υ are used as substitutes for ω and θ or M (Ω is irrelevant for $i = 0$). The line of nodes and the vernal point then become the respective common reference directions in the orbital plane. Note that Ω is measured in the reference plane, while ω and θ are measured in the orbital plane. So, for $i > 0$, the compound angles ϖ, L, and l would be "doglegged" and no longer be true angles, but nevertheless they can be defined as mathematical quantities and treated like that. These doglegged compound angles are frequently used in the literature, but will not be used in this book.

Equinoctial Elements and Coordinate System

One problem of orbit degeneracy with angular elements arises because Ω and ω are measured relative to the ascending line of nodes, which vanishes for $i = 0$. This problem can be removed by rotating the *IJ* reference plane along the line of nodes by the inclination angle into the orbital plane and renaming the axes $K \rightarrow W$, $I \rightarrow E$, $J \rightarrow Q$. This reference frame is called *equinoctial coordinate system EQW* (see Fig. 7.11). By construction, for $i \rightarrow 0$ it crosses over into the *IJK*

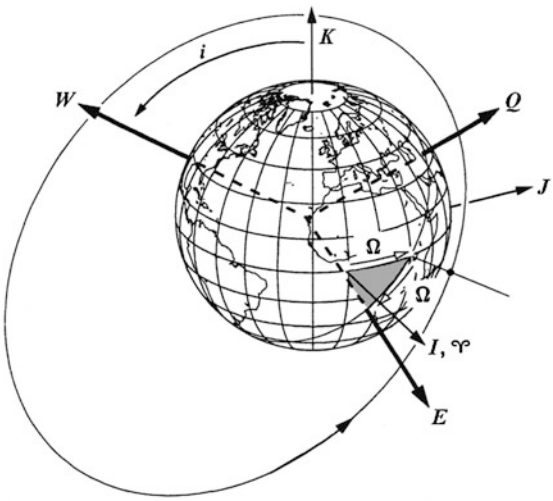


Fig. 7.11 The equinoctial coordinate system EQW. This system is formed from the IJK system by rotating IJK by the inclination angle about the line of nodes. *Credit* Vallado (2007)

coordinate system. The other orbit degeneracy problem, namely that ω and θ (or M) will become undefined for $e = 0$, can be removed by making the eccentricity vector part of the orbital elements. Both strategies are adopted to form a nonsingular set of orbital elements, the so-called **standard equinoctial orbital elements**, which read as follows:

(Standard) Equinoctial Elements (a, h, k, p, q, l)

$\boldsymbol{a}$ Semi-major axis, $-\infty < a < \infty$

Two components of the eccentricity vector:

$\boldsymbol{h} = e \sin(\omega + \Omega)$
$\boldsymbol{k} = e \cos(\omega + \Omega)$

Two components of the rescaled ascending node vector:

$\boldsymbol{p} = \tan\frac{i}{2} \sin\Omega$
$\boldsymbol{q} = \tan\frac{i}{2} \cos\Omega$

The sixth parameter again determines the location of the periapsis in time:

$\boldsymbol{l} = \Omega + \omega + M(t)$, mean longitude, $0 \leq l < 360^\circ$

Note that the RAAN angle Ω now lies in the orbital plane and therefore the mean longitude $l = \Omega + \omega + M$ is no longer doglegged, but a true angle in the orbital plane. Since the equinoctial elements exhibit no singularities they are frequently employed in perturbational equations for equatorial orbits. For more details of equinoctial elements see Battin (1987), and for coordinate transformations to and from the equinoctial coordinate system see Vallado (2007).

In terms of the invariant orbit vectors we can identify

$$\boldsymbol{e} = \begin{pmatrix} h \\ k \end{pmatrix}_{EQ} = e \begin{pmatrix} \sin(\omega + \Omega) \\ \cos(\omega + \Omega) \end{pmatrix}_{EQ} \tag{7.3.28}$$

and the vector (q, p) actually is a rescaled ascending node vector

$$(q, p) = \frac{1}{1 + \cos i} \boldsymbol{n} = \tan\frac{i}{2} \cdot \begin{pmatrix} \cos\Omega \\ \sin\Omega \end{pmatrix}_{EQ}$$

The reason for scaling by $1/(1 + \cos i)$ lies in the better adaptation to Lagrange's planetary equations where the equinoctial elements are frequently used for GEO orbits.

The introduction of the singular free parameters (h, k) and (q, p) is most probably due to Joseph-Louis Lagrange. Today, there are some variants of the standard equinoctial orbital elements, most importantly the so-called **modified equinoctial orbital elements** (Walker et al. 1985) with the only modification $a \to p = a(1 - e^2)$. But also the replacements $\tan\frac{i}{2} \to \tan i$, $\sin i$, or $\sin\frac{i}{2}$ are quite common.

Conversion: State Vector → Orbital Elements
A frequent problem encountered in astrodynamics, in particular with orbit determination, is the conversion of the state vector $(\boldsymbol{r}, \boldsymbol{v})$ into (osculating) Keplerian elements and vice versa. In the following, we will have a look at this problem.

First, we want to convert a state vector $(\boldsymbol{r}, \boldsymbol{v})$ to Keplerian elements. To start with, we compute

$$\boldsymbol{h} = \boldsymbol{r} \times \boldsymbol{v}$$
$$\hat{\boldsymbol{h}} = \boldsymbol{h}/h, \quad \hat{\boldsymbol{r}} = \boldsymbol{r}/r, \quad \hat{\boldsymbol{v}} = \boldsymbol{v}/v$$

The semi-major axis then is determined from the vis-viva Eq. (7.2.14) as

$$a = \frac{\mu r}{2\mu - rv^2} \tag{7.3.29}$$

Observe that the semi-major axis and thus the orbital period $T = 2\pi\sqrt{a^3/\mu}$ can be determined merely from the absolute values of the position vector and orbital velocity. For the eccentricity we have from Eq. (7.3.3) and with Eq. (7.3.6)

$$e = |\boldsymbol{e}| = \sqrt{1 - \frac{h^2}{\mu a}}$$
$$\hat{\boldsymbol{e}} = \boldsymbol{e}/e$$

$$\theta = \begin{cases} \arccos(\hat{\boldsymbol{r}}\hat{\boldsymbol{e}}) @ \ \hat{\boldsymbol{r}}\hat{\boldsymbol{v}} \geq 0 \\ 2\pi - \arccos(\hat{\boldsymbol{r}}\hat{\boldsymbol{e}}) @ \ \hat{\boldsymbol{r}}\hat{\boldsymbol{v}} < 0 \end{cases} \tag{7.3.30}$$

The angular orbital elements i, Ω, ω are given in the geocentric equatorial reference system IJK by Eqs. (7.3.21), (7.3.22), and (7.3.23). To determine the mean anomaly M, we derive from Eq. (7.4.13a)

$$\begin{aligned} E &= \arccos[(1 - r/a)/e] \quad @ \ |\hat{\boldsymbol{r}} \times \hat{\boldsymbol{e}}| > 0 \\ E &= 2\pi - \arccos[(1 - r/a)/e] \quad @ \ |\hat{\boldsymbol{r}} \times \hat{\boldsymbol{e}}| < 0 \end{aligned} \tag{7.3.31}$$

Hence from the Kepler's Eq. (7.4.15) we derive

$$M = E - e \sin E$$

and from Eq. (7.4.14d) we have alternatively

$$\theta = 2 \arctan\left(\sqrt{\frac{1+e}{1-e}} \tan\frac{E}{2}\right)$$

Conversion: Orbital Elements → State Vector
Let the orbital elements $a, e, i, \omega, \Omega, \theta$ or $a, e, i, \omega, \Omega, M$ be given. The orbit radius then is determined as

$$r = \frac{a(1-e^2)}{1+e\,\cos\theta} = a(1-e\,\cos E)$$

where E is the solution of $M = E - e\,\sin E$ (see Newton's method in Sect. 7.4.2). As shown in Sect. 13.1.5, the radial vector and the orbital velocity vector in the *IJK* system (see Sect. 13.1.3) are then determined as

$$\boldsymbol{r} = r \begin{pmatrix} \cos\Omega\,\cos u - \sin\Omega\,\sin u\,\cos i \\ \sin\Omega\,\cos u + \cos\Omega\,\sin u\,\cos i \\ \sin u\,\sin i \end{pmatrix}_{IJK} \tag{7.3.32}$$

$$\boldsymbol{v} = \sqrt{\frac{\mu}{a(1-e^2)}} \begin{pmatrix} -\cos\Omega(\sin u + e\,\sin\omega) - \sin\Omega(\cos u + e\,\cos\omega)\cos i \\ -\sin\Omega(\sin u + e\,\sin\omega) + \cos\Omega(\cos u + e\,\cos\omega)\cos i \\ (\cos u + e\,\cos\omega)\,\sin i \end{pmatrix}_{IJK} \tag{7.3.33}$$

with $u = \omega + \theta$ the argument of latitude. For equatorial orbits Ω is undefined and can be set to zero. The same holds for ω in circular orbits, and hence $u = \theta$.

7.4 Keplerian Orbits

In this chapter we study the detailed properties of the different Keplerian orbits (a.k.a. *conic orbits*) and derive and present their basic results. This includes, in particular, the analytic solutions to $t = t(\theta)$ and the numeric algorithms to calculate $\theta = \theta(t)$ for elliptic and hyperbolic orbits.

7.4.1 Circular Orbit

The most common orbits of spacecraft around celestial bodies are circular or near-circular in shape, since they take on the highest minimum orbital altitude (implies least atmospheric drag) at a given orbital energy (cf. Fig. 7.7) and provide steady orbit conditions. According to Eq. (7.3.18) they also feature maximum angular momentum at a given orbital energy. For $e = 0$, Eq. (7.3.9) can be solved immediately and according to Eq. (7.3.5) we get

$$\begin{aligned} r &= a = p = const \\ \theta &= \sqrt{\frac{\mu}{a^3}} \cdot (t - t_p) = n(t - t_p) \end{aligned} \tag{7.4.1}$$

which is a circular orbit with radius a and with a negative specific orbital energy according to Eq. (7.3.19). From Eq. (7.4.1) we find $2\pi = T\sqrt{\mu/a^3}$ for a full revolution, and hence for the orbital period T (cf. Eq. (7.4.12))

$$\boxed{T = 2\pi\sqrt{\frac{a^3}{\mu}}} \quad \textbf{orbital period} \tag{7.4.2}$$

To determine the orbital velocity, we use vis-viva Eq. (7.2.15) from which with $r = a$ we directly derive

$$\boxed{v = \sqrt{\frac{\mu}{r}} = \sqrt{\frac{\mu}{a}} = const} \tag{7.4.3}$$

The orbital velocity decreases with the root of the orbital radius; for example, the velocity of a body that circles the Earth in the theoretically lowest orbit possible, $r = R_\oplus$, is with $\mu_\oplus = g_0 R_\oplus^2$ (see Eq. (7.1.20))

$$v_\triangleright = \sqrt{g_0 R_\oplus} = 7.905 \text{ km/s} \quad \textbf{first cosmic velocity} \tag{7.4.4}$$

This is the highest possible orbital circular velocity around the Earth, as according to Eq. (7.4.3), the circular velocity decreases with an increasing orbital altitude r. In a typical LEO (such as the ISS) of 400 km, it is only 7.67 km s^{-1}. Though speed decreases for higher orbits you still need more energy to reach higher orbits. This is because from Eq. (7.3.19) $\varepsilon = -\mu/2a = -\mu/2r$, i.e.,

$$2\varepsilon = \varepsilon_{pot} = -2\varepsilon_{kin} = const \tag{7.4.5}$$

The latter follows because $\varepsilon_{kin} = v^2/2 = \mu/2a$. In words, this means that the orbital energy of a circular orbit is negative and its absolute value equals that of the kinetic energy but, most important, that (cf. Eq. (7.3.20))

> At any point in the circular orbit, the absolute value of the potential energy is twice that of the kinetic energy.

Note *Here and in general the potential energy is not the positive energy measured relative to the surface of the celestial body being circled, but the negative energy relative to infinity* (cf. Eq. (7.1.3) *and* Fig. 7.7).

Therefore, the energy for lifting a body into a higher circular orbit is determined as follows:

$$\Delta\varepsilon = \Delta\varepsilon_{kin} + \Delta\varepsilon_{pot} = -\frac{1}{2}\Delta\varepsilon_{pot} + \Delta\varepsilon_{pot} = \frac{1}{2}\Delta\varepsilon_{pot} > 0$$

> Orbit lifting reduces the kinetic energy by a given amount, but it increases potential energy by double that amount. Therefore, although orbit lifting increases the orbital energy the orbital velocity decreases.

On the other hand, this leads to the paradoxical situation (a.k.a. *satellite orbit paradox*) that a S/C in a circular LEO, decelerated due to drag in effect gains velocity because it spirals down to lower altitudes.

7.4.2 Elliptic Orbit

For $0 < e < 1$, the orbit is an ellipse, which is the most general bounded orbit in a gravitational field (**Kepler's first law**). According to Eq. (7.3.19), its orbital energy is negative. According to Eq. (7.3.5) and with Eq. (7.3.6), there is a minimum and a maximum distance to the central body at the focal point (see Fig. 7.12):

$$r_{per} = a(1-e) \qquad \textbf{periapsis} \tag{7.4.6}$$

$$r_{apo} = a(1+e) \qquad \textbf{apoapsis} \tag{7.4.7}$$

The general terms periapsis and apoapsis are also called *pericenter* and *apocenter*. Depending on the central body, the specific terms are peri-/apogee (Earth), peri-/apohelion (Sun), peri-/aposelene (Moon), etc. The line through the periapsis and apoapsis is called line of apsides (a.k.a. *apse line*). From Eqs. (7.4.6) and (7.4.7), the semi-major axis a, the eccentricity e, and the semi-minor axis b are derived as

$$a = \frac{1}{2}\left(r_{apo} + r_{per}\right) \tag{7.4.8}$$

$$e = \frac{r_{apo} - r_{per}}{r_{apo} + r_{per}} \tag{7.4.9}$$

$$b = a\sqrt{1-e^2} = h\sqrt{\frac{a}{\mu}} = \langle r \rangle_\theta$$

Geometrically, the eccentricity is the distance Δ from the center of the ellipse to its focal point in units of a (see Fig. 7.10): $\Delta/a = \left(a - r_{per}\right)/a = ae/a = e$.

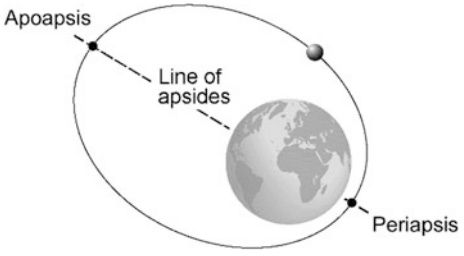


Fig. 7.12 Apoapsis, periapsis, and line of apsides of an elliptic orbit

According to Appendix A.1.2 the semi-minor axis b corresponds to the mean orbit radius $\langle r \rangle_\theta$ averaged over θ.

The velocities at the periapsis and apoapsis follow from Eq. (7.3.14)

$$v_{per} = \frac{\mu}{h}(1+e) = \sqrt{\frac{\mu}{a}}\sqrt{\frac{1+e}{1-e}} \tag{7.4.10}$$

$$v_{apo} = \frac{\mu}{h}(1-e) = \sqrt{\frac{\mu}{a}}\sqrt{\frac{1-e}{1+e}} \tag{7.4.11}$$

According to Kepler's second law, Eq. (7.2.9), the area ΔA swept by the orbit radius during the time Δt is given by $\Delta A = \Delta t \cdot h/2$. If one integrates over a full revolution, the swept area ΔA is the area of the ellipse, πab, and Δt is the orbital period T, and thus $T = 2\pi ab/h$. Because of Eq. (7.4.9), $b = h\sqrt{a/\mu}$, we get

$$\boxed{T = 2\pi\sqrt{\frac{a^3}{\mu}}} \quad \textbf{orbital period (Kepler's third law)} \tag{7.4.12}$$

Remark 1 *Actually, Kepler stated in his third law the relation of two orbit periods around the same central body: $T_1^2/T_2^2 = a_1^3/a_2^3$. This follows from the above equation.*

Remark 2 *It is quite remarkable that the orbital period, just as the specific orbital energy $\varepsilon = -\mu/2a$, does not depend on eccentricity, but only on the semi-major axis.*

Remark 3 *Kepler's third law can be used to determine the mass M of a celestial body: By precise determination of the orbital period and semi-major axis of an object around the celestial body the standard gravitational parameter $\mu = GM$ and hence mass M can be determined. For instance, by measuring the orbits of the stars around the black hole at the center of our Milky Way, its mass was thus determined.*

Given the orbital period we find for the mean motion (see Eq. (7.3.10))

$$n = \frac{2\pi}{T} = \sqrt{\frac{\mu}{a^3}} \tag{7.4.13}$$

Kepler Transformation

We now seek to tackle the Keplerian problem, as already discussed in Sect. 7.3.2, and solve Eq. (7.3.9). To do so, we apply Kepler's method, which transfers the problem to a new angle parameter, the so-called **(elliptic) eccentric anomaly** $\boldsymbol{E}$ (see Fig. 7.13).

Remark *You may be worried that a Roman letter rather than a Greek symbol symbolizes an angle. The answer to such "anomalies" is as always: for historic reason. Introduced by Kepler and getting used to it for centuries nobody dares to switch to modern standards.*

The transformation is performed geometrically by drawing a great circle around the ellipse with radius a and projecting the position vector onto the horizontal and the vertical axis. From analysis in Fig. 7.13 the segments $r\cos\theta = x = a\cos E - ae$ and $r\sin\theta = y = b\sin E = a\sqrt{1-e^2}\sin E$ we get

$$r\cos\theta = a\cos E - ae = a(\cos E - e)$$
$$r\sin\theta = a\sqrt{1-e^2}\sin E$$

Squaring and applying the equations to each other results in

$$\boxed{r = a(1 - e\cos E)} \qquad \textbf{orbit equation}$$

$$\begin{aligned}
\cos\theta &= \frac{\cos E - e}{1 - e\cos E}, & \cos E &= \frac{e + \cos\theta}{1 + e\cos\theta} \\
\sin\theta &= \frac{\sqrt{1-e^2}\sin E}{1 - e\cos E}, & \sin E &= \frac{\sqrt{1-e^2}\sin\theta}{1 + e\cos\theta} \\
\tan\frac{\theta}{2} &= \sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2}
\end{aligned} \tag{7.4.14}$$

The first equation is nothing else than the orbit equation (7.3.5), with the orbit angle substituted by the eccentric anomaly. It is exactly this simplification of the orbit equation that will make it possible to perform the final required integration, which we will see later.

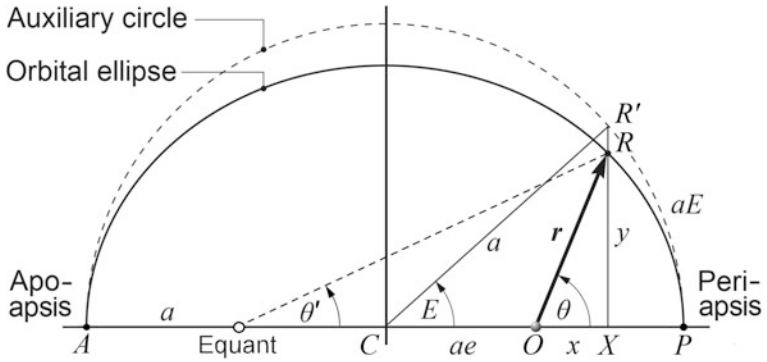


Fig. 7.13 Geometric interpretation of the eccentric anomaly

Note *The mean advancing orbit angle is not E as one would expect intuitively by geometric reasoning from* Fig. 7.13 *but* $M = n(t - t_p)$. *Because* $M = E(t) - e \sin E(t)$(see Eq. (7.4.15)), *M does not have a general simple geometric interpretation. Only if* $e \ll 1$, *M can be shown to be close to the orbit angle* θ' *as measured from the empty focal point* (see Fig. 7.13 *and Historic Remarks below*).

Kepler's Equation

With E as a practical orbit angle, we will now determine $E(t)$ to finally derive via Eq. (7.4.14b) the wanted $\theta(t)$. For this we determine with Eqs. (7.3.12) and (7.4.14)

$$\frac{dM}{dE} = \frac{dM}{d\theta}\frac{d\theta}{dE} = \frac{(1-e^2)^{3/2}}{(1+e\,\cos\theta)^2}\frac{1+e\,\cos\theta}{\sqrt{1-e^2}} = \frac{1-e^2}{1+e\,\cos\theta} = \frac{r}{a} = 1 - e\,\cos E$$

This equation can readily be integrated to deliver

$$\boxed{E(t) - e\,\sin E(t) = M = n\cdot(t - t_p)} \quad \textbf{Kepler's equation} \qquad (7.4.15)$$

with t_p is the time of passage through periapsis, where $E(t_p) = \theta(t_p) = 0$. This equation is written the other way round as in the literature to demonstrate that usually the time t or M is given (right-hand side) and $E(t)$ (left-hand side) needs to be determined from this, which obviously is not easy to achieve, because the relation is transcendental. Because according to Eq. (7.4.14b) E is directly linked to θ, we have in principle achieved our goal of determining the body on its orbit as a function of time.

While the orbit equation (7.3.5) determines the shape of an elliptic orbit, Kepler's equation determines the body's position in the orbit at a given time. Therefore both are key equations celestial mechanics and therefore enframed.

Customizing Kepler's Equation

Kepler's equation is of limited practical use, because it is tied to the time of passage through periapsis t_p, which usually is not known. Rather, if at any given time t_0 the state vector $\boldsymbol{r}_0$, $\boldsymbol{v}_0$ (or a set of orbital elements) is known, then from Eqs. (7.2.15), (7.3.3), and (7.4.14a) it follows that

$$a = \frac{\mu}{2\mu/r_0 - v_0^2}$$

$$\mu\boldsymbol{e} = \mu\left(\frac{1}{r_0} - \frac{1}{a}\right)\boldsymbol{r}_0 - (\boldsymbol{r}_0\boldsymbol{v}_0)\boldsymbol{v}_0 \;\rightarrow\; e = \sqrt{\boldsymbol{e}\boldsymbol{e}}$$

$$E_0 = \arccos\left[\frac{1}{e}\left(1 - \frac{r_0}{a}\right)\right]$$

We now extend the above integration over the equation of lateral motion to the intervall $[t_0, t]$

$$\sqrt{\frac{\mu}{a^3}} \int_{t_0}^{t} dt' = \int_{E_0}^{E} (1 - e \cdot \cos E') \cdot dE' = [(1 - e \cdot \sin E)]_{E_0}^{E}$$

which yields

$$E(t) - e \sin E(t) = \sqrt{\frac{\mu}{a^3}}(t - t_0) + E_0 - e \sin E_0 \tag{7.4.15b}$$

For a given t_0, a, e, E_0 this Kepler's equation can be solved for $E(t)$, which the state vector propagation method in Sect. 7.4.7 makes use of.

If in addition we define $\Delta t = t - t_0$, $\Delta E = E - E_0$, then Kepler's equation can be rewritten (without proof) as

$$\frac{r_0}{a} \sin \Delta E + \frac{r_0 v_0}{\sqrt{\mu a}} (1 - \cos \Delta E) + (\Delta E - \sin \Delta E) = \sqrt{\frac{\mu}{a^3}} \cdot \Delta t \tag{7.4.15c}$$

This a remarkably simple (no eccentricity!) dimensionless equation, on which the widely used *universal variable formulation* described in Sect. 7.4.8 is based on.

Regularization

Kepler's transformation serves two purposes. The first and most obvious is to transform Kepler's problem of integration (see Eq. (7.3.9)) to the simpler problem to finding the root of Kepler's Eq. (7.4.15). The second and often missed benefit is that it regularizes the description of motion. Let us see what regularization means.

As long as the body moves on an almost circular orbit, the motion is obviously quite regular. But if the eccentricity becomes large, there are times when the orbiting body comes close to the central body where it moves very fast and other times where it is far from the central body moving very slowly—the motion is irregular. To demonstrate this mathematically, we set up the time and angular variations of orbit radius. Equation (7.2.11) and differentiating the orbit Eq. (7.3.5) yields

$$\frac{dr}{dt} = \sqrt{\frac{2\mu}{r} - \frac{h^2}{r^2} - \frac{\mu}{a}}$$

$$\frac{dr}{d\theta} = \frac{e}{p} r^2 \sin \theta$$

So, if the eccentricity is high the orbit radius r may change by orders of magnitude and so does the time and angular variations. From a numerical point of view this is very undesirable, because the time steps Δt and angular steps $\Delta\theta$ would need to be adjusted correspondingly. If the steps are nevertheless fixed, then the precision of the determination of the orbit position varies between close approaches and remote motion.

Now let's see what Kepler's transformation does for this problem. From (7.4.14a) we derive

$$\frac{dr}{dE} = ea\,\sin\theta$$

Now the variation of the orbit radius with angle E is quite regular. This is not surprising, because if we position ourselves at the center of the orbit ellipse, radial changes look indeed much more regular than from the position at one of the focal points.

Kepler's transformation does not help to fully regularize angular velocities, though, because both

$$\frac{d\theta}{dt} = \frac{h}{r^2}$$
$$\frac{dE}{dt} = \sqrt{\frac{\mu}{a}}\frac{1}{r}$$

exhibit singularities, although for the eccentric angle velocity, dE/dt, it is less pronounced. This remaining irregular variation is embraced in Kepler's equation and it should be kept in mind that it causes numerical problems if $e \gg 0$. Only the universal solution in Sect. 7.4.6 will also regularize this problem.

So, overall we derive the following rule

> For numerical purposes the use of the eccentric anomaly E over the orbit angle θ as a variable is to be prefered. For high eccentricity orbits E is superior, for very high eccentricity orbits, mandatory.

Later in Sect. 7.4.6 we will see that the so-called universal anomaly G is the best choice to regularize any orbit with $h > 0$, in particular, high eccentricity orbits, and even orbits with $e \to 1$. Analytically, of course, it does not make any difference, except for $e = 1$.

Historic Remarks

Historically, Fig. 7.13 is of major relevance. Beginning with Apollonius (3rd century BC) the Greek astronomers assumed the so-called eccentric model, in which the planets, in particular the Moon and the Sun, move in circles (see auxiliary circles in Fig. 7.13) about Earth, located somewhat off the center of the circle at position O. However, starting with Ptolemy they evaluated all orbital parameters in terms of the so-called *equant* (a.k.a. *punctum aequans*) which is the empty focal point. The equant has the remarkable property that for the true anomaly θ' (angle between the apsidal line to apogee and the radius vector from the equant to the revolving body) holds (see Problem 7.7)

$$\theta' = M - \frac{1}{4}e^2 \sin(2M) + O\left(e^3\right)$$

while for the common true anomaly θ (see Eq. (7.4.19))

$$\theta = M + 2e\, \sin M + \frac{5}{4}e^2 \sin(2M) + O\left(e^3\right).$$

Therefore, as seen from the equant, the angular motions of all planets in our solar system with their small eccentricities look uniform. This fact, which the Greeks knew by observational data and geometrical considerations, together with Plato's doctrine that planets move on circular orbits, made the ancient Greek astronomers, and initially also Kepler, believe in the eccentric model and the epicyclic model, which Ptolemy showed to be equivalent to each other for uniform motion.

Kepler as a geodesist was a mathematician with geometric reasoning. He did not like calculus, a modern science in those days. So, and owing to Copernicus, Kepler applied the eccentric model to the planets (incl. Earth) circling the Sun located in one focal point. But, by studying the precise observational data taken by Tycho Brahe of Mars' orbit, Kepler realized that its orbit must be an ellipse rather than a circle. That the position on the ellipse R at a given time was not on the radius vector CR' to the corresponding position on the auxiliary circle, as he assumed initially, but on the side XR' opposite to E (which is an important ingredient of Kepler's transformation), came to him in a flash of insight by adjusting the precise Mars locations to his elliptical model. model. With this situation in mind it is obvious, how Kepler was able to think up such an ingeniously simple transformation to solve the Keplerian problem: Drawing circles around the ellipse's center arose quite naturally.

It is interesting to see how Kepler was able to solve the Keplerian problem and arrive at his famous solution Eq. (7.4.15) although he could not apply infinitesimal calculus, as we did, because Newton and Leibniz developed this tool only about one century later. He rather studied the equivalent motion of the body on the auxiliary circle, in particular the area the corresponding position vector OR' sweeps from perigee in a given time interval. He knew from his second law Eq. (7.2.9) that this is proportional to the time interval. From Fig. 7.13 one recognizes that this area equals the circle sector $CR'P$ with area $\frac{1}{2}a^2E$ minus the triangle $CR'O$, with area $\frac{1}{2}a^2e\,\sin E$. So he arrived at $E - e\,\sin E \propto t$. The proportionality factor he called mean motion. But, owing to the significance of the equant and in line with Greek geometers he defined his eccentric anomaly relative to the apogee, i.e., $E' = 180^\circ - E$. So in his original work the swept area is the sum of the said two subareas and hence $E' + e\,\sin E' \propto t$.

It was only Leonhard Euler at the beginning of the 18th century, who provided a rigorous analytic derivation of the eccentric anomaly and its relation to the true anomaly. Thus he also regularized the problem and derived Kepler's equation.

Solving Kepler's Equation

Kepler's equation has still the drawback that it cannot be solved analytically for E at a given t. This is achieved only numerically. A common way is Newton's method (a.k.a. Newton-Raphson method). For this one defines the function

$$f(E) = E - e\,\sin E - M$$

which transforms the problem to finding the root of $f(E)$. Newton's method states that you quickly get it in quadratic convergence (that means very fast if you are close to the solution) by the iteration

$$E_{i+1} = E_i - \frac{f(E_i)}{f'(E_i)} = E_i - \frac{E_i - e\,\sin E_i - M}{1 - e\,\cos E_i} \tag{7.4.16}$$

with

$$0 \le M = n \cdot \left(t - t_p\right) \le \pi$$

and

$$E_0 = M + e^2\left[(6M)^{1/3} - M\right] \qquad @ \quad 0 \le M < 0.25$$

$$E_0 = M + \frac{e\,\sin M}{1 - \sin(M+e) + \sin M} \qquad @ \quad 0.25 \le M \le \pi$$

the empirically optimal initial E values for $0 \le e < 1$ (see Esmaelzadeh, Ghadiri (2014)). Observe that Newton's method becomes unstable if the denominator in Eq. (7.4.16) vanishes, i.e. for $e \cos M \to 1$ and therefore care has to be taken for $e > 0.95$ and $M < 0.1$.

We thus have finally reached our goal of determining the orbital position at any time:

Calculation scheme for orbit propagation

For an orbit with mean motion $n = \sqrt{\mu/a^3}$, eccentricity e, and a given time span $t - t_p$ from the last periapsis passage t_p

1. Calculate $M(t)$ from Eq. (7.3.11)
2. If $\pi < M \le 2\pi$ then reduce it to the interval $[0, \pi]$ by $2\pi - M \to M$
3. Calculate $E(t)$ from Newton's method Eq. (7.4.16)
4. Apply this to Eqs. (7.4.14a) and (7.4.14b)

whereby one gets $r = r(t)$ and $\theta = \theta(t)$.

Analytical Solution

Kepler's Eq. (7.4.15) provides a means to find analytical solutions to $r(t)$ and $\theta(t)$. Since from Eqs. (7.4.14a) and (7.4.14c) $\cos E = (r - a)/(ae)$ and $\sin E = (\sqrt{1 - e^2} \cdot \sin\theta)/(1 + e\cos\theta)$, we immediately get from Eq. (7.4.15) the analytical solution to the Keplerian problem

$$M = n \cdot (t - t_p) = \begin{cases} \arcsin\lambda - e\lambda & @ -1 \le \lambda \le 0 \\ \pi - \arcsin\lambda - e\lambda & @ \ 0 \le \lambda \le 1 \end{cases}$$

with

$$\lambda = \sin E = \frac{\sqrt{1 - e^2}\sin\theta}{1 + e\cos\theta} = \operatorname{sgn}(\sin E)\sqrt{1 - \left(\frac{r - a}{ea}\right)^2}$$

or

$$M = n \cdot (t - t_p) = \frac{\pi}{2} + \arcsin\rho - e\sqrt{1 - \rho^2} \quad @ -1 \le \rho \le 1 \tag{7.4.17}$$

with

$$\rho = \cos E = \frac{\cos\theta + e}{1 + e\cos\theta} = \frac{r - a}{ea}$$

$-1 \le \rho \le 1$ is equivalent to $0 \le \theta \le \pi$ or $r_{per} \le r \le r_{apo}$. It should be mentioned that Eq. (7.4.17) can be derived directly by integration of Eq. (7.2.11) (see Problem 7.8).

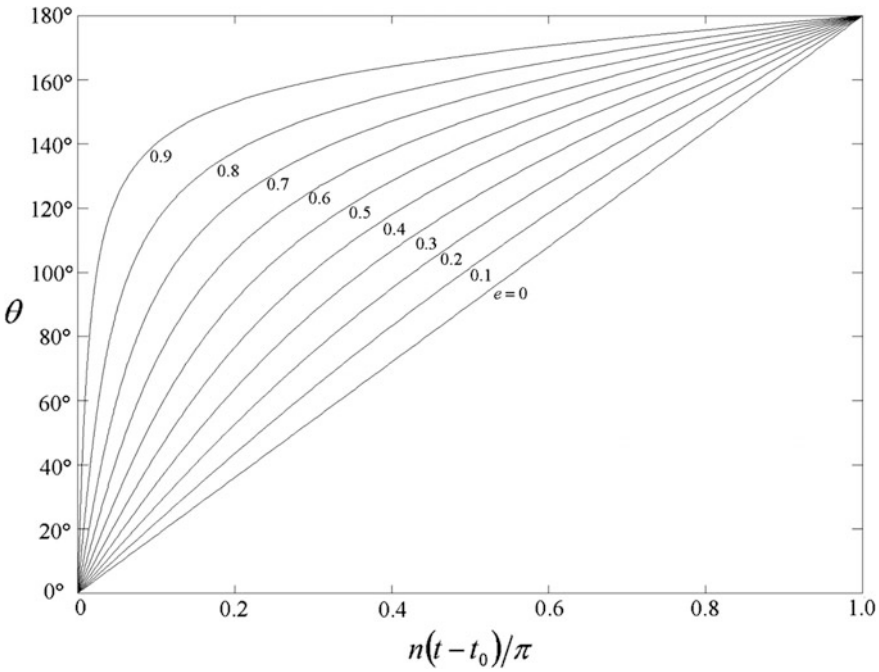


Fig. 7.14 A graphical solution to the Keplerian problem

We recall from Sect. 7.3.2 that this analytical solution is of no practical use. It suffers from the fact that the solution is only implicit. Though, like Kepler's equation, it could also be solved with the Newton method with regard to θ or r, it would be too much effort to calculate the function $f(E)$ and even more so its derivative $f'(E)$, even if done numerically. So when its comes to have a fast algorithm, e.g., an orbit propagator, that every millisecond calculates the exact orbit position of a spacecraft, Kepler and Newton are invincible. If a high accuracy of the result is not decisive, then a graphical depiction of Eq. (7.4.17) such as that given in Fig. 7.14 might be a favorable solution to the Keplerian problem. In addition, it provides a good overview.

Series Expansion of E(M)—Orbit Propagator

Because Newton's method Eq. (7.4.16) converges quadratically, we make use of this to determine a series approximation of $E(t)$ for e. It is easy to show (exercise) that for a more convenient iteration the Newton iteration Eq. (7.4.16) can be rewritten as

$$K_{i+1} = e\frac{\sin(M+K_i) - K_i\cos(M+K_i)}{1 - e\,\cos(M+K_i)}$$

where

$$K := E - M$$

Because this iteration equation is too complicated, we freeze in the slope of the Newton iteration at the starting value $E = M$ (Convince yourself by a depiction of Newton's method, that near the solution, the exact local slope is insignificant for the quadratic convergence.)

$$f' = 1 - e\,\cos E_i \approx 1 - e\,\cos M$$

This delivers the reduced Newton equation

$$E_{i+1} = E_i - \frac{f(E_i)}{f'(E_i)} = E_i - \frac{E_i - e\,\sin E_i - M}{1 - e\,\cos M}$$

i.e.

$$K_{i+1} = \frac{e}{1 - e\,\cos M}\left[\sin(M + K_i) - K_i \cos M\right]$$

or

$$K_{i+1} = s\,\cos K_i + c(\sin K_i - K_i)$$

with

$$s := \frac{e\,\sin M}{1 - e\,\cos M}, \quad c := \frac{e\,\cos M}{1 - e\,\cos M}$$

With this abridged iteration equation and the starting value $K_0 = 0$ we perform three iterations. This delivers

$$K_3 = s\left[1 - \frac{1}{2}s^2\left(1 + \frac{1}{3}c\right) + \frac{1}{24}s^4\left(13 + \frac{51}{5}c\right) + O\left(s^6\right)\right]$$

We therefore obtain with $M = n \cdot \left(t - t_p\right)$ the following result

$$E(t) = M + s\left[1 - \frac{1}{2}s^2\left(1 + \frac{1}{3}c\right) + \frac{1}{24}s^4\left(13 + \frac{51}{5}c\right) + O\left(s^6,\ s^4c^2\right)\right] \quad @\ e < 1 \tag{7.4.18}$$

The convergence of this series is so fast that we have the following small errors

$E(t) \approx M + s$	$\Delta E < 0.03^\circ$ @ $e < 0.1$		
$E(t) \approx M + s - \frac{1}{2}s^3$	$\Delta E < 0.01^\circ$ @ $e < 0.2$;	$\Delta E < 0.07^\circ$	@ $e < 0.3$
Eq.(7.4.18)	$\Delta E < 0.00062^\circ$ @ $e < 0.2$;	$\Delta E < 0.013^\circ$	@ $e < 0.3$

We therefore have the following simple algorithm, a so-called orbit propagator, for elliptic orbits

Orbit Propagator $r(t)$, $\theta(t)$, with $\Delta E < 0.00062°$ for $e < 0.2$
Given a, e, t_0, r_0; $t > t_0$

(a) Derive E_0 from $r_0 = a(1 - e\ \cos E_0)$
(b) Definitions

$$M := \sqrt{\frac{\mu}{a^3}}(t - t_0) + E_0 - e\ \sin E_0$$

$$s := \frac{e\ \sin M}{1 - e\ \cos M}, \quad c := \frac{e\ \cos M}{1 - e\ \cos M}$$

(c) Determine

$$E(t) = M + s\left[1 - \frac{1}{2}s^2\left(1 + \frac{1}{3}c\right) + \frac{1}{24}s^4\left(13 + \frac{51}{5}c\right)\right]$$

$$\rightarrow \quad r(t) = a(1 - e\cos E)$$

$$\rightarrow \quad \theta(t) = 2\arctan\left(\sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2}\right)$$

Function Series Expansions

Other series expansions are provided in the literature (see e.g. Murray and Dermott (1999)). For a derivation see Problem 7.7) for small eccentricities, $e < 0.6627434$. They read

$$\begin{aligned} E &= M + e\sin M + \frac{1}{2}e^2\sin(2M) - \frac{1}{8}e^3[\sin M - 3\sin(3M)] + O(e^4) \\ \frac{r}{a} &= 1 - e\cos M + e^2\sin^2 M + \frac{3}{2}e^3\cos M\sin^2 M + O(e^4) \\ \theta &= M + 2e\sin M + \frac{5}{4}e^2\sin(2M) - \frac{1}{4}e^3\left[\sin M - \frac{13}{3}\sin(3M)\right] + O(e^4) \end{aligned} \tag{7.4.19}$$

Note that these series expansions exhibit only linear convergence and therefore Eq. (7.4.19a) converges much more slowly than Eq. (7.4.18).

Summary

In hindsight the achievement of Kepler's transformation tells us a remarkable story: Time t might be a good world-configuration state sequencer and hence a good state sequencer in physics overall, but it is not a good gravitationally bounded state sequencer. For two reasons: Firstly, it is a linear sequencer while bounded gravitational motion is periodic. Secondly, geometric elliptic motion is highly irregular in terms of time. The true anomaly θ remedies only the first problem, the Kepler transformation remedies both. With it the radial motion becomes remarkably simple. According to the results of Sect. 7.3.4 and Eq. (7.4.14a) the gravitational motion is basically determined by its mean

energy state in the gravitational potential manifested in the semi-mayor axis a. The radial trajectory in terms of E then is simply a sinusoidal swing around this average radius a with amplitude ae, which equals the distance from the center of the ellipse to a focal point. This is why Eq. (7.4.14a) is so important and of practical relevance. The only hassle of this new view, namely the relation between E and t, is encapsuled in Kepler's equation (7.4.15). In Sect. 7.4.5 we will make use of this facilitated approach to find state vector solutions $\boldsymbol{r}(E)$, $\boldsymbol{v}(E)$ for any gravitational motion.

As we will see later, E is a good sequencer, but not a perfect one. With the universal anomaly $G = E\Big/\sqrt{1-e^2}$ we will find in Sect. 7.4.6 the perfect sequencer with the additional benefit that it works equally well for elliptic, parabolic and hyperbolic trajectories.

7.4.3 Hyperbolic Orbit

The most general unbounded trajectory about a celestial body is a flyby hyperbola with $e > 1$. Note that for a hyperbola a is negatively defined, $a < 0$ (see Eq. (7.3.6)). Because of Eq. (7.3.19), the orbital energy is positive, $\varepsilon > 0$.

According to Eqs. (7.3.5) and (7.3.14) a hyperbola has its closest approach to the focal point at the periapsis with

$$r_{per} = \frac{h^2}{\mu(e+1)} = -a(e-1) \tag{7.4.20}$$

$$v_{per} = \frac{\mu}{h}(e+1) = \sqrt{-\frac{\mu}{a}}\sqrt{\frac{e+1}{e-1}} \tag{7.4.21}$$

Therefore its focal point lies at the distance $|a| + r_{per} = e|a| = -ea$ from the origin. A hyperbola does not possess an apoapsis. It reaches infinity at an asymptote, the angle of which is determined from Eq. (7.3.5) for $r \to \infty$ as

$$\cos\theta_\infty = -\frac{1}{e} \tag{7.4.22}$$

The so-called *impact parameter* Δ (a.k.a. *aiming radius*), which is the distance between the focal point and the asymptote measured normal to the asymptote, is found from Fig. 7.15 to be

$$\Delta = -(r_{per} + a)\sin\beta = -ae\ \sin\theta_\infty$$

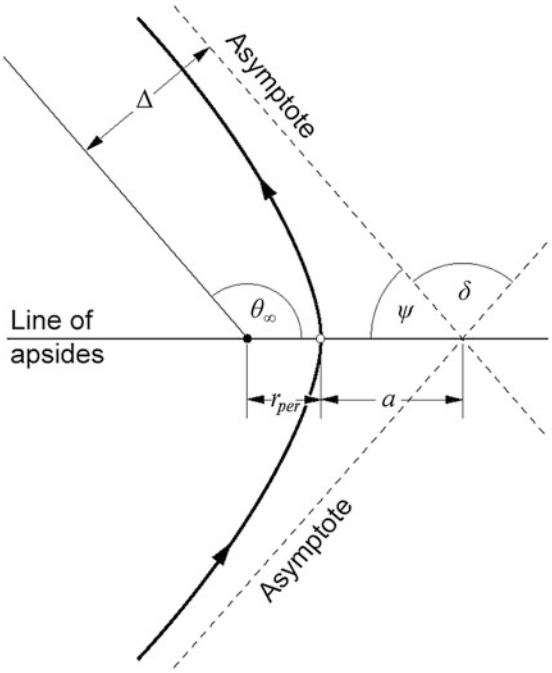


Fig. 7.15 The characteristic parameters of a hyperbola

Here we have chosen a negative sign because Δ should be a positive value, but a is negative. With Eqs. (7.4.20) and (7.4.22) one gets

$$\Delta = -a\sqrt{e^2-1} = r_{per}\sqrt{\frac{e+1}{e-1}} \qquad \textbf{impact parameter} \tag{7.4.23}$$

According to Eq. (7.3.14), the body at infinity has the so-called *hyperbolic excess velocity*

$$v_\infty = \frac{\mu}{h}\sqrt{e^2-1} = \sqrt{-\frac{\mu}{a}} \qquad \textbf{hyperbolic excess velocity} \tag{7.4.24}$$

For interplanetary flight, the parameter

$$C_3 := v_\infty^2 = -\frac{\mu}{a} \qquad \textbf{characteristic energy} \tag{7.4.25}$$

is often used. To attain a given hyperbolic excess velocity, for instance, for an interplanetary transfer, a speed of $v > v_{\triangleright\triangleright}$ is required at the departure orbit, where $v_{\triangleright\triangleright}$ is the escape velocity (see Sect. 7.4.2). It is determined from Eqs. (7.2.14) and (7.4.34) as

$$v^2 = \frac{2\mu}{r} + v_\infty^2 = v_{\triangleright\triangleright}^2 + v_\infty^2 \tag{7.4.26}$$

The related boost to reach this departure velocity is called *Oberth maneuver*. In fact, this equation is nothing other than the energy conservation equation, where only the kinetic energy shows up, because the potential energy vanishes at infinity.

Asymptotic Motion

When approaching $\theta \to \theta_\infty$ the body recedes asymptotically. In this limit it is easy to show from the orbit Eq. (7.3.5) that the following holds for $r(\theta)$

$$r(\theta) = \frac{p}{\sqrt{e^2-1}(\theta_\infty - \theta) + \frac{1}{2}(\theta_\infty - \theta)^2} \quad @\ \theta \to \theta_\infty \tag{7.4.27}$$

To determine $r(t)$ in the asymptotic limit, we recall the radial velocity as given in Eq. (7.2.11), which for $r \to \infty$ in first order perturbation calculation reads

$$\dot{r}^2 = -\frac{\mu}{a} \cdot \left(1 - \frac{2a}{r}\right) \quad @\ r \to \infty$$

With $\sqrt{1-2a/r} \approx 1 - a/r$ this yields the differential equation for asymptotic radial recession (+) or approach (−)

$$\frac{dr}{\sqrt{1-2a/r}} \approx \frac{r \cdot dr}{r-a} = \left(1 - \frac{-a}{r-a}\right)dr = \left(1 - \frac{1}{1-r/a}\right)dr = \pm\sqrt{-\frac{\mu}{a}} \cdot dt$$
$$= \pm v_\infty \cdot dt$$

Solving with the initial condition $r(t=0) = r_0$ we derive

$$\left[r + a\,\ln\left(1 - \frac{r}{a}\right)\right]_{r_0}^{r} = \pm v_\infty (t - t_0)$$

hence

$$r + a\,\ln\left(1 - \frac{r}{a}\right) = \pm v_\infty (t - t_0) + r_0 + a\,\ln\left(1 - \frac{r_0}{a}\right) =: s$$

For $r \to \infty$ we have in 1st order approximation $r = s$, and in 2nd order approximation

$$\begin{array}{ll} r(t) = s - a \cdot \ln\left(1 - \frac{s}{a}\right) & + @\ r \to \infty \\ s = \pm v_\infty (t - t_0) + r_0 + a \cdot \ln\left(1 - \frac{r_0}{a}\right) & - @ \infty \to r \end{array} \tag{7.4.28}$$

with initial condition $r(t_0) = r_0$. We recall that $a < 0$.

Given this result we are able to easily derive $\theta(t \to \infty)$. According to Eq. (7.2.7) we have

$$\int_{\theta}^{\theta_\infty} d\theta' = \int_{t}^{\infty} \frac{h}{r^2(t')} dt'$$

We insert the result Eq. (7.4.28), approximate for $t, s \to \infty$, and make use of

$$h\frac{dt}{ds} = \frac{h}{v_\infty} = \frac{p}{\sqrt{e^2-1}} = -a\sqrt{e^2-1}$$

Thus

$$\theta_\infty - \theta = \int_{t}^{\infty} \frac{h}{[s - a\cdot \ln(-s'/a)]^2} dt' \approx -a\sqrt{e^2-1} \int_{s}^{\infty} \frac{1 + 2a/s' \cdot \ln(-s'/a)}{s'^2} ds'$$

$$\int_{s}^{\infty} \frac{1 + 2a/s' \cdot \ln(-s'/a)}{s'^2} ds' = \int_{s}^{\infty} \left[\frac{1}{s'^2} - \frac{2}{a^2}\frac{\ln(-s'/a)}{(-s'/a)^3}\right] ds' = \frac{1}{s} + \frac{2}{a}\left(-\frac{1}{2}\frac{\ln x}{x^2}\right)\Big|_{-s/a}^{\infty}$$

$$= \frac{1}{s} + \frac{1}{a}\frac{\ln(-s/a)}{(-s/a)^2} + O\left(\frac{1}{s^3}\right)$$

Hence

$$\begin{aligned} \theta(t) &= \theta_\infty + \sqrt{e^2-1}\cdot\frac{a}{s}\left[1 + \frac{a}{s}\ln\left(-\frac{s}{a}\right) + O\left(\frac{1}{s^3}\right)\right] \\ s(t) &= v_\infty(t - t_0) + r_0 + a\cdot\ln\left(1 - \frac{r_0}{a}\right) \qquad @\ t \to \infty \\ \theta_\infty &= \arccos\left(-\frac{1}{e}\right) \end{aligned} \tag{7.4.28b}$$

again with initial condition $r(t_0) = r_0$ and $a < 0$.

Kepler's Equation and Solutions

One can show that the essential Eqs. (7.4.14) and (7.4.15) of the ellipse can be expressed in a similar way for a hyperbola, namely that there is also a hyperbolic eccentric anomaly F for the hyperbola. We can shorten its derivation by the statement that it has the following relation to the elliptic eccentric anomaly E: $E = iF$ with $i = \sqrt{-1}$ (for a proof see Sect. 7.4.6). If one applies this relation to Eqs. (7.4.14), one directly obtains

$$\boxed{r = a(1 - e\,\cosh F)} \qquad \textbf{orbit equation}$$

$$\begin{aligned}
\cos\theta &= \frac{e-\cosh F}{e\,\cosh F-1}, & \cosh F &= \frac{e+\cos\theta}{1+e\,\cos\theta}\\
\sin\theta &= \frac{\sqrt{e^2-1}\cdot\sinh F}{e\,\cosh F-1}, & \sinh F &= \frac{\sqrt{e^2-1}\cdot\sin\theta}{1+e\,\cos\theta}\\
\tan\frac{\theta}{2} &= \sqrt{\frac{e+1}{e-1}}\tanh\frac{F}{2}
\end{aligned} \tag{7.4.29}$$

and

$$M = n\cdot\left(t-t_p\right) = e\ \sinh F(t) - F(t) \tag{7.4.30}$$

with

$$n = \sqrt{-\frac{\mu}{a^3}} \tag{7.4.31}$$

Here, t_p the time of passage through the periapsis, and M the mean anomaly (cf. Eq. (7.3.11)).

To determine the orbit position as a function of time just as with the ellipse, one has to solve Eq. (7.4.30) with Newton's method

$$F_{i+1} = F_i - \frac{e\,\sinh F_i - F_i - M}{1-e\,\cosh F_i} \tag{7.4.32}$$

with $F_0 = e\,\sinh M - M$ the initial value. If the result is inserted into Eqs. (7.4.29a) and (7.4.29b), one obtains the parameterized orbit $r = r(F(t))$ and $\theta = \theta(F(t))$.

Analytical Solution

In analogy to the elliptic case, we find from Eq. (7.4.30) with $\sinh F = \sqrt{e^2-1}\cdot\sin\theta\Big/(1+e\,\cos\theta)$ from Eq. (7.4.29c) an implicit solution for r and θ

$$\begin{aligned}
M = n\cdot\left(t-t_p\right) &= e\lambda - \operatorname{arsinh}\lambda\\
&= e\lambda - \ln\frac{\sqrt{e+1}+\sqrt{e-1}\cdot\tan(\theta/2)}{\sqrt{e+1}-\sqrt{e-1}\cdot\tan(\theta/2)}
\end{aligned}$$

with

$$\lambda = \sinh F = \frac{\sqrt{e^2-1}\cdot\sin\theta}{1+e\,\cos\theta} = \sqrt{\left(\frac{r-a}{ea}\right)^2-1}$$

or

$$M = n \cdot (t - t_p) = e\sqrt{\rho^2 - 1} - \operatorname{arcosh} \rho \\ = e\sqrt{\rho^2 - 1} - \ln\left(\rho + \sqrt{\rho^2 - 1}\right) \tag{7.4.33}$$

with

$$\rho = \cosh F = \frac{e + \cos\theta}{1 + e\cos\theta} = \frac{r - a}{-ea} = \frac{r + |a|}{e|a|}$$

Here again, this analytical solution is of no practical use, because it suffers from the fact that the solution is only implicit.

7.4.4 Parabolic Orbit

For $e = 1$, we get a parabolic orbit. According to Eq. (7.3.18), its orbital energy is $\varepsilon = 0$ and according to Eq. (7.3.7), its semi-major axis is $a = \infty$. However, because the semi-latus rectum $p = a(1 - e^2)$ is still a finite nonzero number, it is used as the sole orbit element to describe the shape of a parabolic orbit—p is the abeam semi-width of the parabola (see Fig. 7.8). Note that the semi-latus rectum, in general, is an excellent orbit element that steadily transforms between the transitions from circle to ellipse to parabola to hyperbola (see Fig. 7.8).

According to the vis-viva Eq. (7.2.15) and to the orbit equation the velocity on a parabolic orbit with $a = \infty$ is

$$v_{esc} = \sqrt{\frac{2\mu}{r}} = 2\sqrt{\frac{\mu}{p}}\cos\frac{\theta}{2} \tag{7.4.34}$$

The parabolic orbit is a limiting orbit where the body is able to just escape the central body reaching infinity with zero velocity, $v(r = \infty) = 0$. Equation (7.4.34), therefore, determines the so-called **escape velocity**, v_{esc}, which is the velocity a body at any position r from the barycenter requires to achieve parabolic escape.

Example

The velocity of a body at the surface of Earth needed to escape Earth's gravitation—the so-called *second cosmic velocity*—is with $\mu_\oplus = g_0 R_\oplus^2$ (see Eq. (7.1.20))

$$v_{\triangleright\triangleright} = \sqrt{2 g_0 R_\oplus} = 11.180 \text{ km s}^{-1} \quad \textbf{second cosmic velocity} \tag{7.3.35}$$

With orbit equation (7.3.5) and Eq. (7.4.34) we have at periapsis, i.e. the closest approach to the barycenter

$$r_{per} = \frac{p}{2}$$
$$v_{per} = 2\sqrt{\frac{\mu}{p}}$$

Observe that the escape velocity at a given orbital radius enables a spacecraft to go to infinity independent of the direction in which the final velocity vector points. In Sect. 8.4.1 we will see how a transfer from any Keplerian orbit to a parabolic escape orbit is performed.

To determine the explicit orbit equations $r(t)$, $\theta(t)$ for a parabolic orbit, we recognize that for $e = 1$ and with the substitution $x := \theta/2$, one is able to directly integrate Eq. (7.3.9)

$$\sqrt{\frac{\mu}{p^3}}(t - t_p) = \int_0^{\theta} \frac{d\theta'}{(1 + \cos\theta')^2} = \int_0^{\theta} \frac{d\theta'}{(2\cos^2\theta'/2)^2} = \frac{1}{2}\int_0^{\theta/2} \frac{dx}{\cos^4 x}$$
$$= \frac{1}{2}\int_0^{\theta/2} \left(\frac{1}{\cos^2 x} + \frac{\tan^2 x}{\cos^2 x}\right) dx = \frac{1}{2}\left(\tan x + \frac{1}{3}\tan^3 x\right)\Big|_0^{\frac{\theta}{2}} = \frac{1}{2}\left(\tan\frac{\theta}{2} + \frac{1}{3}\tan^3\frac{\theta}{2}\right)$$

Thus, the position equation can be provided analytically

$$\sqrt{\frac{\mu}{p^3}}(t - t_p) = \frac{1}{2}\tan\frac{\theta}{2} + \frac{1}{6}\tan^3\frac{\theta}{2} \qquad \textbf{Barker's equation} \qquad (7.4.36)$$
$$= \frac{1}{2}G + \frac{1}{3}G^3$$

where we have defined the universal anomaly (cf. Sect. 7.4.6)

$$G := \tan\frac{\theta}{2} = \sqrt{2\frac{r}{p} - 1}$$

Considering Barker's equation as a cubic function for $\tan(\theta/2)$ we find the roots by applying Cardano's method and Descartes' rule of signs. From this, it can be shown (exercise, Problem 7.5) that there is only one real solution to this equation for θ and r, namely,

$$\tan\frac{\theta}{2} = \left(\sqrt{q^2+1} + q\right)^{1/3} - \left(\sqrt{q^2+1} - q\right)^{1/3} = 2\sinh\left(\frac{1}{3}\operatorname{arsinh} q\right)$$
$$2\frac{r}{p} = \left(\sqrt{q^2+1} + q\right)^{2/3} + \left(\sqrt{q^2+1} - q\right)^{2/3} - 1 = 2\cosh\left(\frac{2}{3}\operatorname{arsinh}|q|\right) - 1 \qquad (7.4.37)$$

with

$$q = 3\sqrt{\frac{\mu}{p^3}}\,(t - t_p)$$

where the later expressions follow from Chebyshev polynomials, which are particularly useful for small time intervals $t - t_p$. So, also for the parabolic case, we were able to solve the Keplerian problem analytically.

Asymptotic Motion

What is the mathematical description of the asymptotic motion, i.e. for $t \to \infty$? With the definition

$$\lambda^3(t) := \frac{1}{2q} = \frac{1}{6\sqrt{\mu/p^3}\,(t - t_p)} \to 0$$

we find for Eq. (7.4.37) the power series expansions

$$\begin{aligned} \tan\tfrac{\theta}{2} &= \tfrac{1}{\lambda} - \lambda + \tfrac{1}{3}\lambda^5 + O(\lambda^7) = (2q)^{1/3} - (2q)^{-1/3} + \cdots \\ 2\tfrac{r}{p} &= \tfrac{1}{\lambda} - 1 + \lambda + \tfrac{2}{3}\lambda^5 + O(\lambda^7) = (2q)^{1/3} - 1 + (2q)^{-1/3} + \cdots \end{aligned} \quad @\ \lambda(t) \to 0 \tag{7.4.38}$$

Because for $\varphi \to \pi/2$ we have the power series expansion

$$\tan\left(\frac{\pi}{2} - \varphi\right) = \frac{1}{\varphi} - \frac{1}{3}\varphi - \frac{1}{45}\varphi^3 - \frac{2}{945}\varphi^5 \quad @\ \varphi \to 0$$

we can remove the divergence of $\tan\theta(\lambda)$ for $\theta \to \pi$ by finding that φ, which satisfies the above $\tan\theta/2$ expansion in λ. We do so by making an ansatz of a power series approximation $\varphi = a\lambda + b\lambda^3 + c\lambda^5 + d\lambda^7$, calculating $1/\varphi - \frac{1}{3}\varphi - \frac{1}{45}\varphi^3 - \frac{2}{945}\varphi^5$ in terms of λ, and equate it to the $\tan\theta/2(\lambda)$ power series expansion. We thus get the result

$$\theta(t) = \pi - 2\varphi = \pi - 2\left(\lambda + \frac{2}{3}\lambda^3 - \frac{1}{45}\lambda^5 - \frac{76}{189}\lambda^7 + \ldots\right) \quad @\ \lambda(t) \to 0 \tag{7.4.39}$$

Near-Parabolic Orbits

The parabolic case is exceptional in that it is not transcendental like the elliptic and hyperbolic cases, where we achieved the $r(t), \theta(t)$ solutions only numerically by solving Kepler's equation. We would like to convey the parabolic non-transcendental nature to both elliptic and hyperbolic orbits close to the parabolic orbit, i.e. for $e \approx 1$. We therefore define a closeness parameter $\varepsilon = 1 - e$. We therewith expand the integral in Eq. (7.3.9) as

$$\int_0^\theta \frac{d\theta'}{\left[1+(1-\varepsilon)\cos\theta'\right]^2} \approx \int_0^\theta \frac{d\theta'}{\left(1+\cos\theta'\right)^2} + 2\varepsilon \int_0^\theta \frac{\cos\theta'}{\left(1+\cos\theta'\right)^3} d\theta' + 3\varepsilon^2 \int_0^\theta \frac{\cos^2\theta'}{\left(1+\cos\theta'\right)^4} d\theta'$$

The solutions to the integrals are (Exercise. Use a symbolic integrator and express results in terms of $\cos^2(\theta/2)$, finally transform into $1/\cos^2(\theta/2) = 1 + \tan^2(\theta/2)$)

$$\int_0^\theta \frac{1}{\left(1+\cos\theta'\right)^2} d\theta' = \frac{1}{2}\tan\frac{\theta}{2}\left(1+\frac{1}{3}\tan^2\frac{\theta}{2}\right)$$

$$\int_0^\theta \frac{\cos\theta'}{\left(1+\cos\theta'\right)^3} d\theta' = \frac{1}{4}\tan\frac{\theta}{2}\left(1-\frac{1}{5}\tan^4\frac{\theta}{2}\right)$$

$$\int_0^\theta \frac{\cos^2\theta'}{\left(1+\cos\theta'\right)^4} d\theta' = \frac{1}{8}\tan\frac{\theta}{2}\left(1-\frac{1}{3}\tan^2\frac{\theta}{2}-\frac{1}{5}\tan^4\frac{\theta}{2}+\frac{1}{7}\tan^6\frac{\theta}{2}\right)$$

With the above definition $G := \tan\theta/2$ Kepler's equation therefore reads

$$\begin{aligned}\frac{q}{3} &:= \sqrt{\frac{\mu}{p^3}}(t-t_p) \\ &\approx \frac{1}{2}G\left(1+\frac{1}{3}G^2\right) + \frac{2\varepsilon}{4}G\left(1-\frac{1}{5}G^4\right) + \frac{3\varepsilon^2}{8}G\left(1-\frac{1}{3}G^2-\frac{1}{5}G^4+\frac{1}{7}G^6\right)\end{aligned} \tag{7.4.40}$$

Because ε is small, we solve the equation for G by one approximation iteration. In the first step we assume $\varepsilon = 0$. The solution is that of the parabolic orbit Eq. (7.4.38). We insert this approximate result into the residual. By defining

$$\frac{s}{3} := \frac{q}{3} - (1-e)\frac{1}{2}G\left(1-\frac{1}{5}G^4\right) - (1-e)^2\frac{3}{8}G\left(1-\frac{1}{3}G^2-\frac{1}{5}G^4+\frac{1}{7}G^6\right)$$

where now $G(q(t))$, we obtain the new Kepler's equation

$$\frac{s(t)}{3} = \frac{1}{2}G\left(1+\frac{1}{3}G^2\right)$$

The solution is again given by Eq. (7.4.38). We therefore have the following solution for near-parabolic orbits

Given $t > t_p$. Determine

$$G = 2\,\sinh\left\{\frac{1}{3}\operatorname{arsinh}\left[3\sqrt{\frac{\mu}{p^3}}(t-t_p)\right]\right\}$$

$$s := \sqrt{\frac{\mu}{p^3}}(t-t_0) - (1-e)\frac{3}{2}G\left(1-\frac{1}{5}G^4\right) - (1-e)^2\frac{9}{8}G\left(1-\frac{1}{3}G^2-\frac{1}{5}G^4+\frac{1}{7}G^6\right) + O\left((1-e)^3\right)$$

Then

$$\tan\frac{\theta}{2} = 2\sinh\left(\frac{1}{3}\operatorname{arsinh}\right)$$
$$\frac{r}{p} = \cosh\left(\frac{2}{3}\operatorname{arsinh}|s|\right) - \frac{1}{2} \tag{7.4.41}$$

7.4.5 ε-Based Transformation

Having identified Kepler's equations as the root of the transcendentality of Kepler's problem, we are now seeking to eliminate transcendentality from Newton's gravitational equation of motion (7.1.19) and thus find for a given initial state $\boldsymbol{r}(t_0)$, $\boldsymbol{v}(t_0)$ the solution $\boldsymbol{r}(t)$, $\boldsymbol{v}(t)$. If $t > t_0$ this problem is called state (vector) propagation. Let us reconsider Kepler's equations. They read

$$n\left(t-t_p\right) = E - e\,\sin E \quad @\ \text{elliptic orbits}$$
$$n\left(t-t_p\right) = e\,\sinh F - F \quad @\ \text{hyperbolic orbits}$$

By differentiating we obtain with Eq. (7.4.14a) and Eq. (7.4.29a)

$$n\frac{dt}{dE} = n\frac{dt}{dF} = \frac{r}{|a|}$$

From this follows

$$dE = dF = \sqrt{\frac{\mu}{|a|}}\frac{1}{r}\cdot dt = \sqrt{2|\varepsilon|}\frac{1}{r}\cdot dt \quad \textbf{ε-based transformation} \tag{7.4.42}$$

The latter expression, which follows from Eq. (7.3.19), is indicative of the fact that the transformation is only suitable as long as the specific orbital energy is not $\varepsilon \approx 0$. This is why we call it energy-based (ε-based) transformation. Comparison with sub-section *Regularization* in Sect. 7.4.2 and with Sect. 7.4.3 reveals that the ε-based transformation is a generalized Kepler transformation, which regularizes the orbit

equation. The expression $dE = \sqrt{\mu/a}/r \cdot dt$ is the (regular) Kepler transformation, which applies to elliptic orbits only.

That the transformation is just bound to the requirement of a finite orbital energy implies two things. On one hand, and unfortunately, it implies that it is not applicable for parabolic or near-parabolic orbits having $\varepsilon \approx 0$. But, according to Eq. (7.3.18), $2\varepsilon \cdot h^2 = \mu^2(e^2 - 1)$. So on the other hand, as long as $\varepsilon \neq 0$, it does apply for orbits exhibiting a vanshing angular momentum, i.e. $h \approx 0$, in case of which $e = 1$. These are the so-called radial trajectories. We will make use of Kepler's equation with $e = 1$ for radial trajectories in Sects. 7.5.1 and 7.5.2. That the Kepler transformation applies for radial motion is a surprising result, because then $\theta = const$, in case of which the native geometrical concepts of mean, eccentric, and hyperbolic anomaly (i.e., orbit angles) are meaningless.

Algebraic Form of Newton's Gravitational Equation of Motion

To obtain an regularized algebraic form of Newton's gravitational equation of motion (7.1.19) we rewrite it as

$$\frac{d^2\boldsymbol{r}}{dt^2} + \frac{\mu}{r^3}\boldsymbol{r} = 0$$

We eliminate transcendentality by applying the ε-based transformation to

$$\frac{d^2\boldsymbol{r}}{dt^2} = \frac{d}{dt}\frac{d\boldsymbol{r}}{dt} = \frac{\mu}{|a|}\frac{1}{r}\frac{d}{dE}\left(\frac{1}{r}\frac{d\boldsymbol{r}}{dE}\right) = \frac{\mu}{|a|}\frac{1}{r}\left(\frac{1}{r}\boldsymbol{r}'' - \frac{r'}{r^2}\boldsymbol{r}'\right)$$

where

$$(\cdot)' := \frac{d}{dE}$$

Thus transformed, Newton's gravitational EoM reads

$$\boldsymbol{r}'' - \frac{r'}{r}\boldsymbol{r}' + |a|\hat{\boldsymbol{r}} = 0$$

This equation is still not solvable analytically. But the unit radial vector $\hat{\boldsymbol{r}} = \boldsymbol{r}/r$ hints at the eccentricity vector, which reads from Eq. (7.3.3) and with $\boldsymbol{r} \cdot \boldsymbol{v} = \boldsymbol{r} \cdot \boldsymbol{v}_r = r\dot{r}$

$$\mu\boldsymbol{e} = \mu\left(\frac{1}{r} - \frac{1}{a}\right)\boldsymbol{r} - (\boldsymbol{r}\boldsymbol{v})\boldsymbol{v} = \mu\hat{\boldsymbol{r}} - \frac{\mu}{a}\boldsymbol{r} - r\dot{r} \cdot \dot{\boldsymbol{r}}$$

We apply Kepler's transformation to this expression and get

$$\frac{r'}{r} \cdot \boldsymbol{r}' = |a|\hat{\boldsymbol{r}} - \mathrm{sgn}(a)(\boldsymbol{r} + a\boldsymbol{e})$$

This result inserted into the above EoM delivers

$$\boldsymbol{r}''(E, F) = -\mathrm{sgn}(a) \cdot (\boldsymbol{r} + a\boldsymbol{e}) \quad \text{Newton's gravitational EoM} \tag{7.4.43}$$

where $\mathrm{sgn}(x)$ is the sign function. This is an algebraic form of Newton's gravitational equation of motion, which is beautifully simple and analytically solvable. To solve it we make a final substitution

$$\boldsymbol{u} := \boldsymbol{r} + a\boldsymbol{e} \rightarrow \boldsymbol{u}'' = \boldsymbol{r}''$$

which delivers

$$\boldsymbol{u}'' = -\mathrm{sgn}(a) \cdot \boldsymbol{u}$$

Solution for Elliptic Orbits

For elliptic orbits with $a > 0$ the EoM delivers the harmonic oscillator $\boldsymbol{u}'' = -\boldsymbol{u}$ having the general solution

$$\boldsymbol{u}(E) = \boldsymbol{r}(E) + a\boldsymbol{e} = \boldsymbol{\alpha} \sin E + \boldsymbol{\beta} \cos E$$
$$\boldsymbol{u}'(E) = \boldsymbol{r}'(E) = \sqrt{\frac{a}{\mu}} r \cdot \boldsymbol{v}(E) = \boldsymbol{\alpha} \cos E - \boldsymbol{\beta} \sin E$$

subject to the initial condition

$$\boldsymbol{r}_0 + a\boldsymbol{e} = \boldsymbol{\alpha} \sin E_0 + \boldsymbol{\beta} \cos E_0$$
$$\sqrt{\frac{a}{\mu}} r_0 \boldsymbol{v}_0 = \boldsymbol{\alpha} \cos E_0 - \boldsymbol{\beta} \sin E_0$$

Solving for the unknow constants $\boldsymbol{\alpha}, \boldsymbol{\beta}$ delivers

$$\boldsymbol{\alpha} = (\boldsymbol{r}_0 + a\boldsymbol{e}) \sin E_0 + \cos E_0 \sqrt{\frac{a}{\mu}} r_0 \boldsymbol{v}_0$$
$$\boldsymbol{\beta} = (\boldsymbol{r}_0 + a\boldsymbol{e}) \cos E_0 - \sin E_0 \sqrt{\frac{a}{\mu}} r_0 \boldsymbol{v}_0$$

From this follows for an ellipse

$$\boldsymbol{r}(E) + a\boldsymbol{e} = (\boldsymbol{r}_0 + a\boldsymbol{e}) \cos(E - E_0) + \sqrt{\frac{a}{\mu}} r_0 \boldsymbol{v}_0 \sin(E - E_0) \tag{7.4.44}$$

Differentiation delivers

$$\boldsymbol{r}' = -(\boldsymbol{r}_0 + a\boldsymbol{e}) \sin(E - E_0) + \sqrt{\frac{a}{\mu}} r_0 \boldsymbol{v}_0 \cos(E - E_0)$$

Applying Eq. (7.4.42) yields

$$\boldsymbol{v}\cdot\sqrt{\frac{a}{\mu}}\cdot\ r=-(\boldsymbol{r}_0+a\boldsymbol{e})\sin(E-E_0)+\sqrt{\frac{a}{\mu}}r_0v_0\cos(E-E_0)$$

hence

$$r\cdot v(E)=-\sqrt{\frac{\mu}{a}}(\boldsymbol{r}_0+a\boldsymbol{e})\sin(E-E_0)+r_0v_0\cos(E-E_0) \tag{7.4.45}$$

Solution for Hyperbolic Orbits
For hyperbolic orbits with $a<0$ the substitution delivers $\boldsymbol{u}''=\boldsymbol{u}$ with the general solution

$$\boldsymbol{u}(F)=\boldsymbol{r}(F)+a\boldsymbol{e}=\boldsymbol{\alpha}e^{F}+\boldsymbol{\beta}e^{-F}$$
$$\boldsymbol{u}'(F)=\boldsymbol{r}'(F)=\sqrt{\frac{|a|}{\mu}}r\cdot\boldsymbol{v}(F)=\boldsymbol{\alpha}e^{F}-\boldsymbol{\beta}e^{-F}$$

By the same token as above for elliptic orbits, it is easy to show that for a hyperbola

$$\boldsymbol{r}(F)+a\boldsymbol{e}=(\boldsymbol{r}_0+a\boldsymbol{e})\cosh(F-F_0)+\sqrt{\frac{|a|}{\mu}}r_0v_0\sinh(F-F_0) \tag{7.4.46}$$

$$r\cdot v(F)=\sqrt{\frac{\mu}{|a|}}(\boldsymbol{r}_0+a\boldsymbol{e})\sinh(F-F_0)+r_0v_0\cosh(F-F_0) \tag{7.4.47}$$

RSW Coordinate System $\boldsymbol{e},\ \boldsymbol{h}\times\boldsymbol{e}$
For highly elliptic, parbolic, or hyperbolic orbits and at large distances from periapsis the vectors $\boldsymbol{r}_0,\boldsymbol{v}_0$ exhibit the property $\angle(\boldsymbol{r}_0,\boldsymbol{v}_0)\to 0$ and hence do not establish a good basis for the propagated state vector $\boldsymbol{r},\boldsymbol{v}$. A better choice then is an orthogonal RSW coordinate system $\boldsymbol{e},\ \boldsymbol{h}\times\boldsymbol{e},\boldsymbol{h}$ that is not derived from $\boldsymbol{r}_0,\boldsymbol{v}_0$ and hence well established. Thus we get the corresponding results by projecting $\boldsymbol{r},\boldsymbol{v}$ on $\boldsymbol{e},\ \boldsymbol{h}\times\boldsymbol{e}$:
For elliptic orbits

$$\begin{aligned}\boldsymbol{r}+a\boldsymbol{e}&=\left[(r_0\boldsymbol{e}+a)\cdot\ \cos(E-E_0)+\boldsymbol{v}_0\boldsymbol{e}\cdot r_0\sqrt{a/\mu}\cdot\ \sin(E-E_0)\right]\cdot\boldsymbol{e}\\&\quad+\left[\boldsymbol{r}_0(\boldsymbol{h}\times\boldsymbol{e})\cdot\ \cos(E-E_0)+\boldsymbol{v}_0(\boldsymbol{h}\times\boldsymbol{e})\cdot r_0\sqrt{a/\mu}\cdot\ \sin(E-E_0)\right]\cdot\boldsymbol{h}\times\boldsymbol{e}\\&=\cos\theta\cdot\boldsymbol{e}+\ \sin\theta\cdot\boldsymbol{h}\times\boldsymbol{e}\\&=\frac{\cos E-e}{1-e\ \cos E}\cdot\boldsymbol{e}+\frac{\sqrt{1-e^2}\sin E}{1-e\ \cos E}\cdot\boldsymbol{h}\times\boldsymbol{e}\end{aligned} \tag{7.4.48}$$

and

$$\begin{aligned} r \cdot \boldsymbol{v} &= \left[-(\boldsymbol{r}_0\boldsymbol{e} + a)\sqrt{\mu/a} \cdot \sin(E - E_0) + \boldsymbol{v}_0\boldsymbol{e} \cdot r_0 \cos(E - E_0)\right] \cdot \boldsymbol{e} \\ &\quad + \left[-\boldsymbol{r}_0(\boldsymbol{h}\times\boldsymbol{e})\sqrt{\mu/a} \cdot \sin(E - E_0) + \boldsymbol{v}_0(\boldsymbol{h}\times\boldsymbol{e}) \cdot r_0 \cos(E - E_0)\right] \cdot \boldsymbol{h}\times\boldsymbol{e} \\ \boldsymbol{v} &= \frac{h}{r^2}(-\sin\theta \cdot \boldsymbol{e} + \cos\theta \cdot \boldsymbol{h}\times\boldsymbol{e}) \\ &= \frac{h}{r^2}\left(-\frac{\sqrt{1-e^2}\sin E}{1 - e\cos E} \cdot \boldsymbol{e} + \frac{\cos E - e}{1 - e\cos E} \cdot \boldsymbol{h}\times\boldsymbol{e}\right) \end{aligned} \tag{7.4.49}$$

and for hyperbolic orbits

$$\begin{aligned} \boldsymbol{r} + a\boldsymbol{e} &= \left[(\boldsymbol{r}_0\boldsymbol{e} + a)\cosh(F - F_0) + \boldsymbol{v}_0\boldsymbol{e} \cdot r_0\sqrt{|a|/\mu} \cdot \sinh(F - F_0)\right] \cdot \boldsymbol{e} \\ &\quad + \left[\boldsymbol{r}_0(\boldsymbol{h}\times\boldsymbol{e}) \cdot \cosh(F - F_0) + \boldsymbol{v}_0(\boldsymbol{h}\times\boldsymbol{e}) \cdot r_0\sqrt{|a|/\mu} \cdot \sinh(F - F_0)\right] \cdot \boldsymbol{h}\times\boldsymbol{e} \\ &= \cos\theta \cdot \boldsymbol{e} + \sin\theta \cdot \boldsymbol{h}\times\boldsymbol{e} \\ &= \frac{e - \cosh F}{e\cosh F - 1} \cdot \boldsymbol{e} + \frac{\sqrt{e^2 - 1} \cdot \sinh F}{e\cosh F - 1} \cdot \boldsymbol{h}\times\boldsymbol{e} \end{aligned} \tag{7.4.50}$$

and

$$\begin{aligned} r \cdot \boldsymbol{v} &= \left[(\boldsymbol{r}_0\boldsymbol{e} + a)\sqrt{\mu/|a|} \cdot \sinh(F - F_0) + \boldsymbol{v}_0\boldsymbol{e} \cdot r_0\cosh(F - F_0)\right] \cdot \boldsymbol{e} \\ &\quad + \left[\boldsymbol{r}_0(\boldsymbol{h}\times\boldsymbol{e})\sqrt{\mu/|a|} \cdot \sinh(F - F_0) + \boldsymbol{v}_0(\boldsymbol{h}\times\boldsymbol{e}) \cdot r_0\cosh(F - F_0)\right] \cdot \boldsymbol{h}\times\boldsymbol{e} \\ \boldsymbol{v} &= \frac{h}{r^2}(-\sin\theta \cdot \boldsymbol{e} + \cos\theta \cdot \boldsymbol{h}\times\boldsymbol{e}) \\ &= \frac{h}{r^2}\left(-\frac{\sqrt{e^2 - 1} \cdot \sinh F}{e\cosh F - 1} \cdot \boldsymbol{e} + \frac{e - \cosh F}{e\cosh F - 1} \cdot \boldsymbol{h}\times\boldsymbol{e}\right) \end{aligned} \tag{7.4.51}$$

7.4.6 h-Based Transformation

The energy-based transformation suffers from the fact that it does not include the parabolic orbit and becomes numerically unstable for near-parabolic orbits, i.e. for $\varepsilon \approx 0$. We therefore seek a transformation that stably covers these cases.

Guided by the Eq. (7.3.18), $2\varepsilon \cdot h^2 = \mu^2(e^2 - 1)$, which claims that $\varepsilon \approx 0 \leftrightarrow e \approx 1$ is sensible if $h = \sqrt{\mu p} \neq 0$, we define a new transformation with variable G as

$$dG := \sqrt{\frac{\mu}{p}}\frac{1}{r} \cdot dt = \frac{\mu}{h}\frac{1}{r} \cdot dt \quad \textbf{h-based transformation} \tag{7.4.52}$$

Because it requires $h \neq 0$ and in line with Eq. (7.4.42) we call it *angular-momentum-based (h-based) transformation*. Comparison with Eq. (7.4.42) reveals that $dE = dF = dG\sqrt{|1-e^2|}$. Therefore we obtain for elliptic und hyperbolic orbits the identities

$$\boxed{G = \frac{E}{\sqrt{1-e^2}} = \frac{F}{\sqrt{e^2-1}}} \quad \textbf{universal anomaly} \tag{7.4.53}$$

Observe that G is an angle, by the same token as E, F are, and therefore it formally is an anomaly (see at the end of Sect. 7.3.1). This is why we call it *universal anomaly*. However, otherwise it lacks a geometrical interpretation—at least I do not know of one, to date. Observe that from this result, it follows

$$E = iF$$

which we made use of by a handwaving argument in Sect. 7.4.3 to derive Kepler's equation for hyperbolic orbits from elliptic orbits. The above h-based transformation retains the regularization property of the ε-based transformation and in addition passes smoothly between elliptic, parabolic, and hyperbolic orbits because it remains numerically stable for $|a| \to \infty$.

From Eq. (7.4.43) we find with $dE = dF = dG\sqrt{|1-e^2|}$ the corresponding Newton's gravitational equation of motion

$$\boldsymbol{r}''(G) = -\left(1-e^2\right)(\boldsymbol{r} + a\boldsymbol{e}) \quad \text{Newton's gravitational EoM} \tag{7.4.54}$$

This equation is also a beautifully simple algebraic form of Newton's gravitational equation of motion, which yields the same results as those in Sect. 7.4.5 for

$$\begin{aligned} E &\to G\sqrt{1-e^2} \quad @ \text{ elliptic orbits} \\ F &\to G\sqrt{e^2-1} \quad @ \text{ hyperbolic orbits} \end{aligned}$$

In order to derive a Kepler equation $t(G)$ applying for any $0 \leq e < \infty$, we integrate the above h-based transformation equation and find with $G(t_p) = 0$, where t_p is the epoch, i.e. the time at periapsis,

$$\sqrt{\frac{\mu}{p}}(t - t_p) = \int_0^G r(G') \cdot dG' \quad @ \; G(t_p) = 0 \tag{7.4.55}$$

To solve this equation we need to find $r(G)$. The road to $r(G)$ is to start with Leibniz's radial differential equation $\ddot{r}(t)$, transform it into $r''(G)$, and solve it to obtain $r(G)$. According to Eq. (7.2.10) Leibniz's equation reads

$$\ddot{r} = -\frac{\mu}{r^2} + \frac{\mu p}{r^3}$$

Applying the h-based transformation yields

$$\ddot{r} = -\frac{\mu}{p}\frac{r''}{r^2} - \frac{\dot{r}^2}{r}$$

From Eq. (7.2.11) we have

$$\dot{r}^2 = \frac{2\mu}{r} - \frac{h^2}{r^2} - \frac{\mu}{a}$$

By inserting these two equations into the above Leibniz's equation we find its G-form to be

$$r''(G) = -\left(1-e^2\right)r + p = -\left(1-e^2\right)(r-a) \quad \text{Leibniz's equation} \tag{7.4.56}$$

We solve this Leibniz's equation by substitution $u := r - a$, which delivers $u'' = r''$ and hence

$$u'' = -\left(1-e^2\right)u$$

The general solution of this equation satisfying the condition $r(G=0) = a(1-e)$ and $r'(G=0) = 0$ at the periapsis reads

$$r - a = -ae\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}G^{2n}\left(1-e^2\right)^n = -ae\,\cos\left(G\sqrt{1-e^2}\right) \tag{7.4.57}$$

From this follows

$$\frac{r}{p} = \frac{1}{1-e^2}\left[1 - e\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}G^{2n}\left(1-e^2\right)^n\right]$$

or

$$\boxed{\begin{aligned}\frac{r}{p} &= \frac{1}{1+e} + e\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+2)!}G^{2k+2}\left(1-e^2\right)^k \\ &= \frac{1}{1+e} + eG^2c_2\left(G^2\left(1-e^2\right)\right)\end{aligned}} \quad @\ 0 \le e < \infty \tag{7.4.58}$$

With this we derive the universal Kepler equation by carrying out the integral in Eq. (7.4.55)

$$\boxed{\begin{aligned}\sqrt{\tfrac{\mu}{p^3}}\left(t-t_p\right) &= \tfrac{G}{1+e} + e\sum_{k=0}^{\infty}\tfrac{(-1)^k}{(2k+3)!}G^{2k+3}\left(1-e^2\right)^k \\ &= \tfrac{G}{1+e} + eG^3c_3\left(G^2\left(1-e^2\right)\right)\end{aligned}} \quad \begin{aligned}&\textbf{universal Kepler equation} \\ &@\ 0 \le e < \infty,\ G\left(t_p\right) = 0\end{aligned} \tag{7.4.59}$$

Here $c_2(x)$, $c_3(x)$ are the so-called *Stumpff functions* of second and third order (see box in Sect. 7.4.8). Observe that in the above two enframed equations the singularity at $e = 1$ has been removed and therefore they apply steadily to all Keplerian orbits $0 \le e < \infty$.

We finally provide the expression for $\theta(G)$. From Eq. (7.3.5) we have

$$e \cdot \cos\theta(G) = \frac{p}{r} - 1 = \frac{\cos G\sqrt{1-e^2} - e}{1 - e\,\cos G\sqrt{1-e^2}} \quad @\ 0 \le e < \infty \tag{7.4.60}$$

or alternatively with Eq. (7.4.14c)

$$\begin{aligned} \tan\tfrac{\theta(G)}{2} &= \sqrt{\tfrac{1+e}{1-e}}\tan\left(\tfrac{1}{2}G\sqrt{1-e^2}\right) \\ &= \frac{1+e}{2}G\sum_{n=0}^{\infty}(-1)^n \cdot \frac{4\left(2^{2n+2}-1\right)}{(2n+2)!}B_{2n+2}G^{2n}(1-e^2)^n \\ &= \frac{1+e}{2}G\left[1 + \tfrac{1}{12}G^2(1-e^2) + \tfrac{1}{120}G^4(1-e^2)^2 + \ldots\right] \end{aligned} \quad @\ 0 \le e < \infty \tag{7.4.61}$$

where

$$\begin{aligned} B_{2n} &= 0, \frac{1}{6}, -\frac{1}{30}, \frac{1}{42}, -\frac{1}{30}, \frac{5}{66} \quad n = 0,1,2,3,4,5 \\ B_{2n} &= (-1)^{n+1}\frac{2(2n)!}{(2\pi)^{2n}}\left(1 + \frac{1}{2^{2n}} + \frac{1}{3^{2n}}\right) \quad @\ n \ge 5,\ \frac{\Delta B_{2n}}{B_{2n}} < 10^{-6} \end{aligned}$$

are the Bernoulli numbers. Observe that in the above power series expansion the singularity at $e = 1$ is removed, as well.

Elliptic and Hyperbolic Orbits

From Eq. (7.4.57) follows

$$\begin{aligned} r(G) &= a\left[1 - e\cos\left(G\sqrt{1-e^2}\right)\right] \quad @\ e < 1 \\ r(G) &= a\left[1 - e\cosh\left(G\sqrt{e^2-1}\right)\right] \quad @\ e > 1 \end{aligned}$$

and from the universal Kepler Eq. (7.4.59)

$$\begin{aligned} \sqrt{\tfrac{\mu}{p^3}}(t - t_p) &= \tfrac{1}{1-e^2}\left[G - e\tfrac{1}{\sqrt{1-e^2}}\sin\left(G\sqrt{1-e^2}\right)\right] \\ \rightarrow\quad \sqrt{\tfrac{\mu}{a^3}}(t - t_p) &= G\sqrt{1-e^2} - e\,\sin\left(G\sqrt{1-e^2}\right) \quad @\ e < 1 \end{aligned}$$

and

$$\begin{aligned} \sqrt{\tfrac{\mu}{p^3}}(t - t_p) &= \tfrac{1}{e^2-1}\left[e\tfrac{1}{\sqrt{e^2-1}}\sinh\left(G\sqrt{e^2-1}\right) - G\right] \\ \rightarrow\quad \sqrt{\tfrac{\mu}{a^3}}(t - t_p) &= e\,\sinh\left(G\sqrt{e^2-1}\right) - G\sqrt{e^2-1} \quad @\ e > 1 \end{aligned}$$

Comparison of this result with Eqs. (7.4.14a), (7.4.29a) and Eqs. (7.4.29a), (7.4.30) corroborates the transformation rule $G\sqrt{1-e^2} \to E$, $G\sqrt{e^2-1} \to F$, and $G \to \theta$ for $e = 0$.

The general solution procedure to find for a given $t > t_p$ the state vector $\boldsymbol{r}(t)$,$v(t)$ for elliptic and hyperbolic orbits is therefore:

Solve Eq. (7.4.59) by Newton's method.

$\to E(t) = G(t)\sqrt{1-e^2}$ @ **elliptic orbits**

$r(t) = a(1 - e\cos E) \to \cos\theta(t) = \frac{1}{e}\left(\frac{p}{r} - 1\right)$

and state vector $\boldsymbol{r}(t)$,$v(t)$ from Eq. (7.4.44), Eq. (7.4.45)

or

$\to F(t) = G(t)\sqrt{e^2-1}$ @ **hyperbolic orbits**

$r(t) = a(1 - e\cosh F) \to \cos\theta(t) = \frac{1}{e}\left(\frac{p}{r} - 1\right)$

and state vector $\boldsymbol{r}(t)$,$\boldsymbol{v}(t)$ from Eqs. (7.4.46), (7.4.47).

Parabolic Orbits

For parabolic orbits with $e = 1$ we derive from Eq. (7.4.59)

$$\frac{q}{3} := \sqrt{\frac{\mu}{p^3}}(t - t_p) = \frac{1}{2}G + \frac{1}{6}G^3 \quad @\ G(t_p) = 0 \tag{7.4.62}$$

From Eq. (7.4.37) follows the solution

$$G(t) = \left(\sqrt{q^2+1} + q\right)^{1/3} - \left(\sqrt{q^2+1} - q\right)^{1/3} = 2\sinh\left(\frac{1}{3}\operatorname{arsinh} q\right)$$

Equation (7.4.58) yields

$$r(t) = \frac{1}{2}p\left(1 + G^2\right) \tag{7.4.63}$$

hence

$$\cos\theta(t) = \frac{p}{r} - 1 = \frac{1 - G^2}{1 + G^2}$$

or

$$\boxed{G = \tan\frac{\theta}{2}} \tag{7.4.64}$$

This agrees with the result derived from Eq. (7.4.61) for $e = 1$. The state vector then is determined by

$$\begin{aligned} \boldsymbol{r}(t) &= \cos\theta \cdot \boldsymbol{e} + \sin\theta(\boldsymbol{h} \times \boldsymbol{e}) \\ \boldsymbol{v}(t) &= \frac{h}{r^2}[-\sin\theta \cdot \boldsymbol{e} + \cos\theta(\boldsymbol{h} \times \boldsymbol{e})] \end{aligned} \tag{7.4.65}$$

Note that with the identity $G \equiv \tan\theta/2$ we recover all the results already derived in Sect. 7.4.4 for the parabolic orbit.

Near-Parabolic Orbits
For near-parabolic orbits with $e \approx 1$ Eq. (7.4.59) delivers

$$\sqrt{\frac{\mu}{p^3}}(t - t_p) = \frac{G}{1+e} + \frac{e}{6}G^3 - \left(1 - e^2\right)\frac{e}{120}G^5 \tag{7.4.66}$$

$\rightarrow$ $G(t)$ by Newton's method

From Eq. (7.4.58) we find

$$r(t) = \frac{1}{2}p\left(G^2 + 1\right) - \frac{1}{4}\left(1 - e^2\right)p\left(\frac{1}{6}G^2 + 1\right)G^2 \tag{7.4.67}$$

$\rightarrow$ $\cos\theta(t) = \frac{1}{e}\left(\frac{p}{r} - 1\right)$

and state vector as given by Eq. (7.4.65).

For $e = 1$ these results obviously pass smoothly into the parabolic case above. For later purposes we solve Eq. (7.4.67) iteratively for G and find

$$\frac{1}{2}p\left(G^2 + 1\right) = r + \left(1 - e^2\right)\frac{p}{24}\left(\frac{2r}{p} + 5\right)\left(\frac{2r}{p} - 1\right)$$

From this follows

$$G = \sqrt{\left(\frac{2r}{p} - 1\right)\left[1 + \frac{1 - e^2}{12}\left(\frac{2r}{p} + 5\right)\right]} \tag{7.4.68}$$

Universal Optimal Solver
Based on the above results we have the following universal solver for state vector propagation, which is optimal in the sense of regularization.

Let $\boldsymbol{r}_0, \boldsymbol{v}_0$ be the state vector given at time t_0.

If the type of orbit and the metric elements of the orbit are not known, determine them by

$$\begin{array}{ll} \boldsymbol{h} = \boldsymbol{r}_0 \times \boldsymbol{v}_0 & \rightarrow p = \boldsymbol{h}^2/\mu \\ \boldsymbol{e} = \frac{1}{\mu}\boldsymbol{v}_0 \times \boldsymbol{h} - \frac{1}{r_0}\boldsymbol{r}_0 & \rightarrow e = \sqrt{\boldsymbol{e}\boldsymbol{e}} \\ \dfrac{1}{a} = \dfrac{2}{r_0} - \dfrac{\boldsymbol{v}_0^2}{\mu} & \end{array}$$

Derive G_0 from the following relevant equation

$$\begin{array}{ll} e\cos\left(G_0\sqrt{1-e^2}\right) = 1 - r_0/a & @\ e<1,\ \text{elliptic orbit} \\ e\cosh\left(G_0\sqrt{e^2-1}\right) = 1 + r_0/|a| & @\ e>1,\ \text{hyperbolic orbit} \\ G_0 = \sqrt{\left(\frac{2r_0}{p}-1\right)\left[1+\frac{1-e^2}{12}\left(\frac{2r_0}{p}+5\right)\right]} & @\ e\approx 1,\ \text{near-parabolic orbit} \end{array}$$

Let $$R(G) = \frac{G}{1+e} + e\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+3)!}G^{2k+3}\left(1-e^2\right)^k$$

$$\rightarrow \quad R'(G) = \frac{1}{1+e} + e\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+2)!}G^{2k+2}\left(1-e^2\right)^k = \frac{r}{p}$$

Then G_0, t_0 solves

$$\sqrt{\frac{\mu}{p^3}}(t_0 - t_p) = R(G_0)$$

With this we eliminate the unknown t_p and derive the customized transcendental equation

$$\sqrt{\frac{\mu}{p^3}}(t - t_0) + R(G_0) = R(G)$$

Therefore any instance of the transcendental equation above is solved for G by the Newton iteration

$$G_{i+1} = G_i - \frac{R(G_i) - R(G_0) - \sqrt{\mu/p^3}(t-t_0)}{R'(G_i)}, \quad i = 1, \ldots$$

Optimal initial values for elliptic orbits with $M = \sqrt{\mu/a^3}(t-t_0) + \left(1-e^2\right)^{3/2}R(G_0)$ are

$$\begin{array}{ll} G_1 = \dfrac{M + e^2\left[(6M)^{1/3} - M\right]}{\sqrt{1-e^2}} & @\ e<1,\ 0 \le M < 0.25 \\ G_1 = \dfrac{1}{\sqrt{1-e^2}}\left[M + \dfrac{e\,\sin M}{1 - \sin(M+e) + \sin M}\right] & @\ e<1,\ 0.25 \le M \le \pi \end{array}$$

and for parabolic or near-parabolic orbits

$$G_1 = \sqrt{\left(\frac{2r_0}{p} - 1\right)\left[1 + \frac{1-e^2}{12}\left(\frac{2r_0}{p} + 5\right)\right]} \quad @\ e \approx 1$$

The denominator of the iteration fraction, $R'(G)$, delivers directly the universal result

$$r(t) = p \cdot R'(G) \rightarrow e\ \cos\theta(t) = \frac{p}{r} - 1$$

and

$$\begin{aligned} E &= G\sqrt{1-e^2} \quad @\ \text{elliptic orbits} \\ F &= G\sqrt{e^2-1} \quad @\ \text{hyperbolic orbits} \end{aligned}$$

This inserted into the relevant solutions $\boldsymbol{r}(E)$, $\boldsymbol{v}(E)$ or $\boldsymbol{r}(F)$, $\boldsymbol{v}(F)$ in Sect. 7.4.5, or Eq. (7.4.65) for near-parabolic orbits, respectively, delivers the propagated state vector.

7.4.7 Conventional State Vector Propagation

A more classical approach for state vector propagation is encapsulated in the following algorithm

Algorithm for Universal State Vector Propagation $\boldsymbol{r}(t_0), \boldsymbol{v}(t_0) \rightarrow \boldsymbol{r}(t), \boldsymbol{v}(t)$

1. Let $\boldsymbol{r}_0, \boldsymbol{v}_0$ be given at time t_0.
2. If the type of orbit and the metric elements of the orbit are not known, determine them by

$$\begin{aligned} \boldsymbol{h} &= \boldsymbol{r}_0 \times \boldsymbol{v}_0 && \rightarrow && p = \boldsymbol{h}^2/\mu \\ \boldsymbol{e} &= \frac{1}{\mu}\boldsymbol{v}_0 \times \boldsymbol{h} - \frac{1}{r_0}\boldsymbol{r}_0 && \rightarrow && e = \sqrt{\boldsymbol{e}\boldsymbol{e}} \\ \frac{1}{a} &= \frac{2}{r_0} - \frac{\boldsymbol{v}_0^2}{\mu} \end{aligned}$$

3. If $e < 1$, then elliptic orbit
   a. Determine E_0 from $e\ \cos E_0 = 1 - r_0/a$
   b. For $t > t_0$ solve $E(t) - e\ \sin E(t) = \sqrt{\mu/a^3} \cdot (t - t_0) + E_0 - \sin E_0$ for $E(t)$, for instance with Newton's method (see Eq. (7.4.16)).
   c. If $e \ll 1$, determine $\boldsymbol{r}(t)$ from Eq. (7.4.44) and $\boldsymbol{v}(t)$ with $r = \sqrt{\boldsymbol{r}\boldsymbol{r}}$ from Eq. (7.4.45).
   If $e \gg 0$ and $\angle(\boldsymbol{r}_0, \boldsymbol{v}_0) \rightarrow 0$ determine $\boldsymbol{r}(t)$ from Eq. (7.4.48) and $\boldsymbol{v}(t)$ with $r = \sqrt{\boldsymbol{r}\boldsymbol{r}}$ from Eq. (7.4.49).

4. If $e > 1$, then hyperbolic orbit
   a. Determine F_0 from $e \cosh F_0 = 1 + r_0/|a|$
   b. For $t > t_0$ solve $e \sinh F(t) - F(t) = \sqrt{\mu/|a|^3} \cdot (t - t_0) + e \sinh F_0 - F_0$ for $F(t)$, for instance with Newton's method (see Eq. (7.4.32)).
   c. Determine $\boldsymbol{r}(t)$ from Eq. (7.4.46) and $\boldsymbol{v}(t)$ with $r = \sqrt{\boldsymbol{r}\boldsymbol{r}}$ from Eq. (7.4.47). If $\angle(\boldsymbol{r}_0, \boldsymbol{v}_0) \to 0$ determine $\boldsymbol{r}(t)$ from Eq. (7.4.50) and $\boldsymbol{v}(t)$ with $r = \sqrt{\boldsymbol{r}\boldsymbol{r}}$ from Eq. (7.4.51).
5. If $e \approx 1$, then parabolic orbit or near-parabolic orbits
   a. Initialization:

$$G_0 = \sqrt{\left(\frac{2r_0}{p} - 1\right)\left[1 + \frac{1 - e^2}{12}\left(\frac{2r_0}{p} + 5\right)\right]}$$
$$R(G) := \frac{G}{1+e} + \frac{e}{6}G^3 - \left(1 - e^2\right)\frac{e}{120}G^5$$

   b. For $t > t_0$ use initial value

$$G_1 = 2 \sinh\left[\frac{1}{3}\operatorname{arsinh}\left(3\sqrt{\frac{\mu}{p^3}}(t - t_0) + 3R(G_0)\right)\right]$$

   and determine $G(t)$ from the Newton iteration

$$G_{i+1} = G_i - \frac{R(G_i) - R(G_0) - \sqrt{\mu/p^3}(t - t_0)}{R'(G_i)}, \quad i = 1, \ldots$$

   c. Determine

$$r(t) = \frac{1}{2}p\left(G^2 + 1\right) - \frac{1}{4}\left(1 - e^2\right)p\left(\frac{1}{6}G^2 + 1\right)G^2$$
$$\cos\theta(t) = \frac{1}{e}\left(\frac{p}{r} - 1\right), \sin\theta(t) = \sqrt{1 - \frac{1}{e^2}\left(\frac{p}{r} - 1\right)^2}$$

   d. Final state vector:

$$\boldsymbol{r}(t) = \cos\theta \cdot \boldsymbol{e} + \sin\theta \cdot (\boldsymbol{h} \times \boldsymbol{e})$$
$$\boldsymbol{v}(t) = \frac{h}{r^2}\left[-\sin\theta \cdot \boldsymbol{e} + \cos\theta \cdot (\boldsymbol{h} \times \boldsymbol{e})\right]$$

7.4.8 Universal Variable Formulation

There exist other propagation methods: for instance, the Lagrange–Gibbs f and g solution method (see Schaub and Junkins (2003) or Chobotov (2002)), but most importantly the so-called universal variable formulation (a.k.a. universal approach). In the following we give only the basic approach without providing the full algorithm (cf. Danby 2003).

The starting point of the universal variable formulation is the Sundman transformation

$$ds = \frac{1}{r} \cdot dt \quad \textbf{Sundman transformation} \tag{7.4.69}$$

Here s is the so-called universal variable of dimension [time]/[length] and therefore it is formally not an anomaly, that is not an angle (see Remark at the end of Sect. 7.3.1). Obviously, the Sundman transformation is essentially the same as the h-based transformation Eq. (7.4.52). Due to its different normalization it delivers a slightly different transformed gravitational equation of motion

$$\boldsymbol{r}'' = -\alpha^2 \boldsymbol{r} - \mu \boldsymbol{e}$$

with

$$\alpha^2 := \frac{\mu}{a} = \frac{2\mu}{r_0} - \boldsymbol{v}_0^2$$

Rather than turning it into an harmonic oscillation EoM by the transformation $\boldsymbol{u} := \boldsymbol{r} + a\boldsymbol{e}$ as done in Sect. 7.4.5, it is differentiated to deliver

$$\boldsymbol{r}''' = -\alpha^2 \boldsymbol{r}' \tag{7.4.70}$$

It can be shown (see Danby 2003, Sect. 6.9) that with initial conditions $\boldsymbol{r}_0, \boldsymbol{v}_0, t_0$ the solution to this vector differential equation reads

$$\begin{aligned} \boldsymbol{r}(s) &= f \cdot \boldsymbol{r}_0 + g \cdot \boldsymbol{v}_0 \\ \boldsymbol{v}(s) &= \dot{f} \cdot \boldsymbol{r}_0 + \dot{g} \cdot \boldsymbol{v}_0 \end{aligned} \tag{7.4.71}$$

with

$$\begin{aligned} f &= 1 - (\mu/r_0) s^2 c_2\left(\alpha^2 s^2\right) \\ \dot{f} &= -[\mu/(r r_0)] s c_1\left(\alpha^2 s^2\right) \\ g &= t - t_0 - \mu s^3 c_3\left(\alpha^2 s^2\right) \\ \dot{g} &= 1 - (\mu/r) s^2 c_2\left(\alpha^2 s^2\right) \end{aligned} \tag{7.4.72}$$

Here the functions $c_n(\alpha^2 s^2)$ are the so-called Stumpff functions as expounded in the box.

Stumpff Functions
The Stumpff functions $c_n(x)$ are defined as

$$c_n(x) := \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{(2k+n)!}, \quad n \geq 0$$

with properties

$$\begin{aligned}
c_n(0) &= \frac{1}{n!} \\
c_n(x) &= \frac{1}{n!} - x \cdot c_{n+2}(x) \\
c_0(4x) &= 2c_0^2(x) = 2[1 - xc_2(x)]^2 \\
c_1(4x) &= c_0(x)c_1(x) = [1 - xc_2(x)][1 - xc_3(x)] \\
c_2(4x) &= \tfrac{1}{2}c_2^2(x) \\
c_3(4x) &= \tfrac{1}{4}c_2(x) + \tfrac{1}{4}c_0(x)c_3(x) = \tfrac{1}{4}c_2(x) + \tfrac{1}{4}[1 - xc_2(x)]c_3(x)
\end{aligned}$$

The latter four equations are useful for two things: First, for reducing an angle x by factors of 4 to a value smaller than a given amount (less than half an orbit). Second, only the power series for c_2 and c_3 need to be used. Explicitely, the first four Stumpff functions read

$$c_0(x) := \begin{cases} \cos\sqrt{x} & x > 0 \\ \cosh\sqrt{-x} & x < 0 \\ 1 & x = 0 \end{cases}$$

$$c_1(x) := \begin{cases} \sin\sqrt{x}/\sqrt{x} & x > 0 \\ \sinh\sqrt{-x}/\sqrt{-x} & x < 0 \\ 1 & x = 0 \end{cases}$$

$$c_2(x) := \begin{cases} (1 - \cos\sqrt{x})/x & x < 0 \\ (\cosh\sqrt{-x} - 1)/(-x) & x < 0 \\ 1/2 & x = 0 \end{cases}$$

$$c_3(x) := \begin{cases} (\sqrt{x} - \sin\sqrt{x})/x^{3/2} & x > 0 \\ (\sinh\sqrt{-x} - \sqrt{-x})\Big/(-x)^{3/2} & x < 0 \\ 1/6 & x = 0 \end{cases}$$

Since $x = \alpha^2 s^2$, the upper of the three expressions holds for elliptic orbits with $\alpha > 0$, the middle for hyperbolic orbits with $\alpha < 0$, and the lower for parabolic orbits with $\alpha = 0$.

Given the Stumpff functions we find

$$\alpha \cdot sc_1 = \begin{cases} \sin(\alpha s) \\ \sinh(\alpha s) \\ \alpha s \end{cases}$$

$$\alpha^2 s^2 c_2 = \begin{cases} 1 - \cos(\alpha s) \\ 1 - \cosh(\alpha s) \\ \alpha^2 s^2/2 \end{cases}$$

$$\alpha^3 s^3 c_3 = \begin{cases} \alpha s - \sin(\alpha s) \\ \alpha s - \sinh(\alpha s) \\ \alpha^3 s^3/6 \end{cases}$$

where again the upper of the three expressions holds for elliptic orbits, the middle for hyperbolic orbits, and the lower for parabolic orbits.

The initial conditions read $\boldsymbol{r}_0, \boldsymbol{v}_0, t_0$. So, to apply Eq. (7.4.72) we still have to determine $s(t)$ for a given time $t > t_0$. For this we have to rewrite Kepler's equation. It is easily verified that any of the above mentioned basic solution functions $f(\alpha^2 s^2)$ exhibit the key property

$$f'''(\alpha^2 s^2) = -\alpha^2 \cdot f'(\alpha^2 s^2)$$

A general solution of Eq. (7.4.70) therefore is

$$\boldsymbol{r}(s) = \boldsymbol{r}_0 + \boldsymbol{r'}_0 sc_1(\alpha^2 s^2) + \boldsymbol{r''}_0 s^2 c_2(\alpha^2 s^2)$$

With this Kepler's equation can be derived by integration of Eq. (7.4.69) as

$$t - t_0 = \int_0^s r(\bar{s}) \cdot d\bar{s} = r_0 sc_1(\alpha^2 s^2) + r_0 \dot{r}_0 s^2 c_2(\alpha^2 s^2) + \mu s^3 c_3(\alpha^2 s^2) \tag{7.4.73}$$

where the constants $r_0, \dot{r}_0, \alpha$ need to be determined from the initial conditions $\boldsymbol{r}_0, \boldsymbol{v}_0, t_0$. This form of Kepler's equation is equivalent to Eq. (7.4.15c), as can easily be verified, and is therefore convenient, because it does not make use of the time of passage through periapsis in the original Kepler's Eq. (7.4.15), which usually is not known, but of the known state vector at a given time t_0. The root of this equation delivers the variable $s(t)$, which has the same property as the eccentric anomaly for the Kepler transformation.

This universal variable formulation is widely discussed in the literature and used in practice. Battin (1964, 1987), Chobotov (2002, Sect. 4.3), and Danby (2003, Sect. 6.9) provide a good overview, and Chobotov (2002) presents in his Sect. 4.5 a practicable algorithm for this propagator; there are many examples and a MATLAB code for this propagator in Curtis (2005, Sect. 3.7).

7.5 Radial Trajectories

For a vanishing angular momentum, $h = 0$, it is inadmissible to take the cross product of $\boldsymbol{h}$ with Eq. (7.1.19) as done in Eq. (7.3.1). Therefore the general orbit Eq. (7.3.5) is not a valid solution in this case. According to Eq. (7.2.7) $h = 0$ happens whenever the transverse velocity vanishes, $v_\theta = 0$ (of course, as $h = r \cdot v_\theta = const$, v_θ is either always zero or never zero)—for instance, if one positions a body at an arbitrary distance from the central body with an initial velocity $v = 0$. To derive the equation of motion we resort to the equations of radial motion (7.2.11) where we set $h = 0$. This results in

$$\begin{aligned} \dot{r}^2 &= \mu\left(\frac{2}{r} - \frac{1}{a}\right) \\ \dot{\theta} &= 0 \end{aligned} \tag{7.5.1}$$

For the lateral motion we hence find the solution

$$\theta = const$$

That is, the body moves on a straight line toward the central body (or away from it) until it crashes into the central body. This is a radial trajectory, a.k.a. *rectilinear orbit*, however, descripively they do not orbit the central mass. Observe that if we convey the definition Eq. (7.3.6) $p := h^2/\mu =: a(1 - e^2)$ to radial trajectories, we may consider them as degenerate elliptic, parabolic or hyperbolic orbits with $e = 1,\ a \neq 0$. Accordingly, we will classify them in the following as radial elliptic, radial parabolic, and radial hyperbolic trajectories. Owing to this degeneration, e and p are no longer discriminatory orbital elements, but a is. We therefore will adapt a as a discriminative orbital parameter.

The equation of radial motion for radial trajectories is just the square root of the above radial differential equation

$$\frac{dr}{dt} = \pm\sqrt{\mu}\sqrt{\frac{2}{r} - \frac{1}{a}} \quad \textbf{equation of radial motion} \tag{7.5.2}$$

The parameter a is the only characteristic orbital element of the radial trajectory shape and signifies its total specific orbital energy ε according to the general Eq. (7.3.19)

$$\varepsilon = -\frac{\mu}{2a}$$

The two signs in Eq. (7.5.2) indicate two different modes of motion of the body: At the plus sign the body moves outbound (increasing dr with increasing dt) while at negative sign it moves inbound. The sign of course depends on the initial conditions r_0, v_0, t_0. If the body initially moves outward, $\dot{r}_0 = v_0 > 0$, then $\dot{r} > 0$ also for $t > 0$ until $v = 0$, and vice versa.

Also the orbital element a of the radial trajectory is determined by the initial conditions. We find from Eq. (7.5.1)

$$\frac{v_0^2}{\mu} = \frac{2}{r_0} - \frac{1}{a}$$

and hence

$$a = \frac{\mu r_0}{2\mu - r_0 v_0^2} \tag{7.5.3}$$

We see from this equation that a may be finite positive or negative, or infinite, depending on the initial conditions. This gives rise to three different types of radial trajectories characterized by a

1. Radial elliptic trajectory: $a > 0, \quad 0 < r \leq 2a$
2. Radial parabolic trajectory: $a = \infty, \quad 0 < r < \infty$
3. Radial hyperbolic trajectory: $a < 0, \quad 0 < r < \infty$

For each type of trajectory, the body may move inward or outward, so we have to discern in total six different cases depending on the initial velocity v_0, which we are now going to investigate.

7.5.1 Radial Elliptic Trajectory

We first assume $a > 0$ and $0 < r \leq 2a$.

Note The radial ellipse is an ellipse with $e = 1$. For this we have from Eq. (7.4.6) *and* (7.4.7) $r_{per} = 0$ *and* $r_{apo} = 2a$: *The focal point coincides with the periapsis at* $r = 0$ *and the empty focus with the apoapsis at* $r = 2a$. *This is the reason for the seemingly odd condition* $r \leq 2a$.

Inward Motion: $v_0 < 0$
From Eq. (7.5.2) the equation of radial motion is

$$\frac{dr}{dt} = -\sqrt{\mu}\sqrt{\frac{2}{r} - \frac{1}{a}} \tag{7.5.4}$$

Separating the variables yields

$$-\frac{dr/a}{\sqrt{2a/r - 1}} = \sqrt{\frac{\mu}{a^3}} \cdot dt$$

Substituting $x = r/a \rightarrow dx = dr/a$ and integrating leads to

$$(t - t_0)\sqrt{\frac{\mu}{a^3}} = -\int_{r_0/a}^{r/a} \frac{dx}{\sqrt{2/x - 1}} = \left[-x\sqrt{\frac{2}{x} - 1} + 2\arcsin\sqrt{\frac{x}{2}}\right]_{r/a}^{r_0/a}$$

With

$$\frac{r_0}{a}\sqrt{\frac{2a}{r_0} - 1} = \frac{r_0 v_0}{\sqrt{\mu a}}$$

we finally get

$$(t - t_0)\sqrt{\frac{\mu}{a^3}} = \frac{r}{a}\sqrt{\frac{2a}{r} - 1} - \frac{r_0 v_0}{\sqrt{\mu a}} - 2\arcsin\sqrt{\frac{r}{2a}} + 2\arcsin\sqrt{\frac{r_0}{2a}} \tag{7.5.5}$$

This is the trajectory equation of the body moving inward. Because this is an implicit solution, $t = t(r)$, it needs to be solved for r, for instance with Newton's method to find $r = r(t)$ explicitly.

It can be shown (see Problem 7.9) that for $a \rightarrow \infty$, $v_0 = 0$, and $t - t_0 \ll \sqrt{\mu/r_0^3}$, the explicit solution to this equation is a near-radial parabolic orbit (see Sect. 7.5.3) with orbit equation

$$r^{3/2}(t) = r_0^{3/2} - \sqrt{\frac{9\mu}{2}} \cdot (t - t_0)\left[1 - \frac{r_0}{4a}\right] \quad @\ a \rightarrow \infty,\ v_0 = 0,\ t - t_0 \ll \sqrt{\mu/r_0^3}$$

Outward Motion: $v_0 > 0$
From Eq. (7.5.2) we get

$$\frac{dr}{dt} = \sqrt{\mu}\sqrt{\frac{2}{r} - \frac{1}{a}} \tag{7.5.6}$$

With just the opposite sign, we derive from the above

$$(t - t_0)\sqrt{\frac{\mu}{a^3}} = -\frac{r}{a}\sqrt{\frac{2a}{r} - 1} + \frac{r_0 v_0}{\sqrt{\mu a}} + 2\arcsin\sqrt{\frac{r}{2a}} - 2\arcsin\sqrt{\frac{r_0}{2a}} \tag{7.5.7}$$

This is the implicit trajectory equation of the body moving outward. Again it needs to be solved by Newton's method for $r = r(t)$.

Numerical Determination
As with all Keplerian equations, Eq. (7.5.7) suffers from the fact that it is implicit for $r(t)$ and this in quite an intricate way. So, for numerical solutions we try to

resort to simpler solutions. In 7.4.5 we have seen that the ε-based transformation is applicable for $h \to 0$. We therefore can determine the trajectory and its velocity as a function of time by applying the corresponding Kepler's Eq. (7.4.15b) with $e = 1$ (see also Herrick (1971)).

$$\begin{aligned} &M = \sqrt{\frac{\mu}{a^3}}(t - t_0), \quad E_0 = \arccos\left(1 - \frac{r_0}{a}\right) \\ &E - \sin E = M + E_0 - \sin E_0 \quad \to \quad E(t) \\ &r = a(1 - \cos E), \quad v = \frac{\sqrt{\mu a}}{r} \sin E \end{aligned} \qquad @ \; r_0 v_0^2 < 2\mu \tag{7.5.8}$$

7.5.2 Radial Hyperbolic Trajectory

We now assume $a < 0$ and $0 < r < \infty$.

Inward Motion: $v_0 < 0$
For $a < 0$ and from Eq. (7.5.2) we get

$$\frac{dr}{dt} = -\sqrt{\mu}\sqrt{\frac{2}{r} + \frac{1}{|a|}} \tag{7.5.9}$$

By the same token as in Sect. 7.5.1, we obtain

$$(t - t_0)\sqrt{\frac{\mu}{|a|^3}} = -\int_{r_0/|a|}^{r/|a|} \frac{dx}{\sqrt{2/x + 1}} = \left[x\sqrt{\frac{2}{x} + 1} - 2\,\text{arcsinh}\sqrt{\frac{x}{2}}\right]_{r/|a|}^{r_0/|a|}$$

and finally for $a < 0$

$$(t - t_0)\sqrt{-\frac{\mu}{a^3}} = \frac{r}{a}\sqrt{1 - \frac{2a}{r}} + 2\,\text{arcsinh}\sqrt{-\frac{r}{2a}} + \frac{r_0 v_0}{\sqrt{-\mu a}} - 2\,\text{arcsinh}\sqrt{-\frac{r_0}{2a}} \tag{7.5.10}$$

A Newton iteration yields $r = r(t)$.

Outward Motion: $v_0 > 0$
From $a < 0$ and

$$\frac{dr}{dt} = \sqrt{\mu}\sqrt{\frac{2}{r} + \frac{1}{|a|}} \tag{7.5.11}$$

we find in analog to the above

$$(t-t_0)\sqrt{-\frac{\mu}{a^3}} = -\frac{r}{a}\sqrt{1-\frac{2a}{r}} - 2\,\operatorname{arsinh}\sqrt{-\frac{r}{2a}} - \frac{r_0 v_0}{\sqrt{-\mu a}} + 2\,\operatorname{arsinh}\sqrt{-\frac{r_0}{2a}} \tag{7.5.12}$$

A Newton iteration yields $r = r(t)$.

Numerical Determination
By the same token as for radial elliptic trajectories, radial hyperbolic trajectories are numerically more easily determined from a customized Eq. (7.4.30) by the procedure

$$\begin{aligned} &M = \sqrt{-\tfrac{\mu}{a^3}}\,(t-t_0), F_0 = \operatorname{arcosh}\left(1-\tfrac{r_0}{a}\right) \\ &\sinh F - F = M + \sinh F_0 - F_0 \quad \rightarrow \quad F(t) \\ &r = a(1-\cosh F), v = \tfrac{\sqrt{-\mu a}}{r}\sinh F \end{aligned} \qquad @\ r_0 v_0^2 > 2\mu \tag{7.5.13}$$

7.5.3 Radial Parabolic Trajectory

Comets that arrive from the border of our solar system (from the so-called Oort cloud) exhibit $e \approx 1$, $a \approx \infty$. This is a parabolic orbit. In addition, if they exhibit $h \approx 0$, we have in this special case $a = \infty$.

Inward Motion: $v_0 < 0$
If $a = \infty$, Eq. (7.5.2) reads for inward motion

$$\frac{dr}{dt} = -\sqrt{\frac{2\mu}{r}} \tag{7.5.14}$$

Separating the variables and integration leads to

$$(t-t_0)\sqrt{2\mu} = -\int_{r_0}^{r}\sqrt{r} = \frac{2}{3}\left(r_0^{3/2} - r^{3/2}\right)$$

where t counts the time from a given initial position $r_0 = r(t_0)$ (for instance, the point of first comet sighting) to a given later trajectory position r. Hence, we obtain

$$r^{3/2}(t) = r_0^{3/2} - \sqrt{\frac{9\mu}{2}}\cdot(t-t_0) \tag{7.5.15}$$

Outward Motion: $v_0 > 0$
The equation of radial motion in this case is

$$\frac{dr}{dt} = \sqrt{\frac{2\mu}{r}} \tag{7.5.16}$$

and by the same token as above we get

$$r^{3/2}(t) = r_0^{3/2} + \sqrt{\frac{9\mu}{2}} \cdot (t - t_0) \tag{7.5.17}$$

Both radial parabolic solutions are explicit in t.

Note that the radial parabolic trajectory is the only solution to Newton's gravitational EoM, which is not regularizable by an ε-based or h-based transformation (see Sect. 7.4.5 and 7.4.6) because in this case $\varepsilon = 0$ and $h = 0$ simultaneously. Fortunatelly, this case does not need regularization because it is not transcendental and therefore exhibits an explicit solution $r(t)$.

7.5.4 Free Fall

Inbound trajectories bear the special situation where the body is placed at a certain distance from the origin and after free fall collides with the surface of the center body or its origin after time t_{col}. To determine t_{col}, let R be the radius of the center body and $t_0 = 0$. Then we get from Eqs. (7.5.5), (7.5.10), and (7.5.15)

$$t_{col} = \begin{cases} \sqrt{\frac{a^3}{\mu}}\left(\frac{R}{a}\sqrt{\frac{2a}{R} - 1} - 2\arcsin\sqrt{\frac{R}{2a}} - \frac{r_0 v_0}{\sqrt{\mu a}} + 2\arcsin\sqrt{\frac{r_0}{2a}}\right) & @ \quad 0 < a < \infty \\ \sqrt{\frac{2}{9\mu}}\left(r_0^{3/2} - R^{3/2}\right) & @ \quad a = \infty \\ \sqrt{-\frac{a^3}{\mu}}\left(\frac{R}{a}\sqrt{1 - \frac{2a}{R}} + 2\,\mathrm{arcsinh}\sqrt{-\frac{R}{2a}} + \frac{r_0 v_0}{\sqrt{-\mu a}} - 2\,\mathrm{arcsinh}\sqrt{-\frac{r_0}{2a}}\right) & @ \quad a < 0 \end{cases} \tag{7.5.18}$$

For the time to collision with the origin we set $R = 0$ and get

$$t_{col} = \begin{cases} \sqrt{\frac{a^3}{\mu}}\left(-\frac{r_0 v_0}{\sqrt{\mu a}} + 2\arcsin\sqrt{\frac{r_0}{2a}}\right) & @ \quad 0 < a < \infty \\ \sqrt{\frac{2}{9\mu}}\, r_0^{3/2} & @ \quad a = \infty \\ \sqrt{-\frac{a^3}{\mu}}\left(\frac{r_0 v_0}{\sqrt{-\mu a}} - 2\,\mathrm{arcsinh}\sqrt{-\frac{r_0}{2a}}\right) & @ \quad a < 0 \end{cases} \tag{7.5.19}$$

Radial Elliptic Trajectory
If $v_0 = 0$ then $a = r_0/2$ and we get for the radial elliptic trajectory

$$t_{col} = \sqrt{\frac{r_0^3}{8\mu}}\left(\frac{2R}{r_0}\sqrt{\frac{r_0}{R} - 1} + 2\arccos\sqrt{\frac{R}{r_0}}\right) \quad @\ v_0 = 0 \tag{7.5.20}$$

If even $R = 0$, then

$$t_{col} = \frac{\pi}{\sqrt{8\mu}} r_0^{3/2} \quad @\ v_0 = 0,\ \ R \equiv r_{col} = 0 \tag{7.5.21}$$

Note that this last result could also be derived if applying the orbiting time of an elliptic orbit (Eq. (7.4.12)) $T = 2\pi\sqrt{a^3/\mu}$ to this case with $a = r_0/2$. This is possible because it is independent of the orbit is eccentricity. The collision time then is half an orbit revolution.

Example

Let us assume that the Earth's motion would abruptly be stopped. When would it crash into the center of the Sun (provided that the total masses of the Sun were combined in the center)? When would it crash onto the surface of the Sun?

From the mean Earth to Sun distance we get $2a = r_0 = 149.6 \times 10^6$ km. Since $\mu_\otimes = 1.327 \times 10^{11}$ km^3 s^{-2}, the time to the center of the Sun according to Eq. (7.5.21) is

$$t_{col} = \pi\sqrt{\frac{(149.6 \times 10^6)^3}{8 \cdot 1.327 \times 10^{11}}}\, s = 64.57 \text{ days}$$

The Sun has a radius of $r = 0.696 \times 10^6$ km. According to Eq. (7.5.20) the body crashes onto the surface of the Sun after

$$t_{col} = \frac{64.57}{\pi}(0.1361 + 2 \cdot 1.5025) \text{ days} = 64.56 \text{ days}$$

According to Eq. (7.5.4), it would have an impact velocity of 41.7 km s^{-1}.

7.5.5 *Bounded Vertical Motion*

A body moving vertically upward with initial velocity $v_0 > 0$ will either leave the gravitational potential for $a < 0$ or come to a halt somewhere and reverse its path for $a > 0$. In the following, we will study this later case.

First the body will move upward on a path with orbital element

$$a = \frac{\mu r_0}{2\mu - r_0 v_0^2} > 0$$

Gravitation will bring the body to a halt at $r_{up} = 2a$. Therefore, the altitude that the body attains is

$$h_{up} = r_{up} - r_0 = \frac{r_0^2 v_0^2}{2\mu - r_0 v_0^2} \tag{7.5.22}$$

If we assume $t_0 = 0$, we get from Eq. (7.5.7) for the time the body moves up to the apex point

$$t_{up} = \sqrt{\frac{a^3}{\mu}}\left(\pi + \frac{r_0 v_0}{\sqrt{\mu a}} - 2\arcsin\sqrt{\frac{r_0}{2a}}\right) = \sqrt{\frac{a^3}{\mu}}\left(\frac{r_0 v_0}{\sqrt{\mu a}} + 2\arccos\sqrt{\frac{r_0}{2a}}\right)$$

At the apex point, the body has velocity $v = 0$. From there on, it will fall backward. According to Eq. (7.5.5), the time it will be at position $r < r_{up}$ is

$$t_{down} = \sqrt{\frac{a^3}{\mu}}\left(\frac{r}{a}\sqrt{\frac{2a}{r} - 1} + 2\arccos\sqrt{\frac{r}{2a}}\right)$$

Note that this "down trajectory" has the same orbital element a as the "up trajectory". So in total the flight time is

$$t_{tot} = \sqrt{\frac{a^3}{\mu}}\left(\frac{r_0 v_0}{\sqrt{\mu a}} + 2\arccos\sqrt{\frac{r_0}{2a}} + \frac{r}{a}\sqrt{\frac{2a}{r} - 1} + 2\arccos\sqrt{\frac{r}{2a}}\right) \tag{7.5.23}$$

Vertical Shot up from a Planetary Surface

We now assume a shot up from the surface of a planet with radius R without an atmosphere, i.e., without drag. In this case $r_0 = r = R$. With

$$a = \frac{\mu R}{2\mu - R v_0^2} > 0$$

Equation (7.5.23) therefore reduces to

$$t_{tot} = 2\sqrt{\frac{a^3}{\mu}}\left(\frac{R v_0}{\sqrt{\mu a}} + 2\arcsin\sqrt{\frac{R v_0^2}{2\mu}}\right) \tag{7.5.24}$$

We finally want a power series approximation of the flight time of a body being shot up from the planet's surface with low kinetic energy, $s_0 := R v_0^2/\mu \ll 2$. Then

$$a = \frac{R}{2}\left(\frac{1}{1 - s_0/2}\right) = \frac{R}{2}\left[1 + \frac{1}{2}s_0 + \frac{1}{4}s_0^2 + \frac{1}{8}s_0^3 + O\left(s_0^4\right)\right]$$

$$\sqrt{\frac{a^3}{\mu}} = \sqrt{\frac{R^3}{8\mu}}\left[1 + \frac{3}{4}s_0 + \frac{15}{32}s_0^2 + \frac{35}{128}s_0^3 + O\left(s_0^4\right)\right]$$

$$\frac{Rv_0}{\sqrt{\mu a}} = \sqrt{2s_0}\left[1 - \frac{1}{4}s_0 - \frac{1}{32}s_0^2 - \frac{1}{256}s_0^3 + O\left(s_0^4\right)\right]$$

$$2\arcsin\sqrt{\frac{s_0}{2}} = \sqrt{2s_0}\left[1 + \frac{1}{12}s_0 + \frac{3}{160}s_0^2 + \frac{15}{2688}s_0^3 + O\left(s_0^4\right)\right]$$

We then find from Eq. (7.5.24) after some calculation

$$t_{tot} = 2\sqrt{\frac{s_0 R^3}{\mu}}\left[1 + \frac{2}{3}s_0 + \frac{2}{5}s_0^2 + \frac{8}{35}s_0^3 + O\left(s_0^4\right)\right]$$

Because $g_0 = \mu/R^2$ is the planet's mean gravitational acceleration at its surface, we finally get

$$t_{tot} = \frac{2v_0}{g_0}\left[1 + \frac{2}{3}s_0 + \frac{2}{5}s_0^2 + \frac{8}{35}s_0^3 + O\left(s_0^4\right)\right] \tag{7.5.25}$$

with

$$s_0 = \frac{Rv_0^2}{\mu} = \frac{v_0^2}{g_0 R}.$$

Since $v_\triangleright = \sqrt{g_0 R}$ is the so-called first cosmic velocity of a planet, Eq. (7.5.25) holds for the condition $(v_0/v_\triangleright)^8 \ll 1$. This result is in line with the classical throw upward where $t_{tot} = 2v_0/g_0$ for $s_0 \to 0$. The power series expansion (7.5.25) diverges for velocities larger than the second cosmic velocity $v \geq v_{\triangleright\triangleright} = \sqrt{2g_0 R}$, i.e., $s_0 \geq 2$.

Example

Let us assume that on the Moon ($R = 1737.4$ km, $\mu = 4.9028 \times 10^3$ km^{3}s^{-2}) a bullet is shot straight up with initial speed $v_0 = 1$ km s^{-1}. What is the altitude and return time it travels?

With $s_0 = Rv_0^2/\mu = 0.35437$ and $a = R/(2 - s_0) = 1055.76$ km, we have according to Eq. (7.5.22) and (7.5.24) $h_{up} = R/(2/s_0 - 1) = 374.13$ km and $t_{tot} = 26.705$ min. Classically, i.e., at a constant gravitational force, we would derive $h_{up} = v_0^2 R^2/2\mu = 307.84$ km and $t_{tot} = 2v_0 R^2/\mu = 20.523$ min.

7.6 Life in Other Universes?

Mathematically, and also physically, it is quite possible that other universes with other spatial dimensions in principle might exist. String theory for example states that at the time of the Big Bang our universe started out with nine spatial

dimensions. According to current beliefs (Brandenberger and Vafa 1989), initially all these nine dimensions were curled up on the Planck scale (10^{-35} m), that is, there were no macroscopic dimensions as what we have today. So-called strings that make up our elementary particles were "living" on these curled-up spaces. Very shortly after the Big Bang, an antistring crashed into one of these rolled up strings and, according to the belief, they eliminated each other and generated an uncurled space dimension: The first macroscopic dimension was born. In one dimension the probability that a string and an antistring meet is still very high. A string and an antistring annihilated anew, leading to a second macroscopic dimension. The question is whether two dimensions offer enough space, so that the strings no longer meet each other. Obviously not: the third dimension was born. Will someday another string and antistring meet again in our three dimensions to open up a fourth dimension? Nobody knows. However, we do know that coincidences play an important role in quantum mechanics. It could have been quite possible that not only two, but even four or five macroscopic dimensions could have formed, especially when the universe was still very small. Could we live in such a universe? Life in our universe, apart from many other factors, decisively depends on whether we have stable planetary orbits around a central star. So if we want to have an answer to the question of whether life would be possible in universes with other dimensions, first of all we would have to find out whether there would be stable planetary orbits. This is exactly what we will figure out now.

7.6.1 Equation of Motion in n Dimensions

First of all, one has to consider that according to Noether's theorem the conservation laws (see Eqs. (7.1.8)–(7.1.10)), especially the law of conservation of angular momentum, are independent of the dimension of the space. They are determined only by the homogeneity and isotropy of spacetime, and not by its dimensionality. The conservation laws are thus valid in all homogeneous n dimensional universes. Also, the law of conservation of energy is valid:

$$E_{kin} + E_{pot} = m\varepsilon = const$$

and the two expressions (see Eq. (7.1.4))

$$E_{kin} = \frac{1}{2}mv^2$$
$$E_{pot} = mU(r)$$

are independent of dimension. The angular momentum is defined as the cross product of position vector and velocity vector. Vectors are one-dimensional entities. That angular momentum is conserved means that these two vectors in a given n dimensional space open up a hyperspace. Two non-collinear vectors open up a plane. So for $n \geq 2$, the gravitationally determined motion in general is restricted to

a plane spanned by the initial vectors. (If by coincidence the initial vectors were collinear, which is inevitably in the case $n = 1$, the motion for $n \geq 1$ would be one dimensional.) Hence, independent of n, the general gravitational motion of a body in an n-dimensional space is always in a plane, and we therefore will measure it by means of polar coordinates (r, θ). As the motion takes place in a plane, the considerations leading to Eq. (7.2.8)

$$v^2 = \dot{r}^2 + \frac{h^2}{r^2}$$

still remain correct. The only thing that changes with space dimensions is the gravitational potential. We find it by solving the corresponding Poisson's Eq. (7.1.1). Its general solution in $n \geq 3$ dimensions is (exercise, Problem 7.1)

$$U(r) = -\frac{\mu}{r^{n-2}} \tag{7.6.1}$$

whereby μ carries the unit $[\mu] = [\mathrm{m}^n/\mathrm{s}^2]$. By applying all the above expressions to the energy conservation equation, we get the vis-viva equation for n dimensions:

$$\dot{r}^2 = \frac{2\mu}{r^{n-2}} - \frac{h^2}{r^2} + 2\varepsilon \qquad \textbf{vis-viva equation in } \boldsymbol{n} \textbf{ dimensions} \tag{7.6.2}$$

We seek for the equation of motion, and differentiate Eq. (7.6.2) to get

$$\ddot{r} = -\frac{(n-2)\mu}{r^{n-1}} + \frac{h^2}{r^3} \tag{7.6.3}$$

A big advantage of this differential equation is the fact that it is not vectorial, but scalar. A disadvantage is the sum of two terms on the right-hand side of the equation, as they both contain r. With this, we are no longer able to find a simple analytical solution for the differential equation by just separating the variables. So we have to look for other approaches. An important feature to solve differential equations in a smart way is to make a solution ansatz or a substitution, that comprises as much advance information as possible. Apparently, we have expressions of the form $1/r$. We assume that the solution is of the same form, and thus we change to the new radial variable: $\rho := 1/r$. (In the literature, this substitution is known as the *Burdet transformation*.) The second piece of previous knowledge we have is the conservation of the angular momentum. Therefore, we select the fixed angular momentum as one coordinate axis z, and for the other coordinates we use the rotating system of polar coordinates $(\rho,\ \theta)$. All in all, we now have a system of cylindrical coordinates $(\rho,\ \theta,\ z)$, which will later prove to be naturally adapted to this problem. We also know that the motion is periodic, whereas the time variable is linear. So it is a good idea to change for $h > 0$ from the time variable t to the orbit angle variable θ. Substituting $\rho = 1/r$ or $r = 1/\rho$, respectively, results with Eq. (7.2.7) in

$$\dot{r} = -\frac{1}{\rho^2}\frac{d\rho}{dt} = -\frac{1}{\rho^2}\frac{d\rho}{d\theta}\frac{d\theta}{dt} = -\frac{1}{\rho^2}\rho'\omega = -\frac{1}{\rho^2}\rho'\frac{h}{r^2} = -h\rho'$$

and

$$\ddot{r} = -h\frac{d\rho'}{dt} = -h\rho''\frac{h}{r^2} = -h^2\rho^2\rho''.$$

With this we obtain for Eq. (7.6.3)

$$\rho'' + \rho = \frac{(n-2)\mu}{h^2}\rho^{n-3}$$

This equation of motion indeed looks easier. But a relation between the second variable θ and ρ is still missing. We get the missing equation from Eq. (7.3.8).

$$\dot{\theta} = \frac{h}{r^2} = h\rho^2$$

This concludes our search for differential equations of motion in n dimensions.

$$\begin{aligned} &\rho'' + \rho = \tfrac{(n-2)\mu}{h^2}\rho^{n-3} \\ &\dot{\theta} = h\rho^2 \\ &\ddot{u}_i = 0 \quad i = 3, \ldots, n \end{aligned}$$

Newton's gravitational EoM in n dimensions (7.6.4)

where u_i are the remaining radial vector components that do not lie in the motion plane.

3-dim Universe

To verify the equation of motion in n dimensions we test for our well-known three dimensions

$$\rho'' + \rho = \frac{\mu}{h^2}$$

To solve it, we rewrite it to

$$\rho'' = -\left(\rho - \frac{\mu}{h^2}\right)$$

With the substitution $\lambda := \rho - \mu/h^2$ we get $\lambda'' = \rho''$, which results in the new simple differential equation $\lambda'' = -\lambda$. It has with the general solution $\lambda = \lambda_0 \cos(\theta - \theta_0)$. By resubstitution we obtain

$$\rho = \frac{\mu}{h^2} + \rho_0 \cos(\theta - \theta_0)$$

The two integration constants ρ_0 and θ_0 are determined by the specific initial conditions. Resubstituting $\rho = 1/r$ results in the well-known orbit equation (see Eq. (7.3.5))

$$r = \frac{p}{1 + e \cdot \cos\theta}$$

with $$p := h^2/\mu,\ e := p\rho_0.$$

7.6.2 4-Dimensional Universe

For $n = 4$ dimensions, the first equation of Eq. (7.6.4) reads

$$\rho'' = -\rho\left(1 - \frac{2\mu}{h^2}\right) = \pm k^2\rho$$

with

$$k := \sqrt{\left|1 - \tfrac{2\mu}{h^2}\right|}.$$

We have to distinguish three cases:

1. Case $1 - \frac{2\mu}{h^2} > 0$

In this case $\rho'' = -k^2\rho$ and the solution is $\rho = \rho_0 \sin(k\theta + \varphi)$ or

$$r = \frac{r_0}{\sin(k\theta)} \tag{7.6.5}$$

where we chose $\varphi = 0^\circ$ as an initial condition at $\theta = \pi/(2k)$. That is for $\theta = 0$ a planet is at infinity and approaches the scene. With $r = r_0$, it attains its smallest distance to the star, and then recedes into infinity. In total, this process can be considered as a flyby at a star as shown in Fig. 7.16.

2. Case $1 - \frac{2\mu}{h^2} < 0$

Here $\rho'' = k^2\rho$ and the solution is $\rho = \rho_0 e^{k\theta + \varphi}$ or

$$r = r_0 e^{-k\theta} \tag{7.6.6}$$

where we again chose an initial $\varphi = 0^\circ$ at $\theta = \pi/2k$. In other words, the planet spirals exponentially toward the star until it crashes into it (see Fig. 7.16).

> **Note**: *The other possible mathematical solution* $r = r_0 e^{k\theta}$ *is unphysical, as it would imply a repelling gravitational force.*

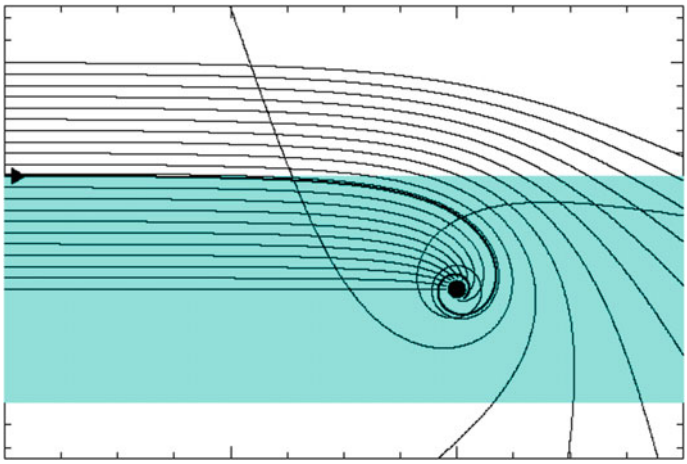


Fig. 7.16 Numerical simulation of the two-body problem in a four-dimensional space. Lightweight bodies with the same momentum, but different impact parameters, approach the central body from the left side. They escape into infinity again or crash into the center, depending on whether their impact parameter is in the shaded area or not. There are no stable orbits. Credit: Tegmark (1997)

3. Case $1 - \frac{2\mu}{h^2} = 0$

   Here, $\rho'' = 0$ and thus $\rho = a\theta + b$ or

$$r = \frac{1}{a\theta} \tag{7.6.7}$$

   Here, $b = 0$ was selected, i.e., $r(\theta = 0) = \infty$. On this borderline orbit the planet spirals toward the star, inversely proportional to θ, until it crashes onto its surface (bold trajectory in Fig. 7.16).

The orbits of three cases are shown in Fig. 7.16 taken from the literature, which were obtained by numerical simulation.

Conclusion: In a four-dimensional universe, none of the three possible orbits are stable or bounded, and therefore no planetary systems can exist and hence no life would be possible.

7.6.3 *Universes with ≥ 5 Dimensions*

In universes with $n \geq 5$ dimensions, the inverse radial acceleration is

$$\rho'' = \frac{(n-2)\mu}{h^2}\rho^{n-3} - \rho = \rho\left[\frac{(n-2)\mu}{h^2}\rho^{n-4} - 1\right] \tag{7.6.8}$$

Let us start our examination with a planet at $r = \infty$, i.e., $\rho = 0$, approaching the star. As long as the distance is large enough, so that

$$r^{n-4} > \frac{(n-2)\mu}{h^2}$$

or

$$\rho^{n-4} < \frac{h^2}{(n-2)\mu}$$

is valid, the expression in brackets of Eq. (7.6.8) is smaller than zero, and therefore $\rho'' < 0$. Let us assume that $\rho'' = -a = const < 0$ for a short period. Then we get the solution

$$\rho = \rho_0 + b\theta - \frac{a}{2}\theta^2$$

Even if the inverse radial velocity b were slightly positive in the beginning, ρ will further decrease after a certain advance of the orbit angle. If ρ decreases, ρ'' will become even more negative, and we get a runaway effect with the limiting value $\rho \to 0$, implying $r \to \infty$. This means that our body is gravitationally not bound to the star. At a certain position, it will approach the star to a minimum distance, depending on the incidence angle and velocity, without falling below the critical radius

$$\rho_{crit} = \frac{1}{r_{crit}} = \left[\frac{h^2}{\mu(n-2)}\right]^{\frac{1}{n-4}}$$

On its further track, it recedes and disappears in the depths of the universe. In total, its track will be more or less deflected. Qualitatively, this case corresponds to the first case in four dimensions.

If the body falls below the critical inverse radius due to its initial conditions, then the term in brackets in Eq. (7.6.8) becomes positive and thus $\rho'' > 0$. Let us assume that $\rho'' = a = const > 0$ for a moment. Then we get the solution

$$\rho = \rho_0 + b\theta + a\theta^2$$

As the body approaches from outside, its inverse radial velocity is $b > 0$, which means that in total ρ increases even faster (i.e., r decreases more). Then the term in brackets in Eq. (7.6.8) attains even larger positive values. Therefore, $\rho'' > 0$ increases further more, and ρ increases even faster. So we get an opposite runaway effect with the limit value $\rho \to \infty$, implying $r \to 0$: The planet approaches the star faster and faster, until it crashes into the star.

Conclusion: Also, in universes with dimensions $n \geq 5$, no stable planetary systems can exist, and thus life is not possible.

How would a $n \geq 4$-dimensional universe then evolve? Probably after its birth and after a very short period, the so-called epoch of radiation, when the masses form, these masses would immediately clot together to form black holes, and they would never form any gravitationally coupled stellar system or even galaxies. The black holes would merge into even bigger holes within a short period because of their large critical radius, and finally there would only be a huge black hole left that would absorb all the radiation and matter of the universe: That would be the quick end of a universe, hardly had it begun to exist.

Remark *Historically, in 1917 Ehrenfest already showed qualitatively, and in 1963 Büchel showed by general energy considerations, that for $n \geq 4$ there is no possibility that stable planetary orbits can exist. They are either deflected by the central body, or crash into it within a very short period of time.*

7.6.4 Universes with ≤ 2 Dimensions

In 1984, Deser and Jackiw and independently from them, Gott and Alpert applied the theory of general relativity to $n \leq 2$ spatial dimensions and found that the space surrounding a point mass would not have a curvature (the Riemann tensor and with it Einstein's curvature tensor would vanish). This means that other particles would not experience any gravitational pull. So, in $n \leq 2$-dimensional spaces there is no gravitational attraction at all, let alone an answer to the question of stable orbits. Classic astrodynamics erroneously has a different point of view. From Poisson's Eq. (7.1.1), it follows that in two dimensions a gravitational potential $U(r) = -\mu \ln r$ with force $F \propto -\mu/r$ exists (exercise, Problem 7.1). Since $U(r)$ diverges for $r \to \infty$, this already shows us that this solution is quite far from reality. The inconsistency between the theory of relativity and Newton's physics can be explained by the fact that for $n \leq 2$ in the theory of general relativity a correspondence to classical physics no longer exists.

Conclusion: Since in universes with $n \leq 2$ dimensions a gravitational force is not existent, planets also cannot exist, let alone life.

So, with our three spatial dimensions, we live on an island of stability, and we can only assume and hope that it is not as coincidental as string theory currently suggests.

7.7 Stellar Orbits

7.7.1 Motion in General Gravitational Potentials

In this section we consider the most general case, the motion of a body in a gravitational potential $U(\boldsymbol{r})$ that is generated by an arbitrary mass distribution $\rho(\boldsymbol{r})$.

To determine $U(\boldsymbol{r})$ from $\rho(\boldsymbol{r})$ we recall from Eq. (7.1.3) that the potential of a point mass M at origin (location 0) is given by

$$U(r) = -\frac{GM}{r} = -G\frac{M}{|\boldsymbol{r}-\boldsymbol{0}|}$$

Therefore, the potential of many masses M_i $(i = 1, \ldots, n)$ at positions $\boldsymbol{r}_i$ is

$$U(\boldsymbol{r}) = -G\sum_i^n \frac{M_i}{|\boldsymbol{r}-\boldsymbol{r}_i|}$$

which entails that the potential may no longer be isotropic. If we assume a celestial body with a continuous mass distribution described by the density distribution function $\rho(\boldsymbol{r})$, then we have to carry out the transition $M_i \to \rho(\boldsymbol{r}) \cdot dV = \rho(\boldsymbol{r}) \cdot d^3\boldsymbol{r}$, whereby the sum becomes a volume integral

$$U(\boldsymbol{r}) = -G\iiint_V \frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|} d^3\boldsymbol{r}' \tag{7.7.1}$$

From Eq. (7.1.5) and Newton's second law Eq. (7.1.12) the motion within this potential is determined by the following equation of motion

$$\ddot{\boldsymbol{r}} = -\frac{d}{d\boldsymbol{r}}U(\boldsymbol{r}) \tag{7.7.2}$$

In view of the application to galaxies having a center of a point symmetric mass distribution, i.e. a location where $U(-\boldsymbol{r}) = U(\boldsymbol{r})$, we place the origin of our inertial reference frame (see Sect. 13.1.1) at this center. Thus

$$\frac{dU}{d\boldsymbol{r}}(\boldsymbol{r}=0) =: \left.\frac{dU}{d\boldsymbol{r}}\right|_0 = 0$$

By employing cylindrical coordinates $\boldsymbol{u}_r, \boldsymbol{u}_\theta, \boldsymbol{u}_z$ and with Eq. (7.2.6) we can decompose the vector equation of motion into

$$\ddot{\boldsymbol{r}} = \begin{pmatrix} \ddot{r} - \omega^2 r \\ 2\omega\dot{r} + \dot{\omega}r \\ \ddot{z} \end{pmatrix}_{r\theta z} = -\frac{dU}{d\boldsymbol{r}} = -\left(\frac{\partial U}{\partial r}, \frac{1}{r}\frac{\partial U}{\partial \theta}, \frac{\partial U}{\partial z}\right)^T_{r\theta z} \tag{7.7.3}$$

The above choice of the origin at the center thus yields

$$\left.\frac{\partial U}{\partial r}\right|_0 = 0, \quad \left.\frac{\partial U}{\partial z}\right|_0 = 0$$

Orbits in Axisymmetric Potentials

Our first restrictive, yet good, assumption (so-called *Wentzel-Kramers-Brillouin approximation*) is that, though billions of stars are moving in a galaxy, they on average generate a gravitational potential that is constant in time. In addition, we assume that the potential is laterally symmetric (axisymmetric), i.e. $\partial U/\partial\theta = 0$, everywhere. From Eq. (7.7.3) we then have $2\omega\dot{r} + \dot{\omega}r = 0$. We multiply this equation by r and find

$$\frac{d}{dt}\left(\omega r^2\right) = \frac{dh}{dt} = 0$$

This implies that in an axisymmetric potential the angular momentum is conserved

$$h = \omega r^2 = const$$

In respect to the radial motion of the body we employ from Sect. 7.2.4 the concept of the effective potential, which is defined as

$$U_{eff}(\boldsymbol{r}) := U(\boldsymbol{r}) + \frac{h^2}{2r^2} = U(\boldsymbol{r}) + \frac{1}{2}\omega^2 r^2 \quad \textbf{effective potential} \tag{7.7.4}$$

For an axisymmetric potential with $h = const$ we have

$$\frac{\partial U_{eff}}{\partial r} = \frac{\partial U}{\partial r} - \frac{h^2}{r^3} = \frac{\partial U}{\partial r} - \omega^2 r$$

Therefore we find from Eq. (7.7.3) two scalar equations of motion

$$\begin{aligned} \ddot{r} &= -\frac{\partial U_{eff}}{\partial r} \\ \ddot{z} &= -\frac{\partial U_{eff}}{\partial z} = -\frac{\partial U}{\partial z} \end{aligned} \quad \textbf{equations of motion} \tag{7.7.5}$$

Circular Guiding Orbits

What is the condition for stable orbits in an axisymmetric potential? For this to happen the effective potential must have a minimum, both in r and z, i.e.

$$0 = \frac{\partial U_{eff}}{\partial r} = \frac{\partial U}{\partial r} - \omega^2 r \tag{7.7.6}$$

$$0 = \frac{\partial U_{eff}}{\partial z} = \frac{\partial U}{\partial z} \tag{7.7.7}$$

We have seen that $\partial U/\partial z = 0$ is fullfilled anywhere in the plane vertical to the symmetry (a.k.a. equatorial plane). At the radial minimum we have $\dot{r} = 0$ and hence a stable circular orbit. Observe that every stable orbit has its individual angular momentum $h = \omega r^2$ and hence there may be infinitely many of them. We denote the radius of a given circular orbit as R and its orbital frequency as $\Omega := \omega(R)$. Thus, we have stable circular orbits in the equatorial plane, which we will call guiding orbits, if the following condition holds

$$\left.\frac{\partial U}{\partial r}\right|_R = R\Omega^2 \quad @ \text{ guiding orbits} \tag{7.7.8a}$$

Since U is a given, this equation in fact is a conditional equation for the angular frequency of a guiding orbit with orbital radius R

$$\Omega = \sqrt{\frac{1}{R}\left.\frac{\partial U}{\partial r}\right|_R} = \frac{h}{R^2} \tag{7.7.8b}$$

Stable Near-Circular Orbits

We want to know whether there also exist stable non-circular orbits in the vicinity of a circular guiding orbit and what are the conditions for them. For a stable near-to-guiding orbit the effective potential

$$U_{\mathit{eff}}(\boldsymbol{r}) = U(\boldsymbol{r}) + \frac{1}{2}\omega^2 r^2$$

needs to have a positive curvature at $r = R$, i.e.

$$\left.\frac{\partial^2 U_{\mathit{eff}}}{\partial r^2}\right|_R = \left.\frac{\partial^2 U}{\partial r^2}\right|_R - \left.\frac{\partial}{\partial r}\frac{h^2}{r^3}\right|_R = \left.\frac{\partial^2 U}{\partial r^2}\right|_R + 3\frac{h^2}{R^4} = \left.\frac{\partial^2 U}{\partial r^2}\right|_R + 3\Omega^2 > 0$$

Owing to Eq. (7.7.8a), we thus find the stability condition at the circular guiding orbit

$$\left.\left(\frac{\partial^2 U}{\partial r^2} + \frac{3}{r}\frac{\partial U}{\partial r}\right)\right|_R > 0 \tag{7.7.9}$$

Since according to Eq. (7.7.6) the local minimum condition $r\omega^2 = \partial U/\partial r$ holds, we have

$$\frac{\partial^2 U}{\partial r^2} = \omega^2 + 2r\omega\frac{\partial \omega}{\partial r}$$

Inserting this into the above stability conditional equation we get the alternative stability condition

$$\left(\frac{r}{\omega}\frac{\partial\omega}{\partial r}\right)\Big|_R = \frac{\partial \ln\omega}{\partial \ln r}\Big|_R > -2 \tag{7.7.10}$$

Let us assume that at the radial distance $r = R$ of the guiding orbit the potential quite generally behaves as $U \propto r^p$ with $p \in \mathbb{R}$. Then

$$\left(\frac{\partial^2 U}{\partial r^2} - \frac{p-1}{r}\frac{\partial U}{\partial r}\right)\Big|_R > 0$$

Inserting this result into Eq. (7.7.9) we find $p > -2$. Because also $U = c \cdot \ln r$ fulfills Eq. (7.7.9), we get for the gravitational potential about the guiding orbit and for attractive forces

$$U = \left\{\begin{array}{ll} cr^p & p > 0 \\ c \cdot \ln r & \\ -c/r^p & -2 < p < 0 \end{array}\right\} \quad @\ r = R \tag{7.7.11}$$

It can be shown (see also below) that all galaxies obey the conditional equation (7.7.9) and Eq. (7.7.10) and hence Eq. (7.7.11). Therefore, as long as stars do not enter the SOI of a neighboring star, which means that they do suffer mutual gravitational interaction, they move on stable near-circular orbits in any galaxy.

7.7.2 Stellar Motion in General Galaxies

We now study stable near-circular orbits in general galaxies and we presume that also some form of epicyclic motion exists. To study that motion in detail, we need to evaluate the equations of motion of the excursions

$$x = r - R$$

and of z from the equilibrium point $(R, 0)$ of the guiding orbit, which according to Eq. (7.7.5) reads

$$\ddot{z} = -\frac{\partial U_{eff}}{\partial z}, \ddot{x} = -\frac{\partial U_{eff}}{\partial x}$$

In view of the small excursions we expand U_{eff} into a Taylor series about the equilibrium point $(R, 0)$

$$U_{eff} = U_{eff}(R,0) + \frac{1}{2}\left(\frac{\partial^2 U_{eff}}{\partial r^2}\right)\Big|_{R,0} x^2 + \frac{1}{2}\left(\frac{\partial^2 U_{eff}}{\partial z^2}\right)\Big|_{R,0} z^2 + O(xz^2) \qquad (7.7.12)$$

and neglect terms of $O(xz^2)$ or higher. Note that the term of $O(xz)$ vanishes, because $U_{eff}(-z) = U_{eff}(z)$. From this we find the equations of a radial and vertical oscillating motion

$$\ddot{x} = -\frac{\partial U_{eff}}{\partial r} = -x \cdot \frac{\partial^2 U_{eff}}{\partial r^2}\Big|_{R,0} =: -\kappa^2 x \qquad \textbf{radial epicycle} \qquad (7.7.13)$$

$$\ddot{z} = -\frac{\partial U_{eff}}{\partial z} = -z \cdot \frac{\partial^2 U_{eff}}{\partial z^2}\Big|_{R,0} =: -\nu^2 z \qquad \textbf{vertical oscillation} \qquad (7.7.14)$$

Relation of Epicycle to Guiding Orbit Oscillation

We now are interested in how κ relates to Ω of the guiding orbit, about which we have the epicyclic oscillations. From Eqs. (7.7.4) and (7.7.8a) we derive

$$\begin{aligned}\kappa^2 &= \frac{\partial^2 U_{eff}}{\partial r^2}\Big|_R = \frac{\partial}{\partial r}\left(\frac{\partial U}{\partial r}\right)\Big|_R + \frac{\partial^2}{\partial r^2}\left(\frac{h^2}{2r^2}\right)\Big|_R = \frac{\partial}{\partial r}\left(r\omega^2\right)\Big|_R + 3\omega^2\big|_R \\ &= \Omega^2 + 2R\Omega\frac{\partial \omega}{\partial r}\Big|_R + 3\Omega^2 = 4\Omega^2\left(1 + \frac{R}{2\Omega}\frac{\partial \omega}{\partial r}\Big|_R\right)\end{aligned}$$

Hence

$$\kappa^2 = 4\Omega^2\left(1 + \frac{1}{2}\frac{\partial \ln \omega}{\partial \ln r}\Big|_R\right) \qquad (7.7.15)$$

Owing to the positive definite value $\kappa^2 > 0$ we again (see Eq. (7.7.10)) derive the condition for a stable near-circular orbit

$$\frac{\partial \ln \omega}{\partial \ln r}\Big|_R > -2.$$

There seem to be two limiting cases. On one hand we have a point mass at the galactic center, which causes $U = -\mu/r$ and thus via the equilibrium condition $r\omega^2 = \partial U/\partial r$ delivers $\omega^2 = \mu/r^3$. Hence $\partial \ln \omega/\partial \ln r = -3/2$, and hence $\kappa = \Omega$. On the other hand $h = \omega r^2 = const$ holds, implying $d\omega/dr < 0$ and hence $d \ln \omega/d \ln r < 0$. So, for any mass distribution we have the limits

$$\Omega \leq \kappa \leq 2\Omega \qquad (7.7.16)$$

Epicyclic Motion
In the above we saw that the guiding orbit at $r = R$ and orbital frequency Ω is superimposed by the radial excursions with EoM $\ddot{x} = -\kappa^2 x$ and hence with path

$$x = x_0 \sin(\kappa t + \varphi)$$

The angular epicylic motion can be derived from the law of the conservation of angular momentum $\omega r^2 = \dot{\theta} r^2 = h = const$ as

$$\dot{\theta} = \frac{h}{R^2} = \frac{\Omega R^2}{(R+x)^2} \approx \Omega\left(1 - \frac{2x}{R} + \ldots\right) = \Omega + \frac{2\Omega}{R} x_0 \sin(\kappa t + \varphi)$$

Direct integration yields

$$\theta(t) = \theta_0 + \Omega \cdot t + \frac{2\Omega}{R\kappa} x_0 \cos(\kappa t + \varphi) \tag{7.7.17}$$

Since $r = R + x$, the radial variation of the stellar orbit about its guiding orbit is given as

$$r = R + x_0 \sin(\kappa t + \varphi) = R\left[1 + \frac{x_0}{R} \sin\left(\frac{\kappa}{\Omega}\theta + \phi\right)\right] \tag{7.7.18}$$

where we have transformed from the time domain into the angular domain employing the mean motion $\theta(t) = \theta_0 + \Omega \cdot t$ from Eq. (7.7.17). Equation (7.7.18) together with Eq. (7.7.17) describes the epicyclic motion.

Orbital Phase Shift
If we compare Eq. (7.7.18) with the general solution for a Keplerian orbit

$$r = \frac{p}{1 + e\,\cos\theta} \approx R(1 + e\,\cos\theta) \quad @\ e \to 0$$

we see that the eccentricity $e = x_0/R$ is given by the relative amplitude of the oscillation, and κ/Ω is the commensurability between epicycle and guiding motion. If we define the angular orbital period θ_P as

$$\theta_P \frac{\kappa}{\Omega} := 2\pi \quad \text{or} \quad \theta_P = 2\pi \frac{\Omega}{\kappa}$$

the orbit phase shift, i.e. the periapsis angle after one epicycle with respect to a full guiding orbit cycle, is

$$\Delta\theta = \theta_P - 2\pi = 2\pi\left(\frac{\Omega}{\kappa} - 1\right) \quad \textbf{orbital phase shift} \tag{7.7.19}$$

Owing to the above restriction $\Omega \leq \kappa \leq 2\Omega$, we have $-180° \leq \Delta\theta \leq 0$. If $\kappa/\Omega = n/m$ the epicycle is commensurable, i.e., the stellar path closes after n epicycles and

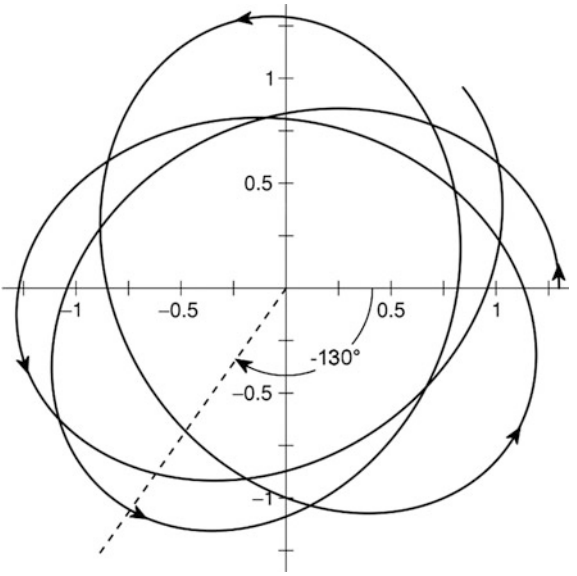


Fig. 7.17 An epicyclic motion with $\kappa = 1.36\,\Omega$ and according orbital phase shift $\Delta\theta = -130°$

m guiding orbit cycles. Usually stellar epicycles are not commensurable. Figure 7.17 depicts the epicyclic motion with $\kappa = 1.56 \cdot \Omega$ and hence $\Delta\theta = -130°$.

Example

For our Sun in the Milky Way we have

$$\kappa = 36.7\ km\ s^{-1}, \mathrm{kpc}^{-1}$$
$$\Omega = 27.2\ \mathrm{km\ s^{-1} kpc^{-1}} = 8.81 \times 10^{-16}\ \mathrm{s^{-1}}$$

So

$$\kappa = (1.35 \pm 0.05)\Omega$$

The Sun therefore makes 1.35 oscillations in and out for every circuit of the galaxy. It therefore takes the Sun $T = 2\pi/\Omega = 226$ Million years to circle the galactic center once. The Sun's orbital phase shift after one epicycle is

$$\Delta\theta = 360° \left(\frac{\Omega}{\kappa} - 1\right) = -93°$$

7.7.3 Stellar Orbits in Globular Cluster Galaxies

Suppose we have a globular cluster galaxy that is a spherical distribution of stars with radius R and mass $M = \frac{4}{3}\pi\rho R^3$. Owing to its billions of stars, it is fair to



Fig. 7.18 The great globular cluster M13 in the constellation Hercules. Credit Marco Burali, Tiziano Capecchi, Marco Mancini/Osservatorio MTM

assume that it is isotropic and homogeneous with constant mass density $\rho = const$ (see Fig. 7.18). We therefore can apply the isotropic form of Poisson's Eq. (7.1.2)

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)U = 4\pi G\rho \quad @\ \rho = const$$

This can be solved by two succeeding direct integrations

$$\begin{aligned} r^2\frac{\partial U}{\partial r} &= \frac{4}{3}\pi G\rho r^3 \\ U(r) &= \frac{2}{3}\pi G\rho r^2 \end{aligned} \tag{7.7.20}$$

with the choice (see Sect. 7.3.4) $U(r \to 0) = 0$.

Since this potential satisfies the stability condition (7.7.11), stable periodic orbits must exist. To determine their orbit equation we apply Eqs. (7.7.2) and (7.1.6) to find the equation of motion

$$\ddot{\boldsymbol{r}} = -\frac{d}{d\boldsymbol{r}}U(\boldsymbol{r}) = -\frac{4}{3}\pi G\rho\boldsymbol{r} = -\frac{GM}{R^3}\boldsymbol{r}$$

With the substitution $\kappa = \sqrt{4\pi G\rho/3}$ this vector equation can be decomposed into

$$\begin{aligned} \ddot{x} &= -\kappa^2 x \\ \ddot{y} &= -\kappa^2 y \end{aligned}$$

The general solution of each these well-known differential equations is the harmonic oscillator

$$u(t) = u_0 + a\ \cos(\kappa t)\cos\varphi + b\ \sin(\kappa t)\sin\varphi$$

This is the equation of an ellipse, where u_0 marks the center and φ is the angle between the coordinate axis and the mayor axis of the ellipse. If we locate the center at the origin and place the x-axis along the major axis ($\varphi = 0$) and the y-axis along the minor axis ($\varphi = 90°$), we obtain

$$\begin{aligned} x &= a\ \cos(\kappa t) \\ y &= b\ \sin(\kappa t) \end{aligned}$$

This solution is an ellipse

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

with orbital frequency $\kappa = \sqrt{4\pi G\rho/3}$, and hence period $T = 2\pi/\kappa$, and centered (not focused) at the center of the galaxy $(0, 0)$. This motion can be considered as an epicyclic motion about a guiding orbit with $\kappa = 2\Omega$, i.e. the epicycle period is in 2:1 correspondence with the guidance orbital period, and orbital phase shift $\Delta\theta = -180°$ (see Sect. 7.7.2).

7.7.4 Stellar Motion in Disk-Shaped Galaxies

Most galaxies such as our Milky Way, however, are thin rotating disks (see Fig. 7.19) with, say, radius a and thickness $h \ll a$. Assuming again a constant mass density $\rho = const$, and assuming no vertical movements of stars within the disk, the gravitational potential in the galactic plane can be shown (Danby 2003, Chap. 5, problem 8) to be

$$\begin{aligned} U &= -2\pi G\rho h a\left[1 - \frac{1}{4}\left(\frac{r}{a}\right)^2 - \frac{3}{4^3}\left(\frac{r}{a}\right)^4 - \frac{5}{4^4}\left(\frac{r}{a}\right)^6 - \frac{7\cdot 5\cdot 3}{4^7}\left(\frac{r}{a}\right)^8 - \cdots\right] \quad @\ r < a \\ \frac{dU}{dr} &= \pi G\rho h\left[\frac{r}{a} + \frac{3}{8}\left(\frac{r}{a}\right)^3 + \frac{15}{64}\left(\frac{r}{a}\right)^5 + \frac{105}{1\,024}\left(\frac{r}{a}\right)^7 + \cdots\right] \\ \frac{d^2U}{dr^2} &= \pi G\rho\frac{h}{a}\left[1 + \frac{9}{8}\left(\frac{r}{a}\right)^2 + \frac{75}{64}\left(\frac{r}{a}\right)^4 + \frac{735}{1\,024}\left(\frac{r}{a}\right)^6 + \cdots\right] \end{aligned}$$



Fig. 7.19 The unbarred spiral galaxy NGC 4565 in the constellation Coma Berenices. Credit Adam Block/Kitt Peak National Observatory

and therefore the force acting on a star with mass m_* is

$$\boldsymbol{F} = -m_* \frac{dU}{d\boldsymbol{r}} = -\pi G m_* \rho h \frac{\boldsymbol{r}}{a}\left[1 + \frac{3}{8}\left(\frac{r}{a}\right)^2 + \frac{15}{64}\left(\frac{r}{a}\right)^4 + \frac{105}{1\,024}\left(\frac{r}{a}\right)^6 + \cdots\right]$$

Because $dU/dr > 0$ and $d^2U/dr^2 > 0$ at any radial distance from the center, the stability condition Eq. (7.7.9)

$$\frac{\partial^2 U}{\partial r^2} + \frac{3}{r}\frac{\partial U}{\partial r} > 0$$

holds and therefore there exist stable near-circular orbits orbits at any radial distance.

The radial value of the gravitational force function is shown in Fig. 7.20. It can be seen that toward the outer regions of the Milky Way, the absolute value of the gravitational force gradually increases over the force $\boldsymbol{F}(\boldsymbol{r}) = -G\rho m_* \boldsymbol{r}$ as in globular cluster galaxies. We therefore expect, in general, also ellipses as stellar orbits centered on the center of the Milky Way. However, when a star on its orbit moves outward, it is subject to an excessive gravitational force, which can be interpreted as an increasing standard gravitational parameter μ. At a given radial distance, this implies an excessive orbital velocity $v + \Delta v = \sqrt{(\mu + \Delta\mu)/r}$, which in turn leads to a prograde rotation of the line of apsides. Therefore and as shown in Sect. 7.7.3, such ellipses in a flattened galaxy do not close themselves, but get offset each time a star completes one revolution. The Sun, for example, takes about 230 million years

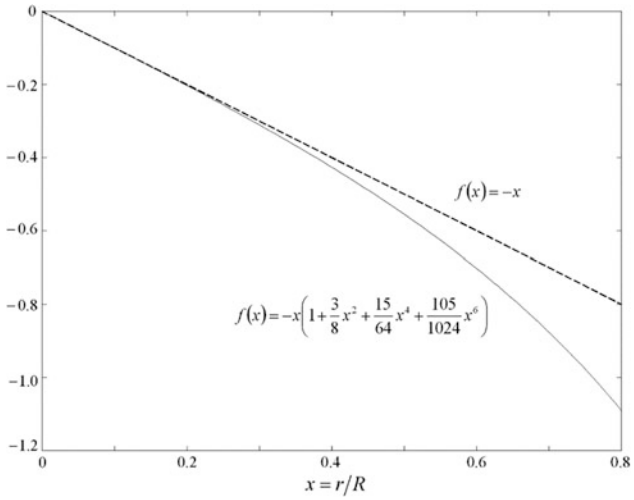


Fig. 7.20 The force function within a flattened galaxy at position $x = r/a$

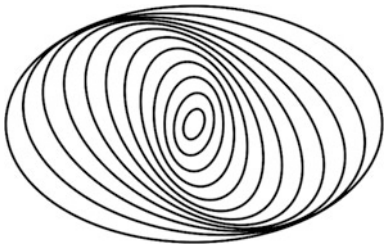


Fig. 7.21 Spiral waves in a flattened galaxy arise when the elliptic star orbits move in unison but are slightly skewed compared to its neighbors. The density of stars is highest where the ellipses crowd together

to go around its elliptic orbit. In that time the orbit gets offset by 105°. In spiral galaxies the elliptic orbits of all stars, though at different mean distances from the center, rotate in lockstep at constant offsets, which causes the spiral arms (see Fig. 7.21). The spiral arms bring about an inhomogeneous galactic density, which via a gravitational instability causes the ellipses to rotate in lockstep.

7.8 Problems

Problem 7.1 *Solutions of Poisson's Equation*
Show that Poisson's equation (7.1.1) in n-dimensional space outside an n-dimensional isotrophic mass distribution has the following solutions for $U(r)$ $(\mu = GM)$:

$$U(r) \propto \begin{cases} -\dfrac{\mu}{r^{n-2}} & @\ n \geq 3 \\ -\mu \ln r & @\ n = 2 \\ \mu \cdot r & @\ n = 1 \end{cases}$$

Consider in particular the case at $r = 0$.

Problem 7.2 *Eccentricity from Eccentricity Vector*
Derive from the eccentricity vector $\boldsymbol{e} = (\dot{\boldsymbol{r}} \times \boldsymbol{h})/\mu - \hat{\boldsymbol{r}}$ directly that for the absolute value of the eccentricity holds $1 - e^2 = h^2/\mu a$.

Problem 7.3 *Virial Theorem of a Two-Body System*
(a) Show that in a two-body system for each body with mass m, which orbits the common barycenter thus having the inertial moment $I = mr^2$,

$$\ddot{I} = 4E_{kin} + 2E_{pot}$$

generally holds.
(b) Then prove the virial theorem Eq. (7.3.20), $2\langle E_{kin}\rangle + \langle E_{pot}\rangle = 0$, for a bounded orbit.

Problem 7.4 *Orbit Equation—Fast Track*
(a) Starting out from $r^2 = \boldsymbol{r} \cdot \boldsymbol{r}$ show with Eq. (7.2.5) that $\ddot{r} = h^2/r^3 - \mu/r^2$
(b) Apply the Burdet transformation $\rho := 1/r$ (see Sect. 7.6.1), solve the equivalent equation of motion in ρ, and show that the orbit equation follows

$$r = \frac{p}{1 + e\cos\theta}$$

with $p := h^2/\mu$ and $e := p\rho_0$, where $\rho_0 = 1/r_0$ is the initial orbital radius.

Problem 7.5 *Solutions to Barker's Equation*
Show with Cardano's method and Descartes' rule of signs that the unique real solutions to Barker's equation from Sect. 7.4.4 are

$$\tan\frac{\theta}{2} = \left(\sqrt{q^2+1} + q\right)^{1/3} - \left(\sqrt{q^2+1} - q\right)^{1/3}$$

and

$$2\frac{r}{p} = \left(\sqrt{q^2+1} + q\right)^{2/3} + \left(\sqrt{q^2+1} - q\right)^{2/3} - 1$$

with

$$q = 3\sqrt{\frac{\mu}{p^3}}(t - t_0)$$

Problem 7.6 *Generalized Solution of Kepler's Problem*
Prove that by the application of the transformation Eqs. (7.4.14) quite generally the following relation holds

$$\int \frac{d\theta}{(1+e\cdot\cos\theta)^n} = \frac{1}{\sqrt{(1-e^2)^{2n-1}}}\int (1-e\cdot\cos E)^{n-1}dE \quad @\ e<1, n\geq 1$$

Problem 7.7 *Series Expansions*
(a) Prove the series expansions (cf. Danby (2003), Sect. 6.14)

$$E = M + e\,\sin M + \frac{e^2}{2}\sin(2M) - \frac{e^3}{8}[\sin M - 3\sin(3M)] + O(e^4)$$
$$\frac{r}{a} = 1 - e\,\cos M + e^2\sin^2 M + \frac{3}{2}e^3\cos M\,\sin^2 M + O(e^4)$$

for an elliptic orbit by applying the Banach fixed point theorem to $E = M + e\,\sin E = f(E)$ under the constraint that f is Lipschitz continuous for $e<0.6627434\ldots$. Then apply the result to $r/a = 1 - e\,\cos E$.

Remark *This solution procedure may sound like elementary mathematics. In fact the solution algorithm, called contraction mapping, is only a generalization of Newton's method. Just the verification that it works is elementary mathematics.*

Note *Contraction mapping is a very convenient method to solve implicit functional relations if the function is Lipschitz continuous. Practically, Lipschitz continuity is not checked beforehand, but contraction mapping is just applied and only then observed whether the series converges. In fact, we made use of the contraction mapping without saying when deriving* Eq. (12.5.15).

(b) Prove that for an elliptic orbit

$$\theta = M + 2e\,\sin M + \frac{5}{4}e^2\sin(2M) + O(e^3)$$

Hint: Apply contraction mapping to the integral equation (7.3.9)

$$\int_0^\theta \frac{dx}{(1+e\cdot\cos x)^2} = \frac{\mu^2}{h^3}(t-t_p) = \frac{M}{(1-e^2)^{3/2}}$$

Note that alternatively holds

$$\cos\theta(t) = \cos M(t) + e(\cos 2M - 1) + O\left(e^2\right)$$

(c) By the same token prove that for the true anomaly θ' (angle between the apsidal line to apogee and the radius vector from the empty focal point to the revolving body) holds

$$\theta' = M - \frac{1}{4}e^2 \sin(2M) + O\left(e^3\right)$$

or alternatively

$$\cos\theta'(t) = \cos M(t) + \frac{1}{8}e^2(\cos M - \cos 3M) + O\left(e^2\right)$$

Problem 7.8 *Radial Position from the Equation of Radial Motion*
Prove that both, Eq. (7.4.17) for the radial position in time of an elliptic orbit and Eq. (7.4.33) for that of a hyperbolic orbit, and Barker's equation for a parabolic orbit, follows from Eq. (7.2.11) by direct integration.

Hint: Show first that (7.2.11) can be rewritten as

$$\begin{aligned} r\cdot\dot{r} &= na^2e\sqrt{1-\rho^2}, & \rho &= \tfrac{r-a}{ae} & @\ \text{ellipse} \\ r\cdot\dot{r} &= na^2e\sqrt{\rho^2-1}, & \rho &= \tfrac{r-a}{-ae} & @\ \text{hyperbola} \\ r\cdot\dot{r} &= \sqrt{\mu p}\sqrt{2\rho-1}, & \rho &= \tfrac{r}{p} & @\ \text{parabola} \end{aligned}$$

Problem 7.9 *Near-Radial Parabolic Orbits*
(a) Show that from the orbit equation for a radial elliptic orbit with initial conditions r_0, v_0, t_0

$$(t-t_0)\sqrt{\frac{\mu}{a^3}} = \frac{r}{a}\sqrt{\frac{2a}{r}-1} - \frac{r_0}{a}\sqrt{\frac{2a}{r_0}-1} - 2\arcsin\sqrt{\frac{r}{2a}} + 2\arcsin\sqrt{\frac{r_0}{2a}}$$

for $a \to \infty$ and $v_0 = 0$ follows the orbit equation for the radial parabolic orbit

$$r^{3/2}(t) = r_0^{3/2} - \sqrt{\frac{9\mu}{2}}\cdot(t-t_0)$$

(b) Show that for a radial elliptic orbit, which is nearly radial parabolic, $a \gg r_0$ and $v_0 = 0$, one gets

$$r^{3/2}\left[1 + \frac{3}{20}\frac{r}{a}\right] \approx r_0^{3/2}\left[1 + \frac{3}{20}\frac{r_0}{a}\right] - \sqrt{\frac{9\mu}{2}}(t-t_0)$$

Because for $t - t_0 \ll \sqrt{\mu/r_0^3}$ from the radial parabolic orbit equation follows

$$\frac{r}{a} \approx \frac{r_0}{a}\left[1 - (t - t_0)\sqrt{\frac{2\mu}{r_0^3}}\right]$$

show that

$$\begin{aligned} r^{3/2}(t) &\approx r_0^{3/2} - \sqrt{\frac{9\mu}{2}}(t - t_0)\left[1 - \frac{r_0}{4a}\left(1 - \frac{12}{10}(t - t_0)\sqrt{\frac{\mu}{2r_0^3}}\right)\right] \\ &\approx r_0^{3/2} - \sqrt{\frac{9\mu}{2}}(t - t_0)\left[1 - \frac{r_0}{4a}\right] \end{aligned}$$

Chapter 8
Orbital Maneuvering

The most important maneuver in space is the one to change the orbit of a space vehicle. Because the initial and final orbits are subject to a central gravitational potential such a S/C will transit between two Keplerian orbits. This is true not only for planetary orbits but also for interplanetary flights with the Sun as the central body. In general, for all these transfers there are two cases to be distinguished:

- *Orbital transfer*
  A S/C is on an initial orbit and just needs to be transferred to another orbit.
- *Orbital rendezvous*
  A S/C on a specific point on the initial orbit needs to be transferred to a specific point on the final orbit (e.g., target object).

For instance, a lunar space probe does an orbital transfer when it is in a circular Earth orbit and heads into a translunar orbit, or, to give another example, when the International Space Station due to drag needs to be transferred to a higher circular orbit. We will treat orbital transfers in Sects. 8.1–8.4. Orbital rendezvous, on the other hand, for instance, is the situation where a Shuttle orbiter performs a maneuver to approach the International Space Station for docking. For methodical reasons we split the orbital rendezvous case into "orbital rendezvous in planetary orbits" (Sect. 8.6) and "interplanetary orbital rendezvous" (Sect. 9.3). The reason that interplanetary rendezvous is treated in a different chapter is because interplanetary flight is a complex three-body problem (Sun, target planet, and S/C), in which orbital rendezvous needs to be embedded.

Orbital Transfer

Let us first consider the orbital transfer case. The two orbits, between which the S/C is to be transferred, may either intercept in one ore two points, or they may not intercept at all.

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Intercepting orbits
If the two orbits intercept, an impulsive thrust maneuver (see Sects. 2.3 and 2.4.1) at any interception point will do to directly attain the target orbit. Such orbital transfers are called **one-impulse maneuvers**, and they are discussed in detail in Sect. 8.1.

Non-intercepting orbits
If the two orbits do not intercept

- a first impulsive maneuver may bring the S/C into a transfer orbit until it hits the target orbit where a second impulsive maneuver transits it into the final orbit. In total, this would be a classical **two-impulse transfer**. The most general case, where the transfer orbit intersects both orbits—the so-called *Lambert transfer*—is discussed in Sect. 8.2. The optimal case when the transfer orbit just touches both orbits—the so-called *Hohmann transfer*—is discussed in Sect. 8.3.
- There are rare cases where it might be useful to transfer first into an outer intermediate orbit and from there into the target orbit. In these cases, three (or more) impulsive firings are required and the subsequent **three-impulse transfers**. This is examined in Sects. 8.4.2 and 8.4.3.
- Alternatively, the S/C might be powered by a continuously fired thruster. In that case the transfer orbit will not be a conic section but usually a spiraling orbit, a **continuous thrust transfer orbit** as explored in Sect. 8.4.5.

Orbital Rendezvous
Orbital rendezvous maneuvers are of high practical relevance and are treated in Sect. 8.6. The specific equation of motions for relative rendezvous are developed in Sect. 8.5.

8.1 One-Impulse Maneuvers

The one-impulse maneuver is the foundation of any maneuvering in space. It ideally is a finite thrust firing with vanishingly short duration, a so-called *boost*, *kick-burn* or just *burn* for short, causing a certain $\Delta \boldsymbol{v}$. We have examined impulsive maneuvers already in Sects. 2.3 and 2.4.1 to derive its extent of validity and the corresponding rocket equation to determine the fuel demand for a given $\Delta \boldsymbol{v}$. This section now will investigate how a $\Delta \boldsymbol{v}$ changes a given orbit and what is the $\Delta \boldsymbol{v}$ for a given change of state vector.

8.1.1 Elementary Maneuvers

We first consider the question of how a maneuver $\Delta \boldsymbol{v}$ at a given orbit position θ changes the orbital elements a, e, i, ω, Ω. Because the evaluation of orbit changes due to arbitrarily large kick-burns becomes quite complex, we consider in the following only small impulse maneuvers $\delta \boldsymbol{v} \ll \boldsymbol{v}$ and their effect, δa, δe, δi, $\delta \omega$, $\delta \Omega$, on the orbital elements. To facilitate the calculation and without loss of generality, we decompose $\delta \boldsymbol{v}$ into the following elementary components (see Fig. 8.1):

$\delta v_{\parallel}$ **tangent maneuver**
kick-burn in the direction, i.e. tangent to the orbital motion (along-track)

$\delta v_{\perp \mathrm{O}}$ kick-burn perpendicular to the direction of motion, but within the orbital plane, outbound

$\delta v_{\perp\perp}$ **plane change maneuver**
kick-burn perpendicular to the orbital direction and perpendicular to the orbital plane, in the direction of the angular momentum vector.

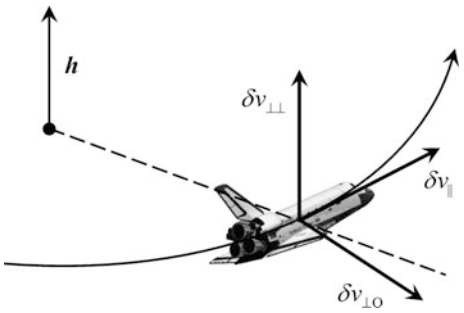


Fig. 8.1 Decomposition of a kick-burn into the along track and two cross-track directions within and outside the orbital plane

Generally, the following applies to these kick-burns:

- Kick-burns perpendicular to the direction of orbital motion can change only the direction of motion and not its speed, and hence not the orbital energy ε. So because $\varepsilon = -\mu/2a$, $\delta v_{\perp \mathrm{O}}$ and $\delta v_{\perp\perp}$ cannot influence the semi-major axis.
- Because $\delta v_{\parallel}$ and $\delta v_{\perp \mathrm{O}}$ are in the orbital plane, they cannot change those orbital elements that determine the orientation of the orbital plane, that is i and Ω. On the other hand, $\delta v_{\perp\perp}$ changes only the orientation of the orbital plane and therefore cannot change a and e. This is why above we called it plane change maneuver.

So, from these general considerations alone we are able to exclude the impact of the kick-burns on some of the orbital elements, which are indicated by empty white boxes in Table 8.1.

Given this decomposition we are now able to analyze the sensitivity of the orbital elements to these elementary kick-burns. The corresponding relations can be derived (exercise, Problem 8.6a) from the Gaussian variational Eq. (12.1.2) with transformations (see Fig. 8.2).

Table 8.1 The effects (matrix elements) of the three different kinds of kick-burns on the orbital elements at the special orbit positions (rightmost column)

	$\delta v_{\parallel}/v_h$	$\delta v_{\perp O}/v_h$	$\delta v_{\perp\perp}/v_h$	
$\delta a/a$	$2/(1\mp e)$			Peri/ Apoapsis $\theta = 0°$ $\theta = 180°$
δe	± 2	–		
$\delta \omega$	–	$\mp 1/e$	○	
δi			$\pm c_{\pm}$	Nodes $\omega + \theta = 0°$ $\omega + \theta = 180°$
$\delta \Omega$			–	
$\delta \omega$	○	○	–	
δi			–	Orthogonal to nodes $\omega + \theta = +90°$ $\omega + \theta = -90°$
$\delta \Omega$			$\pm s_{\pm}/\sin i$	
$\delta \omega$	○	○	$\mp s_{\pm} \cot i$	

White boxes and relations not shown in this table indicate that there are no effects at any orbit position. Dark gray boxes give the dependencies: a dash indicates that for this orbit position the effect vanishes, open circles denote complex dependencies without practical use, and the terms displayed are the factors, that multiplied with the kick-burn of that column gives the change of orbital element of that row. The upper/lower signs correspond to the upper/lower orbit positions in the orbit position column

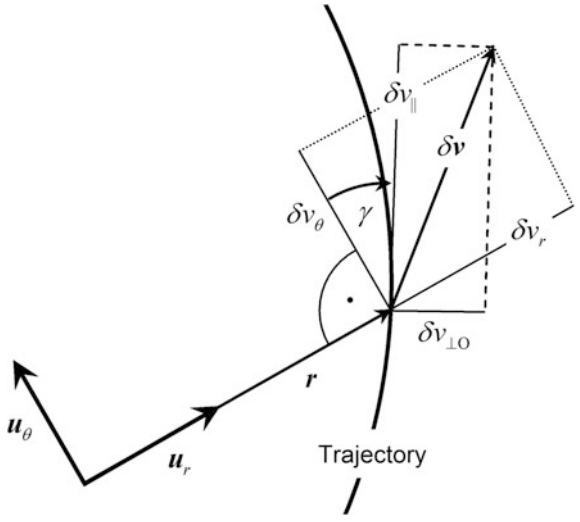


Fig. 8.2 Decomposition of the kick-burn vector in the two different reference systems

$$a_r = \delta v_r = \cos\,\gamma \cdot \delta v_{\perp \mathrm{O}} + \sin\,\gamma \cdot \delta v_{||}$$
$$a_\theta = \delta v_\theta = -\sin\gamma \cdot \delta v_{\perp \mathrm{O}} + \cos\,\gamma \cdot \delta v_{||}$$

and with $\sin\gamma, \cos\gamma$ from Eq. (7.3.16) to get

$$\begin{aligned}
\frac{\delta a}{a} &= \frac{2v}{(1-e^2)v_h}\frac{\delta v_{||}}{v_h} + 0 \cdot \delta v_{\perp \mathrm{O}} + 0 \cdot \delta v_{\perp\perp} \\
\delta e &= 2(e + \cos\theta) \cdot \frac{\delta v_{||}}{v} + \frac{(1-e^2)\sin\theta}{1 + e\cos\theta} \cdot \frac{\delta v_{\perp \mathrm{O}}}{v} + 0 \cdot \delta v_{\perp\perp} \\
\delta i &= 0 \cdot \delta v_{||} + 0 \cdot \delta v_{\perp \mathrm{O}} + \frac{\cos(\theta+\omega)}{1 + e\cos\theta}\frac{\delta v_{\perp\perp}}{v_h} \\
\delta\Omega &= 0 \cdot \delta v_{||} + 0 \cdot \delta v_{\perp \mathrm{O}} + \frac{\sin(\theta+\omega)}{(1 + e\cos\theta)\sin i}\frac{\delta v_{\perp\perp}}{v_h} \\
\delta\omega &= \frac{2\sin\theta}{e} \cdot \frac{\delta v_{||}}{v} - \left(1 + \frac{1}{e}\frac{e + \cos\theta}{1 + e\cos\theta}\right) \cdot \frac{\delta v_{\perp \mathrm{O}}}{v} - \frac{\sin(\theta+\omega)\cot i}{1 + e\cos\theta}\frac{\delta v_{\perp\perp}}{v_h}
\end{aligned} \tag{8.1.1}$$

where

$$v = v_h\sqrt{1 + 2e\,\cos\,\theta + e^2}$$

and

$$v_h := \frac{\mu}{h} = \sqrt{\frac{\mu}{a(1-e^2)}}$$

which corresponds to the orbital velocity at the orbit angle $\cos\theta = -e/2$.

Note *In a circular orbit θ and ω are undefined and therefore the impact of a kick-burn on most orbital elements cannot be determined from* Eq. (8.1.1). *We will treat maneuvers in circular orbits separately in* Sect. 8.1.2.

For practical purposes there are six special orbit positions for optimal firing, which are given in Table 8.1 in the rightmost column. The matrix of the table shows how the three different normalized kick-burns (entries of the three middle columns) at the given special positions affect the different orbital elements (row headings). A dash denotes that the effect just vanishes at this position; a circle denotes that the term describing the effect is more complex and of no practical use; the given terms are the factors, that multiplied by the normalized kick-burn from the column entry deliver the change of orbital element given in the row entry. The important point is, that at the six positions the orbital elements are both selectively and optimally affected, where "optimally" means that for a given change of orbital elements the utilized propulsion mass is minimal. The coefficients shown in the terms are

$$c_{\pm}(\omega) = \frac{1}{1 \pm e \cos \omega} \tag{8.1.2}$$

$$s_{\pm}(\omega) = \frac{1}{1 \pm e \sin \omega} \tag{8.1.3}$$

Note that possible changes of the orbital period are derived via Eq. (7.4.12) to be

$$\frac{\delta T}{T} = \frac{3}{2}\frac{\delta a}{a} \tag{8.1.4}$$

So, only tangential kick-burns effect the orbital period. But tangential burns at the periapsis or apoapsis always change the semi-major axis jointly with the eccentricity according to Table 8.1. The reason is as follows. As for a kick-burn in the peri-/apoapsis, this orbital point is also the peri-/apoapsis for the initial ellipse and the target ellipse, the following is valid:

$$const = r_{apo/per} = a(1 \pm e)$$

Differentiating this equation delivers $0 = \delta a(1 \pm e) \pm a \cdot \delta e$ and hence

$$\frac{\delta a}{a} = -\frac{\delta e}{e \pm 1} \quad @\ \delta v_{||} \quad \text{kick-burn at apo-/periapsis} \tag{8.1.5}$$

Change of Eccentricity

There exists an orbit position not given in Table 8.1, at which with a $\delta v_{\perp O}$ kick-burn the eccentricity can be changed selectively, that is without effecting other orbital elements, namely a or ω. This is from Eq. (8.1.1) position $(e + \cos \theta)/(1 + e \cos \theta) = -e$ or

$$\cos \theta = -\frac{2e}{1 + e^2} \tag{8.1.6}$$

The change at this position amounts to

$$\delta e = \pm\sqrt{1 + e^2}\,\frac{\delta v_{\perp O}}{v_h} \quad \text{where} \quad \begin{matrix} + : & 90^\circ \le \theta < 180^\circ \\ - : & 180^\circ < \theta \le 270^\circ \end{matrix} \tag{8.1.7}$$

However, the following two-impulse maneuver is more efficient to just change the eccentricity and therefore is widely used, in particular in GEO. If one kick-burn $\delta v_{||,\theta}$ is performed at θ in the orbit and another one with $\delta v_{||,180} = -\delta v_{||.\theta}$ at $\theta + 180^\circ$, i.e. at the opposite side of the orbit and in the opposite direction, we get in total

$$\delta e = 4 \cos \theta \left[1 + e \cos \theta + O\left(e^2\right)\right] \cdot \frac{\delta v_{||,\theta}}{v(\theta)}$$

From this we see that eccentricity corrections are most efficient at the peri/apoapsis [or anywhere in a circular orbit, see Sect. 8.1.2)] where we have

$$\delta e = 4 \cdot \frac{\delta v_{||,per}}{\sqrt{\mu/a}} \cdot \left[1 + O\left(e^2\right)\right] \tag{8.1.8}$$

Thus the eccentricity can be increased by one kick-burn $\delta v_{||,per} > 0$ at the periapsis and a second with $-\delta v_{||,per} < 0$ at the apoapsis, and can be decreased by the same procedure with reversed kick-burn directions.

If we assume the same total delta-v as in the one-impulse maneuver above, i.e., $\left|\delta v_{||,per}\right| = \delta v_{\perp \mathrm{O}}/2$, then

$$\delta e \approx \pm 2 \cdot \frac{\delta v_{\perp \mathrm{O}}}{\sqrt{\mu/a}} \left[1 + O\left(e^2\right)\right]$$

which for $e \to 0$ is twice as efficient as the one-impulse maneuver given in Eq. (8.1.7).

Change of Semi-major Axis

We have seen above that it is not possible to separate changes in semi-major axis from changes in eccentricity at one kick-burn. But this is different for a two-impulse maneuver. If one kick-burn $\delta v_{||,\theta}$ is performed at θ in the orbit and the other

$$\frac{\delta v_{||,180}}{v_{180}} = \frac{\cos\,\theta + e}{\cos\,\theta - e} \frac{\delta v_{||,\theta}}{v_\theta} \tag{8.1.9}$$

at $\theta + 180°$, i.e., at the opposite side of the orbit and in the same direction, then $\delta e = 0$ and we get for changes in the semi-major axis

$$\frac{\delta a}{a} = \frac{2}{1 - e^2} \left(\frac{v_\theta^2}{v_h^2} + \frac{\cos\,\theta + e}{\cos\,\theta - e} \frac{v_{180}^2}{v_h^2} \right) \frac{\delta v_{||,\theta}}{v_\theta}$$

and after some expansions

$$\begin{aligned} \frac{\delta a}{a} &= \frac{4\,\cos\,\theta}{\cos\,\theta - e} \frac{\delta v_{||,\theta}}{v_\theta} \\ \delta e &= 0 \end{aligned} \tag{8.1.10a}$$

This reduces for a circular orbit to $\delta v_{||,180} = \delta v_{||,\theta}$ and $\cos\theta = 1$ (see Sect. 8.1.2) and hence

$$\begin{aligned} \delta v_{||,180} &= \delta v_{||,\theta} =: \delta v_{||} \\ \frac{\delta a}{a} &= 4 \frac{\delta v_{||}}{\sqrt{\mu/a}} \\ \delta e &= 0 \end{aligned} \quad @\ e = 0 \tag{8.1.10b}$$

This two-impulsive maneuver is quite common to raise a satellite at its end of life into a higher graveyard orbit.

Change of Periapsis and Apoapsis Radius

We finally note that the periapsis and apoapsis radius r_{per} and r_{apo} can be changed selectively by kick-burns $\delta v_{||,\,apo}$ and $\delta v_{||,\,per}$, respectively, which is highly relevant for orbit maintenance. To show how this comes about we start out again with the above equation

$$r_{apo} = a(1+e)$$

Differentiating this equation yields

$$dr_{apo} = (1+e)\cdot da + a\cdot de$$

hence

$$\frac{dr_{apo}}{r_{apo}} = \frac{da}{a} + \frac{de}{1+e}$$

From Table 8.1 we get

$$\frac{\delta r_{apo}}{r_{apo}} = \left(\frac{2}{1-e} + \frac{2}{1+e}\right)\frac{\delta v_{||,\,per}}{v_h}$$

and finally with $v_h = \sqrt{\mu/[a(1-e^2)]}$

$$\begin{aligned}\frac{\delta r_{apo}}{r_{apo}} &= \frac{4}{\sqrt{1-e^2}}\frac{\delta v_{||,\,per}}{\sqrt{\mu/a}}\\ \frac{\delta r_{per}}{r_{per}} &= \frac{4}{\sqrt{1-e^2}}\frac{\delta v_{||,\,apo}}{\sqrt{\mu/a}}\end{aligned} \tag{8.1.11}$$

where the second equation follows by the same procedure with $r_{per} = a(1-e)$. We hence see that an apse can be lowered or raised by a corresponding kick burn at the apse *opposite* to the one to be changed.

Change of Angular Elements

As seen from Eq. (8.1.1), the angular elements Ω, i can be changed solely by plane change maneuvers $\delta v_{\perp\perp}$

$$\begin{aligned}\delta i &= \frac{\cos(\theta+\omega)}{1+e\,\cos\,\theta}\frac{\delta v_{\perp\perp}}{v_h}\\ \delta\Omega &= \frac{\sin(\theta+\omega)}{(1+e\,\cos\,\theta)\sin\,i}\frac{\delta v_{\perp\perp}}{v_h}\end{aligned}$$

Although the argument of periapsis ω changes for any kick-burn, we neglect it here, because Earth orbits are mostly circular, in case of which ω is irrelevant, or orbits are near-circular and hence the role of ω negligible.

From the above it follows that inclination can be changed selectively and optimally at the two orbital nodes $\omega+\theta=0,\ 180^\circ$

$$\delta i=\pm\frac{1}{1\pm e\,\cos\,\omega}\frac{\delta v_{\perp\perp}}{v_h}\quad @\quad \begin{matrix}+:&\omega+\theta=0^\circ\\-:&\omega+\theta=180^\circ\end{matrix}\tag{8.1.12}$$

Observe that for a given kickburn $\delta v_{\perp\perp}$, the angular elements $\delta\Omega,\delta\omega$ change in the same way except with opposite sign. Therefore a RAAN change by $\delta v_{\perp\perp}$ comes always hand in hand with a change of argument of periapsis, even if two kick-burns in any direction are performed at opposite abeam positions.

RAAN is changed selectively at orbital positions abeam from the nodal points according to

$$\delta\Omega=\pm\frac{1}{(1\pm e\,\cos\,\omega)\sin\,i}\frac{\delta v_{\perp\perp}}{v_h}\quad @\quad \begin{matrix}+:&\omega+\theta=90^\circ\\-:&\omega+\theta=-90^\circ\end{matrix}\tag{8.1.13}$$

Note that is a matter of kick-burn direction $\delta v_{\perp\perp}>0$, $\delta v_{\perp\perp}<0$ to determine whether at a given abeam position RAAN change is positive or negative. Also note that if the orbit has a small eccentricity or acquires it due to gravitational perturbations (see Sect. 12.3.4) with an often unknown argument of periapsis, then two smaller kick-burns with opposite directions and half the magnitude, $\pm\frac{1}{2}\delta v_{\perp\perp}$ at the two different positions yield the simple results

$$\delta i=\frac{\delta v_{\perp\perp}}{v_h}$$
$$\delta\Omega=\frac{1}{\sin\,i}\frac{\delta v_{\perp\perp}}{v_h}\left(1+O\left(e^2\right)\right)$$

8.1.2 *Elementary Maneuvers in Circular Orbits*

The most common orbits around planets are circular orbits, $e=0$, because at a given orbital energy they minimize atmospheric drag and provide steady orbit conditions. But circular orbits do not exhibit a periapsis line, implying an undefined true anomaly θ and argument of periapsis ω. Therefore, the change of orbital elements cannot be determined from Eq. (8.1.1) and Table 8.1. We have to fall back on more basic considerations to treat this problem.

Change of the Semi-major Axis

An exception is δa for which we immediately derive from Eq. (8.1.1), for $e=0$ and hence $v=v_h=\sqrt{a/\mu}$,

$$\frac{\delta a}{a}=2\sqrt{\frac{a}{\mu}}\,\delta v_{\parallel}+0\cdot\delta v_{\perp\mathrm{O}}+0\cdot\delta v_{\perp\perp}$$

To determine $\delta e,\ \delta i$, and $\delta\Omega$ ($\delta\omega$ is irrelevant for circular orbits), we have to take a different approach.

Change of Eccentricity

For δe we go back to the key Eq. (7.3.3) for the eccentricity that reads with $\mu/r = v^2$

$$\mu \boldsymbol{e} = \left(v^2 - \frac{\mu}{a}\right)\boldsymbol{r} - (\boldsymbol{r}\boldsymbol{v})\boldsymbol{v}$$

A $\delta v_{||}$ kick-burn implies $\boldsymbol{\delta v}_{||} \perp \boldsymbol{r}$ in a circular orbit. The velocity after the kick-burn is, therefore, $v = v_0 + \delta v_{||}$, with $v_0^2 = \mu/a = v_h^2$. For the eccentricity after the kick-burn we thus obtain

$$\mu \boldsymbol{e} = \left(v_0^2 + 2\delta v_{||} v_0 - v_0^2\right)\boldsymbol{r} - 0 = 2\delta v_{||} v_0 \boldsymbol{r}$$

Because at kick-burn $\boldsymbol{e} || \boldsymbol{r}$ applies, we have $\theta = 0°$ at that moment. As we had $e = 0$ before the kick-burn we get

$$\delta e = 2\delta v_{||} v_0 \frac{r}{\mu} = 2\delta v_{||} \frac{v_0}{v_0^2} = 2\frac{\delta v_{||}}{v_h}$$

If we perform a $\delta v_{\perp \mathrm{O}}$ kick-burn in a circular orbit, then $\boldsymbol{\delta v}_{\perp \mathrm{O}} || \boldsymbol{r}$ and $\boldsymbol{\delta v}_{\perp \mathrm{O}} \perp \boldsymbol{v}_0$ and therefore $v = v_0$. This renders

$$\mu \boldsymbol{e} = 0 - (\boldsymbol{r}\boldsymbol{v}_0)\boldsymbol{v} - (\boldsymbol{r} \cdot \delta \boldsymbol{v}_{\perp \mathrm{O}})\boldsymbol{v} = -r \cdot \delta v_{\perp \mathrm{O}} \boldsymbol{v}$$

Therefore the S/C right after the kick-burn is at $\theta = 90°$, and because $r = a$, we find

$$\delta e = \frac{a}{\mu} v \cdot \delta v_{\perp \mathrm{O}} = \frac{\delta v_{\perp \mathrm{O}}}{v_h}$$

Change of Inclination

For the inclination change Eq. (8.1.1) yields for a circular orbit

$$di = \cos(\omega + \theta) \cdot \frac{dv_{\perp\perp}}{v_h}$$

On the other hand, according to Fig. 8.4, the change of inclination should be just

$$\delta i = \frac{\delta v_{\perp\perp}}{v} = \frac{\delta v_{\perp\perp}}{v_h}$$

This implies $\omega + \theta = 0$.

Change of RAAN

Finally, we obtain with the condition $\omega + \theta = 0$ for the change of RAAN

$$\delta\Omega = \frac{\sin(\omega+\theta)}{(1+e\,\cos\,\theta)\sin\,i}\frac{dv_{\perp\perp}}{v_h} = 0\cdot\frac{dv_{\perp\perp}}{v_h}$$

Summary
Summing up we find the following expressions of the change of orbital elements of a circular orbit due to kick-burns

$$\begin{pmatrix}\delta a/a\\ \delta e\\ \delta i\\ \delta\Omega\end{pmatrix} = \sqrt{\frac{a}{\mu}}\begin{pmatrix}2 & 0 & 0\\ 2 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}\delta v_{||}\\ \delta v_{\perp \mathrm{O}}\\ \delta v_{\perp\perp}\end{pmatrix} \tag{8.1.14}$$

After $\delta v_{||}$, $\delta v_{\perp \mathrm{O}}$ kick-burns the orbit is elliptic with a true anomaly

$$\theta = \begin{cases} 0^\circ & @\ \delta v_{||}\\ 90^\circ & @\ \delta v_{\perp \mathrm{O}}\end{cases} \tag{8.1.15}$$

Note that this true anomaly plus the position on the circular orbit (argument of latitude u, see Sect. 7.3.5) where the kick-burn was performed determine the induced argument of periapsis ω of the final ellipse: $\omega = u - \theta$.

After a $\delta v_{\perp\perp}$ kick-burn in a circular orbit with initial inclination $i = 0$ the final inclination of the circular orbit is $i = \delta i$ and the position on the orbit is

$$u = \omega + \theta = \begin{cases} 0^\circ & @\ \delta v_{\perp\perp} > 0\\ 180^\circ & @\ \delta v_{\perp\perp} < 0\end{cases} \tag{8.1.16}$$

8.1.3 General Maneuvers

We now want to study arbitrary finite one-impulse maneuvers, which at a given point $\boldsymbol{r}$ in space transfers the state vector $(\boldsymbol{r}, \boldsymbol{v}_1)$ of the initial orbit into the state vector $(\boldsymbol{r}, \boldsymbol{v}_2)$ of the final orbit as depicted in Fig. 8.3. From the vector triangle we hence get very generally

$$\Delta\boldsymbol{v} = \boldsymbol{v}_2 - \boldsymbol{v}_1$$

To determine the corresponding propulsion demand, one has to square this equation and take the root on both sides, resulting in

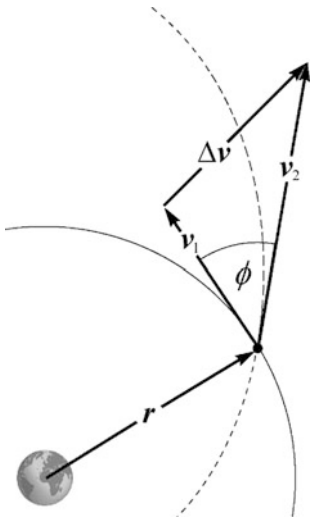


Fig. 8.3 The general one-impulse maneuver

$$\begin{aligned}\Delta v &= \sqrt{v_1^2 + v_2^2 - 2v_1 v_2 \cos\,\phi} \\ &= \sqrt{(v_1 - v_2)^2 + 4v_1 v_2 \sin^2(\phi/2)}\end{aligned} \tag{8.1.17}$$

where ϕ is the **transition angle** enclosed between $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$. This is the most general equation to calculate a single-burn delta-v.

Oberth Effect

There exists a general effect, the Oberth effect, governing some types of orbit transfers. Suppose we move on a Keplerian orbit from any position with state elements r, v to another position with state elements r', v'. Then the vis-viva equation states that the total energy at both positions must be the same

$$v^2 - \frac{2\mu}{r} = v'^2 - \frac{2\mu}{r'}$$

Now let's suppose we do a boost Δv at r. We then want to know: How much is the velocity increase $\Delta v'$ at any other r'? Or in general: How does the increase in velocity impart different radial distances? The according energy equation with boost reads

$$(v + \Delta v)^2 - v^2 = \Delta v(2v + \Delta v) = \Delta v'(2v' + \Delta v') = (v' + \Delta v')^2 - v'^2$$

or

$$\Delta v = \Delta v' \frac{v' + \Delta v'/2}{v + \Delta v/2}$$

This is a telling result. It claims that for a given $\Delta v'$ the required boost Δv to achieve it at radial distances $0 < r \leq r'$, equalling $\infty > v \geq v'$, changes monotonically in the interval

$$0 < \Delta v \leq \Delta v'$$

So, the boost Δv to achieve $\Delta v'$ is minimal at the lowest orbital radius and hence highest velocity v of the orbit. This unexpected effect is called the *Oberth effect*, after Hermann Oberth, an Austro-Hungarian-born rocket scientist who discovered this effect in 1929. It is due to the fact that although momentum changes Δv are key for fuel considerations and thus mission design, energy conservation calling for $(v + \Delta v)^2 - v^2$ governs the motion.

This effect sounds powerful, yet its application is quite limited in space flight, because any boost at a given position in orbit also changes the orbital elements and hence the further trajectory. This is usually not what the mission designer asked for. So, a key phrase describing the Oberth effect above is "How does the increase in velocity impart another radial distance?", rather than "… achieves a given target point". There are three cases where "solely an increase in radial distance" is sufficient. One is for the transfer from an ellipse to a (near)-circular target orbit, which we examine in Sect. 8.3.3. The second is an initial elliptic orbit, usually in LEO, the line of apsides of which is adjustable by the injection conditions or orbit maintenance (keep this important case always in mind); and finally the powered flyby (see Sect. 9.5.3) where the change in flight direction can be counteracted by an adjusted impact parameter, i.e. flyby distance.

Two-Burn Escape Maneuvers

As already suggested by Hermann Oberth in 1928, the consequent application of the Oberth effect gives rise to making escape transfers more efficient. Rather than performing an Oberth maneuver (see Eq. (7.4.26)) at the lowest orbital radius to escape the gravity well, the spacecraft first performs a breaking maneuver (first maneuver) to dive down into the central body's gravity well for a flyby (nicknamed *solar fryby* for a solar flyby). Then at its periapsis an Oberth maneuver (second maneuver) is performed to accelerate the spacecraft to escape speed. According to the Oberth effect, such a two-burn maneuver is more efficient than the direct single-burn Oberth maneuver.

8.1.4 Tangent Plane Maneuvers

A general maneuver changes the shape of an orbit, a, e, its orientation ω within the orbital plane, and the orientation of the plane, i, Ω. Practically, the orientation ω of the orbital plane is largely irrelevant, because most of the spacecraft are in a circular orbit having no ω. In addition, for orbit transitions a Hohmann transfer altering a, e jointly with a plane change i, Ω is essential. We read from Eq. (8.1.1) that to solely change a, e and i, Ω only $\delta v_{||}$ and $\delta v_{\perp\perp}$ kick-burns are relevant. This implies that $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ span a plane, tangent to the orbit at the kick-burn location. This important class of orbit maneuvers is therefore called *tangent plane maneuvers*. Figure 8.4 shows a tangent change maneuver with $v_1 = v_2 =: v$, that is, without a, e changes. Such a particular tangent plane maneuver is hence called *genuine plane change maneuver* (see next section).

Transition Angle in Terms of Orbital Elements

The important property of a tangent plane maneuver is that because the plane spaned by $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ is vertical to the initial and final plane, the maneuver causes a tilt of the initial (index 1) to the final (index 2) orbit about a mutual nodal line, for which $\phi = \angle(\boldsymbol{v}_1, \boldsymbol{v}_2) = \angle(\boldsymbol{I}_1, \boldsymbol{I}_2)$ holds. Here $\boldsymbol{I}$ is the three-dimensional inclination vector (see Eq. (7.3.25)) given as

$$\boldsymbol{I} = \begin{pmatrix} \sin\Omega \sin i \\ -\cos\Omega \sin i \\ \cos i \end{pmatrix}_{IJK}$$

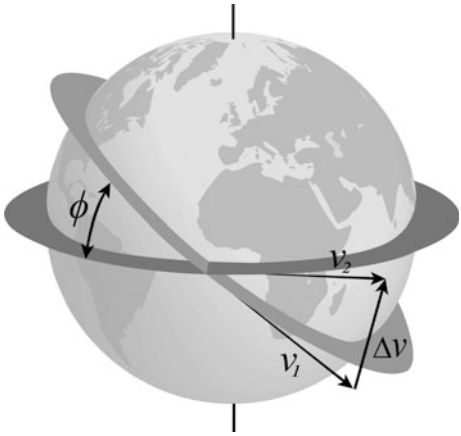


Fig. 8.4 A tangent plane maneuver with $v_1 = v_2$ (a.k.a. genuine plane change maneuver) in a circular orbit. The point where the maneuver takes place is the node of the mutual nodal line between the initial and final orbit plane

With this the transition angle is determined by

$$\cos\phi = \hat{\boldsymbol{v}}_1\hat{\boldsymbol{v}}_2 = \boldsymbol{I}_1\boldsymbol{I}_2 = \sin i_1 \sin i_2 \cos\Delta\Omega + \cos i_1 \cos i_2$$

or after some trigonometric modifications alternatively

$$\sin^2\frac{\phi}{2} = \sin^2\frac{\Delta i}{2} + \sin i_1 \sin i_2 \sin^2\frac{\Delta\Omega}{2} \tag{8.1.18}$$

where $\Delta i := i_2 - i_1$ and $\Delta\Omega := \Omega_2 - \Omega_1$. Obviously, the vector $\boldsymbol{N}$ of the nodal line between the initial and final orbit is given by

$$\boldsymbol{N} = \boldsymbol{I}_1 \times \boldsymbol{I}_2 \tag{8.1.19}$$

Maneuvers with either Inclination Change or RAAN Change
There are two special plane change maneuvers, which are of practical interest. One, in which the RAAN is kept constant, $\Delta\Omega = 0$. We thus obtain from Eq. (8.1.18)

$$\begin{aligned}&\phi = \Delta i\\ &\Delta v = \sqrt{v_1^2 + v_2^2 - 2v_1v_2\cos\Delta i}\end{aligned} \quad @ \quad \Delta\Omega = 0 \quad \textbf{nodal transfer} \tag{8.1.20}$$

According to Table 8.1 this plane tilt is achieved for a $\Delta\boldsymbol{v}$ that lies in either plane tangent to the two nodes of the initial orbit with the reference plane. Such transfers are therefore called *nodal transfers* and are the most frequently used plane change maneuvers.

The other special case is achieved if the inclination is kept constant, $\Delta i = 0$. We thus have from Eq. (8.1.18)

$$\begin{aligned}&\sin(\phi/2) = \sin i \cdot |\sin(\Delta\Omega/2)|\\ &\Delta v = \sqrt{(v_1 - v_2)^2 + 4v_1v_2\sin^2 i \sin^2(\Delta\Omega/2)}\end{aligned} \quad @\ \Delta i = 0 \tag{8.1.21}$$

According to Table 8.1 $\Delta i = 0$ is achieved for a $\Delta\boldsymbol{v}$ that lies in either plane tangent to the two points at 90° abeam from the nodes of the initial orbit with the reference plane.

8.1.5 *Genuine Plane Change Maneuvers*

We finally study just plane change maneuvers without a change in orbital velocity, $v_1 = v_2 =: v$ as depicted in Fig. 8.4. Because we now apply only a $\delta v_{\perp\perp}$ kick-burn, genuine plane change maneuvers are a subset of tangent plane maneuvers, which

according to Eq. (8.1.1) only change i, Ω, ω. From Eq. (8.1.17) we derive for $v = v_1 = v_2$ quite generally

$$\Delta v = 2v \cdot \left| \sin \frac{\phi}{2} \right| \quad @ \quad v := v_1 = v_2 \tag{8.1.22}$$

Because $\Delta v \propto v$ we find the general rule of thumb

Plane Change Rule
An orbit plane change should be performed at the smallest possible orbital velocity, i.e. at best at the apoapsis.

For this reason it might be even preferable to bring the vehicle first into a higher orbit, change the plane there at apoapsis at low speed and low budget, and finally take it back to its original orbit. It is this principle, that the bi-elliptic transfer (Sect. 8.4.2) and the super-synchronous transfer (Sect. 8.4.3) make use of.

Genuine Inclination Change Maneuver
For $\Omega_1 = \Omega_2$ we have from Eq. (8.1.20) $\phi = \Delta i$. We hence obtain from Eq. (8.1.22)

$$\Delta v = 2v \cdot \left| \sin \frac{\Delta i}{2} \right| \quad @ \; v := v_1 = v_2, \; \Delta\Omega = 0 \tag{8.1.23}$$

According to Table 8.1 this tilt is achieved for a $\Delta \boldsymbol{v}$ that lies in either plane tangent to the two nodes of the initial orbit with the reference plane and is orthogonal to the plane bisecting the old and new orbit plane.

Genuine RAAN Change Maneuver
For $\Delta i = 0$ we have from Eq. (8.1.21)

$$\Delta v = 2v \cdot \sin i \cdot \left| \sin \frac{\Delta\Omega}{2} \right| \quad @ \quad v := v_1 = v_2, \Delta i = 0 \tag{8.1.24}$$

This is the effort to rotate the inclination vector by $\Delta\Omega$ about the K-axis in the geocentric equatorial reference frame. According to Table 8.1 this rotation is achieved for a $\Delta \boldsymbol{v}$ that lies in either plane tangent to the two points at 90° abeam from the nodes of the initial orbit with the reference plane. Obviously, a RAAN change by $\Delta\Omega = 180°$ is identical to an inclination change by $\Delta i = 2i$.

Because $\Delta v \propto v \sin i$ we find the general rule of thumb.

RAAN Change Rule
Adjust RAAN at the lowest inclination and/or orbital velocity possible.

8.1.6 Tangent Maneuver

Another quite special case of a tangent plane maneuver is $\phi = 0°$, i.e., the burn is along the direction of motion. We call this a *tangent maneuver* for which from Eq. (8.1.17) follows

$$\Delta v_{||} = |v_2 - v_1| \quad @ \quad \phi = 0° \tag{8.1.25}$$

From Eq. (8.1.1) we see that such a maneuver changes a, e, ω. As we will see in Sect. 8.3.1 this is the most efficient maneuver to increase the size of an orbit and hence to transfer to an outer orbit.

Let us assume we are in an initial orbit with a_1, e_1, ω_1 and perform at $r := r_1 = r_2$ a tangent maneuver $r, v_1, \theta_1 \rightarrow r, v_2, \theta_2$ with

$$v_2 = v_1 \pm \Delta v_{||} \tag{8.1.26}$$

We want to know what is a_2, e_2, ω_2? From the vis-viva Eq. (7.2.15) we immediately find

$$\frac{1}{a_2} = \frac{2}{r} - \frac{v_2^2}{\mu} \tag{8.1.27}$$

To determine the other modified orbital elements we make use of the fact that a tangent maneuver obviously does not change the flight path angle $\gamma := \gamma_1 = \gamma_2$ (see Sect. 7.3.3). From Eq. (7.3.16) we derive the relationships

$$v \cos\gamma = \frac{h}{r}$$
$$\frac{1}{2} v^2 \sin 2\gamma = \frac{\mu}{r} e \sin\theta$$

Applying these equations to the orbits before and after the tangent maneuver we find

$$\frac{v_2}{v_1} = \frac{h_2}{h_1} = \frac{a_2\left(1 - e_2^2\right)}{a_1\left(1 - e_1^2\right)}$$

$$\frac{v_2^2}{v_1^2} = \frac{e_2 \sin\theta_2}{e_1 \sin\theta_1}$$

from which we derive

$$e_2 = \sqrt{1 - \frac{a_1 v_2}{a_2 v_1}\left(1 - e_1^2\right)} \tag{8.1.28}$$

$$\sin\,\theta_2 = \frac{v_2^2}{v_1^2}\frac{e_1}{e_2}\sin\,\theta_1 \tag{8.1.29}$$

To finally determine ω_2 we make use of the fact that a tangent maneuver also does not change the argument of latitude $u = \omega + \theta$ (see Sect. 7.3.5). We hence obtain

$$\omega_2 = \omega_1 + \theta_1 - \theta_2 \tag{8.1.30}$$

Example (GEO Positioning)

Let us assume that a communication satellite is launched with an ATLAS rocket from Cape Canaveral with orbital inclination $i_L = 28°$ after launch and needs to be positioned in GEO (see Berlin (2005)). As we will see in Sect. 12.6.1, due to lunisolar perturbations an optimum initial state in GEO would be $i_{GEO} = 3°$, $\Omega_{GEO} = 280°$. After launch and at the descending node the rocket will make an injection burn at $r_{LEO} = 6578.14$ km into the GTO with apoapsis at $r_{GEO} = 42{,}166$ km. Because the descending node happens to coincide with the Greenwich meridian we have $\Omega_{GTO} = 180°$.

To place the satellite in its final orbit position we have the choice of many different maneuver sequences for all of which are based on the following velocities before (−) and after (+) the kick-burns in LEO and GEO (see Eqs. (7.4.3) and (7.4.10)):

$$\begin{aligned} v_{LEO-} &= 7.784\,\mathrm{km\,s^{-1}} \\ v_{LEO+} &= 10.239\,\mathrm{km\,s^{-1}} \end{aligned} \quad \text{and} \quad \begin{aligned} v_{GEO-} &= 1.597\,\mathrm{km\ s^{-1}} \\ v_{GEO+} &= 3.075\,\mathrm{km\ s^{-1}} \end{aligned}$$

We now study exemplarily four different maneuvering sequences with the following total delta-v:

Sequence 1

- At the descending node after launch an inclination change maneuver ((Eq. (8.1.20) into Eq. (8.1.17)) is made that brings the satellite into a GTO with $i_{GEO} = 3°$, $\Omega_{GTO} = 180°$.
- At apogee a genuine RAAN change maneuver (see Eq. (8.1.24)) changes the orbit plane to $\Omega_{GEO} = 280°$.
- Finally, a tangent maneuver (see Eq. (8.1.25)) brings the satellite into a circular GEO.

$$\begin{aligned}\Delta v &= \sqrt{v_{LEO-}^2 + v_{LEO+}^2 - 2v_{LEO-}v_{LEO+}\cos(28^\circ - 3^\circ)} \\ &\quad + 2v_{GEO-}\sin 3^\circ \cdot \sin[(280^\circ - 180^\circ)/2] + (v_{GEO+} - v_{GEO-}) \\ &= 4.578 + 0.128 + 1.478 = 6.184\,\text{km s}^{-1}\end{aligned}$$

Sequence 2

- At the descending node after launch a tangent maneuver transfers the satellite into a GTO, i.e., $i_{GTO} = 28^\circ,\ \Omega_{GTO} = 180^\circ$.
- At apogee a genuine RAAN change maneuver (see Eq. (8.1.24)) changes the orbit plane to $\Omega_{GEO} = 280^\circ$.
- Finally, a inclination change maneuver (Eq. (8.1.20) into Eq. (8.1.17)) is performed that circularizes the satellite into GEO at $i_{GEO} = 3^\circ$.

$$\begin{aligned}\Delta v &= (v_{LEO+} - v_{LEO-}) + 2v_{GEO-}\sin 28^\circ \cdot \sin[(280^\circ - 180^\circ)/2] \\ &\quad + \sqrt{v_{GEO-}^2 + v_{GEO+}^2 - 2v_{GEO-}v_{GEO+}\cos(28^\circ - 3^\circ)} \\ &= 2.455 + 1.149 + 1.762 = 5.366\,\text{km s}^{-1}\end{aligned}$$

Sequence 3

1. At the descending node after launch a tangent maneuver transfers the satellite into a GTO, i.e., $i_{GTO} = 28^\circ,\ \Omega_{GTO} = 180^\circ$.
2. At apogee a genuine inclination change maneuver (see Eq. (8.1.23)) changes the inclination to $i_{GEO} = 3^\circ$.
3. Still at apogee it is immediately followed by a genuine RAAN change maneuver (see Eq. (8.1.24)) that changes the orbit plane to $\Omega_{GEO} = 280^\circ$.
4. Finally, the orbit is circularized with an apogee kick-burn.

$$\begin{aligned}\Delta v &= (v_{LEO+} - v_{LEO-}) + 2v_{GEO-}\cdot\sin 12.5^\circ \\ &\quad + 2v_{GEO-}\sin 3^\circ \cdot \sin[(280^\circ - 180^\circ)/2] + (v_{GEO+} - v_{GEO-}) \\ &= 2.455 + 0.691 + 0.128 + 1.478 = 4.752\,\text{km s}^{-1}\end{aligned}$$

Sequence 4

- At the descending node the satellite is transferred into a GTO, i.e., $i_{GTO} = 28^\circ, \Omega_{GTO} = 180^\circ$.
- At apogee, an inclination-change maneuver (Eq. (8.1.20) into Eq. (8.1.17)) is performed such that the orbit is circularized at $i_{GEO} = 3^\circ$.
- Finally, a genuine RAAN change maneuver (see Eq. (8.1.24)) positions the satellite at $\Omega_{GEO} = 280^\circ$.

$$\begin{aligned}\Delta v &= (v_{LEO+} - v_{LEO-}) + \sqrt{v_{GEO-}^2 + v_{GEO+}^2 - 2v_{GEO-}v_{GEO+}\cos(28^\circ - 3^\circ)} \\ &\quad + 2v_{GEO+}\sin 3^\circ \cdot \sin[(280^\circ - 180^\circ)/2] \\ &= 2.455 + 1.762 + 0.247 = 4.464\,\text{km s}^{-1}\end{aligned}$$

The essence of comparing the four sequences is that for GEO positioning an inclination change should be combined with the orbit circularization at apoapsis and RAAN should be adjusted at the lowest possible inclination. This is how GEO positioning is actually performed.

This example corroborates the following rule of thumb, which is based on the fact (see Problem 8.1) that a tangent plane maneuver is always more efficient than a tangent maneuver and plane change maneuver performed sequentially at the same point in space:

All-in-One Rule
If possible combine all orbital changes that need to be done into one kick-burn at one orbital position rather than making single burns at successive positions.

8.2 Lambert Transfer

Having explored the characteristics of one-impulse maneuvers between intersecting orbits we are now set to perform orbital transfers between two conic orbits that do not intersect and therefore require a transfer orbit with two orbital transition maneuvers at each end. The most general case is the so-called Lambert transfer, which we will investigate first. It played an important role in the 1960s for Gemini and Apollo orbital rendezvous. Today, Hohmann transfers are more common, which will be studied in Sect. 8.3.

8.2.1 Orbital Boundary Value Problem

Lambert Transfer
Let us assume that we have a S/C at point P_1 on an conic Earth orbit that needs to transfer to point P_2 on a coplanar conic target orbit, both of which do not intersect. (If they intersect we can perform an one-impulse maneuver at the intersection point, as described in Sect. 8.1, and move from there on the target orbit to point P_2.) Because the orbits and hence also the transfer orbit are coplanar they are fully characterized by their form and orientation to each other, i.e., by three orbital elements, namely $a_i, \boldsymbol{e}_i$ for the initial Earth orbit, $a, \boldsymbol{e}$ for the transfer orbit, and $a_f, \boldsymbol{e}_f$

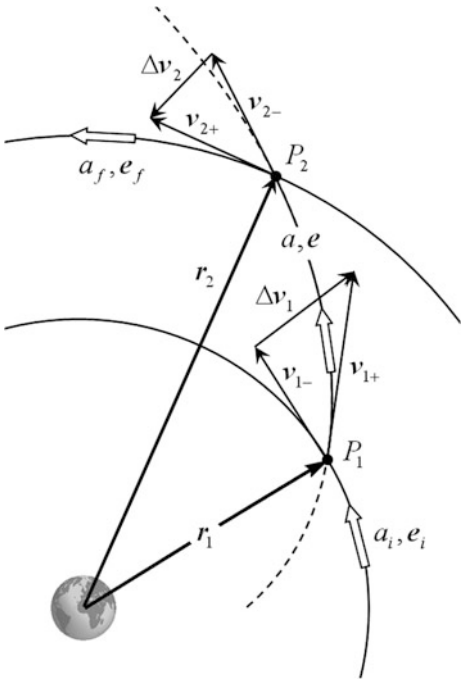


Fig. 8.5 A Lambert transfer (see text for details)

for the final target orbit. The most general description of this problem is that under the influence of a common central gravitational potential a S/C in the initial orbit at P_1 possesses the state vector $(\boldsymbol{r}_1, \boldsymbol{v}_{1-})$ and wants to transfer to P_2 with state vector $(\boldsymbol{r}_2, \boldsymbol{v}_{2+})$. This is a general two-impulse transfer, which we call a *Lambert transfer*, embracing a first impulse maneuver $\Delta\boldsymbol{v}_1$ at P_1 resulting in an unpowered conic coplanar Lambert transfer orbit with orbital elements $a, \boldsymbol{e}$ that ends at P_2 where the second impulse maneuver $\Delta\boldsymbol{v}_2$ takes place.

This can be shown as follows (see Fig. 8.5):

$$(a_i, \boldsymbol{e}_i) \underbrace{[\Delta\boldsymbol{v}_1]}_{P_1} \xrightarrow{a,\boldsymbol{e}} \underbrace{[\Delta\boldsymbol{v}_2]}_{P_2} (a_f, \boldsymbol{e}_f)$$

Lambert transfer

$$(\boldsymbol{r}_1, \boldsymbol{v}_{1-}) \underbrace{-[\Delta\boldsymbol{v}_1]_+}_{P_1} \xrightarrow{a,\boldsymbol{e}} \underbrace{-[\Delta\boldsymbol{v}_2]_+}_{P_2} (\boldsymbol{r}_2, \boldsymbol{v}_{2+})$$

where the $+$ and $-$ signs indicate the situation just before or after an impulse maneuver. Since only the initial and final orbits and P_1 and P_2 are specified, while $\Delta\boldsymbol{v}_1$ and $\Delta\boldsymbol{v}_2$ remain unspecified, there in general exist many different Lambert transfer orbits connecting the two points P_1 and P_2.

Orbital Boundary Value Problem

The problem of finding a transfer orbit connecting two given points P_1 and P_2 is called the *orbital boundary value problem*, which is a special case of the general mathematical *two-point boundary value problem*.

Let us study the orbital boundary value problem by first assuming that just P_1 and P_2 are given through the position vectors $\boldsymbol{r}_1, \boldsymbol{r}_2$. With these we get from the orbit Eqs. (7.3.4) and (7.3.6) the two conditional equations for the three orbital elements $a, \boldsymbol{e}$

$$r_1 + \boldsymbol{e}\boldsymbol{r}_1 = a\left(1 - e^2\right)$$
$$r_2 + \boldsymbol{e}\boldsymbol{r}_2 = a\left(1 - e^2\right)$$

Therefore, two orbital points are not sufficient to unequivocally define a Keplerian transfer orbit, but leave one orbital element undefined. We need a third position vector or an additional scalar specification to concretize the transfer orbit. Solving the orbital boundary value problem by three position vectors is accomplished by Gibbs' method, which will not be investigated here, but see e.g., Vallado (2001) or Curtis (2005). An additional scalar specification might be for instance the transfer time. This famous orbital boundary value problem is called *Lambert's problem* and will be discussed in Sect. 8.2.3.

As we will see in Sect. 8.2.2, even an additional scalar specification generally still leaves the alternative between two so-called *conjugate orbits*: a long path and a short path transfer orbit. Therefore the general orbital boundary value problem for a Lambert transfer to find the orbital elements $a, \boldsymbol{e}$ of the transfer orbit is uniquely defined only by providing the following boundary conditions:

$$\boxed{\begin{array}{l}\boldsymbol{r}_1,\ \boldsymbol{r}_2 \\ +\ \text{an additional scalar specification} \\ +\ \text{a specification for a long or short path transfer orbit}\end{array}} \quad \textbf{boundary conditions} \tag{8.2.1}$$

Given these boundary conditions we are able derive a unique solution $a, \boldsymbol{e}$ to the orbital boundary value problem in the following way. *Suppose we have found the unique transfer orbit by providing p*. According to Battin (1987), the terminal velocities at P_1 and P_2 are then determined as

$$\begin{aligned}\boldsymbol{v}_{1+} &= \frac{\sqrt{\mu p}}{r_1 r_2 \sin(\Delta\theta)}\left[(\boldsymbol{r}_2 - \boldsymbol{r}_1) + \frac{r_2}{p}(1 - \cos\,\Delta\theta)\boldsymbol{r}_1\right] \\ \boldsymbol{v}_{2-} &= \frac{\sqrt{\mu p}}{r_1 r_2 \sin(\Delta\theta)}\left[(\boldsymbol{r}_2 - \boldsymbol{r}_1) - \frac{r_1}{p}(1 - \cos\,\Delta\theta)\boldsymbol{r}_2\right]\end{aligned} \tag{8.2.2}$$

where $\Delta\theta = \angle(\boldsymbol{r}_1, \boldsymbol{r}_2)$ is the so-called *transfer angle*. Having thus found the state vectors $(\boldsymbol{r}_1, \boldsymbol{v}_{1+})$ at point P_1 and $(\boldsymbol{r}_2, \boldsymbol{v}_{2-})$ at point P_2 of the transfer orbit we derive from Eq. (7.3.3) the orientation of the conic transfer orbit as

$$\boldsymbol{e} = \left(\frac{1}{r_1} - \frac{1}{a}\right)\boldsymbol{r}_1 - \frac{1}{\mu}(\boldsymbol{r}_1\boldsymbol{v}_{1+})\boldsymbol{v}_{1+} = \left(\frac{1}{r_2} - \frac{1}{a}\right)\boldsymbol{r}_2 - \frac{1}{\mu}(\boldsymbol{r}_2\boldsymbol{v}_{2-})\boldsymbol{v}_{2-} \tag{8.2.3}$$

Having thus determined the relevant orbital elements $a, \boldsymbol{e}$ of the Lambert transfer orbit, we have found the solution to the orbital boundary value problem. Note that with such methods the orbital boundary value problem is also used to preliminarily determine an unknown orbit. Preliminary orbit determination will be discussed in Chap. 14.

With the state vectors $(\boldsymbol{r}_1, \boldsymbol{v}_{1+})$ and $(\boldsymbol{r}_2, \boldsymbol{v}_{2-})$ we also know the required orbital maneuvers and the total delta-v

$$\begin{aligned} \Delta\boldsymbol{v}_1 &= \boldsymbol{v}_{1+} - \boldsymbol{v}_{1-} \\ \Delta\boldsymbol{v}_2 &= \boldsymbol{v}_{2+} - \boldsymbol{v}_{2-} \\ \Delta\boldsymbol{v} &= |\Delta\boldsymbol{v}_1| + |\Delta\boldsymbol{v}_2| \end{aligned} \tag{8.2.4}$$

Having thus fully determined the Lambert transfer we recognize that everything hinges on the problem of finding the semi-latus rectum p of the Lambert transfer orbit under the given boundary conditions Eq. (8.2.1) to which we turn now.

8.2.2 *Lambert Transfer Orbits*

We will first study the characteristics of the solutions of the orbital boundary value problem if only the position vectors $\boldsymbol{r}_1$, $\boldsymbol{r}_2$ are given. In this case there exists an infinite number of orbits with different a and e connecting P_1 and P_2. Let us first assume that with an additional specification we have picked a particular but otherwise arbitrary a.

Conjugate Orbits: One a, But Two Transfer Orbits

It is a basic property of the orbital boundary value problem that for any particular semi-major axis a in general there exist two different transfer orbits, so-called *conjugate orbits*, with different eccentricity and hence semi-latus rectum, which we will label p and $\tilde{p}$.

From geometrical reasoning, it is enlightening to see why. Suppose there is a transfer orbit with a given a. We define the distances from its empty focus F' to the points P_1 and P_2 as r_1' and r_2', respectively. Then from the definition of an ellipse we have

$$\begin{aligned} r_1 + r_1' &= 2a \\ r_2 + r_2' &= 2a \end{aligned}$$

Since the location of the focal point F is given by the central mass, we can establish the transfer ellipses by determining their empty focal points $F_{1,2}'$. We do so by drawing a circle with radius $r_1' = 2a - r_1$ around P_1 and a circle with radius

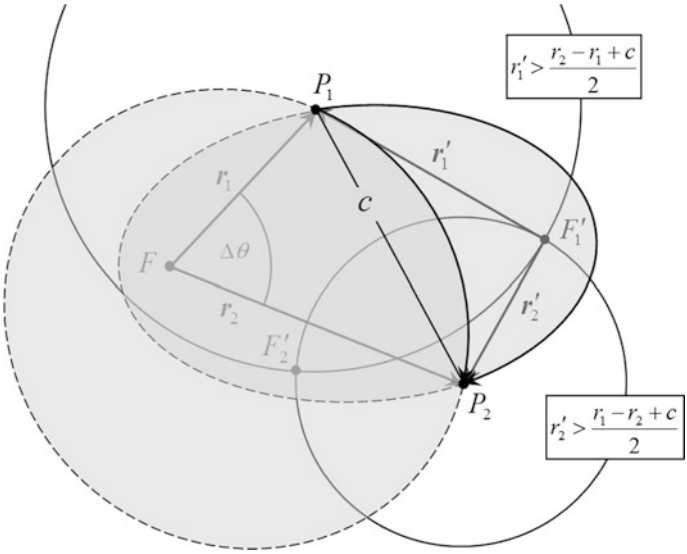


Fig. 8.6 Circles, with radii larger than some critical value, drawn around the points P_1 and P_2 intersect at two points defining the empty focal points F' of the two different transfer ellipses

$r_2' = 2a - r_2$ around P_2. If a is large enough ("large enough" will be shown in a moment to be $a \geq (r_1 + r_2)/2$), the two circles will intersect at two empty focal points on opposite sides of the chord connecting P_1 and P_2 as shown in Fig. 8.6, which define the two transfer ellipses.

A transfer orbit is called a *short path ellipse* if it has the shorter path between P_1 and P_2, and hence a smaller eccentricity, and a shorter transfer time; while the other is called a *long path ellipse*.

Note that their major-axes are indeed of identical length, only their eccentricities and their orientations are different.

When a is known, it can be shown (see Battin (1964, 1987), or Kemble (2006)) that the semi-latus rectum p of the two paths are given as

$$\begin{aligned} p &= a\,\frac{r_1 r_2}{c^2}(1-\cos\Delta\theta)[1-\cos(\alpha+\beta)] \\ &= 2a\,\frac{r_1 r_2}{c^2}(1-\cos\Delta\theta)\sin^2\frac{\alpha+\beta}{2} \end{aligned} \quad @\ \text{short path} \tag{8.2.5a}$$

$$\begin{aligned} \tilde{p} &= \frac{r_1 r_2}{a}\frac{1-\cos\Delta\theta}{1-\cos(\alpha+\beta)} \\ &= 2a\,\frac{r_1 r_2}{c^2}(1-\cos\Delta\theta)\sin^2\frac{\alpha-\beta}{2} \end{aligned} \quad @\text{long path} \tag{8.2.5b}$$

where the chord length c, which is the separation distance between P_1 and P_2, is given from trigonometry by

$$c = \sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos \Delta\theta} \tag{8.2.6}$$

and

$$\sin\frac{\alpha}{2} := \sqrt{\frac{r_1 + r_2 + c}{4a}}, \quad \sin\frac{\beta}{2} := \sqrt{\frac{r_1 + r_2 - c}{4a}}$$

From Eqs. (8.2.5a) and (8.2.5b) we see immediately that

$$p \cdot \tilde{p} = \left[\frac{r_1 r_2}{c}(1 - \cos \Delta\theta)\right]^2 = p_{\min}^2$$

where $p_{\min}$ is the semi-latus rectum of the minimum energy transfer orbit discussed in the following (see Eq. (8.2.9)). We also see that owing to $0 \le \beta \le \alpha \le \pi$

$$c\tilde{p} < r_1 r_2(1 - \cos \Delta\theta) < cp \quad \textbf{long/short path condition} \tag{8.2.7}$$

Minimum Energy Transfer Orbit, But Maximum Transfer Time

Because Eq. (7.3.18) directly relates the magnitude of the semi-major axis to the orbital energy by $\varepsilon = -\mu/2a$, this condition states that among all transfer orbits that

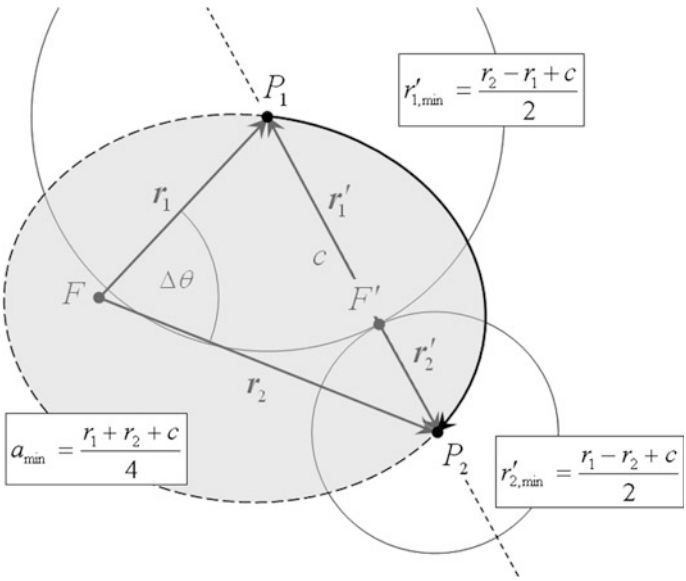


Fig. 8.7 The minimum energy transfer orbit occurs when the empty focal point lies on the chord c connecting point P_1 and P_2

go through P_1 and P_2 there is an orbit with minimum orbital energy. To figure out what its a is, we consider again $r_1 + r_1' = 2a$ and, $r_2 + r_2' = 2a$ from which follows

$$4a = r_1 + r_2 + r_1' + r_2'$$

With $r_1 + r_2$ as a given quantity, a is minimum if $r_1' + r_2'$ becomes minimal, which from Fig. 8.7 is the case if $F_{1,2}'$ lies on the chord c joining P_1 and P_2. Therefore the minimum energy transfer orbit has

$$a_{\min} = \frac{r_1 + r_2 + c}{4} \tag{8.2.8}$$

We recognize that in this limiting situation the two circles around P_1 and P_2 do not intersect, but just touch each other leading to just one empty focal point and therefore to one, and only one, transfer ellipse. If a is smaller than the minimum value the circles will not intersect and no transfer solution exists—the minimum energy orbit is its own conjugate. On the other hand, because $c \leq r_1 + r_2$ always holds, a transfer orbit with $a \geq (r_1 + r_2)/2$ will always be large enough to ensure intersecting circles and hence link the points P_1 and P_2. It can be shown (see, e.g., Battin (1987) or Vallado (2001)) that $p_{\min}$ for the minimum energy orbit, is given by

$$p_{\min} = \frac{r_1 r_2}{c}(1 - \cos\Delta\theta) = \frac{c}{2}\left[1 - \left(\frac{r_1 - r_2}{c}\right)^2\right] \tag{8.2.9}$$

We summarize by stating that

> The **minimum energy ellipse** has a minimum semi-latus rectum given by Eq. (8.2.9) and its empty focal point lies on the chord connecting P_1 and P_2.

Notice that with the additional specification "minimum energy transfer" we have rounded out all boundary conditions and hence settled on a specific solution with an unique $a = a_{\min}$ and $e = \sqrt{1 - p_{\min}/a_{\min}}$.

Note that the occurrence of the minimum energy transfer orbit does not necessarily mean that the total delta-v for the Lambert transfer also becomes a minimum. As seen from Eqs. (8.2.2) and (8.2.4) the total delta-v rather depends in a complex way on the transition from the initial velocity to the transfer velocity at P_1 and back to the final orbital velocity at P_2. But, in general it can be said that any transfer close to a minimum energy transfer is a "good" transfer. Observe that for a minimum energy transfer Δt no longer is a variable that can be freely chosen. The free variable then is the proper configuration of the objects at P_1 and P_2 in their orbits around F that complies with a minimum energy transfer. The search for such an optimal configuration, the so-called orbit phasing, is the objective of Sect. 9.3 for interplanetary transfers.

It can be shown (Problem 8.8) that the minimum energy transfer orbit goes hand in hand with a maximum transfer time. The semi-latus rectum for a minimum energy transfer is given above as $4a_{\min} = r_1 + r_2 + c$. With the transformations introduced for Lambert's problem in Sect. 8.2.3, we have

$$\sin\frac{\alpha_{\min}}{2} = \sqrt{\frac{r_1 + r_2 + c}{4a_{\min}}} = 1,$$
$$\sin\frac{\beta_{\min}}{2} = \sqrt{\frac{r_1 + r_2 - c}{r_1 + r_2 + c}}$$

and hence $\alpha_{\min} = \pi$. With Lambert's equation (see Eq. (8.2.13)) the maximum transfer time of the minimum energy transfer orbit then reads

$$\begin{aligned}\Delta t_{\max} &= \sqrt{\frac{a_{\min}^3}{\mu}}[\pi - \beta_{\min} + \sin\beta_{\min}] \\ &= \sqrt{\frac{a_{\min}^3}{\mu}}\left[\pi - \arccos\left(\frac{c}{a_{\min}} - 1\right) + \sqrt{\frac{c}{a_{\min}}\left(2 - \frac{c}{a_{\min}}\right)}\right]\end{aligned} \tag{8.2.10}$$

The Fundamental Ellipse

There are many transfer orbits with metric orbital elements a, e that connect P_1 and P_2. However, as there is just one transfer orbit with a characteristic a—the minimum energy orbit—there exists just one transfer orbit with a specified characteristic eccentricity: the *fundamental ellipse* with a minimum eccentricity.

To find the orbit with minimum e we first investigate the eccentricity vector as given by the orbit equation Eq. (7.3.4) in vectorial form and apply it to the points P_1 and P_2, obtaining

$$\boldsymbol{e}\boldsymbol{r}_1 = p - r_1$$
$$\boldsymbol{e}\boldsymbol{r}_2 = p - r_2$$

Subtracting these from each other yields

$$\boldsymbol{e}(\boldsymbol{r}_1 - \boldsymbol{r}_2) = r_1 - r_2$$

Since $\boldsymbol{r}_1 - \boldsymbol{r}_2 = c\boldsymbol{u}_c$ is the cord with unit vector $\boldsymbol{u}_c$ and length c, we get

$$\boldsymbol{e}\boldsymbol{u}_c = \frac{r_1 - r_2}{c}$$

This equation states that the eccentricity vector of any transfer orbit between P_1 and P_2 has the same projection on the chord. The fundamental ellipse now is that one of those with the minimum eccentricity. Because the minimum eccentricity is achieved if $\boldsymbol{e}||\boldsymbol{u}_c$, we have for the fundamental ellipse

$$e_F = \frac{r_1 - r_2}{c} = \min \tag{8.2.11}$$

and because $\boldsymbol{e}||\boldsymbol{u}_c$ implicates $2a = r_1' + r_2' = r_1 + r_2$ we have

$$p_F = a\left(1 - e_F^2\right) = \frac{r_1 r_2}{c^2}(r_1 + r_2)(1 - \cos\,\Delta\theta) \tag{8.2.12}$$

Because the fundamental ellipse has a particular p_F it has a particular flight time, which is identical with that of its conjugate orbit having the same semi-latus rectum but different eccentricity.

Note that though the minimum energy transfer ellipse and the fundamental ellipse share the same semi-latus rectum $a_{\min} = a_F = (r_1 + r_2)/2$, they do not share the same position of the empty focal point F'. While F' for the minimum energy orbit lies on the chord, the one for the fundamental ellipse lies on a line through F that runs parallel to the chord (see the "low-e" ellipse in Fig. 8.6 that is close to the fundamental ellipse). Therefore,

> The **fundamental ellipse** has the lowest eccentricity and shares the minimum semi-latus rectum given by Eq. (8.2.12) of all elliptic orbits that lie on P_1 and P_2. Its line of apsides runs parallel to the chord line.

8.2.3 *Lambert's Problem*

Apart from the minimum energy or minimum eccentricity specifications there might be other specifications that settle on a transfer orbit between P_1 and P_2 with a particular a. From a practical and historical point of view the most important, however, is the specification of a transfer time. The problem to find the according transfer orbit is called *Lambert's problem* (Lambert 1761). It can be stated as follows: "the determination of an orbit, having a specified transfer time and connecting two position vectors", or in other words "to solve for the trajectory connecting two position vectors with a given time of flight".

Because in Lambert's problem both transfer time and orbital anomalies are involved, for a solution we start out with Kepler's Eq. (7.4.15) for an elliptic transition and Eq. (7.4.30) for a hyperbolic orbit

$$n \cdot (t_1 - t_2) = n \cdot \Delta t = E_1 - E_2 - e(\sin E_1 - \sin E_2) \qquad @\ \text{elliptic orbit}$$
$$n \cdot (t_1 - t_2) = n \cdot \Delta t = e(\sinh F_1 - \sin F_2) - (F_1 - F_2) \qquad @\ \text{hyperbolic orbit}$$

It can be shown (for a proof see, e.g., Battin (1964, 1987), or Kemble (2006)) that with the substitutions

$$\sin\frac{\alpha - \beta}{2} := \frac{E_2 - E_1}{2}, \cos\frac{\alpha + \beta}{2} := e\,\cos\frac{E_2 + E_1}{2} \qquad @\ \text{elliptic}$$

$$\sinh\frac{\gamma - \delta}{2} := \frac{F_2 - F_1}{2}, \cosh\frac{\gamma + \delta}{2} := e\,\cosh\frac{F_2 + F_1}{2} \qquad @\ \text{hyperbolic}$$

and with the quantities

$$c = \sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos\,\Delta\theta} \qquad @\ 0 \le \Delta\theta \le \pi$$
$$s_\pm := \frac{1}{2}\sqrt{\frac{r_1 + r_2}{c} \pm 1}$$

Kepler's equations can be rewritten to yield *for the short path and hence short transition times and for small transfer angles* $0 \le \Delta\theta \le \pi$ the Lambert's equations

Elliptic Orbit (short path, $0 \le \Delta\theta \le \pi$)

$$\boxed{\Delta t = \sqrt{\frac{a^3}{\mu}}[\alpha - \beta - (\sin\,\alpha - \sin\,\beta)]} \quad \textbf{Lambert's equation} \qquad (8.2.13)$$

with

$$\sin\frac{\alpha}{2} = \sqrt{\frac{r_1 + r_2 + c}{4a}} = \sqrt{\frac{c}{a}}s_+ > 0$$
$$\sin\frac{\beta}{2} = \sqrt{\frac{r_1 + r_2 - c}{4a}} = \sqrt{\frac{c}{a}}s_- > 0 \qquad @ \quad 0 \le \beta \le \alpha \le \pi$$
$$p = a\frac{r_1 r_2}{c^2}(1 - \cos\Delta\theta)[1 - \cos(\alpha + \beta)]$$

Hyperbolic Orbit (short path, $0 \le \Delta\theta \le \pi$)

$$\boxed{\Delta t = \sqrt{\frac{-a^3}{\mu}}[\sinh\gamma - \sinh\delta - (\gamma - \delta)]} \quad \textbf{Lambert's equation} \qquad (8.2.14)$$

with

$$\sinh\frac{\gamma}{2} = \sqrt{-\frac{r_1 + r_2 + c}{4a}} = \sqrt{\frac{c}{-a}}s_+ > 0$$
$$\sinh\frac{\delta}{2} = \sqrt{-\frac{r_1 + r_2 - c}{4a}} = \sqrt{\frac{c}{-a}}s_- > 0 \qquad @\ 0 \le \delta \le \gamma$$
$$p = a\frac{r_1 r_2}{c^2}(1 - \cos\Delta\theta)[1 - \cosh(\gamma + \delta)]$$

where c again is the chord length, i.e., the straight distance between the points P_1 and P_2, and $\Delta\theta = \angle(\boldsymbol{r}_1, \boldsymbol{r}_2)$ is again the so-called *transfer angle*. Obviously, $r_1 + r_2 + c \ge r_1 + r_2 - c$ implies that $0 \le \beta \le \alpha \le \pi$ must hold.

Parabolic and Near-Parabolic Trajectories

If $|a| \to \infty$, i.e., if we have a near-parabolic transfer path, it can be shown (see Problem 8.9) that a can be provided analytically as

$$\frac{c}{a} = \frac{5}{2}\frac{\Delta t\sqrt{\mu/c^3} - \frac{4}{3}\left(s_+^3 - s_-^3\right)}{s_+^5 - s_-^5} \quad @\ \text{near-parabolic path} \tag{8.2.15}$$

Equation (8.2.15) provides also the condition at which $|a| \to \infty$, namely $\frac{3}{4}\Delta t\sqrt{\mu/c^3} \approx s_+^3 - s_-^3$. To determine the semi-latus rectum p for $c/a > 0$ Eq. (8.2.13), and for $c/a < 0$ Eq. (8.2.14) need to be applied.

If

$$\Delta t = \frac{4}{3}\sqrt{\frac{c^3}{\mu}}\left(s_+^3 - s_-^3\right) \quad @\ \text{parabolic path}$$

then $a = \infty$ and we have a parabolic transfer path with a single p given by

$$\begin{aligned} p &= 2\frac{r_1 r_2}{c}(1 - \cos\Delta\theta)(s_+ + s_-)^2 \\ &= 2\frac{r_1 r_2}{c^2}\sin^2\frac{\Delta\theta}{2}\left(r_1 + r_2 + 2\sqrt{r_1 r_2}\cos\frac{\Delta\theta}{2}\right) \end{aligned} \quad @\ \text{parabolic path} \tag{8.2.16}$$

With this the orbit orientation $\boldsymbol{e}$ can be deduced via Eqs. (8.2.2) and (8.2.3).

Lambert's Theorem

From these results Lambert's theorem can be derived.

Lambert's theorem

The transfer time of a body moving between two points on a conic trajectory is a function only of the sum of the distances of the two points from the origin of force, the length c of the chord joining these two positions, and the semi-major axis of the conic: $\Delta t = f(r_1 + r_2, c, a)$

Alternatively, Lambert's theorem can be stated as

$$\Delta t = f(r_1, r_2, a, \Delta\theta)$$

This more explicitly shows that the transfer time does not depend on the individual locations of the starting or arrival point or their true anomalies, but just on their separation angle $\Delta\theta$. This is for instance useful in finding a solution to the orbital rendezvous problem (see Sect. 8.6). It is remarkable that transfer time does not depend on the eccentricity of the transfer orbit. Remarkably, this is the third orbital

quantity, besides the specific orbital energy (Eq. (7.3.19)) and the orbital period of an ellipse (Eq. (7.4.2)), that does not depend on the eccentricity. It is just this independency of e that makes the fundamental ellipse dispensable for practical purposes.

Universal Formulation of Lambert's Equation

Given Lambert's equation we are set to provide a universal formulation of Lambert's equation. With P_1 and P_2 given, we know $\boldsymbol{r}_1, \boldsymbol{r}_2$, $\Delta\theta$, and hence chord length c. For a given Δt we then can derive from Lambert's equation (8.2.13) the semi-major axis a of the corresponding short transfer path numerically, for instance with Newton's method (see Sect. 7.4.2).

In the following we want to provide a numerically robust algorithm for any Keplerian orbit. For that we first define the dimensionless numbers

$$
\begin{aligned}
x &:= c/a \\
\Delta\tau &:= \Delta t \sqrt{\frac{\mu}{c^3}} \\
\boldsymbol{\rho}_1 &= \boldsymbol{r}_1/c,\ \boldsymbol{\rho}_2 = \boldsymbol{r}_2/c \\
s_\pm &= \frac{1}{2}\sqrt{\rho_1 + \rho_2 \pm 1} \\
c_\pm(x) &:= \sqrt{1 - x s_\pm^2}
\end{aligned}
$$

With this and the condition $0 \le \beta \le \alpha \le \pi$ we find for an elliptic orbit

$$
\begin{aligned}
\sin\alpha - \sin\beta &= 2\left(\sin\frac{\alpha}{2}\cos\frac{\alpha}{2} - \sin\frac{\beta}{2}\cos\frac{\beta}{2}\right) \\
&= 2(s_+ c_+ - s_- c_-)\sqrt{x} > 0 \\
\alpha - \beta &= 2\left(\arcsin s_+\sqrt{x} - \arcsin s_-\sqrt{x}\right) \\
&= 2\arcsin\left[(s_+ c_- - s_- c_+)\sqrt{x}\right] > 0
\end{aligned}
$$

and for a hyperbolic orbit with $0 \le \beta \le \alpha$

$$
\begin{aligned}
\sinh\gamma - \sinh\delta &= 2\left(\sinh\frac{\gamma}{2}\cosh\frac{\gamma}{2} - \sinh\frac{\delta}{2}\cosh\frac{\delta}{2}\right) \\
&= 2(s_+ c_+ - s_- c_-)\sqrt{-x} > 0 \\
\gamma - \delta &= 2\left(\operatorname{arsinh} s_+\sqrt{-x} - \arcsin s_-\sqrt{-x}\right) \\
&= 2\operatorname{arsinh}\left[(s_+ c_- - s_- c_+)\sqrt{-x}\right] > 0
\end{aligned}
$$

so that we can rewrite and merge Lambert's equations for both elliptical and hyperbolic orbits into one equation

$$
\frac{\Delta\tau}{2}x = \frac{g\left[(s_+ c_- - s_- c_+)\sqrt{|x|}\right]}{\sqrt{|x|}} - (s_+ c_+ - s_- c_-) \quad @\ \text{Keplerian orbits}
$$

where

$$g\left(\sqrt{|u|}\right) := \begin{cases} \arcsin(\sqrt{u}) & @\ 0 < u \le 1 \\ \operatorname{arsinh}(\sqrt{-u}) & @\ u < 0 \end{cases}$$

The similarity of the power series expansion in both cases gives rise to an universal **L-function**

$$\begin{aligned} L(u) &= 1 + \frac{1}{2\cdot 3}u + \frac{1\cdot 3}{2\cdot 4\cdot 5}u^2 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7}u^3 \ldots \\ &\quad + \frac{(2n-1)!!}{2^n n!(2n+1)}u^n + \ldots \qquad @ -\infty < u \le 1 \\ &= \begin{cases} \dfrac{\arcsin\sqrt{u}}{\sqrt{u}} & @\ 0 \le u \le 1 \\ \dfrac{\operatorname{arsinh}\sqrt{-u}}{\sqrt{-u}} = \dfrac{\ln\left(\sqrt{-u}+\sqrt{1-u}\right)}{\sqrt{-u}} & @\ u < 0 \end{cases} \end{aligned}$$

(For the mathematically inclined reader: Note the relationship between the L-function and Stumpff function $c_2(u)$, see Sect. 7.4.8). We therefore finally obtain

$$\boxed{\frac{\Delta\tau}{2}x = (s_+c_- - s_-c_+)\cdot L(u) - (s_+c_+ - s_-c_-)} \quad @\ \text{Keplerian orbits} \qquad (8.2.17)$$

with $u = (s_+c_- - s_-c_+)^2 x$. Note that for $x = 0$ due to $L(0) = 1$ and $c_\pm(0) = 1$ the right hand side vanishes and so does the left hand side.

Owing to the above dimensionless formulation of Lambert's equation, Lambert's theorem can be reduced to $\Delta\tau = f\left(\frac{c}{r_1+r_2}, \frac{c}{a}\right)$ or $\frac{c}{a} = g\left(\Delta\tau; \frac{c}{r_1+r_2}\right)$, respectively, as displayed in Fig. 8.8. Because $0 \le |r_2 - r_1| \le c \le r_2 + r_1$ the parameter $c/(r_1+r_2)$ is restricted to

$$0 \le \frac{c}{r_1+r_2} \le 1$$

and according to Eq. (8.2.8)

$$4a \ge r_1 + r_2 + c$$

Universal Semi-Latus Rectum

To determine the semi-latus rectum we further define the dimensionless numbers

$$\begin{aligned} \rho &:= (r_1 r_2 - \boldsymbol{r}_1\boldsymbol{r}_2)/c^2 = \frac{1}{2}\left[1 - \left(\frac{r_1 - r_2}{c}\right)^2\right] \\ \lambda &:= 2(s_+c_- + s_-c_+)^2 \end{aligned}$$

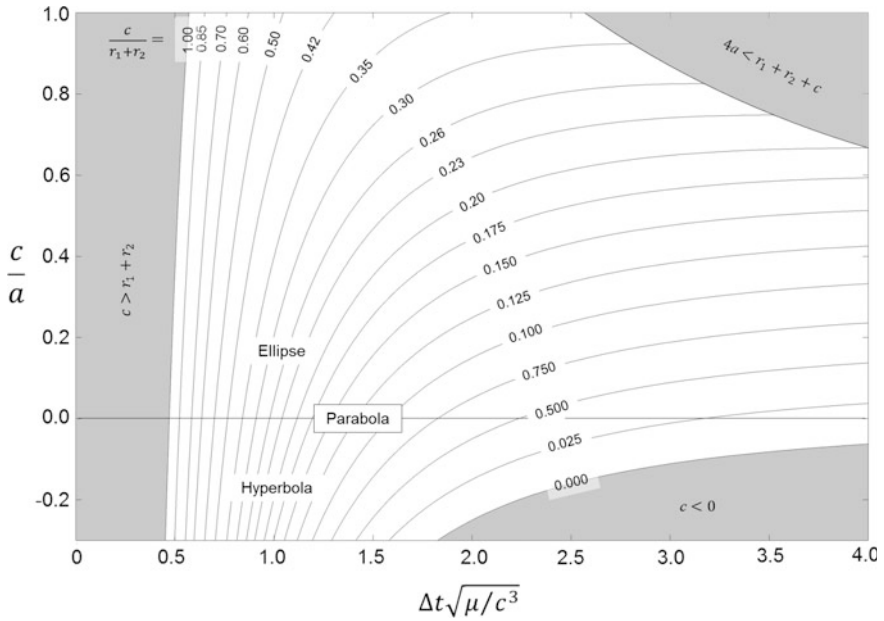


Fig. 8.8 Solution of Lambert's Problem in graphical form

which yields

$$r_1 r_2(1-\cos\Delta\theta) = r_1 r_2 - r_1 r_2\cos\Delta\theta = r_1 r_2 - \boldsymbol{r}_1\boldsymbol{r}_2 = c^2\rho$$

$$\begin{aligned} a[1-\cos(\alpha+\beta)] &= 2a\sin^2\frac{\alpha+\beta}{2} = 2a\left(\sin\frac{\alpha}{2}\cos\frac{\beta}{2} + \cos\frac{\alpha}{2}\sin\frac{\beta}{2}\right)^2 \\ &= 2c(s_+c_- + s_-c_+)^2 = c\lambda \qquad @ \quad x \geq 0 \end{aligned}$$

$$\begin{aligned} a[1-\cosh(\gamma+\delta)] &= -2a\sinh^2\frac{\lambda+\delta}{2} = -2a\left(\sinh\frac{\gamma}{2}\cosh\frac{\delta}{2} + \cosh\frac{\gamma}{2}\sinh\frac{\delta}{2}\right)^2 \\ &= 2c(s_+c_- + s_-c_+)^2 = c\lambda \qquad @ \quad x \leq 0 \end{aligned}$$

We therefore have from Eqs. (8.2.13), (8.2.14), and (8.2.16) for the short path of any Keplerian orbit

$$p = c\rho\lambda \quad @\ \text{Keplerian orbit} \quad \textbf{semi-latus rectum} \tag{8.2.18}$$

With this and some trigonometric expansions we find the following calculation scheme:

Calculation Scheme for an Universal Solution to Lambert's Problem

0. Given $\boldsymbol{r}_1, \boldsymbol{r}_2$, Δt, and μ

1. With $c := \sqrt{r_1^2 + r_2^2 - 2\boldsymbol{r}_1\boldsymbol{r}_2}$ calculate the dimensionless numbers

$$\Delta\tau = \Delta t\sqrt{\frac{\mu}{c^3}}$$

$$s_\pm = \frac{1}{2}\sqrt{\frac{r_1 + r_2}{c} \pm 1}$$

2. With $c_\pm := \sqrt{1 - xs_\pm^2}$ and $u = (s_+c_- - s_-c_+)^2 x$ solve the equation

$$\frac{\Delta\tau}{2} \cdot x + s_+c_+ - s_-c_- = (s_+c_- - s_-c_+)L(u)$$

for x with Newton's method or for difficult hyperbolic orbits $x<0$ with the bisection method. (If $x = c/a<0$ hyperbola, if $x = 0$ parabola, if $x > 0$ ellipse)

3. Calculate the dimensionless number and vectors

$$\lambda = 2(s_+c_- + s_-c_+)^2$$

$$\boldsymbol{\rho}_1 = \boldsymbol{r}_1/c,\ \boldsymbol{\rho}_2 = \boldsymbol{r}_2/c$$

$$\rho = \rho_1\rho_2 - \boldsymbol{\rho}_1\boldsymbol{\rho}_2 = \frac{1}{2}\left[1 - \left(\frac{r_1 - r_2}{c}\right)^2\right]$$

With this the eccentricity vector (see Eq. (8.2.3)) and the dimensionless terminal velocities (see Eq. (8.2.2)) and transfer path eccentricity (see Eq. (8.2.3)) are determined as

$$\boldsymbol{v}_{1+}\sqrt{\frac{c}{\mu}} = \frac{\sqrt{\rho\lambda}}{|\boldsymbol{\rho}_1 \times \boldsymbol{\rho}_2|}\left[\boldsymbol{\rho}_2 - \boldsymbol{\rho}_1 + \frac{1}{\lambda}\hat{\boldsymbol{\rho}}_1\right]$$

$$\boldsymbol{v}_{2-}\sqrt{\frac{c}{\mu}} = \frac{\sqrt{\rho\lambda}}{|\boldsymbol{\rho}_1 \times \boldsymbol{\rho}_2|}\left[\boldsymbol{\rho}_2 - \boldsymbol{\rho}_1 - \frac{1}{\lambda}\hat{\boldsymbol{\rho}}_2\right]$$

$$\boldsymbol{e} = (1 - x\rho_1)\hat{\boldsymbol{\rho}}_1 - \frac{c}{\mu}(\boldsymbol{\rho}_1\boldsymbol{v}_{1+})\boldsymbol{v}_{1+}$$

8.2.4 Minimum Effort Lambert Transfer

Having analyzed the Lambert transfer orbit we will now finally turn to the most important problem from a practical point of view: Given two transfer points P_1 and P_2 with state vectors $(\boldsymbol{r}_1, \boldsymbol{v}_{1-})$ and $(\boldsymbol{r}_2, \boldsymbol{v}_{2+})$ what is the Lambert transfer orbit with the least propulsion effort? So the additional scalar specification here is the least effort equaling a minimum total $\Delta \boldsymbol{v}$ as given in Eq. (8.2.4). In order to determine that transfer orbit, we recall the two orbital equations at P_1 and P_2

$$r_1 + \boldsymbol{e}\boldsymbol{r}_1 = a\left(1 - e^2\right)$$
$$r_2 + \boldsymbol{e}\boldsymbol{r}_2 = a\left(1 - e^2\right)$$

These conditional equations leave one of the three orbital elements undetermined. It is this undetermined element that provides us with the freedom to optimize the transfer orbit. We choose as the free element the orientation of the conic orbit, position vector $\boldsymbol{r}_1$. We denote this angle as

$$\varphi = \angle(\boldsymbol{e}, \boldsymbol{r}_1)$$

We therefore have $\hat{\boldsymbol{e}}\hat{\boldsymbol{r}}_1 = \cos\,\varphi$ and for geometrical reasons $\hat{\boldsymbol{e}}\hat{\boldsymbol{r}}_2 = \cos(\Delta\theta - \varphi)$. From the first equation we then have

$$p = a\left(1 - e^2\right) = r_1(1 + e\,\cos\,\varphi) \tag{8.2.19}$$

Subtracting from this equation the second one from above yields

$$r_1 - r_2 + e r_1 \cos\,\varphi - e r_2 \cos(\Delta\theta - \varphi) = 0$$

from which we derive

$$e = \frac{r_2 - r_1}{r_1 \cos\,\varphi - r_2 \cos(\Delta\theta - \varphi)} \tag{8.2.20}$$

Hence, we have derived the orbital elements a, e and p as a function of the orientation of the transfer orbit $\varphi = \angle(\boldsymbol{e}, \boldsymbol{r}_1)$. We insert this result into Eq. (8.2.2) and find with Eq. (8.2.4) the total Δv as a function of φ. Employing an optimization algorithm that does "minimize $\Delta v(\varphi)$", one finds the optimal φ and from the above equations also the optimal orbital elements.

8.3 Hohmann Transfer

We have already seen that a minimum energy Lambert orbit does not necessarily imply a minimum energy Lambert transfer because for that we also have to take into account the delta-v of the two impulse maneuvers into and out of the transfer orbit. So we are now seeking for a minimum energy Lambert transfer. This is the Hohmann transfer.

The general problem statement is this: What is the transfer orbit between two Keplerian orbits with the least propulsion demand? Because this general problem is quite complex, we will restrict ourselves to the following conditions:

- The two orbits are elliptic and coplanar.
- The orbital bodies revolve around the same central body with the same orientation (co-revolving), so their angular momentum vectors are collinear, only their magnitudes are different.
- The orbital elements are such that the two orbits nowhere touch or cross each other.

Due to the last constraint we can define an inner and an outer orbit, which we will denote by the symbols • and O.

Throughout this section we assume in addition that the transfers are achieved with impulsive thrust maneuvers, so-called *boosts* or *kick-burns* with $F_* \gg F_{ext}$ (see Sect. 2.3) so that the created Δv is determined solely by the thrust characteristics. This assumption is in general valid for today's chemical propulsion engines for orbit control.

8.3.1 The Minimum Principle

As an example for an orbit transfer, let us consider the situation where we are in an elliptic LEO, and we want to get from any point on this orbit into an elliptic geosynchronous orbit, GSO (a GSO is in sync with Earth's rotation only on the average of an orbit) (see Fig. 8.9). A kick-burn in LEO will first take us into an elliptic transfer orbit, the so-called GSO transfer orbit (GTO). This transfer orbit has to cross or touch the GSO at some point. Once we are at the crossing point, a second kick-burn would bring us into GSO.

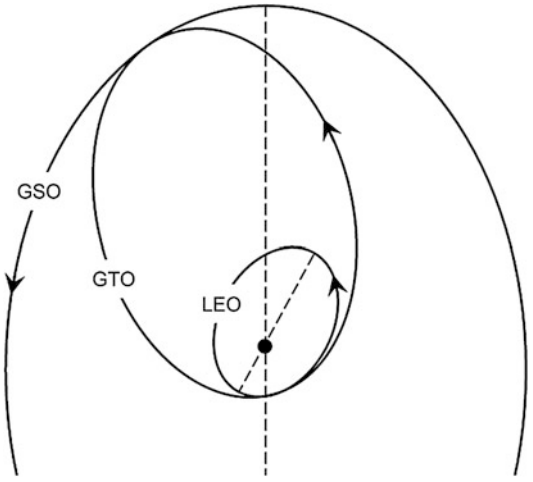


Fig. 8.9 The GTO as a Hohmann transfer orbit from an elliptic LEO to an elliptic GSO (not to scale)

Now, an optimal transfer will minimize the sum of the two Δv at the two transition points. Because a kick-burn $\Delta \boldsymbol{v}$ at a point in space increases the kinetic energy $\varepsilon_{kin}(\boldsymbol{v}+\Delta\boldsymbol{v}) = \frac{1}{2}m(\boldsymbol{v}+\Delta\boldsymbol{v})^2$ but leaves the potential energy unchanged, the orbital energy ε of the S/C increases by the same amount. Owing to $\varepsilon = -\mu/2a$, the upshot is that the semi-major axis expands. In fact, an essential part of the transfer is to enlarge the semi-major axis of the initial LEO to that of the final GSO.

In essence, an optimal transfer will in part maximize the increase of the vehicle's orbital energy at a given amount of Δv. The question is this: How is a kick-burn performed to maximize the increase? The change of the orbital energy due to a kick-burn $\Delta \boldsymbol{v}$ is given by

$$\Delta\varepsilon = \varepsilon_{kin}(\boldsymbol{v}+\Delta\boldsymbol{v}) - \varepsilon_{kin}(\boldsymbol{v}) = \frac{1}{2}(\boldsymbol{v}+\Delta\boldsymbol{v})^2 - \frac{1}{2}\boldsymbol{v}^2 = \boldsymbol{v}\cdot\Delta\boldsymbol{v} + \frac{1}{2}(\Delta v)^2 \qquad (8.3.1)$$

So, for a given amount of boost Δv the orbital energy is maximally increased if the boost $\Delta \boldsymbol{v}$ is parallel to the current velocity vector $\boldsymbol{v}$. In other words,

> A maximum increase in orbital energy is achieved if the transfer boost is in the direction of motion, i.e., tangentially to the initial trajectory.

Since this principle also applies to the second transition point to achieve the final orbit (another increase of a), we immediately obtain the rule for an energetically optimal transfer between to ellipses with only two boosts, the so-called Hohmann transfer:

> A **Hohmann transfer** orbit is an elliptic orbit, that for any transfer with two impulses between any two coplanar, co-revolving, non-crossing elliptic orbits is energetically minimal and therefore has the least propulsion demand in the two-body system. It tangentially touches these orbits at two points where the S/C transits with a kick-burn.

Note *That the Hohmann transfer is the energetically minimal two-impulse transfer holds only in the two-body problem, i.e., for a transfer just between two orbits about a common center mass. If the transfer is to a celestial body on the target orbit (i.e., an orbital rendezvous) or a particular point in the vicinity of a celestial body (e.g., libration point) then we have a three-body problem* (see Chap. 11) *and there are more favorable* (*even more favorable than a three-impulse transfer* (see Sects. 8.4.2 and 8.4.3)*) but more complex transfers* (see Sect. 11.5.2) *possible.*

Such a Hohmann transfer orbit is shown in Fig. 8.9. A Hohmann transfer of course also works the other way round, i.e., for a transfer from an outer to an inner elliptic orbit. So we have arrived at an answer to our optimization problem for a given starting point on the initial orbit. This leaves open the answer to the following question: At which point on the initial orbit should we perform the kick-burn to optimize the overall transfer between two elliptic orbits. One could presume that due to $\Delta\varepsilon \propto \boldsymbol{v} \cdot \Delta\boldsymbol{v}$ a first kick-burn at the highest orbital velocity, i.e., at the periapsis, would always be a good choice. But this neglects the second kick-burn at low speeds at the apoapsis of the transfer ellipse. A survey of the transfer at different orbital elements shows that the ratio of the eccentricities of the inner and the outer orbit is important. In general one can say that if the inner orbit has a higher eccentricity than the outer orbit, the optimal transfer varies, but

> If the inner orbit has a lower eccentricity than the outer orbit, the transfer to or from the apoapsis of the outer orbit requires the least delta-v budget.

The Earth–Mars transfer is a nice example for this rule, because here $e_{earth} = 0.0167$ and $e_{mars} = 0.0934$. Figure 8.10 shows the delta-v budget of the transfer as a function of the orbit angle of the transition point on the Mars orbit.

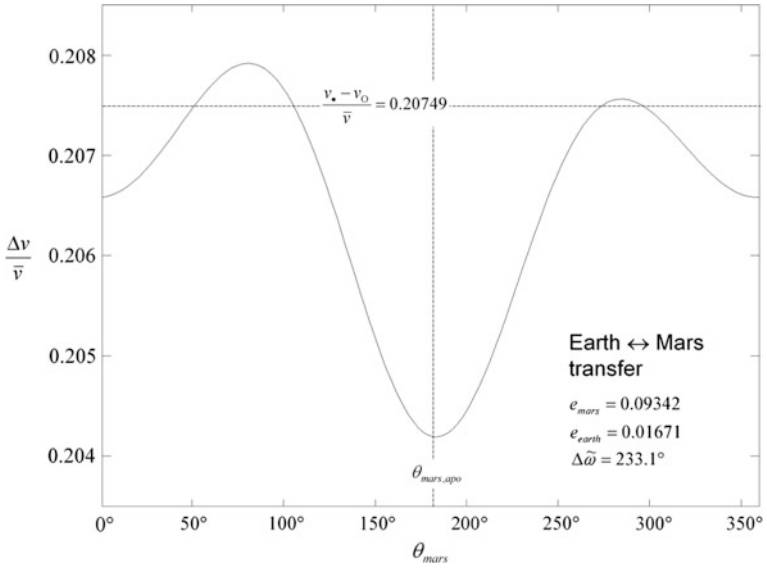


Fig. 8.10 Normalized delta-v budget for Earth ↔ Mars Hohmann transfers as a function of the orbit angle of the intersection with Mars orbit. The angle between their lines of apsis is 233.1°. The horizontal dashed line is the limiting value if both orbits were circular. Note the stretch of the *y*-scale

The transfer optimization problem gets even more intricate if arbitrary angles between the lines of apsides, or non-coplanar, or crossing orbits are considered. Because these cases become too complex to lay them out in a textbook, we skip them. In the following, we rather focus on the transfer between circular and near-circular orbits, which are of great practical value.

8.3.2 Transfer Between Circular Orbits

Hohmann transfers are specifically interesting between two circular orbits. Planetary orbits are mostly circular because at a given orbital energy they minimize atmospheric drag and provide steady orbit conditions, or because in the geostationary orbit the satellite has to rotate exactly in accord with the Earth. For circular orbits $r_\bullet = a_\bullet = const$ and $r_\mathrm{O} = a_\mathrm{O} = const$. It is easy to find out the metric orbital elements of a Hohmann transfer orbit between circular orbits. For the semi-major axis of this Hohmann transfer orbit, the following is obviously applicable:

$$a_H = \frac{1}{2}(a_\bullet + a_\mathrm{O}) \tag{8.3.2}$$

The transfer time is exactly half a period of the transfer ellipse, so according to Eq. (7.4.12)

$$t_H = \pi\sqrt{\frac{a_H^3}{\mu}} \tag{8.3.3}$$

The transfer ellipse with its two degrees of freedom a_H and e_H is completely and unambiguously determined by the boundary condition $r_{H,per} = a_H(1 - e_H)$ (see Eq. (7.4.6)) and by $r_{H,per} = a_\bullet$ and $r_{H,apo} = a_\mathrm{O}$. As for the semi-minor axis $b_H = a_H\sqrt{1 - e_H^2}$ applies, and the other metric orbital elements can be easily derived:

$$e_H = \frac{a_\mathrm{O} - a_\bullet}{a_\mathrm{O} + a_\bullet} \tag{8.3.4}$$

$$b_H = \sqrt{a_\bullet a_\mathrm{O}} \tag{8.3.5}$$

Let us determine the Δv demand for a Hohmann transfer between two circular orbits. According to Eq. (2.4.1), this is calculated for two Hohmann transfer kick-burns as the sum of the individual amounts (see Fig. 8.11):

$$\begin{aligned}\Delta v &= \left\{\begin{array}{ll} |\Delta v_{\bullet\to H}| + |\Delta v_{H\to\mathrm{O}}| & @\ \bullet\to\mathrm{O} \\ |\Delta v_{\mathrm{O}\to H}| + |\Delta v_{H\to\bullet}| & @\ \mathrm{O}\to\bullet \end{array}\right\} \\ &= (v_{H\bullet} - v_\bullet) + (v_\mathrm{O} - v_{H\mathrm{O}})\ @\ \bullet\leftrightarrow\mathrm{O}\end{aligned}$$

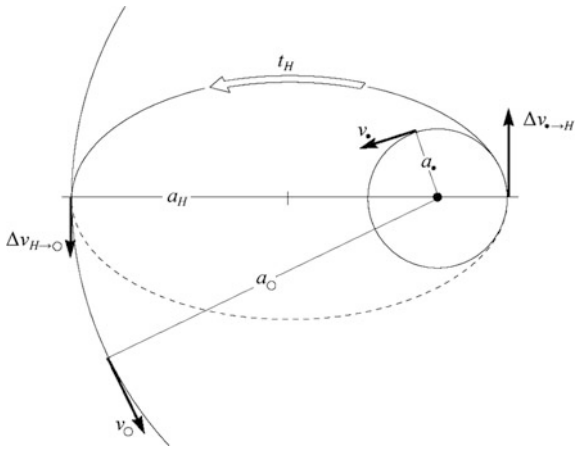


Fig. 8.11 Hohmann transfer between an inner and an outer circular orbit

The latter holds because $|\Delta v_{\bullet\to H}| = |\Delta v_{H\to\bullet}| = v_{H\bullet} - v_\bullet$ and $|\Delta v_{O\to H}| = |\Delta v_{H\to O}| = v_O - v_{HO}$. According to Eq. (8.3.4) $(1+e_H)/(1-e_H) = a_O/a_\bullet$ and therefore from Eq. (7.4.10) follows $v_{H\bullet} = v_{per} = v_\bullet\sqrt{a_O/a_H}$ and $v_{HO} = v_{apo} = v_O\sqrt{a_\bullet/a_H}$. Inserting this into the above equation yields the two contributions

$$\begin{aligned} |\Delta v_{\bullet\leftrightarrow H}| &= v_\bullet\left(\sqrt{\frac{a_O}{a_H}} - 1\right) \\ |\Delta v_{O\leftrightarrow H}| &= v_O\left(1 - \sqrt{\frac{a_\bullet}{a_H}}\right) \end{aligned} \tag{8.3.6}$$

which are shown in Fig. 8.12. So, in total we get

$$\Delta v = |\Delta v_{\bullet\leftrightarrow H}| + |\Delta v_{H\leftrightarrow O}| = v_\bullet\left(\sqrt{\frac{a_O}{a_H}} - 1\right) + v_O\left(1 - \sqrt{\frac{a_\bullet}{a_H}}\right) \tag{8.3.7}$$

If the terms in Eq. (8.3.7) are arranged skillfully and extended by $\sqrt{\mu} = v_\bullet\sqrt{a_\bullet} = v_O\sqrt{a_O}$, we get

$$\boxed{\Delta v = (v_\bullet - v_O)\left(\frac{\sqrt{a_\bullet} + \sqrt{a_O}}{\sqrt{a_H}} - 1\right) < v_\bullet - v_O} \quad @\ \bullet \leftrightarrow O \tag{8.3.8}$$

Equation (8.3.8) is valid for both transfer directions, i.e., $\bullet \to O$ and $O \to \bullet$.

Why is There a Bump?

The two contributions exhibit quite different shapes. Why? And why does $\Delta v_{O\leftrightarrow H}$ have a bump? The first (transfer injection) boost increases continuously with increasing distance between the orbits. This is quite easy to understand, as the semi-major axis a_H of the transfer orbit is determined by this distance, and the

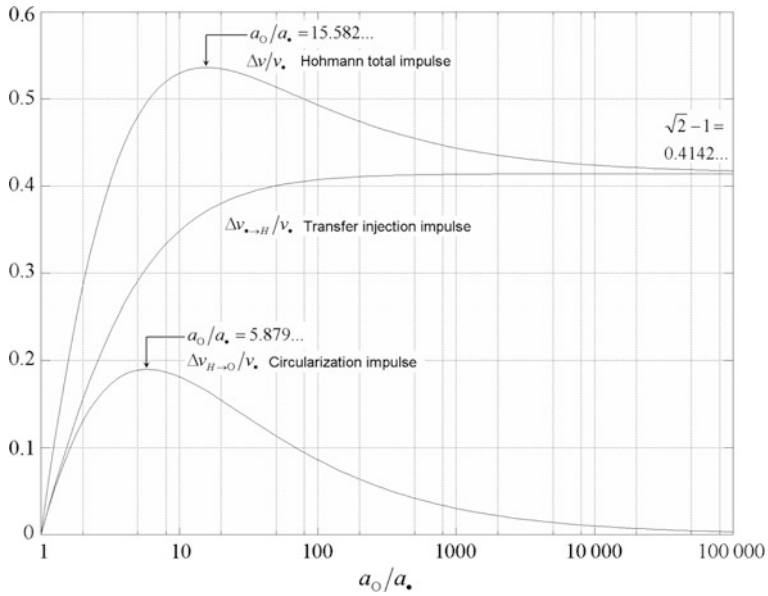


Fig. 8.12 Propulsion demand for Hohmann transfers from an inner to an outer orbit

transfer orbital energy also increases non-linearly with $\varepsilon = -\mu/2a$ according to Eq. (7.3.19). The second (circularization) boost, however, has a maximum at $a_O/a_\bullet = 5.879362...$ (exercise, Problem 8.4a). The reason is as follows. At the one extreme, if both orbits are in close vicinity to each other, i.e., $\Delta a = a_O - a_\bullet \to 0$, we havewith

$$a_H = \frac{1}{2}(a_\bullet + a_O) = a_\bullet\left(1 + \frac{1}{2}\frac{\Delta a}{a_\bullet}\right)$$

and according to Eq. (8.3.6)

$$|\Delta v_{H\leftrightarrow O}| \approx v_O\left(1 - 1\Big/\sqrt{1 + \frac{1}{2}\frac{\Delta a}{a_\bullet}}\right) \approx \frac{1}{4}\frac{\Delta a}{a_\bullet}v_0$$

So, for $\Delta a = 0$ the second boost of course is zero and increases roughly linearly with increasing distance Δa. At the other extreme, when $a_O \to \infty$, then $a_H, \Delta a \to \infty$ as well, and we have for the second boost at infinite distance

$$|\Delta v_{H\leftrightarrow O}| = v_O\left(1 - \sqrt{\frac{a_\bullet}{a_H}}\right) \approx v_0\left(1 - \sqrt{\frac{2a_\bullet}{\Delta a}}\right) \approx v_0 \approx 0$$

So, because at infinity the Hohmann and the target orbital velocity is zero the transfer boost between the two vanishes as well. Because the second boost is

positive between these limiting cases it must exhibit a maximum somewhere in between. Because of this maximum of the second boost, the total Hohmann transfer also has a maximum $\Delta v_{\max} \approx 0.536258 \cdot v_\bullet$ at $a_O/a_\bullet = 15.58172\ldots$ (exercise, Problem 8.4b).

Adjacent Circular Orbits

For a Hohmann transfer between adjacent circular orbits, $a_O \approx a_\bullet$, we can approximate (exercise, Problem 8.2)

$$\frac{\sqrt{a_\bullet}+\sqrt{a_O}}{\sqrt{a_H}} - 1 \approx 1 - \frac{1}{16}\left(\frac{a_O - a_\bullet}{a_\bullet}\right)^2 \to 1 \tag{8.3.9}$$

hence

$$\boxed{\Delta v \approx v_\bullet - v_O} \quad @\ a_O \approx a_\bullet \tag{8.3.10}$$

Example

For $a_O/a_\bullet \leq 2.5$, i.e., for orbits with altitudes up to 10,000 km above Earth's surface, the error due to Eq. (8.3.10) is smaller than 5.1%.

Distant Circular Orbits

For circular orbits that are distant from each other, we rewrite Eq. (8.3.8)

$$\frac{\Delta v}{v_\bullet} = \left(1 - \sqrt{\frac{a_\bullet}{a_O}}\right)\left(\sqrt{2}\frac{1+\sqrt{a_\bullet/a_O}}{\sqrt{1+a_\bullet/a_O}} - 1\right) = \sqrt{2}\frac{1-a_\bullet/a_O}{\sqrt{1+a_\bullet/a_O}} - 1 + \sqrt{\frac{a_\bullet}{a_O}}$$

For these distant orbits $a_\bullet/a_O \ll 1$ and we can approximate

$$\frac{\Delta v}{v_\bullet} = \sqrt{2} - 1 + \sqrt{\frac{a_\bullet}{a_O}} - \frac{3}{\sqrt{2}}\frac{a_\bullet}{a_O} + \cdots \tag{8.3.11}$$

Using $v = \sqrt{\mu/a}$ we therefore obtain in the limit of infinite distances

$$\Delta v = \left(\sqrt{2} - 1\right)v_\bullet + v_O \quad @\ a_O/a_\bullet \to \infty \tag{8.3.12}$$

Between these two limiting cases the expression $\left(\sqrt{a_\bullet} + \sqrt{a_O}\right)/\sqrt{a_H} - 1$ is strictly monotonously decreasing (see Fig. 8.13), and that is why in Eq. (8.3.8) the inequality strictly holds.

Hohmann Transfer as a Special Case of the Minimum Energy Lambert Transfer

Let us consider the case of a Lambert transfer orbit (see Sect. 8.2) between two circular orbits without specified transition points. We examine the minimum energy transfer orbit of Sect. 8.2.2, pick any P_1 on the initial circle, and, as shown in Fig. 8.14, move P_2, because it is arbitrary, until $\Delta\theta = \pi$. In this special case $c = r_1 + r_2$ and both the occupied and empty focal points lie on the chord. This is

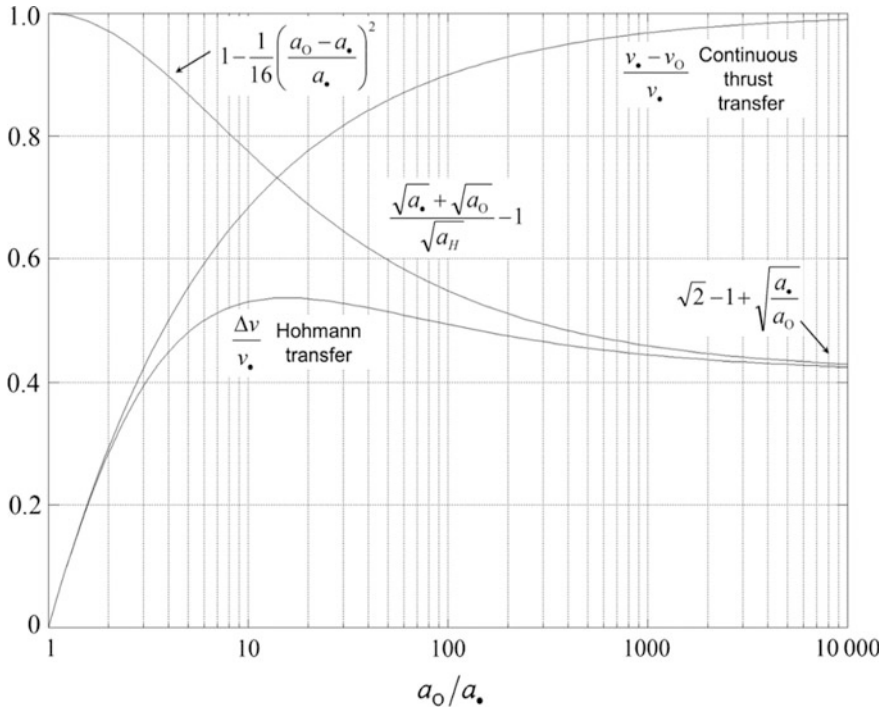


Fig. 8.13 The terms of the Eq. (8.3.8) and their limiting values. The factor $(v_\bullet - v_O)/v_\bullet$ is the propulsion demand of a continuous thrust transfer (see Sect. 8.4.5)

the condition of a minimum energy transfer orbit and it obviously is a Hohmann transfer orbit. By this procedure, we in a different way have verified that the Hohmann transfer is a minimum energy transfer with maximum transfer time.

Note that for this special Lambert transfer the eccentricity is no longer defined. In fact, whatever e might be, the transfer time is independent of e! For the Hohmann transfer, this additional degree of freedom is taken to adjust the tangents of the transfer ellipse to those of the circles at the touching points so that delta-v becomes minimal.

8.3.3 Transfer Between Near-Circular Orbits

We now consider Hohmann transfers between two coplanar, co-revolving, non-crossing ellipses, with small eccentricities $e_\bullet, e_O \ll 1$. The detailed analytical approximations are quite convoluted due to $e_\bullet e_O$ cross-terms, but the general upshot is that the dependency of delta-v as a function of $e_\bullet, e_O$ is very weak. This can be observed, for instance, in Fig. 8.15 where for Hohmann transfers between two apsides points on two orbits with $a_O/a_\bullet = 1.52365$ (Earth–Mars) the normalized delta-v is depicted as a function of $e_\bullet = e_O$ numerically.

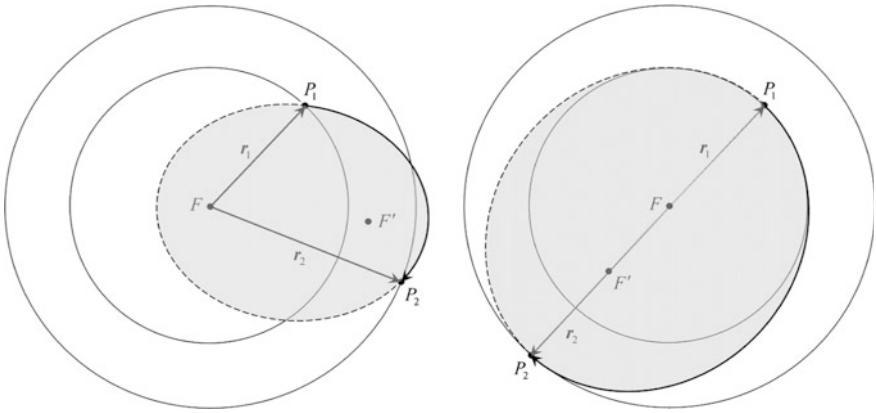


Fig. 8.14 A Lambert transfer between arbitrary points on two circular orbits (left) and the crossover to the Hohmann transfer for $\Delta\theta = \pi$ (right)

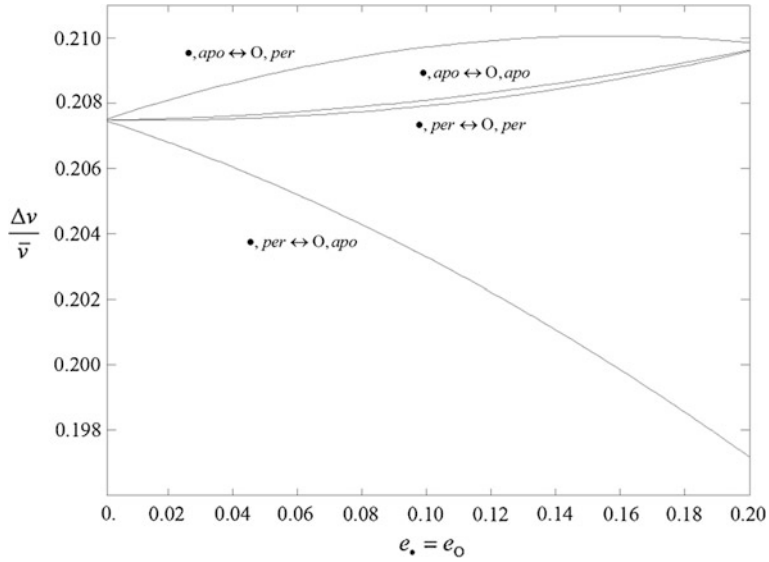


Fig. 8.15 The delta-v budget as a function of the common eccentricities. The dependencies are given for orbits with $a_O/a_\bullet = 1.52365$ (Earth–Mars). Note the stretched Δv-scale

For $e \leq 0.1$ the variations are smaller than 2% of their corresponding circular orbits value $(v_\bullet - v_O)/\bar{v} = 0.20749$. As we can see from Fig. 8.10 this is in accordance with the Earth ↔ Mars transfer, $e_{Mars} = 0.09342$ and $e_{earth} = 0.01671$, even if the transfers would be performed from any position of the orbit. So for all practical purposes, we get the same result as for a transition between two circular orbits.

$$\Delta v \approx (v_\bullet - v_O)\left(\frac{\sqrt{a_O}+\sqrt{a_\bullet}}{\sqrt{(a_O+a_\bullet)/2}} - 1\right) \qquad @\ e_\bullet, e_O \le 0.1 \tag{8.3.13}$$

Nevertheless, owing to the Oberth effect as discussed in Sect. 8.1.3, the general transfer rule is

> The most delta-v efficient Hohmann transfer between a near-circular or even elliptic inner orbit and any outer orbit is always from or to the periapsis of the inner orbit.

8.3.4 Sensitivity Analysis

From an energy point of view Hohmann transfers are the most favorable two-impulse transfer orbits. But they also have some disadvantages. They can be very sensitive to small inaccuracies of the transfer injection impulse. Let us have a closer look at this dependence for transfers between circular orbits.

Let r_{per} and r_{apo} be the periapsis and the apoapsis radius of the transfer orbit respectively (for convenience, we drop the index H to indicate the Hohmann transfer orbit). The periapsis is determined by the initial orbit radius. From the vis-viva Eq. (7.2.15) we get for the velocity on the Hohmann transfer orbit at the periapsis

$$v_{per}^2 = \frac{2\mu}{r_{per}} - \frac{\mu}{a} = \frac{\mu_\oplus}{a}\frac{1+e}{1-e}$$

If the transfer injection burn has a small error dv_{per} at the fixed periapsis $r_{per} = const$ we get

$$2v_{per}dv_{per} = \frac{\mu}{a^2}da$$

from which follows

$$\frac{da}{a} = 2\frac{av^2}{\mu}\frac{dv_{per}}{v_{per}} = 2\frac{1+e}{1-e}\frac{dv_{per}}{v_{per}} \tag{8.3.14}$$

So, a thrust error generates a certain variation of the semi-major axis. Now we want to know how this affects the position of the apoapsis. We start with Eq. (7.4.7)

$$r_{apo} = a(1+e)$$

Its change in position is determined by differentiation

$$\frac{dr_{apo}}{r_{apo}} = \frac{da}{a} + \frac{de}{1+e} \tag{8.3.15}$$

Both parameters a and e are not independent of each other, but linked by the constancy of the periapsis of the initial orbit:

$$const = r_{per} = a(1-e)$$

After differentiating this equation, we see how their changes depend on each other

$$de = (1-e)\frac{da}{a} \tag{8.3.16}$$

So together with Eq. (8.3.15), we get

$$\frac{dr_{apo}}{r_{apo}} = \frac{2}{1+e}\frac{da}{a}$$

and finally with Eq. (8.3.14), we get

$$\frac{dr_{apo}}{r_{apo}} = \frac{4}{1-e}\frac{dv}{v} = \frac{4a}{r_{per}}\frac{dv_{per}}{v_{per}} = 2\left(1+\frac{r_{apo}}{r_{per}}\right)\frac{dv_{per}}{v_{per}} \tag{8.3.17}$$

That is, for a given injection burn error dv_{per}, the relative target point accuracy decreases with increasing transfer distances.

Example 1

Let us examine the Hohmann transfer from an initial LEO orbit (h = 400 km) to GEO. Because of $r_{GEO}/r_{LEO} = 6.232$, we get

$$\frac{dr_{apo}}{r_{apo}} = 14.46\frac{dv_{per}}{v_{per}}$$

A relatively small burn error of just 0.5% would lead to an inaccuracy in the target distance of 7.2%. That is, 3000 km deviation from the GEO orbit!

Example 2

Let us have a look at the Hohmann transfer from an initial LEO orbit (h = 400 km) to the Moon. Because of $r_{moon}/r_{earth} = 56.654$, we get

$$\frac{dr_{apo}}{r_{apo}} = 115.3\frac{dv_{per}}{v_{per}}$$

This means that the same small burn error of just 0.5% would lead to an inaccuracy in the target distance of 58%. That would just bring us to nirvana!

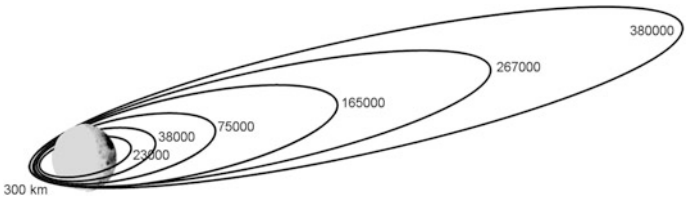


Fig. 8.16 The transfer trajectories of Chandrayaan-1 from Earth to Moon. The apogee (given in km) was increased step by step by successive kick-burns at 300 km perigee

Summary
Plain Hohmann transfers may be the most favorable transfer orbits from an energy point of view, but they implicate long transfer times and the large possible target errors.

However, already small deviations from an ideal Hohmann transfer and in-flight corrections strongly reduce these drawbacks as discussed in Sect. 9.3.2 for near-Hohmann transfers to Mars. In addition, if transfer time is not an issue, one can get a grip on the high sensitivity of highly elliptic transfer orbits by gradually raising the apogee of the transfer orbit to the final Hohmann transfer orbit. Thereby after each raising maneuver the next maneuver can be adjusted to the actual apogee. Figure 8.16 shows the transfer trajectory of the Indian Chandrayaan-1 Moon mission, which was launched on October 22, 2008 and arrived at the Moon 13 days, later where it was captured by Moon's gravitation into a highly elliptic Moon orbit.

8.4 Other Transfers

In this section we study some orbit transfers which are non-Hohmann, but nevertheless of practical value.

8.4.1 Parabolic Escape Transfer

In Sect. 7.4.4. we have studied parabolic orbits. They have the special property that the body reaches infinity at zero velocity. Therefore the simplest escape orbit is a parabolic orbit. We also saw that at a given orbital radius the escape velocity v_{esc} enables a spacecraft to go to infinity independent of the direction in which the final velocity vector points. Yet, the propulsion effect to achieve the escape velocity strongly depends on this direction (see Sect. 8.1.3). In Sect. 8.3.1 we saw that at a given orbital position with $\boldsymbol{r}_0, \boldsymbol{v}_0$ the least effort to achieve a predetermined $\boldsymbol{v}_{esc}$ is by the impulse maneuver $\Delta\boldsymbol{v}||\boldsymbol{v}_0$, i.e. a boost into the instantaneous flight direction in the initial orbit. From Eq. (7.3.16) this implies that the flight angle λ of an initial elliptic orbit with a, e and final parabolic orbit with p_p then is identical

$$\cos\gamma = \sqrt{\frac{a(1-e^2)}{r_0(2-r_0/a)}} = \sqrt{\frac{p_p}{2r_0}}$$

From this it follows that the semi-latus rectum of the parabolic escape orbit is

$$p_p = 2\frac{a(1-e^2)}{2-r_0/a} \quad @\ \Delta\boldsymbol{v}||\boldsymbol{v}_0 \tag{8.4.1}$$

The orientation of the escape orbit, i.e. its line of apsides is given from the parabolic orbit equation as

$$\angle(\boldsymbol{e}_p, \boldsymbol{r}_0) = \cos\,\theta_{p,0} = \frac{p_p}{r_0} - 1 \tag{8.4.2}$$

The initial elliptic orbit and the final parabolic orbit are coplanar.

Finally it would be interesting to know at which point on the ellipse an escape would be most efficient. We calculate Δv from the vis-viva equation to be

$$\Delta v = \sqrt{\mu}\left(\sqrt{\frac{2}{r_0}} - \sqrt{\frac{2}{r_0} - \frac{1}{a}}\right) \tag{8.4.3}$$

From this equation it is obvious that the required Δv lessens with smaller orbit radius. Therefore it is minimal at its periapsis and maximal at its apoapsis, although the differences are only slight for $e<0.1$. Nevertheless, this result is a good example of the Oberth effect decribed in Sect. 8.1.3.

But, most importantly, the boost position solely determines the asymptotic direction of motion. So, the escape procedure is to first figure out the direction of asymptotic motion and measure it as an orbit angle relative to the line of apsides of the initial elliptic orbit. Call this angle θ_∞. Then $\theta_0 = \theta_\infty + \theta_{p,0}$ or

$$\cos(\theta_0 - \theta_\infty) = \cos\,\theta_{p,0} = \frac{p_p}{r_0} - 1 = \frac{2(1+e\,\cos\,\theta_0)^2}{1+2e\,\cos\,\theta_0+e^2} - 1 \tag{8.4.4}$$

So, for a given θ_∞ the orbit angle θ_0 of the boost position is the root of this equation.

Example

We assume a circular orbit with $r = r_0 = a,\ e = 0$. At any point in orbit we want to transfer into an escape parabola with the least Δv effort. What then is the escape parabola?

From Eq. (8.4.1) follows $p = 2r_0$ and from Eq. (7.3.16) $\sin\gamma = \sin 0^\circ = 0 = \sin\theta \rightarrow \theta = 0^\circ$. This means that the boost location constitutes the periapsis of the escape parabola.

8.4.2 Bi-elliptic Transfer

The maximum of the total delta-v for Hohmann transfers (see Fig. 8.12) at $a_O/a_\bullet = 15.582$ occurs because $\Delta v_{H\to O}$ achieves its maximum at $a_O/a_\bullet \approx 5.88$ for turning into the target orbit. This second delta-v contribution however vanishes for $r \to \infty$. This gives rise to the assumption that it might be possible to save propulsion demand with a total of three impulses by first escaping from the initial orbit to far out or even to infinity, and then turning back again to the coplanar target orbit (see Fig. 8.17).

We want to determine the propulsion demand for such a three-impulse transfer (a.k.a. *bi-elliptic transfer*). According to Eq. (8.3.12) the delta-v budget to a remote circular orbit with orbit radius $a_\times \gg a_O > a_\bullet$ and orbital velocity $v_\times = \sqrt{\mu/a_\times}$ is

$$|\Delta v_{\bullet\to\times}| \approx \left(\sqrt{2}-1\right)v_\bullet + v_\times$$

The delta-v budget for a kick-burn that brings us back into the target orbit in turn amounts to

$$|\Delta v_{\times\to O}| \approx \left(\sqrt{2}-1\right)v_O + v_\times$$

So, in total we have

$$\Delta v = |\Delta v_{\bullet\to\times}| + |\Delta v_{\times\to O}| \approx \left(\sqrt{2}-1\right)(v_\bullet + v_O) + 2v_\times \quad @\, a_\times \gg a_O \qquad (8.4.5)$$

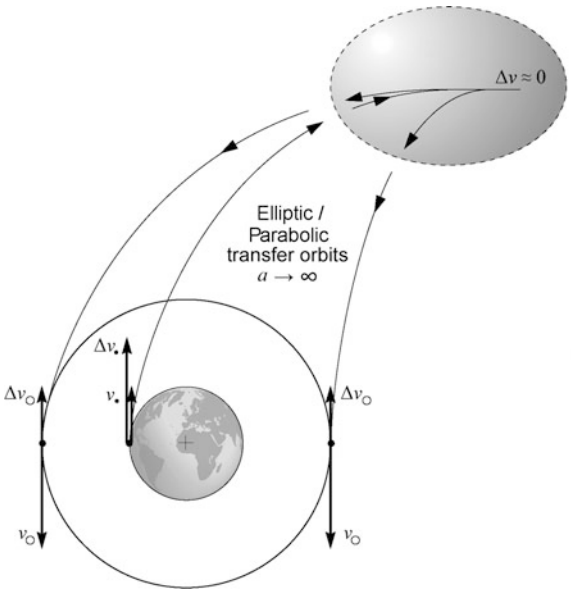


Fig. 8.17 Schematic of a bi-elliptic transfer with an infinite intermediate orbit

For a bi-parabolic transfer, when the intermediate orbit is infinitely far away, then $v_\times = 0$ and the total delta-v budget becomes minimal

$$\Delta v = \left(\sqrt{2} - 1\right)(v_\bullet + v_O) \quad @\, a_\times / a_O \to \infty \tag{8.4.6}$$

Equation (8.4.6) is illustrated in Fig. 8.18. For $a_O/a_\bullet > 11.938765\ldots$ the propulsion demand is indeed more favorable with bi-parabolic transfers than with Hohmann transfers (exercise, Problem 8.5). A practical and hence important benefit of a bi-parabolic transfer is that at the outermost position, where $v \approx 0$, a change of the orbital plane, flight direction, or even the direction of rotation of the orbital curve may be done virtually without any propulsion demand. The serious drawback is that it takes increasingly longer to get farther out.

Break-Even Point with Hohmann Transfer

To ease this problem, it would be interesting to know at which intermediate orbit distance the bi-elliptic transfer starts to be advantageous to the Hohmann transfer. From Eqs. (8.4.5) and (8.3.11) we get the conditional equation $\left(\text{note that } O(v_\times) = O(v_O^2/v_\bullet)\right)$

$$\Delta v_{bi} \approx \left(\sqrt{2} - 1\right)(v_\bullet + v_O) + 2v_\times < \left(\sqrt{2} - 1\right)v_\bullet + v_O - \frac{3}{\sqrt{2}}\frac{v_O^2}{v_\bullet} \approx \Delta v_H$$

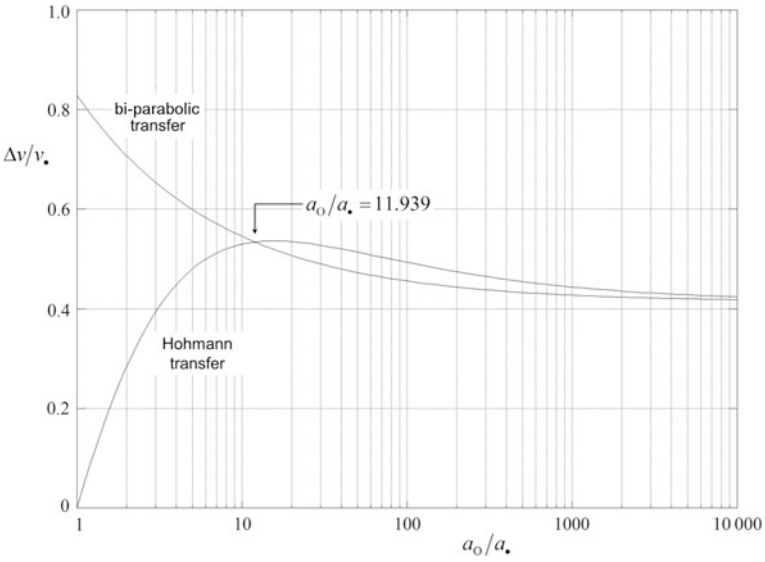


Fig. 8.18 Comparison between an infinite bi-parabolic transfer and a Hohmann transfer

This implies

$$\frac{v_\times}{v_O} < 1 - \frac{1}{\sqrt{2}} - \frac{3}{2\sqrt{2}}\frac{v_O}{v_\bullet}$$

and therefore

$$\frac{a_\times}{a_O} > \frac{8}{\left[2\left(\sqrt{2}-1\right) - 3\sqrt{a_\bullet/a_O}\right]^2} \tag{8.4.7}$$

For example let us assume three ratios $a_O/a_\bullet = 20,\ 100,\ 500$. For this we find $a_\times > (322,\ 28.6,\ 16.60) \cdot a_O = (6441,\ 2860,\ 8299) \cdot a_\bullet$.

Finally, one might question at which $a_O/a_\bullet$ the bi-elliptic transfer is most advantageous to a Hohmann transfer, i.e.: When does $v_\times/v_O$ become maximal, or when does the relation $a_\times/a_\bullet$ become minimal? The answer is easily calculated from Eq. (8.4.7) to be $a_O/a_\bullet|_{opt} = \left[3/\left(\sqrt{2}-1\right)\right]^2 = 52.456$ and $a_\times/a_\bullet|_{\min} = 2445.8$. This optimum case corresponds roughly to a transition from an Earth parking orbit at altitude $h = 250\ km$ to the Moon, $a_{Moon}/a_{park} = 57.3$. Even at this near optimal situation one would need to travel to $a_\times = 2455 \cdot a_{park} = 42.8 \cdot a_{moon}$ to break even with a Hohmann transfer, not to talk about the travel time to get there and back to the Moon. This impractical situation, providing only humble delta-v advantage, is why bi-elliptic transfers are academic cases that are rarely employed.

8.4.3 Super-Synchronous Transfer Orbits

This section is about some special types of transfer orbits to GEO. A classical transfer orbit is the GEO transfer orbit (GTO), which is Hohmann transfer from a low Earth orbit to GEO. In practice, though, there is no LEO. The launch vehicle injects directly from its launch ascent trajectory into an elliptic orbit with apogee at GEO.

There are cases where other transfer orbits, so-called *super-synchronous (GEO) transfer orbit* (SSTO), a.k.a. Super-GTO, or GTO+, are superior to a regular GTO. A SSTO generally is an orbit that rather than a GEO synchronous orbit (GSO), which is any elliptic orbit (incl. GEO) with period of a sideral day (see Sect. 13.2), T_{GSO}, features $T_{SSTO} > T_{GSO}$. According to Eq. (7.4.12) this implies $a_{SSTO} > a_{GSO}$. There exist the following two types of SSTOs.

Bi-Elliptic Transfer

The delta-v benefit of a bi-elliptic transfer increases with the inclination to be changed at the apoapsis. This benefit is decisive for launches from launch sites at high latitude β, which equals the launch orbit inclination i (see Sect. 8.6.1), to GEO. For such a bi-elliptic transfer the upper stage of the rocket injects the satellite into an SSTO at the descending node of the launch trajectory, i.e. when crossing the equator. This injection burn includes the so-called minimum residual shutdown strategy (MRS) where any excess propellant is expended to reduce the inclination

as far as possible, usually by 2–3°. The apogee of the SSTO usually is chosen to be twice to three times the GEO radius (see Fig. 8.19). At SSTO apogee an apogee boost places the spacecraft into a so-called interim transfer orbit (ITO) to GEO and at the same time zeroing the residual inclination. When arriving at GEO, a final burn brings the spacecraft into a so-called circular drift orbit, which is just little smaller than GEO to let the S/C drift from the burn position to its final longitude position. Having arrived there, tiny adjustment burns bring the S/C into its operational orbit state.

The key point with such SSTOs is that, even though the total Δv of all three burns may be larger, the satellite's propellant spending for the SSTO apogee boost and the final burns are lower than for a standard GTO. So, as long as the additional Δv of the rocket upper stage for SSTO is within its propellant budget, any reduction of the satellite's Δv is welcome to invest the saved propellant into later station-keeping and thus extend the satellite's orbital life time. This is why bi-elliptic SSTOs for transfering satellites into GEO are quite frequent for high latitude launch sites such as Cape Canaveral (Atlas) and Baikonur (Proton).

Common to all such SSTOs is the fact that the SSTO injection burn is performed at the descending node of the launch trajectory, which implies $\omega = 180°$ for the SSTO. Hence the tangent plane maneuver (see Sect. 8.1.4) at SSTO apogee into a ITO is a nodal transfer. The RAAN of the ITO is irrelevant because the inclination is usually nullified at SSTO apogee.

Hence both Δv_1 and Δv_2 are nodal transfers as determined by Eq. (8.1.20). The delta-v for the three burns are therefore as follows

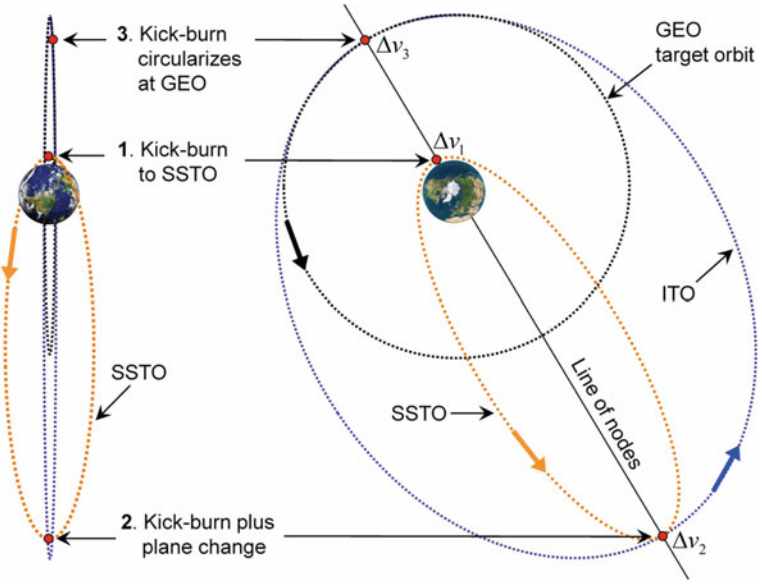


Fig. 8.19 A bi-elliptical super-synchronous transfer with its three kick-burns. Views are in Earth's equatorial plane (left) and normal to it (right)

$$\begin{aligned}\Delta v_1 &= \sqrt{v_{LEO-}^2 + v_{LEO+}^2 - 2v_{LEO-}v_{LEO+}\cos\Delta i_1}\\ \Delta v_2 &= \sqrt{v_{ITO-}^2 + v_{ITO+}^2 - 2v_{ITO-}v_{ITO+}\cos\Delta i_2}\\ \Delta v_3 &= v_{GEO-} - v_{GEO+}\end{aligned} \tag{8.4.8}$$

where the subscripts − and + indicate the values before and after the burn, respectively.

Case Study: The Super-Synchronous Transfer Orbit Mission of *Orion 1*

The Telstar 11 telecommunications satellite, operated by Loral Skynet, provided video for broadcast and business television, high-speed internet access and multimedia services, broadband and data networking to Europe, the United States to the Rocky Mountains and parts of Canada and Mexico.

It was launched as Orion 1 (a.k.a. Orion F1) on November 29, 1994, on an Atlas IIA rocket from Cape Canaveral (latitude 28.5°). The standard Atlas IIA GTO foresees a nodal transfer at the descending node, i.e. a GTO injection burn at 185 km altitude including an inclination change of about 2.8° (depending on the wind conditions during ascent and hence on the decisive propellant excess just before main engines cut off, MECO). A transfer burn at apogee, which is GTOs ascending node and a circularization burn including an inclination change of $\Delta i = 25°$ at the point where GTO touches the GEO, delivers the satellite into a so-called drift orbit. There the satellite drifts with a rate of 0.3°W per day into its final station-keeping box at 37.5°W. This earlier and standard Atlas IIA LEO→GTO→GEO sequence requires a GTO-injection boost of $\Delta v_{rocket} = 2.497\,\mathrm{km/s}$ for the rocket's last stage and a $\Delta v_{sat} = 1.764\,\mathrm{km/s}$ for the spacecraft. The latter would imply a lifetime in GEO to be less than 7 years, much less than required.

It was therefore decided that at the descending node the launcher should inject the spacecraft into a SSTO, in order to significantly reduce the later spacecraft's $\Delta v_{sat} = \Delta v_2 + \Delta v_3$ at the expense of a relatively small Δv_1 increase of the launcher. The SSTO chosen had the properties

$$\begin{aligned} r_{apo} &= 129{,}885\,\mathrm{km}\\ r_{per} &= 6563.1\,\mathrm{km}\\ i &= 25.7°\\ \Omega &= 173.6°\\ \omega &= 179.98° \end{aligned}$$

To bring the spacecraft from SSTO apogee into GEO, one apogee burn would change the inclination $\Delta i = 25^\circ$ and bring the S/C into the interim transfer orbit (ITO) to GEO, while a final burn at ITO perigee would circularize it into GEO. For safety reasons each burn was split into two, which in view of delta-v, however, are equal to the said two burns. According to Eq. (8.4.8) the delta-v for the three burns are as follows

$$\begin{aligned} v_{LEO-} &= 7.793\,\text{km/s}, \\ v_{LEO+} &= 10.753\,\text{km/s} \\ \Delta v_1 &= \sqrt{v_{LEO-}^2 + v_{LEO+}^2 - 2v_{LEO-}v_{LEO+}\cos(28.5^\circ - 25.7^\circ)} \\ &= 2.993\ \text{km/s} \end{aligned}$$

$$v_{ITO-} = 0.5434\ \text{km/s}, \quad v_{ITO+} = 1.227\ \text{km/s}$$

$$\begin{aligned} \Delta v_2 &= \sqrt{v_{ITO-}^2 + v_{ITO+}^2 - 2v_{ITO-}v_{ITO+}\cos(25.7^\circ - 0.6^\circ)} \\ &= 0.770\ \text{km/s} \end{aligned}$$

and

$$\begin{aligned} v_{GEO-} &= 3.778\ \text{km/s}, \quad v_{GEO+} = 3.075\ \text{km/s} \\ \Delta v_3 &= v_{GEO-} - v_{GEO+} = 0.703\ \text{km/s} \end{aligned}$$

So, the Atlas rocket had to provide a $\Delta v_1 = 2.993$ km/s and hence an additional

$$\Delta v_{rocket+} = 2.993 - 2.497\ \text{km/s} = 0.496\ \text{km/s}$$

which was within the fuel budget. On the other hand, the total delta-v for the Orion satellite was $\Delta v_1 + \Delta v_2 = 0.770 + 0.703 = 1.473$ km/s thus providing a benefit of

$$\Delta v_{sat+} = 1.473\,\text{km/s} - 1.764\,\text{km/s} = -291\ \text{m/s}$$

This delta-v benefit extended Orion's lifetime for an additional 5 years, sufficient to meet the mission requirements.

Because any reduction of the satellite's total delta-v, $\Delta v_{sat} = \Delta v_2 + \Delta v_3$, is essential for the satellite's lifetime, Fig. 8.20 plots $\Delta v_2 + \Delta v_3$ for $\Delta i_1 = 0$ and different $\Delta i \equiv \Delta i_2$ as a function of SSTO apogee distance.

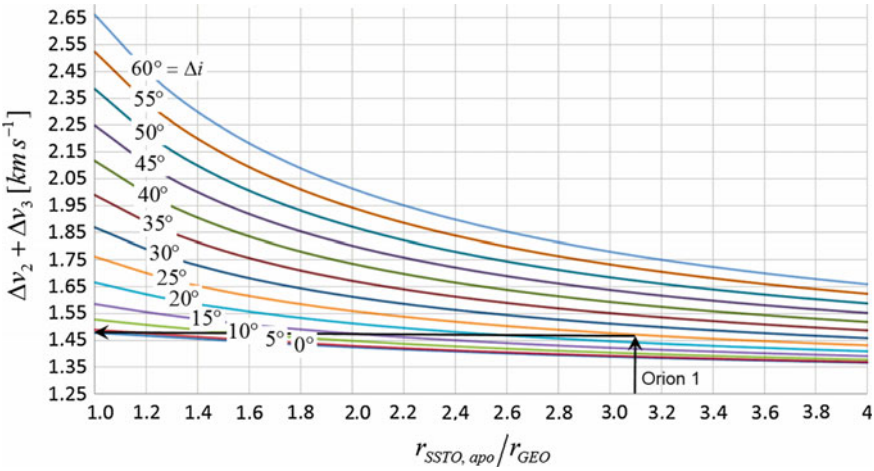


Fig. 8.20 The total delta-v of the satellite for a LEO → SSTO → ITO → GEO transfer as a function of relative SSTO apogee distance and for different inclination changes at the second kick-burn (SSTO apogee) and vanishing inclination change at the first and third kick-burn

Continuous Thrust Transfer

With today's trend to all-electric propulsion platforms (cf. Sect. 5.2.2, Subsection *Comparsion with other Thrusters*) there is the need for an efficient electric propulsion transfer from a SSTO, as provided by the launch vehicle, to GEO.

Early transfers were based on GTOs using a thrust arc strategy around apogee, similar to the chemical apogee-thrusts. Such a transfer requires about 150 days. In 1995, A. Spitzer proposed a thrust arc strategy based on SSTO (so-called Spitzer's scheme) with 16% less delta-v and a significantly reduced transfer time of about 110 days. Immediately thereafter it was found by C.R. Koppel that a continuous thrust strategy (CTS, see Sect. 8.4.5), i.e. continuously thrusting over all orbits, was even more practical. Although CTS requires a 17% larger delta-v than *Spitzer's scheme*, it reduces the transfer time to about 90 days, with an equivalent reduction of orbits passing the radiation of the Van Allen belts. This is why CTS for all-electric propulsion S/C today is the means of choice.

For a CTS the launch vehicle injects the S/C into a SSTO with typically 200 km × 60,000 km altitude and with a launch-specific non-zero RAAN and inclination. The electric thrust vector then is inertially oriented (star mode) with some out-of-plane component, such that in course of the transfer the in-plane component gradually removes the eccentricity (see Eq. (8.1.1b)) and the out-of plane component decreases the inclination to zero (see Eq. (8.1.1c), RAAN hence becomes irrelevant). The orbit thus gradually crosses over to the target GEO (see Fig. 8.21).

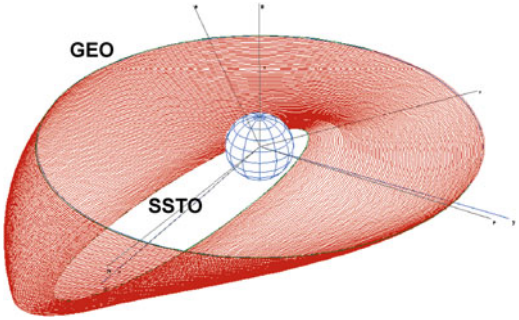


Fig. 8.21 Continuous thrust transfer from a SSTO (185 km × 6000 km, $i = 28°$) to GEO (*Credit* C.R Koppel, 1999)

8.4.4 n-Impulse Transfers

Very generally it can be shown that for a transition between *any* two elliptic orbits the total delta-v budget for a three-impulse transfer (not necessarily bi-elliptic) might be smaller, but does not have to be, than that with the two-impulse Hohmann transfer. The above bi-elliptic transfer for $a_O/a_\bullet > 11.94$ is an example for this. In addition, it can also be shown that the total delta-v budget minimized by Hohmann or a three-impulse transfer cannot be further minimized by maneuvers with more than three impulses. So the Hohmann transfer or the minimum three-impulse transfer represents the absolute minimum for the propulsion demand in a two-body problem. But note that things are different in the three-body problem (see Note in Sect. 8.3.1).

8.4.5 Continuous Thrust Transfer

If we have electrical propulsion engines such as ion thrusters, thrust is low and continuous, so impulse transfers are impossible. An ion engine would rather have to fire permanently in the direction of motion (recall: $\Delta\varepsilon \propto \boldsymbol{v} \cdot \Delta\boldsymbol{v}$) to optimally but slowly spiral the satellite into higher and higher orbits. We now want to calculate the transfer orbit and the delta-v for a continuous tangential thrust maneuver for an initially circular orbit. How does an infinitesimal small tangential thrust $dv := \delta v_{||}$ alter a circular orbit? The answer is given by Eq. (8.1.14):

$$\frac{dr}{r} = 2\sqrt{\frac{r}{\mu}}\, dv$$

To find the total delta-v, we have to integrate the differentially small velocity increase between an initial $r_\bullet$ and final orbit radius r_O

$$\Delta v = \int dv = \frac{\sqrt{\mu}}{2} \int_{r_\bullet}^{r_O} \frac{dr}{r^{3/2}} = -\sqrt{\frac{\mu}{r}}\Big|_{r_\bullet}^{r_O} = v_\bullet - v_O \tag{8.4.9}$$

Of course it does not make a difference for the propulsion demand Δv whether we spiral up from $r_\bullet$ to r_O or down from r_O to $r_\bullet$. If we compare this result with Eq. (8.3.8), we see from Fig. 8.13 that the Hohmann transfer is always more favorable than a continuous thrust transfer. But with ion propulsions and their very tiny thrusts there is no alternative to that.

To calculate the transfer time between an initial circular orbit with orbit radius $r_\bullet$ and the instantaneous circular orbit with radius r, we need the explicit trajectory $r = r(t)$. As the circular condition $v = \sqrt{\mu/r}$ is valid for each point of the orbital curve, it is sufficient to find $v = v(t)$. We find it with the help of the thrust equation $F_* = m \cdot \dot{v} = \dot{m} v_*$. To be able to apply it, we have to consider the mass reduction $\dot{m}$ due to the propellant consumption. We assume that the vehicle with the mass m is accelerated by a constant thrust F with constant mass flow rate $\dot{m}_p = -\dot{m} = const$, so: $m = m_0 + \dot{m}t$. With this relation, we integrate the corresponding equation $dv = \dot{m} v_* / m \cdot dt$

$$\int_{v_\bullet}^{v} dv' = \int_0^t \frac{\dot{m} v_*}{m}\, dt'$$

to get

$$v - v_\bullet = v_* \int_0^t \frac{dt'}{m_0/\dot{m} + t'} = v_* \ln\left(\frac{m_0/\dot{m} + t}{m_0/\dot{m}}\right) = v_* \ln\left(1 - \frac{\dot{m}_p}{m_0} t\right) \qquad (8.4.10)$$

Because of $v = \sqrt{\mu/r}$, this results in the following spiral trajectory (see Fig. 8.22)

$$r(t) = \mu \left[v_\bullet + v_* \ln\left(1 - \frac{\dot{m}_p}{m_0} t\right)\right]^{-2} \quad @\ \dot{m}_p = const \qquad (8.4.11)$$

By solving for t we get the following from Eq. (8.4.10) for the transfer time

$$t_{CT} = \frac{m_0}{\dot{m}_p}\left[1 - \exp\left(\frac{v - v_\bullet}{v_*}\right)\right] = \frac{m_0 v_*}{F_*}\left\{1 - \exp\left[\frac{\sqrt{\mu}}{v_*}\left(\frac{1}{\sqrt{r}} - \frac{1}{\sqrt{r_\bullet}}\right)\right]\right\} \qquad (8.4.12)$$

Example

A satellite is released from the Space Shuttle payload bay at an altitude of 300 km, and it is supposed to spiral with an ion thruster ($F_*/m_0 = 100\,\mu g = 10^{-3} m\ s^{-2}$ and $v_* = 10\,000\ m\,s^{-1}$) to GEO. Because $v_\bullet = 7.72\,km\ s^{-1}$ and $v_{GEO} = v_O = 3.07\,km\ s^{-1}$, the transfer time is calculated to be $t_{CT} = 43$ days.

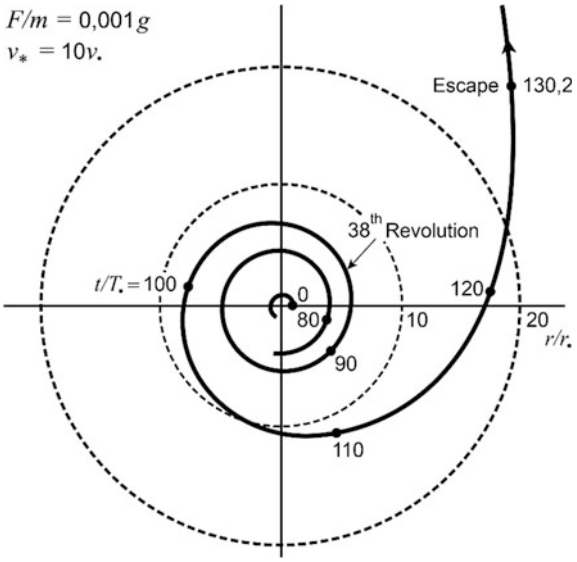


Fig. 8.22 Continuous thrust trajectory as a result of a continuous tangential thrust. Index • refers to the initial orbit. Because the spiral is very narrow at the beginning, only the last two revolutions are shown

8.5 Relative Orbits

Having explored the conditions for a general transfer between orbits, the more delicate problem is to meet a target point in the final orbit, a so-called orbital rendezvous. In this and the next section we will focus on orbital rendezvous in Earth orbits though the described methods apply to any planetary orbit. For interplanetary orbital rendezvous we refer the reader to Sect. 9.3.

In orbital rendezvous, there is generally a passive target object to rendezvous with an interceptor (a.k.a. *chaser*) as the active part that performs the rendezvous maneuvers. Rather than describe the required rendezvous maneuvers in an inertial reference frame such as a geocentric reference frame, it is very convenient to describe them relative to the target. It is convenient because if the interceptor moves in the vicinity of the target we can linearize the equations of motion, which will enable us to solve them analytically. On the other hand, the target in a conical motion about the center of gravity constitutes a non-inertial reference system, a fact that will complicate our equation of motion somewhat. Nevertheless, the description of relative motion about a target object, which is done in the following, is a prerequisite for analyzing rendezvous maneuvers, which is the objective of Sect. 8.6.

8.5.1 General Equation of Motion

Suppose there is a target object in a planetary orbit and a S/C (usually dubbed chaser) wants to navigate in its vicinity and with respect to it. Good examples are the approach maneuvers of a Soyuz vehicle to the ISS or the re-docking of the ascending Lunar Module to the Command/Service Module in low Lunar orbit during the Apollo moon missions. For this mission we need to know the relative motion to the target and how to maneuver to reach the target. This is the subject of the next two sections.

Let us assume the target (for instance, we assume to be the ISS) is in an arbitrary terrestrial conic orbit with position vector $\boldsymbol{R}$ in the geocentric reference frame. For this orbit Newton's gravitational EoM (7.1.19) must hold

$$\ddot{\boldsymbol{R}} = -\frac{\mu}{R^3}\boldsymbol{R} \tag{8.5.1}$$

Of course, Newton's gravitational EoM must equally hold for the chaser's orbit with position vector $\boldsymbol{\rho}$ in the geocentric reference frame, i.e.

$$\ddot{\boldsymbol{\rho}} = -\frac{\mu\boldsymbol{\rho}}{\rho^3}$$

We start deriving the chaser's equation of motion relative to the ISS by defining the relative vector $\boldsymbol{r} = \boldsymbol{\rho} - \boldsymbol{R}$ as shown in Fig. 8.23. The specific reference frame is the ISS-centric coordinate system RSW as described in Sect. 13.1.4, Fig. 13.4, with unit vectors $\boldsymbol{u}_x, \boldsymbol{u}_y, \boldsymbol{u}_z = \boldsymbol{S}, \boldsymbol{W}, \boldsymbol{R}$: The z-axis points along the radial vector while the x-axis points along the moving direction of the target. The y-axis completes the right-handed reference system. Our goal is to find the equation of motion of the relative vector

$$\boldsymbol{r} = \boldsymbol{\rho} - \boldsymbol{R} = x\boldsymbol{u}_x + y\boldsymbol{u}_y + z\boldsymbol{u}_z =: (x, y, z)$$

expressed in the co-moving coordinate system $(\boldsymbol{u}_x, \boldsymbol{u}_y, \boldsymbol{u}_z)$. Since we seek for a differential equation for $\boldsymbol{r}$, we first have to express $\boldsymbol{\rho}$ in terms of $\boldsymbol{R}$ and $\boldsymbol{r}$. Because $r \ll R$ we can expand the term $1/\rho^3$ as follows:

$$\begin{aligned}\frac{1}{\rho^3} &= \frac{1}{\left[x^2 + y^2 + (z+R)^2\right]^{3/2}} = \frac{1}{\left[r^2 + 2zR + R^2\right]^{3/2}} \approx \frac{1}{R^3}\frac{1}{(1+z/R)^3} \\ &\approx \frac{1}{R^3}\left(1 - 3\frac{z}{R}\right)\end{aligned}$$

and hence

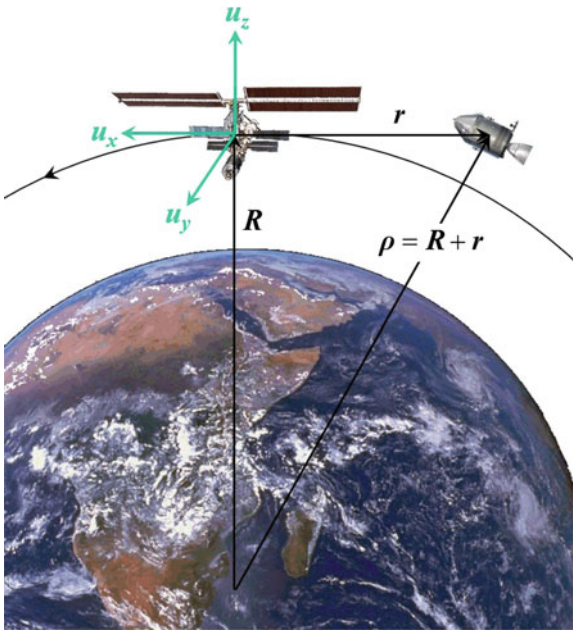


Fig. 8.23 Definition of relative vectors and RSW coordinate system in the target object (ISS), that moves on a circular orbit. The z-axis points along the radial vector while the x-axis points along the moving direction of the target object. The y-axis completes the right-handed reference system

$$\ddot{\boldsymbol{\rho}} = -\frac{\mu\boldsymbol{\rho}}{\rho^3} \approx -\frac{\mu}{R^3}(\boldsymbol{R}+\boldsymbol{r})\left(1-3\frac{z}{R}\right) \approx -\frac{\mu}{R^3}(\boldsymbol{R}+\boldsymbol{r}-3z\boldsymbol{u}_z)$$

Because from the definition of $\boldsymbol{\rho}$ and from Eq. (8.5.1) also $\ddot{\boldsymbol{\rho}} = \ddot{\boldsymbol{R}} + \ddot{\boldsymbol{r}} = -\mu\boldsymbol{R}/R^3 + \ddot{\boldsymbol{r}}$ holds, we get

$$-\frac{\mu}{R^3}\boldsymbol{R} + \ddot{\boldsymbol{r}} \approx -\frac{\mu}{R^3}(\boldsymbol{R}+\boldsymbol{r}-3z\boldsymbol{u}_z)$$

and finally

$$\ddot{\boldsymbol{r}} = -\frac{\mu}{R^3}(\boldsymbol{r}-3z\boldsymbol{u}_z) \quad @\, x, y, z \ll R \tag{8.5.2}$$

We want to solve this vector equation in the non-inertial co-moving target RSW coordinate system $(\boldsymbol{u}_x, \boldsymbol{u}_y, \boldsymbol{u}_z)$. Now, as $\boldsymbol{r} = x\boldsymbol{u}_x(t) + y\boldsymbol{u}_y + z\boldsymbol{u}_z(t)$ is already given in RSW, how is $\ddot{\boldsymbol{r}}$ expressed in it?

The answer is provided by Eq. (7.2.2) for $\boldsymbol{a} \equiv \boldsymbol{r} = x\boldsymbol{u}_x(t) + y\boldsymbol{u}_y + z\boldsymbol{u}_z(t) =: (x, y, z)_{xyz}$ and for the angular velocity of the target $\boldsymbol{\omega} = \omega(0, 1, 0)_{xyz}$ as

$$\begin{aligned}\ddot{\boldsymbol{r}} &= \ddot{\boldsymbol{r}}_{xyz} + 2\boldsymbol{\omega} \times \dot{\boldsymbol{r}}_{xyz} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}) + \dot{\boldsymbol{\omega}} \times \boldsymbol{r} \\ &= \left(\ddot{x} + 2\omega\dot{z} - \omega^2 x + \dot{\omega} z\right)\boldsymbol{u}_x + \ddot{y}\boldsymbol{u}_y + \left(\ddot{z} - 2\omega\dot{x} - \omega^2 z - \dot{\omega} x\right)\boldsymbol{u}_z \\ &= \begin{pmatrix} \ddot{x} + 2\omega\dot{z} - \omega^2 x + \dot{\omega} z \\ \ddot{y} \\ \ddot{z} - 2\omega\dot{x} - \omega^2 z - \dot{\omega} x \end{pmatrix}_{xyz}\end{aligned}$$

where

$$\dot{\boldsymbol{r}}_{xyz} := (\dot{x}, \dot{y}, \dot{z})_{xyz},\ \ddot{\boldsymbol{r}}_{xyz} := (\ddot{x}, \ddot{y}, \ddot{z})_{xyz}$$

If we insert this into Eq. (8.5.2), we finally obtain the equation of motion

$$\ddot{\boldsymbol{r}} = \begin{pmatrix} \ddot{x} + 2\omega\dot{z} - \omega^2 x + \dot{\omega} z \\ \ddot{y} \\ \ddot{z} - 2\omega\dot{x} - \omega^2 z - \dot{\omega} x \end{pmatrix}_{xyz} = -\frac{\mu}{R^3} \begin{pmatrix} x \\ y \\ z - 3z \end{pmatrix}_{xyz}$$

Because this vectorial equation must hold for each orthogonal component separately, we finally find the three differential equations

$$\begin{aligned} &\ddot{x} + 2\omega\dot{z} - \left(\omega^2 - \tfrac{\mu}{R^3}\right)x + \dot{\omega} z = 0 \\ &\ddot{y} \qquad\qquad + \tfrac{\mu}{R^3} y \qquad\quad = 0 \\ &\ddot{z} - 2\omega\dot{x} - \left(\omega^2 + 2\tfrac{\mu}{R^3}\right)z - \dot{\omega} x = 0 \end{aligned} \quad @\, x, y, z \ll R \quad \textbf{equations of motion} \qquad (8.5.3)$$

This are the equations of motion of a chaser, not being subject to external forces, relative to a target with coordinates (x, y, z) in the target's topocentric RSW coordinate system, which quite generally is in a conic orbit about a center of gravity and, due to its conical trajectory, rotates with angular velocity $\omega(t)$. Note that from Eqs. (7.2.7), (7.3.7) and (7.4.13) follows

$$\frac{\mu}{R^3} = \sqrt{n_e}\omega^{3/2}$$

with

$$n_e := \frac{n}{(1 - e^2)^{3/2}} = \sqrt{\frac{\mu}{a^3(1 - e^2)^3}}$$

Observe that only the first and third equations are coupled. If initially $y(0) = 0$ and $\dot{y}(0) = 0$, i.e., if the chaser initially is in the orbital plane, then $\ddot{y} = 0$ and hence $y(t) = 0$, which means that it will always stay in that plane. Otherwise and if $R \approx const$ it will oscillate about this plane with frequency $\sqrt{\mu/R^3}$.

Elliptic Target Orbit

To get more practical and to find a solution to Eq. (8.5.3) we now assume an elliptic orbit with orbital elements a, e about a planet. For this, we get according to Eq. (7.2.7)

$$\omega = \frac{\sqrt{\mu a(1-e^2)}}{R^2}$$

and with Eqs. (7.3.15a) and (7.4.14c)

$$\dot{\omega} = -2\frac{\mu e \sin\theta}{R^3} = -\frac{2\mu a e}{R^4}\sqrt{1-e^2}\sin E$$

with

$$R(t) = a[1 - e\cos E(t)]$$

where the eccentric anomaly $E(t)$ is the solution of Kepler's Eq. (7.4.15)

$$n \cdot t = t\sqrt{\frac{\mu}{a^3}} = E(t) - e\sin E(t)$$

This transcendental equation can be solved by Newton's iteration method (Eq. (7.4.16))

$$E_{i+1} = E_i - \frac{E_i - e\sin E_i - n\cdot t}{1 - e\cos E_i}$$

Of course, similar equations can easily be derived for a hyperbolic trajectory (cf. Sect. 7.4.3).

Given these time dependent quantities, the differential Eq. (8.5.3) for a rendezvous on an elliptic target orbit can be solved numerically for instance by a Runge–Kutta method.

8.5.2 *Circular Orbits*

We now come to the most common situation where the target is in a circular orbit. Circular orbits are so common because for a given minimum altitude they have the lowest orbital energy, entailing the least delta-v to achieve. Therefore, nearly every target in a low Earth orbit will attain a circular or quasi-circular orbit. For instance, the ISS has an eccentricity of typically $e = 0.0001 - 0.001$, that can be shown to be induced by the anisotropies of Earth's gravitational potential (see Sect. 12.3.2).

For a circular orbit we have $\ddot{\boldsymbol{R}} = -\mu\boldsymbol{R}/R^3 = -n^2\boldsymbol{R}$ and $n = \sqrt{\mu/R^3} = 2\pi/T = const$. If be insert this into the general equation of motion Eq. (8.5.3) we

obtain

$$\boxed{\begin{aligned}\ddot{x}+2n\dot{z}&=0\\ \ddot{y}+n^2y&=0\\ \ddot{z}-2n\dot{x}-3n^2z&=0\end{aligned}}\quad @\, x,y,z \ll R \quad \textbf{Hill's equations} \tag{8.5.4}$$

These are the famous *Hill's equations* (a.k.a. *Clohessy–Wiltshire equations*) governing the motion of a S/C, not being subject to external forces, with coordinates (x, y, z) in the topocentric RSW coordinate system of a target object circling a planet with constant orbital velocity $\omega = n$.

µg-Forces on Space Stations

A modest application of Hill's equation in a circular orbit is the residual forces acting on parts of a satellite or a space station. Such residual forces are important for experiments in space, which contribute some unwanted artificial gravity to the otherwise weightlessness in space. To determine their level, let us assume we have an experimental drawer in a science rack in one of the ISS laboratories located at the distance $\boldsymbol{r} = (x, y, z)$ from the center of mass (CM) of the ISS, which orbits at orbital frequency $n = \sqrt{\mu/R^3}$. Since the drawer is fixed relative to CM, we have $\dot{\boldsymbol{r}} = (\dot{x}, \dot{y}, \dot{z}) = 0$. If the sample in the drawer is further out in the z-direction (see Fig. 8.24) than CM, it experiences an enhanced centrifugal force, but smaller gravitational force. If it is further toward Earth it experiences the opposite effect: a smaller centrifugal force but a larger gravitational force. In total, it thus experiences some residual force, a so-called *tidal force*. We can determine this tidal force if we interpret the terms $\ddot{x}, \ddot{y}, \ddot{z}$ in Eq. (8.5.4) as the residual accelerations making up the tidal force. In doing so we end up with the tidal acceleration in terms of *Earth's mean gravitational acceleration at its surface* (see Sect. 7.1.2), $g_0 := \mu/R_\oplus^2 \approx \mu/R^2 = 9.80\,\mathrm{m\,s^{-2}}$, as

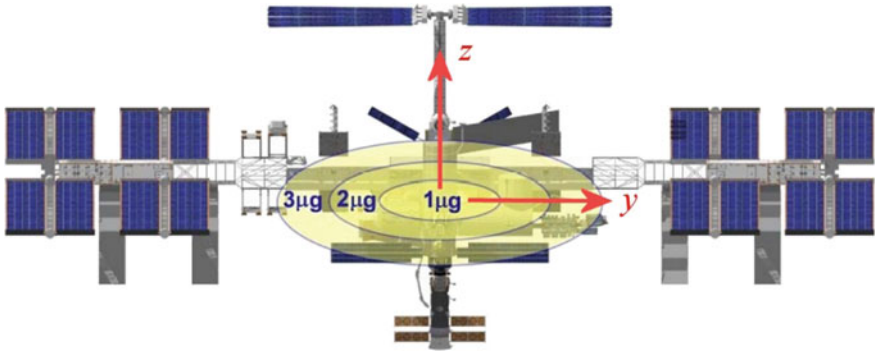


Fig. 8.24 The levels of residual µg-forces on the International Space Station at positions off the center of mass

$$\frac{\ddot{x}}{g_0} = 0, \quad \frac{\ddot{y}}{g_0} = -\frac{y}{R}, \quad \frac{\ddot{z}}{g_0} = 3\frac{z}{R}$$

So, depending on the distances y, z from CM we get different levels of tidal forces. By analyzing Eq. (8.5.3) we realize that in the z-direction there are two gravitational tide contributions and just one centrifugal tide contribution in the overall total of 3. In the y-direction there is just one contribution from the gravitational (lateral) tide. In general the tidal forces are of order $10^{-6}g = 1\mu g$. This is why experiments on space stations are commonly called μg-experiments. For instance, on the ISS, having a typical altitude of 350 km, a level of $1\mu g$ is obtained at a horizontal distance of $y = 10^{-6} \times 6.73 \times 10^6\,\mathrm{m} = 6.73\,\mathrm{m}$ and at a vertical distance of $z = \frac{1}{3} \times 10^{-6} \times 6.73 \times 10^6\,\mathrm{m} = 2.24\,\mathrm{m}$ from CM. Overall, we have μg-ellipses as depicted in Fig. 8.24.

Solution of Hill's Equations

To describe the explicit motion $\boldsymbol{r}(t) = (x(t), y(t), z(t))$ of a chaser, we have to solve Hill's equations. This can be done straightforwardly with only little effort. To do so efficiently, we first replace time by the dimensionless mean anomaly and define both it and its derivatives.

$$\begin{aligned} M &= nt \\ (.)' &:= \frac{d}{dM} = \frac{1}{n}\frac{d}{dt}(.) \\ (.)'' &:= \frac{d^2}{dM^2} = \frac{1}{n^2}\frac{d^2}{dt^2}(.) \end{aligned} \tag{8.5.5}$$

and initial conditions at $t = 0$ as

$$\begin{array}{ll} x(0) = x_0, & x'(0) = x'_0 \\ y(0) = y_0, & y'(0) = y'_0 \\ z(0) = z_0, & z'(0) = z'_0 \end{array} \quad \textbf{initial conditions} \tag{8.5.6}$$

With this Hill's equations read

$$\begin{aligned} x'' + 2z' &= 0 \\ y'' + y &= 0 \\ z'' - 2x' - 3z &= 0 \end{aligned} \tag{8.5.7}$$

First observe that the second Hill equation is decoupled from all others. It has the form of a harmonic oscillator. We therefore find its solution as

$$y = y_0 \cos M + y_0' \sin M$$

We now integrate the first Hill equation directly. By taking the initial conditions into account we get

$$x' = -2z + 2z_0 + x_0'$$

We insert this result into the third Hill equation getting

$$z'' = 2x' + 3z = -\left[z - \left(4z_0 + 2x_0'\right)\right]$$

This, too, is the equation of a harmonic oscillator for the expression $z - \left(4z_0 + 2x_0'\right)$ yielding

$$z - \left(4z_0 + 2x_0'\right) = z_0' \sin M - \left(3z_0 + 2x_0'\right) \cos M$$

under consideration of the initial conditions $z(0) = z_0, z'(0) = z_0'$. We solve for z and find

$$z = 4z_0 + 2x_0' + z_0' \sin M - \left(3z_0 + 2x_0'\right) \cos M$$

Inserting this into the above equation $x' = -2z + 2nz_0 + x_0'$ we get

$$x' = -\left(6z_0 + 3x_0'\right) - 2z_0' \sin M + \left(6z_0 + 4x_0'\right) \cos M$$

This can be integrated directly to deliver with initial condition $x(0) = x_0$

$$x = x_0 - 2z_0' - \left(6z_0 + 3x_0'\right)M + 2z_0' \cos M + \left(6z_0 + 4x_0'\right) \sin M$$

We summarize the results as

$$\begin{pmatrix} x \\ z \\ x' \\ z' \end{pmatrix} = \begin{bmatrix} 1 & 6\sin M - 6M & 4\sin M - 3M & 2\cos M - 2 \\ 0 & -3\cos M + 4 & -2\cos M + 2 & \sin M \\ 0 & 6\cos M - 6 & 4\cos M - 3 & -2\sin M \\ 0 & 3\sin M & 2\sin M & \cos M \end{bmatrix} \begin{pmatrix} x_0 \\ z_0 \\ x_0' \\ z_0' \end{pmatrix} \tag{8.5.8}$$

and in convenient terms

$$\underbrace{\begin{pmatrix} x \\ z \\ \dot{x} \\ \dot{z} \end{pmatrix}}_{\substack{final \\ state\ vector}} = \underbrace{\begin{bmatrix} 1 & 6(\sin nt - nt) & (4\sin nt)/n - 3t & 2(\cos nt - 1)/n \\ 0 & 4 - 3\cos nt & 2(1 - \cos nt)/n & (\sin nt)/n \\ 0 & 6n(\cos nt - 1) & 4\cos nt - 3 & -2\sin nt \\ 0 & 3n\ \sin\ nt & 2\sin nt & \cos nt \end{bmatrix}}_{time-evolution\ \ matrix} \underbrace{\begin{pmatrix} x_0 \\ z_0 \\ \dot{x}_0 \\ \dot{z}_0 \end{pmatrix}}_{\substack{initial \\ state\ vector}} \tag{8.5.9}$$

and

$$\boxed{\begin{pmatrix} y \\ \dot{y} \end{pmatrix} = \begin{bmatrix} \cos nt & (\sin nt)/n \\ -n \sin nt & \cos nt \end{bmatrix} \begin{pmatrix} y_0 \\ \dot{y}_0 \end{pmatrix}} \tag{8.5.10}$$

These are the basic equations for orbital motion in the vicinity of the reference object and with respect to its co-moving topocentric coordinate system RSW.

Application to Orbital Rendezvous

A typical application of Eq. (8.5.9) is the design of an orbital rendezvous. If at time $t_0 = 0$ a S/C is at an initial point $\boldsymbol{r}_0 = (x_0, y_0, z_0)$ the problem to solve is: What should be the initial velocity $\boldsymbol{v}_0 = (\dot{x}_0, \dot{y}_0, \dot{z}_0)$ to meet after time t a given target point $\boldsymbol{r} = (x, y, z) = (0, 0, 0)$? The solution can be derived from the equations for x, y, and z of Eqs. (8.5.9) and (8.5.10) by setting $x = y = z = 0$ and solving for $\dot{x}_0, \dot{y}_0, \dot{z}_0$. If this is done, one straightforwardly obtains

$$\begin{aligned} \frac{\dot{x}_0}{n} &= \frac{2(\cos nt - 1)x_0 + (4 \sin nt - 3nt \cos nt)z_0}{3nt \sin nt + 8(\cos nt - 1)} \\ \frac{\dot{z}_0}{n} &= \frac{\sin nt \cdot x_0 - [6nt \sin nt + 14(\cos nt - 1)]z_0}{3nt \sin nt + 8(\cos nt - 1)} \\ \frac{\dot{y}_0}{n} &= -\frac{y_0}{\tan nt} \end{aligned} \tag{8.5.11}$$

or in matrix expression

$$\begin{pmatrix} \dot{x}_0 \\ \dot{z}_0 \end{pmatrix} = \frac{n}{3nt \sin nt + 8 \cos nt - 8} \begin{pmatrix} 2 \cos nt - 2 & 4 \sin nt - 3nt \cos nt \\ \sin nt & 14 - 14 \cos nt - 6nt \sin nt \end{pmatrix} \begin{pmatrix} x_0 \\ z_0 \end{pmatrix}$$

$$\dot{y}_0 = -\frac{n}{\tan nt} y_0$$

The required delta-v for these maneuvers then is given by

$$\Delta v = \sqrt{(\dot{x}_0 - \dot{x}_{0-})^2 + (\dot{y}_0 - \dot{y}_{0-})^2 + (\dot{z}_0 - \dot{z}_{0-})^2} \tag{8.5.12}$$

where $\boldsymbol{v}_{0-} = (\dot{x}_{0-}, \dot{y}_{0-}, \dot{z}_{0-})$ is the velocity of the S/C just before the impulse maneuver at (x_0, y_0, z_0).

It is an easy exercise to show that in the limit $t \to 0$ Eq. (8.5.11) pass over into

$$\begin{aligned} \dot{x}_0 &= -\frac{x_0}{t} - n z_0 - \frac{1}{6} n^2 x_0 t + O(n^3 z_0 t^2) \\ \dot{z}_0 &= -\frac{z_0}{t} + n x_0 - \frac{5}{6} n^2 z_0 t + O(n^3 x_0 t^2) \qquad @\, t \to 0 \\ \dot{y}_0 &= -\frac{y_0}{t} + \frac{1}{3} n^2 y_0 t + O(n^4 y_0 t^3) \end{aligned}$$

Therefore, for $(nt)^2 \ll 1$, which implies roughly $t < 0.02 \cdot T$, we can write

$$\begin{aligned} \dot{x}_0 &= -\tfrac{x_0}{t} - n\, z_0 \\ \dot{z}_0 &= -\tfrac{z_0}{t} + n\, x_0 \\ \dot{y}_0 &= -\tfrac{y_0}{t} \end{aligned} \qquad @\ (nt)^2 \ll 1 \tag{8.5.13}$$

These expressions will later be useful to explore maneuvers in the vicinity of the target object.

8.5.3 *Flyaround Trajectories*

In order to fly in a controlled way in the vicinity of a target object we need to know: What does a flyaround trajectory of a S/C look like? We will now explore the two different types of trajectories because they are essential for understanding the general behavior of such a S/C.

The Prolate Cycloid

On November 18, 2008, the astronaut Heidemarie Stefanyshyn-Piper lost her toolbox in space when carrying out repair work outside the International Space Station. She accidentally and gently touched the toolbox (which should have been secured by a line either to her or the ISS, but was not) giving the box a push such that it slowly drifted away from the station (see Fig. 8.25).

This accident is a good example to study the trajectory of slowly moving objects near a reference system, which for our purposes is the ISS at an average altitude of 350 km. So, what was the trajectory of the toolbox? It is important to know in which direction the push was relative to the flight direction of the ISS. Because we do not know, we will assume two different situations. First, Heidemarie pushed the toolbox along the flight direction, with an initial velocity of, say, $\boldsymbol{v}_0 = (\dot{x}_0, \dot{y}_0, \dot{z}_0) = (v_0, 0, 0)$.



Fig. 8.25 The toolbox (top right) of astronaut Stefanyshyn-Piper slowly drifting away from the ISS truss structure (bottom left). *Credit* NASA

We study the trajectory in the RSW coordinate system of Heidemarie, so $\boldsymbol{r}_0 = (x_0, y_0, z_0) = (0, 0, 0)$. Under these initial assumptions, we read from Eq. (8.5.9)

$$\begin{aligned} x(t) &= \frac{v_0}{n}(4\sin nt - 3nt) \approx v_0 t \\ \dot{x}(t) &= v_0(4\cos nt - 3) \\ z(t) &= 2\frac{v_0}{n}(1 - \cos nt) \approx n v_0 t^2 \\ \dot{z}(t) &= 2v_0 \sin nt \end{aligned} \tag{8.5.14}$$

where the approximations are for early times on the trajectory, $nt \ll 1$. If we solve the approximation for x and insert it in the approximation for z, we obtain a parabolic trajectory

$$z = \frac{nx^2}{v_0}$$

To be more concrete, let us assume that Heidemarie pushed the toolbox backward, against flight direction, with an initial velocity of, say, $v_0 = -0.1\,\mathrm{m\,s^{-1}}$. Because v_0 is negative we get a down curved parabola (see Fig. 8.26): The toolbox dives below the orbit of the ISS, which first seems to be quite surprising. Interestingly the toolbox comes to a halt in $-x$-direction when $\cos nt = 3/4$ and thereafter moves back toward the ISS, yet at increasingly lower altitudes. After half an orbit, it reaches the lowest point (perigee), and then it climbs up to reach the initial altitude of 350 km after a full revolution of the ISS, however at a distance of 1.65 km in front of the ISS. The toolbox will repeat this loopy motion indefinitely thereby moving away from the ISS—the toolbox will be lost. Such a winding trajectory is mathematically called a *prolate cycloid*.

What is the reason for such a weird trajectory? When the toolbox is kicked off backward it has a lower velocity as it should have for a circular orbit at this altitude.

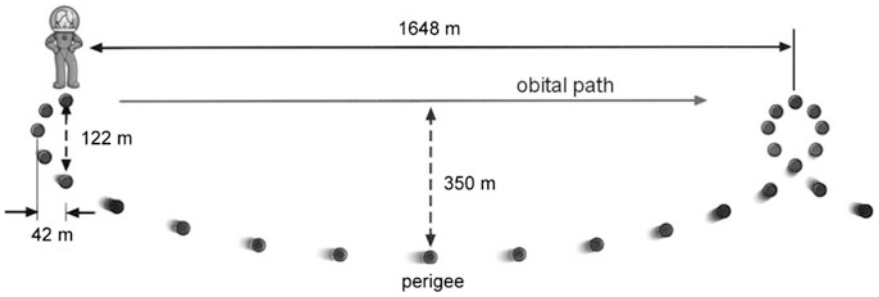


Fig. 8.26 The prolate cycloid trajectory of the toolbox that Heidemarie lost, if its initial motion, $v_0 = -0.1\,\mathrm{m\,s^{-1}}$, was reverse to the flight direction of the ISS. ISS altitude is $h = 350\,\mathrm{km}$

So the centrifugal force $F_c = mv^2/r$ is lower than the gravitational force, which rapidly pulls the box down. But upon dropping it gains speed, which brings it to a lateral halt at 42 m away from the ISS. With increasing speed the centrifugal force increases that after half a revolution at perigee brings the decline to a halt at 350 m below and 824 m ahead of the ISS. Since the speed now is too large for this altitude, the box dives up and decelerates until it "kisses" the orbit to iterate its dive.

The size of the cycloid depends linearly on the initial velocity while its orientation reverses when v_0 is with or against the flight direction of the ISS. Figure 8.27 gives an overview of this behavior for different initial x_0.

The reason for this weird motion can be understood when seen in a geocentric reference system (see Fig. 8.28). An object initially slower than the ISS at point 0 will have less orbital energy and hence move on an elliptic trajectory with smaller major-axis than the ISS. This implies that the object will drop below the ISS and will have a smaller orbital period, so showing up in front of the ISS after one revolution at the initial point 0. An object with a larger initial velocity will move on a larger ellipse with reversed behavior.

This view in an inertial reference frame makes clear that the prolate cycloid is the trajectory of a Hohmann transfer as observed in the target's reference frame. This should have been clear from the very beginning, because the initial kick-burn is tangential to the initial orbit. This is the attribute of a Hohmann transfer.

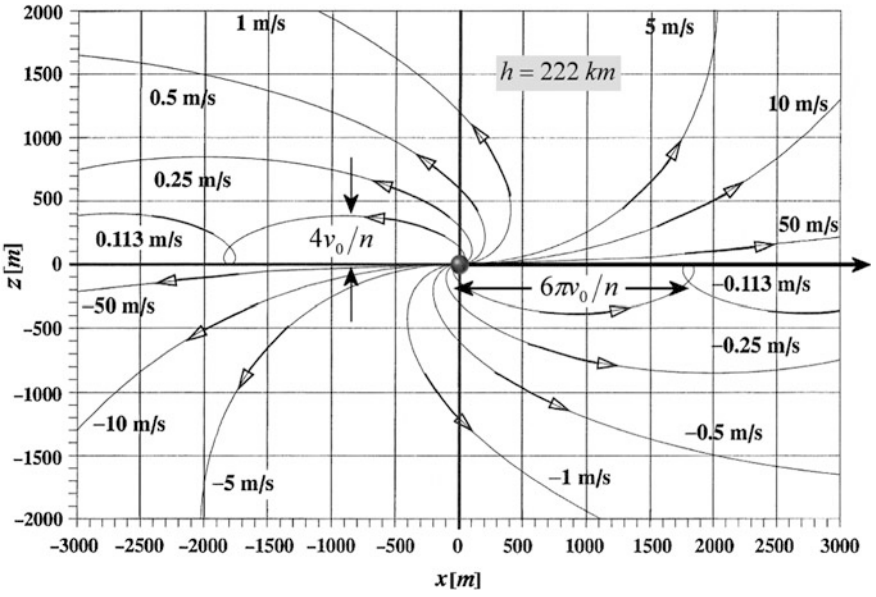


Fig. 8.27 Shown are for different v_0 in x-direction the trajectories (prolate cycloid) of the object moving relative to a reference point [center dot, which itself moves on an orbit at altitude $h =$ 222 km to the right (bold arrow)]. *Credit* Vallado (2007)

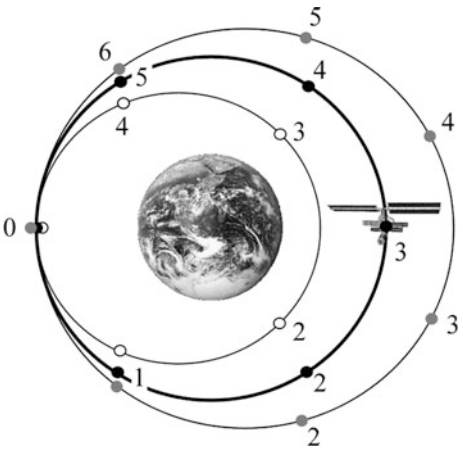


Fig. 8.28 A schematic sketch of the prolate cycloid motion as viewed from the inertial reference frame of Earth. The smaller ellipse is for an object with smaller velocity and the larger ellipse with a larger velocity at point 0. The numbered points give the positions on each orbit after constant time intervals

The Ellipse

We now assume that Heidemarie pushed the toolbox vertically to the flight direction, with initial velocity $\boldsymbol{v}_0 = (\dot{x}_0, \dot{y}_0, \dot{z}_0) = (0, 0, v_0)$. Interestingly, the trajectory of the toolbox would be totally different. From Eq. (8.5.9) we derive the trajectory to be

$$\begin{aligned} x &= 2\frac{v_0}{n}(\cos\, nt - 1) \approx -nv_0t^2 \\ z &= \frac{v_0}{n}\sin\, nt \approx v_0t \end{aligned} \tag{8.5.15}$$

again with approximations for early times on the trajectory, $nt \ll 1$. By removing the time parameter for $nt \ll 1$, we again find a parabola, that, however, opens to the rear side of the ISS.

$$z = \sqrt{-\frac{v_0}{n}x}$$

To derive the full trajectory we recognize that we can solve both equations for $\cos\, nt$ and $\sin\, nt$ and via the trigonometric equation $\sin^2 x + \cos^2 x = 1$, we find

$$\frac{(x + 2v_0/n)^2}{(2v_0/n)^2} + \frac{z^2}{(v_0/n)^2} = 1$$

which is the equation of an ellipse with semi-major axis $a = 2v_0/n$ along the x-axis and semi-minor axis $b = v_0/n$ along the z-axis, and therefore eccentricity $e = \sqrt{1 - b^2/a^2} = \sqrt{3/4} = 0.8660\ldots$. Details of such ellipses with $v_0 = 0.1\,\mathrm{m\,s^{-1}}$ are shown in Fig. 8.29 and for varying initial velocities in Fig. 8.30.

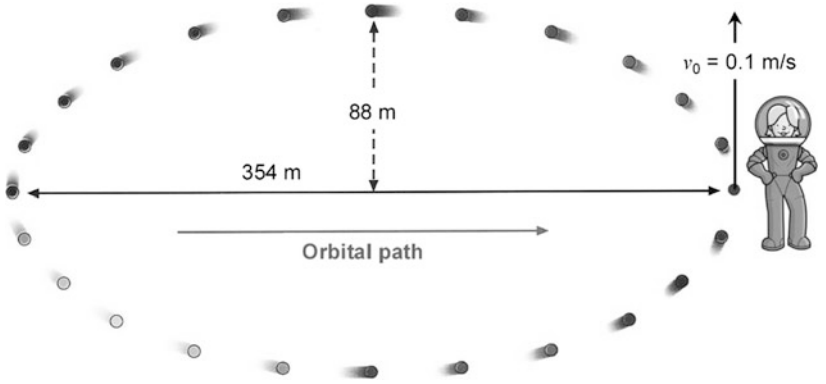


Fig. 8.29 The path of Heidemarie's toolbox if pushed with $v_0 = 0.1$ m s^{-1} along the positive z-direction. The box follows a closed elliptic path and returns to Heidemarie after one orbital revolution of the ISS. ISS altitude is $h = 350$ km

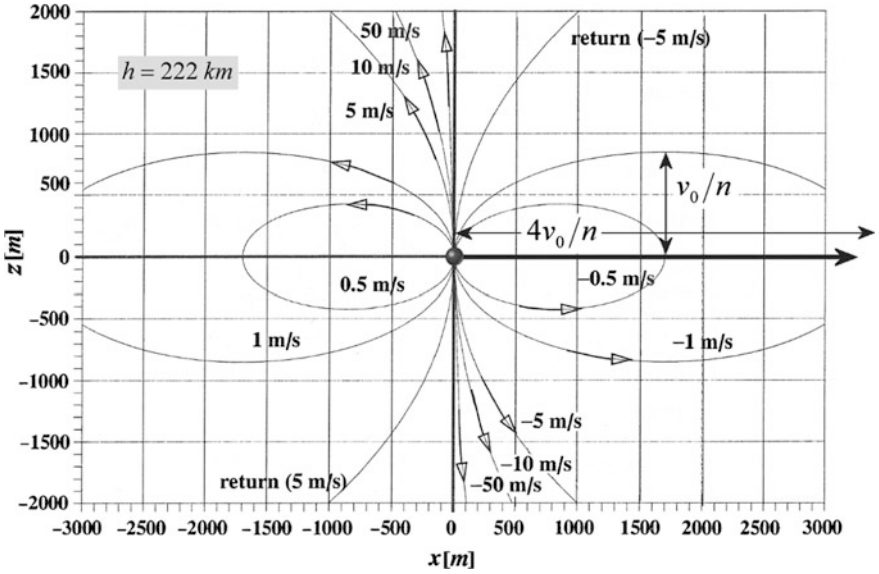


Fig. 8.30 Trajectories (ellipses) of an object moving relative to a reference point [center dot, which itself moves on an orbit at altitude $h = 222$ km to the right (bold arrow)] for different v_0 in z-direction. *Credit* Vallado (2007)

We can summarize the behavior of the toolbox as follows: When initially moving up, the orbital velocity remains constant. However, the angular velocity $\omega = v/r$ and the centrifugal force $F_c = mv^2/r$ decrease. Owing to this, the toolbox falls behind the ISS and also loses vertical speed, because the gravitation pull

becomes bigger than the centrifugal force. After a quarter of an orbital revolution of the ISS, the toolbox drops down and crosses the orbit after half an orbital revolution, but now with downward velocity. The inverse behavior thereafter would bring the toolbox back to Heidemarie.

If a S/C relative to a target object starts its trajectory from the orbital path of the target object and initially moves along the x- or z-direction, the ellipse and the prolate cycloid are the two possible forms of this trajectory. The trajectories become more intricate if $\boldsymbol{v}_0$ is not pointing into the x- or z-direction, or the initial location is displaced from the orbital path, or both. In any of these cases the main features of such convoluted trajectories are the ellipse and the prolate cycloid. The reason is, whatever the initial conditions of the S/C might be, it will move on an ellipse around the Earth. If its semi-major axis a is the same as that of the target object it will just fly around the target object periodically. If its a is different, its orbital period will be different and therefore the periodic motion will be superimposed by a motion that let the S/C and the target object drift relative to each other. These two features are extensively used for applied orbital rendezvous maneuvers to which we will turn now.

8.5.4 Near-Circular Orbits

Differential Equations

We now want to convey these results to near-circular orbits, i.e. orbits with $e \ll 1$. For small eccentricities it is easy to show (exercise, Problem (8.9)) that Eq. (8.5.3) pass over into

$$\begin{aligned} \ddot{x} + 2n\dot{z} &= en(nx\cos nt + 2nz\sin nt - 4\dot{z}\cos nt) \\ \ddot{y} + n^2 y &= -3en^2 y\cos nt \qquad @\ e \ll 1 \quad (8.5.16) \\ \ddot{z} - 2n\dot{x} - 3n^2 z &= 2en(-nx\sin nt + 5nz\cos nt + 2\dot{x}\cos nt) \end{aligned}$$

or in our dimensionless notation (see Sect. 8.5.2)

$$M = nt, \quad (.)' := \frac{d}{dM} = \frac{1}{n}\frac{d}{dt}, \quad (.)'' = \frac{d^2}{dM^2} = \frac{1}{n^2}\frac{d^2}{dt^2}$$

they read

$$\begin{aligned} x'' + 2z' &= e(x\cos M + 2z\sin M - 4z'\cos M) \\ y'' + y &= 3e(-y\cos M) \qquad @\ e \ll 1 \quad (8.5.17) \\ z'' - 2x' - 3z &= 2e(-x\sin M + 5z\cos M + 2x'\cos M) \end{aligned}$$

Here M is measured relative to the periapsis of the elliptical target orbit.

Solution to Differential Equations

It is quite intricate to solve these differential equations. We therefore leave this as an exercise to the reader (see Problem 8.10) and provide here only the exact results:

$$x(M) = \left[\begin{pmatrix} 1 \\ 6\sin M - 6M \\ 4\sin M - 3M \\ 2\cos M - 2 \end{pmatrix}^T + e \begin{pmatrix} -\cos M + 1 \\ \sin M \cos M + 12 \sin M - 6M \cos M - 15M \\ 6\sin M \cos M - 3M \cos M - 3M \\ 3\cos^2 M - 2\cos M - 1 \end{pmatrix}^T \right] \begin{pmatrix} x_0 \\ z_0 \\ x_0' \\ z_0' \end{pmatrix}$$

$$y(M) = \left[\begin{pmatrix} \cos M \\ \sin M \end{pmatrix}^T + e \begin{pmatrix} \cos^2 M + \cos M - 2 \\ \sin M \cos M - \sin M \end{pmatrix}^T \right] \begin{pmatrix} y_0 \\ y_0' \end{pmatrix} \tag{8.5.18}$$

$$z(M) = -\left[\begin{pmatrix} 0 \\ 3\cos M - 4 \\ 2\cos qM - 2 \\ -\sin M \end{pmatrix}^T + e \begin{pmatrix} 0 \\ 6\cos^2 M + 10\cos M + 6M\sin M - 16 \\ 4\cos^2 M + 2\cos M + 3M\sin M - 6 \\ -2\sin M \cos M + 2\sin M \end{pmatrix}^T \right] \begin{pmatrix} x_0 \\ z_0 \\ x_0' \\ z_0' \end{pmatrix}$$

We recall that M is measured relative to the periapsis of the elliptical target orbit. Observe that these solutions satisfy our standard initial conditions

$$\begin{aligned} x(0) &= x_0, \quad x'(0) = x_0' \\ y(0) &= y_0, \quad y'(0) = y_0' \\ z(0) &= z_0, \quad z'(0) = z_0' \end{aligned}$$

Flyaround Trajectories

At the periapsis let us first place a S/C at $(x_0, 0, 0)$ behind the ISS with initial zero relative velocity, $\boldsymbol{v}_0 = (0, 0, 0)$, and see what happens. From Eq. (8.5.18) we obtain

$$\begin{aligned} x(t) &= [1 + e(1 - \cos nt)]x_0 \\ y(t) &= 0 \\ z(t) &= 0 \end{aligned}$$

If the reference orbit were circular we would get a steady distance, $x(t) = x_0 = const$ to the target because the orbit velocity is always constant. For an elliptical target the distance slightly changes over one orbit in the interval $x_0 \le x(t) \le (1 + 2e) \cdot x_0$. Where $x = (1 + 2e) \cdot x_0$ happens at the apoapsis (Fig. 8.31).

The Prolate Cycloid

We now start with an initial longitudinal velocity $\boldsymbol{v}_0 = (\dot{x}_0, \dot{y}_0, \dot{z}_0) = (v_0, 0, 0)$ and again study the trajectory in the RSW coordinate system of the ISS, so $\boldsymbol{r}_0 = (x_0, y_0, z_0) = (0, 0, 0)$. Under these initial assumptions, we read from Eq. (8.5.18) after some minor modification

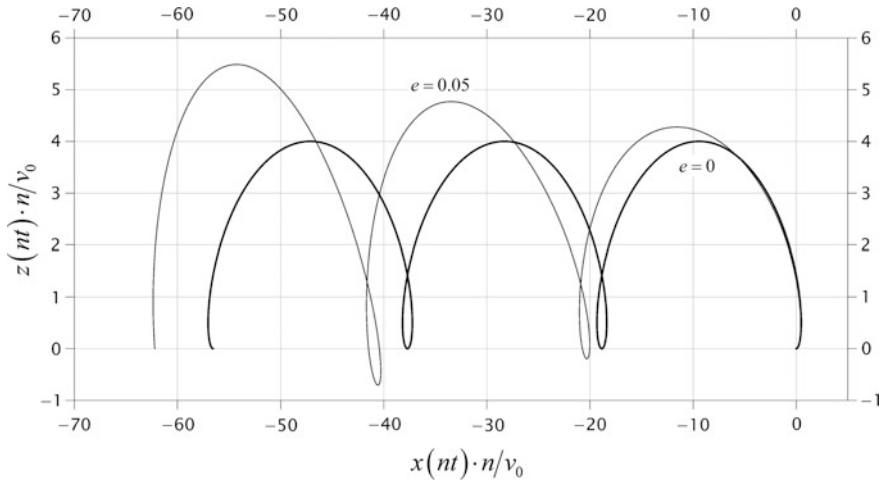


Fig. 8.31 The prolate cycloid as given by Eq. (8.5.19) with $e = 0.05$ and with $e = 0$ for reference

$$\begin{aligned} x(t) &= \frac{v_0}{n}[4\sin nt - 3nt + 3e(2\sin nt\cos nt - nt(1+\cos nt))] \\ \dot{x}(t) &= v_0[4\cos nt - 3 + 3e(2\cos 2nt - \cos nt + nt\sin nt - 1)] \end{aligned} \tag{8.5.19}$$

and

$$\begin{aligned} z(t) &= -\frac{v_0}{n}\left[2\cos nt - 2 + e\left(4\cos^2 nt + 2\cos nt + 3nt\sin nt - 6\right)\right]\dot{z}(t) \\ &= v_0[2\sin nt + e(4\sin 2nt - \sin nt - 3nt\cos nt)] \end{aligned}$$

The Ellipse

As in Sect. 8.5.2 we finally examine the flyaround trajectory with an initial transversal velocity $\boldsymbol{v}_0 = (\dot{x}_0, \dot{y}_0, \dot{z}_0) = (0, 0, v_0)$. In this case we find

$$\begin{aligned} x(t) &= \frac{v_0}{n}\left[2\cos nt - 2 + e\left(3\cos^2 nt - 2\cos nt - 1\right)\right] \\ z(t) &= \frac{v_0}{n}\sin nt[1 + 2e(\cos nt - 1)] \end{aligned} \tag{8.5.20}$$

We eliminate the time dependency by using $\sin^2 nt + \cos^2 nt = 1$ and find the trajectory

$$\left[\frac{x + 2v_0/n[1 - \frac{e}{2}(3\cos^2 nt - 2\cos nt - 1)]}{2v_0/n}\right]^2 + \left[\frac{z - 2ev_0\sin nt(\cos nt - 1)/n}{v_0/n}\right]^2 = 1$$

This is also an ellipse, with an center offset by $2e$ as shown in Fig. 8.32.

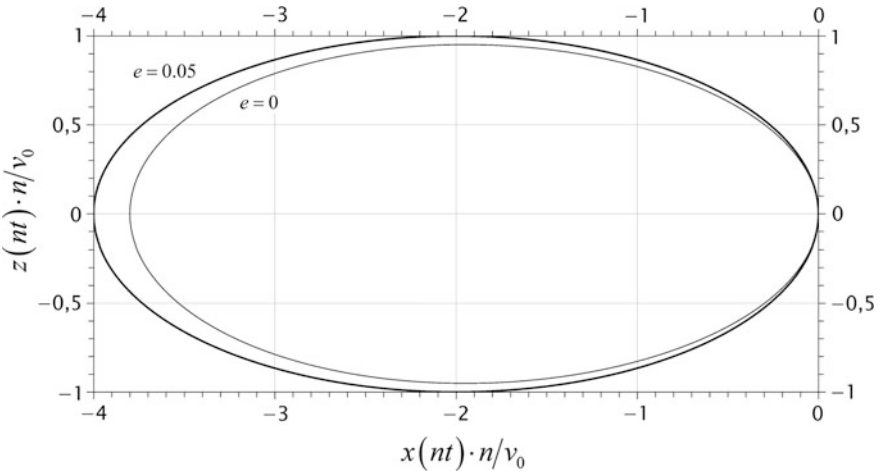


Fig. 8.32 The ellipse as given by Eq. (8.5.20) with $e = 0.05$ and $e = 0$ for reference

8.6 Orbital Rendezvous

Rendezvous and docking (R&D) form some of the most complex maneuvers to be carried out by a spacecraft in spaceflight. To accomplish successful rendezvous and mating of two spacecraft, absolute and relative navigation on orbit, sensing of the target object, precise attitude determination and control, maneuver planning, and the use of highly complex mechanisms must be mastered. The development of these skills was the purpose of the US Gemini program and the early Soviet Soyuz program, and they were further refined in the era of the Space Shuttle and the ISS. In recent years, the desire for autonomous robotic on-orbit servicing systems and plans for more ambitious human and robotic exploration of the solar system create new challenges for technology and mission designers.

The purpose of any R&D mission is to establish physical contact between two or more spacecraft to establish electrical, material, and crew exchange. Therefore R&D is a prerequisite for the construction and maintenance of space stations such as the ISS, as well as any servicing mission to satellites in Earth orbit. Such missions can be manned (e.g., the Hubble servicing missions) or unmanned (e.g., future On-Orbit Servicing). R&D is also a mission-enabling skill in human and robotic exploration of the solar system. Without successful R&D, the Apollo missions would not have been possible.

As the established term *rendezvous and docking* implies, the involved operations are divided into two distinct parts, each with a particular set of goals.

Rendezvous: During the rendezvous part of the mission, the involved spacecraft are guided to meet in the same volume of space, at the same time. In most applications, the target object (also often referred to as *resident space object*) is inert, and the

interceptor (a.k.a. chaser) performs all maneuvers to meet the position and time requirements. However, as an exception to this rule, in the so-called *control box rendezvous*, it is the target spacecraft that executes a number of maneuvers to meet the interceptor after it was launched. This reduces interceptor vehicle propellant consumption, naturally at a cost to the target. It therefore can only be performed with targets having orbit maneuvering capabilities, which usually rules out space stations and a large number of satellites. Nonetheless, this type of rendezvous was performed on some Space Shuttle missions (e.g., STS-49 to service Intelsat VI) and was also planned for the contingency rescue mission STS-400 to Atlantis' Hubble Servicing Mission 4.

Docking: The goal of docking in a mission is to establish physical contact between the involved spacecraft. Although commonly the term *docking* is used, there actually exist two distinct cases: *docking*, and *capture and berthing*. In

(1) *Docking*. The interceptor approaches the target with non-zero relative velocity, brings its docking tool into alignment with the target's counterpart, and establishes a firm structural connection by using its own momentum. Docking therefore relies only on maneuvering capabilities of the two spacecraft and on properly functional docking tools. This approach was used during Gemini and Apollo and still is in use with Soyuz/Progress, the Space Shuttle, and ESA's Automated Transfer Vehicle (ATV) missions to ISS.
(2) *Capture and berthing*. The interceptor is maneuvered into close proximity of the target and an initial mechanical connection between both is established by a robotic manipulator. This manipulator can be either situated on the interceptor (as is the case with the Shuttle Robotic Manipulator System (RMS) used for capturing the Hubble Space Telescope), or on the target vehicle. This is the approach taken with the Space Station Remote Manipulator System (SSRMS) on ISS capturing the Japanese H-II Transfer Vehicle (HTV). After capture, the captured spacecraft is then moved by the manipulator to a berthing position, which is a device similar to a docking port.

The choice of the manipulator's location is, on the one hand, dependent on the sizes and masses of the spacecraft. The attitude of a heavier spacecraft is less influenced by the disturbance torques caused by the movement of the manipulator. On the other hand, a space manipulator is a very complex and hence an expensive mechanism. It will therefore be mounted on the spacecraft with the longer lifetime and/or reentry capability, and not on the disposable spacecraft like HTV.

The combination of both rendezvous and docking is not an end to itself, but serves to fulfill the purpose of a mission. It therefore must always be carefully planned and designed within the larger mission context. This influences not only design decisions such as launch windows, approach trajectories, and the selection of sensors, but also the general approach modes, be it operator-in-the-loop or autonomous robotic. To understand the challenges, choices, and trade-offs involved, the following section provides details about R&D mission design.

Mission Phases

Generally, R&D missions involve both an interceptor spacecraft, which begins the mission on the launch pad, and a target spacecraft, which is usually already in orbit by the time of the interceptor's launch. The following sections will discuss the mission phases of typical R&D missions in Low Earth Orbit (LEO), which are in the following order:

1. Launch
2. Phasing
3. Homing
4. Closing
5. Final Approach
6. Docking/Capture

This sequence of mission steps results in a typical R&D mission profile as shown in Fig. 8.33.

This R&D mission profile typically takes two days to accomplish. This is why it is sometimes called a *2-day rendezvous profile*. It is a standard profile used for R&D missions in particular for ISS R&D by Soyuz, Shuttle or ESA ATV.

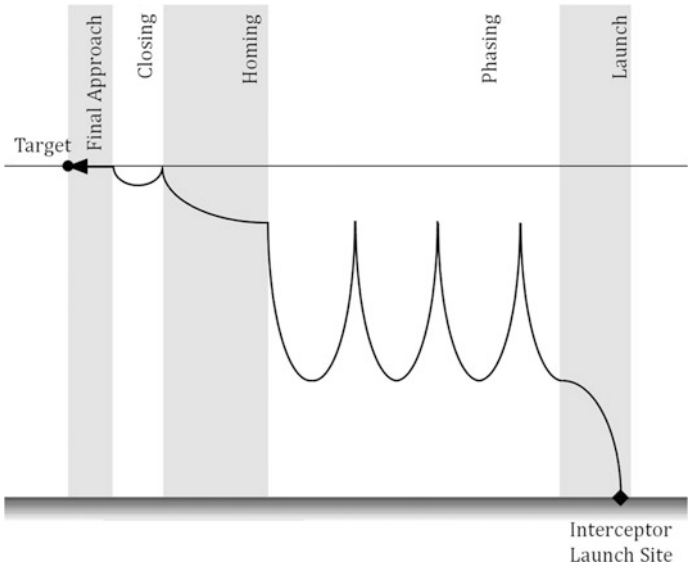


Fig. 8.33 Sketch of a typical R&D mission profile consisting of launch phase, phasing maneuvers, homing, and close-range rendezvous (closing and final approach) including docking

8.6.1 Launch Phase

The launch phase comprises the injection of the interceptor into the orbital plane of the target, as well as achieving stable orbital conditions. To directly meet the plane of the target, the interceptor must be launched inside a narrow launch window. This launch window is derived in the following steps.

First, we derive the launch azimuth φ, which is the angle between the launch trajectory and the geographic North, i.e., the local direction in which to launch. If we denote the target's orbit inclination as i and the launch site latitude as β, we have from Fig. 8.34 and according to Napier's rules for spherical angles $\cos i = \cos\beta \cdot \sin(180^\circ - \varphi) = \cos\beta \cdot \sin\varphi$ from which we obtain the following two solutions:

$$\begin{aligned} \varphi_1 &= \arcsin\frac{\cos i}{\cos\beta} \quad @\ \text{ascending pass} \\ \varphi_2 &= 180^\circ - \arcsin\frac{\cos i}{\cos\beta} \quad @\ \text{descending pass} \end{aligned} \tag{8.6.1}$$

The reason is, a launch site passes twice a day through a given orbital plane: Once on the ascending pass of the target orbit with launch azimuth φ_1, the other on the descending pass with launch azimuth φ_2. So, if for a launch site the launch azimuth is not limited there are two launch opportunities every day.

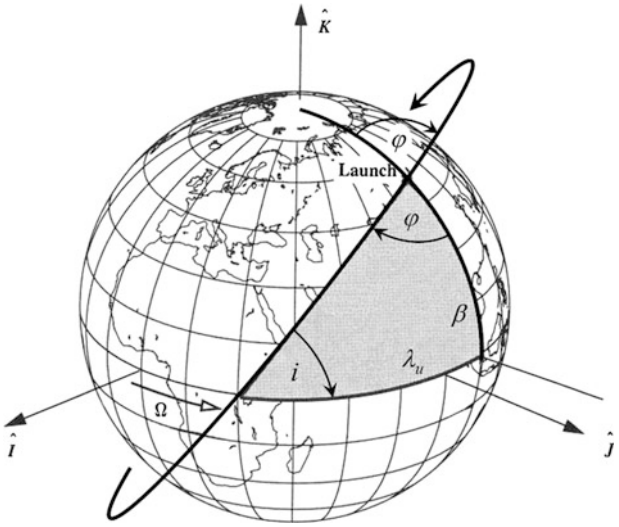


Fig. 8.34 Launch window trigonometry. Illustrated are the target orbit with RAAN Ω and inclination i, launch site latitude β, launch azimuth φ, and the auxiliary angle λ_u. *Credit* Vallado (2007)

From $\cos i = \cos \beta \cdot \sin \varphi$ we can read that because $|\sin \varphi| \leq 1$, we have $\cos i \leq \cos \beta$ implying $i \geq \beta$. So there exists no launch azimuth to achieve an $i < \beta$, or in other words: Orbits with $i < \beta$ cannot be reached directly. This seemingly paradoxical situation is elucidated in Fig. 8.35. For example, launchers from Kennedy Space Center with $\beta = 28.47°$ cannot reach orbits with $i < 28.47°$. Only in the limiting case $\varphi = 90°,\ 270°$, i.e., if the launch azimuth is East or West (into a retrograde orbit), we obtain $i = \beta$. If orbits $i < \beta$ need to be reached, the launcher is first launched with $i = \beta$. When its orbit intersects the desired target orbit having $i < \beta$, a plane change maneuver (a.k.a. dogleg maneuver) is performed that, however, is associated with large propellant consumption and thus reduced payload for a given target orbit.

What is the right time to launch? The Universal Time of launch, T_{UT}, defines the moment in time when the launch site is in the plane of the International Space Station's orbit, the so-called *in-plane time*. At this launch time the interceptor reaches the targeted orbital plane with the least effort (delta-v). It is given by

$$T_{UT} = \frac{\theta_{GMST} - \theta_{GMST0}}{\omega_{\oplus}} \quad \textbf{in-plane time (launch time)} \tag{8.6.2}$$

where $\omega_{\oplus} = 7.2921150 \times 10^{-5}\ \mathrm{s}^{-1}$ is Earth's sidereal rotation rate $\omega_{\oplus}$. The hour angle θ_{GMST}, representing the Greenwich Mean Sidereal Time (GMST) for the launch to occur at, is calculated as

$$\theta_{GMST} = \Omega + \lambda_u - \lambda$$

where the auxiliary angle λ_u (see Fig. 8.34), which is the difference between the launch site longitude λ and the target orbit is Right Ascension of Ascending Node (RAAN) Ω, is calculated from Napier's rules as

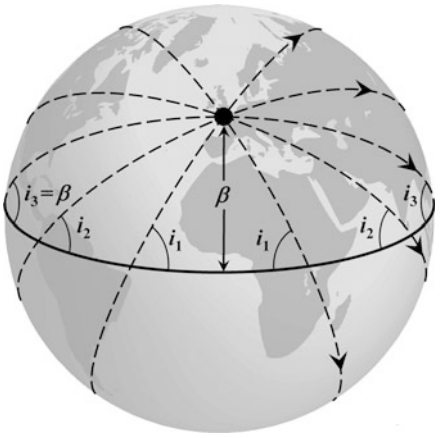


Fig. 8.35 Whatever the launch azimuth is, the accessible orbit inclination i is always larger than the latitude of the launch site β. If the launch azimuth is $\varphi = 90°$ then $i = \beta$ is achieved

$$\lambda_u = \arccos \frac{\cos\ \varphi_{1,2}}{\sin\ i} = \pm\ \arcsin \frac{\sqrt{1 - (\cos\ i/\cos\ \beta)^2}}{\sin\ i}$$

The required hour angle θ_{GMST0} of a particular launch day at hour 00:00 h can be obtained from the Astronomical Almanac of the given year or from

$$\begin{aligned}\theta_{GMST0} =& 100.4606184^\circ + 3600.77005361^\circ\,\mathrm{s}^{-1} \cdot T_{UT1} \\ &+ 0.00038793^\circ\,\mathrm{s}^{-2} \cdot T_{UT1}^2 - 2.6 \times 10^{-8}\ ^\circ\,\mathrm{s}^{-3} \cdot T_{UT1}^3\end{aligned}$$

The term T_{UT1} denotes the number of *Julian centuries* of the launch day at 00:00:00 h elapsed since the *standard epoch* J2000, which is given as

$$T_{UT1} = \frac{JD - 2\ 451\ 545.0}{36\ 525}$$

where in turn the Julian date JD of the given launch day at 00:00:00 h, which should be provided in terms of years $\langle yyyy \rangle$, months $\langle mm \rangle$, and days $\langle dd \rangle$, is computed by

$$\begin{aligned}JD = 367 \cdot \langle yyyy \rangle - INT\left\{1.75 \cdot \left[\langle yyyy \rangle + INT\left(\frac{\langle mm \rangle + 9}{12}\right)\right]\right\} \\ + INT\left(\frac{275}{9}\langle mm \rangle\right) + \langle dd \rangle + 1\ 721\ 013.5\end{aligned}$$

where the function $INT(x)$ truncates the real number x to the next lower integer number.

Launch Windows

Any deviation from the in-plane launch time would implicate a dogleg maneuver and hence an additional propulsion demand for a plane-change maneuver (see Eq. (8.1.22) and Fig. 8.4 for details). On the other hand, since this precision is impractical to achieve both for organizational and technical reasons, one allows a small amount of time for the launch of the interceptor on both sides of the ideal launch time T_{UT}. This is the so-called **planar launch window**. The width of the planar launch window depends on the launch azimuth. If the launch azimuth is $\varphi \approx 90^\circ$, i.e., if the inclination matches the launch site latitude, the launch window is typically up to 1 h because no later plane adjustment would be necessary. With decreasing launch azimuth any deviation of the launch time from the in-plane time will cause increasing plane differences and hence an increasing propulsion demand to correct them. For Shuttle launches to the ISS ($i = 51.6^\circ$) the planar launch window decreases to only 10 min. Targets above 57° inclinations have planar launch windows of less than 5 min. In missions to ISS, in practice the Shuttle launch time is appointed to the opening of the launch window, i.e., 5 min before the in-plane time. In case of a launch delay due to a possible countdown problem there

remain 10 min to fix the problem. If no problem occurs, the Shuttle is put on hold for 5 min and is launched on in-plane time.

Apart from orbital mechanics there are other restrictions defining other kinds of launch windows on different time scales. An important consideration is the *sun angle*, which is the angle between the direction to the Sun and the targeted orbital plane. The sun angle is important for visibility conditions during final approach (cf. Fig. 8.45 where the Shuttle needs to see the ISS during daytime conditions, in particular during final approach) and for solar power generation of the docked spacecraft. For the Space Shuttle another concern is the ability to monitor its ascent and to visually check the external tank for damages during launch and ascent.

When launching toward ISS, traffic conditions also impact launch window planning, since multiple spacecraft such as Soyuz, Progress, ATV, HTV, or Shuttle want to approach the station, during beneficial lighting conditions. All these factors must be considered in mission planning, which leads to the small number of launch windows available to ISS or Hubble per year. This is the reason why the failure to launch a mission during the originally intended window can cause launch delays of months, instead of hours or days as dictated by orbital mechanics. Also the so-called *phase window* (see next section) restricts the launch opportunities.

For a given launch site the range of permitted launch azimuths is usually restricted due to safety concerns of flying a launch vehicle over densely populated areas. For instance, the launch azimuth restrictions at Kennedy Space Center are $35^\circ \leq \varphi \leq 120^\circ$, where the lower bound is due to the US West Coast and the upper bound to the Caribbean islands. This restricts the Shuttle to have only one launch opportunity per day to the ISS.

8.6.2 Phasing

After successful completion of the launch phase, the interceptor spacecraft achieves a stable orbit within the same plane as the target. The two orbits are thus coplanar and typically near circular. (Alternatively, the interceptor is in a plane from which the target plane can be reached within the capabilities of the orbital maneuvering system of the interceptor.)

However, the target might be anywhere on its orbit. Therefore, the first part of the target rendezvous, the so-called *far range rendezvous*, requires first a reduction in the distance to the target, until it can be acquired by the sensors of the interceptor, and then a transfer to a stable holding point on the trailing side of the target.

This first part of far range rendezvous phase is called *phasing* because it is to reduce the so-called *(orbital) phase angle* ϑ , which is the difference in true anomaly as measured in the flight direction from the target to the interceptor. Phasing is typically conducted in absolute navigation, i.e., with reference to an inertial reference frame, and guided by ground control. As an example, Space Shuttle phasing maneuvers are planned by mission control using orbit determination data obtained by ground radar and *Tracking and Data Relay Satellite System*

(TDRSS) Doppler measurements. By processing both the target and interceptor tracking data, the orbital phase angle is determined.

Usually, the interceptor needs to chase the target, so only negative initial phase angles are permitted. This is why the interceptor is often called *chaser*. In addition, only certain initial phase angles are permitted. Owing to performance limitations and constrained crew activities this so-called **phase window** (window accounting for phase angle makeup capability) varies for the Shuttle between $40° \leq -\vartheta_i \leq 360°$. The relationship of the phase window to the planar launch window changes each day and depends on the target's orbital period and inclination.

Owing to a given initial phase angle to be made up, the interceptor will have finalized its launch trajectory on a generally slightly elliptic or circular orbit with its semi-major axis a_I smaller than the a_T of the target orbit. Whether the initial apogee coincides with the target orbit depends on the particular rendezvous strategy, but usually the interceptor orbit nowhere crosses the target orbit. Because $a_I < a_T$ the orbital frequency (mean motion) is larger than that of the target, which implies that the orbital phase angle is reduced continuously—the goal of phasing.

How much is the phase angle reduced in course of one orbital revolution? Assuming small differences in the semi-major axes, which is a quite good approximation for LEO and since $n = \sqrt{\mu/a^3} = 2\pi/T$ of the target the phase reduction after one orbital revolution is

$$\Delta\vartheta = -\Delta n \cdot T = -\left(\frac{dn}{da} \cdot \Delta a\right) \cdot T = \frac{3}{2}\frac{n}{a} \cdot \Delta a \cdot T$$

Therefore

$$\Delta\vartheta = 3\pi\frac{\Delta a}{a} \text{ per orbit} \tag{8.6.3}$$

So, the phase angle from the interceptor (behind the target) to the target decreases if $a_I < a_T$, i.e., if $\Delta a = a_I - a_T < 0$. This phase angle reduction relates to a closing distance of

$$\Delta x = \Delta\vartheta \cdot a = -3\pi \cdot \Delta a \quad \text{per orbit}$$

To give an example, the ISS orbits Earth at an altitude of about $h = 350$ km. Owing to drag becoming too excessive, the lowest chaser altitude is limited to about $h = 250$ km. This implies $\Delta a \geq -100$ km.

So, for instance the Soyuz is able to approach the ISS typically with $\Delta\vartheta_{\max} = 7°$ per orbit equaling $\Delta x_{\max} = 820$ km per orbit. To limit the physical and fatigue stress to astronauts in the small S/C one typically sets a limit of two days to get to ISS docking. Accounting also for the first checkout orbit, the homing, closing, and final approach phases, the number of Soyuz phasing orbits are typically limited to 21 orbits equaling a maximum initial phase angle of $\Delta\vartheta_i \leq 150°$. This of course has a

grave impact on the possible launch windows. To ease this problem and to even further reduce the stress to astronauts the trajectory department of RSC Energia in March 2013 implemented a new rendezvous profile for Soyuz TMA-08M and for later Soyuz missions, where the phasing orbits were reduced to 5 orbits or less (typically 4), which has become known as the *fast rendezvous profile* a.k.a. *four-orbit rendezvous profile*. This fast profile was tested first on the unmanned ISS supply mission Progress M-16M in August 2012, while it was first applied to the manned mission Soyuz TMA-08M in March 2013 and to all manned Soyuz mission since then. This new 4-orbit rendezvous profile implies and initial phase angle of $\Delta\vartheta_i 15^\circ - 20^\circ$, which is achieved by adjusting the ISS orbit raising maneuvers months before launch day such that the ISS falls within this initial phase angle range.

8.6.3 Homing Phase

The interceptor is now on a phasing orbit about 50 km away from the target drifting slowly toward the target. The objective of the upcoming homing transfer, the second part of the far range rendezvous, is to transfer the interceptor to a stable holding and aiming point in the vicinity of the target (see Figs. 8.33 and 8.36). A prerequisite of the transfer is that the target must be acquired by the relative navigation sensors of the interceptor. For the Space Shuttle, mission control hands over rendezvous guidance to the orbiter's crew at 74 km from the target. At this point a target like the ISS can be tracked using star trackers or radar.

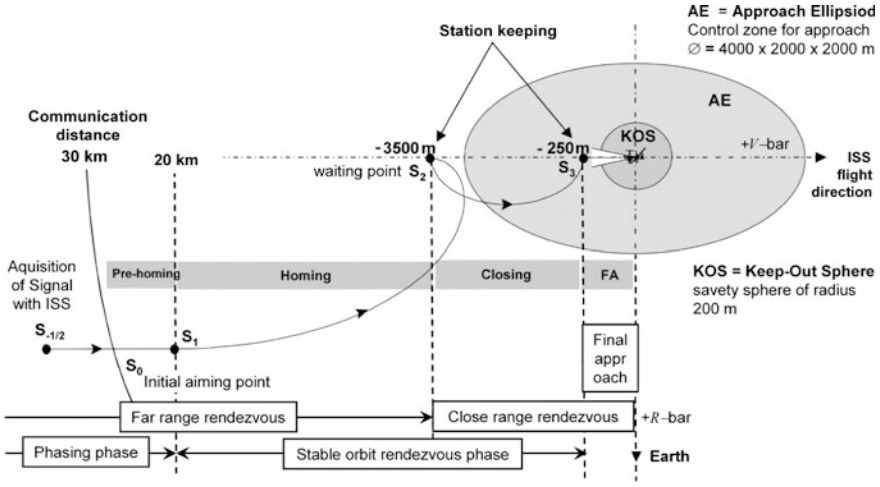


Fig. 8.36 Homing, closing, and final approach profile and phases for ISS rendezvous. ISS safe-approach procedures require station-keeping points S on the V-bar, an approach corridor, and a keep-out sphere around ISS that approaching spacecraft must use. *Credit* Wartenberg and Amadieu/ESA

With the homing maneuver also the relative approach velocity must be reduced to a safe level. In addition, the dispersions in position, orientation, and angular rate must be reduced to meet the conditions required for the upcoming close range rendezvous. This includes the synchronization of the motion timeline of the two spacecraft.

LVLH Reference Frame

For the discussion of the now following rendezvous approaches, the *Local Vertical Local Horizontal* (LVLH) reference frame is defined (see Fig. 8.37). The origin of LVLH is located at the center of mass of the target. Its $+x$-axis, also called the +V-bar, points along the target's velocity vector. The $-z$-axis, referred to as +R-bar, points antiparallel to the target's radial vector. The $-y$-axis, also called +H-bar, completes the right-handed system and thus points along orbit normal. In the following a "+V-bar approach" means that the interceptor approaches the target on the target's +V-bar (into −V-bar direction). Accordingly, a "−R-bar approach" is on the target's −R-bar (into +R-bar direction), etc.

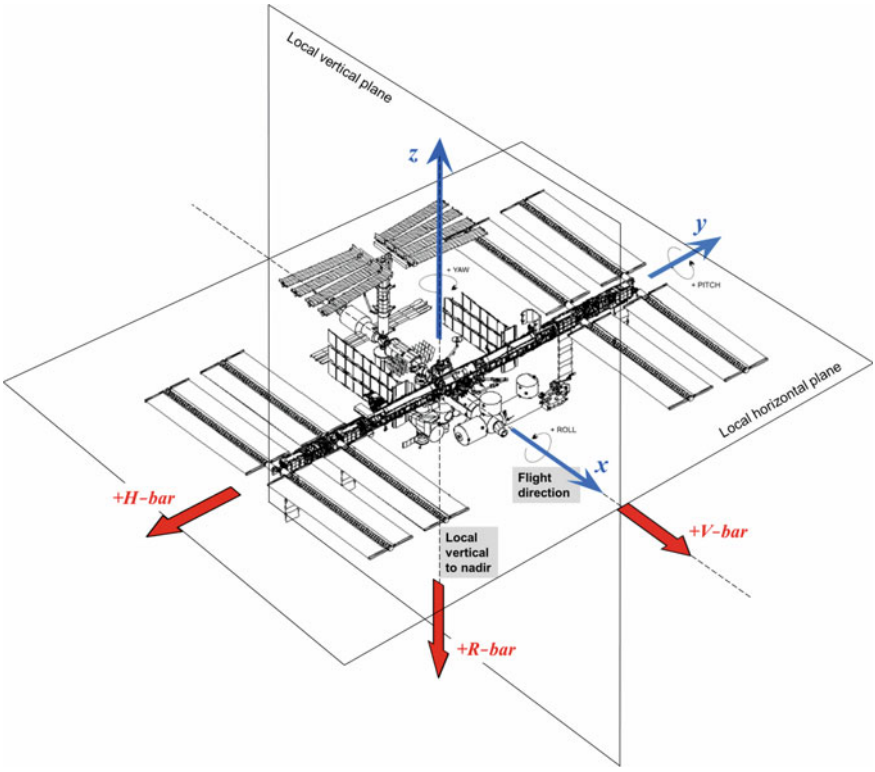


Fig. 8.37 Local vertical local horizontal (LVLH) reference frame: +V-bar ($+x$-axis) is in the direction of the spacecraft's velocity vector, +R-bar ($+z$-axis) is in the direction of the negative radius vector, and +H-bar ($-y$-axis) completes the right-handed system

Homing Transfer

The homing transfer, which commences at point S1 in Fig. 8.36, is a classical Hohmann transfer as described in Sect. 8.3. The principle situation is shown in Fig. 8.38, where the orbits, however, are not to scale because at the end of the phasing phase, the two orbits with $a \approx 6750\,\text{km}$ in LEO have a radial distance of typically only about 10 km. Owing to this, the phase angle at the beginning of the Hohmann maneuver is practically zero while its complement to 180°, the so-called *lead angle*, is $\alpha_L = 180° - \vartheta \approx 180°$.

Let ϑ_i be the initial phase angle and let ϑ_f be the final phase angle at S2 behind the target. The key question is: At a given a, Δa, and ϑ_f, what is the right ϑ_i and the right tangential kick-burn Δv to get to S2? To find an answer we apply from Sect. 8.3.2 the essential results of a Hohmann transfer to adjacent circular orbits as for rendezvous orbits

$$a_H = \frac{1}{2}(a_I + a) = a\left(1 + \frac{\Delta a}{2a}\right)$$

$$t_H = \pi\sqrt{\frac{a_H^3}{\mu}}$$

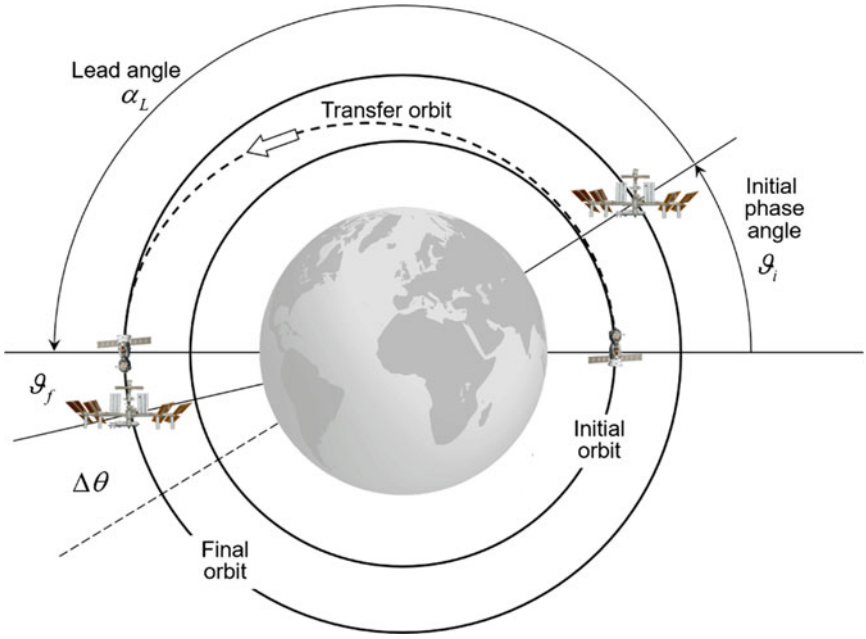


Fig. 8.38 General homing transfer of an interceptor (Soyuz) to meet a target point behind a target (ISS) in a circular coplanar orbit. Initially the required phase angle is ϑ and the interceptor begins the phasing maneuver by applying a thrust. After a half revolution on the transfer orbit, the interceptor reaches the target point, where a thrust maneuver brings the interceptor to a steady relative position behind the target and thus finalizes the Hohmann transfer

Now, while the interceptor transits over an orbit angle of 180° on the Hohmann transfer orbit, the target with $n = \sqrt{\mu/a^3}$ covers the orbit angle (see Fig. 8.38)

$$180^\circ - \Delta\theta = n \cdot t_H = 180^\circ \cdot \left(\frac{a_H}{a}\right)^{3/2} = 180^\circ \cdot \left(1 + \frac{\Delta a}{2a}\right)^{3/2} \approx 180^\circ \cdot \left(1 + \frac{3}{4}\frac{\Delta a}{a}\right)$$

Because in LEO $\Delta a/a \approx 10^{-3}$, we could safely neglect terms of higher order. Therefore, we have for the initial phase angle (see Fig. 8.38)

$$\vartheta_i = \vartheta_f + \Delta\theta = \vartheta_f - 180^\circ \cdot \frac{3}{4}\frac{\Delta a}{a}$$

or

$$\vartheta_i = \vartheta_f - 135^\circ \frac{\Delta a}{a} \tag{8.6.4}$$

For the required delta-v, we find with a circular orbital velocity $v = \sqrt{\mu/a}$

$$\Delta v = v_I - v_T = v_T \left(\sqrt{\frac{a}{a + \Delta a}} - 1\right) = v_T \left(\sqrt{1 - \frac{\Delta a}{a}} - 1\right)$$

and hence

$$\Delta v = -\frac{1}{2}\frac{\Delta a}{a}\sqrt{\frac{\mu}{a}} \tag{8.6.5}$$

Example

The Shuttle shall perform a homing maneuver from $\Delta a = -10\,\text{km}$ to a waiting point $S2 = 3\,\text{km}$ behind the ISS at $h = 350\,\text{km}$.

We have $\vartheta_f = (180^\circ/\pi) \cdot (10/6728) = 0.085^\circ$ and $v_T = 7.697\ \text{km s}^{-1}$. With this we get $\vartheta_i = \vartheta_f + 0.201^\circ = 0.286^\circ$. This means the Shuttle has to perform a burn with $\Delta v = 5.72\ \text{m s}^{-1}$ into flight direction at a position 33.6 km behind the ISS. Because the Shuttle initially is 10 km below the ISS, the viewing distance to the ISS at burn is $\sqrt{10^2 + 33.6^2} = 35.0\,\text{km}$

NASA's Space Rendezvous History

At NASA there exist two general approaches for homing and closing maneuvers: The historical *coelliptic rendezvous* and today's *stable orbit rendezvous* (SOR).

Coelliptic Rendezvous

Coelliptic orbits are coplanar elliptic (including circular) orbits with a common occupied focus (see Fig. 8.39). The arguments of perigee ω are equal, meaning that the lines of apsides of the orbits are congruent. In addition to this, the differences in perigee and apogee radii are equal. In a spacecraft-fixed reference frame, coelliptic orbits appear as two parallel lines. These particular orbits allow for easy, intuitive,

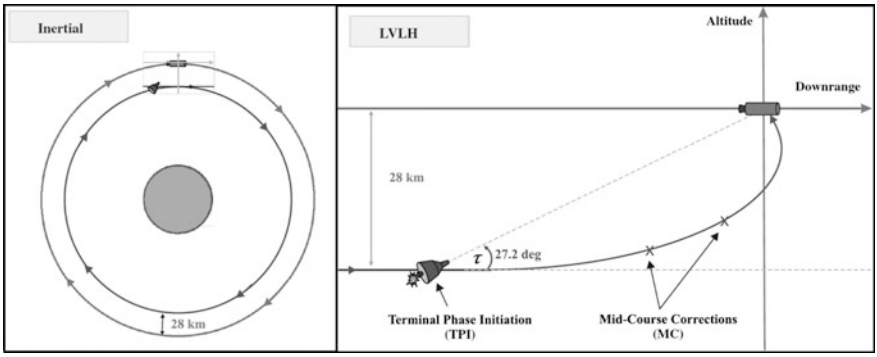


Fig. 8.39 Gemini coelliptic rendezvous in inertial and spacecraft reference frames; τ denotes the trigger angle between the direction to the target and the local horizontal plane at which the transfer maneuver is triggered. *Credit* Woffinden and Geller (2007)

and robust maneuver planning by means of so-called *trigger angle targeting*. This technique was developed during the Gemini program. It allows astronaut pilots to reliably achieve rendezvous by pointing the interceptor spacecraft at the target at a certain trigger angle τ (see Fig. 8.39) relative to the direction of flight and then engaging the orbital maneuvering thrusters, the so-called *Terminal Phase Initiation* (TPI). The trigger angle can be measured with simple cueing devices similar to a sextant. During the transfer, the target will move along its orbit within the so-called *transfer angle* α (the orbit angle covered during transfer). If rendezvous is achieved within a single revolution of the target, thus with $\alpha < 360°$, it is called a *direct rendezvous*. Any case with multiple target revolutions is referred to as *indirect rendezvous*. The point aimed at by the interceptor, in front of or behind the target, is referred to as the *downrange targeting location*.

It can be shown (see Woffinden (2007)), and this is a key property of coelliptic rendezvous, that for direct rendezvous, the elevation trigger angle τ is independent of the angular velocity ω and the relative altitude of the coelliptic trajectory. Hence, the same trigger angle applies for all orbits, regardless of the coelliptic height differential. For indirect rendezvous, the trigger angles show dependency on the ratio between the downrange targeting location and the relative altitude of the coelliptic trajectory. In recent numerical simulations it was determined that the optimal trigger angle for a minimum-Δv intercept maneuver is 27.0°, with a transfer angle of 163.1°. Interestingly, this is not equal to a Hohmann transfer with a trigger angle of 0° and a transfer angle of 180°. In reality, this optimal angle will not be perfectly achievable. A range between 26.8° and 27.3° was therefore identified as providing optimal combinations of required Δv, Line-of-Sight (LOS) approach rates, and positioning accuracy.

Interestingly, without running numerical computer simulations, NASA selected a trigger angle of 27° with a transfer angle of 130° for Gemini/Apollo rendezvous operations. For these manned rendezvous missions trigger angle targeting was particularly attractive. It allowed the use of the astronauts' eyes and simple elevation cueing for maneuver triggering in case of the failure of the

rendezvous radar system. After applying the initial Δv along LOS, the pilot performed one or two mid-course correction maneuvers, before finally approaching the target for docking. During final approach, the pilot benefited from a low inertial LOS approach rate during final breaking and approach, as well as from good visibility of the target against the star background. This was important since LOS closing rates are difficult to judge visually without ambient references.

Coelliptic rendezvous remained NASA's rendezvous approach of choice throughout Gemini and Apollo. Its major strength was the backup capability to perform the TPI burn manually. It was then modified into a *dual coelliptic rendezvous* profile for the Skylab missions. In this profile, the interceptor flew a coelliptic transfer onto a holding orbit below the target. From there another coelliptic maneuver finalized rendezvous. The modifications were applied to improve the final approach lighting conditions for manual piloting, as well as the quality of long-range optical tracking using reflected sunlight. This dual coelliptic profile was then also the baseline for Shuttle R&D missions. Given the characteristics of typical Shuttle rendezvous targets, there existed concerns regarding the usability and quality of optical tracking of small target objects using reflected sunlight in the presence of Earth's illuminated surface and bright celestial objects. Another issue was the depletion of the Shuttle's *Reaction Control System* (RCS) propellant due to high relative approach velocities. This initially led to the adoption of a so-called *tuned coelliptic rendezvous* (TCR) profile.

Stable Orbit Rendezvous (SOR)

For current space station operations all these coelliptic rendezvous approaches were replaced by the so-called *stable orbit rendezvous*. Such a SOR was first flown on Gemini XI and was later suggested to address the concerns over target tracking and propellant consumption for Space Shuttle R&D. The reason is that it supports inertial approaches with lower relative velocity than the inertial approaches from the Apollo legacy coelliptic profile. In addition, a stable orbit profile desensitizes the mission timeline from trajectory considerations, as the interceptor could theoretically remain at the waiting point for indefinite periods of time. Stable orbit station-keeping at multiple kilometers of distance to the target (15 km for Space Shuttle ISS approaches) was also preferable to the close-range (at distances of tens of meters) station-keeping associated with coelliptic approaches. In such close proximity, continuous crew monitoring and frequent correction maneuver are needed, resulting in high propellant expenditure. Therefore, the advantages of SOR profiles over coelliptic approaches are lower propellant consumption and stable station-keeping points on V-bar, leading to less demand on crew position monitoring and correction. Hence, the SOR has become the standard for ISS operations for Space Shuttle, Soyuz and ATV, as well as for other rendezvous operations, such as with the Hubble Space Telescope.

For more details on coelliptic and Space Shuttle R&D missions refer, e.g., to Goodman (2006).

8.6.4 Closing Phase

Once on the target orbit at S2 (see Fig. 8.36) jets are fired to bring the interceptor to a hold at a safe distance about 3 km behind the target. The target now is within range of the interceptor sensors and thus relative navigation can commence. This station-keeping point S2 is essential to assess the situation and plan the upcoming closing maneuver.

The closing maneuver depends on the type of final approach: For a final ±R-bar approach the interceptor needs to get to the ±R-bar below/above the target. For a final +V-bar approach the interceptor needs to proceed further on the −V-bar closer to the target to the station-keeping point S3, and for a +V-bar approach the interceptor has to fly around the target to approach it from the leading end. Anyway, with the closing maneuver we ingress the *Approach Ellipsoid* of the ISS. All operations inside the Approach Ellipsoid are "combined operations" involving the mission control authorities in Houston and Moscow. From here on safety as not to collide with the ISS has the highest priority.

−V-Bar Approach

Let us assume that the docking port is on the trailing end of the target and therefore an approach further on the −V-bar is favorable. What are the options to carry it out? In Sect. 8.5.3 we have seen that we may approach the target on the V-bar either via a prolate cycloid (which is a Hohmann trajectory; see end of section "Ellipse" in Sect. 8.5.3) or via an ellipse. The decision is based on safety versus efficiency: If in course of the prolate cycloid we would lose control over the interceptor we would drift away infinitely and if it is a "flat" cycloid we might even hit the target. However, if safety is not paramount then one could traverse the distance from S2 to S3 by one or several cycles keeping the momentum along the V-bar and hence save fuel. At each reversal point one could even stop the approach, assess the situation, and fine-tune the further approach. This option is chosen by NASA for the Shuttle closing phase (see Fig. 8.45). If safety is top priority then the ellipse trajectory is the choice, because if control over the interceptor would be lost, it would automatically return to S2. However, this method requires to fire the jet at every intersection with the V-bar, thus coming to a halt and thereafter repeat the whole cycle procedure. "Safety-first" requirement of course comes at the expense of a higher propulsion demand.

So, for +V-bar approach of a "safety-first" rendezvous with the ISS as shown in Figs. 8.36 and 8.40 the ellipse maneuver is used. As shown in Sect. 8.5.3, in particular Figs. 8.29 and 8.30, a vertical (radial) burn v_0 will bring us on an ellipse to the next waiting point S3 that lies by $\Delta x = \mathrm{S2} - \mathrm{S3} = 4v_0/n < 0$, with $\mathrm{S2} < 0,\ \mathrm{S3} < 0$, closer to the target. When arriving at S3 a reverse burn must be fired to bring the interceptor to a halt. For the required delta-v for such a step of width Δx we find from Fig. 8.30

$$\Delta v = \pm 2 \times \frac{\Delta x}{4}\sqrt{\frac{\mu}{a^3}} \quad @\ \text{elliptic trajectory} \tag{8.6.6}$$

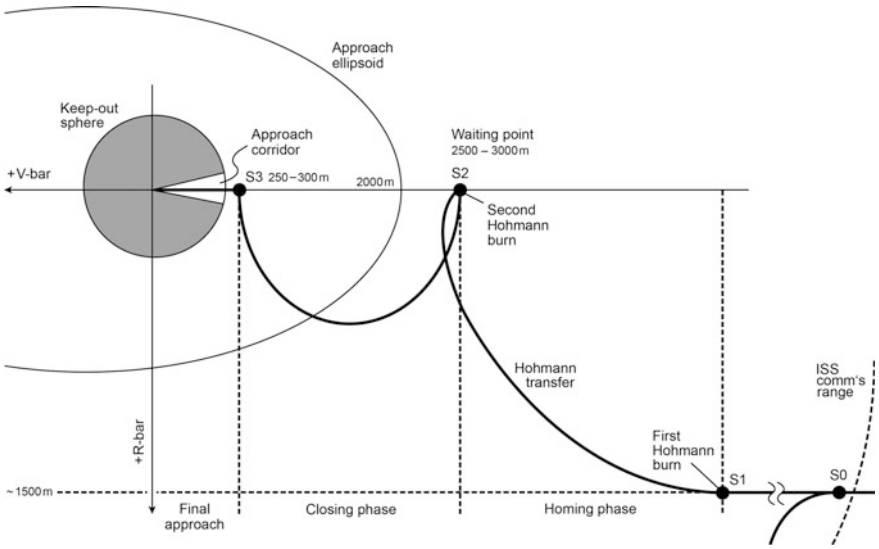


Fig. 8.40 ISS stable orbit approach on −V-bar as typically adopted by an ATV rendezvous. First, a Hohmann transfer brings the interceptor to the waiting point S2. Then, it approaches the ISS on an elliptic trajectory to waiting point S3. *Credit* Fehse (2003) and U. Walter

where the factor 2 indicates that we need two burns for the entire approach maneuver and the ± sign that the elliptic trajectory might be on either side of the V-bar. This maneuver can be performed at any step size and as often as wanted to get to S3. Observe that it does not make any difference for the total delta-v if a given distance is covered with more or less steps because $\Delta v \propto \Delta x$. How many increments are to be chosen is just a matter of safety and time (every step lasts one orbital period). So, if Δx is the total distance between S2 and S3, Eq. (8.6.6) provides the delta-v for the entire closing transfer independent of the number of steps. To approach the ISS, every incremental distance of 1 km requires $\Delta v = 2 \cdot 0.286\,\mathrm{m\,s^{-1}}$.

If the cycloidal approach would be chosen, we have from Fig. 8.27 $\Delta x = 6\pi v_0/n$ and hence

$$\Delta v = \pm 2 \times \frac{\Delta x}{6\pi \cdot k}\sqrt{\frac{\mu}{a^3}} \quad @\ \text{cycloidal trajectory} \tag{8.6.7}$$

where Δx is the total distance between S2 and S3 and k is the number of cycloidal revolutions to traverse this distance. Here again the factor 2 indicates that we need a initiation burn at S2 plus a stop burn of equal absolute value at S3. We therefore see

that the cycloidal approach is by a factor $6\pi \cdot k/4 = 4.7 \cdot k$ more efficient that the elliptic approach—at the expense of safety. We recall that every cycloidal or elliptic cycle takes a full orbital period.

Approach to +V-Bar

If one wants to approach on the –V-bar from S2 to the waiting point S3 on the +V-bar, i.e., to the other side of the target, one has to apply the same maneuvers (elliptic or cycloidal trajectories) as above, the only difference being that for $\Delta x = \mathrm{S2} - \mathrm{S3} < 0$ holds $\mathrm{S2} < 0,\ \mathrm{S3} > 0$.

Approach to ±R-Bar

If the final approach is on +R-bar or −R-bar the closing phase must bring the interceptor in a loop from −V-bar to ±R-bar (see Fig. 8.41) at below/above the target. Here again both the cycloidal and elliptic trajectory would do, but here both approaches are equally safe. This is because if control is lost for a cycloidal trajectory it would just pass by the target at the aimed distance and after that drift away from the target. In the elliptic trajectory case the interceptor would just orbit the target on an ellipse. Because we have seen that the cycloidal trajectory is much more efficient it is the preferred approach method.

The cycloidal loop is performed in two (or one) steps. First, a tangential burn Δv brings the interceptor on a prolate cycloid (Hohmann trajectory) to the transition point S3_ where a burn Δv is fired against the direction of movement. This brings the interceptor into a circular Earth orbit below/above the ISS. On this orbit it slowly drifts to the point S3 below the ISS from where the final approach commences. Note that neither S3_ nor S3 are stable points with respect to the ISS. If the first burn is accurate enough it can be set such that S3_ coincides with S3.

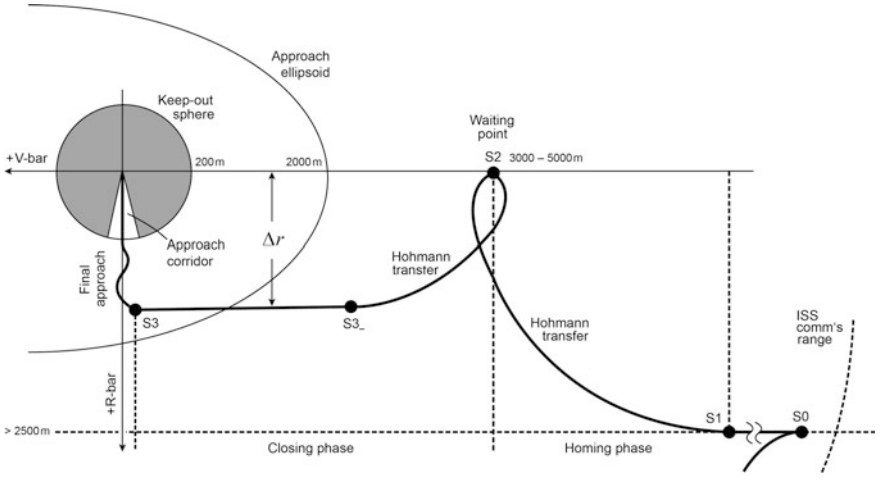


Fig. 8.41 From −V-bar to +R-bar approach to ISS. A Hohmann transfer brings the interceptor first to the intermediate point S3_ where it crosses over into a circular orbit on which it drifts to the final point S3. *Credit* Fehse (2003) and U. Walter

What needs to be determined is: Given the initial point S2 and a radial distance Δr (counted positively outward along the z-axis) the interceptor shall dive below ($\Delta r < 0$) or above ($\Delta r > 0$) the ISS, what is the delta-v to perform the entire maneuver? From Fig. 8.27 we derive that the diving distance is $\Delta r \equiv \Delta z = 4v_0/n$. We therefore find for the initial delta-v

$$\Delta v_i = \frac{\Delta r}{4}\sqrt{\frac{\mu}{a^3}} \quad @\ \text{initial burn} \tag{8.6.8}$$

Upon diving down/up, the interceptor speeds up and according to Eq. (8.5.14) achieves at the lowest/highest point of the trajectory, which is S3$_-$, the relative velocity $\dot{x} = v_0(4\cos\pi - 3) = -7v_0$. This corresponds in an Earth centered system to

$$v_- = \sqrt{\frac{\mu}{a}} - 7v_0$$

At $r = a + \Delta r$, the anticipated circular orbit has an orbital velocity of

$$v_+ = \sqrt{\frac{\mu}{a+\Delta r}} = \sqrt{\frac{\mu}{a}}\left(1 - \frac{1}{2}\frac{\Delta r}{a}\right) = \sqrt{\frac{\mu}{a}} - 2v_0$$

where the latter follows by applying Eq. (8.6.8). For the delta-v of the braking burn at S3$_-$, we therefore obtain

$$\Delta v_f = v_- - v_+ = -5v_0 \tag{8.6.9}$$

So the total delta-v is

$$\Delta v = |\Delta v_i| + |\Delta v_f| = 6 \times \frac{\Delta r}{4}\sqrt{\frac{\mu}{a^3}}$$

Thus, while for the $-$V-bar approach our effort to get to S3 is maximally $\Delta v = 2|v_0|$ (elliptic trajectory), it is much bigger, namely $\Delta v = 6|v_0|$, in the case of an approach to $\pm$R-bar.

8.6.5 *Final Approach*

The interceptor is now on the waiting point S3 just outside of the Keep-Out Sphere, about 200 m away from the ISS (see Fig. 8.42). This is where the *final approach* (a.k.a. *proximity operations or terminal phase*) begins. Other than during the stable orbit rendezvous where the interceptor approached the target via stable orbits (i.e., cycloidal trajectories that can be stopped at the reversal points or elliptic trajectories that even back off from the target if rendezvous control is lost and therefore are "safe" orbits), the trajectories now are more or less straight to directly intercept the

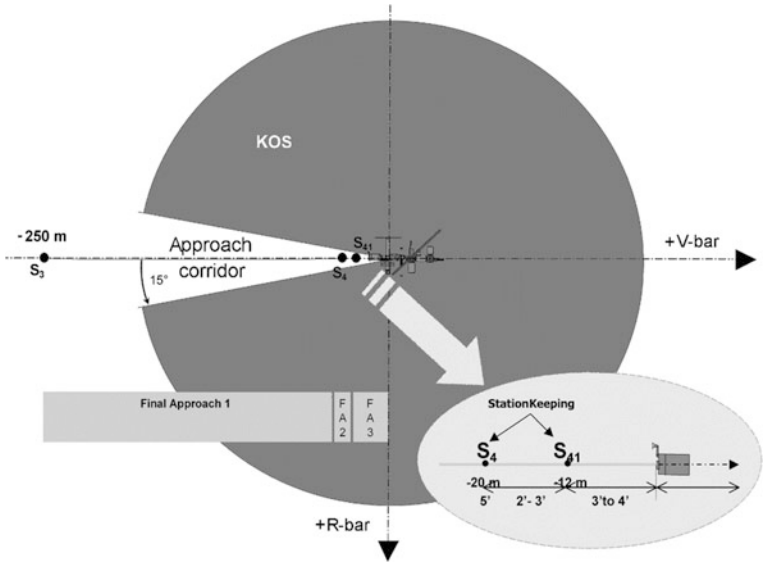


Fig. 8.42 ISS approach corridor for V-bar approach. *Credit* Wartenberg and Amadieu/ESA

target and therefore are on "collision course" with it. The final approach ends at a distance of a few meters upfront the ISS, either when docking is imminent or when the target is within capture distance of the manipulator. During this phase the spacecraft usually maneuvers autonomously, i.e., without intervention by ground control. The spacecraft control loop must therefore be closed locally, either by the crew or by the automatic controllers. Final approach can thus be considered the most critical part of the R&D mission. During this phase, minor errors can cause accidents.

±V-Bar approach

First, we consider the approach on +V-bar or on −V-bar (see definition of "±V-bar approach" in the section *LVLH Reference Frame* above) from waiting point S3 through the conical approach corridor with half angle 15° to the next waiting point S4 (the initial waiting point for docking approach). To penetrate the cone one could continue to apply the ellipse maneuvers. However, every incremental step takes one orbital period of 91.5 min in case of the ISS, which is far too much.

One therefore switches to another approach mode called *forced translation.* Figure 8.43 shows its principle. Let us assume a burn is performed aiming directly at the target. According to Fig. 8.27, the higher speed would force the interceptor to drift upward (centrifugal force is bigger than the gravitational force at this circular altitude), violate the approach cone, and never encounter S4 (see Fig. 8.43). However, we can counteract the updrift by providing in addition to the forward translation an initial small downward force. If this is done properly, it forces the

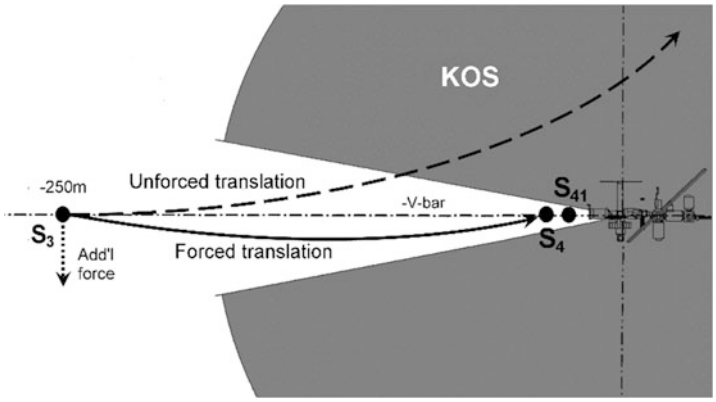


Fig. 8.43 A straight trajectory would turn up and leave the approach corridor. A downward forced translation enforces the trajectory to stay in the corridor and hit the waiting point

initial trajectory slope down just that the trajectory at its end hits S4. The forward translation with this extra little downward force is called *forced translation*.

It needs to be determined how big the additional delta-v, equivalent to the downward force, is. We therefore revisit Sect. 8.5.2 where we have shown that for translation times $(nt)^2 \ll 1$ Eq. (8.5.13) holds. These equations give the answer to the question what the initial velocity $\boldsymbol{v}_0 = (\dot{x}_0, \dot{y}_0, \dot{z}_0)$ at the initial point (x_0, y_0, z_0) should be in order to meet after time t a given target point at the origin $(0, 0, 0)$. In our case the vector from the target point to the initial distance of the interceptor is $(x_0, y_0, z_0) = (\pm\Delta x, 0, 0)$ for a $\pm$ V-bar approach. Therefore, Eq. (8.5.13) reduces to

$$\boldsymbol{v}_0 = \left(v_{0x},\ v_{0y},\ v_{0z}\right) = \left(\mp\frac{\Delta x}{t},\ 0,\ \pm n \cdot \Delta x\right) \quad @\ \pm \text{V-bar approach} \qquad (8.6.10)$$

$$\Delta\boldsymbol{v} = \boldsymbol{v}_0 - \boldsymbol{v}_i$$

where $n = \sqrt{\mu/a^3}$, $\Delta\boldsymbol{v}$ is the initiation burn vector, and $\boldsymbol{v}_i$ is the interceptor's velocity incident of the starting point S3 (if the starting point was a waiting point, $\boldsymbol{v}_i = 0$). Of course, for braking the forced translation at S4 the same amount of Δv is required, however, for v_{0x} into the opposite direction. The initial speed in x-direction, v_{0x}, is easy to grasp: The required speed is distance divided by flight time. The delta-v in negative z-direction (radial thrust downward) is just the forced part. Assuming that the delta-v in each direction is generated by separate thrusters (as usually the case), the absolute value of the total approach delta-v is

$$\Delta v = 2\left(\frac{\Delta x}{t} + n \cdot \Delta x\right) = 2 \cdot \Delta x \cdot \left(\frac{1}{t} + n\right)$$

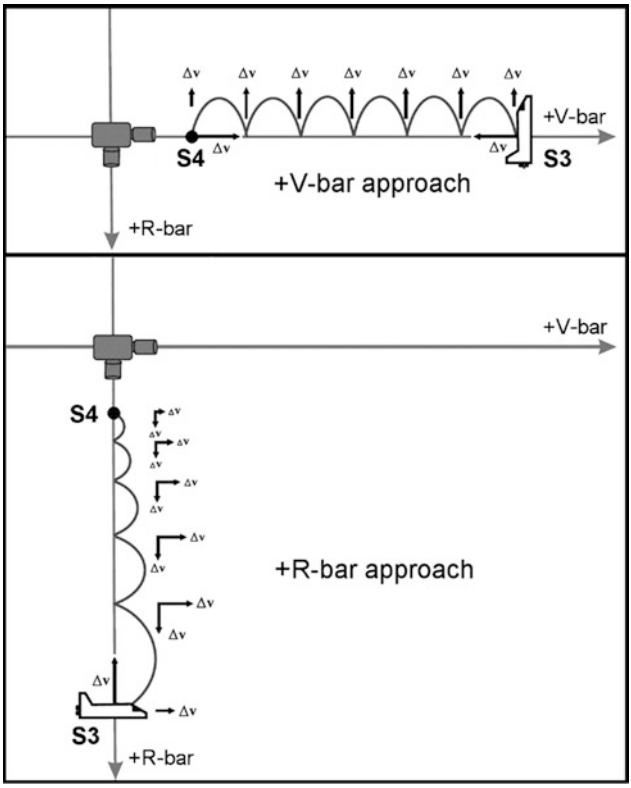


Fig. 8.44 Final approach hopping profiles of a Shuttle final approach along the +V-bar (above) and the +R-bar (below, including deceleration burns). *Credit* Woffinden and Geller (2007)

It would not be wise to cover the final approach distance in one move. Initial point and/or firing errors would jeopardize encountering exactly the target point. One rather splits the distance into two or more parts and whenever the trajectory meets the V-bar a new forced delta-v of $\pm 2 \times n \cdot \Delta x$ is applied starting the forced translation anew. This leads to a hopping approach as depicted in Fig. 8.44 (top) for a Shuttle +V-bar approach. This is why it is called *hopping trajectory*. Because the total delta-v has the linear dependency $\Delta v \propto \Delta x$, it does not make any difference in terms of delta-v effort to split the final approach distance into k shorter hops of length ε, $\Delta x = k \cdot \varepsilon$, but still cover the total distance in the same time t, or not. However, owing to boost errors, it is preferable to make more shorter hops.

±R-Bar Approach

Forced translation can also be applied on the +R-bar or −R-bar, which for instance is performed by a Soyuz or Progress docking to the radial port (Pirs docking compartment) of the ISS. This approach was originally designed for docking the Shuttle with *MIR*. It was also used on Hubble servicing mission STS-82. To

determine the initiating delta-v for the forced translation from S3 to S4 we have from Eq. (8.5.13) with the initial distance vector $(0, 0, \mp\Delta z)$ for ±R-bar approach

$$\boldsymbol{v}_0 = \left(v_{0x},\ v_{0y},\ v_{0z}\right) = \left(\pm n \cdot \Delta z,\ 0,\ \pm\frac{\Delta z}{t}\right) \quad @\ \pm\text{R-bar approach} \tag{8.6.11}$$

$$\Delta\boldsymbol{v} = \boldsymbol{v}_0 - \boldsymbol{v}_i$$

where $n = \sqrt{\mu/a^3}$, $\Delta\boldsymbol{v}$ the initiation burn vector, and $\boldsymbol{v}_i$ is the interceptor's velocity incident to the starting point S3. The total approach delta-v is by the same token as above

$$\Delta v = 2 \cdot \Delta z\left(\frac{1}{t} + n\right)$$

And also in this case owing to the linear dependency $\Delta v \propto \Delta z$ it does not make any difference in terms of delta-v effort to split the distance into more shorter hops. Thus, we might have also hopping trajectories as shown in Fig. 8.44 (bottom).

Proximity Operations

From an orbital mechanics point of view the last few meters starting out from S4, the docking approach in the proximity of the target, is the most easy part of the rendezvous. This is because for very short distances $x,\ z,\ \tau \to 0$ and hence

$$\begin{aligned} v_{0x} &= -\frac{x_0}{t} - nz_0 & v_{0x} &= -\frac{x_0}{t} \\ v_{0z} &= -\frac{z_0}{t} + nx_0 \ \rightarrow & v_{0z} &= -\frac{z_0}{t} \\ v_{0y} &= -\frac{y_0}{t} & v_{0y} &= -\frac{y_0}{t} \end{aligned} \tag{8.6.12}$$

$$\Delta\boldsymbol{v} = \left(v_{0x},\ v_{0y},\ v_{0z}\right) - \boldsymbol{v}_i$$

where $\boldsymbol{v}_i$ is the interceptor's velocity incident to the starting point S4. Therefore, if on this docking approach the commander or pilot navigates the interceptor, steering becomes intuitively easy because the required momentary speed is just distance per time. However, in order not to crash into the docking port the approach speed has to be continuously reduced (see Fig. 8.44, bottom).

Docking/Capture Phase

At the end of the final approach phase, the interceptor is in position in front of the target's docking port or capture interface and all thrusting has ceased. The interceptor's relative velocity is either zero for capture & berthing, or slightly above zero for docking. On one hand, this approach rate must be great enough to prevent the vehicles from bouncing off each other without capture being achieved. On the other hand, it must be low enough to prevent structural damage or loss of control and/or

the ability to attenuate the momentum. The exact position, velocity, orientation, and angular rate tolerances depend on the specific docking or capture tools being used.

The docking/capture phase is the conclusion of a R&D mission. It encompasses the following activities:

- docking/capture of the target by the interceptor (or vice versa)
- establishment of a rigid structural connection
- connection of fluid, gas, electrical, propellant and communication lines
- establishment of a pressurized passageway, if crew transfer is part of the mission goals.

Docking means that the active spacecraft positions itself and establishes the physical connection using its own momentum. In capture and berthing, either the target or interceptor is captured, positioned and connected by a robotic manipulator to a berthing mechanism. Berthing thus allows contact to be made at a near-zero cloture rate, which means a higher level of control for the operator and avoids the process of one vehicle basically flying into the other. It is therefore the generally preferable approach but comes at the cost of requiring a complex, heavy and expensive *Remote Manipulator System* (RMS).

8.6.6 Shuttle-ISS Rendezvous

The Shuttle's close range rendezvous with the ISS is somewhat different, though, because the docking port is on the leading side (+V-bar) of the ISS. Therefore, beginning at S2 the Shuttle flies in two cycloidal steps and within two orbits a homing approach from S2 to S3 such that it goes up just below the ISS (see Fig. 8.45). Note that the launch window was chosen such that daytime (i.e., sunlight) condition is at those parts of the trajectory, including the final approach and docking (see Figs. 8.46 and 8.47), where the ISS needs to be seen from the Shuttle and vice versa. Just before arriving at S3 the commander of the Shuttle takes over manual control for the remainder of the approach and docking. He will stop the Shuttle at S3 some 180 m below the ISS and will maneuver the Shuttle through a 9 min, 360° backflip (a.k.a. *Rendezvous Pitch Maneuver*, RPM) that allows the station crew to take pictures of the Shuttle's heat shield to see whether it was damaged during launch. The Commander then will move the Shuttle at a so-called TORVA maneuver from the +R-bar to the +V-bar in a position about 120 m directly in front of the station in preparation for the final approach to the pressurized mating adapter PMA-2 located at the leading end of the US utility hub *Harmony*.

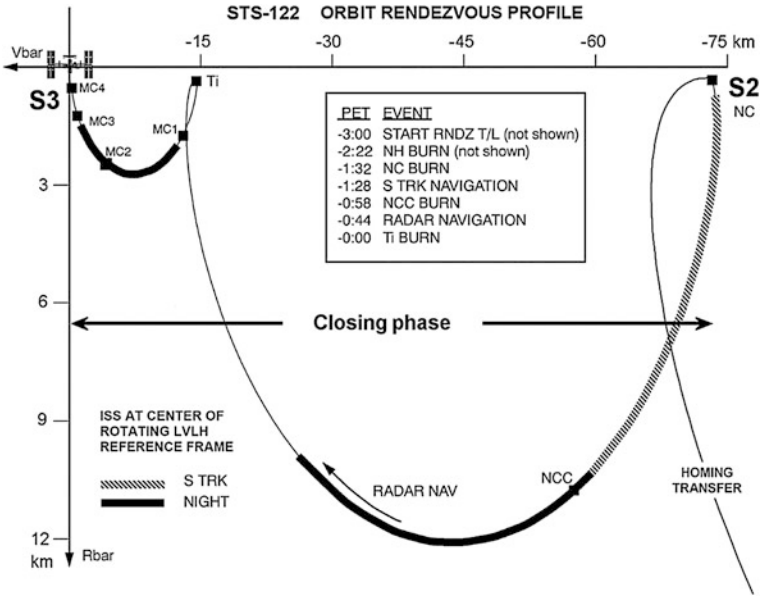
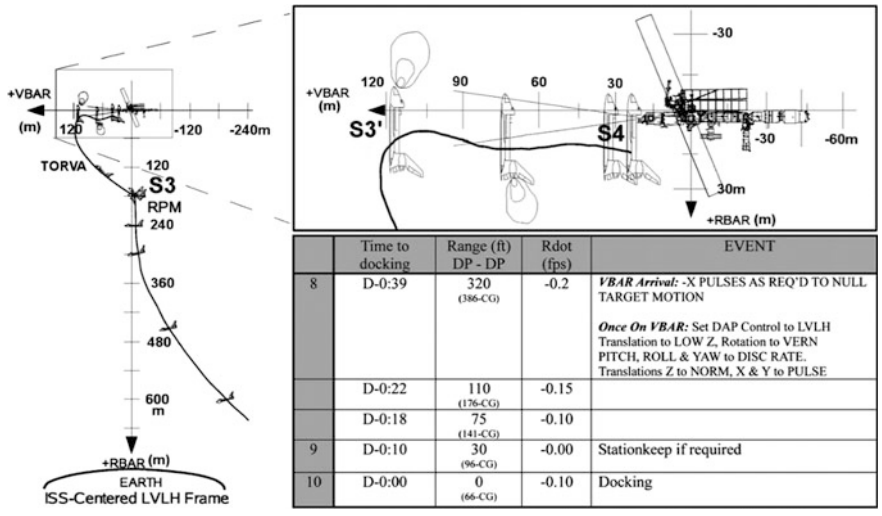


Fig. 8.45 The cycloidal trajectory in the closing phase of a Shuttle STS-122 rendezvous with the ISS. *Credit* NASA



	Time to docking	Range (ft) DP - DP	Rdot (fps)	EVENT
8	D-0:39	320 (386-CG)	-0.2	***VBAR Arrival:*** -X PULSES AS REQ'D TO NULL TARGET MOTION ***Once On VBAR:*** Set DAP Control to LVLH Translation to LOW Z, Rotation to VERN PITCH, ROLL & YAW to DISC RATE. Translations Z to NORM, X & Y to PULSE
	D-0:22	110 (176-CG)	-0.15	
	D-0:18	75 (141-CG)	-0.10	
9	D-0:10	30 (96-CG)	-0.00	Stationkeep if required
10	D-0:00	0 (66-CG)	-0.10	Docking

Fig. 8.46 Final approach trajectory of the Shuttle STS-122 approaching the ISS. *Credit* NASA

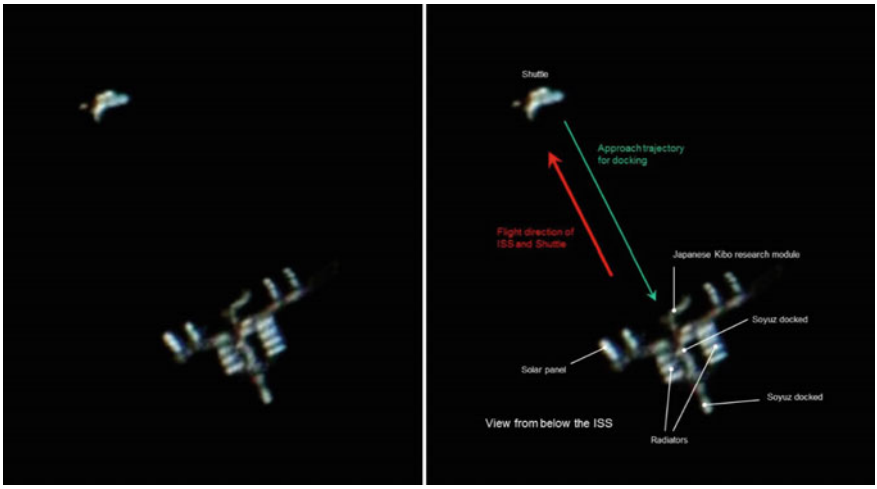


Fig. 8.47 Final approach trajectory of Shuttle STS-133 approaching ISS (left raw, right annotat ed picture). Picture taken with ISO 800, 1/800 s, through a 8.5″ Newtonian on February 26, 2011, 18:35:54 h GMT, over UK. *Credit* Rob Bullen

Acronyms for NASA Rendezvous Maneuvers

NC	(Phasing correction burn) Performed to hit a range relative to the target at future times
NH	(Height adjust burn) Performed to hit a delta-height relative to the target at future times
NPC	(Plane change burn) Performed to remove planar errors relative to the target at future times
NCC	(Corrective combination burn) First on-board targeted burn in the rendezvous sequence to reduce phasing and height errors relative to the target at Ti
Ti	(Terminal intercept burn) Second on-board targeted burn in the rendezvous sequence to place the orbiter on a trajectory to intercept the target in one orbit
MC-1,2,3,4	(Midcourse correction burns) On-board targeted burns to correct the post Ti trajectory in preparation of the final approach phase
RPM	(Rendezvous Pitch Maneuver) A 360° backflip that allows the station crew to take pictures from the Zvezda Service Module of the Shuttle's heat shield
TORVA	(Twice Orbital Rate R-bar to V-bar Approach) This manually performed maneuver brings the orbiter from the +R-bar to the +V-bar

8.6.7 Plume Impingement

The term *plume impingement* covers all effects exerted on the target object if it is impacted by the exhaust gases of the interceptor's reaction control system (RCS, the maneuvering control system of an interceptor) thrusters. One of these effects is the plume pressure force acting on the target and causing position and attitude disturbances. Another is the heat load placed on the target's structure by the hot gases. This can lead to overheating of parts of the surface and the underlying structure. The third effect is the contamination of the target's surfaces by combustion products and unburned propellant components. This can cause contamination of not only sensitive elements on the target's surfaces, particularly optical elements such as camera lenses, solar arrays or docking sensors, but also of sealing elements of the docking mechanism. This risk of contamination must be considered in particular during orbit servicing missions such as Hubble servicing, where these considerations impacted the design of final approach trajectories.

Therefore, plume impingement is one of mission planners' major concerns during proximity operations, apart from collision avoidance and maneuver precision. It can only be avoided if thruster activity near the target is minimized. This in turn means that the interceptor's relative velocity must diminish below a threshold value as it approaches the target. During Gemini and Apollo, plume impingement never became a significant issue due to the thrust magnitude, the position and canting of the RCS nozzles, as well as the roughly equal sizes of interceptor and target and the absence of large appendages such as solar arrays. This changed during the Skylab missions. During Skylab 2, the Apollo *Command and Service Module* (CSM) was maneuvered within close proximity so that a crewman standing in the hatch could reach the stuck solar array with a deployment tool. The CSM thrusting to null the closing velocity triggered Skylab AOCS to fire its jets in order to maintain its attitude. This resulted in an opening rate between the two vehicles. On the later Apollo-Soyuz Test Mission, four of the CSM's thrusters were inhibited 2 s prior to docking contact in order to prevent plume loading of the Soyuz solar arrays.

Space Shuttle

These lessons were carried into the Space Shuttle design process. The massive orbiter is designed to assemble and maintain large space stations and service comparatively small and light satellites. These are equipped with large solar arrays and antennas or sensitive optics. Plume impingement therefore is a prime concern. The size of the Shuttle was predetermined by the payload it was designed to carry and the location of RCS thrusters by its shape, which in turn was determined by the requirements of re-entry and atmospheric flight. Plume impingement concerns could therefore be addressed only by careful design of R&D approaches.

The underlying assumptions were as follows: The target spacecraft could not be designed with features preventing contamination (e.g., movable sensor covers as

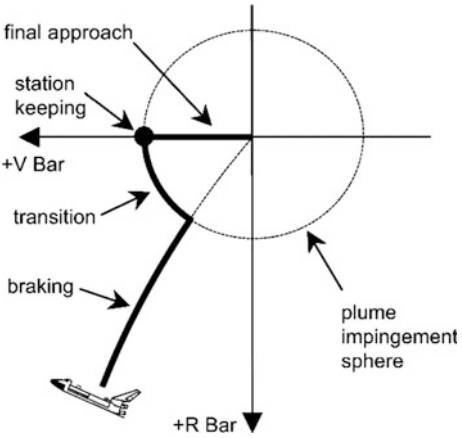


Fig. 8.48 Approach profile to avoid plume impingement on target. When the approaching interceptor reaches the border of a plume impingement sphere, it performs a transition maneuver to a station-keeping point on the V-bar following the borders of the plume impingement sphere. After the station-keeping point is reached, the interceptor flies a forced translation final approach. *Credit* Goodmann (2006)

found on Hubble), and the control of the target attitude could not prevent contamination. Therefore, on each mission a target-dependent minimum range existed, at which the thrusters could still be fired in the direction of the target without contamination concern. At the minimum range, the orbiter was to transition from a direct approach trajectory to a station-keeping point on the V-bar (see Fig. 8.48). From this point on the final approach would be flown in forced translation. A number of such approaches were planned for the Shuttle's *Long Duration Exposure Facility* (LDEF) mission. Simulations showed that an Apollo-type inertial approach and braking technique would cause LDEF to tumble. In addition, plume impingement induced dynamics at grapple ranges that could make both LDEF deployment and retrieval difficult.

Another countermeasure for plume impingement issues specific to the Space Shuttle was the development of the *Low-Z* approach. In this approach, all forward firing RCS jets are inhibited, with all thrust thus acting primarily along the spacecraft's longitudinal x-axis (see Fig. 8.49). All braking thrust in the z-direction therefore results from the canting of the longitudinal thrusters. This provides minimal RCS braking capability while minimizing RCS plume impingement. It is also expensive in terms of propellant use. Notwithstanding its limitations, *Low-Z* mode has been employed on satellite servicing missions, including Hubble servicing, and the missions to MIR and ISS.

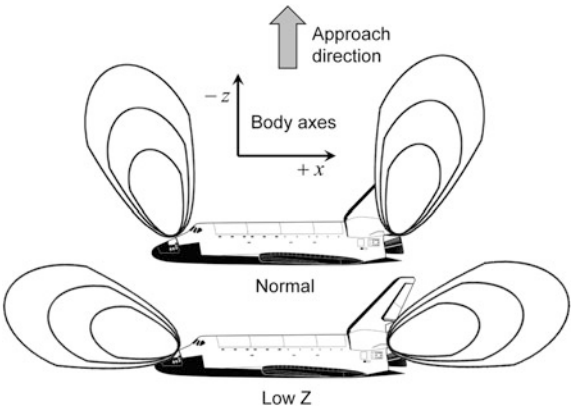


Fig. 8.49 Space shuttle low-Z RCS mode. The orbiter's thrusters are fired only in the Shuttle's longitudinal axis (x-axis). This significantly reduces plume impingement. On the other hand, owing to the canting of the thrusters, this provides minimal braking capability in the forward z-direction

8.7 Problems

Problem 8.1 *Efficiency of Combined Orbital Maneuvers*
Prove the statement in Sect. 8.1.3 that in terms of delta-v an orbit transition with $\boldsymbol{v}_1 || \boldsymbol{v}_2$ combined with a change in orbit orientation $v_1 = v_2,\ \phi \neq 0$ is always more efficient than performing the two changes sequentially at the same point in space.

Problem 8.2 *Adjacent Circular Orbit Approximation*
Prove Eq. (8.3.9)

$$\frac{\sqrt{a_\bullet} + \sqrt{a_O}}{\sqrt{a_H}} - 1 \approx 1 - \frac{1}{16}\left(\frac{a_O - a_\bullet}{a_\bullet}\right)^2 \quad @\ a_O \to a_\bullet$$

Problem 8.3 *Transfer between Aligned Ellipses*
Consider a Hohmann transfer between two coplanar and coaxial ellipses. Show that the propulsion demand for the transition between the periapsis of the inner ellipse and the apoapsis of the outer ellipse is

$$\Delta v = \sqrt{\mu}\left(\frac{1}{\sqrt{r_{\bullet,per}}} - \frac{1}{\sqrt{r_{O,apo}}}\right)\left(\frac{\sqrt{r_{O,apo}} + \sqrt{r_{\bullet,per}}}{\sqrt{a_T}} - \frac{\sqrt{r_{O,apo}}\sqrt{1+e_\bullet} - \sqrt{r_{\bullet,per}}\sqrt{1-e_O}}{\sqrt{r_{O,apo}} - \sqrt{r_{\bullet,per}}}\right)$$
$$\leq \sqrt{\mu}\left(\frac{1}{\sqrt{r_{\bullet,per}}} - \frac{1}{\sqrt{r_{O,apo}}}\right)$$

Remark *For circular orbits this expression passes over to Eq. (8.3.8).*

Problem 8.4 *Hohmann Transfer Maxima*
(a) Prove that the maximum of the circularization impulse $\Delta v_{HO\to O}$ (see Fig. 8.12) is the root of the equation $x^3 - 5x^2 - 5x - 1 = 0$. Find the root $x = r_O/r_\bullet = 5.879362\ldots$ by Newton's method.
(b) By the same token, prove that the propulsion demand of a Hohmann transfer achieves a maximum at $x = a_O/a_\bullet = 15.58172\ldots$, which is the root of $x^3 - 15x^2 - 9x - 1 = 0$.

Problem 8.5 *Hohmann Versus Bi-Elliptic Transfer*
Prove that for $x = r_O/r_\bullet > 11.9387654724\ldots$ the bi-elliptic transfer has a lower propulsion demand than the Hohmann transfer. Show that $x = r_O/r_\bullet$ is the root of the equation $x - 1 = \sqrt{1+x}(\sqrt{x} - \sqrt{2} + 1)$.

Problem 8.6 *Variations of Orbital Elements by Kick-Burns*
(a) Prove Eq. (8.1.1) as described in the text.
(b) Prove Table 8.1 by applying Eq. (8.1.1) at the given positions.

Problem 8.7 *Orbit Phasing*
Suppose two satellites are flying in a close formation on the same orbit at relative distance s and orbital period T. Show that, if s needs to be corrected, a kick-burn $\delta v_{||}$ at the periapsis will cause a position shift of

$$\delta s_{per} = 3T\frac{1+e}{1-e}\delta v_{||}$$

at the periapsis after one orbit, while a kick-burn $\delta v_{||}$ at the apoapsis will cause a position shift of

$$\delta s_{apo} = 3T\frac{1-e}{1+e}\delta v_{||}$$

at the apoapsis after one orbit.

Problem 8.8 *Maximum Transfer Time for Minimum Energy Transfer*
Given the minimum energy transfer orbit between two points P_1 and P_2 having $4a_{\min} = r_1 + r_2 + c$. Show that for any slightly larger transfer orbit with $a = a_{\min} + \Delta a$ the transfer time is

$$\Delta t \approx \Delta t_{\max} - 4a_{\min}\sqrt{\frac{\Delta a}{\mu}} \quad @\ \beta < \pi$$

$$\Delta t \approx \Delta t_{\max} - a_{\min}\sqrt{\frac{\Delta a}{\mu}} \quad @\ \beta \approx \pi$$

and therefore the minimum energy transfer orbit indeed has a maximum transfer time. We will explore the characteristic square root behavior $\delta(\Delta t) \propto -\sqrt{\Delta a}$ near

the maximum transfer time for a Hohmann transfer a bit further in Sect. 9.3.2 before Eq. (9.3.15).

Problem 8.9 *Lambert's Problem for $a \to \infty$*

Prove that from Lambert's Eq. (8.2.13) for $a \to \infty$ follows

$$a = \frac{3}{40} \cdot \frac{\Delta_+^{5/2} - \Delta_-^{5/2}}{6\Delta t\sqrt{\mu} - \left(\Delta_+^{3/2} - \Delta_-^{3/2}\right)}$$

with $\Delta_\pm = r_1 + r_2 \pm \sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos\Delta\theta}$. Since $\sqrt{\Delta_\pm} = 2s\sqrt{c}$ this delivers Eq. (8.2.15).

Problem 8.10 *Solution of the Differential Equations for Elliptic Orbits*
It is the objective of this exercise to solve the differential Eq. (8.5.17). According to the theory of differential equations the general solution is the sum of the solution of the homogeneous differential equations given by Eq. (8.5.8) and a special solution to the inhomogeneous differential equations. We therefore just have to seek for this special solution.

(a) Before doing so we make use of the fact that the right hand side of the differential solution is small, namely of order e. We therefore can insert the homogeneous solutions on the right hand side. Show that the differential equations then read

$$x'' + 2z' = e \begin{pmatrix} \cos M \\ -6M\,\cos M + 8\sin M - 6\sin 2M \\ -3M\,\cos M + 4\sin M - 4\sin 2M \\ -2\cos M - 2\cos 2M \end{pmatrix}^T \begin{pmatrix} x_0 \\ z_0 \\ x_0' \\ z_0' \end{pmatrix}$$

$$y'' + y = -3e \begin{pmatrix} \cos^2 M \\ \sin M\,\cos M \end{pmatrix}^T \begin{pmatrix} y_0 \\ y_0' \end{pmatrix}$$

$$z'' - 3z - 2x' = 2e \begin{pmatrix} -\sin M \\ 6M\,\sin M + 8\cos M + \frac{3}{2}\cos 2M - \frac{9}{2} \\ 3M\,\sin M + 4\cos M + \cos 2M - 3 \\ +2\sin M - \frac{1}{2}\sin 2M \end{pmatrix}^T \begin{pmatrix} x_0 \\ z_0 \\ x_0' \\ z_0' \end{pmatrix}$$

(b) Define the second equation as

$$y'' + y = f(M)$$

The homogeneous differential equation is known to have the fundamental system of solutions $y_{1,2}(M) = (\sin M,\ \cos M)$. Now, according to the theory of differential equations a special solution is given as

$$y^*(M) = -\sin M \int_{M_0}^{M} \frac{f(\mu)\cos\,\mu}{W(\mu)} \cdot d\mu + \cos M \int_{M_0}^{M} \frac{f(\mu)\sin\,\mu}{W(\mu)} \cdot d\mu$$

where $W(M)$ is the Wronskian of the fundamental system of solutions. Show that this delivers

$$y^*(M) = -e \begin{pmatrix} \cos^2 M + \cos M - 2 \\ \sin M \cos M - \sin M \end{pmatrix}^T \begin{pmatrix} y_0 \\ y_0' \end{pmatrix}$$

(c) By the same token as in Sect. 8.5.2 derive the differential equation $z'' + z = g(M)$. Show that

$$u'' + u = e \begin{pmatrix} 0 \\ -12\cos M + 9\cos 2M \\ -6\cos M + 6\cos 2M \\ -3\sin 2M \end{pmatrix}^T \begin{pmatrix} x_0 \\ z_0 \\ x_0' \\ z_0' \end{pmatrix} =: g(M)$$

with $u = z - (4 + 13e)z_0 - (2 + 4e)x_0'$. Then apply the above integral expression for the special solution $z^*(M)$. Finally integrate the differential equation $x' + 2z = h(M)$ directly to derive $x(M)$.

Chapter 9
Interplanetary Flight

In the previous chapter, we had a look at the transfer between two Keplerian orbits, and we saw in Sect. 8.1 how an orbit can purposefully be altered with one-impulse maneuvers. So we know how to head for targets with as little effort as possible, and we are generally prepared to embark on flights to other planets in our solar system. This entails two problems:

1. Shortly after launch we will mainly move in the gravitational field of the Earth. However, the further away we get from Earth, the smaller its force becomes, until during the transit flight we reach the influence of the Sun. Moreover, when we approach the planet of destination, we will enter the domain of its gravitational field, and there we will move on totally different orbits compared to the orbits during transit. How can we describe our orbits under these changing gravitational influences?
2. The second problem is that we no longer simply have to get from one planetary orbit to another. As for orbital rendezvous we also want to meet the planet on the target orbit. This phasing problem is a further difficulty that has to be considered.

In fact, the first problem—to determine the orbit in the spheres of influence of different celestial bodies—is so serious that we cannot solve it exactly with analytical means. Hence, in practice, all interplanetary flights are determined only by complex numerical simulations. This enables one also to take even more complex situations into account, such as the so-called *gravity-assist*, *weak stability boundary* maneuvers, which we will discuss later, or even *invariant manifolds* (see Sect. 11.5.2). But as the important goal here is the basic understanding of orbit mechanics, we are seeking for a method to essentially describe the processes, albeit not precisely. This is indeed possible. The method is called "patched conics."

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9.1 Patched Conics

Patched conics is a sequential domain-by-domain method. It simply separates the problem of a transition between two gravitational fields into two independent spatial domains: One domain where one gravitational field dominates the other, and thus the other is neglected, and a second domain where the situation is the other way round. The transition between them is, of course, not abrupt in reality, but it is rather very gradual. If you take the whole flight trajectory however, the transition phase is rather short, and this is why the patched-conics method works so well in practice: It can be shown that albeit the precision of the orbit trajectories derived by this method is only mediocre, the derived delta-v budget precision is very good, so it is possible to carry out a fairly good mission planning with patched conics, which is later merely refined by numerical methods. Now let us have a closer look at the patched-conics method.

The baseline of the patched-conics method is that in any space domain the trajectory of a vehicle is determined by only one gravitational field, namely, the one that dominates. According to patched conics, if we start in LEO, we exclusively move in the gravitational field of the Earth, and we neglect the gravitational field of the Sun, which is about 1600 times weaker. The further we go into interplanetary space, the more the gravitational influence ratio is shifted toward the Sun, until we reach a point where both gravitational forces, and thus also the two accelerations affecting our spacecraft, have the same strength. That is where we transit from the so-called sphere of influence (SOI) of the Earth to the SOI of the Sun. The practical simplification is that the orbit calculation takes into account only one gravitational field and then the other. It is, of course, important to connect steadily and differentiably, that is to patch, the orbital conic segment in one SOI with the one in the other. That is no problem as long as one knows where the transition point is. Where do we find the edge r_{SOI} of a SOI?

9.1.1 Sphere of Influence

To calculate the edge, let us first consider a test mass m near a planet (index p), jointly orbiting the Sun (index sun). The test mass is exposed to the planet's and Sun's gravitational forces, which are given by

$$F_p(r) = \frac{Gm_p}{r^2} \tag{9.1.1}$$

and

$$F_{sun}(R) = \frac{Gm_{sun}}{R^2} \tag{9.1.2}$$

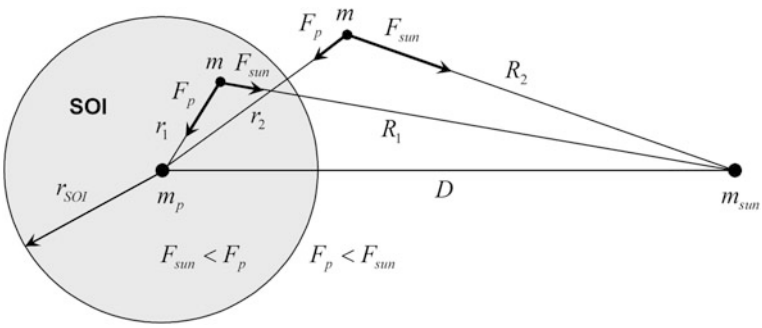


Fig. 9.1 The gravitational force $\underline{F}$ of a planet m_p and of the Sun m_{sun} effecting a test mass m at two points near the SOI boundary

where we denote distances with regard to the planet by r and those with regard to the Sun by R. Because the test mass together with the planet orbit the Sun, we have to consider, in addition the gravitational forces, the centrifugal forces. At the center of mass of the planet its centrifugal force just cancels out the gravitational force of the Sun. A planetary orbit is just in balance with these two forces. Any departure r of the test mass m from the planet's orbit in the direction to the Sun leads to a reduction in the centrifugal force and at the same time to an increase in the gravitational force of the Sun, which for small distances effectively comes to a factor $\sqrt{r/R}$ for F_{sun}, and thus Eq. (9.1.2) must be rewritten as

$$F_{sun}(R) = \frac{Gm_{sun}}{R^2}\sqrt{\frac{r}{R}}$$

We now consider two points near the edge of the SOI (see Fig. 9.1). At the point closer to the planet, the gravitational force of the planet dominates that of the Sun, $F_{sun} < F_p$, and at the other point it is just the other way round, $F_p < F_{sun}$. Right on the edge of the SOI, the equilibrium equation

$$F_{sun}(R) = F_p(r_{SOI})$$

must hold. If one inserts the above expressions into this equilibrium equation one obtains

$$r_{SOI} = R\left(\frac{m_p}{m_{sun}}\right)^{2/5}$$

We now consider the point on the SOI edge, which lies on the line connecting the planet and the Sun at mutual distance D. Then $R = D - r_{SOI}$ and we find

$$r_{SOI} = \frac{D}{\left(m_p/m_{sun}\right)^{-2/5} + 1} \tag{9.1.3}$$

If $m_p = m_{sun}$ then for symmetry reasons $r_{SOI} = \frac{1}{2}D$ must hold, which is also derived from this equation. Since usually $m_p \ll m_{sun}$, we arrive at the approximate result

$$\boxed{r_{SOI} = D\left(\frac{m_p}{m_{sun}}\right)^{2/5}} \quad @\ m_p \ll m_{sun} \tag{9.1.4}$$

So far, our argumentation was based on forces on the connecting line between planet and Sun. Because the test mass experiences slight variation in the Sun's gravitation and centrifugal force when being at off-line positions on the SOI boundary, the SOI of a planet actually is a rotational ellipsoid with the connecting line as its symmetry axis. However, this induced anisotropy is so small that for all practical cases the ellipsoid of influence in good approximation is a sphere with radius r_{SOI}.

Equation (9.1.4) was already derived by the French mathematician Lagrange around 1800, and it is still the best practical estimate for the SOI of a small celestial body orbiting another. If one applies the well-known planetary parameters of our solar system, one gets the SOI radii as given in Table 9.1.

Table 9.1 The radii of the planets' spheres of influence

Planet	r_{SOI} (10^6 km)
Mercury	0.113
Venus	0.618
Earth	0.932
Mars	0.579
Jupiter	48.35
Saturn	54.71
Uranus	51.91
Neptune	87.01
Pluto	3.36

9.1.2 Patched Conics

Let us suppose we make an interplanetary flight. As long as the S/C is within the SOI of the departure planet, the impact of the Sun can be neglected, and we get a hyperbolic departure orbit. When passing the SOI boundary the planetocentric hyperbolic orbit crosses over into a heliocentric transit trajectory. Because the SOI of a planet is much smaller than its distance to the Sun, the transit trajectory is a Hohmann ellipse between the two planetary orbits. Finally, the S/C approaches the target planet on a planetocentric hyperbola in its SOI. For the patched-conics process each of these three Keplerian orbits has to be tuned so that they smoothly pass into each other with regard to location and orbital velocity (Fig. 9.2).

Moon's SOI

To derive the SOI radius of the Moon in the gravitational field of the Earth, one has to consider that $m_{earth}/m_{moon} = 81.300$. So $(m_{moon}/m_{earth})^{-2/5} = 5.808$ is no more much bigger than unity. Therefore, the Moon's SOI radius has to be calculated from Eq. (9.1.3) to be $r_{SOI} = 56,500$ km.

> **Note** *In the literature, one finds* $r_{SOI} = 66\,200$ *km, which is derived from Eq. (9.1.4). However, Eq. (9.1.3) is more suited in this case.*

Though we can provide a SOI radius, the patched-conics method (a Hohmann ellipse around the Earth patched with a hyperbola around the Moon) is not appropriate for lunar trajectories. The reasons are as follows:

1. The lunar SOI is no longer negligibly small compared to the Earth–Moon distance. Therefore, the elliptic orbit no longer is a Hohmann orbit to the Moon.
2. Earth and Moon move around a common center of mass, which is 4670 km away from the geocenter, and thus the position of the Earth considerably shifts in the course of time.

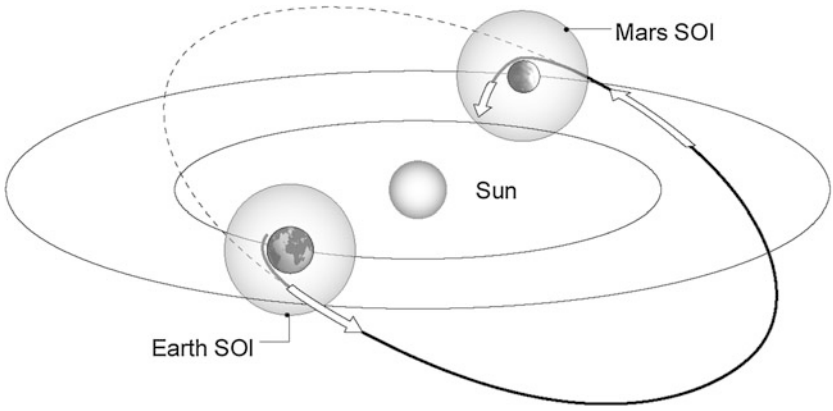


Fig. 9.2 Patched-conics trajectory for a transit between Earth and Mars

3. The impact of the Sun cannot be neglected in the transfer zone.

These additional complications can be described by an extended patched-conics method with a so-called *Michielsen chart*. For details, refer to Kaplan (1976, Sect. 3.5). So-called *rapprochement orbits* as well as the "free return orbit" for Moon missions will be explained in detail in Sect. 11.4.4.

9.2 Departure Orbits

To elaborate the patched-conics method for interplanetary flights, let us take a flight from Earth to Mars as a running example. Without much loss of generality we assume that both planetary orbits are circular and in the same orbit plane. As with every interplanetary flight, the departure of the S/C is from a circular LEO parking orbit with radius r_0 (see Fig. 9.3). Hence, its velocity is

$$v_0 = \sqrt{\frac{\mu_\oplus}{r_0}} \tag{9.2.1}$$

To get the S/C from here to the heliocentric transfer orbit, a matching impulse maneuver $\Delta \boldsymbol{v} = \boldsymbol{v}_1 - \boldsymbol{v}_0$ is required at the right position θ_∞ of the LEO, such that the departure velocity $\boldsymbol{v}_1$ causes an excess velocity $\boldsymbol{v}_\infty$ that at the edge of the SOI is parallel to Earth's trajectory. This $\boldsymbol{v}_\infty$ is the initial velocity for the predetermined heliocentric Hohmann transfer orbit. So, for a flight to an outer planet, $\boldsymbol{v}_\infty$ needs to

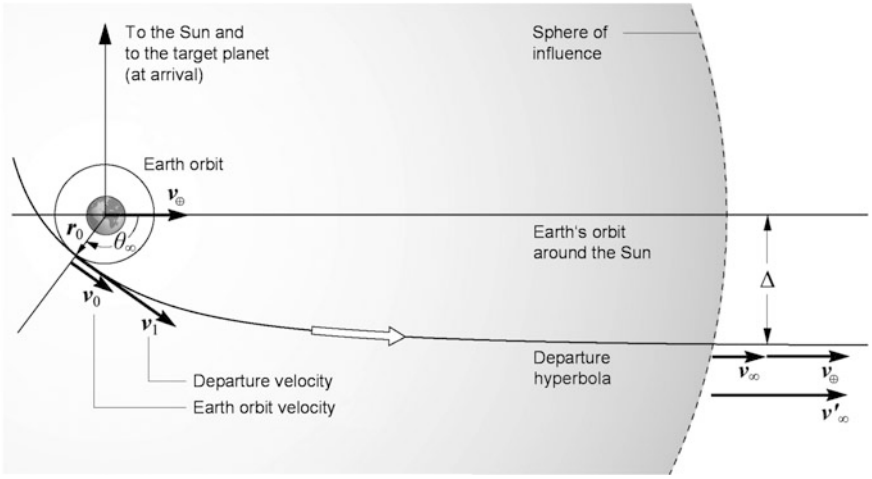


Fig. 9.3 Orbit characteristics at departure (parking orbit and departure hyperbola) to an outer planet (cf. Fig. 9.5a for the big view)

be in the direction of Earth's trajectory, for a flight to an inner planet, $\boldsymbol{v}_\infty$ needs to be opposite to the direction of Earth's trajectory.

Supposing $\boldsymbol{v}_\infty$ is given, which we will determine later, we first want to calculate the required Δv and θ_∞. From a given excess velocity $\boldsymbol{v}_\infty$ the departure velocity for a hyperbola can be calculated from Eq. (7.4.26) as

$$v_1 = \sqrt{\frac{2\mu_\oplus}{r_0} + v_\infty^2} = \sqrt{2v_0^2 + v_\infty^2} \tag{9.2.2}$$

So for the injection burn, we get

$$\Delta v = v_1 - v_0 = \sqrt{2v_0^2 + v_\infty^2} - v_0 \tag{9.2.3}$$

As mentioned with Eq. (7.4.26), the parameter $C_3 := v_\infty^2$ is called **characteristic energy**. The eccentricity of the departure hyperbola is calculated with $v_1 = \mu(e+1)/h$ from Eq. (7.4.26) and $v_\infty = \mu\sqrt{e^2-1}/h$ from Eq. (7.4.26) and with Eq. (9.2.2) as

$$e = 1 + \frac{v_\infty^2}{v_0^2} = 1 + \frac{r_0 v_\infty^2}{\mu_\oplus} \tag{9.2.4}$$

To determine the right timing angle θ_∞ for the injection burn, we use Eq. (7.4.26) and get

$$\theta_\infty = \arccos\frac{1}{e} = \arccos\frac{v_0^2}{v_0^2 + v_\infty^2} \tag{9.2.5}$$

Figure 9.3 shows that this angle is measured relative to the direction of orbital movement of the Earth around the Sun.

Sensitivity Analysis

In practice, the injection burn can never be carried out exactly, but only with certain thrust errors $\delta v_{||}$ and $\delta v_{\perp O}$ (see Sects. 8.1.1 and 8.3.4). What is the impact of these thrust errors on the hyperbolic excess velocity with regard to both its value and the asymptotic direction?

To figure out the excess velocity error δv_∞, we examine Eq. (9.2.3), which we rewrite as

$$v_\infty^2 + 2v_0^2 = (\Delta v + v_0)^2$$

If we differentiate and rearrange the result, we get

$$\frac{\delta v_\infty}{v_\infty} = \frac{v_1(v_1 - v_0)}{v_\infty^2}\frac{\delta v_{||}}{\Delta v} \tag{9.2.6}$$

For the error $\delta\theta_\infty$ there are two contributions: one because of Eqs. (9.2.6) and (9.2.5) $\delta v_{||} \to \delta v_\infty \to \delta\theta_\infty$, and the other because a perpendicular thrust error leads to a rotation of the line of apsides of the hyperbola, that is, $\delta v_{\perp O} \to \delta\omega = \delta\theta_\infty$. The first contribution is

$$\delta\theta_\infty = -\frac{1}{e}\sqrt{\frac{e-1}{e+1}}\frac{\delta v_\infty}{v_\infty} = -\frac{v_0^2}{v_0^2 + v_\infty^2}\frac{v_\infty}{v_1}\frac{\delta v_\infty}{v_\infty} = -\frac{v_0^2(v_1 - v_0)}{v_\infty\left(v_0^2 + v_\infty^2\right)}\frac{\delta v_{||}}{\Delta v}$$

For the second we get according to Table 8.1 and because of $h = r_0 v_1 = \mu_\oplus v_1 / v_0^2$:

$$\delta\theta_\infty = \delta\omega = -\frac{h}{e\mu_\oplus}\delta v_{\perp O} = -\frac{v_1}{e v_0^2}\delta v_{\perp O}$$

With Eqs. (9.2.2) and (9.2.4) we finally get

$$\delta\theta_\infty = -\frac{v_1(v_1 - v_0)}{v_0^2 + v_\infty^2}\frac{\delta v_{\perp O}}{\Delta v}$$

So, in total we have

$$\delta\theta_\infty = -\frac{v_1(v_1 - v_0)}{v_0^2 + v_\infty^2}\left(\frac{v_0^2}{v_1 v_\infty}\frac{\delta v_{||}}{\Delta v} + \frac{\delta v_{\perp O}}{\Delta v}\right) \tag{9.2.7}$$

Example

For an Earth → Mars transit with $v_\infty = 3.040$ km s^{-1}, $v_0 = 7.76$ km s^{-1} (300 km parking orbit), and $v_1 = 11.34$ km s^{-1}, we get

$$\frac{\delta v_\infty}{v_\infty} = 4.4\frac{\delta v_{||}}{\Delta v} \text{ and } \delta\theta_\infty = -0.59\left(1.7\frac{\delta v_{||}}{\Delta v} + \frac{\delta v_{\perp O}}{\Delta v}\right)$$

A 1% error in thrust direction and perpendicular to it leads to an error of $\delta v_\infty / v_\infty = 4.4\%$ and $\delta\theta_\infty = -0.91°$.

Departure Hyperbolas

To pass over into a tangential heliocentric Hohmann transfer orbit at the edge of the Earth SOI, the orbital plane of the departure hyperbola has to include the instantaneous velocity vector of the Earth (i.e., the asymptotic velocity vector $\boldsymbol{v}_\infty$ has to be along $\boldsymbol{v}_\oplus$). But apart from that, the departure orbit plane may have arbitrary

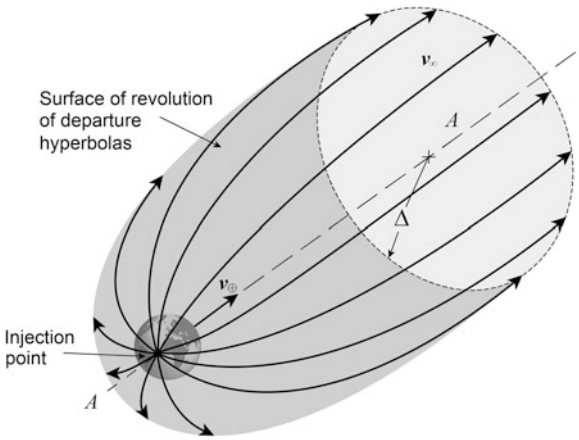


Fig. 9.4 Possible orientations of the departure hyperbola

orientation (see Fig. 9.4). So there are many possible departure hyperbolas. However, to include Earth's velocity vector, the parking orbit has to be included as well. Such a matching parking orbit can be reached only twice a day for a specific launch site (see Sect. 8.6.1) if no dogleg maneuver shall be performed. So there are only two launch windows per day for interplanetary flights.

Once the S/C is in the right parking orbit, there is one injection burn opportunity per orbit. According to Fig. 9.3 this opportunity lies on the night side of the Earth for flights to outer planets, that is, for $\boldsymbol{v}_\infty$ parallel to $\boldsymbol{v}_\oplus$ and for prograde orbits, and on the day side of the Earth for flights to inner planets, that is, for $\boldsymbol{v}_\infty$ antiparallel to $\boldsymbol{v}_\oplus$.

9.3 Transfer Orbits

9.3.1 Hohmann Transfers

We have now reached the edge of the Earth's SOI. From here, the trajectory is a heliocentric conic, so we use the heliocentric reference frame, which we indicate by a prime. In this primed reference frame Earth has orbital velocity $\boldsymbol{v}_\oplus$, and the velocity of our S/C is

$$\boldsymbol{v}'_\infty = \boldsymbol{v}_\infty + \boldsymbol{v}_\oplus$$

As we had selected our entry conditions so that for flights to outer planets $\boldsymbol{v}_\infty$ is parallel to $\boldsymbol{v}_\oplus$, and for inner planets $\boldsymbol{v}_\infty$ antiparallel to $\boldsymbol{v}_\oplus$, this implies

$$v'_\infty = \begin{cases} v_\oplus + v_\infty & @ \rightarrow \textit{outer planets} \\ v_\oplus - v_\infty & @ \rightarrow \textit{inner planets} \end{cases} \tag{9.3.1}$$

We still have to determine the required transfer orbit, that is, the excess velocity v_∞, which ensures to meet the target orbit. In Sect. 8.3, we saw that a Hohmann transfer would be energetically most favorable. If we select a Hohmann transfer, the transfer injection burn to an outer or an inner planet reads from Eq. (8.3.6) as

$$v_\infty = \Delta v_{\bullet \rightarrow H} = v_\oplus \left[\sqrt{\frac{a_{planet}}{a_H}} - 1 \right]$$

$$v_\infty = \Delta v_{O \rightarrow H} = v_\oplus \left[1 - \sqrt{\frac{a_{planet}}{a_H}} \right]$$

So for both types of missions we get with $a_H = (a_\oplus + a_{planet})/2$ (Mars : $a_H = 1.2618\ a_\oplus$),

$$v_\infty = v_\oplus \left| 1 - \sqrt{\frac{2a_{planet}}{a_\oplus + a_{planet}}} \right| \tag{9.3.2}$$

a_H is the semi-major axis of the Hohmann transfer orbit, and $v_\oplus = 29.78$ km s^{-1}. For Mars, we get $v_\infty = 2.972$ km s^{-1}. Note that this excess velocity is unprimed, so it is valid in the geocentric reference frame, not in the primed heliocentric reference frame. For the Hohmann transfer time we find from Eq. (8.1)

$$t_H = \pi \sqrt{\frac{a_H^3}{\mu_{sun}}} = \frac{T_\oplus}{4\sqrt{2}} \left(\frac{a_\oplus + a_{planet}}{a_\oplus} \right)^{3/2} \tag{9.3.3}$$

with $T_\oplus = 365.256\ d$ the period of the Earth orbit.

Orbit Phasing

With a Hohmann transfer, we would just touch the planetary target orbit. But to hit the target planet, Earth and target planet have to be in a specific mutual configuration. To determine this configuration, we make use of our assumption that both planetary orbits are circular and lie in the same plane (ecliptic plane). This restriction is insignificant, but it eases to explain the principle of configuration determination. According to Fig. 9.5a, the angle $\theta_{\bullet O}$ between the initial configuration of Earth and the target planet and $\theta_{O\bullet}$ between their final configuration (the so-called *phase angle*) are, respectively,

$$\theta_{\bullet O} = 180^\circ - n_O t_H = 180^\circ \left(1 - \sqrt{\frac{a_H^3}{a_O^3}} \right) \tag{9.3.4a}$$

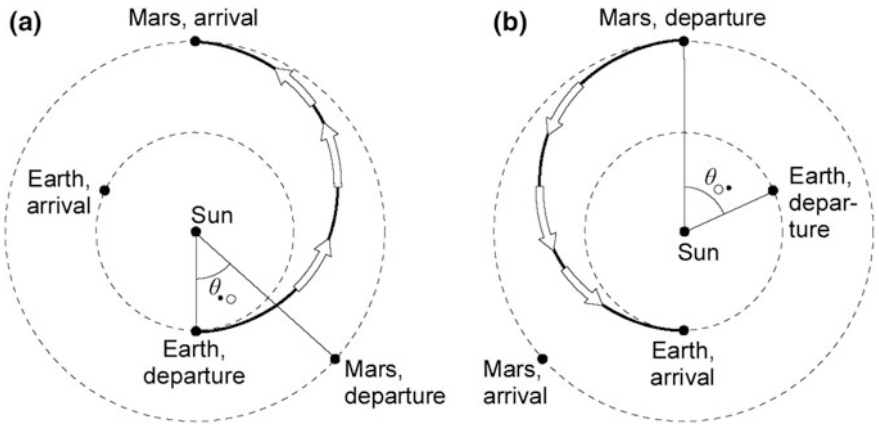


Fig. 9.5 Relative configuration of Earth and Mars for forward (**a**) and return transits (**b**)

$$\theta_{\mathrm{O}\bullet} = 180^\circ - n_\bullet t_H = 180^\circ \left(1 - \sqrt{\frac{a_H^3}{a_\bullet^3}}\right) \tag{9.3.4b}$$

where n_O, $n_\bullet$ are the mean motion of the outer (Mars) and the inner planet (Earth), respectively. So for a Mars transit, we get: $t_H = 258.86\,d$, $\theta_{\bullet\mathrm{O}} = 44.3^\circ$, $\theta_{\mathrm{O}\bullet} = -75.1^\circ$.

When do these constellations come up again? This question is essential for interplanetary missions, because the narrow launch window is open only in this interval. With regard to a given direction in the ecliptic plane, let us take the vernal point (index 0), the orbit positions evolve according to Eq. (7.4.1b)

$$\theta_\bullet = \theta_{\bullet,0} + n_\bullet t$$
$$\theta_\mathrm{O} = \theta_{\mathrm{O},0} + n_\mathrm{O} t$$

and thus the phase angle is

$$\theta_{\bullet\mathrm{O}} = \theta_\mathrm{O} - \theta_\bullet = \theta_0 + (n_\mathrm{O} - n_\bullet)t$$

The phase angle recurs after time $t = T_{syn}$, the so-called *synodic period*, when $\theta_{\bullet\mathrm{O}} \to \theta_{\bullet\mathrm{O}} - 2\pi$, that is,

$$(n_\mathrm{O} - n_\bullet)T_{syn} = -2\pi$$

From this follows

$$\frac{1}{T_{syn}} = \frac{n_\bullet}{2\pi} - \frac{n_\mathrm{O}}{2\pi}$$

or

$$\frac{1}{T_{syn}} = \frac{1}{T_\bullet} - \frac{1}{T_O} \quad \textbf{synodic period} \tag{9.3.5}$$

and $T_O, T_\bullet$ are the orbital periods of the outer and inner planets, respectively. For Earth–Mars, we have $T_{syn} = 2.135\ yr$.

In contrast to Moon missions it is not possible to fly back to Earth at just any time, but one has to wait for the proper planetary constellation, just as for the forward flight (see Fig. 9.5b). What is the waiting time after arrival at the planet to get the proper return flight constellation? The answer can be derived from similar phase considerations as above (exercise, Problem 9.1). It is

$$t_{wait} = T_{syn}\left(k - \left|1 - \frac{2t_H}{T_\oplus}\right|\right) \tag{9.3.6}$$

where k is any natural number for which $t_{wait} > 0$. The shortest waiting time therefore occurs with the smallest k, when $t_{wait} > 0$ for the first time. Finally, the minimum total flight time is calculated with forward and return flight time plus minimum waiting time as

$$t_{trip} = 2t_H + t_{wait} \tag{9.3.7}$$

Table 9.2 gives an overview of the derived characteristic times for the relevant planets in the solar system.

Table 9.2 The characteristic times for Hohmann transfers from Earth to other planets in our solar system

Planet	T [yr]	T_{syn} [d]	t_H [d]	t_{wait} [d]	t_{trip} [yr]
Mercury	0.2408	115.9	105.5	66.9	0.761
Venus	0.6152	584.0	146.1	467.1	2.079
Earth	1.0000	–	–	–	–
Mars	1.8808	779.9	258.9	454.2	2.661
Jupiter	11.863	398.9	997.5	209.8	6.036
Saturn	29.447	378.1	2.208	344.0	13.03
Uranus	84.017	369.7	5.857	343.6	33.01
Neptune	164.79	367.5	11.182	283.6	62.00

9.3.2 *Non-Hohmann Transfers*

Hohmann orbits may be the most favorable transfer orbits from an energetic point of view, but in Sect. 8.3.4 we already found out that Hohmann transfer orbits are very sensitive to initial thrust errors, and also that they take the longest time. So just a little more thrust would make sure that, with small thrust errors, the transfer orbit still intersects the target orbit, while transition time drastically decreases (cf. Fig. 9.7).

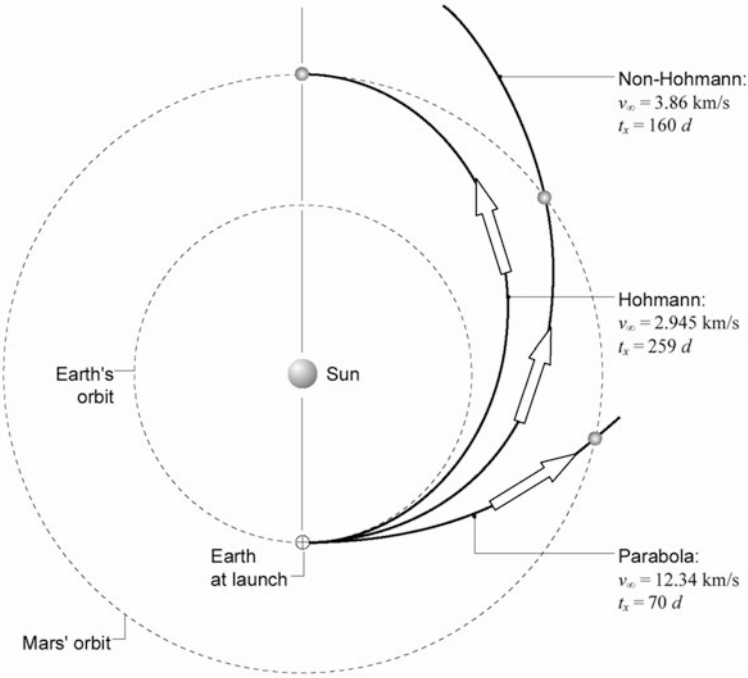


Fig. 9.6 Various Earth–Mars transfer orbits

But how does the crossing point and with it the transition time change with some transfer excess velocity in the initial parking orbit?

Let us consider a flight from an inner to an outer target orbit (running example Earth → Mars), for which $v'_\infty = v_\oplus + v_\infty$ holds according to Eq. (9.3.1). We recall that we assumed coplanar and circular planetary orbits. An elliptic transfer orbit is of the form

$$r = \frac{a(1-e^2)}{1+e\cos\theta} \tag{9.3.8}$$

The ellipse touches the initial orbit with radius $a_\bullet$ at its periapsis. Therefore

$$a_\bullet = r_{per} = a(1-e)$$

and the initial speed at periapsis is

$$v'_\infty = \sqrt{\frac{\mu}{a}\frac{1+e}{1-e}} = \sqrt{\frac{\mu}{a_\bullet}(1+e)}$$

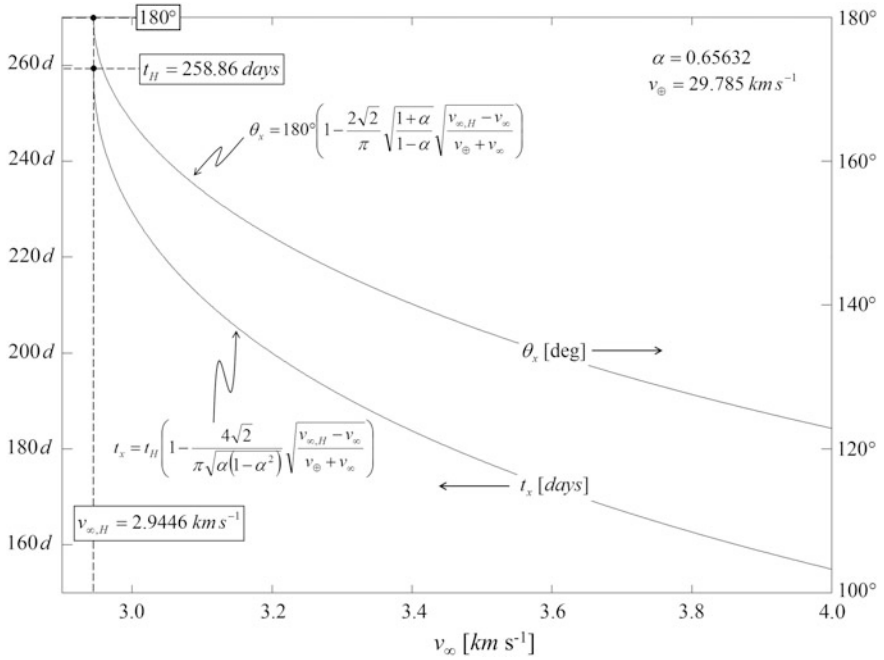


Fig. 9.7 Dependence of the transition time $t_\times$ and the orbit crossing angle $\theta_\times$ as a function of the excess velocity v_∞ for an Earth → Mars transit

From this we derive

$$1 + e = \frac{a_\bullet v_\infty'^2}{\mu} \tag{9.3.9}$$

and

$$a = \frac{a_\bullet}{1 - e} \tag{9.3.10}$$

The crossing (index $\times$) with the target orbit at $r = a_O$ determines the true anomaly of the crossing point via Eq. (9.3.8)

$$\cos\theta_\times = \frac{1}{e}\left[\frac{a(1-e)(1+e)}{a_O} - 1\right]$$

From this we find with Eqs. (9.3.9) and (9.3.10)

$$\cos\theta_\times = \frac{1}{e}\left[\frac{a_\bullet(1+e)}{a_O} - 1\right] \quad @\ \bullet \to O \tag{9.3.11}$$

According to Eq. (7.4.14c), we can translate this into an eccentric anomaly

$$\sin E_\times = \frac{\sqrt{1-e^2}\cdot \sin\theta_\times}{1+e\,\cos\theta_\times} \tag{9.3.12}$$

to find from Kepler's Eq. (7.4.15) the transition time

$$\sqrt{\frac{\mu}{a^3}}\cdot t_\times = E_\times - e\,\sin E_\times \tag{9.3.13}$$

Figure 9.7 shows how, according to Eqs. (9.3.11) and (9.3.13), the transition time and the true anomaly of the crossing point decrease with increasing excess velocity, $a_\bullet v_\infty'^2/\mu$, for an Earth → Mars transit.

For missions from an outer to an inner planet, for instance Earth → Venus, one has to substitute the indices $\bullet \leftrightarrow \mathrm{O}$ in Eqs. (9.3.9–9.3.11). Furthermore $\cos\theta_\times = [a_\mathrm{O}(1-e)/a_\bullet - 1]/e$ and $a_\mathrm{O}v_\infty'^2/\mu = 1-e$ holds. Otherwise Eqs. (9.3.12) and (9.3.13) remain unchanged.

Approximate Solution near Hohmann Transfers

If we convey the general elliptic transfer orbit into the specific Hohmann transfer orbit, the derivatives of the transition time and of the true anomaly diverge. These are the two drawbacks of a Hohmann transfer we already mentioned in the sensitivity analysis in Sect. 8.3.4. So for practical purposes one will choose a transfer orbit away from the Hohmann transfer so that despite injection burn errors an intersection with the target orbit is ensured, but otherwise close enough that one still profits from the energetic advantage of a Hohmann transfer. What is the impact of injection burn errors on the intersection time $t_\times$ and angle $\theta_\times$ near a Hohmann transfer. Specifically we want to trace the error impacts $\Delta v_1 \to \Delta\theta_\times$ and $\Delta v_1 \to \Delta t_\times$ mathematically.

We begin with the error chain $\Delta v_1 \to \Delta v_\infty \to \Delta a \to \Delta E_\times \to \Delta t_\times$. For this we work in the chain backward and start with $\Delta E_\times \to \Delta t_\times$. The time on the transfer ellipse is given by the above Kepler's equation. To determine the effects of tiny variations, we linearize Kepler's equation

$$t_\times \cdot dn + n\cdot dt_\times = (1-e\,\cos E_\times)dE - \sin E_\times \cdot de$$

Because $n = \sqrt{a^3/\mu}$ and therefore $dn = 3/2\sqrt{a/\mu}\cdot da$, we obtain for small variations of the orbital elements at $E_\times \approx \pi$

$$n\cdot \Delta t_\times = (1+e)\cdot \Delta E_\times - \frac{3}{2}t_\times\sqrt{\frac{a}{\mu}}\cdot \Delta a \tag{9.3.14}$$

We now have to derive the chain link $\Delta a \to \Delta E_\times$ to arrive at $\Delta a \to \Delta t_\times$. Let us have a look at the orbit Eq. (7.4.14a)

$$r = a(1 - e\,\cos E)$$

We differentiate this equation by bearing in mind that the radius of the intersecting point remains constant, $r = a_{target} = const$, and that according to Eq. (8.3.16) $de = (1 - e)da/a$. After some rearrangements we obtain

$$\frac{dE}{da} = -\frac{1}{a_H e_H}\tan\frac{E}{2}$$

Let us check: For $E < \pi$ we get with increasing semi-major axis a decreasing eccentric anomaly, as expected. Because we are close to the Hohmann transfer $\tan E/2 \to \infty$. So to get finite variations, we have to integrate this equation. We do so by separating its variables

$$\int_{\pi}^{\pi-\Delta E_\times} \frac{dE}{\tan(E/2)} = -\frac{1}{e_H}\int_{a_H}^{a_H+\Delta a}\frac{da}{a}$$

In the left integral, we substitute $E = \pi - 2\delta$ from which follows $dE = -2 \cdot d\delta$, and we get

$$\int_{\pi}^{\pi-\Delta E_\times} \frac{dE}{\tan(E/2)} = -2\int_{0}^{\Delta E_\times/2}\frac{d\delta}{\tan(\pi/2 - \delta)}$$

For $\delta \to 0$ the following is valid:

$$\frac{1}{\tan(\pi/2 - \delta)} = \frac{1}{\cot\,\delta} = \tan\delta \approx \delta$$

With this we get for the integral on the left-hand side of the equation

$$\int_{\pi}^{\pi-\Delta E_x} \frac{dE}{\tan(E/2)} \approx -2\int_{0}^{\Delta E_x/2}\delta \cdot d\delta = -\frac{(\Delta E)^2}{4}$$

and for that on the right-hand side

$$-\frac{1}{e_H}\int_{a_H}^{a_H+\Delta a}\frac{da}{a} = -\frac{1}{e_H}\ln\left(1 + \frac{\Delta a}{a_H}\right) \approx -\frac{1}{e_H}\frac{\Delta a}{a_H}$$

In summary, we obtain for the chain link $\Delta a \rightarrow \Delta E_\times$

$$\Delta E_\times \approx -\frac{2}{\sqrt{e_H}}\sqrt{\frac{\Delta a}{a_H}}$$

So, any $\Delta E_\times$ decreases with the square root of Δa for $\Delta a \rightarrow 0$. Compared to this rapid variation we can neglect the linear dependency Δa in Eq. (9.3.14) and obtain with $n = 2\pi/T \approx 2\pi/2t_H = \pi/t_H$

$$\frac{\Delta t_\times}{t_H} \approx -\frac{2}{\pi}\frac{1+e_H}{\sqrt{e_H}}\sqrt{\frac{\Delta a}{a_H}}$$

We now work one more step backward in our error chain and look for the dependency $\Delta v_\infty \rightarrow \Delta a$. Because of Eq. (8.3.4) $e_H = (a_O - a_\bullet)/(a_O + a_\bullet)$ and since from Eq. (8.3.16) for a Hohmann transfer follows

$$\frac{da}{a_H} = 2\frac{a_O}{a_\bullet}\frac{dv'_\infty}{v'_\infty} = 2\frac{a_O}{a_\bullet}\frac{dv_\infty}{v_\oplus + v_\infty}$$

any excess velocity error Δv_∞ can be converted into the transition time error

$$\frac{\Delta t_\times}{t_H} \approx -\frac{4\sqrt{2}}{\pi\sqrt{\alpha(1-\alpha^2)}}\sqrt{\frac{|v_{\infty,H} - v_\infty|}{v_\oplus \pm v_\infty}} = -\frac{4\sqrt{2}}{\pi\sqrt{\alpha(1-\alpha^2)}}\sqrt{\frac{v_\infty}{v_\oplus \pm v_\infty}}\sqrt{\frac{\Delta v_\infty}{v_\infty}} \tag{9.3.15}$$

with

$$\alpha := \frac{a_\bullet}{a_O}$$

with $v_{\infty,H}$ the excess velocity for a Hohmann transfer. According to Eq. (9.3.1) the $\pm$ signs denote the transit to an outer/inner planet. We now make the final step backward to derive $\Delta v_1 \rightarrow \Delta v_\infty$. Differentiating Eq. (9.2.2), we get

$$\frac{\Delta v_1}{v_1} = \frac{v_\infty^2}{v_1^2}\frac{\Delta v_\infty}{v_\infty}$$

This inserted into Eq. (9.3.15) finally delivers the transition time from the transplanetar injection burn to the intersection with the target orbit in dependence of the injection burn error in the LEO parking orbit Δv_1. From $t_x = t_H \pm \Delta t_x$ we get

$$t_\times \approx t_H\left[1 \mp \frac{4\sqrt{2}}{\pi\sqrt{\alpha(1-\alpha^2)}}\frac{v_1}{\sqrt{v_\infty(v_\oplus \pm v_\infty)}}\sqrt{\frac{\Delta v_1}{v_1}}\right] \tag{9.3.16}$$

with $\alpha = a_\bullet/a_O$.

By the same token the true anomaly of the intersection point is calculated (exercise, Problem 9.4) by the error chain $\Delta v_1 \rightarrow \Delta v_\infty \rightarrow \Delta a \rightarrow \Delta E_\times \rightarrow \Delta\theta_\times$ to be

$$\theta_\times \approx 180^\circ \left[1 \mp \frac{2\sqrt{2}}{\pi}\sqrt{\frac{1+\alpha}{1-\alpha}}\frac{v_1}{\sqrt{v_\infty(v_\oplus \pm v_\infty)}}\sqrt{\frac{\Delta v_1}{v_1}}\right] \tag{9.3.17}$$

In both equations, the $\mp$ signs denote temporally the first and the second intersection of the transfer ellipse with the target orbit, respectively, and the $\pm$ signs the transit to an outer or inner planet, respectively. The divergences close to the Hohmann transfer are of the form $\sqrt{\Delta v_\infty} \propto \sqrt{\Delta v_1}$, in accordance with Fig. 9.7.

Example

For an Earth $\rightarrow$ Mars transit, $v_\infty = 2.972$ km s^{-1}, $v_1 = 11.32$ km s^{-1}s (300 km parking orbit), $v_\oplus = 29.78$ km s^{-1} and $\alpha = 0.65632$. From this we derive for the transition time to the first intersection point

$$t_\times \approx t_H\left(1 - 3.380\sqrt{\frac{\Delta v_1}{v_1}}\right)$$

Because for Mars $t_H = 259$ d, this means that for an injection burn error of $\Delta v_1 = 0.01$ km s^{-1} in LEO surplus to the regular Hohmann injection burn, $v_1 = 11.32$ km s^{-1} (that is a surplus of only 0.1%) we achieve a transition time reduction of $\Delta t = 26.0$ d, that is, 10%! The first intersection angle is at

$$\theta_\times \approx 180^\circ\left(1 - 2.268\sqrt{\frac{\Delta v_1}{v_1}}\right)$$

and the reduction of the intersection angle is $\Delta\theta_\times = 12.1^\circ$.

9.4 Arrival Orbit

After the heliocentric transition phase, the S/C enters the SOI of the target planet and crosses it on a hyperbola, which according to Eq. (9.2.4) has the eccentricity

$$e = 1 + \frac{r_{per}v_\infty^2}{\mu} \tag{9.4.1}$$

For Hohmann transfers, the approach hyperbola has to include only the line of movement of the target planet. Otherwise and in analogy to departure (see Fig. 9.4), the orientation of the hyperbolic target orbit can be chosen arbitrarily (see Fig. 9.8).

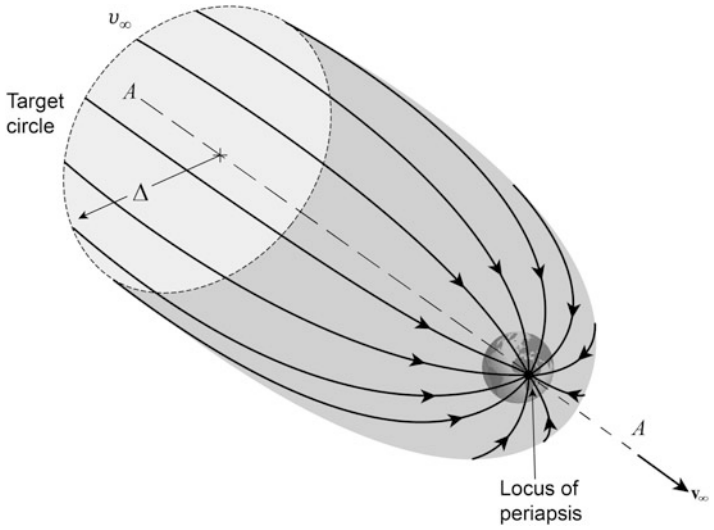


Fig. 9.8 Possible orientations of the approach hyperbola

According to Eq. (7.4.23), the impact parameter, which we are free to choose by adjustment burns in course of the heliocentric transit, is related to the closest approach distance r_{per} to the target planet by

$$\Delta = r_{per}\sqrt{\frac{e+1}{e-1}} = r_{per}\sqrt{1+\frac{2\mu}{r_{per}v_\infty^2}} \tag{9.4.2a}$$

or conversely

$$r_{per} = \sqrt{\Delta^2 + \left(\frac{\mu}{v_\infty^2}\right)^2} - \frac{\mu}{v_\infty^2} \tag{9.4.2b}$$

Optimal Planetary Capture

It is now our goal to transfer the S/C with minimum delta-v effort from its unbounded hyperbolic approach (index h) orbit into an elliptically bound capture orbit with orbital elements a_c, e_c. We know from Eq. (8.1.11) that the required injection burn to catch the S/C (to lower the infinite apoapsis to a finite value) preferably takes place at the periapsis where the hyperbolic speed is according to Eqs. (7.4.20) and (7.4.21)

$$v_{h,per} = \sqrt{\frac{\mu(e_h+1)}{r_{per}}} = \sqrt{v_\infty^2 + \frac{2\mu}{r_{per}}}$$

The latter equation results from Eq. (9.4.1). For the elliptic capture orbit the periapsis velocity is according to Eqs. (7.4.10a) and (7.4.6a)

$$v_{c,per} = \sqrt{\frac{\mu(1+e_c)}{r_{per}}}$$

So the required kick-burn at the joint periapsis is

$$\Delta v = \sqrt{v_\infty^2 + \frac{2\mu}{r_{per}}} - \sqrt{\frac{\mu(1+e_c)}{r_{per}}}$$

Note that the delta-v depends only indirectly (via the periapsis) on the semi-major axis of the capture orbit. Now, if we want to minimize the Δv demand at a given eccentricity of the capture orbit, we can do this by determining the optimal periapsis distance. The result of this quite simple minimizing task (exercise) is

$$r_{per,opt} = \frac{2\mu}{v_\infty^2}\frac{1-e_c}{1+e_c} \tag{9.4.3}$$

So we get

$$\Delta v_{\min} = v_\infty\sqrt{\frac{1-e_c}{2}} \tag{9.4.4}$$

as the minimum kick-burn effort. According to Eq. (9.4.2a), the optimal periapsis distance is adjusted by the following optimal impact parameter:

$$\Delta_{opt} = 2\sqrt{2}\frac{\sqrt{1-e_c}}{1+e_c}\frac{\mu}{v_\infty^2} \tag{9.4.5}$$

If aerocapture (capture from an hyperbolic into a bounded orbit) is possible, such as at Mars or Venus, then one first aims at a highly elliptic orbit $e \approx 1$, which is achieved by a very small injection burn (Eq. (9.4.4)) with a very low periapsis distance (Eq. (9.4.3)) (Of course, one has to bear in mind that the impact parameter always has to be larger than the radius of the planet.) As we will see in Sect. 12.7.2, the captured S/C will be circularized by aerobraking because of the atmospheric drag at the periapsis, and finally after many orbits depending on the drag, it will be turned down to a circular orbit with radius $a = r_{per}$. This is the most effective way to achieve a circular target orbit, and thus it is also regularly done on Mars missions, as for instance with NASA's Mars Reconnaissance Orbiter in March 2006.

9.5 Flyby Maneuvers

9.5.1 Overview

The exploration of the planets in our solar system is one of the most attractive undertakings in astronautics. However, the propulsion demands for missions to these planets are considerable. We had already seen that for Mars we would need $v_1 = 11.32$ km s^{-1}. If we wanted to fly to Jupiter on a Hohmann transfer orbit, according to Eqs. (9.3.2) and (9.2.2) we would already need $v_1 = 14.04$ km s^{-1}. If we even wanted to leave the solar system, we would need at least $v_\infty = (\sqrt{2} - 1)v_\oplus = 12.34$ km s^{-1} and therefore $v_1 = 16.48$ km s^{-1}. These propulsion demands are impossible to achieve with chemical propulsions, even with just a minimum payload. But despite this fact, the probes Voyager and Pioneer flew to the most remote planets of our solar system, already in the 1970s, and by now they have actually left our solar system. How did NASA manage to accomplish that? The trick is called "gravity-assist maneuver", also called "swing-by" or "flyby maneuver". Such a maneuver makes use of a near flyby past a planet to pick up additional momentum. You could compare it with a roller-blader, who just for a short time hitches up to a passing bus to gain speed. The flyby maneuver is not understandable in the two-body system: If an S/C enters the SOI of a planet it is deflected and leaves the SOI with exactly the same escape velocity that it had when it entered the SOI. Only if we watch the flyby in the heliocentric system, we see that the S/C indeed picks up speed: The approached planet has an orbital velocity, and because the S/C gravitationally "hitches up" to the planet during flyby it is able to pick up part of the planet's orbital velocity. It is a special feature of a many-body system that its bodies can exchange energy and impulse. Note that momentum gain by flyby works only because the planet loses the same momentum that the S/C gains. However, as the mass of a planet is much bigger than the mass of the S/C, the velocity change of the planet is insignificant and undetectable.

In fact, Pioneer and Voyager not only undertook one flyby maneuver, but several in a row. And with each flyby the S/Cs gained speed until the escape velocity of the solar system was exceeded. This trick is frequently used today to voyage to other planets. Even when traveling only to the next planet Mars, you could save delta-v for Earth excess velocity if you pick up momentum before from Moon flybys (see Sect. 9.6). If you are clever, you could pick up this momentum and gain velocity even several times. But of course it all has its price, and here the price is time. Every flight to the Moon and back to the Earth requires a lot of time, and thus prolongs the flight to the real target, that is, Mars. However, sometimes planets are constellated such that there is an intermediate planet to flyby right on the trajectory to the target planet. In very rare cases, and they occur only once every few decades, there are several planets in a row to swing along to the outermost planets in our solar system.

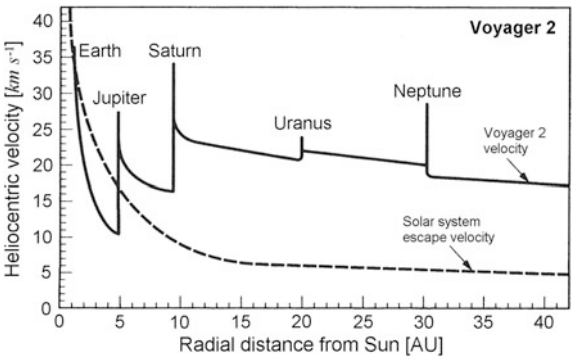


Fig. 9.9 The heliocentric velocity gain of Voyager 2 at its flybys at Jupiter, Saturn, Uranus, and Neptune. The flyby at Neptune was to leave the ecliptic plane and flyby Neptun's moon Triton. Thereby it lost heliocentric speed. *Credit* NASA

Voyager 2 flew along Jupiter, Saturn, Uranus, and Neptune (see Fig. 9.9), a planetary constellation that only occurs once every 189 years. This situation came along in the middle of the 1970s, and thus NASA took on this very tricky mission already at that early time of interplanetary space flight.

9.5.2 Flyby Framework

Let us assume a spacecraft approaches a planet with velocity $\boldsymbol{v}_p$ in its orbital plane from a large distance for a flyby. At the time of entering the planet's SOI (cf. Fig. 9.2), it may have incoming velocity $\boldsymbol{v}_{in}$ in the heliocentric reference frame. Because we only know to describe the flyby in the "planet–spacecraft" two-body system with their center of mass (which is essentially the planet) at rest, we have to map the incoming velocity into the planetocentric reference frame, where $\boldsymbol{v}_p = 0$. According to Fig. 9.10 the S/C enters the SOI with the mapped asymptotic velocity $\boldsymbol{v}_\infty^-$

$$\boldsymbol{v}_\infty^- = \boldsymbol{v}_{in} - \boldsymbol{v}_p \tag{9.5.1}$$

The subscript ∞ of the entry velocity denotes that it is a hyperbolic asymptotic velocity. The superscript "–" means that we are before flyby. We are now in the SOI and switch to the planetary reference frame. Figure 9.11 illustrates the most general situation for the flyby of a S/C within the SOI. The flyby takes place on a

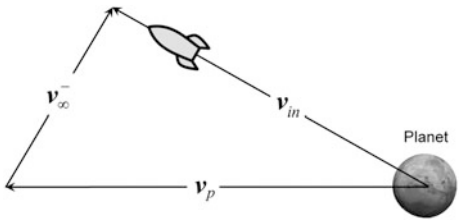


Fig. 9.10 The velocity triangle in the heliocentric reference frame at entry into the planet's SOI

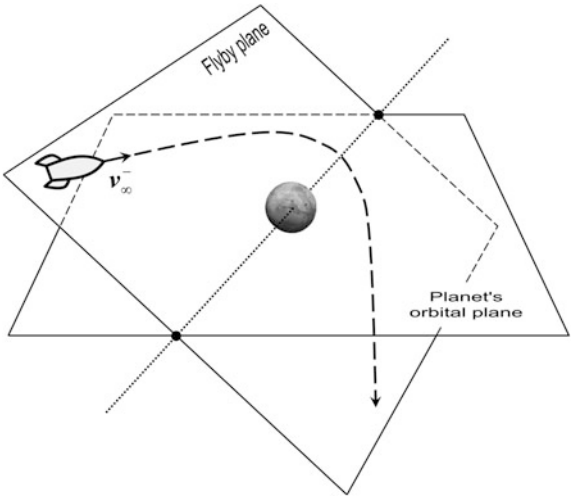


Fig. 9.11 The hyperbolic flyby path of the S/C in the flyby plane relative to the planet's orbital plane

hyperbola in the flyby plane, which needs not to be coplanar with planet's orbital plane. This non-coplanarity complicates the mathematical description somewhat, which we will however neglect for the time being, by assuming that the flyby plane is coplanar with planet's orbital plane. This, of cause, implies that the incoming velocity $\boldsymbol{v}_{in}$ and the flyby trajectory are also in the planet's orbital plane.

The S/C then passes through the SOI, whereby it is deflected on a hyperbolic trajectory, and leaves the SOI with asymptotic exit velocity $\boldsymbol{v}_\infty^+$. As we are in a two-body system, the principle of conservation of energy holds, and according to the vis-viva Eq. (7.2.15), the asymptotic entry and exit velocities must be identical

$$v_\infty^- = v_\infty^+ =: v_\infty \tag{9.5.2}$$

The absolute value of the exit velocity of the S/C is thus identical to its entry velocity, only the flight direction changes. This velocity turn δ may be counter-clockwise, $\delta > 0$, or clockwise $\delta < 0$ depending on whether the S/C intersects the planet's orbit "from inside" or "from outside" and whether it initially crosses the planet's orbit "before" or "behind" the planet (see Fig. 9.12). "From inside/outside" means that the S/C intersects the planet orbit from inside/outside. "Before" and "behind" refer to the direction of movement of the planet. As seen, either a before or a behind flyby can cause clockwise and counterclockwise turns. But from Fig. 9.13 it generally can be said that, if the S/C flies by behind a planet, it is "carried along" the orbit by the planet, so it gains speed, $v_{out} > v_{in}$. If it moves before the planet, then its flight momentum is "hampered" and the transferred momentum leads to a reduction of its velocity, $v_{out} < v_{in}$. This is not strictly true, but it is mostly true. The latter case may seem to be academic, as you usually want to gain velocity. But this is only the case with flights to outer planets. To get to inner planets, one has to reduce the velocity of the S/C after it leaves the Earth. This can

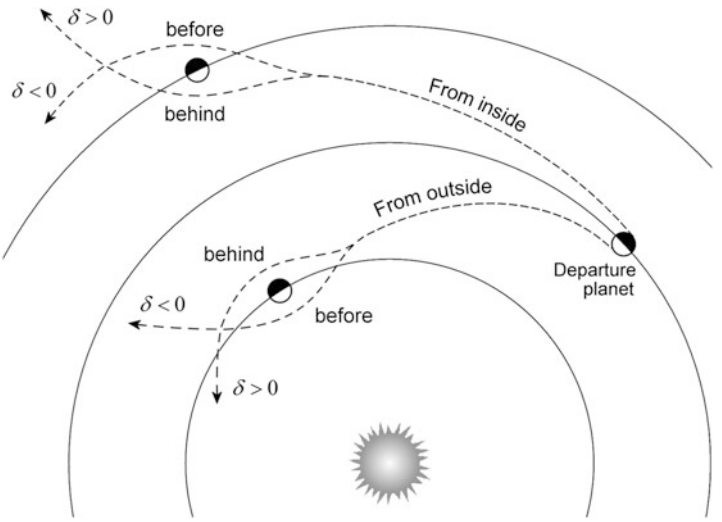


Fig. 9.12 Before/behind flybys in the "from inside" case (above) and in the "from outside" case (below). Note the different turn directions for these two cases

be done by flybys to the Moon, or, if you want to go to Mercury, by flying by Venus—if the planetary constellation admits it.

It will be the main undertaking in the next section to determine the exact deflection angle δ. Let us assume for a moment that we have found δ. Because $v_\infty^- = v_\infty^+ =: v_\infty$, the deflection can be described as a rotation of the velocity vector (see Fig. 9.13) with angle δ

$$\boldsymbol{v}_\infty^+ = \boldsymbol{R}_\delta \boldsymbol{v}_\infty^- \tag{9.5.3}$$

with rotation matrix

$$\boldsymbol{R}_\delta = \begin{pmatrix} \cos\delta & -\sin\delta \\ \sin\delta & \cos\delta \end{pmatrix}$$

Once we know the asymptotic exit velocity vector $\boldsymbol{v}_\infty^+$, we can derive the outgoing vector $\boldsymbol{v}_{out}$ in the heliocentric frame according to Fig. 9.13 and in analogy to Eq. (9.5.1) as

$$\boldsymbol{v}_{out} = \boldsymbol{v}_\infty^+ + \boldsymbol{v}_p \tag{9.5.4}$$

So, to determine the final outgoing vector, everything hinges on the determination of the deflection angle, to which we turn now.

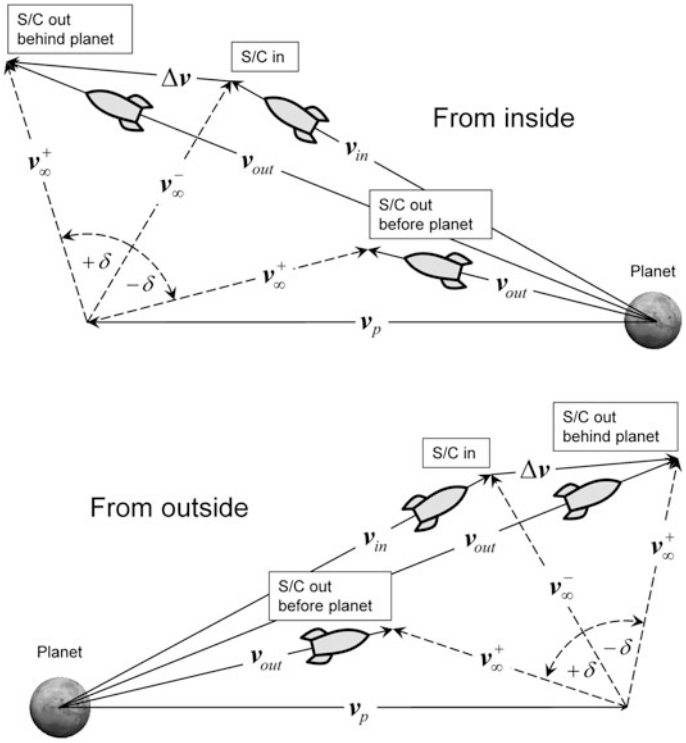


Fig. 9.13 The vector diagram of a flyby from a heliocentric point of view (full vectors) for the "from inside" case (top) and for the "from outside" case (bottom). The S/C enters the SOI of the planet with velocity $\boldsymbol{v}_{in}$. During flyby behind or before the planet, it changes flight direction by angle δ and then leaves the SOI with the velocity $\boldsymbol{v}_{out}$

9.5.3 *Planetocentric Flyby Analysis*

In this section we want to determine quantitatively the changes of flight direction and speed the S/C undergoes during flyby in the planetocentric reference frame.

Deflection Angle

We first calculate the deflection angle. According to Fig. 9.14 it is given by

$$\delta = \pi - 2\psi$$

The angle ψ in turn is, according to Fig. 7.15, the reverse angle of the escape angle θ_∞, for which $\cos\theta_\infty = -1/e$ holds (see Eq. (7.4.22)). Hence

$$\tan\psi = \tan(\pi - \theta_\infty) = -\tan\theta_\infty = \sqrt{e^2 - 1} = \frac{\Delta}{-a}$$

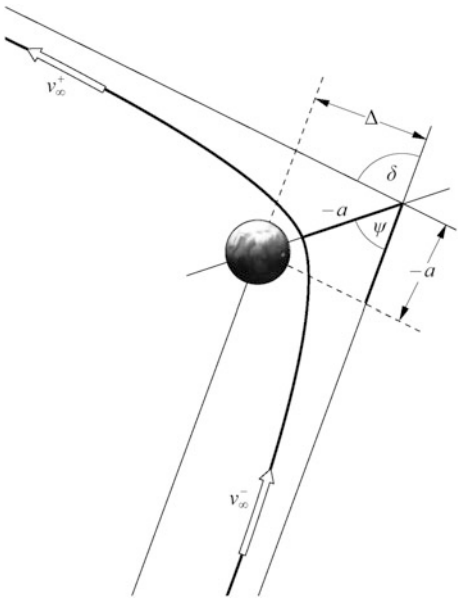


Fig. 9.14 Flyby hyperbola and flyby parameters in the planetocentric reference frame

where the latter follows from Eq. (7.4.23). While the impact parameter Δ (see Sect. 7.4.3) is our freely adjustable parameter, the hyperbolic parameter $-a > 0$ and eccentricity is given by Eqs. (7.4.23) and (7.4.24) as

$$v_\infty^2 = -\frac{\mu_p}{a}, \Delta = -a\sqrt{e^2-1} = r_{per}\sqrt{\frac{e+1}{e-1}}$$

Inserting this and $\delta = \pi - 2\psi$ into the above equation leads to the result

$$\delta = \pi - 2\arctan\frac{\Delta \cdot v_\infty^2}{\mu_p}$$

or

$$\tan\frac{\delta}{2} = \frac{\mu_p}{\Delta \cdot v_\infty^2} \tag{9.5.5}$$

So the deflection depends on the adjustable impact parameter Δ and the given entry velocity v_∞. Observe that for $\Delta \to 0$ the deflection angle tends to $\delta \to 180°$, which is the limiting turn a S/C theoretically could perform if it could fly at zero distance past the planet's center of mass.

Impact parameter

The impact parameter Δ is a key parameter to control a flyby maneuver. How is it determined when entering the SOI? At the edge of the planet's SOI the position vector from the planet to the entering S/C is

$$\boldsymbol{r}_{\infty}^{-} = \boldsymbol{r}_{S/C} - \boldsymbol{r}_{p} \tag{9.5.6}$$

With this we find from Fig. 9.15 for the impact parameter vector

$$\boldsymbol{\Delta} = \boldsymbol{r}_{\infty}^{-} + \hat{\boldsymbol{v}}_{\infty}^{-}\left(-\hat{\boldsymbol{v}}_{\infty}^{-} \cdot \boldsymbol{r}_{\infty}^{-}\right) = \boldsymbol{r}_{\infty}^{-} - \hat{\boldsymbol{v}}_{\infty}^{-}\left(\hat{\boldsymbol{v}}_{\infty}^{-} \cdot \boldsymbol{r}_{\infty}^{-}\right) \tag{9.5.7}$$

where $\hat{\boldsymbol{v}}_{\infty}^{-}$ is the unit vector of $\boldsymbol{v}_{\infty}^{-}$. With this expression we can mathematically define before/behind flybys.

$$\begin{aligned} &\text{Before flyby} \quad \boldsymbol{\Delta} \cdot \boldsymbol{v}_{p} > 0 \\ &\text{Behind flyby} \quad \boldsymbol{\Delta} \cdot \boldsymbol{v}_{p} < 0 \end{aligned}$$

In the literature the so-called **B-plane** is frequently used, which is defined as the plane through the center of the planet and perpendicular to $\boldsymbol{v}_{\infty}^{-}$, i.e., the incoming asymptote of the S/C trajectory. Therefore, the impact parameter vector $\boldsymbol{\Delta}$ lies in the B-plane.

Turn direction

To determine the direction in which the S/C turns during flyby, we have to consider the relation between $\boldsymbol{r}_{\infty}^{-}$ and $\boldsymbol{\Delta}$ (see Fig. 9.15). The S/C will make a positive turn if $\boldsymbol{\Delta}$ points to the right (as see from the S/C toward the planet) of the planet. This orientation can be expressed by the orientation of the vector cross product relative to the plane's normal vector $\boldsymbol{n}$. Of the two possible normal vectors $\boldsymbol{n}$, we choose the vector that is parallel to the angular momentum of a counterclockwise turn (right-hand rule)

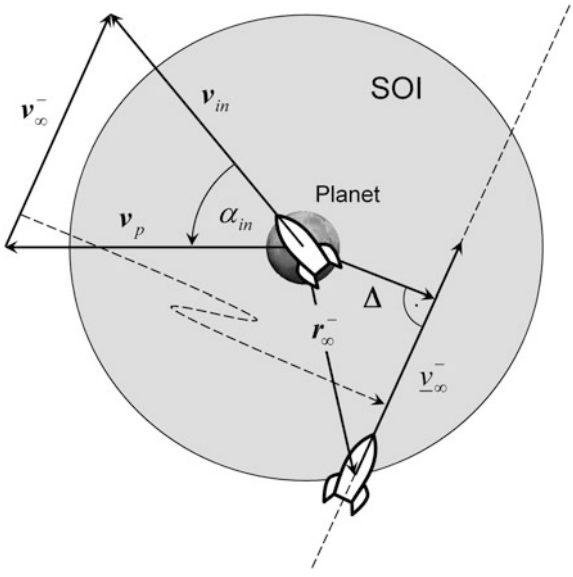


Fig. 9.15 Entry triangles: The left triangle is the velocity triangle from Fig. 9.10. Its entry velocity is taken at the SOI entry point (right triangle) to determine the impact parameter vector

$$\operatorname{sgn}(\delta) = \operatorname{sgn}\left[(\boldsymbol{r}_\infty^- \times \boldsymbol{\Delta}) \cdot \boldsymbol{n}\right] = \operatorname{sgn}\left[(\boldsymbol{r}_\infty^- \times \boldsymbol{\Delta})_z\right] = \operatorname{sgn}\left(r_{\infty,x}^- \Delta_y - r_{\infty,y}^- \Delta_x\right)$$

The vector components x, y lie in the flyby plane and z is vertical along $\boldsymbol{n}$. Rather than assigning this orientation to the deflection angle δ, we prefer to assign it already to the value of Δ because the deflection can then be described to only two parameters v_∞ and Δ_p. This can be done, because our deflection Eq. (9.5.5) is sensitive to the sign of Δ. So

$$\Delta_p = \operatorname{sgn}\left(r_{\infty,x}^- \Delta_y - r_{\infty,y}^- \Delta_x\right) \frac{v_p^2}{\mu_p} |\boldsymbol{\Delta}| \quad \textbf{normalized impact parameter} \tag{9.5.8}$$

and from Eq. (9.5.5)

$$\delta = 2 \arctan \frac{v_p^2}{\Delta_p \cdot v_\infty^2} \quad \textbf{deflection angle} \tag{9.5.9}$$

where the impact parameter is normalized to a dimensionless form for reasons we will see later. These two equations describe the wanted amount and direction of the deflection angle.

Velocity Change

From Eq. (9.5.3) with Eq. (9.5.9) we immediately find for the velocity change vector

$$\Delta \boldsymbol{v} = \boldsymbol{v}_\infty^+ - \boldsymbol{v}_\infty^- = \boldsymbol{R}_\delta \boldsymbol{v}_\infty^- - \boldsymbol{v}_\infty^-$$

To determine its value let us have a look at the velocity triangle shown in Fig. 9.16. As it is an isosceles triangle, the following is valid:

$$\sin \frac{\delta}{2} = \frac{\Delta v/2}{v_\infty}$$

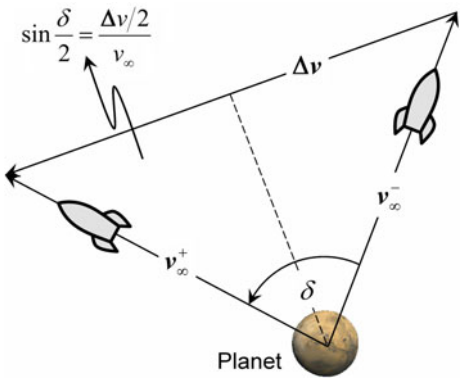


Fig. 9.16 The velocity transfer triangle from a planet point of view

Inserting this into Eq. (9.5.5) yields

$$\frac{\Delta v}{v_\infty} = \frac{2}{\sqrt{1 + \left(\Delta \cdot v_\infty^2/\mu_p\right)^2}} \quad \textbf{flyby delta-v} \tag{9.5.10}$$

This is the wanted delta-v of the flyby maneuver, which of course and according to Fig. 9.13 is the same in the heliocentric reference frame.

Powered Flyby

During a flyby an additional boost may be done to increase or decrease the exit velocity. According to the *Oberth effect* in Sects. 8.1.3 and 8.4.1, such a boost is most efficient at the periapsis of the hyperbolic flyby orbit. From

$$\frac{1}{2}v_{per}^2 - \frac{\mu}{r_{per}} = \frac{1}{2}v_\infty^2$$
$$\frac{1}{2}(v_{per} + \Delta v)^2 - \frac{\mu}{r_{per}} = \frac{1}{2}v_\infty'^2$$

follows

$$\begin{aligned} v_\infty' &= v_\infty\sqrt{1 + 2\frac{\Delta v}{v_\infty}\sqrt{\frac{e+1}{e-1}} + \left(\frac{\Delta v}{v_\infty}\right)^2} \\ &\approx (v_\infty + \Delta v)\left[1 + \frac{\Delta v}{v_\infty}\left(\sqrt{\frac{e+1}{e-1}} - 1\right) + O\left(\frac{\Delta v}{v_\infty}\right)^2\right] \end{aligned}$$

This is significantly more than a boost at the SOI leading to $v_\infty' = v_\infty + \Delta v$. This boost-gain effect is the Oberth effect discussed in Sect. 8.1.3. From Eq. (7.4.21) we have

$$v_{per} = \sqrt{\frac{\mu}{r_{per}}}\sqrt{e+1}$$

Therefore

$$v_{per} + \Delta v = \sqrt{\frac{\mu}{r_{per}}}\sqrt{e'+1}$$

From which follows for the eccentricity of the powered hyperbolic exit orbit

$$\sqrt{e'+1} = \Delta v\sqrt{\frac{r_{per}}{\mu}} + \sqrt{e+1}$$

and hence the altered asymptotic angle is

$$\cos\theta'_\infty = -\frac{1}{e'}$$

This change in asymptotic flight direction may be either desirable or may be adjusted by a change in the impact parameter.

Conclusions

It is remarkable that there is only one lumped parameter, $\Delta \cdot v_\infty^2/\mu_p$, that is decisive for both the amount of the deflection and the amount of the velocity change. The smaller it is, the larger are Δv and δ, and the other way round. In summary, we can say

> The delta-v and the deflection angle of a flyby increase with
>
> - decreasing asymptotic entry velocity v_∞^2
> - decreasing impact parameter Δ
> - increasing mass of the target planet $\mu_p = Gm_p$.

Thus, a sharp flyby with a small entry velocity past a massive planet such as Jupiter changes velocity and flight direction far more than a flyby at a large distance and high entry velocity past a small planet. This is intuitively clear, but it is good to know how it works out quantitatively.

As mentioned before, the key parameter to tune a flyby is the impact parameter Δ. Because it is usually the goal of a flyby to maximize the delta-v (consistently with the change of the other orbital elements), Δ is to be chosen as small as possible. Quite naturally $r_{per} = R_p$ is the limit, in which case the S/C would scrape the surface of the planet with radius R_p. According to Eq. (9.4.2a), this limit translates into the constraint for the impact parameter

$$\Delta > R_p\sqrt{1+2\frac{\mu_p}{R_p v_\infty^2}} \tag{9.5.11}$$

Even this limit should not be exhausted, first because the true impact parameter might deviate from the determined one by errors in position measurement, and second because nearly all planets possess an atmosphere. If the S/C dives too deeply into it, the drag might annihilate the anticipated delta-v gain. In most cases, though, not the delta-v gain alone is decisive, but the outgoing direction is also important, because it must match the direction to the target. Therefore, for a detailed mission planning the above constraint and drag have to be taken into account just as side constraints.

> *Example*
>
> We apply our results to the Mars flyby of the Rosetta probe, which took place on February 25, 2007. According to ΔDOR measurements (see *Deep Space Tracking* in Sect. 14.2.1) by ESA, the asymptotic velocities

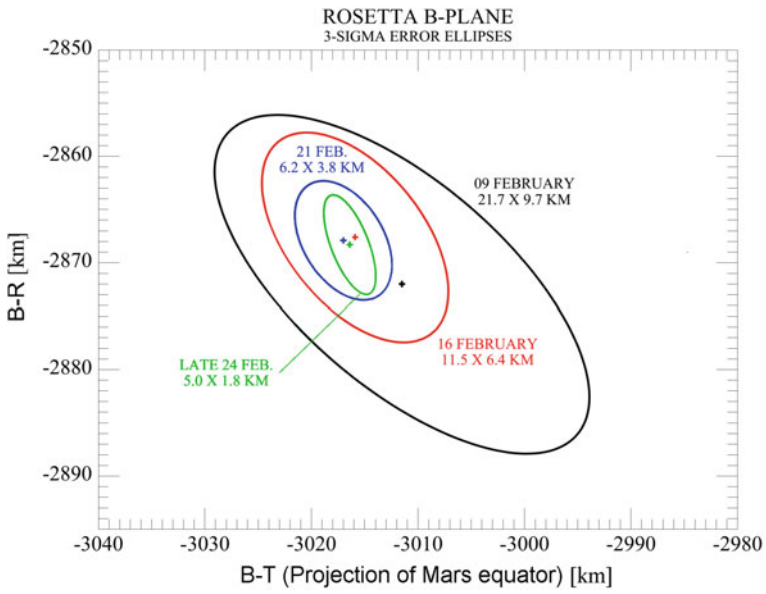


Fig. 9.17 The B-plane of Rosetta's Mars flyby with error ellipses measured at different times before flyby. The T-axis is defined to be the projection of the Mars equator of date, while the R-axis is vertical to this, and the origin of the T-R coordinate system lies in Mars' center of mass. *Credit* ESA/ ESOC

were $v_\infty = 8.808$ km s^{-1}. From ESA's B-plane data, as shown in Fig. 9.17, we derive $\Delta = \sqrt{(2869\text{ km})^2 + (3016\text{ km})^2} = 4163$ km. With $\mu_{Mars} = 4.2828 \times 10^4$ km^3 s^{-2} from Table A.1 in Appendix A, we find from Eq. (9.5.5) the deflection angle to be $\delta = 15.10°$ and from Eq. (9.5.10) the flyby delta-v to be $\Delta v = 2.315$ km s^{-1}. These are exactly the results of the actual flyby. From Eq. (9.4.2a) we derive that the closest approach distance was $r_{per} = 3648$ km, which equals a lowest altitude of only 252 km. This risky flyby of the 1.3 billion Euro Rosetta probe was nicknamed the "billion Euro gamble".

9.5.4 *Heliocentric Flyby Analysis*

Flyby in the Orbital Plane

Now that we know the upshot of a flyby in the planetocentric reference frame, it is quite easy to determine the wanted outgoing velocity $\boldsymbol{v}_{out}$ of the S/C in the heliocentric reference frame. We need to map only the incoming velocity $\boldsymbol{v}_{in}$ into the planetocentric frame to obtain $\boldsymbol{v}_\infty^-$. We then turn this vector by the angle δ to receive $\boldsymbol{v}_\infty^+$, which we map back into the heliocentric frame to finally get $\boldsymbol{v}_{out}$. If these steps

are performed with Eqs. (9.5.1), (9.5.3), and (9.5.4) and some trigonometric relations are applied, we get

$$\boldsymbol{v}_{out} = \boldsymbol{R}_\delta(\boldsymbol{v}_{in} - \boldsymbol{v}_p) + \boldsymbol{v}_p \tag{9.5.12}$$

where

$$\boldsymbol{R}_\delta = \begin{pmatrix} \cos\delta & -\sin\delta \\ \sin\delta & \cos\delta \end{pmatrix} = \frac{1}{1+\chi^2}\begin{pmatrix} \chi^2-1 & -2\chi \\ 2\chi & \chi^2-1 \end{pmatrix}$$

with

$$\delta = 2\arctan\frac{1}{\chi}$$

$$\chi := \Delta_p\left(\frac{\boldsymbol{v}_{in} - \boldsymbol{v}_p}{v_p}\right)^2$$

Generalized Flyby

We finally generalize the calculations for $\boldsymbol{v}_{out}$ to a flyby that does not take place in the planet's orbital plane (see Fig. 9.11). Let $\boldsymbol{v}_{in}$ be any incoming vector that hits the planet's SOI at S/C position $\boldsymbol{r}_{S/C}$ as measured from the heliocentric origin. It is mapped into the planetocentric frame to become the entry vector $\boldsymbol{v}_\infty^- = \boldsymbol{v}_{in} - \boldsymbol{v}_p$. According to Fig. 9.15 and Eq. (9.5.7) $\boldsymbol{\Delta} = \boldsymbol{r}_\infty^- - \hat{\boldsymbol{v}}_\infty^-\left(\hat{\boldsymbol{v}}_\infty^- \cdot \boldsymbol{r}_\infty^-\right)$ still holds. Now that the flyby plane's normal vector $\boldsymbol{n}$ is no longer in the direction of the z-axis, we have from the above the general expression

$$\Delta_p = \operatorname{sgn}\left[(\boldsymbol{r}_\infty^- \times \boldsymbol{\Delta}) \cdot \boldsymbol{n}\right]\frac{v_p^2}{\mu_p}|\boldsymbol{\Delta}| \tag{9.5.13}$$

The next step would be the deflection described by the rotation matrix. Because this is given only in the flyby plane spanned by $(\hat{\boldsymbol{v}}_\infty^-, \hat{\boldsymbol{\Delta}})$, we have to transform the vectors into a coordinate system in this plane. Let $\boldsymbol{T}_{P\to F}$ be the matrix made up from two Euler rotations that transforms the Cartesian planetocentric coordinate system into the flyby plane (first rotation along z-axis to bring the x-axis along the nodal line of the intersecting planes (see Fig. 9.11)) and then (second rotation along this new x-axis) brings the z-axis along $\hat{\boldsymbol{v}}_\infty^- \times \hat{\boldsymbol{\Delta}}$. Then the outgoing vector can be written as

$$\boldsymbol{v}_{out} = \boldsymbol{T}_{P\to F}^{-1}\boldsymbol{R}_\delta\boldsymbol{T}_{P\to F}(\boldsymbol{v}_{in} - \boldsymbol{v}_p) + \boldsymbol{v}_p \tag{9.5.14}$$

In conclusion the following calculation scheme for a general flyby can be given

Flyby Calculation Scheme

1. For the S/C at the edge of the SOI of the flyby planet determine
   - planet $\boldsymbol{r}_p$ and S/C position $\boldsymbol{r}_{S/C}$,
   - instantaneous orbital velocity of the planet $\boldsymbol{v}_p$, and
   - incoming velocity $\boldsymbol{v}_{in}$ of S/C.
2. Determine:
$$\boldsymbol{r}_\infty^- = \boldsymbol{r}_{S/C} - \boldsymbol{r}_p$$
$$\boldsymbol{v}_\infty^- = \boldsymbol{v}_{in} - \boldsymbol{v}_p$$
$$v_\infty = \sqrt{\boldsymbol{v}_\infty^- \boldsymbol{v}_\infty^-} \text{ and } \hat{\boldsymbol{v}}_\infty^- = \boldsymbol{v}_\infty^- / v_\infty$$
3. Determine the impact parameter vector and the normalized impact parameter
$$\boldsymbol{\Delta} = \boldsymbol{r}_\infty^- - \hat{\boldsymbol{v}}_\infty^- \left(\hat{\boldsymbol{v}}_\infty^- \cdot \boldsymbol{r}_\infty^-\right)$$
   From this follows
   $\Delta_p = \operatorname{sgn}\left[(\boldsymbol{r}_\infty^- \times \boldsymbol{\Delta}) \cdot \boldsymbol{n}\right] \frac{v_p^2}{\mu_p} \sqrt{\boldsymbol{\Delta}\boldsymbol{\Delta}}$ (dimensionless)
4. Determine the rotation matrix
$$\boldsymbol{R}_\delta = \frac{1}{1+\chi^2} \begin{pmatrix} \chi^2 - 1 & -2\chi & 0 \\ 2\chi & \chi^2 - 1 & 0 \\ 0 & 0 & 1+\chi^2 \end{pmatrix} \text{ with } \chi = \Delta_p \frac{v_\infty^2}{v_p^2}$$
5. Determine the transformation matrix $\boldsymbol{T}_{P\to F}$ that by two Euler rotations transforms the planetocentric Cartesian coordinate system into the flyby plane $\left[\hat{\boldsymbol{v}}_\infty^-, \hat{\boldsymbol{\Delta}}\right]$. For flybys in the planet's orbital plane, $\boldsymbol{T}_{P\to F} = \boldsymbol{1}$ holds.
6. Determine $\boldsymbol{v}_{out} = \boldsymbol{T}_{P\to F}^{-1} \boldsymbol{R}_\delta \boldsymbol{T}_{P\to F} (\boldsymbol{v}_{in} - \boldsymbol{v}_p) + \boldsymbol{v}_p$.

Numerical Calculations

To illustrate the effects of a flyby, we assume a flyby in the planet's orbital plane and choose $\boldsymbol{v}_p$ as a basis for the Cartesian coordinate system, $\boldsymbol{v}_p = v_p(1, 0)$, relative to which the incoming and outgoing velocity vector is measured, $\boldsymbol{v}_{in} = v_{in}(\cos\alpha_{in}, \sin\alpha_{in})$, $\boldsymbol{v}_{out} = v_{out}(\cos\alpha_{out}, \sin\alpha_{out})$, cf. Fig. 9.15. The results are shown in Figs. 9.18 and 9.19 for $v_{in}/v_p = 1$. Because at flyby a S/C has about the same heliocentric orbital speed as the planet, we chose $v_{in}/v_p = 1$ as a good ballpark figure. A special situation occurs when $\alpha_{in} = 60°$ and $\Delta_p = \sqrt{3} = 1.732$, which is elaborated in Fig. 9.20. In this case $\Delta v = v_p = v_{in}$ (see Fig. 9.21). But because $\delta = -60°$ as well, it follows that

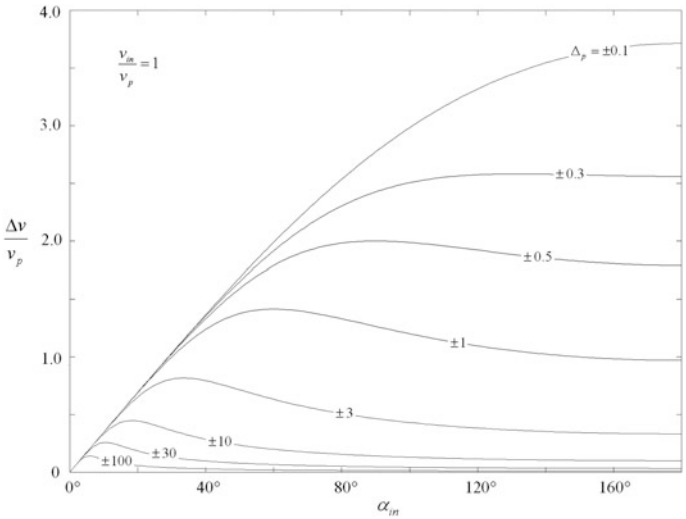


Fig. 9.18 Normalized delta-v as a function of the incoming angle α_{in} in the heliocentric system for $\Delta_p < 0$ and $\Delta_p > 0$

$v_{out} = 0$: The S/C comes to a full stop in the heliocentric system. The outgoing angle α_{out} therefore becomes undefined and hence also the deflection angle $\Delta\alpha = \alpha_{out} - \alpha_{in}$.

9.5.5 *Transition of Orbital Elements*

We finally want to know how the orbital elements of the interplanetary heliocentric orbit are modified by a flyby maneuver. If the received Δv is small, we can consider the induced momentum transfer formally as a kick-burn, $\delta \boldsymbol{v} = \boldsymbol{v}_{out} - \boldsymbol{v}_{in} = (\delta v_{||}, \delta v_{\perp O}, \delta v_{\perp\perp})$, for which the orbital element changes in the heliocentric reference frame are given by Eq. (8.1.1). If the momentum transfer is quite big, then we have to determine the orbital elements through the outgoing state vector $(\boldsymbol{r}_{out}, \boldsymbol{v}_{out})$, where $\boldsymbol{r}_{out}$ is the position vector at the time the S/C leaves the SOI. This procedure is described in Sect. 7.3.5, Subsection *Conversion: State Vector → Orbital Elements*.

At any rate, if the flyby is in the planet's orbital plane then the S/C receives a $\Delta \boldsymbol{v}$ with components $\Delta v_{||}$ and $\Delta v_{\perp O}$, only. According to Table 8.1, in this case only the orbital elements a, e, ω are modified. If the flyby is not in the orbital plane, then the S/C also receives a $\Delta v_{\perp\perp}$ and the orbital elements i, Ω are also effected. If the flyby plane is perpendicular to the planet's orbital plane, then only i, Ω, ω are effected. So the decisive reason for in-plane flybys is to alter the flight direction via ω and/or the semi-major axis, which because of $\varepsilon = -\mu/2a$ correspond to a change in orbital energy. This maneuver was chosen for the Voyager and Pioneer space probe at Jupiter to escape from the solar system. Out-of-plane flybys make use of the inclination change. The Ulysses space probe for instance flew in February 1992

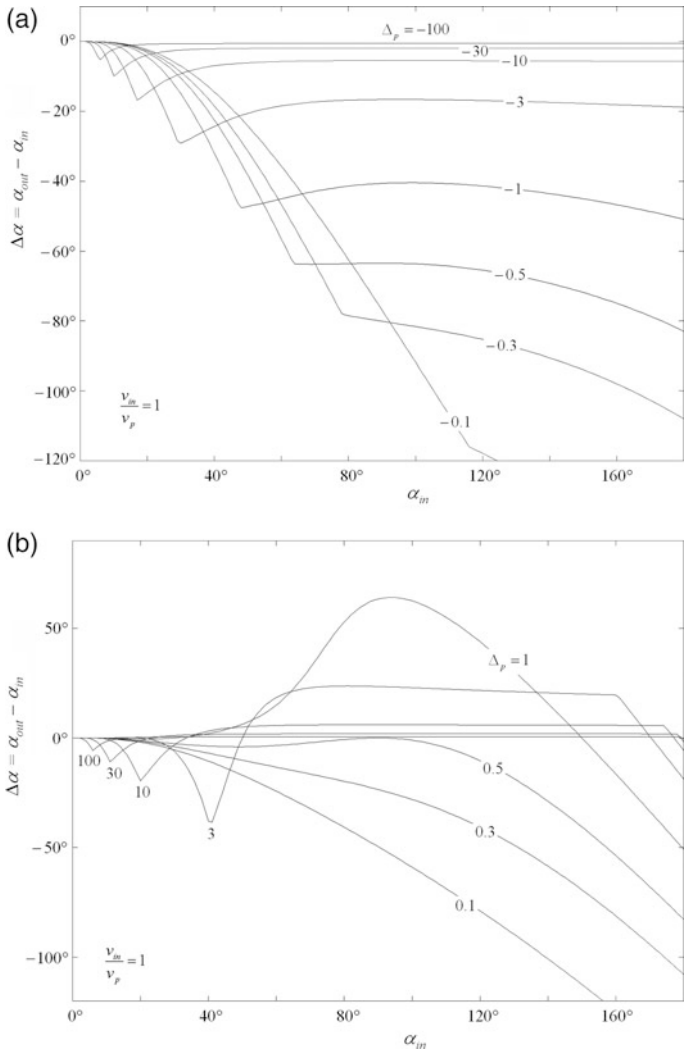


Fig. 9.19 **a** Deflection angle $\Delta\alpha$ as a function of the incoming angle α_{in} in the heliocentric system for a flyby with $\Delta_p < 0$. **b** Deflection angle $\Delta\alpha$ as a function of the incoming angle α_{in} in the heliocentric system for a flyby with $\Delta_p > 0$

right above Jupiter to be propelled on a polar orbit around the Sun with inclination $i = 80.2°$. Any of these maneuvers would otherwise require a lot of delta-v.

Tisserand's Relation

According to Sects. 11.3 and 11.4, a fly of a S/C by a planet can also be interpreted as a trajectory in the restricted three-body system Sun–planet–S/C. We want to apply those results to a planetary flyby in the heliocentric inertial (sidereal) system.

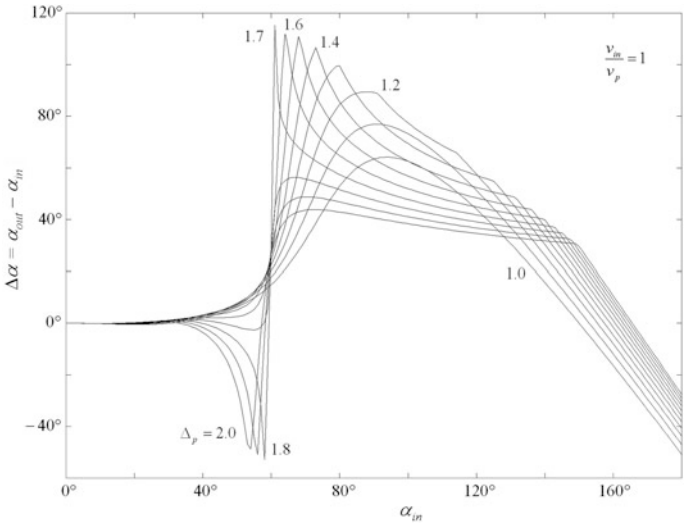


Fig. 9.20 Deflection angle $\Delta\alpha$ as a function of the incoming angle α_{in} in the critical interval $\Delta_p = 1.0 - 2.0$

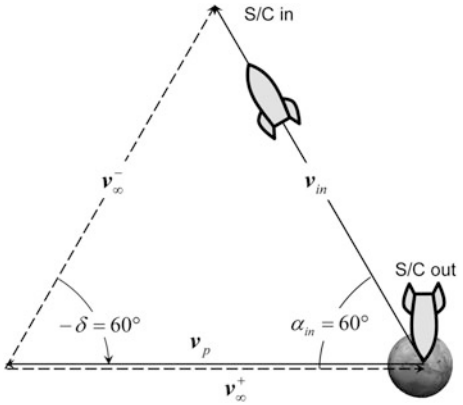


Fig. 9.21 Flyby before a planet for $v_{out} = 0$

We therefore transform Jacobi's integral (11.4.7) in the synodic system of the revolving planet into the heliocentric inertial system by a simple Eulerian rotation. By doing so, the centrifugal force term transforms as (see Murray and Dermott 1999, p. 71),

$$\frac{\xi^2+\eta^2}{2}=\frac{n^2(x^2+y^2)}{2}\rightarrow \boldsymbol{hn}$$

where, $\boldsymbol{n}=const$ is the orbital frequency vector of the planet's circular orbit around the Sun with $n^2=G(m_{sun}+m_p)/a_p^3\approx\mu_{sun}/a_p^3$ the constant orbital frequency (mean motion), and orbital radius a_p. Thus we obtain

$$\frac{E_{tot}}{m}=\varepsilon_{tot}=-\left(\frac{\mu_{sun}}{\Delta r_{sun}}+\frac{\mu_p}{\Delta r_p}\right)-\boldsymbol{hn}+\frac{1}{2}v^2=const \tag{9.5.15}$$

Here Δr_{sun} and Δr_p are the distances of the S/C to the Sun and the planet, respectively, and $\boldsymbol{h}$, v, and i are the S/C's angular momentum, speed, and inclination in the heliocentric system, respectively, i.e. $\boldsymbol{hn}=hn\cos i$.

Note *Eq. (9.5.15) expresses an energy conservation that is quite peculiar. Other than the kinetic and gravitational energy, the new term* $-\boldsymbol{hn}$ *shows up, which we can interpret physically as a flyby potential. As long as the S/C is outside the planetary SOI,* $\varepsilon_{flyby}=-\boldsymbol{hn}=const$ *and therefore according to Newton's third law, Eq. (7.1.11),* $-d\varepsilon_{flyby}/d\boldsymbol{r}=\boldsymbol{F}_{flyby}=0$*. So, the S/C does not experience any deflection force outside the SOI, in accordance with our expectation. When the S/C dives into the SOI,* $\boldsymbol{h}=\boldsymbol{r}\times\boldsymbol{v}$ *changes (the angular momentum is not constant in the heliocentric system!) and thus the flyby energy changes as well, which can be interpreted as a deflection force. The energy transferred by a complete flyby to the S/C is* $\Delta\varepsilon_{flyby}=-(\boldsymbol{h}_{out}\boldsymbol{n}-\boldsymbol{h}_{in}\boldsymbol{n})=-[\boldsymbol{r}\times(\boldsymbol{v}_{out}-\boldsymbol{v}_{in})]\boldsymbol{n}=-(\boldsymbol{r}\times\Delta\boldsymbol{v})\boldsymbol{n}$.

We now apply Eq. (9.5.15) to a position external to the SOI of the planet. In this case $m_{planet}/r_{planet}\ll m_{sun}/r_{sun}$, so we are effectively in a two-body system where $v^2=\mu_{sun}(2/r_{sun}-1/a)$ (see Eq. (7.2.15)) holds. Therefore, we find

$$-\frac{\mu_{sun}}{r_{sun}}-h\sqrt{\frac{\mu_{sun}}{a_p^3}}\cos i+\frac{\mu_{sun}}{r_{sun}}-\frac{\mu_{sun}}{2a}=const$$

so

$$\frac{h}{\sqrt{\mu_{sun}a_p^3}}\cos i+\frac{1}{2a}=const$$

We finally apply this equation to the S/C orbit before and after the flyby at the planet. With Eq. (7.3.7), $h^2=\mu_{sun}a(1-e^2)$, we arrive at

$$\frac{a_p}{a_{in}}+2\cos i_{in}\sqrt{\frac{a_{in}}{a_p}(1-e_{in}^2)}=\frac{a_p}{a_{out}}+2\cos i_{out}\sqrt{\frac{a_{out}}{a_p}(1-e_{out}^2)}=T_P=const \tag{9.5.16}$$

This is the so-called **Tisserand Relation** (Tisserand 1896) with Tisserand's parameter T_p. As a flyby invariant it relates the unknown orbital elements of the exiting S/C to those known of the entering S/C. For a deflection in the orbital plane, $i_{in} = i_{out} = 0$. The Tisserand relation holds for any flyby, also for flights of unknown comets and asteroids by planets. Because their orbital elements change during flyby, it would be hard to decide whether a newly observed celestial body is identical to a known one or not. The Tisserand relation is here a helpful decision criterion.

9.6 Weak Stability Boundary Transfers

Apart from flyby maneuvers there is another possibility to reduce the propulsion demand of an interplanetary mission in a many-body system: the so-called *weak stability boundary transfer*. Weak stability boundary, often abbreviated by WSB, refers to the transitional area between two gravitational fields, that is, the area around the edge of a SOI. For a WSB transfer, you purposefully inject a S/C into a WSB—as shown in Fig. 9.22 with the example of the Sun–Earth–Moon system for the Sun–Earth WSB—such that it arrives there with $v \approx 0$ as determined in the co-rotating system. Under these circumstances the energy demand (equals propulsion demand) $\Delta\varepsilon \approx \boldsymbol{v} \cdot \Delta\boldsymbol{v}$ (see Eq. (8.3.1) for any $\Delta\boldsymbol{v}$, which carries the vehicle into a transfer orbit to the target, becomes arbitrarily small. So, effectively, a WSB transfer is a bi-elliptic transfer (see Sect. 8.4.2). If the bi-elliptic transfer is already advantageous to the Hohmann transfer with regard to delta-v, the WSB transfer is even more favorable than the bi-elliptic transfer, as the gravitational saddle point between Sun and Earth—also called L_1 point (see Fig. 11.11)—is energetically lower than at infinite distance. In summary, one saves time and propulsion effort compared to the bi-elliptic transfer.

The WSB transfer from Earth to Moon, shown in Fig. 9.22, was calculated by numerical integration of the equation of motion in the limited four-body system Sun–Earth–Moon–S/C. It also makes use of another feature of this many-body system: After the S/C has been transferred from a 400 km LEO to a highly elliptic transfer orbit to the Earth–Sun WSB, it remains there for a quite long period of time because of $v \approx 0$. During that time, the position of the Sun changes because of the orbital movement of the Earth, and thus the gravitational force of the Sun affects the S/C more and more from a lateral direction. This gives the S/C the correct and necessary Δv to transfer it into a transfer orbit to the Moon, where it approaches its surface at a minimum 111 km altitude with very low relative velocity without engine ignition (we note that this is the underlying effect of the stable invariant manifolds as shown in Fig. 11.24 in Sect. 11.5.2 for the circular restricted three-body problem). Because of the combination of these three-body effects—WSB transfer plus solar acceleration—the delta-v of the trajectory shown in Fig. 9.22 is more than 200 m s^{-1} less than that for the Apollo flight path. Instead,

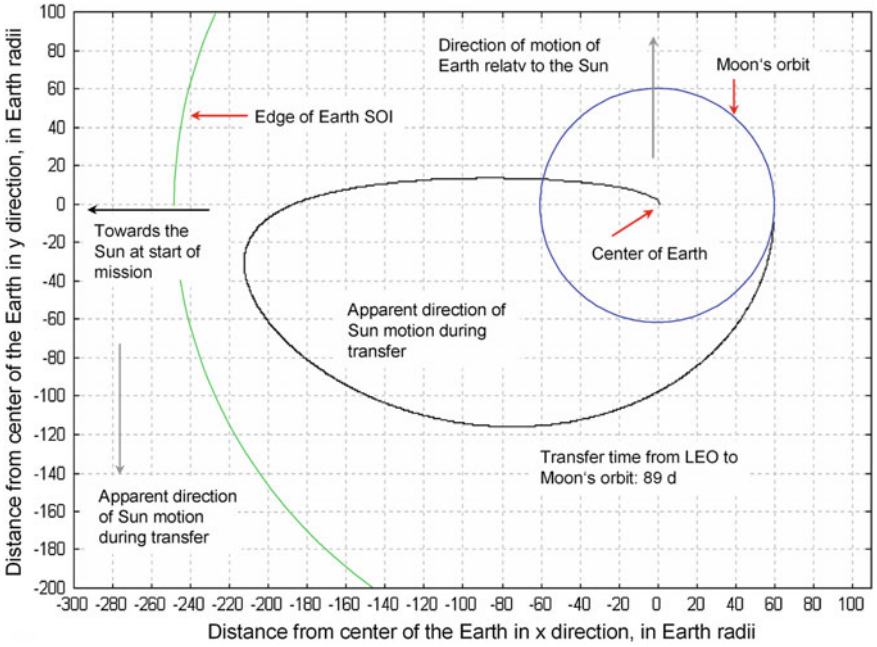


Fig. 9.22 Earth → Moon transfer orbit exploiting solar gravitation at Earth–Sun WSB (perigee altitude = 400 km above the Earth's surface; perigee velocity = 1.410 times the circular orbital velocity)

the transfer time increases from 3 to 90 days, which is however of minor importance for unmanned missions.

Lowest Thrust Transfers

Flyby maneuvers and WSB transfers are the ingredients for ultimate low-thrust missions to the Moon and to the planets in our solar system. The Japanese mission HITEN demonstrated how to get to the Moon with as little propellant as possible. It first made use of a Moon flyby (see Fig. 9.23) to carry the spacecraft to the WSB between Earth and Sun. Then, without engine ignition, just by acceleration of the Sun, it was transferred via the L_5 libration point to the Moon, where the Moon caught the S/C in a highly elliptic orbit by ballistic capture. If you take a closer look, you will realize that this trajectory is impossible with a patched-conics approximation, as it would violate energy conservation; in a three or four body system it is, however, possible.

A similar dexterous sequence of maneuvers in the Earth–Moon region was conceived for the Japanese NOZOMI mission to Mars. Figure 9.24 shows the maneuvers preceding the injection into the transfer orbit. Just as with HITEN, the S/C was first supposed to reach a highly elliptic transfer orbit to the Moon. Then a Moon flyby was used to get to the WSB between Earth and Sun where the acceleration of the Sun turned it back to the Moon. A second Moon flyby brought

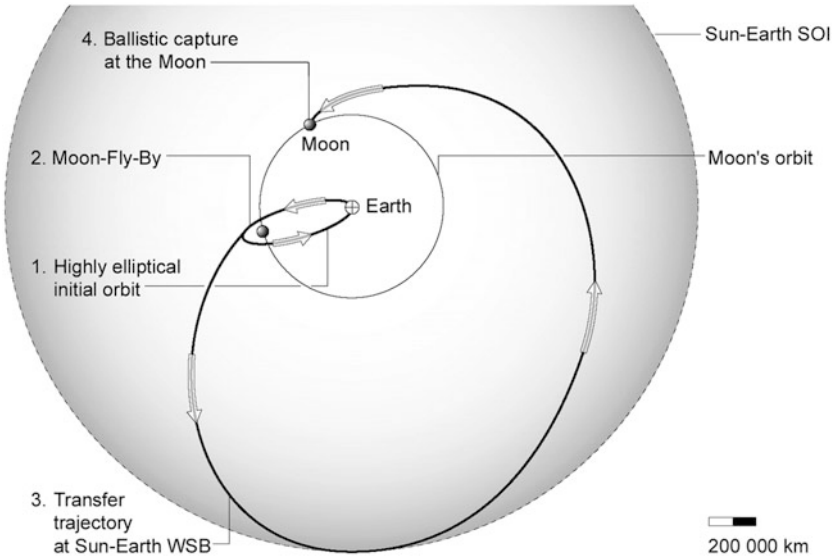


Fig. 9.23 The Moon flyby and WSB transfer trajectory of the Japanese Moon probe HITEN

the S/C back to Earth, where at the closest approach distance a trans-Mars injection burn initiated the transfer flight to Mars.

That was at least the mission planning. However, due to an unfortunate construction the propellant froze in the pipeline to the main engine, and thus this journey could not be carried out as planned. Consequently, Japanese' mission control postponed the trans-Mars injection for 2 years to wait for the next favorable Earth–Mars constellation. During that time, the NOZOMI S/C gained enough speed by ever repeated Moon flybys and WSB transfers that its weak reaction control engines sufficed to give the S/C the final small injection burn to Mars.

9.7 Problems

Problem 9.1 *Planet Waiting Time*
Prove Eq. (9.3.6) from basic considerations.

Hint: Prove first that for flights to outer planets

$$t_{wait} = T_{syn}\left(k + 1 - \frac{2t_H}{T_\oplus}\right) \quad @ \oplus \leftrightarrow \mathrm{O}$$

and for flights to inner planets

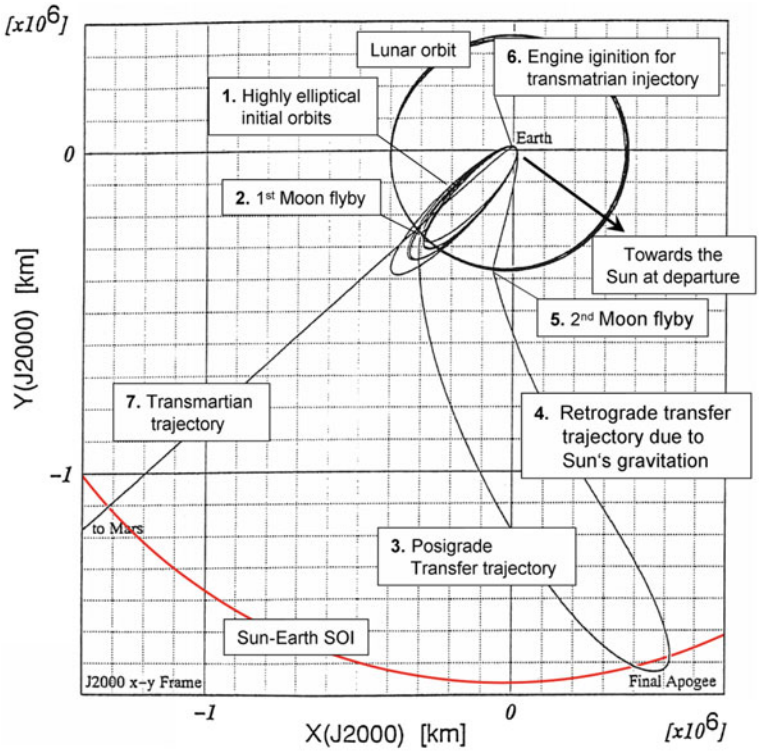


Fig. 9.24 The sequence of Moon flyby and WSB maneuvers of the Japanese Mars probe NOZOMI preceding the trans-Mars injection in the co-rotating Earth–Moon system. *Credit* ISAS

$$t_{wait} = T_{syn}\left(k - 1 + \frac{2t_H}{T_\oplus}\right) \quad @ \oplus \leftrightarrow \bullet$$

Problem 9.2 *Forgetful Asteroid Prospector*
Consider two asteroids that move in circular coplanar heliocentric orbits with the following elements:

Asteroid A: $a = 2.0\,AU, \quad \omega + \Omega = 139^\circ, \quad t_0 = 2025, January\ 1.0$
Asteroid B. $a = 3.5\,AU, \quad \omega + \Omega = 271^\circ, \quad t_0 = 2025, January\ 1.0.$

An absent-minded asteroid prospector working on A decides on 2025, *January* 1.0, to move his ship, with the greatest economy in fuel, to B.

(a) Show that the first available take-off time is 2026 January 26.8.
(b) When he arrives at B he discovers that he has left his Geiger counter on A and has to go back for it. Show that his minimum waiting time on B if the return journey is also made under the fuel economy condition is 1.930 years (neglect the asteroids' gravitational fields).

(Adopted from Roy 2005, Problem 12.10).

Problem 9.3 *Flyby Distances*
Prove that the distances indicated with $-a$ in Fig. 9.14 indeed have this value.

Problem 9.4 *Near Hohmann Transfer I*
Prove Eq. (9.3.17).

Problem 9.5 *Near Hohmann Transfer II*
In Sect. 9.3.2 we found for near Hohmann transfers for the dependence for the chain link $\Delta a \rightarrow \Delta E_{\times}$ in lowest approximation

$$\Delta E_{\times} = -\frac{2}{\sqrt{e_H}}\sqrt{\frac{\Delta a}{a_H}} =: -2\sqrt{\alpha}$$

Show that in the next higher approximation in Δa

$$\Delta E_x = -2\sqrt{\alpha}\left(1 - \frac{\alpha}{4}\right)$$

with corresponding impacts on $\Delta t_{\times}$ and $\Delta\theta_{\times}$.

Chapter 10
Planetary Entry

10.1 Introduction

After a spaceflight, the planetary entry (a.k.a. reentry for entry into Earth's atmosphere) of a spacecraft is subject to the same aerodynamic and physical laws and equations (see Eqs. (6.3.6) and (6.3.7)) as ascent. One might therefore infer that the circumstances of both situations are the same. But they actually differ vastly due to the initial and boundary conditions: At launch we have $h = 0, v = 0$ at flight path angle $\gamma = 90°$ and full thrust during ascent, while at reentry it is exactly the other way round, $h \approx 350$ km, $v \approx 7.9$ km/s, $\gamma \approx 0°$, and no thrust. Owing to these converse initial conditions the S/C in LEO prior to reentry possesses a high amount of potential and kinetic energy of approximately 33 MJ kg^{-1}. This energy has to be annihilated during reentry in a controlled way and in a relatively short period of time, while the structural load on the vehicle needs to be kept within limits. In face of this problem there are four critical parameters to be considered when designing a vehicle for atmospheric reentry to avoid damage to the S/C and the crew:

- Peak heat flux
- Heat load
- Peak deceleration
- Peak dynamic pressure

Peak heat flux (heat per unit area and unit time = heat flow density) selects the thermal protection material, while heat load selects the thickness of the protection material stack. Peak deceleration is of major importance for the crew and should not exceed 8 g. Dynamic pressure causes aerodynamic stress load to the vehicle and is significant in particular for winged bodies: The Shuttle was designed for 2.5 g load while Apollo for a 12 g load. In total, these constraints impose boundaries on the reentry trajectory, which are depictured for a Shuttle reentry in Fig. 10.1.

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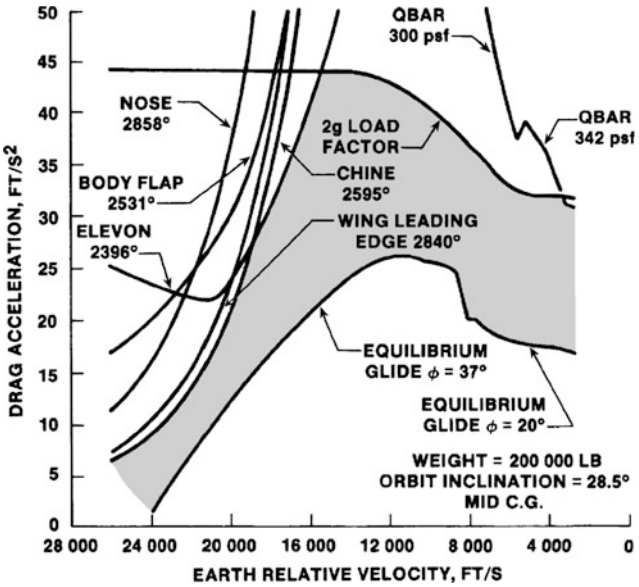


Fig. 10.1 Representative constraint boundaries (English units, temperature in Fahrenheit) for a Space Shuttle reentry shaping a reentry corridor (gray). *Credit* N. Chaffee, NASA/JSC

In summary, the deceleration force causing stress load to the vehicle structure and crew plus the heat load to the thermal protection system are the essential physical quantities, which need to be examined for a reentry as done in this chapter.

10.1.1 Aerothermodynamical Challenges

Aerothermodynamics investigates the heat flux onto the vehicle and the subsequent heating of the spacecraft's surface. During reentry this is caused by the annihilation of the low Earth orbital energy, which essentially is the kinetic energy $E_{tot} \approx \frac{1}{2}mv^2 = \frac{1}{2}mg_0\, R_\oplus \approx 33$ MJ kg^{-1}. The following rough estimate shows that this heat flux and surface heating is tremendous, and keeping the heat in given limits is not easy to accomplish.

A capsule in the so-called ballistic reentry (see Sect. 10.3.2) for aerobraking converts its orbital energy into frictional heat, which is released as a heat flow rate $\dot{Q} = E_{tot}/\tau$ within typically $\tau \approx 0.5$ min. Usually 99.9% of that is released via heat convection to the air flow. The rest, sometimes called convective heat flux, which is quantified by the so-called Stanton number St and having roughly the universal value of $St \approx 0.1\%$, is taken up by the surface of the S/C. The thermal shield of the S/C with an area of A and with emissivity of typically $\varepsilon \approx 0.85$ then radiates away

the absorbed heat with a heat flux $\dot{q}_{S/C}$, which according to the Stefan-Boltzmann law is related to the shield temperature T

$$T^4 = \frac{\dot{Q}}{\varepsilon\sigma A} = \frac{\dot{q}_{S/C}}{\varepsilon\sigma} \tag{10.1.1}$$

with the Stefan–Boltzmann constant $\sigma = 5.6705 \times 10^{-8}$ W m^{-2} K^{-4}. When the thermal shield is in thermal balance, we get the following from the above considerations:

$$St\frac{E_{tot}}{\tau} = St\frac{mg_0R_\oplus}{2\tau} = A\varepsilon\sigma T^4 \tag{10.1.2}$$

Example 1

The Mercury capsule had a mass of $m = 1450$ kg and a thermal shield with area $A = 2.8$ m^2. From Eq. (10.1.2) we derive that during reentry a shield temperature of roughly $T \approx 1890$ K $= 1610$ °C occurred. These temperatures are at the limit today's heat insulations can withstand.

Example 2

If a Space Shuttle with a mass $m \approx 100{,}000$ kg and an effective stagnation point area of $A \approx 5$ m^2 would have also reentried on a ballistic trajectory, it would have been exposed to $T \approx 3680$ °C and hence would have not survived, as the thermal tiles at the nose and at the front edge of the Shuttle wings were designed for a maximum of 1750 °C. Actually, the Shuttle with its high lift reduced the peak heat load by drastically extending the path through the atmosphere and thus spreading the heat over a much longer period of time about 10 min (equals blackout time, see Sect. 10.7.4). The so-called angle of attack (see Sect. 6.2.5) also displaced part of the generated heat to the lower side, which we will account for by increasing the effective stagnation point area by the factor of 2. According to Eq. (10.1.2) these measures reduced the peak temperature to $T \approx 1600$ °C. This is a conservative estimate. In fact, the measured maximal reentry temperatures at the critical wing leading edge did not exceed 1400 °C.

Heat Flux Models

These results are only rough estimates. To get more accurate results, we have to calculate the time-dependent heat flux as well as its maximum. The total heat flow rate generated by the air impinging with velocity v and mass flow rate $\dot{m} = \rho vA$ (see continuity Eq. (1.2.8)) is given by Newtonian flow theory (see Sect. 6.2.2), which holds in the free molecular flow regime at extremely low flow densities or at hypersonic speeds when the shock layer is so thin that it practically coincides with the body surface, as $\dot{Q}_{tot} = \dot{m}h_{tot}$. The total heat flux therefore is

$$\dot{q}_{tot} = \frac{\dot{Q}_{tot}}{A} = \frac{\dot{m} h_{tot}}{A} = \frac{\rho v A}{A} \frac{v^2}{2} = \frac{1}{2} \rho v^3$$

where h_{tot} is the total enthalpy per unit mass of the free streaming air, which at hypersonic speeds essentially equals the specific kinetic energy of the air flow. The portion that by diffusion mechanisms is transferred to the S/C is

$$\dot{q}_{S/C} = St \cdot \dot{q}_{tot} = \frac{St}{2} \rho v^3$$

This is the elementary theoretical result for the heat flux of a flat plate with area A.

The more sophisticated and most common one-dimensional, steady, inviscid, adiabatic flow theory of dissociating gas in thermochemical equilibrium was shown by Fay and Riddell to result in a stagnation-point heat transfer rate equation, so-called *Fay–Riddell equation* (Bertin 1994, Eq. 5-36), which for a cold wall approximation reads (Bertin 1994; Hirschel 2004)

$$\begin{aligned} \dot{q}_{S/C} &= \frac{St}{2} \sqrt{\rho \rho_q \frac{R_0}{R_n}} \cdot v^3 \\ &= \dot{q}_e \sqrt{\frac{\rho}{\rho_e} \frac{R_0}{R_n}} \cdot \left(\frac{v}{v_\triangleright} \right)^3 \end{aligned} \tag{10.1.3}$$

with

$$\begin{aligned} &\rho_q = 0.121 \text{ kg m}^{-3} = 0.121 \text{ W s}^3 \text{ m}^{-5} \quad @ \ St = 1.00 \times 10^{-3} \\ &\rho_e = 1.80 \times 10^{-8} \text{ kg m}^{-3} \\ &R_0 = 1 \text{ m} \\ &v_\triangleright = 7.905 \text{ km s}^{-1} \text{ first cosmic velocity} \\ &\dot{q}_e = \frac{St}{2} \sqrt{\rho_q \rho_e} \cdot v_\triangleright^3 = 1.15 \text{ W cm}^{-2} \end{aligned}$$

where ρ_q corresponds to the so-called Sutton-Graves value of the stagnation point heat transfer coefficient, ρ_e is the mean atmospheric density at entry interface according to MSIS-E-90, R_n the radius of the surface curvature at that (nose) point where stagnation occurs, and $\dot{q}_e$ the standard heat flux at entry interface.

According to highly sophisticated one-dimensional adiabatic steady-state heat flux models the heat flux at the stagnation points of an aeroshell at hypersonic speeds was generally determined to be $\dot{q}_{S/C} \propto \sqrt{\rho / R_n} \cdot v^x$ with $x = 3.00 - 3.22$ depending on details of the model with $x = 3$ if the viscosity of the medium is proportional to $\sqrt{T}$. Because it can be shown that the essential results of this chapter vary at maximum by 1% depending on x, we assume in the following the Fay–Riddell equation with $x = 3$.

Example

Upon reentry of the Space Shuttle through Earth's atmosphere the maximum stagnation point heating occured at 68.9 km altitude, where $\rho = 1.075 \times 10^{-4}$ kg m^{-3} and $v = 6.61$ km s^{-1}. At that trajectory point the Shuttle had a 40° angle of attack, which brings about an effective nose radius of 1.29 m. What is the stagnation point heat flux?

According to Eq. (10.1.3) we find $\dot{q}_{S/C} = 45.7$ W cm^{-2}. Based on experimental data from the Space Shuttle, Zoby (1982) quotes a maximum stagnation point heat flux of $\dot{q}_{S/C} = 45$ W cm^{-2}, very close to our result.

At a given entry velocity profile $v = v(\rho)$ this heat flux at the stagnation point achieves its maximum for $d\dot{q}/d\rho = 0$. This is the conditional equation, which we will use in this chapter to calculate the peak heat flux, which in turn can be inserted into Eq. (10.1.1) to derive the peak surface temperature of the S/C.

10.1.2 Entry Interface

Let us have a closer look at the reentry process into the atmosphere. Reentry formally commences at the so-called entry interface.

> According to international agreements, the entry interface is located at an altitude of 400,000 ft = 122 km, i.e., at the border between heterosphere and homosphere.

The atmosphere of course does not abruptly set in at this altitude, but drag and lift start there to have an influence on the entering vehicle. According to Eq. (6.1.5) the homosphere below 120 km the atmospheric density obeys the barometric formula

$$\rho(r) = \rho_0 \exp\left(-\frac{r-R}{H}\right) = \rho_0 \exp\left(-\frac{h}{H}\right)$$

with $\rho_0 = 1.752$ kg/m^3 and $H = 6.7 \pm 0.1$ km average scale height for the entire homosphere.

At the entry interface the orbit parameters take on the values r_e, v_e, γ_e. According to standard conventions (for details see Sect. 10.2) the flight path angle γ of a reentry vehicle (a.k.a. *entry angle* or *velocity angle*) is to be understood as a positive angle, and hence also $\gamma_e > 0$.

10.1.3 Deorbit Phase

Before we study the reentry into the atmosphere it needs to be known how the S/C gets from its preceding trajectory to the entry interface. Usually the starting point is a circular Earth orbit, the radius of which we denote by r_i (initial). Deorbit is initiated by a deorbit burn at a certain position on this orbit, which transfers the S/C onto an entry ellipse with a low-lying periapsis, which intersects the entry interface at a predetermined flight path angle γ_e (see Fig. 10.2). The deburn position has to be chosen such that the entry point is at the right distance to the anticipated landing site. We now want to evaluate the questions: What is the required delta-v for the deorbit? At which position is the entry interface attained? What is the entry velocity at entry interface? It is now our objective to determine these three values.

At deorbit burn the S/C is positioned at the apoapsis of the entry ellipse with the still to be determined orbital elements a, e. If this ellipse is supposed to intersect the entry interface at position r_e, θ_e with entry velocity v_e the ellipse is unequivocally determined because Eqs. (7.3.16b), (7.3.14), and (7.3.5), state that the following holds:

$$\begin{aligned} \cos\gamma_e &= \frac{1+e\cos\theta_e}{\sqrt{1+2e\cos\theta_e+e^2}} \\ v_e &= \frac{\mu}{h}\sqrt{1+2e\cos\theta_e+e^2} \\ r_e &= \frac{a(1-e^2)}{1+e\cos\theta_e} = R+h_e = 6500\ \text{km} \end{aligned} \tag{10.1.4}$$

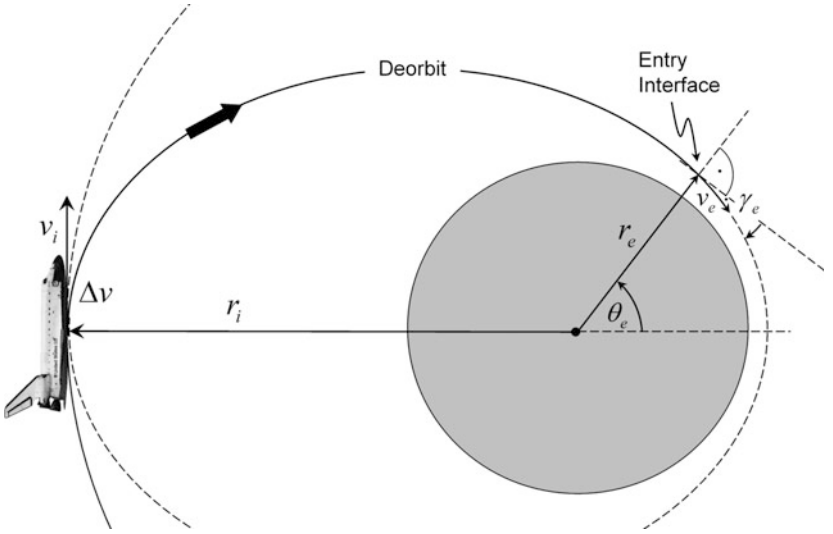


Fig. 10.2 Deorbit phase: from deorbit burn to entry interface

The orbital elements are not independent from each other because we have at the deorbit position at apogee

$$r_i = a(1+e) \tag{10.1.5}$$

From this condition and from Eq. (10.1.4) we derive

$$e\cos\theta_e = \frac{a(1-e^2)}{r_e} - 1 = \alpha(1-e) - 1 \tag{10.1.6}$$

with

$$\alpha := \frac{r_i}{r_e} = \frac{r_i}{6500\ \text{km}} > 1 \tag{10.1.7}$$

and

$$h = \sqrt{\mu a(1-e^2)} = \sqrt{\mu r_i(1-e)}$$

We insert these results back into Eq. (10.1.4) and find after some trivial steps

$$\begin{aligned}
\cos\gamma_e &= \alpha\sqrt{\frac{1-e}{2\alpha-1-e}} \\
v_e &= v_i\sqrt{2\alpha-1-e} \\
e\cos\theta_e &= \alpha(1-e)-1 \\
\Delta v &= \sqrt{\frac{\mu}{r_i}} - \sqrt{\frac{\mu}{a}\frac{1-e}{1+e}} = v_i\left(1-\sqrt{1-e}\right)
\end{aligned}$$

with

$$v_i = \sqrt{\frac{\mu}{r_i}}$$

the orbital velocity of the initial orbit and Δv the propulsion demand for the deorbit burn. These equations have to be interpreted as follows. At a given r_i, r_e, γ_e the first equation delivers the eccentricity e of the entry ellipse. With this we find from the other equations the wanted entry velocity and the propulsion demand for deorbit. This is exactly what we are going to do now. From the first equation we derive after some rearrangements

$$e = \frac{\alpha^2 - (2\alpha-1)\cos^2\gamma_e}{\alpha^2 - \cos^2\gamma_e} \tag{10.1.8}$$

We insert this into the other three equations and find

$$v_e = \alpha v_i \sqrt{\frac{2\alpha_-}{\alpha^2 - \cos^2\gamma_e}} = \sqrt{\frac{2\mu}{r_e}\frac{\alpha\alpha_-}{\alpha^2 - \cos^2\gamma_e}} \tag{10.1.9a}$$

$$\Delta v = v_i\left(1 - \cos\gamma_e\sqrt{\frac{2\alpha_-}{\alpha^2 - \cos^2\gamma_e}}\right) = v_i - \frac{\cos\gamma_e}{\alpha}v_e \tag{10.1.9b}$$

$$\cos\theta_e = \frac{\alpha_-^2\cos^2\gamma_e - \alpha^2\sin^2\gamma_e}{\alpha_-^2 + \alpha^2\sin^2\gamma_e} \tag{10.1.9c}$$

with $\alpha := \frac{r_i}{r_e} = \frac{r_i}{6500\text{ km}}, \alpha_- := \alpha - 1 = \frac{r_i}{r_e} - 1, v_i = \sqrt{\frac{\mu}{r_i}}.$.

These are the wanted expressions. There are only two cases of practical interest: $\alpha \approx 1$ and $\alpha \to \infty$, which we will investigate now.

Interplanetary Reentry ($\alpha \to \infty$)
The S/C approaches Earth from infinity, for instance from the Moon or from a planet, for aerocapture and subsequent reentry. In this case we simply get

$$\begin{aligned} v_e &\approx \sqrt{\tfrac{2\mu}{r_e}\left(1 - \tfrac{1}{\alpha}\right)} \approx \sqrt{\tfrac{2\mu}{r_e}} \\ \Delta v &\approx v_i\left(1 - \cos\gamma_e\sqrt{\tfrac{2}{\alpha}}\right) \approx v_i \approx 0 \quad @\ \ \gamma_e^2 \ll 1 \\ \theta_e &= 2\gamma_e \end{aligned} \tag{10.1.10}$$

As expected, the entry velocity is the second cosmic velocity (see Sect. 7.4.4), and the deorbit burn effort becomes arbitrarily small. Of course the burn cannot be performed so precisely as to hit exactly the entry interface. Therefore, the entry trajectory usually must be adjusted several times during approach to pass through the so-called entry corridor to ensure aerocapture (see also Sect. 10.5.2). Figure 10.3 displays the entry corridor for Apollo 11. Three adjusting maneuvers were planned for this Moon mission to enter the corridor.

LEO Reentry ($\alpha \to 1$)
The S/C initially is in a LEO. For $\gamma_e \leq 10°$ the trigonometric functions can be approximated and because then $\gamma_e^4 \ll 20 \cdot \alpha_-$, Eqs. (10.1.9a) and (10.1.9b) can be written as (exercise, Problem 10.1)

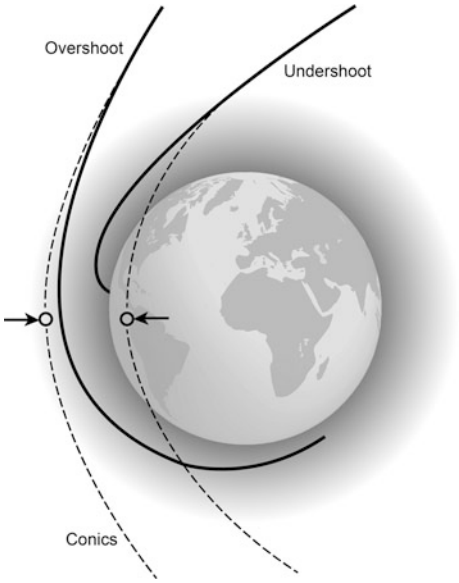


Fig. 10.3 Limiting trajectories of the Apollo reentry corridor

$$\begin{aligned} v_e &= v_i\left(1+\tfrac{3}{4}\alpha_- - \frac{\gamma_e^2}{8\alpha_-}\right) \\ \Delta v &= \tfrac{v_i}{4}\left[\alpha_- + \frac{\gamma_e^2}{2\alpha_-}\right] \end{aligned} \quad @ \quad \gamma_e \le 10^\circ \qquad (10.1.11)$$

from which follows

$$\begin{aligned} v_e < v_i \quad @ \quad \gamma_e > \sqrt{6}\cdot\alpha_- \\ v_e > v_i \quad @ \quad \gamma_e < \sqrt{6}\cdot\alpha_- \end{aligned}$$

For θ_e no simpler expression can be derived than that in Eq. (10.1.9c).

Example

After undocking from the ISS at 400 km altitude ($\alpha = 1.0428, v_i = 7.669$ km s^{-1}) the Space Shuttle had to deorbit such that it encounters the entry interface with a standard flight path angle $\gamma_e = 1^\circ$. According to Eqs. (10.1.9c) and (10.1.11) we obtain: $\theta_e = 46.1^\circ, v_e = 1.031 \cdot v_i = 7.91$ km s^{-1}, and $\Delta v = 0.0116 \cdot v_i = 88.8$ m s^{-1} (cf. Shuttle Reentry, Sect. 10.7.1).

On the other hand the Soyuz capsule after undocking from the ISS usually acquires a flight path angle of $\gamma_e = 3^\circ$ at entry interface. We find from Eqs. (10.1.9c) and (10.1.11): $\theta_e = 76.2^\circ, v_e = 1.024 \cdot v_i = 7.85$ km s^{-1}, and $\Delta v = 0.0187 \cdot v_i = 143$ m s^{-1}.

Note *For all flat entries from LEO the entry velocity is* $v_e \approx v_\triangleright$, *where* $v_\triangleright = \sqrt{g_0 R_\oplus} = 7.905\ \mathrm{km\ s^{-1}}$ *is the first cosmic velocity, while for entries from infinity (celestial bodies)* $v_e \approx v_{\triangleright\triangleright}$ *with* $v_{\triangleright\triangleright} = \sqrt{2 g_0 R_\oplus} = 11.18\ \mathrm{km\ s^{-1}}$ *the second cosmic velocity. Therefore, the assumptions, which we are going to be used in the following reentry investigations,* $\varepsilon_e = v_e^2/v_\triangleright^2 = 1$ *and* $\varepsilon_e = v_e^2/v_\triangleright^2 = 2$, *respectively, are excellent assumptions for these two cases.*

10.2 Equations of Motion

We are now at entry interface, where atmospheric reentry commences. As reentry is subject to the same physical laws as ascent, we adopt the general orbit Eqs. (6.3.6)–(6.3.12). But in contrast to ascent, no propulsion is required, and this is why we set thrust and also the mass change rate $\dot{m} = 0$ to zero.

According to our current definition a descending S/C would have negative flight path angle. But to be in line with standard conventions it should be positive (see Fig. 10.4). So we formally apply the transformation $\gamma \to -\gamma$, and $\dot{\gamma} \to -\dot{\gamma}$ to Eqs. (6.3.6)–(6.3.12), which results in

$$\dot{v} = -\frac{D}{m} + g \sin\gamma \tag{10.2.1}$$

$$v\dot{\gamma} = -\frac{L}{m} + \left(g - \frac{v^2}{r}\right)\cos\gamma \tag{10.2.2}$$

$$\begin{aligned} \dot{h} &= -v \sin\gamma \\ \dot{x} &= v \cos\gamma \end{aligned} \tag{10.2.3}$$

with altitude $h = r - R$, radius of the planet R, downrange distance x, and (see Sects. 6.2.3 and 7.1.2)

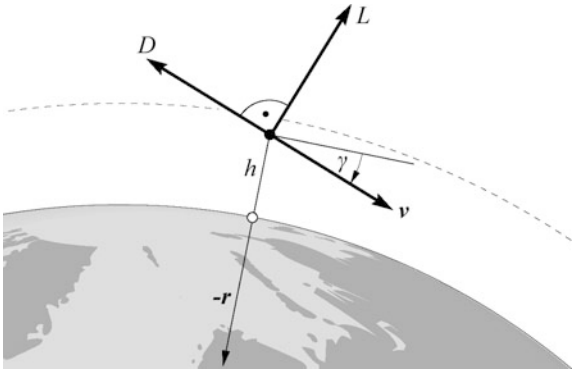


Fig. 10.4 Definition of the reentry variables

$$D = \frac{1}{2}\rho(r)v^2 C_D A_\perp = mv^2 \frac{\kappa_D}{H} e^{-\frac{h}{H}} \tag{10.2.4}$$

$$L = \frac{1}{2}\rho(r)v^2 C_L A_\perp = mv^2 \frac{\kappa_L}{H} e^{-\frac{h}{H}} \tag{10.2.5}$$

$$g(r) = g_0 \frac{R^2}{r^2} \tag{10.2.6}$$

We recall from Sect. 6.3 that we assume lift forces to lie in the motion plane and the S/C therefore remains in this plane and we have no cross motion.

10.2.1 Normalized Equations of Motion

To study the behavior of reentry we need to solve the equations of motion numerically in their time-dependent representation. Because Eqs. (10.2.1)–(10.2.3) are afflicted by physical dimensions, they are not suitable for that. We therefore revert to the dimensionless form of the equations of motion as given by Eq. (6.3.13). For reentry we set the normalized thrust $\Phi = 0$ and again apply the formal transformation $\gamma \to -\gamma$, and $\dot{\gamma} \to -\dot{\gamma}$, thus getting the following result

$$\boxed{\begin{aligned} \mu' &= -\mu^2 \kappa_D \frac{R}{H} e^{-\eta} + \sin\gamma \\ \mu\gamma' &= -\mu^2 \kappa_D \frac{R}{H}\frac{L}{D} e^{-\eta} + (1-\mu^2)\cos\gamma \\ \eta' &= -\mu \sin\gamma \\ \chi' &= \mu \cos\gamma \end{aligned}} \tag{10.2.7}$$

with

$$\boxed{\begin{aligned} &\mu = \frac{v}{v_\triangleright}, \eta = \frac{h}{H}, \chi = \frac{x}{H} \\ &\tau = \frac{g_0}{v_\triangleright} t, \ (\ldots)' = \frac{d}{d\tau}, \\ &v_\triangleright = \sqrt{g_0 R} \end{aligned}}$$

We recall that the first three equations are coupled, and that η', χ' are the normalized altitude and downrange velocities, respectively, which enable us to derive the time-dependent solutions $h(t), x(t)$ and hence the ascent and reentry trajectory $h = h(x)$. The equations of motion in form of Eq. (10.2.7) are optimally adapted to be coded and solved numerically such as by a Runge-Kutta method. For specific problems more elaborate equations without the approximations made here (see beginning of Sect. 10.3) are used. The relatively simple equations above however capture general entry behavior, so we will limit ourselves to them.

Significance of *L/D* Ratio

Equations (10.2.7) make clear that there is only one characteristic S/C variable, L/D, to control the reentry trajectory. We recall from Sect. 6.2.5 that for hypersonic velocities L/D largely depends on the angle of attack (AOA) α. For Apollo capsules we had (see Eq. 6.2.26)

$$\frac{L}{D} = 0.0143 \cdot \alpha[^\circ] \quad @ \text{ Apollo capsule, } \alpha = 0 - 40^\circ \tag{10.2.8}$$

In the following we will assume $\alpha = 20^\circ$ and hence $L/D_{capsule} = 0.32$ (Apollo 4: L/D = 0.369). For winged bodies L/D was given in Eq. (6.2.28) as

$$\frac{L}{D} = \frac{\sin\alpha \sin 2\alpha}{2\sin^3\alpha + C_f} \quad @ \text{ winged body} \tag{10.2.9}$$

where C_f is the skin friction drag coefficient. Specifically for the Space Shuttle we found

$$C_f = 0.021\left(\frac{\rho_0 v_0}{\rho v}\right)^{1/5}, \qquad \rho_0 v_0 = 1.2041 \text{ kg m}^{-3}\, Ma \quad v \geq 3\, Ma \tag{10.2.10}$$

For the typical AOA $\alpha = 20^\circ - 42^\circ$ of a Shuttle reentry we therefore get $L/D_{Shuttle} = 1.0 - 1.8$. An average $L/D_{Shuttle,ave} = 1.3$, which we will use occasionally in the following, is therefore attained at $\alpha = 35^\circ$.

As the horizontal lift and hence the bank angle (see Sect. 6.2.5) controls the out of flight plane motion, we summarize by stating that

> The L/D ratio and hence the angle-of-attack α is the only and hence crucial parameter in controlling the powerless reentry trajectory in the flight plane, while the bank angle μ is the only control parameter for out of flight plane motion (cross-range capability).

Numerical Solutions

To get a first overview of the entry behavior, Figs. 10.5, 10.6 and 10.7, with reentry trajectories in the upper part, describe the reentry of a capsule with a typical $L/D = 0.3$ for three different entry angles: a steep $\gamma_e = 45^\circ$, a medium $\gamma_e = 10^\circ$, and a very flat entry with $\gamma_e = 2^\circ$. They were calculated with a step-size controlled Runge-Kutta method solving the above normalized system of equations. The two illustrations on the same page belong to the same entry angle. The time-dependent velocities, altitudes, and decelerations are shown in the lower illustrations. The different entry profiles as a function of the entry angle first attracts attention. Note the quite different scales of the downrange x-axes, so visually the depicted entry angles are not to scale. Only in Fig. 10.5a the x- and y-scales are the same, so the depicted profiles have accurate contours.

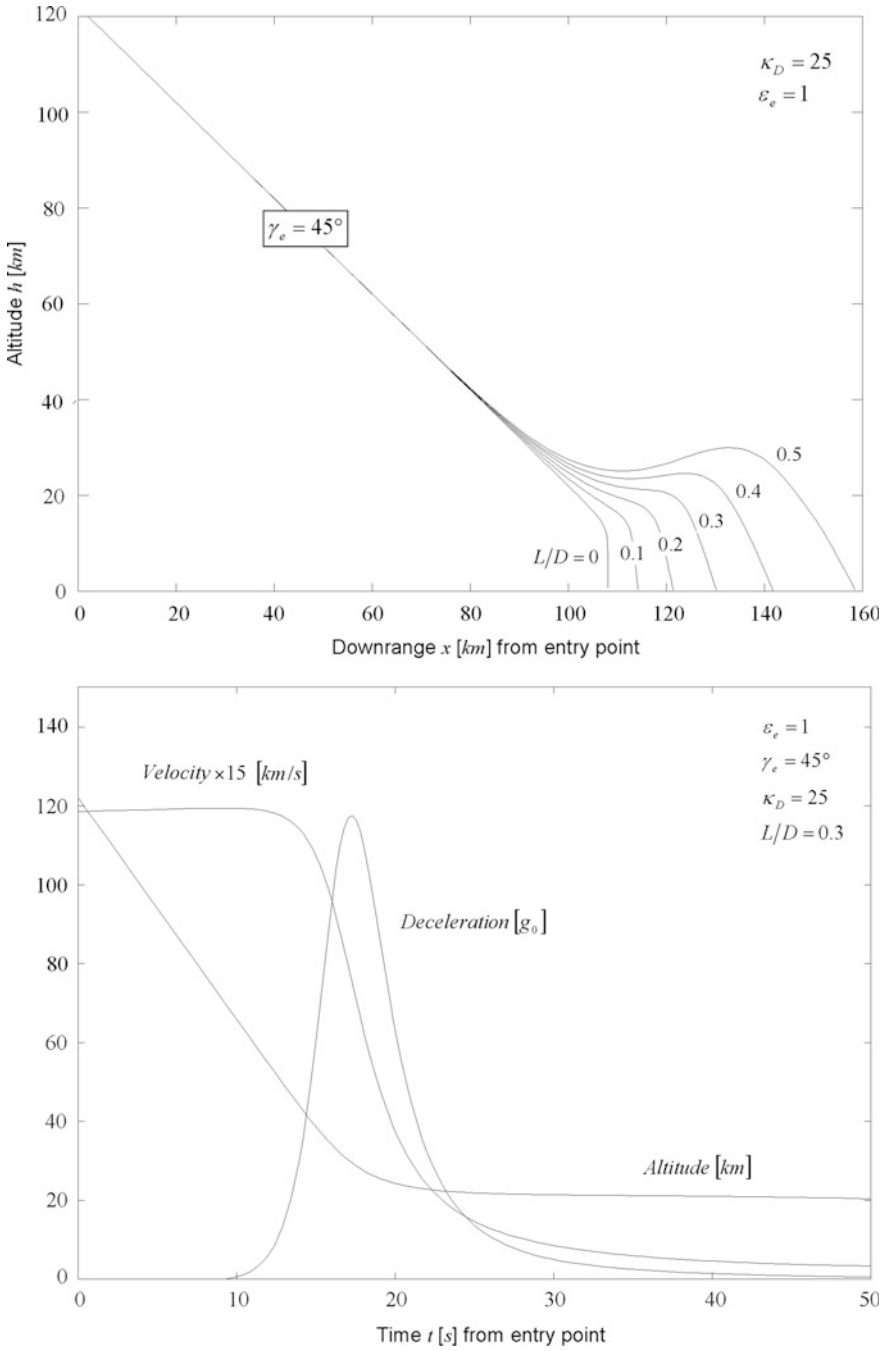


Fig. 10.5 Entry profiles of a spacecraft with $\gamma_e = 45°$

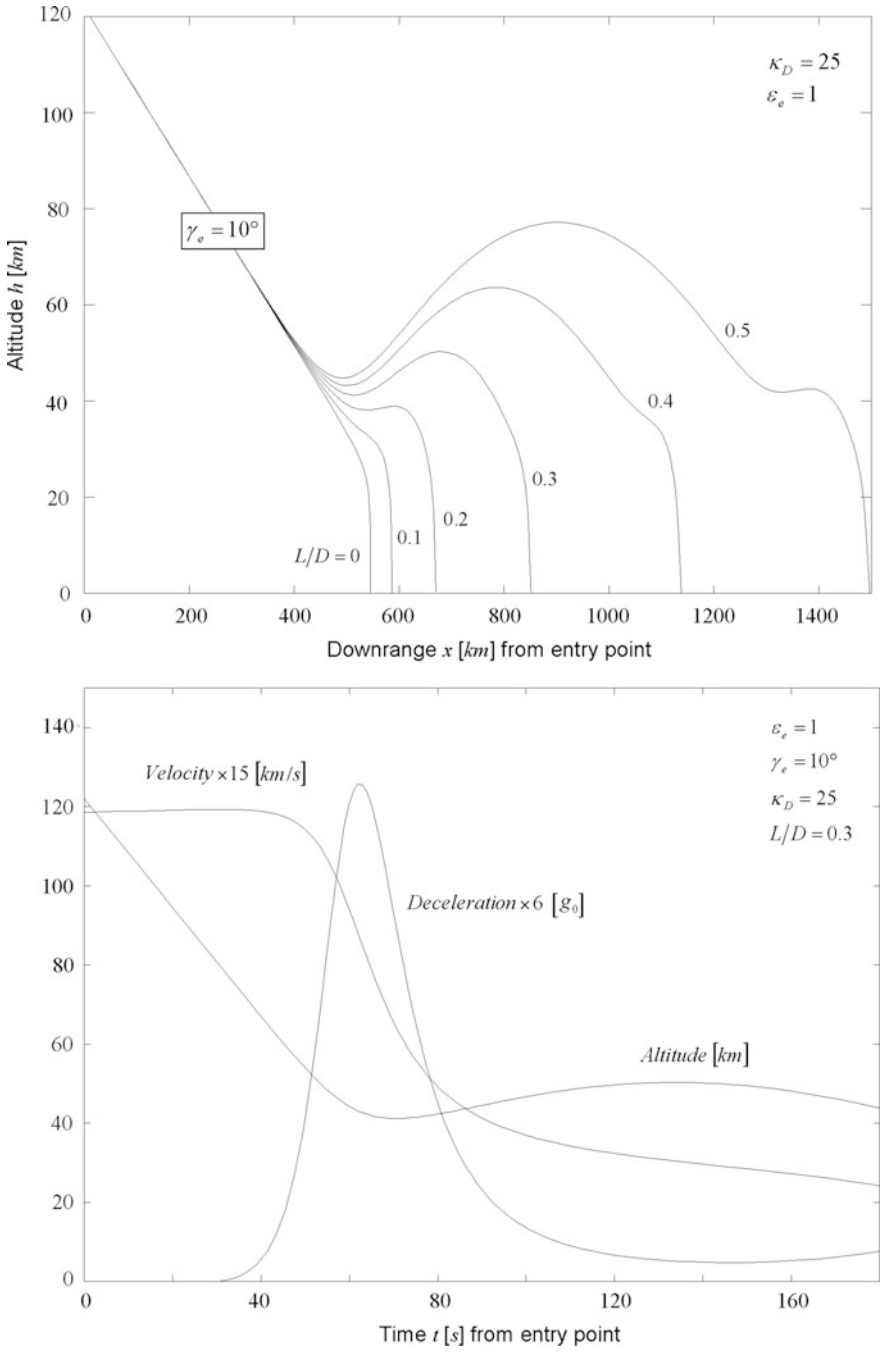


Fig. 10.6 Entry profiles of a spacecraft with $\gamma_e = 10°$

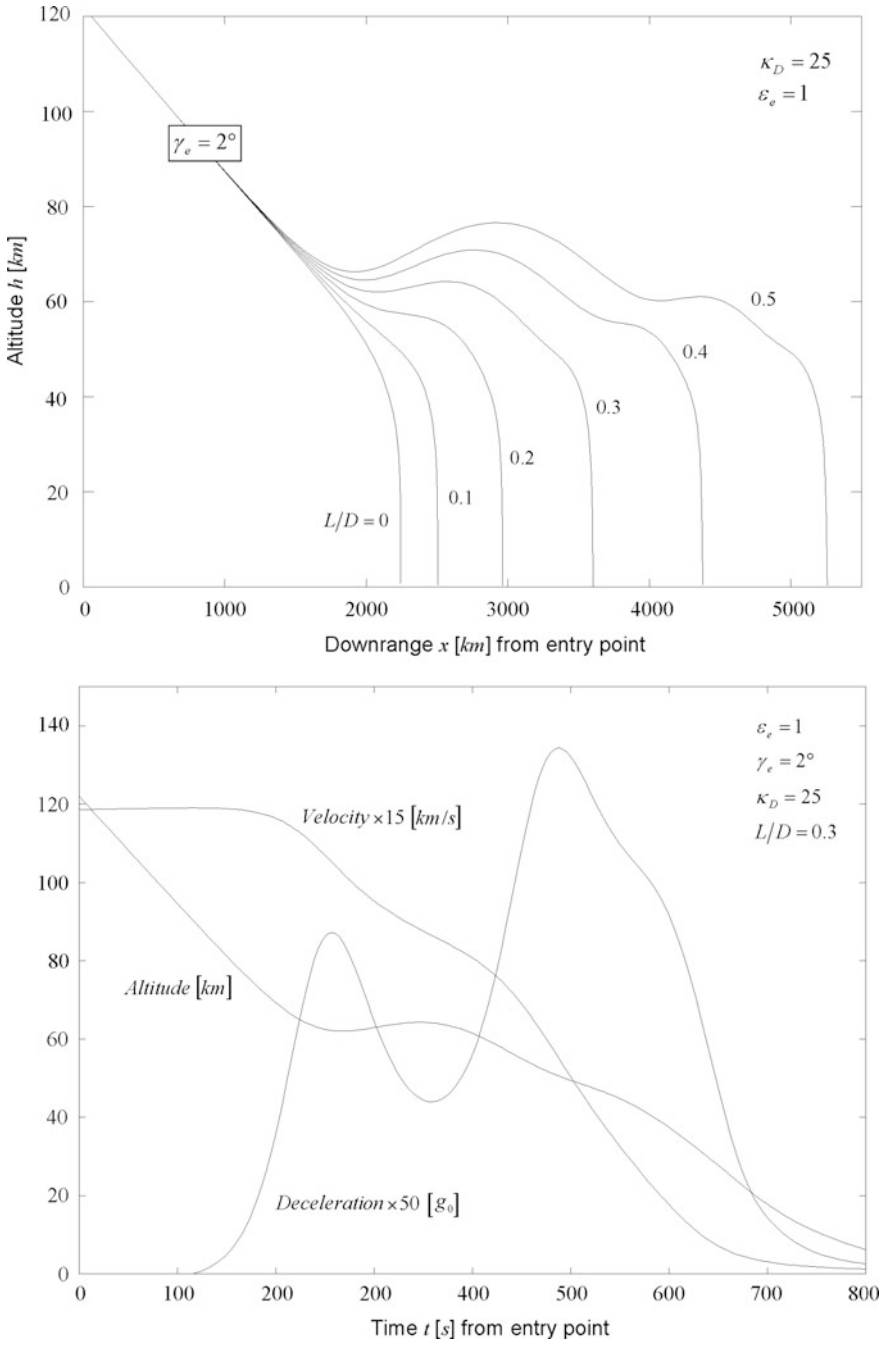


Fig. 10.7 Entry profiles of a spacecraft with $\gamma_e = 2°$

The most serious effect of flat entry angles are the increasing ranges and entry durations: A S/C with $\gamma_e = 45°$ hits the ground only 150 km downrange of the entry point, with $\gamma_e = 10°$ this comes to approximately 1000 km, and with $\gamma_e = 2°$ approximately 4000 km with correspondingly increased entry durations. Apart from that the entry profiles are the same at the beginning: With an altitude of down to 60 km, with steeper ones even down to 40 km, the reentering body moves on a straight line. Only then do lifting forces become significant. Lifts with $L/D > 0.2$ have dramatic consequences: Instead of going down continuously, the S/C literally rebounds off the atmosphere, falls back, dips in again, and then for big lifts and very flat entries slightly rises again, until it completely goes down. If the excursions are small, they are called "reflections" and "skips" for larger excursions with a subsequent ballistic coast phase. They are typical for common capsules with $L/D \approx 0.3$, and they are very pronounced for winged bodies $L/D > 1$ if there are no countermeasures. These effects, which are of quite practical interest, will be treated in detail in Sect. 10.5. Winged bodies with their more fragile structures should not enter with an angle that is too steep, because the decelerations already attain 20 g at 10°. Neither the crew nor the system would be able to endure that. These winged bodies have to enter with a very flat angle with typically 2° (Fig. 10.7), so that during the second reflection they only have to endure a maximum deceleration of merely 3 g, which is tolerable. The reason for the reduction of the deceleration is that the reduction of the velocity, i.e., the acceleration, is spread over a longer period of time, and thus it is reduced at any point in time. The critical decelerations with steeper entries $\gamma_e \geq 10°$ are always just before the reflection, and they drastically increase with an increasing entry angle. For $\gamma_e = 45°$ and $L/D = 0.3$ it already amounts to 118 g!

10.2.2 Reduced Equations of Motion

Numerical solutions are imperative for real missions, but they furnish no insight into why the entry profiles are as they are. For our goal of understanding, we need to find solutions or at least partial solutions, which mathematically relate entry parameters and entry trajectory and thus show up the limits of applicability. The above dimensionless equations of motion are still too complicated for analytical solutions, so we are looking for simplifications.

ε and γ Equations

The third differential equation relates the altitude with time. This relation allows us to substitute the time variable by the altitude variable. Thereby we get rid of the time dependence of the entry trajectory and obtain the entry variables as a function of altitude $v(h), \gamma(h)$. This is exactly what we want. So we are looking for differential equations with h as the independent variable. We again make use of Eqs. (10.2.4) and (10.2.5); this time however we substitute the entry velocity by the new dimensionless variable

$$\varepsilon := \frac{v^2}{v_\triangleright^2} = \frac{E_{kin}(v)}{E_{kin,0}}. \tag{10.2.11}$$

with $v_\triangleright = \sqrt{g_0 R_\oplus} = 7.905$ km s^{-1} which is the first cosmic velocity (see Eq. (7.4.4)). We are now set to replace the time variable t by the altitude variable. We do so by introducing the *dimensionless altitude variable* λ, which quite naturally occurs in the differential equations

$$\lambda := \frac{2\kappa_D}{\sin\gamma_e} e^{-\frac{h}{H}} = \frac{BH}{\sin\gamma_e}\rho \tag{10.2.12}$$

This altogether allows us (exercise, Problem 10.3) to transform the equations of motion into the following dimensionless form of only two coupled differential equations

$$\frac{d(\ln\varepsilon)}{d\lambda} = \underbrace{-\frac{\sin\gamma_e}{\sin\gamma}}_{drag} \underbrace{+\frac{2H}{\varepsilon\lambda R}}_{gravity} \quad \textbf{ε equation} \tag{10.2.13a}$$

$$\frac{d(\cos\gamma)}{d\lambda} = \underbrace{\frac{\sin\gamma_e}{2}\frac{L}{D}}_{lift} \underbrace{-\left(\frac{1}{\varepsilon}-1\right)\frac{H\cos\gamma}{\lambda R}}_{gravity-centrifugal} \quad \textbf{γ equation} \tag{10.2.13b}$$

Let us have a closer look at these equations. They describe the state changes of the entry body (velocity, equals kinetic energy ε, and flight path angle γ) as a function of the instantaneous altitude λ. If we compare them with Eqs. (10.2.1) and (10.2.2) we see the following. On the right-hand side of Eq. (10.2.13a) the first term is the modified drag term, and the second term is the modified gravitational term. On the right-hand side of Eq. (10.2.13b) we have the modified lift term as the first term, and the gravitational term ($1/\varepsilon$) as the second reduced by the centrifugal force (-1), the so-called *reduced gravitational term*. To be able to distinguish between the two equations later, we call the first one ε *equation* and the second one γ *equation*.

Observe that in line with Eqs. (10.2.7) the lift-to-drag ratio L/D turns out to be the only S/C variable to control the reentry trajectory.

Deceleration

Equations (10.2.13) permit to directly derive the deceleration, which is an important figure for crew and vehicle structure. By considering $\dot{h} = -v\sin\gamma$ from Eq. (10.2.3) we get

$$a = \frac{dv}{dt} = \frac{1}{d(\ln \varepsilon)/dv} \frac{d(\ln \varepsilon)}{d\lambda} \frac{d\lambda}{dh} \frac{dh}{dt} = \frac{\varepsilon v_0^2}{2v} \left[-\frac{\sin \gamma_e}{\sin \gamma} + \frac{2H}{\varepsilon \lambda R} \right] \left(-\frac{\lambda}{H} \right) (-v \sin \gamma)$$

from which because of $v^2 = \varepsilon v_\triangleright^2 = \varepsilon g_0 R$ follows

$$a = -\frac{v^2 \lambda}{2H} \sin \gamma_e + g_0 \sin \gamma = -g_0 \left(\frac{\varepsilon \lambda R}{2H} \sin \gamma_e - \sin \gamma \right) \tag{10.2.14}$$

So, except a short period of time after entry (drag-free phase, see Sect. 10.3.1), when the S/C accelerates due to gravitation and negligible drag ($\lambda \ll R/H$), the expression in the brackets is positive and the vehicle decelerates. Once we have determined the entry profiles $v(h)$ and $\gamma(h)$ for any reentry case by solving the ε and γ equations, Eq. (10.2.14) will provide us the answer to the crucial question: "What is the deceleration and in particular the maximum and hence critical deceleration a_{crit} of the spacecraft and hence also for the crew?"

Chapman's Theory

There exist other and even more sophisticated transformations to simplify the equations of motion, most notably by Chapman (1959, see also Vinh et al. 1980). He defines the normalized horizontal speed as the independent variable

$$\bar{u} := \frac{v}{v_\triangleright} \cos \gamma \quad \text{independent variable}$$

and

$$Z := \frac{1}{2} \lambda \bar{u} \sin \gamma_e \sqrt{\frac{R}{H}} \quad \text{dependent variable}$$

This yields a single second-order differential equation of motion

$$\bar{u} \frac{d}{d\bar{u}} \left(\frac{dZ}{d\bar{u}} - \frac{Z}{\bar{u}} \right) = \frac{1 - \bar{u}^2}{\bar{u} Z} \cos^4 \gamma - \sqrt{\frac{R}{H}} \frac{L}{D} \cos^3 \gamma$$

Chapman (1961) (see also Vinh et al. 1980) published the expression $\bar{u} \cdot Z(\bar{u})$ as a function of $\bar{u}$ for various entry profiles $\gamma_e, L/D$. Note that

$$\bar{u} \cdot Z(\bar{u}) = \frac{a \cdot \cos^2 \gamma}{\sqrt{1 - (L/D)^2}} \sqrt{\frac{H}{R}}$$

So, Chapman's theory essentially provides the acceleration a at a given velocity, but gives no insight into the reentry trajectory $\gamma(h)$ and velocity profile $v(h)$. In the following reentry analysis we therefore stick to the ε and γ equations as given by Eqs. (10.2.13a) and (10.2.13b), which provides these two revealing reentry profiles.

10.3 Elementary Results

From the above numerical solutions of the complex entry profiles it is evident, that there are no global analytical solutions. We will therefore focus on certain entry phases, which allow approximate analytical solutions, in particular in the critical deceleration phase.

General Approximations
To do so we have to make some gentle approximations to simplify the equations:

- Gravitation $g = g_0 = const.$
  According to Eq. (10.2.6), this assumption entails an error of $(6370/6248)^2 \leq 4.0\%$. If one chooses for g the mean value of $\langle g \rangle_h = 9.62$ m/s^2 at an altitude of 61 km, the error is even reduced to $(6370/6309)^2 \leq 2.0\%$.
- $v^2/r \approx v^2/R \approx v^2/r_e$
  This assumption entails an error of $6370/6248 \leq 2.0\%$.
- κ_D is assumed to be constant during the whole reentry process.
  The actual deviations from this constant for a S/C with a constant angle of attack are no more than $\pm 10\%$. If the angle of attack slightly changes, the important parameter L/D is still within a $\pm 10\%$ range.

The first two errors are negligible with regard to the third assumption, to all prior assumptions (e.g., a constant scale height for the barometric Formula (6.1.5), to the assumption that the Earth is a non-rotating inertial system), and other qualitative assumptions we will make later on.

10.3.1 Drag-Free Phase

The S/C is now at the entry interface at 122 km altitude with state vector (r_e, γ_e). Reentry can be roughly divided into two different phases as illustrated in Fig. 10.8.

Directly after entry, the aerodynamic drag is so low that drag can practically be neglected. So the body descends with the entry angle almost in free fall toward Earth. The motion Eqs. (10.2.1)–(10.2.3) therefore and because of $v^2 \approx gR^2/r \approx gr$ (circular orbital velocity) reduce to

$$\begin{aligned} \dot{v} &\approx g \sin\gamma \\ \dot{\gamma} &\approx 0 \\ \dot{h} &= -v \sin\gamma \end{aligned}$$

in this reentry phase. Integration results in

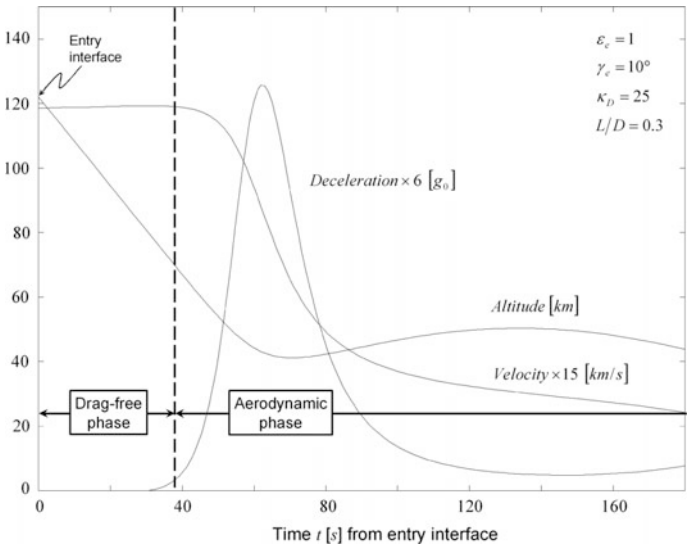


Fig. 10.8 The onset of deceleration departs the aerodynamic phase from the drag-free phase

$$\begin{aligned} \gamma &= \gamma_e \\ v &= v_e + g \sin\gamma_e \cdot t \approx const \\ h &= h_e - v_e \sin\gamma_e \cdot t \end{aligned} \tag{10.3.1}$$

This is exactly the behavior we observe also in the numerical results in Figs. 10.4, 10.5 and 10.6. As v_e is still very big, altitude decreases strongly while the velocity increases only slightly.

Note *The reason for the absolutely straight reentry trajectory, rather than one which is bent downward due to gravity as one might expect, is the centrifugal force, which like in a circular orbit still counterbalances the gravitational force.*

Below approximately 70–80 km, the impinging air behaves like a free molecular flow with a rapidly increasing aerodynamic drag. This is where the crucial aerodynamic phase starts. The transition between the two phases takes place when $d\varepsilon/d\lambda = 0$ and is characterized by the onset of a deceleration. According to Eq. (10.2.13a), this implies that

$$\frac{\sin\gamma_e}{\sin\gamma} = \frac{2H}{\varepsilon\lambda R}$$

Because of Eq. (10.2.12), $\varepsilon \approx 1$, and $\gamma \approx \gamma_e$, this determines the altitude of transition between the two phases to be

$$h_{trans} = H \ln\left(\frac{\kappa_D R}{\sin\gamma_e H}\right) \tag{10.3.2}$$

It mainly depends on γ_e and reaches for flat entries, $\gamma_e = 2^\circ - 10^\circ, h_{trans} = 85 - 95$ km, and for steep entries, $\gamma_e = 10^\circ - 45^\circ, h_{trans} = 75 - 85$ km (cf. Figs. 10.5, 10.6 and 10.7). Observe that the condition $d\varepsilon/d\lambda = 0$ marks also the maximum kinetic energy and hence velocity of the vehicle. Therefore, at the phase transition altitude h_{trans} the vehicle acquires its peak reentry velocity.

Adjusted Scale Height
Because the decisive action of the following reentry happens in the altitude range $30\text{ km} < h < 70$ km we adjust the scale height of the barometric formula for further calculations to this altitude, which is (see Table 6.2)

$$\boxed{H \approx 7.6\text{ km}} \quad @ \quad 30\text{ km} < h < 70\text{ km} \tag{10.3.3}$$

or we will later adjust it to the particular altitude under consideration.

10.3.2 Ballistic Reentry

The above numerical calculations show that the entry profiles in the aerodynamic phase may be quite different and complex. So it would be foolish to tackle the equations of motion head on. First of all, we need to understand the basic behavior of the solutions without the perturbing gravitation and lift terms L/D and H/R. This determines our approach, step by step, from simple approximations of equations to the more difficult ones. So we start out by neglecting the perturbation terms L/D and H/R to find solutions for a non-disturbed reentry, and later, we will also take into account lift and gravitational perturbations.

We first assume that the S/C does not have any lift, $L/D = 0$. This is the so-called *ballistic reentry*. The expression "ballistic" refers to "like the flight of a ball", which does not have any lift. Except for truly spherical reentry bodies, which do not exist in practice, $L = 0$ is only valid for axially symmetric bodies with absolutely no angle of attack (AOA, which is the angle between the axis of symmetry and oncoming flow). In reality even small AOAs produce differences in the position of the center of mass and the center of aerodynamic pressure and therefore significant lift. If this is not desired (because skipping can easily occur (see Sect. 10.5), and the resulting difficulty to determine the landing site), so if a true ballistic trajectory is wanted, one can slowly roll the S/C to average out such lifts. The Mercury capsule for instance had a nominal roll rate of 15° per second.

LEO Reentry Profile with Large Drag

For a rough result and as the second assumption we assume that compared to drag all other forces acting on the vehicle are negligible, which is true for altitudes below about 70 km (see Fig. 10.8). So setting the gravitational term to zero in Eq. (10.2.13a) and in Eq. (10.2.13b) $L = 0$ and $\varepsilon = \varepsilon_e = 1$ for an LEO reentry, the corresponding differential equations read

$$\begin{aligned} \frac{d(\ln \varepsilon)}{d\lambda} &= -\frac{\sin \gamma_e}{\sin \gamma} \\ \frac{d(\cos \gamma)}{d\lambda} &= 0 \end{aligned} \tag{10.3.4}$$

This set of equation can be solved easily. The second equation directly yields $\cos \gamma = \cos \gamma_e = const$ and therefore

$$\boxed{\gamma = \gamma_e = const} \tag{10.3.5}$$

Because the reentry body is subject just to drag, it entries on a straight line (cf. Figs. 10.5 and 10.6) and decelerates. This deceleration is described by the term on the right-hand side of the first Eq. (10.3.4). We can solve this equation as well by inserting Eq. (10.3.5)

$$\int_{e_e}^{\varepsilon} d(\ln \varepsilon') = -\int_{\lambda e}^{\lambda} d\lambda'$$

Integration delivers

$$\ln \frac{\varepsilon}{\varepsilon_e} = -(\lambda - \lambda_e)$$

and hence

$$\varepsilon = \varepsilon_e e^{-(\lambda - \lambda_e)} \approx \varepsilon_e e^{-\lambda} \tag{10.3.6}$$

The latter holds because $\lambda_e \approx 10^{-7}$. With Eq. (10.2.11) we find from Eq. (10.3.6) for the velocity

$$v = v_e e^{-(\lambda - \lambda_e/2)} \approx v_e e^{-\lambda/2} \tag{10.3.7}$$

As we will see later Eqs. (10.3.5) and (10.3.7) are useful descriptions for any early entry phase where drag exceeds lift and gravitational forces. This is why these results despite their simplicity are of general significance even for very flat lifting reentries (cf. Sect. 10.6; Eq. (10.6.4)).

General Reentry Profile

We now relax the large drag assumption to include also the pull of gravity for a ballistic reentry not only from LEO, but possibly also beyond. Gravity will firstly change the flight path angle. We therefore insert the above first-order result into Eq. (10.2.13b) for an iterative refinement and get

$$\frac{d(\cos\gamma)}{\cos\gamma} = -\frac{H}{\varepsilon_e R}\frac{e^{\lambda}-\varepsilon_e}{\lambda}d\lambda$$

Since

$$\int \frac{e^x}{x}dx = Ei(x) \approx \gamma + \ln x + x$$

where $\gamma = 0.57721\ldots$ is Euler's constant, we get

$$\ln\frac{\cos\gamma}{\cos\gamma_e} = -\frac{H}{\varepsilon_e R}\left[\lambda - (\varepsilon_e - 1)\ln\frac{\lambda}{\lambda_e}\right]$$

Because $\frac{H}{\varepsilon_e R}\left[\lambda - (\varepsilon_e - 1)\ln\frac{\lambda}{\lambda_e}\right] < 0.01$ we can approximate

$$\frac{\cos\gamma}{\cos\gamma_e} = 1 - \frac{H}{\varepsilon_e R}\left[\lambda - (\varepsilon_e - 1)\ln\frac{\lambda}{\lambda_e}\right]$$

From this follows after a short and straight forward auxiliary calculation

$$\frac{\sin\gamma}{\sin\gamma_e} = 1 + \frac{H}{\varepsilon_e R}\lambda\cot^2\gamma_e$$

We insert this result into Eq. (10.2.13a) to find

$$d(\ln\varepsilon) = -\left(1 - \frac{H}{\varepsilon_e R}\lambda\cot^2\gamma_e\right)d\lambda + \frac{2H}{\varepsilon_e R}\frac{e^{\lambda}}{\lambda}d\lambda$$

Integration and using $\varepsilon_e = v_e^2/(gR)$ from Eq. (10.2.11) delivers

$$\begin{aligned}\ln\frac{v(\lambda)}{v_e} &= \underbrace{-\frac{\lambda}{2}}_{drag} \underbrace{+\frac{gH}{v_e^2}\ln\frac{\lambda}{\lambda_e}}_{gravity} \underbrace{+\frac{gH}{v_e^2}\frac{1}{4}\lambda^2\cot^2\gamma_e}_{FPA\ change} \\ &\approx -\frac{\lambda}{2} + \frac{gH}{v_e^2}\ln\frac{\lambda}{\lambda_e}\end{aligned} \quad @\ L = 0 \tag{10.3.8}$$

where the latter follows from $\frac{1}{4}\lambda^2\cot^2\gamma_e \ll \ln\lambda/\lambda_e$ for any practical FPA. This is the more refined result of Eq. (10.3.7) of the second order.

To also find a more useful $v(t)$ rather than the above $v(\lambda)$, we need to determine $\lambda(t)$, that is $h(t)$ because according to Eq. (10.2.12)

$$\lambda := \frac{2\kappa_D}{\sin\gamma_e} e^{-\frac{h}{H}}$$

We do this by integrating $\dot{h} = -v\sin\gamma \approx -v\sin\gamma_e$ yielding with Eq. (10.3.8)

$$h - h_e = -v_e \sin\gamma_e \int_0^t \exp\left(-\frac{\lambda}{2} + \frac{gH}{v_e^2}\ln\frac{\lambda}{\lambda_e}\right) \cdot dt$$

and since the value of the expression in the exponent is less than 0.1

$$h(t) \approx h_e - v_e \sin\gamma_e \int_0^t \left(1 - \frac{\lambda}{2} + \frac{gH}{v_e^2}\ln\frac{\lambda}{\lambda_e}\right) \cdot dt \quad @\ L = 0$$

Consider this result as an equation to iteratively determine and hence improve $h(\lambda(h(t)))$. As a first iteration we assume $\lambda = \lambda_e < 10^{-5} \approx 0$ and hence find

$$h(t) \approx h_e - v_e \sin\gamma_e \cdot t$$

or equivalently

$$\frac{\lambda}{2} = \frac{\lambda_e}{2}\exp\left(\frac{v_e \sin\gamma_e}{H} t\right)$$

In a second iteration we insert this result to obtain

$$h(t) = h_e \underbrace{-v_e \sin\gamma_e \cdot t}_{\text{initial speed}} \underbrace{-\frac{1}{2} g \sin^2\gamma_e \cdot t^2}_{\text{gravity}} \underbrace{+ H\frac{\lambda_e}{2}\exp\left(\frac{v_e \sin\gamma_e}{H} t\right)}_{\text{drag}} \quad @\ L = 0 \tag{10.3.9}$$

The drag term has a positive algebraic sign because drag reduces speed. Therefore, at constant flight path angle the vehicle descents in a given time period a shorter distance than in free fall. With the definition

$$\Delta\eta(t) := \frac{h_e - h(t)}{H} = \frac{v_e}{H}\sin\gamma_e \cdot t + \frac{1}{2}\frac{g}{H}\sin^2\gamma_e \cdot t^2 - \frac{\lambda_e}{2}\exp\left(\frac{v_e \sin\gamma_e}{H} t\right)$$

Equation (10.3.9) yields $\lambda/2 = \lambda_e/2 \cdot \exp(\Delta\eta)$, which inserted into Eq. (10.3.8) finally delivers

$$\boxed{\ln\frac{v(t)}{v_e} = -\frac{\lambda_e}{2}\exp(\Delta\eta) + \frac{gH}{v_e^2}\Delta\eta} \quad @ \quad L = 0 \tag{10.3.10}$$

Critical Deceleration

According to Eq. (10.3.6) the kinetic energy decreases exponentially with λ. But because initially $\lambda \ll 1$, the velocity at the beginning of the drag-free phase decreases only slowly (see Fig. 10.8). The maximum velocity reduction, that is the maximum and hence critical deceleration, happens according to Eq. (10.2.14) and because of $\gamma = \gamma_e$ when

$$\frac{da}{d\lambda} \propto \frac{d}{d\lambda}\left(\lambda e^{-\lambda}\right) = (1-\lambda)e^{-\lambda} = 0$$

and therefore at

$$\lambda_{crit} = 1 \tag{10.3.11}$$

Remark *In hindsight we recognize that the choice of the factor 2 in the definition* Eq. (10.2.12) *was motivated to have* λ_{crit} *unity.*

From Eq. (10.2.12) follows that $\lambda_{crit} = 1$ corresponds to the critical deceleration altitude

$$h_{crit} = H\ln\frac{2\kappa_D}{\sin\gamma_e} \tag{10.3.12}$$

From Eq. (10.2.14) we finally obtain for the critical deceleration

$$a_{crit} = \left(g_0 - \frac{v_e^2}{2eH}\right)\sin\gamma_e \approx -\frac{v_e^2\sin\gamma_e}{2eH} \tag{10.3.13}$$

The last three equations mark the ball-park of the results for reentries with lift and gravitational perturbations to which we turn after the next section.

10.3.3 Heat Flux

As stated in our thermal problem setting in Sect. 10.1.1, it is our goal to determine the peak heat flux in the course of reentry. Quite generally, the heat flux on a S/C at the stagnation point is according to Eq. (10.1.3)

$$\dot{q}_{S/C} = \frac{St}{2}\sqrt{\rho\rho_q\frac{R_0}{R_n}}\cdot v^3$$

To find the maximum $\dot{q}_{S/C}(v)$ we need to have the dependency of ρ from the speed. From Eq. (10.2.12)

$$\lambda = \frac{BH}{\sin\gamma_e}\rho$$

where $B = C_D A_\perp / m$ is the ballistic coefficient (see Eq. (6.2.19)). We know from the numerical calculations that the maximum deceleration occurs where the body first deviates from the straight trajectory. Because we expect the maximum heating around maximum deceleration we apply for a ballistic reentry the basic solution Eq. (10.3.7), $v = v_e e^{-\lambda/2}$. The atmospheric density as a function of v then is determined to be

$$\rho = 2\frac{\sin\gamma_e}{BH}\ln\frac{v_e}{v}$$

Hence

$$\dot{q}_{S/C} = \frac{St}{2}\sqrt{2\rho_q\frac{R_0}{R_n}\frac{\sin\gamma_e}{BH}}\cdot\sqrt{\ln\frac{v_e}{v}}\cdot v^3$$

As $\dot{q}$ is monotonous in v, this results in the condition equation for a maximum $\dot{q}$

$$\frac{1}{\dot{q}}\frac{d\dot{q}}{dt} \propto \frac{1}{\dot{q}}\frac{d\dot{q}}{dv} = \frac{1}{v\ln(v/v_e)}\left(\frac{1}{2} - 3\ln\frac{v}{v_e}\right) = 0$$

From this follows that

$$v_{\max\dot{q}} = \frac{v_e}{e^{1/6}} = 6.3\ \mathrm{km\ s^{-1}}\cdot\sqrt{\varepsilon_e} \approx 6.3\ \mathrm{km\ s^{-1}} \tag{10.3.14}$$

Remark *If we would have assumed the Detra and Hidalgo heat flux model with* $\dot{q}_{S/C} \propto \sqrt{\rho}\, v^{3.15}$ *we would have derived for the critical velocity* $v_{\max\dot{q}} = v_e e^{-1/6.3}$, *which deviates by only 1% from the above result.*

So the S/C experiences its maximum heating even far before its critical deceleration at $v_{crit} = 4.5\,\mathrm{km\,s^{-1}}$ (see Eq. (10.4.17)). The altitude where the maximum heating is reached is derived from $v = v_e e^{-\lambda/2}$ as $\lambda_{\max\dot{q}} = 1/3$ and because of Eq. (10.2.12)

$$h_{\max\dot{q}} = H\ln\frac{6\kappa_D}{\sin\gamma_e} = h_{crit} + 1.10H \approx h_{crit} + 8.8\ \mathrm{km} \tag{10.3.15}$$

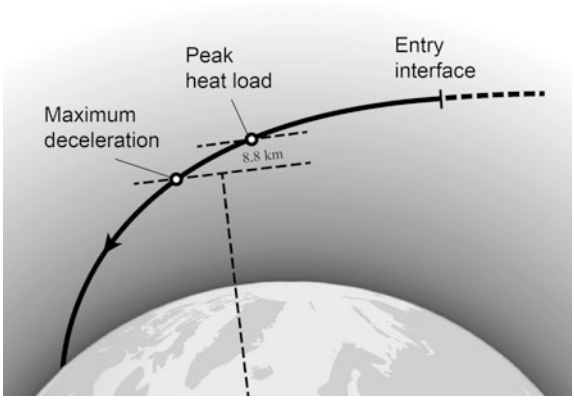


Fig. 10.9 Occurrence of peak heat load and maximum deceleration for a ballistic reentry

with $H = 8.0$ km. This relation is qualitatively depicted in Fig. 10.9. For a typical ballistic entry angle $\gamma_e \approx 3°$ (see Soyuz reentry in Sect. 10.1.3) we have $h_{\max \dot{q}} \approx 60$ km. For $\gamma_e = 3° - 10°$ maximal heating therefore occurs at an altitude of about 55 km and for high entry angles, $\gamma_e \approx 45°$, at 41 km.

Finally, the peak heat flux at the stagnation point of the S/C is found to be

$$\dot{q}_{S/C\,\max} = \frac{St}{2} \cdot v_e^3 \sqrt{\frac{\rho_q \sin\gamma_e}{3e \cdot BH} \frac{R_0}{R_n}} = \dot{q}_e \left(\frac{v_e}{v_\triangleright}\right)^3 \sqrt{\frac{\sin\gamma_e}{3e\rho_e BH} \frac{R_0}{R_n}} \tag{10.3.16}$$

with $\dot{q}_e = \frac{St}{2}\sqrt{\rho_q \rho_e} v_\triangleright^3 = 1.15$ W cm^{-2} the standard heat flux at entry interface, where $St = 0.001, v_\triangleright = 7.905$ km s^{-1}, $R_0 = 1$ m, $\rho_q = 0.121$ kg m^{-3} and $\rho_e = 1.80 \times 10^{-8}$ kg m^{-3} the mean atmospheric density at entry interface according to MSIS-E-90. Observe that the heat flux is reduced by a larger radius of curvature, R_n, and by a larger ballistic coefficient $B = C_D A_\perp / m$. Note that the expression $\rho_e BR$ and hence the entire square root is dimensionless and that $\rho_e BH$ relates to Bstar, B^*, of NORAD's TLE (see Remark in Sect. 12.7.3) as

$$\rho_e BH = 2B^* \frac{\rho_e}{\rho_{120}} \frac{H}{R} = 1.46 B^* \frac{H}{R}$$

where the latter holds only if the actual atmospheric density at entry interface is the standard value $\rho_{120} \equiv \rho_0 = 2.461 \times 10^{-8}$ kg m^{-3} as assumed in the Simplified General Perturbation Model SGP of NORAD.

With this and Eq. (10.1.1) the peak temperature at this point is calculated to be

$$T_{\max}^4 = \frac{T_e^4}{\varepsilon} \cdot \left(\frac{v_e}{v_\triangleright}\right)^3 \sqrt{\frac{\sin\gamma_e}{3e\rho_e BH} \frac{R_0}{R_n}} \tag{10.3.17}$$

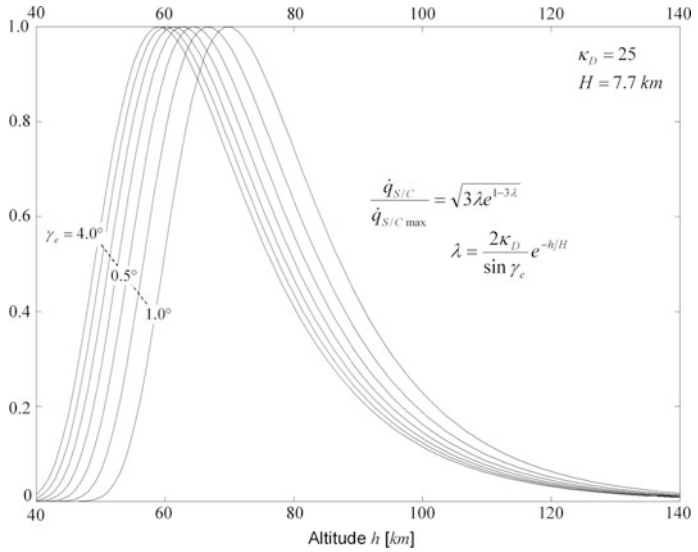


Fig. 10.10 Normalized ballistic reentry heat flux at the stagnation point of a spacecraft as a function of altitude h and entry angle γ_e in steps of 0.5°

with

$$T_e = \left(\frac{\dot{q}_e}{\sigma}\right)^{1/4} = 671\ \mathrm{K}$$

the standard stagnation point temperature at entry interface and $\varepsilon \approx 0.85$ the approximate emissivity of the heat shield. With Eq. (10.3.16) we can rewrite Eq. (10.1.3) to a normalized heat flux on the S/C as

$$\frac{\dot{q}_{S/C}}{\dot{q}_{S/C\,\max}} = \sqrt{3\lambda e^{1-3\lambda}} \tag{10.3.18}$$

This function is plotted in Fig. 10.10 for various entry angles.

As will be shown at the end of Sect. 10.4.2 all considerations in this section for the calculation of the heat flux for a ballistic reentry are also valid for the general reentry case. Therefore, all findings in this Sect. 10.3.3 are elementary results (except for lifting reentry, see Sect. 10.6.3).

10.4 Reentry with Lift

10.4.1 Lift-Only Case

We now make a step forward to solve the basic differential Eq. (10.2.13) for reentry by allowing for a lift of the reentry vehicle, but still neglecting gravitational and centrifugal forces via $H/R = 0$ (see explications following Eq. (10.2.13)). The differential equations then read

$$\frac{d(\ln \varepsilon)}{d\lambda} = -\frac{\sin \gamma_e}{\sin \gamma}$$
$$\frac{d(\cos \gamma)}{d\lambda} = \frac{\sin \gamma_e}{2} \frac{L}{D}$$

We recall from Eq. (10.2.13) that we assume $L/D \approx const$. In fact, L and D depend somewhat differently on speed and angle of attack, so that L/D slightly depends on speed and AOA. Within our approximations (see Sect. 10.3) $L/D = const$, however, is a good assumption, which we will adopt from now. This permits to solve the second equation directly by separating the variables

$$\cos \gamma = (1 + b\lambda) \cos \gamma_e \tag{10.4.1}$$

where the constant

$$b := \frac{\tan \gamma_e}{2} \frac{L}{D} \tag{10.4.2}$$

describes the lift power (buoyancy) and $1 + b(\lambda - \lambda_e) \approx 1 + b\lambda$. What does this quite important equation tell us? The vehicle entries the atmosphere at the entry interface $\lambda = \lambda_e$ with $\gamma = \gamma_e$. Since it descends, λ increases. According to Eq. (10.4.1) a positive lift, $b > 0$, decreases the flight path angle steadily until $\cos \gamma = 1$, when a horizontal flight with $\gamma = 0$, is attained. Of course, lift continues to act on the vehicle, which leads now to an increase in altitude and hence a decreasing λ, which via Eq. (10.4.1) in turn implies $\cos \gamma < 1$ anew, but this time with a negative flight path angle $\gamma < 0°$. In total, the positive lift results in a steadily upward curved trajectory (cf. the numerical calculations in Sect. 10.2.1). If lift is negative, that is if the reentry body turns upside down, then the trajectory steadily turns down. From Eq. (10.4.1) we find after some minor trigonometric conversions

$$\sin \gamma \approx \sin \gamma_e \sqrt{1 - 2c\lambda} \tag{10.4.3}$$

with constant

$$c := b\cot^2\gamma_e = \frac{\cot\gamma_e}{2}\frac{L}{D}$$

where we have neglected the term of order $b^2\lambda^2$ in the root, which is equivalent to $\lambda L/D \ll 4\cot\gamma_e$, and which typically holds for altitudes $h > H\ln(100L/D)$. We insert this expression into the first differential equation to find

$$d(\ln\varepsilon) = -\frac{\sin\gamma_e}{\sin\gamma}d\lambda = -\frac{d\lambda}{\sqrt{1-2c\lambda}}$$

We can solve this equation analytically to arrive at

$$\ln\frac{v}{v_e} = \frac{1}{2c}\left(\sqrt{1-2c\lambda}-1\right) \quad @ \quad \lambda\frac{L}{D} \ll 4\cot\gamma_e \tag{10.4.4}$$

This solution holds for any L/D values, even for a high-lift reentry, as long as $\lambda L/D \ll 4\cot\gamma_e$. There even exists a fully analytical solution without the approximation $\lambda L/D \ll 4\cot\gamma_e$. But because this is much more complex and because it does not help to understand we pursue it in an exercise (Problem 10.6). As expected, Eq. (10.4.4) passes over into Eq. (10.3.6) for $L/D \propto c \to 0$.

10.4.2 General Results

After these introductory considerations we now make the final step forward in solving the reentry equations of motion by allowing for the perturbative terms of gravitation and centrifugal forces

$$\frac{d(\ln\varepsilon)}{d\lambda} = -\frac{\sin\gamma_e}{\sin\gamma} + \frac{2H}{\varepsilon\lambda R}$$
$$\frac{d(\cos\gamma)}{d\lambda} = \frac{\sin\gamma_e}{2}\frac{L}{D} - \left(\frac{1}{\varepsilon}-1\right)\frac{H\cos\gamma}{\lambda R}$$

Because $H/R \approx 0.001$ we assume the H/R-terms to be gravitational perturbations of first order with respect to the terms considered so far. We will take these perturbations fully into account. However, because they are small, it will suffice to apply for $\cos\gamma$ and ε the undisturbed terms of Sect. 10.3.2 in these perturbative terms, i.e., we will not consider perturbations of perturbations. With this so-called second-order perturbation analysis we are looking for solutions. These solutions will not be globally exact (we already know that there are no globally exact solutions), but they will be applicable for a quite extended region of λ.

Flight Path Angle

For $\cos\gamma$ and ε we insert the unperturbed expressions from Eqs. (10.3.5) and (10.3.6). The γ equation then reads

$$d(\cos\gamma) = \left[\frac{\sin\gamma_e}{2}\frac{L}{D} - \left(\frac{e^\lambda}{\varepsilon_e} - 1\right)\frac{H}{\lambda R}\cos\gamma_e\right]d\lambda$$

from which by direct integration follows

$$\cos\gamma = \left[1 + b\lambda - \frac{H}{\varepsilon_e R}\int_{\lambda_e}^{\lambda}\frac{e^x - \varepsilon_e}{x}dx\right]\cos\gamma_e$$

where again $1 + b(\lambda - \lambda_e) \approx 1 + b\lambda$. The integral that comprises the perturbation can be solved analytically

$$\int_{\lambda_e}^{\lambda}\left(\frac{e^x}{x} - \frac{\varepsilon_e}{x}\right)dx = Ei(\lambda) - Ei(\lambda_e) - \varepsilon_e \ln\frac{\lambda}{\lambda_e}$$

We find from any special formulary that the exponential integral $Ei(x)$ can be expressed as

$$Ei(\lambda) - Ei(\lambda_e) = f^\lambda - f^{\lambda_e} + \ln\frac{\lambda}{\lambda_e} \approx f^\lambda + \ln\frac{\lambda}{\lambda_e} \tag{10.4.5}$$

where we have defined the exponential-like function

$$f^x := \int_0^x \frac{e^y - 1}{y}dy = \sum_{n=1}^{\infty}\frac{x^n}{nn!} \tag{10.4.6}$$

Due to the global convergence of this power series this function can easily be calculated numerically and is depicted in Fig. 10.11. We therefore find with Eq. (10.2.12)

$$\chi(\lambda, \varepsilon_e) := \int_{\lambda_e}^{\lambda}\frac{e^x - \varepsilon_e}{x}dx = f^\lambda - (\varepsilon_e - 1)\ln\frac{\lambda}{\lambda_e} \approx f^\lambda - (\varepsilon_e - 1)\frac{h_e - h}{H} \tag{10.4.7}$$

where we have defined the chi function $\chi(x, \varepsilon)$ that is related to the f function by $\chi(x, 1) = f^x$.

The solution for the flight path angle therefore is

$$\cos\gamma = [1 + b\lambda - p\chi(\lambda, \varepsilon_e)]\cos\gamma_e \quad \textbf{FPA equation} \tag{10.4.8}$$

where

$$p := \frac{H}{\varepsilon_e R} \approx 0.001 \ll 1$$

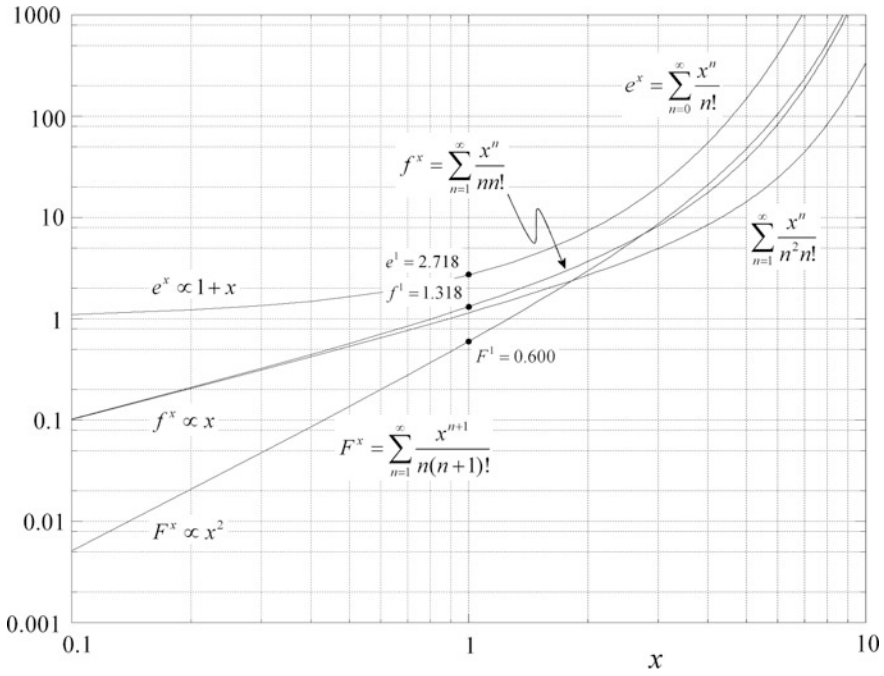


Fig. 10.11 Representation of the functions f^x, F^x as defined in the text and e^x

signifies the strength of the gravitational perturbation (see gravity term in Eq. (10.2.13a)). The FPA Eq. (10.4.8) is the full-fledged version of Eq. (10.4.1) for a general reentry and hence is of high significance for understanding reentry. It indirectly describes the course of the reentry trajectory with altitude for any entry condition. Because of these properties we will make use of it for equilibrium reentries.

Let us have a closer look at the reentry behavior with the FPA equation. For $\varepsilon_e = 1$ at high altitudes $\chi(\lambda, 1) \approx \lambda \ll 1$. Therefore, $\cos\gamma \approx \cos\gamma_e$: the entry vehicle descends on a straight line. With decreasing altitude λ and hence $\chi(\lambda, 1)$ increases exponentially. Since $b = 0.01 - 0.1$ while $p \approx 0.001$, the lift term $b\lambda$ incipiently is more significant than $p\chi$. It increases $\cos\gamma$, so γ decreases: The vehicle will slowly deviate upward from the straight line. At lower altitudes, $\lambda > 1, \chi$ increases exponentially with λ (therefore double-exponentially with decreasing h) so the gravity term quickly becomes significant. The specific value of $b = L \cdot \tan\gamma_e/(2D)$ depends on the entry angle and the lift. If it is quite substantial, the right-hand side of Eq. (10.4.8) at some point becomes unity where the vehicle flies horizontally. But because the lift continuously lifts the vehicle it begins to ascent ($\gamma < 0$), implying a decreasing $\cos\gamma$, because also λ decreases. So, if b if sufficiently big, the vehicle may reverse the flight path angle and ascent before gravity overcomes this excursion. If b is too small, then there is just an indication of an upturn, but gravity will beat it soon. This is exactly what we see in the numerical calculations Figs. 10.5, 10.6 and 10.7.

Critical Deceleration
The critical deceleration a_{crit}, that is the maximum deceleration during reentry, can be derived from Eq. (10.2.14). We rearrange this equation to

$$a = -g_0 \frac{R \sin\gamma_e}{2H} \left(\varepsilon\lambda - \frac{2H}{R} \frac{\sin\gamma}{\sin\gamma_e} \right)$$

We now apply second-order perturbation analysis: Because H/R is a perturbation $\sin\gamma \approx \sin\gamma_e$, and hence

$$a = -g_0 \sin\gamma_e \left(\frac{\varepsilon\lambda R}{2H} - 1 \right) \tag{10.4.9}$$

This expression is no longer exact, but because even for extremely flat entry angles $\varepsilon_{crit}\lambda_{crit} \geq 0.1$ and because $2H/R = 0.0021$, this still is a very good approximation for all practical purposes. From the maximum condition $da/d\lambda = 0$, we derive from Eq. (10.4.9) in a few steps the critical λ to be

$$\lambda_{crit} = \left[1 + \frac{2H}{\varepsilon_{crit} R} \right] \frac{\sin\gamma_{crit}}{\sin\gamma_e} \tag{10.4.10}$$

We now could insert this into Eq. (10.4.9) to find the wanted a_{crit}. This, however, would in principle be not permissible because ε_{crit}, γ_{crit} in Eq. (10.4.10) themselves depend on λ_{crit}. But for the upcoming special cases this will be not a problem and Eq. (10.4.10) will therefore be of great value.

Heat Flux
What is the peak heating in the case of a reentry with lift? From Eq. (10.4.10) we see that the critical deceleration takes place at $\lambda_{crit} \approx 1$. Since we saw that the critical heating happens before that point, all the results of Sect. 10.3.3 for the peak heating for a ballistic reentry holds also for this general case.

10.4.3 Near-Ballistic Reentry

Up to this point we have solved the γ equation of motion including gravitational forces in second-order perturbation analysis. Now that we move on to solve also the ε equation this is no longer possible. If we still want to take gravitation into account, we can do so only by applying first-order perturbation analysis. So the following solutions will apply only for more restricted altitudes.

Entry Velocity
The problem is that we can no longer exactly integrate the expression $1/\sin\gamma = 1/\sqrt{1-\cos^2\gamma}$ with $\cos\gamma = [1 + b\lambda - p\chi(\lambda, \varepsilon_e)]\cos\gamma_e$. We can do so only with

linear approximations by assuming

$$c^2\lambda^2 \ll 1 \quad \text{and} \quad q^2\chi^2(\lambda, \varepsilon_e) \ll 1$$

with constants

$$c := b\cot^2\gamma_e = \frac{\cot\gamma_e}{2}\frac{L}{D}$$
$$q := p\cot^2\gamma_e = \frac{H}{\varepsilon_e R}\cot^2\gamma_e$$

which is equivalent to assuming a relatively steep reentry with low lift, i.e., a near-ballistic case. If these approximations hold we obtain

$$\sin\gamma \approx \sin\gamma_e\sqrt{1 - 2c\lambda + 2q\chi} \tag{10.4.11}$$

from which follows

$$\frac{\sin\gamma_e}{\sin\gamma} \approx \frac{1}{\sqrt{1 - 2c\lambda + 2q\chi}} \approx 1 + c\lambda - q\chi$$

We insert this result into the ε equation and separate the variables

$$d(\ln\varepsilon) = \left[-(1 + c\lambda - q\chi) + \frac{2H}{\varepsilon\lambda R}\right]d\lambda$$

Because the second term on the right-hand side is the perturbation we can adopt for ε the unperturbed expression from Eq. (10.3.6). With Eq. (10.4.11) and the approximations $c^2\lambda^2 \ll 1$ and $q^2\chi^2(\lambda, \varepsilon_e) \ll 1$ we then derive

$$\ln\frac{\varepsilon}{\varepsilon_e} = -\int_{\lambda_e}^{\lambda}(1 + c\lambda - q\chi)d\lambda + 2p\int_{\lambda_e}^{\lambda}\frac{e^x}{x}dx$$
$$\approx -\lambda - \frac{1}{2}c\lambda^2 + q\int_0^{\lambda-\lambda_e}\chi(x, \varepsilon_e)dx + 2p[Ei(\lambda) - Ei(\lambda_e)]$$

So we finally find with Eq. (10.4.5)

$$\ln\frac{v}{v_e} \approx -\frac{\lambda}{2} - \frac{1}{4}c\lambda^2 + \frac{q}{2}\mathrm{X}(\lambda, \varepsilon_e) + p\chi(\lambda, 0) \tag{10.4.12}$$

with

$$\mathrm{X}(\lambda, \varepsilon_e) := \int_0^{\lambda} \chi(x, \varepsilon_e) \cdot dx = F^{\lambda} - (\varepsilon_e - 1)\lambda\left(\ln\frac{\lambda}{\lambda_e} - 1\right)$$

where we have introduced the X function and F function

$$F^x := \int_0^x f^y dy = \sum_{n=1}^{\infty} \frac{x^{n+1}}{n(n+1)!}$$

which like f^x can easily be calculated numerically by the power series expansion. It is also depicted in Fig. 10.11.

Entry Trajectory

Within the first-order perturbation analysis, that is for $c^2\lambda^2 \ll 1$ and $q^2\chi^2(\lambda, \varepsilon_e) \ll 1$, it can be shown easily (exercise, Problem 10.4) that the course of the trajectory with altitude can be described analytically by

$$x \approx \cot\gamma_e \left\{(h_e - h) + \frac{H}{\sin^2\gamma_e}[b\lambda - p \cdot \Theta(\lambda, \varepsilon_e)]\right\} \tag{10.4.13}$$

with

$$\Theta(\lambda, \varepsilon_e) := \int_0^{\lambda-\lambda_e} \frac{\chi(x, \varepsilon_e)}{x} dx = \sum_{n=1}^{\infty} \frac{\lambda^n}{n^2 n!} - \frac{1}{2}(\varepsilon_e - 1)\ln^2\frac{\lambda}{\lambda_e}$$

Here x is the downrange distance relative to the entry point. This dependency is illustrated in Fig. 10.12 for an entry with $\gamma_e = 45°$. We recognize the straight entry line $x = \cot\gamma_e(h_e - h)$. The actual entry trajectory deviates from this for a positive lift by an upturn and for negative lift by a downturn. The trajectory representation ends where $c^2\lambda^2 \ll 1$ is no longer valid. The numerical solutions of the full equations of motion for this case show that for $L/D = 0.3$ the vehicle in the further course of the trajectory flies horizontally at 21 km altitude for a moment to finally descend. For $L/D = 0.4$, there is a reflection point at 23.5 km altitude, a maximum at 24.6 km altitude, and thereafter a final descend. For $L/D = 0.5$ the reflection point is at 25 km and the maximum at 30 km altitude.

Critical Deceleration

To determine λ_{crit} we employ the approximate solutions of $\varepsilon(\lambda)$ and $\sin\gamma(\lambda)$ for the unperturbed reentry from Eqs. (10.3.6) and (10.4.3) and obtain for this first-order perturbation analysis

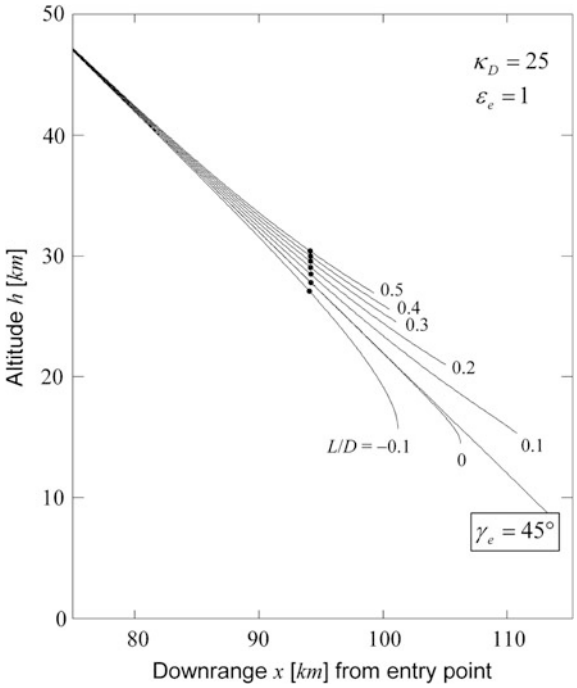


Fig. 10.12 Reentry trajectories for $\gamma_e = 45°$ as given by Eq. (10.4.13) for different L/D ratios. On the x-axis is the downrange distance from entry point. The full dots denote the critical altitudes

$$\lambda_{crit} = (1 + 2ep)\sqrt{1 - 2c + 2q\chi(1, \varepsilon_e)}$$

with

$$\chi(1, \varepsilon_e) = 1.318 - (\varepsilon_e - 1) \ln \frac{1}{\lambda_e}$$

and

$$\ln \frac{1}{\lambda_e} = 18.2 - \ln \frac{2\kappa_D}{\sin \gamma_e}$$

Because $2ep = 2eH/\varepsilon_e R < 0.0065$, this term is negligible and therefore we find with the definition of c from Eq. (10.4.3)

$$\lambda_{crit} = \sqrt{1 - 2c + 2q\chi(1, \varepsilon_e)} = \sqrt{1 - \cot \gamma_e \frac{L}{D} + 2 \cot^2 \gamma_e \frac{H}{\varepsilon_e R} \chi(1, \varepsilon_e)} \quad (10.4.14)$$

Note *Because we have assumed* $c, p \ll 1$, *essentially* $\lambda_{crit} \approx 1$. *Lift and gravity cause only minor variations from this value.*

The critical altitude at which the critical deceleration happens is determined from Eq. (10.2.12) as

$$h_{crit} = H \ln \frac{2\kappa_D}{\lambda_{crit} \sin \gamma_e} \qquad \textbf{critical deceleration altitude} \tag{10.4.15}$$

Note *Because* $\lambda_{crit} \approx 1$ *also holds for very flat lifting reentries* (see Sect. 10.6; Eq. (10.6.7)), Eq. (10.4.15) *applies quite generally.*

Together with Eq. (10.4.14) this equation describes how the critical altitude changes as a function of lift and entry angle. It increases with increasing lift (see Fig. 10.12) and with decreasing entry angle. For $L \approx 0$ and $\gamma_e = 3^\circ - 10^\circ$ critical altitudes are about 43–52 km. We recall that for the above derivation of λ_{crit} we used approximate differential equations that do not reproduce the intricate trajectory of flat reentries as depictured in Figs. 10.6 and 10.7. The critical deceleration therefore always refers to the first deceleration phase at the commencement of reentry, which for very flat reentries might be lower than decelerations on later reentries after skipping (see Figs. 10.7 and 10.13b).
For the critical deceleration we apply Eq. (10.4.9) to the critical point. Inserting Eq. (10.4.14) yields for $c^2\lambda^2 \ll 1$ and $q^2\chi^2(\lambda, \varepsilon_e) \ll 1$

$$a_{crit} = -\frac{v_e^2 \sin \gamma_e}{2eH}\left[1 - \frac{1}{4}\cot\gamma_e \frac{L}{D} + \frac{H}{\varepsilon_e R}(\xi - 2e)\right] \qquad \textbf{critical deceleration} \tag{10.4.16}$$

with constant

$$\xi(\varepsilon_e) := 2.636 + (\varepsilon_e - 0.400) \cdot \cot^2 \gamma_e - \left[\cot^2 \gamma_e(\varepsilon_e - 1) - 2\right] \ln \frac{1}{\lambda_e}$$

The velocity at which the deceleration becomes maximal is determined from Eq. (10.4.12) to be

$$v_{crit} = \frac{v_e}{\sqrt{e}}\left[1 - \frac{\cot\gamma_e}{8}\frac{L}{D} + \frac{H}{2\varepsilon_e R}\xi(\varepsilon_e)\right] \approx \frac{v_e}{\sqrt{e}} = 4.5 \text{ km/s} \cdot \sqrt{\varepsilon_e} \tag{10.4.17}$$

For the latter we have chosen $v_e = 7.44$ km/s, which is more realistic, because due to the Earth's rotation the entry velocity with respect to the atmosphere is effectively reduced.

Reentry from LEO
For a reentry from LEO we have $\varepsilon_e = 1$ and therefore

$$\xi - 2e = 0.600 \cdot \cot^2 \gamma_e + 33.6 - 2 \ln \frac{2\kappa_D}{\sin \gamma_e} \approx 0.600 \cdot \cot^2 \gamma_e + 21.0$$

and therefore

$$a_{crit} = -\frac{v_0^2 \sin \gamma_e}{2eH} \left(1.025 - \frac{1}{4} \cot \gamma_e \frac{L}{D} + 6.30 \times 10^{-4} \cot^2 \gamma_e \right) \quad @ \quad \varepsilon_e = 1$$

Because $v_0^2/(2eH) = Rg_0/(2eH) = 154 \cdot g_0$, we finally obtain

$$\boxed{a_{crit} = -154 \cdot g_0 \sin \gamma_e \left(1.025 - \tfrac{1}{4} \cot \gamma_e \tfrac{L}{D} + 6.30 \cdot 10^{-4} \cot^2 \gamma_e\right)} \quad @ \quad \varepsilon_e = 1 \tag{10.4.18}$$

For decreasing entry angles the critical deceleration deviates more and more from the simple relationship $a_{crit} = -154 \cdot g_0 \varepsilon_e \sin \gamma_e \cdot 1.025 = -158 \cdot g_0 \varepsilon_e \sin \gamma_e$ (see Fig. 10.13a) to larger values. This is counteracted by a positive lift. In Fig. 10.13a, the critical deceleration for $\varepsilon_e = 1$ is plotted according to Eq. (10.4.18) for different L/D.

For $\gamma_e \leq 1.5°$ our approximations definitely no longer apply, because $\cot \gamma_e$ diverges. On the other hand $\sin \gamma_e \to 0$. One could presume that overall a_{crit} would converge for $\gamma_e \to 0$. Numerical calculations that continue the analytical solutions for $\gamma_e \leq 1.5°$ (see Fig. 10.13) corroborate this supposition. With a semi-analytical ansatz V. A. Yaroshevsky in 1964 could even show that for $L = 0$ and with $H = 7.6$ km (see Eq. (10.3.3)), the critical deceleration for $\gamma_e \to 0$ converges to $a_{crit} = 0.277 \cdot g_0 \sqrt{R/H} = 8.0 g_0$. This is in excellent agreement with our numerical calculations. This limiting case, however, is of no practical interest, since for $\gamma_e \to 0$ the downrange distance becomes infinite. It is just the other way round that, to determine precisely the landing site of a capsule, the downrange distance should be as small and hence the entry angle as large as possible.

These contradictory requirements can only be resolved by a capsule with lift. We therefore provide in Fig. 10.13b a fully numerical calculation for $-0.5 \leq L/D \leq 0.6$ based on the MSIS-E-90 atmospheric model.

Example

For manned missions the reentry trajectory is chosen such that the critical deceleration never exceeds the maximum tolerable value of 10 g. From Fig. 10.13b it follows that for ballistic entries, $L = 0$, from LEO $\gamma_e < 3.1°$. Of course no astronaut wants to pull 10 g. Therefore, reentry with the Soyuz capsule is limited to 4.5 g. According to Fig. 10.13a this corresponds to an entry angle of 3.9° at a lift of $L/D = 0.3$, which is typical for capsules with a heat shield.

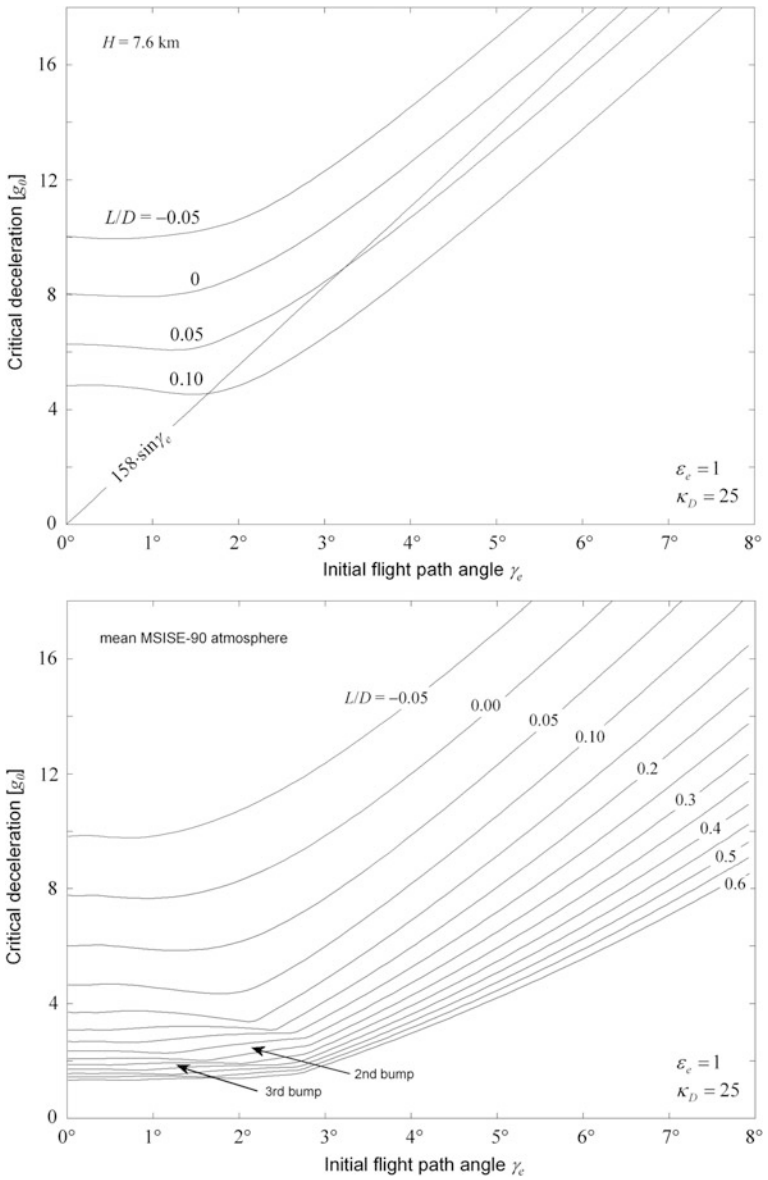


Fig. 10.13 The critical deceleration as a function of the initial flight path angle (entry angle) and lift. **a** Approximate calculations as from Eq. (10.4.18), for small $\cot\gamma_e L/D$, i.e., for $L/D \leq 0.1$ and $\gamma_e > 1.5°$. Results for $\gamma_e \leq 1.5°$ are numerical solutions. **b** Fully numerical calculations with a mean MSIS-E-90 atmospheric model (cf. Sect. 6.1.3) for $L/D \leq 0.6$. The critical deceleration occurs at the first deceleration maximum, except for areas indicated by "2nd bump" and "3rd bump" where it happens at the second maximum (cf. Fig. 10.7) and third maximum, respectively. Differences between (**a**) and (**b**) result from the different atmospheric models used

10.5 Reflection and Skip Reentry

10.5.1 Reflection

From the discussion of the FPA Eq. (10.4.8) we saw that it nicely reproduces the upturn of the reentry trajectory for positive lift. In fact, we can take the FPA equation to determine the point—the reflection point—where the vehicle turns back to increasing altitudes. From the reflection condition $\cos\gamma = 1$ we derive from (10.4.8) for the reflection altitude λ_r

$$1 + \frac{\tan\gamma_e}{2}\frac{L}{D}\lambda_r - \frac{H}{\varepsilon_e R}\chi(\lambda_r, \varepsilon_e) - \frac{1}{\cos\gamma_e} = 0 \tag{10.5.1}$$

Reflections typically take place at $h_r > 20\,\text{km} \rightarrow \lambda_r < 3$, where according to Fig. 10.11 $\chi(3, \varepsilon_e \approx 1) = f^3 \approx 10$. Therefore, $H\chi/\varepsilon_e R \ll 1$, which implies that Eq. (10.5.1) can be reduced to

$$1 + \frac{\tan\gamma_e}{2}\frac{L}{D}\lambda_r - \frac{1}{\cos\gamma_e} = 0 \tag{10.5.2}$$

From this and with Eq. (10.2.12) it follows for the reflection altitude

$$h_r = H\ln\left(\frac{L}{D}\frac{\kappa_D}{1-\cos\gamma_e}\right) \quad \textbf{reflection altitude} \tag{10.5.3}$$

Figure 10.14 shows the reflection altitudes as a function of the entry angle for a given L/D and entry angle as calculated from Eq. (10.5.1). The results are almost identical to those from Eq. (10.5.3) except for $L/D < 0.2$ and $\gamma_e < 20°$, because then the gravity term is no longer negligible compared to the lift term.

> **Note** *The reflection altitude is independent from the entry velocity! One would have expected that it decreases with increasing entry speed because a higher entry momentum defies the ability to turn the vehicle up. But, on the other hand, the lifting force, which does the turn, increases quadratically with speed (cf. Eq. (10.2.5)), which just compensates the higher inertia of the vehicle.*

What are the entry parameters for which a reflection occurs? The condition derived from Eq. (10.5.1) reads

$$\frac{L}{D} \geq \left(\frac{1}{\cos\gamma_e} - 1\right)\frac{1}{\lambda_r} + \frac{H}{\varepsilon_e R}\frac{\chi(\lambda_r, \varepsilon_e)}{\lambda_r} \tag{10.5.4}$$

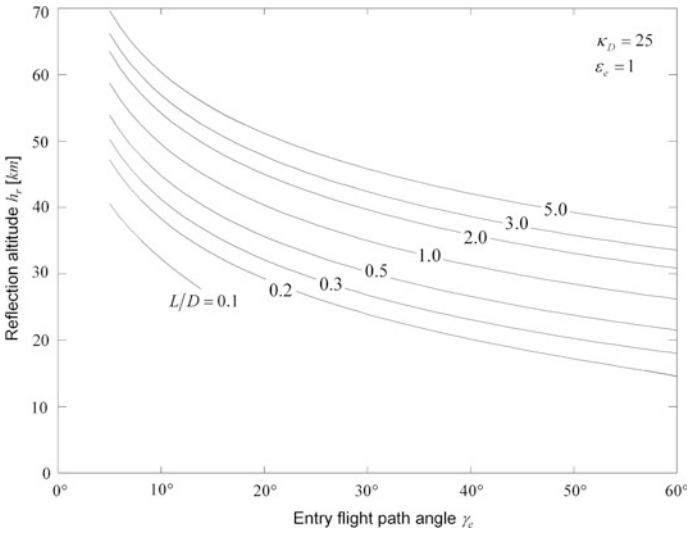


Fig. 10.14 Reflection altitudes as a function of the entry angle and lift for $\varepsilon_e = 1$ (as derived from Eq. (10.5.1))

This equation states that the vehicle reflects at given λ if L/D satisfies this equation. We are now seeking that minimal L/D for which reflection sets in at altitude $\lambda_{r,hor}$, that is, where it just flies horizontally for the first time. Minimizing the term on the right-hand side of Eq. (10.5.4) delivers the condition equation

$$\left(\frac{1}{\cos\gamma_e} - 1\right) + \frac{H}{\varepsilon_e R}\left[e^{\lambda_{r,hor}} - \varepsilon_e - \chi\left(\lambda_{r,hor}, \varepsilon_e\right)\right] = 0 \tag{10.5.5}$$

The root of this equation for a given entry angle, which can be determined numerically, delivers $\lambda_{r,hor}$ and hence $h_{r,hor}$. Inserting it into Eq. (10.5.4) one obtains the wanted minimal L/D for a given entry angle. This dependency is displayed in Fig. 10.15 for $\varepsilon_e = 1$ and $\varepsilon_e = 2$.

If reflection is not desirable at all, then $L/D < 0.1$ has to be ensured. This can be achieved either by a continuous rotation of the capsule (which was done with Mercury), which ensures $\langle L/D\rangle_t = 0$ on a time average, or by turning the capsule sideways or even upside down such that the lift vector points down implying $L/D < 0$.

In course of the trajectory after reflection the vehicle speed quickly diminishes so that gravitation, $p\chi \approx 1$, outweighs the lift and therefore the S/C descends. This second entry phase, induced by the relentless gravitation, is not reflected by the term χ, though, because we assumed in the perturbation term of the γ equation just $\varepsilon = \varepsilon_e \exp(-\lambda)$. What in fact happens is as follows. Because in the ε equation $d\varepsilon \propto -d\lambda/\sin\gamma$, the velocity always decreases, independently whether the S/C ascends $(d\lambda < 0, \sin\gamma < 0)$ or descends $(d\lambda > 0, \sin\gamma > 0)$ and therefore also its

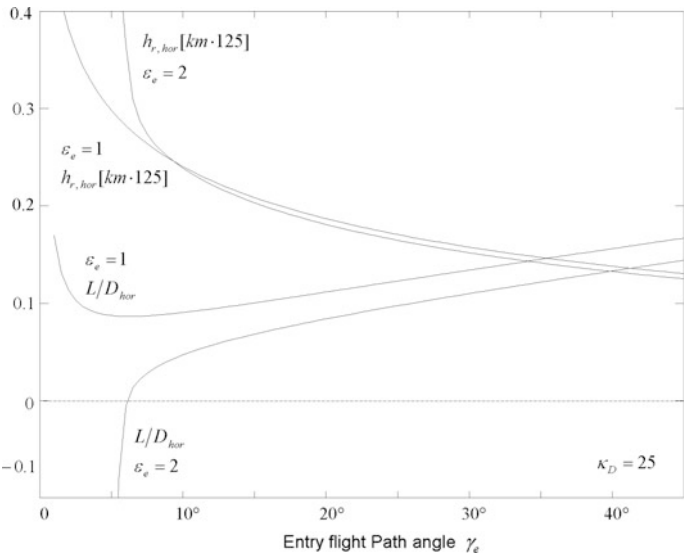


Fig. 10.15 Minimum values for *L/D* and corresponding altitudes at which reflection of a reentering vehicle sets in, for $\varepsilon_e = 1$ and $\varepsilon_e = 2$

energy $\varepsilon \propto v^2$ decreases steadily. Therefore, the term $-H\cos\gamma/\varepsilon\lambda R$ in the γ equation steadily increases, and at some time becomes of the order unity, independently from the flight path angle. It then dominates the lift and after the reflection at negative flight path angles $\gamma < 0$ forces the vehicle to turn down $d(\cos\gamma) \propto -d\lambda > 0$, and when it begins to descend again, $\gamma > 0$, to turn down even more rapidly $d(\cos\gamma) \propto -d\lambda < 0$. This behavior that we have derived from the equations of motion just reflects the fact that lift declines quadratically with decreasing speed so that gravity takes over and makes the vehicle sink in the long run. In order to verify this behavior in detail from the equations of motions, one has to solve the time-dependent form Eqs. (10.2.7) and (10.2.14) numerically. To apply for the analysis the reduced equations of motions Eq. (10.2.13) with λ as the independent variable, which has to progress per definition, would be useless, because after reflection λ actually decreases. So the λ-dependent γ Eq. (10.2.13b) cannot provide us a trajectory after reflection.

Moderate reflections are usually desirable for capsule reentries because they decrease speed without an increase of deceleration. We now will see how reflections can be driven into the extreme to utilize them for achieving moderate reentry decelerations for manned missions even at very high entry speeds as from interplanetary missions.

10.5.2 Skip Reentry

A skip reentry is the smart utilization of reflections to purposefully reduce the critical deceleration. It was used for the first time for the reentries of Apollo capsules after the return from the Moon. Here the entry velocity roughly corresponded to the second cosmic velocity of 11 km/s, and thus the entry energy was $\varepsilon_e = 2$. So the mission managers encountered the big problem to handle twice the amount of kinetic energy, as compared to the preceding Mercury and Gemini low Earth orbit missions and this seemed to be a special challenge for thermal protection and critical deceleration. The problem was solved by skip reentry (see Fig. 10.16). For reentry $\gamma_e \approx 6.5^\circ$ was chosen and the design of the capsule was laid out such that the center of mass of the capsule did not coincide with the center of pressure to obtain a $L/D \approx 0.3$. From Fig. 10.15 it can be seen that for $\varepsilon_e = 2$ and $\gamma_e \approx 6.5^\circ$ reflections occur for $L/D > 0$. $L/D = 0.3$ was therefore a sure choice. Such an intentional reflection with a subsequent ballistic coast is called a "skip". By means of skipping the initial speed can be reduced to such an extent that in a second dip reentry the deceleration forces are tolerable (see Figs. 10.6 and 10.7). The key purpose of skipping is a recurring stepwise speed reduction.

Exit Velocity

How big is the speed reduction by one skip? To determine it we examine the equations of motions (10.2.1) and (10.2.2). From Sect. 10.3.1 we know that at critical altitudes 30 km $\leq h \leq$ 70 km, when drag is about maximum, the gravitational term can be neglected.

> **Note** *In the γ equation the reduced gravitational term has for $1 \leq \varepsilon_e \leq 2$ a more sustainable effect because of the surplus centrifugal force of the approaching S/C. This centrifugal force effectively increases the lift resulting in higher reflection altitudes, as will be found from the following calculations. So we are on the safe side.*

Neglecting the gravitational terms we derive from Eqs. (10.2.1) and (10.2.2)

$$\dot{v} = -\frac{D}{m}$$
$$v\dot{\gamma} = -\frac{L}{m}$$

In order to determine the speed reduction we need the dependency $v(\gamma)$ of the symmetric trajectory around the reflection point to apply the condition $\gamma_{out} = -\gamma_{in}$ at the entry interface. So we need to get rid of the time dependency. We do this by dividing the above equations yielding

$$\frac{\dot{v}}{\dot{\gamma}} = \frac{dv}{d\gamma} = v\frac{D}{L}$$

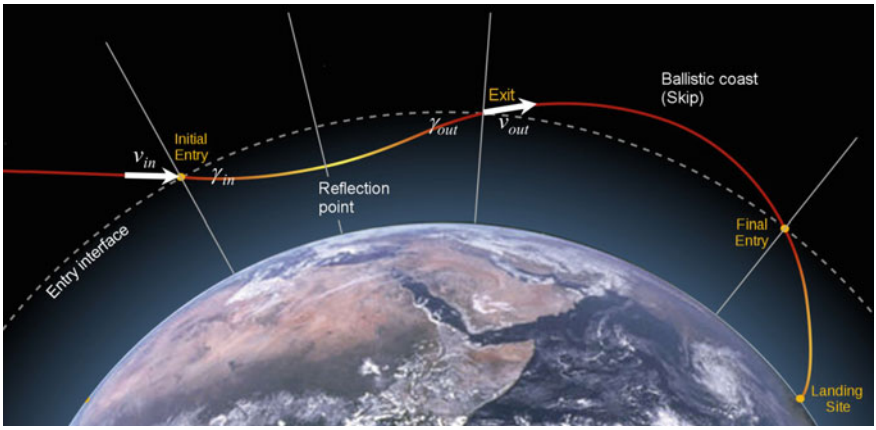


Fig. 10.16 Trajectory and trajectory parameters of a skip reentry. *Credit* Clem Tillier, Wikimedia Commons and U. Walter

From this follows with the initial conditions $v_{in} = v_e$, $\gamma_{in} = \gamma_e > 0$

$$\int_{v_{in}}^{v} \frac{dv}{v} = \frac{D}{L} \int_{\gamma_{in}}^{\gamma} d\gamma$$

and

$$v = v_{in} \exp\left[(\gamma - \gamma_{in})\frac{D}{L}\right]$$

As reflection implies an almost symmetrical flight path (see Fig. 10.16) about the reflection point, the condition

$$\gamma_{out} = -\gamma_{in}$$

has to be valid. So we get for the reduced exit velocity after reflection

$$v_{out} = v_{in} \exp\left(-2\gamma_{in}\frac{D}{L}\right) \tag{10.5.6}$$

Example

For the Apollo missions returning from the Moon the return velocity, and so the entry velocity of the capsule was $v_e = 11.0$ km s^{-1}. The selected entry angle was $\gamma_{in} = \gamma_e = 6.5°$ and the L/D ratio was $L/D = 0.3$. Therefore, and according to Eq. (10.5.6) the exit velocity after the reflection was $v_{out} = 0.470 \cdot v_e = 5.2$ km s^{-1}. With approximately this velocity the capsule was dipped again for a double dip (see below).

"Double Dip" Reentry with Apollo Flights

For reentries from outside LEO, there exists the so-called *entry corridor* for aerocapture, which for Apollo was only $5.0° < \gamma_e < 7.0°$ wide (see Figs. 10.17 and 10.3). For $\gamma_e > 7.0°$ the maximum admissible deceleration of 12 g would have exceeded during the skips. For $\gamma_e < 5.0°$ the splash down point would have been too imprecise due to an extended recoils from the atmosphere, or the risk of not or too weakly grazing the atmosphere for a skip would have been too large. This narrow corridor required a very precise approach from the Moon. In order to better determine the landing point for the Apollo and the Soviet Zond Moon flights, the so-called "double dip" reentry was used during the return process with a lift reversion: During the first reentry the lift vector was flown upward, as described above. After the reflection, however, the capsule was rotated so that the lift vector pointed downward (see Sect. 6.2.5), so that the negative lift kept the flight altitude roughly at the reflection altitude. With this maneuver it was possible to avoid bouncing back and the increasing inaccuracy of the landing point coming with this. In addition, the deceleration could be kept at lower altitudes to a more constant level, which altogether led to a safer landing.

Figure 10.18 depicts the historical reentry trajectory of Apollo 11. Judged against a comparable reentry trajectory from LEO (see Fig. 10.6) we recognize that the maximum after the first reflection is less developed. This is just the result of the negative lift. The reflection altitude of 185,000 ft = 56.5 km can easily be verified by applying Eq. (10.5.3) with the entry flight inclination angle of 6.5° and scale height $H = 8.0$ km at that altitude. This proves that the reflection altitude formula Eq. (10.5.3) is reliable and indeed is independent from the entry speed.

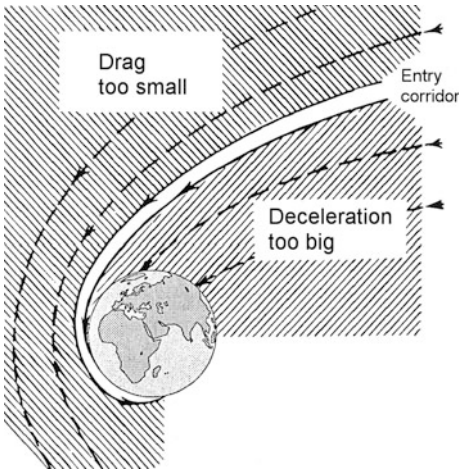


Fig. 10.17 Entry corridor for the return of the Apollo capsules from the Moon. *Credit* NASA

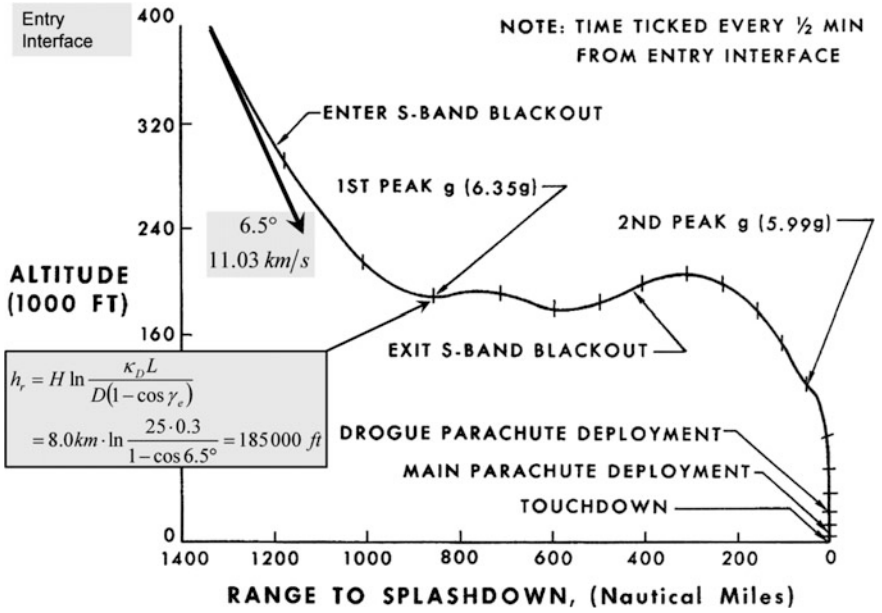


Fig. 10.18 The original reentry trajectory of Apollo 11 with an entry flight inclination angle of 6.5° and reflection at 185,000 ft altitude. *Credit* NASA

10.5.3 Phugoid Mode

It should be obvious now to drive skipping into the extreme and skip not only once but at a very shallow entry angle again and again to slowly but steadily reduce the speed. This indeed would be possible. Any reentry body with $L > 0.5$ at $\varepsilon_e = 1$ and $\gamma_e < 2°$ will experience multiple or continuous skipping. This can be easily verified numerically and is done and displayed in Fig. 10.19 for $L/D = 1.3$ and $\gamma_e = 1.2°$, which were typical entry parameters for the Space Shuttle. The state of such shallow continuous ups and downs is called *hypersonic phugoid mode*. These are oscillatory variations of altitude, where the flight path angle periodically oscillates with decreasing amplitude around zero.

The characteristic feature of a phugoid motion is that the S/C at very high speed and at very flat flight path angle, i.e., $\cos\gamma \approx 1$ and $\sin\gamma \approx \gamma$, oscillates around a mean flight path angle, $\gamma_D = const$, so $\langle\dot{\gamma}\rangle_t \approx 0$. We therefore can approximate the equations for a phugoid motion from the general equations of motion (10.2.1), (10.2.2), and (10.2.3) as

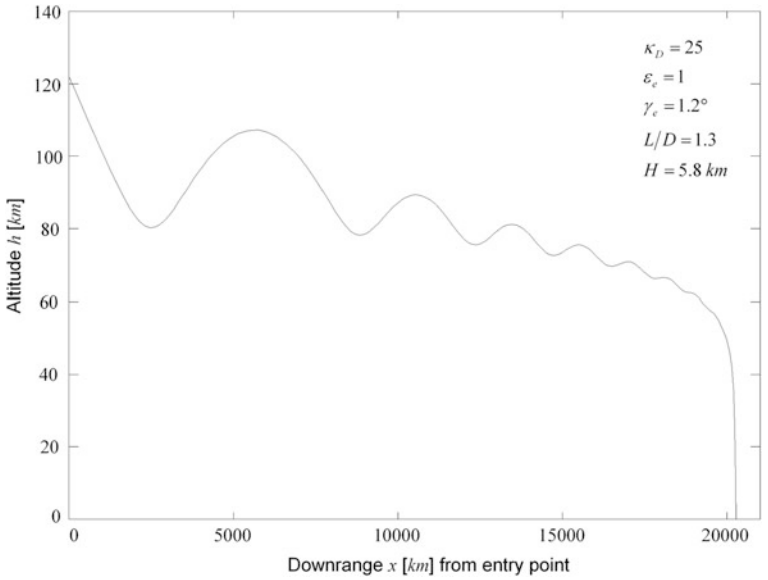


Fig. 10.19 The phugoid oscillation that would result from an uncontrolled lifting reentry of a Space Shuttle ($\gamma_e = 1.2°, L/D = 1.3$). The scale height $H = 5.8$ km is adjusted to the altitude where the phugoid oscillation happens

$$\dot{\gamma} = -\frac{L/v}{m} + \left(\frac{g}{v} - \frac{v}{r}\right)$$
$$\dot{v} = -\frac{D}{m} + g\gamma$$
$$\dot{h} = -v\gamma$$

The oscillations should be noticeable in variations of the flight path angle. In seeking for a differential equation for that we differentiate the above γ equation with respect to time

$$\ddot{\gamma} = -\frac{1}{m}\left(\frac{\partial(L/v)}{\partial h}\dot{h} + \frac{\partial(L/v)}{\partial v}\dot{v}\right) + 0 \cdot \dot{h} - \left(\frac{g}{v^2} + \frac{1}{r}\right)\dot{v}$$

where we have assumed that the oscillations take place at about a constant altitude, $r \approx const$. From Eqs. (10.2.4) to (10.2.5) follows

$$\frac{\partial(L/v)}{\partial h} = -\frac{(L/v)}{H} \quad \text{and} \quad \frac{\partial(L/v)}{\partial v} = \frac{L}{v^2}$$

and therefore

$$\ddot{\gamma} = -\frac{1}{m}\left[\frac{L}{H}\gamma + \frac{L}{v^2}\left(g\gamma - \frac{D}{m}\right)\right] - \left(\frac{g}{v^2} + \frac{1}{r}\right)\left(g\gamma - \frac{D}{m}\right)$$

from which follows

$$\ddot{\gamma} = -\left[\frac{L}{mH} + g\left(\frac{g}{v^2} + \frac{1}{r} + \frac{L}{mv^2}\right)\right]\gamma + \frac{D}{m}\left(\frac{g}{v^2} + \frac{1}{r} + \frac{L}{mv^2}\right)$$

Because it follows from the γ equation with $\langle\dot{\gamma}\rangle_t \approx 0$ and $\cos\gamma \approx 1$ that $v^2 \approx \langle v\rangle_t^2 = r(g - L/m)$, we find

$$\begin{aligned}\frac{g}{v^2} + \frac{1}{r} + \frac{L}{mv^2} &\approx \frac{g}{r(g - L/m)} + \frac{1}{r} + \frac{L}{r(mg - L)} = \frac{1}{r}\frac{mg + mg - L + L}{mg - L} \\ &= \frac{1}{r}\frac{2mg}{mg - L}\end{aligned}$$

Hence

$$\ddot{\gamma} = -\left(\frac{L}{mH} + \frac{2g}{r}\frac{mg}{mg - L}\right)\gamma + \frac{2D}{mr}\frac{mg}{mg - L} \tag{10.5.7}$$

This is the differential equation of a somewhat odd linear oscillator, which we write in the general form

$$\ddot{\gamma} = -\left(\omega_0^2 + \omega_1^2\right)\gamma + c$$

Obviously there are two contributions to the phugoid oscillation with angular frequencies

$$\omega_0 = \sqrt{\frac{L}{mH}} \quad \text{and} \quad \omega_1 = \sqrt{\frac{2g}{r}\frac{mg}{mg - L}} \tag{10.5.8}$$

If we trace back these contributions in the derivation, we see that the oscillation with ω_1 is caused by the causal chain (mode): declining S/C → increasing speed → increasing centrifugal force + increasing lift → decreasing flight path angle → upturn. On the other hand, the ω_0 oscillation stems from the chain: declining S/C → decreasing altitude → exponentially increasing atmospheric pressure → strongly increasing lift → quickly decreasing flight path angle → immediate upturn. We could interpret the latter process also as a bouncing off the atmosphere. These two oscillations happen on quite different time scales. Because

$$\frac{L}{mg} \approx \frac{RL}{mv^2} = \kappa_D \frac{L}{D}\frac{R}{H}e^{-h/H} \approx 0.2$$

it follows

$$\frac{\omega_1^2}{\omega_0^2} \approx \frac{2g}{r} \Big/ \frac{L}{mH} = 2\frac{mg}{L}\frac{H}{r} \approx 2 \cdot 0.2 \cdot 0.001 = 0.0004$$

The short-term ω_0-mode therefore is the more forceful mode by orders of magnitude, which is why we can neglect ω_1. We therefore can simplify Eq. (10.5.7)

$$\ddot{\gamma} = -\omega_0^2 \gamma + 2\frac{D}{mr}\frac{mg}{mg - L} = -\omega_0^2(\gamma - \gamma_D)$$

with

$$\gamma_D = 2\frac{D}{L}\frac{H}{r}\frac{mg}{mg - L} \approx 0.1^\circ \tag{10.5.9}$$

To solve the equation we substitute $x := \gamma - \gamma_D$. This implies $\ddot{\gamma} = \ddot{x}$ we therefore get the differential equation $\ddot{x} = -\omega_0^2 x$ with the solution $x = x_0 \cos(\omega_0 t + \varphi)$. By resubstitution and because $\gamma(t = 0) = \gamma_e$ we finally get

$$\gamma = \gamma_D + (\gamma_e - \gamma_D)\cos\omega_0 t \tag{10.5.10}$$

The drag-induced offset $\gamma_D \approx 0.1^\circ$ is the time averaged value of the FPA. It determines the long-term decline of the mean altitude of the vehicle and is easily recognized as such in Fig. 10.19. The period of the phugoid oscillation is determined via Eqs. (10.2.5) and (6.2.17) to be

$$T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{mH}{L}} = 2\pi\frac{H}{v}e^{\frac{h}{2H}}\sqrt{\frac{D}{\kappa_D L}} \quad \textbf{phugoid period} \tag{10.5.11}$$

So the period decreases exponentially with the altitude at which the phugoid motion takes place. This exponential dependency is nicely depicted in Fig. 10.19. Although the mean flight altitude decreases only slightly with the phugoid motion, the phugoid period (and the mode amplitude) decreases quickly. For a phugoid motion at an altitude of typically $h = 80 - 90$ km ($H \approx 5.8$ km, see Table 6.2) and for $\kappa_L = \kappa_D L/D = 32.5$ the period is $T = 15 - 35$ min.

Because permanent skipping causes a gentle velocity reduction, it may seem ideal, at first glance. However it has a serious drawback: Without any lift control, the downrange distance and with it the landing site can virtually not be determined. Note that in the case of Fig. 10.19 the distance traveled until landing is about

20,000 km, that is half around the globe! This is unacceptable neither for winged bodies and even less for capsules (they have to land on ground or on water with a good accuracy to pick them up), so phugoid modes have to be avoided at any rate. But one can turn the objective upside down. According to an idea of the famous Austrian space engineer Eugen Sänger (1905–1964), it would be possible to design an intercontinental transport high-lift vehicle, the so-called *Antipodal Bomber*, with say, $L/D \approx 2.5$ without any propulsion by accelerating it to LEO speeds and then using wave-like gliding along the surface of the atmosphere to reach a given target point anywhere on the globe with a controllable lift in the late phase. The time required for orbiting the Earth would then be about 1 h 45 min. The critical deceleration at the first reflection would be only 0.1 g, increasing for the following skips until about only 0.4 g in the final phase.

10.6 Lifting Reentry

We have seen that for $\gamma < 2°$ the approximations for a ballistic reentry are no longer valid. For heavy S/Cs, however, reentries at such flat angles are the only way to keep the peak heat load and peak structural load below tolerable values by spreading the deceleration and hence friction over a much longer time period. This can be achieved by an increased L/D of the S/C. This is why all the larger S/Cs are so-called winged bodies, such as the Space Shuttle.

We will now analyze this limiting case of a lifting reentry, where an adjustable lift is utilized to maintain a constant flight path angle, which is typically of the order of 1°. Figure 10.20 shows for instance the target area for the flight path angle at entry interface for a Shuttle reentry. So

$$\dot{\gamma} = 0 \quad \textbf{lifting reentry condition} \tag{10.6.1}$$

Hence $\gamma = \gamma_e \approx 1° = const$. Because of this condition we have: $\sin\gamma = \sin\gamma_e \approx \gamma_e$ and $\cos\gamma \approx 1$. Therefore, the left-hand sides of Eqs. (10.2.2) and (10.2.13b) vanish. In order to have also the right sides zero the following must hold (approximation: $g \approx g_0, r \approx R$)

$$\frac{L}{m} = g - \frac{v^2}{r} \approx g_0\left(1 - \frac{v^2}{v_\triangleright^2}\right) \quad \text{with } v_\triangleright^2 = g_0 R \tag{10.6.2}$$

This equation implies that in course of reentry lift has to be constantly adjusted such that together with the centrifugal force as the other "pulling up" force, they just balance the gravitational force, which "pulls down". This is where the expression *equilibrium glide* (a.k.a. *lifting reentry* or *gliding reentry*) stems from.

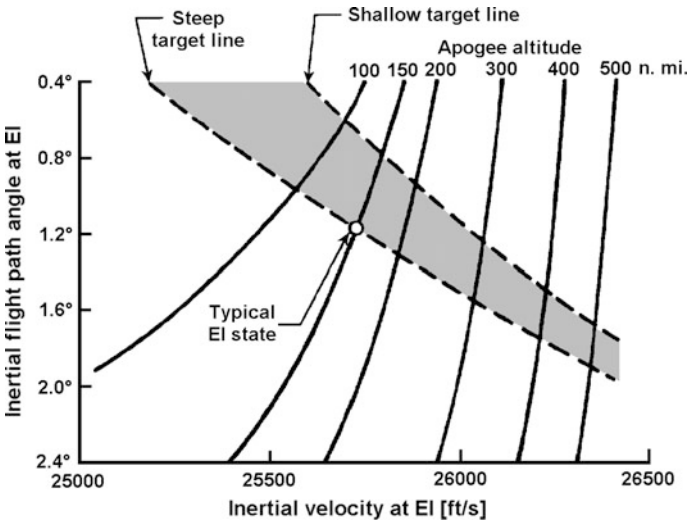


Fig. 10.20 Target area (gray) for the flight path angle at entry interface, EI, for the Space Shuttle. *Credit* N. Chaffee, NASA/JSC

Uncontrolled Lifting Reentry

We already came across lifting reentry when we investigated the phugoid mode. In that case, however, the FPA was not constant at any point in time, but constancy was achieved only on time average:

$$\langle \gamma \rangle_t = \gamma_D = 2\frac{D}{L}\frac{H}{r}\frac{mg}{mg - L} = const$$

Furthermore, the FPA oscillated around this mean value, because we had assumed an unregulated constant L/D ratio.

Controlled Lifting Reentry

The phugoid mode is undesirable for a steady lifting reentry. Therefore a reentry vehicle is required, with $L/D_{\max} \geq 0.7$ and which can adjust lift and drag separately. Lift must be adjustable to such an extent that the phugoid oscillation can be compensated and therefore $\gamma = \gamma_e = const$ can be enforced at any time. This indeed is possible via the so-called *angle of attack* (AOA) α and the so-called *roll angle* (a.k.a. *bank angle*) μ. For the Space Shuttle Fig. 10.23 depicts how drag and lift can be varied jointly by the AOA such that the L/D ratio changes in a way characteristic for the S/C as given in Eq. (10.2.9). Of course for $\alpha = 90°$ $L/D = 0$ always holds, because vertical lift must vanish at that angle. Independent from the AOA a bank angle can be set. The cosine of the bank angle determines the lift component, which points into the z-direction, i.e., upward as shown in Fig. 6.8. It is only this component that balances the gravitational and centrifugal force, and which is

decisive in our equations. If the bank angle is $\mu = 90°$, then the S/C is tilted fully sideways and no upward lift is generated. In summary:

> For a controlled lifting reentry the angle of attack and bank angle of the vehicle are adjusted such that the generated upward lift plus the centrifugal force exactly compensates the gravitational force and hence $\gamma = const$.

10.6.1 Reentry Trajectory

How must L/D be adjusted at a given altitude such that Eq. (10.6.2) is fulfilled? Since we do not know the velocity as a function of altitude for a lifting reentry Eq. (10.6.2) is not a control law for L/D in itself—rather, we have to solve the equation of motion. Applying the condition Eq. (10.6.1) to the basic equations of motion Eq. (10.2.13) we find

$$\begin{aligned} \frac{d(\ln \varepsilon)}{d\lambda} &= -1 + \frac{2H}{\varepsilon \lambda R} \\ \frac{d(\cos \gamma)}{d\lambda} &= \frac{\sin \gamma_e}{2}\frac{L}{D} - \left(\frac{1}{\varepsilon} - 1\right)\frac{H \cos \gamma}{\lambda R} \end{aligned} \qquad \textbf{equations of motion} \qquad (10.6.3)$$

We know already the solution of the γ equation. It is the FPA Eq. (10.4.8). The solution of the decoupled ε equation can be easily derived. We first separate the variables

$$d(\ln \varepsilon) = \left(-1 + \frac{2H}{\varepsilon \lambda R}\right) d\lambda$$

This equation of motion is the same as the one in Sect. 10.4.2 except that here $\gamma = \gamma_e = const \rightarrow c = q = 0$. Therefore, we derive from Eq. (10.4.12) the second-order solutions

$$\begin{aligned} \ln \frac{v}{v_e} &= -\frac{\lambda}{2} + \frac{H}{\varepsilon_e R}\chi(\lambda, 0) \\ \cos \gamma &= \left[1 + \frac{\tan \gamma_e}{2}\frac{L}{D}\lambda - \frac{H}{\varepsilon_e R}\chi(\lambda, \varepsilon_e)\right] \cos \gamma_e \end{aligned} \qquad (10.6.4)$$

The second equation is the FPA Eq. (10.4.8), which also holds for lifting reentry as long as $\tan \gamma_e \cdot \lambda L/(2D) \ll 1$, that is $\exp(h/H) \gg L\kappa_D/D$, i.e., down to about $h = 45$ km. With the FPA equation we have the answer to the question, how to

adjust the lift. Because if for an equilibrium glide $\cos\gamma = \cos\gamma_e = const$ must hold, the last two terms in the square bracket must cancel each other. This condition provides the L/D control law

$$\frac{L}{D} = \frac{2H}{\varepsilon_e R \tan\gamma_e}\frac{\chi(\lambda,\varepsilon_e)}{\lambda} \qquad \textbf{L/D control law} \tag{10.6.5}$$

At very high altitudes, that is for $\lambda_e \ll \lambda < 1$, we can approximate in Eq. (10.6.5) $\chi(\lambda,1) = f^{\lambda} \approx \lambda$ and therefore with $H \approx 7.6$ km and $\varepsilon_e = 1$ we find

$$\frac{L}{D} \approx \frac{2H}{\varepsilon_e R \tan\gamma_e} \approx 0.10 - 0.14$$

This value is much smaller than the typical lift of a winged body, such as the Space Shuttle, which had $L/D \approx 1.0$ at high altitudes, where $\alpha = 40°$ (see Fig. 10.25; Eq. (10.2.10)). Drag has to be as big as possible at very high altitudes, but not to an extent that the flight attitude becomes unstable. Therefore, the AOA is set to the limiting value $\alpha \approx 45°$, while the vertical lift is reduced by a bank angle μ (see Sect. 6.2.5 and note that $L \equiv L_v$ so far, see convention after Eq. (10.2.6)). The bank angle produces horizontal lift, which moves the S/C sideways out of the motion plane, additionally providing cross-range capability (cf. Sect. 10.7.1). To achieve the value $\cos\mu \cdot L/D = 0.10 - 0.14$ we have

$$\mu \approx \arccos(0.12 \mp 0.2) = 83° \pm 1° \tag{10.6.6}$$

Because this is too close to the critical value $\mu = 90°$, where the Space Shuttle would have plunged down, NASA limited the bank angle to $\mu = 80°$ (see Fig. 10.24).

10.6.2 Critical Deceleration

Having found the solutions of the equations of motions it is now straightforward to determine the critical deceleration and those trajectory parameters, namely critical altitude and critical speed, at which it is achieved. Because $\gamma = \gamma_e$, Eq. (10.4.9) is exact and therefore also Eq. (10.4.10). This is true for any entry phases down to the smallest velocities as long as $\gamma = \gamma_e = \gamma_{crit}$. From this we derive the critical altitude

$$\lambda_{crit} = 1 + \frac{2H}{\varepsilon_{crit} R} \quad @ \quad \gamma = \gamma_e = const$$

How big is ε_{crit}? From Eq. (10.6.4) we find that with $H \cdot \chi(\lambda_{crit} \approx 1.0, 0)/R \approx 10^{-3} \ll \lambda_{crit}/2$ follows $\varepsilon_{crit} \approx \varepsilon_e \exp(-\lambda_{crit})$. So for $\lambda_{crit} \approx 1$ we have $\varepsilon_{crit} = \varepsilon_e/e$ and therefore

$$\lambda_{crit} = 1 + \frac{2eH}{\varepsilon_e R} \approx 1 \tag{10.6.7}$$

Thereby we have proven self-consistently that Eq. (10.6.7) is correct. According to Eq. (10.4.15) we find for the critical altitude

$$h_{crit} = H \ln \frac{2\kappa_D}{\sin \gamma_e} \approx 60 \text{ km} \tag{10.6.8}$$

With Eq. (10.6.4) the critical velocity is found to be

$$\ln \frac{v_{crit}}{v_e} = -\frac{1}{2} + \frac{H}{\varepsilon_e R} \chi(1,0)$$

Because

$$\frac{H}{\varepsilon_e R} \chi(\lambda,0) \approx \frac{H}{R} \cdot \frac{h_e - h}{H} = \frac{h_e - h}{R} \approx 0.01$$

we obtain

$$v_{crit} = \frac{v_e}{\sqrt{e}} \left(1 + \frac{H}{\varepsilon_e R} \chi(1,0)\right) \approx \frac{v_e}{\sqrt{e}} = 4.5 \text{ km s}^{-1} \cdot \sqrt{\varepsilon_e} \approx 4.5 \text{ km s}^{-1} \tag{10.6.9}$$

From Eq. (10.4.9) we finally derive the critical deceleration

$$a_{crit} = -\frac{v_\triangleright^2 \sin \gamma_e}{2H} \left[\varepsilon_{crit} \lambda_{crit} - 2\frac{H}{R}\right] \approx -\frac{v_\triangleright^2 \sin \gamma_e}{2H} \varepsilon_{crit} \lambda_{crit} \approx -\frac{v_\triangleright^2 \sin \gamma_e}{2H} \frac{\varepsilon_e}{e}$$

from which follows

$$a_{crit} \approx -\frac{R}{2eH} \sin \gamma_e \cdot g_0 \approx -154 \cdot \sin \gamma_e \cdot g_0 \tag{10.6.10}$$

This equation states that the critical deceleration may take on virtually any value by adjusting the entry angle accordingly. For $\gamma_e = 1.0°$ the deceleration amounts to a modest $a_{crit} = -2.7 \cdot g_0$.

10.6.3 Heat Flux

The structural stress load capacity of the Space Shuttle was about $|a_{crit}| \approx 2.5 \cdot g_0$. Other than this the peak heat load, which via the heat emission corresponds to a peak surface temperature of the heat tiles, is another critical parameter that has to be

taken care of. To determine the peak heat flux we consider Eq. (10.6.2) by applying Eq. (10.2.5)

$$\rho = \rho_0 \frac{H}{\kappa_L R}\left(\frac{v_\triangleright^2}{v^2} - 1\right) = \frac{2}{BR}\frac{D}{L}\left(\frac{v_\triangleright^2}{v^2} - 1\right)$$

where $B = C_D A_\perp / m$ is the ballistic coefficient (see Eq. (6.2.19)). Inserting this into Eq. (10.1.3) we get for the heat flux onto the stagnation point of the vehicle

$$\dot{q}_{S/C} = \frac{St}{2}\sqrt{\rho\rho_q \frac{R_0}{R_n}} \cdot v^3 = \frac{St}{2}\sqrt{\frac{2\rho_q}{BR}\frac{D}{L}\frac{R_0}{R_n}}\sqrt{\frac{v_\triangleright^2}{v^2} - 1} \cdot v^3$$

From the maximum condition

$$\frac{1}{\dot{q}}\frac{d\dot{q}}{dv} = \frac{1}{v\left(v_\triangleright^2/v^2 - 1\right)}\left(2\frac{v_\triangleright^2}{v^2} - 3\right) = 0$$

we finally derive with the critical velocity

$$v_{\max \dot{q}} = v_\triangleright \sqrt{\frac{2}{3}} = 6.1 \text{ km/s} \tag{10.6.11}$$

Remark *If we would have assumed the Detra and Hidalgo heat flux model with* $\dot{q}_{S/C} \propto \sqrt{\rho}\, v^{3.15}$ *we would have derived for the critical velocity* $v_{\max \dot{q}} = v_\triangleright \sqrt{2.15/3.15}$*, which deviates by only 1% from the above result.*

To find the altitude at which the peak heat load is achieved we consider Eq. (10.6.4). Since $v_{\max \dot{q}}$ is about the same as for the ballistic reentry we expect the peak heat load at about 50 km altitude at which λ is of order unity. Because from the above the term $H\chi(\lambda, 0)/(\varepsilon_e R)$ is negligible, $v = v_e e^{-\lambda/2}$ holds at these altitudes. We therefore get for a lifting reentry from LEO, $v_e \approx v_\triangleright$, $\lambda_{\max \dot{q}} = -\ln(2/3)$ and because of Eq. (10.2.12)

$$h_{\max \dot{q}} = H \ln \frac{4.933 \cdot \kappa_D}{\sin \gamma_e} = h_{crit} + 0.90H \quad @ \quad v_e \approx v_0 \tag{10.6.12}$$

For a typical $\gamma_e = 1^\circ$ and $\kappa_D = 25$ we find $h_{\max \dot{q}} \approx 68$ km $\approx h_{crit} + 6.9$ km (cf. Eq. (10.3.15) and Fig. 10.10 for a ballistic reentry).

For the wanted peak heat flux to the S/C we find

$$\dot{q}_{S/C,\max} = \frac{St}{2}\sqrt{\frac{2\rho_q}{BR}\frac{D}{L}\frac{R_0}{R_n}}\frac{2}{3\sqrt{3}}v_\triangleright^3 = \dot{q}_e\frac{2}{3}\sqrt{\frac{2}{3\rho_e BR}\frac{D}{L}\frac{R_0}{R_n}} \tag{10.6.13}$$

with $\dot{q}_e = \frac{St}{2}\sqrt{\rho_q\rho_e}v_\triangleright^3 = 1.15$ W cm^{-2} the standard heat flux at entry interface where $St = 0.001, v_\triangleright = 7.905$ km s^{-1}, $R_0 = 1$ m, $\rho_q = 0.121$ kg m^{-3}, and $\rho_e = 1.80 \times 10^{-8}$ kg m^{-3} the mean atmospheric density at entry interface according to MSIS-E-90. Observe that the heat flux at the stagnation point is reduced by a larger radius of curvature, R_n. However, contrary to the ballistic case (see Sect. 10.3.3) the heat flux no longer depends on the drag coefficient, but on the lift coefficient because the ballistic coefficient $B \propto C_D \propto D$ (see Eqs. (6.2.19) and (10.2.4)). Note that the expression $\rho_e BR$ and hence the entire square root is dimensionless. Note also that $\rho_e BR$ relates to the Bstar B^* of NORAD's TLE (see Remark in Sect. 12.7.3) as

$$\rho_e BR = 2B^*\frac{\rho_e}{\rho_{120}} = 1.46B^*$$

where the latter holds only if the actual atmospheric density at entry interface is the standard value $\rho_{120} \equiv \rho_0 = 2.461 \times 10^{-8}$ kg m^{-3} as assumed in the Simplified General Perturbation Model SGP of NORAD.

With this and with Eq. (10.1.1) the peak temperature load at the stagnation point of the S/C can be determined to be

$$\boxed{T_{\max}^4 = \frac{T_e^4}{\varepsilon}\cdot\frac{2}{3}\sqrt{\frac{2}{3\rho_e BR}\frac{D}{L}\frac{R_0}{R_n}}} \tag{10.6.14}$$

with

$$T_e = \left(\frac{\dot{q}_e}{\sigma}\right)^{1/4} = 671 \text{ K}$$

the standard stagnation point temperature at entry interface and $\varepsilon \approx 0.85$ the approximate emissivity of the heat shield. With Eq. (10.6.13) we can rewrite Eq. (10.1.3) as a normalized equation for the heat flux onto the stagnation point of the S/C as

$$\frac{\dot{q}_{S/C}}{\dot{q}_{S/C\,\max}} = \frac{3\sqrt{3}}{2}\sqrt{(e^\lambda - 1)e^{-3\lambda}} \tag{10.6.15}$$

This function is plotted in Fig. 10.21 for various entry angles.

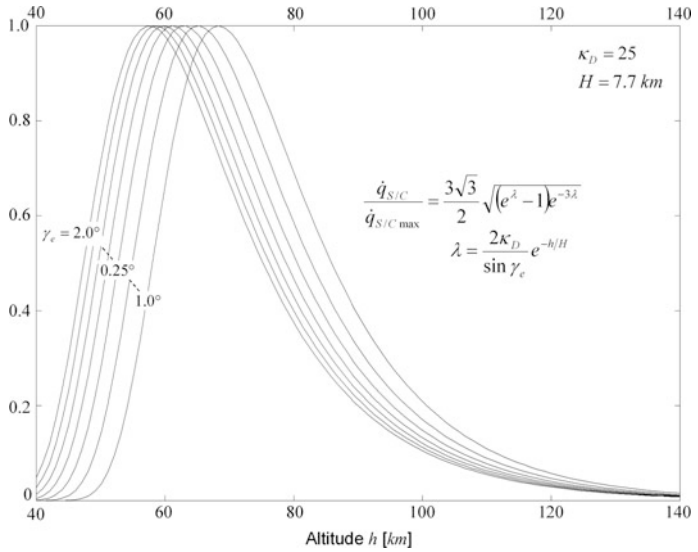


Fig. 10.21 Normalized heat flux at the stagnation point of a spacecraft with lifting reentry as a function of altitude h and entry angle γ_e in steps of 0.25°

10.7 Space Shuttle Reentry

Having investigated so far the challenges of a reentry, we now want to exemplify the practical solution in a case study, namely the Space Shuttle reentry. The detailed figures of the reentry profile described hereafter are taken from the reentry of my Shuttle mission STS-55 on May 6, 1993. However, the reentry profile applies quantitatively to all Shuttle reentries and qualitatively to any winged body's reentry.

The preferred diagram to represent reentry constraint boundaries and hence to formulate guidance control would be the dynamic pressure state versus velocity state space. However, this would require deriving the dynamic pressure from vehicle acceleration and altitudes and to have lift and drag coefficient models for all angles of attack and Mach numbers. Because such relationships would include considerable uncertainties, the constrained boundaries are usually reformulated into a drag acceleration state versus Earth relative velocity state space diagram (see Fig. 10.22). This formulation requires only an estimate of L/D as a function of AOA, which is well known (see Eq. (10.2.9)), and drag acceleration, which is easily measured as an acceleration force normal to the trajectory.

Note that here, in the following, and quite generally in NASA terminology "drag acceleration" means "vehicle deceleration due to drag", i.e. $-\dot{v}_D$, which according to Eq. (10.2.1) is $-\dot{v}_D = D/m$, i.e. mass-specific drag itself.

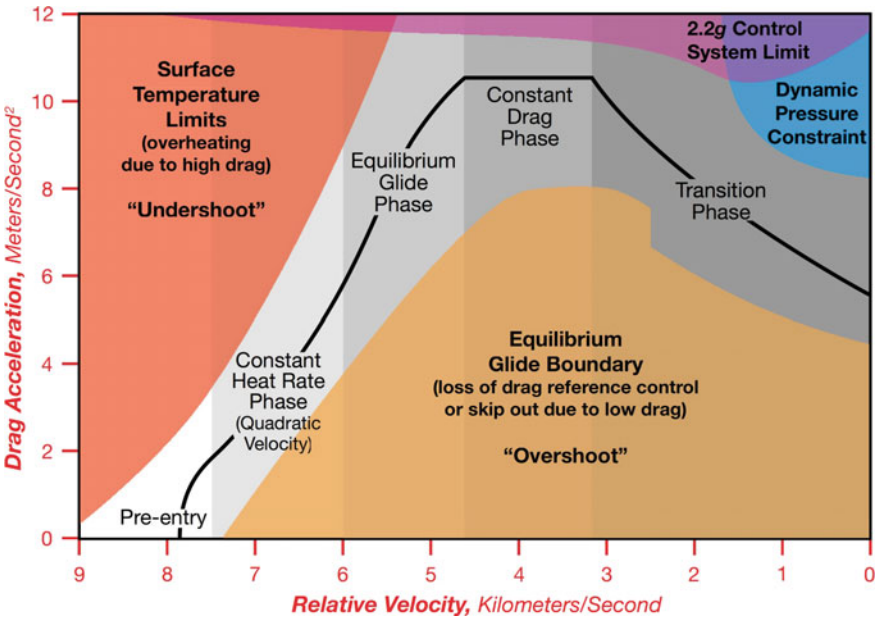


Fig. 10.22 The Shuttle reentry profile in the drag acceleration/Earth relative velocity state space (cf. Fig. 10.1). The Shuttle entered the diagram at high velocity and crossed it on the bold line from left to right within the so-called "flight corridor" (gray area). *Credit* M. Tigges, R. Rohan/NASA

Starting from the deorbit burn, the Shuttle traversed seven different reentry phases: the deorbit phase; the five aerodynamic phases as depictured in Fig. 10.22: pre-entry phase, constant heat phase, equilibrium phase, constant drag phase, and transition phase; and finally the Terminal Area Energy Management (TAEM) phase. We will walk through all these phases in the following sub-sections. The reentry profile described hereafter is taken from the reentry of my mission STS-55 on May 6, 1993, but is typical for the lifting reentry of any kind of winged body.

Deorbit Phase

Typically, a Shuttle resides in an circular LEO orbit at altitude $h_i = 300$ km. Sixty minutes and forty seconds before touchdown, and with it 170° west of the touchdown point, it executes a deorbit burn of 176 s duration to reduce the orbital velocity of 7.73 km s^{-1} by just 0.0885 km s^{-1} (i.e., by only 1.1%!—see Shuttle reentry example in Sect. 10.1.3). This brings the Shuttle on an entry ellipse, on which it descends. After 28 min it crosses the entry interface at an altitude of 122 km with $\gamma_e = 1.2°$ and $v_e = 7.86$ km s^{-1} (generally 1.0°–1.5° and 7.82 − 7.89 km s^{-1}).

10.7.1 Reentry Flight Design and Pre-entry Phase

Beyond the entry point down to Mach 14, the primary goal is to avoid unacceptable aerothermodynamic heating, especially on the upper surface and wing leading edge. To accomplish this, the Shuttle is placed at an AOA (angle of attack) of about $\alpha = 40°$ (see Fig. 10.23). The air molecules, which can be considered ballistic at these altitudes, collide with the large bottom side of the Shuttle thus transferring more momentum and hence causing also more desired drag.

But at the same time and according to Eq. (6.2.27) they cause increased lift. This is undesirable as in this phase of almost free fall gravitational force and centrifugal force balance each other. So, an upward lift would bring about skipping. To avoid this, the S/C is tilted sideways (see Sect. 6.2.5) with a bank angle of $\mu \leq 80°$ (see Fig. 10.24) at an unaltered AOA, so that the Shuttle is in a sloping lateral attitude with very low vertical lift. This is in line with our *L/D* control law from Eq. (10.6.5). NASA limits the bank angle to 80°, because any uncontrollable slight increase beyond 80° would lead to the Space Shuttle plunging straight down. The technical term of this roll maneuver is "roll reversals" or "bank reversals" because they are carried out alternately to the left and right side.

During a bank reversal, which can last up to 1 min, the lift may become very large, in particular at the moment when the bank angle is about zero. This would immediately induce a skipping action. To avoid this, the AOA is simultaneously increased during the bank reversal so that lift increases not too much (cf. Fig. 10.25). Thereby the drag increases insignificantly.

An additional benefit of roll reversals is that they allow lateral steering of the Space Shuttle, which at this very early reentry phase increases the cross-range capability to about 2000 km. This is highly desirable to ensure that the Shuttle reaches the landing site (see Fig. 10.26) and still has sufficient range capability to properly align with the runway heading (see TAEM phase Sect. 10.7.6). Roll maneuvers are continually carried out during reentry right down to Mach 2.5.

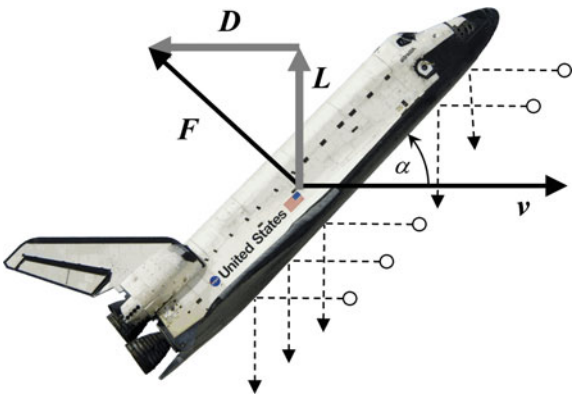


Fig. 10.23 Drag and lift of a Space Shuttle at a typical $AOA = \alpha = 40°$ in the early reentry phases

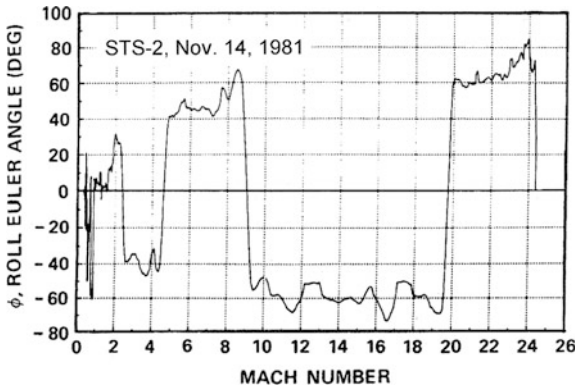


Fig. 10.24 Bank angle as a function of entry velocity. *Credit* P. Romere, C. Young (1983), NASA/AIAA

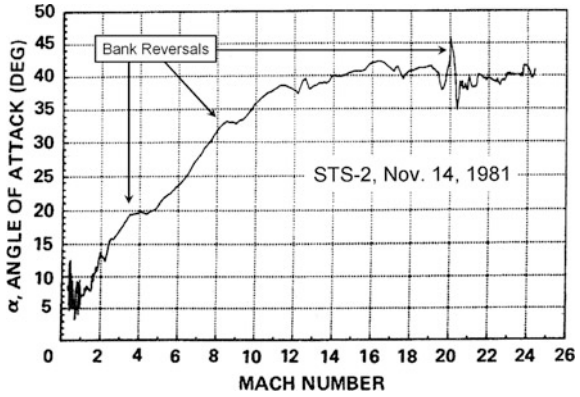


Fig. 10.25 AOA profile as a function of entry velocity. *Credit* P. Romere, C. Young (1983), NASA/AIAA

The guidance control algorithm has two objectives: To guide the Shuttle through the flight corridor (see Fig. 10.22) and at any given moment to compute how far the vehicle could fly. i.e. whether it could reach the runway. Any difference between the analytically computed range and the required range would trigger an adjustment in the drag-velocity references to remove that range error. Thus it made use of the tolerances of the flight corridor to remove range errors.

Pre-entry Phase

In this sloping lateral attitude the Shuttle flies at a roughly constant velocity of $v = 7.9$ km s^{-1} (cf. Fig. 10.7) down to an altitude of about 80 km. So, drag merely reduces its potential energy. In this phase the two primary parameters to control the vehicle's attitude and thus lift and drag and also the desired range and cross-range during reentry are AOA (set by the body flap) and bank angle (set by the ailerons and aft yaw Reaction Control System jets).

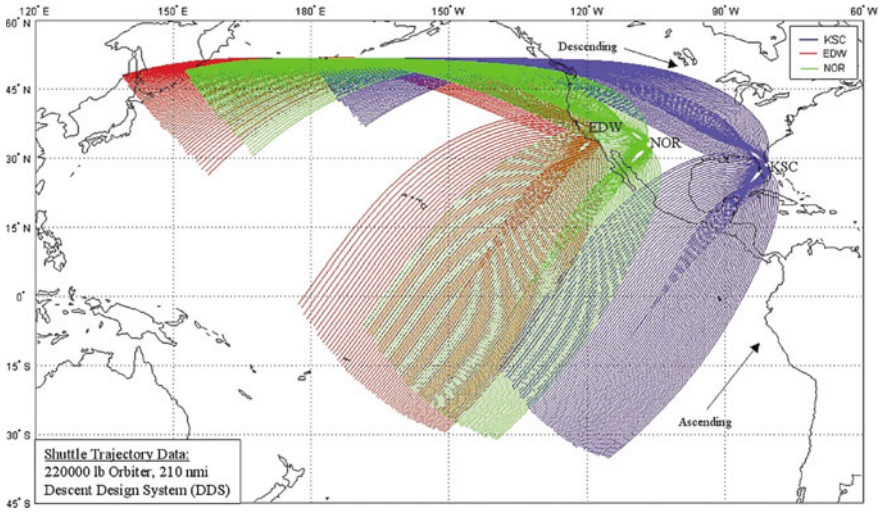


Fig. 10.26 Possible reentry ground tracks from 51.6° orbit inclination (ISS). Blue lines are landing at Kennedy Space Center (KSC), green at White Sands Space Harbor, Northrup Strip (NOR), and red at Edwards Air Force Base (EDW). *Credit* NASA

10.7.2 Constant Heat Rate Phase (Thermal Control Phase)

At about 80 km altitude the heat flux is 70% of its maximum value (see Fig. 10.21), so that from now on the heat rate needs to be controlled by vehicle guidance. This is the constant heat rate phase of the reentry profile, which extends from altitudes 80–50 km and in which the Shuttle retains its *AOA* of about 40°. At these altitudes the thermal shield of the Shuttle heats up so much that the impacting air around the Shuttle ionizes forming a plasma, which is impermeable for electromagnetic waves. For about 15 min there is no radio communication with Mission Control. That is why this phase is called the "blackout phase". The drag is now so strong that velocity is fiercely reduced.

A constant heat rate was achieved by assuming two consecutive segments with a heuristic quadratic velocity profile for drag acceleration (= drag)

$$D = c_1 + c_2 v + c_3 v^2$$

with constants c_1, c_2, c_3 that empirically determine the bold line in the flight corridor of Fig. 10.22. Deviations from this reference drag profile are counteracted by modulations of the *AOA* of around 40°. This heat control procedure limits the heat flux such that it does not peak as depicted in Fig. 10.21 for an equilibrium glide, but plateaus the heat flux and hence also the maximum temperature at the stagnation point. This effect can be seen from the temperature profile of the nose cap of the Shuttle in Fig. 10.27.

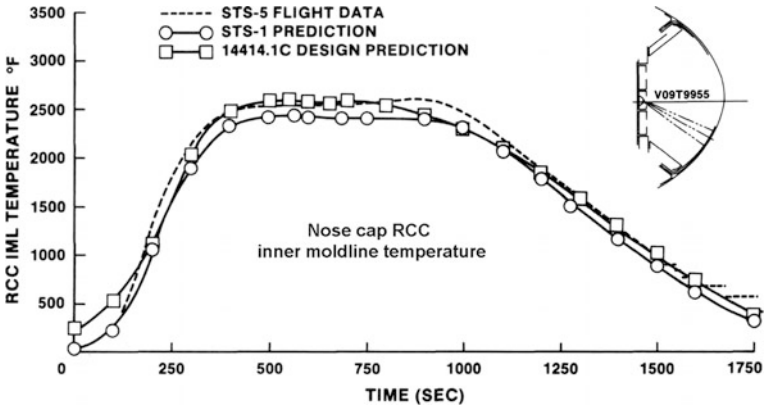


Fig. 10.27 Nose cap RCC inner moldline temperature as a function of time after entry interface. *Credit* NASA

10.7.3 Equilibrium Glide Phase

As the Shuttle speed drops below 6.2 km/s, the constant heat rate phase ends and the equilibrium glide phase begins. This should be considered as an intermediate phase between heating and rapidly increasing deceleration forces as the vehicles penetrated deeper into the atmosphere. It is an uncritical phase because according to Fig. 10.22 the flight corridor widens. Nevertheless, the glide slope guidance is designed to reserve range capability.

Apart from the altitude-dependent flight profile, which we had discussed in Sect. 10.4.2, it is possible to provide for this flight phase a time-dependent expression for the key trajectory parameters. To do so we consider the equations of motion in the time-dependent form as given by Eq. (10.2.1) with $\cos\gamma \approx 1, \sin\gamma \approx 0$ and Eq. (10.6.2)

$$\dot{v} = -\frac{D}{m} \tag{10.7.1}$$

$$\frac{L}{m} = g_0\left(1 - \frac{v^2}{v_\triangleright^2}\right), v_\triangleright = 7.905 \text{ km s}^{-1}$$

and therefore the drag-velocity profile for guidance and control

$$D = \frac{mg_0}{L/D}\left(1 - \frac{v^2}{v_\triangleright^2}\right)$$

We eliminate m from both equations, and get

$$\dot{v} = a = -g_0 \frac{D}{L}\left(1 - \frac{v^2}{v_\triangleright^2}\right) \tag{10.7.2}$$

Note that $v < v_\triangleright$ is mandatory, as for $v > v_\triangleright$ we would get $\dot{v} > 0$, i.e., the Shuttle would accelerate rather than decelerate. Separating the variables in Eq. (10.6.17) results in

$$\int_{v_e/v_0}^{v/v_0} \frac{dx}{1-x^2} = \operatorname{arctan\,h}\frac{v}{v_\triangleright} - \operatorname{arctan\,h}\frac{v_e}{v_\triangleright} = -\frac{g_0}{v_\triangleright}\int_0^t \frac{D}{L}dt = -\frac{Dg_0}{Lv_\triangleright}t$$

Because of the constant AOA, $D/L = const$ is valid during that flight phase, and we can extract this term from the integral. After some modifications, we get

$$v = v_e \frac{1 - \frac{v_\triangleright}{v_e}\tanh\left(\frac{\alpha t}{v_\triangleright}\right)}{1 - \frac{v_e}{v_\triangleright}\tanh\left(\frac{\alpha t}{v_\triangleright}\right)} \quad \text{with } \alpha := g_0\frac{D}{L} \tag{10.7.3}$$

To see that v indeed decreases, we expand this equation into a power series of $\alpha t \ll v_\triangleright$ for small time periods (exercise, Problem 10.7) and for $v_e < v_\triangleright$

$$v = v_e - \alpha t\left(1 - \frac{v_e^2}{v_\triangleright^2}\right)\left[1 + \frac{v_e}{v_\triangleright^2}\alpha t - \frac{\alpha^2 t^2}{v_\triangleright^2}\left(\frac{1}{3} - \frac{v_e^2}{v_\triangleright^2}\right) + \cdots\right] \tag{10.7.4}$$

We recognize that velocity decreases at an increasing rate. The corresponding deceleration is found by differentiating Eq. (10.7.4)

$$a = -\alpha\left(1 - \frac{v_e^2}{v_\triangleright^2}\right)\left[1 + \frac{2v_e}{v_\triangleright^2}\alpha t - \frac{1}{v_\triangleright^2}\alpha^2 t^2\left(1 - 3\frac{v_e^2}{v_\triangleright^2}\right) + \cdots\right] \tag{10.7.5}$$

The acceleration increases monotonously over time. For a winged body with $L/D = 1.3$, such as the Shuttle, Eqs. (10.7.4) and (10.7.5) have an inaccuracy of about 10% after 15 min, so they are sufficiently accurate over the entire blackout phase.

10.7.4 Constant-Drag Phase

The equilibrium phase passes into the constant drag phase when the maximum drag acceleration of 10 m/s^2 occurs, or if a velocity of 4.6 km/s is achieved, which typically happens at an altitude of about 50 km. Because of the further increasing atmospheric density, the Space Shuttle comes now into a fully aerodynamic state. With a still decreasing velocity the absolute value of the deceleration increases to $1.5g_0$ according to Eq. (10.7.2). In order to limit the stress load to the Shuttle and the crew, the drag deceleration now is curbed to $D/m = -a = 1.5g_0$ This is achieved at steadily decreasing velocities, according to Eq. (10.2.9) and Eq. (10.7.2), by increasing L/D and hence decreasing AOA = α according to

$$\frac{\sin\alpha\sin 2\alpha}{2\sin^3\alpha+0.045}=\frac{L}{D}=\frac{1}{1.5}\left(1-\frac{v^2}{v_{\triangleright}^2}\right)\quad @\quad D/m=-\dot{v}=1.5g_0 \tag{10.7.6}$$

Thus the AOA is constantly reduced from 40° down to about 33° at Mach 9. It is during this phase that the velocity falls below Mach 11 where the Shuttle leaves the blackout phase.

10.7.5 Transition Phase

Owing to the strongly increasing atmospheric density the dynamic pressure and hence the load factor on the Shuttle increases so that starting at a speed of 3.2 km/s the drag deceleration has to be reduced. This is achieved by further reducing the AOA to 8° at Mach 1 (see Fig. 10.25). According to Eq. (10.2.9) a decreasing AOA increases L/D. Due to a much reduced velocity, the flight-path angle becomes significantly steeper and the flight strongly aerodynamic. This influences the formulation of the Shuttle AOA guidance from a drag-velocity to a drag-energy profile of the form

$$D=D_F+c_5(E-E_F)$$

where $E=\frac{1}{2}mv^2+mgh$ is the total vehicle energy, c_5 is an empirical parameter derived from range requirements, and index F indicates final values at the transition to the TAEM phase, namely an altitude of about 25 km at Mach 2.5. This delivers the spacecraft to the desired energy state at TAEM interface. Up to this point the Shuttle has covered a distance of 8000 km in 54 min.

10.7.6 TAEM Phase

The Shuttle enters the final phase, the so-called Terminal Area Energy Management (TAEM) phase (see Fig. 10.28), with Mach 2.5 at 25 km altitude and a distance of about 90 km from the runway. During the remaining 6 min the Shuttle undergoes flight maneuvers with changing $AOA=4^\circ-10^\circ$ to align velocity and heading to the approach cone of the landing strip. At Mach 1, about 4 min prior to touch down, the commander takes manual control of the spacecraft's approach. The glide path angle ($\gamma=17^\circ-19^\circ$) in the landing phase is six times bigger than that of a commercial aircraft, and the touch down velocity $v=335$ km h^{-1} is about twice as high.

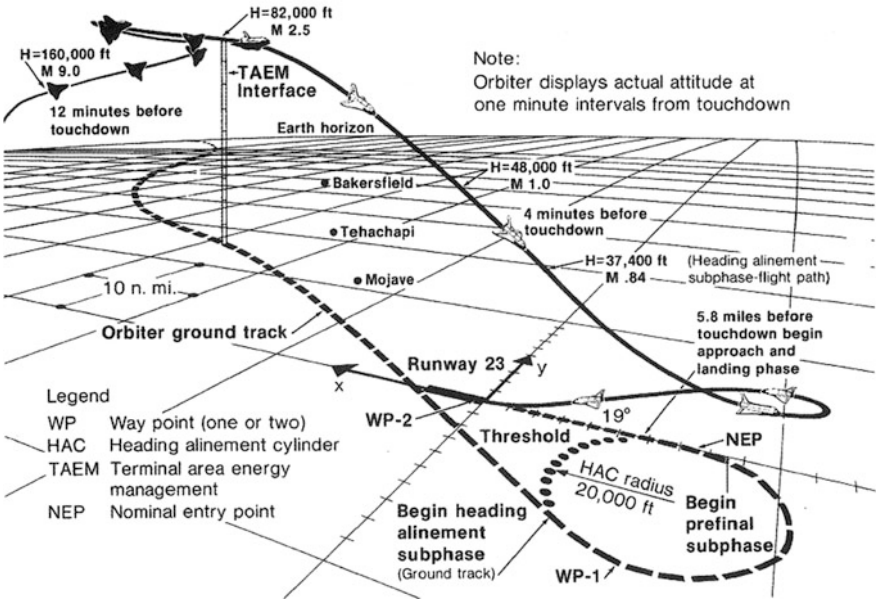


Fig. 10.28 STS-3 TAEM entry profile to Edwards Airforce Base. *Credit* NASA

10.8 Problems

Problem 10.1 *LEO Deorbit*
Prove Eq. (10.1.11).

Problem 10.2 *Normalized Equations of Motion*
From the equations of motions (10.2.1), (10.2.2), and (10.2.3) derive the normalized equations of motion (10.2.7).

Problem 10.3 *Reduced Equations of Motion*
From the equations of motions (10.2.1), (10.2.2), and (10.2.3) derive the reduced equations of motions (10.2.13) through the variable substitution $dt \rightarrow d\lambda$.

Problem 10.4 *Low-Lift Reentry Trajectory*
Prove the low-lift reentry trajectory, Eq. (10.4.13).

Hint: Derive the first-order differential equation of the trajectory equation from $\dot{h} = -v\sin\gamma$ and $\dot{x} = v\cos\gamma$.

Problem 10.5 *Subsonic Gliding Angle*
Show from the equations of motion that for reentry vehicles with a substantial lift the glide path angle in the terminal subsonic gliding phase where $\dot{v} \ll g\sin\gamma$ becomes

$$\tan\gamma = \frac{D}{L}$$

Problem 10.6 *High-Lift Reentry*
We assume a high-lift reentry.

(a) Show that for the reflection phase where gravitation is negligible and hence $H/R \to 0$ the exact solutions to the equations of motion

$$\frac{d\varepsilon}{d\lambda} = -\frac{\sin\gamma_e}{\sin\gamma}\varepsilon$$
$$\frac{d(\cos\gamma)}{d\lambda} = \frac{\sin\gamma_e}{2}\frac{L}{D}$$

read

$$\cos\gamma = \cos\gamma_e\left[1 + \frac{\tan\gamma_e}{2}\frac{L}{D}(\lambda - \lambda_e)\right]$$

and

$$\ln\frac{\varepsilon}{\varepsilon_e} = -2\frac{D}{L}\left[\arcsin\left(\cos\gamma_e + \frac{\sin\gamma_e}{2}\frac{L}{D}(\lambda - \lambda_e)\right) - \arcsin(\cos\gamma_e)\right]$$

(b) Using the functional approximation

$$\arcsin(\cos\alpha + x\sin\alpha) - \arcsin(\cos\alpha) = x + \frac{1}{2}\cot\alpha \cdot x^2$$
$$+ \frac{1}{2}\left(\frac{1}{3} + \cot^2\alpha\right)x^3 + \cdots$$

for $x \to 0$ prove that the latter solution passes over into

$$\ln\frac{\varepsilon}{\varepsilon_e} = -(\lambda - \lambda_e) - \frac{1}{4}\frac{L}{D}\cot\gamma_e(\lambda - \lambda_e)^2$$

and hence is in accordance with Eq. (10.4.4) for $c \to 0$ and Eq. (10.4.12).

Problem 10.7 *Deceleration in Blackout Phase*
Prove Eq. (10.7.4) from Eq. (10.7.3) for $\alpha t \ll v_\triangleright$.

Chapter 11
Three-Body Problem

11.1 The N-Body Problem

11.1.1 Integrals of Motion

In Chap. 7, we have looked at two point masses that were moving under their mutual gravitational influence. Formally speaking we were dealing with two bodies each with six degrees of freedom (three position vector components and three velocity vector components). To describe their motion, in total 12 quantities had to be determined, specified by six coupled equations of motion of second order (see Eq. (7.1.14)) or 12 coupled equations of motion of first order:

$$\dot{\boldsymbol{v}}_1 = +\frac{Gm_2}{|\boldsymbol{r}_1 - \boldsymbol{r}_2|^3}(\boldsymbol{r}_1 - \boldsymbol{r}_2), \quad \boldsymbol{v}_1 = \dot{\boldsymbol{r}}_1$$
$$\dot{\boldsymbol{v}}_2 = -\frac{Gm_1}{|\boldsymbol{r}_1 - \boldsymbol{r}_2|^3}(\boldsymbol{r}_1 - \boldsymbol{r}_2), \quad \boldsymbol{v}_2 = \dot{\boldsymbol{r}}_2$$

By transforming the origin of the reference system into the center of mass of the two bodies (see Sect. 7.1.5) we were able to split the differential equations into two independent sets with three coupled equations of second degree each, namely $\ddot{\boldsymbol{r}} = -\mu\boldsymbol{r}/r^3$ and $\ddot{\boldsymbol{r}}_{CM} = 0$. We succeeded to directly integrate them, thus finding unambiguous analytical solutions.

A world with just two bodies is too idealistic in most cases. The motion of the Moon, for example, which circles the Earth, and at the same time is subject to the influence of the Sun, cannot be described adequately by just a two-body system. For these three bodies and for the general case of n bodies, one has to go back to the $6n$ coupled differential equations of first order, analogous to the above, which describe the acceleration and velocity of each body under the gravitational forces of all other bodies. The specific motion of these bodies is determined by the $6n$ quantities $(\boldsymbol{r}_1, \boldsymbol{v}_1), (\boldsymbol{r}_2, \boldsymbol{v}_2), \ldots, (\boldsymbol{r}_n, \boldsymbol{v}_n)$, which follow from integrating the

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differential equations. Their motion and hence the $(\boldsymbol{r}_1, \boldsymbol{v}_1)$, $(\boldsymbol{r}_2, \boldsymbol{v}_2), \ldots, (\boldsymbol{r}_n, \boldsymbol{v}_n)$ is, however, restricted due to the earlier discussed conservation laws. Mathematically, the conservation laws are 10 constraint equations for these $6n$ quantities, namely

- Six (i.e., 2×3) equations for the conservation of momentum of the center of mass (corresponds to the non-accelerated motion of the center of mass vector or equivalently to the initial values $\boldsymbol{v}_0$ and $\boldsymbol{r}_0$ of the center of mass, see Eq. (7.1.18).
- Three equations for the conservation of the total angular momentum.
- One equation for total energy conservation.

Each constraint equation defines a conserved quantity—a so-called "integral of motion" (here "integral" means a quantity that is independent of the motion and thus constant). So in essence there are $6n - 10$ degrees of freedom, which entails that $6n - 10$ quantities remain to be determined. Already in 1896 the French mathematician and physicist Henri Poincaré showed that for the general n-body problem (i.e., $n \geq 3$ bodies with arbitrary masses and arbitrary initial conditions), there cannot exist any further algebraic integrals of motion. So the general n-body problem is analytically not integrable, and thus cannot be solved analytically. Because in general the effective gravitational force on a body is no longer central, its trajectory is non-periodic, and because energy conservation applies to the entirety of bodies, a single body might gain or loose energy, such that even unbounded solutions may exist (see Sect. 11.1.2).

The n-body problem might not be solved analytically, but it is possible to solve it by other means. One possibility is to approximate the solution by convergent function series expansion. This mathematical method is rather complex, so we do not want to go into details here. We just mention that the expansion Eqs. (7.4.19) are function series expansions of the solution to Kepler's Equation (7.4.15) and Eq. (7.4.14) for elliptic orbits. In addition, with today's computers it is quite simple to get point-by-point solutions with arbitrary accuracies by solving the differential equations numerically for instance by Cowell's method described in Sect. 12.2.4. Despite the superior numerical capabilities, which today are exclusively applied for specific space missions, also the so-called patched-conics method, which we already got to know in Sect. 9.1, is regularly used to solve n-body problems by approximation. This method is used for interplanetary flights to gain preliminary insight into possible trajectories, which is indispensable to handle the complex calculation models of a detailed mission design.

11.1.2 Stability of an N-Body System

The basic reason why we are in general not able to analytically describe the orbits of n interacting bodies is that the energy and angular momentum of each single body are not conserved, but only those of the total system. This implies that any body may gain or lose energy and angular momentum in a random way by

gravitational interaction between the bodies. In this section we want to explore the stability of an n-body system and its conditions, i.e. the question whether a body may gain or lose energy indefinitely and thus is able to escape the system. In such a case we would call the system unstable.

Condition of System Instability
In a system with n bodies, $i = 1, \ldots, n$, having masses m_i and orbit radii r_i measured relative to the common barycenter consider the function

$$I := \sum_{i=1}^{n} m_i r_i^2 \quad \textbf{polar moment of inertia}$$

By construction I is always positive and also finite for bounded orbits. It becomes infinite if one or more of the n bodies escapes from the system. Therefore I may be considered as a good indicator for the stability of an n-body system. In this system we have for the total kinetic and total potential energy

$$E_{pot}(t) = -\sum_i \frac{\mu m_i}{r_i} \qquad \text{total potential energy of the system}$$

$$E_{kin}(t) = \frac{1}{2}\sum_i m_i \dot{r}_i^2 \qquad \text{total kinetic energy of the system}$$

$$E_{tot} = E_{kin} + E_{pot} = const \qquad \text{total energy of the system}$$

with μ the relevant gravitational parameter of the system. We now differentiate I twice and find

$$\ddot{I} = 2\sum_i m_i\left(\dot{r}_i^2 + r_i\ddot{r}_i\right)$$

With (see Eq. (7.1.5))

$$\sum_i m_i r_i \ddot{r}_i = \sum_i r_i F_i = -\sum_i r_i \frac{\partial}{\partial r_i} E_{pot} = -\sum_i \frac{\mu m_i}{r_i} = E_{pot}$$

we finally obtain

$$\ddot{I}(t) = 4E_{kin} + 2E_{pot} = 2E_{kin} + 2E_{tot} \tag{11.1.1}$$

Per definition E_{kin} is positive, while E_{tot} might be positive or negative. If $E_{tot} > -E_{kin}(t)$ for all times then from Eq. (11.1.1) it follows that $\ddot{I}(t) > 0$ and hence $I \to \infty$. This implies that at least one $r_i \to \infty$, meaning that at least one body escapes the system, or the entire system coherently disintegrates. On the other hand,

if the n-body system is stable then $E_{tot} < -E_{kin}$ must hold. Observe, however, that the reverse conclusion "$E_{tot} < -E_{kin} \rightarrow$ no body escapes the system", does not hold.

Virial Theorem

We will now prove that the virial theorem (7.3.20) also holds for the total kinetic and potential energy of an n-body system, regardless whether it is bounded or unbounded. In preparation for the proof, we formerly integrate Eq. (11.1.1)

$$\dot{I} = 4 \int_0^t E_{kin} \cdot dt' + 2 \int_0^t E_{pot} \cdot dt' + \dot{I}_0$$

We now let $t \rightarrow \infty$ and define P to be the longest period of all bounded bodies in the system. Thus we have with the time average of the system energies

$$\langle E \rangle := \frac{1}{P} \int_0^P E \cdot dt$$

$$\dot{I} = 4\langle E_{kin} \rangle t + 2\langle E_{pot} \rangle t + \dot{I}_0$$

or

$$\frac{1}{2}\dot{I}/t = 2\langle E_{kin} \rangle + \langle E_{pot} \rangle + \dot{I}_0/t$$

Hence

$$\frac{1}{2} \lim_{t \rightarrow \infty} \dot{I}/t = 2\langle E_{kin} \rangle + \langle E_{pot} \rangle$$

Applying l'Hôpital's rule $\lim_{t \rightarrow \infty} (\dot{I}/t) = 2 \lim_{t \rightarrow \infty} (I/t^2)$ we finally arrive at the result

$$2\langle E_{kin} \rangle + \langle E_{pot} \rangle = \lim_{t \rightarrow \infty} (I/t^2)$$

The term on the right hand side determines whether the system is bounded or not. As discussed above, as long as all bodies in the system stay bounded, $I < \infty$, and hence $\lim_{t \rightarrow \infty} (I/t^2) = 0$. So, for a bounded system and for times much longer than the longest body period we get the virial theorem

$$\boxed{\langle E_{pot} \rangle = -2\langle E_{kin} \rangle} \quad \textbf{virial theorem} \text{ for } \textbf{bounded} \text{ systems} \qquad (11.1.2)$$

Now let's consider an unbounded system, in which one or many bodies recede asymptotically to infinite distance from the barycenter (c.f. Pollard (1976, p. 67ff)).

At infinite distances the system can be considered as a single total point mass. Thus the motion of the detaching body becomes that of a 2-body system. From Eq. (7.4.28) we know that this is an asymptotic hyperbolic trajectory of the form

$$r(t) = v_\infty t + O(\ln\, t) + const \quad @\ t \to \infty$$

where $v_\infty = \sqrt{-\mu/a}$ is the hyperbolic excess velocity (see Sect. 7.4.3). Hence we get for an escaping body with mass m

$$\lim_{t\to\infty} \frac{I}{t^2} = \lim_{t\to\infty} \frac{mr^2}{t^2} = mv_\infty^2$$

The virial theorem for an unbounded system with j bodies having escaped the system and for times much longer than the longest body period in the bounded system part therefore reads

$$\boxed{2\langle E_{kin}\rangle + \langle E_{pot}\rangle = \sum_{i=1}^{j} m_i v_{i,\infty}^2} \quad \textbf{virial theorem} \text{ for } \textbf{unbounded} \text{ systems} \tag{11.1.3}$$

This result is conclusive because the left hand side of the equation is twice the energy of the remaining bounded bodies, while the right hand side is twice the hyperbolic excess kinetic energy of the escaped bodies.

System Collapse

We now wonder whether a n-body system might collapse. To find a stability condition we study the absolute value of the total angular momentum

$$L = \left|\sum_i m_i \boldsymbol{r}_i \times \boldsymbol{v}_i\right| \le \sum_i m_i |\boldsymbol{r}_i \times \boldsymbol{v}_i| \le \sum_i m_i r_i v_i = \sum_i (\sqrt{m_i} r_i)(\sqrt{m_i} v_i)$$

By setting $a_i := r_i\sqrt{m_i}$ and $b_i := v_i\sqrt{m_i}$ and applying Cauchy's inequality

$$\left(\sum\nolimits_i a_i b_i\right)^2 \le \left(\sum\nolimits_i a_i^2\right)\left(\sum\nolimits_i b_i^2\right)$$

we obtain

$$L^2 \le \sum_i m_i r_i^2 \sum_i m_i v_i^2 = 2I \cdot E_{kin}$$

which delivers Sundman's inequality

$$E_{kin} \ge \frac{L^2}{2I} \quad \textbf{Sundman's inequality} \tag{11.1.4}$$

From Sundman's inequality follows directly.

Sundman's theorem
A many-body system made up of point masses (in fact any planetary system) that at any point in time has a finite kinetic energy and total angular momentum, i.e. $I \neq 0$, can never fully collapse.

Since for our solar system today $E_{kin} > 0$, $L > 0$, it will never fully collapse. Good to know. Note, however, that partial collapses in the form of binary collisions and collisions between up to $n - 1$ bodies are possible.

Caution is in place with accepting Sundman's theorem for describing the long-term fate of our solar system and any stellar system in general. The planets will not endlessly circulate the Sun. In about 5 billion years our Sun will expand, become a so-called Red Giant, and swallow successively first Mercury, then Venus und finally even Earth. In addition, the solar wind made up of mostly protons constantly sweeps out of our solar system, thus loosing mass and with it angular momentum and kinetic energy. By the same token solar radiation makes the Sun loose energy. Finally gravitational waves caused by the orbiting planets make their orbits shrink over very long time scales. And, last but not least, there is a high probability that our solar system may collide with another star system, by which either system is broken up or at least some of the planets are slung out of these systems. So the key point about the fate of our solar system is that it is not a closed system, which revolves forever, but its fate is determined by its interactions with the universe through radiation and by collisions.

11.1.3 N-Body Choreographies

There are many-body systems, so-called *n-body choreographies*, that show a synchronous and concerted motion of all bodies under very special conditions. One condition is that all masses are exactly the same. The other is that they have to have very special initial positions and velocities. In an *n*-body choreography all bodies move either on a single winding two-dimensional orbit or they move on several winding orbits, each of which may not be flat, thus these orbits cover three-dimensional space and are generally intertwined.

It was only in 2000 that a solution was found where three bodies revolve around each other in an orbit of the form of a figure 8 (see Fig. 11.1). In contrast to many other similar, symmetrical, co-orbital orbits with more than three bodies (see Figs. 11.2 and 11.3), which are not stable, this figure-eight orbit is dynamically stable. The so-called stability domain, that is the range of admissible deviations

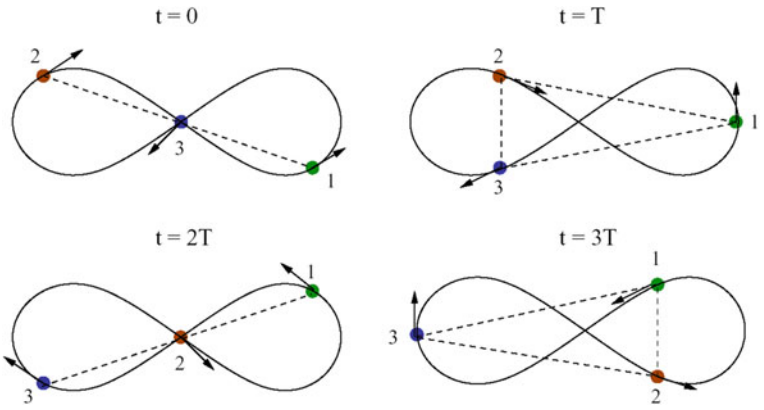


Fig. 11.1 The figure-eight orbit. A stable co-orbital motion of three masses on an 8-shaped orbit at different time intervals. *Credit* A. Chenciner and R. Montgomery

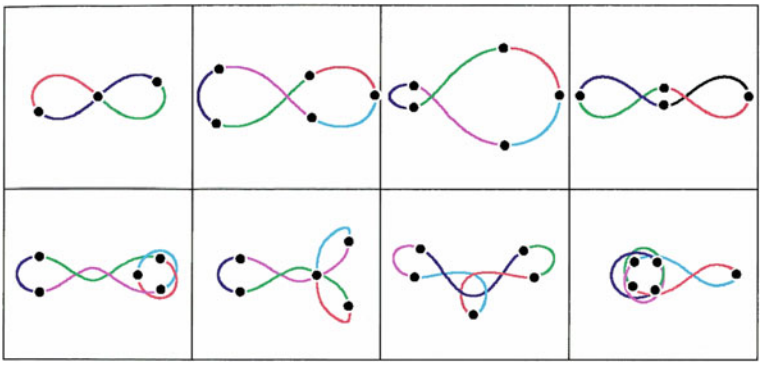


Fig. 11.2 Planar n-body choreographies of three to five masses. Colors indicate the orbits of the masses from their current position to their next symmetrical position. *Credit* Robert Jenkins and Carles Simó

from the ideal orbit or mass, however, is so small that numerical simulations predict only 1–100 instances of figure-eight orbits in the observable universe.

See www.maths.manchester.ac.uk/∼jm/Choreographies/ for many more animations on n-body choreographies.

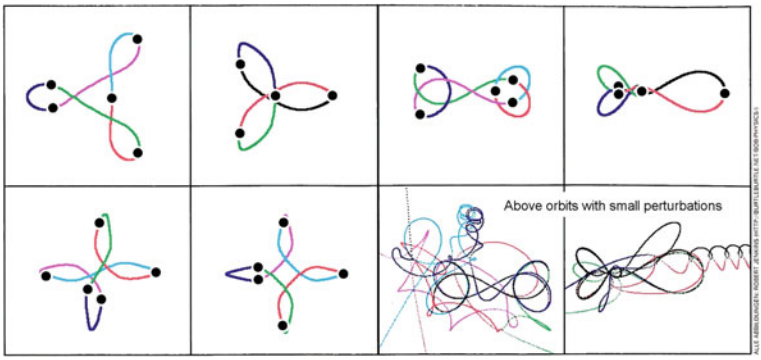


Fig. 11.3 More planar n-body choreographies of four and five masses. Colors indicate the orbits of the masses from their current position to their next symmetrical position. *Credit* Robert Jenkins and Carles Simó

11.2 Synchronous 3-Body Orbits

In this section we examine two special 3-body choreographies with possibly different masses, and which have already been known for 250 years. The property of the general *n*-body problem, that the trajectories of the bodies are non-periodic and may even be unbounded, does not hold for two other special cases of 3-body systems where all bodies always display a constant motion-pattern with either synchronous unbounded or bounded periodic trajectories:

1. *Collinear configuration*—three arbitrary masses move synchronously in a linear configuration with fixed relative distances on Keplerian orbits (ellipses, parabolas, or hyperbolas).
2. *Equilateral configuration*—three arbitrary masses move synchronously in an equilateral triangle configuration with fixed relative distances on coplanar Keplerian orbits (ellipses, parabolas, or hyperbolas).

We shall now look at these two special constant motion-patterns.

11.2.1 Collinear Configuration

Already in 1765 the Swiss mathematician Leonhard Euler showed that if the three bodies with arbitrary masses adopt certain positions on a straight line—called *collinear libration points,* a.k.a. *Eulerian points*—their joint motion can be described as a rotation of this straight line where their mutual distances change such that the distance ratios, and hence the entire configuration, remain constant

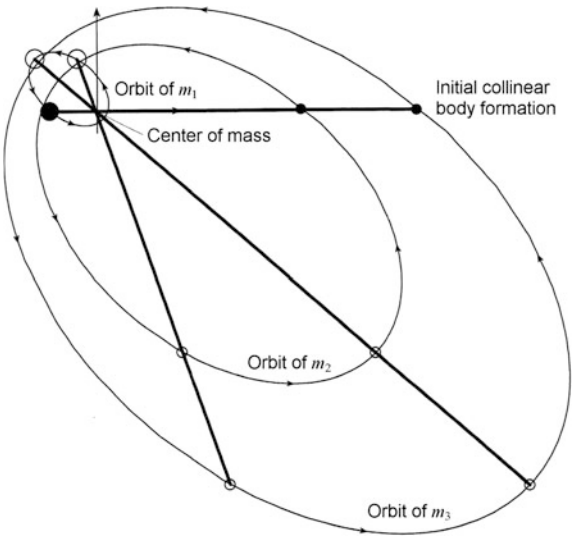


Fig. 11.4 Dynamics of the collinear masses configuration with $m_1' = 1/10$ Earth mass, $m_2' =$ Moon mass, $m_3' = 1/2$ Moon mass

(see Fig. 11.4). Because Euler's collinear configuration is of high relevance for the R3BP, we will take a closer look at its mathematical description.

Let m_1', m_2', m_3' be three collinear masses of arbitrary size, where, without loss of generality, the mass labeled m_2' is located between the other two, m_1' and m_3', and $m_1' \geq m_3'$. With regard to their common center of mass their relative distances and coordination can therefore described by the two constants α, β relating their position vectors $\boldsymbol{r}_i$, $i = 1, 2, 3$ as

$$\begin{aligned} \boldsymbol{r}_3 &= \alpha \boldsymbol{r}_2, \quad \alpha > 1 \\ \boldsymbol{r}_1 &= \beta \boldsymbol{r}_2, \quad \beta < 0 \end{aligned} \tag{11.2.1}$$

With this we get from the center-of-mass equation (cf. Eq. (7.1.17)) $\sum_i m_i' \boldsymbol{r}_i = 0$

$$\beta m_1' + m_2' + \alpha m_3' = 0 \tag{11.2.2}$$

Given this, it can be shown (see Guthmann (2000)) that in an inertial reference frame with origin in their common center of mass each rotating mass obeys one of the following three interrelated equations of motion

$$\ddot{\boldsymbol{r}}_i + \frac{\mu_i}{r_i^3} \boldsymbol{r}_i = 0, \quad i = 1, 2, 3 \tag{11.2.3}$$

with

$$\begin{aligned}\frac{\mu_1}{G} &= \frac{\beta^2 m_2'}{(1-\beta)^2} + \frac{\beta^2 m_3'}{(\alpha-\beta)^2} \\ \frac{\mu_2}{G} &= -\frac{m_3'}{(\alpha-1)^2} + \frac{m_1'}{(1-\beta)^2} \\ \frac{\mu_3}{G} &= \frac{\alpha^2 m_2'}{(\alpha-1)^2} + \frac{\alpha^2 m_1'}{(\alpha-\beta)^2}\end{aligned} \tag{11.2.4}$$

It can be shown (Exercise, Problem 11.2, cf. Guthmann (2000)) that α is the unequivocal *positive* root and β the unequivocal *negative* root of

$$\begin{aligned}\mu_3 &= \alpha^3 \mu_2 \\ \mu_1 &= -\beta^3 \mu_2\end{aligned} \tag{11.2.5}$$

Because according to Eq. (11.2.3) each mass is subject to a central Newtonian force, the orbits must be Keplerian orbits (conic sections). With a_i being their semi-major axes it follows immediately from Eqs. (11.2.1) and (11.2.2) that

$$\sqrt{\frac{\mu_i}{a_i^3}} = n = const \tag{11.2.6}$$

The three orbits therefore have the same mean motion and thus the same orbital frequency and hence orbital period.

Initial Conditions
Let us assume that we start the motion such that the three masses are in a collinar configuration with α and β given by Eq. (11.2.5), i.e., all three Keplerian orbits have a common true anomaly θ. In addition, if we choose the initial velocities of each mass to be

1. proportional in magnitude to its distance to the center of mass, and
2. their velocity vectors form equal angles with their corresponding radial position vectors

From Eqs. (11.2.1) and (11.2.6) follows that $r_1/a_1 = r_2/a_2 = r_3/a_3$. This implies with Eq. (7.3.5) that

$$e_1 = e_2 = e_3 =: e \tag{11.2.7}$$

Given these results, all three Keplerian orbits obey

$$r_i = \frac{a_i(1-e^2)}{1+e\,\cos\theta} \tag{11.2.8}$$

Rotation Dynamics

We now show that under the said initial conditions the motion of the masses conserves their collinearity. To do so we prove that the orbital frequency of each of the three bodies is the same at any time. With $e_i = e$ and Eq. (7.2.7) we get

$$\omega_i = \frac{h_i}{r_i^2} = \frac{\sqrt{\mu_i a_i (1-e^2)}}{r_i^2}$$

With Eqs. (11.2.1) and (11.2.5) we find

$$\omega_3 = \frac{\sqrt{\mu_3 a_3 (1-e^2)}}{r_3^2} = \frac{\sqrt{\alpha^3 \mu_2 \alpha a_2 (1-e^2)}}{\alpha^2 r_2^2} = \omega_2$$

and

$$\omega_1 = \frac{\sqrt{\mu_1 a_1 (1-e^2)}}{r_1^2} = \frac{\sqrt{-\beta^3 \mu_2 (-\beta) a_2 (1-e^2)}}{\beta^2 r_2^2} = \omega_2$$

This proves that all three orbits have the equal instantaneous orbital frequency

$$\omega(\theta) = \frac{\sqrt{\mu_i a_i (1-e^2)}}{r_i^2} = \sqrt{\frac{\mu_i}{a_i}} \frac{(1+e\,\cos\theta)^2}{a_i (1-e^2)^{3/2}} = \frac{2\pi}{T} \frac{(1+e\,\cos\theta)^2}{(1-e^2)^{3/2}} \qquad (11.2.9)$$

> The collinear masses rotate on a straight line, with variable absolute distances, but constant relative distances between the masses—like a rotating rubber band.

As an example, Fig. 11.5 shows the dynamics of three collinear, rotating masses with $m_1' = 1/10$ Earth mass, $m_2' =$ Moon mass, $m_3' = 1/2$ Moon mass. The relative distances on the rotating configuration line are obviously retained, and the individual masses co-rotate on ellipses with a common line of apsides, and with the focus in the joint center of mass.

One may be surprised that the motion of the masses is indeed on a straight line, as this seems to contradict $n = \sqrt{\mu/a^3}$, which infers that the mean orbital motion

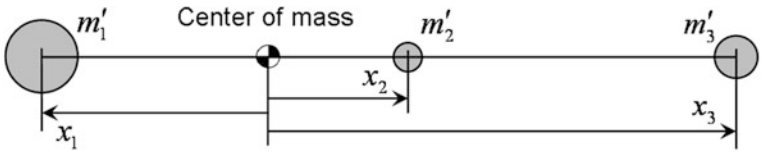


Fig. 11.5 Definitions for the collinear configuration with arbitrary masses, where the mass in the middle is denoted as m_2' and without loss of generality $m_1' \geq m_3'$

n of a mass is different for different semi-major axes. But one has to consider that according to Eq. (11.2.4) every orbit has a different μ_i from which Eq. (11.2.6), $n = \sqrt{\mu_i/a_i^3}$ follows.

Reference Frame

Because the points are collinear and the distance ratios of the masses are fixed by the constants α and β, we choose the configuration line as our co-rotating coordinate x-axis so that the masses are located at positions $\boldsymbol{r}_i = (x_i, 0, 0)$ (see Fig. 11.5). Then by the above definition, $x_3 = \alpha x_2$ and $x_1 = \beta x_2$. The collinear configuration can be characterized by the configuration parameter χ, which is defined in terms of the relative distances

$$\begin{aligned} x_{12} &:= x_2 - x_1 \\ x_{13} &:= x_3 - x_1 = (1+\chi)x_{12} \\ x_{23} &:= x_3 - x_2 = \chi x_{12} \end{aligned} \tag{11.2.10}$$

$$\chi := \frac{x_{23}}{x_{12}} = \frac{x_3 - x_2}{x_2 - x_1} = \frac{\alpha - 1}{1 - \beta} > 0 \tag{11.2.11}$$

With χ we relate the distances of the masses to the reference distance x_{12}, which may be chosen freely. The parameter χ is solely determined by the three masses, the details of which will be studied in a moment.

Lagrange's Quintic Equation

We now want to determine the constant, relative positions of the masses on the line, as described by χ, from the given masses. Because the three masses are configuration-invariant on a straight line we need not to solve the equation of motion, but it suffices to derive a conditional equation for χ determined by the three masses. Equation (11.2.11) results in

$$\alpha - \beta = (1+\chi)(1-\beta) = \frac{1+\chi}{\chi}(\alpha - 1)$$

With this and Eq. (11.2.4) we can rewrite the conditional Eq. (11.2.5), $\mu_3 = \alpha^3 \mu_2$, as

$$\alpha\left[-\frac{m_3'}{(\alpha-1)^2} + \frac{m_1'}{(1-\beta)^2}\right] = \frac{m_2'}{(\alpha-1)^2} + \frac{m_1'}{(\alpha-\beta)^2}$$

and

$$\alpha\left(\chi^2 m_1' - m_3'\right) = m_2' + m_1' \frac{\chi^2}{(1+\chi)^2}$$

On the other hand, from the center-of-mass equation $\beta m_1' + m_2' + \alpha m_3' = 0$ and the above follows that

$$1 - \beta = \frac{m_1' + m_2' + \alpha m_3'}{m_1'} = \frac{\alpha - 1}{\chi}$$

Hence

$$\alpha = \frac{m_1' + \chi(m_1' + m_2')}{m_1' - \chi m_3'}$$

Inserting this into the above equation leads to

$$\frac{m_1' + \chi(m_1' + m_2')}{m_1' - \chi m_3'} \left(\chi^2 m_1' - m_3'\right) = m_2' + m_1' \frac{\chi^2}{(1+\chi)^2} \tag{11.2.12}$$

After some basic algebra, this equation can be transformed into **Lagrange's quintic equation**

$$\begin{aligned}&(m_1' + m_2')\chi^5 + (3m_1' + 2m_2')\chi^4 + (3m_1' + m_2')\chi^3 \\ &\quad - (m_2' + 3m_3')\chi^2 - (2m_2' + 3m_3')\chi - (m_2' + m_3') = 0\end{aligned} \tag{11.2.13}$$

for the given masses m_1', m_2' and m_3', where m_2' is located in the middle. The single positive root χ of Lagrange's quintic equation determines via the given reference distance x_{12} and by means of Eq. (11.2.10) all the other distances.

Note 1 *Because the coefficients of the powers of χ change sign only once, from Descartes' rule of signs follows that there is only a single positive root of Lagrange's quintic equation.*

Note 2 *Equation (11.2.13) holds for any type of collinear Keplerian orbit that obeys Eq. (11.2.3), be it bounded or unbounded.*

The synchronous motion of the collinear configuration only takes place if there are no perturbations. In the presence of even the tiniest perturbation of the configuration the collinear configuration is always *unstable*, even in the R3BP limit: The masses then will run away from this configuration.

Circular Orbits

If the initial conditions are such that the three bodies move on circular orbits, then the absolute distances, and in particular x_{12}, are even constant. The uniform period for these circular orbits turns out (exercise, Problem 11.3) to be

$$\omega = n = \frac{2\pi}{T} = \sqrt{\frac{\mu}{x_{12}^3}\frac{m_1'\chi^2 - m_3'}{m_1' - m_3'\chi}} = \sqrt{\frac{\mu_i}{x_i^3}} = const \tag{11.2.14}$$

with

$$\mu = GM = G\left(m_1' + m_2' + m_3'\right)$$

In this case Euler's collinear configuration is not only configuration-invariant, but also form-invariant. The fixed positions in the co-rotating synodic reference frame are called *stationary points*.

11.2.2 Equilateral Configuration

We are seeking for all three-body configurations, where the three co-moving masses take up a fixed configuration. It can be shown that other than the collinear configuration there can exist only one more such configuration, which was found in 1772 by the French mathematician Joseph Lagrange and sometimes named after him. The results will merely be summarized here without proof. Lagrange's configuration is a configuration of three bodies with arbitrary masses m_1, m_2, m_3, which without loss of generality we order by their mass quantity, $m_3 < m_2 < m_1$. They show an equilateral triangular formation and obey one of the corresponding three Newton's gravitational EoM

$$\ddot{\boldsymbol{r}}_i + \frac{\mu_i}{r_i^3}\boldsymbol{r}_i = 0 \quad \text{for} \quad i = 1, 2, 3$$

with

$$\begin{aligned} \mu_1 &= \tfrac{G}{M^2}\left(m_2^2 + m_3^2 + m_2 m_3\right)^{3/2} \\ \mu_2 &= \tfrac{G}{M^2}\left(m_1^2 + m_3^2 + m_1 m_3\right)^{3/2} \\ \mu_3 &= \tfrac{G}{M^2}\left(m_1^2 + m_2^2 + m_1 m_2\right)^{3/2} \end{aligned} \tag{11.2.15}$$

and

$$M = m_1 + m_2 + m_3$$

If the initial conditions for each mass are such that

1. The net resultant force on a mass is a position vector through the system center of mass.
2. The velocity of a mass is proportional in magnitude to its distance to the center of mass.
3. The velocity vectors form equal angles with their corresponding radial position vectors.

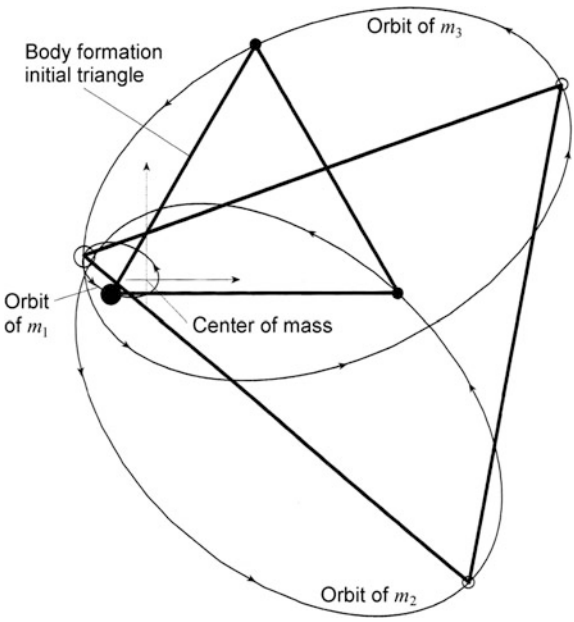


Fig. 11.6 Dynamics of an equilateral configuration with elliptic orbits

it can be shown that the equilateral configuration, just like the collinear configuration, does not change its symmetry, but merely is stretched in space, that is, its distances change, but their ratios remain constant (see Fig. 11.6). We state this property as follows:

> The equilateral configuration while rotating continuously changes its size—it is "breathing".

If the total energy of the system is negative, zero, or positive, this results in bounded ellipses (or circles), or unbounded parabolas or hyperbolas, revolving around a common center of mass. The size of their semi-axis depends on the size of the individual mass. Figure 11.6 shows a bounded system with the masses: 1/10 Earth mass, Moon mass, and half the mass of the Moon.

Though Lagrange's equilateral configuration has a remarkable symmetry, it is generally unstable: it disintegrates after a certain period of time. It is only dynamically stable in the limit of a restricted three-body configuration (see next section) when one primary mass is significantly bigger than the other (see Eq. (11.5.12)).

Circular Orbits

If the initial conditions are such that the three bodies move in circles, then, like the collinear configuration, the equilateral configuration is not only configuration-invariant,

but also form-invariant. It can be shown (exercise, Problem 11.4) that in this case the orbital frequency is given as

$$\omega = n = \frac{2\pi}{T} = \sqrt{\frac{\mu_i}{r_i^3}} = \sqrt{\frac{\mu}{r_{ij}^3}} = const \tag{11.2.16}$$

with the distance between the bodies

$$r_{ij} := r_{12} = r_{23} = r_{13}$$

and

$$\mu = GM = G(m_1 + m_2 + m_3)$$

11.3 Restricted Three-Body Problem

The general collinear and equilateral configurations are quite academic cases. From a practical point of view, only the special case of the *restricted three-body problem* (R3BP) plays a role. Here one of the three masses, denoted m, e.g. a spacecraft, is negligibly small compared with the other two, the so-called *primary bodies*. As an example the primaries might be Earth–Moon or Sun–Earth. As we have seen before, the collinear and equilateral configurations hold for any masses m_1, m_2, m_3 and therefore also for the R3BP.

Without restriction to generality we label the lighter of the two primaries, the so-called **minor primary**, as m_2, and the **major primary** as m_1, that is

$$m \ll m_2 < m_1$$

In addition and for later convenience we normalize the masses of the two primaries by their total mass $M = m_1 + m_2$

$$\mu := \frac{m_2}{m_1 + m_2} < \frac{1}{2} \quad \text{reduced minor primary mass} \tag{11.3.1}$$

$$1 - \mu = \frac{m_1}{m_1 + m_2} < 1 \quad \text{reduced major primary mass}$$

The two primary masses are rotating in general on elliptic orbits with varying angular velocity $\omega(t)$ and varying mutual distance $d(t)$ about their barycenter.

Euler defined the R3BP in 1772, after he had already discovered in 1765 the collinear equilibrium points, the results of which also apply to the R3BP. Also in 1772 Lagrange discovered all five equilibrium points in the R3BP including the two equilateral equilibrium points. Ever since then, the collinear equilibrium points of

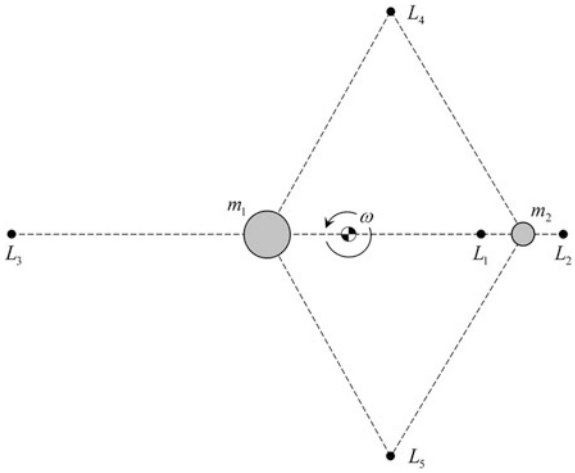


Fig. 11.7 Location of the collinear points L_1 to L_3 and equilateral points L_4 and L_5 in the R3BP

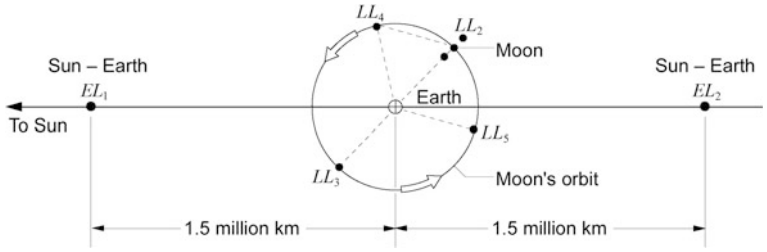


Fig. 11.8 Libration points in the vicinity of Earth

mass m in the R3BP are also known as *Eulerian points* L_1, L_2, L_3, and the equilateral equilibrium points are also known as *Lagrangian points* L_4 and L_5 (see Figs. 11.7 and 11.8). All five points together are the so-called *libration points*, but quite frequently and confusingly they are also called *Lagrangian points*. We evade this equivocality by labeling them as (Euler's) collinear libration points and (Lagrange's) equilateral libration points.

Note *In literature the libration points are not uniquely labeled. In this book we adopt the most frequent labeling:* L_1 *is the point between the two primaries,* L_2 *the point beyond the minor primary, and* L_3 *the point opposite to the minor primary. One often finds a reverse labeling of* L_1 *and* L_2*. Unfortunately, an inconsistent labeling is used also for the equilateral libration points. As usual we call* L_4 *the leading and* L_5 *the trailing equilateral libration point with respect to motion of the minor primary* m_2 *(But see for instance Wiesel (1997) and Roy (2005) for a reverse labeling.).*

Terminology In short notation Sun-Earth libration points are called EL, e.g. Sun-Earth L_1 is EL_1. Earth–Moon (lunar) libration points are called LL, e.g. Earth-Moon L_2 is LL_2 (see Fig. 11.8).

We now take a look at the characteristics of the libration points.

11.3.1 Collinear Libration Points

First of all, we introduce appropriate distance variables: Let Δ_i be the distance from a collinear libration point L_i to the center of the major primary body normalized to the distance between the two primary bodies. According to Fig. 11.4 and Eq. (11.2.10) the following holds for these normalized distances

$$\begin{aligned} \Delta_3 &:= \frac{x_{23}}{x_{12}} = \chi \\ \Delta_2 &:= \frac{x_{13}}{x_{12}} = 1+\chi \\ \Delta_1 &:= \frac{x_{12}}{x_{13}} = \frac{1}{1+\chi} \end{aligned} \tag{11.3.2}$$

As our considerations in Sect. 11.2.2 were valid for collinear configurations with any masses, we get from Eq. (11.2.12) for $m_3' = m \approx 0$ the following conditional equation of the configuration parameter χ for the collinear points L_2 and L_3

$$m_1' + \chi\left(m_1' + m_2'\right) = \frac{m_2'}{\chi^2} + \frac{m_1'}{(1+\chi)^2} \tag{11.3.3}$$

We recall that the primed masses are labeled according to their position (m_2' is in the middle, see Fig. 11.5), and unprimed masses according to their masses, $m \ll m_2 < m_1$.

L_3 point
Now, if $m_2' > m_1'$, then $m_2' = m_1$, $m_1' = m_2$, and $m_3' = m$ is located at the L_3 point. In this case, we get from Eq. (11.3.1)

$$\mu = \frac{m_2}{m_1+m_2} = \frac{m_1'}{m_2'+m_1'} = \frac{1}{1+m_2'/m_1'}$$

from which follows

$$\frac{m_2'}{m_1'} = \frac{1-\mu}{\mu}$$

With this and with Eq. (11.3.2a) we get from Eq. (11.3.3) the conditional equation for Δ_3

$$\Delta_3 + \mu = \frac{\mu}{(1+\Delta_3)^2} + \frac{1-\mu}{\Delta_3^2} \quad @\ \boldsymbol{L_3}\ \textbf{point} \tag{11.3.4}$$

***L_2* point**

If $m_1' > m_2'$, then $m_2' = m_2$, $m_1' = m_1$, and $m_3' = m$ is located at the L_2 point near m_2. In this case, we get

$$\mu = \frac{m_2}{m_1 + m_2} = \frac{m_2'}{m_1' + m_2'} = \frac{1}{1 + m_1'/m_2'}$$

from which follows

$$\frac{m_1'}{m_2'} = \frac{1-\mu}{\mu}$$

With this and with Eq. (11.3.2b) we get from Eq. (11.3.3) the conditional equation for Δ_2

$$\Delta_2 - \mu = \frac{\mu}{(1-\Delta_2)^2} + \frac{1-\mu}{\Delta_2^2} \quad @\ \boldsymbol{L_2}\ \textbf{point} \tag{11.3.5}$$

***L_1* point**

Finally, if $m_1' > m_3'$ then $m_1' = m_1$, $m_3' = m_2$, and $m_2' = m \approx 0$ is located at the L_1 point near m_2. Because of Eq. (11.2.12), we get for the conditional equation of the configuration parameter χ

$$\chi^2 m_1' - m_3' = \frac{\chi^2}{(1+\chi)^3}\left(m_1' - \chi m_3'\right)$$

and

$$\mu = \frac{m_2}{m_1 + m_2} = \frac{m_3'}{m_1' + m_3'} = \frac{1}{1 + m_1'/m_3'}$$

from which follows

$$\frac{m_1'}{m_3'} = \frac{1-\mu}{\mu}$$

With this and Eq. (11.3.2c) we get as the conditional equation for Δ_1

$$\Delta_1 - \mu = -\frac{\mu}{(1-\Delta_1)^2} + \frac{1-\mu}{\Delta_1^2} \quad @\ \boldsymbol{L_1}\ \textbf{point} \tag{11.3.6}$$

Location of Collinear Libration Points

Since $\mu < 1$ the solutions of Eqs. (11.3.4)–(11.3.6) can be determined (exercise, Problem 11.5) by power series approximations (cf. Szebehely (1967, p.134ff))

$$\begin{aligned}
\Delta_1 &= 1 - \lambda + \frac{1}{3}\lambda^2 + \frac{1}{9}\lambda^3 - \frac{58}{81}\lambda^4 + \frac{11}{243}\lambda^5 + \frac{4}{9}\lambda^6 + \cdots < 1 \\
\Delta_2 &= 1 + \lambda + \frac{1}{3}\lambda^2 - \frac{1}{9}\lambda^3 + \frac{50}{81}\lambda^4 + \frac{43}{243}\lambda^5 - \frac{4}{9}\lambda^6 + \cdots > 1 \\
\Delta_3 &= 1 - \frac{7}{12}\mu - \frac{1127}{20736}\mu^3 - \cdots < 1
\end{aligned} \tag{11.3.7}$$

with

$$\lambda := \left(\frac{\mu}{3}\right)^{1/3} = \left(\frac{1}{3}\frac{m_2}{m_1+m_2}\right)^{1/3}$$

For the Sun–planet or planet–Moon systems $\mu \ll 1$ and hence Eq. (11.3.7) provides very good approximations. Only for the Earth–Moon system $\lambda = 0.159401$ is relatively big, leading to a correspondingly worse convergence. In Eq. (11.3.7), the collinear distances are given with regard to the distance between the two primary bodies. If both primary orbits are circular then the collinear libration points also have fixed distances relative to the center of mass. If their orbits are elliptic, their mutual distance changes, and with it also the absolute distance to the collinear libration points.

From a physical point of view the positions of the collinear libration points result from the sum of the gravitational forces from both primaries, plus the centrifugal force of its rotation around the center of mass. This is why, for example, the L_1 point is not located where the gravitational forces of m_1 and m_2 just cancel each other out, but further in the direction to the major primary, to balance the centrifugal force by a somewhat larger gravitational force.

11.3.2 Equilateral Libration Points

The two equilateral libration points are determined by the equilaterality property of the triangle m_1, m_2, m. That is,

$$\begin{aligned} \frac{\Delta x}{d} &= 0.5, \quad \frac{\Delta y}{d} = +\frac{\sqrt{3}}{2}, \quad \Delta_4 = \frac{\Delta x^2 + \Delta y^2}{d^2} = 1 \quad @\ L_4 \\ \frac{\Delta x}{d} &= 0.5, \quad \frac{\Delta y}{d} = -\frac{\sqrt{3}}{2}, \quad \Delta_5 = \frac{\Delta x^2 + \Delta y^2}{d^2} = 1 \quad @\ L_5 \end{aligned} \tag{11.3.8}$$

where Δy measures the distance perpendicular to the configuration line of the two primaries and d is their mutual distance.

Table 11.1 shows specific examples: the exact position of all libration points in the Earth–Moon system with $m_{Earth}/m_{Moon} = 81.30094$.

11.4 Circular Restricted Three-Body Problem

Up to now we have examined the R3BP where the test mass m is negligibly small compared to the other two primary masses m_1, m_2, which are rotating on elliptic orbits about their barycenter implying varying angular velocity ω and varying mutual distance d. We now assume that the initial conditions are such that their orbits are circular so that $d = const$ and hence according to Eq. (7.4.1)

$$\omega = n = \sqrt{\frac{G(m_1 + m_2)}{d^3}} = const$$

In this section we want to study the situation where the test mass is free to move with respect to the primary masses. This test mass, for instance, might be a Moon probe on a quite convoluted trajectory from Earth to the Moon. The determination

Table 11.1 Libration points in the Earth–Moon system

Libration points	$\Delta x/d$	$\Delta y/d$	C
L_1	0.849068	0	−1.6735
L_2	1.167830	0	−1.6649
L_3	0.992912	0	−1.5810
L_4	0.500000	0.866025	−1.5600
L_5	0.500000	−0.866025	−1.5600

Here Δx is the distance of a libration point to the Earth on the x-axis, which is the configuration line between Earth and Moon; Δy is the distance of a libration point to Earth perpendicular to the configuration line; and d the distance between Earth and Moon. C is the Jacobi constant (see Sect. 11.4.2) for $v = 0$

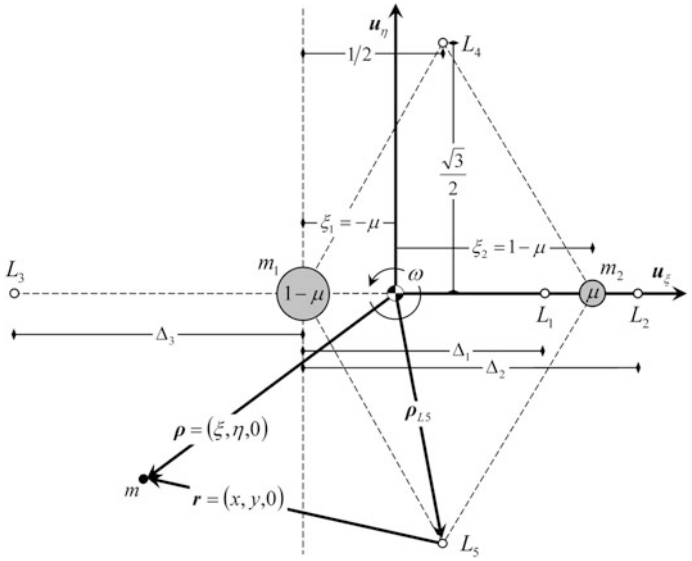


Fig. 11.9 Positions, distances, and vectors in the synodic reference frame of a CR3BP. Vector $\boldsymbol{r}$ as defined in Sect. 11.5.1

of the motion of the test mass between these circularly rotating primaries is called the *circular restricted three-body problem*, CR3BP.

The Synodic Reference Frame

To determine the trajectory of m we define the co-rotating (synodic) reference frame by the orthonormal axes $(\boldsymbol{u}_\xi, \boldsymbol{u}_\eta, \boldsymbol{u}_\zeta)$ (see Fig. 11.9) where $\boldsymbol{u}_\xi$ points from the barycenter to the minor primary m_2, $\boldsymbol{u}_\zeta$ points along the total angular momentum $\boldsymbol{h}$ being vertical to the common motion plane, and $\boldsymbol{u}_\eta$, lying in the plane of rotation and pointing to the half-plane where L_4 is located, completes the right-handed reference system. Within this synodic reference frame we define the normalized position vector to m as

$$\boldsymbol{\rho} := \xi \boldsymbol{u}_\xi + \eta \boldsymbol{u}_\eta + \zeta \boldsymbol{u}_\zeta = \boldsymbol{r}/d \tag{11.4.1}$$

where the coordinates ξ, η, ζ are the axis intercepts normalized by d and therefore dimensionless. Due to $m \approx 0$, the two primaries have the fixed positions $\boldsymbol{\rho}_i = (\xi_i, 0, 0)$ in this synodic reference frame with $\xi_2 - \xi_1 = 1$ following from the normalizing condition. With this relation and from the center-of-mass conditions (see Sect. 7.1.5) $(1-\mu)\xi_1 + \mu\xi_2 = 0$ we find that the minor primary with mass μ is located at $\xi_2 = 1-\mu$ while the bigger one with mass $1-\mu$ is at $\xi_1 = -\mu$. Hence

$$\boldsymbol{\rho}_i = (\xi_i, 0, 0) = \begin{pmatrix} -\mu & 0 & 0 \\ 1-\mu & 0 & 0 \end{pmatrix} \quad \left\{ \begin{array}{l} i = 1, \text{ major primary} \\ i = 2, \text{ minor primary} \end{array} \right\} \tag{11.4.2}$$

Therefore, the relative distances of m to the primaries are

$$\begin{aligned} \Delta\rho_1 &= |\boldsymbol{\rho} - \boldsymbol{\rho}_1| = \sqrt{(\xi+\mu)^2 + \eta^2 + \zeta^2} \\ \Delta\rho_2 &= |\boldsymbol{\rho} - \boldsymbol{\rho}_2| = \sqrt{[\xi-(1-\mu)]^2 + \eta^2 + \zeta^2} \end{aligned} \tag{11.4.3}$$

Thus, the gravitational potential for the test mass m can be rewritten as

$$\begin{aligned} U &= -\frac{Gm_1}{|\boldsymbol{r} - \boldsymbol{r}_1|} - \frac{Gm_2}{|\boldsymbol{r} - \boldsymbol{r}_2|} \\ &= -\frac{G(m_1+m_2)}{d}\left(\frac{1-\mu}{\Delta\rho_1} + \frac{\mu}{\Delta\rho_2}\right) =: d^2 n^2 U' \end{aligned} \tag{11.4.4}$$

with

$$U' = -\frac{1-\mu}{\Delta\rho_1} - \frac{\mu}{\Delta\rho_2}$$

11.4.1 Equation of Motion

Quite generally, the equation of motion is derived from Newton's second law of motion, Eq. (7.1.12). Due to $m \approx 0$ the barycenter of the three bodies is identical to the barycenter of the two primaries and hence the acting force stems by virtue of Eq. (7.1.5) from the gravitational potential U as

$$m\ddot{\boldsymbol{r}} = \boldsymbol{F} = -m\frac{\partial U}{\partial \boldsymbol{r}}$$

We cross over to the synodic reference frame by inserting Eqs. (11.4.1) and (11.4.4), which leads to

$$\ddot{\boldsymbol{\rho}} = -n^2 \frac{\partial U'}{\partial \boldsymbol{\rho}}$$

To study the motion in this non-inertial synodic reference frame rotating with a constant angular velocity $\boldsymbol{n} = (0, 0, n)$ we have to transform $\ddot{\boldsymbol{\rho}}$ according Eq. (7.2.2) as

$$\begin{aligned}\ddot{\boldsymbol{\rho}} &= \ddot{\boldsymbol{\rho}}|_{syn} + \overbrace{2\boldsymbol{n}\times\dot{\boldsymbol{\rho}}|_{syn}}^{Coriolis\ force} + \overbrace{\boldsymbol{n}\times(\boldsymbol{n}\times\boldsymbol{\rho})}^{centrifugal\ force} + \overbrace{\dot{\boldsymbol{n}}\times\boldsymbol{\rho}}^{=0} \\ &= \ddot{\boldsymbol{\rho}}|_{syn} + 2n\Big(-\dot{\eta},\dot{\xi},0\Big) - n^2(\xi,\eta,0) + 0\end{aligned} \tag{11.4.5}$$

with

$$\begin{aligned}\dot{\boldsymbol{\rho}}|_{syn} &:= \dot{\xi}\boldsymbol{u}_\xi + \dot{\eta}\boldsymbol{u}_\eta + \dot{\zeta}\boldsymbol{u}_\zeta \\ \ddot{\boldsymbol{\rho}}|_{syn} &:= \ddot{\xi}\boldsymbol{u}_\xi + \ddot{\eta}\boldsymbol{u}_\eta + \ddot{\zeta}\boldsymbol{u}_\zeta\end{aligned}$$

leading to the equations of motion in the synodic reference frame

$$\begin{pmatrix}\ddot{\xi} - 2n\dot{\eta} - n^2\xi \\ \ddot{\eta} + 2n\dot{\xi} - n^2\eta \\ \ddot{\zeta}\end{pmatrix} = -n^2\frac{\partial U'}{\partial\boldsymbol{\rho}}$$

Defining the dimensionless time and the differential operator

$$\begin{aligned}\tau &:= n\,t \\ (\,)' &:= \frac{d}{d\tau}\end{aligned}$$

yields the dimensionless form

$$\begin{aligned}&\begin{pmatrix}\xi'' - 2\eta' - \xi \\ \eta'' + 2\xi' - \eta \\ \zeta''\end{pmatrix} = -\frac{\partial U'}{\partial\boldsymbol{\rho}} \\ &= -\frac{1-\mu}{\Delta\rho_1^3}\begin{pmatrix}\xi+\mu \\ \eta \\ \zeta\end{pmatrix} - \frac{\mu}{\Delta\rho_2^3}\begin{pmatrix}\xi-(1-\mu) \\ \eta \\ \zeta\end{pmatrix}\end{aligned} \quad \textbf{EoM in the CR3BP} \tag{11.4.6}$$

where the latter follows by differentiating U' from Eqs. (11.4.4) with (11.4.3). This is the differential equation of motion (EoM) (actually three coupled scalar equations) of a small body at $\boldsymbol{\rho} = (\xi,\eta,\zeta)$ moving in the normalized gravitational potential U' (see Eq. (11.4.4)) of a minor primary with reduced mass μ at $\xi_2 = 1-\mu$ plus a major primary with reduced mass $1-\mu$ at location $\xi_1 = -\mu$ expressed in the synodic (co-rotating) barycentric reference frame.

Because the equations of motion are coupled via $\Delta\rho_i^{-3} = \Big[(\xi-\xi_i)^2 + \eta^2 + \zeta^2\Big]^{-3/2}$ they are too complex to be solved analytically. Only for specific regions, namely in the vicinity of the libration points, we are able to linearize the equations and thus derive analytical solutions. This will be done in Sects. 11.5.1 and 11.5.2. Although not being able to solve the general case analytical, we are capable to furnish general conclusions about the motion of the test mass to which we turn now.

11.4.2 Jacobi's Integral

Though we are not able to solve the above equations of motion it might be surprising to see that we can at least integrate them once. For that we first rewrite Eq. (11.4.6) in vectorial form

$$\boldsymbol{\rho}'' + 2\hat{\boldsymbol{n}} \times \boldsymbol{\rho}' - (\xi, \eta, 0) + \frac{\partial U'}{\partial \boldsymbol{\rho}} = 0$$

We then multiply this equation by the velocity vector $\boldsymbol{\rho}' = (\xi', \eta', \zeta')$. Because $(\hat{\boldsymbol{n}} \times \boldsymbol{\rho}') \cdot \boldsymbol{\rho}' = (\boldsymbol{\rho}' \times \boldsymbol{\rho}') \cdot \hat{\boldsymbol{n}}$ we get

$$\boldsymbol{\rho}' \, \boldsymbol{\rho}'' - \left(\xi\xi' + \eta\eta'\right) + \frac{\partial U'}{\partial \boldsymbol{\rho}} \boldsymbol{\rho}' = 0$$

This equation is suited to be integrated directly yielding

$$\frac{1}{2} v^2 - \frac{1}{2}\left(\xi^2 + \eta^2\right) + U' = C \quad \textbf{Jacobi's integral} \tag{11.4.7}$$

with $v^2 = \boldsymbol{v}^2 := \boldsymbol{\rho}'^2 = \xi'^2 + \eta'^2 + \zeta'^2$. C is the integration constant, which is called **Jacobi constant**. *Jacobi's integral of motion* or *Jacobi's integral* for short is a normalized conservation of energy equation for the moving test mass because it relates all energies of m normalized to dimensionless numbers: The first term on the left side corresponds to its kinetic energy, the second term to its rotational energy, and U', of course, is the normalized potential energy. Finally, the Jacobi constant C corresponds via $C = \varepsilon_{tot}/(d^2 n^2)$ to the specific total energy of m.

Conservation of energy for the test mass holds, because in the synodic system the primaries maintain fixed positions and the test mass just moves in their conservative gravitational potentials and the conservative rotational potential of the rotating synodic system. Note that in an inertial reference frame the primaries move and exchange energy with the test mass via the gravitational interaction. Therefore, the energy of the test mass is not conserved in the inertial reference frame where it constantly gains or looses energy. On this energy transfer property hinges the flyby maneuver (see Sect. 9.5).

Rotational Potential in the Synodic System

It may be confusing that in the energy conservation Eq. (11.4.7) the rotational energy and with it the rotational potential $U_\omega := E_\omega/m = -\frac{1}{2}\omega^2(x^2 + y^2) = -\frac{1}{2}\omega^2 r^2$ is negative, exactly opposite to what we would expect (see Eq. (7.2.16)).

The reason is the following. The centrifugal force of course always points outward, i.e., $\boldsymbol{F}_\omega = m\omega^2\boldsymbol{r}$. According to Eq. (7.1.5) the corresponding rotational potential is determined by

$$U_\omega = -\frac{1}{m}\int \boldsymbol{F}_\omega \cdot d\boldsymbol{r} = -\int \omega^2 r \cdot dr$$

In an inertial (sidereal) reference frame $h = \omega \cdot r^2$ (see Eq. (7.2.7)) is constant and therefore

$$U_\omega = -\int \omega^2 r \cdot dr = -h^2 \int \frac{dr}{r^3} = \frac{1}{2}\frac{h^2}{r^2} = \frac{1}{2}\omega^2 r^2 \quad @ \text{ inertial reference frame}$$

However, in a co-rotating synodic reference frame $\omega = const$ and therefore

$$U_\omega = -\omega^2 \int r \cdot dr = -\frac{1}{2}\omega^2 r^2 \quad @ \text{ synodic reference frame}$$

So, while in the inertial system the rotational energy $E_\omega = m \cdot U_{rot} = 1/2\, mh^2/r^2$ is positive, it is negative in the synodic system: $E_\omega = -1/2\, m\omega^2 r^2$ (see Fig. 11.10).

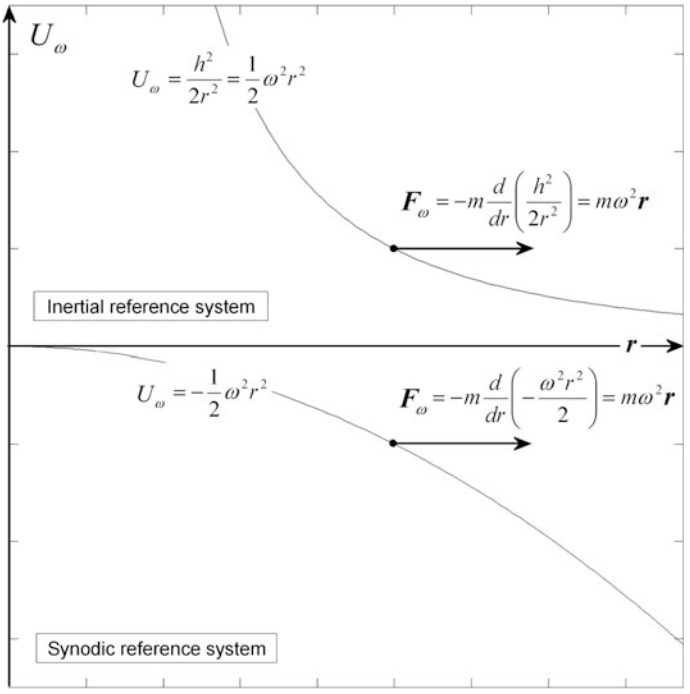


Fig. 11.10 Centrifugal potential and centrifugal force in an inertial and a synodic system

In both cases the energy decreases with increasing distance from the origin, in a way that the gradient always forms the same centrifugal force. Because the rotational potential (energy) is negative in the synodic system, one speaks of a potential field—comparable to a gravitational field, which is negative as well—which creates a fictitious force (centrifugal force). Here "fictitious" does not mean that the force is not real, but a real force that fictitiously acts from the outside. In rotating reference frames fictitious forces occur as centrifugal or Coriolis forces (the latter if a body is moving in this frame). In an inertial reference frame both types can be understood to be caused by inertial forces.

Effective Potential

In view of Eq. (11.4.7) and taking Eq. (11.4.4) into account we define for future convenience and in line with literature the *positive* effective potential

$$\Omega := \frac{1}{2}\left(\xi^2 + \eta^2\right) + \frac{1-\mu}{\Delta\rho_1} + \frac{\mu}{\Delta\rho_2} > 0 \quad \textbf{effective potential} \tag{11.4.8}$$

Its important shape is depictured in Fig. 11.11 together with the location of the libration points.

Remark W*ith a positively defined potential we dissent from physical conventions, see Eq.* (7.1.3)*, which defines potentials U negatively. We hence use with* Ω *a greek symbol rather than U. In literature the positively defined effective potential (a.k.a. "pseudopotential" due to the fictitious centrifugal force) is frequently denoted as U, too.*

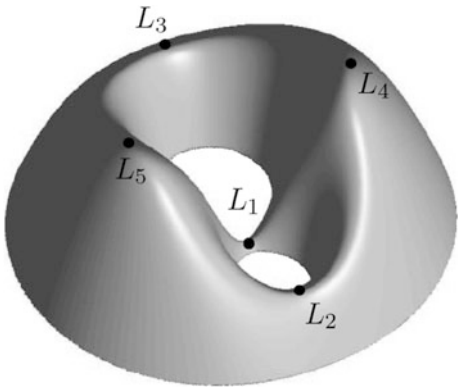


Fig. 11.11 Effective potential Ω and libration points in the CR3BP. MATLAB plot

With the effective potential we can rewrite Jacobi's integral in the form

$$\frac{1}{2}v^2 - \Omega = C \quad \textbf{Jacobi's integral} \tag{11.4.9}$$

For bounded orbits, as considered in the following, the total energy ε_{tot} and with it also C are negative, $C<0$, otherwise for an unbounded motion $C \geq 0$. Its actual value is determined from the initial conditions. It can be shown (see Problem 11.6) that for a stationary test mass at the libration points and for $m \to 0$ the Jacobi constant is given as $C_{L1} = C_{L2} = C_{L3} = -3/2$.

Note *In literature the Jacobi constant for bounded orbits often is positively defined, i.e.,* $C := \Omega - \frac{1}{2}v^2 > 0$*, or even* $C := 2\Omega - v^2 > 0$.

Because $\partial\Omega/\partial\boldsymbol{\rho} = -(\xi, \eta, 0) + \partial U'/\partial\boldsymbol{\rho}$ we can rewrite the EoM (11.4.6) as

$$\begin{aligned}\begin{pmatrix} \xi'' - 2\eta' \\ \eta'' + 2\xi' \\ \zeta'' \end{pmatrix} &= \frac{\partial\Omega}{\partial\boldsymbol{\rho}} \\ = \begin{pmatrix} \xi \\ \eta \\ 0 \end{pmatrix} &- \frac{1-\mu}{\Delta\rho_1^3}\begin{pmatrix} \xi+\mu \\ \eta \\ \zeta \end{pmatrix} - \frac{\mu}{\Delta\rho_2^3}\begin{pmatrix} \xi-(1-\mu) \\ \eta \\ \zeta \end{pmatrix}\end{aligned} \tag{11.4.10}$$

Remark *The positions of the libration points in the CR3BP of course may also be derived from the static equilibrium conditions* $\boldsymbol{\rho}'_L = \boldsymbol{\rho}''_L = 0$ *and thus according to the above equation from* $(\partial\Omega/\partial\boldsymbol{\rho})_L = 0$*. In Sects. 11.3.1 and 11.3.2 we rather adopted the derivation from the motion in the inertial system, because then it becomes clear that the libration points are just special cases of the general collinear and equilateral configurations and therefore also bear their characteristics.*

11.4.3 Stability at Libration Points

From Sect. 11.2 we already know that collinear and equilateral libration points quite generally are not stable, and hence in particular not in the CR3BP. From Fig. 11.11 we can see why and even in which way. Because the test body m always tends to decrease its potential energy (and in return gains speed) it will "move down the hills". The corresponding acceleration forces are depicted in Fig. 11.12. For the two equilateral points the situation is clear: Any direction is downhill (blue arrows). Therefore these points must be statically unstable.

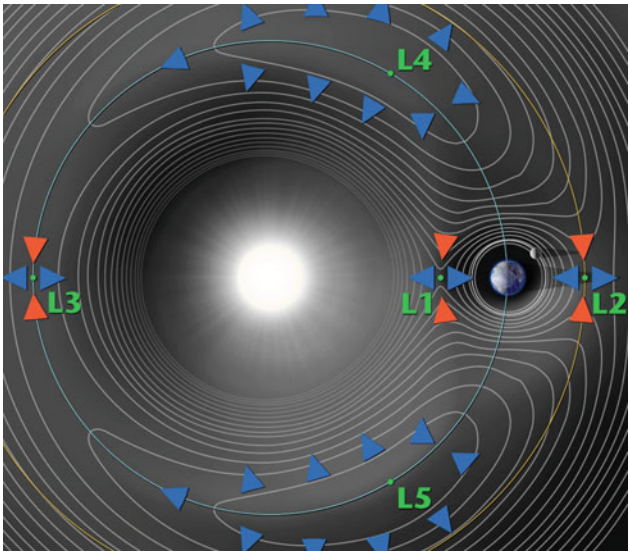


Fig. 11.12 Equipotential lines of the effective potential Ω and the stabilizing (red) and destabilizing forces (blue) at the libration points in the Sun-Earth system resulting from it. *Credit* NASA

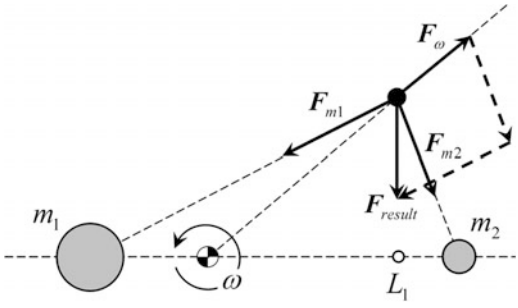


Fig. 11.13 The restoring force (resultant arrow F_{result}) of a body near L_1 resulting from the gravitational forces (thick arrows F_{m1} and F_{m2}) and the centrifugal force (thick arrow F_ω)

For the collinear points the destabilizing forces (blue arrows in Fig. 11.12) point along the direction of the configuration line. Therefore they are unstable. However, perpendicular to this line there are restoring forces (red arrows, both in the Moon's motion plane and vertically to it, not shown here) that push the mass m back to the collinear points. This restoring effect is comprehensible, as the resultant from the gravitational forces and the centrifugal force points in the direction of L_1 (see Fig. 11.13). So the shapes of the potential at the three collinear points are saddles (see Fig. 11.11).

In Sect. 11.5.2 it will be shown that if you bring a spacecraft to a collinear libration point L_1 or L_2 and leave it to its own without any station-keeping measures, small initial deviations Δx_0 along the configuration line would exponentially increase according to

$$\Delta x = \Delta x_0 \cdot \exp(t/\tau) \tag{11.4.11}$$

The e-folding relaxation time τ can be derived from Eq. (11.5.8) with $c_2 = 4$ (see Eq. (11.5.6)) to be

$$\tau = \frac{T}{2\pi\sqrt{1+2\sqrt{7}}} = 0.06345 \cdot T \quad @\ L_1, L_2 \tag{11.4.12}$$

where T is the orbital period of the primary bodies. So, in the Earth–Moon system, a deviation e-folds successively every $\tau = 1.73$ days and in the Sun–Earth system $\tau = 23.2$ days.

The collinear libration point L_3 turns out to be more stable. Applying Eq. (11.5.8) with $c_2 = 1 + \frac{7}{8}\mu$ (see Eq. (11.5.6)), we obtain

$$\tau = \frac{T}{\pi}\sqrt{\frac{2(m_1+m_2)}{21m_2}} \quad @\ L_3 \tag{11.4.13}$$

For the system Earth–Moon this results in $\tau = 24.4$ days. So, compared to any other fixed position in the Earth–Moon system, considerably less station-keeping effort along the configuration line and hence less propellant is required.

11.4.4 General System Dynamics

In Sect. 11.1.1 we already mentioned that for the general n-body problem (n bodies with arbitrary mass and with arbitrary initial conditions) there exist no algebraic integrals of motion other than the classical conservation laws. In this general case we would not have any information on the motion of the test body. The only possibility would be to solve the equations of motion numerically. In the special case of the CR3BP we have found with Jacobi's integral Eq. (11.4.9) an additional conditional equation that restricts the motion of the test body and thus provides some general information about its motion.

To figure out what this additional integral brings about we recall the motion of a body in a central effective potential U_{eff} in the two-body problem as given by Eq. (7.2.19) and depicted in Fig. 7.7. There we saw that the specific total energy restricts the orbital motion and thus defines a class of orbits having the same characteristics, namely elliptic, parabolic, and hyperbolic orbits. By relating the specific total energy to Jacobi's integral we now want to convey this property to the motion of a body in the effective potential Ω the CR3BP.

We therefore define the class of all trajectories having a given Jacobi constant C, i.e., a given total energy, as **invariant manifolds**. Manifolds of dynamical systems can be pictured quite generally as 2D topological subspaces in our common 3D space. A specific manifold represents the phase space of a system's dynamics. Owing to the property of constant total energy, invariant manifolds have the

additional property that a body's trajectory in this phase space that starts out in an invariant manifold remains in this manifold for the duration of its dynamical evolution. Mathematically speaking, the body's state flows in the manifold and therefore the manifold is said to be invariant under this flow.

The objective of this section is to study the invariant manifolds of a CR3BP in general in and Sect. 11.5 those about libration points. Generally, there are two types of invariant manifolds, namely *center manifolds*, which are the phase space of periodic and quasi-periodic orbits, and two variants of *hyperbolic manifolds, a.k.a. saddle manifolds*, namely *stable manifolds* and *unstable manifolds,* which are the phase space of orbits, which wind on or off periodic and quasi-periodic orbits, respectively, as will be discussed in Sect. 11.5.2.

Remark *Much of the nomenclature used here, such as "invariant manifold" or "flow", stems from dynamical systems theory, which deals with the behavior of non-linear systems as described by non-linear ordinary differential equations. An excellent introduction and account of dynamical system theory is given by Ledermann (1990).*

Zero-velocity Curves

We now use Jacobi's integral to determine the space that is accessible to the test mass. It ends where its velocity becomes zero. So, if we set $v = 0$ in Eq. (11.4.9), we get a curve that envelopes the space in which the body can possibly move. The test body cannot cross the envelope curve, it can touch it with velocity $v = 0$ at selected points. For a given C of the test body, the envelope curve is the line, the coordinates of which satisfy the equation $\Omega(\xi, \eta, \zeta) = -C$ and is apparently the contour line of the effective potential in Fig. 11.11. The envelope curve is also called *zero-velocity curve*, a.k.a. *Hill curve* after the astronomer Hill, who studied it in detail in the 19th century.

Let us examine in detail the zero-velocity curve and how it depends on the energy of the test mass. For a given negative C (total energy) and because kinetic energy is positive, the body can move only in those spatial areas where $-\Omega \leq C$. According to Eq. (11.4.8), this is the case whenever $\Delta\rho_1$ or $\Delta\rho_2$ is very small, i.e., when m is close to one of the bodies m_1 or m_2 (large negative gravitational energy), or when m is far a way from both (large negative rotational energy). These areas are indicated in white in Fig. 11.14 for the Earth m_1 and the Moon m_2. The inaccessible area $-\Omega > C$ in between is indicated in gray. If the energy of the test body and hence C gradually increases, the test body is able to access more and more space (Fig. 11.14). In Fig. 11.14b for $C = -1.6735$ a transit from Earth to the Moon via L_1 is possible for the first time. It shows that the flight to Moon via L_1 is energetically most favorable. If the energy of the test body is further increased, L_2 and L_3 also become accessible (Fig. 11.14c, d). The equilateral libration points L_4 and L_5 are potentially the highest points in Earth–Moon system, and are achieved in the end for $C = -1.5600$ (Fig. 11.14e, f).

The zero-velocity curves merely define the limits of motion of the test body. They do not tell us anything about how the test body moves within the permissible

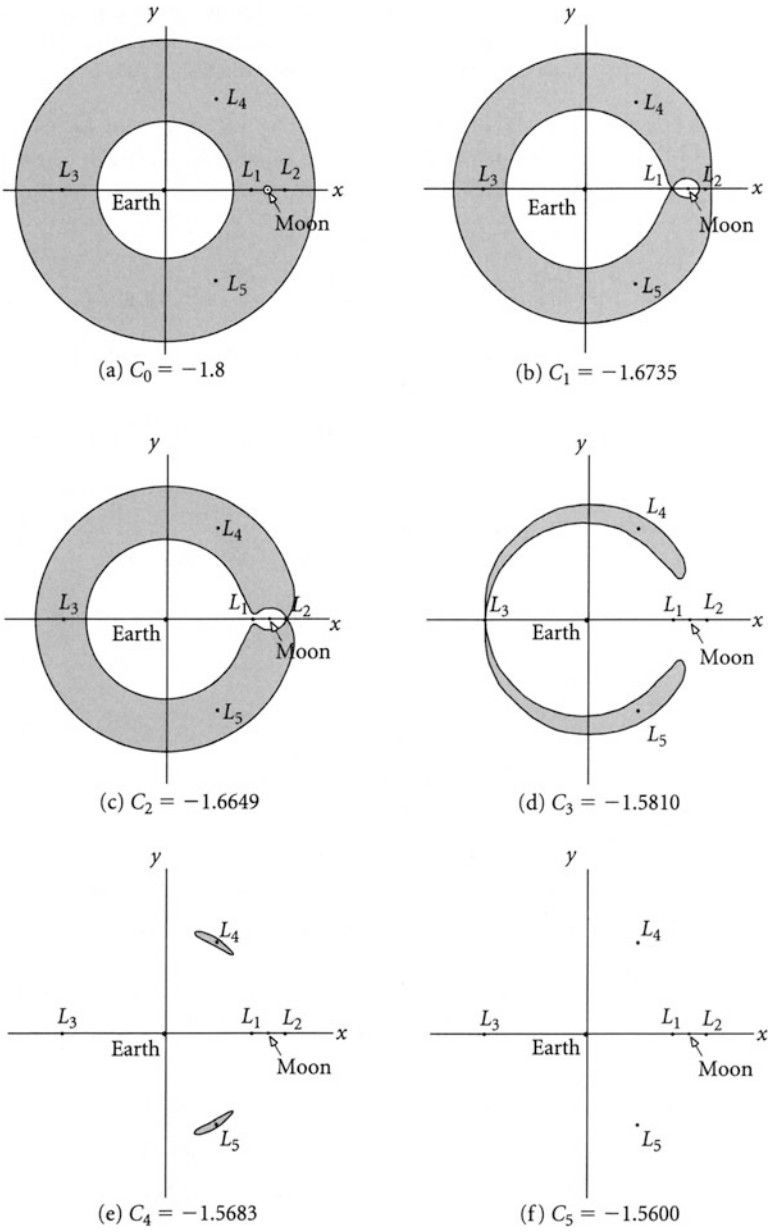


Fig. 11.14 Zero-velocity curves in the Earth–Moon system with increasing Jacobi constant C (total energy). *Credit* Curtis (2005)

areas, that is, about the distinct properties of the invariant manifolds. For instance, we know that the most favorable trajectory from Earth to Moon is via L_1, but we do not know what a corresponding trajectory looks like. To find out we have to solve the equations of motion with the initial conditions defining the energy of the test body. This will be done in the next three sections (see Fig. 11.27 for a particular Earth–Moon trajectory via L_1).

Chaotic Trajectories

Apart from the conservation of Jacobi's integral, which accounts for the zero-velocity curves, there are no limits on the trajectory of a body. So, to determine a trajectory from a given initial state vector, Eq. (11.4.10) must be solved numerically. A typical result shown in Fig. 11.15 for the Earth–Moon system. From the intricate shape of the trajectory we conclude that any tiny change in the initial state will cause increasing deviations from the original trajectory that will end up in a totally different path. Such a behaviour, being highly sensitive to the initial condition, is typical for a so-called *deterministic chaos*. It is chaotic because it is nondeterministic in the long run but deterministic in the short. Deterministic chaotic behavior is very common for motion governed by non-linear differential equations, such as the equation of motion (11.4.10). The science of deterministic chaos, the so-called *chaos theory*, is a special but important branch of *dynamical systems theory*. We do not want to go into details of chaos theory, but only mention two main characteristics.

The sensivity of the long term evolution of the system manifests itself as an exponential growth of perturbations in the initial conditions. This is explicitely demonstrated by the equation $\Delta x = \Delta x_0 \cdot \exp(t/\tau)$ of Sect. 11.4.3, which determines the stability at the libration points. The relaxation time τ is nothing else than the inverse of the so-called Lyapunov exponent in chaos theory.

Although deterministic chaotic systems are inherently unpredictable, they are easily controlable. *Control theory* deals with influencing the behavior of dynamical systems. Basic methods of controlling a chaotic orbit are the *OGY* (Ott, Grebogi and

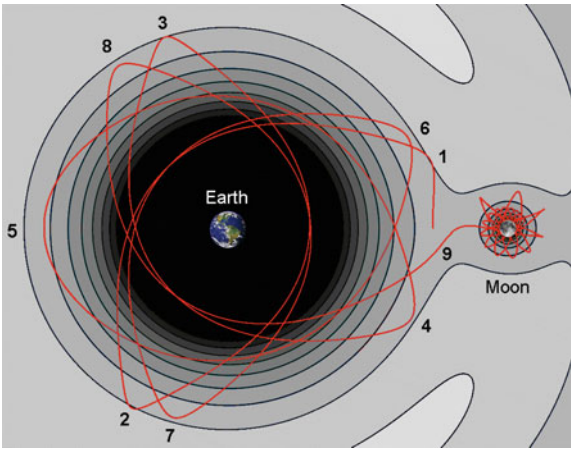


Fig. 11.15 A chaotic trajectory in the Earth–Moon system with a final moon capture. *Credit* Andreas Sandberg, Creative Commons, and U. Walter

Yorke) *method*, and *Pyragas continuous control method*. In the OGY method swift and tiny kick-burns once per cycle are applied to keep a spacecraft on a periodic target trajectory. Such kick-burns would keep for instance the spacecraft infinitely at a libration point, or adjust the trajectory in Fig. 11.15 such that the turning points $6 \rightarrow 1, 7 \rightarrow 2, 8 \rightarrow 3$, and $9 \rightarrow 4$ would coincide, leading to a closed periodic orbit. The same can be achieved by the Pyragas method, employing continuous thrust. As long as the system evolves close to the desired periodic orbit, the orbit-keeping thrust is very small, but increases rapidly when it drifts away from the target orbit.

Rapprochement Orbits

Such a closed periodic orbit belongs to the numerous class of highly symmetrical periodic orbits. Among those there is a special class of highly symmetrical periodic orbits with the additional feature that the body approaches one or both primaries repeatedly and very closely. This is why this type of orbit is called a rapprochement orbit. French mathematicians extensively studied these orbits, in particular for the Earth–Moon case.

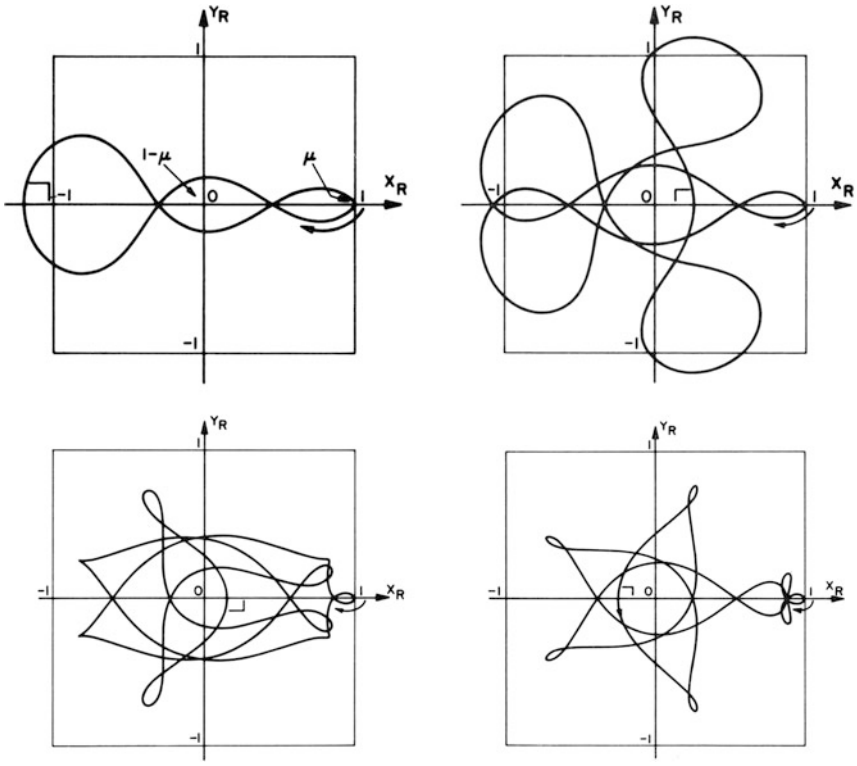


Fig. 11.16 Four typical rapprochement orbits in the Earth–Moon system. *Credit* Hoelker R.F., NASA TN D-5529

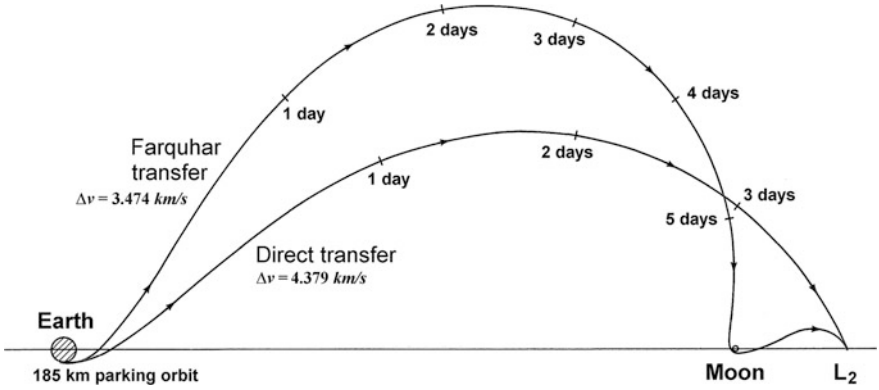


Fig. 11.17 A three-impulse Farquhar transfer orbit based on a rapprochment orbit from a 185 km Earth parking orbit to the LL_2 point (with stopping) with $\Delta v = 3.474\,km/s$ taking 212 hours compared to a two-impulse direct transfer with $\Delta v = 4.379\,km/s$ taking 96 hours. (*Credit* Farquhar, R. W. / NASA-GSFC (1971))

Figure 11.16 shows four typical rapprochement orbits. The transfer body infinitely shuttles on a symmetrically closed curve between Earth and Moon. The practical application of rapprochement orbits is their use to commute to a primary or between the primaries. There is an immense number of rapprochement orbits, which, however, share the property of all chaotic systems: Any tiny initial perturbation, if uncorrected, amplifies until the orbit is no longer symmetrical–periodical, but takes on a chaotic course. However, as we have learned above, keeping a rapprochement orbit in symmetrical shape requires little effort.

There are variants of these orbits that are interesting from another practical point of view, e.g., low energetic transfer orbits to the libration points between Earth and Moon. Figure 11.17 shows such a low energy orbit to libration point L_2, which was studied by Farquhar and coworkers and which is anticipated to be utilized for the SLS Exploration Mission One (EM-1) in 2020. It takes advantage of a special flyby past the Moon to swing by to L_2.

Free-Return Trajectories

When mission planning for the first manned US missions to the Moon was at issue, the safety of the crew in case of a main engine failure played a crucial role. Thus a trajectory was selected that assured the return of the astronauts to the Earth even with a total main engine failure. This special circumlunar trajectory was called "free-return" trajectory and it is depicted in Fig. 11.18. It is a symmetrical-periodical trajectory, and thus is a rapprochement orbit. It passes the surface of the Moon with a minimum distance of 111 km, which corresponds exactly to the

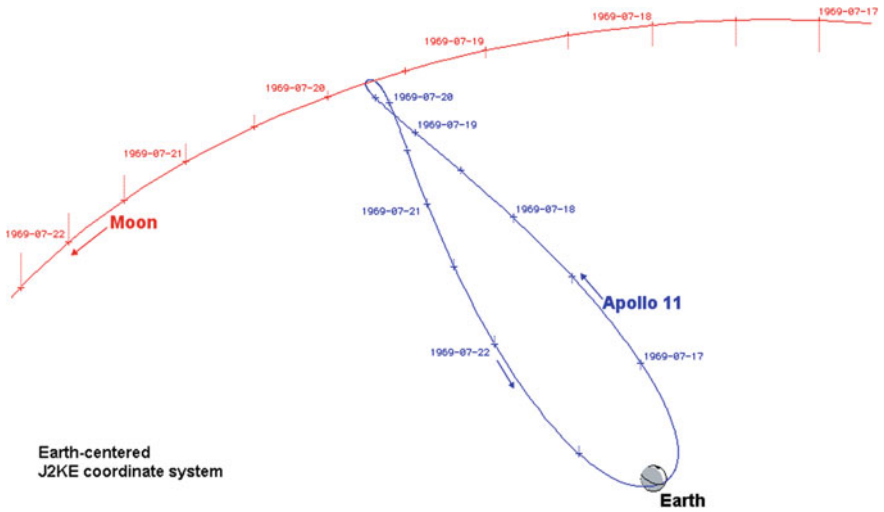


Fig. 11.18 The reconstructed circumlunar, free-return trajectory of Apollo 11 to the Moon. *Credit* Simulation by Daniel R. Adamo

periselene altitude of the missions Apollo 8, 10, and 11. Later Moon-landing missions entered into a circular Moon orbit at this point, from which it was then possible to descent to the Moon's surface. If the mission had to be aborted for any reason, the engine ignition required for braking behind the Moon into the circular Moon orbit would not have happened, and the astronauts would automatically have returned to the Earth. This indeed happened with Apollo 13.

The trajectory shown in Fig. 11.18 is fully symmetric with regard to the Earth–Moon connecting line. The actual trajectories of the Apollo missions were slightly asymmetric, such that on return the spacecraft would touch the Earth's atmosphere in order to guarantee an automatic reentry. This asymmetry was achieved by a slight shift of the position of the periselene.

During the outward as well as return flight the free-return trajectory is clearly elliptical in the proximity of the Earth, as the gravitational influence of the Earth is dominating. Approaching the edge of the Moon's SOI (see Sect. 9.1) the trajectory becomes more and more a straight line: here the orbital velocity has already decreased quite a lot, and the gravitational influence of the Earth and the Moon plus centrifugal force just cancel out each other. In this area the real trajectory deviates utmost from the patched-conics approximation. In the surroundings of the Moon the trajectory is bent into a hyperbola.

11.5 Dynamics About Libration Points

11.5.1 Equation of Motion

In the following we want to explore the trajectories in the vicinity of the five libration points L, the location of which we denote in the barycenter system (see Fig. 11.9) by

$$\boldsymbol{\rho}_L = (\xi_L, \eta_L, 0)$$

From the results of Sects. 11.3.1 and 11.3.2, and because $\xi_L = \Delta x/d + \xi_1 = \Delta x/d - \mu$ we have the results in Table 11.2. In order to derive the equations of motion about libration points we define the position vector $\boldsymbol{r} = (x, y, z)$ relative to a libration point under consideration (see Fig. 11.9)

$$\boldsymbol{\rho} =: \boldsymbol{\rho}_L + \boldsymbol{r} \tag{11.5.1}$$

Note that we reuse the notation $\boldsymbol{r}$ (cf. Eq. (11.4.1)) for convenience and because it is also used quite often in literature. For the position of the primaries relative to this libration point

$$\boldsymbol{r}_i = (x_i,\, y_i,\, 0) = (\xi_i - \xi_L,\, \eta_i - \eta_L,\, 0) \quad @\ i = 1, 2 \tag{11.5.2}$$

we have with $\xi_1 = -\mu$, $\xi_2 = 1 - \mu$, and ξ_L, μ_L from Table 11.2 the results listed also in Table 11.2.

Because from Eq. (11.5.1) $0 = \eta_i = \eta_L + y_i$ and hence $\eta = \eta_L + y = y - y_i$, we have $\Delta\rho_i = \sqrt{(\xi - \xi_i)^2 + \eta^2 + \zeta^2} = \sqrt{(x - x_i)^2 + (y - y_i)^2 + z^2}$. This gives rise to the definition

$$\Delta r_i := \sqrt{(x - x_i)^2 + (y - y_i)^2 + z^2} \tag{11.5.3}$$

Table 11.2 Positions ξ_L, μ_L of the libration points in the barycentric synodic reference frame and positions x_1, y_1 and x_2, y_2 of the two primaries relative to the libration points

	L_1	L_2	L_3	L_4	L_5
ξ_L	$\Delta_1 - \mu$	$\Delta_2 - \mu$	$-\Delta_3 - \mu$	$0.5 - \mu$	$0.5 - \mu$
η_L	0	0	0	$+\sqrt{3}/2$	$-\sqrt{3}/2$
x_1	$-\Delta_1$	$-\Delta_2$	Δ_3	–0.5	–0.5
y_1	0	0	0	$-\sqrt{3}/2$	$+\sqrt{3}/2$
x_2	$1 - \Delta_1$	$-(\Delta_2 - 1)$	$1 + \Delta_3$	+0.5	+0.5
y_2	0	0	0	$-\sqrt{3}/2$	$+\sqrt{3}/2$

and therefore

$$\frac{\partial\Omega}{\partial\boldsymbol{\rho}} = \frac{\partial\Omega}{\partial\boldsymbol{r}} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} + \boldsymbol{\rho}_L + \frac{\partial}{\partial\boldsymbol{r}}\left(\frac{1-\mu}{\Delta r_1} + \frac{\mu}{\Delta r_2}\right)$$

Inserting these results into our master Eq. (11.4.10) we obtain

$$\begin{aligned} &\begin{pmatrix} x'' - 2y' - x \\ y'' + 2x' - y \\ z'' \end{pmatrix} = \frac{\partial\Omega}{\partial\boldsymbol{r}} - \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} \\ &= \boldsymbol{\rho}_L + \frac{\partial}{\partial\boldsymbol{r}}\left(\frac{1-\mu}{\Delta r_1} + \frac{\mu}{\Delta r_2}\right) \\ &= \boldsymbol{\rho}_L - \left[\frac{1-\mu}{\Delta r_1^3}\begin{pmatrix} x - x_1 \\ y - y_1 \\ z \end{pmatrix} + \frac{\mu}{\Delta r_2^3}\begin{pmatrix} x - x_2 \\ y - y_2 \\ z \end{pmatrix}\right] \end{aligned} \qquad \textbf{EoM about a libration point} \qquad (11.5.4)$$

These are the equations of motion of a small moving body at the normalized distance $\boldsymbol{r} = (x, y, z)$ from any of the five libration points, with Δr_1, Δr_2 given by Eq. (11.5.3) and x_1, y_1, x_2, y_2 given by Table 11.2 expressed in the synodic (co-rotating) barycentric reference frame.

In the following we will study the dynamics in the vicinity of libration points (For more details the book Perozzi (2010) is recommended). To do so we have to distinguish between collinear and equilateral libration points.

11.5.2 Collinear Libration Points

We first consider the detailed dynamics in the vicinity the collinear libration points L_1, L_2, L_3.

Equation of Motion

In the vicinity of a collinear libration point $\boldsymbol{r} \approx 0$ and therefore we are able to evaluate the upper right side of Eq. (11.5.4) in powers of r. To do so, and because for any collinear libration point $y_i = 0$ holds, we first write

$$\begin{aligned} \Delta r_i &= \sqrt{(x - x_i)^2 + y^2 + z^2} = |x_i|\sqrt{\left(\frac{x}{x_i} - 1\right)^2 + \left(\frac{y}{x_i}\right)^2 + \left(\frac{z}{x_i}\right)^2} \\ &= |x_i|\sqrt{1 - 2\frac{x}{r}\frac{r}{x_i} + \left(\frac{r}{x_i}\right)^2} \end{aligned}$$

Now, according to mathematical tables, $1/\sqrt{1-2\alpha u+u^2}$ is the generating function of the Legendre polynomials P_n of degree n

$$\frac{1}{\sqrt{1-2\alpha u+u^2}}=\sum_{n=0}^{\infty}u^nP_n(\alpha)$$

Identifying $\alpha\equiv x/r$ and $u\equiv r/x_i$ we therefore find for the power series in r

$$\frac{1}{\Delta r_i}=\frac{1}{|x_i|}\sum_{n=1}^{\infty}\left(\frac{r}{x_i}\right)^nP_n\left(\frac{x}{r}\right)$$

Inserting this and x_i from Table 11.2 into Eq. (11.5.4) we find for the equation of motion

$$\begin{pmatrix}x''-2y'-x\\ y''+2x'-y\\ z''\end{pmatrix}=\boldsymbol{\rho}_L+\frac{\partial}{\partial\boldsymbol{r}}\left(\frac{1-\mu}{\Delta r_1}+\frac{\mu}{\Delta r_2}\right)=\boldsymbol{\rho}_L+\frac{\partial}{\partial\boldsymbol{r}}\sum_{n=1}^{\infty}c_nr^nP_n\left(\frac{x}{r}\right)\quad(11.5.5)$$

with

$$c_n=\operatorname{sgn}(x_1)\frac{1-\mu}{x_1^{n+1}}+\operatorname{sgn}(x_2)\frac{\mu}{x_2^{n+1}}$$

For $n=0$ we have $\partial/\partial\boldsymbol{r}(c_0P_0)=\partial/\partial\boldsymbol{r}(c_0\cdot1)=0$. Evaluating the term for $n=1$ we have with $P_1(\alpha)=\alpha$ and Eqs. (11.4.3), (11.4.4), (11.4.5) and Table 11.2 $c_1=-\xi_L$ and hence

$$\frac{\partial}{\partial\boldsymbol{r}}\left[c_1rP_1\left(\frac{x}{r}\right)\right]=\frac{\partial}{\partial\boldsymbol{r}}(c_1x)=\begin{pmatrix}c_1\\0\\0\end{pmatrix}=-\boldsymbol{\rho}_L$$

With this and applying for $n=2$ the same procedure to $P_2(\alpha)=\frac{1}{2}(3\alpha^2-1)$ we can rewrite the EoM (11.5.4) as

$$\begin{pmatrix}x''-2y'-(1+2c_2)x\\ y''+2x'\quad-(1-c_2)y\\ z''\qquad\qquad+c_2z\end{pmatrix}=\frac{\partial}{\partial\boldsymbol{r}}\sum_{n=3}^{\infty}c_nr^nP_n\left(\frac{x}{r}\right)\qquad\textbf{EoM near }\boldsymbol{L_1,L_2,L_3}\quad(11.5.6)$$

with

$$c_2=\begin{cases}\frac{1-\mu}{\Delta_1^3}+\frac{\mu}{(1-\Delta_1)^3}=4+6\lambda+O(\lambda^2) & @\ L_1\\ \frac{1-\mu}{\Delta_2^3}+\frac{\mu}{(\Delta_2-1)^3}=4-6\lambda+O(\lambda^2) & @\ L_2\\ \frac{1-\mu}{\Delta_3^3}+\frac{\mu}{(1+\Delta_3)^3}=1+\frac{7}{8}\mu+O(\mu^2) & @\ L_3\end{cases},$$

and

$$\lambda=\left(\frac{\mu}{3}\right)^{1/3},\quad\mu=\frac{m_2}{m_1+m_2}$$

where the latter follows from employing the power series expansions of Eq. (11.3.7). These are the equations of motion of a small moving body at $\boldsymbol{r} = (x, y, z)$ (dimensionless) in the vicinity of and relative to a collinear libration point in the synodic reference frame.

Invariant Manifolds

In celestial mechanics the investigation of these coupled equations of motion and the study of their general behavior is currently an ongoing research in order to design for instance space telescope missions to L_1 or L_2 points in the Sun–Earth and Earth–Moon systems. The differential Eqs. (11.5.6) are quite complex and can generally be solved only numerically. However, without knowing the detailed solution we can study the types of solutions and their fundamental behavior. This fundamental behavior is already captured by the linear part, which is the left sides of Eq. (11.5.6). The right sides are in the vicinity of the collinear libration points minor, non-linear modifications. We therefore study in the following the linearized equations

$$\boxed{\begin{array}{lll} x'' - 2y' - (1+2c_2)x = 0 \\ y'' + 2x' \quad - (1-c_2)y = 0 \\ z'' \qquad\qquad + c_2 z = 0 \end{array}} \quad \textbf{linearized EoM} \tag{11.5.7}$$

Upon solving these linearized differential equations one notices that the third equation is decoupled from the other two and displays the behavior of a harmonic oscillator with frequency $\omega_z = \sqrt{c_2}$. For the other two equations we make the standard ansatz for linear differential equations, $x = x_0 \exp(\lambda\tau)$, $y = y_0 \exp(\lambda\tau)$, and find for the characteristic polynomial

$$\lambda^4 + (2 - c_2)\lambda^2 + (1 + 2c_2)(1 - c_2) = 0$$

of the matrix

$$\boldsymbol{A} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1+2c_2 & 0 & 0 & 2 \\ 0 & 1-c_2 & -2 & 0 \end{pmatrix}$$

which satisfies $\boldsymbol{X}' = \boldsymbol{A}\boldsymbol{X}$, where $\boldsymbol{X} = (x, y, x', y')$. Considered as a quadratic characteristic equation for λ^2 we find that, because of $c_2 > 1$ (see Eq. (11.5.6)) the sign changes only once between consecutive terms, which implies from Descartes' rule of signs that λ^2 has a positive and a negative root, namely

$$\lambda_-^2 = \frac{1}{2}\left(c_2 - 2 - \sqrt{9c_2^2 - 8c_2}\right) < 0$$
$$\lambda_+^2 = \frac{1}{2}\left(c_2 - 2 + \sqrt{9c_2^2 - 8c_2}\right) > 0$$

By assigning the solutions $\lambda_{xy} = \pm i\sqrt{-\lambda_-^2} =: \pm i\omega_{xy}$ and $s = \pm\sqrt{\lambda_+^2}$ we find that the solutions of the linearized EoM (11.5.7) quite generally can be written as

$$\begin{aligned} x &= A_1 e^{s\tau} + A_2 e^{-s\tau} + A_x \cos\left(\omega_{xy}\tau + \phi\right) \\ y &= a(A_1 e^{s\tau} - A_2 e^{-s\tau}) + bA_x \sin\left(\omega_{xy}\tau + \phi\right) \\ z &= A_z \cos(\omega_z \tau + \psi) \end{aligned} \tag{11.5.8}$$

where

$$a = \frac{s^2 - 1 - 2c_2}{2s}, \quad b = -\frac{1 + 2c_2 + \omega_{xy}^2}{2\omega_{xy}}$$

Note that owing to the normalization of the EoM the angular frequencies ω_{xy}, ω_z are actually in units $n = \sqrt{G(m_1 + m_2)/d^3} = 2\pi/T$, i.e., the orbital frequency of the synodic system. To give an example: In the Sun–Earth synodic system we have at L_1 and L_2

	c_2	ω_{xy}	ω_z	s	a	b
L_1	4.061012	2.07184	2.00025	2.50869	−0.53982	−3.20839
L_2	3.940522	2.05701	1.98507	2.48432	−0.54526	−3.18723

The coefficients A_1, A_2, A_{xy}, A_z and phase angles ϕ, ψ are determined from the initial conditions $x_0, y_0, z_0, x_0', y_0', z_0'$. So, the initial conditions that determine the Jacoby constant C, plus Eq. (11.5.8) are the clue to finding the C-invariant manifolds, i.e., the classes of trajectories.

Before we now turn into a discussion of the trajectories we have to recall that the equations of motion Eq. (11.5.6) in fact have a non-linear right-hand side, which becomes increasingly significant as the motion departs from the collinear libration point. This non-linear part effectively modifies all the above trajectory parameters $\omega_{xy}, \omega_z, s, a, b$. Most importantly, the frequencies ω_{xy}, ω_z start to vary with larger amplitudes A_x, A_z of their orbits.

Keeping this in mind we can read off Eq. (11.5.8) the following general behavior: Depending on the initial conditions there are unbounded solutions for $A_1, A_2 \neq 0$ and stable periodic solutions for $A_1, A_2 = 0$. This is in accordance with the *center manifold theorem* stating that associated with each imaginary eigenvalue of the Jacobian matrix $\boldsymbol{A}$ there exists a bounded so-called center manifold and to each real eigenvalue and an unbounded hyperbolic manifold (saddle). So we have at

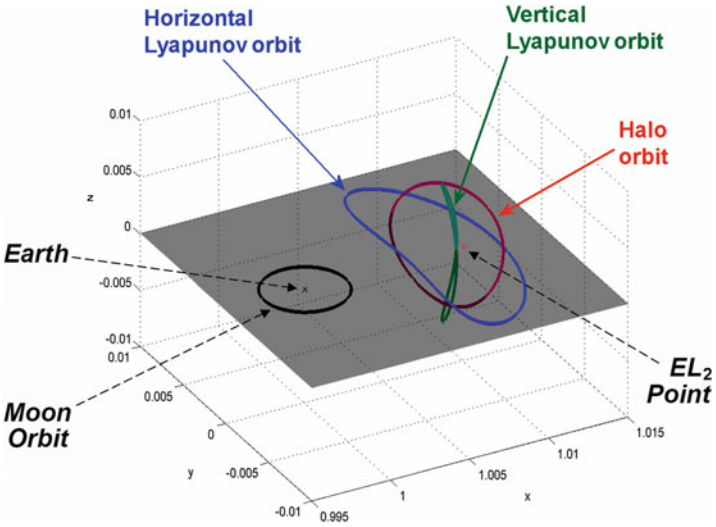


Fig. 11.19 Three of four periodic orbits about the EL_2 point in the Sun–Earth system: the two Lyapunov orbits and a southern halo orbit. *Credit* Egemen Kolemen (2012)

each equilibrium point a center × center × saddle type of solutions with accordingly bounded and unbounded orbits.

Center Manifold Orbits

When the oscillatory amplitudes are very small, i.e., when we are in the linear limit, there are two types of periodic orbits (see Fig. 11.19):

1. **Vertical Lyapunov orbits** for $A_1, A_2, A_x = 0$ and $A_z \neq 0$, which are vertical periodic motions with frequency ω_z along the z-axis, and a
2. **Horizontal Lyapunov orbits** (a.k.a. planar Lyapunov orbit) for $A_1, A_2, A_z = 0$ and $A_x \neq 0$, which are near-elliptic motions in the x–y plane with frequency ω_{xy} and with semi-major axis along the y-axis being $b \approx 3.2$ times the semi-minor axis along the x-axis.

For $A_x, A_z \neq 0$ amplitudes we find quasi-periodic 2D tori orbits, so-called **Lissajous orbits**, with motion amplitudes in all three axis revolving either Lyapunov orbit (see Fig. 11.20). Depending on the concrete initial A_x, A_z, Lissajous orbits occur in many different sizes and shapes that can even occur as crossovers between vertical and horizontal Lyapunov orbits. Lissajous orbits in general do not close and exhibit $\omega_{xy} > \omega_z$ (see above table). If we increase the energy of the orbits by increasing the amplitudes, and via the non-linear terms of the EoM thus also ω_{xy}, ω_z, there might be cases when ω_{xy}/ω_z becomes a rational value, in which case we get closed periodic Lissajous orbits.

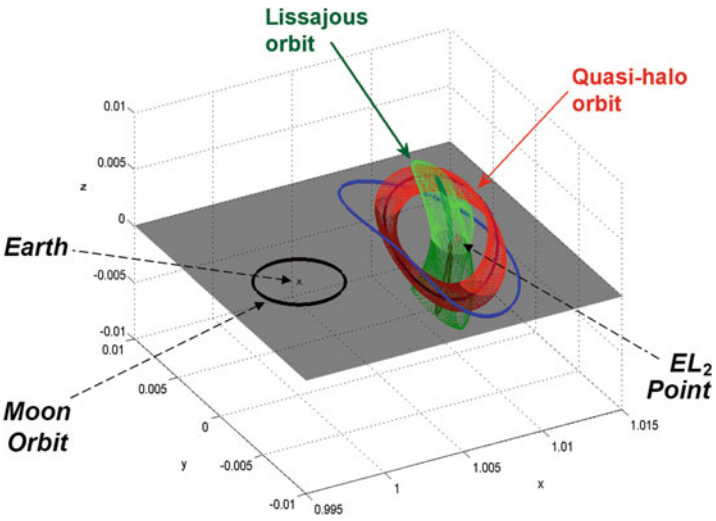


Fig. 11.20 Two quasi-periodic orbits about the EL_2 point in the Sun–Earth system: a Lissajous orbit revolving the vertical Lyapunov orbit and a southern quasi-halo orbit. *Credit* Egemen Kolemen (2012)

When the very special case $\omega_{xy} = \omega_z$ is attained (For a motion about the L_2 point in the Sun–Earth system this happens for $A_x \approx 200\,000$ km) two new periodic orbits, the so-called northern and southern **halo orbits** (for the definition of *northern* and *southern* see below) bifurcate from the horizontal Lyapunov orbit (see Fig. 11.19). For $\omega_{xy} \approx \omega_z$ we have quasi-periodic orbits around these halo orbits, so-called **quasi-halo orbits** (see Fig. 11.20). In between the Lissajous and quasi-halo orbits chaotic behavior occurs (see Fig. 11.22). Because the c_2 coefficient in the linearized EoM (11.5.6) is nearly constant for the three collinear points, the periodic orbits about the different collinear points are very similar. The sets of phase states of the above bounded periodic and quasi-periodic orbits form the so-called *center manifold* (cf. Sect. 11.4.4), which need to be categorized from the unbounded motion to which we will return below.

Note *Precise periodic halo orbits exist only in the exact circular restricted problem. In the real world the minor primary mass (e.g. Moon revolving around the Earth) has a somewhat elliptic orbit and therefore "halo orbits" are only close to true periodic halo orbits. For this reason the term "halo orbit" or "periodic orbit" in the literature (see for instance "L1 periodic orbit" in Fig. 12.24b) usually and sloppily includes any periodic and quasi-periodic orbits, or center manifold orbits in general. In particular, the "halo orbit" of the Genesis mission as depicted in Fig. 11.28 was actually a Lissajous orbit.*

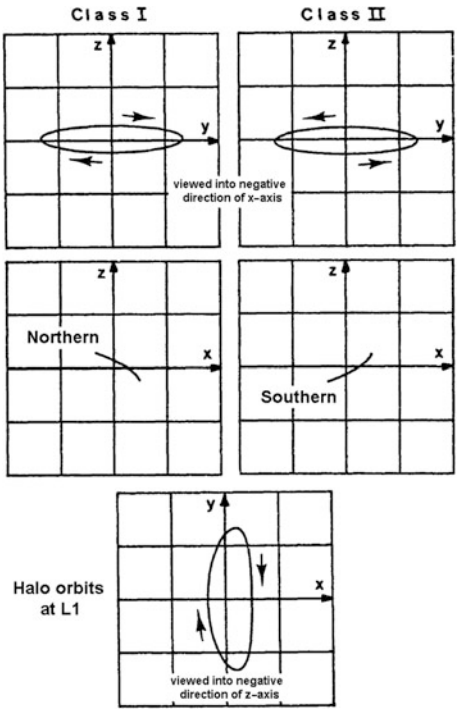


Fig. 11.21 Class I and Class II halo orbits at a L_1 point. *Credit* D. L. Richardson (1980)

Classification of Halo Orbits

According to Eq. (11.5.8) the bounded periodic or quasi-periodic motion can be written as

$$x = A_x \cos\left(\omega_{xy}\tau + \phi\right)$$
$$y = bA_x \sin\left(\omega_{xy}\tau + \phi\right)$$
$$z = A_z \cos(\omega_z\tau + \psi)$$

The periodicity of the halo orbits with $\omega_{xy} = \omega_z$ implies that the phase angles ϕ, ψ obey the relation

$$\psi = \phi + k\frac{\pi}{2}, \quad k = \pm 1$$

From this relation two types of halo orbits result:

Class I halo orbit: If $k = +1$ the sense of revolution of the body about the libration point is clockwise as viewed into the negative direction of the x-axis.

Class II halo orbit: If $k = -1$ the sense of revolution is counterclockwise as viewed into the negative direction of the x-axis.

Class I / Class II halo orbits have the property that they can be viewed as mirror images across the x–y plane as seen in the x–z plane where Class I / Class II halos have their upper turning point (in z-direction) lying on the more negative/positive side of the x-axis (see Fig. 11.21).

However, they are asymmetric about the other two planes. The reason being that the effective potential around a collinear point deviates slightly from a paraboloid, obviously being shallower and hence the halo orbit more elongated outbound from the closest primary. For instance, in the middle part of Fig. 11.23 the halo orbit is more elongated inbound to the Sun. Of course, a body on a halo orbit spends more than half of its period on the more elongated part of the orbit. This gives rise to another classification of halo orbits called

Northern/Southern halo orbit
A body on a Northern/Southern halo orbit spends more than half of its period above/below the x–y plane (in z-direction), i.e., North/South of it.

Whether a Class I or Class II halo is a Northern or Southern halo depends on whether the halo orbit is elongated inbound or outbound. Table 11.3 summarizes this mapping between the two halo classifications and gives the sense of rotation of the halo orbit as seen from the major and minor primary.

For instance, in Fig. 11.19 we have a Northern (Class II) halo orbit at L_2. This is because the upper turning point of the halo orbit lies more on the positive side of the x-axis (i.e., Class II type) and because the orbit is elongated outbound (positive x-axis) at ML_2, which is above the x–y plane (i.e., Northern type). The sense of rotation as seen from Earth (major primary) is clockwise. On the other hand we have in Fig. 11.23 for the SOHO probe at the EL_1 point a Southern (Class II) halo

Table 11.3 Relation of class-type and Northern/Southern-type at the three collinear points and sense of rotation (SOR) as seen from the major and minor primary

	L_1	L_2	L_3
Class I	**Northern**	**Southern**	**Northern**
SOR seen from major primary	Anti-clockwise	Anti-clockwise	Clockwise
SOR seen from minor primary	Clockwise	Anti-clockwise	Clockwise
Class II	**Southern**	**Northern**	**Southern**
SOR seen from major primary	Clockwise	Clockwise	Anti-clockwise
SOR seen from minor primary	Anti-clockwise	Clockwise	Anti-clockwise

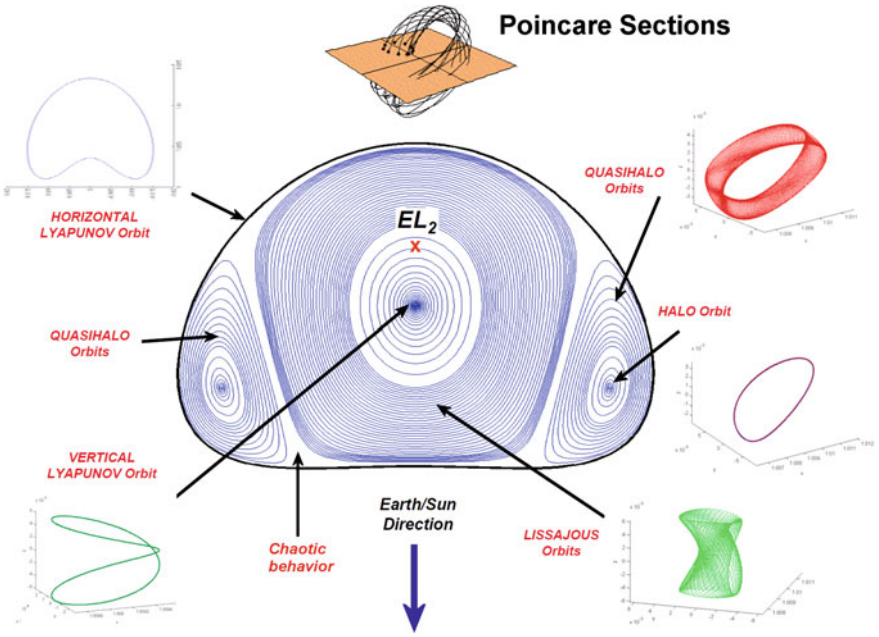


Fig. 11.22 The Poincaré maps of center manifolds orbits about a collinear libration point. *Credit* Egemen Kolemen (2012)

orbit, because the upper turning point lies closer to Earth (Fig. 11.23, middle), i.e., more on the positive side of the x-axis and its elongated part being on the Southern side. The sense of rotation as seen from Earth (minor primary) is anti-clockwise (Fig. 11.23, bottom).

Summary of Center Manifolds

Figure 11.22 gives an overview of the various periodic and quasi-periodic orbits forming the center manifolds about a collinear libration point by presenting their Poincaré maps. A Poincaré map of an orbit is the set of all points where the orbit intersects a given plane, in this case the x–y plane. This allows viewing a three-dimensional orbit in only two dimensions while preserving the essential features of the orbit.

Case Study: SOHO's Halo Orbit around EL_1 For a probe that resides in a halo orbit the SOHO mission about EL_1 is a good example (see Fig. 11.23). SOHO (Solar and Heliospheric Observatory) is an ESA/NASA space probe to study the Sun in the optical and ultraviolet spectrum at the Sun–Earth L_1 point, a.k.a. EL_1. Launched on December 2, 1995, it was planned as a 2-year mission, but mission extension lasting at least until December 2012 was approved in 2009.

From a communication point of view positioning SOHO at EL_1 would have been favorable, because there the distance to Earth is constant, as well as the S-band antenna would need merely a constant pointing to Earth. Being directly at the L_1 point, though, SOHO would be located in the line of view as seen from Earth. Therefore, the Sun radiation would jam the S-band communication. Therefore residing in this solar exclusion zone (SEZ) would render impossible any communication with Earth. The chosen Southern Class II-type halo orbit around EL_1 is a perfect solution to this problem because its extension is just large enough to circumscribe the SEZ (see Fig. 11.23, bottom) but on the other hand is small enough to require a one-dimensional antenna pointing angle of only maximal $\pm 30^\circ$ in the y-direction to Earth.

We will now draw on the SOHO mission to understand the basic physical background of its halo motion rather than just accept it as a result of the equations of motion. Let us assume that SOHO moves at position vector $\boldsymbol{\rho}$ on the halo orbit in the synodic system of the Earth rotating with ω around the Sun. SOHO's motion in the configuration plane causes in-plane Coriolis forces $\boldsymbol{F}_{Cor} = -2m(\boldsymbol{\omega} \times \dot{\boldsymbol{\rho}})$ vertical to $\boldsymbol{\rho}$. To visualize the results of this, we examine Fig. 11.23. The projection of the halo motion in the plane perpendicular to the configuration line (y–z plane) at L_1 has a component in y-direction that is perpendicular to the Sun–Earth configuration line. Let us assume this sideway motion is initially in the direction of the revolving Earth equaling the upward arrow of the SOHO orbit in the upper part of Fig. 11.23. Then this slightly increased orbit speed yields a slightly increased centrifugal force and hence a deflection in x-direction, outward toward the Earth (This is just the Coriolis effect.). According to Eq. (7.2.15) $v = \sqrt{\mu(2/r - 1/a)}$. So an increasing distance r to the Sun implies a reduction of the orbital speed until the satellite reverses its motion in y-direction. Beyond the L_1 distance its centrifugal force is smaller than needed to balance the Sun's gravitational force. The satellite therefore begins to move toward the Sun, which in turn again increases the orbital speed. This brings the satellite back to its initial point. In summary, the body has described a nearly elliptic orbit in the x–y plane. In total we therefore have a three-dimensional halo orbit: The orbit in the y–z is an ellipse and via the Coriolis forces its y-component induces in the x–y plane an orbit with an almost elliptic shape. Thus Coriolis forces are responsible for the stability of the two center-type solutions. The saddle-type solution brings about the 1-dimensional instability of a halo orbit (see next sub-section) rooted in the 1-dimensional instability of the corresponding collinear equilibrium point.

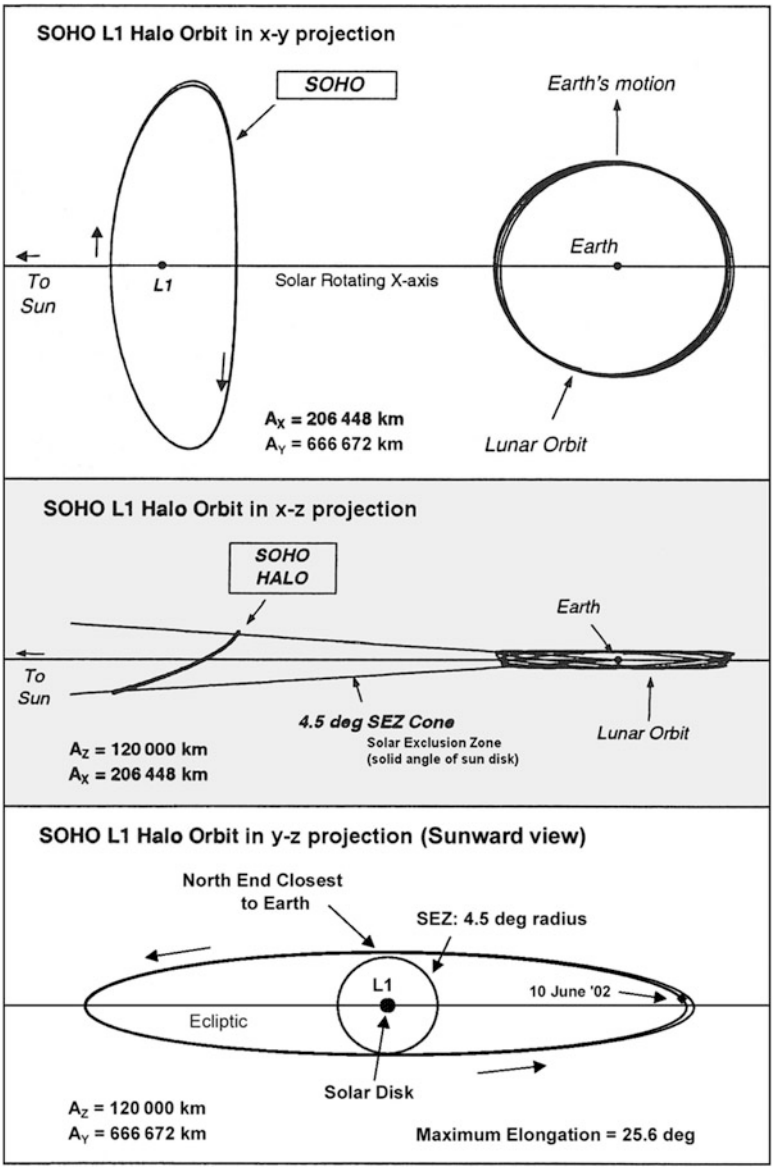


Fig. 11.23 The Southern Class II-type halo orbit of SOHO around the L_1 point in the Sun–Earth system spanning 413 × 1333 × 240 milllion m in x-, y-, and z-direction, respectively, and its projection onto the x–y plane (ecliptic plane, above), x–z plane (side view, middle), and y–z plane (sunward view, below). *Credit* ESA

For small deflection amplitudes the motions in the two perpendicular planes are not synchronous and Lissajous orbits result. For larger amplitudes $A_x \approx$ 200.000 km the increasing non-linearity of the effective potential lead to a synchronization of the two elliptic orbits, which results in a single elliptic orbit forming an certain angle with the configuration plane (see middle part of Fig. 11.23), whereby the line of nodes is the y-axis. This is the halo orbit.

Stable and Unstable Hyperbolic Manifolds

In the following explications we restrict the investigation of invariant manifolds mostly to trajectories in the x–y plane, i.e., assume that $A_z = 0$, because it already captures all their characteristics. For $A_z \neq 0$ the trajectories would exhibit just an additional independent vertical oscillation.

All the above periodic and quasiperiodic orbits of the center manifolds inherit the instability of the corresponding collinear equilibrium point (see Sect. 11.4.3). The instability is reflected in the saddle solutions $s = \pm\lambda_+$ of the linearized EoM (11.5.7) and occurs if $A_x \neq 0$. If in addition $A_1 = 0$, $A_2 \neq 0$ we have so-called **stable manifolds**, i.e., trajectories winding onto periodic center manifolds about L (see green lines in Fig. 11.25 left and right). They never truly arrive on that center manifold in finite time, but rather come asymtotically close to it as $A_2 e^{-s\tau} \to 0$. For $A_1 \neq 0$, $A_2 = 0$ we get **unstable manifolds**, i.e. trajectories winding asymptotically off center manifolds, as $A_1 e^{s\tau} \to \infty$ (see red lines in Fig. 11.25 left). Stable and unstable hyperbolic manifolds are obviously unbounded orbits.

As a specific example of hyperbolic manifolds we take those connecting to the halo orbit at the EL_1 point. For a stable manifold, i.e., a transition to that orbit, the spacecraft initially is either in a parking orbit around Earth from which a small injection burn (delta-v) will move it into the stable manifold, thus arriving at the halo orbit without any additional maneuver. Or, and quite frequently, the S/C is placed initially, as shown in Fig. 11.24, in a highly eccentric orbit about the Earth with perigee typically about 500 km and apogee at about 1.2 million km and hence at the Earth's SOI (so-called weak stability boundary, see Sect. 9.6, Fig. 9.22). In the synodic Sun–Earth system the apse line of this initial elliptic orbit rotates with the orbital period of the Earth around the Sun. The solar gravity disturbs this orbit by lifting its apogee. The extent of this perturbation strongly depends on the orientation of the apse line relative to the Sun–Earth direction. For every initial apse line orientation we thus find a different stable manifold trajectory (red lines in Fig. 11.24) onto the halo orbit at EL_1.

Transit and Non-transit Orbits

From Fig. 11.24 it is obvious that all possible stable manifold trajectories form a tube. This holds also for unstable manifold trajectories. Thus, hyperbolic manifolds geometrically are tubes, sometimes called tunnels, that partition the energy manifold and act as separatrices for the flow through the equilibrium region as shown in Fig. 11.25: Those inside the tubes transit from one side of the L region to the other and hence are called **transit orbits**, and those outside the tubes, so-called

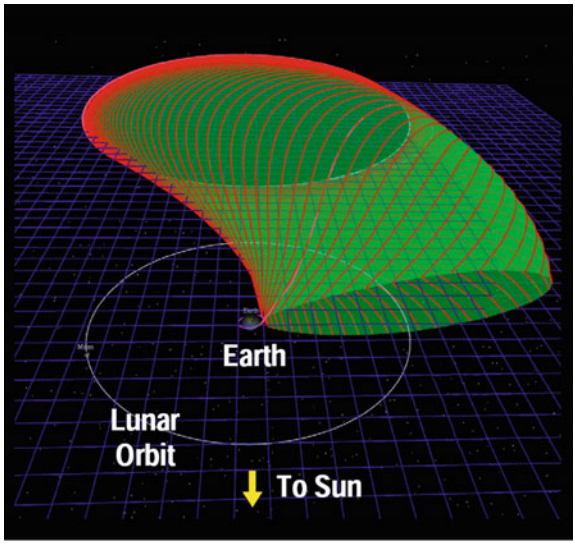


Fig. 11.24 Example of stable manifold trajectories (red lines) winding onto a halo orbit about the Sun–Earth libration point EL_2. Under the influence of the Sun's gravitational force an initially high elliptic orbit (lower end of tube) with perigee at about 500 km and apogee at about 1.2 million km will undergo a transition onto the halo orbit. The concrete trajectory depends on the slightly different initial conditions of the initial orbit. *Credit* Martin W. Lo (2001)

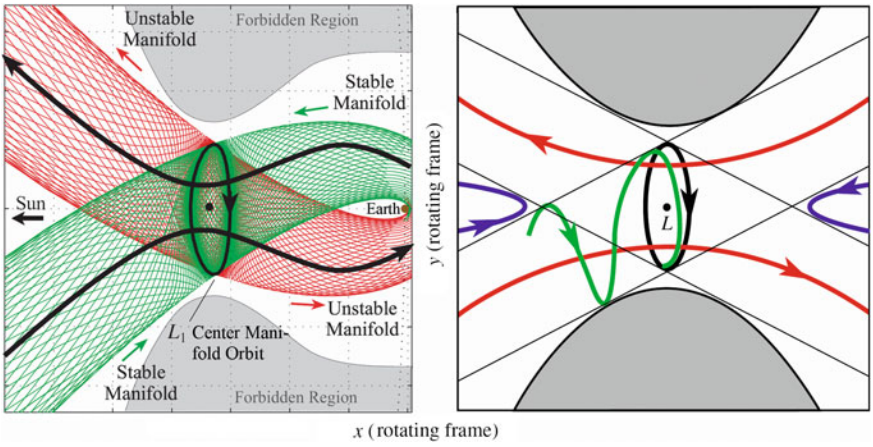


Fig. 11.25 Left: Shown are the projections of trajectories of the stable manifold (green) onto, and unstable manifold trajectories (red) off a center manifold orbit about EL_1. Right: Unbounded orbits in the vicinity of a L region. Shown are transit orbits (red), non-transit orbits (blue) and a trajectory of a stable manifold (green). The parallel lines delineate the manifold tubes. In the left picture the black arrows indicate smooth transit orbits. *Credit* Fig. 11.25 left, Martin W. Lo (2001); Fig. 11.25 right, reproduced from Koon (2000) with the permission of AIP Publishing

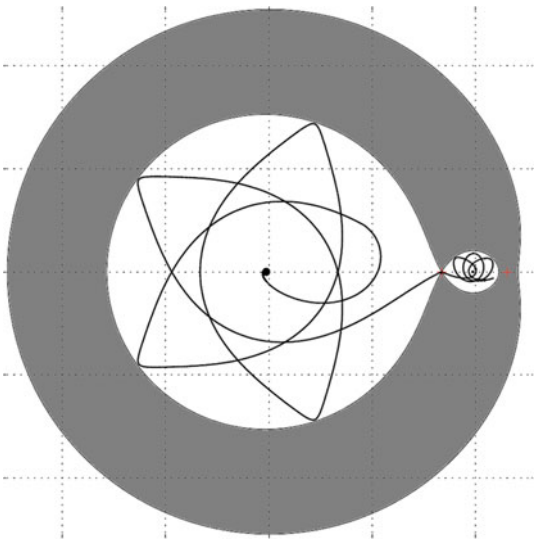


Fig. 11.26 Example for an L_1 transit Earth–Moon in the synodic reference frame

non-transit orbits, are reflected from the L region. If even $A_x = 0$, we get stable transit and non-transit orbit that smoothly transit the L region (see Fig. 11.25 right) or are smoothly reflected from it.

As an interesting example, Fig. 11.26 shows an Earth–Moon transit orbit at L_1 with a ballistic capture at the Moon. Since $v \approx 0$ at L_1, the trajectory has $A_x \approx 0$ at that point, meaning a smooth transition from the inner region to the Moon-bound region.

Once in the center manifold orbit, a small thrust (kick-burn) will bring the S/C into an unstable manifold drifting away from EL_2. The tubes of stable and unstable manifolds passing through the center manifold orbit about EL_1 are shown on the left side of Fig. 11.25.

Because all center manifolds inherit the instability of the corresponding collinear equilibrium point, some minor station-keeping effort is always required, which typically is of the order of 50 cm/s per year.

Heteroclinic and Homoclinic Orbits

Of course, similar windings exist onto and off center manifold orbits at EL_2. Therefore, there exist commuting orbits, as shown in Fig. 11.27, that transit on an unstable manifold out of a center manifold orbit about L_1, transverse Earth's region via a so-called heteroclinic (meaning "connecting") orbit, and converge on a stable manifold into a center manifold orbit about L_2—and back again. Such trajectories around the major primary that convey halo orbits from one libration point to another

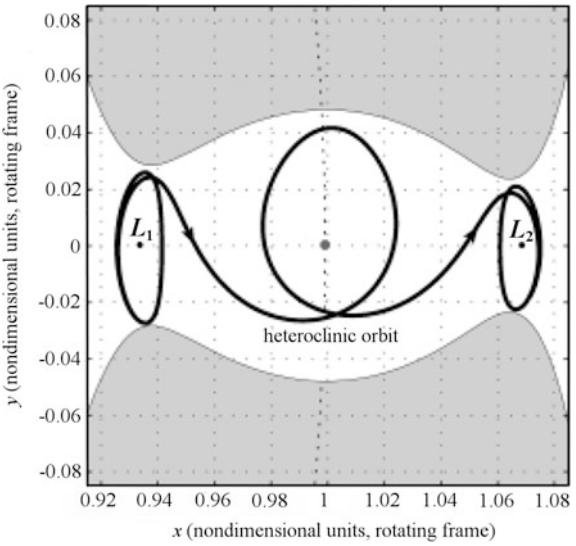


Fig. 11.27 A heteroclinic orbit connecting center manifold orbits at EL_1 and EL_2 via a stable and unstable manifold orbit. Reproduced from Koon (2000) with the permission of AIP Publishing

are called **heteroclinic orbits**. On the other hand, trajectories that wind off a Class I-type halo orbit, circle around the major primary, and wind onto a Class II-type halo orbit (or vice versa, see above) about the *same* libration point, i.e., which just change the sense of rotation in an halo orbit, are called **homoclinic orbits**.

Case Study: Genesis Mission to the Halo Orbit about EL_1 An interesting case in point was the Genesis mission where the invariant manifolds were utilized for a nearly powerless mission to and from the EL_1 point. The Genesis probe was launched on August 8, 2001, with the mission objective to collect a sample of solar wind atomic particles at EL_1 and return it to Earth for analysis. After about 3 years it returned to Earth where it unfortunately crashed into the desert floor due to a design flaw in a deceleration sensor, which should have triggered the parachute deployment. Figure 11.28 depicts the Genesis mission trajectory into a "halo orbit" (actually a Lissajous orbit) about EL_1 and then via a heteroclinic orbit into a near halo orbit at EL_3 to adjust for a daylight reentry into Earth's atmosphere.

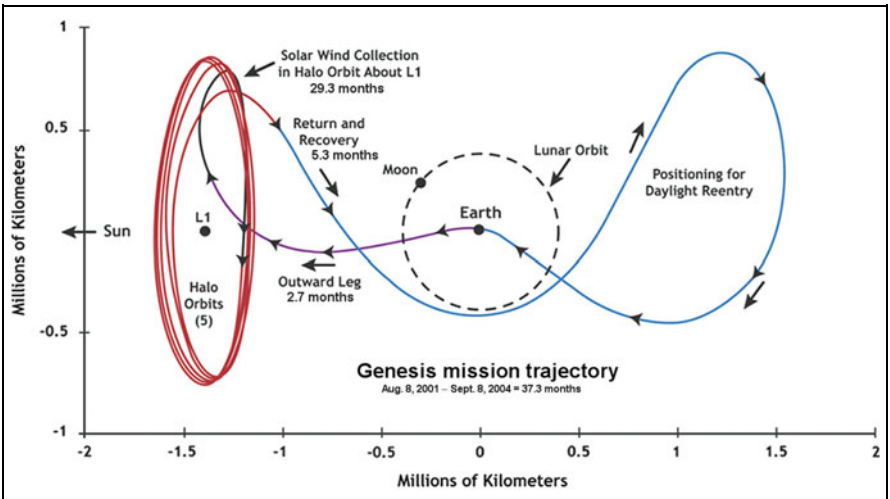


Fig. 11.28 The Genesis mission trajectory projected onto the x–y plane. *Credit* NASA

11.5.3 Equilateral Libration Points

We now consider the detailed dynamics in the vicinity of equilateral libration points L_4, L_5.

Equation of Motion Near Equilateral Libration Points

By the same token as in Sect. 11.5.2 we want to study the dynamics in the vicinity of the equilateral libration points and evaluate the right-hand side of Eq. (11.5.4) in powers of r. We therefore expand the expression for the distances to the equilateral libration points

$$\Delta r_i = \sqrt{(x - x_i)^2 + (y - y_i)^2 + z^2} = \sqrt{x_i^2 + y_i^2 - 2(x_i x + y_i y) + u^2}$$

with $u = \sqrt{x^2 + y^2 + z^2}$. According to Table 11.2 for equilateral libration points holds $x_i^2 + y_i^2 = (\mp 1/2)^2 + \left(\pm\sqrt{3}/2\right)^2 = 1$. Identifying $\alpha_i \equiv (x_i x + y_i y)/u$ the expansion reads

$$\frac{1}{\Delta r_i} = \frac{1}{\sqrt{1 - 2\alpha_i u + u^2}} = 1 + \sum_{n=1}^{\infty} u^n P_n(\alpha_i)$$

The equation of motion (11.5.4) then reads

$$\begin{pmatrix} x'' - 2y' - x \\ y'' + 2x' - y \\ z'' \end{pmatrix} = \boldsymbol{\rho}_L + \frac{\partial}{\partial \boldsymbol{r}} \left(\frac{1-\mu}{\Delta r_1} + \frac{\mu}{\Delta r_2} \right)$$
$$= \boldsymbol{\rho}_L + \frac{\partial}{\partial \boldsymbol{r}} \sum_{n=1}^{\infty} u^n \left[(1-\mu) P_n \left(\frac{x_1 x + y_1 y}{u} \right) + \mu P_n \left(\frac{x_2 x + y_2 y}{u} \right) \right]$$

For $n = 1$ $P_1(\alpha) = \alpha$ and considering Table 11.2 we have

$$\frac{\partial}{\partial \boldsymbol{r}} [(1-\mu)(x_1 x + y_1 y) + \mu(x_2 x + y_2 y)]$$
$$= \begin{pmatrix} (1-\mu)x_1 + \mu x_2 \\ (1-\mu)y_1 + \mu y_2 \\ 0 \end{pmatrix} = \begin{pmatrix} -0.5 + \mu \\ \mp\sqrt{3}/2 \\ 0 \end{pmatrix} = -\boldsymbol{\rho}_L$$

When calculating also the term with $n = 2$ (exercise) we get

$$\begin{aligned} &x'' - 2y' - \tfrac{3}{4}x \mp \tfrac{3\sqrt{3}}{4}(1-2\mu)y = 0 \\ &y'' + 2x' - \tfrac{9}{4}y \mp \tfrac{3\sqrt{3}}{4}(1-2\mu)x = 0 \\ &z'' + z = 0 \end{aligned} \qquad \textbf{EoM near } \boldsymbol{L_4, L_5} \qquad (11.5.9)$$

These are the linearized equations of motion in the vicinity of the two equilateral libration points, whereby the upper minus sign applies for L_4 and the lower plus sign for L_5. We recall that in literature L_4 and L_5 might be labeled inversely.

Invariant Manifolds Near Equilateral Libration Points

Upon solving these equations of motion we realize that as in the collinear equilibrium case the third equation is decoupled and describes a harmonic oscillator about the x–y plane, with frequency $\omega_z = n = \sqrt{G(m_1 + m_2)/d^3} = 2\pi/T$ i.e., in step with the synodic revolution frequency. For the two coupled equations we again make the standard ansatz $x = x_0 \exp(\lambda\tau)$, $y = y_0 \exp(\lambda\tau)$ and find for the characteristic polynomial

$$\lambda^4 + \lambda^2 + \frac{27}{4}\mu(1-\mu) = 0$$

of the differential equation matrix

$$\boldsymbol{A} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 3/4 & c & 0 & 2 \\ c & 9/4 & -2 & 0 \end{pmatrix}$$

which satisfies $\boldsymbol{X}' = \boldsymbol{A}\boldsymbol{X}$, where $\boldsymbol{X} = (x, y, x', y')$, with $c = \pm 3\sqrt{3}(1-2\mu)/4$, where the upper plus sign is for the L_4 point and the lower minus sign is for the L_5 point. Considered as a quadratic characteristic equation for λ^2 the terms in the characteristic equation do not exhibit any sign changes. According to Descartes' rule of signs this implies that there are no positive roots and hence only two negative roots, $\lambda_{1,2}^2 < 0$. By defining $\omega_{1,2} = \sqrt{-\lambda_{1,2}^2}$ we get

$$\omega_{1,2} = \frac{n}{\sqrt{2}}\sqrt{1 \pm \sqrt{1 - 27\mu(1-\mu)}} \tag{11.5.10}$$

with $n = \sqrt{G(m_1 + m_2)/d^3} = 2\pi/T$ the orbital frequency of the synodic system. Because the solutions of the characteristic polynomial are all imaginary the general solutions of the linearized EoM (11.5.9) can be generally written as (see also Murray and Dermott (1999) or Roy (2005))

$$\begin{aligned} x(t) &= \alpha_0 + \alpha_1 \sin(\omega_1 t + \varphi) + \alpha_2 \sin(\omega_2 t + \varphi) \\ y(t) &= \beta_0 + \beta_1 \cos(\omega_1 t + \phi) + \beta_2 \cos(\omega_2 t + \phi) \end{aligned} \tag{11.5.11}$$

From this we recognize that $\omega_{1,2}$ are the angular frequencies of two modes, $i = 1, 2$, with amplitudes α_i, β_i and phases φ, ϕ, which are determined from the initial conditions. As in the case of the collinear equilibrium the amplitudes are coupled via $\omega_{1,2}$ and the curvature of the potential mirrored by c, reducing the number of free parameters to be determined by the initial conditions to four. The two modes are periodic motions about the equilateral libration points, if and only if both angular frequencies are truly real. This is the case if in Eq. (11.5.10) the radicand $1 - 27\mu(1-\mu) \geq 0$, implying $\mu \leq \left(27 - 3\sqrt{69}\right)/54 = 0.0385$, in turn implying

$$m_1 \geq \frac{25 + 3\sqrt{69}}{2} m_2 = 24.96\, m_2 \tag{11.5.12}$$

This condition corresponds to a minimal curvature of the effective potential at the equilateral libration points, which is necessary to cause enough acceleration and hence speed and Coriolis force to curve the body on a bounded periodic orbit. As all the Sun–planet and planet–Moon constellations in our solar system fulfill condition Eq. (11.5.12), all equilateral libration points in our solar system are dynamically stable. This is true even for the Earth–Moon system where $m_{Earth} = 81.30094\, m_{Moon}$. Actually, as of January 2011, 4790 asteroids, so-called Trojans, were found at the equilateral libration points of the Sun–Jupiter (4779), Sun–Mars (4) and Sun–Neptune (7) systems. The first discovered and most famous is Achilles, which moves in a bounded orbit about Jupiter's L_4 point.

How do these two periodic modes look like? Because $\mu \ll 1$, we can approximate Eq. (11.5.10) to

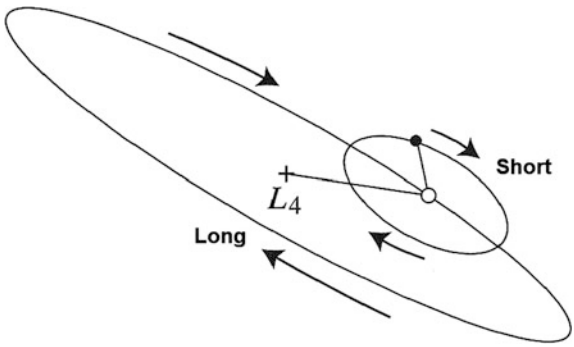


Fig. 11.29 The epicyclic orbit of a body (full dot) around L_4, which may be considered as motion on a short-term ellipse, the center of which (open dot) in turn moves on a long-term ellipse around L_4. *Credit* Murray and Dermott (1999)

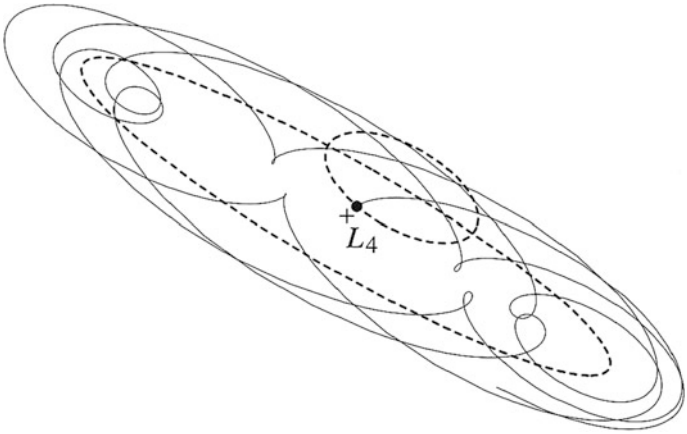


Fig. 11.30 The epicyclic motion (full line) of a body with Jacobi constant $C = -1.563$ around the equilateral libration point L_4 in the Earth–Moon system, as a composite of the two basic modes (dashed line) over 13 orbital periods. *Credit* Murray and Dermott (1999)

$$\omega_1 = n\left(1 - \frac{27}{8}\mu\right) \approx n$$
$$\omega_2 = n\sqrt{\frac{27}{4}\mu}$$

Hence there is one short-term mode with period $2\pi/\omega_1 \approx T$ and semi-major axis α_1, β_1 and a long-term mode with period $2\pi/\omega_2 = T\sqrt{4/(27\mu)}$ and semi-major axis α_2, β_2. The total movement can be considered as a short-term elliptic epicycle with semi-axis ratio $\alpha_1/\beta_1 = \sqrt{3\mu}$, moving on a long-term ellipse with semi-axis ratio $\alpha_2/\beta_2 = 1/2$ around the equilateral libration point (see Fig. 11.29). This composite motion is displayed in Fig. 11.30. The trajectory may (but does not

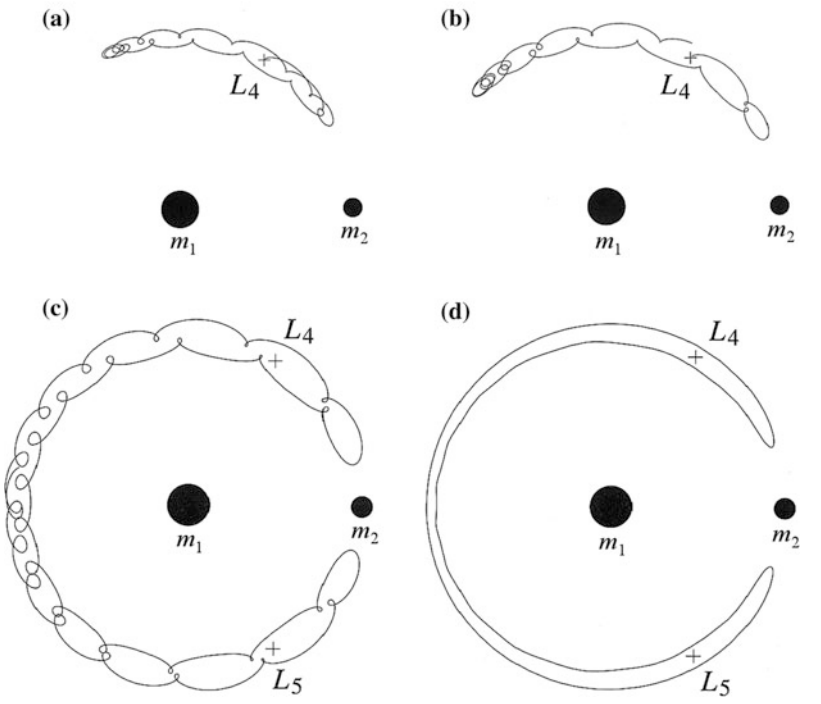


Fig. 11.31 Librations about L_4/L_5 points in the synodic frame with varied initial conditions (see text): (a and b) tadpole orbits over 13 synodic periods for $\mu = 0.001$ matching almost the Sun–Jupiter system; and (c and d) horseshoe orbits for $\mu = 9.53875 \times 10^{-4}$ equaling the Sun–Jupiter system. *Credit* Murray and Dermott (1999)

have to) touch the zero-velocity curve at some points with zero velocity, but cannot cross it. The smaller the excluded Hill zone around the equilateral libration point, the closer is the trajectory around the equilateral libration point.

In summary, bounded periodic orbits affected by Coriolis forces exist both about collinear and equilateral libration points. In physics a periodic motion around a equilibrium point is denoted as "libration", the term "libration points" just stems from the existence of such periodic orbits around these points.

Tadpole and Horseshoe Orbits

The epicyclic motion described above occurs only if it does not deviate too far from the equilateral libration point. How does an orbit look like if its excursions become bigger? In this case approximate analytical solutions cannot be provided any more. But numerical solutions of the EoM (11.4.10) show that orbits with decreasing orbital energies and increasingly initial tangential velocities become more and more elongated. Figure 11.31a, b depictures the trajectories of a body starting in Fig. 11.31a

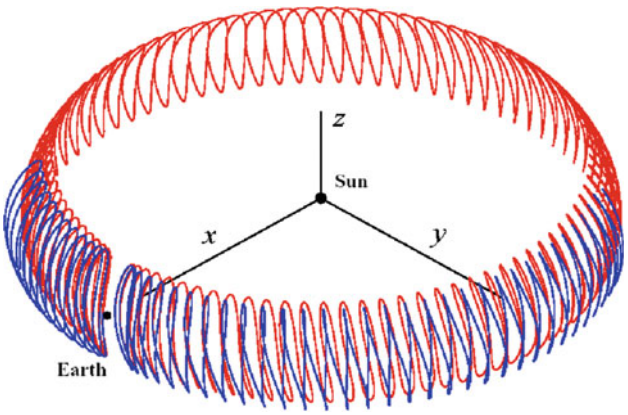


Fig. 11.32 Three-dimensional representation of the horseshoe orbit of asteroid 2002 AA29 in the synodic Sun–Earth system. The looping is caused by a finite value of the z-component of the initial velocity. The blue trajectory is the horseshoe orbit after reversal at the end points of the horseshoe. *Credit* M. Connors, Athabasca University, and © Wiley

with $\dot{\boldsymbol{\rho}}_0 = 0$ at $\boldsymbol{\rho}_0 = (\xi_{L4} + 0.0065,\ \eta_{L4} + 0.0065,\ 0)$ and in Fig. 11.31b with $\dot{\boldsymbol{\rho}}_0 = 0$ at $\boldsymbol{\rho}_0 = (\xi_{L4} + 0.008,\ \eta_{L4} + 0.008,\ 0)$, i.e., close to L_4 with $C \approx -\Omega_{L4}$. Because they depart fairly far from the equilibrium points and owing to their shape they are called *tadpole orbits*. While the short-term mode of motion is preserved, the semi-major axis of the long-term ellipses stretches to a circular arc along the circular orbit of m_2. If the Jacobi constant is decreased and/or increasing initial velocity is more tangential the tadpole orbits elongates and arcs until both meet at the L_3 point and then merge: A so-called *horseshoe orbit* has formed as depicted Fig. 11.31c for $\dot{\boldsymbol{\rho}}_0 = (0,\ -0.06118,\ 0)$ at $\boldsymbol{\rho}_0 = (\xi_{L3} + 0.02372,\ 0,\ 0)$, i.e., close to L_3. For suitable initial tangential velocities the amplitude of the short-term mode can be suppressed forming a smooth horseshoe orbit as shown in Fig. 11.31d for $\dot{\boldsymbol{\rho}}_0 = (0,\ -0.04032,\ 0)$ at $\boldsymbol{\rho}_0 = (\xi_{L3} - 0.02705,\ 0,\ 0)$. In this case the object moves with a nearly constant velocity on an equipotential line in the system of the two primaries (see e.g., Fig. 11.11), which encompasses the L_3, L_4, and L_5 points. If the initial condition is such that the suitable tangential velocity has a component in z-direction that is out of the configuration plane then vertically looping horseshoe orbits occur as depicted in Fig. 11.32.

All Jupiter and Mars Trojans known so far move on tadpole orbits. But only recently the asteroids 3753 Cruithne, 2002 AA29, and 2010 SO16 were found in the Sun–Earth system as three examples of horseshoe orbits. However, 3753 Cruithne moves on a horseshoe orbit with a high eccentricity, $e = 0.515$, and high inclination, $i = 19.81°$, to the ecliptic, which is why it is sometimes not counted as a horseshoe object. In addition, in 1980 Voyager 1 found the two equally massive asteroids Janus and Epimetheus with identical orbital radii around Saturn, which move on horseshoe orbits relative to each other. Trojans on tadpole orbits, asteroids on horseshoe orbits,

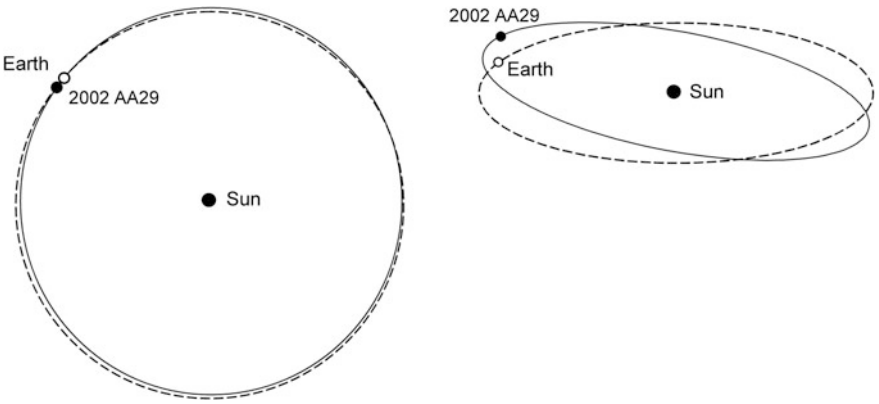


Fig. 11.33 The orbit of asteroid 2002 AA29 in the heliocentric system in the ecliptic (left) and viewed oblique (above)

and so-called quasi-satellites are called **co-orbital objects**, because they move in the same or nearly the same orbit of a celestial body around a central body.

With all these complex co-orbital objects it should be remembered that in the inertial barycentric system of the primaries the test mass always moves on elliptic orbits around the barycenter, which deviate just slightly from the minor primary. Only the transition into the synodic system results in the complex relative motions just discussed. In this way the horseshoe orbit from Fig. 11.32 corresponds to the full elliptic orbits in the inertial heliocentric system in Fig. 11.33.

11.6 Problems

Problem 11.1 *Alternative Proof of Sundman's Inequality*
Alternatively to Sect. 11.1.2 prove Sundman's inequality Eq. (11.1.4)

$$E_{kin} \geq \frac{L^2}{2I}$$

by looking for the minimum of

$$E_{kin,\theta} = \frac{1}{2}\sum_i m_i v_{i,\theta}^2 \leq \frac{1}{2}\sum_i m_i \left(v_{i,\theta}^2 + v_{i,r}^2\right) = E_{kin}$$

with respect to the angular momentum of each body.

Problem 11.2 *Collinear Configuration*
Let Eqs. (11.2.1) and (11.2.2) be given. Prove that α is the unequivocal *positive* root of $\mu_3 = \alpha^3 \mu_2$.

Hint: cf. Guthmann (2000, pp. 242ff)

Problem 11.3 *Circularly Rotating Collinear Configuration*
Show that for a circularly rotating collinear configuration the orbital period is

$$T = 2\pi\sqrt{\frac{x_{12}^3}{\mu}\frac{m_1' - m_3'\chi}{m_1'\chi^2 - m_3'}} \quad \text{with } \mu = GM = G\left(m_1' + m_2' + m_3'\right)$$

Hint: Starting from the center of mass equation $m_1'x_1 + m_2'x_2 + m_3'x_3 = 0$ show that

$$x_2 = x_{12}\frac{m_1' - m_3'\chi}{M}$$

and finally because of $n^2 = \mu_i/x_i^3$ the wanted result.

Problem 11.4 *Circularly Rotating Equilateral Configuration (hard)*
Show that for a circularly rotating equilateral configuration $\Delta r = r_{12} = r_{23} = r_{13} = const$ Eq. (11.2.11)

$$n = \frac{2\pi}{T} = \sqrt{\frac{\mu_i}{r_i^3}} = \sqrt{\frac{\mu}{(\Delta r)^3}}$$

holds.

Hint: First show $n = \sqrt{\mu_i/r_i^3}$, and then $\mu_i/r_i^3 = \mu/(\Delta r)^3$ by using the center of mass equation $m_1\boldsymbol{r}_1 + m_2\boldsymbol{r}_2 + m_3\boldsymbol{r}_3 = 0$.

Problem 11.5 *Collinear Libration Points (tedious but good practice)*
(a) For the L_3—point the condition equation for Δ_3 is Eq. (11.3.4). By making the ansatz $\Delta_3 = 1 + a\mu + c\mu^3$ and assuming that the term of order $b\mu^2 = 0$, prove that

$$a = -\frac{7}{12}, \quad c = -\frac{12103}{13824}$$

(b) For L_1 and L_2 points make the ansatz $\Delta_i = 1 + a_i\lambda + b_i\lambda^2 + c_i\lambda^3$ with $\lambda = (\mu/3)^{1/3}$ and show that by inserting into the according condition Eqs. (11.3.5) and (11.3.6) one gets $a_1 = -a_2 = -1$, $b_1 = b_2 = 1/3$, and $c_1 = -c_2 = 1/9$.

Problem 11.6 *Jacobi Constant Approximations*
By applying Eq. (11.3.7), show that for the Jacobi constant of a stationary test mass in the synodic system at the libration points the following power series expansions up to order $O(\mu)$ hold

$$C_{L1} = -\frac{3}{2} - \frac{3}{2}\lambda + 5\lambda^3 + \ldots$$
$$C_{L2} = -\frac{3}{2} - \frac{3}{2}\lambda + 7\lambda^3 + \ldots$$
$$C_{L3} = -\frac{3}{2} - \frac{3}{2}\lambda^3 + \ldots$$
$$C_{L4} = C_{L5} = -\frac{3}{2} + \frac{3}{2}\lambda^3 + \ldots$$

with

$$\lambda = \left(\frac{\mu}{3}\right)^{1/3}, \quad \mu = \frac{m_2}{m_1 + m_2}$$

Hint: Consider Eq. (11.4.9) for $v = 0$.

Chapter 12
Orbit Perturbations

12.1 Problem Setting

12.1.1 Origins of Perturbations

So far, we have studied the two-body problem, in which one body moves under a central Newtonian force, as given by a gravitational field of a second body, a point mass. This led to Keplerian orbits as the solutions of the respective equations of motion. However, in reality there are many external forces acting on the body, which are neither point symmetric nor Newtonian. For instance, a massive central body usually is not quite homogeneous and isotropic, which in general gives rise to non-central and non-Newtonian forces. In addition, the gravitational forces of other celestial bodies, in particular neighboring planets, or interactions with the space environment will perturb the Keplerian orbit around the central body. In total there exist the following major disturbing forces:

- gravitational forces resulting from the non-spherical geometry and mass distribution of the central body
- gravitational forces of other celestial bodies (such as the Sun, Moon, planets)
- acceleration force resulting from the solar radiation pressure
- acceleration force resulting from the drag of the remaining atmosphere

Figure 12.1 provides a graphical representation of all relevant perturbations acting on an Earth-orbiting S/C as a function of altitude. Obviously Earth's anisotropy generates various perturbation terms J_{nm} of different strength, and atmospheric drag decreases rapidly with increasing altitude. In addition, Table 12.1 gives an overview of the essential external perturbations giving rise to accelerations of the S/C in a LEO at 500 km altitude and in GEO for comparison.

Judged from their magnitudes the first four disturbing forces (except drag for GEO) have to be taken into account for real missions. The perturbations cause the

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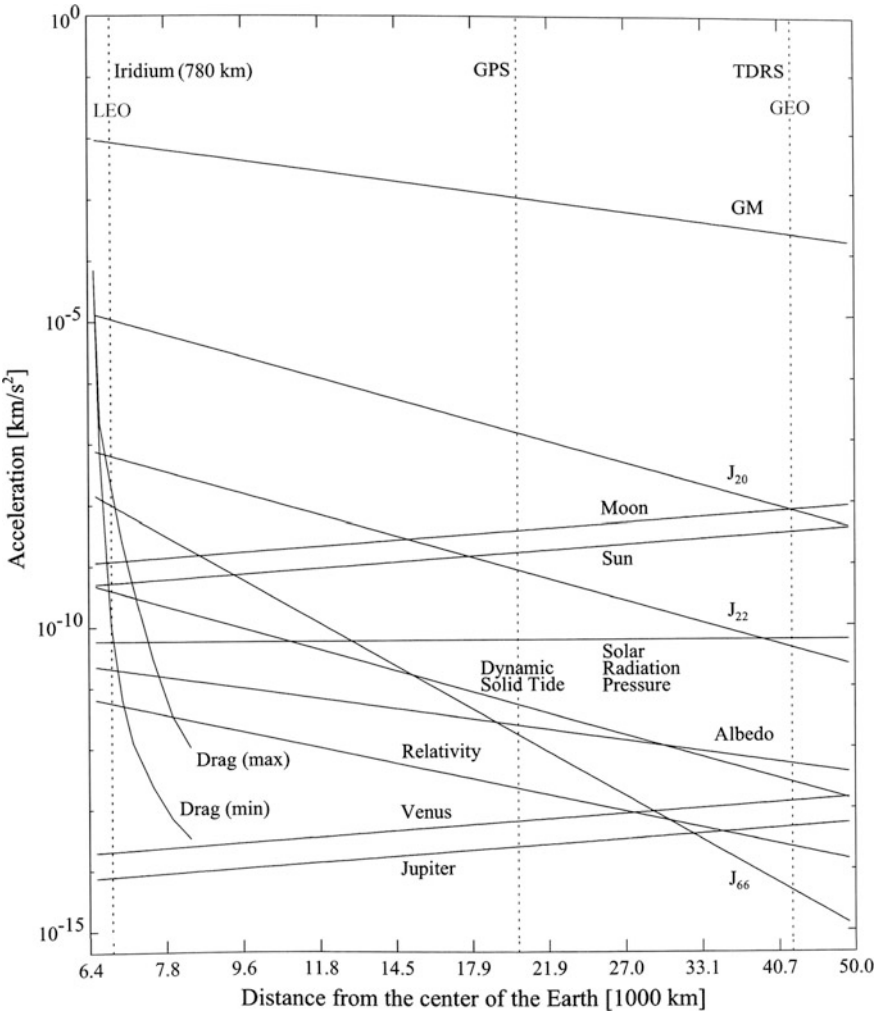


Fig. 12.1 Magnitudes of different perturbations of a satellite orbit: GM = regular gravitational force of the Earth; J_{nm} = gravitational multipoles; relativity = relativistic deviations; and the satellites Iridium, Lageos, GPS, TDRS at altitudes between LEO and GEO. *Credit* Montenbruck (2000)

orbits to be no longer Keplerian, but as long as the perturbations are small, which we can safely assume in the following, an orbit can be linearly approximated by a sequence of confocal Keplerian orbits, implying that the set of orbital elements

$$\vartheta_i \in (a, e, i, \Omega, \omega, M - n \cdot t)$$

Table 12.1 External perturbational accelerations on a S/C with a given $A_\perp/m$, where $A_\perp$ is the effective surface area perpendicular to the impinging force and m the mass of the S/C

Source of perturbation	Acceleration [m s^{-2}] in 500 km altitude	Acceleration [m s^{-2}] in GEO
Drag (mean)	$6 \times 10^{-5}\ A_\perp/m$	$1.8 \times 10^{-13}\ A_\perp/m$
Solar pressure	$4.67 \times 10^{-5}\ A_\perp/m$	$4.67 \times 10^{-5}\ A_\perp/m$
Sun (mean)	5.6×10^{-7}	3.5×10^{-6}
Moon (mean)	1.2×10^{-6}	7.3×10^{-6}
Jupiter (maximum)	8.5×10^{-12}	5.2×10^{-11}

changes with time from one Keplerian orbit to the adjacent one according to

$$\vartheta_i(t) \approx \vartheta_{i,0} + \dot{\vartheta}_i \cdot t \tag{12.1.1}$$

12.1.2 Osculating Orbits

Let us consider a concrete example. As we will see at the end of Sect. 12.3.3 the line of apsides rotates under the influence of Earth's equatorial bulge. By adopting a *PQW* coordinate system (see Fig. 13.3), where we orient the *P*-axis along the line of nodes, the orbit Eq. 7.3.5 then reads for a prograding apsidal line

$$r = \frac{p}{1 + e \cdot \cos[\theta - \omega(\theta)]} \tag{12.1.2}$$

So, at a given orbit angle θ_0, as measured against the line of nodes, the trajectory of this example can be approximated by the instantaneous Keplerian orbit

$$r = \frac{p}{1 + e \cdot \cos[\theta - \omega(\theta_0)]} \quad \textbf{osculating orbit} \tag{12.1.3}$$

which is called an *osculating orbit* (Latin "osculare" means "to kiss", see Fig. 12.2). Thus, any disturbed orbit is a continuous sequence of confocal osculating orbits. This osculation analysis reduces the perturbation problem to determining at a given point in time t_0 the elements $\vartheta_{i,0} \in (a_0, e_0, i_0, \Omega_0, \omega_0, M_0)$ of the osculating orbit (a.k.a. **osculating elements**), as we did in the above example, and finding the time derivatives $\dot{\vartheta}_i \in \left(\dot{a}, \dot{e}, \dot{i}, \dot{\Omega}, \dot{\omega}, \dot{M} - n\right)$ from theory.

Deriving the time derivatives for all major external perturbations in LEO and GEO as given in Table 12.1 is the objective of this chapter. To do so we need a general rule for how to calculate the derivatives from the perturbing forces. This is provided in the following section.

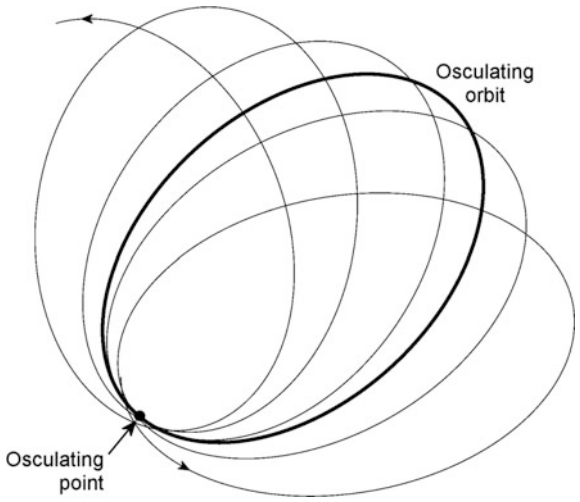


Fig. 12.2 An osculating ellipse (bold line) attached to an intermediate periapsis of a disturbed elliptic orbit (thin line, given by Eq. (12.1.2)) showing a progression of the apsidal line

12.1.3 Gaussian Variational Equations

Any perturbing force $\boldsymbol{F}_{perturb}$, or alternatively the corresponding acceleration $\boldsymbol{a}_{perturb} = \boldsymbol{F}_{perturb}/m$, acting on an orbiting body can be decomposed in the *RSW* coordinate system (see Fig. 13.4 in Sect. 13.1.4) into a radial component a_r, a cross-radial component a_θ, and into a component a_h perpendicular to the orbital plane, i.e., along the angular momentum $\boldsymbol{h}$, which perturbs the equation of motion according to

$$\ddot{\boldsymbol{r}} + \frac{\mu}{r^3}\boldsymbol{r} = \boldsymbol{a}_{perturb} = (a_r, a_\theta, a_h)_{RSW}$$

Most generally $\dot{\boldsymbol{\vartheta}}$ is derived by starting out from the general solution of the above differential equation having the functional form $\boldsymbol{r} = \boldsymbol{r}[\vartheta_i(t), t]$. Differentiation leads to $\dot{\boldsymbol{r}} = \boldsymbol{v} + \boldsymbol{\Phi}$, with $\boldsymbol{v} = \partial \boldsymbol{r}/\partial t$ the velocity of the perturbed orbit and $\boldsymbol{\Phi} = \sum_i \frac{\partial \boldsymbol{r}}{\partial \vartheta_i}\dot{\vartheta}_i$. It can be shown (see, e.g., Gurfil (2007) or Efroimsky (2006)) that by differentiating again and substituting the result into the above perturbed equation of motion one can derive $\dot{\boldsymbol{\vartheta}}$. Because this way of derivation is tedious and mathematically quite demanding (see, e.g., Vallado (2001), Schaub and Junkins (2003) or Danby (2003)) we will not attempt it here but defer to an exercise (Problem 12.1) for an ab initio approach. From any of these approaches one finds the so-called **Gaussian variational equations (GVEs)**

$$
\begin{aligned}
\dot{a} &= \frac{2a^2}{h}\left[e\,\sin\theta\,\cdot\,a_r + (1+e\,\cos\theta)a_\theta\right] \\
\dot{e} &= \frac{h}{\mu}\left[\sin\theta\,\cdot\,a_r + \left(\frac{e+\cos\theta}{1+e\,\cos\theta} + \cos\theta\right)a_\theta\right] \\
\dot{i} &= \frac{r\,\cos(\theta+\omega)}{h}a_h \\
\dot{\omega} &= \frac{h}{e\mu}\left[-\cos\theta\,\cdot\,a_r + \frac{2+e\,\cos\theta}{1+e\,\cos\theta}\sin\theta\,\cdot\,a_\theta\right] - \frac{r\,\sin(\theta+\omega)\cos i}{h\,\sin i}\,\cdot\,a_h \\
\dot{\Omega} &= \frac{r\,\sin(\theta+\omega)}{h\,\sin i}a_h \\
\dot{M} - n &= -\frac{h\sqrt{1-e^2}}{\mu}\left[\left(\frac{2}{1+e\,\cos\theta} - \frac{\cos\theta}{e}\right)a_r + \frac{\sin\theta}{e}\left(\frac{2+e\,\cos\theta}{1+e\,\cos\theta}\right)a_\theta\right] \\
&= -\sqrt{1-e^2}\left(\frac{2r}{h}a_r + \dot{\omega} + \dot{\Omega}\,\cos i\right)
\end{aligned}
\tag{12.1.4}
$$

Note *The elements $\dot{\vartheta}_i$ of the Gaussian variational equations become infinite or zero if accelerations act on orbits with $h \to 0$, i.e., on radial orbits or orbits close to those. In addition $\dot{\omega}$ and $\dot{M}$ become infinite if a_r or a_θ accelerations act on a circular orbit, $e \to 0$, and $\dot{\omega}$ and $\dot{\Omega}$ become infinite if an a_h acceleration acts on an orbit with $i \to 0$.*

Remark *The vector function $\mathbf{\Phi}$ is called* gauge function*, because it constitutes a gauge freedom, which means it can be chosen freely without having an impact on the result $\dot{\vartheta}_i$. As Lagrange already did, the so-called* Lagrange constraints *or* osculation constraints, $\mathbf{\Phi} = 0$*, are usually chosen, implying that the velocity vector of the perturbed orbit equals the one of the generating Keplerian orbit. This assumption, however, is fully arbitrary. Removing these constraints leads to the so-called* gauge-generalized equations*. Their evaluation is a very new and ongoing research and out of the scope of this book.*

In the following sections, we want to determine the perturbation accelerations listed above and calculate the corresponding variations of the orbital elements according to the Gaussian variational equations.

12.2 Gravitational Perturbations

12.2.1 Geoid

The gravitational field of the Earth is not absolutely isotropic, but has slightly different values in different directions. The reason is its non-spherical shape and its density variations within. The geometric body representing the corresponding asymmetrical gravitational field in the different directions is called *geoid* as depicted in Fig. 12.3. Strictly speaking, the geoid is the body representing the equipotential surface of the gravity potential at sea level, i.e., the surface to which the gravity vector g is always perpendicular. Graphically, the geoid is the equilibrium shape of the Earth, if its surface were fully covered with stationary water. In zero-order approximation Earth's masses are distributed evenly and the geoid is a sphere. The largest deviation from the sphere is caused by the rotation of the Earth, which displaces the masses to the equator due to the centrifugal forces. Earth's radius at the equator is 6378.14 km and hence is some 21.4 km longer than through the poles. Deducting a sphere from the geoid therefore results in a bumpy oblate spheroid, which never deviates more than 25 m from the true oblate spheroid (see Fig. 12.4, left).

Deducting the sphere and the oblate spheroid from the geoid results in a potato shaped figure (see Fig. 12.4, right, and color plates at the end of Sect. 12.2.2). Observe that its heights and downs do not follow the contours of the continents. Rather the Earth's interior is viscous with a high proportion of iron, magnesium, nickel, silicon, and oxygen. Temperature gradients between the hot interior of about

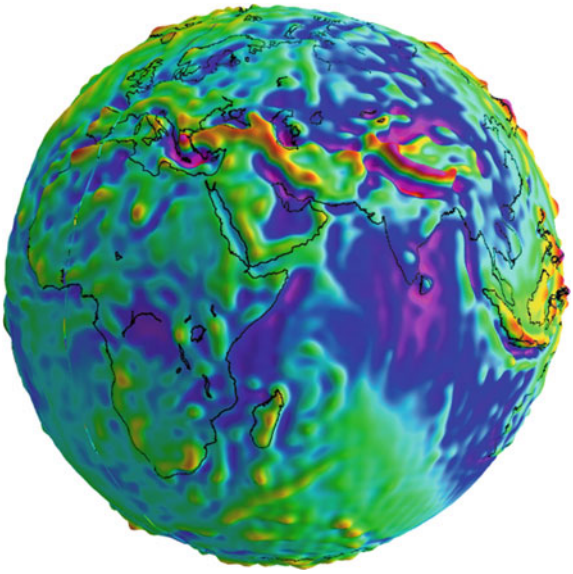


Fig. 12.3 Geoid of the Earth. *Credit* GFZ Potsdam

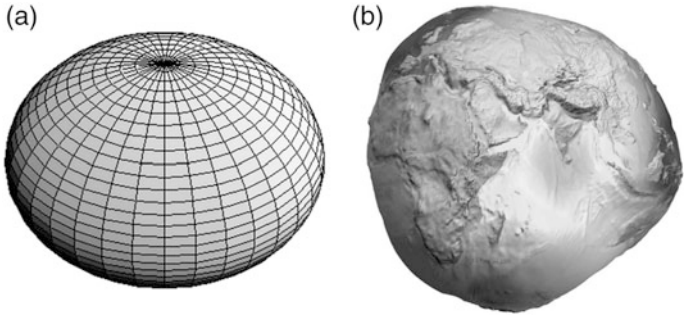


Fig. 12.4 (**a**, left) Earth has a bulge of 21.4 km height around the equator deforming the geoid to an oblate spheroid (excessively represented). (**b**, right) There are only small deviations from this spheroid (+20 m to −25 m), which make up a shape like a potato (= geoid – spheroid). For the entire shape of the "potato" see color plates at the end of Sect. 12.2.2. *Credit* GFZ Potsdam

4500 °C and the surface of the Earth cause vertical convection flows on the one hand, and slightly asymmetrical density distributions on the other, which is reflected in the potato shape of the geoid. Note that Fig. 12.3 depicts the real geoid, whereas Fig. 12.4 only shows the anisotropies, if the sphere and the spheroid are successively deducted. The deviations from the spheroid caused by the Earth's mass inhomogeneities are called *geoid undulations*.

As mountains embody a large amount of mass, the fine-structured geoid undulations clearly reflect the mountain ranges, which mainly account for the more complicated structure beyond the potato shape of the geoid. Because the finest details of the geoid reflect the mass proportions below the Earth's surface, one is able to determine from these structures for instance oil fields.

12.2.2 Gravitational Potential

The orbit of a spacecraft circling the Earth is influenced by these gravitational anisotropies, leading to slightly deformed Keplerian orbits. According to Eq. (7.7.1) the gravitational potential of an arbitrary mass distribution $\rho(\boldsymbol{r})$ is given by

$$U(\boldsymbol{r}) = -G \iiint_V \frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} d^3\boldsymbol{r}' \tag{12.2.1}$$

Let us assume we know Earth's mass distribution and we want to evaluate the above integral. To do so it can be shown that the expression $1/|\boldsymbol{r} - \boldsymbol{r}'|$ can be approximated by a series of spherical orthogonal functions, the so-called Legendre polynomials $P_n(x)$ of degree n,

$$\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}=\frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\gamma)$$

where $\gamma=\angle(\boldsymbol{r},\boldsymbol{r}')$. We apply this result to the International Terrestrial Reference Frame (ITRF) (see Sect. 13.1.3), which is an Earth-centered Earth-fixed reference frame. Locations on Earth's surface are familiarly referenced by the geocentric latitude β (the spherical arc—i.e., without taking into account Earth's flattening—measured from the equator to the positive end of the z-axis) and geographic longitude λ (the arc from Greenwich meridian measured eastward, that is counterclockwise looking toward the origin from the positive end of the z-axis), i.e.,

$$\begin{aligned}\boldsymbol{r}&=r\begin{pmatrix}\cos\beta\,\cos\lambda\\ \cos\beta\,\sin\lambda\\ \sin\beta\end{pmatrix}_{ITRF}\\ &=r\begin{pmatrix}\cos(\Omega-\theta_{GMST})\cos u-\sin(\Omega-\theta_{GMST})\sin u\,\cos i\\ \sin(\Omega-\theta_{GMST})\cos u+\cos(\Omega-\theta_{GMST})\sin u\,\cos i\\ \sin u\,\sin i\end{pmatrix}_{ITRF}\end{aligned}$$

where the latter follows from Eq. (13.1.8b), θ_{GMST} is the *Greenwich Mean Sidereal Time*, which in the equatorial plane is the hour angle from the vernal point (First Point of Aries) to the Greenwich Meridian, and $u=\omega+\theta$ is the argument of latitude.

With these definitions the *spherical harmonic addition theorem* can be applied in the form

$$P_n(\cos\gamma)=P_n(\sin\beta)P_n(\sin\beta')+2\sum_{m=1}^{n}\frac{(n-m)!}{(n+m)!}P_n^m(\sin\beta)P_n^m(\sin\beta')\cos m(\lambda-\lambda')$$

where $P_n^m(x)$ are the so-called *(unnormalized) associated Legendre polynomials* of degree n and order m and $P_n^0\equiv P_n$. If the volume integral with the dashed coordinates is carried out over these functions weighted with the density $\rho(\boldsymbol{r}')$, we obtain (see, e.g., Kaplan (1976, p. 273)) expressions of the form $C_n^m P_n^m(\sin\beta)\cos m\lambda$ and $S_n^m P_n^m(\sin\beta)\sin m\lambda$, where the multipole coefficients C_n^m und S_n^m represent the performed integrals. By this procedure Eq. (12.2.1) can be written in the standard form, adopted by the International Astronomical Union (IAU) in 1961, as

$$U(r,\beta,\lambda)=-\frac{\mu_\oplus}{r}\left\{1+\sum_{n=1}^{\infty}\left(\frac{R_\oplus}{r}\right)^n\sum_{m=0}^{n}P_n^m(\sin\beta)\left[C_n^m\cos m\lambda+S_n^m\sin m\lambda\right]\right\}\tag{12.2.2}$$

with Earth's standard gravitational parameter $\mu_\oplus=3.9860044105\times10^5\ \mathrm{km^3\,s^{-2}}$ and Earth's mean equatorial radius $R_\oplus=6378.1363$ km (both values as from the Earth gravitational model EGM96). Note, from the condition in Sect. 7.1.1 that $r\geq$ maximal radial mass extension = mean equatorial radius $R_\oplus$ follows that the expansion Eq. (12.2.2) must be terms of $R_\oplus/r$. Because the anisotropic terms of the

gravitational potential are small, it is convenient to separate them from the spherical potential $-\mu_\oplus/r$ and lump them into the residual perturbational potential R

$$U(r,\beta,\lambda) = -\frac{\mu_\oplus}{r} - R(r,\beta,\lambda) \tag{12.2.3}$$

with

$$R(r,\beta,\lambda) = \frac{\mu_\oplus}{r}\sum_{n=2}^{\infty}\left(\frac{R_\oplus}{r}\right)^n\left[-J_n P_n(\sin\beta) + \sum_{m=1}^{n} J_{nm} P_n^m(\sin\beta)\cos m(\lambda-\lambda_{nm})\right]$$

Here the conventional definitions and substitutions for studying planetary perturbations

$$J_n := J_{n0} := -C_n^0 \quad m = 0 \tag{12.2.4}$$

$$\left.\begin{aligned} J_{nm}\cos m\lambda_{nm} &:= C_n^m \\ J_{nm}\sin m\lambda_{nm} &:= S_n^m \end{aligned}\right\} \quad m \geq 1$$

were applied resulting for $m \geq 1$ in

$$J_{nm} = \sqrt{\left(C_n^m\right)^2 + \left(S_n^m\right)^2} > 0 \quad \textbf{harmonic coefficients} \tag{12.2.5a}$$

$$\begin{aligned} m\lambda_{nm} &= \arctan\frac{S_n^m}{C_n^m} + \frac{\pi}{2}\left[1 - \operatorname{sgn}\left(C_n^m\right)\right]\cdot\operatorname{sgn}\left(S_n^m\right) \\ &= 2\arctan\frac{S_n^m}{C_n^m + J_{nm}} \end{aligned} \quad \textbf{equilibrium longitudes} \tag{12.2.5b}$$

with J_{nm} the so-called *harmonic coefficients* and λ_{nm} the *equilibrium longitudes*. The term with $\pi/2$ in Eq. (12.2.5a) and the alternative expression thereafter, derived from half the equilibrium longitude angle, ensures that, as required by Eq. (12.2.4), $\operatorname{sgn}(\cos m\lambda_{nm}) = \operatorname{sgn}\left(C_n^m\right)$ and $\operatorname{sgn}(\sin m\lambda_{nm}) = \operatorname{sgn}\left(S_n^m\right)$ and therefore $-180^\circ \leq m\lambda_{nm} \leq +180^\circ$.

Note *For the sake of clarity we recall that in this textbook a potential designated by the letter U is the classical physical potential, which satisfies the equation of motion* $\ddot{\boldsymbol{r}} = -\nabla U = \nabla(\mu/r) + \nabla R$. *In the literature, U is often defined with inverse sign, while the residual potential R is usually defined uniquely as in* Eq. (12.2.3). *The terms with* J_n *have opposite signs to the terms with* J_{nm} *that has historical reasons and is done throughout the literature and here.*

Legendre Polynomials

The Legendre polynomials $P_n(x) \equiv P_n^0(x)$ and associated Legendre polynomials $P_n^m(x)$ can be derived analytically and numerically via the recurrence relations

Recurrence iteration to derive $P_n(x)$ and $P_n^m(x)$

$s := \sin\beta = x$

$c := \cos\beta = \sqrt{1-x^2}$

$P_0 \equiv P_0^0 = 1$

Do $n = 1,\ \infty$

$nP_n = (2n-1)sP_{n-1} - (n-1)P_{n-2}$

$P_n^n = -(2n-1)cP_{n-1}^{n-1} = (-c)^n(2n-1)!!$

Do $m = n-1,\ 0,\ -1$

$(n-m)sP_n^m = cP_n^{m+1} + (n+m)P_{n-1}^m$

End Do

End Do

With this iteration we derive in the following the relevant associated Legendre polynomials $P_n^m(x)$ up to $n = 5$

$$
\begin{aligned}
&P_2 \equiv P_2^0 = \frac{1}{2}\left(3s^2-1\right),\ P_2^1 = -3sc,\ P_2^2 = 3c^2\\
&P_3 \equiv P_3^0 = \frac{1}{2}s\left(5s^2-3\right),\ P_3^1 = -\frac{3}{2}c\left(5s^2-1\right),\ P_3^2 = 15sc^2,\ P_3^3 = -15c^3\\
&P_4 \equiv P_4^0 = \frac{1}{8}\left(35s^4-30s^2+3\right),\ P_4^1 = -\frac{5}{2}sc\left(7s^2-3\right),\ P_4^2 = \frac{15}{2}c^2\left(7s^2-1\right),\\
&\quad P_4^3 = -105sc^3,\ P_4^4 = 105c^4\\
&P_5 \equiv P_5^0 = \frac{1}{8}s\left(63s^4-70s^2+15\right),\ P_5^1 = -\frac{15}{8}c\left(21s^4-14s^2+1\right),\\
&\quad P_5^2 = \frac{105}{2}sc^2\left(3s^2-1\right),\ P_5^3 = -\frac{105}{2}c^3\left(9s^2-1\right),\ P_5^4 = 945sc^4,\ P_5^5 = -945c^5
\end{aligned}
\tag{12.2.6}
$$

Earth's Gravitational Multipoles

Because the detailed density distribution of the Earth is beyond our knowledge, the multipole coefficients C_n^m and S_n^m are available only by measuring them with particular satellites. The most famous and most precise of them is the GRACE satellite mission, which operated from 2002 to 2017. The values thereby obtained are given in Table 12.2. In this representation the origin of the polar reference frame is at the center of Earth's mass and the polar axis is Earth's axis of rotation.

$$
\begin{aligned}
J_2 &= 1.0826266 \times 10^{-3} & J_3 &= -2.5326 \times 10^{-6}\\
J_4 &= -1.6196 \times 10^{-6} & J_5 &= -2.2730 \times 10^{-7}\\
J_6 &= 5.4068 \times 10^{-7} & J_7 &= -3.5236 \times 10^{-7}\\
J_8 &= -2.0480 \times 10^{-7} & J_9 &= -1.2062 \times 10^{-7}
\end{aligned}
$$

Table 12.2 Multipole coefficients of Earth's gravitational potential

C_n^m	$m = 0$	1	2	3	4	5
$n = 0$	+1.000000					
1	0.00	0.00				
2	−1.082627 × 10−3	0.00	1.5745×10^{-6}			
3	2.5326×10^{-6}	-2.1926×10^{-6}	3.0899×10^{-7}	-1.0055×10^{-7}		
4	1.6196×10^{-6}	5.0880×10^{-7}	7.8418×10^{-8}	$^{-}5.9210 \times 10^{-8}$	-3.9841×10^{-9}	
5	-2.2730×10^{-7}	5.3180×10^{-8}	1.0559×10^{-7}	1.4930×10^{-8}	-2.2993×10^{-9}	-4.3082×10^{-10}
S_n^m	$m = 0$	1	2	3	4	5
$n = 0$	0.00					
1	0.00	0.00				
2	0.00	0.00	-9.0380×10^{-7}			
3	0.00	-2.6843×10^{-7}	-2.1144×10^{-7}	-1.9722×10^{-7}		
4	0.00	4.4914×10^{-7}	1.4818×10^{-7}	1.2008×10^{-8}	6.5257×10^{-9}	
5	0.00	8.0859×10^{-8}	-5.2329×10^{-8}	7.0973×10^{-9}	3.8671×10^{-10}	1.6482×10^{-9}

J_{nm}/λ_{nm}	$m = 0$	1	2	3	4	5
$n = 2$	**1.082627×10^{-3}**	0.0	1.8155×10^{-6}			
		–	−14.928°			
3	-2.5326×10^{-6}	2.2090×10^{-6}	3.7441×10^{-7}	2.2137×10^{-7}		
		−173.01°	−17.192°	−39.005°		
4	-1.6196×10^{-6}	6.7868×10^{-7}	1.6765×10^{-7}	6.0415×10^{-8}	7.6458×10^{-9}	
		41.436°	31.056°	56.180°	30.351°	
5	-2.2730×10^{-7}	9.6780×10^{-8}	1.1784×10^{-7}	1.6531×10^{-8}	2.3316×10^{-9}	1.7036×10^{-9}
		56.667°	−13.182°	8.4751°	42.613°	20.930°

$J_2 = 1.0826266 \times 10^{-3}$	$J_3 = -2.5326 \times 10^{-6}$	$J_4 = -1.6196 \times 10^{-6}$	$J_5 = -2.2730 \times 10^{-7}$
$J_6 = 5.4068 \times 10^{-7}$	$J_7 = -3.5236 \times 10^{-7}$	$J_8 = -2.0480 \times 10^{-7}$	$J_9 = -1.2062 \times 10^{-7}$

Note *The multipole coefficients* $\overline{C}_{nm}$, $\overline{C}_{nm}$ *as used in geodesy (see e.g., Earth Gravitational Model 1996 (EGM96),* http://cddis.gsfc.nasa.gov/926/egm96/getit.html*) are related to the more common coefficients* C_n^m, S_n^m *adopted also here through*

$$\begin{Bmatrix} C_n^m \\ S_n^m \end{Bmatrix} = (-1)^m \sqrt{\frac{(2-\delta_{0m})(2n+1)(n-m)!}{(n+m)!}} \begin{Bmatrix} \overline{C}_{nm} \\ \overline{S}_{nm} \end{Bmatrix}$$

with Kronecker delta

$$\delta_{0m} = \begin{cases} 1 & @\ m = 0 \\ 0 & @\ m > 0 \end{cases}$$

Table 12.2 is incomplete. In fact, multipole coefficients are known today of order up to $n = 2159$. Some of them vanish for the following reasons:

- $S_n^0 = 0$, because for $m = 0$ $S_n^0 \sin(0 \cdot \lambda) = 0$. Therefore the coefficients S_n^0 are undetermined and can be set to zero;
- $C_1^0 = C_1^1 = S_1^1 = 0$, because the center of mass is chosen to be at the origin of the reference frame, which is why the sum in Eq. (12.2.3) starts with $n = 2$;
- $C_2^1 = S_2^1 = 0$, because the z-axis points along Earth's principal moment of inertia;
- S_2^2 would vanish if the coordinate axes would coincide with the principal axes of inertia (see Sect. 15.1.1). But traditionally the x-axis is chosen to lie in the Greenwich meridian.

Physically speaking, the successive terms of the sums in Eq. (12.2.2) correspond to a stepwise spherical approximation (multipole approximation) of the anisotropic gravitational potential, i.e., of the geoid. The terms $P_n^m(\sin\beta)\cos(m\lambda)$ and $P_n^m(\sin\beta)\sin(m\lambda)$ are so-called *spherical harmonics* (see Figs. 12.5 and 12.6). They are the spherical distribution functions (multipoles) of order (n, m), and are

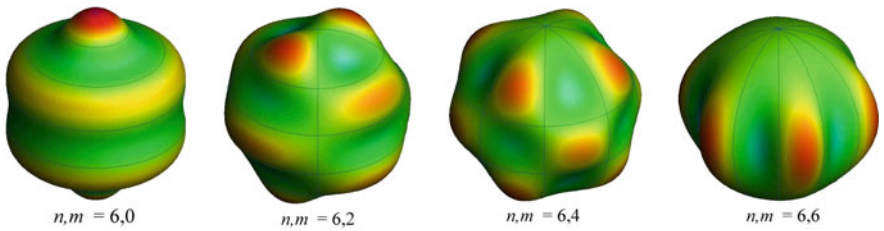


Fig. 12.5 The spherical harmonics of degree 6 and even order. *Credit* GFZ Potsdam

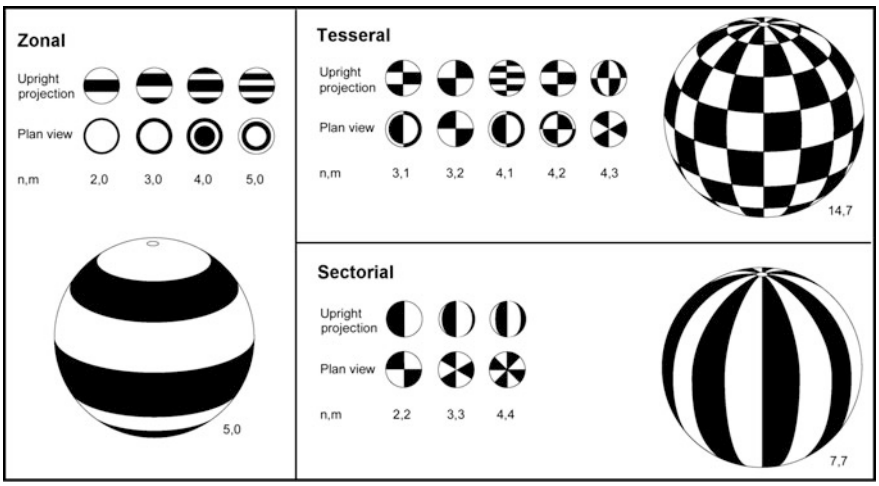
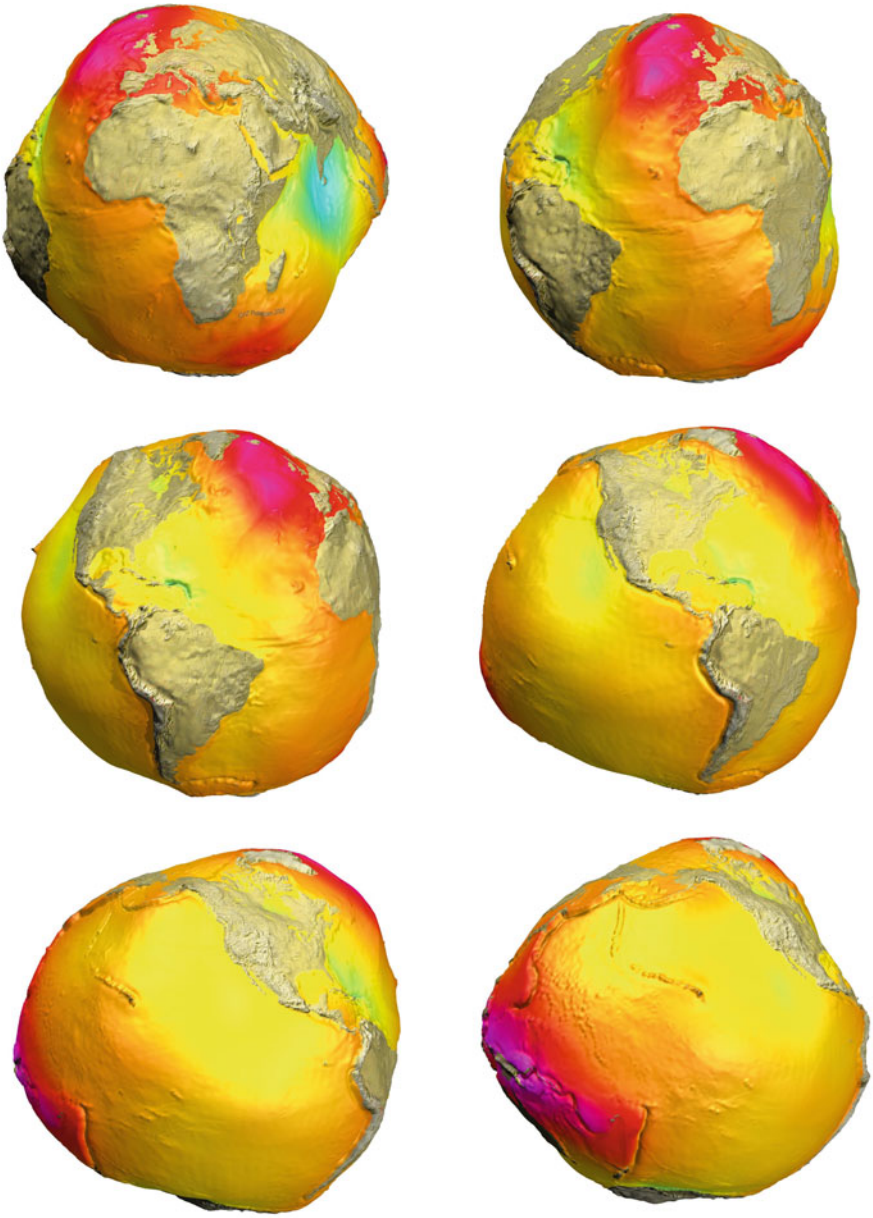


Fig. 12.6 The nodal surfaces of spherical harmonics of degree n and order m: zonal harmonics, tesseral harmonics, and sectorial harmonics

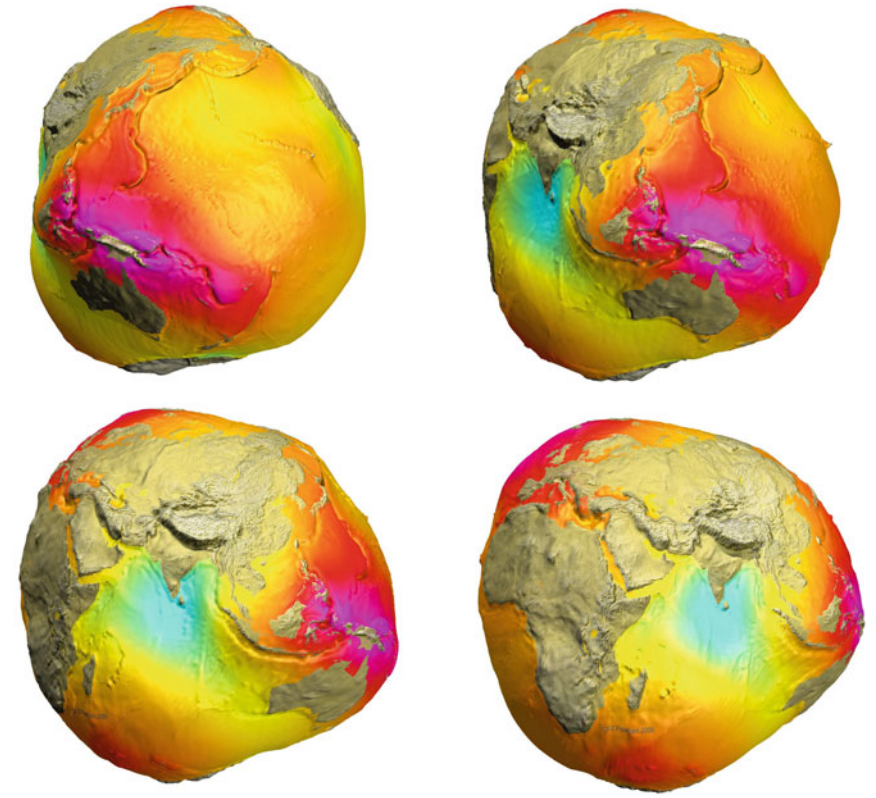
called *zonal harmonics* for $(n, 0)$ because they describe just latitudinal variations, *sectorial harmonics* for (n, n) describing only longitudinal variations, and *tesseral harmonics* for $(n,\ 0 < m < n)$ describing mixed variations. The multipole coefficients J_{nm} as given in Table 12.2 determine the strength of these multipoles: $J_0 = -C_0^0 = 1$ causes the predominant geoid sphere, $J_2 = -1.082627 \times 10^{-3}$ the **oblate spheroid**, and the J_{nm} of order 10^{-6} the general shape of the potato (see figures of the potato on color plates on page 568 and 569).

Note *In earlier literature one may find the statement that Earth is pear shaped (symmetry axis = polar axis, pear stalk at south pole) beyond the spheroid. This discovery dates back to a publication of J. A. O'Keefe, A. Eckles, and R. Squires from 1959. They derived a pear shape from the analysis of long-periodic effects of orbital eccentricities of the first US satellite Vanguard I. Because such effects stem only from zonal multipoles (see below), from this method only the C_n^0 coefficients (in particular the pear shaped C_3^0,) are derivable and not the C_2^2 and C_3^1 coefficients, which extend the pear to a potato.*

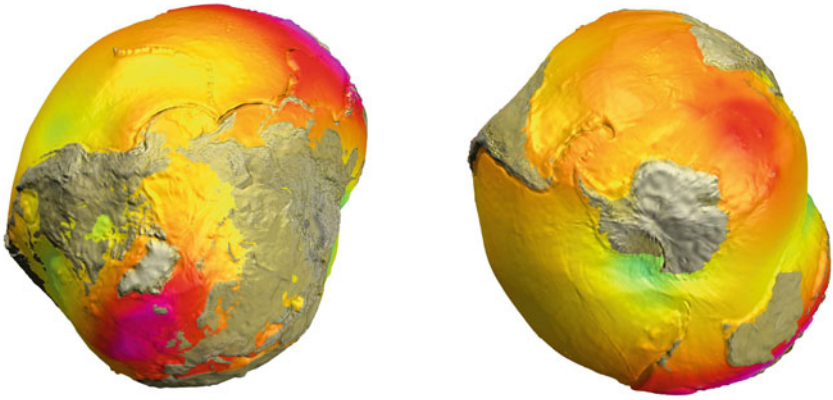


The potato residual of the geoid at different view angles in the equatorial plane

Credit GFZ Potsdam



The potato residual of the geoid at different view angles in the equatorial plane (continued)



The potato residual of the geoid as viewed from the North Pole (left) and the South Pole (right)

Credit GFZ Potsdam

12.2.3 Lagrange's Planetary Equations

In order to understand the key effects of the gravitational anisotropy on orbital elements, we shall now look for analytic perturbation solutions. This is facilitated by the special property that the gravitational perturbation can be expressed in terms of a potential R, and hence the perturbational force and the perturbational acceleration $\boldsymbol{a}_p = (a_r, a_\theta, a_h)_{RSW}$ (see Sect. 12.1.2) is just the gradient of it (see Eq. 7.1.5)

$$\boldsymbol{a}_p = \frac{1}{m}\boldsymbol{F}_p = -\frac{dR}{d\boldsymbol{r}}$$

If this gradient is inserted into the Gaussian variational Eqs. (12.1.4), then the time-derivatives of the orbital elements can simply be expressed by partial derivatives of the perturbation function R. Linear perturbation theory thus delivers the so-called *Lagrange's planetary equations* (see, e.g., Schaub and Junkins 2003, p. 508f)

$$\begin{aligned}
\dot{a} &= \frac{2a}{h}\sqrt{1-e^2}\,\frac{\partial R}{\partial M} \\
\dot{e} &= \frac{1-e^2}{he}\left[\sqrt{1-e^2}\,\frac{\partial R}{\partial M} - \frac{\partial R}{\partial \omega}\right] \\
\dot{i} &= \frac{1}{h\,\sin i}\left[\cos i\,\frac{\partial R}{\partial \omega} - \frac{\partial R}{\partial \Omega}\right] \\
\dot{\omega} &= \frac{1}{h}\left[\frac{1-e^2}{e}\frac{\partial R}{\partial e} - \cot i\,\frac{\partial R}{\partial i}\right] \\
\dot{\Omega} &= \frac{1}{h\,\sin i}\frac{\partial R}{\partial i} \\
\dot{M} - n &= -\frac{\sqrt{1-e^2}}{h}\left[\frac{1-e^2}{e}\frac{\partial R}{\partial e} + 2a\frac{\partial R}{\partial a}\right]
\end{aligned}$$

Lagrange's planetary equations (LPEs) (12.2.7)

These are the basic equations by which we will qualitatively study the perturbational effects in Sect. 12.3.

12.2.4 Numerical Perturbation Methods

In this section we consider the different numerical methods to determine a perturbed orbit.

Cowell's Method by Recurrence Iteration
According to Eq. (7.7.2) the equation of motion for a general gravitational potential U reads

$$\ddot{\boldsymbol{r}} = -\frac{dU}{d\boldsymbol{r}} \tag{12.2.8}$$

Employing Eq. (12.2.2) for U and choosing an Earth-fixed Cartesian reference system, in which the trajectory vector is $\boldsymbol{r} = (x, y, z)$, this equation can be solved numerically to determine the perturbed orbit. This is known as **Cowell's method**. Owing to its high accuracy it is widely used today although at the cost of high computing time.

For practical purposes it can be shown (see Montenbruck and Gill (2000)) that the terms $V_{n,m}(\boldsymbol{r})$, $W_{n,m}(\boldsymbol{r})$; $n = 0, \ldots, n_{\max}+1$; $m = 0, \ldots, n$ can be calculated recursively by the following iteration:

Recurrence Iteration

$$\alpha_x := \frac{xR_\oplus}{r^2}, \alpha_y := \frac{yR_\oplus}{r^2}, \alpha_z := \frac{zR_\oplus}{r^2}, \alpha_R := \frac{R_\oplus^2}{r^2}$$

$$V_{0,0} = \frac{R_\oplus}{r}, \quad W_{0,0} = 0$$

Do $m = 0, n_{\max}$

$$V_{m-1,m} = W_{m-1,m} = 0$$

Do $n = m, n_{\max}$

$$V_{n+1,m} = \alpha_z \frac{2n+1}{n-m+1} \cdot V_{n,m} - \alpha_R \frac{n+m}{n-m+1} \cdot V_{n-1,m}$$

$$W_{n+1,m} = \alpha_z \frac{2n+1}{n-m+1} \cdot W_{n,m} - \alpha_R \frac{n+m}{n-m+1} \cdot W_{n-1,m}$$

End Do

$$V_{m+1,m+1} = (2m+1)\left(\alpha_x V_{m,m} - \alpha_y W_{m,m}\right)$$

$$W_{m+1,m+1} = (2m+1)\left(\alpha_x W_{m,m} + \alpha_y V_{m,m}\right)$$

End Do

Remark: *Observe that the recurrence iteration implies* $W_{n,0} = 0$

With these quantities Eq. (12.2.8) can be rewritten as follows:

$$\begin{aligned}
\ddot{x} &= -\frac{\mu_\oplus}{R_\oplus^2}\sum_{n=0}^{n_{\max}}\left[C_{n0}V_{n+1,1} + \frac{1}{2}\sum_{m=1}^{n}\left\{C_{nm}V_{n+1,m+1} + S_{nm}W_{n+1,m+1}\right.\right. \\
&\quad \left.\left. - \frac{(n-m+2)!}{(n-m)!}\left(C_{nm}V_{n+1,m-1} + S_{nm}W_{n+1,m-1}\right)\right\}\right] \\
\ddot{y} &= -\frac{\mu_\oplus}{R_\oplus^2}\sum_{n=0}^{n_{\max}}\left[C_{n0}W_{n+1,1} + \frac{1}{2}\sum_{m=1}^{n}\left\{C_{nm}W_{n+1,m+1} - S_{nm}V_{n+1,m+1}\right.\right. \\
&\quad \left.\left. + \frac{(n-m+2)!}{(n-m)!}\left(C_{nm}W_{n+1,m-1} - S_{nm}V_{n+1,m-1}\right)\right\}\right] \\
\ddot{z} &= -\frac{\mu_\oplus}{R_\oplus^2}\sum_{n=0}^{n_{\max}}\sum_{m=1}^{n}(n-m+1)\left(C_{nm}V_{n+1,m} + S_{nm}W_{n+1,m}\right)
\end{aligned} \tag{12.2.9}$$

with

$$\left\{\begin{matrix} C_{nm} \\ S_{nm} \end{matrix}\right\} = \sqrt{\frac{(2-\delta_{0m})(2n+1)(n-m)!}{(n+m)!}}\left\{\begin{matrix} \overline{C}_{nm} \\ \overline{S}_{nm} \end{matrix}\right\} = (-1)^m\left\{\begin{matrix} C_n^m \\ S_n^m \end{matrix}\right\}$$

and

$$\delta_{0m} = \begin{cases} 1 & @\ m = 0 \\ 0 & @\ m > 0 \end{cases}$$

and $\overline{C}_{nm}, \overline{S}_{nm}$ given at ftp://cddis.gsfc.nasa.gov/pub/egm96/general_info/egm96_to360.ascii plus ftp://cddis.gsfc.nasa.gov/pub/egm96/general_info/readme.egm96, and

$$\begin{aligned}
\mu_\oplus &= 3.986004415 \times 10^5\ \mathrm{km^3\,s^{-2}} \\
R_\oplus &= 6378.1363\ \mathrm{km}
\end{aligned}$$

If one takes into account only the biggest perturbation, the spheroid, $n_{\max} = 2$ and $m = 0$, the explicit equations read:

$$\begin{aligned}
\ddot{x} &= -\frac{x\mu_\oplus}{r^3}\left[1 + C_2^0\frac{3}{2}\frac{R_\oplus^2}{r^2}\left(5\frac{z^2}{r^2} - 1\right)\right] \\
\ddot{y} &= -\frac{y\mu_\oplus}{r^3}\left[1 + C_2^0\frac{3}{2}\frac{R_\oplus^2}{r^2}\left(5\frac{z^2}{r^2} - 1\right)\right] = \frac{y}{x}\ddot{x} \\
\ddot{z} &= -\frac{z\mu_\oplus}{r^3}\left[1 + C_2^0\frac{3}{2}\frac{R_\oplus^2}{r^2}\left(5\frac{z^2}{r^2} - 3\right)\right]
\end{aligned} \tag{12.2.10}$$

So, for calculating a trajectory, first the maximum order $n_{\max}$ of the perturbation is chosen depending on the accuracy of the initial data and of the orbit needed. Then the coupled differential Eqs. (12.2.9) or (12.2.10) are solved, whereby one of the currently best solvers is the Runge–Kutta–Nyström algorithm RKN12(10)17M (see Brankin et al. 1989), which can be found in the NAG Library under the name of D02LAF. The initial step size should be $h = 0.1$ for all Keplerian problems. Note that also other perturbations like neighboring planets can easily be taken into account by Eq. (12.2.9) by just adding the perturbational terms in the Cartesian coordinate form. Observe that the solution is given in an Earth-fixed reference system. In order to find the result in an inertial (sidereal) system an appropriate coordinate transformation has to be applied to the solution $\boldsymbol{r} = (x, y, z)$.

Encke's Method

If the perturbational acceleration $\boldsymbol{a}_p$ is very small, as in our case of gravitational perturbation, it can be separated and calculated as such. The undisturbed Keplerian orbit $\boldsymbol{r}_0(t)$ then is called *osculating orbit*, and the residual $\boldsymbol{\delta}(t) := \boldsymbol{r} - \boldsymbol{r}_0$. Figure 12.7 illustrates the defined values. It can be shown that for this residual the following equation of motion holds (see, e.g., Schaub and Junkins 2003)

$$\ddot{\boldsymbol{\delta}} = -\frac{\mu}{r_0^3}\left(3q\frac{1+q+q^2/3}{1+(1+q)^{3/2}}\boldsymbol{r}+\boldsymbol{\delta}\right)+\boldsymbol{a}_p \approx \frac{\mu}{r_0^3}\left(6\frac{\boldsymbol{\delta}\cdot\boldsymbol{r}}{r^2}\boldsymbol{r}-\boldsymbol{\delta}\right)+\boldsymbol{a}_p \qquad (12.2.11)$$

with

$$q = \frac{\boldsymbol{\delta}\cdot\boldsymbol{\delta} - 2\boldsymbol{\delta}\cdot\boldsymbol{r}}{r^2}$$

This differential equation is also solved with a high-quality Runge–Kutta algorithm. This approach of separating the perturbation from the osculating orbit is called **Encke's method**. It is a very accurate method, as the numerical integration only treats the perturbation in question, and does not have to "drag along" the full Keplerian orbit. However, because it is not well documented and tested in literature

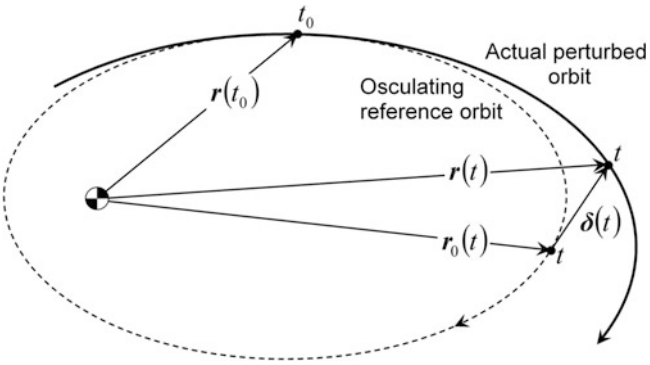


Fig. 12.7 Illustration of Encke's method

and due to the increased numerical precision of today's CPUs it is more convenient to apply the above recurrence type of Cowell's method.

Solving the LPEs
Another method is to directly integrate the LPEs (12.2.7) by applying the residual gravitational term R of Eq. (12.2.3). Although this method is fast and yields a good accuracy it is not widely used in the space community because it is not well documented (see Herrick (1972) for method details) and the dominance of Cowell's method.

12.3 Gravitational Perturbation Effects

12.3.1 Classification of Effects

General Considerations
The gravitational potential U as given by Eq. (12.2.3) and depicted by the geoid in Fig. 12.3 is solely position-dependent. Therefore (see insert Box *Conservative Force* in Sect. 7.1.2 and Fig. 7.3) Earth's gravitational force can still be derived as a gradient of U, $\boldsymbol{F} = -m \cdot dU/d\boldsymbol{r}$, and hence is a conservative field, implying that the work done by Earth's gravitational field on a closed orbit is zero. So, in any gravitational potential the orbital energy over one closed orbital revolution, which essentially is one orbital period (strictly speaking: one period of long-periodic variations; see Eq. (12.3.6)), remains unchanged and according to Eq. (7.3.19) the mean semi-major axis therefore must remain constant

$$\bar{a} \equiv \langle a \rangle_P = const \quad @ \text{ geoid} \tag{12.3.1}$$

However, the gravitational force derived from the geoid is no longer simply radial

$$\boldsymbol{F}(\boldsymbol{r}) = -m\frac{dU}{d\boldsymbol{r}} \neq -mg(r)\hat{\boldsymbol{r}} \quad @ \text{ geoid}$$

and therefore is no longer a central force field, though still being conservative. This is because the gravitational potential residual $R(r, \beta, \lambda)$ as given in Eq. (12.2.3), and hence $\boldsymbol{F}$, does not depend solely on r, as did the mass point potential $-\mu/r$, but also on β, λ. We saw from Sect. 7.2.2 that a non-central force field quite generally exerts a torque on the orbit and thus changes its angular momentum both in orientation and absolute value. In particular, the orbital plane will tilt. We therefore expect that

> The small non-spherical fraction of Earth's gravitational field slightly varies all orbital elements over time. This holds even if the elements are averaged over one orbital period, except for the semi-major axis.

This general behavior can be seen also from Lagrange's planetary equations (12.2.7).

Mean Orbital Elements and Short-Periodic Variations

Generally, a gravitational perturbation affects the trajectory momentarily at any point along its path. Therefore, the orbital elements $\vartheta_i \in (a, e, i, \Omega, \omega, M - n \cdot t)$ vary periodically and significantly over each orbit, which is on the order of an hour (LEO) to a day (GEO). Usually, we are not interested in such recurring periodic effects in the course of an orbit. Therefore we average the effects over one orbital period T and define the mean orbital elements as

$$\bar{\vartheta}_i(t) := \langle \vartheta_i \rangle_P = \frac{1}{T} \cdot \int_0^T \vartheta_i \cdot dt = \frac{1}{2\pi} \int_0^{2\pi} \vartheta_i \cdot dM \quad \textbf{mean orbital elements} \tag{12.3.2}$$

The latter holds because $M = n(t - t_0)$ and hence $dt = dM/n = T/2\pi \cdot dM$. Given the mean orbital elements we define the short-periodic deviations and *short-periodic variations* in the course of an orbit as

$$\vartheta_{i,short}(M) := \vartheta_i - \bar{\vartheta}_i \quad \textbf{short-periodic deviations} \tag{12.3.3}$$

$$\dot{\vartheta}_{i,short}(M) := \dot{\vartheta}_i - \dot{\bar{\vartheta}}_i \quad \textbf{short-periodic variations} \tag{12.3.4}$$

which vary with the **fast variable** true anomaly θ or mean anomaly M, equivalently (see for instance Eq. (12.3.22)).

Secular Elements and Long-Periodic Variations

Though short-periodic effects are significant, they often are opposite on the two halves of an orbit, such that overall they fully cancel out over one orbital period. However, as we will see in Sect. 12.3.3, oblateness perturbations constantly vary the argument of periapsis ω such that the line of apsides constantly rotates. A slowly rotating apsidal line implies that over one orbit the perturbational effects at any θ are little different from those at $-\theta$. This imperfect cancellation of opposite contributions accumulates and leads to slow variations of the mean orbital elements. Therefore, $\bar{\omega}$ is a **slow variable**, which may modulate the mean orbital elements in pace with the rotation of the line of apsides, the period being typically a couple of months for Earth. By the same token $\bar{\Omega}$ in general is also a slow variable, and hence $\dot{\bar{\vartheta}}_i(\bar{\omega}, \bar{\Omega})$. We can also remove these long-period variations by integration over both one revolution of the line of apsides and the line of nodes, which results in the so-called secular variations

$$\dot{\vartheta}_{i,\mathrm{sec}}(t) := \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \dot{\bar{\vartheta}}_i \cdot d\bar{\omega} \cdot d\bar{\Omega} \quad \textbf{secular variations} \tag{12.3.5}$$

Secular variations of orbital elements usually vary steadily and monotonically (usually linearly) with time so that secular orbital elements $\vartheta_i = \vartheta_{i,0} + \dot{\vartheta}_i \cdot t$ grow progressively and unbounded.

Variations owing to slow variables, so-called **long-periodic** variations, can thus be separated by

$$\dot{\vartheta}_{i,long}(\bar{\omega}, \bar{\Omega}) \equiv \dot{\bar{\vartheta}}_i - \dot{\vartheta}_{i,\text{sec}} \quad \textbf{long-periodic variations} \tag{12.3.6}$$

In summary we have

$$\dot{\vartheta}_i = \dot{\bar{\vartheta}}_i + \dot{\vartheta}_{i,short} = \dot{\vartheta}_{i,\text{sec}} + \dot{\vartheta}_{i,long}(\bar{\Omega}, \bar{\omega}) + \dot{\vartheta}_{i,short}(M) \tag{12.3.7}$$

Note that according to Eq. (7.3.11) $M = n(t - t_p) = \langle \omega \rangle_P (t - t_p)$ and hence is already a quantity averaged over one orbital period.

12.3.2 Removing Short-Periodic Effects

In the following we are not interested in the recurring short-periodic effects in the course of one orbit, but only in the variations of the mean orbital elements. From the LPEs (12.2.7) we see that $\dot{\vartheta}_i \propto \partial R / \partial \vartheta_i$. Hence the variation of the mean orbital elements are determined as

$$\dot{\bar{\vartheta}}_i = \frac{1}{2\pi} \int_0^{2\pi} \dot{\vartheta}_i \cdot dM \propto \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial R}{\partial \vartheta_j} \cdot dM = \frac{\partial \bar{R}}{\partial \vartheta_j} \tag{12.3.8}$$

with

$$\bar{R} := \frac{1}{T} \int_0^T R \cdot dt = \frac{1}{2\pi} \int_0^{2\pi} R \cdot dM =: \langle R \rangle_{P,M}$$

From Eq. (12.2.3) we obtain the mean residual perturbational potential

$$\begin{aligned} \bar{R}(r, \beta) &= -\left\langle \frac{\mu_\oplus}{r} \sum_{n=2}^{\infty} \left(\frac{R_\oplus}{r} \right)^n \sum_{m=0}^{n} J_{nm} P_n^m (\sin \beta) \cos m(\lambda - \lambda_{nm}) \right\rangle_{P,M} \\ &= -\frac{\mu_\oplus}{a} \left(\frac{R_\oplus}{a} \right)^n \sum_{n=2}^{\infty} \sum_{m=0}^{n} J_{nm} \left\langle \left(\frac{a}{r} \right)^{n+1} P_n^m (\sin \beta) \cos m(\lambda - \lambda_{nm}) \right\rangle_{P,M} \end{aligned}$$

In order to carry out the averaging integration, we recall from Napier's rules of spherical trigonometry that $\sin \beta = \sin i \cdot \sin(\theta + \omega)$ holds between inclination i, argument of periapsis ω, and true anomaly θ. Owing to this and because the orbit

equation (7.3.5) also relates r and θ we switch the variables $dM \to d\theta$ to carry out the integral. We have from Eq. (7.3.12)

$$dM = \left(\frac{r}{a}\right)^2 \frac{1}{\sqrt{1-e^2}} d\theta$$

With this we get for the average residual perturbational potential

$$\bar{R}(r,\beta) = -\frac{\mu_\oplus}{a\sqrt{1-e^2}} \left(\frac{R_\oplus}{a}\right)^n \sum_{n=2}^{\infty} \sum_{m=0}^{n} J_{nm} \left\langle \left(\frac{a}{r}\right)^{n-1} P_n^m(\sin\beta) \cos m(\lambda - \lambda_{nm}) \right\rangle_{P,\theta} \tag{12.3.9}$$

Note that in this notation actually $a \equiv \bar{a}$, $e \equiv \bar{e}$ for convenience, which holds for the rest of this chapter. Substituting the orbit Eq. (7.3.5)

$$\frac{a}{r} = \frac{1 + e\,\cos\theta}{1-e^2}$$

delivers

$$\begin{aligned} \bar{R}(r,\beta) = &-\frac{\mu_\oplus}{a}\sqrt{1-e^2}\left[\frac{R_\oplus}{a(1-e^2)}\right]^n \\ &\times \sum_{n=2}^{\infty}\sum_{m=0}^{n} J_{nm} \left\langle (1+e\cos\theta)^{n-1} P_n^m(\sin\beta)\cos m(\lambda-\lambda_{nm}) \right\rangle_{P,\theta} \end{aligned}$$

For further convenience (see Eqs. (12.3.15) and (12.3.20)) we now define the *reduced harmonic coefficients*

$$j_{nm} := \frac{3}{2} J_{nm} \left[\frac{R_\oplus}{a(1-e^2)}\right]^n \quad \textbf{reduced harmonic coefficients} \tag{12.3.10}$$

The common factor $3/2$ is a remnant of orbit averaging. Observe that for the derivatives $\partial R/\partial e$ in the LPEs, Eq. (12.2.7), it needs to be considered that the reduced harmonic coefficients are a function of eccentricity, i.e. $j_{nm}(e)$. With this we finally obtain the mean residual perturbational potential

$$\begin{aligned} \bar{R} &= \sum_{n=2}^{\infty}\sum_{m=0}^{n} \bar{R}_{nm} \\ \bar{R}_{nm} &:= -\frac{2}{3}\frac{\mu_\oplus}{a}\sqrt{1-e^2}\, j_{nm}(e) \cdot \bar{P}_n^m \\ \bar{P}_n^m &:= \left\langle (1+e\,\cos\theta)^{n-1} P_n^m(\sin\beta)\cos m(\lambda-\lambda_{nm}) \right\rangle_{P,\theta} \end{aligned} \tag{12.3.11}$$

12.3.3 Oblateness Perturbation

Table 12.2 shows that Earth's oblateness, given by $(n,m)=(2,0)$, $C_2^0=-J_2=1.0826266\times 10^{-3}$ and depicted in Fig. 12.3a, constitutes by far the strongest perturbation of the gravitational potential. We therefore want study the effects of this perturbation in detail.

Relation to Earth's Polar Flattening
It can be shown (exercise, Problem 12.2) that

$$J:=\frac{3}{2}J_2\approx f\left(1-\frac{1}{2}f\right)-\frac{\omega^2R_\oplus^3}{2\mu_\oplus}\left(1-\frac{3\omega^2R_\oplus^3}{2\mu_\oplus}-\frac{2}{7}f\right)\approx f-\frac{\omega^2R_\oplus^3}{2\mu_\oplus} \tag{12.3.12}$$

where the defined quantity J is frequently used in literature and has the indicated relation to Earth's polar flattening $f:=\left(R_\oplus-R_{polar}\right)/R_\oplus\approx 1/298.264$ with $R_\oplus\equiv R_{equatorial}$.

Average Oblateness Potential
The oblateness perturbation is a enlightening case to see how the removal of the short-periodic effects works. With $P_2\equiv P_2^0=\frac{1}{2}\left(3\sin^2\beta-1\right)$ from Eq. (12.2.6) we have from Eq. (12.3.11)

$$\begin{aligned}\bar{P}_2&=-\frac{1}{2}\left\langle(1+e\,\cos\theta)\left(1-3\sin^2\beta\right)\right\rangle_{P,\theta}\\&=-\frac{1}{2}\left[\left\langle(1+e\,\cos\theta)\right\rangle_{P,\theta}-3\left\langle(1+e\,\cos\theta)\sin^2\beta\right\rangle_{P,\theta}\right]\end{aligned}$$

In order to perform the averaging over one period, we need to express β in terms of θ. This is given by Napier's rule $\sin\beta=\sin i\cdot\sin(\theta+\omega)$, from which follows

$$\sin^2\beta=\sin^2 i\cdot\sin^2(\theta+\omega)=\frac{1}{2}\sin^2 i[1-\cos 2(\theta+\omega)]$$

Because $\cos\theta$ and $\cos 2(\theta+\omega)$ are periodic functions of θ, integrals comprising these two terms vanish over one period. Thus we have

$$\begin{aligned}\left\langle(1+e\,\cos\theta)\right\rangle_{P,\theta}&=\left\langle 1\right\rangle_{P,\theta}+e\left\langle\cos\theta\right\rangle_{P,\theta}=1+0=1\\\left\langle(1+e\,\cos\theta)\sin^2\beta\right\rangle_{P,\theta}&=\left\langle\sin^2\beta\right\rangle_{P,\theta}=\frac{1}{2}\sin^2 i\left\langle 1-\cos 2(\theta+\omega)\right\rangle_{P,\theta}=\frac{1}{2}\sin^2 i\end{aligned}$$

hence

$$\bar{P}_2=-\frac{1}{2}\left(1-\frac{3}{2}\sin^2 i\right) \tag{12.3.13}$$

We finally get for the average oblateness potential

$$\bar{R}_2 = \frac{\mu_\oplus j_2(e)}{3a}\sqrt{1-e^2}\left(1-\frac{3}{2}\sin^2 i\right) \tag{12.3.14}$$

Variation of Orbital Elements

We now carry out the partial derivatives according to Eq. (12.3.8), which can be done straightforwardly, and get from Eq. (12.2.7) the variations of the orbital elements averaged over one period, i.e., $\dot{\vartheta} \equiv \bar{\dot{\vartheta}} := \langle \dot{\vartheta} \rangle_P$, $\dot{\vartheta} = \dot{a}, \dot{e}, \dot{i}, \dot{\omega}, \dot{\Omega}, \dot{M}$

$$\begin{aligned}
&\dot{\Omega}_{\text{sec}} = -nj_2 \cos i && \textbf{regression of nodes}\\
&\dot{\omega}_{\text{sec}} = 2nj_2\left(1-\frac{5}{4}\sin^2 i\right) && \textbf{progression of line of apsides}\\
&\dot{M}_{\text{sec}} - n = nj_2\sqrt{1-e^2}\left(1-\frac{3}{2}\sin^2 i\right) && \textbf{progression of epoch}\\
&\dot{a}_{\text{sec}} = \dot{e}_{\text{sec}} = \dot{i}_{\text{sec}} = 0
\end{aligned} \tag{12.3.15}$$

with

$$n = \sqrt{\frac{\mu_\oplus}{a^3}},\; j_2 = \frac{3}{2}J_2\left[\frac{R_\oplus}{a(1-e^2)}\right]^2 > 0$$
$$J_2 = -C_2^0 = 0.0010826266.$$

First we note that all these variations of the mean orbital elements are not periodic but **secular variations**, i.e., they constantly increase with time. Thus, after one orbital period T the node has shifted by $\Delta\Omega = \dot{\Omega}_{\text{sec}}\, T$, the periapsis by $\Delta\omega = \dot{\omega}_{\text{sec}}\, T$ (see Figs. 12.9 and 12.2), and the orbital phase by $\Delta M = \left(\dot{M}_{\text{sec}} - n\right) T$. The progression or regression of the line of apsides due to $\Delta\omega$ is called *orbital* or *apsidal precession*.

Second, the averaged variations vanish at critical inclinations, namely

$$\begin{aligned}
\dot{\Omega}_{\text{sec}} &= 0 \quad @\; i_{crit} = 90^\circ\\
\dot{\omega}_{\text{sec}} &= 0 \quad @\; i_{crit} = \arcsin\frac{2}{\sqrt{5}} = 63.43^\circ\\
\dot{M}_{\text{sec}} &= n \quad @\; i_{crit} = \arcsin\sqrt{\frac{2}{3}} = 54.74^\circ
\end{aligned}$$

Expanding the term n by $R_\oplus^{3/2}$ and with $\sqrt{\mu_\oplus/R_\oplus^3} = 1.23945 \times 10^{-3}\text{s}^{-1}$ the term nj_2 can be expressed for Earth in the convenient form

$$nj_2 = 9.96404 \cdot \frac{(R_\oplus/a)^{7/2}}{(1-e^2)^2}\;\left[^\circ\ \text{day}^{-1}\right] \quad @\ \text{Earth} \tag{12.3.16}$$

So, for low Earth orbits $nj_2 \approx 7^\circ\ \text{day}^{-1}$. Therefore in LEO $\Delta\Omega$ and $\Delta\omega$ are typically of order 0.5° per revolution or a couple of degrees per day and are therefore of high significance for orbit determination.

The shift of nodes caused by the oblateness of the Earth can be explained as follows: the oblateness can be considered a bulge around the equator of a spherical Earth (see Fig. 12.8), which attracts the body on its orbit. This causes a torque triggering the rotation of the orbital plane (see Fig. 12.9), just as with a spinning top.

The total effect of the equatorial bulge on the orbit can be understood as follows: if its gravitational pull is considered as continuous kick-burns of type $\delta v_{\perp\perp} \propto \sin(\theta+\omega)$ and type $\delta v_{\perp\text{O}} \propto \cos(\theta+\omega)$. According to Eq. (8.1.1) this causes $\delta e(-\theta) = -\delta e(\theta)$ and $\delta i(-\theta) = -\delta i(\theta)$, i.e., the effects cancel out over one period, and therefore $\langle \delta a \rangle_P = \langle \delta e \rangle_P = \langle \delta i \rangle_P = 0$. On the other hand $\delta\omega(-\theta) = \delta\omega(\theta)$ and $\delta\Omega(-\theta) = \delta\Omega(\theta)$, i.e., they sum up and hence $\langle \delta\Omega \rangle_P \neq 0$, implying a shift of the node, and $\langle \delta\omega \rangle_P \neq 0$, a shift of the periapsis and thus also a shift of the epoch, δt_0. For inclinations $i < 63.4^\circ$ the periapsis shifts along the line of motion (progression of the line of apsides). Therefore the orbital period increases and accordingly the mean anomaly and the epoch. The two bordering inclinations, at which the shift of line of apsides and the shift of epoch change direction, do not coincide, as one would expect, is due to the minor contribution of the shift of nodes to the orbital period. In conclusion it can be said that

> The oblateness of the Earth—the by far biggest contribution to a variation of orbital elements—only changes the orientation of the elliptic orbit, but not its size and its shape.

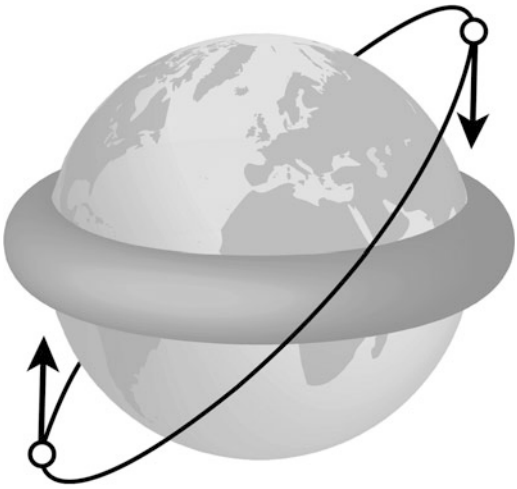


Fig. 12.8 Oblateness of the Earth interpreted as an equatorial bulge attracting the orbiting body

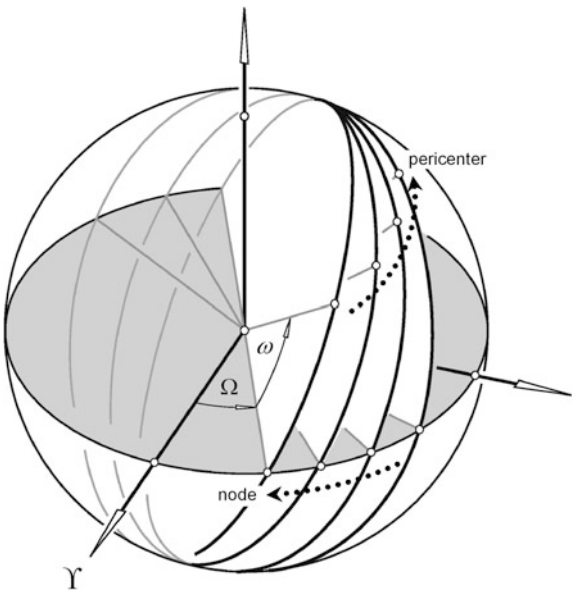


Fig. 12.9 The joint orbital regression of nodes and progression of the line of apsides

The Perturbed Elliptic Orbit

Having derived the variation of orbital elements under oblateness perturbation, we are now able to determine the trajectory in the orbital plane. To include oblateness perturbations in the orbit equation we have to reconsider the motion in the orbit. Under gravitational perturbations the body in the orbit suffers the positional shift $\left(\dot{M} - n\right)t$ and the rotation of the orbit's line of apsides is $\dot{\omega}\, t$. For the radial position only the position relative to the periapsis matters. By assuming for the sake of simplicity $t_0 = 0$ and $\omega(t = 0) = 0$, we have for the argument of periapsis $\omega(t) = \dot{\omega} \cdot t$ with $\dot{\omega} = const$. According to Sect. 12.1.2 the orbit equation 7.3.5 therefore reads

$$r = \frac{p}{1 + e \cdot \cos(\theta - \omega)} = \frac{p}{1 + e \cdot \cos(\theta - \dot{\omega} \cdot t)}$$

We now assume $e \ll 1$ and because $\dot{\omega}t = 2j_2\left(1 - \frac{5}{4}\sin^2 i\right)nt \ll 1$ we can safely approximate $nt = M \approx \theta$ and therefore

$$\frac{p}{r(\theta)} = 1 + e \cdot \cos\left\{\left[1 + 2j_2\left(1 - \frac{5}{4}\sin^2 i\right)\right] \cdot \theta\right\}$$

Such a rotating ellipse is depicted in Fig. 12.2. We recall that

$$j_2 = \frac{3}{2}J_2\left[\frac{R_\oplus}{a(1-e^2)}\right]^2 \approx \frac{3}{2}J_2\left[\frac{R_\oplus}{a}\right]^2 > 0$$

and because $\dot{\omega}t = 2j_2\left(1 - \frac{5}{4}\sin^2 i\right)nt \ll 1$, only an approximate value of a needs to be known.

For Kepler's equation (7.4.15) only positional shift matters and therefore $M(t) = \left[n + \left(\dot{M} - n\right)\right]\left(t - t_p\right)$. Hence from Eq. (12.3.15) follows

$$E(t) - e\ \sin E(t) = \left[1 + j_2\sqrt{1-e^2}\left(1 - \frac{3}{2}\sin^2 i\right)\right] \cdot n\left(t - t_p\right)$$

Note that E by definition measures the eccentric angle relative to the periapsis (see Fig. 7.13), which itself shifts owing to the apsidal precession $\dot{\omega}$. The combined shift is the draconitic motion as given in Eq. (12.4.1).

12.3.4 Higher-Order Perturbations

According to Lagrange's planetary equations (12.2.7) and (12.2.3) the variations of orbital elements caused by a perturbation R of degree n are of magnitude $R/h \propto 1/hr^{n+1} \approx 1/a^{n+3/2}$. Therefore the impact of higher-order perturbations on an orbit quickly diminishes with increasing orbital radii and increasing degree n as can be observed in Fig. 12.1. For instance, the radius of a geostationary orbit, $r = a \approx 42{,}000$ km, is larger by a factor 6.2 than those of LEO. Hence the secular variations of the orbital elements caused by the biggest perturbation, i.e., Earth's oblateness (see Fig. 12.4a), are smaller by a factor of $6.2^{-3.5} \approx 0.17\ \%$ and therefore negligible in the short-term. According to Table 12.2 this is even more so because the corresponding multipole coefficients also decline with increasing degree n and also with increasing order m. Thus the most considerable perturbation contribution stems from the first-order oblateness perturbations. Nonetheless, for high-precision long-term orbit propagation calculations sectorial perturbations of second order and the perturbations of at least third order have to be considered. In practice, this can be done numerically (see Sect. 12.2.4). In the following we want to explore the impact of higher-order perturbations at least qualitatively.

Higher-order perturbations can be distinguished into:

- **Higher-order multipole perturbations**
  They are due to higher-order multipoles of the gravity field with terms of $J_{22}, J_{3m}, J_{4m} \ldots$.

- **Higher-order oblateness perturbations**
  The LPEs (12.2.7) are the result of first-order perturbation theory. Applying second-order perturbation theory leads to non-linear LPEs (The non-linearity arises from short-period perturbations $\Delta i_{short}, \Delta e_{short}$ due to J_2 oblateness perturbations, which in turn cause both secular and long-periodic perturbations of order J_2^2 via the terms $\partial R/\partial i, \partial R/\partial e$ on the right-hand side of the LPEs (12.2.7)).

Because according to Table 12.2, second part

$$J_2 = O\left(10^{-3}\right)$$
$$J_{22}, J_3, J_{31}, J_4, J_2^2 = O\left(10^{-6}\right)$$

the oblateness perturbation is by far the leading effect, while all of the latter perturbations are equally relevant (gray shaded in Table 12.2, second part). In the following we will determine their corresponding average perturbation potentials. We conclude that the higher-order oblateness perturbations are of the same order of magnitude as the higher-order multipole perturbations, namely $O\left(10^{-6}\right)$, and hence have taken into account in general, which often is not done in the literature.

Average Zonal Harmonics
We first consider the average of the zonal harmonics over one orbit

$$\bar{P}_n \equiv \bar{P}_n^0 = \left\langle (1 + e\cos\theta)^{n-1} P_n(\sin\beta) \right\rangle_{P,\theta}$$

with $P_n(x)$ given in Eq. (12.2.6). In general, the averaging integrals can be treated by applying the power reduction formulas

$$\sin^n x = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} \cos\left[(n-2k)\left(x - \frac{\pi}{2}\right)\right]$$

$$\cos^n x = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} \cos[(n-2k)x]$$

$$x = \sin\beta = \sin i \cdot \sin(\theta + \omega)$$

to reduce the $P_n(x)$ to trigonometric expressions with terms $\sin m(\theta+\omega)$, $\cos m(\theta+\omega)$, $m \le n$. These in turn can be expressed through trigonometric angle sum identities by the terms $\sin m\theta$, $\cos m\theta$, $m \le n$. By expanding the integrands it is easily verified that quite generally

$$\left\langle (1+e\,\cos\theta)^{n-1}\sin m\theta\right\rangle_{P,\theta}=0 \quad @\ 0\le m\le n$$

$$\left\langle (1+e\,\cos\theta)^{n-1}\cos m\theta\right\rangle_{P,\theta}=0 \quad @\ m=n$$

while for $n=2-5$ the following relevant averaging integrals hold exactly:

$$\langle (1+e\,\cos\theta)\rangle_{P,\theta}=1, \qquad \langle (1+e\,\cos\theta)\cos\theta\rangle_{P,\theta}=\frac{1}{2}e$$

$$\left\langle (1+e\,\cos\theta)^2\right\rangle_{P,\theta}=1+\frac{1}{2}e^2, \qquad \left\langle (1+e\,\cos\theta)^2\cos\theta\right\rangle_{P,\theta}=e,$$
$$\left\langle (1+e\,\cos\theta)^2\cos 2\theta\right\rangle_{P,\theta}=\frac{1}{4}e^2$$

$$\left\langle (1+e\,\cos\theta)^3\right\rangle_{P,\theta}=1+\frac{3}{2}e^2, \qquad \left\langle (1+e\,\cos\theta)^3\cos\theta\right\rangle_{P,\theta}=\frac{3}{8}e\left(4+e^2\right),$$
$$\left\langle (1+e\,\cos\theta)^3\cos 2\theta\right\rangle_{P,\theta}=\frac{3}{4}e^2, \qquad \left\langle (1+e\,\cos\theta)^3\cos 3\theta\right\rangle_{P,\theta}=\frac{1}{8}e^3$$

$$\left\langle (1+e\,\cos\theta)^4\right\rangle_{P,\theta}=1+3e^2+\frac{3}{8}e^4, \qquad \left\langle (1+e\,\cos\theta)^4\cos\theta\right\rangle_{P,\theta}=\frac{1}{2}e\left(4+3e^2\right),$$
$$\left\langle (1+e\,\cos\theta)^4\cos 2\theta\right\rangle_{P,\theta}=\frac{1}{4}e^2\left(6+e^2\right), \quad \left\langle (1+e\,\cos\theta)^4\cos 3\theta\right\rangle_{P,\theta}=\frac{1}{2}e^3,$$
$$\left\langle (1+e\,\cos\theta)^4\cos 4\theta\right\rangle_{P,\theta}=\frac{1}{16}e^4$$

Note that

$$\left\langle (1+e\,\cos\theta)^{n-1}\begin{bmatrix}\sin m\theta\\ \cos m\theta\end{bmatrix}\right\rangle_{P,\theta}=\left(1-e^2\right)^{n-1}\left\langle \left(\frac{a}{r}\right)^{n-1}\begin{bmatrix}\sin m\theta\\ \cos m\theta\end{bmatrix}\right\rangle_{P,\theta}$$

The latter expressions are usually provided in literature. Explicitely we derive by the help of a symbolic integrator the following exact equations

$$\begin{aligned}
\bar{P}_2&=-\frac{1}{2}\left(1-\frac{3}{2}\sin^2 i\right)\\
\bar{P}_3&=-\frac{3}{2}e\,\sin i\,\sin\omega\left(1-\frac{5}{4}\sin^2 i\right)\\
\bar{P}_4&=\frac{3}{8}\left\{\begin{array}{l}\left[1-5\sin^2 i\left(1-\frac{7}{8}\sin^2 i\right)\right]\left(1+\frac{3}{2}e^2\right)\\ -\frac{15}{4}e^2\cos 2\omega\cdot\ \sin^2 i\left(1-\frac{7}{6}\sin^2 i\right)\end{array}\right\}\\
\bar{P}_5&=\frac{15}{4}e\,\sin i\left\{\begin{array}{l}\sin\omega\left[1-\frac{7}{2}\sin^2 i\left(1-\frac{3}{4}\sin^2 i\right)\right]\left(1+\frac{3}{4}e^2\right)\\ +\ \frac{7}{24}e^2\sin 3\omega\cdot\ \sin^2 i\left(1-\frac{9}{8}\sin^2 i\right)\end{array}\right\}
\end{aligned} \tag{12.3.17}$$

We recall that all orbital elements in these equations and from now on in Sect. 12.3 are meant to be mean orbital values, i.e., $a \equiv \bar{a}$, $e \equiv \bar{e}$, $i \equiv \bar{i}$, $\omega \equiv \bar{\omega}$, $\Omega \equiv \bar{\Omega}$, $M \equiv \bar{M}$.

Average Tesseral and Sectorial Harmonics

Although tesseral and sectorial harmonics do not contribute to secular perturbations, as we will see later, they have a significant effect on long-periodic dynamics. Rather than expounding the derivation of the mean perturbational potentials (see Zhong and Gurfil 2013) we just provide them here

$$\begin{aligned}
\bar{P}_2^2 &= -\frac{3}{2}\sin^2 i \cdot \cos 2(\Omega_{22} - \Omega) \\
\bar{P}_3^1 &= \frac{3}{2}e\left\{\sin(\Omega_{31} - \Omega)\sin\omega\cos i\left(1 - \frac{15}{4}\sin^2 i\right) + \cos(\Omega_{31} - \Omega)\cos\omega\left(1 - \frac{5}{4}\sin^2 i\right)\right\} \\
\bar{P}_3^2 &= -\frac{15}{2}e\sin^2 i\left\{\sin 2(\Omega_{32} - \Omega)\cos\omega\cos i - \cos 2(\Omega_{32} - \Omega)\sin\omega\left(1 - \frac{3}{2}\sin^2 i\right)\right\} \\
\bar{P}_3^2 &= -\frac{15}{2}e\sin^2 i\left\{\sin 2(\Omega_{32} - \Omega)\cos\omega\cos i - \cos 2(\Omega_{32} - \Omega)\sin\omega\left(1 - \frac{3}{2}\sin^2 i\right)\right\}
\end{aligned} \tag{12.3.18}$$

with

$$\Omega_{nm} := \lambda_{nm} + \theta_{GMST}$$

where λ_{nm} are given in Table 12.2, second part, and θ_{GMST} is the *Greenwich Mean Sidereal Time*, which is the hour angle from the vernal point (First Point of Aries) to the Greenwich Meridian in the equatorial plane. We recall from Eq. (12.3.11) that the averaged perturbation potentials are derived from these as

$$\bar{R}_{nm} = -\frac{2}{3}\frac{\mu_\oplus}{a}\sqrt{1 - e^2}\, j_{nm}(e) \cdot \bar{P}_n^m$$

Second-Order Oblateness Perturbations

We finally consider the average potential of the second-order oblateness perturbation (indicated by a double dash). This is given by (see e.g. Zhong and Gurfil 2013) as

$$\begin{aligned}
\bar{P}_2'' &= -\frac{33}{16}\left(1 - \frac{21}{11}\sin^2 i + \frac{19}{22}\sin^4 i\right) + O(e^4) \\
&\quad - \frac{3}{32}e^2\left[5\left(1 - 2\sin^2 i + \frac{23}{20}\sin^4 i\right) + 7\sin^2 i\left(1 - \frac{15}{14}\sin^2 i\right)\cos 2\omega\right] \\
\bar{R}_2'' &= -\frac{4}{9}\frac{\mu_\oplus}{a}\sqrt{1 - e^2}\, j_2^2(e) \cdot \bar{P}_2''
\end{aligned} \tag{12.3.19}$$

Mean Variation of Orbital Elements

Inserting the above averaged perturbation potentials into the LPEs (12.2.7) would yield in extension to Eq. (12.3.15) the full-fledged set for the long-period variation of orbital elements (Zhong et Gurfil 2013; Chao 2005, p. 22; Liu 1980; Cook 1966; Shapiro 1995). This, however, not only is quite long-winded but also unreasonable because most satellites in Earth's orbits have small eccentricities. We therefore provide here the set of equations reduced to $e \ll 1$. They read (recall that here $a \equiv \bar{a},\ e \equiv \bar{e},\ i \equiv \bar{i},\ \omega \equiv \bar{\omega},\ \Omega \equiv \bar{\Omega},\ M \equiv \bar{M}$ and $J_{21} = 0$)

$$\dot{a} = 0 \tag{12.3.20}$$

$$\begin{aligned}\frac{1}{n}\dot{e} = &-\left[j_{n,odd}\right]\cos\omega + j\begin{bmatrix}\sin(\Omega_{31}-\Omega)\cos\omega\,\cos i\left(1-\frac{15}{4}\sin^2 i\right)\\ -\cos(\Omega_{31}-\Omega)\sin\omega\left(1-\frac{5}{4}\sin^2 i\right)\end{bmatrix}\\ &+O\left(j_2^2 e\,\sin 2\omega, j_4 e\,\sin 4\omega, j_{41} e\,\sin\Omega\,\cos\omega\right)\end{aligned}$$

$$\begin{aligned}\frac{1}{n}\dot{i} = &-2j_{22}\sin 2(\Omega_{22}-\Omega)\sin i\\ &+O(j_3 e\,\cos 3\omega, j_{31} e\,\cos\omega\,\sin\Omega, j_{41} e\,\cos\omega\,\sin\Omega)\end{aligned}$$

$$\begin{aligned}\frac{1}{n}\dot{\omega} = &\,2j_2\left(1-\frac{5}{4}\sin^2 i\right) - 2j_{22}\cos 2(\Omega_{22}-\Omega)\left(1-\frac{5}{2}\sin^2 i\right)\\ &+\frac{1}{e}\left\{\left[j_{n,odd}\right]\sin\omega - j_{31}\begin{bmatrix}\sin(\Omega_{31}-\Omega)\sin\omega\,\cos i\left(1-\frac{15}{4}\sin^2 i\right)\\ +\cos(\Omega_{31}-\Omega)\cos\omega\left(1-\frac{5}{4}\sin^2 i\right)\end{bmatrix}\right\}\\ &+j_2^2\left[6\left(1-\frac{79}{72}\sin^2 i\right)\left(1-\frac{5}{4}\sin^2 i\right) - \frac{7}{12}\sin^2 i\left(1-\frac{15}{14}\sin^2 i\right)\cos 2\omega\right]\\ &-5j_4\left[\left(1-\frac{31}{8}\sin^2 i+\frac{49}{16}\sin^4 i\right)+\frac{3}{8}\sin^2 i\left(1-\frac{7}{6}\sin^2 i\right)\cos 2\omega\right]\\ &+O(j_6, j_3 e\,\cos 3\omega, j_{31} e\,\cos\omega\,\sin\Omega, j_{41} e\,\cos\omega\,\sin\Omega)\end{aligned}$$

$$\begin{aligned}\frac{1}{n}\dot{\Omega} = &-\cos i\left[j_2+\frac{5}{2}j_2^2\left(1-\frac{19}{15}\sin^2 i\right)-\frac{5}{2}j_4\left(1-\frac{7}{4}\sin^2 i\right)\right]\\ &+2j_{22}\cos 2(\Omega_{22}-\Omega)\cos i\\ &+O(j_6, j_3 e\,\cos 3\omega, j_{31} e\,\cos\omega\,\sin\Omega, j_{41} e\,\cos\omega\,\sin\Omega)\end{aligned}$$

$$\begin{aligned}\frac{1}{n}(\dot{M}-n) &= j_2\left(1-\frac{3}{2}\sin^2 i\right)\sqrt{1-e^2}+3j_{22}\cos 2(\Omega_{22}-\Omega)\sin^2 i\\ &\quad -\frac{1}{e}\left\{\left[j_{n,odd}\right]\sin\omega - j_{31}\left[\begin{array}{l}\sin(\Omega_{31}-\Omega)\sin\omega\,\cos i\left(1-\frac{15}{4}\sin^2 i\right)\\ +\cos(\Omega_{31}-\Omega)\cos\omega\left(1-\frac{5}{4}\sin^2 i\right)\end{array}\right]\right\}\\ &\quad +\frac{3}{2}j_2^2\left[\left(1-\frac{49}{18}\sin^2 i+\frac{137}{72}\sin^4 i\right)+\frac{7}{18}\sin^2 i\left(1-\frac{15}{14}\sin^2 i\right)\cos 2\omega\right]\\ &\quad +\frac{15}{8}j_4\sin^2 i\left(1-\frac{7}{6}\sin^2 i\right)\cos 2\omega\\ &\quad -\frac{35}{24}j_6\left(1-\frac{21}{2}\sin^2 i+\frac{189}{8}\sin^4 i-\frac{231}{16}\sin^6 i+O(\cos 4\omega)\right)\\ &\quad +O(j_8, j_3 e\,\cos 3\omega, j_{31}e\,\cos\omega\,\sin\Omega, j_{41}e\,\cos\omega\,\sin\Omega)\end{aligned}$$

with

$$\Omega_{nm} := \lambda_{nm} + \theta_{GMST}$$

where λ_{nm} are given in Table 12.2, second part, and θ_{GMST} is the *Greenwich Mean Sidereal Time*, which is the hour angle from the vernal point (First Point of Aries) to the Greenwich Meridian in the equatorial plane, and the sum of odd zonals terms

$$\begin{aligned}\left[j_{n,odd}\right] &:= -\frac{2}{3}\sum_{n\ odd} j_n\frac{n-1}{n(n+1)}P_n^1(0)P_n^1(\cos i)+O\left(j_3 e^2\right)\\ &= \sin i\left[\begin{array}{l}j_3\left(1-\frac{5}{4}\sin^2 i\right)-\frac{5}{2}j_5\left(1-\frac{7}{2}\sin^2 i+\frac{21}{8}\sin^4 i\right)\\ +\frac{35}{8}j_7\left(1-\frac{27}{4}\sin^2 i+\frac{99}{8}\sin^4 i-\frac{429}{64}\sin^6 i\right)\\ -\frac{105}{16}j_9\left(1-11\sin^2 i+\frac{143}{4}\sin^4 i-\frac{715}{16}\sin^6 i+\frac{2431}{128}\sin^8 i\right)\\ +O(j_{11})+O(j_3 e^2)\end{array}\right]\end{aligned} \tag{12.3.21}$$

where

$$P_n^1(\cos i) = \frac{\sin i}{2^n}\sum_{k=0}^{(n-1)/2}(-1)^k\frac{(2n-2k)!}{(n-1-2k)!(n-k)!k!}(\cos i)^{n-1-2k}$$

The variations listed above include long-periodic variations (terms with ω, Ω) plus secular variations (see below). Observe that the higher-order oblateness perturbations j_2^2 contribute only to $\dot{\omega}, \dot{M}$.

Short-Periodic Deviations

We recall that in Eq. (12.3.20) all the orbital elements and their variations are meant to be values averaged over one orbital period. The osculating elements exhibit the following small short-periodic deviations (see Eq. (12.3.3)) from these averaged values in the course of one orbital period (Kozai 1959, see Zhong et Gurfil 2013)

$$\begin{aligned}
\frac{a_{short}}{a} &= j_2 \sin^2 i \cdot \cos 2(\omega+\theta) \\
e_{short} &= -\frac{1}{2} j_2 \sin^2 i \left[\begin{array}{l} \cos(2\omega+\theta) + \frac{1}{3}\cos(2\omega+3\theta) \\ -\,3\cos\theta \cdot \cos(2\omega+2\theta) \end{array} \right] \quad @\ e \ll 1 \qquad (12.3.22) \\
i_{short} &= \frac{1}{4} j_2 \sin 2i \cdot \cos 2(\omega+\theta)
\end{aligned}$$

and hence they all are of the order $j_2 \approx 10^{-3}$. Observe that although the gravitational potential is a conservative one and hence $\langle a \rangle_P = const$ (see Box *Conservative Force* in Sect. 7.1.2 and Eq. (12.3.1)) the osculating a varies slightly over one orbit.

Secular Variations

From Eq. (12.3.20) and with Eq. (12.3.2) we can directly read off the total secular variations

$$\begin{aligned}
\dot{a} \equiv \dot{a}_{\text{sec}} &= 0, \\
\dot{e}_{\text{sec}} &= 0, \\
\dot{i}_{\text{sec}} &= 0
\end{aligned}$$

$$\begin{aligned}
\frac{1}{n}\dot{\omega}_{\text{sec}} &= 2j_2\left(1-\frac{5}{4}\sin^2 i\right) + 6j_2^2\left(1-\frac{79}{72}\sin^2 i\right)\left(1-\frac{5}{4}\sin^2 i\right) \\
&\quad - 5j_4\left(1-\frac{31}{8}\sin^2 i + \frac{49}{16}\sin^4 i\right) + O(j_6) \qquad (12.3.23) \\
&= 2j_2\left(1-\frac{5}{4}\sin^2 i\right)\left[1+O\left(10^{-3}\right)\right]
\end{aligned}$$

$$\begin{aligned}
\frac{1}{n}\dot{\Omega}_{\text{sec}} &= -\cos i\left[j_2 + \frac{5}{2}j_2^2\left(1-\frac{19}{15}\sin^2 i\right) - \frac{5}{2}j_4\left(1-\frac{7}{4}\sin^2 i\right) + O(j_6)\right] \\
&= -j_2\cos i\left[1+O\left(10^{-3}\right)\right]
\end{aligned}$$

$$\begin{aligned}
\frac{1}{n}\left(\dot{M}-n\right) &= j_2\left(1-\frac{3}{2}\sin^2 i\right)\sqrt{1-e^2} + \frac{3}{2}j_2^2\left\{\left(1-\frac{49}{18}\sin^2 i + \frac{137}{72}\sin^4 i\right)\right\} \\
&\quad - \frac{35}{24}j_6\left(1-\frac{21}{2}\sin^2 i + \frac{189}{8}\sin^4 i - \frac{231}{16}\sin^6 i\right) + O(j_8) \\
&= j_2\left(1-\frac{3}{2}\sin^2 i\right)\sqrt{1-e^2}\left[1+O\left(10^{-3}\right)\right]
\end{aligned}$$

This means that the secular effects of the gravitational potential are essentially due to the oblateness perturbation. Observe that only even zonal harmonics contribute to secular effects, with the peculiarity that the fourth zonal harmonic does not contribute to $\dot{M}$ but only higher even orders of zonal harmonics. All other terms in Eq. (12.3.20) are long-periodic variations of the slow variables ω and Ω.

Near-Circular Orbits

According to Eq. (12.3.20) the variations of the orbital elements seem to be highly complicated. Though the upshot is quite easy: The variations are governed by the oblateness variations with factor $j_2 = O(10^{-3})$. All other effects owing to higher order harmonics are of order $j_{22}, j_2^2, j_3, j_{31}, j_4, j_6 = O\left(10^{-6}\right)$. They all contribute with either a small secular effect or a modulation of the oblateness effects. There exists, however, one exception from this rule, namely for near-circular orbits, when $e < 0.01$. Then the braces with the factor $1/e$ become significant, i.e., $O\left(\left[j_{n,odd}\right]/e, j_{31}/e\right) \approx O(10^{-3}) \approx O(j_2)$. Yet, the divergence of the $1/e$-term for $e \to 0$ is of no practical significance because in this limit there is no well-defined periapsis. Thus any tiny change of the orbital shape due to a gravitational anisotropy brings about an increasing variation in the position of the periapsis, which causes a diverging $\dot{\omega}$ and $\dot{M} - n$. This is a theoretical artifact that can also be seen from the fact that the orbital motion relative to Earth's surface, the so-called mean draconitic motion n_Ω (see Sect. 12.4.1), should be unaffected for $e \to 0$ and indeed in the expression $n_\Omega = n + \dot{\omega} + \left(\dot{M} - n\right)$ the j_3-terms cancel out each other. This holds for all higher perturbations that can be seen from the LPEs Eq. (12.2.7) where for $\dot{\omega} + \left(\dot{M} - n\right)$ the critical terms $e^{-1}\partial R/\partial e$ cancel out each other for $e \to 0$.

Nevertheless, we are interested how the odd harmonics affect the orbital elements $\dot{e}, \dot{\omega}$ for near-circular orbits. To study this we rewrite Eq. (12.3.20)

$$\begin{aligned}
\dot{e} &= -n\left[j_{n,odd}\right]\cos\omega \\
&\quad + nj_{31}\left[\begin{array}{l}\sin(\Omega_{31}-\Omega)\cos\omega\cos i\left(1-\frac{15}{4}\sin^2 i\right) \\ \qquad -\cos(\Omega_{31}-\Omega)\sin\omega\left(1-\frac{5}{4}\sin^2 i\right)\end{array}\right] \\
\dot{\omega} &= \dot{\omega}_{\text{sec}} + \underbrace{\dot{\omega}_{long}(\omega,\Omega)}_{negligible} \\
&\quad + \frac{n}{e}\left\{\left[j_{n,odd}\right]\sin\omega - j_{31}\left[\begin{array}{l}\sin(\Omega_{31}-\Omega)\sin\omega\cos i\left(1-\frac{15}{4}\sin^2 i\right) \\ \quad +\cos(\Omega_{31}-\Omega)\cos\omega\left(1-\frac{5}{4}\sin^2 i\right)\end{array}\right]\right\}
\end{aligned} \tag{12.3.24}$$

Since $\dot{\omega}_{long}(\omega,\Omega) = O\left(10^{-6}\right) \ll 1/e\left\{O\left(10^{-6}\right)\right\}$ we neglect this long-periodic term.

Effect of Odd Zonal Harmonics

For a preparatory treatment of these differential equations we also neglect the long-periodic j_{31}-terms for the time being arriving at

$$\begin{aligned}
\dot{e} &= -n\left[j_{n,odd}\right]\cos\omega \\
\dot{\omega} &= \dot{\omega}_{\text{sec}} + \frac{n}{e}\left[j_{n,odd}\right]\sin\omega
\end{aligned} \tag{12.3.25}$$

Observe that from Table 12.2, second part and Eq. (12.3.10) $O(j_3) = 10^{-6}$, but $O(j_5, j_7, j_9) = 10^{-7}$, and therefore

$$\dot{\omega}_{\text{sec}} \approx 2nj_2\left(1 - \frac{5}{4}\sin^2 i\right)$$
$$\left[j_{n,odd}\right] \approx j_3 \sin i\left(1 - \frac{5}{4}\sin^2 i\right)$$

We therefore define the more physical parameters

$$\begin{aligned} n_\omega &:= \dot{\omega}_{\text{sec}} = const \approx 2nj_2\left(1 - \frac{5}{4}\sin^2 i\right) \\ e_\omega &:= -\frac{n\left[j_{n,odd}\right]}{n_\omega} \approx -\frac{1}{2}\frac{j_3}{j_2}\sin i \\ &= 1.170 \times 10^{-3}\frac{R_\oplus}{a}\sin i \approx 10^{-3}\sin i < 0.001 \end{aligned} \tag{12.3.26}$$

With these we rewrite the differential equations

$$\begin{aligned} \dot{e} &= e_\omega n_\omega \cos\omega \\ \dot{\omega} &= n_\omega\left(1 - \frac{e_\omega}{e}\sin\omega\right) \end{aligned} \tag{12.3.27}$$

We treat these coupled differential equations by showing that from these follows

$$\frac{d}{dt}(e\,\sin\omega) = \dot{e}\,\sin\omega + e\,\dot{\omega}\,\cos\omega = n_\omega \cdot\, e\,\cos\omega$$

This gives rise to the substitution

$$x := e\,\sin\omega$$
$$y := e\,\cos\omega$$

With this we rewrite the above differential equations

$$\dot{x} = n_\omega y$$
$$\dot{y} = -n_\omega(x - e_\omega)$$

Thus we have transformed the problem into a set of coupled non-divergent linear differential equations. The solution is the well-known harmonic oscillator in 2 dimensions

$$x - e_\omega = r\,\sin(n_\omega t + \varphi)$$
$$y = r\,\cos(n_\omega t + \varphi)$$

where the parameters r, φ are given by the initial conditions. Since $(x - e_\omega)^2 + y^2 = r^2$, we recognize that the vector $(e \sin\omega,\ e \cos\omega)$ in the x, y reference frame is rooted at the point $(e_\omega,\ 0)$ with its tip rotating at orbital frequency n_ω on a circle with radius r. In the orbital plane $(e \sin\omega,\ e \cos\omega)$ is the eccentricity vector with component $e \sin\omega$ along the line of apsides. Due to the offset e_ω between both reference frames the angular velocity $\omega(t)$ and length $e(t)$ of the eccentricity vector varies in the course of its rotation. We recall that factually nothing rotates, but only the line of apsides of the virtual osculating orbit. To determine $e(t), \omega(t)$ we make use of $e = \sqrt{x^2 + y^2}$ and $\cos\omega(t) = y/e$. Hence we finally arrive at the solution of the differential equations (12.3.27)

$$\begin{aligned} e(t) &= \sqrt{e_\omega^2 + r^2 + 2e_\omega r \sin(n_\omega t + \varphi)} \\ \cos\omega(t) &= \frac{r}{e(t)} \cos(n_\omega t + \varphi) \end{aligned} \tag{12.3.28}$$

We define $e(0) =: e_0$ and $\omega(0) =: \omega_0$ as the initial conditions. This implies

$$\begin{aligned} r \sin\varphi &= e_0 \sin\omega_0 - e_\omega \\ r \cos\varphi &= e_0 \cos\omega_0 \end{aligned} \tag{12.3.29}$$

from which follows

$$\begin{aligned} r &= \sqrt{e_0^2 + e_\omega^2 - 2e_0 e_\omega \sin\omega_0} \\ \tan\varphi &= \tan\omega_0 - \frac{e_\omega}{e_0 \cos\omega_0} \end{aligned} \tag{12.3.30}$$

We now apply this general result to two significant and limiting cases.

In the first case, the initial eccentricity is $e_0 > e_\omega$, implying $e_0 \approx r > e_\omega$ and $\phi \approx \omega_0$. Then we derive from Eq. (12.3.28) and the general relation $\cos(\alpha + \varepsilon \cos\alpha) \approx \cos\alpha \cdot (1 - \varepsilon \sin\alpha)$ for $\varepsilon \ll 1$

$$\begin{aligned} e(t) &= e_0 + e_\omega \sin(n_\omega t + \omega_0) \\ \omega(t) &= \omega_0 + n_\omega t + \tfrac{e_\omega}{e_0}\left[\cos(n_\omega t + \omega_0) - \cos\omega_0\right] \end{aligned} \quad @ \begin{array}{l} e_0 > e_\omega \\ \approx 0.001 \end{array} \tag{12.3.31}$$

This result can be also found by directly integrating Eq. (12.3.27) assuming initially $\omega = \omega_0 + n_\omega t$. Hence the odd zonal harmonics, mainly the j_3-term, periodically modulate the eccentricity e_0 with modulation amplitude $10^{-3} \sin i$ and frequency n_ω (see Fig. 12.10, top). Note that very close to the critical inclination, $i = 63.435° \pm 0.06°$, where $1 - \frac{5}{4}\sin^2 i < 10^{-3}$, we have $n_\omega = \dot{\omega}_{\text{sec}} = n \cdot O\left(j_2^2, j_4\right) \approx n \cdot 10^{-6}$ and $\left[j_{n,odd}\right] \approx O(j_5) \approx 10^{-6}$, and hence the induced eccentricity becomes a maximum with $e_\omega = -n\left[j_{n,odd}\right]/n_\omega \approx 1$.

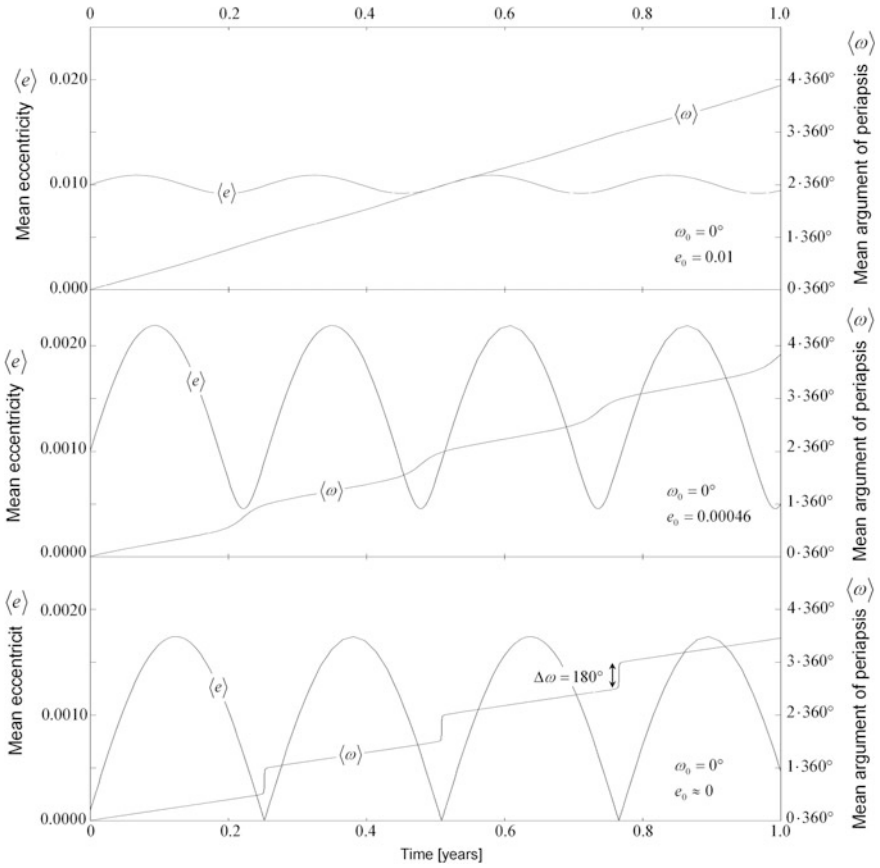


Fig. 12.10 Numerical computation of the evolution of the mean eccentricity and of the mean argument of periapsis of the ISS with $\iota = 51.6°$, $h = 350$ km for different e_0

The second significant case is when $e_0 < e_\omega$ and hence the second term in the brackets in Eq. (12.3.27) becomes significant. When $e \ll e_\omega$ the mean angular motion of the periapsis $\dot{\omega}$ becomes very big, i.e., the mean periapsis moves rapidly on the near-circular orbit. When the periapsis has turned by 180° the j_3-perturbational forces having decreased so far e will act inversely and now increase e. But once $e > e_\omega$ the excessive rotation stops and we are back to "normal". So, we recognize that the term $e_\omega / e \cdot \sin\omega$ ensures that the eccentricity is always positive. Mathematically speaking, if $e_0 = 0$, then we derive from Eq. (12.3.29) $\varphi = 270°$ and $r = e_\omega$ and we obtain from Eq. (12.3.28)

$$\begin{aligned} e(t) &= 2e_\omega \left|\sin\frac{n_\omega t}{2}\right| \\ \cos\omega(t) &= \frac{1}{2}\sin(n_\omega t)/\left|\sin\frac{n_\omega t}{2}\right| \end{aligned} \qquad @\ e_0 < e_\omega \tag{12.3.32}$$

The behavior of $e(t), \omega(t)$ in this case is depicted in Fig. 12.10 (bottom). Observe that when the eccentricity vanishes the periapsis jumps by 180° ensuring that $\sin\omega$ does not change sign when traversing null.

Example

The International Space Station at $h = 350$ km and $i = 51.6°$ and an assumed $e_0 = 0$ has $e_\omega = 8.66 \times 10^{-4}$ and $n_\omega = 7.68°\ \text{day}^{-1}$. Therefore the j_3-induced eccentricity is $e = 0.00173 \cdot \sin\omega$ and the cycle time is $T_c = 2\pi/n_\omega = 46.9$ days. The secular behavior of such e with the corresponding ω is depicted in Fig. 12.10 (bottom).

Effect of Odd Zonal Plus Tesseral Harmonics

We now consider also the contribution of the tesseral j_{31}-term in Eq. (12.3.24). It is an easy exercise to show that by the same line of treatment as above we get the following results

$$\begin{aligned}\dot{x} &= n_\omega y - n_{\cos\Omega}\cos(\Omega_{31} - \Omega)\\ \dot{y} &= -n_\omega(x - e_\omega) + n_{\sin\Omega}\sin(\Omega_{31} - \Omega)\end{aligned}$$

where

$$\begin{aligned}n_{\cos\Omega} &:= nj_{31}\left(1 - \frac{5}{4}\sin^2 i\right)\\ n_{\sin\Omega} &:= nj_{31}\cos i\left(1 - \frac{15}{4}\sin^2 i\right)\end{aligned} \tag{12.3.33}$$

We now make use of the fact that due to Earth's oblateness $\Omega(t)$ with high accuracy varies linearly, namely

$$\Omega(t) \approx \Omega_0 - n_\Omega \cdot t$$

$$n_\Omega := nj_2\cos i$$

The solution of the above set of differential equations then is

$$\begin{aligned}x - e_\omega &= r\ \sin(n_\omega t + \varphi) + r'\sin(n_\Omega t + \lambda)\\ y &= r\ \cos(n_\omega t + \varphi) + r''\cos(n_\Omega t + \lambda)\end{aligned}$$

with

$$\begin{aligned}&\lambda := \Omega_{31} - \Omega_0\\ &r' := \frac{n_\omega n_{\sin\Omega} + n_\Omega n_{\cos\Omega}}{n_\omega^2 - n_\Omega^2} = \frac{j_{31}\cos i\left(1 - \frac{5}{2}\sin^2 i\right)}{j_2\left(1 - 3\sin^2 i + \frac{25}{12}\sin^4 i\right)} = O\left(10^{-3}\right)\\ &r'' := \frac{n_\omega n_{\cos\Omega} + n_\Omega n_{\sin\Omega}}{n_\omega^2 - n_\Omega^2} = \frac{j_{31}\left(1 + \frac{13}{4}\sin^2 i + \frac{55}{24}\sin^4 i\right)}{j_2\left(1 - 3\sin^2 i + \frac{25}{12}\sin^4 i\right)} = O\left(10^{-3}\right)\end{aligned} \tag{12.3.34}$$

Since $O(e_\omega) = O(r') = O(r'') = 0.001$ while $O(r)$ depends on the initial orbit conditions we rewrite the solution

$$x = r\,\sin(n_\omega t + \varphi) + e_{\omega\Omega}$$
$$y = r\,\cos(n_\omega t + \varphi) + e_\Omega$$

as

$$\begin{aligned} e_{\omega\Omega}(t) &= r'\sin(n_\Omega t + \lambda) + e_\omega = O\big(10^{-3}\big) \\ e_\Omega(t) &= r''\cos(n_\Omega t + \lambda) = O\big(10^{-3}\big) \end{aligned} \tag{12.3.35}$$

and thus obtain

$$\begin{aligned} e(t) &= \sqrt{r^2 + 2r[e_{\omega\Omega}\sin(n_\omega t + \varphi) + e_\Omega\cos(n_\omega t + \varphi)] + e_\Omega^2 + e_{\omega\Omega}^2} \\ \cos\omega(t) &= \frac{r\cos(n_\omega t + \varphi) + e_\Omega}{e(t)} \end{aligned} \tag{12.3.36}$$

where r, φ are given by the initial conditions

$$\begin{aligned} r\,\sin\varphi &= e_0\sin\omega_0 - r'\sin\lambda - e_\omega \\ r\,\cos\varphi &= e_0\cos\omega_0 - r''\cos\lambda \end{aligned} \tag{12.3.37}$$

We now reconsider the above two limiting cases. For $e_0 > e_\omega$, implying $e_0 \approx r > e_{\omega\Omega}, e_\Omega$ and $\varphi \approx \omega_0$, we obtain by Eq. (12.3.36) and iterative direct integration of Eq. (12.3.24)

$$\begin{aligned} e(t) &= e_0 + [e_{\omega\Omega}(t)\sin(n_\omega t + \omega_0) + e_\Omega(t)\cos(n_\omega t + \omega_0)] \\ \omega(t) &= \omega_0 + n_\omega t + \tfrac{e_\omega}{e_0}[\cos(n_\omega t + \omega_0) - \cos\omega_0] \\ &\quad + \tfrac{n_{\sin\Omega} + n_{\cos\Omega}}{2(n_\Omega - n_\omega)e_0}[\sin[(n_\Omega - n_\omega)t + \lambda - \omega_0] - \cos(\lambda - \omega_0)] \\ &\quad - \tfrac{n_{\sin\Omega} - n_{\cos\Omega}}{2(n_\Omega + n_\omega)e_0}[\sin[(n_\Omega + n_\omega)t + \lambda + \omega_0] - \cos(\lambda + \omega_0)] \end{aligned} \quad @ \begin{array}{l} e_0 > e_\omega \\ \approx 0.001 \end{array} \tag{12.3.38}$$

We therefore recognize that

Complex Modulation of e_0, ω_0 for a Near-Circular LEO
For near-circular LEO orbits, $0.001 < e_0 < 0.01$, the constant eccentricity, $e = e_0$, and linearly increasing argument of periapsis, $\omega = \omega_0 + n_\omega t$, are complexly modulated with amplitudes of $O(10^{-3})$ chiefly owing to j_3, j_{31}-perturbations.

In the case of a close-circular orbit, $e_0 < e_\omega \approx 0.001$, we have from Eq. (12.3.37) for $e_0 = 0$

$$r = \sqrt{r''^2 \cos^2\lambda + (r'\sin\lambda + e_\omega)^2} = O(10^{-3}) \quad \begin{matrix} @\ e_0 < e_\omega \\ \approx 0.001 \end{matrix} \tag{12.3.39}$$

We therefore recognize that

Wild and Chaotic Modulation of e_0, ω_0 for a Close-Circular LEO
For initially close-circular LEO orbits, $e_0 < 0.001$, the eccentricity and the argument of periapsis wobble around an average value of $e_{ave}(t) \approx 0.001$ wildly in a very complex way with amplitude $O(10^{-3})$, chiefly owing to j_3, j_{31} perturbations.

In summary we see that no satellite is able to establish zero eccentricity over a finite fraction of time. Mainly j_3- and j_{31}-induced perturbations will at least bring about $e \approx 0.001$. This is why satellites often have a declared $e \approx 0.001$, though they should have $e = 0$. See, e.g., GPS satellites in Sect. 12.4.4.

Observe that as a result of a weak non-vanishing eccentricity, precise orbits, such as those of Earth observation satellites, in the long run need inclination maintenance, because we have from (12.3.20) $\dot{i}_{\text{sec}} = O(j_3 e)$.

Summary
In closing we note that if the argument of periapsis would not rotate by gravitational perturbations but would remain constant, then all orbital elements would grow unlimited. So, by and large it can be stated that the rotation of ω, primarily owing to Earth's oblateness and also to j_3- and j_{31}-perturbations, has a stabilizing influence on orbits.

Numerical simulations with higher multipole moments also reveal that beyond these results no new effects occur and that their contributions are very small. Further analysis of the influence of higher-order perturbations can be found in Chao (2005), Kaula (1966), Beutler (2005b), Vallado (2001), Groves (1960), Campan et al. (1995), Liu (1974), and Fortesque et al. (2003).

12.3.5 Sun-Synchronous Orbits

Is orbit perturbation good or bad for a mission? Well, it depends. To have an orbit shape and orientation devoid of perturbations might be good for some missions. However, remote sensing or communication satellites in LEO require orbits that remain steady relative to Earth's rotating surface or exactly recurr after some days. So, actually an orbit would be useful that is in sync with Earth's rotation. This begs

the question: Can orbit perturbation be utilized to achieve such a sync? The answer in many cases is "yes". In the following we study these cases.

Sun-synchronous orbits, SSOs, are orbits that maintain a constant orientation of the orbit plane toward the Sun in the course of a year. In this case the so-called *beta angle* β, which is the angle at which the sunlight strikes the orbital plane (see Sect. 14.1.1), is constant over the year. This is a useful property since the lighting conditions for the satellite's solar cells then are constant. Because the Earth is revolving around the Sun, sun-synchronicity implies that the orbit plane has to revolve by 360° within a year. Now, a constant beta angle can be achieved by employing the regression of nodes effect such that the nodal rate as given in first order by Eq. (12.3.15) equals that of the Earth's revolution around the Sun, i.e., if

$$\dot{\Omega} = -\sqrt{\frac{\mu_\oplus}{a^3}}\frac{3J_2R_\oplus^2}{2a^2(1-e^2)^2}\cos i = \frac{2\pi}{365.256363}\text{day}^{-1} = 1.7202 \times 10^{-2}\text{day}^{-1}$$

which implies

$$\cos i = -(1-e^2)^2\left(\frac{a}{R_\oplus}\right)^{7/2}\sqrt{\frac{R_\oplus^3}{\mu_\oplus}}\frac{2}{3J_2}1.7202 \times 10^{-2}\text{day}^{-1}$$

$$\cos i = -0.098917 \cdot (1-e^2)^2\left(\frac{a}{R_\oplus}\right)^{7/2} \tag{12.3.40}$$

The negative sign implies that SSOs are retrograde, $i > 90°$. In practice, for remote sensing missions circular or near-circular LEOs are chosen frequently with the semi-major axis determined by other mission constraints, such as a repeating ground track (see Sect. 12.4.1). If for such a remote sensing mission the inclination is chosen as given above, this orbit will be sun-synchronous. Sun-synchronous, repeat ground track orbits have typical altitudes of h = 700–900 km and therefore have inclinations 98.2°–99.0°. So-called *magic orbits* are sun-synchronous and type I frozen (see below) at $i = 116.57°$ and with an high eccentricity of $e = 0.345$.

Note that in particular those SSOs will receive continuous solar irradiation that have a beta angle of $\beta = 90°$. In this case the satellite's ground track is just along Earth's terminator. However, this does not provide good lighting conditions for remote sensing, which is why the beta angle is chosen smaller to have good lighting condition in the morning over the areas of interest. It can be shown from geometrical considerations that even these SSOs will never be in eclipse and hence are also permanently irradiated by the Sun if their beta angles obey the relation $\beta > \arcsin(R_\oplus/a)$ see Eq. (14.1.3). It should ne noted that lunisolar perturbations induce an inclination change, which for a typical Sun-synchronous orbit with $h =$ 850 km and $i = 98.8°$ amounts to the small value of $\dot{i}_{\text{sec}} \approx 0{-}0.047°$ year^{-1} depending on the beta angle β.

12.3.6 Frozen Orbits

For Earth observation, which is the most important application in LEO, it is of paramount importance to have observational consistency, i.e., to revisit any observation site under nearly identical circumstances. This can be achieved by a spacecraft in a polar, repeating Sun-synchronous orbit (this ensures constant lighting conditions), and with constant orbit geometry keeping the periapsis at the same latitude. We will now explore how such geometrical constancy can be maintained despite orbital perturbations.

There is nothing that can be done to suppress the regression of nodes $\dot{\Omega}$. But this does not impede observation, it only impacts the so-called access area, i.e., the surface area seen at a particular time from orbit. So only $\dot{a},\ \dot{e},\ \dot{i},\ \dot{\omega} \approx 0$ need to be achieved. Orbits that aside from $\dot{a},\ \dot{i} \approx 0$ also exhibit the properties $\dot{e}, \dot{\omega} \approx 0$ are called **frozen orbits** because the rotation of the apse line is stopped. From Eq. (12.3.20) for $\dot{\omega}$ it can be seen that frozen orbits can be obtained in two ways:

Type I Frozen Orbits
If the expression $1 - \frac{5}{4}\sin^2 i = 0$, i.e. if an orbit maintains the critical inclination

$$i_f = \arcsin\sqrt{4/5} = 63.43^\circ \text{ or } 116.57^\circ$$

the orbit is frozen. These are the co-called *Type I frozen orbits*. This frozen condition is utilized by the so-called *Magic, Cobra*, *Molniya,* and *Tundra orbits*. All these are highly elliptic orbits with orbital periods of exactly 3, 8, 12, and 24 h and with perigee/apogee at altitudes 525/7800, 800/27 000, 1000/39 358, and 5370/66 400 km, respectively, which are employed for communication services. The latter are frequently used by Russia. Due to their low pace over Russia at apogee they cover the high latitudes of Russian territory over most of the orbital period. Note that although the eccentricities of all these orbits are very large, $e = 0.3{-}0.7$, and $\dot{i},\ \dot{\Omega} \propto O(j_3 e) \approx 0$ still holds, because $J_3 \approx -2.339 \times 10^{-3} J_2$ (see Table 12.2).

Type II Frozen Orbits
Conditions $\dot{e}, \dot{\omega} \approx 0$ in Eq. (12.3.20) can also be achieved if $\omega_f := \omega = 90^\circ$ or 270° and when $\dot{\omega} = 0$. This implies that

$$0 = 2j_2\left(1 - \frac{5}{4}\sin^2 i\right) + \frac{1}{e}\left[j_{n,odd}\right]\sin\omega + 6j_2^2\left\{\left(1 - \frac{79}{72}\sin^2 i\right)\left(1 - \frac{5}{4}\sin^2 i\right)\right\} - 5j_4\left\{\left(1 - \frac{31}{8}\sin^2 i + \frac{49}{16}\sin^4 i\right)\right\}$$

We first consider only the second order contribution j_3. In this case

$$\left[j_{n,odd}\right] \approx j_3 \sin i\left(1-\frac{5}{4}\sin^2 i\right)$$

and

$$0 = 2j_2 + \frac{\sin i \, \sin\omega}{e} j_3$$

The eccentricity at which for $\omega_f = 90°$ the bracket vanishes is

$$e_{f0} = -\frac{1}{2}\frac{j_3}{j_2}\sin i = -\frac{1}{2}\frac{J_3 R_\oplus}{J_2 a}\sin i = 1.170 \times 10^{-3}\frac{R_\oplus}{a}\sin i \tag{12.3.41}$$

Because of $0 < e_f \ll 1$ it is always possible to achieve, orbits with these conditions, which are called *Type II frozen orbits*.

Taking into account higher-order multipole perturbation as well, we first neglect j_2^2, j_4 contributions. In this case and for $\omega_f = 90°$ we get

$$\begin{aligned} e_f &= \frac{\sin i}{2j_2\left(1-\frac{5}{4}\sin^2 i\right)}\left\{\begin{array}{l} -j_3\left(1-\frac{5}{4}\sin^2 i\right)+\frac{5}{2}j_5\left(1-\frac{7}{2}\sin^2 i+\frac{21}{8}\sin^4 i\right) \\ -\frac{35}{8}j_7\left(1-\frac{27}{4}\sin^2 i+\frac{99}{8}\sin^4 i-\frac{429}{64}\sin^6 i\right)+O(j_{11}) \\ +\frac{105}{16}j_9\left(1-11\sin^2 i+\frac{143}{4}\sin^4 i-\frac{715}{16}\sin^6 i+\frac{2431}{128}\sin^8 i\right) \end{array}\right\} \\ &= e_{f0}[1-O(j_5/j_3,)] = e_{f0}\left[1-O\left(10^{-1}\right)\right] \end{aligned} \tag{12.3.42}$$

Therefore, and as will be shown in the next section, these higher order perturbations j_n, $n \geq 5$, modify the coarse result as given in Eq. (12.3.41) by up to 20%.

To now also account for the j_2^2, j_4 contributions we define

$$e_{f+} = e_f(1+\varepsilon)$$

With this we derive with $\dot{\omega} = 0$ and $\cos\omega_f = \cos 2\omega_f = 0$ from Eq. (12.3.20) iteratively the conditional equation for ε

$$\begin{aligned} 0 = &-2j_2\left(1-\frac{5}{4}\sin^2 i\right)\varepsilon + 6j_2^2\left(1-\frac{79}{72}\sin^2 i\right)\left(1-\frac{5}{4}\sin^2 i\right) \\ &-5j_4\left(1-\frac{31}{8}\sin^2 i+\frac{49}{16}\sin^4 i\right) \end{aligned}$$

hence

$$\varepsilon = 3j_2\left(1-\frac{79}{72}\sin^2 i\right) - \frac{5}{2}\frac{j_4}{j_2}\left(1-\frac{31}{8}\sin^2 i+\frac{49}{16}\sin^4 i\right)/\left(1-\frac{5}{4}\sin^2 i\right)$$

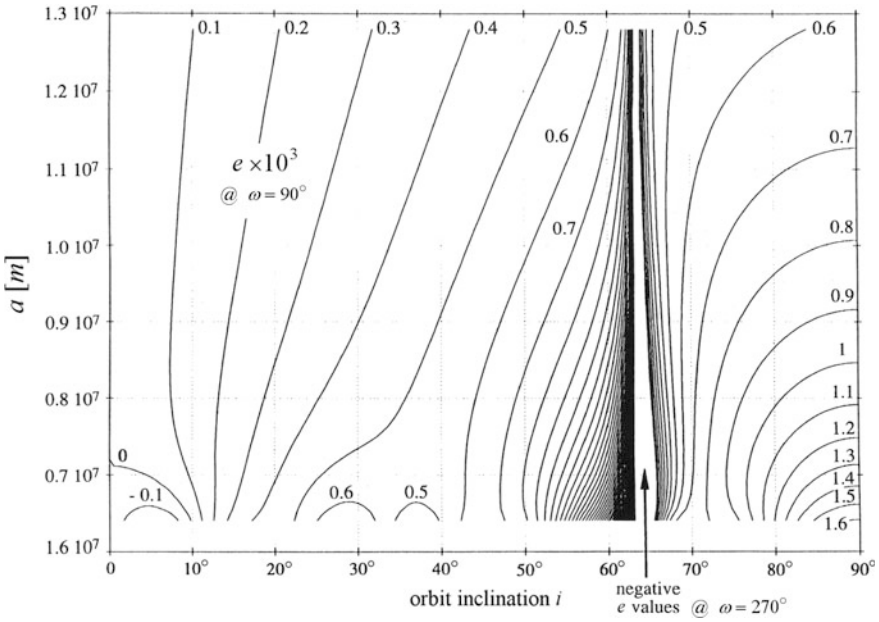


Fig. 12.11 Eccentricity e of type II frozen orbits multiplied by 1000 for prograde orbits, $0 \leq i \leq 90°$, and $R_\oplus \leq a \leq 2R_\oplus$. *Credit* Micheau (1995)

The magnitude of ε therefore is

$$O(\varepsilon) = O(j_2, j_4/j_2) = 10^{-3}$$

Hence, the j_2^2, j_4 contributions can be neglected. We therefore conclude that

> Only odd higher-order multipole perturbations and not even higher-order multipole perturbations or higher-order oblateness perturbations, j_2^2, affect type II frozen orbits significantly.

Since j_n are functions of a, the value of e_f may be parametrized as a function of a and i. Figure 12.11 depicts these dependencies for $0 \leq i \leq 90°$ and $R_\oplus \leq a \leq 2R_\oplus$. There are several things to note. First, for nearly all cases $e_f < 90°$, i.e. type II frozen orbits are all near-circular. Second, at the type I frozen condition $\frac{5}{4}\sin i = 1$, the type II eccentricity becomes ill-defined. Therefore, at the critical inclinations $i_f = \arcsin\sqrt{4/5} = 63.43°$ or $116.57°$, the orbit is naturally frozen (Type I frozen) for the entire range of eccentricity values e. Third, the majority of flown type II frozen orbits have $\omega_f = 90°$.

A good example of a type II frozen orbit is the orbit of the Topex/Poseidon satellite and its follow-ups *Jason 1* and *Jason 2* with $e = 9.5 \times 10^{-5}$, $i = 66.04°$, and $a = 7714.43$ km.

It may be questioned whether under the influence of drag and other perturbations in LEO frozen orbits remain frozen. The answer is affirmative concerning drag. However, solar radiation will gradually destroy the frozen condition. Therefore periodic orbit adjustments are needed to maintain the frozen orbital state.

12.3.7 Frozen Sun-Synchronous Orbits

By applying Eqs. (12.3.40) and (12.3.42) simultaneously it is possible to have a type II frozen Sun-synchronous orbit. From Eq. (12.3.40) follows

$$\sin^2 i = 1 - \cos^2 i = 1 - 0.0097845\left(\frac{a}{R_\oplus}\right)^7 =: 1 - \varepsilon$$

where $\varepsilon \approx 0.01$. Inserting this into Eq. (12.3.42) and expanding ε results for $\omega_f = 90^\circ$ in

$$e_f = -\frac{\sin i}{2j_2}\left\{j_3 + \frac{5}{4}\left[\left(j_5 + \frac{35}{32}j_7 + \frac{147}{128}j_9\right) - \left(9j_5 + \frac{385}{16}j_7 + \frac{5733}{128}j_9\right)\varepsilon\right]\right\}$$

Inserting $\varepsilon = 0.0097845(a/R_\oplus)^7$ and the j-values as given in Eq. (12.3.10) we finally get

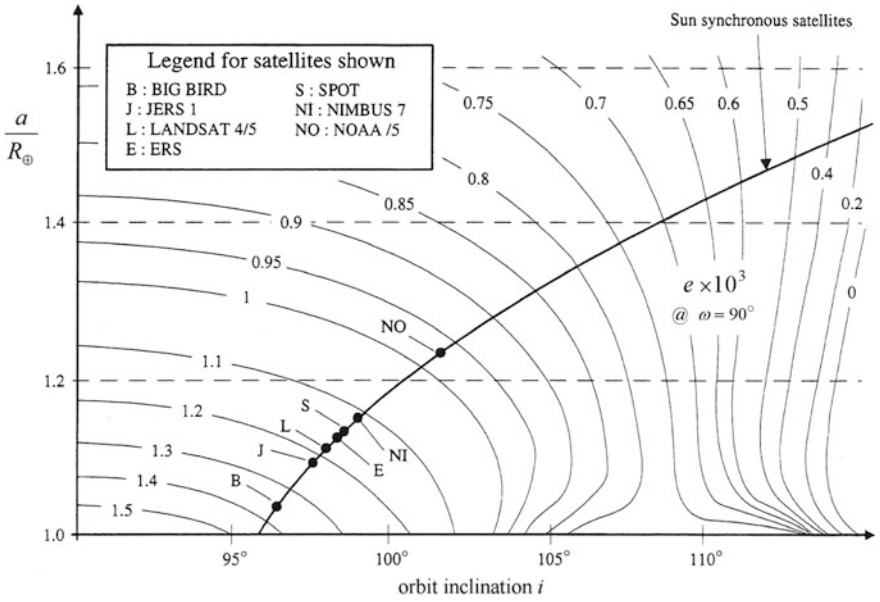


Fig. 12.12 Eccentricity e of Sun-synchronous orbits multiplied by 1000 for retrograde orbits, $90^\circ \le i \le 115^\circ$, and $R_\oplus \le a \le 1.6R_\oplus$. *Credit* Micheau (1995)

$$e_f = 1.170 \times 10^{-3}\gamma\sqrt{1 - 0.0097845\ \gamma^{-7}} \times \left\{1 + \frac{1}{8}\left[\begin{array}{c}\left(0.8975\ \gamma^2 + 1.5217\ \gamma^4 + 0.5470\ \gamma^6\right) \\ -\left(0.0791\ \gamma^{-5} + 0.328\ \gamma^{-3} + 0.208\ \gamma^{-1}\right)\end{array}\right]\right\} \tag{12.3.43}$$

where

$$\gamma = \frac{R_\oplus}{a} < 1$$

The "Sun-synchronous satellites"-line in Fig. 12.12 depicts the above equation in terms of $e_f(a/R_\oplus)$. In addition, the frozen condition can be read off for $90° \leq i \leq 115°$ and $R_\oplus \leq a \leq 1.6R_\oplus$ and thus extends Fig. 12.11 to retrograde orbits.

Nearly all Earth observation satellites, most prominently the US Landsat satellites and the French SPOT satellites, are frozen Sun-synchronous. For instance we have for SPOT 1-5: $e = 1.25 \times 10^{-3}$, $i = 98.7°$, and $a = 7010$ km; for SPOT 6/7: $e = 1.20 \times 10^{-3}$, $i = 98.2°$, and $a = 7072$ km; and for LANDSAT 4/5/7 $e = 1.20 \times 10^{-3}$, $i = 98.2°$, and $a = 7083$ km.

12.4 Resonant Orbits

A resonant orbit is a special type of orbit that spatially "resonates" with the modulations of the gravitational potential. This means that its ground track, i.e., the projection of the orbit vertically onto the surface of the Earth, recurrently passes only over decisive areas. Because this is equivalent to a repeating pattern, resonant orbits are also called *repeat ground track orbits*; and *commensurate orbits* because the recurrence implies that the ground track is in lock-step with Earth's rotation, meaning that its pattern recurs after a certain number of terrestrial revolutions matching a certain number of orbital revolutions.

Resonant orbits exhibit a property that makes them very special. Because their ground tracks recur exactly thus producing a thin line pattern, they do not cover every point of the surface within their repetition period. This is why those tesseral harmonic perturbations J_{nm} with $m \neq 0$ that do not match the ground patterns are insignificant, while those that do match are not averaged out over time. The resonant orbit rather repeatedly picks them up and by this they are amplified leading to significant orbital perturbations, so-called *resonant perturbations.* This type of perturbation due to repeated pattern effects is quite different from the type of perturbation, evaluated in Sect. 12.3, due to single instance effects. For a resonant orbit we rather have to answer the following three questions:

1. What are the key orbital elements of a resonant orbit?
2. For a given resonant orbit which are the relevant spherical harmonics of the gravitational potential?

3. How do these spherical harmonics perturb the resonant orbit?

In the following sections we will give answers to the first two questions for the most important cases of circular and near-circular resonant orbits, $\varepsilon \approx 0$, and we will exemplify for a geostationary orbits and GPS orbits how these are perturbed.

Draconitic Motion

Before we go into the details of resonant orbits first we have to find a measure for recurrence. As the reference point of recurrence one usually chooses the ascending node in the geocentric equatorial reference frame. The orbital period relative to this node is the so-called *draconitic*[1] *period*, T_Ω. To determine it we recall from Sect. 12.3.3—and noting that $\dot{\vartheta} \equiv \langle \dot{\vartheta} \rangle_P$, $\dot{\vartheta} = \dot{a}, \dot{e}, \dot{i}, \dot{\omega}, \dot{\Omega}, \dot{M}$, i.e., orbital elements averaged over one period—that due to the oblateness perturbation the orbit after one revolution suffers the positional shift $(\dot{M} - n)T_\Omega$ in the orbit and the rotation of the orbit's line of apsides $\dot{\omega}\, T_\Omega$. Therefore, the mean draconitic motion, which is the mean motion on the orbit between two passes of the (drifting) ascending node, is

$$\begin{aligned} n_\Omega &= n + \dot{\omega} + (\dot{M} - n) = \dot{\omega} + \dot{M} \qquad \textbf{mean draconitic motion} \\ &= \sqrt{\frac{\mu}{a^3}}\left\{1 + j_2\left[3 - 4\sin^2 i + \left(1 - \frac{3}{2}\sin^2 i\right)\left(\sqrt{1-e^2} - 1\right)\right]\right\} \\ &= \sqrt{\frac{\mu}{a^3}}\left\{1 + j_2\left[3 - 4\sin^2 i + O(e^2)\right]\right\} \quad @\ e \approx 0 \end{aligned} \tag{12.4.1}$$

where the latter relation follows for circular or near-circular orbits from Eq. (12.3.15) as a result of the oblateness perturbation. The n_Ω is nearly unaffected by higher-order perturbations. This follows from our considerations in Sect. 12.3.2 where we have shown that the significant J_3 effect cancels out for the mean draconitic motion. Observe that for $i < \arcsin\sqrt{3/4} = 60^\circ$ a near-circular Earth-orbiting satellite progresses faster between two ascending nodes, or slower otherwise.

The mean draconitic motion deviates from the orbital mean motion n. This can be understood if we interpret the second and third line of Eq. (12.4.1) in terms of a modified standard gravitational parameter $\mu' = \mu\{1 + j_2[\ldots]\}^2$, which is due to the different gravitational pull for on a satellite from Earth's bulge. This pull is positive (negative) for $i < 60^\circ$ ($i > 60^\circ$).

From the mean draconitic motion the draconitic period is easily determined as

$$T_\Omega = \frac{2\pi}{n_\Omega} = 2\pi\sqrt{\frac{a^3}{\mu}}\frac{1}{1 + j_2\left[3 - 4\sin^2 i + O(e^2)\right]} \quad @\ e \approx 0 \quad \textbf{draconitic period} \tag{12.4.2}$$

which is the time between the succeeding passes of the drifting ascending node.

[1]The term *draconitic* (a.k.a. draconic) derives from the ascending node of the Moon's orbit around Earth, which in traditional astrology is called *dragon's head*. The dragon is *draco* in Latin.

12.4.1 Resonance Conditions

In this section we want to figure out the answer to the first question: When does an orbit achieve resonance? We do not want to go into the sophisticated results of resonant perturbations, which are basically laid out originally in Kaula's resonant perturbation theory (Kaula 1966) and summarized in Vallado (2001, Sect. 9.6) but simplify the analysis by restricting ourselves for the rest of Sect. 12.4 to the practically relevant circular or near-circular orbits, $e \approx 0$.

We start our resonance considerations with an arbitrary Earth orbit. The longitude of its ascending node λ_Ω is shifted after one draconitic period T_Ω by the so-called *orbital track interval*

$$\Delta\lambda_\Omega = \left(\omega_\oplus - \dot{\Omega}\right) T_\Omega$$

to westerly longitudes, with $\omega_\oplus = 2\pi/T_{sidereal} = 7.2921150 \times 10^{-5}\ \mathrm{s}^{-1}$. Now, resonance, i.e., repetition of the ground track pattern, is achieved if after k orbital revolutions the total shift $k \cdot \Delta\lambda_\Omega$ equals a multiple l of a terrestrial revolution of 360°, i.e., if

$$k\left(\omega_\oplus - \dot{\Omega}\right) T_\Omega = 2\pi \cdot l$$

where k, l are any non-divisible natural numbers. We call such an orbit a ***k*:*l* commensurate orbit**. With Eqs. (12.4.2) and (12.4.1) we obtain from the above the following resonance condition equation

$$n_\Omega = \frac{k}{l}\left(\omega_\oplus - \dot{\Omega}\right) = \dot{\omega} + \dot{M} \quad \textbf{resonance condition} \tag{12.4.3}$$

Safely neglecting higher perturbational terms j_2^2, j_4 as shown in Sect. 12.3.3 we can insert the expression from Eq. (12.4.1) for n_Ω and from Eq. (12.3.15) for $\langle\dot{\Omega}\rangle$, and obtain the conditional equation for the semi-major axis of a commensurate orbit, a_c,

$$\frac{k}{l}\sqrt{\frac{a_c^3}{\mu}}\,\omega_\oplus = 1 + j_2\left[3 - 4\sin^2 i - \frac{k}{l}\cos i + O\left(e^2\right)\right] \quad @\ e \approx 0$$

with $j_2 = \dfrac{3J_2R_\oplus^2}{2a_c^2} + O\left(e^2\right)$.

Note that only the semi-major axis and not the eccentricity is specified by the resonance condition because the mean motion of an orbit does not depend on its eccentricity. In order to solve for the self-consistent a_c we recall that j_2 is only of order $j_2 \approx 10^{-5}$, which permits us to apply perturbation theory and find the solution by iteration. In zeroth order we set $j_2 = 0$ and with $\sqrt[3]{\mu/\omega_\oplus^2} = 42\,164.17$ km we find

$$a_{c0} = 42\,164.17 \cdot \left(\frac{l}{k}\right)^{2/3} \text{km}$$

Applying this to j_2 we get

$$j_2 = \frac{3}{2}\frac{J_2 R_\oplus^2}{\left(\mu/\omega_\oplus^2\right)^{2/3}}\left(\frac{k}{l}\right)^{4/3} = 3.71595 \times 10^{-5}\left(\frac{k}{l}\right)^{4/3}$$

For a first-order approximation we insert this into the above equation and finally find

$$a_c = a_{c0}(1+\Delta)^{2/3} = 42\,164.17\left[\frac{l}{k}(1+\Delta)\right]^{2/3}\ \text{km} \tag{12.4.4}$$

with

$$\Delta = 3.71595 \times 10^{-5}\left(\frac{k}{l}\right)^{4/3}\left[3 - 4\sin^2 i - \frac{k}{l}\cos i + O\left(e^2\right)\right] \quad @\ e \approx 0$$

This equation provides the semi-major axis of a $k : l$ commensurate orbit subject to gravitational perturbation, which is the key orbital element of a resonant orbit. For low elliptic orbits with altitudes $h \leq 1300$ km we find $\Delta \approx -10^{-2}$, which is considerable. That is to say, the regression of nodes and the progression of the line of apsides due to Earth's oblateness and the orbital inclination have to be taken into account to determine the exact altitude via Eq. (12.4.4).

Repeat Ground Track Orbits in LEO

A good case of commensurate orbits are remote sensing satellites, i.e. Earth observation satellites in low Earth orbits (LEO). Remote sensing mission conditions usually require that the satellite needs to retrace the ground track after a given number of days and hence periodically revisit predetermined areas of interest. Such an orbit is called

> ***Repeat ground track* (RGT) *orbit***, a.k.a. *phased orbit*, denominated as *l*D*k*R. This means that after k integer orbital revolutions and l integer days a RGT satellite will begin to retrace its ground track pattern.

For example a satellite in a 2D14R orbit repeats its identical ground track after 2 days, during which it performs 14 revolutions. *l*D*k*R orbits are therefore $k : l$ commensurate orbits with resonance condition Eq. (12.4.3). In Fig. 12.13 each dot depicts a possible RGT in LEO. So-called

> ***Phased Sun-synchronous orbits*** are RGTs, which in addition are Sun-synchronous and hence are near-polar RGTs.

Today's Earth observation satellites are phased Sun-synchronous having repeat cycles of typically 16 days or more. Good examples are Spot 4 (26D365R) or Landsat 7/8 (16D233R).

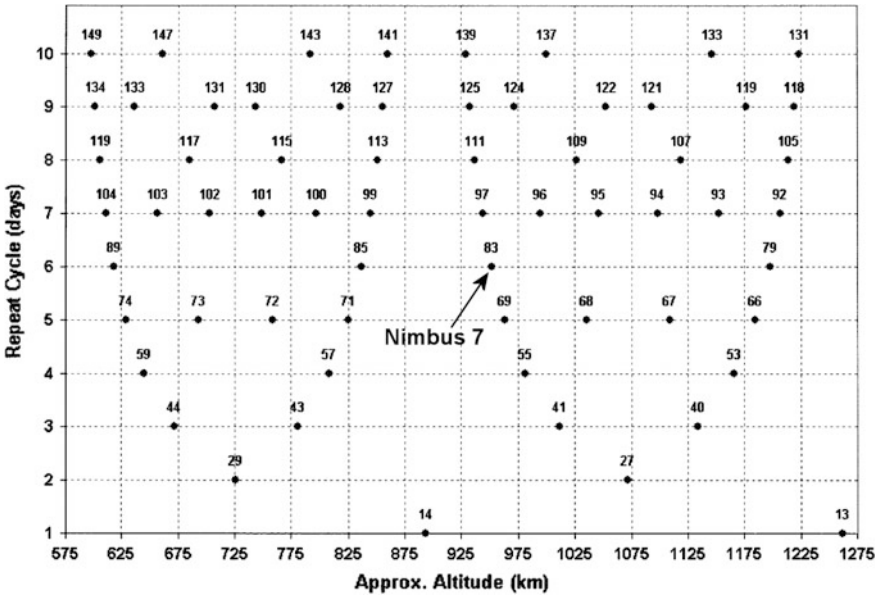


Fig. 12.13 Repeat ground tracks in LEO with altitude given on the horizontal axis. Each dot designates an RGT with repeat cycle l days (vertical axis) after k revolutions (number on top of each RGT dot). Note the degeneracies, e.g. $l{:}k$ = 1:14 (1D14R) = 2:28 (2D28R) = 3:42 (3D42R) = ⋯ *Credit* NASA/Ronald J. Boain

Example

Design a phased Sun-synchronous Earth observation satellite having a circular orbit at altitude of about 700 km.

According to the above we derive from $a_{c0} \approx 7078$ km $= 42\,164.17 \cdot (l/k)^{2/3}$ km a commensurability of $l = 16$, $k = 232.63 \approx 233$. Because Sun-synchronous orbits are at about $i \approx 98°$ we find from Eq. (12.4.4) $a_c \approx 7077.4$ km or $h_c \approx 699.3$ km. To ensure Sun-synchronicity we apply Eq. (12.3.40) and obtain $i = 98.19°$. These are exactly the orbital data of Landsat 7 and 8 satellites having 16D233R.

12.4.2 Resonance Dynamics

Coupling Multipoles

Having found the orbits that revisit certain areas of the Earth's surface regularly we now turn to the next question, namely: Which spherical harmonics are in unison with these specific areas and hence may dynamically affect the resonant orbit? In other words, which are the perturbing multipoles J_{nm} with degree n and order m this commensurate (near-)circular orbit couples to? The answer, again given by Kaula's resonant perturbation theory, turns out to be

$$\frac{k}{l} = \frac{m}{n-2p}, \quad p = 0, 1, 2, 3, \ldots \quad @\ e \approx 0$$

This equation can be fulfilled for all those n, m, p for which holds

$$\begin{array}{l} m = jk \geq 1 \\ n = jl + 2p \geq 2 \end{array} \quad \begin{array}{l} j = 1, 2, 3, 4, \ldots \\ p = 0, 1, 2, 3, \ldots \end{array}, \quad m \leq n \quad @\ e \approx 0 \tag{12.4.5}$$

The equation states that an orbit with $k : l$ resonance will couple to only those multipoles J_{nm} that satisfy Eq. (12.4.5).

Note *Mere zonal multipoles J_{n0} are unable to cause resonant perturbations, because they lack any sectors to cause longitudinal accelerations. Therefore* $1 \leq m \leq n$ holds. Be reminded that $J_{11} = J_{21} = 0$ due to Table 12.2.

Equation (12.4.5) arises from the procedure to match the orbital ground track to the multipole pattern in such a way that the induced acceleration forces add up to a non-vanishing value over one revolution. An example is given in Fig. 12.14 for the ground track for GPS satellites (details see Sect. 12.4.4), which are 2:1 commensurate with Earth's rotation. If the GPS orbit would be circular, only the acceleration forces due to J_{32}, J_{44}, which follow from Eq. (12.4.5), are in lock-step leading to a drift of the resonant orbit. But because GPS satellites exhibit $e = 0.01 - 0.02$ (for details see Sect. 12.4.4) the acceleration forces for the synchronous

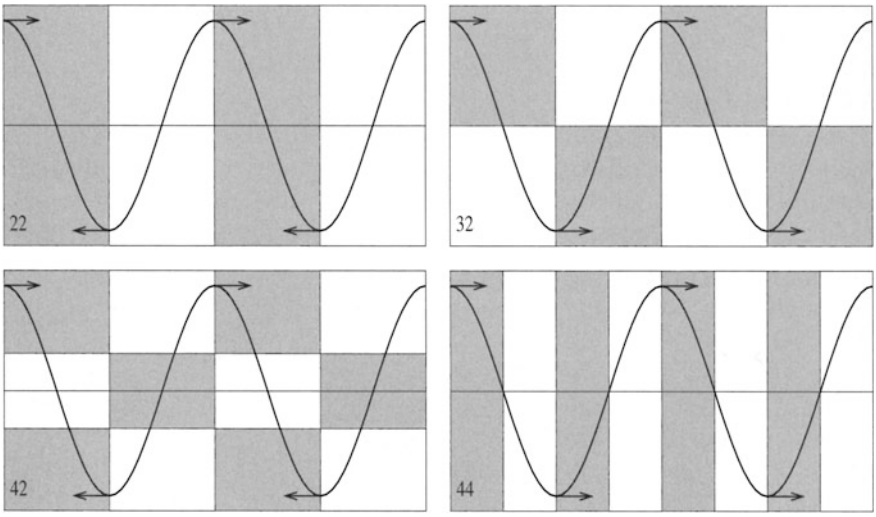


Fig. 12.14 Periodic drift forces indicated as arrows acting on a circular 2:1 commensurate orbit (GPS orbit). The Mercator projection shows gravitational multipole patterns (rectangles) of the Earth with higher (marked gray) and lower (marked white) gravitational attraction. *Credit* Beutler (2005b) and Urs Hugentobler (1998)

apsides J_{22}, J_{42} mutually do not exactly annihilate and hence also have a small impact on the orbit. We will discuss the interaction between a multipole and the resonant orbit in Sect. 12.4.5 for GEO in detail.

Equation of Motion

The resonant gravitational multipoles perturb the commensurate orbit such that tiny forces (see arrows in Fig. 12.14) accelerate or decelerate the body on the orbit along its track. The resulting modified orbital velocity implies an altered centrifugal force, which in turn leads to a variation of the orbital radius equaling the osculating semi-major axis. Assuming near-circular orbits with $e \approx 0$ we get from Eq. (12.2.7) for these variations quite generally

$$\frac{da}{dt} = \frac{2}{an}\frac{\partial R}{\partial M} \quad @\ e \approx 0$$

Owing to $n = \sqrt{\mu/a^3}$ a variation in the semi-major axis translates into variation of mean motion as

$$\dot{n} = -\frac{3}{2}\frac{n}{a}\dot{a} = -\frac{3}{a^2}\frac{\partial R}{\partial M} \quad @\ e \approx 0 \tag{12.4.6}$$

This is quite a telling equation. It states that

In effect we have a repulsive force, from which we overall expect an oscillating behavior of the perturbed body on its Keplerian orbit.

According to Eq. (12.4.3) $\dot{n}$ translates into a drift rate of the longitude of the ascending node of a commensurate orbit as

Dynamics at Stable and Unstable Positions

At positions where the perturbational acceleration force $\partial R/\partial M > 0$ the body is driven to larger orbital radii, $\dot{a} > 0$. This in turn leads to a slow down, $\dot{n} < 0$, of the orbital velocity and hence, and contrary to first expectations, to a drift of the body (see outer green arrows in Fig. 12.19) opposite to the acceleration forces (minus sign, red arrows in Fig. 12.19) and therefore back to a stable position. At positions with $\partial R/\partial M < 0$ we have the opposite dynamics and hence unstable (metastable) positions.

$$\ddot{\lambda}_\Omega = \frac{l}{k}\dot{n}_c = -\frac{3}{a^2}\frac{l}{k}\frac{\partial R}{\partial M} \quad @\ e \approx 0 \tag{12.4.7}$$

where according to Eq. (12.2.3)

$$R(a, \beta, \lambda) \ = a^2 n^2 \sum_{n=2}^{\infty} \left(\frac{R_\oplus}{a}\right)^n \sum_{m=1}^{n} J_{nm} P_n^m(\sin\beta) \cos m(\lambda - \lambda_{nm})$$

with $1 \leq m$ (see Note following Eq. (12.4.5)). From this we already can observe that the resonant terms R_{nm} will detune the resonant orbit. The induced motion in phase space (λ_Ω, a) is shown in Fig. 12.15 for a GEO satellite.

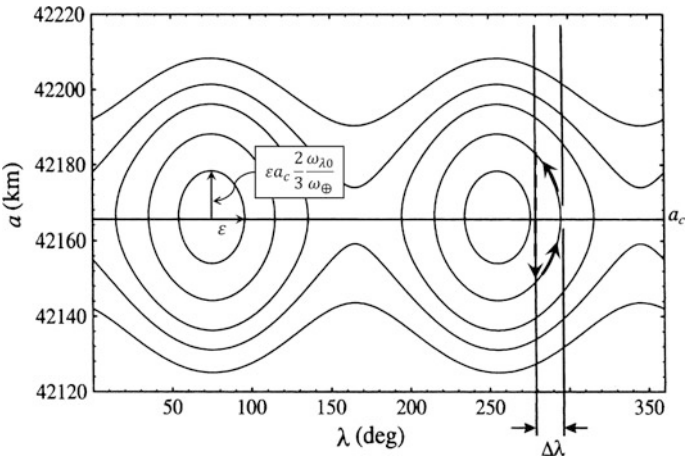


Fig. 12.15 Motion profiles in the (λ_Ω, a) phase space of spacecraft in GEO. The right hand side shows the classical east-west station-keeping strategy in the $\Delta\lambda$ dead-band (cf. Fig. 12.22)

To determine the motional details of resonance dynamics, we have to derive the equation of motion by evaluating $\partial R/\partial M$, which is quite complicated. We therefore again resort to Kaula's resonant perturbation theory where he showed that (see, e.g., Chao (2005)) for near-circular orbits with $e<0.01$

$$R = \frac{\mu}{r}\sum_{n,m}\left(\frac{R_\oplus}{r}\right)^n J_{nm}P_n^m(\sin\beta)\cos m(\lambda-\lambda_{nm})$$
$$= \frac{\mu}{r}\sum_{n,m,p}\left(\frac{R_\oplus}{a}\right)^n J_{nm}F_{nmp}(i)\left\{\begin{array}{ll}\cos m(\lambda_\Omega-\lambda_{nm}) & @\ n-m=even\\ \sin m(\lambda_\Omega-\lambda_{nm}) & @\ n-m=odd\end{array}\right\} \quad @\ e<0.01$$

where

$$\lambda_\Omega = \frac{l}{k}(\omega+M)+\Omega-\theta_{GMST},$$

θ_{GMST} is the *Greenwich Mean Sidereal Time*, which in the equatorial plane is the hour angle from the vernal point (First Point of Aries) to the Greenwich Meridian, and $F_{nmp}(i)$ are the so-called *inclination functions* given, e.g., by Chao (2005, Table 4.2).

Remark *For considerably eccentric orbits, i.e., if* $e>0.01$*, an additional index q plus in the right-hand side of the above equation an additional q-sum over eccentricity functions* $G_{npq}(e)$ *shows up in the resonance condition Eq. (12.4.5). For details see e.g., Chao* (2005, *Chapter 4*).

This may hold even for orbits with weak eccentricity if they couple via *the q index to Earth's oblateness which is much more pronounced than any higher-order deformations. However, it can be shown that this applies only to* $k:l=2:1$*, i.e., to commensurate GPS orbits, which is why we treat this problem separately there* (see Sect. 12.4.4).

With $r = a$ we can differentiate the above expression and by defining the symbol

$$\begin{bmatrix} x_{nm} \\ y_{nm} \end{bmatrix} := \begin{Bmatrix} x_{nm}, & n-m = even \\ y_{nm}, & n-m = odd \end{Bmatrix}$$

we finally obtain from Eq. (12.4.7)

$$\ddot{\lambda}_\Omega = 3n_c^2 \left(\frac{l}{k}\right)^2 \sum_{n/m/p} \left(\frac{R_\oplus}{a_c}\right)^n m J_{nm} F_{nmp} \begin{bmatrix} \sin m(\lambda_\Omega - \lambda_{nm}) \\ -\cos m(\lambda_\Omega - \lambda_{nm}) \end{bmatrix} \quad @\ e < 0.01\ \textbf{EoM} \tag{12.4.8}$$

This is the equation of motion of the longitude of the ascending node of a $k:l$ circular or near-circular commensurate orbit with semi-major axis a_c as given in Eq. (12.4.4), $n_c = \sqrt{\mu_\oplus / a_c^3}$, $J_{n,m}$, λ_{nm} given by Eq. (12.3.15), and triples $n/m/p$ as obtained from Eq. (12.4.5). It tells us the dynamical behavior of a satellite about a resonant orbit. Note that from Eq. (12.4.3) $n_c \frac{l}{k} \approx \omega_\oplus$.

By multiplying the above equation on both sides with $\dot{\lambda}_\Omega$ we can directly integrate it to get the drift velocity

$$\dot{\lambda}_\Omega^2 = 6n_c^2 \left(\frac{l}{k}\right)^2 \sum_{n/m/p} \left(\frac{R_\oplus}{a_c}\right)^n J_{nm} F_{nmp} \begin{bmatrix} \cos m(\lambda_\Omega - \lambda_{nm}) \\ \sin m(\lambda_\Omega - \lambda_{nm}) \end{bmatrix} + \dot{\lambda}_{\Omega 0}^2$$

Of course the drift velocity becomes maximum or minimum at longitudes, which are the roots of Eq. (12.4.8). This is a differential equation from which $\lambda_\Omega(t)$ might be derived as well.

Dynamics About Stable Positions

According to Eq. (12.4.8) $\ddot{\lambda}_\Omega$ is periodic and thus will have at least $2 \times \min(m)$ roots, half of which have a negative slope (see Fig. 12.16 for GPS and Fig. 12.21 for GEO orbits) dynamics about these longitudes are periodic oscillations. We denote these stable positions as λ_0. To circumstantiate these oscillations we approximate the equation of motion about the stable positions as

$$\ddot{\lambda}_\Omega = \left.\frac{d\ddot{\lambda}_\Omega}{d\lambda_\Omega}\right|_0 (\lambda_\Omega - \lambda_0) =: -\omega_{\lambda 0}^2 (\lambda_\Omega - \lambda_0) \tag{12.4.9}$$

with

$$\omega_{\lambda 0}^2 = -3n_c^2 \left(\frac{l}{k}\right)^2 \sum_{n/m/p} \left(\frac{R_\oplus}{a_c}\right)^n \mathrm{m}^2 J_{nm} F_{nmp} \begin{bmatrix} \cos m(\lambda_0 - \lambda_{nm}) \\ \sin m(\lambda_0 - \lambda_{nm}) \end{bmatrix}$$

The solution is a harmonic oscillator of general type

$$\lambda_\Omega = \lambda_0 + \varepsilon \sin(\omega_{\lambda 0} t + \varphi)$$

where the amplitude ε and phase φ are given by the initial conditions.

We now consider the drift rate of the semi-major commensurate axis, which according to Eqs. (12.4.6) and (12.4.7) is

$$\dot{a} = -\frac{2}{3}\frac{a_c}{n_c}\frac{k}{l}\dot{\lambda}_\Omega = -2a_c n_c \frac{l}{k}\sum_{n/m/p}\left(\frac{R_\oplus}{a_c}\right)^n mJ_{nm}F_{nmp}\begin{bmatrix}\sin m(\lambda_\Omega - \lambda_{nm})\\ -\cos m(\lambda_\Omega - \lambda_{nm})\end{bmatrix}$$

We want to study this drift rate for small oscillations about the stable points. We therefore apply Eq. (12.4.9) and get with $\lambda_\Omega - \lambda_0 = \varepsilon\,\sin(\omega_{\lambda 0}t + \varphi)$ and $n_{c\frac{l}{k}} \approx \omega_\oplus$

$$\dot{a} = \frac{2}{3}\frac{a_c}{n_c}\frac{k}{l}\omega_{\lambda 0}^2\varepsilon\,\sin(\omega_{\lambda 0}t + \varphi) \approx a_c\varepsilon\frac{2}{3}\frac{\omega_{\lambda 0}^2}{\omega_\oplus}\sin(\omega_{\lambda 0}t + \varphi)$$

We are able to integrate this equation to yield with the initial conditions $\lambda_\Omega(0) = \lambda_0 + \varepsilon$, i.e., $\varphi = \pm 90°$, and $a(0) = a_c$

$$a(t) = a_c\varepsilon\frac{2}{3}\frac{\omega_{\lambda 0}}{\omega_\oplus}\cos(\omega_{\lambda 0}t \pm 90°) + a_c$$

and hence

$$\begin{aligned}\lambda_\Omega(t) &= \lambda_0 + \varepsilon\,\cos\omega_{\lambda 0}t\\ a(t) &= a_c\left(1 \pm \varepsilon\tfrac{2}{3}\tfrac{\omega_{\lambda 0}}{\omega_\oplus}\sin\omega_{\lambda 0}t\right)\end{aligned} \quad @\ e < 0.01,\ \lambda_\Omega(0) = \lambda_0 + \varepsilon \qquad (12.4.10)$$

with $\omega_\oplus = 6.300\ \text{day}^{-1}$ and ε the small amplitude of the longitudinal oscillation. These equations describe an elliptic clock- or anticlockwise harmonic motion in the (λ_Ω, a) plane, the so-called *phase space*, about the center point (λ_0, a_c) with amplitudes $\left(\varepsilon, \varepsilon a_c\frac{2}{3}\omega_{\lambda 0}/\omega_\oplus\right)$ as shown in Fig. 12.15 for GEOs.

Having derived the basic dynamics of resonant orbits we will consider in the following two sections the GPS orbits and geostationary as typical and instructive examples of resonant orbits.

12.4.3 Low Earth Orbits

In LEOs only repeat ground track orbits (see Sect. 12.4.1) with resonances of type $k:1$ (1DkR) suffer considerable resonance dynamics (see dots on bottom line in Fig. 12.14). Higher-order resonances $k:2$, $k:3$, … tend to become less relevant, because then there is an increasing number of sectors "out of resonance sequence", which effectively decreases the overall resonant acceleration, even though the multipole coefficients of the relevant multipoles ($k/2$, m), ($k/3$, m), … are by a factor of about $2^{3/2} = 2.8$, $3^{3/2} = 5.2$, … larger ("Kaula's rule of thumb", Kaula 1966). Therefore the work-horses of remote sensing, Landsat 7/8 with 16D233R = 233:16, are far from suffering any resonance dynamics.

From the rule Eq. (12.4.5) we derive that for these the following multipoles are effective:

$a_{c0} - R_\oplus$ [km]	k:1 resonance	Relevant multipoles (n,m,p)
1248.2	13:1	(13,13,6), (15,13,7), (17,13,8), (19,13,9), …
880.6	14:1	(15,14,7), (17,14,8), (19,14,9), (21,14,10), …
554.2	15:1	(15,15,7), (17,15,8), (19,15,9), (21,15,10), …
262.3	16:1	(17,16,8), (19,16,9), (21,16,10), (23,16,11), …

Note that the 16 : 1 resonance at altitude $h = 262.3$ km is already strongly affected by drag, which continuously lowers the orbit (see Sect. 12.7.3) and so detunes the resonance if no orbit maintenance is performed.

12.4.4 GPS Orbits

The GPS (Global Positioning System) is a US space-based global navigation satellite system where between 24 and 32 (usually 31) active satellites circle Earth equally spaced in six orbital planes (designated A through F) with $\Omega_A = 10^\circ$ and separated by $\Delta\Omega = 60^\circ$. Their specific orbital elements are $i = 55^\circ$, $e = 0.001$, and they exhibit a revolutionary period of exactly half a sidereal day, i.e., they are in a deep $k : l = 2 : 1$ resonance with Earth's rotation. From Eq. (12.4.4) we derive $\Delta = -7.783 \times 10^{-5}$ or a resonant orbital radius of

$$a_{GPS} := a_c = 26\,560.38 \text{ km}$$
$$n_{GPS} = \sqrt{\frac{\mu_\oplus}{a_{GPS}^3}} = 12.602 \text{ day}^{-1}$$
$$\frac{R_\oplus}{a_{GPS}} = 0.24014$$

From Eq. (12.4.5) we infer that for $n \leq 5$ the triples $n/m/p = 321,\ 441,\ 522$ need to be considered. With this resonance condition we obtain from Eqs. (12.4.8) and (12.2.5a), and Table 12.2

$$\ddot{\lambda}_\Omega = \frac{3}{4} n_{GPS}^2 \left[\begin{array}{l} -2\left(\frac{R_\oplus}{a_{GPS}}\right)^3 J_{32} F_{321} \cos 2(\lambda_\Omega - \lambda_{32}) \\ +4\left(\frac{R_\oplus}{a_{GPS}}\right)^4 J_{44} F_{441} \sin 4(\lambda_\Omega - \lambda_{44}) \\ -2\left(\frac{R_\oplus}{a_{GPS}}\right)^5 J_{52} F_{522} \cos 2(\lambda_\Omega - \lambda_{52}) \end{array} \right]$$

with

$$
\begin{aligned}
J_{32} &= 3.74408 \times 10^{-7}, \quad \lambda_{32} = -17.19^\circ \\
J_{44} &= 7.64577 \times 10^{-9}, \quad \lambda_{44} = 30.35^\circ \\
J_{52} &= 1.17815 \times 10^{-7}, \quad \lambda_{52} = -13.18^\circ \\
F_{321} &= \frac{15}{8}\sin i\left(1 - 2\cos i - 3\cos^2 i\right) = -1.7419 \\
F_{441} &= \frac{105}{4}\sin^2 i(1 + \cos i)^2 = 43.615 \\
F_{522} &= -\frac{105}{16}\sin i\left(1 - 2\cos i - 3\cos^2 i\right) \\
&\quad + \frac{315}{32}\sin^3 i\left(1 - 2\cos i - 5\cos^2 i\right) = -3.6000
\end{aligned}
$$

Hence for $e < 0.01$

$$
\ddot{\lambda}_\Omega = \frac{1}{2}\omega_\lambda^2[\cos 2(\lambda_\Omega - \lambda_{32}) + 0.2456 \cdot \sin 4(\lambda_\Omega - \lambda_{44}) + 0.0375 \cdot \cos 2(\lambda_\Omega - \lambda_{52})] \tag{12.4.11}
$$

with $\omega_\lambda = n_{GPS}\sqrt{3J_{32}|F_{321}|\frac{R_\oplus^3}{a_{GPS}^3}} = 0.7577 \text{ year}^{-1}$.

This drift rate as a function of longitudinal position is depicted in Fig. 12.16. Equation (12.4.11) has four roots at $\lambda_\Omega = 25.9°\text{E}, 118.7°\text{E}, 205.9°\text{E}, 298.7°\text{E}$ with

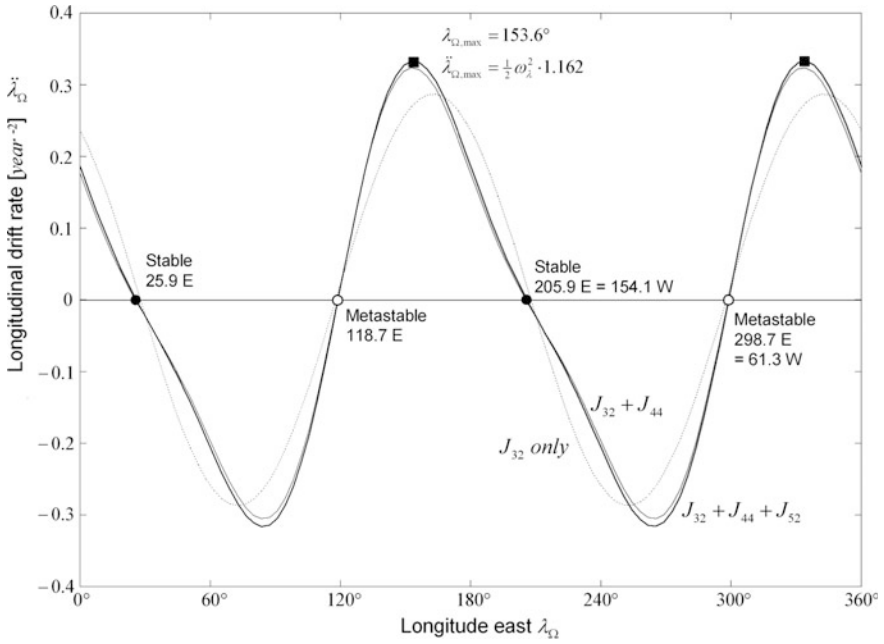


Fig. 12.16 Drift rate of the longitude of the ascending node for GPS satellites

negative slopes (indicating stable positions) at $\lambda_0 = 25.9°\text{E}, 205.9°\text{E}$ and positive slopes (indicating metastable positions) at the other two positions.

The λ_Ω of the GPS satellites are more or less equally distributed over the range 0°–360° and therefore every one of them has an individual drift rate.

Dynamics About Stable Positions

In order to evaluate the dynamic behavior at the stable points we apply Eq. (12.4.9)

$$\begin{aligned}\ddot{\lambda}_\Omega &= \left.\frac{d\ddot{\lambda}_\Omega}{d\lambda_\Omega}\right|_{\lambda_0} (\lambda_\Omega - \lambda_0) \\ &= -\omega_\lambda^2[\sin 2(\lambda_0 - \lambda_{32}) - 2 \cdot 0.2456 \cos 4(\lambda_0 - \lambda_{44}) + 0.0375 \sin 2(\lambda_0 - \lambda_{52})](\lambda_\Omega - \lambda_0) \\ &= -0.5668 \cdot \omega_\lambda^2 \cdot (\lambda_\Omega - \lambda_0) = -\omega_{\lambda 0}^2 \cdot (\lambda_\Omega - \lambda_0)\end{aligned}$$

This is the equation for an oscillating motion with angular frequency

$$\omega_{\lambda 0} = \omega_\lambda \sqrt{0.5668} = 0.5704 \text{ year}^{-1}$$

which equals a period of $T_{\lambda 0} = 2\pi/\omega_{\lambda 0} = 11.0$ years. The negative sign confirms that the longitudes $\lambda_0 = 25.9°\text{E}$, $205.9°\text{E}$ are dynamically stable positions. We determine from Eq. (12.4.10) that the harmonic motion in the phase space (λ_Ω, a) is

$$\begin{aligned}\lambda_\Omega(t) &= \lambda_0 + \varepsilon \cos \omega_{\lambda 0} t \\ a(t) &= a_c\left(1 \pm 1.65 \times 10^{-4} \varepsilon \sin \omega_{\lambda 0} t\right)\end{aligned}$$

The actual anti-clockwise harmonic motion (plus sign) is depicted in Fig. 12.17 for some GPS satellites. Observe that all motions are elliptic, although the longitudinal excursions ε of some satellites exceed the linear range about the stable points. This, however, only implies that the motion is no longer harmonic, i.e., the motion on the ellipse is no longer uniform.

Eccentric GPS Orbits

We note that a more detailed evaluation of the GPS motion would need to consider the fact that due to lunisolar perturbations the long-term eccentricity increases linearly and after 10 years is about e = 0.01–0.02. This implies that according to Eq. (12.4.4) the quantity of the resonant semi-major axis remains practically unaffected by this, but according to the Remark in Sect. 12.4.2 now the J_{22} term has to be taken into account, which contributes the additional two terms

$$\frac{a_{GPS}}{R_\oplus} \frac{J_{22}}{J_{32}|F_{321}|} \{F_{221} G_{211} \sin[2(\lambda_\Omega - \lambda_{22}) - \omega] + F_{220} G_{20-1} \sin[2(\lambda_\Omega - \lambda_{22}) + \omega]\}$$

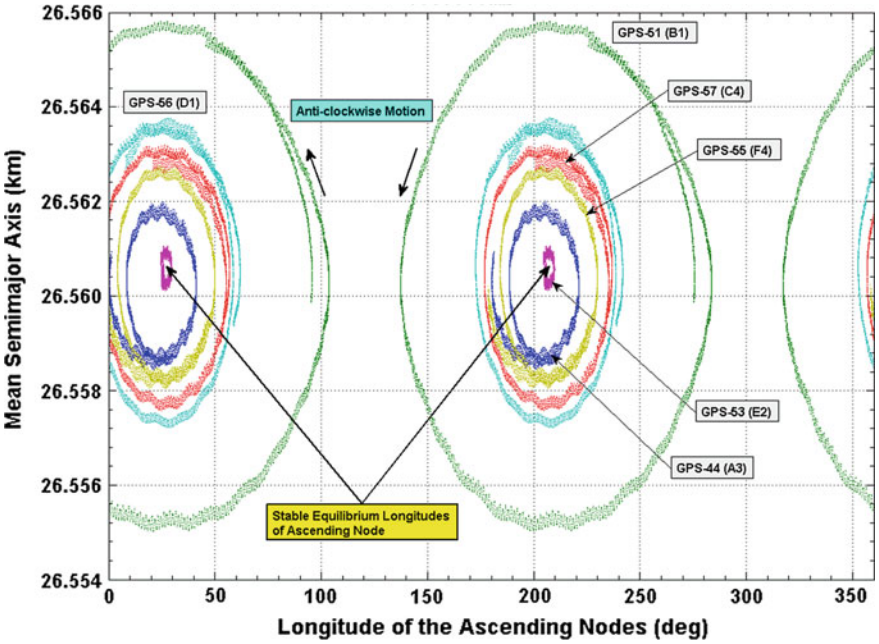


Fig. 12.17 The harmonic motion of selected GPS satellites in the (λ_Ω, a) phase space. *Credit* L. Anselmo and C. Pardini, ISTI/CNR

to the square bracket of Eq. (12.4.11). From Chao (2005, Tables 4.1 and 4.2) we find for the inclination functions $F_{nmp}(i)$ and eccentricity functions $G_{npq}(e)$ with lowest order in e

$$F_{221} = \frac{3}{2}\sin^2 i = 1.0065, \qquad G_{211} = \frac{3}{2}e$$
$$F_{220} = \frac{3}{4}(1+\cos i)^2 = 1.8571, \qquad G_{20-1} = -\frac{1}{2}e$$

With $J_{22} = 1.8155 \times 10^{-6}$ and $\lambda_{22} = -14.93^\circ$ from Eq. (12.2.5a) we then have

$$\ddot{\lambda}_\Omega = \frac{1}{2}\omega_\lambda^2 \left\{ \begin{array}{l} \cos 2(\lambda_\Omega - \lambda_{32}) + 0.2456 \cdot \sin 4(\lambda_\Omega - \lambda_{44}) + 0.0375 \cdot \cos 2(\lambda_\Omega - \lambda_{52}) \\ +e \cdot 17.501 \cdot \sin[2(\lambda_\Omega - \lambda_{22}) - \omega(t)] \\ -e \cdot 10.764 \cdot \sin[2(\lambda_\Omega - \lambda_{22}) + \omega(t)] \end{array} \right\}$$

According to Sect. 12.3.3 $\omega(t) \approx 2n_{GPS}j_2\left(1 - \frac{5}{4}\sin^2 i\right) \cdot t = \dot{\omega}_{GPS}t$ for $e > 0.001$ with $\dot{\omega}_{GPS} = 7.964^\circ \cdot \text{year}^{-1} = 4.405 \times 10^{-9}\ \text{s}^{-1}$. Due to this time dependence the equation of motion then can only be solved numerically.

GLONASS, Compass, and Galileo
The Russian Global Navigation Satellite System, GLONASS, with a target constellation of 24 satellites is in a mild 17 : 8 resonance orbit at $i = 64.8°$, while the 27 MEO satellites of the Chinese satellite navigation system *Compass* is in an insignificant 41 : 22 resonance at $i = 55°$, and the European satellite navigation system *Galileo* with 30 spacecraft (including 3 spares) is chosen to be in a mild 17 : 10 resonance at $i = 56°$. According to Eq. (12.4.4) their semi-major axes hence are $a_{GLONASS} = 25507.60$ km, $a_{Compass} = 27840.96$ km, and $a_{Galileo} = 29600.27$ km. They are thus sufficiently far away from the 2 : 1 resonance and do not suffer from considerable resonance perturbations.

12.4.5 Geostationary Orbit

A geostationary orbit, or GEO for short, with $e = i = 0$ is a prime example of a resonant orbit exhibiting $k = l = 1$, i.e., the orbit is in perfect pace with Earth's rotation. Hence a satellite in GEO seems to be at rest as seen from any position on Earth. This makes GEO a perfect orbit for telecommunication services, in particular broadcasting services, because this ensures a continuous service while the alignment of the satellite dish remains fixed. This is why there are so many communications satellites in GEO, about 450 satellites as of 2018.

From the above properties follows with Eq. (12.4.4) that $\Delta = 2 \times 3.71595 \times 10^{-5} = 7.4319 \times 10^{-5}$. With this we get $a_c = 42\,164.17 \times 1^{2/3}[1 + \Delta]^{2/3}$ km and hence

$$a_{GEO} := a_c = 42\,166.26 \text{ km}$$
$$n_{GEO} = \sqrt{\mu_\oplus / a_{GEO}^3} = \omega_\oplus = 7.2921150 \times 10^{-5} \text{ s}^{-1}$$

for the commensurate GEO radius. We recall from the discussion after Eq. (12.4.1) that rather than having the radius at $a = \sqrt[3]{\mu/\omega_\oplus^2} = 42{,}164.17$ km the somewhat elevated radius is due to an increased gravitational pull of the equatorial bulge to get in pace with Earth's rotational speed $\omega_\oplus$. From Eq. (12.4.5) we derive that J_{22}, J_{31}, J_{32}, and J_{33} are relevant, to name the most significant contributions.

The resonant perturbation of GEO is sometimes called *triaxiality*. "Triaxiality" refers to the triaxiality of the "potato's" potential (see Fig. 12.4, right, and color plates on pages 568 and 569 and Fig. 12.19), which includes zonal, sectorial, and tesseral harmonics (bold multipole coefficients in Table 12.2). The most prominent part of the "potato" is the multipole J_{22}, which looks like a dumbbell as depicted in Fig. 12.18. It stretches along the 15°W–165°E direction.

Because it is this dumbbell that most significantly acts upon GEO we will learn most of GEO's resonant perturbation by explaining how the dumbbell interacts with a satellite's orbit. The masses on both sides of the dumbbell cause a constant lateral gravitational pull on the body in GEO. Its impact on the body depends on its initial

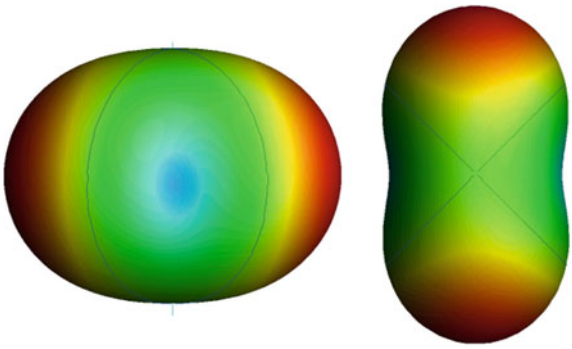


Fig. 12.18 Earth's "dumbbell" multipole J_{22}. (Left) Observed from the terrestrial equatorial plane. (Right) Observed from the terrestrial pole. *Credit* GFZ Potsdam

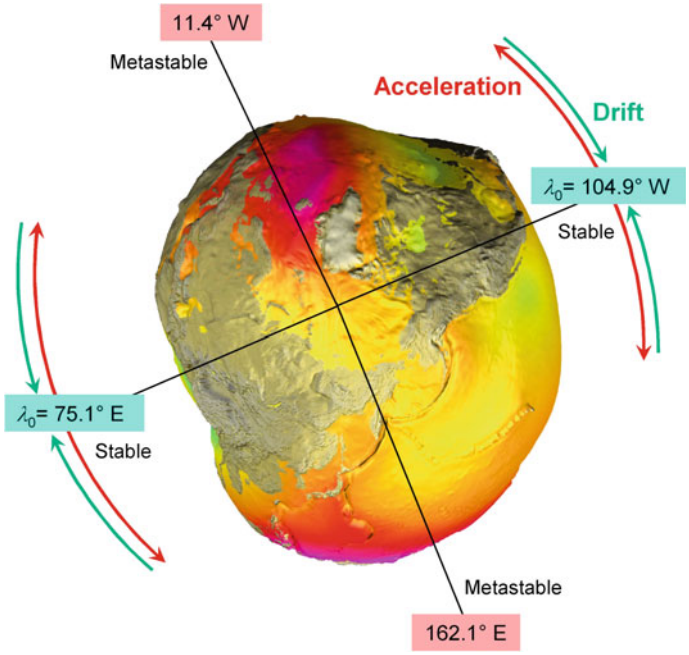


Fig. 12.19 Forces on a body in geostationary orbit at the abeam points of the "triaxial potato". *Credit* GFZ Potsdam

position. If the position is exactly on the dumbbell's lateral axis, i.e., displaced by 90° from the longitudinal 15°W–165°E axis at 75°E or 105°W, then it is subject to the same gravitational pull from both ends of the dumbbell. So the effective gravitational pull is indifferent, and its position would be metastable. If it is displaced by 45° with regard to the longitudinal axis, it is subject to a stronger force from the closer end of the dumbbell. This leads to the motion dynamics as described in sub-section *Dynamics about Stable Positions* in Sect. 12.4.2, depictured in phase space in Fig. 12.16, and plotted in real space in Fig. 12.20.

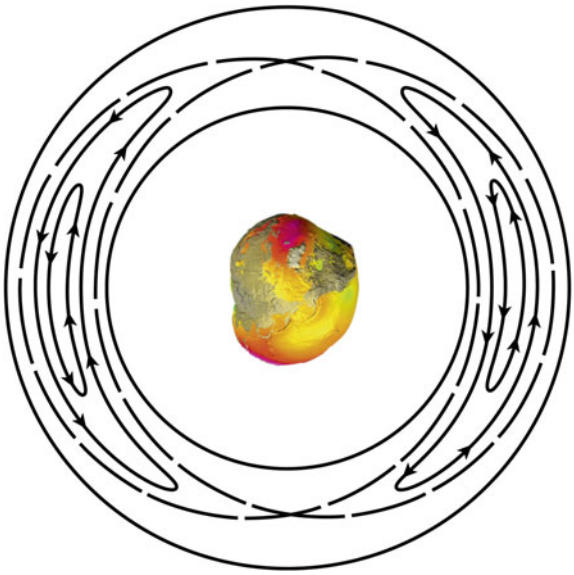


Fig. 12.20 Motion dynamics of a body in an oblate ellipsoidal gravitational potential, which resembles that in a dumbbell potential

So we see that the abeam positions at 75°E and 105°W, though being statically unstable, are actually dynamically stable. This situation is similar to the dynamical stability of a S/C near the statically unstable L_4 and L_5 equilateral libration points (see Sect. 11.5.3). If the body is positioned right above the ends of the dumbbell this would also be an indifferent position. But just a small deviation would cause a tangential acceleration back to the indifferent position, causing a drift even further away from the initial position according to Eq. (12.4.7). The positions above the ends of the longitudinal axis hence turn out to be metastable. The actual values of the stable and metastable positions are slightly different from these, which is due to the J_{31} and J_{33} perturbations (see below).

Equation of Motion

We now want to describe the orbit drifts on mathematical grounds and therefore resort to the results of Sect. 12.4.2. But rather than to use Eq. (12.4.8) (there exists no ascending node for GEOs anyway) we start out with Eq. (12.4.6) and take into account that any variation of the mean motion due to resonance perturbations equals the drift rate of the longitudinal position of the satellite, $\ddot{\lambda} = \dot{n}_{GEO}$. Owing to the circularity of the orbit we also have $dM = d\theta = d\lambda$. This is applied to Eq. (12.4.6), which yields the equation of longitudinal motion

$$\ddot{\lambda} = -\frac{3}{a^2}\frac{\partial R}{\partial \lambda} \tag{12.4.12}$$

with

$$R(a,0,\lambda) = a^2 n^2 \sum_{n=2}^{3} \left(\frac{R_\oplus}{a}\right)^n \sum_{m=1}^{n} J_{nm} P_n^m(0) \cos m(\lambda - \lambda_{nm})$$

where $\beta = 0$ because $i = 0$. From Eq. (12.2.6) with $x = \sin\beta = s = 0$, $c = \cos\beta = 1$ we get

$$P_2^1(0) = 0,\ P_2^2(0) = 3,\ P_3^1(0) = 3/2,\ P_3^2(0) = 0,\ P_3^3(0) = -15$$

which corroborates the finding from Sect. 12.4.2 that only J_{22}, J_{31}, and J_{33} are relevant resonance terms in GEO. It is now easy to evaluate $\partial R/\partial\lambda$ and we obtain for the GEO drift rate

$$\begin{aligned}\ddot{\lambda} &= 3n^2\left\{6\frac{R_\oplus^2}{a^2}J_{22}\sin 2(\lambda-\lambda_{22}) + \frac{R_\oplus^3}{a^3}\left[\frac{3}{2}J_{31}\sin(\lambda-\lambda_{31}) - 45J_{33}\sin 3(\lambda-\lambda_{33})\right]\right\} \\ &= \frac{18R_\oplus^2}{a^2}n^2 J_{22}\left[\sin 2(\lambda-\lambda_{22}) + \frac{1}{4}\frac{R_\oplus}{a}\frac{J_{31}}{J_{22}}\sin(\lambda-\lambda_{31}) - \frac{45}{6}\frac{R_\oplus}{a}\frac{J_{33}}{J_{22}}\sin 3(\lambda-\lambda_{33})\right]\end{aligned}$$

From Eq. (12.2.5a) and Table 12.2 we derive

$$\begin{aligned} J_{22} &= 1.8155\times10^{-6}, \quad \lambda_{22} = -14.93^\circ \\ J_{31} &= 2.2090\times10^{-6}, \quad \lambda_{31} = -173.02^\circ \\ J_{33} &= 0.22137\times10^{-6}, \quad \lambda_{33} = -39.00^\circ \end{aligned}$$

If we insert these values and $a \equiv a_{GEO}$ and $n \equiv n_{GEO}$ for GEO, we finally arrive at the GEO drift rate (cf. Campan 1995)

$$\ddot{\lambda} = \frac{1}{2}\omega_\lambda^2 \cdot [\sin 2(\lambda-\lambda_{22}) + 0.0460\cdot\sin(\lambda-\lambda_{31}) - 0.1383\cdot\sin 3(\lambda-\lambda_{33})] \tag{12.4.13}$$

with angular frequency

$$\omega_\lambda = 6\frac{R_\oplus n_{GEO}}{a_{GEO}}\sqrt{J_{22}} = 2.814\ \text{year}^{-1}$$

This drift rate as a function of longitudinal position is depicted in Fig. 12.21. Equation (12.4.13) has four roots at $\lambda = 75.1^\circ$E, 162.1°E, 255.1°E, 348.6°E with negative slopes (indicating stable positions) at $\lambda_0 = 75.1^\circ$E, 255.1°E and positive slopes (indicating metastable positions) at the other two positions (see Fig. 12.19). Because the perturbation terms J_{31} and J_{33} are an order of magnitude smaller than J_{22}, Eq. (12.4.13) can roughly be approximated just by the J_{22} term

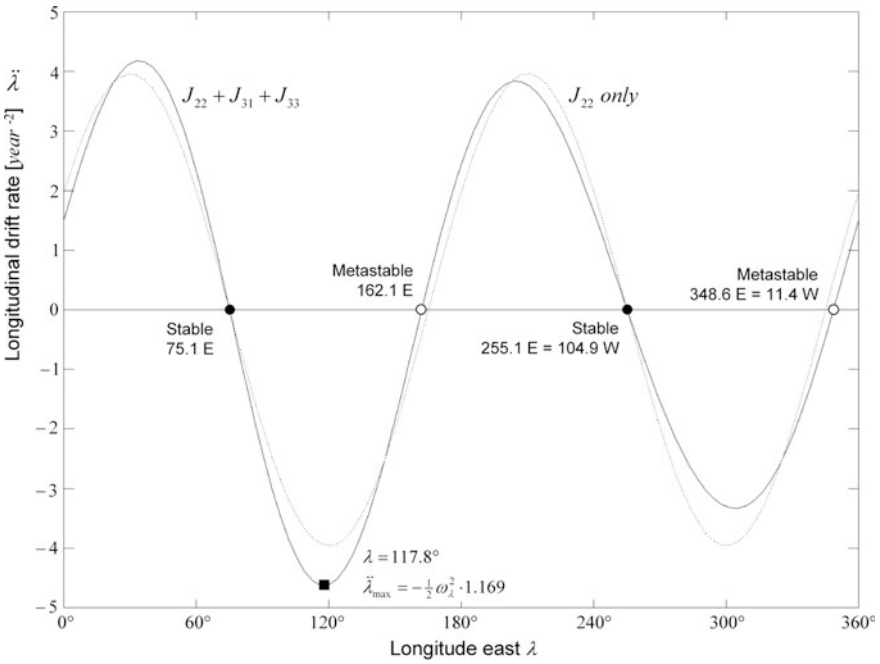


Fig. 12.21 Longitudinal drift rates of a GEO satellite as given by Eq. (12.4.13)

$$\ddot{\lambda} = \frac{1}{2}\omega_\lambda^2 \cdot \sin 2(\lambda - \lambda_{22}) = -\frac{1}{2}\omega_\lambda^2 \cdot \sin 2(\lambda - \lambda_0) \tag{12.4.14}$$

where the latter follows from $\lambda_0 - \lambda_{22} = 90°$.

Dynamics Near Stable Positions

If the body is near a stable position λ_0, in Eq. (12.4.14) the sine for small arguments can be approximated linearly to give

$$\ddot{\lambda} = -\omega_\lambda^2 \cdot (\lambda - \lambda_0)$$

This is a harmonic oscillation around the stable positions with period $T_{\lambda 0} = 2\pi/\omega_\lambda = 2.23$ years. We may improve this calculation by applying Eq. (12.4.9)

$$\begin{aligned}\ddot{\lambda}_\Omega &= \left.\frac{d\ddot{\lambda}_\Omega}{d\lambda_\Omega}\right|_{\lambda_0} (\lambda_\Omega - \lambda_0) \\ &= 0.5 \cdot \omega_\lambda^2 [2\cos 2(\lambda_0 - \lambda_{22}) + 0.0460\cos(\lambda_0 - \lambda_{31}) \\ &\quad + 3 \cdot 0.1383\cos 3(\lambda_0 - \lambda_{33})](\lambda_\Omega - \lambda_0) \\ &= -0.8109 \cdot \omega_\lambda^2 \cdot (\lambda_\Omega - \lambda_0)\end{aligned}$$

This yields a period of $T_{\lambda 0} = 2\pi/\left(\sqrt{0.8109} \cdot \omega_\lambda\right) = 2.48$ years.

Dynamics at Unstable Positions

Let us have a closer look at the time behavior when the body is far away from one of the two equilibrium positions, which usually is the case. For small time periods the instantaneous position λ can be considered almost constant. So the right-hand side of the differential Eq. (12.4.14) is a constant, and the equation can simply be integrated directly

$$\lambda = \lambda_i + \frac{1}{2}\ddot{\lambda} \cdot t^2 \tag{12.4.15}$$

So the body, initially in a resting position, moves away quadratically with time from its initial position λ_i. This is the so-called *east–west drift*.

GEO satellites are required to be kept within a dead-band slot of typical width $\Delta\lambda = 0.1^\circ$ at the center of an allocated standard longitudinal position box of size $\Delta\lambda = 0.1^\circ$ for satellites with inclination limits $i < 1^\circ$ and size $\Delta\lambda = 0.2^\circ$ for satellites with inclination limits $i < 5^\circ$. To counteract the drift, and as depictured in Fig. 12.22 correction burns, so-called *east–west station-keeping maneuers* (E/W drift maneuvers), need to be fired at that side of the box to which the satellite tends to drift freely, which cause the satellite to drift to the other side of the dead-band and back again. The recurrence time and the delta-v demand for one kick-burn can be derived from Eq. (12.4.15) to be

$$\Delta t_{sk} = 2\sqrt{\frac{2 \cdot \Delta\lambda}{|\ddot{\lambda}|}} \geq 20.1 \text{ days}$$

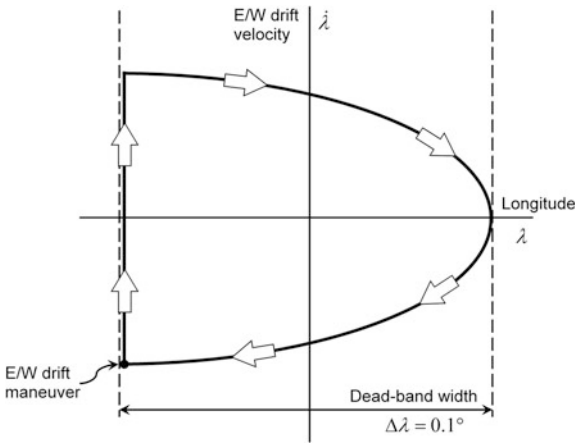


Fig. 12.22 Satellite drift pattern inside its longitude dead-band box

and

$$\Delta v_{sk} = |a_{||}| \cdot \Delta t_{sk} = \frac{2}{3} r \sqrt{2 \cdot \Delta\lambda \cdot |\ddot{\lambda}|} \leq 0.113 \text{ m s}^{-1}$$

The limits are derived from the position $\lambda = 117.8°$ where the longitudinal acceleration has its maximum absolute value with $|\ddot{\lambda}|_{\max} = \frac{1}{2}\omega_{\lambda}^2 \cdot 1.169$ (see Fig. 12.21). In total for east–west station-keeping a yearly delta-v of

$$\boxed{\Delta v = \tfrac{1}{3} r |\ddot{\lambda}| \cdot 1 \text{ year} \leq 2.06 \text{ ms}^{-1}} \text{ per year} \qquad (12.4.16)$$

needs to be taken into account. East–west station-keeping strategies and maneuvers, which also include eccentricity corrections owing to solar radiation pressure (see Sect. 12.5) are discussed at the end of Sect. 12.5.3.

12.5 Solar Radiation Pressure

A S/C orbiting a planet at a distance of r from the Sun will be affected by solar radiation unless it happens to be in the shadow of the planet. A light particle does not possess mass, but according to quantum mechanics it still carries a linear momentum h/λ (h is Planck's constant, λ is the wavelength of the light particle), which, depending on the **surface reflectivity** ρ of the S/C, transfers momentum $\rho h/\lambda$. If the surface is absorbing, then $\rho = 1$; if it is reflecting, $\rho = 2$; and if it is transparent, $\rho = 0$. The solar radiation thus produces a total radiation pressure $p_{S/C}$ on the S/C, which via the mass of the S/C creates the acceleration a_{sun}

$$\frac{m a_{sun}}{A_{\perp}} = p_{S/C} = \rho\, p_{sun}$$

where

$$p_{sun} = \frac{E}{c} = p_0 \left(\frac{r_{\oplus}}{r}\right)^2 N \text{ m}^{-2} \quad \textbf{radiation pressure of the Sun} \qquad (12.5.1)$$

with

c	velocity of light
$E = (1372 \pm 45) \left(\frac{r_{\oplus}}{r}\right)^2 \text{ W m}^{-2}$	intensity of solar radiation (seasonal)
$p_0 = (4.58 \pm 0.15) \times 10^{-6} \text{ N m}^{-2}$	seasonal
$r_{\oplus} = 1.4962 \times 10^8$ km	mean radius of Earth's orbit
$A_{\perp}$	surface of the S/C projected onto the direction of radiation

This results in the following acceleration of the S/C

$$a_{sun} = p_{sun} \cdot B_r \tag{12.5.2}$$

where we have defined the ballistic radiation coefficient

$$B_r = \rho \frac{A_\perp}{m} \tag{12.5.3}$$

similar to the ballistic drag coefficient in Sect. 6.2.3.

12.5.1 Effects of Solar Radiation

Qualitative Considerations

To derive the effects of solar radiation on the orbital elements, Let us assume a circular or a weakly elliptic orbit of the S/C at any altitude, i.e., $e \approx 0$ in addition to $i \approx 0$. The direction of radiation shall be in the orbital plane and perpendicular to the line of apsides, i.e., the Sun shines "laterally" onto the orbit (see Fig. 12.23). (As we will see in a moment, if we start out with a circular orbit, the solar pressure will cause an eccentricity with the line of apsides perpendicular to the radiation direction. So this assumption always holds.) Due to $e \approx 0$, $i \approx 0$ we can neglect $\dot{\omega}$, $\dot{\Omega}$ effects, and since the radiation force does not have a component vertical to the orbital plane, Table 8.1 tells us that no inclination changes result. So we only have to focus on $\dot{a}$, $\dot{e}$ effects. Table 8.1 also tells us that forces in the orbital plane perpendicular to the path do not change a, yet Eq. (8.1.1) tell us that such forces

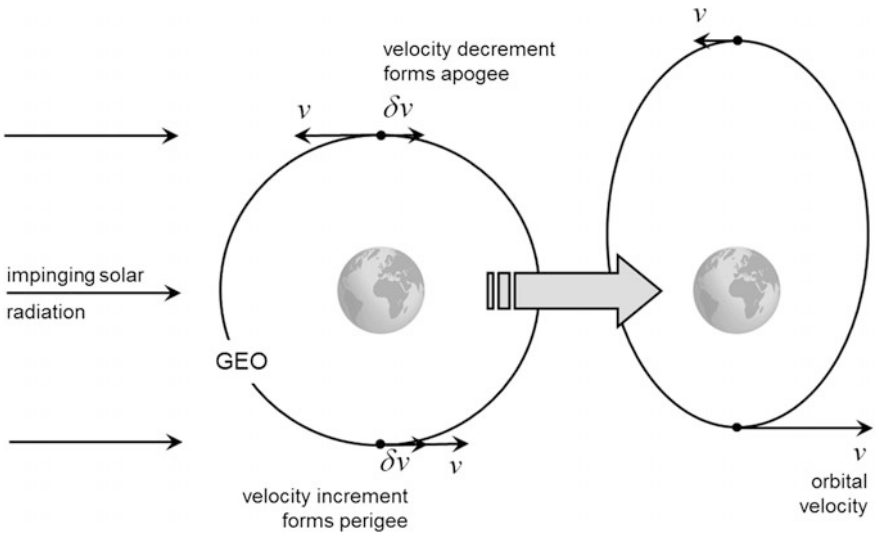


Fig. 12.23 The solar pressure with acceleration effect δv deforms a circular geostationary orbit to an ellipse

change e, but, because the effect of a force with constant direction is opposite on the two sides of the ellipse, they cancel out. So we only have to consider along-track forces at apoapsis and periapsis affecting $\dot{a}$, $\dot{e}$.

Let us assume that the orientation of the orbit is such that the radiation produces a minute push of δv along-track at the periapsis, and the same push in opposite direction at apoapsis. Then the induced change of the semi-major axis and the eccentricity according to Table 8.1 is

$$\frac{\delta a}{a} = \frac{2}{1+e}\frac{(-\delta v)}{v_h} \quad \text{and} \quad \delta e = -2\frac{(-\delta v)}{v_h} \quad @ \text{ apoapsis}$$
$$\frac{\delta a}{a} = \frac{2}{1-e}\frac{(+\delta v)}{v_h} \quad \text{and} \quad \delta e = +2\frac{(+\delta v)}{v_h} \quad @ \text{ periapsis}$$

with

$$v_h = \frac{\mu}{h} = \sqrt{\frac{\mu}{a(1-e^2)}} \approx \sqrt{\frac{\mu}{a}} = v_0$$

where the approximation holds for small eccentricities. So we get

$$\frac{\delta a}{a} \approx 2(e-1)\frac{\delta v}{v_0} \quad \text{and} \quad \delta e = 2\frac{\delta v}{v_0} \quad @ \text{ apoapsis}$$
$$\frac{\delta a}{a} \approx 2(e+1)\frac{\delta v}{v_0} \quad \text{and} \quad \delta e = 2\frac{\delta v}{v_0} \quad @ \text{ periapsis}$$

If we add up both impacts, we get the following changes

$$\frac{\delta a}{a} = 4e\frac{\delta v}{v_0} \quad \text{per revolution} \tag{12.5.4a}$$

$$\delta e = 4\frac{v}{v_0} \quad \text{per revolution} \tag{12.5.4b}$$

The result $\delta e > 0$ positively feeds back our assumption that the solar radiation is parallel to the orbital velocity vector at periapsis. So starting out with a circular orbit, solar radiation will increase the orbital speed on one lateral side of the orbit and decrease is on the opposite side, thus inducing an eccentricity with a lateral line of apsides. With this constellation, the eccentricity will constantly increase with each revolution. So, we can say the following:

Impact of Solar Radiation on a Circular Orbit
For solar radiation impinging a circular or near-circular orbit the semi-major axis and hence the orbital period remains unaffected. However, an eccentricity laterally to the radiation direction develops, which constantly increases.

Quantitative Perturbation Calculation

We start out to determine the variation of the eccentricity quantitatively by refining the above considerations. Because the change of eccentricity comes from the along-track accelerations around the periapsis and apoapsis we have to estimate the integral of their impact over one revolution around these points. To estimate this, we apply Eq. (12.5.4a) with $\delta v = a_{sun} \cdot \delta t$, where δt is the effective impact time per revolution, which we estimate to be $\delta t \approx T/2$. As $T = 2\pi\sqrt{a^3/\mu}$ holds for an elliptic and a circular orbit, with $v_0 = \sqrt{\mu/a}$ we get

$$\delta e = 4\pi \cdot p_{sun} B_r \frac{a^2}{\mu} \quad \text{per revolution}$$

This is a first rough estimate. In order to derive the orbit changes exactly, we need to determine the components of the solar force in radial direction and vertically to it and then integrate their effects over one orbit according to the Gaussian variational equations (12.1.4). Because the line of apsides is lateral to the radiation, we find from Fig. 12.24 for the acceleration components due to the solar pressure

$$\boldsymbol{a}_{sun} = (a_r, a_\theta, a_h) = a_{sun}(\sin\theta, \cos\theta, 0)$$

and with Eq. (12.1.4) for the variations of the semi-major axis and eccentricity per revolution

$$\delta a = \frac{2a^2}{h} a_{sun} \int_0^T \left(e \sin^2\theta + \cos\theta + e \cos^2\theta\right) dt$$

$$\delta e = \frac{a}{h} a_{sun} \int_0^T \left(\sin^2\theta + e \cos\theta + \cos^2\theta - e \cos^3\theta + \cos^2\theta\right) dt$$

where we have linearized all terms because $e \ll 1$. From Eq. (7.3.8) we find

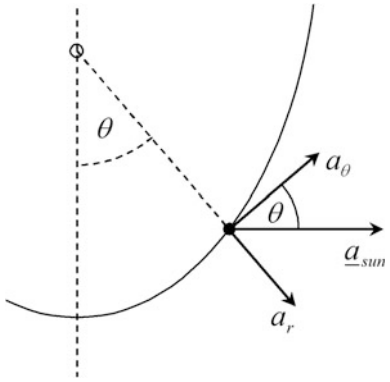


Fig. 12.24 Solar radiation acceleration components

$$dt = \frac{h^3/\mu^2}{(1+e\,\cos\theta)^2}d\theta \approx \frac{h^3}{\mu^2}(1-2e\,\cos\theta)d\theta$$

from which with $h^2 = \mu a(1-e^2) \approx \mu a$ we obtain

$$\delta a = \frac{2a^3}{\mu}a_{sun}\int_0^{2\pi}\left(e+\cos\theta+e\,\cos^2\theta\right)d\theta$$

$$\delta e = \frac{a^2}{\mu}a_{sun}\int_0^{2\pi}\left(1+e\,\cos\theta+\cos^2\theta-e\,\cos^3\theta-2e\,\cos\theta-2e\,\cos^3\theta\right)d\theta$$

$$= \frac{a^2}{\mu}a_{sun}\int_0^{2\pi}\left(1-e\,\cos\theta+\cos^2\theta-3e\,\cos^3\theta\right)d\theta$$

Because from symmetry considerations $\langle\cos\theta\rangle_\theta = \langle\cos^3\theta\rangle_\theta = 0$ and because

$$\int_0^{2\pi}\left(1+\cos^2\theta\right)d\theta = \left(\theta+\frac{1}{2}\sin\theta\,\cos\theta+\frac{\theta}{2}\right)\Big|_0^{2\pi} = 3\pi$$

we finally derive with Eq. (12.5.2)

$$\begin{aligned}\delta a &= 6\pi\,p_{sun}B_r\frac{a^3}{\mu}e+O(e^2)\approx 0\\ \delta e &= 3\pi\,p_{sun}B_r\frac{a^2}{\mu}+O(e^2)\end{aligned}\quad \text{per revolution, @ } e\ll 1 \qquad (12.5.5)$$

This nicely agrees with our qualitative and rough quantitative considerations above. Because δe increases quadratically with the semi-major axis, this effect is 50 times bigger in GEO than in LEO.

We shall now concern ourselves with the other orbital elements. Applying $(a_r, a_\theta, a_h) = p_{sun}B_r(\sin\theta, \cos\theta, 0)$ for $e \approx 0$ to Eq. (12.1.2) we find

$$\begin{aligned}\delta i &= \delta\Omega = 0\\ \delta\omega &= -\delta(M-nT) = \frac{hp_{sun}B_r}{2e\mu}\sin 2\theta\end{aligned}$$

Note that from this follows that in a geocentric-equatorial reference frame the mean position on the orbit, i.e., the mean longitude (see Sect. 7.3.5),

$$l = \Omega + \omega + M = \Omega_0 + \delta\Omega + \omega_0 + \delta\omega + M_0 + \delta(M - nT)$$
$$= \Omega_0 + \omega_0 + M_0$$

remains unaffected by solarization. This implies that the mean position of a GEO satellite relative to an observer on Earth remains unchanged, but will only begin to oscillate sideways about its mean position (details see Sect. 12.5.3) with increasing eccentricity. This is good news from an east-west station-keeping point of view.

In summary, for $e \approx 0$ solar radiation essentially increases the eccentricity and the line of apsides rotates. So, effectively only the eccentricity vector $\boldsymbol{e}$ changes.

12.5.2 Orbital Evolution

We want to determine the temporal evolution of the eccentricity vector $\boldsymbol{e}$. To do so we need an equation of motion for it. We first introduce an appropriate reference frame with x-axis along the line of apsides and the y-axis vertical to it. The above considerations dealt with e_x, for which we get the equation of motion

$$\frac{de_x}{dt} = \frac{\delta e}{T} = \frac{3\pi}{2\pi} \cdot p_{sun} B_r \sqrt{\frac{\mu}{a^3}} \frac{a^2}{\mu} = \frac{3}{2} \cdot p_{sun} B_r \sqrt{\frac{a}{\mu}} = \frac{3 p_{sun} B_r}{2na} =: \kappa B_r$$

with

$$\kappa = \frac{3 p_{sun}}{2na} = (2.23 \pm 0.07) \times 10^{-9}\,\mathrm{kg\,m^{-2}} s^{-1} \quad @\ \mathrm{GEO}$$

if the Sun's rays are vertically to the line of apsides of a GEO. In case they are not vertical, the angle between the solar radiation direction and the line of apsides is defined as the mean longitude of the Sun: $\lambda_* = n_* t + \lambda_0$ with

$$n_* = 0.9856^\circ\,\mathrm{day}^{-1} = 0.01720\,\mathrm{rad\,day}^{-1}$$

the mean motion of the Sun. With this we obtain for the Sun-angle-dependent variation of the eccentricity component along the line of apsides

$$\frac{de_x}{dt} = -\kappa B_r \sin(n_* t + \lambda_0) \tag{12.5.6}$$

It can be shown (see Campan et al. (1995)) that the line of apsides follows the motion of the Sun angle. We therefore obtain for the e_y component

$$\frac{de_y}{dt} = \kappa B_r \cos(n_* t + \lambda_0) \tag{12.5.7}$$

The solutions to the above equations of motion are easy to find:

$$\begin{aligned} e_x(t) &= e_{x0} + \frac{\kappa B_r}{n_*} \cos(n_* t + \lambda_0) \\ e_y(t) &= e_{y0} + \frac{\kappa B_r}{n_*} \sin(n_* t + \lambda_0) \end{aligned} \tag{12.5.8}$$

The tip of the eccentricity vector with initial value $\boldsymbol{e}_0 = \big(e_x(t_0),\ e_y(t_0)\big)$ therefore describes a circle with radius (see Fig. 12.25).

$$\frac{\kappa B_r}{n_*} = (0.0112 \pm 0.0004)\,\mathrm{kg\,m^{-2}} \cdot B_r \quad @\ \mathrm{GEO}$$

This description is based on the assumption that the Sun moves on a circle in the equatorial plane of the Earth, which is not quite true. Its inclination (angle between equatorial plane and ecliptic) causes the circle to be actually a weak ellipse. In addition other perturbations (J_{20} term of the geoid, lunisolar perturbation) lead to rosette-type deviations from the circle including the effect that the initial and final points no longer coincide (see Fig. 12.26).

Eccentricity Evolution

We first want to know the eccentricity evolution of an initially circular GEO, i.e., $\boldsymbol{e}(t = 0) = \boldsymbol{0}$ and $\lambda_0 = 90°$ (see Fig. 12.23). Applying this to Eq. (12.5.8) yields

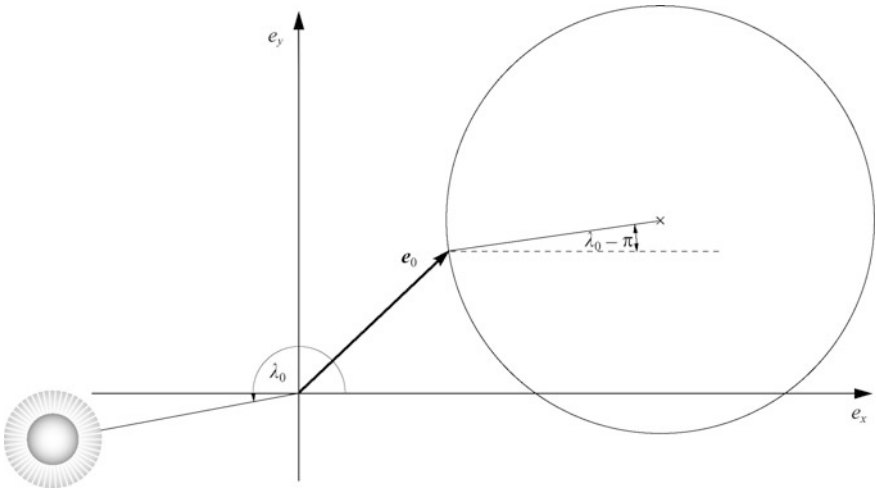


Fig. 12.25 The circular motion of the tip of the GEO eccentricity vector with initial value e_0 within a year. *Credit* Campan (1995), © CNES/CÉPADUÈS

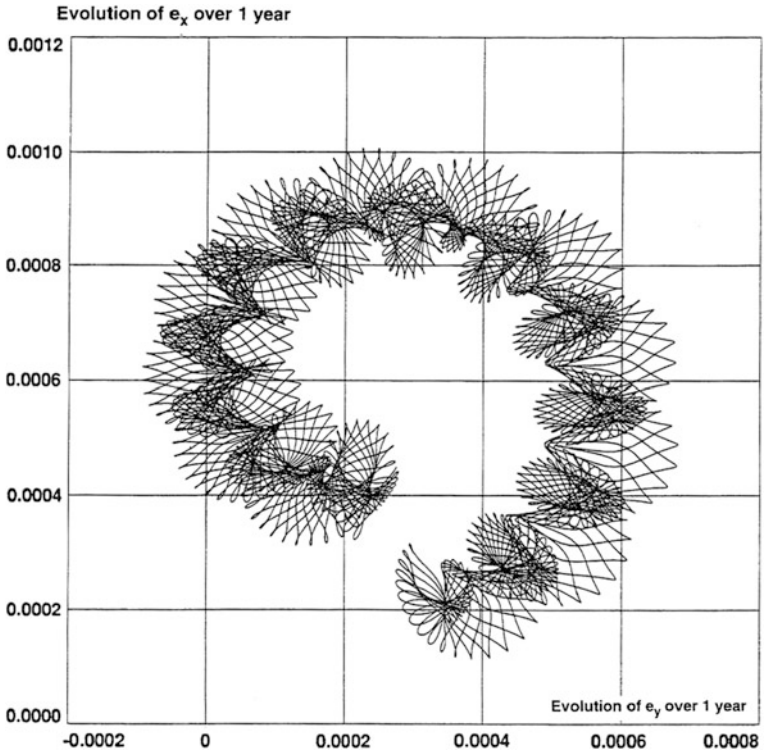


Fig. 12.26 The yearly motion of the GEO eccentricity vector under the action of all orbital perturbations. *Credit* Campan (1995), © CNES/CÉPADUÈS

$$\begin{aligned} e_x(t) &= -\frac{\kappa B_r}{n_*}\sin n_* t \\ e_y(t) &= -\frac{\kappa B_r}{n_*}(1-\cos n_* t) \end{aligned} \tag{12.5.9}$$

For the absolute value of the eccentricity we therefore obtain

$$e(t) = 2\frac{\kappa B_r}{n_*}\sin\frac{n_* t}{2} = \kappa B_r t\left[1-\frac{1}{24}(n_* t)^2 + O\left((n_* t)^4\right)\right] \quad @\ e_0 = 0 \tag{12.5.10}$$

and

$$\frac{de}{dt} = \kappa B_r \cos\frac{n_* t}{2} = \kappa B_r\left[1-\frac{1}{8}(n_* t)^2 + O\left((n_* t)^4\right)\right] \tag{12.5.11}$$

A communication relay satellite in GEO typically has $B_r \approx 0.05\ \mathrm{m}^2/\mathrm{kg}$ implying an initial

$$\delta e \approx 1 \times 10^{-5} \quad \text{per revolution} \tag{12.5.12}$$

Alternatively we could choose the initial condition such that the center point of the circle described by the eccentricity vector is at $\boldsymbol{e} = 0$. In this case

$$\begin{aligned} e_x(t) &= \frac{\kappa B_r}{n_*}\cos(n_* t + \lambda_0) \\ e_y(t) &= \frac{\kappa B_r}{n_*}\sin(n_* t + \lambda_0) \end{aligned} \tag{12.5.13}$$

implying that the absolute value of the eccentricity equaling the radius of the circle is constant

$$e = e_0 = \frac{\kappa B_r}{n_*} = const \tag{12.5.14}$$

This strategy is adopted for the European communications relay satellite ARTEMIS, which exhibits $B_r = 0.0369\ \mathrm{m}^2/\mathrm{kg}$ and therefore has a constant eccentricity of $e = 4.13 \times 10^{-4}$.

12.5.3 Correction Maneuvers

The yearly drag-induced motion of the eccentricity vector $\boldsymbol{e}$ however is of no relevance, because in GEO the absolute amount of eccentricity has to be regularly erased by correction maneuvers. To understand why, we have to know what the implications of a non-vanishing eccentricity are. Let us examine the periodic deviations of the orbital radius and the true anomaly caused by a body in an orbit with small eccentricity relative to a body in a circular orbit (so-called *guiding center*) with the same semi-major axis. This deviation is the apparent periodic horizontal motion of the position of the body in GEO as observed from the rotating Earth. To do this, we recall Eq. (7.3.9)

$$\frac{\mu^2}{h^3}(t - t_p) = \int_0^\theta \frac{d\theta'}{(1 + e \cdot \cos\theta')^2}$$

Because $e \ll 1$, $h^2 = \mu a(1 - e^2) \approx \mu a$ and $n = \sqrt{\mu/a^3}$, and by choosing $t_p = 0$, we get

$$nt \approx \int_0^\theta (1 - 2e \cdot \cos\theta')d\theta' = \theta - 2e\ \sin\theta$$

So, in zero-order approximation we get for $\theta(t)$ a circular orbit with $\theta = nt$ and in first-order approximation we have

$$\theta \approx nt + 2e\,\sin(nt) =: nt + \Delta\theta$$

i.e., compared with a circular orbit with $r = a$ the deviations are

$$\begin{aligned} \Delta\theta(t) &= 2e\,\sin(nt) \\ \Delta r(t) &= -e\,a\,\cos(nt) \end{aligned} \tag{12.5.15}$$

The latter is obtained from the orbit Eq. (7.4.14a) with $E \approx M = nt$ (see Eq. (7.4.15)). Because for small lateral deviations Δs the relation $\Delta s = a \cdot \Delta\theta$ holds, Eq. (12.5.15) describes an elliptic motion in the orbit plane (see Fig. 12.27) around the guiding center. Its semi-minor axis is ea in the radial direction and its semi-major axis is $2ea$ in the lateral direction, both of which increase with growing eccentricity. For the lateral oscillation range we find $\delta\theta = \Delta\theta_{\max} - \Delta\theta_{\min} = 4e$. As geostationary satellites are kept in an assigned dead-band slot of typical width $\Delta\theta = 0.1^\circ = 1.745 \times 10^{-3}$ rad , we get the following limit for the slowly increasing eccentricity

$$e_{\max} \leq \frac{1.745 \times 10^{-3}}{4} = 4.36 \times 10^{-4} \tag{12.5.16}$$

In the case when the orbit strategy is to always reset the eccentricity to zero when it reaches this limiting value, a correction maneuver has to be carried out at that point, which occurs after about 45 days for a typical $B_r \approx 0.05\ \mathrm{m^2\,kg^{-1}}$. Since the correction maneuver should not change the semi-major axis and with it the orbiting time, this has to be done according to Eq. (8.1.8) by two eccentricity correction maneuvers at both the peri- and apoapsis.

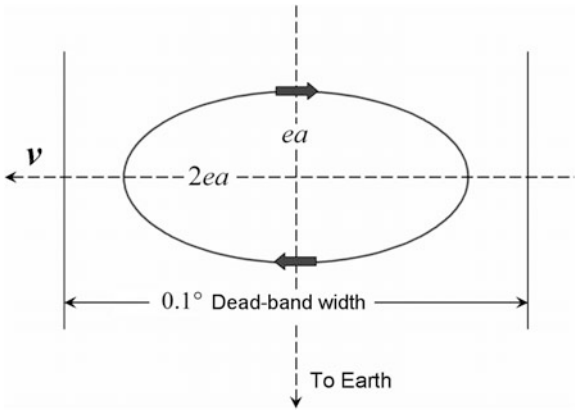


Fig. 12.27 Apparent motion of a geostationary satellite with eccentricity e in the guiding center system, i.e., the motion as seen in an Earth-fixed reference frame

$$2 \times \left|\Delta v_{||}\right| = \frac{1.745 \times 10^{-3}}{16} \sqrt{\frac{\mu}{a}} = 0.67 \text{ m/s} \tag{12.5.17}$$

Note that in this case Δv adds up to about $5\,\mathrm{m\,s^{-1}}$ within a year and hence makes up a significant share of the station-keeping effort.

In the other strategy case when the eccentricity vector rotates with a constant absolute value, the need for a correction maneuver depends on the amount of this value. For ARTEMIS $e = 4.13 \times 10^{-4} < e_{\max}$ and therefore in principle no correction maneuver would be needed to be performed.

East–West Station-Keeping Strategies

In practice triaxiality plus solar radiation together make up the two most effectual sources (and lunisolar perturbations a minor source) of east–west excursions. Depending on which of the sources is dominating (B_r and hence eccentricity versus longitudinal position and hence drift (see Fig. 12.7)) there are two basic strategies for east–west station-keeping (a.k.a. *longitude station-keeping*; see Soop (1994) or Chao (2005)):

1. *The drift due to triaxiality is dominating*.
   In this case *longitude control* is performed with correction strategy as laid out in Fig. 12.22. By doing so one gets the eccentricity control as a free by-product. The behavior of such a satellite is shown in Fig. 12.28. East-west

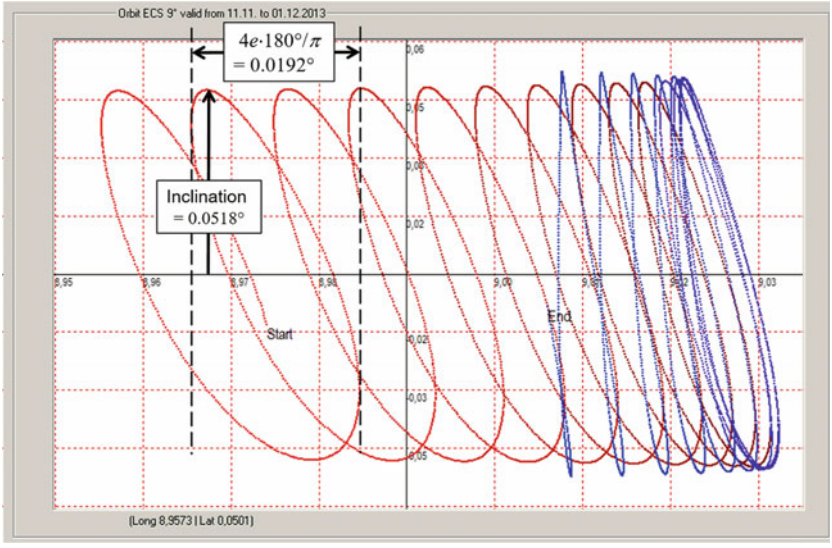


Fig. 12.28 Inclination and longitude (mean longitude 9°) of Eutelsat's ECS-9 GEO satellite under longitude control during 11-Nov-2013 to 01-Dec-2013 measured in the guiding center system. Every loop corresponds to one day. While drifting from West to East (red path) and back (blue path) its inclination of initially 0.052° increases slowly to 0.056° while its eccentricity decreases. (© Chr. Gleich/TUM)

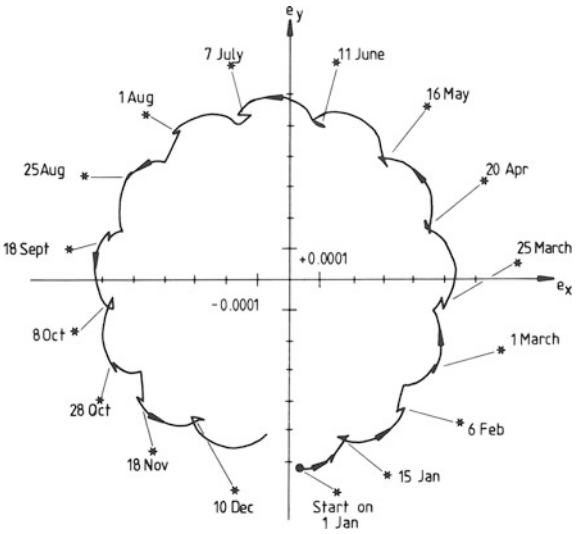


Fig. 12.29 The eccentricity vector under a SPPS-strategy over one year of a GEO satellite with B = 0.058 m^2/kg. *Credit* Soop (1994), © ESA

station-keeping maneuvers of this type are performed for broadcasting satellites (typically about once a week).

2. *The eccentricity-induced excursions due to solar radiation are dominating*. In this case eccentricity needs to be counteracted by the so-called *Sun-pointing perigee strategy SPPS*. This method minimizes the eccentricity variation around a small induced value by keeping the orbit's perigee pointing to the Sun as closely as possible. This is achieved by performing the longitude station-keeping maneuvers at some location away from perigee or apogee. Thereby the longitudinal drift compensation is provided as a free by-product. The path of the eccentricity vector of a SPPS-controlled satellite is shown in Fig. 12.29.

Most telecommunication satellites with their large solar panels usually belong to the second category. But ARTEMIS positioned at $\lambda = 21.40°$E and hence exhibiting a strong triaxiality drift but moderate B_r belongs to the first one.

12.6 Celestial Perturbations

12.6.1 Lunisolar Perturbations

We recall from Table 12.2 that apart from perturbations by the Earth's asymmetrical gravitational potential, also the Sun and the Moon perturb Earth orbits. Their effects are noticeable in particular in GEO because they are no more concealed by the gravitational perturbations as in LEO. The analytical description of these lunisolar perturbations is very complicated and complex (see Campan et al. (1995) or Noton (1998, p.70, and Appendix A.1)). This is why we will treat them only qualitatively.

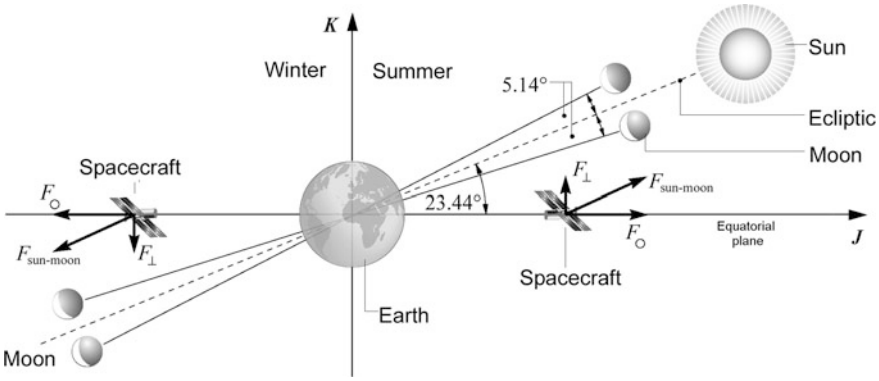


Fig. 12.30 Lunisolar perturbation force $F_{sun-moon}$ acting on a S/C in an equatorial orbit decomposed into a force component F_O in the orbital plane and $F_\perp$ perpendicular to the orbital plane

Because the terrestrial equatorial plane makes an inclination of 23.44° against the ecliptic and of 23.5 ± 5.1° against the lunar plane, the gravitational force of the Sun and the Moon can be decomposed into a component acting in the equatorial plane, i.e., in the GEO orbital plane, F_O, and one perpendicular to the equatorial plane $F_\perp$ (see Fig. 12.30).

In-Plane Force Effects

Let us consider first the effect of a revolving constant in-plane force F_O of a perturbing celestial body with standard gravitational parameter μ_P and position $\boldsymbol{R}$ from the center of the S/C orbit. The perturbation acceleration $\boldsymbol{a}_O$ on the S/C with position vector $\boldsymbol{r}$ is

$$\boldsymbol{a} = \mu_p \frac{\boldsymbol{R} - \boldsymbol{r}}{|\boldsymbol{R} - \boldsymbol{r}|^3} \tag{12.6.1}$$

Because in the orbital plane $\hat{\boldsymbol{R}} = \hat{\boldsymbol{r}}\cos\theta - \boldsymbol{e}_\theta \sin\theta = \boldsymbol{e}_r \cos\theta - \boldsymbol{e}_\theta \sin\theta$, where and $\cos\theta = \hat{\boldsymbol{r}}\hat{\boldsymbol{R}}$, we have for the in-plane acceleration $\boldsymbol{a}_O = a_r \boldsymbol{e}_r + a_\theta \boldsymbol{e}_\theta$ with

$$a_r = \mu_p \frac{R\cos\theta - r}{|\boldsymbol{R} - \boldsymbol{r}|^3}, \quad a_\theta = -\mu_p \frac{R\sin\theta}{|\boldsymbol{R} - \boldsymbol{r}|^3}$$

$$|\boldsymbol{R} - \boldsymbol{r}|^{-3} = \left(R^2 + r^2 - 2rR\cos\theta\right)^{-3/2}$$

$$\approx \frac{1}{R^3}\left(1 - 2\frac{r}{R}\cos\theta\right)^{-3/2} \approx \frac{1}{R^3}\left(1 + 3\frac{r}{R}\cos\theta\right)$$

We hence have the symmetry properties

$$a_r(-\theta) = a_r(\theta), \quad a_\theta(-\theta) = -a_\theta(\theta)$$

Applying these properties to the Gaussian variational equations (12.1.4) we see that $\dot{i} = \dot{\Omega} = 0$, and $\dot{a}(-\theta) = -\dot{a}(\theta)$, $\dot{e}(-\theta) = -\dot{e}(\theta)$. Because $\dot{\omega}$ is not relevant for a circular orbit, we arrive at the result that a revolving constant in-plane force F_{O} does not have a secular effect on the orbital elements of a body that circles Earth. Because the symmetry properties do not change if the Moon or Sun are in opposite locations around the Earth, the variation of orbital elements over one revolution of the Moon or Sun around the Earth also averages out. So there are no secular effects whatsoever.

Out-of-Plane Force Effects

The component $F_{\perp}$ perpendicular to the orbital plane, on the other hand, does have secular effects. To see which, we recognize that $F_{\perp}$ causes an perturbational acceleration a_h normal to the orbit plane. From Eq. (12.1.4) we see that such an a_h affects only the orbital elements i, ω and Ω. Due to $\dot{e} = 0$ and therefore a steady $e = 0$ for a GEO, ω and hence $\dot{\omega}$ are not relevant.

We therefore only have to evaluate $i(t)$, $\Omega(t)$. We do so by employing the concept of the three-dimensional inclination vector. According to Eq. (7.3.25) it is given by

$$\boldsymbol{I} \equiv \hat{\boldsymbol{h}} = \begin{pmatrix} \sin\Omega \sin i \\ -\cos\Omega \sin i \\ \cos i \end{pmatrix}_{IJK}$$

According to the above, the motion of this vector fully describes the effect of lunisolar perturbations. Figure 12.31 shows a typical example of the track of the inclination vector of a S/C in a GEO on the celestial sphere under the influence of the Sun and the Moon.

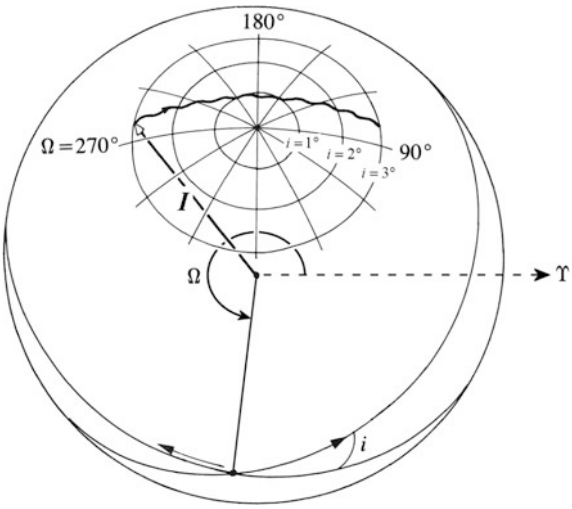


Fig. 12.31 Trace of the tip of the inclination vector of a GEO on the celestial sphere. *Credit* Berlin (1988)

To understand how this motion comes about, we assume that the S/C initially is in a circular orbit with $i = 0$. According to Eq. (8.1.14), a_h, which is locally equivalent to the acceleration due to a $\delta v_{\perp\perp}$ kick-burn, initiates an inclination with an argument of latitude of the S/C

$$u = \begin{cases} 0^\circ & @\ a_h > 0 \\ 180^\circ & @\ a_h < 0 \end{cases}$$

Because the nodal line of the ecliptic plane and equatorial plane points to the First point of Aries Υ, Moon and Sun cause no vertical perturbation when they cross the equatorial plane, i.e. when they are in the direction of Υ or on the opposite side. We therefore expect the initial inclination δi when $a_h > 0$ is maximum abeam from the nodal line and hence at $\Omega = u + 90^\circ = 90^\circ$. As the S/C revolves once in its orbit for a steady position of the Sun and Moon, i.e., $a_h \approx const \rightarrow \delta v_{\perp\perp} \approx const$, the S/C experiences a $-\delta i$ at the opposite location, i.e. at $u = 180^\circ$, which cancels the first one. Thus the average over one S/C orbit produces no secular inclination.

However, the situation is different at other positions on the orbit as the Moon and Sun move around the Earth. According to the Gaussian variational equations (12.1.4), when they are at -90° from the nodal line with $a_h > 0$ they cause an $\delta i > 0$ and when they are at 90° from the nodal line with $a_h < 0$ they cause the same $\delta i > 0$. Therefore the revolution of the Moon and the Sun about the Earth constantly increase the inclination at $\Omega = 90^\circ$. The inclination vector hence initially displays the following motion $\boldsymbol{I}(t) \approx (i(t),\ 0,\ 0)_{IJK}$, that is, it tilts towards the First point of Aries Υ. When the Sun or Moon are just off the abeam positions they also tilt the inclination vector minimally to the side. However, because this is not a secular effect, the evolution of the inclination as shown in Fig. 12.32 just displays small wiggles with twice the frequency of the revolution of the corresponding celestial body about the Earth. Therefore the Sun causes two wiggles while the Moon causes 25 wiggles per year.

The actual motion of the inclination vector over the years depends on the initial RAAN and is depicted for a GEO in Fig. 12.33.

The inclination vector is also subject to the oblateness perturbation of Earth's gravitational field (see Sect. 12.3.3), which gives rise to a regression of nodes $\dot{\Omega} < 0$

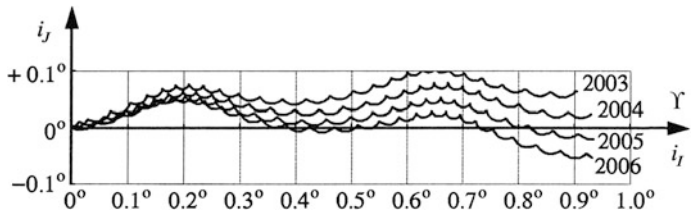


Fig. 12.32 Evolution of the inclination vector components for a GEO over a year for 2003–2006 with inclination set to zero at the beginning of each year. *Credit* Soop (1994), © ESA

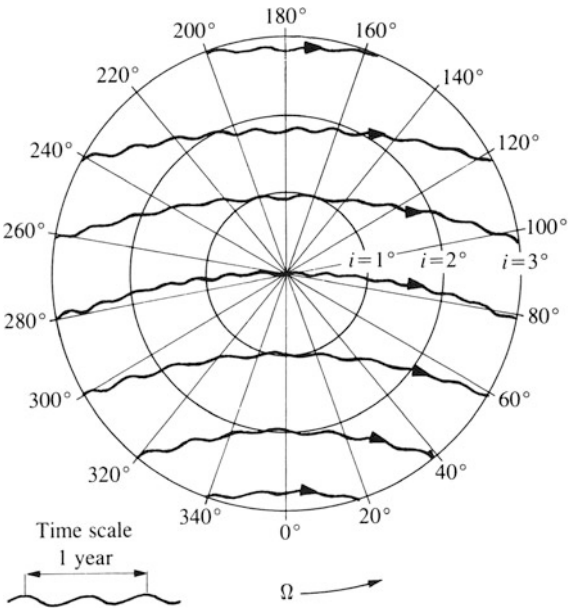


Fig. 12.33 Evolution of the inclination vector of a GEO in the i-Ω diagram depending on the initial RAAN. *Credit* Berlin (1988)

(see Eq. (12.3.15)), which effectively curves the trace of the inclination vector in the i-Ω diagram as seen in Fig. 12.33 (for more details see Berlin (1988, p. 60ff) and Soop (1994, p. 86)).

For the determination of the quantitative secular variation of inclination we refer to Campan (1995), who shows that in first-order perturbation calculations the mean secular inclination change is

$$n \cdot \dot{i}_{\text{sec}} = \frac{3}{8} n_b^2 \mu_b \sin 2\langle i_b \rangle = \begin{cases} 0.533^{\circ 2}\text{day}^{-2} & @\ b = Moon \\ 0.268^{\circ 2}\text{day}^{-2} & @\ b = Sun \end{cases} \tag{12.6.2}$$

with $\langle i_b \rangle$, its inclination with regard to the equatorial plane; $n_b = 360^\circ / T_b$, its mean motion; n the mean motion of the perturbed satellite; and

$$\mu_b = \frac{m_b}{m_b + m_\oplus} = \left\{ \begin{matrix} 1 & @\ b = Sun \\ 1/82.3 & @\ b = Moon \end{matrix} \right\}$$

If the inclination is not controlled, as for dead satellites in the graveyard orbit, the lunisolar perturbations act upon the inclination while Earth's oblateness causes a precessional motion of the orbital plane. As shown in Fig. 12.34 both together over long time spans cause an oscillation of inclination in the range $0 \le i \le 15^\circ$ jointly with a precessional motion of the orbital plane inclination within the limits $0 \le \Omega \le 100^\circ$ or $260^\circ \le \Omega \le 360^\circ$.

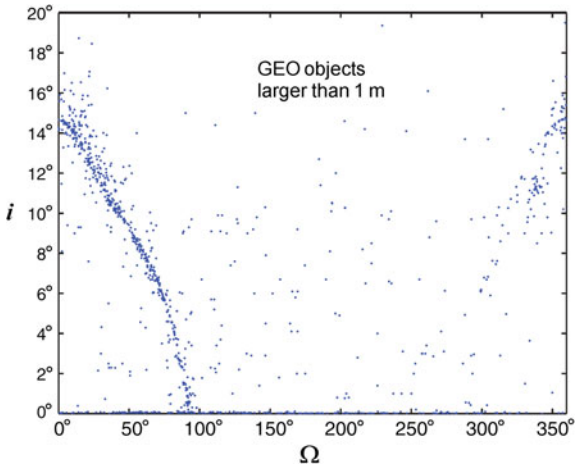


Fig. 12.34 Distribution of space objects at GEO in i-Ω space. *Credit* A. Rossi, IFAC-CNR, 2010

Low Earth Orbits

From this we find for equatorial LEOs at $h = 600$ km, equaling $n = 5362^\circ\ \text{day}^{-1}$, the yearly inclination change rate of

$$\dot{i}_{\text{sec}} = \begin{cases} 0.036^\circ\ \text{yr}^{-1} & @\ \textit{Moon} \\ 0.018^\circ\ \text{yr}^{-1} & @\ \textit{Sun} \end{cases} \quad @\ \text{equatorial LEO}$$

and hence for both perturbations $\dot{i}_{\text{sec}} \approx 0.054^\circ\ \text{yr}^{-1}$. Taking all planetary perturbations into account we have

$$\dot{i}_{\text{sec}} \approx 0.062^\circ\ \text{yr}^{-1} \quad @\ \text{equatorial LEO} \tag{12.6.3}$$

which is a typical value for LEOs in general. This is very small but nevertheless bigger than that caused by gravitational perturbations for near-circular orbits (see Sect. 12.3.4).

Geostationary Orbits

GEOs are more prone to lunisolar perturbations because bodies on such orbits move slower and hence the time-averaging effect is lower. With $n = 360.99^\circ\ \text{day}^{-1}$

$$\dot{i}_{\text{sec}} = \begin{cases} 0.539^\circ\ \text{yr}^{-1} & @\ \textit{Moon} \\ 0.271^\circ\ \text{yr}^{-1} & @\ \textit{Sun} \end{cases} \quad @\ \text{GEO}$$

If one allows also higher orders of perturbation $k \cdot n_b$, then one obtains for the yearly rate of the inclination variation $\dot{i}_{\text{sec}} = 0.478^\circ\ \text{yr}^{-1}$ by lunar and $\dot{i}_{\text{sec}} \approx 0.319^\circ\ \text{yr}^{-1}$ by solar perturbations, whereby the latter varies from year to year somewhat. So, for both perturbations together we find $\dot{i}_{\text{sec}} = 0.797^\circ\ \text{yr}^{-1}$. Again, taking all planetary perturbations into account we have

$$\dot{i}_{\text{sec}} = 0.854^\circ \pm 12\%\ \text{yr}^{-1} \quad @\ \text{GEO} \tag{12.6.4}$$

The 12% variation is due to the periodic variation of the Moon's orbit inclination relative to the ecliptic with a period of 18.6 years. Because from Earth the movement of a body in an inclined geostationary orbit looks like a vertical oscillation, it is called *North–South drift*. The **North–South station-keeping** (a.k.a. *inclination station-keeping*) effort depends on the station-keeping maneuvers applied. A straightforward maneuver is to remove the inclination by a direct maneuver performed at one of the nodes (see Fig. 8.4). The mean delta-v effort can be calculated from Eq. (8.1.23) to be

$$\Delta v = 2v_{GEO} \sin\frac{\Delta i}{2}$$

With $\Delta i = \dot{i}_{\text{sec}} \cdot 1\ \text{yr} = 0.854^\circ$ and $v_{GEO} = 3.066\ \text{km s}^{-1}$ and therefore obtain

$$\Delta v = 45.7 \pm 12\%\ \ \text{m s}^{-1}\,\text{yr}^{-1} \quad @\ \text{inclination removal maneuver}$$

Rather than just eliminate the induced inclination it is obvious that it is better (see Fig. 12.33) to overact and place the inclination vector at a small $i < 1^\circ$ with $\Omega = 270^\circ$ and let it drift through $i = 0^\circ$ to the same i with $\Omega = 90^\circ$. At this point a RAAN adjustment maneuver is performed, which places the S/C again at i with $\Omega = 270^\circ$, where the cycle starts anew. The delta-v effort for such a maneuver is given by Eq. (8.1.21) with $v_1 = v_2 = v_{GEO}$ as

$$\Delta v = 2v_{GEO} \sin i \cdot \sin 90^\circ$$

and hence

$$\Delta v = 6.132 \cdot \sin i\ \ \text{km s}^{-1} \quad @\ \text{cyclic RAAN adjustment}, i < 1^\circ \tag{12.6.5}$$

With an mean annual secular perturbation of $\dot{i}_{\text{sec}} = 0.854^\circ \pm 12\%\ \text{yr}^{-1}$ and adjustments for $i \leq 0.917^\circ$ and accounting for the phase of Moon's orbit inclination relative to the ecliptic we have

$$\boxed{\Delta v = 45.7 - 5.2 \sin\left(2\pi \frac{yy - 1992}{18.6}\right)\ \text{m s}^{-1}\,\text{yr}^{-1}} \quad @\ \text{cyclic RAAN adjustment}, i < 1^\circ \tag{12.6.6}$$

where *yy* is the Gregorian year in question. This result of course is identical to the inclination removal maneuver because turning the inclination vector by 180° is the same as applying the inclination removal twice.

However, the cyclic RAAN adjustment maneuver becomes slightly more efficient if there are no drift constraints in North–South direction, because such drifts do not interfere with adjacent satellites. In this case the yearly adjustment effort drops because the trace of the inclination vector is curved. For instance and as

shown in Fig. 12.33, the inclination vector takes 7.5 years to drift from $i = 3°$, $\Omega = 280°$ $\rightarrow$ $i = 0$ $\rightarrow$ $i = 3°$, $\Omega = 80°$ from where it is placed with a cyclic RAAN maneuver back at $i = 3°$, $\Omega = 280°$. According to Eq. (8.1.21) the delta-v effort for this maneuver is

$$\Delta v = 2v_{GEO} \sin 3° \cdot \sin[360° - (280° - 80°)]/2 = 0.316 \text{ km s}^{-1}$$

and we have a yearly delta-v effort of only

$$\Delta v = 42.1 \pm 12\% \text{ m s}^{-1} \text{ yr}^{-1} \text{ @ cyclic RAAN adjustment}, i = 3°$$

Yet, this is still more than one order of magnitude bigger than the east–west station–keeping in the worst case (see Eq. (12.4.16)) and hence more decisive for the propulsion demand for orbit control. This result confirms the rule of thumb that inclination changes in astronautics always imply high propulsion efforts.

North-South station-keeping maneuvers are performed about once or twice every two weeks, often alternating with an East-West station-keeping maneuver (see end of Sect. 12.5.3).

12.6.2 Relativistic Perturbations

In Sect. 12.6.1 we have seen that lunisolar perturbations cause a constant shift of the true anomaly of an elliptic Earth orbit and hence a progression of line of apsides. This effect of course holds also for planetary elliptic orbits that are affected by gravitational fields of the Sun and all other planets in the solar system leading to a perihel motion for all solar planets. For Mercury, where this effect is most pronounced, this can be calculated to amount to 1.28 arcsec per revolution, equaling 532 arcsec per century.

It was a puzzle for astronomers at the beginning of the twentieth century that the observed perihel motion of Mercury was actually found to be 575 arcsec per century. So, 43 arcsec were missing. It was one of the great triumphs of Albert Einstein to show that his theory of general relativity is able to fully account for the missing gap and hence to prove that General Relativity Theory is true. Let us see how general relativity adds perihel motion to planetary orbits.

In General Relativity Theory the equation of motion for an orbit is modified in terms of Sect. 7.6.1, subsection "3-dim Universe", to be (see, e.g., Beutler 2005a, p. 148)

$$\rho'' + \rho = \frac{\mu}{h^2} + \frac{3\mu}{c^2}\rho^2 \tag{12.6.7}$$

We recall from Sect. 7.6.1 that $\rho = 1/r$ and $\rho' = d\rho/d\theta$, and c is the speed of light. Since for a near-circular orbit $v \approx \sqrt{\mu/r}$ is its orbital speed, the ratio of the two terms on the right-hand side of the equation is

$$\frac{3\mu\rho^2/c^2}{\mu/h^2} \approx \frac{3\mu a}{c^2 r^2} \approx \frac{3\mu}{c^2 r} \approx 3\frac{v^2}{c^2} \ll 1$$

Therefore the contribution due to relativity is very small. This is why we can perform a calculation of perturbation in that we first neglect the relativistic perturbation. We thus find according to Eq. 7.3.5 the solution of the Keplerian orbit

$$\rho = \frac{\mu}{h^2}(1 + e\,\cos\theta)$$

We insert this solution into Eq. (12.6.7) and neglecting $O(e^2)$ we find the equation of motion

$$\rho'' + \rho = \frac{1}{p}[1 + \varepsilon(1 + 2e\,\cos\theta)] \tag{12.6.8}$$

with semi-latus rectum $p = h^2/\mu = a(1 - e^2)$ and $\varepsilon := 3\mu/(c^2 p)$. Because we expect the perturbed solution close to the unperturbed we make the ansatz

$$\rho = \frac{1}{p}[1 + e\,\cos\theta + f(\theta)]$$

Inserting this into Eq. (12.6.8) leads to equation of motion for the residual $f(\theta)$

$$f'' + f = \varepsilon(1 + 2e\,\cos\theta)$$

A particular solution to this equation is

$$f = \varepsilon(1 + e\theta\,\sin\theta)$$

as can be easily verified by insertion. The general solution therefore is

$$p\rho = (1 + \varepsilon) + e(\cos\theta + \varepsilon\theta\,\sin\theta) \approx (1 + \varepsilon) + e\,\cos(\theta - \varepsilon\theta)$$

The last step holds if we neglect terms of ε^2. Hence we finally find the orbit equation

$$\frac{p}{1 + \varepsilon}\rho = 1 + \frac{e}{1 + \varepsilon}\cos(\theta - \varepsilon\theta) \tag{12.6.9}$$

If we compare this result with the unperturbed orbit $p\rho = 1 + e\,\cos\theta$ we recognize that the relativistic orbit suffers a tiny shrinkage $a \to a/(1 + \varepsilon)$, a marginally reduced eccentricity $e \to e/(1 + \varepsilon)$, and a small prograde perihel motion per revolution of

$$\Delta\omega = 2\pi\varepsilon = \frac{6\pi\mu}{c^2 p} = \frac{6\pi\mu}{a(1-e^2)c^2} \text{ per revolution} \tag{12.6.10}$$

Since the orbital period is $T = 2\pi\sqrt{a^3/\mu}$ we have for the perihel motion per century (100 *y*)

$$\Delta\omega_{100y} = \Delta\omega\frac{100\ y}{T} = \frac{3}{a(1-e^2)c^2}\left(\frac{\mu}{a}\right)^{3/2} 100\ y$$

From Table A.1 in Appendix A we insert the orbital elements of Mercury, which due to its closest solar orbit is expected to suffer mostly from relativistic gravitational effects, and find

$$\Delta\omega_{100y} = 42.98 \text{ arcsec} \quad @ \text{ Mercury}$$

This is the celebrated relativistic contribution, which perfectly explains the missing gap of 43 arcsec.

Remark *In the 1960s there has been a considerable controversy over whether the solar oblateness J_2, which must also contribute to Mercury's perihel motion and was not considered to that date, might prove general relativity wrong. From* Eq. (12.3.15) *we can derive the oblateness' contribution to be* $\Delta\omega_{100y} = 2.54 \times 10^5 \cdot J_2$ arcsec. *With today's observational value of* $J_2 = (1.9 \pm 0.3) \times 10^{-7}$ *we arrive at* $\Delta\omega_{100y} = 0.05$ arcsec, *which is negligible compared to the inaccuracies of the planet's contributions. The missing gap therefore is solely due to general relativity.*

It should be mentioned that in general an orbit about a central body is perturbed by other relativistic effects. In particular, if the central body rotates its mass currents produce a so-called gravitomagnetic dipole field, which via relativistic space frame dragging (so-called Lense-Thirring effect) in general causes a retrograde motion of the periapsis plus a precession of the orbital plane. These two effects, however, are so weak that they could be observed to date only in the very strong gravitational fields of binary pulsars.

12.7 Drag

In low Earth orbits the atmospheric density cannot be neglected and therefore may exert a marked resistance on a circulating spacecraft. In this chapter we want to determine how the orbit—specifically, how the orbital elements—will be affected by atmospheric drag.

12.7.1 Drag Perturbations

According to Sect. 6.2.3 the atmospheric deceleration can be described in dependence on the drag D as

$$a_D = \frac{D}{m} = \frac{C_D}{2}\frac{A_\perp}{m}\rho v_a^2 = \frac{B}{2}\rho v_a^2 \quad @\ h > 150\ \text{km} \tag{12.7.1}$$

We now want to determine the impact of atmospheric drag on the orbital elements. We identify the drag force as an orbital perturbation, which we split into radial, cross-radial, and vertical components and derive from Fig. 12.35 with the above result

$$\boldsymbol{a}_D = (a_r, a_\theta, a_h) = -\frac{B}{2}\rho(r) \cdot v_a^2(\sin\gamma, \cos\gamma, 0) \tag{12.7.2}$$

The negative sign denotes that the S/C is decelerated by the drag. According to Eq. (7.3.16) the flight path angle γ is related to the true anomaly by

$$\sin\gamma = \frac{e\ \sin\theta}{\sqrt{1 + 2e\ \cos\theta + e^2}}$$
$$\cos\gamma = \frac{1 + e\ \cos\theta}{\sqrt{1 + 2e\ \cos\theta + e^2}}$$

We insert this perturbation into the Gaussian variational equations (12.1.4) and find for the wanted variations of orbital elements

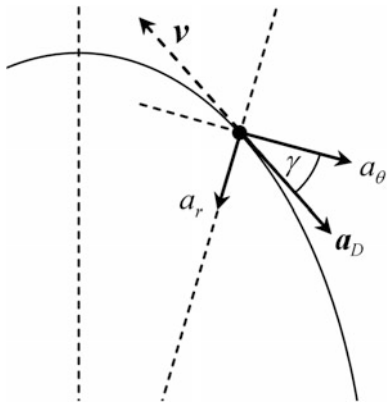


Fig. 12.35 Decomposition of the atmospheric drag into radial and lateral components

$$\begin{aligned}
\dot{a} &= -\frac{B\rho v_a^2}{n}\frac{1}{\xi(e,\theta)} \\
\dot{e} &= -\frac{B\rho v_a^2}{na}\xi(e,\theta)(e+\cos\theta) \\
\dot{\omega} &= -\frac{B\rho v_a^2}{nae}\xi(e,\theta)\sin\theta \\
\dot{M}-n &= \frac{B\rho v_a^2}{nae}\xi(e,\theta)\frac{\sin\theta\cdot(1+e\cos\theta+e^2)}{(1+e\cos\theta)}\sqrt{1-e^2} \\
\dot{i} &= \dot{\Omega} = 0
\end{aligned} \tag{12.7.3}$$

with

$$\xi(e,\theta) = \frac{\sqrt{1-e^2}}{\sqrt{1+2e\cos\theta+e^2}} = \sqrt{\frac{\mu}{a}}\frac{\sqrt{1-e^2}}{v}$$

In the following we are only interested in the mean variations of the orbital elements over one orbital period. So, in the absence of slow variables we only have to integrate over one orbit. From the above equations we establish $\dot{\omega}(-\theta) = -\dot{\omega}(\theta)$ and $\dot{M}(-\theta) = -\dot{M}(\theta)$, implying that each periodic variation is compensated by an equally negative variation on the other side of the orbit. We therefore find

$$\dot{\omega}_{\text{sec}} = \frac{1}{2\pi}\int_{-\pi}^{\pi}\dot{\omega}(r,\theta)\cdot dM = 0$$

$$\dot{M}_{\text{sec}} - n = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\dot{M}(r,\theta)-n\right]\cdot dM = 0$$

We therefore arrive at the following important result:

Impact of Drag on Orbital Elements
Drag secularly affects only the eccentricity and the size of the semi-major axis.

12.7.2 Orbit Circularization

We now want to figure out how large the mean variations of a and e are. The mean variation is the integral over the total atmospheric drag of one orbit. Due to the exponential dependence of the atmospheric density, drag is by far the strongest around the periapsis, $\theta \approx 0$, for an elliptic orbit. In order to be able to carry out the integration, we need to know the θ dependences of all factors in Eq. (12.7.3). This is particularly true for the atmospheric density $\rho(\theta)$. To find it out we expand the orbit equation $r = a(1-e^2)/(1+e\cos\theta)$ for small angles at the periapsis

$$r \approx r_{per}\left[1+\frac{e}{2(1+e)}\theta^2\right] = r_{per} + a\frac{1-e}{1+e}\frac{e}{2}\theta^2$$

With this expression $\rho(\theta)$ can be written as

$$\rho(\theta) = \rho_{per}\exp\left(-\frac{r-r_{per}}{H_{per}}\right) \approx \rho_{per}\exp\left(-\frac{ae}{2H_{per}}\frac{1-e}{1+e}\theta^2\right)$$

hence

$$\rho(\theta) = \rho_{per}\exp\left(-\frac{\theta^2}{2\sigma^2}\right) \tag{12.7.4}$$

with

$$\sigma = \sqrt{\frac{H_{per}}{ea}\frac{1+e}{1-e}}$$

where $\rho_{per} = \rho\left(r_{per}\right)$ is the atmospheric density at periapsis. By the same token we expand the other terms in Eq. (12.7.3) and find with Eq. (7.4.11)

$$v_a^2 = \frac{\mu(1+2e\,\cos\theta+e^2)}{a(1-e^2)} \approx \frac{\mu}{a}\frac{1+e}{1-e}\left[1-\frac{e}{(1+e)^2}\theta^2\right]$$

and

$$\xi(e,\theta) \approx \sqrt{\frac{1-e}{1+e}\left[1+\frac{e}{2(1+e)^2}\theta^2\right]}$$

We insert these expressions into Eq. (12.7.3) and obtain

$$\dot{a} \approx -B\rho_{per}\sqrt{\mu a}\left(\frac{1+e}{1-e}\right)^{3/2}\exp\left(-\frac{\theta^2}{2\sigma^2}\right)\left[1-\frac{3e}{2(1+e)^2}\theta^2\right]$$

$$\dot{e} \approx -B\rho_{per}\sqrt{\frac{\mu}{a}}\frac{(1+e)^{3/2}}{(1-e)^{1/2}}\exp\left(-\frac{\theta^2}{2\sigma^2}\right)\left[1-\frac{1+2e}{2(1+e)^2}\theta^2\right]$$

We now carry out the integration over one orbital period by assuming that $\sigma<\theta \ll 1$, i.e., $e \gg H_{per}/a$. This condition ensures that the density decays within the θ-range where our expansion is valid. Note that for the Earth $0.001 \le H_{per}/a < 0.01$, where the lower limit holds for $h<120$ km and the upper limit for $h>300$ km. We then get with

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\exp\left(-\frac{\theta^2}{2\sigma^2}\right)\cdot d\theta \approx \frac{\sigma}{\sqrt{2\pi}}$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\exp\left(-\frac{\theta^2}{2\sigma^2}\right)\theta^2\cdot d\theta \approx \frac{\sigma^3}{\sqrt{2\pi}}$$

finally the wanted equations of mean variation:

$$\begin{aligned}\langle \dot{a} \rangle &= -B\rho_{per}\left(\tfrac{1+e}{1-e}\right)^2 \sqrt{\tfrac{\mu H_{per}}{2\pi e}}\left[1 - \tfrac{3H_{per}}{2a(1-e^2)}\right] \\ &\approx -B\rho_{per}\left(\tfrac{1+e}{1-e}\right)^2 \sqrt{\tfrac{\mu H_{per}}{2\pi e}} = \langle \dot{a}_{circle} \rangle \tfrac{\sigma}{\sqrt{2\pi}} \ll \langle \dot{a}_{circle} \rangle \qquad @\ e \gg \frac{H_{per}}{a} \\ \langle \dot{e} \rangle &= -B\rho_{per} \tfrac{(1+e)^2}{1-e} \tfrac{1}{a} \sqrt{\tfrac{\mu H_{per}}{2\pi e}}\left[1 - \tfrac{H_{per}}{2ea} \tfrac{1+2e}{1-e^2}\right] \\ &\approx -B\rho_{per} \tfrac{(1+e)^2}{1-e} \tfrac{1}{a} \sqrt{\tfrac{\mu H_{per}}{2\pi e}} = (1-e)\tfrac{\langle \dot{a} \rangle}{a}\end{aligned} \tag{12.7.5}$$

where $\langle \dot{a}_{circle} \rangle$ is the mean variation of a circle with $a_{circle} = r_{per}$. So, the impact of drag on the semi-major axis and the eccentricity of an elliptic orbit is about the same. However, because for an elliptic orbit the drag impacts the body only around the periapsis, and is much less than that for a circular orbit, which is continuously exposed to the drag, the elliptic semi-major axis does not decrease as fast as the radius of a comparable circular orbit.

Decoupling the Differential Equations

These coupled differential equations can be solved by separating the variables in the second equation (because we will deal only with mean variations of orbital elements in the following, we drop the average sign $\langle . \rangle$ for convenience)

$$\frac{de}{1-e} = \frac{da}{a}$$

with the solution (see Eq. (7.4.6))

$$r_{per} = a(1-e) = const \tag{12.7.6}$$

The periapsis hence remains unaffected, while the apoapsis

$$\dot{r}_{apo} = \dot{a}(1+e) + a\dot{e} = \dot{a}(1+e) + \dot{a}(1-e) = 2\dot{a} < 0$$

decreases. This behavior can be seized quite easily if we consider the temporary drag at the periapsis as a deceleration kick-burn. According to Eq. (8.1.11) such a kick-burn lowers the apsis on the opposite side of the orbit. With that we arrive at the important result:

How Drag Removes Eccentricity
Drag circularizes elliptic orbits by lowering the apoapsis, but maintaining the periapsis.

This aerobraking property as displayed in Figs. 12.36 and 12.37 is often used after planetary or aerocapture at Mars or Venus to turn the highly elliptic initial orbit down to a circular target orbit without any propulsion effort.

By making use of Eq. (12.7.6) one finally obtains from Eq. (12.7.5) the decoupled differential equations for the mean orbital elements

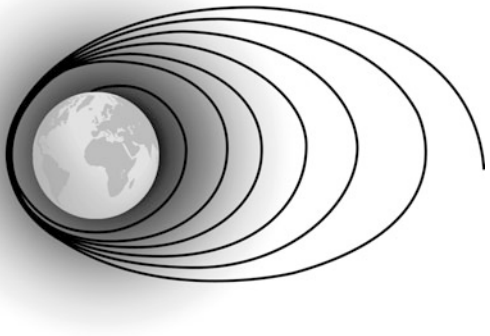


Fig. 12.36 Circularization and decay of an elliptic orbit due to drag

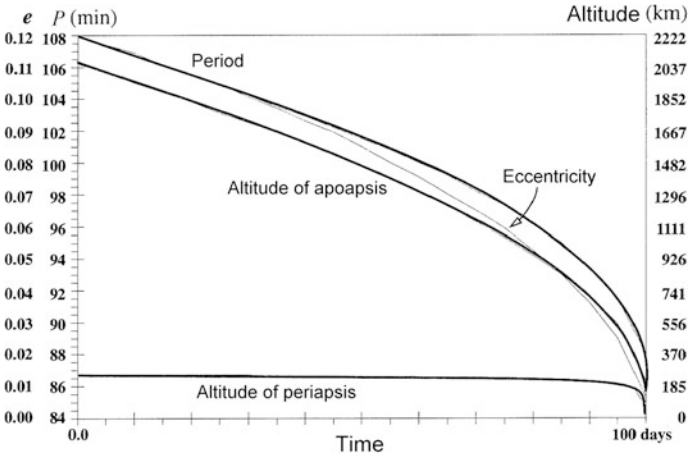


Fig. 12.37 Drag-induced decay of orbit parameters with $e_0 = 0.12$ and $h_{per} = 200$ km. P is the orbital period. *Credit* Vallado (2007)

$$\begin{aligned} \dot{a} &= -B\rho_{per}\sqrt{\frac{\mu H_{per}}{2\pi}}\left(\frac{2a}{r_{per}}-1\right)^2\sqrt{\frac{1}{1-r_{per}/a}} \\ \dot{e} &= -\frac{B\rho_{per}}{r_{per}}\sqrt{\frac{\mu H_{per}}{2\pi}}\frac{(1+e)^2}{\sqrt{e}} \end{aligned} \quad @\ e \gg \frac{H_{per}}{a} \tag{12.7.7}$$

Circularization Time

We are now seeking for solutions of the decoupled differential equations with the initial conditions $a_0 = a(t_0),\ e_0 = e(t_0)$. In the second equation we separate the variables

$$\frac{\sqrt{e}}{(1+e)^2}de = -\frac{B\rho_{per}}{r_{per}}\sqrt{\frac{\mu H_{per}}{2\pi}}dt$$

and find from relevant mathematical tables of integrals the following time-dependence of the eccentricity

$$\arctan\sqrt{e} - \frac{\sqrt{e}}{1+e} = \arctan\sqrt{e_0} - \frac{\sqrt{e_0}}{1+e_0} - \frac{B\rho_{per}}{r_{per}}\sqrt{\frac{\mu H_{per}}{2\pi}}(t-t_0) \qquad (12.7.8)$$

Since

$$\arctan\sqrt{e} - \frac{\sqrt{e}}{1+e} = e^{3/2}\left(\frac{2}{3} - \frac{4}{5}e + \frac{6}{7}e^2 - \cdots\right)$$

we can approximate

$$e^{3/2}(t) = e_0^{3/2} - \frac{3B\rho_{per}}{2r_{per}}\sqrt{\frac{\mu H_{per}}{2\pi}}(t-t_0) \quad @ \quad \frac{H_{per}}{a} \ll e < 0.02$$

Because we have $a(1-e) = r_{per} = const$, and with Eq. (7.4.9) it follows

$$e(t) = 1 - \frac{r_{per}}{a(t)} = \frac{r_{apo}(t) - r_{per}}{r_{apo}(t) + r_{per}}$$

from which we obtain by insertion the analytical orbit equation also for $a(t)$ and $r_{apo}(t)$.

How long would it take to circularize an elliptic orbit? This can be determined quite easily. The circularization time t_{cir} is the time to $e = 0$, i.e., $t_{cir} = t(e = 0)$. Inserting this condition into the above equations we find with $t_0 = 0$

$$t_{cir} = \frac{r_{per}}{B\rho_{per}}\sqrt{\frac{2\pi}{\mu H_{per}}}\left(\arctan\sqrt{e_0} - \frac{\sqrt{e_0}}{1+e_0}\right) \quad \textbf{circularization time} \qquad (12.7.9)$$

or

$$t_{cir} = \frac{2r_{per}e_0^{3/2}}{3B\rho_{per}}\sqrt{\frac{2\pi}{\mu H_{per}}} \quad @ \quad \frac{H_{per}}{a} \ll e < 0.02 \qquad (12.7.10)$$

with e_0 the initial eccentricity, and ρ_{per} the atmospheric density and H_{per} the scale height at periapsis. When tracing back the cause for the $e_0^{3/2}$ dependence we find its origin in the fact that, for increasing eccentricities, the stretch s within which the S/C dives into the dense portion of the atmosphere decreases with $s \approx 2\sigma \propto 1/\sqrt{e}$ (see Eq. (12.7.4)). The integral of this dependence then leads to the $e_0^{3/2}$ dependence.

12.7.3 Circular Orbits

When the ellipse is circularized down to $e < H(a)/a$, the body encounters a constant drag upon circling the planet with radius a. To determine the element changes of a circular orbit, we have to re-examine the second expression of Eq. (12.7.3). To derive the mean changes of e we have to average it with $e = 0$ and hence $\xi(e, \theta) = 1$ over one orbit. Because of $\langle \cos\theta \rangle_\theta = 0$ and as in a circular orbit $v^2 = \mu/a$ (Eq. 7.4.3) and $n^2 = \mu/a^3$(Eq. 7.3.10), we get $v_a^2 = n^2 a^2$ and thus

$$\begin{aligned} \dot{a} &= -\frac{B\rho\, v_a^2}{n} = -B\rho(a)\sqrt{\mu a} \\ \dot{e}_{\text{sec}} &= 0 \end{aligned} \qquad @\ e < \frac{H(a)}{a} \tag{12.7.11}$$

So we can make the following statement:

Impact of Drag on Semi-Major-Axis
Drag constantly decreases the radius of a circular orbit, without changing its eccentricity.

Ballistic Coefficient B from NORAD TLE
Quite generally, it is hard to come by the ballistic coefficient B of a S/C. From Eq. (12.7.11) one could presume that if the momentary orbit radius a of a S/C and atmospheric density $\rho(a)$ is known one would be able to determine it from its orbital decay $\dot{a}$ as

$$B\rho(a) = -\frac{\dot{a}}{\sqrt{\mu a}}$$

As a matter of fact, the NORADS TLE (two-line elements (see e.g. www.space-track.org) provide the figures $n\ \left[\text{rev}\cdot\text{day}^{-1}\right]$ and $\dot{n}/2\ \left[\text{rev}\cdot\text{day}^{-2}\right]$ in the said units. The orbital decay rate $\dot{a}$ can be derived via $a^3 = \mu_\oplus/n^2$ and hence $3\dot{a}/a = -2\dot{n}/n$ from this. With this and with $\sqrt{\mu a} = (\mu^2/n)^{1/3}$ we find

$$B\rho(a)\ \left[\text{km}^{-1}\right] = 5.024 \times 10^{-6} \frac{\dot{n}/2\ \left[\text{rev}\cdot\text{day}^{-2}\right]}{\left(n\ \left[\text{rev}\cdot\text{day}^{-1}\right]\right)^{4/3}} \qquad @\ \text{Earth}, e < \frac{H(a)}{a} \tag{12.7.12}$$

To determine from this the ballistic coefficient B, the atmospheric density averaged over one elliptic orbit at the altitude $h = a - R_\oplus$, where the S/C suffered the decay, needs to be known. This can, for instance, be derived from Fig. 6.3 in Sect. 6.1.2 by applying current solar flux figures $F_{10.7}$ from https://spawx.nwra.com/spawx/f10.html.

Example

What is the ballistic coefficient of the International Space Station in its torque equilibrium attitude (TEA)?

TEA is an "airplane like" attitude maintained relative to the Local Vertical Local Horizontal (LVLH, see Sect. 15.1.3). The pitch angle is $-10° \pm 5°$ depending on the altitude and hence on the residual atmosphere. For this attitude the residual drag torque counterbalances the gravity–gradient torque (see Sect. 15.4.1). The TLE of the ISS on April 6, 2018, 05:45:34 UTC was given as

1 25544*U* 98067*A* 18096.20365559 .00002236 00000−0 40882−4 0 9998
2 25544 51.6441 17.5650 0001462 307.6006 167.7216 15.54202230107329

From this we find $n = 15.5420\,\text{rev} \cdot \text{day}^{-1}$ and $\dot{n}/2 = 2.24 \times 10^{-5}\,\text{rev} \cdot \text{day}^{-2}$. This yields an altitude of

$$h[\text{km}] = a - a_{\oplus} = 4.224 \times 10^4 \left(n\left[\text{rev} \cdot \text{day}^{-1}\right]\right)^{-2/3} - 6378\,\text{km} = 404\,\text{km}$$

and with Eq. (12.7.12) $B_{ISS}\rho(404\,\text{km}) = 2.90 \times 10^{-12}\text{km}^{-1}$. At that time the atmospheric density was given as $\rho(404\,\text{km}) = 4.76 \times 10^{-13}\text{kg m}^{-3}$ (low solar activity, $F_{10.7} = 67$). With this we derive the ISS ballistic coefficient in the TEA attitude mode to be B = 0.0061 $\text{m}^2\ \text{kg}^{-1}$.

Remark (NORAD's Bstar) *This procedure in fact is done by the North American Aerospace Defense Command, NORAD. It publishes the orbital elements as the so-called two-line elements derived from their observations as two line elements, TLEs, at celestrak.com/NORAD/elements. The third last entry in the first line of a TLE, the so-called* B^* *(Bstar, dimensionless), reflects the decay and is related to B* via $B^* = R_{\oplus}\rho_{120}B/2$, *where* $R_{\oplus} = 6378.13$ *km and* $\rho_{120} = \rho(h = 120\ km)$. *NORAD in its so-called Simplified General Perturbation Model SGP, in which* B^* *is derived from the observed orbital decay data, employs a simplified static atmospheric model (Lane's model), which assumes a standard value* $\rho_{120} \equiv \rho_0 = 2.461 \times 10^{-8}\ kg\ m^{-3}$. *Because* ρ_{120} *actually may vary by more than an order of magnitude depending on the solar and geomagnetic activities it is useless to determine B from* B^* with $\rho_{120} = 2.461 \times 10^{-8}\ kg\ m^{-3}$ *as sometimes recommended. For instance, at 400 km altitude (about ISS orbit) the ballistic coefficients* B_{TLE} *derived from* B^* *related to the true ones are* $B_{TLE}/B_{true} = 1.83,\ 7.08,\ 0.080$ *for mean, extremely high, and low solar activities, respectively. Therefore* B^* *should be considered only as a fudge parameter to adequately describe the momentary orbit decay with the SGP propagator algorithm and its derivatives.*

Orbit Decay

To describe the orbit decay quantitatively we need to solve the differential Eq. (12.7.11). Separating the variables in the first equation results in

$$dt = -\frac{1}{B\sqrt{\mu}}\frac{da}{\sqrt{a}\rho(a)}$$

from which by integration follows:

$$t - t_0 = -\frac{1}{B\sqrt{\mu}}\int_{a_0}^{a}\frac{da}{\sqrt{a}\rho(a)} = \frac{1}{B\sqrt{\mu}}\int_{a}^{a_0}\frac{da}{\sqrt{a}\rho(a)} \tag{12.7.13}$$

where a_0, t_0 are the initial values of the orbit. To further evaluate the integral analytically, $\rho(a)$ has to be expressed, according to Eq. (6.1.8), in a piecewise exponential form

$$\rho(a) = \rho_i \exp\left(-\frac{h-h_i}{H_i}\right) = \rho_{i+1}\exp\left(-\frac{h-h_{i+1}}{H_i}\right) \quad @\ h_i < h < h_{i+1} \tag{12.7.14}$$

where $h = a - R_\oplus$, h_i is the base altitude, and H_i the scale height for the ith altitude interval as given in Table 6.2. Correspondingly we also achieve only a piecewise description of the orbit trajectory. For the initial part of the decaying orbit we therefore make the ansatz

$$\rho(a) = \rho_0 \exp\left(-\frac{h-h_0}{H_0}\right) \quad @\ h < h_0$$

where ρ_0 is the atmospheric mass density at the initial altitude h_0 and H_0 the mean scale height over the integration interval just below the initial altitude. If integration is performed over the interval $h_i < h < h_{i+1} = h_0$ as given by Table 6.2 in Sect. 6.1.4 then the values of $\rho_0 = \rho_{i+1}$ and $H_0 = H_i$ (cf. Eq. (12.7.14)) can be taken from Table 6.2. With this and with the substitution $a = h + R_\oplus =: x + R_\oplus$ we get from Eq. (12.7.13)

$$t - t_0 = \frac{1}{B\rho_0\sqrt{\mu}}\int_{h}^{h_0}\frac{e^{(x-h_0)/H_0}}{\sqrt{x+R_\oplus}}dx \tag{12.7.15}$$

Because by far the biggest contributions to the integral come from the initial altitude, we can safely approximate

$$\begin{aligned}\int_h^{h_0} \frac{e^{(x-h_0)/H_0}}{\sqrt{x+R_\oplus}}dx &= \frac{1}{\sqrt{a_0}}\int_h^{h_0} \frac{e^{(x-h_0)/H_0}}{\sqrt{1+(x-h_0)/a_0}}dx \\ &\approx \frac{1}{\sqrt{a_0}}\int_h^{h_0} e^{(x-h_0)/H_0}\left(1-\frac{x-h_0}{2a_0}\right)dx \\ &= \frac{H_0}{\sqrt{a_0}}e^{(x-h_0)/H_0}\left(1-\frac{x-h_0+H_0}{2a_0}\right)\Bigg|_h^{h_0} \\ &= \frac{H_0}{\sqrt{a_0}}\left[\left(1+\frac{H_0}{2a_0}\right)-e^{(h-h_0)/H_0}\left(1-\frac{h-h_0}{2a_0}+\frac{H_0}{2a_0}\right)\right] \\ &\approx \frac{H_0}{\sqrt{a_0}}\left[1-e^{(h-h_0)/H_0}\left(1-\frac{h-h_0}{2a_0}\right)\right]\end{aligned} \tag{12.7.16}$$

The last term arises when $H_0/2a_0 < 0.0001$.

Orbit Trajectory
For LEOs with $h_0 \leq 1000$ km we have $h_0/2a_0 < 0.07$. Therefore

$$t - t_0 \approx \frac{H_0}{B\rho_0\sqrt{\mu a_0}}\left[1-e^{(h-h_0)/H_0}\right]$$

With this and with $h - h_0 = a - a_0$ we derive for the initial trajectory of a circular orbit

$$a(t) = a_0 + H_0 \ln\left[1-\frac{B\rho_0\sqrt{\mu a_0}}{H_0}(t-t_0)\right] \qquad e < \frac{H(a)}{a} \tag{12.7.17}$$

As an example, Fig. 12.38 depicts $a(t)$ of the International Space Station, from which the decay and the reboosts are clearly visible.

12.7.4 Orbit Lifetime

When does a S/C in a circular or near-circular low Earth orbit without orbit maintenance burn up in the Earth's atmosphere? In the following, we want to determine this so-called orbit life t_L of a circular orbit from a given initial altitude. As we only want to estimate the orbit lifetime we apply Eq. (12.7.16) and use it with the initial values right down to $h = 0$, whereby we only slightly overestimate the orbit lifetime. With this approximation and setting $t_0 = 0$ we get

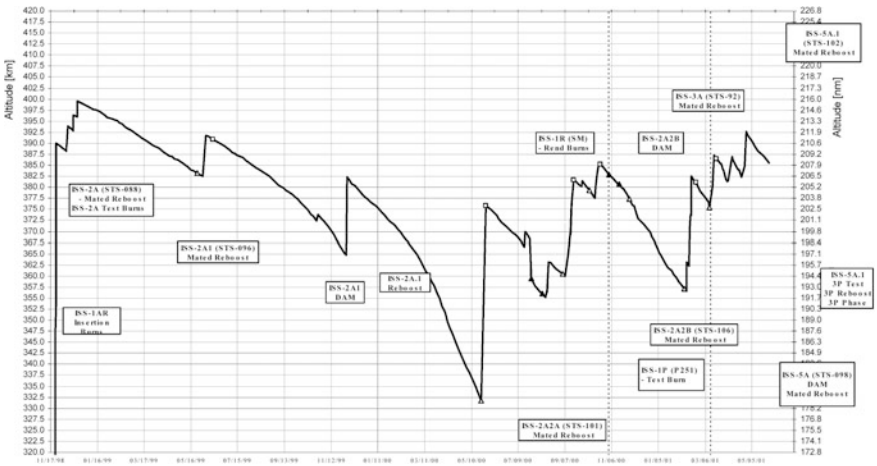


Fig. 12.38 The as-flown altitude profile of the International Space Station between its first assembly flight in November 1998 and May 2001. *Credit* NASA

$$t_L = \frac{H_0}{B\rho_0\sqrt{\mu a_0}}\left[1 - e^{-h_0/H_0}\left(1 + \frac{h_0}{2a_0}\right)\right]$$

Because $h_0/H_0 > 10$ we get with Eq. (12.7.11) in excellent approximation

$$\boxed{t_L = \frac{H_0}{B\rho_0\sqrt{\mu a_0}} = \frac{H_0}{|\dot{a}_0|}} \quad @ \quad e < H_0/a_0 \tag{12.7.18}$$

with

$a_0 = h_0 + R_\oplus$ initial radius of the circular orbit

$\rho_0 = \rho(h_0)$ atmospheric density at the initial altitude h_0

$H_0 = H(h_0)$ atmospheric scale height just below the initial altitude h_0

If the initial altitude coincides with a base altitude $h_0 = h_i$ as given in Table 6.2 in Sect. 6.1.4 then $\rho_0 = \rho_i$ and $H_0 = H_{i-1}$ as given in Table 6.2. Note that according to the end of Sect. 6.2.4 the ballistic coefficient B can vary by as much as 80% over a wide range of altitudes and solar activity phases. Therefore, the mean ballistic coefficient at the initial altitude is decisive.

Example

What is the mean orbit lifetime of the International Space Station in TEA attitude mode at its common altitude of 350 km and at a mean solar activity of $F_{10.7} = 140$?

From the Example in Sect. 12.7.3 we have for the ISS $B_{ISS} = 0.0061\ m^2\ kg^{-1}$. From the MSISE-90 model follows for a mean solar activity at altitude 350 km that $\rho(350\ km) = 9.80 \times 10^{-12}\ kg\ m^{-3}$ and $H_0 = 53.1$ km. With $\mu = 3.986 \cdot 10^5\ km^3 s^{-2}$ Eq. (12.7.18) furnishes $t_L \approx 200$ days.

Alternatively, we find from Fig. 12.38 for the time frame May 27, 2000 to July 15, 2000 a decay rate of $\dot{a}_0 = -0.18$ km day^{-1} at an altitude of $h = 370$ km. From $H_0 = 53.3$ km we find with Eq. (12.7.18) $t_L \approx 300$ days. Note that only six months later, in the time frame December 6, 2000 to Janunary 5, 2001, the ISS at the same altitude decayed at a rate $\dot{a}_0 = -0.28$ km day^{-1} implying an expected orbit lifetime of only $t_L \approx 190$ days. This big difference reflects the variation of the mean daily atmospheric density at such altitudes (see also Remark NORAD's Bstar in Sect. 12.7.3).

Remark *To maintain the space station at an altitude of 350–400 km, routine reboosts of currently about once a month are performed. The required propellant for this is about* 7500 kg year^{-1}.

If one evaluates the integral in Eq. (12.7.13) numerically with the atmospheric density as given by Eq. (6.1.8) one obtains the mean orbit lifetime as displayed in Fig. 12.39 for a spacecraft with various ballistic coefficients. $B = 0.005$ m^2/kg is a good average value.

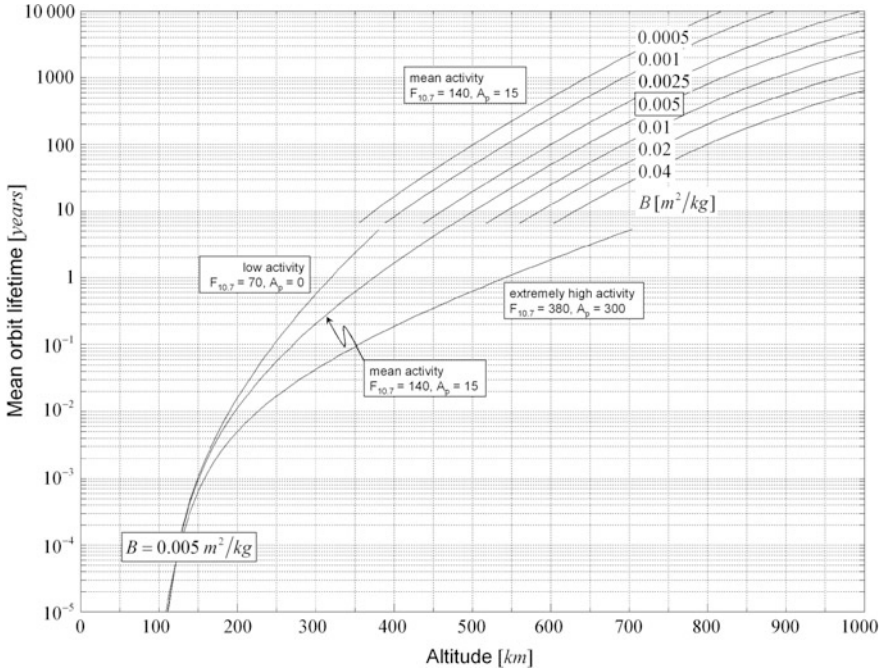


Fig. 12.39 In the upper part the mean orbit lifetime of a circular orbit as a function of altitude for various ballistic coefficients B is given. If the orbit life time of a S/C is less than 6 years it varies drastically by more than one order of magnitude due to the actual solar activity. In the lower part these variances are given for a mean $B = 0.005$ m^2 kg^{-1}

Rule of Thumb

From Table 6.2 we observe that above an altitude of 200 km roughly $H \approx 0.14 \cdot h$ holds. Because the atmospheric density about an altitude h_0 varies exponentially as given by Eq. (12.7.14) and therefore $t_L \approx t_{L0}\, e^{(h-h_0)/H_0}$ an increase of the orbital radius by $\Delta h = H_0 \ln 2 \approx 0.1 \cdot h_0$, i.e., an about 10% altitude increase, doubles the orbit life time in LEO. This in turn gives rise to the assumption that the orbit life time $t_L(h)$ is subject to a power law. We therefore make the ansatz $t_L = t_{L0}(h/h_0)^x$, fit this ansatz to the numerical data, and arrive at the approximate equation

$$t_L \approx 1.7 \frac{0.005}{B} \left(\frac{h}{400}\right)^{8.4} \quad @\ 300\ \text{km} < h < 900\ \text{km},\ e < H_0/a_0 \tag{12.7.19}$$

Here the ballistic coefficient B is to be provided in units $m^2 kg^{-1}$, h in km, and t_L then is given in *years*. Compared to Fig. 12.39, Eq. (12.7.19) has an error of only up to 10% in the range 350–650 km increasing to 30% at the altitudes 300 and 900 km.

If we set $h = h_0(1+\varepsilon)$ and solve Eq. (12.7.19) for $t_L = 2t_{L0}$ we find $\varepsilon = 2^{1/8.4} - 1 = 0.086$. This yields the key rule of thumb:

> **Rule of Thumb for Orbit Life Time**
> An increase of a circular orbital altitude in LEO by 10% roughly doubles the orbit life time.

Orbit Lifetime from NORAD TLE

The initial orbital decay rate $\dot{a}_0$ in Earth's atmosphere can be derived via $a^3 = \mu_\oplus / n^2$ and hence $3\dot{a}/a = -2\dot{n}/n$ as

$$\dot{a}_0 \left[\text{km day}^{-1}\right] = -1.769 \times 10^5 \frac{\dot{n}_0/2 \left[\text{rev day}^{-2}\right]}{\left(n_0 \left[\text{rev} \cdot \text{day}^{-1}\right]\right)^{5/3}} \quad @\ \text{Earth}$$

Since n_0 and $\dot{n}_0/2$ are given by NORAD TLE in the said units the orbit lifetime of a NORAD catalog object can immediately be derived from Eq. (12.7.18) as

$$t_L[\text{days}] = \frac{H_0}{|\dot{a}_0|} = 5.652 \times 10^{-6} \frac{\left(n_0 \left[\text{rev day}^{-1}\right]\right)^{5/3}}{\dot{n}_0/2 \left[\text{rev day}^{-2}\right]} H_0[\text{km}] \quad @\ \text{Earth} \tag{12.7.20}$$

Example

What would be the mean orbit lifetime of the International Space Station in TEA attitude mode on April 6, 2018, 05:45:34 UTC?

According to the Example in Sect. 12.7.3 we have for the ISS at that point in time $n_0 = 15.5420\,\text{rev} \cdot \text{day}^{-1}$ and $\dot{n}_0/2 = 2.24 \times 10^{-5}\,\text{rev} \cdot \text{day}^{-2}$, and for low activity from the MSISE-90 model H_0 = 41.95 km. With Eq. (12.7.20) we therefore derive t_L = 1024 days = 2.8 years. This very high value is due to the low activity of the Sun in 2018, i.e., a low atmospheric density, plus the high altitude of the ISS of h = 404.47 km.

Total Orbit Lifetime of Elliptic Orbits

If the initial orbit is elliptic, the total orbit lifetime is the circularization time plus circular orbit lifetime. Which of the two is prevailing? Since, at the transition between the two phases, $r_0 = r_{per}$, we derive from Eqs. (12.7.10) and (12.7.18) the following ratio of the two contributions:

$$\frac{t_{cir}}{t_L} = \sqrt{2\pi}\left(\frac{r_{per}}{H_{per}}\right)^{3/2}\left(\arctan\sqrt{e_0} - \frac{\sqrt{e_0}}{1+e_0}\right) \approx \sqrt{2\pi}\left(\frac{r_{per}}{H_{per}}\right)^{3/2}\frac{2}{3}e_0^{3/2}$$

From $e_0 \gg \frac{H_{per}}{a} \approx \frac{H_{per}}{r_{per}}$ we get

$$\frac{t_{cir}}{t_L} \gg \frac{2}{3}\sqrt{2\pi} = 1.7 \tag{12.7.21}$$

So, circularization time is much bigger than the circular orbit lifetime. This has the following practical consequence that for a planetary capture and a subsequent circularization to a circular target orbit, the periapsis has to be chosen lower than the target radius to more rapidly turn down the elliptic orbit. When the apoapsis attains the target orbit radius, a kick-burn at the apoapsis (see Eq. (8.1.11)) will increase the periapsis to the target radius. Though this maneuver requires some propulsion effort, it is much less demanding than to circularize the ellipse by propulsion only without making use of the atmospheric drag.

Example

On October 28, 2010, the European satellite operator Eutelsat lost its communications satellite W3B due to a sizeable leak in its fuel tank just after it was placed with Ariane 5 into the highly elliptic transfer orbit with a perigee of 249.2 km and an apogee of 35,907 km. After considering moving W3B to a graveyard orbit or guiding the satellite to destruction over the Pacific Ocean, managers ultimately concluded their only option was to leave the satellite where it was. Under the given ballistic coefficient of $B = 0.015\ \mathrm{m^2\,kg^{-1}}$ when will W3B been circularized and burn up in the atmosphere?

With the given perigee and apogee of the transfer orbit we find from Eq. (7.4.9) that the orbit's initial eccentricity is $e_0 = 0.729$. At the perigee altitude of 250 km we have from Table 6.2 $H \approx 40$ km and $\rho = 7.25 \times 10^{-11}\ \mathrm{kg\,m^{-3}}$. From Eq. (12.7.9) we therefore derive a circularization time of $t_{cir} = 25.8$ yrs. Compared to this and according to Fig. 12.39 the orbital life time is $t_L < 0.01$ yr. As a result, W3B will burn up in about 26 years.

12.8 Problems

Problem 12.1 *Gaussian Variational Equations*
To prove the Gaussian Variational Equations (12.1.4) proceed as follows: First prove Eq. (8.1.1) and then apply the reverse transformation as given before in Eq. (8.1.1).

1. Prove Eq. (8.1.1)

Do this by first considering kick-burns only within the orbital plane and then out of the plane:

(a) To derive the change of orbital elements for in-plane kick-burns first introduce the angle changes:

$$d\gamma := \frac{dv_{\perp \mathrm{O}}}{v}, \quad d\phi := \frac{dv_{\perp\perp}}{v} \quad \text{and} \quad d\eta := \frac{dv_{\parallel}}{v} = \frac{dv}{v}$$

To simplify calculations we define

$$s := \frac{rv^2}{\mu}$$

With this rewrite the vis-viva Eq. (7.2.15) to

$$r = (2 - s)a \tag{a}$$

From Eqs. (13.1.8) and (13.1.9) follows

$$\boldsymbol{v} = \left(\frac{e\mu}{h}\sin\theta, \frac{h}{r}\right) = v(\sin\gamma, \cos\gamma)$$

Therefore $h = rv\cos\gamma$. Show that from $h^2 = \mu a(1 - e^2)$, Eq. (7.3.7), follows

$$e^2 = 1 - s(2 - s)\cos^2\gamma \tag{b}$$

and

$$\sin\theta = \frac{s}{2e}\sin 2\gamma \tag{c}$$

Convince yourself by a drawing that if a kick-burn takes place at a certain position $\boldsymbol{r}$ in space, which remains constant during the kick-burn maneuver, then any change in θ corresponds to a negative change in ω, i.e., $d\omega = -d\theta$. With this and from Eqs. (a) to (c) prove with the relations $s\cos^2\gamma = 1 + e\cos\theta$, $s\cos 2\gamma = 1 + 2e\cos\theta - e\cos E$, and $s\sin 2\gamma = 2e\sin\theta$ the differential equations

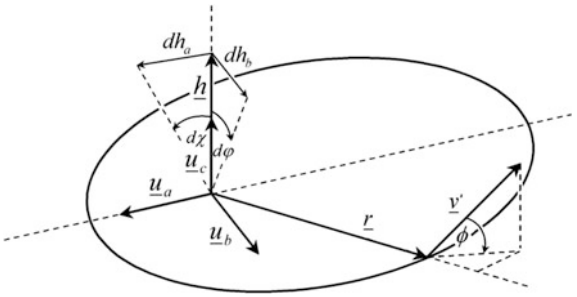


Fig. 12.40 Definition of angles for out-of-the-plane kick-burns

$$da = \frac{a}{2-s}ds = \frac{2as}{2-s}d\eta$$
$$de = 2(e + \cos\theta) \cdot d\eta + \frac{r}{a}\sin\theta \cdot d\gamma$$
$$e \cdot d\omega = 2\sin\theta \cdot d\eta - (e + \cos E)d\gamma$$

(b) To derive the changes of orbital elements for out-of-plane kick-burns consider the definitions made in Fig. 12.40

Show that from

$$d\boldsymbol{v} = \frac{\partial \boldsymbol{v}}{\partial \gamma}d\gamma + \frac{\partial \boldsymbol{v}}{\partial \phi}d\phi + \frac{\partial \boldsymbol{v}}{\partial \eta}d\eta$$

follows

$$d\boldsymbol{v} = v \cdot [\cos(\theta-\gamma) \cdot d\gamma,\ \sin(\theta-\gamma) \cdot d\gamma,\ d\phi] + v \cdot [-\sin(\theta-\gamma),\ \cos(\theta-\gamma),\ 0]d\eta$$

and therefore

$$d\boldsymbol{h} = d(\boldsymbol{r} \times \boldsymbol{v}) = \boldsymbol{r} \times d\boldsymbol{v}$$
$$= rv[\sin\theta \cdot d\phi, -\cos\theta \cdot d\phi, -\sin\gamma \cdot d\gamma + \cos\gamma \cdot d\eta]$$

Because on the other hand $d\boldsymbol{h} = (dh_a,\ dh_b,\ dh_c) = (h \cdot d\chi,\ h \cdot d\varphi,\ dh)$ follows

$$d\chi = \frac{\sin\theta}{\cos\gamma}d\phi, \quad d\varphi = -\frac{\cos\theta}{\cos\gamma}d\phi, \quad \frac{dh}{h} = -\tan\gamma \cdot d\gamma + d\eta$$

Let $d\sigma_a, d\sigma_b, d\sigma_c$ be the positive deflections around the coordinate axes $(\boldsymbol{u}_a, \boldsymbol{u}_b, \boldsymbol{u}_c)$ for which holds (see, e.g., Kaplan 1976, Eq. (1.28))

$$\begin{pmatrix} d\Omega \\ di \\ d\omega \end{pmatrix} = \frac{1}{\sin i} \begin{bmatrix} \sin\omega & \cos\omega & 0 \\ \cos\omega \sin i & -\sin\omega \sin i & 0 \\ -\sin\omega \cos i & -\cos\omega \cos i & \sin i \end{bmatrix} \begin{pmatrix} d\sigma_a \\ d\sigma_b \\ d\sigma_c \end{pmatrix}$$

Show that

$$\begin{pmatrix} d\sigma_a \\ d\sigma_b \\ d\sigma_c \end{pmatrix} = \begin{pmatrix} -d\varphi \\ d\chi \\ -d\theta \end{pmatrix}$$

from which the desired change of orbital elements are derived.

2. Apply the reverse transformation of (see equations before Eq. (8.1.1))

$$a_r = \cos\gamma \cdot \delta v_{\perp O} + \sin\gamma \cdot \delta v_{||}$$
$$a_\theta = -\sin\gamma \cdot \delta v_{\perp O} + \cos\gamma \cdot \delta v_{||}$$

to finally derive the Gaussian Variational Eq. (12.1.2).

Problem 12.2 *Earth's Oblateness and J*
Prove by the use of $P_2^0 = P_2 = \frac{1}{2}(3\sin^2\beta - 1)$ the relation

$$J \approx f - \frac{\omega^2 R_\oplus^3}{2\mu_\oplus}$$

between the harmonic coefficient

$$J := \frac{3}{2} J_2 := -\frac{3}{2} C_2^0 = 0.001623945$$

and Earth's oblateness

$$f := \frac{R_\oplus - R_{polar}}{R_\oplus} \approx \frac{1}{298.2564}$$

Hint: The shape of the Earth's surface forms under the physical *principle of least action*. Due to this principle the so-called *Lagrangian* $L = U_\omega - U_{pot}$ must be minimal and constant over the entire Earth surface. Use Eq. (7.2.16) to determine U_ω at different latitudes and Earth's gravitational potential $U = U_{pot}$ up to order P_2^0.
Note: When also taking P_4^0 into account one derives the refined solution (see Kaula (1966))

$$J = f\left(1 - \frac{1}{2}f\right) - \frac{\omega^2 R_\oplus^3}{2\mu_\oplus}\left(1 - \frac{3\omega^2 R_\oplus^3}{2\mu_\oplus} - \frac{2}{7}f\right)$$

Problem 12.3 *Triaxial Motion*
Show that the equation of motion (12.4.14) for a satellite at an unstable GEO position

$$\ddot{\lambda} = -\frac{1}{2}\omega_\lambda^2 \cdot \sin 2(\lambda - \lambda_0)$$

with initial condition $\lambda = \lambda_i \neq \lambda_0$ has the approximate solution

$$\lambda = \lambda_i - \frac{1}{4}\omega_\lambda^2 \sin[2(\lambda_i - \lambda_0)] \cdot t^2 + \frac{1}{96}\omega_\lambda^4 \sin[4(\lambda_i - \lambda_0)] \cdot t^4 - O(t^6)$$

Problem 12.4 *Orbit Changes by Atmospheric Maneuvers*

(1) After the Columbia accident on February 1, 2003, NASA administrator O'Keefe canceled any Hubble repair mission, because from the Hubble telescope the Space Shuttle was not supposed to be able to reach the ISS as a safe haven. Given the orbit elements of ISS (*altitude* = 400 km, $i = 51.63°$, $e \approx 0$) and the Hubble telescope (*altitude* = 590 km, $i = 28.5°$, $e \approx 0$) and the fact that the OMS engines of a shuttle can only provide a delta-v of about $\Delta v = 200$ m/s, show that O'Keefe was right.

(2) We have seen in Sect. 12.7.2 that it is possible to change the semi-major axis of an orbit by dragging through the atmosphere at the periapsis of an elliptic orbit. Suppose the vehicle has also lift. The vehicle then receives the delta-v changes $\delta v_{||}$, $\delta v_{\perp O}$ or $\delta v_{||}$, $\delta v_{\perp\perp}$ depending on the orientation of the lift vector. Show by a similar procedure as in Sect. 12.7.2 that the delta-v change due to one fly-through the periapsis of an ellipse is given by

$$|\delta v_{||}| \approx \frac{C_D}{2}\frac{\rho_{per} A_\perp}{m}(1+e)\sqrt{\frac{2\pi\mu H_{per}}{e}}$$

$$|\delta v_{\perp O}| = |\delta v_{\perp\perp}| \approx \frac{C_L}{2}\frac{\rho_{per} A_\perp}{m}(1+e)\sqrt{\frac{2\pi\mu H_{per}}{e}}$$

(3) Now consider a life threatening situation onboard the Space Shuttle during Hubble repair. Show that, though an inclination turn is in principle possible with an atmospheric maneuver, the following orbit maneuver would not be feasible: lower one side of the shuttle orbit by a small deorbit burn such that the now slightly elliptic orbit would touch the atmosphere at its periapsis. The shuttle would not only receive a delta-v of $\delta v_{||}$, $\delta v_{\perp\perp}$ thereby decelerating but also change the inclination $28.5° \rightarrow 51.63°$ due to its lift, $C_L/C_D = L/D \approx 1.3$. Finally the elliptic orbit would be raised to a circular ISS LEO and the shuttle maneuvered to the ISS by some negligible rendezvous maneuvers.

Hint: Consult Table 8.1.

Problem 12.5 *King-Hele's Orbit Lifetime*
In his reputed book King-Hele (1987, p. 60ff) provides the following expression for the orbit life time of a satellite in a LEO

$$t_L \approx -\frac{3e_0T_0}{4\dot{T}_0}\frac{I_0(z_0)}{I_1(z_0)}\left(1+2e_0\frac{I_1(z_0)}{I_0(z_0)}-\frac{9e_0z_0}{40}+\frac{H}{2a_0}\right) \quad @ \quad z_0=\frac{a_0e_0}{H_0}<3$$

with

$$I_0(z)=1+\frac{z^2}{4}+\frac{z^4}{64}+\frac{z^6}{2304}+\cdots \quad \text{Bessel function of the first kind and order 0}$$

$$I_1(z)=\frac{z}{2}\left(1-\frac{z^2}{8}+\frac{z^4}{192}-\frac{z^6}{9216}+\cdots\right) \quad \text{Bessel function of the first kind and order 1}$$

and $T_0=2\pi\sqrt{a_0^3/\mu}$.

Show that for an elliptic orbit from the above follows

$$t_L \approx \frac{H_0}{B\rho_0\sqrt{\mu a_0}}\left(1+\frac{H_0}{2a_0}\right) \approx \frac{H_0}{B\rho_0\sqrt{\mu a_0}}\left(1+\frac{H_0}{2R_\oplus}\right)$$

and therefore concurs with Eq. (12.7.18).

Chapter 13
Reference Frames

13.1 Space Frames

To delineate the general trajectory of a body mathematically or graphically one needs an observational space reference frame. In physics a *reference frame* is meant to be a concrete realization of a conceptional *reference system* with a choice of a *coordinate system*—meaning: a system of coordinates, such as rectangular (Cartesian), polar, or cylindrical coordinates, adapted to the symmetry of the problem—having a specific origin and orientation. (Note that "reference frame" and "coordinate system" are used in the literature almost synonymously.)

There is no restriction to the choice of the reference frame. Any frame in principle will do. However, mathematical equations take on different forms in different frames. Inappropriate choices cause inefficient mathematical representations, which may even become untreatable, while being straightforward in a suitable frames. So, choosing the right reference frame is of paramount importance. In the sections below the most common reference frames are presented.

Terminology of Reference Systems In celestial mechanics the many reference systems are classified according to the position of the point of origin, as well as the x–y plane, and the orientation of the x- and y-axes. The corresponding terms are given in the following table.

	Denomination
Position of the point of origin	
Center of the Sun	Heliocentric
Center of Earth	Geocentric
Center of mass	Barycentric
Position of observation	Topocentric

(continued)

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(continued)

	Denomination
Orientation of the x–y plane	
Ecliptic plane	Ecliptical
Equatorial plane	Equatorial

Transformation between Arbitrary Reference Frames
Frequently the problem arises to transform between two reference frames. The most general transformation between two vectors r and r' in the reference frames F and F', respectively, is

$$\boldsymbol{r}' = \boldsymbol{R}_\Delta \boldsymbol{r} + \boldsymbol{T}_\Delta$$

with an arbitrary translation vector $\boldsymbol{T}_\Delta$ and rotation matrix $\boldsymbol{R}_\Delta$, both of which may be time-dependent. To find the coordinate transformation equations between two arbitrary reference frames is an art in its own, which however is utterly mathematical. Because the derivation of these transformations does not add to the understanding of astrodynamics, we merely refer to the work of Vallado (2001, Chap. 3), who treats coordinate transformations extensively.

Fictitious Forces
Reference frames, which are rotating, that is where $\boldsymbol{R}_\Delta(t)$ is time-dependent, give rise to centrifugal and Coriolis accelerations according to

$$\ddot{\boldsymbol{r}}' = \underbrace{\ddot{\boldsymbol{R}}_\Delta \boldsymbol{r}}_{\text{centrifugal}} + \underbrace{2\dot{\boldsymbol{R}}_\Delta \boldsymbol{r}}_{\text{Coriolis}} + \boldsymbol{R}_\Delta \ddot{\boldsymbol{r}}$$

These accelerations cause the body's trajectory to curve in F'. An observer in the rotating reference frame perceives these accelerations due to forces that fictitiously acts from outside and hence are called *fictitious forces*.

13.1.1 Inertial Reference Frames

Although any reference frame has its special benefits, there exist a class of reference frames that are very special in their own: the inertial reference frames. Suppose we have a body initially at $\boldsymbol{r}_0 := \boldsymbol{r}(t_0)$ that is not exposed to any external force (field). According to Newton's second law (Eq. 7.1.12), it will not be accelerated. That is $\ddot{\boldsymbol{r}} = 0$, the general solution of which is $\boldsymbol{r} = \boldsymbol{v}_0 t + \boldsymbol{r}_0$ and hence a straight line (Newton's first law, cf. Sect. 7.1.4). This uniform motion is caused by the inertia of the body. However, this statement holds only in a special kind of reference frame. This gives rise to the physical definition of an *inertial reference frame*:

An **inertial reference frame** is a reference frame, in which a body unaffected by external forces moves in a straight line.

Remark *The precision of this definition is slightly sacrificed to its simplicity. There might be non-inertial reference frames, in which one or more unaffected bodies move in straight lines. The sufficient condition and hence precise definition is: An inertial reference frame is one in which three unaffected bodies that at any point in time were at the same place in space thereafter move linearly independently in straight lines.*

According to a postulate of the Austrian physicist Ernst Mach (1838–1916), so-called *Mach's principle*, inertia of a body emerges from the influence of all masses in our universe on that body: An inertial force occurs if a body is accelerated relative to the mean of these universal masses. Conversely, if its inertial force is zero, i.e., if it is non-accelerated because it is in a force-free state, the corresponding inertial reference frame must be fixed relative to the mean of these universal masses. The fixed stars are our reference to the universe, which is why inertial systems are also called "sidereal systems".

How are inertial reference frames related to each other? Let F and F' be two inertial reference frames. A straight line in frame F then is also a straight line in F' if their reference frames are translationally shifted by $\boldsymbol{r}_\Delta = const$, or rotationally shifted by $\boldsymbol{R}_\Delta = const$, or moves with a mutual velocity $\boldsymbol{v}_\Delta = const$, constant in time, or any combination of this, i.e., if $\boldsymbol{r}' = \boldsymbol{R}_\Delta \boldsymbol{r} + \boldsymbol{v}_\Delta t + \boldsymbol{r}_\Delta$. This so-called Galilean transformation is the most general transformation that preserves inertial motion. By differentiating the Galileian transformation equation twice it is easily verified that it implies: If $\ddot{\boldsymbol{r}} = 0$ then $\ddot{\boldsymbol{r}}' = 0$. This equals the property that a straight line $\boldsymbol{r} = \boldsymbol{v}_0 t + \boldsymbol{r}_0$ in F is also a straight line in F', which can be verified by inserting $\boldsymbol{r} = \boldsymbol{v}_0 t + \boldsymbol{r}_0$ into $\boldsymbol{r}' = \boldsymbol{R}_\Delta \boldsymbol{r} + \boldsymbol{v}_\Delta t + \boldsymbol{r}_\Delta$. Compared to the above general transformation equation $\boldsymbol{r}' = \boldsymbol{R}_\Delta \boldsymbol{r} + \boldsymbol{T}_\Delta$ we see that for inertial reference frames $\boldsymbol{R}_\Delta(t) = const$ and $\boldsymbol{T}_\Delta = \boldsymbol{v}_\Delta t + \boldsymbol{r}_\Delta$ must hold**.**

In summary, we can state that:

Inertial (sidereal) reference frames are reference frames with a fixed relation to the entirety of the masses in the universe. They are related to each other through the transformation $\boldsymbol{r}' = \boldsymbol{R}_\Delta \boldsymbol{r} + \boldsymbol{v}_\Delta t + \boldsymbol{r}_\Delta$, where $\boldsymbol{R}_\Delta$, $\boldsymbol{v}_\Delta$, $\boldsymbol{r}_\Delta$ are constant in time.

So, inertial reference frames have the unique property that no fictitious forces emerge. Owing to this, physical laws and equations take on their standard and most simple form only in inertial reference frames. Moreover, identical experiments only give identical results when carried out in different inertial frames.

The International Celestial Reference Frame (ICRF)
What are the fixation points in our universe, to which we need to fix—according to *Mach's principle*—an inertial reference frame? They are stars so far away that their relative motion remains undetectable even with the best of our telescopes. The International Celestial Reference Frame (ICRF) is an internationally standardized quasi-inertial Cartesian reference frame. It is based on the Sixth Fundamental Catalog (FK6), which in Part I defines 878 compact extragalactic (distance!) radio sources. At least 340 of them are adopted as fixation points to establish the ICRF coordinate axes with an accuracy of 0.35 milliarc-seconds per year. The origin of this frame is the barycenter (center of mass) of our Solar System. The ICRF base plane (normal to the z-axis) is the equatorial plane at January 1, 2000, 12.00 h, (so-called J2000, today's reference (standard) epoch) and the x-axis is the direction to the quasar 3C273, which is the optically brightest quasar in our sky. The y-axis is determined from $\boldsymbol{y} = \boldsymbol{z} \times \boldsymbol{x}$.

13.1.2 Heliocentric Reference Frames

The Ecliptic Coordinate System
The ICRF is a perfect physical set-up, but almost useless in practice, because the measurement of extragalactic radio sources is not an astronomer's daily work. But what can easily be measured from the observation of the Sun's movement is the ecliptic plane, in which the Earth moves around the Sun and that in good approximation is fixed in space. To establish a convenient Cartesian reference frame we place the origin at the center of the Sun and take the normal to this plane as the z-axis. What is still needed is a distinguishing direction in the ecliptic. A convenient choice is its intersection with the terrestrial equatorial plane that creates a line of nodes (see Fig. 13.1). This line is chosen as the x-axis, with the direction from the origin (Sun) to the ascending node (see Sect. 7.3.5) of Earth's orbit as its positive part. As seen from the Earth, the Sun at the vernal equinox—around March 21 each year—points along this x-axis to the so-called *First Point of Aries* Υ (a.k.a. *vernal point*), which is an imaginary point on the celestial sphere. The final y-axis is again determined from $\boldsymbol{y} = \boldsymbol{z} \times \boldsymbol{x}$. This Cartesian reference frame is called the *(heliocentric) ecliptic coordinate system* (a.k.a. *heliocentric ecliptic reference frame or heliocentric coordinate system*).

> **Note** *Unfortunately and confusingly, rather than* First point of Aries *or* vernal point *sometimes the notion "vernal equinox of epoch" or just "vernal equinox" is used.*

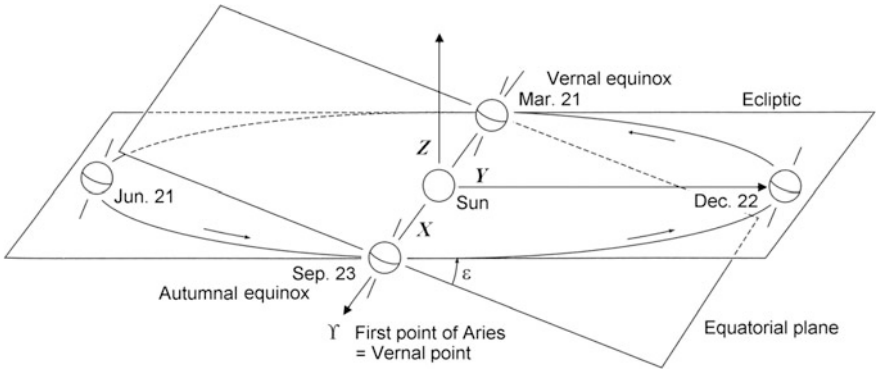


Fig. 13.1 The ecliptic coordinate system *XYZ*. *First Point of Aries* ♈ is the imaginary point on the celestial sphere (as seen from the Sun) of the ascending node at the intersection of the terrestrial equatorial plane and the ecliptic, which equals the direction in which the Sun is seen from Earth at the vernal equinox. *Credit* Montenbruck (2000)

Historical Remark Today when Earth is at vernal equinox the Sun as seen from Earth is positioned in the Sign of Pisces. 4000 years ago—at the dawn of western astronomy—the Sun at vernal equinox was positioned in the first constellation (or Sign) of the Zodiac: Aries, as seen from the Earth. This is why the direction at vernal equinox is still marked by Aries' symbol ♈. This direction is the same as that at which the Earth is positioned at the autumnal equinox as seen from the origin (Sun).

Because Earth's axis nutates and precesses (see Sects. 15.2.2 and 15.3), the equatorial plane, and thus the x-axis, is not fixed. So, the ecliptic coordinate system is not perfectly inertial. But because the precessional motion is very slow (one rotation within 27,500 years) and the nutational motion very small (9.2″ amplitude) and both are known very precisely, this is still a good reference frame for most practical purposes, if the variations are taken into account.

This example nicely shows how Cartesian reference frames in general are constructed for practical purposes: Take a rotational movement of a body (origin)–e.g., terrestrial rotation or orbit of the Earth around the Sun. It defines a rotation axis. Owing to the conservation of angular momentum, this axis is inertial (as long as the body is not subject to perturbative forces). This is the z-axis (polar axis). This axis defines an invariable plane (orbital plane, equatorial plane) in which a distinguished direction makes up the x-axis. The y-axis finally results from $\boldsymbol{y} = \boldsymbol{z} \times \boldsymbol{x}$.

13.1.3 Terrestrial Reference Frames

Since Earth is rotating, there are two types of Earth-centered reference frames. The

1. Earth-centered inertial (ECI) reference system. Here the x-y plane coincides with Earth's equatorial plane. The x-axis points in a fixed direction relative to the celestial sphere. (Note that the ECI reference system is not truly inertial since it revolves with Earth around the Sun.)
2. Earth-centered Earth-fixed (ECEF) reference system where the coordinate system co-rotates with Earth and hence is fixed to Earth's surface.

In the ECEF reference system, the *ecliptic longitude* λ measures the angle from the reference meridian to the object under consideration in the base plane and the *latitude* φ, δ, or β measures the angle vertical to the base plane.

The Equatorial Coordinate System (*IJK*)
The *(geocentric) equatorial coordinate system* (reference frame) as shown in Fig. 13.2 is the standard realization of the ECI reference system. Its x-, y-, z-axes are denoted in celestial mechanics by *I*, *J, K*. Since the line of nodes made up by the ecliptic and the equatorial plane is common to both planes, the vernal point can be chosen for both geocentric equatorial and heliocentric ecliptical systems.

For long-term celestial observations, it is necessary to have a temporally fixed reference frame. The *IJK* frame however is not a fixed frame. As already mentioned, the gravitational perturbation of neighboring planets causes the Earth's axis to precess (see Sect. 15.3.3), and thus the vernal point shifts 1° in 72 years. So today, the vernal point is 56° further in the direction of the sign of Pisces. This implies that for practical applications one has to define a specific reference frame by

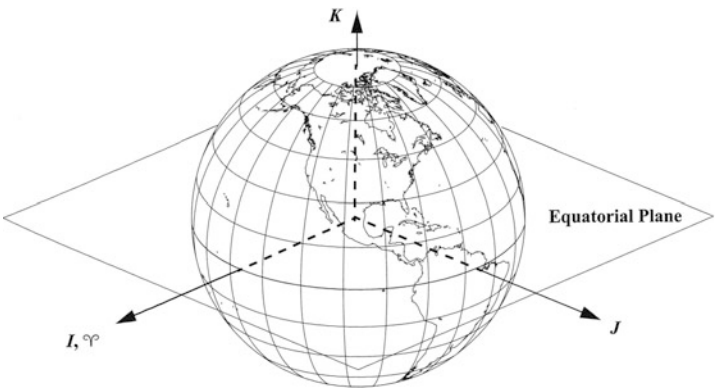


Fig. 13.2 The Cartesian *equatorial coordinate system IJK* is the geocentric equatorial reference frame with Cartesian coordinates *I, J, K*. *Credit* Vallado (2007)

choosing an epoch (reference time). At present, the epoch January 1, 2000, 12.00 h (J2000) is commonly used and named (astronomic) *standard epoch*. Astronomical data that refer to this standard epoch are indexed with J2000, for instance, α_{J2000} for the right ascension.

The International Terrestrial Reference Frame (ITRF)
The International Terrestrial Reference frame with the current realization ITRF2014 is an instantiation of the ECEF reference system with a equatorial base plane (z-axis is the direction of the Earth's pole) and the reference meridian essentially being the Greenwich Meridian.

The ITRF is linked to the ECI equatorial coordinate *IJK* system via the *Greenwich Mean Sidereal Time* θ_{GMST}, which is the hour angle from the vernal point (First Point of Aries) to the average Greenwich Meridian.

The tie between the ICRF and ITRF is provided by the IERS Earth Orientation Parameters (EOP), which describe the orientation of the Celestial Ephemeris Pole (CEP) in the terrestrial and celestial systems.

13.1.4 Orbital Reference Frames

The three reference frames commonly employed for satellite applications are shown in Figs. 13.3 and 13.4: The *perifocal coordinate system PQW* and the two *satellite coordinate systems NTW and RSW*. In all three systems the orbital plane naturally is the x–y plane and therefore $\hat{\boldsymbol{W}} = \hat{\boldsymbol{h}}$, which is why it does not enter the denomination

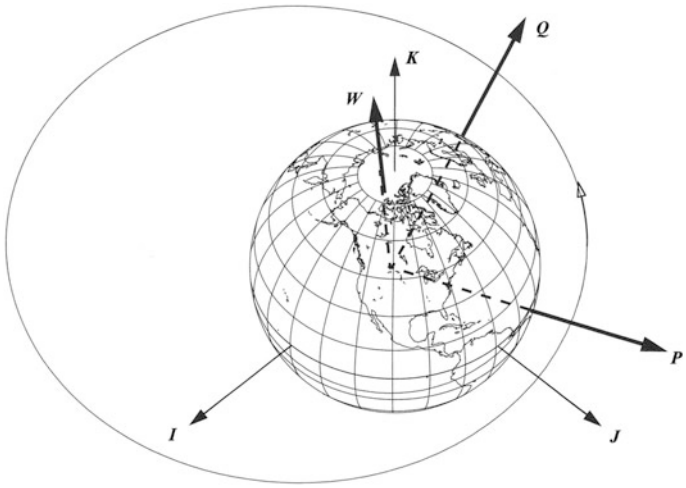


Fig. 13.3 The geocentric Cartesian *perifocal coordinate system PQW* is based on a satellite's orbit. The P–Q plane is the orbital plane and the P-axis points to the perigee. *Credit* Vallado (2007)

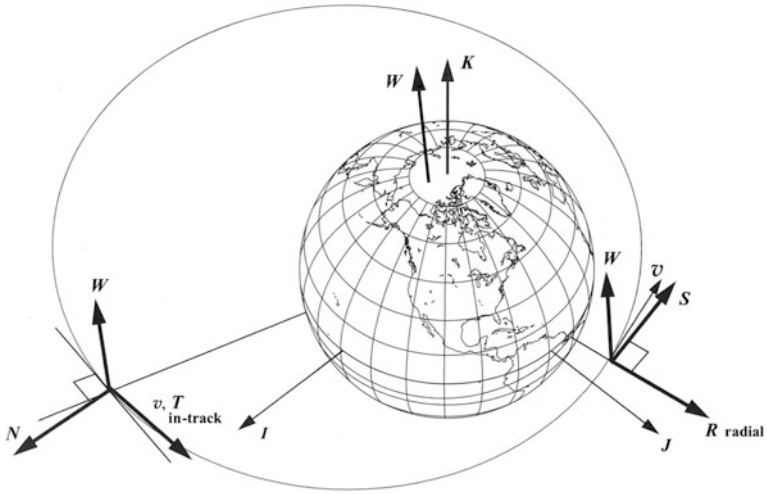


Fig. 13.4 The topocentric Cartesian *satellite coordinate systems NTW* and *RSW* co-rotate with a satellite. The *N-T* plane (= *R–S* plane) is the orbital plane with *T* being the normalized vector pointing in the direction of instantaneous motion and *R* being the normalized vector to the position of the satellite. *Credit* Vallado (2007)

for these reference frames. In the *PQW* system $\hat{\boldsymbol{P}} = \hat{\boldsymbol{e}}$ and $\hat{\boldsymbol{Q}} = \hat{\boldsymbol{h}} \times \hat{\boldsymbol{e}}$. A detailed account of these and some other topocentric reference frames is given in Vallado (2001). Note that the auxiliary reference frame of Fig. 7.6 actually is a *RSW* system.

A not so common choice is the equinoctial coordinate system *EQW* of Fig. 7.11 in Sect. 7.3.5.

Transformations

There are instances when a vector needs to be transformed between the *IJK*, *PQW*, and *RSW* systems. In the following, we will derive and provide the transformation equations.

All of the following transformations are a sequence of elementary rotations. A general elementary rotation in a plane by a positive (in terms of the right-hand rule) angle α is known to be described by the rotation matrix

$$\boldsymbol{R}(\alpha) = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$$

for which holds

$$\boldsymbol{R}^{-1}(\alpha) = \boldsymbol{R}^{T}(\alpha) = \boldsymbol{R}(-\alpha)$$

We apply this to the transformation between the co-rotating coordinate system *RSW* and the quasi-inertial perifocal coordinate system *PQW*. The transformation $PQW \rightarrow RSW$ is just a rotation about the 3rd axis $\boldsymbol{u}_K$ by $\alpha = \theta$. For any vector $\boldsymbol{a}$, which we denominate as $\boldsymbol{a}_{PQW}$, $\boldsymbol{a}_{RSW}$, and $\boldsymbol{a}_{NTW}$ in the corresponding systems, the transformation between the three systems therefore is

$$\boldsymbol{a}_{RSW} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \boldsymbol{a}_{PQW} \tag{13.1.1}$$

$$\boldsymbol{a}_{NTW} = \begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \boldsymbol{a}_{RSW} \tag{13.1.2}$$

$$\boldsymbol{a}_{NTW} = \begin{pmatrix} \cos(\gamma-\theta) & -\sin(\gamma-\theta) & 0 \\ \sin(\gamma-\theta) & \cos(\gamma-\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \boldsymbol{a}_{PQW} \tag{13.1.2a}$$

We are now seeking equations for the most common transformation $PQW \rightarrow IJK$. The transformation can be expressed as three sequential elementary rotations $(-\omega)_3 \rightarrow (-i)_1 \rightarrow (-\Omega)_3$ around the 3-1-3 axes (cf. Vallado 2007; Pisacane 2005, or Battin 1964). This results in

$$\boldsymbol{a}_{IJK} = \boldsymbol{R}_{313}(-\omega, -i, -\Omega) \cdot \boldsymbol{a}_{PQW} \tag{13.1.3a}$$

with the compound rotation matrix

$$\boldsymbol{R}_{313}(-\omega,-i,-\Omega) = \begin{pmatrix} \cos\Omega & -\sin\Omega & 0 \\ \sin\Omega & \cos\Omega & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos i & -\sin i \\ 0 & \sin i & \cos i \end{pmatrix} \begin{pmatrix} \cos\omega & -\sin\omega & 0 \\ \sin\omega & \cos\omega & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\Omega\cos\omega - \sin\Omega\sin\omega\cos i & -\cos\Omega\sin\omega - \sin\Omega\cos\omega\cos i & \sin\Omega\sin i \\ \sin\Omega\cos\omega + \cos\Omega\sin\omega\cos i & -\sin\Omega\sin\omega + \cos\Omega\cos\omega\cos i & -\cos\Omega\sin i \\ \sin\omega\sin i & \cos\omega\sin i & \cos i \end{pmatrix} \tag{13.1.3b}$$

The inverse transformation reads

$$\boldsymbol{a}_{PQW} = \boldsymbol{R}_{313}(\Omega, i, \omega) \cdot \boldsymbol{a}_{IJK} \tag{13.1.4a}$$

$$\boldsymbol{R}_{313}(\Omega, i, \omega) = \begin{pmatrix} \cos\omega & \sin\omega & 0 \\ -\sin\omega & \cos\omega & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos i & \sin i \\ 0 & -\sin i & \cos i \end{pmatrix} \begin{pmatrix} \cos\Omega & \sin\Omega & 0 \\ -\sin\Omega & \cos\Omega & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} \cos\Omega\cos\omega - \sin\Omega\sin\omega\cos i & \sin\Omega\cos\omega + \cos\Omega\sin\omega\cos i & \sin\omega\sin i \\ -\cos\Omega\sin\omega - \sin\Omega\cos\omega\cos i & -\sin\Omega\sin\omega + \cos\Omega\cos\omega\cos i & \cos\omega\sin i \\ \sin\Omega\sin i & -\cos\Omega\sin i & \cos i \end{pmatrix} \tag{13.1.4b}$$

For equatorial orbits Ω is undefined and can be set to zero. The same holds for ω in circular orbits.

13.1.5 Vector Representations

We have already seen in Sect. 7.3.5 that the state vector $(\boldsymbol{r}, \boldsymbol{v})$ is an equivalent representation of the full information of an orbit. As such the radial vector and the orbital velocity vector often need to be determined in terms of the reference frames *IJK* and the *PQW*. This is the objective of this section.

We start out with a description of orbital vectors in a reference frame attached to the orbital plane. Section 13.1.4 presents the two important orbital reference frames *PQW* and *RSW,* and the transformations between these and the equatorial coordinate system *IJK*.

We begin with the two invariant vectors angular momentum $\boldsymbol{h}$ and eccentricity vector $\boldsymbol{e}$. For these we obviously have $\boldsymbol{h} = h(0,\ 0,\ 1)^T_{PQW}$ and $\boldsymbol{e} = e(1,\ 0,\ 0)^T_{PQW}$. With Eqs. (13.1.1) and (13.1.3a) we therefore obtain

$$\boldsymbol{h} = h\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}_{PQW} = h\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}_{RSW} = h\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}_{NTW} = h\begin{pmatrix} \sin\Omega\sin i \\ -\cos\Omega\sin i \\ \cos i \end{pmatrix}_{IJK} \tag{13.1.5}$$

$$\begin{aligned} \boldsymbol{e} &= e\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}_{PQW} = e\begin{pmatrix} \cos\theta \\ -\sin\theta \\ 0 \end{pmatrix}_{RSW} = e\begin{pmatrix} \cos(\gamma-\theta) \\ \sin(\gamma-\theta) \\ 0 \end{pmatrix}_{NTW} \\ &= e\begin{pmatrix} \cos\Omega\cos\omega - \sin\Omega\sin\omega\cos i \\ \sin\Omega\cos\omega + \cos\Omega\sin\omega\cos i \\ \sin\omega\sin i \end{pmatrix}_{IJK} \end{aligned} \tag{13.1.6}$$

For the radial vector holds $\boldsymbol{r} = r(1,\ 0,\ 0)^T_{RSW}$ and therefore

$$\begin{aligned}\boldsymbol{r} &= r\begin{pmatrix}1\\0\\0\end{pmatrix}_{RSW} = r\begin{pmatrix}\cos\gamma\\ \sin\gamma\\ 0\end{pmatrix}_{NTW} = r\begin{pmatrix}\cos\theta\\ \sin\theta\\ 0\end{pmatrix}_{PQW}\\ &= r\begin{pmatrix}\cos\Omega\cos u - \sin\Omega\sin u\cos i\\ \sin\Omega\cos u + \cos\Omega\sin u\cos i\\ \sin u\sin i\end{pmatrix}_{IJK}\end{aligned} \tag{13.1.7}$$

with the argument of latitude $u = \omega + \theta$. We recall that for equatorial orbits Ω is undefined and can be set to zero. The same holds for ω in circular orbits, and hence $u = \theta$.

From Sect. 12.2.2 we have seen that

$$\boldsymbol{r} = r\begin{pmatrix}\cos\beta\cos\lambda\\ \cos\beta\sin\lambda\\ \sin\beta\end{pmatrix}_{ITRF} = \begin{pmatrix}\cos\beta\cos(\lambda+\theta_{GMST})\\ \cos\beta\sin(\lambda+\theta_{GMST})\\ \sin\beta\end{pmatrix}_{IJK}$$

where θ_{GMST} is the *Greenwich Mean Sidereal Time*, which in the equatorial plane is the hour angle from the vernal point (First Point of Aries) to the Greenwich Meridian. Identifying this with Eq. (13.1.7) we obtain the relations between the angular elements $i, \Omega, \omega, \theta$ and geocentric latitude β and geographic longitude λ

$$\begin{aligned}\cos\beta\cos(\lambda+\theta_{GMST}) &= \cos\Omega\cos u - \sin\Omega\sin u\cos i\\ \cos\beta\sin(\lambda+\theta_{GMST}) &= \sin\Omega\cos u + \cos\Omega\sin u\cos i\\ \sin\beta &= \sin u\sin i\end{aligned} \tag{13.1.8a}$$

or alternatively

$$\begin{aligned}\cos\beta\cos\lambda &= \cos(\Omega-\theta_{GMST})\cos u - \sin(\Omega-\theta_{GMST})\sin u\cos i\\ \cos\beta\sin\lambda &= \sin(\Omega-\theta_{GMST})\cos u + \cos(\Omega-\theta_{GMST})\sin u\cos i\\ \sin\beta &= \sin u\sin i\end{aligned} \tag{13.1.8b}$$

According to Eq. (7.2.5) the velocity vector is given as $\boldsymbol{v} = \left(\dot{r},\ r\dot{\theta},\ 0\right)^T_{RSW}$. For evaluating its elements in terms of $a,\ e,\ \theta$ we recall from Eq. (7.3.15a) that

$$\dot{r} = \frac{e\mu}{h}\sin\theta$$

Because of Eq. (7.2.7), we find that

$$r\dot{\theta} = r\frac{h}{r^2} = \frac{h}{r} = \frac{\mu}{h}(1 + e\,\cos\theta)$$

With $\dot{r}^2 = v^2 - r^2\omega^2$ from Eq. (7.2.8) and with Eqs. (13.1.1) and (13.1.3a) we therefore obtain

$$\begin{aligned}
\boldsymbol{v} &= \frac{\mu}{h}\begin{pmatrix} \sqrt{v^2 - r^2\omega^2} \\ r\omega \\ 0 \end{pmatrix}_{RSW} = \frac{\mu}{h}\begin{pmatrix} e\,\sin\theta \\ 1 + e\,\cos\theta \\ 0 \end{pmatrix}_{RSW} \\
&= \frac{\mu}{h}\begin{pmatrix} -e\,\sin(\gamma - \theta) - \sin\gamma \\ e\,\cos(\gamma - \theta) + \cos\gamma \\ 0 \end{pmatrix}_{NTW} = \frac{\mu}{h}\begin{pmatrix} -\sin\theta \\ e + \cos\theta \\ 0 \end{pmatrix}_{PQW} \\
&= \frac{\mu}{h}\begin{pmatrix} -\cos\Omega(\sin u + e\,\sin\omega) - \sin\Omega(\cos u + e\,\cos\omega)\cos i \\ -\sin\Omega(\sin u + e\,\sin\omega) + \cos\Omega(\cos u + e\,\cos\omega)\cos i \\ (\cos u + e\,\cos\omega)\,\sin i \end{pmatrix}_{IJK}
\end{aligned} \tag{13.1.9}$$

with $u = \omega + \theta$. Note that one may substitute $r\omega = \sqrt{\mu p}/r$. The velocity vector may also be expressed in terms of the flight path angle γ (see Sect. 7.3.3). According to Fig. 7.9, Eqs. (13.1.2), and (13.1.3a)

$$\begin{aligned}
\boldsymbol{v} &= v\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}_{NTW} = v\begin{pmatrix} \sin\gamma \\ \cos\gamma \\ 0 \end{pmatrix}_{RSW} = v\begin{pmatrix} \sin(\gamma - \theta) \\ \cos(\gamma - \theta) \\ 0 \end{pmatrix}_{PQW} \\
&= v\begin{pmatrix} \cos\Omega\,\sin(\gamma - u) - \sin\Omega\,\cos(\gamma - u)\,\cos i \\ \sin\Omega\,\sin(\gamma - u) + \cos\Omega\,\cos(\gamma - u)\,\cos i \\ \cos(\gamma - u)\,\sin i \end{pmatrix}_{IJK}
\end{aligned} \tag{13.1.10}$$

with $u = \omega + \theta$. We recall again that for equatorial orbits Ω is undefined and can be set to zero. The same holds for ω in circular orbits, and hence $u = \theta$.

13.2 Time Frames

In the following, we want to set up a time reference frame to determine a point in time unequivocally. To do this, we have to

1. make time measurable, and
2. determine a point of origin.

Measuring Time

The simplest way to measure time is with steady, periodic events, as periods are countable. Measuring time is thus reduced to counting periods of uniform length. The most obvious period for humans has always been a day. If you divide it into 24 smaller and these in another 60 even smaller subperiods, etc., then you get the well-known units of time: hour, minute, and second. The problem of this type of timekeeping starts with the question: What exactly is a day? If it is the time period between two successive culminations of the Sun, then we are talking about the *solar day* with duration of 24 h. If it is the time period between two successive culminations of a star, then we are talking about a *sidereal day* with duration of only 23.934472 h = 23 h 56 min 4.099 s. The small, but important, difference comes about because of the additional rotation angle of about 1° per day of the Earth owing to the motion of the Earth around the Sun (see Fig. 13.5).

However, this ambiguity is only the beginning of the problem to find the right measure for time. Because of the irregularity of the orbital motion of the Earth (slight eccentricity) and the inclination of the ecliptic, a solar day does not always have the same duration. The time difference between the longest (November 4) and the shortest (February 12) day of a year is almost 31 min. In 1925, there was an attempt to level out this flaw by defining a *mean solar day*, the average of all the days the Earth needs to circle around the Sun once. The time determined by the mean solar day is called *Universal Time* (UT) or *Greenwich Mean Time* (GMT), and it refers to the Greenwich Meridian. But even UT is not constant. Because of the constant deceleration of the terrestrial rotation by means of the tidal impact of the Moon, the mean length of a day is reduced by about 2.1 ms per century. This

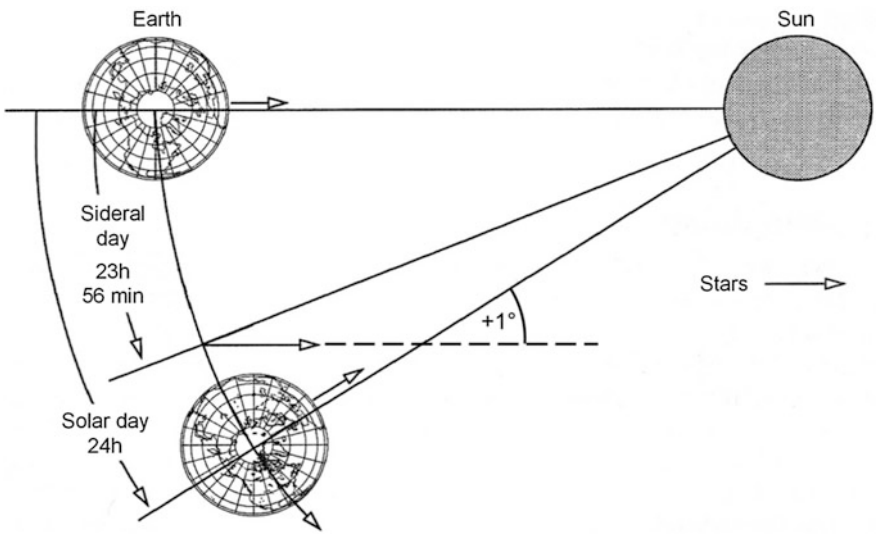


Fig. 13.5 Difference between solar day and sidereal day. *Credit* Vallado (2007)

flaw and irregularities due to Earth's polar motion are considered in the time coordinate UT1 derived from UT.

Owing to this unsteadiness of Earth's rotation, in 1972 it was agreed to use a new time standard that is not based on astronomical observations but on the extremely steady oscillation of atoms, the so-called *International Atomic Time* (TAI = Temps Atomique International). According to TAI, 1 s is the time elapsed during 9,192,631,770 periods of the radiation of cesium-133 atoms corresponding to the transition between the two hyperfine structural levels of their ground state. To get this time in line with UT1, the *Coordinated Universal Time* (UTC, also called Zulu time) was introduced. UTC is a measure of time, which is based on TAI. To be in accordance with the season, represented by UT1, deviations to UT1 are considered by introducing leap seconds on December 31 of and, if necessary, on June 30, if UTC and UT1 differ more than ± 0.9 s. Today UTC is the standard civil time and broadcasted by radio stations worldwide.

Referencing Time

As time can be measured now very exactly, the only open issue is to define an arbitrary point of origin in time. Once this point has been defined, any event in time (in astronomy the moment (event) of a special position of a celestial body is also called an epoch) before or after that can be determined by counting the days, hours, minutes, etc. from the origin. The counted number is the date for the event. The time distance between two dates is a time interval. A list of successive dates is called a calendar. All calendars have one thing in common: a date is counted in units of days. Their only difference is that they have different points of origin, as they can be selected arbitrarily.

The worldwide distributed civil calendar today is the Gregorian calendar. The point of origin of the Gregorian calendar is at the birth of Christ. The Gregorian calendar in addition maps the days on years. Per definition, the mapping function is 1 year = 365.2425 days, and vernal equinox is on March 21. So a Gregorian year basically has 365 days. The additional incommensurate 0.2425 days per year equaling 97 additional days per 400 years are introduced by so-called leap days. A leap day, February 29, occurs every 4 years, but is omitted in years divisible by 100 years, and not omitted by years divisible by 400. The birth of Christ is on January 1 of the year 1. The day before that was of December 31, –1 (i.e., December 31, 1 BC). So there is no year 0.

The Julian Date

In astronomy, on the other hand, the so-called *Julian date JD*, is used as the calendar—not to be confused with the ancient Roman *Julian calendar*. Its point of origin is 12 h noon on January 1, 4713 BC, which marks the beginning of Julian day 0. A Julian date is the decimal number of the time interval between this origin and the given point in time in unit of days. So a Julian date does not have weeks, months and years and therefore does not have the consistency problems, which arise in civil calendars due to leap days, leap months and various calendar reforms. A date in the Gregorian calendar given in years (*yr*), months (*mo*), days (*d*), hours

(h), minutes (min), and seconds (s) for the years $1901 \leq yr \leq 2099$ can be converted into JD as follows:

$$JD = 367 \cdot yr - INT\left\{1.75 \cdot \left[yr + INT\left(\frac{mo+9}{12}\right)\right]\right\} + INT\left(\frac{275 \cdot mo}{9}\right) + d + 1\,721\,013.5 + \frac{\frac{s}{60} + \text{min}}{60} + h \quad (13.2.1)$$

where the function $INT(x)$ truncates the real number x to the next lower integer number. Conversion algorithms between a Gregorian date and a Julian date can be found in the literature as, for instance, in Montenbruck and Gill (2000, Appendix A.2).

Remark *The odd origin 4713 BC was adopted from the so-called* Julian period *proposed in 1583 AD by the French scholar Joseph Justus Scaliger to merge three ancient calendars (15 years indiction cycle, 19 years Metonic cycle, and 28 years Solar cycle) into one calendar system as their multiples. The 4713 BC falls at the last time when all three cycles were in their first year together. 12.00 h noon was chosen in order not to have a day leap by using the ephemerides, because the ancient astronomers observed the sky at night.*

Other Astronomical Calendars

As nowadays the number of days of a Julian date is pretty high, two versions of the Julian date are commonly used today:

1. The modified Julian calendar (*Modified Julian date*, *MJD*) is defined as

$$MJD = JD - 2,400,000.5.$$

   with the epoch November 17, 00:00 h, 1858. The additional half-day was introduced so that the beginning of a day (midnight) is in line with UTC.
2. For numerical calculations in astronomy, *Julian centuries* T with origin J2000 = 2 451 545.0 = January 1, 2000, 12.00 h (*standard epoch*, see Sect. 13.2 and Sect. 7.3.1, Subsection *The Periapsis*) are widely used:

$$T = \frac{JD - 2451\ 545.0}{36,525}$$

A more detailed account on time coordinates and their conversions can be found in Vallado (2001).

Chapter 14
Orbit Geometry and Determination

14.1 Orbit Geometry

In this section we derive some useful formulas for the design of satellite missions.

14.1.1 Eclipse Duration

Satellites in planetary orbits are generally subject to eclipse. Because they are usually powered by solar cells, their electric power system needs to be designed such that even throughout the eclipse the satellite is sufficiently powered. It is therefore crucial to know the eclipse duration of the satellite. This is derived in the following for a circular orbit.

As it turns out, a key parameter is the so-called *beta angle* (a.k.a. *orbit beta angle*) as shown in Fig. 14.1. The beta angle is the angle between the solar vector and the orbit plane. If the solar vector lies in the orbit plane $\beta = 0°$, if it is normal to it $\beta = 90°$. Hence $0° \leq \beta \leq 90°$.

From trigonometry it can be shown that the beta angle is given by the orbital elements as

$$\sin\beta = \sin i \cdot \cos\delta_S \cdot \sin(\Omega - \Omega_S) + \cos i \cdot \sin\delta_S \tag{14.1.1}$$

with

- i Orbit inclination
- Ω Right ascension of ascending node (RAAN) of the orbit
- Ω_S Angle in the ecliptic between the point of Aries and the direction to the Sun (right ascension of the Sun)
- δ_S Declination of the Sun. $\sin\delta_S = \sin\phi \cdot \sin\Omega_S$ for circular planetary orbits
- ϕ Inclination of planet's equatorial plane relative to the ecliptic. $\phi = 23.44°$ for Earth

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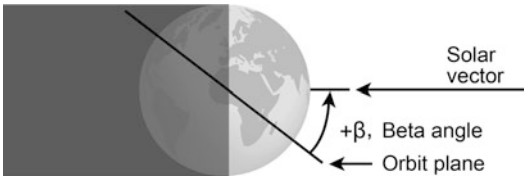


Fig. 14.1 The beta angle β is the angle between the solar vector and the orbit plane

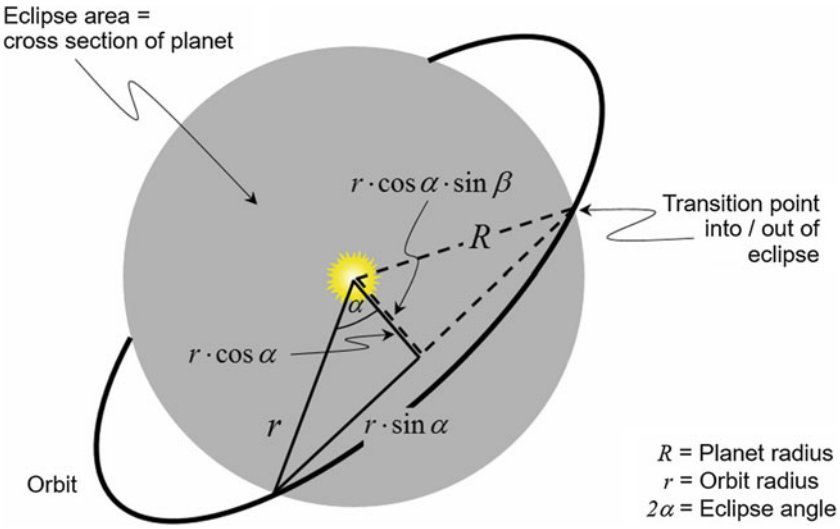


Fig. 14.2 Geometry of a circular orbit with radius r about a spherical planet with radius R and its path through the eclipse. Because the view is through the planet's center towards the Sun, the gray area is the dark side of the planet. The bold triangle with eclipse half angle α lies in the orbit plane, while the dashed triangle is its projection onto the plane perpendicular to the direction of view

Observe that Ω is a rapid variable relative to δ_S (see Sect. 12.3.3). Therefore $-1 \leq \sin(\Omega - \Omega_S) \leq 1$ and hence the beta angle varies for one RAAN revolution between

$$i - \phi \leq i - \delta_S \leq \beta \leq i + \delta_S \leq i + \phi \tag{14.1.2}$$

We now consider an arbitrary circular orbit with orbit radius r about a spherical planet with radius R. Figure 14.2 displays the orbit and eclipse geometry as viewed through the planet's center towards the Sun.

If we denote t_E to be the eclipse duration and 2α the eclipse angle, then

$$t_E = \frac{2\alpha}{360^\circ} T$$

where T is the orbital period. To determine the eclipse duration, we have to derive the eclipse angle. From Fig. 14.2 we find

$$R^2 = r^2 \cos^2\alpha \cdot \sin^2\beta + r^2 \sin^2\alpha = r^2\left(\sin^2\alpha \cdot \cos^2\beta + \sin^2\beta\right)$$

where α is half the eclipse angle and the latter expression follows after some trigonometrical expansions. Solving for α yields

$$\sin\alpha = \frac{\sqrt{(R/r)^2 - \sin^2\beta}}{\cos\beta} =: \frac{\sqrt{\sin^2\beta_* - \sin^2\beta}}{\cos\beta}$$

Here we have defined the limiting angle β_*. Obviously, only for $\beta < \beta_* := \arcsin(R/r)$ is the radicand positive and an eclipse exists. So, the condition for an eclipse reads

$$\sin\beta \le \frac{R}{r} \quad \textbf{eclipse condition} \tag{14.1.3}$$

From the above equation we alternatively find

$$\cos\alpha = \frac{\sqrt{r^2 - R^2}}{r\,\cos\beta}$$

which finally yields for the eclipse duration

$$t_E = \frac{T}{180^\circ} \arccos\frac{\sqrt{r^2 - R^2}}{r\cos\beta} \quad \textbf{eclipse duration} \tag{14.1.4}$$

Example

What is the time of the year, when a geostationary satellite is in eclipse and what is the maximal eclipse duration?

A GEO satellite experiences maximal eclipse if the Sun, the center of Earth, and the satellite lie on a line. This happens twice a year, namely at the vernal and autumnal equinoxes (see Fig. 13.1), which are on March 21 and September 23. We then have $\beta = 0$ and hence

$$t_E = \frac{23.934\ h}{180^\circ} \arccos\frac{\sqrt{42166^2 - 6378.1^2}}{42166} = 69.4\,\text{min}$$

Because we treat the Sun as a point source, this is the time between the middle of the entry and exit penumbras. Because the penumbra in GEO lasts 6 min, the duration of the umbra is 63.4 min and the total eclipse time 75.4 min. The condition for eclipse limit in GEO reads, with $i = 0$

$$\sin\beta = \sin 23.44^\circ \cdot \sin\Omega_S = R/r = 0.15126$$

Therefore, the days in eclipse are

$$\text{equinox} \pm \frac{365.2425\,\text{days}}{360^\circ} \arcsin\Omega_S = \text{equinox} \pm 22.7\,\text{days}$$

We finally determine those orbits that within a year and over several revolutions are steadily exposed to sunlight, i.e. do not suffer eclipse. This is the so-called *hot case* for a satellite (see Chap. 16, Problem 3). From Eqs. (14.1.2) and (14.1.3) it follows that this happens if

$$\max(\sin\beta) = \sin(i+\phi) > \frac{R}{r} \tag{14.1.5}$$

with $\phi = 23.44^\circ$ for Earth.

14.1.2 Access Area

The true outer horizon marks the theoretical limit of the observable area (access area) as seen from space, which ends at the horizon. The knowledge of the distance to the horizon is an important piece of information for Earth observation satellites, but also for astronauts in space. Even space travelers before a suborbital flight have asked me the question: How far can I see on Earth? The answer is given here.

For the sake of simplicity, we assume a spherical planet, which for practical purposes is quite a good approximation. Figure 14.3 depicts the geometry of a satellite above a planet.

Let R be the planetary radius and h the altitude of the satellite above the subsatellite point (nadir). Then for the rectangular triangle with the right angle at the true outer horizon, we have

$$\cos\lambda_0 = \frac{R}{R+h}$$

Let b_0 be the segment of a circle from the subsatellite point (a.k.a. nadir) to the horizon. Then

$$b_0 = \lambda_0 R = R \cdot \arccos\frac{R}{R+h} \quad \textbf{distance to true outer horizon} \tag{14.1.6}$$

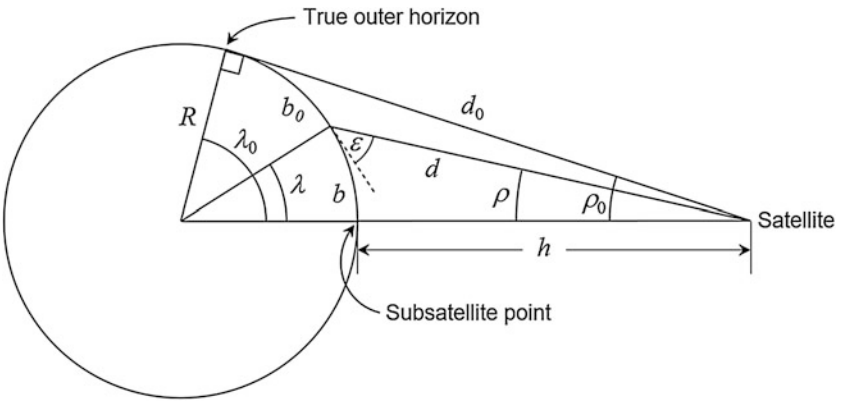


Fig. 14.3 Geometry of a satellite at altitude h above a planet's surface

For LEO satellites with $h \leq 1000\,\text{km} \ll R_\oplus$ we are looking for an approximation for b_0. From

$$\begin{aligned}\arccos\frac{R}{R+h} &= \arcsin\sqrt{1-\left(\frac{1}{1+h/R}\right)^2} \\ &= \sqrt{2\frac{h}{R}}\left[1-\frac{5}{12}\frac{h}{R}+\frac{42}{156}\left(\frac{h}{R}\right)^2+O\left(\frac{h^3}{R^3}\right)\right]\end{aligned}$$

we obtain the following approximations for LEOs

$$\begin{aligned} b_0 &\approx \sqrt{2hR_\oplus}\cdot\left(1-\tfrac{5}{12}\tfrac{h}{R_\oplus}\right) \quad @ \quad \text{error } < 1\% \\ &\approx \sqrt{2hR_\oplus} \qquad\qquad\quad @ \quad \text{error } = 6.5\%\cdot\tfrac{h}{1000\,\text{km}} \end{aligned} \qquad @\ \text{LEO} \qquad (14.1.7)$$

The so-called *instantaneous access area* (IAA) is the spherical segment (expressed in solid angle) of a planet as seen by a satellite at an instantaneous point on the orbit. With the above it is given as

$$\Omega_{IAA} = 2\pi(1-\cos\lambda_0) = 2\pi\frac{h}{R+h} \quad \textbf{instantaneous access area} \qquad (14.1.8)$$

Note that the actually observable part for Earth's surface might be much smaller due to atmospheric impairments such as fog or clouds, or twilight, etc.

Examples

These are the distances b_0 from nadir to Earth's horizon for some typical altitudes:

Eye level	$h = 1.7\,\text{m}$,	$b_0 = 4.7\,\text{km}$
Top of a house	$h = 10\,\text{m}$,	$b_0 = 11.3\,\text{km}$
Top of a hill	$h = 100\,\text{m}$,	$b_0 = 35.7\,\text{km}$
Top of a mountain	$h = 1000\,\text{m}$,	$b_0 = 113\,\text{km}$
Edge of space(suborbital flight)	$h = 100\,\text{km}$,	$b_0 = 1122\,\text{km}$
International Space Station:	$h = 350\,\text{km}$,	$b_0 = 2066\,\text{km}$

Suborbital space travelers should be aware of the fact that the horizon is too oblique to see anything there. Therefore, and for the above atmospheric reasons, at an altitude of 100 km surface details will be seen only for distances up to about 700 km.

14.2 Orbit Determination

This section deals with the problem of how to determine orbit elements, so that the path of a satellite is known and can be propagated (see Sects. 7.4.7 and 7.4.8 for orbit propagation). The most convenient way, surely, is to look them up in NORAD's two-line elements (TLE, see e.g. www.space-track.org). But sometimes they need to be determined faster or better, for instance, when a satellite has been launched into space and its orbit needs to be known immediately. So, how are the orbital elements of an actual orbit determined? This section is aimed at giving an answer to this question.

To be able to specifically determine the six orbital elements for a given S/C, we need to observe at least six suitable components of $\boldsymbol{r}$ or $\boldsymbol{v}$ of the S/C, which we call *orbital parameters*. As observables always include errors of observation, they also bring about errors of the derived orbital elements. So, to improve the accuracy of the derived orbital elements, far more than six observations are usually taken. Usually some orbit parameters are determined several times successively. Apart from observational errors, orbit perturbations occur due to gravitational asymmetries, solar winds, drag, etc. To determine the exact orbit under all these constraints, we need specific methods, which will be explained in the following.

14.2.1 Orbit Tracking

Orbit determination comes in two steps. First the satellite needs to be detected in its orbit by orbit tracking. The concept of tracking is considered in this sub-section. Thereafter the tracking data need to be converted by mathematical means into the orbital elements. Such methods will be presented in the remaining sub-sections.

Radar Tracking
Orbit tracking is usually done with ground-based parabolic radar antennas, which are directed toward the satellite in question (see Fig. 14.4). A signal is sent to the

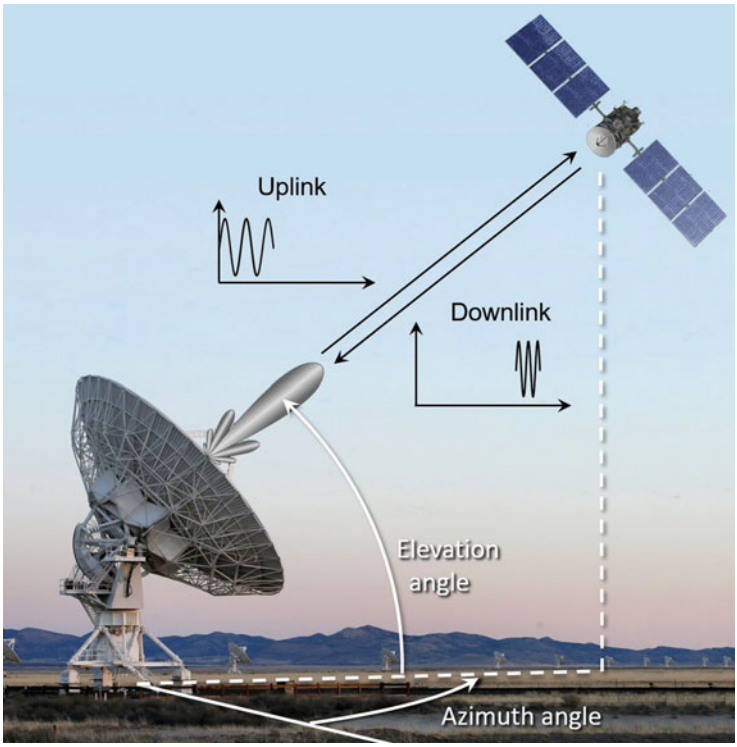


Fig. 14.4 Classical measurements of pointing angles, 2-way slant range, and Doppler shift to determine the range rate of a satellite

satellite and either a passively reflected signal is detected or a transponder on-board the satellite returns the signal with a well-known response time (Fig. 14.5). Either way, radar tracking provides us with the following orbital parameters:

- *Azimuth* and *elevation* (pointing angles) of the receiving antenna in the topocentric system of the ground station by tracking the direction of the maximum of the received signal. An antenna with the diameter of 15 m has an angle resolution of typically 0.1°.
- *Distance* (two-way ranging) from the ground station to the S/C by measuring the runtime of the returned signal with accuracy between 1 and 20 m.
- *Radial velocity* (range rate) with regard to the ground station by measuring the Doppler shift of the returned signal with an accuracy of $0.1-1\,\mathrm{mm\,s^{-1}}$.

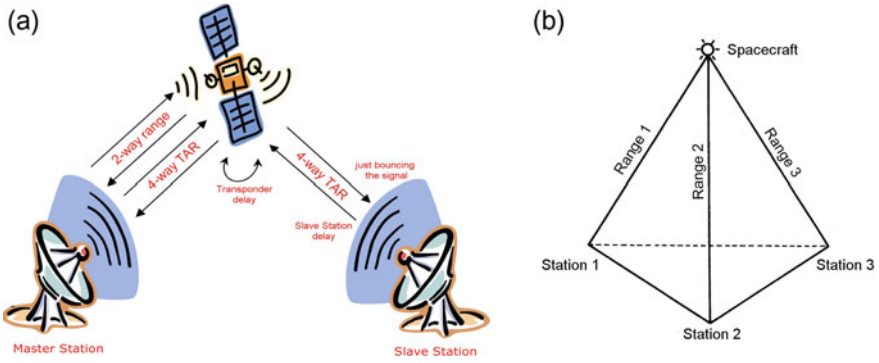


Fig. 14.5 **a** With 4-way TAR a signal from the master ground station is sent to the satellite, which redirects it to a slave station, which bounces the signal back to the satellite and back to the master ground station. **b** Having measured the ranges from at least three ground stations the exact orbit position can be determined by triangulation. *Credit* ESA

Turn-Around Ranging

In 2-way turn-around ranging (TAR) the satellite redirects the signal from a master ground station to a slave station, which by means of precise timing measures the runtime of the signal. The operational costs of such a (usually manned) slave station is quite high. Therefore 4-way TAR is often employed with an unmanned slave station that just bounces the signal back to the satellite, which redirects the signal back to the master ground station (Fig. 14.5a). From both, 2-way TAR and 4-way TAR, the ranges between the two stations and the satellite can be derived. By including two or more slave stations three or more different ranges can be determined. By triangulation this delivers the orbit position (Fig. 14.5b) with an accuracy of typically 2–3 m.

Deep Space Tracking

For deep space missions with many perturbation effects on the trajectory it is vital to know and hence to determine the position of the spacecraft regularly, in particular with high accuracy for fly-by maneuvers. NASA and ESA therefore augment the conventional ranging and doppler tracking by a the so-called ΔDOR (a.k.a. Delta-DOR, Delta-Differential One-Way Ranging) technique. The ΔDOR principle is simple (see Fig. 14.6). The deep space spacecraft transmits a signal, which is received by two ground stations, having a baseline as large as possible, with a certain delay time due to the slightly different distances to the ground stations. However, the delay time is affected by some sources of error: The radio waves travelling through the troposphere, ionosphere and solar plasma are diffracted differently. These errors are corrected by ΔDOR by tracking a quasar—an active galatic nucleus—that is seen in a direction close to the spacecraft (less than 10°) for calibration. Quasar positions are known extremely accurately through astronomical measurements, typically to a couple of nanoradians. The delay time of the quasar is subtracted from that of the spacecraft to provide the ΔDOR measurement.

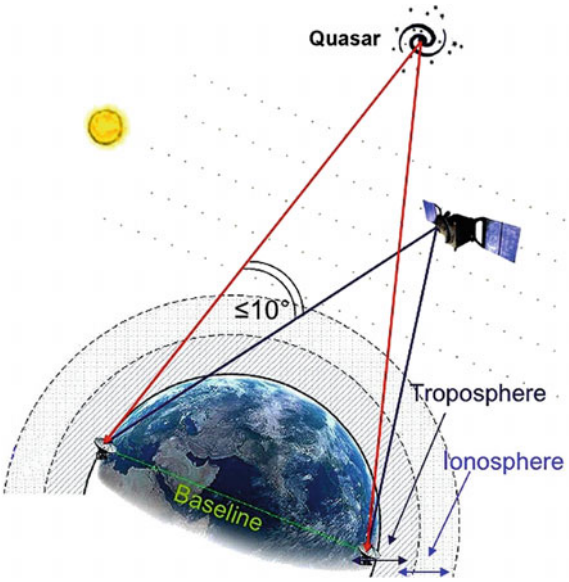


Fig. 14.6 The ΔDOR measuring principle. *Credit* ESA

To provide the full information of the spacecraft's position, two ΔDOR measurements at two baselines involving at least three groundstations are performed. This determines the accurate direction, which together with the classical ranging measurement determines the accurate position.

Other Tracking Systems

Tracking from ground stations has the disadvantage that the satellite can be observed only during the short time of overpass, so it is relatively inaccurate. NASA solved this problem by taking orbital data with its *Tracking and Data Relay Satellites* (TDRS) in GEO. Currently nine satellites are in use clustered near 41°W and 171°W and one satellite at 275°W. So they are separated from each other by 130° longitude and centered around White Sands Ground Terminal in New Mexico. However only two are active at the same time. Thus they cover 85–100% of all LEO satellites.

Today optical tracking systems, such as the Satellite Laser Ranging of the US Natural Environment Research Council (NERC) (see Fig. 14.7), or imaging systems, such as the US Ground-based Electro-Optical Deep Space Surveillance Telescope (GEODSS), are also used. Imaging systems reach an angle resolution of typically one arcsecond, and thus are far more accurate than radars. Laser ranging systems also have a better angle resolution and have the additional advantage that they do not require a satellite transponder. On the other hand, they do not work when it is cloudy. They determine the range from the runtime of a signal reflected from the satellite's surface quite accurately to just about 1 cm.

All ground-based systems suffer from the refraction (changes of the ray's path due to varying atmospheric density) of signals by the atmosphere (atmospheric

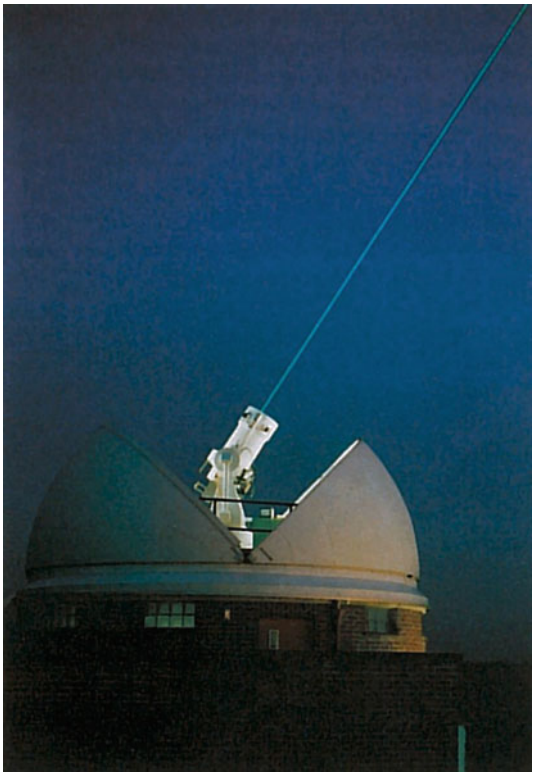


Fig. 14.7 Satellite laser ranging at NERC. *Credit* Montenbruck (2000)

refraction and ionospheric refraction), which moreover is time-dependent. If the orbits are to be tracked accurately, these effects have to be accounted for, which is a quite complex task. So, the complexity of tracking a satellite increases with the required accuracy. This general rule seems to have been turned upside down lately by the advent of space-qualified GPS receivers on-board a spacecraft. These receivers not only are relatively cheap, but also allow position and velocity determination on-board the S/C in real time. The advantage here is that orbit determination and possible required orbit corrections can be determined onboard. Orbit determination efforts are thus transferred from the ground station to the spacecraft, which considerably reduces mission control efforts on ground.

14.2.2 Generalized Orbit Determination Method

In the following, we first expound a general orbit determination method to determine the orbital elements independent of the type of measurements (position vectors, and/

or angles-only measurements, and/or radial velocity measurements, all those with or without time stamps). Section 14.2.3 will give an example for GEO determination. Only thereafter, will we outline three other established simple and elegant methods in Sects. 14.2.4, 14.2.5, and 14.2.6.

Overall, the orbit determination (measurement model) approach comprises two steps: Determination of the

1. orientation of the orbital plane, i.e. determine i, Ω,
2. shape of the orbit and orbit alignment, i.e. the metric elements a, e and the argument of periapsis ω, respectively.

The first step is straightforward and actually requires only the measurement of two sets of tracking data angles. The second step is the difficult part of the problem, which will be generalized.

Orientation of the Orbital Plane

Suppose we have two (= minimum requirement) point angles (i.e. angles-only) measurements or more in time sequence

$$\hat{\boldsymbol{r}}_1, \ldots, \hat{\boldsymbol{r}}_n, \; n \geq 2 \quad \text{pointing angles}$$

measured in a reference frame, which generally is the Cartesian equatorial coordinate system IJK of a planet having coordinate axes with unit vectors $\boldsymbol{u}_I$, $\boldsymbol{u}_J$, $\boldsymbol{u}_K$. From $\hat{\boldsymbol{r}}_1, \ldots, \hat{\boldsymbol{r}}_n$ we choose two vectors $\hat{\boldsymbol{r}}_i$, $\hat{\boldsymbol{r}}_j$, from which the angular momentum unit vector

$$\hat{\boldsymbol{h}} = \frac{\hat{\boldsymbol{r}}_i \times \hat{\boldsymbol{r}}_j}{\left|\hat{\boldsymbol{r}}_i \times \hat{\boldsymbol{r}}_j\right|}$$

is determined and with it the invariant orbital plane. Because $\hat{\boldsymbol{h}}$ is key to the following calculations, it is clear that owing to numerical stability reasons $\hat{\boldsymbol{r}}_i$, $\hat{\boldsymbol{r}}_j$ have to be chosen such that $\left|\hat{\boldsymbol{r}}_i \times \hat{\boldsymbol{r}}_j\right| = \sin \angle\left(\hat{\boldsymbol{r}}_i, \; \hat{\boldsymbol{r}}_j\right) = \max$, i.e. they should be perpendicular to each other as far as possible. This directly yields the inclination of the orbit

$$i = \arccos\left(\boldsymbol{u}_K \hat{\boldsymbol{h}}\right) \tag{14.2.1}$$

In addition, we have for the unit vector to the ascending node of the orbit

$$\boldsymbol{u}_n = \frac{1}{\sin i}\left(\boldsymbol{u}_K \times \hat{\boldsymbol{h}}\right) \quad \text{ascending node vector}$$

Therefore and because $\boldsymbol{u}_K(\boldsymbol{u}_I \times \boldsymbol{u}_n) = \boldsymbol{u}_J \boldsymbol{u}_n$ we have for RAAN

$$\Omega = \begin{cases} \arccos(\boldsymbol{u}_I \boldsymbol{u}_n) & @ \ \ \boldsymbol{u}_J \boldsymbol{u}_n \geq 0 \\ 2\pi - \arccos(\boldsymbol{u}_I \boldsymbol{u}_n) & @ \ \ \boldsymbol{u}_J \boldsymbol{u}_n < 0 \end{cases} \tag{14.2.2}$$

such that $0 \leq \Omega \leq 2\pi$. Needless to say that, by the same token as above, the node vector and with it RAAN becomes indeterminate with vanishing inclination. We further have per definition

$$\boldsymbol{u}_n \hat{\boldsymbol{r}}_i = \cos u_i \quad @ \ \ i = 1, \ldots, n$$

where

$$u = \theta + \omega \qquad \text{argument of latitude}$$

is the so-called *argument of latitude* (see Sect. 7.3.5) with $0 \leq u \leq 2\pi$, and θ, ω are the orbit angle and the argument of periapsis, respectively. Because $\hat{\boldsymbol{h}}(\boldsymbol{u}_n \times \hat{\boldsymbol{r}}_i) = \hat{\boldsymbol{r}}_i\left(\hat{\boldsymbol{h}} \times \boldsymbol{u}_n\right) = \hat{\boldsymbol{r}}_i\left[\hat{\boldsymbol{h}} \times \left(\boldsymbol{u}_K \times \hat{\boldsymbol{h}}\right)\right] = \boldsymbol{u}_K \hat{\boldsymbol{r}}_i$ we determine u_i from the pointing angles as

$$u_i = \begin{cases} \arccos(\boldsymbol{u}_n \hat{\boldsymbol{r}}_i) & @ \ \ \boldsymbol{u}_K \hat{\boldsymbol{r}}_i \geq 0 \\ 2\pi - \arccos(\boldsymbol{u}_n \hat{\boldsymbol{r}}_i) & @ \ \ \boldsymbol{u}_K \hat{\boldsymbol{r}}_i < 0 \end{cases} \tag{14.2.3}$$

In summary, the orientation of the orbital plane can be determined straight away from merely two angles-only measurements.

Orbit Shape and Orbit Alignment

We now come to the cornerstone of the generalization of orbit determination, namely to derive the other orbital elements from the tracking data. There may be a number of k measurements, which provide the radial vectors $\boldsymbol{r}_1, \ldots, \boldsymbol{r}_k$, i.e. in addition to the angles also the orbit radii. In this case, the following orbit equations must apply

$$r_i = \frac{a(1 - e^2)}{1 + e \cos \theta_i} \quad @ \ \ i = 1, \ldots, k$$

Applying $\theta_i = u_i - \omega$, we derive from these the f-functions

$$f_i = 1 + e \cos(u_i - \omega) - \frac{a(1 - e^2)}{r_i} \quad @ \ \ i = 1, \ldots, k \tag{14.2.4}$$

The rationale of the f-functions is, only if we find the right orbital elements that fit the measured data, then $f_i = 0$.

Some measurements may also come with time stamps $t_1, \ldots, t_l$. For those Kepler's equation must apply

$$n \cdot \left(t_i - t_p\right) = E_i - e \, \sin E_i \quad @ \quad i = 1, \ldots, l$$

where $n = \sqrt{\mu/a^3}$ and t_p is the unknown epoch, i.e. the time at passage through the periapsis. To apply the measured μ_i we make use of Eq. (7.4.14b)

$$\sin E_i = \frac{\sqrt{1-e^2}\,\sin(u_i - \omega)}{1 + e\,\cos(u_i - \omega)}$$

and insert it into Kepler's equation, which we rewrite as

$$\sin E_i = \sin\left[n \cdot \left(t_i - t_p\right) + e \, \sin E_i\right]$$

This gives rise to the g-functions

$$g_i = \sin\left[\sqrt{\frac{\mu}{a^3}}\cdot\left(t_i - t_p\right) + e\frac{\sqrt{1-e^2}\,\sin(u_i - \omega)}{1 + e\,\cos(u_i - \omega)}\right] - \frac{\sqrt{1-e^2}\,\sin(u_i - \omega)}{1 + e\,\cos(u_i - \omega)} \quad @ \quad i = 1, \ldots, l \qquad (14.2.5)$$

serving the same purpose as the f-functions.

Finally, there may be a number of m radial velocity (range rate) measurements $\dot{r}_i$. From Eq. (7.3.15a) we directly obtain the according h-functions

$$h_i = \dot{r}_i - e\sqrt{\frac{\mu}{a(1-e^2)}}\,\sin(u_i - \omega) \quad @ \quad i = 1, \ldots, m$$

Now that we have developed the body of our generalized method, we are flexible enough to accept any type of measurement to derive the wanted quantities a, e, ω from the roots of the applying functions f_i, g_i, h_i. Here we list some typical cases: Given

- 3 measurements of the position vector: $\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3$.
  $\rightarrow f_1 = f_2 = f_3 = 0$. From these three equations a, e, ω can derived.
- 2 measurements of the position vector and their time stamps: $\boldsymbol{r}_1, \boldsymbol{r}_2 | \; t_1, t_2$.
  $\rightarrow f_1 = f_2 = g_1 = g_2 = 0$. From these four equations a, e, ω, t_p can derived.
- 4 measurements of pointing angles and their time stamps: $\hat{\boldsymbol{r}}_1, \hat{\boldsymbol{r}}_2, \hat{\boldsymbol{r}}_3, \hat{\boldsymbol{r}}_4$ $|t_1, t_3, t_3, t_4$. This is the re-known and so-called *angles-only orbit determination*.
  $\rightarrow g_1 = g_2 = g_3 = g_4 = 0$. From these four equations a, e, ω, t_p can derived.
- 2 position vectors plus their range rates.
  $\rightarrow f_1 = f_2 = h_1 = h_2 = 0$. From these four equations a, e, ω can be derived.

The method of choice to find the root of n variables from n nonlinear f, g, and h functions is Newton's method for n dimension, plus applying unitary Householder transformations for solving the set of linear equations $\boldsymbol{A}_{3\times 3}\boldsymbol{x} = \boldsymbol{b}$ in the first case, as part of the method, and $\boldsymbol{A}_{4\times 4}\boldsymbol{x} = \boldsymbol{b}$ in the second and third case. Unitary Householder transformations have the unique property that they do not degrade the numerical condition of the problem if, for instance the variables exhibit different orders of magnitude.

Finally, let us assume that we have more f, g, and h functions than the 4 variables a, e, ω, t_p

$$\begin{array}{l} f_1, \ldots, f_k = 0 \\ g_1, \ldots, g_l = 0 \\ h_1, \ldots, h_m = 0 \end{array} \qquad @ \quad k + l + m > 4$$

which is the case for instance when there are a great many measurements. To solve such an over-determined system is called the *least squares problem*. The prefered way for solving for a, e, ω, t_p is also by Newton's method, this time by a 4-dimensional type and again applying unitary Householder transformations to $\boldsymbol{A}_{4\times(k+l)}\boldsymbol{x} = \boldsymbol{b}$. The unitarity of the Householder transformations is here particularly useful if the overdetermined set of equations comes along with some ill-conditioned measurements.

14.2.3 GEO Orbit from Angles-Only Data

As a sample application of the generalized orbit determination method, suppose the orbital elements of a GEO satellite need to be determined by angles-only measurements with time stamps. As a first step the elements i, Ω and the quantities u_i are derived as described in the above generalized orbit determination method. The specific property of a GEO orbit is that

$$a_{GEO} = 42\,166.26\,\text{km}$$

is well-known and hence $n = \sqrt{\mu/a_{GEO}^3} = 1/(3.80957\,\text{h})$, and that $e \ll 1$. Owing to the latter peculiarity we can simplify the g-functions. To do so, we revert to Eq. (7.4.19c) from which follows

$$\theta = u - \omega \approx M + 2e\,\sin M \tag{14.2.6}$$

We apply this key equation to the measurement differences $\Delta\theta_{ji} = \theta_j - \theta_i$, $\Delta u_{ji} = u_j - u_i$, $\Delta t_{ji} = t_j - t_i$, and

$$M_k := n(t_k - t_p) =: nt_k - \tau_p$$

We thus get

$$\begin{aligned} \Delta u_{21} &\approx \Delta t_{21} + 2e[\sin M_2 - \sin M_1] \\ \Delta u_{32} &\approx \Delta t_{32} + 2e[\sin M_3 - \sin M_2] \end{aligned}$$

$$\rightarrow \quad \begin{aligned} 2e[\sin M_2 - \sin M_1] &\approx \Delta u_{21} - n \cdot \Delta t_{21} \\ 2e[\sin M_3 - \sin M_2] &\approx \Delta u_{32} - n \cdot \Delta t_{32} \end{aligned}$$

Now we eliminate the eccentricity by deviding both equations

$$\frac{\sin M_2 - \sin M_1}{\sin M_3 - \sin M_2} \approx \frac{\Delta u_{21} - n \cdot \Delta t_{21}}{\Delta u_{32} - n \cdot \Delta t_{32}}$$

which yields the wanted g-function

$$g(\tau_p) = (\Delta u_{32} - n \cdot \Delta t_{32})[\sin M_2 - \sin M_1] - (\Delta u_{21} - n \cdot \Delta t_{21})[\sin M_3 - \sin M_2]$$

The Newton-iteration $\tilde{\tau}_p = \tau_p - f(\tau_p)/f'(\tau_p)$ to derive τ_p as the root of $g(\tau_p)$ reads in our case

$$\tilde{\tau}_p = \tau_p + \frac{(\Delta u_{32} - n \cdot \Delta t_{32})[\sin M_2 - \sin M_1] - (\Delta u_{21} - n \cdot \Delta t_{21})[\sin M_3 - \sin M_2]}{(\Delta u_{32} - n \cdot \Delta t_{32})[\cos M_2 - \cos M_1] - (\Delta u_{21} - n \cdot \Delta t_{21})[\cos M_3 - \cos M_2]}$$

where $M_i = nt_i - \tau_p$. With the iterative solution $\tau_p = nt_p$ and from the above equations, modified for better numerical stability, the eccentricity is obtained as

$$e = \frac{1}{2} \frac{\Delta u_{31} - n \cdot \Delta t_{31}}{\sin M_3 - \sin M_1} \tag{14.2.7}$$

Finally, Eq. (14.2.6) yields for any measurement i

$$\omega = u_i - M_i - 2e \sin M_i \tag{14.2.8}$$

In conclusion, we see that for GEO orbit determination only 3 pointing angles, plus corresponding time stamps are needed. This is one less measurement than in the general case because a is known a priori.

The determination of GEO elements becomes even simpler if the satellite motion in the guiding center system is constantly recorded as shown in Fig. 12.28. Then, according to Sect. 12.5.3, subsection *East–West Station-Keeping Strategies*, the inclination and eccentricity of a GEO can be read directly from the record: The North-South motion

width of one daily cycle is twice the inclination and the width of the East-West motion width equals $4e$. With the eccentricity e thus derived, there is only ω left to be determined. For that we recall Eq. (14.2.6) and make use of the fact that for the first order term $M \approx \theta = u - \omega$ holds. Hence

$$n(t - t_p) = (u - \omega) - 2e \sin(u - \omega)$$

From two point angles u_1, u_2 with time stamps t_1, t_2 and by taking differences of two equations, thus eliminating t_0, we get

$$n \cdot \Delta t_{21} = \Delta u_{21} - 2e[\sin(u_2 - \omega) - \sin(u_1 - \omega)]$$

By applying the trigonometric relation

$$\sin(u_2 - \omega) - \sin(u_1 - \omega) = 2 \sin \frac{\Delta u_{21}}{2} \cos \frac{u_1 + u_2 - 2\omega}{2}$$

we can solve for ω and finally get

$$\omega = \frac{1}{2}\left\{ u_1 + u_2 - 2 \arccos\left[\frac{\Delta u_{21} - n \cdot \Delta t_{21}}{4e \sin(\Delta u_{21}/2)}\right]\right\}$$

14.2.4 Simple Orbit Estimation

For a preliminary estimation of the state vector, the following method is suitable. It is based on the position vectors $\boldsymbol{r}_0, \boldsymbol{r}_+$ measured at the small time interval Δt, typically in the course of a LEO satellite pass. From Eqs. (14.2.1) and (14.2.2) we immediately derive i, Ω.

Now, let $\boldsymbol{r}_0 := \boldsymbol{r}(0)$ be the position vector at the first observation time $t_0 = 0$. Any time later it can be expressed as a Taylor series

$$\boldsymbol{r} \approx \boldsymbol{r}_0 + \dot{\boldsymbol{r}}_0 t + \frac{1}{2}\ddot{\boldsymbol{r}}_0 t^2 + \frac{1}{6}\dddot{\boldsymbol{r}}_0 t^3 \tag{14.2.9}$$

From this follows by differentiation that

$$\ddot{\boldsymbol{r}} = \ddot{\boldsymbol{r}}_0 + \dddot{\boldsymbol{r}}_0 t \tag{14.2.10}$$

We now apply Newton's equation of motion $\ddot{\boldsymbol{r}} = -\gamma \boldsymbol{r}$ with $\gamma := \mu/r^3$

$$-\gamma \boldsymbol{r} = \ddot{\boldsymbol{r}}_0 + \dddot{\boldsymbol{r}}_0 t$$

from which with Eq. (14.2.10) and with the definition $\boldsymbol{r}_+ := \boldsymbol{r}(t_+) = \boldsymbol{r}(t_0 + \Delta t) = \boldsymbol{r}(\Delta t)$ it follows that

$$-\gamma_0 \boldsymbol{r}_0 = \ddot{\boldsymbol{r}}_0$$
$$-\gamma_1 \boldsymbol{r}_+ = \ddot{\boldsymbol{r}}_0 + \dddot{\boldsymbol{r}}_0 \cdot \Delta t$$

Solving these equations for $\ddot{\boldsymbol{r}}_0$ and $\dddot{\boldsymbol{r}}_0$ and inserting the results into Eq. (14.2.9) for $\boldsymbol{r}_+ = \boldsymbol{r}(\Delta t)$ one obtains for $\dot{\boldsymbol{r}}_0 \equiv \boldsymbol{v}_0$

$$\boldsymbol{v}_0 = \frac{\boldsymbol{r}_+ - \boldsymbol{r}_0}{\Delta t} + \frac{\mu}{6}\left(\frac{2\boldsymbol{r}_0}{r_0^3} + \frac{\boldsymbol{r}_+}{r_+^3}\right) \cdot \Delta t \tag{14.2.11}$$

If three subsequent measurements $\boldsymbol{r}_-(t_-), \boldsymbol{r}_0(t_0), \boldsymbol{r}_+(t_+)$ are taken at small time intervals, then it can be shown (Herrick and Gibbs) and $\Delta t_{ij} = t_i - t_j > 0$ that

$$\begin{aligned}\boldsymbol{v}_0 = {} & \Delta t_{+0}\left[\frac{\mu}{12r_-^3} - \frac{1}{\Delta t_{0-}\Delta t_{+-}}\right]\boldsymbol{r}_- + (\Delta t_{+0} - \Delta t_{0-})\left[\frac{\mu}{12r_0^3} + \frac{1}{\Delta t_{0-}\Delta t_{+0}}\right]\boldsymbol{r}_0 \\ & + \Delta t_{0-}\left[\frac{\mu}{12r_+^3} + \frac{1}{\Delta t_{+-}\Delta t_{+0}}\right]\boldsymbol{r}_+\end{aligned}$$

Thus the state vector $(\boldsymbol{r}_0, \boldsymbol{v}_0)$ is determined at time t_0. This so-called *Herrick-Gibbs method* works best for small geocentric angles, typically less than 15° for LEO and less than 6° for GSO, i.e. for successive measurements from one groundstation.

If the geocentric angles of the three subsequent measurements $\boldsymbol{r}_-, \boldsymbol{r}_0, \boldsymbol{r}_+$ are more appart than 15° for LEO and 6° for GSO, then the following *Gibbs method* is preferable

$$\begin{aligned}\boldsymbol{N} &:= r_-(\boldsymbol{r}_0 \times \boldsymbol{r}_+) + r_0(\boldsymbol{r}_+ \times \boldsymbol{r}_-) + r_+(\boldsymbol{r}_0 \times \boldsymbol{r}_+) \\ \boldsymbol{D} &:= \boldsymbol{r}_0 \times \boldsymbol{r}_+ + \boldsymbol{r}_+ \times \boldsymbol{r}_- + \boldsymbol{r}_- \times \boldsymbol{r}_0 \\ \boldsymbol{S} &:= \boldsymbol{r}_-(r_0 - r_+) + \boldsymbol{r}_0(r_+ - r_-) + \boldsymbol{r}_+(r_- - r_0) \\ \boldsymbol{v}_0 &= \sqrt{\frac{\mu}{ND}}\left(\frac{\boldsymbol{D} \times \boldsymbol{r}_0}{r_0}\right) + \boldsymbol{S}\end{aligned}$$

As shown in Subsection *Conversion: State Vector → Orbital Elements* in Sect. 7.3.5 the orbital elements can easily obtained from these state vectors.

14.2.5 Modified Battin's Method

We present here an elegant method essentially described by Battin in his book *Astronautical Guidance* (Battin 1964, p. 22f) to determine the orbital elements directly from the measurements of three successive orbit radii r_1, r_2, r_3 and the angles between them $\Delta\theta_{ji} = \theta_j - \theta_i$. From the orbit equation we have

$$r_i = \frac{p}{1 + e\,\cos\theta_i} \quad @\ \ i = 1,\ 2,\ 3 \tag{14.2.12}$$

from which follows

$$e\,\cos\theta_i = \frac{p}{r_i} - 1 \qquad @\ \ i = 1,\ 2,\ 3 \tag{14.2.13}$$

$$e\,\sin\theta_i = \frac{r_j(p - r_i)\cos\Delta\theta_{ji} - r_i\big(p - r_j\big)}{r_{ji} r_j \sin\Delta\theta_{ji}} \quad @\ \ j \neq i \tag{14.2.14}$$

with

$$\Delta\theta_{ji} = \theta_j - \theta_i$$

Obviously, from any two position vectors one can derive p and e from Eq. (14.2.12). However, we seek for a numerically more stable solution where all three position vectors equally contribute, and which, in addition, merely makes use of intermediate angles. We therefore pick in Eq. (14.2.14) any position vector i, equate the two equations for $j \neq i$, and solve it for p. We thus get

$$p = \frac{r_1 r_2 r_3(\sin\Delta\theta_{23} + \sin\Delta\theta_{31} + \sin\Delta\theta_{12})}{r_2 r_3 \sin\Delta\theta_{23} + r_1 r_3 \sin\Delta\theta_{31} + r_1 r_2 \sin\Delta\theta_{12}} = a\big(1 - e^2\big) \tag{14.2.7}$$

To determine e, rather than making use of Eq. (14.2.13) as Battin does, we seek an equation for e, in which again only intermediate angles are required. We therefore make use of Eqs. (14.2.13) and (14.2.14) and for numerical stability choose those two vectors $\boldsymbol{r}_k$, $\boldsymbol{r}_l$, with property $\sin\angle\big(\boldsymbol{r}_i,\ \boldsymbol{r}_j\big) = \max$, i.e. they should be perpendicular to each other as much as possible. We thus determine $e^2 \sin\Delta\theta_{lk} = (e\,\sin\theta_l)(e\,\cos\theta_k) - (e\,\cos\theta_l)(e\,\sin\theta_k)$. From this follows

$$e^2 \sin\Delta\theta_{lk} = \left(\frac{p}{r_k} - 1\right)^2 + \left(\frac{p}{r_l} - 1\right)^2 - 2\left(\frac{p}{r_k} - 1\right)\left(\frac{p}{r_l} - 1\right)\cos\Delta\theta_{lk} \tag{14.2.15}$$

Thus, from Eqs. (14.2.7) and (14.2.15) the metric elements p, e, a can be determined.

To determine the orientation of the orbit, we make use of the fact that the eccentricity vector lies in the orbit plane spanned by the two position vectors $\boldsymbol{r}_k$, $\boldsymbol{r}_l$ with the said optimal property. Therefore

$$\boldsymbol{e} = \alpha\,\hat{\boldsymbol{r}}_k + \beta\,\hat{\boldsymbol{r}}_l$$

From this follows

$$\boldsymbol{er}_k = \alpha\, r_k + \beta\, r_k \cos\Delta\theta_{lk}$$
$$\boldsymbol{er}_l = \alpha\, r_l \cos\Delta\theta_{lk} + \beta\, r_l$$

Equating these equations with the two results from the orbit equation

$$\boldsymbol{er}_k = er_k \cos\theta_{ki} = p - r_k$$
$$\boldsymbol{er}_l = er_l \cos\theta_l = p - r_l$$

finally delivers

$$\begin{aligned}\alpha \sin^2\Delta\theta_{lk} &= \left(\frac{p}{r_k} - 1\right) - \left(\frac{p}{r_l} - 1\right)\cos\Delta\theta_{lk}\\ \beta \sin^2\Delta\theta_{lk} &= \left(\frac{p}{r_l} - 1\right) - \left(\frac{p}{r_k} - 1\right)\cos\Delta\theta_{lk}\end{aligned} \tag{14.2.16}$$

from which the wanted α, β can be extracted. Observe that this method is numerically stable for any conic orbit, even for $|a| \to \infty$. However, if the eccentricity is small, the direction of the eccentricity vector is not well established.

14.2.6 Advanced Orbit Determination

Considering Oblateness Perturbations

The methods expounded so far in Sect. 14.2 are based on the concept of the ideal 2-body problem, which is a fair approximation for just one orbit. However, any external force such as gravitational perturbations, lunisolar perturbations, or drag change the orbit over many revolutions, so that derived orbit elements apply only for the instantaneous osculating orbit (see Sect. 12.1.2). This implies that tracking data need to be taken over just one orbit. Even in the course of one orbit a low Earth orbit changes orientation slightly. As shown in Sect. 12.3.3 the oblateness perturbation induced changes $\dot\omega, \dot\Omega, \dot M$ in LEO are of order $10(R_\oplus/a)^{7/2}\left[^\circ\,\mathrm{day}^{-1}\right] \approx 0.5^\circ$ per orbit. Since tracking data have an angular resolution of typically 0.1°, oblateness perturbation matters in particular when tracking data are taken over more than one orbit.

A first remedy for LEO orbits that do not suffer under too great drag is to take the oblateness perturbations into account. According to Sect. 12.3.3 RAAN change between two data takes i, j with $t_j > t_i$ is

$$\Delta\Omega_{ji} = \Omega_j - \Omega_i = -n \cdot \Delta t_{ji} \cdot j_2 \cos i$$

with

$$j_2 = \frac{3}{2} J_2 \left[\frac{R_\oplus}{a(1-e^2)}\right]^2, \quad J_2 = 1.0826266 \times 10^{-3}$$

Therefore, of the two vectors $\hat{\boldsymbol{r}}_i$, $\hat{\boldsymbol{r}}_j$ with $t_j > t_i$, from which the angular momentum unit vector is determined, $\hat{\boldsymbol{r}}_j$ needs to be rotated back about the $\boldsymbol{u}_K$-axis by the RAAN change angle according to

$$\hat{\boldsymbol{r}}'_j = \begin{pmatrix} \cos(-\Delta\Omega_{ji}) & \sin(-\Delta\Omega_{ji}) & 0 \\ -\sin(-\Delta\Omega_{ji}) & \cos(-\Delta\Omega_{ji}) & 0 \\ 0 & 0 & 1 \end{pmatrix} \hat{\boldsymbol{r}}_j = \begin{pmatrix} \cos\Delta\Omega_{ji} & -\sin\Delta\Omega_{ji} & 0 \\ \sin\Delta\Omega_{ji} & \cos\Delta\Omega_{ji} & 0 \\ 0 & 0 & 1 \end{pmatrix} \hat{\boldsymbol{r}}_j$$

The rest of the generalized orbit determination procedure in Sect. 14.2.2 remains the same, namely

$$\hat{\boldsymbol{h}} = \frac{\hat{\boldsymbol{r}}_i \times \hat{\boldsymbol{r}}'_j}{\left|\hat{\boldsymbol{r}}_i \times \hat{\boldsymbol{r}}'_j\right|}, \quad \boldsymbol{u}_n = \frac{1}{\sin i}\left(\boldsymbol{u}_K \times \hat{\boldsymbol{h}}\right)$$

and

$$\Omega_i = \begin{cases} \arccos(\boldsymbol{u}_I \boldsymbol{u}_n) & @ \ \boldsymbol{u}_J \boldsymbol{u}_n \geq 0 \\ 2\pi - \arccos(\boldsymbol{u}_I \boldsymbol{u}_n) & @ \ \boldsymbol{u}_J \boldsymbol{u}_n < 0 \end{cases}$$

which now is the RAAN at the first data take i. Note that owing to $j_2 \cdot n \cdot \Delta t \ll 1$ for j_2 only an approximate value of a needs to be known. We recall that the inclination is not subject to change under oblateness perturbations.

According to the results at the end of Sect. 12.3.3 the expressions $u_i - \omega$, $\Delta\theta$, $(t_i - t_0)$ and Δt in all key equations (in particular in the f-functions, g-functions, and h-functions) in Sects. 14.2.2, 14.2.3 and 14.2.5 need to be substituted by

$$\begin{aligned} n(t_i - t_0) \quad &\rightarrow \quad \left[1 + j_2\left(1 - \tfrac{3}{2}\sin^2 i\right)\right] \cdot n(t_i - t_0) \\ n \cdot \Delta t \quad &\rightarrow \quad \left[1 + j_2\left(1 - \tfrac{3}{2}\sin^2 i\right)\right] n \cdot \Delta t \end{aligned}$$

$$\begin{aligned} u_i - \omega \quad &\rightarrow \quad u_i - \omega_k - 2j_2\left(1 - \tfrac{5}{4}\sin^2 i\right) \cdot n(t_i - t_k) \approx u_i - \omega_k - 2j_2\left(1 - \tfrac{5}{4}\sin^2 i\right) \cdot \Delta\theta_{ik} \\ \Delta\theta_{ik} \quad &\rightarrow \quad \Delta\theta_{ik} \cdot \left[1 + 2j_2\left(1 - \tfrac{5}{4}\sin^2 i\right)\right] \end{aligned}$$

This improved calculation determines the orientation of the apse line, ω_k, at the time where data with index k were taken.

Proficient Orbit Determination

A procedure that proficiently derives the orbital elements takes all orbit perturbations into account by solving the the 3-D equation of motion at each time a data take is taken, p for instance by Cowell's Method by Recurrence Iteration (see Sect. 12.2.4).

The solution then provides the time-dependent state vector, $(\boldsymbol{r}(t), \boldsymbol{v}(t))$, representing a comprehensive description of the orbit. The solution of the orbit tracking problem therefore is as follows:

1. Carry out a sufficient number of measurements of orbital parameters.
2. Use these tracking data to determine with a least-square method (for batch operation on ground) or Kalman filtering method (for on-board sequential real-time processing) the position and velocity of the S/C (so-called *orbit estimation*). This generates the so-called *measurement model* $(\boldsymbol{r}, \boldsymbol{v})_m$ of the state vector.
3. Use this measurement model as initial values to solve with numerical methods the equations of motion (see Sect. 12.2.4), the precision of which is chosen corresponding to the accuracy of the measurement model. The solution propagates the orbital state into the future and is called the *theoretical trajectory model* of the state vector $(\boldsymbol{r}, \boldsymbol{v})_t$.
4. Compare the predictions of the theoretical trajectory model with the measurement model updated by further measurements. To minimize the deviations of the two models–the so-called *residuals*—vary still unknown model parameters (such as drag coefficient or remaining atmospheric density). By this procedure, the still unknown model parameters are determined.
5. When propagation limits are reached, noticeable by increasingly unresolved residuals, a new iteration starts. The measurement model is updated by measurements (the so-called d*ifferential correction*). It is then used as updated initial values for the solver of the equations of motion, whereby the theoretical prediction of the orbit is improved.

This procedure clearly shows that this orbit determination method does not only accurately determine the trajectory, but that by adjusting the parameters of unknown perturbations one can also determine their characteristics. This is exactly the way by which the coefficients of the terrestrial potential (see Sect. 12.2.2) were determined with missions in the past, such as CHAMP, GRACE, or GOCE by high precision measurements of their orbits.

A detailed explication of how all these five steps are implemented, in practice, would be far beyond the scope of this book. For details, the interested reader should consult the books of Montenbruck and Gill (2000), Tapley et al. (2004), Vallado (2007), or Escobal (1965).

Chapter 15
Spacecraft Attitude Dynamics

The general motion of a perfectly rigid body is the superposition of translation and rotation. We talked about translation under the influence of a gravitational field and possible external perturbations in the previous chapters. Now we want to have a look at the rotational behavior of a body.

A satellite in an Earth orbit has certain tasks to perform, which usually impose requirements on its attitude and rotational behavior. The attitude control of an Earth observation satellite, for instance, has to ensure that in course of its orbit the optical sensor always points to nadir or maybe somewhat obliquely. At the same time the satellite, say with a pushbroom sensor, is not allowed to have any rotational motions, because this would lead to distorted images. So it is important that rotational motions can be controlled. But there are question to be asked: What do we have to control? What is the attitudinal and rotational behavior of a satellite in space?

15.1 Fundamentals of Rotation

A spacecraft can well be considered as a rigid body. Here "rigid" means that any in-flight change of distances between the components (mass particles) of the satellite does not have an impact on rotary dynamics. If for instance the satellite vibrates or performs other cyclic motions, e.g., thermal expansion, at such an amplitude that it alters its inertial matrix (see below) significantly, the satellite is not truly rigid. The vibrational dynamics of a satellite is a case of special importance, which we do not want to consider here.

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15.1.1 Elementary Physics

Principles of Rotation

The rotation of a rigid body forces all mass particles to circle around a common line through the center of mass, the rotation axis, with the same angular velocity ω. This circling motion can be described as the change of the orientation angle φ of the rigid body in the time t. The instantaneous angular velocity is thus $\omega(t) := \dot{\varphi}(t) = d\varphi/dt$. In general, the body spins around an axis, but also this spin axis may turn around one or even two axes perpendicular to it. Hence the angular velocity vector is made up of three components: $\boldsymbol{\omega} = (\omega_x, \omega_y, \omega_z)$. For the mathematical description of such a rotation it is very convenient to place the origin of the reference system on the rotation line at the center of mass. Such a system is called a *topocentric system*. We still leave open whether the system co-rotates with the body, or whether it is an inertial system, or else.

Angular Momentum

Just as there is the linear momentum $\boldsymbol{p}$ in translational physics, we have the angular momentum $\boldsymbol{L}$ in rotational physics. According to Fig. 15.1 the angular momentum of a mass particle i with the mass m_i is defined as

$$\boldsymbol{L}_i := \boldsymbol{r}_i \times \boldsymbol{p}_i = m_i \boldsymbol{r}_i \times \boldsymbol{v}_i \tag{15.1.1}$$

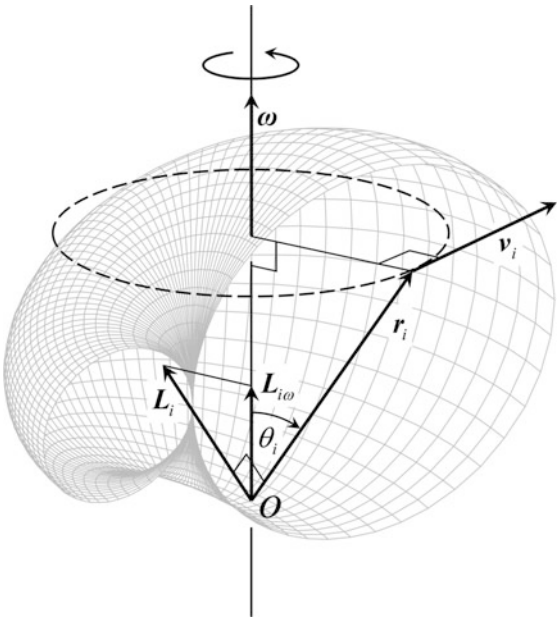


Fig. 15.1 Angular momentum L_i of a mass particle i of a rotating irregular rigid body. *Credit* Ag2gaeh [CC BY-SA 4] & U. Walter

As $\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{r}$ holds quite generally for a rotation, we get the following for the total angular momentum of the body

$$\boldsymbol{L} = \sum_{all\ i} m_i \boldsymbol{r}_i \times (\boldsymbol{\omega} \times \boldsymbol{r}_i) \tag{15.1.2}$$

where the sum comprises all particles of the rotating body. Observe that $\boldsymbol{\omega}$ does not have an index i, as all the particles circle around a common axis with the same angular velocity.

Moment of Inertia

At first we are interested only in the component of the angular momentum $L_{i\omega}$ of a particle along the rotation axis. Let $\theta_i = \angle(\boldsymbol{\omega}, \boldsymbol{r}_i)$ with the origin O lying on the rotation axis (see Fig. 15.1). The absolute value of $\boldsymbol{v}_i = \boldsymbol{\omega} \times \boldsymbol{r}_i$ then is

$$v_i = \omega \cdot r_i \sin\theta_i =: \omega \cdot R_{i\omega}$$

where $R_{i\omega}$ is the distance to the rotation axis. The z component of $\boldsymbol{L}_i$ with regard to $\boldsymbol{\omega}$ is

$$\begin{aligned} L_{i\omega} &= m_i r_i v_i \cdot \cos(\pi/2 - \theta_i) \\ &= m_i (r_i \sin\theta_i)(\omega R_{i\omega}) = m_i R_{i\omega}^2 \omega \end{aligned}$$

So the component L_ω of the total angular momentum along the rotation axis is

$$L_\omega = \sum_{al\ i} L_{i\omega} = \omega \cdot \sum_{all\ i} m_i R_{i\omega}^2$$

We now define the moment of inertia with regard to the rotation axis $\boldsymbol{\omega}$

$$I_\omega := \sum_{all\ i} m_i R_{i\omega}^2$$

resulting simply in $L_\omega = \omega \cdot I_\omega$. So the moment of inertia is a key quantity for angular motion.

Inertia Tensor

After these preliminary remarks to illustrate the significance of the moment of inertia with regard to a given rotation axis, we want to derive a general expression for the moment of inertia with regard to any rotation axis. If one decomposes the expression $\boldsymbol{r} \times (\boldsymbol{\omega} \times \boldsymbol{r}) = \boldsymbol{\omega}(\boldsymbol{r}\boldsymbol{r}) - \boldsymbol{r}(\boldsymbol{r}\boldsymbol{\omega})$ in Eq. (15.1.2) into its axial components and performs the transition to infinitesimally small particles $\sum m_i \to \int dm$, one gets

$$\boldsymbol{L} = \begin{pmatrix} \int (y^2 + z^2) dm & -\int xy \cdot dm & -\int xz \cdot dm \\ -\int yx \cdot dm & \int (x^2 + z^2) dm & -\int yz \cdot dm \\ -\int zx \cdot dm & -\int zy \cdot dm & \int (x^2 + y^2) dm \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

We now define the matrix in the equation as

$$\boldsymbol{I} := \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \quad \textbf{inertia tensor} \tag{15.1.3a}$$

with

$$I_{kl} = \int\limits_{total\ body} \left(r^2 \delta_{kl} - x_k x_l\right) dm \quad \textbf{moments of inertia} \tag{15.1.3b}$$

where δ_{kl} is the *Kronecker delta.* (The Kronecker delta is defined as $\delta_{kl} = 1$ if $k = l$, and $\delta_{kl} = 0$ if $k \neq l$.) This results in

$$\boldsymbol{L} = \boldsymbol{I}\boldsymbol{\omega} \tag{15.1.4}$$

Obviously the inertia tensor is real and symmetrical $(I_{kl} = I_{lk})$ with nine components. Because of the symmetry, only six of them are independent. The moments of inertia depend on the selected reference frame and its coordinate system. Mathematics holds that as $\boldsymbol{I}$ is symmetrical, it is always possible to find a reference frame with a Cartesian coordinate system, where the tensor takes on a diagonal form:

$$\boldsymbol{I}_P = \begin{pmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{pmatrix} \tag{15.1.5}$$

The Cartesian axes of such a reference system are called *principal axes (of inertia)*, and the corresponding diagonal components $I_x,\ I_y,\ I_z$ are called *principal moments of inertia*.

Note *Because the choice of the reference system in body dynamics is important, we indicate the chosen system as an index. If the index is omitted then no choice is necessary.*

Remark *For a general, non-diagonal inertia tensor* $\boldsymbol{I}$*, the principal moments of inertia are the eigenvalues of* $\boldsymbol{I}$*, and the principal axes of inertia are its eigenvectors.*

In general the three principal moments of inertia are unequal. We call the principal axis with the biggest moment of inertia *major principal axis* and the one with the smallest moment of inertia *minor principal axis*. The principal axes are to the largest extent axes of body symmetry, as illustrated in Fig. 15.2 for some simple bodies.

According to Eqs. (15.1.4) and (15.1.5) we can write the total angular momentum in this principal axes system as

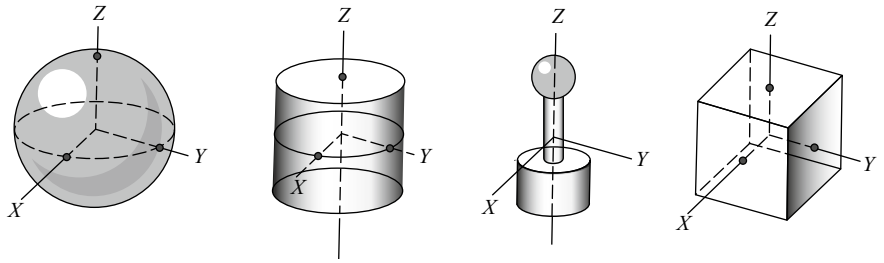


Fig. 15.2 Principal axes of a few symmetrical bodies. Credit Springer Int. Pub.

$$\boldsymbol{L}_P = \left(I_x\omega_x,\ I_y\omega_y,\ I_z\omega_z\right)_P \tag{15.1.6}$$

Note that when no force is impacting from outside, the orientation of the angular momentum vector is only conserved in an inertial system. If we choose a reference system co-rotating with the body, such as the principal axis system, the angular momentum vector in this non-inertial reference system rotates as well (see Sect. 15.2.3 for details) and therefore is not conserved, i.e. it is not constant. Just its amount

$$L^2 = L_P^2 = I_x^2\omega_x^2 + I_y^2\omega_y^2 + I_z^2\omega_z^2 = const \tag{15.1.7}$$

remains constant, because rotation does not change the length of a vector.

Why Rotation is so Odd

The angular momentum equation $\boldsymbol{L} = \boldsymbol{I}\boldsymbol{\omega}$ has the same meaning and importance in rotational kinematics as the momentum equation $\boldsymbol{p} = m\boldsymbol{v}$ in linear kinematics. Because of momentum and angular momentum conservation, both are the hub of mechanics. This brings us for the first time to the important equivalence between rotational and translational dynamics. As we will see in Sect. 15.1.4 this equivalence is even more profound. The equivalence is a 1:1 equivalence: $\boldsymbol{L} \leftrightarrow \boldsymbol{p},\ \boldsymbol{\omega} \leftrightarrow \boldsymbol{v},\ \boldsymbol{I} \leftrightarrow m$. So far, however, there are two flaws in this equivalence:

1. Mass is a scalar, whereas inertia is a tensor.
2. Mass m is independent of coordinates. Unfortunately, this is not true for the moment of inertia $\boldsymbol{I}$. The definition in Eq. (15.1.3a) clearly exhibits the dependence of the components I_{kl} on the coordinates $x,\ y,\ z$.

The first flaw complicates the mathematical treatment of rotational physics quite a lot. But that is the nature of rotation, and we cannot get around it. We can live with this flaw, because it is manageable, even though matrix algebra may be tedious.

The second flaw is a heftier problem. Let us see what it implies. Writing $\boldsymbol{L} = \boldsymbol{I}\boldsymbol{\omega}$ for each component reads

$$L_i = I_{ii}\omega_i + I_{ij}\omega_j + I_{ik}\omega_k, \quad i = 1, 2, 3$$

In an inertial reference frame—which is the most natural choice of a reference system (see Sect. 13.1.1), which is why it sometimes is called *physical system*—each of the $L_i = const$. But this restriction is not grave. It leaves the body the freedom to continuously change $\boldsymbol{\omega} = (\omega_i, \omega_j, \omega_k)$ according to

$$I_{ii}\omega_i(t) + I_{ij}\omega_j(t) + I_{ik}\omega_k(t) = const, \quad i = 1, 2, 3$$

that is, the way it rotates. So, an unsymmetrical body not only has no marked rotation axis—it just rotates around any given axis—but the rotation rates even change in time. This is why the motion of an unsymmetrical body looks so odd. Even worse, when the $\omega_i, \omega_j, \omega_k(t)$ change, the coordinates $x_i,\ y_i,\ z_i$ of the parts of the body change as well, and thus all moments of inertia $I_{ij}(t)$. This time dependence would practically render it impossible to calculate dynamic rotations, and thus is a real problem.

The Body System

But we are free to choose the reference system we want. So, for specific mathematical calculations in rigid body dynamics, we always use a reference system where the moments of inertia are constant with regard to time: a so-called *body system*. A body system is a special case of a topocentric system with its coordinate axes fixed relative to the geometry of the body. It co-rotates with the body.

We gain this advantage at the expense of intuitive clarity. This is because our understanding of the world around us is based on observations as an external inertial observer. Although mathematics in a body system is much more easy, this does not help our intuition: As a rotating observer we in general barely have a feeling for the pointing direction of the angular momentum vector and angular velocity vector (though theoretically we could determine it by evaluating the felt centrifugal forces), but we do not have the slightest idea, how much the vectors are drifting, and why in which direction. The changing rotational rates would only make us quite dizzy due to the forces induced into the vestibular system.

This is why one considers the circumstances of a body dynamics first in an illustrative reference system. Then one jumps to a body system in which the problem can be solved mathematically (but with little understanding why things behave as they are). The solution is finally transformed back into the earlier system to visualize how the body behaves and grasp an idea of why this is so. Because in the following we want to solve dynamical problems, we do not get around these transformations (see Eqs. (15.1.14) and (15.1.15)) and the temporary loss of intuitive understanding. In contrast to current literature, however, we will limit the presentation of transformation equations to the absolute minimum, as they do not help us to understand rotational dynamics.

Types of Reference Systems

In order to distinguish the reference systems from now on,

We denote or index body system coordinates by x, y, z and the coordinates of non-body systems by 1, 2, 3.

Physical parameters, presented in a body system, will have the index B (body axis) for a general body system. The principal axes system is a special body system with index P (principal axis). Inertial (physical) systems usually do not get an index, only when we want to explicitly distinguish them from others, and in this case, they get the index I. So now we have

$$\begin{aligned}
&\boldsymbol{I} \equiv \boldsymbol{I}_I = \left(I_{ij} \neq const\right)\\
&\boldsymbol{I}_B = \left(I_{kl} = const\right)\\
&\boldsymbol{\omega}_B = \left(\omega_x, \omega_y, \omega_z\right)\\
&\boldsymbol{\omega}_{else} = \left(\omega_1, \omega_2, \omega_3\right)
\end{aligned}$$

Remark *For matrix calculation a vector* $\boldsymbol{a}$ *has to be written as a column. From a formal point of view, a vector written as a line would be the transpose, i.e.,* $\boldsymbol{a} = \left(a_x, a_y, a_z\right)^T$. *For the sake of simplicity, we will do without this detail, and we will just write* $\boldsymbol{a} = \left(a_x, a_y, a_z\right)$ *hereafter.*

The transformation of a constant vector equation between reference systems is straightforward. Let $\boldsymbol{C}$ be the transformation matrix (rotation matrix with the three Euler angles) of the transformation $I \rightarrow B$ between the two reference systems, $\boldsymbol{a}_B = \boldsymbol{C}\boldsymbol{a}_I$. Then from $\boldsymbol{L} = \boldsymbol{I}\boldsymbol{\omega}$ just follows $\boldsymbol{L}_B = \boldsymbol{C}\boldsymbol{I}\boldsymbol{\omega} = \boldsymbol{C}\boldsymbol{I}\boldsymbol{C}^{-1}\boldsymbol{C}\boldsymbol{\omega} = \boldsymbol{I}_B\boldsymbol{\omega}_B$. The transformation of a time-dependent vector equation is more complex and will be treated below.

Rotational Energy For a rotating body the rotational energy can be derived as

$$\begin{aligned}
E_{rot} &= \frac{1}{2}\sum_i m_i \dot{v}_i^2 = \frac{1}{2}\sum_i m_i(\boldsymbol{\omega} \times \boldsymbol{r}_i) \cdot (\boldsymbol{\omega} \times \boldsymbol{r}_i) = \frac{1}{2}\sum_i m_i \boldsymbol{\omega} \cdot [\boldsymbol{r}_i \times (\boldsymbol{\omega} \times \boldsymbol{r}_i)]\\
&= \frac{1}{2}\boldsymbol{\omega} \cdot \sum_i m_i \boldsymbol{r}_i \times (\boldsymbol{\omega} \times \boldsymbol{r}_i)
\end{aligned}$$

With Eqs. (15.1.2) and (15.1.4) we thus get

$$E_{rot} = \frac{1}{2}\boldsymbol{\omega}\boldsymbol{L} = \frac{1}{2}\boldsymbol{\omega}(\boldsymbol{I}\boldsymbol{\omega}) \quad \textbf{rotational energy} \tag{15.1.8}$$

Because of Eq. (15.1.6), the following is valid in the principal axes system (cf. Eq. (7.2.18))

$$E_{rot} = \frac{1}{2}\left(I_x\omega_x^2 + I_y\omega_y^2 + I_z\omega_z^2\right) \tag{15.1.9}$$

Because rotational energy is a conserved scalar quantity independent on the choice of reference system, we can drop the reference system index.

15.1.2 Equations of Rotational Motion

To be able to specifically calculate the motions of a rotating body, we are now looking for equations of motion that correspond to the Newton equations of motion in linear dynamics. Newton's law $\boldsymbol{F} = d\boldsymbol{p}/dt$, which by $\boldsymbol{p} = m\boldsymbol{v}$ leads to the linear equation of motion $\boldsymbol{F} = m\dot{\boldsymbol{v}}$, and the equivalence $\boldsymbol{p} \leftrightarrow \boldsymbol{L}$ inspires us to determine $d\boldsymbol{L}/dt$. We use Eq. (15.1.1) to obtain

$$\frac{d\boldsymbol{L}}{dt} = \sum_i m_i\left(\frac{d\boldsymbol{r}_i}{dt} \times \boldsymbol{v}_i + \boldsymbol{r}_i \times \frac{d\boldsymbol{v}_i}{dt}\right) = \sum_i \left[m_i(\boldsymbol{v}_i \times \boldsymbol{v}_i) + \boldsymbol{r}_i \times \frac{d(m_i\boldsymbol{v}_i)}{dt}\right]$$

First we have $\boldsymbol{v}_i \times \boldsymbol{v}_i = 0$. Also, as $m_i\boldsymbol{v}_i = \boldsymbol{p}_i$ is an instantaneous tangential momentum and $d(m_i\boldsymbol{v}_i)/dt$ corresponds, according to Newton, to an external force on the particle i:

$$\frac{d(m_i\boldsymbol{v}_i)}{dt} = \frac{d\boldsymbol{p}_i}{dt} = \boldsymbol{F}_i$$

and because

$$\sum_i \boldsymbol{r}_i \times \boldsymbol{F}_i = \boldsymbol{T} \tag{15.1.10}$$

is an external torque $\boldsymbol{T}$, we finally get

$$\frac{d}{dt}\boldsymbol{L}_I = \boldsymbol{T}_I \tag{15.1.11}$$

As the conservation of each angular momentum component is only valid in an inertial reference frame, this is the equation of motion for the angular momentum of a body in an inertial reference frame (indicated by the index I). This equation is equivalent to Newton's law $d\boldsymbol{p}/dt = \boldsymbol{F}$ in linear dynamics.

Equations (15.1.10) and (15.1.11) imply the following: External forces $\boldsymbol{F}_i$ that affect every point of the body make up a total torque $\boldsymbol{T}$ on the body, which causes a motion of the angular momentum, and via $\boldsymbol{L} = \boldsymbol{I}\boldsymbol{\omega}$ a change of the rotation of the body. If there are no external forces, the angular momentum $\boldsymbol{L}$ is constant in the inertial system. External forces in astrodynamics are the gravitational effects of other celestial bodies such as the Sun and Moon, the oblate spheroid form J_{20} and

the atmospheric drag of the Earth, and radiation pressure of the Sun, but also internal torques caused by engine propulsion or internal magnetic fields interacting with the magnetic field of the Earth.

In order to evaluate the motion equation $\dot{\boldsymbol{L}}_I = \boldsymbol{T}_I$ any further, we would have to differentiate $\boldsymbol{L} = \boldsymbol{I}\boldsymbol{\omega}$. Because in an inertial system the inertia tensor is not constant, we switch as announced to the body system for mathematical simplicity. To do so, we need to know how the time derivation of a vector is transformed between an inertial system and a rotating system, i.e., $I \rightarrow B$. Let $\boldsymbol{a}$ be an arbitrary vector that transforms between the two systems as $\boldsymbol{a}_B = \boldsymbol{C}\boldsymbol{a}_I$, where $\boldsymbol{C}$ is the rotation matrix with the three Euler angles of the three rotation steps. Then we find from the chain rule for the differentiation of a product

$$\dot{\boldsymbol{a}}_B = \frac{d}{dt}(\boldsymbol{C}\boldsymbol{a}_I) = \dot{\boldsymbol{C}}\boldsymbol{a}_I + \boldsymbol{C}\dot{\boldsymbol{a}}_I = -\boldsymbol{\omega}_B \times \boldsymbol{a}_B + \boldsymbol{C}\dot{\boldsymbol{a}}_I$$

where we made use of the identity $\dot{\boldsymbol{C}}\boldsymbol{a}_I = -\boldsymbol{\omega}_B \times (\boldsymbol{C}\boldsymbol{a}_I) = -\boldsymbol{\omega}_B \times \boldsymbol{a}_B$ (see, e.g., Pisacane and Moore (2005, Eq. 5.3.14), and cf. Sect. 7.2.1). We now apply this transformation to $\boldsymbol{a} \equiv \boldsymbol{L}$. Thereby we consider that from $\boldsymbol{L} = \boldsymbol{I}\boldsymbol{\omega}$ with $\boldsymbol{I}_B = const$ we simply get

$$\dot{\boldsymbol{L}}_B = \dot{\boldsymbol{I}}_B\boldsymbol{\omega}_B + \boldsymbol{I}_B\dot{\boldsymbol{\omega}}_B = \boldsymbol{I}_B\dot{\boldsymbol{\omega}}_B$$

This is the crucial simple equation, why we switched to a body system. Applying the Euler rotation to $\dot{\boldsymbol{L}}_I = \boldsymbol{T}_I$ yields

$$\boldsymbol{C}\dot{\boldsymbol{L}}_I = \boldsymbol{C}\boldsymbol{T}_I = \boldsymbol{T}_B$$

With this we finally get

$$\boldsymbol{I}_B\dot{\boldsymbol{\omega}}_B = \boldsymbol{T}_B - \boldsymbol{\omega}_B \times (\boldsymbol{I}_B\boldsymbol{\omega}_B) \quad \textbf{Euler's equation} \text{ (vectorial)} \tag{15.1.12}$$

for an arbitrary body system. If we choose the body system to be the principal axes system x, y, z, in which $\boldsymbol{I}_P = diag\left(I_x, I_y, I_z\right)$ is diagonal, then this equation can be expressed for each component as

$$\boxed{\begin{aligned} I_x\dot{\omega}_x &= T_x + \left(I_y - I_z\right)\omega_y\omega_z \\ I_y\dot{\omega}_y &= T_y + \left(I_z - I_x\right)\omega_z\omega_x \\ I_z\dot{\omega}_z &= T_z + \left(I_x - I_y\right)\omega_x\omega_y \end{aligned}} \quad \textbf{Euler's equations} \tag{15.1.13}$$

where we dropped index P for the sake of simplicity. These are the equations of rotational motion for $\omega_x, \omega_y, \omega_z$ in the principal axes system x, y, z with constant I_x, I_y, I_z.

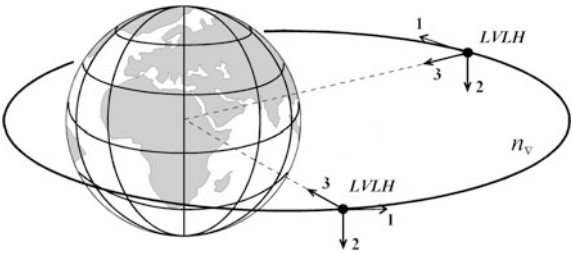


Fig. 15.3 The common orbital *LVLH* system (123 reference system) of a satellite circling the Earth with orbital frequency n_∇. It emerges from the *RSW* system of Fig. 13.4 by turning the 3-axis by 180° around the 1-axis

15.1.3 Coordinate Systems

The principal axis system of a body is a given (except the *x*/*y*-axis pointing ambiguity for a fully rotational-symmetric *z*-axis). The only choice left is their labeling: *x*, *y*, or *z*. We will do this with respect to the external orbital reference system, into which we have to transform back the angular rates $\omega_x, \omega_y, \omega_z$, derived from the equations of solution, to make the motion representation particularly vivid. Figure 15.3 shows an orbital reference frame, the so-called LVLH coordinate system, which is very suitable to describe a satellite in a circular planetary orbit with mean motion n_∇, and it is frequently used to study attitude dynamics in such orbits. Its 3-axis points toward nadir and the 2-axis is normal to the orbital plane but opposite to the orbital momentum vector, so that in a circular orbit the 1-axis points in flight direction: $\boldsymbol{u}_1 = \boldsymbol{u}_2 \times \boldsymbol{u}_3$. In astronautics this system is commonly denoted as "local vertical, local horizontal" (see Fig. 8.37), or *LVLH system* for short. This system emerges from the *RSW* system of Fig. 13.4 by turning the 3-axis by 180° around the 1-axis.

We now overlay the principal axes system with the *LVLH* system by matching the axes according to $x, y, z \leftrightarrow 1, 2, 3$. By this procedure the principal axes x, y, z are determined such that the *x*-axis points into flight direction, the *z*-axis points "down" (see Fig. 15.4).

Coordinate Transformations

If we now allow the body to swerve around slightly, its principal axes will somewhat deviate from the *LVLH* system. The corresponding small rotation angles of the body with its principal axes *x, y, z* are usually denoted (as derived from aviation, see Fig. 15.4) as $\underline{roll} = (\phi)_1$, $\underline{pitch} = (\theta)_2$, and $\underline{yaw} = (\psi)_3$. From Fig. 15.5 we recognize that the rate vector of a small body rotation around the 1-axis is given by $\boldsymbol{\omega}_{rpy,P} \approx \boldsymbol{\omega}_{rpy,LVLH} = \left(\dot{\phi}, 0, 0\right)_{LVLH}$ and therefore for small general body rotations around the axes 1, 2, 3 by

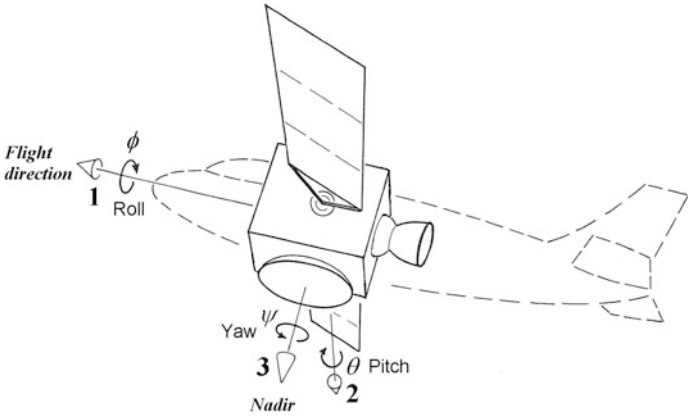


Fig. 15.4 Body axis alignment and satellite rotations *roll*, *pitch*, and *yaw* as defined in the *LVLH* orbital system. *Credit* Berlin (1988)

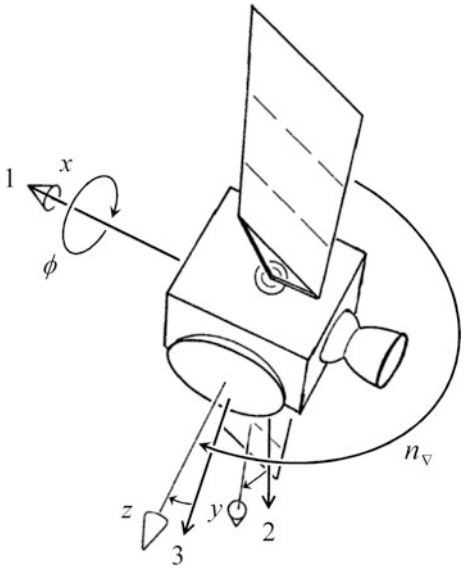


Fig. 15.5 Overlay of the 123-system *LVLH* and *x, y, z* principal axes body system displaying a small roll ϕ around the 1-axis, plus orbital revolution of the *y*-axis around the 2-axis with orbital frequency n_∇. *Credit* Berlin (1988) and U. Walter

$$\boldsymbol{\omega}_{rpy,P} \approx \boldsymbol{\omega}_{rpy,LVLH} = \left(\dot{\phi}, \dot{\theta}, \dot{\psi}\right)_{LVLH}$$

In addition we assume that the satellite's orbit around Earth is circular. So, as shown in Fig. 15.5 the satellite turns with constant orbital frequency n_∇ around the 2-axis. The vector of this rotation for small yaw and roll angles is written as

$$\boldsymbol{n}_{\nabla,P} = n_\nabla(\sin\psi \cdot \boldsymbol{u}_1 + \boldsymbol{u}_2 - \sin\phi \cdot \boldsymbol{u}_3) \approx n_\nabla(\psi, 1, -\phi)_{LVLH}$$

The combined rotation rates of the body expressed in the *LVLH* orbital system therefore are

$$\boldsymbol{\omega}_P = \boldsymbol{\omega}_{rpy,P} + \boldsymbol{n}_{\nabla,P} = \left(\dot{\phi} + n_\nabla\psi, \dot{\theta} + n_\nabla, \dot{\psi} - n_\nabla\phi\right)_{LVLH} \tag{15.1.14}$$

from which follows

$$\dot{\boldsymbol{\omega}}_P = \left(\ddot{\phi} + n_\nabla\dot{\psi}, \ddot{\theta}, \ddot{\psi} - n_\nabla\dot{\phi}\right)_{LVLH} \tag{15.1.15}$$

These are the wanted kinematic transformation equations for the Euler angles (so-called *Euler rate equations*) between the two selected reference frames. They will be needed to analyze the solutions $\omega_x, \omega_y, \omega_z$ of the satellite motion equations in the *LVLH* orbital system.

15.1.4 Rotation-to-Translation Equivalence

So far we have been deriving equations for different physical parameters that are characteristic for rotational mechanics. If you compare these with those of translational mechanics (see Table 15.1), you will see that the already mentioned equivalence $\boldsymbol{L} \leftrightarrow \boldsymbol{p}$, $\boldsymbol{\omega} \leftrightarrow \boldsymbol{v}$, $\boldsymbol{I} \leftrightarrow m$ holds without exception for the basic physical quantities. We therefore have found a fundamental symmetry between these two fields of physics.

But be careful by applying this equivalence. From the equation of translational motion $\boldsymbol{F} = m\dot{\boldsymbol{v}}$ does not follow the rotational form $\boldsymbol{T} = \boldsymbol{I}\dot{\boldsymbol{\omega}}$. We rather have from $\boldsymbol{L} = \boldsymbol{I}\boldsymbol{\omega}$ the derivation $\boldsymbol{T} = \dot{\boldsymbol{L}} = \dot{\boldsymbol{I}}\boldsymbol{\omega} + \boldsymbol{I}\dot{\boldsymbol{\omega}}$. This written in linear dynamics reads $\boldsymbol{F} = \dot{m}\boldsymbol{v} + m\dot{\boldsymbol{v}}$, and only because $\dot{m} = 0$ we find $\boldsymbol{F} = m\dot{\boldsymbol{v}}$. But $\dot{\boldsymbol{I}}$ is too complicated in rotational physics. This is why the equation $\boldsymbol{T} = \boldsymbol{I}\dot{\boldsymbol{\omega}} + \dot{\boldsymbol{I}}\boldsymbol{\omega}$, though being valid, is useless. Only by switching to a body system we found Euler's equation $\boldsymbol{T}_B = \boldsymbol{I}_B\dot{\boldsymbol{\omega}}_B + \boldsymbol{\omega}_B \times (\boldsymbol{I}_B\boldsymbol{\omega}_B)$, which by $B = P$ is a treatable equation of motion.

Table 15.1 Equivalencies between physical laws of translation and rotation

Translation		Rotation	
Momentum	$\boldsymbol{p} = m\boldsymbol{v}$	Angular momentum	$\boldsymbol{L} = \boldsymbol{I}\boldsymbol{\omega}$
Force	$\boldsymbol{F} = d\boldsymbol{p}/dt$	Torque	$\boldsymbol{T} = d\boldsymbol{L}/dt$
Kinetic energy	$E = 1/2\,\boldsymbol{v}\boldsymbol{p} = 1/2\,m\boldsymbol{v}^2$	Rotational energy	$E = 1/2\,\boldsymbol{\omega}\boldsymbol{L} = 1/2\,\boldsymbol{\omega}\boldsymbol{I}\boldsymbol{\omega}$
Power	$P = \boldsymbol{F}\boldsymbol{v}$	Power	$P = \boldsymbol{T}\boldsymbol{\omega}$

Rotational Power

We still have to complete the table with the equation for the rotational power. Under the influence of an external torque, the rotational energy changes together with the angular momentum according to Eq. (15.1.8). The power P_{rot}, which the external torque has to bring up to change the rotational energy, is defined as $P_{rot} = dE_{rot}/dt$. From Eq. (15.1.8) we then have

$$P_{rot} = \frac{dE_{rot}}{dt} = \frac{1}{2}\left(\dot{\boldsymbol{\omega}}_B \boldsymbol{L}_B + \boldsymbol{\omega}_B \dot{\boldsymbol{L}}_B\right)$$

Taking Eq. (15.1.11) into account we derive

$$\dot{\boldsymbol{\omega}}_B \boldsymbol{L}_B = \dot{\boldsymbol{\omega}}_B (\boldsymbol{I}_B \boldsymbol{\omega}_B) = \left(\dot{\boldsymbol{\omega}}_B^T \boldsymbol{I}_B\right)\boldsymbol{\omega}_B = (\boldsymbol{I}_B \dot{\boldsymbol{\omega}}_B)\boldsymbol{\omega}_B = \dot{\boldsymbol{L}}_B \boldsymbol{\omega}_B = \boldsymbol{\omega}_B \dot{\boldsymbol{L}}_B = \boldsymbol{\omega}_B \boldsymbol{T}_B$$

Therefore the rotational power induced by an external torque is

$$P_{rot} = \frac{1}{2}(\boldsymbol{\omega}_B \boldsymbol{T}_B + \boldsymbol{\omega}_B \boldsymbol{T}_B) = \boldsymbol{\omega} \boldsymbol{T} \quad \textbf{rotational power} \tag{15.1.16}$$

The latter generalization holds because the rotational power is a scalar and therefore is independent from any chosen reference system.

15.2 Attitude Kinematics

In this section we assume that the body is not subject to any external force, i.e., $T_x, T_y, T_z = 0$ (such freely rotating bodies are also called *free gyros*), and want to study the essential characteristics of its consequent rotational kinematics by applying the torque-free Euler's equation of motion.

Let us have a look at an arbitrarily shaped body from the point of view of an external observer in an inertial reference frame. Equation $\boldsymbol{L} = \boldsymbol{I}\boldsymbol{\omega}$ states that in general the rotation $\boldsymbol{\omega}$ is not parallel to $\boldsymbol{L}$ and hence (see above) not constant—the body tumbles. What is the more profound reason for this? The reason is as follows. An arbitrary mass particle i of the body has the instantaneous tangential velocity v_i. If the particle with position vector $\boldsymbol{r}_i$ from the center of mass (origin) by chance does not lie on the plane vertical to the rotation axis through the center of mass, then its angular momentum $\boldsymbol{L}_i = m_i \boldsymbol{r}_i \times \boldsymbol{v}_i$ is not parallel to the rotation axis (see Fig. 15.1). Due to the body's rigid bonding, the particle is forced on a circular orbit around the rotation axis (Fig. 15.1, dashed circle), $\boldsymbol{L}_i$ rotates around the rotation axis. The total angular momentum of the rigid body is $\boldsymbol{L} = \sum_i \boldsymbol{L}_i$. If the mass particles are arranged symmetrically with regard to the rotation axis, that is if the rotation axis is a body symmetry axis (principal axis), all $\boldsymbol{L}_i$ add up to a total momentum parallel to the rotation axis. However, this is generally not the case, so

the total angular momentum $\boldsymbol{L}$ is generally not parallel to the rotation axis $\boldsymbol{\omega}$. The implication on the movement of the body in these two cases is as follows:

1. **One of the principal axes of inertia is initially aligned parallel to $\boldsymbol{L}$.**
   In this case $\boldsymbol{I} \equiv \boldsymbol{I}_P = diag\left(I_x, I_y, I_z\right) = const$. This implies $\boldsymbol{L} || \boldsymbol{\omega}$. The law of conservation angular momentum, $\boldsymbol{L} = const$, directly results in $\boldsymbol{\omega} = const$. So the body continues to rotate around this principal axis of inertia.
2. **Not one of the principal axes of inertia is initially aligned parallel to $\boldsymbol{L}$.**
   As mentioned earlier we then have

$$L_i = I_{ii}(t)\omega_i(t) + I_{ij}(t)\omega_j(t) + I_{ik}(t)\omega_k(t) = const, \quad i = 1, 2, 3$$

   This gives the body the freedom to change continuously the rotational rates. So $\boldsymbol{\omega} \neq const$. This implies that it may tumble wildly.

Our objective in the next three sections is to find out for a given inertial characteristic of the body and with the equations of motion how $\boldsymbol{\omega} = \boldsymbol{\omega}(t)$ looks like near a principal axis. In the last section we are going to qualitatively describe general rotational motions.

15.2.1 Stability

We start out with the first case $\boldsymbol{L} || \boldsymbol{\omega}$, where the body rotates about one of its principal axes. Theoretically we have $\boldsymbol{\omega} = const$. However, this does not necessarily imply that this rotation is stable under the impact of a small perturbation. We therefore want to find stability criteria and modes of motion for rotations around one of the three principal axes of inertia.

Stability Criteria

Let us assume the body has an initial rotation of ω_0 around an arbitrary principal axis of inertia, which we denote as z-axis, i.e., $\omega_z = \omega_0$ and $\omega_x = \omega_y = 0$. This choice is in accordance with Euler's equation of motion, Eq. (15.1.13), without external torques. Note that by employing Euler's equation we automatically switch into the principal axes system. So we observe the body to be stationary while the world around turns, including $\boldsymbol{L}$ and $\boldsymbol{\omega}$.

Now we give the body a kick sideways and thereby perturb the rotation by the tiny amount ε, which causes small rotation rates around the x- and y-axis with the amount $\omega_x, \omega_y \propto \varepsilon$, while $\omega_z \approx \omega_0 \approx const$ roughly remains constant. For this perturbed case, the Euler's equation (15.1.13) take on the form

$$I_x\dot{\omega}_x = \left(I_y - I_z\right)\omega_y\omega_0$$
$$I_y\dot{\omega}_y = \left(I_z - I_x\right)\omega_x\omega_0$$
$$I_z\dot{\omega}_z = \left(I_x - I_y\right)\omega_x\omega_y \approx 0$$

Terms of the order ε^2 can be neglected with regard to ε. This is why $(I_x - I_y)\omega_x\omega_y \propto \varepsilon^2 \approx 0$. Therefore from the last equation follows $\dot{\omega}_z \approx 0$, which in retrospect confirms our initial assumption $\omega_z \approx \omega_0 \approx const$. We rewrite the first two equations as

$$\dot{\omega}_x = \frac{I_y - I_z}{I_x}\omega_0\omega_y \tag{15.2.1a}$$

$$\dot{\omega}_y = \frac{I_z - I_x}{I_y}\omega_0\omega_x \tag{15.2.1b}$$

These are coupled differential equations, which we decouple by differentiating Eq. (15.2.1a) and inserting the result into Eq. (15.2.1b), or the other way round. This gives us

$$\ddot{\omega}_x = -\lambda^2\omega_x \tag{15.2.2a}$$

$$\ddot{\omega}_y = -\lambda^2\omega_y \tag{15.2.2b}$$

with

$$\lambda := \omega_0\sqrt{\frac{I_z - I_y}{I_y}\frac{I_z - I_x}{I_x}}$$

The general solutions of these linear differential equations are known to be

$$\omega_x(t) = \omega_{x0}e^{i(\lambda t + \phi)}$$
$$\omega_y(t) = \omega_{y0}e^{i(\lambda t + \phi)}$$

From this we can read off the stability criterion of the perturbed rotation: We get no exponentially diverging solutions only if λ is real. This means that, to achieve a stable rotation around the z-axis $(\omega_z = const)$, the radicand of the root in Eq. (15.2.2a) has to be positive. This is the case if

$$I_z > I_x, I_y \quad \text{or} \quad I_z < I_x, I_y$$

For a symmetrical gyro, i.e., if the two principal axes $I_x = I_y \neq I_z$ we get $\lambda = \omega_0\left(I_z - I_y\right)/I_y$. In this case, λ is always real and thus the three principal axes are always stable. We summarize these important results:

A rotation is stable only if it takes place along the major or minor principal axis. The rotation around the principal axes with mean inertia moment is not stable. If two principal moments of inertia coincide (symmetrical gyro), all principal axes are stable.

This feature is often used to give a symmetrical satellite a fixed orientation in space by spinning it around a principal axis of inertia (spin stabilization).

15.2.2 Nutation

We now want to know how the spin axis of a body moves. If λ is real, the general solution for Eq. (15.2.2a) is

$$\omega_x(t) = \omega_{x0} \sin(\lambda t + \varphi) \tag{15.2.3a}$$

and by inserting it into Eq. (15.2.1b) and integrating we get for the other component

$$\omega_y(t) = -\omega_{y0} \cos(\lambda t + \varphi) \tag{15.2.3b}$$

with

$$\omega_{y0} = \omega_{x0} \sqrt{\frac{I_z - I_x}{I_z - I_y} \frac{I_x}{I_y}}.$$

In the body-fixed principal axes system and for arbitrary principal moments of inertia, the tip of the rotation vector therefore describes an elliptic motion with semi-axes $\omega_{x0} \neq \omega_{y0}$. That is, rotation axis and rotation speed both change continuously. If $I_z \gg I_x, I_y$ (flat body), we get $\omega_{y0} = \omega_{x0} \sqrt{I_x / I_y}$, and for $I_z \ll I_x, I_y$ (elongated body), we get $\omega_{y0} = \omega_{x0} \cdot I_x / I_y$.

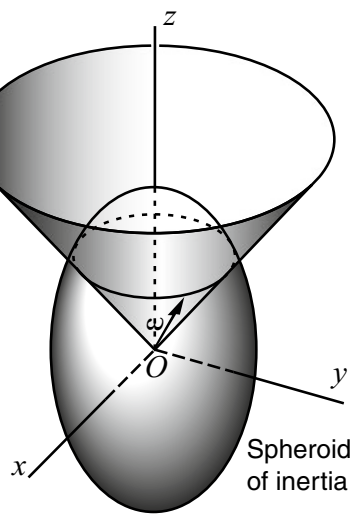


Fig. 15.6 The motion of $\boldsymbol{\omega}$ of a symmetrical gyro in the principal axis system with its spheroid of inertia. Credit Springer Int. Pub.

In the important case of *a symmetrical gyro* where $I_x = I_y$, we get $\omega_{x0} = \omega_{y0}$, i.e., the tip rotates on a circle (see Fig. 15.6). In addition, the following is valid:

$$\omega_{xy}^2 := \omega_x^2 + \omega_y^2 = \omega_{x0}^2 = const$$

with ω_{xy} as the turning component in the x–y plane perpendicular to the x-axis. In total we have

$$|\boldsymbol{\omega}| = \sqrt{\omega_x^2 + \omega_y^2 + \omega_z^2} = \sqrt{\omega_{xy}^2 + \omega_0^2} = const \tag{15.2.4}$$

So, a symmetrical gyro maintains a constant rate of rotation, only the rotation axis moves conically.

With this we get the following stable motion as viewed from a body-fixed reference system. If initially the rotation $\boldsymbol{\omega}$ is along one of the stable principal axes, $\boldsymbol{\omega}$ is maintained and the world around turns evenly around this axis. A small external perturbation (slight touch of the body = torque), provokes the vector $\boldsymbol{\omega}$ to generally describe an elliptic (with changing length), or for symmetrical gyros a conic (with constant length) rotation around the principal axis. From an external point of view, the body first rotates evenly about the stable principal axis (= body axis)—it rotates stationary. The kick provokes the rotation axis to turn elliptically (with changing rate of rotation) or conically (with constant rotation rate) around the angular momentum vector now fixed in space. This motion is called *nutation*, so it pitches or, in colloquial English, it tumbles. But this nutation of the rotation axis is not what we see, because the rotation axis is imperceptible for us. What we see is the motion of the body axis $\boldsymbol{u}_z$. To illustrate its motion we need to know what the relation of $\boldsymbol{u}_z$ with $\boldsymbol{L}, \boldsymbol{\omega}$ is, to which we come now.

Nutation of a Torque-Free Symmetrical Gyro

The torque-free symmetrical gyro, $I_x = I_y \neq I_z$, is a case in point to study the motion of the body axis. Due to its conspicuous symmetrical form we call the $\boldsymbol{u}_z$ body axis the *figure axis*. For an external observer (inertial frame) this figure axis moves on a cone, the so-called *nutation cone* around the constant angular momentum vector (see Fig. 15.7) with frequency (see Eq. (15.2.2a))

$$\nu_N = \frac{\lambda}{2\pi} = \frac{\omega_0}{2\pi}\frac{I_z - I_x}{I_x} \quad \textbf{nutation frequency} \tag{15.2.5}$$

In addition, it can be shown that $\boldsymbol{L}, \boldsymbol{u}_z$ and the rotation axis $\boldsymbol{\omega}$ are coplanar. Therefore $\boldsymbol{\omega}$ also rotates with nutation frequency around $\boldsymbol{L}$ on the so-called *space cone*. The two cone angles (see Fig. 15.7) are given as:

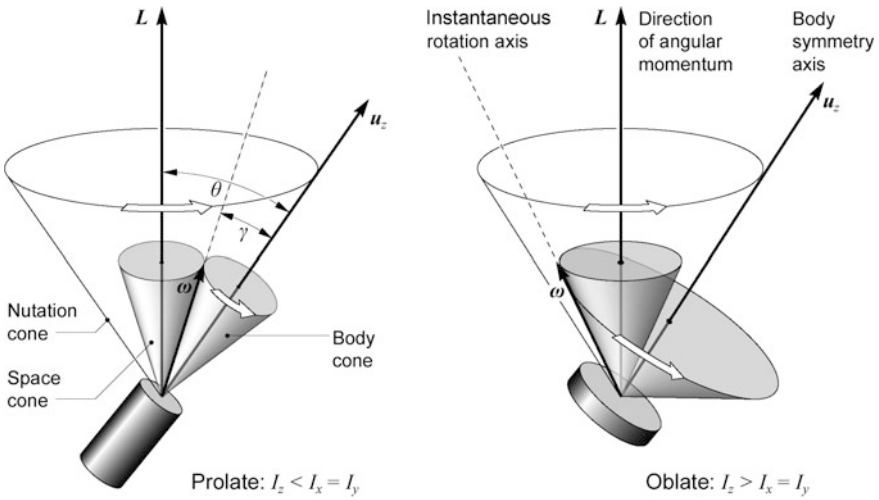


Fig. 15.7 Nutation of a free symmetrical gyro for a prolate (left) and oblate body (right) as observed in an inertial system

$$\tan\theta := \tan\angle(\boldsymbol{L},\boldsymbol{u}_z) = \frac{L_{xy}}{L_z} = \frac{I_x\omega_{xy}}{I_z\omega_z} = const \quad \textbf{nutation angle}$$
$$\tan\gamma := \tan\angle(\boldsymbol{\omega},\boldsymbol{u}_z) = \frac{\omega_{xy}}{\omega_y} = const \tag{15.2.6}$$

Hence $\tan\theta = \frac{I_x}{I_z}\tan\gamma$. So, for these two angles the following is valid:

$$\theta > \gamma \quad @\ I_x = I_y > I_z \quad (\text{elongated ``prolate'' body})$$
$$\theta < \gamma \quad @\ I_x = I_y < I_z \quad (\text{flat ``oblate'' body})$$

These properties make it possible to illustrate the nutation for a prolate and oblate symmetric gyro as follows: The figure axis of the spinning gyro turns with frequency ν_N and nutation angle θ around the angular momentum axis $\boldsymbol{L}$, which is fixed in space. The instantaneous rotation axis, which lies in between (prolate) or on the opposite side (oblate), turns with the same rate. These conical rotations can be also interpreted as if two cones, space cone and body cone, either side by side (prolate) or inside each other (oblate) mutually rolled on each other (Fig. 15.7) without slip, whereby the line of contact is just the instantaneous rotation axis.

This example shows clearly why it is so difficult to understand the rotational kinematics of a body. When we look at a rotating body our eyes are fixed on the conspicuous figure axis. From a physical point of view however, this figure axis does not play a role, because Euler's equations assign a physical meaning only to the rotation. But the location of the rotation axis in space is non-perceptible to us. Merely the angular momentum axis can be imagined fictitiously as a steady axis,

because the figure axis co-rotates with the rotation axis around it. It is this co-rotation around this fictitious axis, which makes rotational kinematics become clear to us.

Clarification of Terms

The unusual terms "prolate" and "oblate" call for some explanation. One has to distinguish between a prolate/oblate geometry of a symmetrical body (see Fig. 15.8) and a prolate / oblate rotation of such a body. A prolate (oblate) geometry means that the body is elongated (flat) with regard to its axis of symmetry. A prolate (oblate) rotation on the other hand means that a body rotates along the minor (major) principal axis. If a cigarette rolls down a slope, its prolate shape also rotates in a prolate way. If you put the cigarette flat on a table and turn it, the prolate cigarette rotates in an oblate way. On the other hand, a coin rotating in a perpendicular position on a table is an oblate body rotating in a prolate way. The friction with the table causes the rotation to slowly turn over into a flat rotation along the outer edge of the coin, i.e., an oblate rotation, until the coin comes to rest.

Confusion may arise also from the terms *nutation* and *precession*. The pitch around an axis is generally denoted as precession (see Sect. 15.3.3). But precession in physics has the special meaning of the pitch of the angular momentum around the axis of an applying external force. Though precession and nutation for a spinning top may look alike, the physics is much different. Nevertheless the term "precession" is often used for nutation in literature, which adds to the confusion of these two effects. Therefore we refrain here from using the term "precession" or "precess" for the nutational motion.

15.2.3 General Torque-Free Motion

Until now we examined only the rotational motions near principal axes. We finally want to investigate the most general case: the torque-free motion of an arbitrarily

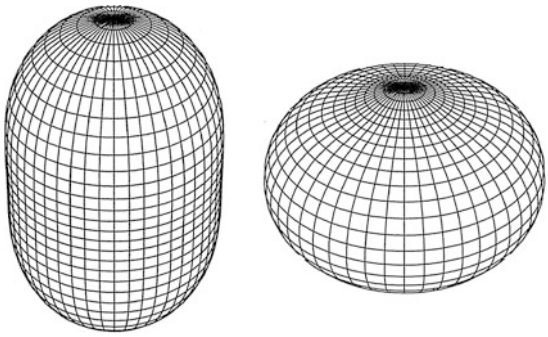


Fig. 15.8 Prolate (left) and oblate (right) geometry

shaped body. Because its mathematical description is far too complex we want to restrict the discussion to an only qualitative geometrical representation developed by Poinsot, which instead is very clear.

For our representation we adopt the body-fixed reference system of the principal axes. We have shown in Eq. (15.1.6) that the orientation of the angular momentum vector described in a body system is not constant. What can we say about the motion of $\boldsymbol{L}$ and the rotation vector $\boldsymbol{\omega}$ in this system in the most general case? Well, their motion is restricted only by to constraints:

- The first is the *conservation of angular momentum*, which is written in the principal axes system as

$$\frac{L_x^2}{L^2} + \frac{L_y^2}{L^2} + \frac{L_z^2}{L^2} = 1 \quad L\,\text{sphere} \tag{15.2.10}$$

- The second is the *conservation of energy*, which reads in the principal axes system according to Eqs. (15.1.9) with (15.1.6)

$$\frac{L_x^2}{2E_{rot}I_x} + \frac{L_y^2}{2E_{rot}I_y} + \frac{L_z^2}{2E_{rot}I_z} = 1 \quad \textbf{Poinsot ellipsoid}\,(E\,\text{ellipsoid}) \tag{15.2.11}$$

The first equation is the functional equation for the surface of a sphere with radius L, on which the tip of the angular momentum vector is free to move. The second functional equation describes a triaxial ellipsoid (Poinsot ellipsoid or E ellipsoid) with the semi-axes $\sqrt{2E_{rot}I_x}$, $\sqrt{2E_{rot}I_y}$, $\sqrt{2E_{rot}I_z}$. Because the tip of $\boldsymbol{L}$ has also to move on the surface of the ellipsoid, $\boldsymbol{L}$ can only move on the intersection line of the two surfaces, the so-called *polhode* (see Fig. 15.9a).

We are now going to describe the possible paths of $\boldsymbol{L}$ under energy dissipation. If the energy is maximal then the E ellipsoid encloses the L sphere and touches it

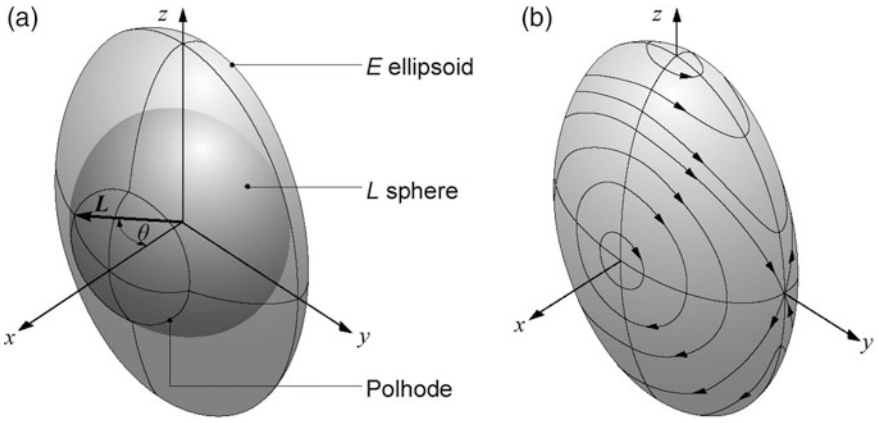


Fig. 15.9 The fixed L sphere and E ellipsoid in the principal axes system on the left (**a**). Their intersection line makes up the polhode, which is depicted for different rotational energies on the right (**b**)

tangentially in the direction of the minor principal axis—the polhode is just a point. In this case the body rotates nutation-free along the principal axis, which we will denote as the x-axis, with the smallest moment of inertia. When the rotational energy lowers, the polhode point quickly becomes a circle. Now $\boldsymbol{L}$ turns on this circle around the x-axis. (Or reciprocally, the x-axis nutates around $\boldsymbol{L}$ with nutation angle θ. This latter view was adopted in Fig. 15.7 where for an external observer $\boldsymbol{L}$ was fixed and the principal body axis nutated around it.) This situation is depicted in Fig. 15.9a and in addition in Fig. 15.9b with the direction of motion. The more the energy decreases the larger becomes the polhode (nutation circle). In addition it slowly buckles until it transits into a diagonal circle limit when $\boldsymbol{L}$ travels through the mean principal axis. In this case the E ellipsoid touches the L sphere at the middle principal axis. But because $\boldsymbol{L}$ smoothly passes through this principal y-axis, this rotation is not stable (cf. Sect. 15.2.2): the body tumbles maximally. When the energy is further reduced, $\boldsymbol{L}$ begins to nutate at large angles around the major principal axis (z-axis). The nutation angle shrinks with further decline of the energy until at minimal energy the body spins nutation-free around this major principal axis. In this limiting case the L sphere encloses the E ellipsoid and touches it tangentially at the crossing with the z body axis.

15.3 Attitude Dynamics Under External Torque

Attitude dynamics is the physics of rotational dynamics of a body under the influence of external forces that apply torques on a body. In the next section we will study the types of external forces and the torque they apply on a spacecraft. Thereafter we study the consequent attitude dynamics.

15.3.1 External Torques

We now assume that a body in an Earth orbit is subject to an external torque, which may be either due to the impact of residual atmospheric molecules, to Earth's magnetic field, or to the inhomogeneities of Earth's gravitational field, which cause gravity gradient forces, so-called tidal forces. In the following we will discuss these external forces and the resulting torques on the body qualitatively, but explain quantitatively the gravity-gradient torques in a separate Sect. 15.4.

Earth's Magnetic Field

If the surface of the spacecraft is electrically conducting, a rotation of its body in Earth's magnetic field induces closed eddy currents via magnetic induction in the conducting surface. These currents interact with Earth's magnetic field and cause Lorentz forces, which in turn generate a uniform torque, dampening the initial rotation. The calculation of this effect for the geometry of a real satellite is quite

complicated in nature and can be solved only numerically. We therefore give here only a qualitative description.

Let $\boldsymbol{B}$ be Earth's magnetic field and let us assume that the spacecraft is a conducting spherical shell of radius R and thickness $d \ll R$, which spins with vector $\boldsymbol{\omega}$ making the angle $\lambda = \angle(\boldsymbol{\omega}, \boldsymbol{B})$ with Earth's magnetic field. As early as 1880 the physicist Heinrich Rudolf Hertz showed that the torque on the spacecraft, resulting from the Lorentz force of eddy currents in its body system, is given as

$$\boldsymbol{T}_{ec} = -\frac{3}{2}\pi \frac{B^2}{\kappa} \sin^2 \lambda \cdot R^4 d \cdot \boldsymbol{\omega} \quad @\, d \ll R \tag{15.3.1}$$

where κ is the electrical resistivity of the shell and the negative sign indicates that the torque reduces the rotation. In LEO $B^2\omega \sin^2 \lambda/\kappa \approx 1\, Nm^{-4}$.

Atmospheric Drag

In the heterosphere above 150 km the atmospheric density is so small that the motion of the atmospheric molecules can be considered to be ballistic (free molecular flow regime). From Sect. 6.2.3 we derive that the force caused by the impingement of the molecules on an infinitesimal surface element dA_i is given by

$$d\boldsymbol{F}_i = \frac{1}{2}\rho v_a^2 C_D \hat{\boldsymbol{v}}_a \cdot dA_{\perp i} = \frac{1}{2}\rho v_a^2 C_D (\boldsymbol{n}_i \hat{\boldsymbol{v}}_a)\hat{\boldsymbol{v}}_a \cdot dA_i$$

where $\boldsymbol{n}_i$ is the outward normal vector on the surface element. This impingement causes two effects.

First, the impingement on dA_i at radial distance $\boldsymbol{r}_i$ from the center of gravity causes a torque on the body according to

$$\boldsymbol{T}_i = \frac{1}{2}\rho v_a^2 C_D \int_A (\boldsymbol{n}_i \hat{\boldsymbol{v}}_a)(\hat{\boldsymbol{v}}_a \times \boldsymbol{r}_i) \cdot dA_i \tag{15.3.2}$$

For a symmetric or rotating spacecraft we have on average $<\boldsymbol{T}_i>_{ave} \approx 0$ due to the anti-symmetric contributions of the integrand. If the spacecraft is not symmetric and is always aligned by means of attitude control, as for instance the International Space Station, this impingement torque can be considerable and even be oscillating because the atmospheric density at a given altitude varies between the dayside and nightside of the Earth.

Second, if the body is rotating, this rotation is dampened by the viscosity of the residual atmosphere. That is in molecular terms: upon impingement the atmospheric molecules receive—apart from elastic and diffuse scattering from the surface—an additional linear momentum, tangential to the surface, so they are dragged along with the rotating surface. The corresponding rotational drag torque is effectively described by the term

$$\boldsymbol{T}_d = -f_r \boldsymbol{\omega} \tag{15.3.3}$$

where f_r is the rotational frictional drag coefficient, the body spins with $\boldsymbol{\omega}$, and again the negative sign indicates that the torque reduces the rotation. Classically, we have $f_r = 8\pi\eta R^3$ for a sphere of radius R with η the *dynamic viscosity* (a.k.a. *absolute viscosity*). However, in the free molecular flow regime the concept of viscosity as such is no longer viable and we have to generally assume a rotational drag coefficient of the form

$$f_r \propto \rho A_\perp R \propto \rho R^3$$

Though this rotational drag torque is always existent, its quantity varies in line with the atmospheric density and thus with the rotational drag coefficient by orders of magnitude depending on the orbital altitude.

Insolation Torque

Insolation acts in the same way as atmospheric drag, in that it generates a torque

$$\boldsymbol{T}_s \propto \int_A \rho_i (\boldsymbol{n}_i \hat{\boldsymbol{s}})(\hat{\boldsymbol{s}} \times \boldsymbol{r}_i) \cdot dA_i \tag{15.3.4}$$

where $\hat{\boldsymbol{s}}$ is the unit vector of the impinging solar radiation. Because the direction of the incoming solar radiation changes in an orbital reference frame and because the torque is strongly dependent on the surface reflectivity ρ_i (see Sect. 12.5) of each surface element, the insolation torque is much harder to determine. On the other hand, it usually is much smaller than the atmospheric impingement torque so that it is often neglected.

Given the different external torques, we now want to explore how an external torque affects the attitude dynamics of a satellite. In the next two sections we will focus on eddy current torque and rotational drag torque, while in Sect. 15.4 we will go into the full rotational dynamics under the influence of the very important gravity tidal forces.

15.3.2 Road to Flat Spin

Both eddy current torque and rotational drag torque decrease the body's rotational energy. This loss of energy is called *energy dissipation*. The rotational energy of a satellite can also be dissipated by internal torques, for instance by fuel sloshing or deliberately by nutation dampers (see Fig. 15.10).

A rotating body subject to energy dissipation will first change its rotational state by maintaining its angular momentum. This process will be examined in this section. If this process is no longer possible it has to lower its angular momentum either by transferring angular momentum to the impinging atmospheric molecules

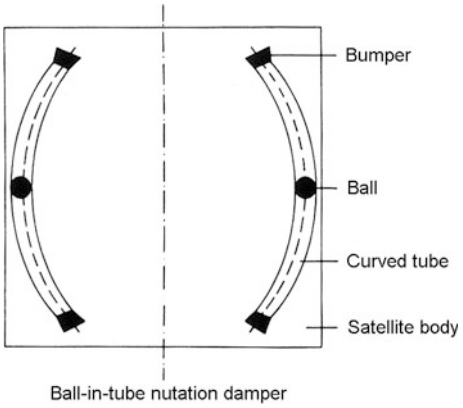


Fig. 15.10 Rotations around the axis perpendicular to the image plane are damped by this nutation damper due to the friction of the accelerated ball with the viscous fluid in the tube

via the scattering process, or to the terrestrial globe via interaction with Earth's magnetic field. This process is the subject of Sect. 15.3.3.

Energy Dissipation under Constant Angular Momentum

A rotating body can dissipate energy at constant angular momentum by changing its nutation. To see why, let us have a look at the kinetic energy of rotation around an arbitrary instantaneous rotation axis $\boldsymbol{\omega}$. According to Eq. (15.1.9) and $L_\omega = \omega \cdot I_\omega = L = const$ (see "Moment of Inertia" in Sect. 15.1.1) it amounts to

$$E_{rot,\omega} = \frac{1}{2} I_\omega \omega^2 = \frac{1}{2} \frac{L^2}{I_\omega} \tag{15.3.5}$$

As the moments of inertia depend on the choice of rotation axis, so does the corresponding rotational energy. We also recall from the "Moment of Inertia" in Sect. 15.1.1 that $I_\omega = \sum_i m_i R_{i\omega}^2$, where $R_{i\omega}$ is the distance between the body particle with mass m_i and the rotation axis. This shows that the moment of inertia is smallest and, because of Eq. (15.3.5), its energy is largest when the body rotates prolate. (Strictly speaking: when it rotates along the minor principal axis. But because most satellites are symmetrical gyros, we adopt this denomination from now on.) For a body rotating in an oblate way the corresponding quantities are of course exactly the opposite. In summary,

$$E_{\max} = \frac{1}{2} \frac{L^2}{I_{\min}} \quad @ \text{ prolate rotation}$$

$$E_{\min} = \frac{1}{2} \frac{L^2}{I_{\max}} \quad @ \text{ oblate rotation}$$

So, if a satellite loses rotational energy at $L = const$ it has to pass from a rather prolate to a more oblate nutational rotation. This implies nothing other than changing its nutation angle (see Fig. 15.9a).

Nutation Angle Rate

To be able to determine the nutation angle rate quantitatively, for a symmetrical gyro we are looking for a relation between E_{rot} and θ. We use Eqs. (15.1.7) and (15.1.9) to form the expression

$$L^2 - 2E_{rot}I_x = \omega_z^2 I_z(I_z - I_x) = L^2 \cos^2\theta \frac{I_z - I_x}{I_z} \tag{15.3.6}$$

The latter results from (15.2.6) because of

$$\cos\theta = \frac{1}{\sqrt{1 + \tan^2\theta}} = \frac{L_z}{\sqrt{L_z^2 + L_{xy}^2}} = \frac{L_z}{L} = \frac{I_z\omega_z}{L}$$

Differentiating Eq. (15.3.6) with regard to time delivers

$$\dot{\theta} = \frac{2}{L^2 \sin 2\theta} \frac{I_z I_x}{I_z - I_x} \dot{E}_{rot} \tag{15.3.7}$$

Modes of Motion

A rotating body dissipating energy at constant angular momentum can therefore behave in two different ways:

1. Initially, the body rotates in a prolate way. In this case $I_z < I_x$. Let us assume that initially there is no nutation along the minor principal axis, $\theta = 0$, so the body has maximal rotational energy. Then for even the slightest energy dissipation $\Delta E_{rot} < 0$, we get $\Delta\theta > 0$: The body starts to nutate and increases its nutation amplitude further and further.
2. Initially, the body rotates or nutates in a nearly oblate way. Here $I_z > I_x$. The rotational energy then is close to minimal at the beginning. According to (15.3.7) the body can achieve a further reduction of the rotational energy $\Delta E_{rot} < 0$ only by reducing the nutation, $\Delta\theta < 0$: Nutation is dampened. When the nutation has at some time died out this way, the satellite has reached its state of minimal energy at constant angular momentum. By spinning evenly along the major principal axis (so-called *flat spin*) there is no way to dissipate energy internally any further. Only external residual atmosphere can further reduce rotational energy by decelerating the spin by simultaneously reducing the angular momentum. But there will never be nutation again.

We therefore derive the important satellite design rule

When a satellite shall be spin-stabilized, its spin should be around its major principal axis. In this case, the spin is not only stable, but any energy dissipation such as deceleration by the remaining atmosphere dampens out any possible initial nutation: The satellite spins evenly also in the long term.

NASA unfortunately had to learn this major-principal-axis rule, when on February 1, 1958, Explorer 1 was launched into space as the first US satellite. The highly elongated (prolate) satellite was spin-stabilized in a prolate way. Because of its four long wire aerials causing energy dissipation due to atmospheric drag, the satellite unexpectedly turned to a flat spin after just a few hours. This was only noted because it interfered with the communication of the satellite.

The behavioral patterns derived here are only valid for fully rigid bodies. Dual-spin satellites, where two parts of the body are rotatively decoupled, have a partially different behavior. They can be spin-stable also along their prolate axis. As the significance of dual-spin satellites is rather small today, we do not want to go into details here.

15.3.3 Flat Spin Dynamics

Even in this full flat spin, external torques are acting on the satellite. So, what are the attitude dynamics of a flat spin under such external torques and what is its final fate?

Decay of Rotation

Let us first assume the above case where the external torque is anti-parallel to the rotation axis and hence in the flat spin case anti-parallel to the angular momentum, $\boldsymbol{T} \propto -\boldsymbol{\omega} \propto -\boldsymbol{L}$. As $d\boldsymbol{L}/dt = \boldsymbol{T}$ is valid in an inertial reference frame (see Eq. (15.1.11), we have $d\boldsymbol{L} = \boldsymbol{T} \cdot dt \propto -\boldsymbol{L}$, and hence also $d\boldsymbol{L}$ is anti-parallel to $\boldsymbol{L}$, i.e., the angular momentum and hence the angular rate is reduced. From $\boldsymbol{L} \propto \boldsymbol{\omega}$ and according to Eq. (15.1.16) we see that the change of rotational power is maximal with $P_{rot} = \omega T$.

To explore this effect quantitatively, we consider Euler's equation (15.1.13) with a frictional torque of general type $\boldsymbol{T} = -\alpha\boldsymbol{\omega}$ (see Sect. 15.3.1) acting on the system. Assuming the z-axis as the flat spin body-axis, and imposing the flat spin conditions $\omega_x = \omega_y = 0$, we get the following single differential equation in the body system

$$\dot{L} = I_z \dot{\omega}_z = -\alpha\omega_z \tag{15.3.8}$$

The mathematical solution to this equation is an exponential decay of the angular velocity $\omega_z = \omega_{z0} \exp(-t/\tau)$ with $\tau = I_z/\alpha$ the mean decay time. In physical terms this means that the satellite no longer can dissipate energy at constant angular

momentum, but has to release angular momentum to the external world via the interaction process.

If the frictional torque is an eddy current torque, then according to Eq. (15.3.1) $\alpha = 3\pi B^2 \sin^2 \lambda \cdot R^4 d/2\kappa$ and the decay time is known to be generally of the order of 100 days. For example, the rotation of the many H10 upper stages of Ariane 4 with highly prolate shape, which are still in Sun-synchronous orbits at altitudes between 500 and 800 km, decayed within 20 days if rotating prolate initially and the decay rate was more around 200 days if initially rotating oblate. So any rotating spacecraft with a conducting surface will show very low angular rates after a few years in LEO.

If the frictional torque is frictional drag torque (see Eq. (15.3.3)) that acts on the system, the rotation also decays. This time, however, the decay varies by orders of magnitude because the atmospheric density and thus the rotational drag coefficient exponentially decreases with altitude (see Sect. 6.1.2; Fig. 6.3 therein).

Precession

Quite generally, the external torque is of type $\boldsymbol{T} \times \boldsymbol{L} = I_z(\boldsymbol{T} \times \boldsymbol{\omega}) \neq 0$, i.e., it makes an angle with the angular momentum. This situation is similar to a symmetrical $\left(I_x = I_y \neq I_z\right)$ gyro rotating stably along its major or minor principal axis, while an external force $\boldsymbol{F}$ (such as gravity) impacting with a lever arm at distance $\boldsymbol{r}$ from its center of mass causes the external torque $\boldsymbol{T} = \boldsymbol{r} \times \boldsymbol{F}$ on the gyro (see Fig. 15.11).

Since in the flat spin case $\boldsymbol{T}$ and therefore $d\boldsymbol{L}$ are perpendicular to $\boldsymbol{L}$, $\boldsymbol{L}$ changes only its direction, but not its absolute value. The induced rotation of $\boldsymbol{L}$ as depicted in Fig. 15.11 is called the **precession** of the body. And since $\boldsymbol{L} \propto \boldsymbol{\omega}$ the change of rotational power is $P_{rot} = 0$, i.e. the rotational energy does not change.

Let us have a closer look at the precession of a gyro. According to Eq. (15.1.10) the external force $\boldsymbol{F}_G$ (in Fig. 15.11 it is the gravitational force) produces an external torque $\boldsymbol{T} = \boldsymbol{r} \times \boldsymbol{F}_G$ that is perpendicular to $\boldsymbol{F}_G$ and $\boldsymbol{r}$. As, because of $d\boldsymbol{L} = \boldsymbol{T} \cdot dt$, the change of the torque is always in the direction of the plane perpendicular to $\boldsymbol{F}_G$, $\boldsymbol{L}$ will also always move in circles in this plane. This is the precession motion.

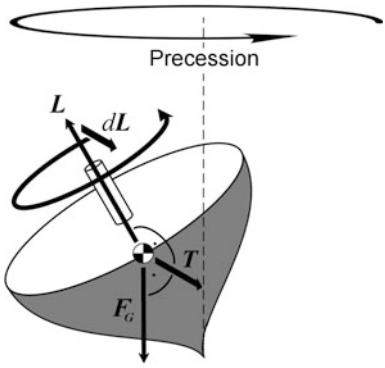


Fig. 15.11 Precession of a gyro under the impact of an external force (here gravitational force F_G)

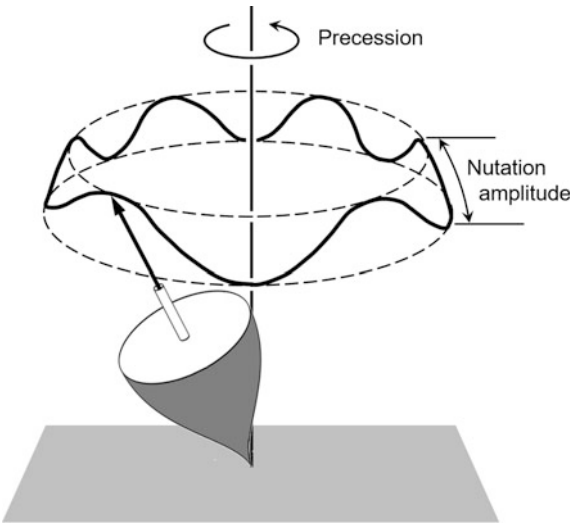


Fig. 15.12 Superimposed nutation and precession of a symmetrical gyro

If the gyro is not in a fully flat rotation but exhibits a small nutation, then nutation and precession are superimposed and its figure axis produces a motion, which may look like to the one depicted in Fig. 15.12.

Let us take a concrete example. The Earth's polar axis nutates around its angular moment axis with a nutation amplitude of 19 arcseconds and a period of

$$T_N = \frac{2\pi}{\omega_N} = 18.6\,\text{years}$$

Because of their gravitational impact on the equatorial bulge, which is inclined by 23.5° to the ecliptic, Sun and Moon create a cross torque on Earth's rotation axis that induces an additional precession with the period

$$T_P = \frac{2\pi}{\omega_P} = 25,800\,\text{years}$$

where the precession angle of course is the angle between the plane in which the torque applies and the polar axis and hence is $\phi = 23.5°$. The polar axis under these circumstances describes a compound motion much close to that in Fig. 15.12.

15.4 Gravity-Gradient Stabilization

In low Earth orbit the gravitation gradient of the Earth may have a significant impact on the spatial orientation of the satellite and its rotational behavior, which may be either undesirable, or which can be harnessed skillfully as we will see in this

section. If the satellite fairly elongates in the radial direction to the Earth, those satellite parts that are further away from the center of mass, but nevertheless move with the same orbital frequency of the satellite as a whole, are subject to the additional centrifugal force, and additionally to a smaller gravitational force. Those parts of the satellite that are further toward Earth by the same distance from the center of mass experience just the opposite effect: a smaller centrifugal force plus a larger gravitational force. All in all, this results in a gravity-gradient (*GG*) torque, which tends to align the satellite along the position vector (i.e., in the direction of the geocenter). The differential forces within an aligned body are called *tidal forces*. The question here is: What are the effects of this GG torque on the attitude and the rotational behavior of the satellite?

To answer this question, first of all we have to calculate the torque of the gravity gradient on the S/C. This torque will then be applied to Euler's equation (15.1.13) to study the dynamic behavior and the dynamic stability of the S/C.

15.4.1 Gravity-Gradient Torque

According to Fig. 15.13 any infinitesimal component dm of an orbiting body having orbital frequency $n_\nabla = \sqrt{\mu/R^3}$ located at distance $\boldsymbol{r}$ from the center of mass (CM) experiences a centrifugal acceleration, $\ddot{\boldsymbol{r}}_\omega$, and gravitational acceleration, $\ddot{\boldsymbol{r}}_G$, and thus causes a small torque on the component according to

$$d\boldsymbol{T} = \boldsymbol{r} \times (\ddot{\boldsymbol{r}}_G + \ddot{\boldsymbol{r}}_\omega) \cdot dm$$

with Eq. (7.2.17) we have

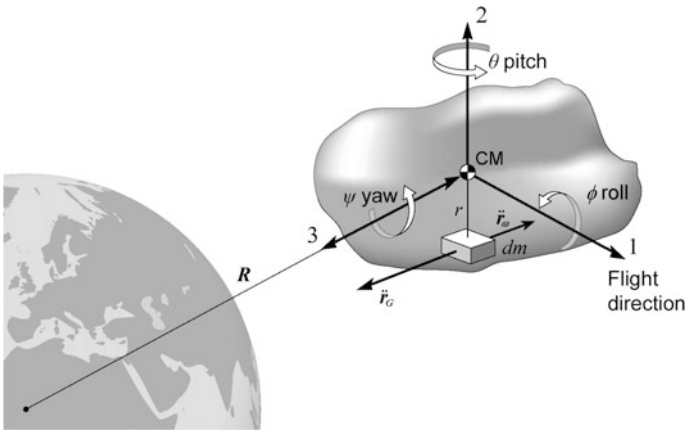


Fig. 15.13 Rotational motion of an arbitrary body in the *RSW* system (cf. Figs. 15.3 and 15.4) where CM is its center of mass

$$\boldsymbol{r} \times \ddot{\boldsymbol{r}}_{\omega} = \boldsymbol{r} \times \left[n_{\nabla}^2 (\boldsymbol{R} + \boldsymbol{r}) - n_{\nabla}^2 \boldsymbol{R} \right] = \boldsymbol{r} \times n_{\nabla}^2 \boldsymbol{r} = 0$$

and

$$\boldsymbol{r} \times \ddot{\boldsymbol{r}}_G = \boldsymbol{r} \times \left[-\frac{\mu}{|\boldsymbol{R} + \boldsymbol{r}|^3} (\boldsymbol{R} + \boldsymbol{r}) + \frac{\mu}{R^3} \boldsymbol{R} \right] = \boldsymbol{r} \times \left[-\frac{\mu}{|\boldsymbol{R} + \boldsymbol{r}|^3} \boldsymbol{R} + \frac{\mu}{R^3} \boldsymbol{R} \right]$$

where the latter follows because $\boldsymbol{r} \times \boldsymbol{r} = 0$. Therefore

$$d\boldsymbol{T} = \frac{\mu \cdot dm}{R^3} \left[1 - \frac{R^3}{|\boldsymbol{R} + \boldsymbol{r}|^3} \right] (\boldsymbol{r} \times \boldsymbol{R})$$

As $R \gg r$ holds very well, the expression in the square brackets can be approximated with very high accuracy as

$$\begin{aligned} 1 - \frac{R^3}{|\boldsymbol{R} + \boldsymbol{r}|^3} &= 1 - \frac{R^3}{(R^2 + 2\boldsymbol{R} \cdot \boldsymbol{r} + r^2)^{3/2}} \approx 1 - \frac{1}{(1 + 2\boldsymbol{R} \cdot \boldsymbol{r}/R^2)^{3/2}} \\ &\approx 1 - \left(1 - 3\frac{\boldsymbol{R} \cdot \boldsymbol{r}}{R^2} \right) = 3\frac{\boldsymbol{R} \cdot \boldsymbol{r}}{R^2} \end{aligned}$$

Thus, the gravity-gradient torque on the entire body is

$$\boldsymbol{T}_{GG} = \int d\boldsymbol{T} = \frac{\mu}{R^3} \int 3\frac{\boldsymbol{R} \cdot \boldsymbol{r}}{R^2} (\boldsymbol{r} \times \boldsymbol{R}) \cdot dm$$

Hence

$$\boldsymbol{T}_{GG} = -\frac{3\mu}{R^5} \boldsymbol{R} \times \int (\boldsymbol{R} \cdot \boldsymbol{r}) \boldsymbol{r} \cdot dm$$

Because of the vector relation $\boldsymbol{r} \times (\boldsymbol{R} \times \boldsymbol{r}) = (\boldsymbol{r} \cdot \boldsymbol{r})\boldsymbol{R} - (\boldsymbol{R} \cdot \boldsymbol{r})\boldsymbol{r}$ we get

$$\boldsymbol{T}_{GG} = \frac{3\mu}{R^5} \boldsymbol{R} \times \int [\boldsymbol{r} \times (\boldsymbol{R} \times \boldsymbol{r}) - (\boldsymbol{r} \cdot \boldsymbol{r})\boldsymbol{R}] \cdot dm$$

As $\boldsymbol{R} \times \boldsymbol{R} = 0$, the second term of the integral is zero, and we get

$$\boldsymbol{T}_{GG} = \frac{3\mu}{R^5} \boldsymbol{R} \times \int \boldsymbol{r} \times (\boldsymbol{R} \times \boldsymbol{r}) \cdot dm$$

According to Eqs. (15.1.2) and (15.1.3a) we have

$$\int \boldsymbol{r} \times (\boldsymbol{R} \times \boldsymbol{r}) \cdot dm = \boldsymbol{I}\boldsymbol{R}$$

with $\boldsymbol{I}$ as the inertia tensor. So for the gravity-gradient torque, we finally get the simple expression

$$\boldsymbol{T}_{GG} = \frac{3\mu}{R^5}\boldsymbol{R} \times \boldsymbol{I}\boldsymbol{R} = 3n_{\triangledown}^2 \boldsymbol{u}_R \times \boldsymbol{I}\boldsymbol{u}_R \tag{15.4.1}$$

with $\boldsymbol{u}_R$ as the unit vector from the geocenter to the center of mass of the S/C and the mean motion $n_{\triangledown} = \sqrt{\mu/R^3}$ of the satellite.

15.4.2 Gravity-Gradient Dynamics

Now that we know the torque on the S/C, we want to know if there are stable attitudes of the S/C under this influence. To figure this out we employ the principal axes body system x, y, z in which the inertia tensor is diagonal, $\boldsymbol{I}_P = diag\left(I_x, I_y, I_z\right)$. In this reference system with the body at rest and oriented toward the Earth $(x, y, z = 1, 2, 3)$ the vector from the geocenter to the body center of mass is given as $\boldsymbol{R} = \left(R_x, R_y, R_z\right)_P^T$. With this Eq. (15.4.1) results in

$$\boldsymbol{T}_{GG} = \frac{3n_{\triangledown}^2}{R^2}\begin{pmatrix} R_x \\ R_y \\ R_z \end{pmatrix}_P \times \begin{pmatrix} I_x R_x \\ I_y R_y \\ I_z R_z \end{pmatrix}_P = \frac{3n_{\triangledown}^2}{R^2}\begin{bmatrix} R_y R_z\left(I_z - I_y\right) \\ R_x R_z\left(I_x - I_z\right) \\ R_x R_y\left(I_y - I_x\right) \end{bmatrix}_P \tag{15.4.2}$$

From this we recognize

1. The S/C is not subject to a torque around the principal i axis if the S/C is rotationally symmetrical around this axis, i.e., if $I_j = I_k$. This is reasonable because, as the axis is rotationally symmetrical, there is no imbalance on which the force can act to turn the body.
2. The S/C is not subject to a torque around any principal axis, if
   - the S/C is fully symmetrical, i.e., $I_x = I_y = I_z$, or
   - one of the principal axes points exactly to nadir, i.e., $\boldsymbol{R} = R(\pm 1, 0, 0)_P^T$ or $\boldsymbol{R} = R(0, \pm 1, 0)_P^T$ or $\boldsymbol{R} = R(0, 0, \pm 1)_P^T$.

Usually both these conditions are not fulfilled and therefore a GG torque $\boldsymbol{T}_{GG}$ occurs, leading inevitably to a turn and hence to a rotational dynamics of the S/C. There are a number of questions to be asked: What do these rotations look like? Are there any stable oscillations around certain axes, and, if this is the case, under what conditions? Or are the rotations all unstable? "Stability of an oscillation" expressed in simplified terms means that, with a small perturbation (external torque), the body carries out stable harmonic oscillations around this axis. If the oscillation is unstable, the state of motion exponentially degrades and passes to a stable oscillation around another axis, or even turns into a chaotic state of rotation between all the axes.

Let us investigate whether stable oscillations exist, and which ones are stable.

Equations of Motion

We now permit small roll (ϕ), pitch (θ), and yaw (ψ) rotations as perturbations. With these the unit vector in R direction in the principal axes system is, according to Fig. 15.5,

$$\begin{aligned}\boldsymbol{u}_{R,P} &= -\sin\theta\cdot\boldsymbol{u}_1 + \sin\phi\,\cos\theta\cdot\boldsymbol{u}_2 + \cos\phi\,\cos\theta\cdot\boldsymbol{u}_3\\ &= (-\sin\theta, \sin\phi\,\cos\theta, \cos\phi\,\cos\theta)_{LVLH}\end{aligned}$$

Inserting this into Eq. (15.4.1) results in

$$\boldsymbol{T}_{GG,P} = \frac{3}{2}n_\nabla^2\begin{bmatrix}(I_z-I_y)\cos^2\theta\,\sin 2\phi\\ -(I_x-I_z)\cos\phi\,\sin 2\theta\\ -(I_y-I_x)\sin\phi\,\sin 2\theta\end{bmatrix}_{LVLH} \approx 3n_\nabla^2\begin{bmatrix}(I_z-I_y)\phi\\ -(I_x-I_z)\theta\\ 0\end{bmatrix}_{LVLH} \tag{15.4.3}$$

where the latter holds for $\phi,\theta,\psi \to 0$. If we now apply Eq. (15.4.3) to Euler's equations (15.1.13), and consider from Eqs. (15.1.14) and (15.1.15) the Euler rate equations

$$\begin{aligned}\boldsymbol{\omega}_P &= \left(\dot\phi + n_\nabla\psi, \dot\theta + n_\nabla, \dot\psi - n_\nabla\phi\right)_{LVLH}\\ \dot{\boldsymbol{\omega}}_P &= \left(\ddot\phi + n_\nabla\dot\psi, \ddot\theta, \ddot\psi - n_\nabla\dot\phi\right)_{LVLH}\end{aligned}$$

one gets the following equations of motion for rotations in the *LVLH* system

$$\ddot\phi + n_\nabla(1-k_x)\dot\psi + 4n_\nabla^2 k_x\phi = 0 \tag{15.4.4a}$$

$$\ddot\psi + n_\nabla(k_z-1)\dot\phi + n_\nabla^2 k_z\psi = 0 \tag{15.4.4b}$$

$$\ddot\theta + 3n_\nabla^2 k_y\theta = 0 \tag{15.4.4c}$$

$$\text{with}\quad k_x := \frac{I_y-I_z}{I_x},\ k_y := \frac{I_x-I_z}{I_y},\ k_z := \frac{I_y-I_x}{I_z} \tag{15.4.5}$$

The first two equations describe a coupled roll-yaw motion (cf. Fig. 15.4). The last equation is decoupled and describes a pitch oscillation.

Pitch Oscillation

According to Eq. (15.4.4a) the S/C oscillates like a harmonic oscillator with frequency

$$\omega_2 = n_\nabla\sqrt{\frac{3(I_x-I_z)}{I_y}} \tag{15.4.6}$$

around the 2-axis, i.e., a pitch oscillation in the plane, which is set up by the flight direction and nadir. This frequency is real, and thus the oscillation is stable, if the root is real, i.e., if

$$\boxed{I_z < I_x} \quad \textbf{stability condition for pitch oscillation} \tag{15.4.7}$$

So we have a satellite that circles the planet with its z-axis always swinging around the nadir direction (see Fig. 15.14). This remarkable effect is called *gravity-gradient stability*. We get the ideal state of an S/C circling the Earth by constantly pointing toward the geocenter as depicted in Fig. 15.15, if we dampen the oscillation (see Fig. 15.16) so that it disappears. This stable state arouses a lot of practical interest, as it is possible to achieve a constant nadir orientation very easily without any control systems. Such a constant orientation can be used to align, for instance, a communication antenna or an Earth sensor always to the Earth's nadir point.

This situation in which a circling body rotates about its own axis synchronously with its orbit period—a so-called a 1:1 orbital resonance—is by the way absolutely identical to the one of the Moon circling Earth. Due to the gradient force of the Earth it is a bit elongated along the connecting line Earth–Moon. This prolate shape is aligned to Earth by the gravity-gradient force leading to a libration coupled 1:1 with its revolution around Earth.

Coupled Roll-Yaw Oscillation

We are now looking for the stability criteria for rotations about the other two axes. If the S/C is rotationally symmetric, $I_x = I_y \neq I_z$, then according to Eq. (15.4.2) the body experiences no torque around the z-axis, and the GG torque does not contribute to the stability along this axis: the satellite will freely rotate around its z-axis. For the case of a nonsymmetrical S/C, $I_x \neq I_y \neq I_z$, we are looking for solutions for the first two coupled linear differential Eqs. (15.4.3), which are known to be solved by the general ansatz

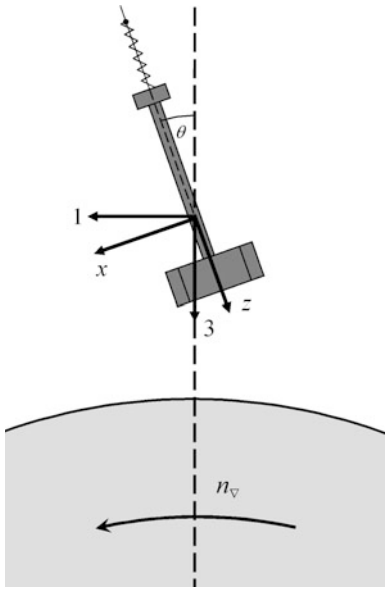


Fig. 15.14 Pitch oscillation of a body in gravity-gradient mode

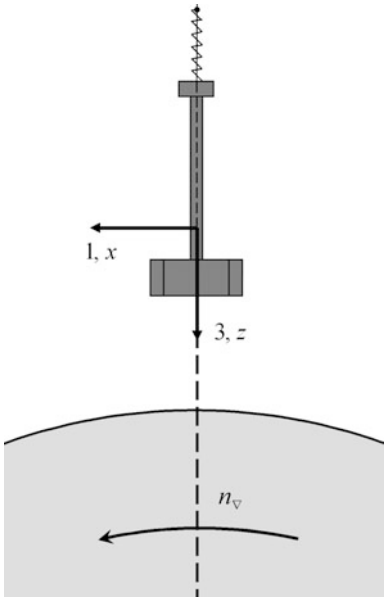


Fig. 15.15 Stable gravity-gradient orientation of a rotationally symmetrical prolate body

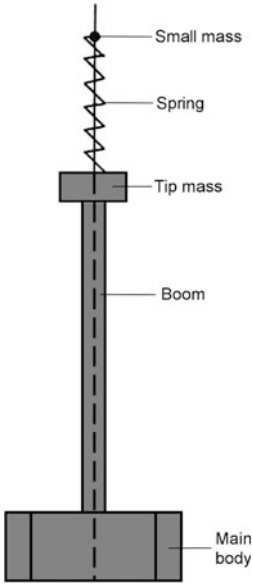


Fig. 15.16 Spring libration damper for a gravity-gradient stabilized body

$$\psi = \psi_0 e^{sn_\nabla t}$$
$$\phi = \phi_0 e^{sn_\nabla t}$$

We insert this ansatz into Eqs. (15.4.4a) and (15.4.4b) and thus get the characteristic quartic equation

$$s^4 + s^2(1 + 3k_z + k_x k_z) + 4k_x k_z = 0$$

The solutions are

$$s_{1,2}^2 = -a^2 \pm \sqrt{a^2 - 4k_x k_z}$$

with $$a = \tfrac{1}{2}(1 + 3k_z + k_x k_z).$$

To get periodic oscillations, all the solutions have to be imaginary. This is fulfilled if the root above is real, and $s_{1,2}^2 < 0$. These two conditions imply

$$\begin{aligned} 1 + 3k_z + k_x k_z &> 4\sqrt{k_x k_z} \\ k_x k_z &> 0 \end{aligned} \tag{15.4.8}$$

Because, as can easily be shown, $k_y = (k_z - k_x)/(1 - k_x k_z)$ and $1 - k_x k_z > 0$, the above pitch stability condition $I_z < I_x$, which is equivalent to $k_y > 0$, transforms into

$$k_x < k_z \tag{15.4.9}$$

The three stability conditions from Eqs. (15.4.8) and (15.4.9) are fulfilled simultaneously only in the two white regions I and II of Fig. 15.17.

In reality, region II, called *DeBra-Delp region*, is not stable. One can see that from the stability conditions of region II: $I_z \geq I_y$, $I_x \geq I_y$. Because of Eq. (15.3.5)

$$E_{rot} = \frac{1}{2} I_\omega \omega^2 = \frac{1}{2} \frac{L^2}{I_\omega}$$

this pitch oscillation around the axis with minimal moment of inertia is a state of maximum oscillation energy. Even the smallest dissipation of energy (see Sect. 15.2.3), e.g., by residual atmosphere, transfers to a state of lowest energy, i.e., to the axis with the largest moment of inertia, and thus to region I. Therefore, for coupled roll-yaw oscillations of real dampened systems only region I, called *Lagrange region*, is stable. This area is characterized by the condition

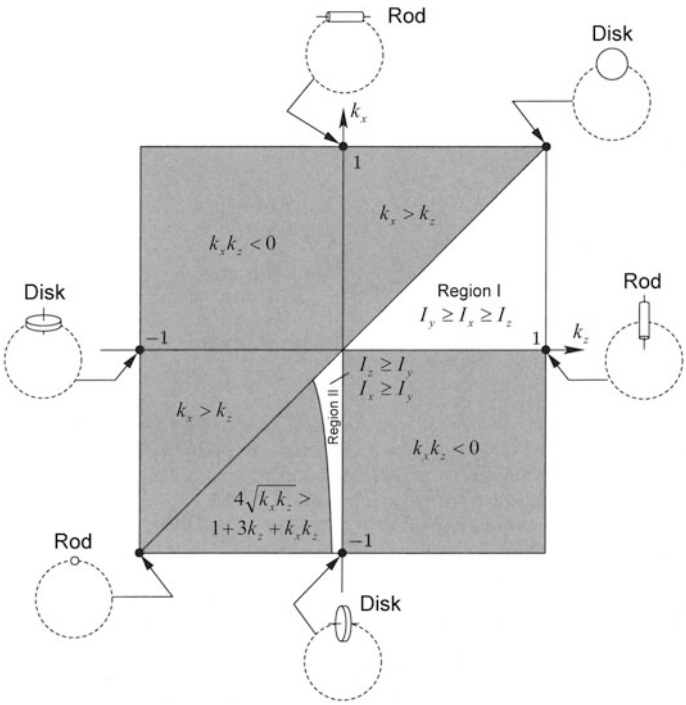


Fig. 15.17 The two regions of stability of a roll-yaw oscillation in gravity-gradient mode. *Credit* Steiner (2004)

$$\boxed{I_z \leq I_x \leq I_y} \quad \textbf{stability condition for roll - yaw oscillations} \tag{15.4.10}$$

If the coupled roll-yaw oscillation is stable, it oscillates according to our ansatz $\psi = \psi_0 e^{sn_\nabla t}, \phi = \phi_0 e^{sn_\nabla t}$ with the angular frequency $|s| n_\nabla \leq \sqrt{6} \cdot n_\nabla$, that is of order orbital frequency, and hence very slowly.

Chapter 16
Thermal Radiation Physics and Modeling

Symbols and Terminology

Subscripts

a	Absorber (absorbed)
alb	Albedo
B	Body under consideration
E	Earth
eff	Effective (temperature)
ext	External (to satellite)
int	Internal (to satellite)
IR	Infrared
r	Receiver (received)
S	Solar
sat	Satellite
sol	Solar
t	Transmitter (transmitted)
λ	Spectral
Ω	Directional
$\bullet$	Black body
$\perp$	(Surface) projection onto a given direction

Latin Symbols

a	Albedo
A	Size of surface area
E	Irradiance
$F_{i\to j}$	View factor (a.k.a. configuration factor) from surface i to surface j
G_{ij}	Heat conduction or heat convection coefficient (a.k.a. conductor) between node i and j
h	Planck constant

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I_Ω	Radiant intensity (directional)
k_B	Boltzmann constant
L_Ω	Radiance (directional)
$L_{\lambda\Omega}$	Spectral radiance (directional)
M	Radiant exitance
M_λ	Spectral exitance
Q	Radiant energy
R	Radiative conduction coefficient
s	Distance between emitting and absorbing surface points
S/C	Spacecraft

Greek Symbols

α	Absorptivity (*a.k.a.* absorption coefficient)
ε	Emissivity (*a.k.a.* emission coefficient, emittance)
θ	Polar angle (see Fig. 16.1)
λ	Wavelength
Φ	Radiant flux (*a.k.a.* radiative heat flux)
σ	Stefan-Boltzmann constant
Ω	Solid angle

Abbreviations

BOL	Beginning of life
CFRP	Carbon fiber reinforced plastics
ECSS	European cooperation for space standardization
EOL	End of life
FDM	Finite differences model
FEM	Finite element model
GMM	Geometrical mathematical model
IR	Infrared
ISS	International space station
MLI	Multi-layer insulation
OSR	Optical surface reflector
PCB	Printed circuit board
RHU	Radioisotope heater units
RTG	Radioisotope thermoelectric generator
S/C	Spacecraft
SSM	Second surface mirror
TCS	Thermal control system
TMM	Thermal mathematical model
TRP	Temperature reference point

Glossary for Thermal Modeling

Terms (TM = Thermal Model)

TM-specific term (here: CAPITAL LETTERS)	TM symbol	Conventional term (here: regular letters)	Conventional symbol
CONDUCTANCE		Coupling coefficient	
Linear CONDUCTOR	G_L	Coupling coefficient	G_{ij}
Fluid CONDUCTOR	G_F	Coupling coefficient	G_{ij}
Radiative CONDUCTOR	G_R	Coupling coefficient	R_{ij}
CONTACT CONDUCTANCE		Coupling coefficient	
CAPACITANCE or thermal capacity	C	Heat capacity	C
Heat flux	Q	Heat flux	Φ

The ultimate goal of this chapter is to determine the temperature equilibrium distribution inside a spacecraft (S/C) as a result of the thermal equilibrium with its space environment. Knowing these inside temperatures is essential when designing a S/C, as most components only work reliably within certain temperature ranges: batteries loose capacity and propellants may freeze.

The heat exchange and thermal balance problem of the S/C can be split into a S/C internal and external one, which must be also in equilibrium with each another. An internal thermal balance analysis covers the heat exchanged between the passive parts and surfaces within the S/C, as well as internal heat sources like heaters, batteries etc. This internal heat balance is complemented by the external one as the S/C interacts with external heat sources (Sun, planets, etc.) and sinks (deep space). The boundary between those two "cavities" is the outer hull of the S/C.

In Sect. 16.1 we start out with the radiometric concepts of radiation and the physics of some idealized radiation surfaces. Then we look in Sect. 16.2 at the radiation exchange between surfaces and how to possibly affect it, before we finally come in Sect. 16.3 to the thermal modeling and simulation as the basis for thermal spacecraft design.

16.1 Radiation Properties

Let us first start with the external heat balance. It is driven by the radiative heat exchange of the outer S/C surfaces with the environment. On the one hand these surfaces can absorb heat in different wavelengths, while on the other they emit heat, usually in the infra-red (IR) range. There are three key sources for the heat flux onto a S/C that is in an orbit in proximity to a planet: direct sunlight, sunlight reflected by the planet (albedo), and infrared radiation (IR) from the planet. As the density of Earth's atmosphere in space is extremely low, there is almost no heat transfer by

heat convection. The only transfer mechanism between the S/C and the environment is thermal radiation.

For the sake of analytical simplicity let us assume for now that the S/C is isothermal, i.e. that it has a uniform surface temperature and that the S/C is in thermal equilibrium, so there is more or less the same temperature everywhere within the S/C. If this is not the case, our calculations are valid for the thermally averaged S/C. In this way let us have a look at the thermal radiation heat transfer.

16.1.1 Radiometric Concepts

When radiation is transferred, electromagnetic waves (photons), each with energy $E_v = hv$, where h is Planck's constant and ν its frequency, are emitted and transmitted by a radiator (subscript t) to an absorbing receiver (subscript r). These indices are consistent with those in radio and microwave technology. In the following, rather than photometric notation we will use radiometric notation and nomenclature, which is more common in thermal radiation analysis.

Photon Flux
The total number N of photons that are emitted, reflected, or received by a surface is the so-called *radiant energy*

$$Q = N \cdot E_v \ [\mathrm{J}] \quad \textbf{radiant energy} \tag{16.1.1}$$

measured in joule $[\mathrm{J}] = [\mathrm{W} \cdot \mathrm{s}]$. The radiant energy per time unit dt is called the *radiant flux* (a.k.a. *radiative heat flux*)

$$\Phi := \frac{dQ}{dt} \ [\mathrm{W}] \quad \textbf{radiant flux} \tag{16.1.2}$$

The radiant flux is the only physically relevant quantity for thermal radiation, because it describes the total flow of photons per time unit. So, it is the photon flux in question and hence is the starting point for all further considerations.

> **Note** *The radiometric term flux used here is not consistent with flux in the physics of transport phenomena (heat transfer, mass transfer, and fluid dynamics), where flux is defined as the rate of flow of a quantity per unit area.*

Photons may have different wavelengths λ and may flow in different directions $\Omega = (\theta, \varphi)$ (see Fig. 16.1). Therefore, we differentiate the photon flux into a flux density per $d\lambda$, $d\Omega$, and $dA_{t,\perp}$ of an emitting, receiving, or reflecting, surface

$$d^3\Phi = L_{\lambda\Omega} \cdot d\lambda \, d\Omega \, dA_{\perp} \tag{16.1.3}$$

where the flux density $L_{\lambda\Omega}$ is the so-called *spectral radiance* (a.k.a. *spectral luminance* (in photometry) or *spectral (radiation) intensity*)

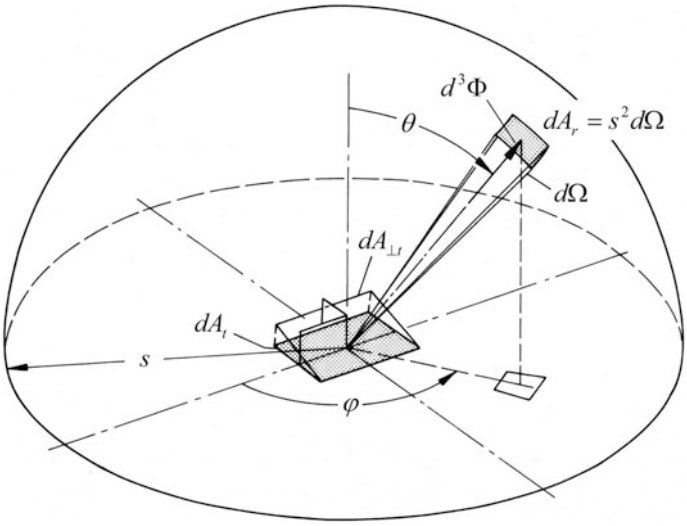


Fig. 16.1 Unit surface area dA_t transmitting radiant flux $d^3\Phi$ into direction $\Omega = (\theta, \varphi)$ (θ = polar angle) with aperture angle $d\Omega$, and an infinitesimal receiving surface $dA_r = s^2 d\Omega$ at distance s

$$L_{\lambda\Omega}(\lambda, \Omega) := \frac{d^3\Phi}{d\lambda\, d\Omega\, dA_\perp} \quad \left[\frac{W}{m^2\, sr\, \mu m}\right] \quad \textbf{spectral radiance} \qquad (16.1.4)$$

The subscripts λ and Ω indicate the corresponding density. It is crucial to this definition that the radiance is defined per dA projected onto the transmitting or receiving direction, i.e. per

$$dA_\perp = dA \cdot \cos\theta$$

where θ is the polar angle of the radiation's direction (see Fig. 16.1).

In summary, The basic physical quantity $d^3\Phi$ describes a photon flux having wavelength λ, flowing in direction $\Omega = (\theta, \varphi)$, and being related to any given projected surface $dA_\perp$ that emits, reflects, or receives the radiation.

Terminology and Notations

- When sloppily saying "the radiation", we actually mean, here and in the following, the important quantity $d^3\Phi$, the differential photon flux.
- All photon-flux-derived quantities (above and in the following) are defined relative to a differential (emitting, receiving, or reflecting) surface area dA (as part of a possibly total radiating body). Hence, dA can radiate only into the hemisphere above it, indicated by the symbol $\cap$.

- Quantities that are derived by integration over all directions hence will sometimes be termed "hemispherical" and those derived by integration over all wavelengths as "total".
- Spectral (λ) and/or directional (Ω) densities, such as $L_{\lambda\Omega}(\lambda, \Omega)$, generally have according spectral and directional dependencies. We therefore will furthermore drop the dependency specification (λ, Ω) and just write $L_{\lambda\Omega}$. In the case when a relevant dependency does not apply, we cross it out, e.g. $L_{\Omega}(\not{\Omega})$ for a directionally independent radiance.
- We sometimes will make use of the more compact notation of integrals

$$[f(\lambda)]_{\lambda} := \int_0^{\infty} f(\lambda) \cdot d\lambda, \quad [f(\theta, \varphi)]_{\Omega} := \int_{\cap} f(\theta, \varphi) \cdot d\Omega$$

Surface-Specific Radiation Densities

Obviously, the transmitted radiation $d^3\Phi_t$ must be proportional to and dependent on the emitting surface area of the radiator. Therefore, radiometry defines the so-called *radiant exitance* (a.k.a, *emissive power*) M as

$$M := \frac{d\Phi_t}{dA_t} \left[\frac{W}{m^2}\right] \quad \textbf{radiant exitance} \tag{16.1.5}$$

Note that in contrast to radiance, exitance is defined per actual emitting unit surface area dA_t. Because by definition M applies only for transmitting (emitting) surfaces, its subscript t is always dropped. Most generally, the so-called *spectral and directional radiant exitance* is defined as

$$M_{\lambda\Omega} := \frac{d\Phi_t}{d\lambda\, d\Omega_t\, dA_t} \left[\frac{W}{m^2\, sr\, \mu m}\right] \tag{16.1.6}$$

By comparison with Eq. (16.1.4) and because $dA_{\perp} = dA \cdot \cos\theta$ we find

$$M_{\lambda\Omega} = L_{\lambda\Omega,t} \cos\theta_t \tag{16.1.7}$$

Another prominent quantity is the *spectral (radiant) exitance*

$$M_{\lambda} := \frac{dM}{d\lambda} \left[\frac{W}{m^2 \mu m}\right] \quad \textbf{spectral exitance} \tag{16.1.8}$$

By the same token we define the quantity *irradiance*, which is the radiant flux incident on a receiving unit area

$$E := \frac{d\Phi_r}{dA_r} \quad \textbf{irradiance} \tag{16.1.9}$$

By definition, E applies only for receiving surfaces, which is why its subscript r is also dropped regularly. By the same token as above we have

$$E_{\lambda\Omega}(\lambda, \Omega) = L_{\lambda\Omega,r} \cos\theta_r \tag{16.1.10}$$

Total Directional Radiation

For thermal radiation exchange between two surfaces, to which we come later, only the total and not the spectral exchanged photon flux is of interest. This results in directional flux densities, which are integral in λ and are therefore sometimes called "total".

So, for an emitting, reflecting, or receiving surface element $dA_\perp$ and in accordance with Eq. (16.1.4) we have the so-called *radiance* (a.k.a. *luminance* (in photometry) or *radiation intensity*)

$$L_\Omega := \frac{d^2\Phi}{d\Omega \cdot dA_\perp} \left[\frac{W}{m^2 sr}\right] \quad \textbf{radiance} \tag{16.1.11}$$

For a given emitting surface this directional photon flux density in total defines the fraction of hemisphere above the surface that receives the incoming irradiation (cf. Fig. 16.1). This is the significance of the quantity *radiation*.

Another directional quantity is the so-called *radiant intensity* (a.k.a. *total intensity*) defined as the radiant flux per solid angle Ω

$$I_\Omega(\Omega) := \frac{d\Phi}{d\Omega} \left[\frac{W}{sr}\right] \quad \textbf{radiant intensity} \tag{16.1.12}$$

In conclusion, we find from Eqs. (16.1.5), (16.1.12), and (16.1.11) and with $dA_\perp = dA \cdot \cos\theta$

$$\frac{dM}{d\Omega_t} = \frac{dI_{\Omega,t}}{dA_t} = L_{\Omega,t} \cos\theta_t \quad @ \text{ transmitter} \tag{16.1.13}$$

$$\frac{dE}{d\Omega_r} = \frac{dI_{\Omega,r}}{dA_r} = L_{\Omega,r} \cos\theta_r \quad @ \text{ receiver} \tag{16.1.14}$$

16.1.2 Diffuse Radiators

Although all real radiators exhibit wavelength-dependent emission and absorption, there are many radiators, particular those with very rough surfaces, showing nearly direction-independent (isotropic) emission and absorption. Therefore, assuming isotropic radiator properties is often a good approximation. This is widely utilized

and will be essential for radiation exchange in Sect. 16.2.1. Isotropic radiation is also called *diffuse radiation* and a diffusely emitting surface accordingly is called a *diffuse radiator, diffuse surface,* or *Lambertian surface*.

What are the radiation properties of a diffuse surface? According to Eq. (16.1.7) for every radiator $M_{\lambda\Omega} = L_{\lambda\Omega,t} \cos\theta_t$ holds. If a surface radiates isotropically (diffusively), then $L_{\lambda\Omega,t}(\not{\Omega})$, meaning $L_{\lambda\Omega,t}$ is independent of direction Ω i.e., it has the same radiance when viewed from any angle. We thus can write

$$M_{\lambda\Omega} = L_{\lambda\Omega,t}(\not{\Omega}) \cos\theta_t \quad @ \text{ diffuse} \quad \textbf{Lambert's cosine law} \tag{16.1.15}$$

This is the so-called *Lambert's cosine law*. We recall that the $\cos\theta$ dependence is merely of geometrical origin, namely due to the fact that $dA_\perp = dA \cdot \cos\theta$.

We now integrate Lambert's cosine law over a hemisphere and make use of the isotropy:

$$M_\lambda = \int_{\cap} M_{\lambda\Omega} \cdot d\Omega = \int_{\cap} L_{\lambda\Omega}(\not{\Omega}) \cos\theta \cdot d\Omega = L_{\lambda\Omega}(\not{\Omega}) \int_{\cap} \cos\theta \cdot d\Omega$$

Because $d\Omega = \sin\theta \cdot d\theta \cdot d\varphi$, the last integration delivers

$$\int_{\cap} \cos\theta \cdot d\Omega = \int_0^{\pi/2} \int_0^{2\pi} d\varphi \cdot \cos\theta \sin\theta \cdot d\theta = 2\pi \left(\frac{1}{2} \sin^2\theta \right) \Big|_0^{\pi/2} = \pi$$

So, the integral does not result in the solid angle of a hemisphere, 2π, but only in π, because the effective radiation surface $dA_\perp = dA \cdot \cos\theta$ decreases with increasing polar angle, thus diminishing the exitance towards the azimuthal plane. We therefore obtain for a diffuse radiator

$$M_\lambda = \pi \cdot L_{\lambda\Omega}(\not{\Omega}) \quad @ \text{ diffuse} \tag{16.1.16a}$$

and by integration over all wavelengths finally

$$M = \pi \cdot L_\Omega(\not{\Omega}) \quad @ \text{ diffuse} \tag{16.1.16b}$$

Observe that directional density L_Ω is in units $[\mathrm{W}/(\mathrm{m}^2 sr)]$ and π in $[sr]$, which correctly delivers M in $[\mathrm{W}/\mathrm{m}^2]$. According to Eq. (16.1.14), a diffuse surface hence emits the radiant intensity

$$I_\Omega(\not{\Omega}) = \int L_\Omega(\not{\Omega}) \cos\theta_t \cdot dA_t = \frac{M}{\pi} \int \cos\theta_t \cdot dA_t$$

By defining

$$A_\perp := \int_A \cos\theta \cdot dA \quad \text{effective (projected) surface} \tag{16.1.17}$$

we obtain

$$I_\Omega(\not{\Omega}) = \frac{M}{\pi} A_{\perp,t} \quad @ \text{ diffuse} \tag{16.1.18}$$

16.1.3 Black-Body Radiator

Spectral and directional exitance is key for thermal radiation physics. By definition, a so-called *black-body radiator* (subscript •) is a radiator (radiating body, surface or surface element) with the following idealized spectral and directional properties:

1. A black body absorbs all incident radiation, regardless of wavelength and direction.
2. At a given temperature and wavelength no body emits more energy than a black body.
3. A black body is a diffuse radiator, i.e. the emitted radiation and hence $L_{\lambda\Omega,\bullet}(\not{\Omega})$ is independent of direction (indicated by $\not{\Omega}$).

Because the black body is a diffuse radiator, all results from Sect. 16.1.2 apply.

Owing to internal statistical quantum processes in thermal equilibrium the absorbed radiation is reemitted with a characteristic spectral distribution, to which we come in a moment. For instance, a black-body at room temperature will absorb all visible sunlight and turn it into an infrared (IR) radiation spectrum. Because IR radiation is outside the visible range, such a black body will look totally black from outside—the reason it has been given that name. A black body is an idealization in most cases. But as we will see in the upcoming Sect. 16.1.4 real radiators and absorbers can be traced back to a black-body by means of correction factors. This is why the physics of a black-body radiator is so important. Let us have a closer look at it.

In 1900 the famous physicist Max Planck was able to show—for which he later got a Nobel prize—that the spectral exitance of a black-body radiator can be written as

$$M_{\lambda,\bullet} := \frac{dM_\bullet}{d\lambda} = \frac{2\pi hc^2}{\lambda^5}\left[\exp\left(\frac{hc}{\lambda k_B T}\right) - 1\right]^{-1} \quad \textbf{Planck's law} \tag{16.1.19}$$

with c the velocity of light, k_B Boltzmann constant, and h Planck's constant. It is depicted in Fig. 16.2. The substitution

$$x := \frac{hc}{\lambda k_B T}$$

yields the normalized form

$$M_{\lambda,\bullet} = \frac{2\pi k_B^5 T^5}{c^3 h^4} \frac{x^5}{e^x - 1}$$

Observe that it has the consistent shape $x^5/(e^x - 1)$, and it is merely stretched in size by T^5. So, $M_{\lambda,\bullet}$ has a consistent maximum at $x_{\max} = 4.965114232\ldots$

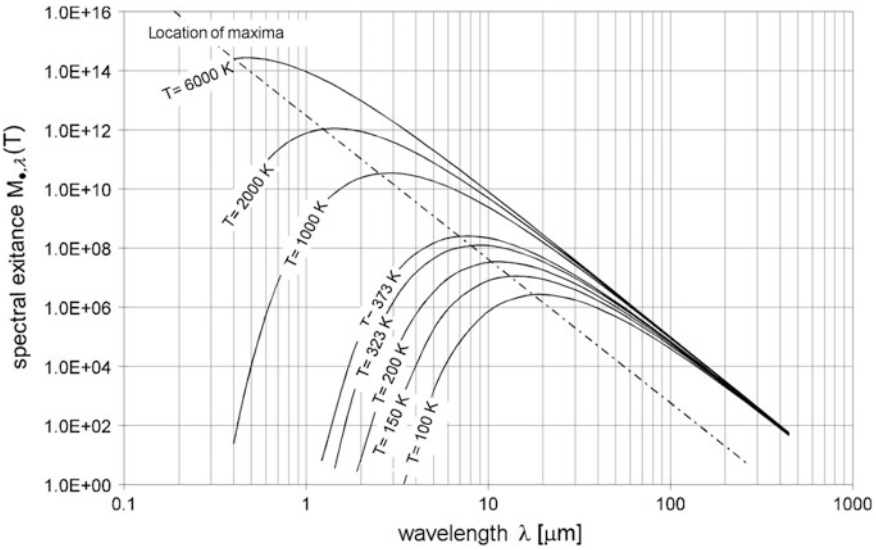


Fig. 16.2 The spectral exitance $M_{\lambda,\bullet}$ of a black body as a function of wavelength and for different body temperatures

(see Problem 16.1) and thus

$$\lambda_{\max} \cdot T = \frac{hc}{k_B x_{\max}} = 2897.752\ \mu\text{m} \cdot \text{K} \quad \textbf{Wien's displacement law} \qquad (16.1.20)$$

Example

The radiation temperature of the Sun is T = 5778 K and its spectral maximum hence is at $\lambda_{\max} = 502$ nm (green light). Earth on the other hand radiates at temperature T = 254 K (see Problem 16.2), which corresponds to a radiation maximum at $\lambda_{\max} = 11.4$ μm, i.e. infrared radiation.

Remark *The variable $x = hc/\lambda k_B T = h\nu/k_B T$, with photon frequency ν, actually has physical relevance: The energy of a photon is $E_\nu = h\nu$ and thus $E_\nu = x(\nu)k_B T$. Therefore, most photons are radiated with the energy $E_{\max} = x_{\max} k_B T \approx 5 k_B T$. But, as the function $x^5/(e^x - 1)$ is strongly asymmetrical, $E_{\max}$ does not correspond to the mean energy E_{ave} of all emitted photons, but rather $E_{ave} = 3.83\ k_B T$. In other words: A black-body radiator at temperature T has internal energy $k_B T$ and thus generates photons with mean energy 3.83 $k_B T$, i.e. of order $k_B T$. So a black-body radiates its internal energy $k_B T$ away by photons of the same energy.*

For the radiant exitance of a black body we find with the help of mathematical tables

$$M_\bullet = \int_0^\infty M_{\lambda,\bullet} \cdot d\lambda = \frac{2\pi k_B^4 T^4}{c^2 h^3} \int_0^\infty \frac{x^3}{e^x - 1} \cdot dx = \frac{2\pi k_B^4 T^4}{c^2 h^3} \frac{\pi^4}{15}$$

Hence

$$\boxed{M_\bullet = \sigma T^4} \quad \textbf{Stefan–Boltzmann law} \tag{16.1.21}$$

with

$$\sigma := \frac{2\pi^5 k_B^4}{15 c^2 h^3} = 5.670400 \times 10^{-8} \frac{\mathrm{W}}{\mathrm{m}^2\ \mathrm{K}^4} \quad \textbf{Stefan–Boltzmann constant}$$

The Stefan-Boltzmann law is an important result, which we will make use of frequently in the upcoming sections.

Since a black body is a diffuse radiator, we derive from Eq. (16.1.16b)

$$M_\bullet = \pi L_{\Omega,\bullet}(\not{\Omega}) = \sigma T^4 \tag{16.1.22}$$

16.1.4 Selective Surfaces

Real radiators generally differ from the ideal of a black-body radiator by being spectrally selective and also differ from a diffuse surface by being directionally selective.

Emitting Surfaces

Based on the properties of a black body, spectrally and directionally selective emitting surfaces can be described as surfaces that modulate the ideal spectral radiance of a black-body radiator by a characteristic emission factor, the so-called *directional spectral emissivity* (a.k.a. *directional spectral emission coefficient*), defined by

$$\varepsilon_{\lambda\Omega}(T) := \frac{L_{\lambda\Omega,t}}{L_{\lambda\Omega,\bullet}} \leq 1 \quad \textbf{directional spectral emissivity} \tag{16.1.23}$$

Note that $\varepsilon_{\lambda\Omega}(T)$ is dimensionless and generally temperature-dependent. From this definition we derive with Eq. (16.1.7)

$$M_{\lambda\Omega,t} = \varepsilon_{\lambda\Omega} L_{\lambda\Omega,\bullet}(\not{\Omega}) \cos\theta_t \tag{16.1.24}$$

Because for further thermal radiation exchange between surfaces only directional spectral radiation flow is of interest, we define the *directional total emissivity* as

$$\varepsilon_\Omega(T) := \frac{L_{\Omega,t}}{L_{\Omega,\bullet}} = \frac{\pi}{\sigma T^4} \int_0^\infty \varepsilon_{\lambda\Omega} L_{\lambda\Omega,\bullet}(T) \cdot d\lambda \le 1 \quad \begin{matrix}\textbf{directional}\\ \textbf{total emissivity}\end{matrix} \tag{16.1.25}$$

where the latter follows from Eq. (16.1.22). We further define the important relation

$$\boxed{M =: \varepsilon(T) \cdot M_\bullet = \varepsilon(T) \cdot \sigma\, T^4} \tag{16.1.26}$$

This determines the *hemispherical total emissivity*, which reads with Eqs. (16.1.24) and (16.1.25) (for the notation $[\ldots]_x$ see insert *Terminology and Notations* in Sect. 16.1.1) after Eq. (16.1.4))

$$\varepsilon(T) := \frac{M}{M_\bullet} = \frac{1}{M_\bullet} \left[\left[\varepsilon_{\lambda\Omega} L_{\lambda\Omega,\bullet}(\not{\Omega}) \cos\theta_t \right]_{\Omega_t} \right]_\lambda < 1 \quad \begin{matrix}\textbf{hemispherical}\\ \textbf{total emissivity}\end{matrix}$$

This can be transposed either as

$$\varepsilon(T) = \frac{1}{M_\bullet} \left[\left[\varepsilon_{\lambda\Omega} \cos\theta_t \right]_{\Omega_t} L_{\lambda\Omega,\bullet} \right]_\lambda = \frac{1}{M_\bullet} \left[\varepsilon_\lambda L_{\lambda\Omega,\bullet} \right]_\lambda$$

or

$$\varepsilon(T) = \frac{1}{M_\bullet} \left[\left[\varepsilon_{\lambda\Omega} L_{\lambda\Omega,\bullet} \right]_\lambda \cos\theta_t \right]_{\Omega_t} = \frac{1}{M_\bullet} \left[\varepsilon_\Omega L_{\Omega,\bullet}(\not{\Omega}) \cos\theta_t \right]_{\Omega_t} = \frac{L_{\Omega,\bullet}}{M_\bullet} \left[\varepsilon_\Omega \cos\theta_t \right]_{\Omega_t}$$

From the latter we finally obtain with $M_\bullet = \pi\, L_{\Omega,\bullet}$ (see Eq. (16.1.22))

$$\varepsilon(T) = \frac{1}{\pi} \int_\cap \varepsilon_\Omega(T) \cos\theta_t \cdot d\Omega_t \le 1 \tag{16.1.27}$$

Absorbing Surfaces

Equivalently, the radiant flux $d^2\Phi_a$ absorbed by a body is not the entire received radiant flux $d^2\Phi_r$ (as for a black body), but is a spectrally and directionally selective fraction, $\alpha_{\lambda\Omega}$, of that. This fraction is named *directional spectral absorptivity* (a.k.a. *directional spectral absorption coefficient*)

$$\alpha_{\lambda\Omega}(T) := \frac{d^3\Phi_a}{d^3\Phi_r} = \frac{L_{\lambda\Omega,a}}{L_{\lambda\Omega,r}} = \frac{E_{\lambda\Omega,a}}{E_{\lambda\Omega,r}} \le 1 \quad \begin{matrix}\textbf{directional}\\ \textbf{spectral absorptivity}\end{matrix} \tag{16.1.28}$$

where the latter identities follow from Eqs. (16.1.4) to (16.1.10). Note that $\alpha_{\lambda\Omega}(T)$, like $\varepsilon_{\lambda\Omega}(T)$, is dimensionless and generally also temperature-dependent. The subscripts just indicate the spectral and directional dependences.

By the same token as for the emissivity we define the *directional total absorptivity* as

$$\alpha_\Omega(T) := \frac{L_{\Omega,a}}{L_{\Omega,r}} = \frac{\left[L_{\lambda\Omega,a}\right]_\lambda}{\left[L_{\lambda\Omega,r}\right]_\lambda} = \frac{\left[\alpha_{\lambda\Omega} L_{\lambda\Omega,r}\right]_\lambda}{\left[L_{\lambda\Omega,r}\right]_\lambda} \le 1 \quad \begin{matrix}\textbf{directional}\\ \textbf{total absorptivity}\end{matrix} \tag{16.1.29}$$

(for the notation $[\ldots]_x$ see insert *Terminology and Notations* in Sect. 16.1.1 after Eq. (16.1.4)) and the *hemispherical total absorptivity* as

$$\alpha(T) := \frac{E_a}{E_r} = \frac{\left[\left[\alpha_{\lambda\Omega} L_{\lambda\Omega,r}\right]_\lambda \cos\theta_r\right]_\Omega}{\left[\left[L_{\lambda\Omega,r}\right]_\lambda \cos\theta_r\right]_\Omega} \le 1 \quad \begin{matrix}\textbf{hemisperical}\\ \textbf{total absorptivity}\end{matrix} \tag{16.1.30}$$

With Eq. (16.1.29) we derive

$$\alpha = \frac{\left[\left[\alpha_{\lambda\Omega} L_{\lambda\Omega,r}\right]_\lambda \cos\theta_r\right]_\Omega}{\left[\left[L_{\lambda\Omega,r}\right]_\lambda \cos\theta_r\right]_\Omega} = \frac{\left[\alpha_\Omega L_{\Omega,r} \cos\theta_r\right]_\Omega}{\left[L_{\Omega,r} \cos\theta_r\right]_\Omega}$$

Equation (16.1.30) can we written with Eq. (16.1.9) as

$$d^2\Phi_a = \alpha(T_t) \cdot d^2\Phi_r \tag{16.1.31}$$

Observe that the total absorptivity depends on the received spectral distribution and hence on the temperature of the transmitter (radiator) and not on that of the absorber.

Real Radiators and Absorbers

Spectrally selective surfaces are the rule. Figure 16.3 for instance shows the spectral absorptivity of some metals and paints used in S/C thermal design. In particular the emissivity of metallic surfaces is reduced due to their strong absorption with an absorption edge in the infrared (see Fig. 16.3) and varies drastically depending on the type of metal and its surface properties (e.g. oxidation).

Most dielectric materials and metals have also directionally selective surfaces. Figure 16.4 shows typical angle-dependent emission figures.

If surfaces are no longer smooth (i.e. roughness > wavelength ≈ 0.5 μm) as assumed so far, or if they are oxidized, contaminated, or even painted, their emissivity and absorptivity may exhibit quite different values. They also show a strong dependency on brightness, wavelength and temperature. For $T < 500$ K and $\lambda > 5$ μm all metals exhibit the relation

$$\varepsilon(T, \Omega) \propto T \quad @ \text{ metals}$$

and therefore

$$M = \varepsilon(T, \Omega) \cdot \sigma T^4 \propto T^5 \quad @ \text{ metals}$$

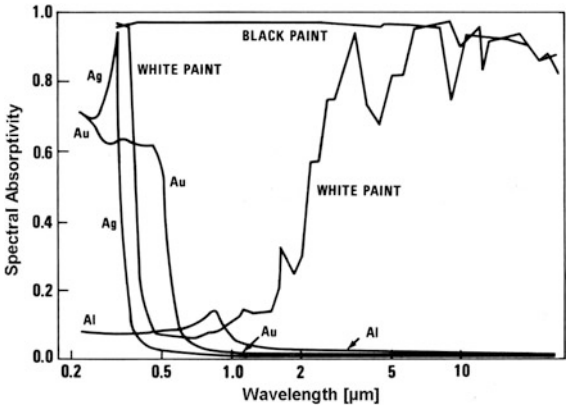


Fig. 16.3 Wavelength-dependent absorptivity of some spectrally selective surfaces. *Credit* NASA

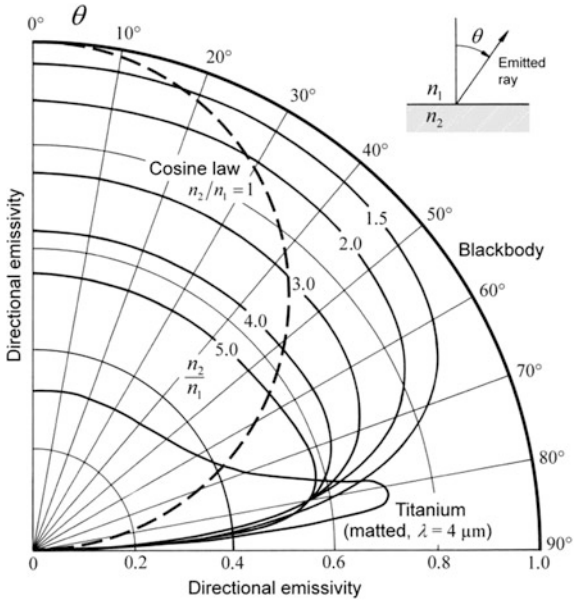


Fig. 16.4 Directional emissivity of selective plane surfaces: dielectric materials with $n_2/n_1 > 1$, matted titanium as an example of metals, compared to Lambert's cosine law, and a diffuse black-body radiator

Because with increasing/decreasing brightness temperature T the spectral exitance shifts towards lower/higher wavelengths via Wien's displacement law Eq. (16.1.20) this implies that for metals ε_λ strongly increases with decreasing wavelength, typically by a factor of 3 (iron, platinum, nickel, titanium) in the range $\lambda = 1{-}10$ μm. This is why the wavelength is given for the emissivity of matted

titanium in Fig. 16.4. A notable exception from this rule is copper, with a nearly wavelength- and thus brightness-temperature independent emissivity, thus depending only on directivity and surface properties.

So, Lambert's cosine law is no longer valid for dielectric or metallic surfaces. In the literature, this is quite often overlooked, and metals are treated the same way as diffuse radiators. In practice, this may not make much a difference. In principle, however, it is not admissible and has to be justified in each individual case.

16.1.5 Kirchhoff's Law

We now consider a receiver in radiative equilibrium with its environment. What does "radiative equilibrium" mean? As a matter of fact, every body in our world is exposed to radiation from its environment, and the irradiated body reemits radiation to its environment. We have seen in Sect. 16.1.3 that a body at temperature T has internal energy $k_B T$ and emits heat with the same energy. So, bodies share their internal energy with energies of other bodies by thermal radiation. If radiation is the only means for a body to exchange energy, it is in radiative equilibrium if all received energy equals all emitted energy.

For our theoretical analysis let us assume the simplest case: A radiatively absorbing body is completely surrounded by a black-body radiator. In radiative equilibrium its temperature has to be identical to the temperature T of the black-body radiator. That is, the spectral radiance of its emitted and absorbed radiation must be identical. This conclusion is Kirchhoff's law of thermal radiation:

> *Any body at a given temperature T emits in every solid angle element and in every wavelength interval the same radiative power as it absorbs there from the radiation of a black body having the same temperature.*

In our notation this law reads

$$d^3\Phi_t(T) = \alpha_{\lambda\Omega}(T) \cdot d^3\Phi_\bullet(T) \quad \textbf{Kirchhoff's law}$$

When applying Eq. (16.1.3), this expression translates into

$$L_{\lambda\Omega,t}(T) = \alpha_{\lambda\Omega}(T) L_{\lambda\Omega,\bullet}(T) \tag{16.1.32}$$

It tells us that the spectral radiance $L_{\lambda\Omega}$ of any radiating body in radiative equilibrium with its environment can be traced back to the spectral radiance of a black body. This is the significance of Kirchhoff's law.

$\alpha = \varepsilon$ Identities

When applying Eq. (16.1.23), $L_{\lambda\Omega,t} = \varepsilon_{\lambda\Omega}(T) L_{\lambda\Omega,\bullet}$, we get an alternative formulation of Kirchhoff's law

$$\alpha_{\lambda\Omega}(T) = \varepsilon_{\lambda\Omega}(T) \quad @ \quad T_t = T_r =: T \tag{16.1.33}$$

So, as long as a body is in radiative equilibrium with any environment, Eq. (16.1.33) is always valid.

We now want to find similar identities as Eq. (16.1.33) for the directional total emissivity/absorptivity and the hemispherical total emissivity/absorptivity.

From Eqs. (16.1.27) to (16.1.29) we get the identity

$$\varepsilon_\Omega(T) = \frac{\left[\varepsilon_{\lambda\Omega} L_{\lambda\Omega,\bullet}\right]_\lambda}{L_{\Omega,\bullet}} \overset{!}{=} \frac{\left[\alpha_{\lambda\Omega} L_{\lambda\Omega,r}\right]_\lambda}{\left[L_{\lambda\Omega,r}\right]_\lambda} = \alpha_\Omega(T)$$

This inner equation holds, if

- $\left[\varepsilon_{\lambda\Omega} L_{\lambda\Omega,\bullet}\right]_\lambda = \varepsilon_{\lambda\Omega}\left[L_{\lambda\Omega,\bullet}\right]_\lambda = \varepsilon_{\lambda\Omega} L_{\Omega,\bullet}$ plus $\left[\alpha_{\lambda\Omega} L_{\lambda\Omega,r}\right]_\lambda = \alpha_{\lambda\Omega}\left[L_{\lambda\Omega,r}\right]_\lambda$, or
- $L_{\lambda\Omega,r}(\lambda) \propto L_{\lambda\Omega,\bullet}(\lambda)$, i.e. if the incident radiation has a spectral distribution proportional to that of a black body at T. In other words:

$$\varepsilon_\Omega(T) = \alpha_\Omega(T) \quad @ \quad T_t = T_r =: T \tag{16.1.34}$$

holds if

- $\varepsilon_{\lambda\Omega} = \alpha_{\lambda\Omega}$ is independent of the wavelength, or
- The incident radiation has a spectral distribution proportional to that of a black body at T.

From Eqs. (16.1.27) to (16.1.29) we get the identity

$$\varepsilon(T) = \frac{\left[\varepsilon_\Omega \cos\theta_t\right]_\Omega}{\pi} \overset{!}{=} \frac{\left[\alpha_\Omega L_{\Omega,r} \cos\theta_r\right]_\Omega}{\left[L_{\Omega,r} \cos\theta_r\right]_\Omega} = \alpha(T)$$

The inner equation holds only if $\varepsilon_\Omega(T) = \alpha_\Omega(T)$ and if

- $\varepsilon_\Omega, \alpha_\Omega$ are and hence $\varepsilon_{\lambda\Omega} = \alpha_{\lambda\Omega}$ is independent of angle, or
- $L_{\Omega,r}(\not{\Omega})$, i.e. the incident radiation is independent of incident angle, i.e. it is diffuse.

This implies:

$$\varepsilon(T) = \alpha(T) \quad @ \quad T_t = T_r =: T \tag{16.1.35}$$

which holds if

- $\varepsilon_{\lambda\Omega} = \alpha_{\lambda\Omega}$ is independent of wavelength and angle (= gray body); or
- the incident radiation is diffuse and has a spectral distribution proportional to that of a black body at T (= gray environment); or
- $\varepsilon_{\lambda\Omega} = \alpha_{\lambda\Omega}$ is independent of angle plus the incident radiation has a spectral distribution proportional to that of a black body at T.

Note, from the above derivation it also follows that if the spectral temperatures of the incident and emanating radiation are NOT the same, i.e. $T_t \neq T_r$, then

$$\boxed{\alpha(T_t) \neq \varepsilon(T_r)} \quad @ \quad T_t \neq T_r \tag{16.1.36}$$

Usually, this is the case for a body in an extreme environment, such as a spacecraft in space receiving solar radiation with radiation temperature $T_{sol} \approx 5778$ K and emitting into deep space that has a radiation temperature $T_{space} = 2.73$ K.

Gray-Body Radiator

The problem of determining radiation exchange between surfaces within a spacecraft, as approached in Sect. 16.2, is greatly simplified if Eq. (16.1.35) holds for a given surface. One of the conditions for Eq. (16.1.35) reads "$\varepsilon_{\lambda\Omega} = \alpha_{\lambda\Omega}$ is independent of wavelength and angle", In other words, a surface that inherits all the properties of a black-body radiator, except that it radiates with a spectrally and directionally constant fraction of the black-body radiator exhibits the property

$$\varepsilon(T) = \alpha(T) < 1 \quad @ \quad \text{gray body} \tag{16.1.37}$$

Such bodies are called *gray bodies* or *gray surfaces*. In particular they radiate diffusively and hence are diffuse surfaces. Therefore they sometimes are also called *diffuse-gray surfaces.*

16.2 Radiation Exchange

Now that we know the emission and absorption characteristics of different surfaces, especially those of diffuse surfaces, we want to know how much radiation energy is exchanged between two surfaces.

16.2.1 Transmitted and Absorbed Flux

Let us have a closer look at a radiating surface dA_t that transmits radiant flux to a receiver surface dA_r, each of both having orientation θ_t and θ_r between their surface normal and the interconnecting ray line. According to Eq. (16.1.11), this transmitted flux can be written as

$$d^2\Phi_{t\to r} = L_{\Omega,t} \cdot dA_{t,\perp} d\Omega_r = \frac{1}{s^2} L_{\Omega,t} \cdot dA_{\perp,t} dA_{\perp,r} \tag{16.2.1}$$

where s is the distance from the transmitter to the receiver and the latter follows from the definition of the solid angle (see Fig. 16.1)

$$s^2 \cdot d\Omega_r = dA_{\perp,r} = \cos\theta_r \cdot dA_r$$

In summary, we get

$$d^2\Phi_{t\to r} = \frac{1}{s^2} L_{\Omega,t} \cos\theta_t \cos\theta_r \cdot dA_r dA_t \tag{16.2.2}$$

This radiant flux in *Watt* units describes the transmission of photons per unit time emitted from the transmitter in direction Ω to the receiver at distance s.

Absorbed Radiant Flux
The emitted radiant flux will somewhere hit a receiving surface, which absorbs at least a fraction of it. To determine the absorbed radiant flux let us assume that the transmitter is a diffuse (Lambertian) radiator as described in Sect. 16.1.2, which, on average over all radiators, is a good assumption. According to Eqs. (16.1.31), (16.2.2), and (16.1.16b) we then have

$$d^2\Phi_a = \alpha_r \cdot d^2\Phi_{t\to r} = \alpha_r \frac{M}{\pi s^2} \cos\theta_t \cos\theta_r \cdot dA_r dA_t \tag{16.2.3}$$

Remark *For physical correctness, in the above we should have written* $s^2 \cdot d\Omega_r = \cos\theta_r \cdot dA_r \cdot 1[sr]$ *and* Eq. (16.2.2) *therefore should read* $d^2\Phi_{t\to r} = L_{\Omega,t} \cos\theta_t \cos\theta_r / s^2 \cdot dA_r dA_t \cdot 1[sr]$. *However, since in* Eq. (16.2.3) π *is in units* $[sr]$ (*see comment after* Eq. (16.1.16b)), *the unit* $[sr]$ *cancels out. So, from now on* π *is dimensionless.*

The total radiant flux between emitter surface A_t and absorber surface A_r is the double integral over both sides of this equation. By carrying out this operation and with Eq. (16.1.26) the absorbed radiant flux at the receiver becomes

$$\boxed{\Phi_a = \alpha_r \varepsilon_t A_t F_{t\to r} \sigma T_t^4} \tag{16.2.4}$$

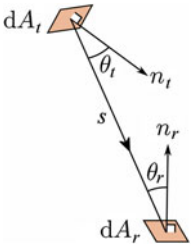
where $F_{t\to r}$ is the so-called *view factor* as discussed in the next section. We recall that the receiver absorbs the transmitted spectrum and hence $\alpha_r(T_t)$.

16.2.2 View Factor

In Eq. (16.2.4) we have introduced the dimensionless *view factor* (a.k.a. *configuration factor, shape factor, angle factor, or exchange factor*)

$$F_{t\to r} := \frac{1}{A_t} \int_{A_r} \int_{A_t} \frac{\cos\theta_t \cos\theta_r}{\pi s^2} \cdot dA_t dA_r \le 1 \quad \textbf{view factor} \tag{16.2.5}$$

A_t, A_r transmitter surface or receiver surface respectively
θ_t, θ_r angle between radiation line and the surface normal
s distance between the two surface points.



The view factor is the proportion of radiation emitted from A_t that reaches A_r. So

$$F_{t\to r} = \frac{\Phi_{t\to r}}{\Phi_t} = \frac{\Phi_{t\to r}}{M_L(T_t)A_t}$$

The view factor describes the vision geometry between transmitter and receiver surfaces: What does surface A_r see of surface A_t? By definition, the view factor is non-dimensional and smaller than or equal 1. The following three properties hold for the view factor: Since the sum of all surfaces A_r that are external with regard to the transmitter have to fill the total visible solid angle, the emitted radiation will hit any surface around the transmitter. Therefore, the following rule must be valid

$$\sum_{all\ r} F_{t\to r} = 1 \quad \textbf{summation rule} \tag{16.2.6}$$

Second, because of the commutability of the integration variables in Eq. (16.2.5), we directly get

$$A_t F_{t\to r} = A_r F_{r\to t} \quad \textbf{reciprocity relation} \tag{16.2.7}$$

Thirdly, because for the total absorbed radiant flux between two emitters 1, 2 and two absorber surfaces 3, 4 the following has to be valid

$$\Phi_{a,1+2\to 3+4} = \Phi_{a,1\to 3} + \Phi_{a,1\to 4} + \Phi_{a,2\to 3} + \Phi_{a,2\to 4}$$

This results with Eq. (16.2.4) in the so called *addition theorem*

$$\begin{aligned}(A_1 + A_2)F_{1+2\to 3+4} = {} & A_1F_{1\to 3} + A_1F_{1\to 4} \\ & + A_2F_{2\to 3} + A_2F_{2\to 4} \quad \textbf{addition theorem}\end{aligned} \tag{16.2.8}$$

Compilation of View Factors

It is quite easy to calculate the visual factor for two flat plates at a distance of s, and with $A_t, A_r \ll \pi s^2$. In this case, the angles θ_t, θ_r are more or less constant along the integration area. So we get

$$F_{plate\ A_t \rightarrow plate\ A_r} = \cos\theta_t \cos\theta_r \frac{A_t}{\pi s^2} \tag{16.2.9}$$

If a spacecraft is considered a plate or a small sphere we can also get relatively simple expressions for the view factors for a spacecraft circling a spherical celestial body (planet)

$$F_{plate \rightarrow sphere} = \mu^2 \cos\phi \quad \text{for} \quad \phi + \arcsin\mu < \frac{\pi}{2} \tag{16.2.10}$$

$$F_{small\ sphere \rightarrow big\ sphere} = \frac{1}{2}\left(1 - \sqrt{1-\mu^2}\right) \tag{16.2.11}$$

with

$$\mu = R/(R+h)$$

- h orbit altitude above the surface of the planet
- R radius of the planet
- ϕ angle between the orbital position vector and the surface normal of the S/C.

Compilations of these and many different geometries can be found on Howell's website www.thermalradiation.net/tablecon.html.

Multiple Reflection

For thermal radiation between two surfaces we also have to consider multiple reflection as well as the mutual proximity of the surfaces. This is taken into account by the *form factor* $\Im_{i \leftrightarrow j}$. Hence, the form factor for two parallel plates with infinite extent and the emission parameters ε_i and ε_j is no longer (see Eq. (16.2.9)) $F_{i \rightarrow j} = F_{j \rightarrow i} = 1$, but (see Howell 2011)

$$\Im_{i \leftrightarrow j} = \frac{1}{1/\varepsilon_i + 1/\varepsilon_j - 1} \tag{16.2.12}$$

If even the surface i is able to see itself (concave curvature) the following is valid

$$\Im_{i \leftrightarrow (i,j)} = \frac{F_{i \rightarrow i}}{1/\varepsilon_i + (1 - F_{i \rightarrow i})(1/\varepsilon_j - 1)} \tag{16.2.13}$$

This shows how complicated such calculations may become.

If surfaces become arbitrarily complex, the only way to handle this problem is to use the so-called ***ray tracing process***. Hereby, rays are generated by Monte Carlo statistics and the path of each individual ray is traced between given surfaces with

their given emissivities and absorptivities. These ray tracing tools are highly complicated, but also highly accurate (see Subsection *Monte Carlo Ray Tracing* in Sect. 16.3.4).

16.2.3 Point Radiators

Point radiators, such as the Sun, are specific for compiling the absorbed radiant flux as given in Eq. (16.2.4). On one hand, owing to a high surface temperature their radiant exitance is extremely large. Yet, due to the large distance only a very small amount of this radiation hits a receiver on a distant planet, implying that the view factor is very small. So we need to investigate the expression $A_t F_{t\to r}\sigma T_t^4$.

Let us assume the point-like radiator has a circular effective (projected) transmitter surface $A_{t,\perp} = \pi R_t^2$. Because the source appears to be a point, it must be very far away, i.e. $s \gg R_t$. Thus and applying Eq. (16.1.17) we can simplify the view factor Eq. (16.2.5) as

$$\begin{aligned} A_t F_{t\to r} &= \int_{A_r}\int_{A_t} \frac{\cos\theta_t \cos\theta_r}{\pi s^2}\cdot dA_t dA_r = \frac{1}{\pi s^2}\int_{A_r}\int_{A_t} \cos\theta_t \cos\theta_r \cdot dA_t dA_r \\ &= \int_{A_r} \cos\theta_t \cdot dA_t \times \int_{A_t} \cos\theta_r \cdot dA_r = \frac{1}{\pi s^2} A_{t,\perp} A_{r,\perp} = \left(\frac{R_t}{s^2}\right)^2 A_{r,\perp} \end{aligned}$$

For the absorbed radiant flux Eq. (16.2.4) we hence obtain

$$\Phi_a = \alpha_r \varepsilon_t A_t F_{t\to r}\sigma T_t^4 = \alpha_r \varepsilon_t \sigma T_t^4 \left(\frac{R_t}{s}\right)^2 A_{r,\perp}$$

With the definition

$$T_{t,eff} := T_t\sqrt{R_t/s} \quad \textbf{effective temperature} \tag{16.2.14}$$

we finally get

$$\Phi_a = \alpha_r(T_t)\varepsilon_t A_{r\perp}\sigma T_{t,eff}^4 \tag{16.2.15}$$

This expression for the radiant flux is obviously more practical than Eq. (16.2.4). A comparison of Eq. (16.2.4) with Eq. (16.2.15) reveals that the introduction of the effective temperature accomplishes the following formal substitution

$$A_t F_{t\to r} T_t^4 \to A_{r\perp} T_{t,eff}^4 \tag{16.2.16}$$

Note *The effective temperature merely simplifies the computation of the radiant flux onto a receiver. However, the radiation temperature is still the temperature of the actual radiating surface. In the case of the Sun it is* $T_{sun} = 5778$ K *and the absorptivity of received solar radiation is thus* $\alpha_r(5778\ \mathrm{K})$.

Sun as a Point Radiator

The Sun is an excellent example of a point radiator (black-body radiator). It has $T_{sun} = 5778$ K, an apparent $R_{sun} = 695{,}000$ km, and mean distance $s = 1.496 \times 10^8$ km to Earth. The effective temperature for a solar irradiation perpendicular to a surface at the Earth's distance therefore is

$$T_{sun,eff} = 5778\ \mathrm{K}\sqrt{\frac{0.6959}{149.6}} = 394.4\ \mathrm{K} \tag{16.2.17}$$

Let us do a quick cross check to make sure the result is correct. The seasonal irradiation of Earth by the Sun is given by

$$E_{sun\to earth\perp} = S_0 \cdot \left[1 + 0.034 \cdot \cos\left(360^\circ \frac{n}{365}\right)\right] \frac{W}{m^2} = 1361.5 \pm 46 \frac{W}{m^2}$$

where $S_0 = 1361.5\ W/m^2$ is the solar constant (mean irradiation) and n = day of the year (January 1: $n = 1$). For the effective temperature of the Sun this results in

$$T_{sun,eff} = \left(\frac{E_{sun\to earth\perp}}{\sigma}\right)^{1/4} = \left(\frac{1361.5 \pm 46}{5.6704 \times 10^{-8}\ \mathrm{K}^4}\right)^{1/4} = 394 \pm 3\ \mathrm{K}$$

16.2.4 Radiation Exchange Between Two Bodies

We are now prepared to derive the expression for an effective radiant flux exchange Φ_{ij} between two surfaces i and j. According to Eq. (16.2.4) we get

$$\Phi_{i\to j} = \alpha_j(T_i)\varepsilon_i(T_i)A_iF_{i\to j}\sigma T_i^4 - \alpha_i(T_j)\varepsilon_j(T_j)A_jF_{j\to i}\sigma T_j^4$$

From Eq. (16.2.7) we have $A_iF_{i\to j} = A_jF_{j\to i}$ and therefore

$$\Phi_{i\to j} = A_iF_{i\to j}\sigma\left[\alpha_j(T_i)\varepsilon_i(T_i)T_i^4 - \alpha_i(T_j)\varepsilon_j(T_j)T_j^4\right] \tag{16.2.18}$$

If for the transmitter and receiver surfaces $\alpha_j(T_i)\varepsilon_i(T_i) \approx \alpha_i(T_j)\varepsilon_j(T_j)$, we finally get for the effective radiant flux $i \to j$

$$\boxed{\Phi_{i\to j} = \varepsilon_i \alpha_j A_i F_{i\to j} \sigma\left(T_i^4 - T_j^4\right)} \quad @ \quad \alpha_j(T_i)\varepsilon_i(T_i) \approx \alpha_i\left(T_j\right)\varepsilon_j\left(T_j\right) \tag{16.2.19}$$

The condition holds in particular if $\alpha_j(T_i) \approx \varepsilon_j\left(T_j\right)$ plus $\alpha_i\left(T_j\right) \approx \varepsilon_i(T_i)$, which according to Eq. (16.1.35) is the case

- if $T_i \approx T_j$, or
- for black bodies with $\alpha = \varepsilon = 1$, or
- for gray surfaces having $\alpha = \varepsilon = const < 1$,

Multiple Reflection Between Large Parallel Flat Plates

In the case where we have two large parallel flat plates, both of size A, but different emissivities ε_i, ε_j, with multiple reflections between them, we have seen from Eq. (16.2.12) that we have to consider the form factor $\Im_{i\leftrightarrow j}$. Since a ray between these two plates will be absorbed eventually it seems conclusive (see Howell 2011) that

$$\varepsilon_i \alpha_j F_{i\to j} \to \frac{1}{1/\varepsilon_i(T_i) + 1/\varepsilon_j\left(T_j\right) - 1}$$

and therefore

$$\boxed{\Phi_{i\to j} = \frac{A\sigma\left(T_i^4 - T_j^4\right)}{1/\varepsilon_i(T_i) + 1/\varepsilon_j\left(T_j\right) - 1}} \quad @ \text{ large parallel flat plates} \tag{16.2.20}$$

Case in Point: Multi-Layer Insulation

We want to explore the effect of shielding layers on insulation properties as exploited by so-called multi-layer insulation blankets. MLI blankets are extensively used in spacecraft as lightweight thermal protection systems. An example for a commonly used outer layer is a 25–50 μm Kapton film (which give the MLI its typical gold color) backed with a few Angstrom thick layer of vapor deposited aluminum (VDA), The multiple layers in the blanket are usually 6 μm Mylar (or Kapton) with a few Angstrom thick layer of VDA on each side, which makes them look like aluminum foil. The layers are separated by a spacer netting ("bridal veil") or felt-like structure, made for example from dacron, which prevents contact of the adjacent foils (see Fig. 16.5). Heat transfer analysis has shown that the radiant flux at room temperature dominates the conductive flux through a MLI blanket by about one order of magnitude. We will therefore focus here on the radiative insulation property of shielding layers.

We assume that a body with a flat surface and a given temperature T_0 exchanges radiation with a "flat" external environment having effective temperature T_∞. According to Eq. (16.2.20) the radiant exitance to the external environment then is

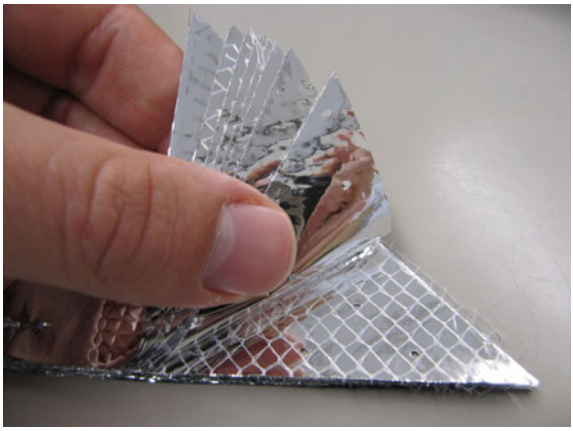


Fig. 16.5 A typical MLI blanket with VDA Kapton layers and dacron netting in between. *Credit* IRS/Uni Stuttgart

$$\Phi_{0\to\infty} = \frac{A\sigma\left(T_0^4 - T_\infty^4\right)}{1/\varepsilon_0 - 1/\varepsilon_\infty - 1}$$

We now add n thermal shielding layers between the body and its environment and assume that both sides of the layers have the same emissivity ε. Since the radiant exitance (heat flow), $M = \Phi/A$, has to be identical from layer to layer we therefore get the following set of $n + 1$ equations

$$\begin{gathered} M(1/\varepsilon_0 + 1/\varepsilon - 1) = \sigma\left(T_0^4 - T_1^4\right) \\ M(2/\varepsilon - 1) = \sigma\left(T_1^4 - T_2^4\right) \\ \cdots \\ M(2/\varepsilon - 1) = \sigma\left(T_i^4 - T_j^4\right) \\ \cdots \\ M(1/\varepsilon + 1/\varepsilon_\infty - 1) = \sigma\left(T_n^4 - T_\infty^4\right) \end{gathered}$$

Adding all these equations each side by itself results in

$$M\left[\frac{1}{\varepsilon_0} + \frac{1}{\varepsilon_\infty} - 1 + n\left(\frac{2}{\varepsilon} - 1\right)\right] = \sigma\left(T_0^4 - T_\infty^4\right)$$

Typically $\varepsilon_0 \approx \varepsilon_\infty \approx 0.8$, while an MLI has tens of layers with $\varepsilon \approx 0.068$. So, we do not make a big error if we neglect the $\varepsilon_0, \varepsilon_\infty$ terms. Hence we finally find

$$\Phi_{n\to\infty} = \frac{\Phi_{0\to\infty}}{1/\varepsilon_0 + 1/\varepsilon_\infty - 1 + n(2/\varepsilon - 1)} \approx \frac{\Phi_{0\to\infty}}{n(2/\varepsilon - 1)} \quad @\ n \text{ shielding layers} \tag{16.2.21}$$

where $\Phi_{0\to\infty}$ was the unshielded radiant flux. Verbalized:

> A multi-layer insulation with n shielding layers reduces the effective radiant flux between a given body and its environment typically by a factor of $0.035/n$.

Example

Assume we have a deep space probe shielded by MLI with the internal MLI layer of $T_0 = 300$ K and the outer layer (facing deep space) at a temperature of $T_0 \approx 150$ K. Then, with a 30 layer MLI we have an emitted radiant exitance

$$M_{rad} \approx \frac{0.035}{30}\sigma\left(300\text{K}^4 - 150\text{K}^4\right) = 0.50\ \frac{\text{W}}{\text{m}^2}$$

For comparison, the conduction heat transfer (cf. Eq. (16.3.3)) through the layers, $M_{n\to\infty} = G(T_0 - T_\infty)$, is measured to be very roughly $G \approx 5 \times 10^{-3}$ W m^{-2} K^{-1}. Therefore

$$M_{cond} \approx 0.005\ \frac{\text{W}}{\text{m}^2\ \text{K}} \cdot (300\ \text{K} - 150\ \text{K}) = 0.75\ \frac{\text{W}}{\text{m}^2}$$

In this example, at these temperatures (inner layer 300 K, outer layer 150 K), the conduction heat transfer would be of about the same magnitude as the radiation heat transfer, and hence well-balanced.
In reality, both the total heat transfer and the share between conduction and radiation heat transfer through MLI is temperature dependent. Generally, the radiative heat transfer gains significance with increasing temperature and vice versa. If, for example, the outer layer is not 150 K but 450 K due to the exposure to the Sun the radiative share is $M_{rad} = -2.18$ W/m^2 and the conductive $M_{cond} = -0.75$ W/m^2, i.e. no longer well-balanced. Therefore, in practical applications the heat transfer through an MLI must be generally considered as a function of the MLI's temperature.
In addition, the efficiency of MLI layers and the share between radiative and conductive heat transfer also vary with the number of layers, layup type, and even from layup to layup due to different workmanship.

16.2.5 Spacecraft Thermal Balance

We now consider an arbitrary body B without internal radiation source exchanging radiation with many surrounding bodies $1, 2, 3, \ldots$ all having different temperatures. Therefore and in contrast to Eq. (16.1.35) we have $\alpha_B(T_i) \neq \varepsilon_B\left(T_j\right)$ and $\alpha_B(T_i) \neq \alpha_B\left(T_j\right)$. Because the radiation of the different radiation sources is not

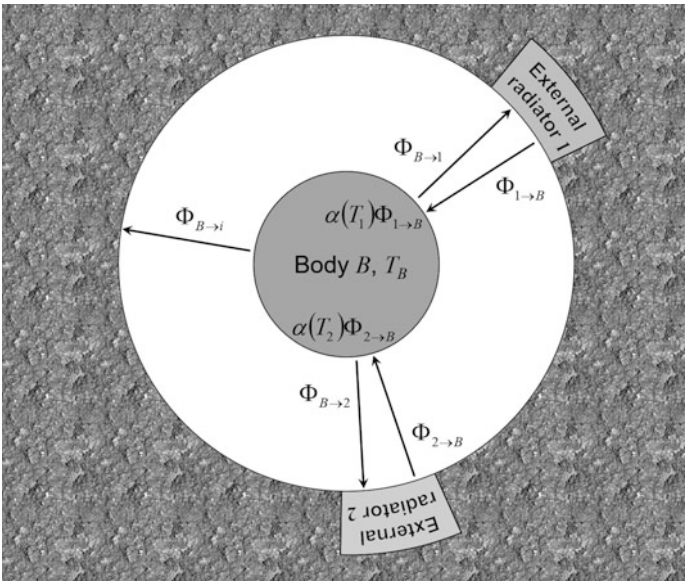


Fig. 16.6 Radiation balance between a body B and its environment with two representative external radiators

necessarily diffuse (see Fig. 16.6), we have to take into account the radiant flux balance with the radiators involved. First, there is the radiant flux emitted by the body B. According to Eqs. (16.1.5) and (16.1.21), it is

$$\Phi_B = M_B(T_B) \cdot A_B = \varepsilon_B A_B \sigma T_B^4 \tag{16.2.22}$$

On the other hand, there is the incoming radiation of the surrounding bodies i. With Eqs. (16.2.4) and (16.2.7) it can be written as

$$\begin{aligned} \Phi_a &= \sum_i \alpha_B(T_i) \cdot \Phi_{i\to B} = \sum_i \alpha_B(T_i) A_i F_{i\to B} \varepsilon_i(T_i) \sigma T_i^4 \\ &= A_B \sum_i \alpha_B(T_i) F_{B\to i} \varepsilon_i(T_i) \sigma T_i^4 \end{aligned} \tag{16.2.23}$$

Equating Eq. (16.2.22) with Eq. (16.2.23) results in

$$\varepsilon_B T_B^4 = \sum_i \alpha_B(T_i) \varepsilon_i(T_i) F_{B\to i} T_i^4 + \frac{\Phi_{\text{int}}}{A_B \sigma} \quad \textbf{radiative equilibrium law} \tag{16.2.24}$$

where we now consider also an internal source with heat flux Φ_{int}.

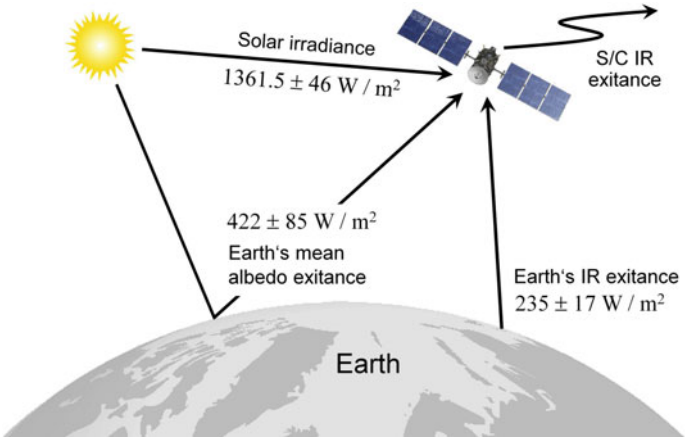


Fig. 16.7 Radiation balance of a satellite in Earth orbit

With Eq. (16.2.24) we are able to set up a thermal balance equation for any spacecraft in space as depicted in Fig. 16.7 and to calculate from it the wanted equilibrium body temperature T_B.

Here we assume a satellite (sat) in Earth orbit exchanging radiation with the Sun, Earth, and space. It is a valid assumption that each of the three bodies radiates like a black-body $(\varepsilon_i = 1)$. Applying this to Eq. (16.2.23), and applying the effective temperature concept, Eq. (16.2.16), to the Sun as a point-source we get

$$\begin{aligned}\varepsilon_{IR} T_{sat}^4 = {} & \alpha_{sol}\delta_{sun}\frac{A_{sat\perp sun}}{A_{sat}} T_{sun,eff}^4 + \alpha_{sol}\delta_a(\gamma) F_{sat\to earth} a T_{sun,eff}^4 \\ & + \alpha_{IR} F_{sat\to earth} T_{earth}^4 + \alpha_{IR} F_{sat\to\infty} T_\infty^4 + \frac{\Phi_{\text{int}}}{A_{sat}\sigma}\end{aligned} \tag{16.2.25}$$

The first term on the right side of the equation corresponds to the absorbed solar radiation, the second term to the absorbed albedo from the Earth, the third term to the absorbed thermal radiation from the Earth, the fourth term to the absorbed radiation from the depths of space, and the last term to the heat generated by the satellite itself.

The given quantities take on the following values:

T_{sat} Temperature of the satellite (or any S/C).

T_{earth} Mean radiation temperature of the Earth = 254 K. This corresponds to an IR radiant exitance at Earth's surface of $0.25(1-a)\cdot 1361.5\ \text{W m}^{-2} = 235 \pm 17\ \text{W m}^{-2}$. Note: Terrestrial IR exitance from geothermal heat flow of merely $0.082\ \text{W m}^{-2}$ is negligible compared to the total terrestrial IR exitance.

T_∞ Black-body temperature of space = cosmic background radiation temperature = 2.73 K.

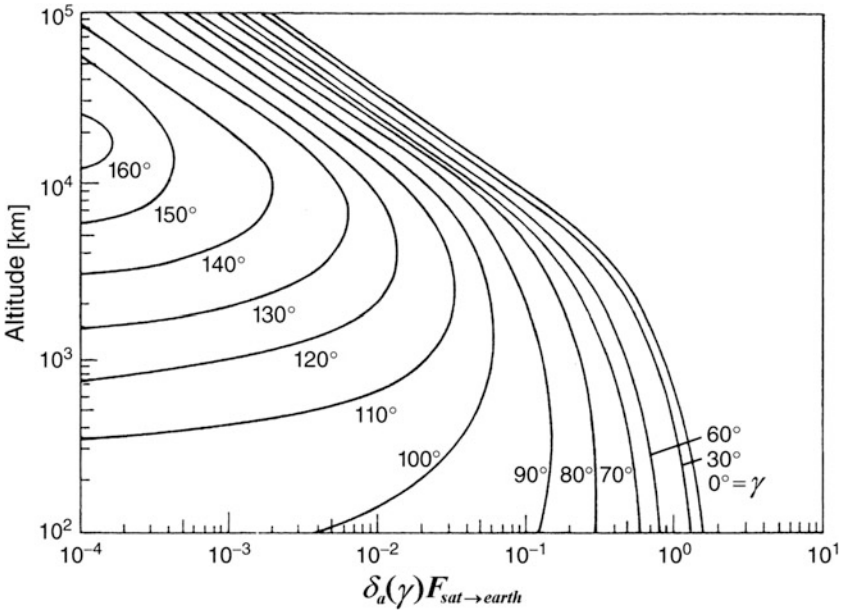


Fig. 16.8 The factor $\delta_a(\gamma)F_{sat\rightarrow earth}$ as a function of orbit altitude and angle γ (see text). *Credit* Fortescue (2003)

$T_{sun,eff}$ Sun's effective temperature at distance 1 AU, $T_{sun,eff} = (394 \pm 3)$ K, where ± 3.3 K is the seasonal change (Jan. 3/July 4).

Φ_{int} Heat flux generated within the satellite (internally).

A_{sat} Total surface of the satellite.

$A_{\text{sat}\perp sun}$ Surface of the spacecraft projected onto the direction to the Sun. The following is valid for a plate (sphere): $A_{sat\perp sun} = 0.5 \cos\theta \cdot A_{sat} (0.25 \cdot A_{sat})$.

α_{sol} $= \alpha_{sat}(T_{sun})$, the absorptivity of the surface of the S/C (see also Eq. (16.1.31) with $T_{sun} = 5762$ K) for solar radiation in the visible spectrum.

ε_{IR} $= \varepsilon_{sat}(T_{sat})$, the infrared emissivity of the surface of the S/C (see Eq. (16.1.26) with T $\approx$ 300 K).

a Albedo of the Earth: a = 0.07 − 0.85, annual average value: 0.31 [0.22 (equator) − 0.69 (south pole)].

δ_{sun} Sun visibility factor. δ_{sun} = 1: Sun shines on satellite. δ_{sun} = 0: Satellite is in the Earth's shadow.

$\delta_a(\gamma)$ Day-side visibility factor, i.e. the day-side proportion of Earth's surface visible at the satellite, $0 \leq \delta_a \leq 1$. It depends on the angle γ, which is the angle between the local vertical (nadir direction) and the direction of the incident solar radiation (not to be confused with Sun's beta angle β, see Sect. 14.1.1). Figure 16.8 depicts the product $\delta_a(\gamma)F_{sat\rightarrow earth}$.

Note *The above values are mean values with regard to space and time, and the error indications only refer to time variations. Spatial variations have not been considered. The latter can be quite considerable for the IR radiation of the Earth (poles!) and for the albedo. The albedo at the subsolar point (when the surface of the Earth is between the geocenter and the Sun) is 0.31 and decreases on either side towards the limb, until it becomes 0.0 at the terminator. This is because albedo represents a reflectivity, and the corresponding thermal flow decreases with the cosine of the radiation angle of the reflected ray. For more precise numerical calculations, one has to consider the beta angle β of the orbit and the current deviation from the subsolar point (see Clawson 2002).*

In the albedo radiation term the Earth always behaves like a black-body radiator with the temperature $T_{sun,eff}$ and an intensity reduced to the albedo. Apart from that, the bright daytime side does not always face the satellite. So the factor δ_a is the relation between the visible part of the daytime surface to the total visible surface of the Earth. As the fourth power of the temperature of space can be neglected with regard to all other temperatures, we finally get

$$T_{sat}^4 = \frac{\alpha_{sol}}{\varepsilon_{IR}} \left(\delta_{sun} \frac{A_{sat\perp sun}}{A_{sat}} T_{sun,eff}^4 + a\delta_a(\gamma) F_{sat\to earth} T_{sun,eff}^4 \right) + F_{sat\to earth} T_{earth}^4 + \frac{\Phi_{\text{int}}}{\varepsilon_{IR} A_{sat} \sigma} \tag{16.2.26}$$

Spherical Satellites Orbiting Earth

Next, we want to derive a simple formula to estimate the S/C temperature near any planet. For that and making the rough (but mostly sound) approximation that a spherical satellite (see Eq. (16.2.11)) is orbiting a planet with radius R at altitude h, then $A_{sat\perp sun} = 0.25 A_{sat}$ holds, and the above thermal balance equation can be reduced to

$$T_{sat}^4 = \frac{\alpha_{sol}}{4\varepsilon_{IR}} (\delta_{sun} + 2a\delta_a F_{sat\to Earth}) T_{sun,eff}^4 + \frac{1}{2} F T_{planet}^4 + \frac{\Phi_{\text{int}}}{\varepsilon_{IR} A_{sat} \sigma} \tag{16.2.27}$$

where

$$F = 1 - \sqrt{1 - \left(\frac{R}{R+h} \right)^2}$$

R radius of the planet

h orbit altitude of the S/C above the surface of the planet.

For a spherical satellite that orbits Earth we specifically get

$$T_{sat} = 279\ \text{K} \cdot \left[\frac{\alpha_{sol}}{\varepsilon_{IR}} (\delta_{sun} + 0.68 \cdot \delta_a F_{sat\to earth}) + 0.328F + \varphi \right]^{1/4} \tag{16.2.28}$$

with

$$\phi = \frac{\Phi_{\text{int}}}{\varepsilon_{IR} A_{sat} \sigma (279\text{K})^4} = \frac{\Phi_{\text{int}}}{\varepsilon_{IR} A_{sat} \cdot 344 \ \left[\text{W/m}^2\right]}$$

Note that Eqs. (16.2.25) to (16.2.28) are just coarse estimates of the actual temperature conditions. If more accurate temperatures are required, numerical modeling and calculations are indispensable, which are usually done jointly with the modeling of the S/C as described in the following sections. Nonetheless, the results so far achieved are a good first guess to obtain an overview of the overall temperature conditions.

16.2.6 α/ε Materials

In the thermal balance equation the factor $\alpha_{sol}/\varepsilon_{IR}$ is the key parameter to control the satellite's temperature. As shown in Fig. 16.9 there exist many materials spanning the range $0.1 \leq \alpha_{sol}/\varepsilon_{IR} \leq 10$. Accordingly, temperature reduction can be easily

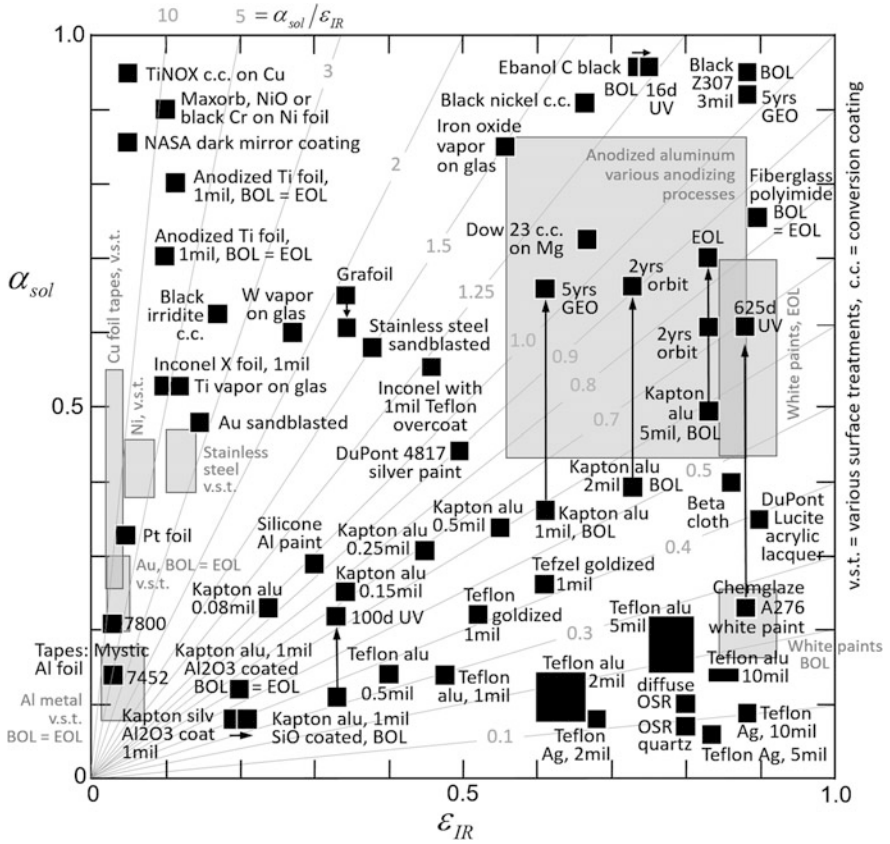


Fig. 16.9 Solar absorption coefficients α_{sol} and infrared emission rates ε_{IR} of different materials. *Data source* Gilmore (2002), Appendix A; © U. Walter

achieved by employing strongly emitting white paint (degrades significantly in space, though) or, even better, so-called **optical solar reflectors** (OSR) or **second surface mirrors** (SSM), namely **silver-coated Teflon** $\alpha_{sol}/\varepsilon_{IR} = 0.12$ or **aluminized Teflon** $\alpha_{sol}/\varepsilon_{IR} = 0.20$. They are made of ultraclear fluorinated ethylene propylene (FEP) Teflon with a thin vapour-deposited layer (0.04–0.4 mm) of silver or aluminum on one side. Silver/Teflon hardly changes under UV radiation, however it degrades rapidly with particle radiation $>10^{15}$ electrons/cm^3 with 5 keV up to 1 meV. SSMs owe their distinct thermal property to the fact that Teflon is a good thermal radiator, whereas the silver coating reflects the incoming sunlight efficiently. As Teflon on the exterior side is relatively inert, the advantage of silver-coated Teflon in relation to other SSMs (such as quartz over silver) is that it hardly degrades.

Outgasing is a special problem of thermal surfaces. Adhesives, elastomers, PC boards, lubricants, and paints as well as the jets of spacecraft reaction control systems generate gases in the vacuum of outer space, which deposit rapidly on surfaces. This can considerably change the absorption and reflection behavior. Apart from that, UV radiation also changes this undesired coating toward larger absorptivity. These impurities are probably the reason for long-term degradation of α/ε material. It can only be corrected by considering the corresponding safety margins.

Atomic Oxygen (AO) is a main cause of surface degradation in LEO. Metals are relatively immune to AO, but other standard thermal materials such as polyester, polyimides and paints are very sensitive to AO.

Electric Charging may occur at higher altitudes. Many thermal materials are excellent insulators, which may lead to sparks and thus damage of the surfaces. Conducting surfaces can remedy the problem. This is another reason why aluminized/ silver-plated Teflon (linked with satellite mass) is a good solution.

Surface Degradation Generally, in thermal design the degradation of the selected coating is a key parameter and a lot of effort is spent by thermal paint developers to characterize the so-called end-of-life (EOL) properties of thermal paints, i.e. their $\alpha_{sol}/\varepsilon_{IR}$ at the end of the spacecraft's life. Sometimes paint samples are sent into space in so-called exposure experiments, to let the paint experience the space environment and to measure the resulting thermo-optical properties after a given exposure time. Other samples can be exposed to accelerated life tests where the samples are exposed to a mix of different radiation sources and atomic oxygen to simulate the aging process.

16.3 Thermal Modeling

P. Hager and M. Czupalla

In Sect. 16.2 the thermal equilibrium and heat exchange between surfaces were derived quite generally from basic physical considerations. From this the heat balance and thus the resulting temperatures of a S/C in space can in principle be derived. This analytical approach provides temperatures of entire bodies or surfaces in thermal equilibrium under a given steady thermal environment.

In reality, the temperatures of a spacecraft are not uniform over its geometry and the thermal environment changes constantly and heavily over an orbit. For instance, a body orbiting a planet is exposed to the Sun on the day-side and not on the night-side. In order to keep critical S/C subsystems within given temperature limits, it is key to determine the temperature distribution of a S/C under such a changing thermal environment.

Pivotal to the determination of these actual S/C temperatures is the thermal modeling process, which on one hand is based upon the above derived analytical considerations, but on the other hand is a quite elaborate process on its own. The purpose of thermal modeling is to verify through a thermal analysis that the designed *thermal control system* (TCS) of a space system meets a set of thermal requirements. The thermal control system is the entirety of thermal control hardware and concepts to control S/C temperatures, temperature gradients and temperature stability.

Based on the results of Sect. 16.2, we introduce in this section the thermal modeling process as implemented in today's thermal design tools for space applications. The approach is as follows. In the first Sect. 16.3.1 we outline the typical requirements for a thermal control system that must be achieved by the thermal design, along with the boundary conditions limiting the solution space.

The subsequent Sect. 16.3.2 extends the simple analytical approach derived for an overall spacecraft heat balance to a series of heat balance equations capable of capturing spatial and temporal temperature gradients. This section is the starting point for the numerical approach behind thermal models.

Next, Sect. 16.3.3 explains how thermal models are set up and how a space system is partitioned into multiple nodes, each having its own heat equation with specific boundary conditions.

The two core parts of a thermal model, the so-called *Geometrical Mathematical Model* and the *Thermal Mathematical Model*, are introduced in two Sects. 16.3.4 and 16.3.5 thereafter.

In Sect. 16.3.6, we show how thermal models are applied in a thermal analysis process, in order to fully validate the thermal design.

The last Sect. 16.3.7 demonstrates by some simple case studies how thermal mathematical models are set up and analyzed.

16.3.1 Thermal Requirements and Boundary Conditions

To assure a specified thermal behavior of a S/C, the thermal modeling of a S/C in its entirety and its thermal environment are required. Thermal requirements in turn are derived from functional and/or performance requirements. The key requirements for thermal analysis are the temperature limits of the individual components, spatial temperature gradients, and temporal temperature gradients. Temporal temperature gradients are more commonly referred to as *temperature stability*. Temperature limits are distinguished into operational and non-operational temperature limits and applied if recurring or 'off-the-shelf' components are used. A typical example for recurring components with a narrow temperature range are batteries. For batteries, particularly Li-ion, the operational design temperature-range is approximately −10 to +30 °C while their non-operational temperature design temperature-range approximately is −20 to +60 °C. For new developments or for entire instruments and at the beginning of a project, predefined temperature limits most often do not exist. In such cases, the boundary values are a result of the thermal analysis at higher system levels.

Spatial gradients are specified to forestall thermo-elastic problems, which might lead to functional loss or performance reduction, such as pointing performance reduction of a spaceborne telescope. Typical spatial gradients at the interface between satellite platform and an instrument vary between 3 and 1 °C/m for missions with a high pointing performance requirement. For interfaces between other components of the spacecraft, requirements typically concern interface temperatures and transient conductive as well as radiative heat fluxes.

Temporal gradients chiefly, but not exclusively, impact performance of components, such as optical detectors. An example for a temporal thermal stability requirement is 0.1 °C per orbit. Because thermal stability sometimes is transformed into the frequency domain, it then is specified in units $\mathrm{K}/\sqrt{\mathrm{Hz}}$.

Further requirements to the thermal control system, amongst others, are allocated heater power, mass requirements, or restrictions due to the available volume.

Requirements and boundary conditions are imposed primarily by the environment, in which a space-system is operated (see Sect. 16.2.5), but also by different environments on the ground. For instance, consider a spacecraft that is transported by an aircraft, truck, or ship, sits on the launch pad, or is stored in a clean room. Then it might be exposed at one extreme to tropical and at the other to tundra-like conditions. Although such additional requirements derived from launch campaign activities usually do not drive the S/C thermal design, they need to be taken into account.

Yet other thermal requirements and boundary conditions apply for components being transported in a pressurized vessel to the ISS, along with the then existing convective heat exchange. These peculiarities will not be addressed here.

The thermal requirements and boundary conditions are either verified through thermal analysis, or tests, or a combination thereof. Thermal tests will not be addressed here. For a detailed discussion of tests see for example Gilmore (2002) or

ECSS-ST-E-31C. The verification through thermal analysis is exclusively achieved through the application of thermal modeling.

In summary, the objective of thermal design is the balancing of the incoming and outgoing heat fluxes in a way that keeps the resulting temperatures as well as spatial and temporal temperature gradients in the ranges required to fulfill mission performance. One should keep in mind, though, that the root cause of a temperature problem, is not a given temperature distribution, but the underlying heat fluxes, which must be examined. So, adjustments of the thermal design are achieved by modification of heat fluxes.

16.3.2 Heat Equation

Having sketched the thermal requirements to be verified, we now address the question of how to determine the temperatures inside a spacecraft. The fundamental approach, introduced in Sects. 16.1 and 16.2, allows the computation of heat fluxes and temperatures for simple geometries and steady state conditions on the outside of a spacecraft or for an idealized one-piece-of-spacecraft model (a.k.a. *one-node model*). Yet, the temperatures thus derived are only qualitative and by no means representative of the temperatures of individual components. This is because the temperatures of individual components are also affected by the internal heat dissipation, internal thermal radiation, internal heat convection (convection of gases or liquids), and internal heat conduction (in solid bodies or liquids). Internal temperatures therefore must be calculated by solving the physical heat equation potentially including all these factors

$$\rho c_p \frac{\partial T}{\partial t} = -\nabla \cdot \boldsymbol{\phi} + \phi_{\text{int}}(\boldsymbol{r}, t) \quad \textbf{heat equation} \tag{16.3.1}$$

Here $\boldsymbol{\phi}$ is the 3-dimensional heat flux vector field, i.e. the vector field of heat flow rates per unit area, the term $\phi_{\text{int}}(\boldsymbol{r}, t)$ describes contingent heat flux sources at position $\boldsymbol{r}$ within the S/C, $\nabla = (\partial/\partial x, \partial/\partial y, \partial/\partial z)$ is the nabla operator, ρ is the local mass density, and c_p is the mass-specific heat capacity.

> **Note** *Because the heat equation is a key equation in the physics of transport phenomena, flux here is defined as flow rate per unit area, different from the terminology of flux in radiometry* (*see Note in* Sect. 16.1.1). *This is why here we use the symbol ϕ for flux rather than Φ as in* Sects. 16.1 and 16.2.

Discretized Heat Equation

In view of the upcoming modeling approach we discretize the system subject to heat transfer (here S/C) into small finite elements (a.k.a. *nodes*). The heat equation for a single node i then reads

$$C_i \frac{dT_i}{dt} = \sum_{j \neq i} \Phi_{j \to i} + \Phi_{i,\text{int}} \tag{16.3.2}$$

Here $C_i = c_{p,i}\rho_i V_i$ is the heat capacity of node i, $\Phi_{i,\text{int}}$ is the internally generated heat flux (equals heat flow rate in radiometric terminology), and $\Phi_{j \to i}$ are the radiometric heat fluxes to node i from all other nodes j linked to i.

To solve the node equations for a system we need to determine the $\Phi_{j \to i}$. There are two different types of fluxes with quite different temperature behavior.

Heat Flux by Convection and Conduction

For gas-based heat convection and solid-body-based heat conduction between two point masses at the temperatures T_i and T_j the following linear relation is valid

$$\Phi_{j \to i} = G_{ij}\left(T_j - T_i\right) \tag{16.3.3}$$

with the so-called *coupling coefficient*

$$G_{ij} = \begin{cases} \kappa A_{ij}/l_{ij} & @ \text{ heat conduction} \\ KA_{ij} & @ \text{ heat convection} \end{cases}$$

where

κ heat conduction coefficient
K heat convection coefficient
A_{ij} cross-section area of interface
l_{ij} distance between nodes.

In most thermal analysis software the coupling coefficient is termed *CONDUCTOR* (capital letters denote thermal-model-specific terms).

Radiative Heat Flux

The simple and linear form of Eq. (16.3.3) is not valid for thermal radiation. For that we rather have to resort to Eq. (16.2.19). It reads

$$\Phi_{j \to i} = \alpha_i \varepsilon_j A_j F_{j \to i} \sigma\left(T_j^4 - T_i^4\right) = R_{ij}\sigma\left(T_j^4 - T_i^4\right) \tag{16.3.4}$$

Here the *radiation coupling coefficient* in thermal analysis is defined as

$$R_{ij} = \alpha_i \varepsilon_j A_j F_{j \to i}$$

We recall that Eq. (16.3.4) strictly holds only for $\alpha_j(T_i) \approx \varepsilon_j\left(T_j\right)$ plus $\alpha_i\left(T_j\right) \approx \varepsilon_i(T_i)$. This holds for nodes within a S/C where $T_i \approx T_j$ (see Eq. (16.1.35)) and for heat exchange with external sources (Sun, albedo, Earth IR, space), which in good approximation can be considered as black or gray bodies (see end of Sect. 16.1.5). Therefore, Eq. (16.3.4) can be generally applied in thermal modeling.

Node Equation

Inserting these results for heat convection/conduction and thermal radiation into Eq. (16.3.2) we finally arrive for a single node i at the so-called *node equation*

$$\boxed{C_i \frac{dT_i}{dt} = \sum_{j \neq i} G_{ij}\left(T_j - T_i\right) + \sum_{j \neq i} R_{ij}\sigma\left(T_j^4 - T_i^4\right) + \Phi_i} \quad \textbf{node equation} \qquad (16.3.5)$$

Here Φ_i may now be the sum of node's heat fluxes due to internal heat sources or due to boundary conditions, i.e. heat flows from the environment external to the node.

In conclusion we have established a simple relation for the thermal conductivity Eq. (16.3.3) (linear) and for thermal radiation Eq. (16.3.4), which have to be inserted into the node Eq. (16.3.2). In Sect. 16.3.5 the conductive and radiative coupling coefficient will be addressed in more detail.

Boundary Conditions

In order to solve Eq. (16.3.5), boundary conditions need to be provided, either temperature or heat flux boundary conditions. Mathematically, a temperature is a boundary condition of type 1, i.e. a Dirichlet boundary condition, while heat flux is of type 2, i.e. a Neumann boundary condition. Physically, a temperature boundary (type 1) condition implies a sink (or source) of infinite capacity, as for example the background temperature of deep space or the temperature of a celestial body. A fixed temperature as boundary in a spacecraft has to be treated with care, as it requires a tight control of input power and duty cycle. A heat flux boundary condition for example is a heat flux from the Sun, from Earth, or from any other celestial body. Internally to a spacecraft, a constant heat flux is generated by heaters or by dissipation of electric energy in electronic boxes. Heat fluxes are the prevailing boundary conditions.

The simplified approach expounded above hits its limits already in design phase A of a space mission project. Complex numerical efforts are required to solve a huge set of dependent linear differential equations as given by Eqs. (16.3.3)–(16.3.5) with a given set of boundary conditions. Therefore, a structured mathematical approach of modeling the thermal network in a S/C becomes a necessity. This approach will be addressed in Sects. 16.3.4 and 16.3.5.

16.3.3 Thermal Model Setup

We recall that at a given position $\boldsymbol{r}$ within a given body and time t the heat Eq. (16.3.1) describes the incoming and outgoing heat fluxes (right hand side) and for flux imbalances the resulting temperature changes per time unit (left hand side), whereby the rate values are determined by the local heat capacity. A negative heat imbalance means cooling, while a positive means heating. The heat may enter or leave the body via multiple paths. Hence and independent of the paths, the driver for cooling or heating is the total heat flux.

However, the heat balance equation in its time-invariant form (Eq. 16.2.26) yields only the temperature for the total body at heat balance, that is the average temperature of that entity. Temperature changes and distribution or gradients along any dimension of the body cannot be captured by this approach. The nature of the thermal space environment, however, dictates that the temperature of a satellite cannot be uniform and constant when the Sun illuminates the hull of an orbiting S/C on one side, while the coldness of space acts as a heat sink on the other side. Hence, we are in need of a method that enables us to predict spatially and temporally distributed temperatures as caused by the environment acting via different parts on the S/C.

Model Discretization

The answer to this challenge is a spatial and temporal discretization with the finite difference method. The total S/C is dissected into small and virtually isothermal parts, which can exchange heat with each other and with their environment. The discretized isothermal parts are called *nodes*. A thermal node is defined by the isothermal temperature and the heat capacity (CAPACITANCE) of the part it represents. Nodes do not carry any information about the heat transport properties of the represented part.

The nodes are connected by so-called *CONDUCTORS*, which represent the heat transport capabilities between the two thermal nodes the CONDUCTOR connects. CONDUCTORS can be either conductive, radiative, or convective (not covered here). They are derived from a combination of the heat transport properties of both connected nodes.

In order to manage the network of nodes and CONDUCTORS, a customized Finite Difference Method (FDM) approach is used in space thermal engineering, the so-called *lumped parameter method*. A good introduction to this method is found in Gilmore (2002) and *ESATAN-TMS* (2010). The lumped parameter method has its origin in the analogy to electronic circuits where current flows are modeled by equivalent resistors and capacitors. In the customized FDM approach, nodes and CONDUCTORS have split functionalities and are not necessarily bound to a geometrical representation, though often simulating one. The FDM nodes, representing the heat capacity property CAPACITANCE, describe the transient temperature evolution driven by the heat imbalance, while the CONDUCTORS determine the heat transport network between the nodes and as such drive the spatial temperature distribution. A schematic visualization of the FDM discretization concept used for thermal modeling is shown in Fig. 16.10.

Note that the concept of split functionalities distinguishes the finite difference method (FDM) from the finite element method (FEM), which is used for instance in mechanical engineering. The FEM approach is based on the concept of elements and nodes. Elements are simple finite geometric forms that make up the FEM mesh of the body under consideration. Nodes are the vertices of an element and hold the parameters (deformation, stress, etc.) for the discretized function acting on an element. Being bound to geometry, FEM methods are well suited for conductive problems and quite complicated for radiative ones. In the FDM approach, however,

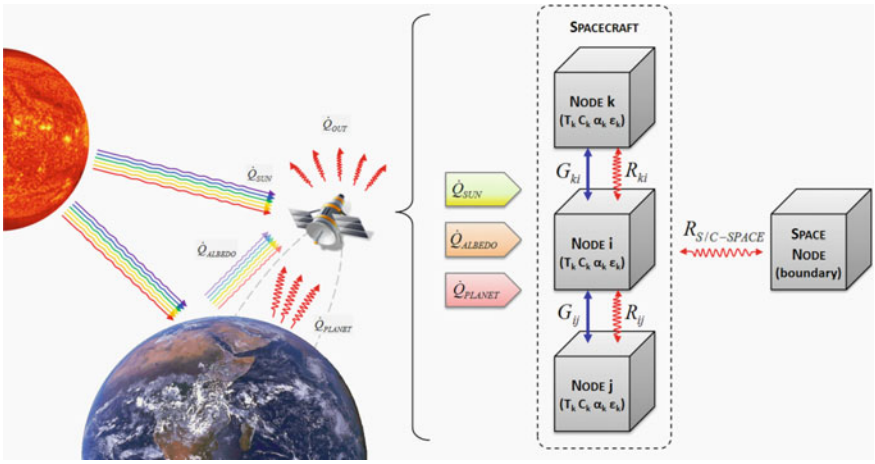


Fig. 16.10 Schematic of the finite differences method applied in S/C thermal modeling

any node can be connected to any other by means of CONDUCTORS. The connection is not geometry dependent which simplifies the modeling of radiative CONDUCTORS.

Thermal Mathematical Model Versus Geometrical Mathematical Model
How is the mesh set up for a specific spacecraft, how are the coupling coefficients (CONDUCTORS) G_{ij} and R_{ij} determined, and how is the discretized heat equation solved? In this paragraph we introduce the setup of thermal models. Thermal models typically consist of a *geometrical mathematical model* (GMM) as pre-processor for the *thermal mathematical model* (TMM). The GMM often comprises also mission specific functionalities for example orbital features that impact heat fluxes or boundary conditions. The logic of this model setup and the connections and interdependencies between the GMM and TMM are explained hereafter and are schematically visualized in Fig. 16.11.

A discretization of a given body or part into nodes yields as many heat equations as there are nodes. The node Eq. (16.3.5) is a first order differential equation. A system with n discretized nodes hence yields n coupled first order differential equations, the solution of which determines the nodal temperatures as enforced by the applied boundary conditions. Such a thermodynamic network with its boundaries is referred to as the *thermal mathematical model* (TMM). It will be discussed in more detail in Sect. 16.3.5. The TMM is the backbone of a thermal analysis.

The TMM can be set up without the need of a geometric model of the system. The discretization can be performed manually, assigning parts of a modeled entity to different nodes and by applying their connections as CONDUCTORS and the necessary boundary conditions in the form of temperatures or heat loads. Particularly for small conductive problems, such as the case study discussed in Sect. 16.3.7 a TMM-only approach may be sufficient.

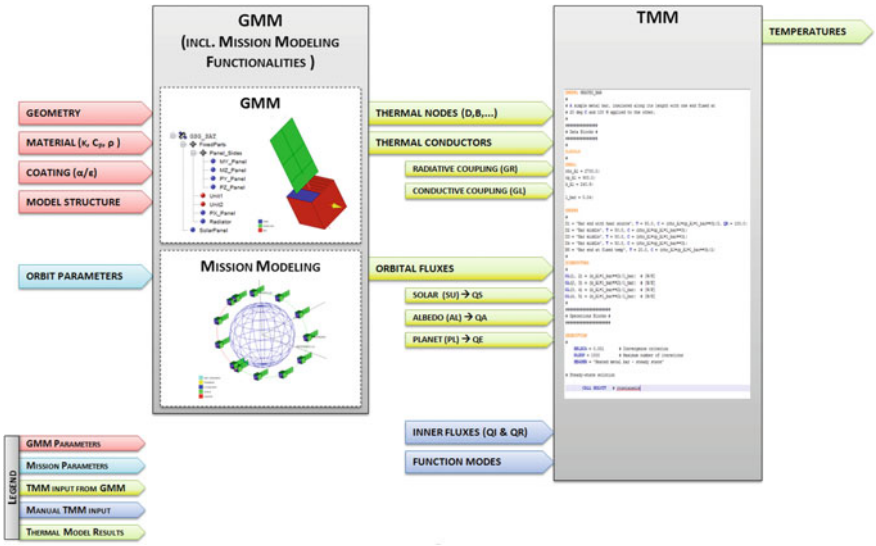


Fig. 16.11 Dependencies between geometry and mission modeling functionalities of a GMM and TMM as modeled and depictured by ESATAN-TMS

However, as soon as radiative heat couplings matter, the manual TMM setup is tiresome and error prone. In particular, for actual S/C geometries the analytical calculation of radiative couplings with their view factors is rarely feasible. In addition, a S/C usually continuously changes its orientation relative to the external environment (planet, Sun, etc.) and also the orientation between the S/C subsystems may vary (e.g. rotating solar arrays). These facts impose dynamic boundary conditions, which must be captured in the thermal network simulated by the TMM.

The generation of these elaborate radiative couplings and environmental radiative heat fluxes is the main purpose of the GMM. With evolving software capabilities the GMM nowadays is also employed to determine the environmental heat flux parameters between the nodes of the discretized system. It can also provide the thermal nodes and the conductive heat links to the TMM. All thermal nodes of a GMM then are also used in the TMM. However, not all TMM thermal nodes are represented in the GMM. From this it should be clear that it is not possible to compute temperatures of any thermal node or the entire S/C with the GMM alone. It merely provides the necessary inputs for the thermal network simulated by the TMM. It depends on the specific problem as to which parameters of the thermal network are derived from the GMM and which are included manually into the TMM by the thermal engineer.

In today's thermal engineering software packages such as ESATAN-TMS, SYSTEMA THERMICA, THERMAL DESKTOP, or NX I-deas TMG, the GMM acquires more and more abilities and thus importance as established by its graphical user interface (GUI). However, manual intervention in the TMM is still necessary to work out particular features of a S/C thermal control system, such as heater

regulation, fluid networks, or functional modes switching. Different from the GMM, which usually is handled via a graphical user interface, a list of semantic code lines is the only input to TMM calculation, which can be run in batch mode. The list can be manipulated by any text editor. The physical background, the mathematical representation, and thus the semantics for the TMM is independent of the used software package. Yet, in order to be compiled and interpreted, the TMM needs to follow a defined syntax. The syntax varies somewhat between thermal software packages. It is far beyond the scope of this book to describe differences in the syntax. The interested reader is referred to the software manuals.

Let us now have a closer look at GMM and TMM.

16.3.4 Geometric Mathematical Model (GMM)

As shown above, the GMM supports the discretization of a modeled structure into thermal nodes and CONDUCTORS. It is mainly used to determine the radiative couplings between the nodes within a S/C and between the S/C and the environmental heat sources and sinks (Sun, planets, space). However, it is also a useful method of generating the thermal nodes and the conductive heat links.

From Meshing to Thermal Network

How do we get from a given geometry of a body to the geometrical nodes? Usually, satellites are made of thin plates and foils. Therefore, in the thermal network the 3D volumetric parts of a satellite are approximated by 2D shells. The 2D shells are further discretized in 2D polygonal elements (see Fig. 16.12). The thickness of the volumetric body is represented by a parameter in the 2D polygons. The process is called *meshing*. An example of a meshed satellite is shown in Fig. 16.13. Each 2D element typically represents one thermal node. However, in most software packages it is also possible to represent multiple surfaces by just one thermal node. The nodes then contain the combined information from all the assigned surfaces, while all assigned surfaces are given the same thermal node number.

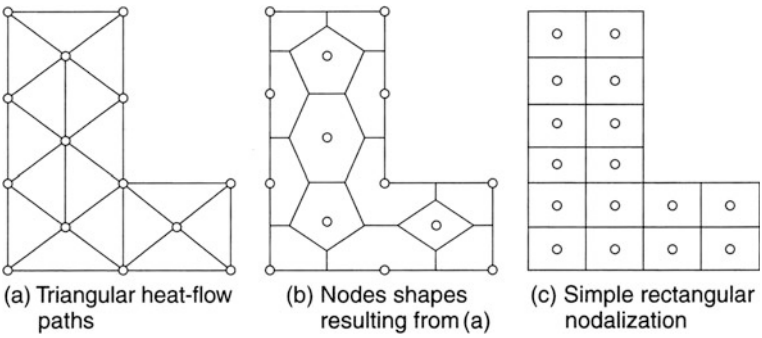


Fig. 16.12 Discretization of a 2D shell into three different types of polygonal elements that are adapted to the shell geometry. *Credit* Gilmore (2002)

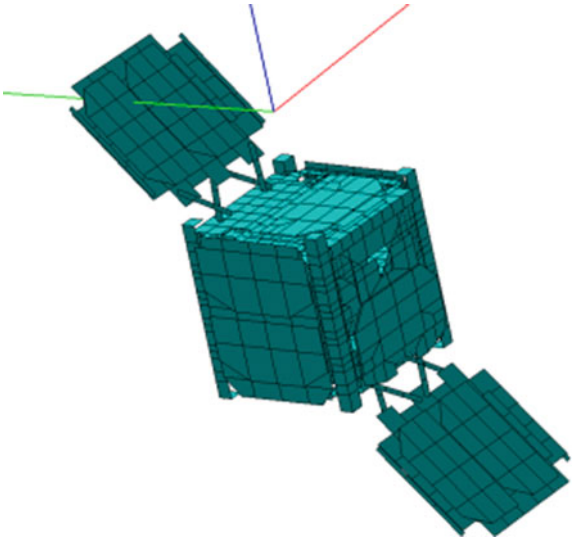


Fig. 16.13 Example geometrical mathematical model of a CubeSat

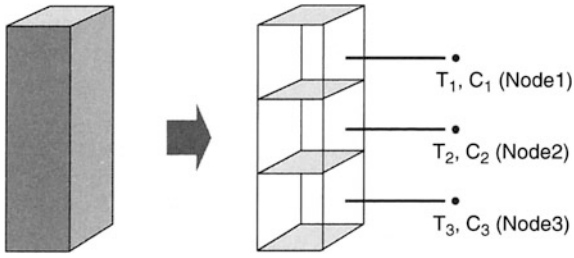


Fig. 16.14 A cuboid meshed into three thermal nodes with assigned temperatures and heat capacities. *Credit* Gilmore (2002)

For volumetric bodies some software packages offer 3D elements for more precise modeling, for example thermal isolation of cryogenic launcher upper stages. Because the vast majority of the thermal meshing is performed with 2D shells, while 3D shells are only used for rather special problems, we do not treat 3D elements in the following.

For a thermal node each assigned surface is given a specific geometry (defined by points), thickness and material (with defined specific heat and mass density). The thermal properties are then assigned to each surface to make up the corresponding thermal node (see Fig. 16.14). The heat couplings derived from the geometry establish the total conductive network between all surfaces.

In addition to the thermal nodes and conductive network, each surface of the GMM has a set of coating-dependent thermo-optical properties. These properties cover the full range of possible physical parameters. They require absorption values α and emission values ε, which can be wavelength dependent. The surfaces can be opaque or transparent and can act as diffuse Lambertian or specular reflectors. The geometrical information provided by the surface is utilized to determine the view factors $F_{t\to r}$ between the surfaces and thus between the different geometries, subsystems, etc. Here index t denotes the transmitting surface and the index r denotes the receiving surface.

With these view factors and the thermo-optical properties the radiative links between two thermal nodes are derived as given in Eq. (16.3.4). Note that it is sufficient to derive $F_{t\to r}$ while $F_{r\to t}$ can be derived according to the reciprocity relation Eq. (16.2.7). Nevertheless and even for coarse meshes a huge number of radiative links is typically generated. This results from the fact that by definition all surfaces, and as such all thermal nodes in sight of one another, are connected radiatively. Each radiative connection results in one corresponding radiative link.

In most software packages it is also possible to assign two thermal nodes (not necessarily spatially neighboring nodes) to one surface, namely the common surface of two adjacent nodes. Because each node can have a custom set of properties (type of material, thickness, thermo-optical properties), the conductive or radiative link between the nodes on different sides then is provided as a parameter of the surface. This approach is widely used to simulate surfaces covered by multi-layer isolation (MLI) blankets, or can be used for honeycomb structures.

GMM De-featuring

A GMM is typically much more simple than a geometric model for FEM. Many details are omitted, as for example bore holes or chamfers. Complicated geometric shapes are simplified, as in most cases they do not play a significant role for both the conductive and the radiative model. The process in which such geometry details are suppressed or simplified is called *de-featuring*. The amount of de-featuring and hence the fidelity level of the applied mesh strongly varies with the modeled problem and with the experience of the thermal engineer.

The GMM must at least comprise elements representing large surfaces with significant temperature differences or with a significant view factor to the external environment. Small surfaces with similar temperatures have very small radiative links to each other and therefore are negligible. However, dismissing insignificant surfaces from the GMM does not necessarily imply that the corresponding thermal nodes can be omitted from the thermal network. Those nodes and the conductive links between them can still be implemented manually in the TMM.

> **Note** *In some software packages GMM body surfaces are permitted to only conduct and not radiate to the environment. In that case they are considered to only generate thermal nodes and conductive links and do not participate in the view factor calculations.*

Unexperienced users of thermal software tend to generate large numbers of nodes without necessity. However, to reduce computation time, it is good practice to reduce the number of generated surfaces and thermal nodes as much as possible. Distributed thermal nodes are only needed when spatial temperature gradients matter. Almost isothermal surfaces should be modeled coarsely rather than finely meshed. It is the art of thermal node discretization to find the right model complexity for sensible and easily interpretable results with the lowest computation time.

GMM Node Versus TMM Node Generation

Thermal nodes can be generated in the GMM or TMM. Today, the vast majority of thermal nodes are generated in the GMM. A GMM includes at least all relevant radiative surfaces, which quite often also cover the conductive elements. However, elements which predominantly interact conductively (e.g. cables, thermal straps, heat pipes) are

typically not captured in the GMM. They are either added as thermal nodes with conductive links in the TMM or sometimes added just in the form of conductive links without an associated thermal node. A classic example are thermal straps. With their high conductivities they effectively exchange heat between the thermal nodes they connect to. But they are rarely included as thermal nodes in GMMs as their temperature rarely matters and their contribution to radiative exchange is negligible due to their small size.

For certain S/C parts, such as payloads, it is essential to capture fine-grained spatial temperature gradients and thus temperature maps to analyze thermo-elastic deformations. The GMM mesh then typically requires a higher fidelity implying more thermal nodes, the temperatures of which are transferred to finite element models to provide the required thermo-elastic deformations.

GMM and the External Environment

The basic GMM just contains the meshed geometry of the S/C, which is sufficient to establish the thermal nodes and the S/C-internal conductive and radiative network. However, as shown above, the TMM also requires boundary conditions to derive the temperatures for a given S/C mission.

Important boundary conditions are the radiative links between the S/C and the external environment, i.e. emission to space and absorption from Sun, planet IR, and albedo. These radiative links and hence heat fluxes continuously change with position and attitude of the S/C in its orbit. Because these external boundary conditions change with orbital position and S/C attitude, the GMM must be likewise adjustable. Therefore, beyond the basic GMM, which just supports the meshing of geometries, GMM software packages often also provide corresponding mission modeling functionalities. These allow the user to virtually put the S/C GMM (its meshed geometry) into an orbit around a planet or on a planetary surface. According to mission requirements, the S/C then is oriented with respect to the Sun, planet, and flight direction. Finally, the radiative heat exchange between the S/C and the external environment is derived for a finite number of sampling points in time. For every sampling point a radiative analysis is performed and the resulting radiative couplings (to space, and within the S/C) and the absorbed radiant fluxes (Sun, planet IR, albedo) are stored in lookup tables. The GMM mission modeling functionalities allow the simulation of closed orbits, chained orbital arcs, as well as open transfer trajectories with varying S/C attitudes to the external environment. The S/C attitude can be predefined, for example certain surfaces (e.g. solar arrays) are always oriented to the Sun, while other surfaces (e.g. an instrument) point to the "nadir"—to the planet.

> **Note** *In GMM the number of orbital sample points is the driving factor for the sampling frequency in the resulting TMM. Even though the TMM might utilize many more time steps over an orbit to derive the temperatures, the number of sampling points in the GMM lookup tables for the radiant flux will always limit the thermal precision over an orbit. For TMM time points in between the GMM table points, radiant fluxes will be interpolated.*

Also the external environment can be customized to some extent. For example, the thermal properties of planets can vary with orbital longitude and altitude. Some software packages also vary the planet's albedo accordingly. Also, the direction of solar radiation can be simulated to be either parallel at planetary distances or to be converging for missions close to the Sun. A recent novelty in environmental simulation is the capability to cover three body problems, such as a S/C on the surface of a planet with the Sun or moon passing over it.

A typical effect implemented in a GMM is S/C aging. S/C's thermo-optical properties usually vary (degrade) with age. Most importantly, absorptivity in the UV and visible spectrum increases with age yielding a colder S/C at the beginning and a hotter S/C at the end of a mission. Therefore, analyses are usually performed for the beginning of the mission, the so-called *beginning of life* (BOL) and for the end of the mission, the so-called *end of life* (EOL). Correspondingly, the mission feature software allows to mimic lowest (at EOL) and highest (at BOL) incident radiant fluxes by varying the solar constant as well as the albedo and the temperature of a planet. These varying thermo-optical properties of the S/C and of the external environment (Sun, albedo, etc.) are often combined to so-called *radiative cases*, to verify the thermal design of the analyzed S/C. Classically, a cold-BOL case and hot-EOL case serve as such extremes.

Monte Carlo Ray Tracing

In summary, a GMM determines the radiative heat exchange between the S/C and its environment and inside a S/C. In Eq. (16.2.4) we have seen how the radiant flux emitted from a transmitter surface and absorbed by a receiving surface is derived. Equation (16.2.19) tells us which parameters are needed to describe the radiative heat exchange between two surfaces, in particular thermal nodes. In both expressions the view factor $F_{t\to r}$ between the affected geometries (receiver and transmitter) plays a major role. In fact, it is the most difficult factor to derive, which historically was the driving force to develop GMMs. To determine view factors analytically between surfaces in 3D (cf. Sect. 16.2.2), which may even change their mutual orientation (moving parts of a S/C) and orientation (attitude) with respect to the environment (Sun, planet, etc.), would be extremely challenging and prohibitively lengthy for a S/C.

Therefore, GMMs typically utilize the numerical approach called *Monte Carlo ray tracing* (Fig. 16.15). The approach is very intuitive and can be described as follows. We first recall that the view factor $F_{t\to r}$ effectively is that fraction of the hemisphere (dome) of a transmitting plane surface, which is occupied by the receiving surface. Also remember that a typical Lambertian (diffuse) transmitter emits electromagnetic waves uniformly in all directions. These two facts are the foundation of the Monte Carlo ray tracing method.

In essence, this method simulates light beams emitted from random spots on a transmitting surface into random directions (hence the name Monte Carlo). Each beam is supposed to carry a given defined total number of emitted rays (photons). For example, if 1000 rays are fired from a sphere that emits 1 W s, each ray carries 1/1000 W s.

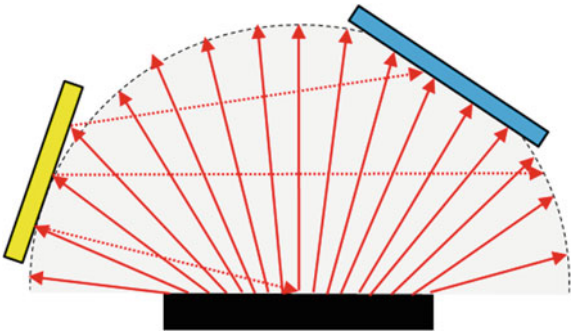


Fig. 16.15 Visualization of the Monte Carlo ray tracing method of a diffuse emitter radiating towards a blue receiver and a yellow with specular reflection

The rays "hit" absorbing surfaces that occupy fractions of the hemisphere. The Monte Carlo simulation records the angle under which a ray hits an absorbing surface (as it is part of the view factor) and according to the thermo-optical properties of the hit surface determines whether it is absorbed, transmitted, or reflected, or any combination thereof. Reflectance can be either specular or diffuse. For specular reflection the ray is simply "forwarded" according to the reflection law, while for diffuse reflections each ray becomes the source of a new ray burst of many rays carrying together the total energy remaining in the ray after the reflection.

The reflected or transmitted ray propagates further until it hits the next surface. This method allows the capturing of view factors of an emitting surface to surfaces that may not even be in direct sight. This intricate effect may be decisive for the thermal design of a S/C and is not captured with the so-called *Gebhart factor method* applied by some radiative coupling software codes.

Each ray is tracked until a given lower limit value of its virtual energy is reached, whereupon the ray is no longer considered or truncated. Overall, the ray tracer records how much energy of a given transmitter is deposited at hit receiver surfaces (direct, reflected, and transmitted rays). The view factor $F_{t\rightarrow r}$ of a receiving surface then is the total energy of all absorbed incoming rays under the corresponding angle versus the energy of all rays fired from the transmitting surface.

This approach is applied to all radiative heat exchanges between nodes within the S/C and between the S/C and the external environment. In the software the external environment is also approximated as surfaces around the S/C, however autonomously by the software not seen by the user. From the derived view factors the GMM creates radiative couplings between all surfaces and with the external environment. Because ray tracing is performed for every surface and for every orbital position and attitude of the S/C, this can result in extensive computation times and extremely big data arrays as inputs for the TMM.

To limit the data exported from the GMM to the TMM, often a filter is applied to the arrays, which removes extremely small radiative couplings. This approach, though, becomes dangerous when fine meshes are applied, because they result in a huge number

of extremely small radiative couplings. If the absolute limit then is not carefully adjusted, a noticeable error in the radiative heat exchange network may be introduced.

To further limit computation time and the size of the exported radiative couplings, the GMM of a S/C is often split in so-called *cavities*. Cavities are S/C sections, particularly inside a S/C, whose surfaces never change their orientation relative to each other and are never exposed to the external environment. Within such static cavities ray tracing need to be applied only once, while for surfaces exposed to the external environment it must be performed for any orbit position and change in S/C attitude.

16.3.5 Thermal Mathematical Model (TMM)

How are external heat fluxes, as determined by the GMM, interrelated to the internal heat fluxes due to temperature differences and to internal heat dissipation? Let us have a closer look at the Thermal Mathematical Model (TMM). In Sect. 16.3.3 we introduced the general concept behind the TMM. The TMM is a mathematical representation of the thermodynamic network, as given by the discretization of the spacecraft. This thermodynamic network consists of thermal nodes. Each thermal node in a TMM is represented by the node Eq. (16.3.5)

$$C_i \frac{dT_i}{dt} = \sum_{j \neq i} G_{ij}\left(T_j - T_i\right) + \sum_{j \neq i} R_{ij}\sigma\left(T_j^4 - T_i^4\right) + \Phi_i$$

The transient part of the thermal network is represented by the capacitors C_i of the thermal nodes, i.e. the lumped finite volumes. Their key physical parameter is the temperature T_i, which is a state variable describing the level of stored energy within a node. By definition, the temperature is constant within the volume of that thermal node. The flow of energy from one finite volume to another is driven by differing temperatures, and its amount is given by the conductive or convective, G_{ij}, and radiative, R_{ij}, coupling coefficients, which were introduced in Sect. 16.3.2. In addition, Φ_i captures heat fluxes due to internal heat sources and external heat fluxes due to the space environment.

The node equations for all nodes, which are a system of coupled linear differential equations, are key for a TMM. Several numerical solvers are available to determine the temperatures of all nodes from the node equations. Prior to solving the node equations the following steps need to be carried out for all nodes, in this sequence:

1. **Determine conductive and convective coupling coefficients** G_{ij}
   In aerospace thermal engineering, (linear) conductive coupling coefficients (CONDUCTORS) are usually denoted by $G_{L,ij}$ and convective (fluid) coupling coefficient by $G_{F,ij}$. In this book we follow the convention that G_{ij} stands for either a conductive or convective coupling coefficient.

Although the physics to determine the coupling coefficients is unequivocal, in practice many details of how the heat is transferred are vague and therefore their determination requires a lot of testing or expert knowledge. We will focus on conductive coupling coefficients here, because they occur most often. Convective coupling coefficients are only necessary for fluid loops or if heat transport in a pressurized vessel such as the ISS needs to be computed. The interested reader is referred for example to Gilmore (2002).

The thermal conductivity of a material (internal) and the CONTACT CONDUCTANCE (at the interface) between two components, although both are denoted in TMM as $G_{L,ij}$, behave physically different. The thermal conductivity of a material generally is temperature-dependent. For composite materials, such as carbon fiber reinforced plastics (CFRP), printed circuit boards (PCB), or honeycomb structures thermal conductivity can even differ significantly between different directions in the material (anisotropy), while for metals one can safely assume isotropic material properties. On the other hand, CONTACT CONDUCTANCES at the interface between conductive elements in practice vary from application to application and therefore have to be determined experimentally. Consequently, their determination is costly and hence often proprietary information of companies. More common values can be found in textbooks.

Conductive coupling coefficients and CONTACT CONDUCTANCE can be merged into one quantity in the same way as ohmic resistors in an electrical network. For n parallel conductive coupling coefficients it reads

$$G_{ij} = \sum_{n}^{1} G_{ijn} \tag{16.3.7}$$

The according expression for n serial conductive coupling coefficients reads

$$\frac{1}{G_{ij}} = \sum_{n}^{1} \frac{1}{G_{ij_n}} \tag{16.3.8}$$

2. **Determine the radiative coupling coefficients R_{ij}**
In aerospace thermal engineering radiative coupling coefficients (CONDUCTORS) are mostly denoted as $G_{R,ij}$. In this book we however stick to our convention R_{ij}. Calculations of R_{ij} with rough values for view factors $F_{i\to j}$ and optical surface properties α, ε are a good first step to quantify radiative heat exchanges. Values for α, ε for typical thermal control surfaces are given e.g. in Gilmore (2002), Appendix A, and Sect. 16.2.6, Fig. 16.9. View factors between any pair of thermal nodes i and j can be determined analytically. For analytical expressions of view factors see for example Howell (2011), also available in the online catalog www.thermalradiation.net/indexCat.html.

Yet, with increasing complexity of the geometry (see Sect. 16.3.4), ray tracing is necessary to determine view factors and thus the radiative coupling coefficients R_{ij}. An additional complication for radiative exchanges are directional selective surfaces (see Sect. 16.1.5), a.k.a. specularity and opacity. Specularity describes the type of angular reflection (specular or diffuse) of a surface at certain wavelengths and can be temperature-dependent. Opacity is a quantity to model transmission through surfaces (transparency) in the visible part of the electromagnetic spectrum. It is cumbersome to determine specularity and opacity analytically. R_{ij} values are determined in the GMM (see Sect. 16.3.4). In the TMM the radiative coupling coefficients R_{ij} are usually provided as an external input without further processing. In principle, it is possible to manipulate R_{ij} in the TMM. This might be necessary for example to rotate S/C components, for which different sets of R_{ij} apply for different rotation states. For every time step in the TMM, for example based on the operation mode of an instrument or based on the position in orbit, an individual set of R_{ij} can be loaded by an external call.

3. **Determine internal heat fluxes $\Phi_{i,\text{int}}$**
The internal heat flux $\Phi_{i,\text{int}}$ (see Eq. (16.3.5)) generated within a node i depends on the S/C design and operational modes. It is usually caused by the heat dissipation of electronic equipment. In general and as a first approximation, it is appropriate to assume 'consumed' electric power equals 'dissipated' heat. But for a number of electronic components this statement is not true. For example, some electronic components emit a fraction of the consumed electric power in form of electromagnetic waves. Dissipated heat can also originate from exothermal chemical processes, such as the discharging of batteries or phase change of materials, as well as from radioactive decay in radioisotope thermoelectric generators (RTGs) or radioisotope heat units (RHUs). Finally, heaters, which are an electrical "equipment" to intentionally dissipate heat, are important heat sources.

   In terms of thermal modeling, the internal heat flux $\Phi_{i,\text{int}}$ can be attached to a single thermal node or a group of thermal nodes.

4. **Retrieve external heat fluxes $\Phi_{i,ext}$**
Unlike the standard notation used in Sects. 16.1 and 16.2, external heat fluxes in the TMM are usually denoted as Φ_S (a.k.a. Q_S) for direct solar heat flux, Φ_A (a.k.a. Q_A) for albedo heat flux, and Φ_E (a.k.a. Q_E) for planetary heat flux. Usually, the TMM calls them as an external input without further processing. As for radiative coupling coefficients different sets of external heat fluxes are also available, for example for a rotating part or different orbit segments. For the modeling of external fluxes see Sect. 16.3.4.

   Some thermal nodes might be exposed to environmental and/or internal heat fluxes, others not. For the node equation it is irrelevant whether a heat flux is internally dissipated heat from an electronic box or a flux from an external

source. The total heat flux on a given thermal node is always the sum of internal $\Phi_{i,\text{int}}$ and external heat fluxes $\Phi_{i,\,ext}$, whatever contributes. Therefore, for thermal nodes on the surface of a spacecraft, i.e. the boundary to space, external sources can formally be considered as neighboring nodes.

5. **Determine the heat capacity C_i for all thermal nodes i**
   The heat capacity C_i of a thermal node is its ability to store energy. It is proportional to the mass of the lumped volume and its specific heat capacity.

$$C_i = c_{p,i} m_i = c_{p,i} \rho_i V_i$$

   Determination of the node volume is straightforward and its density is a material property provided by textbooks. Be aware that there are many different types of materials such as alloys or carbon composites, each having a distinct density. Generally, the specific heat capacity is temperature-dependent and increases with increasing temperature. Values for the specific heat capacity at room temperature can also be found in textbooks or material data bases (see for example Gilmore (2002), Appendix B, or online material databases such as http://www.matweb.com). Given these values, the determination of the heat capacity (a.k.a. CAPACITANCE or thermal capacity) is usually carried out by the GMM meshing process, which forwards these heat capacities to the TMM. The temperature-dependent behavior is often more challenging to obtain. If the value of a peculiar specific heat capacity or its temperature dependence is not available, it has to be measured in dedicated material tests.

6. **Include control functionalities and variables**
   Finally, possible time-dependent heat dissipation functionalities are added to a TMM, such as a simple thermostat, a PID control logic, or the physical function of thermoelectric elements. They must reproduce the behavior of the modeled component. Also active thermal control components, such as pumped fluid loops, can be modeled and included in the TMM.

   Another feature of TMMs is the possibility to include variables to parameterize the model. This allows switching between operational modes or the study of the impact of single parameters on the results of the entire model.

The set of node Eq. (16.3.5) is finally solved with a standard finite difference method. The solution provides the temperatures of all nodes of the modelled body. A mistake commonly made by inexperienced thermal engineers is to thoughtlessly accept the results as a given fact. Results may look credible, although the numerical solvers have not converged to the given limit. So, a check whether the convergence criteria of the solver are achieved is always in order. Be critical with the temperature results you obtain and don't fall in love with your model—be self-critical.

16.3.6 Applied Thermal Design and Analysis

We have seen above that S/C thermal design is analyzed by a discretized thermal model with many thermal nodes. Thus the thermal behavior of a real satellite, i.e. its temperatures at different orbital positions and attitudes is determined. Analytical considerations (see Sect. 16.2) can provide only a very basic idea of the average temperatures of entire bodies at special orbital conditions. In this chapter we examine how a discretized thermal model must be set up and which kind of analyses are necessary at which levels. At the end, the vital question is tackled of how precise thermal models and their temperature predictions can be and how the remaining temperature uncertainties are handled to obtain robust and reliable results.

Global Versus Local Analyses

Thermal design of an entire satellite is usually broken down into a subset of smaller thermal design problems. On the other hand S/C subsystems or units with similar temperatures are usually merged into a so-called *temperature enclosure,* which is then designed to attain the predefined temperature.

To create local enclosures, local thermal design methods are applied to balance, couple, or isolate the enclosures from each other. The task of the thermal engineer then is to balance the heat flows between the different enclosures in such a way as to maintain the desired temperatures throughout the mission in each of them. This is commonly achieved by thermal control hardware such as: distributed heater patches, heat pipe networks, customized MLI blankets, or thermally isolating washers or thermal fillers for interfaces.

As in other disciplines thermal engineering also applies a combination of global and local analyses, which are often, but not always, split over different contractual levels. Global analyses are utilized to check the compatibility of the overall thermal design with mission requirements and to check interface temperatures between the S/C and all its subsystems by means of temperature reference points (TRPs). TRPs are selected locations within a S/C in particular within a subsystem, at which the global analysis guarantees a given temperature. This information is used by the local analysis as a boundary I/F temperature.

Local analyses are performed on subsystem and on component level in environments derived from global analyses. Depending on the complexity of the subsystem, multiple levels of local analyses can be necessary to justify a given thermal design. On the component level this can lead to thermal meshes with a very high number of thermal nodes. At all levels, the critical cases that drive the thermal designs of components or subsystems may be quite different.

Capturing Spatial Temperature Gradients

For a subsystem, temperature requirements dictate the temperature range for all parts, including all hot and cold spots. Thermal models with too few nodes, down to one node per subsystem, can provide a first estimate of the scale of temperatures to be expected. But they fail to reliably predict the temperatures for any location of a modeled subsystem because they average out extreme temperatures. So, to capture

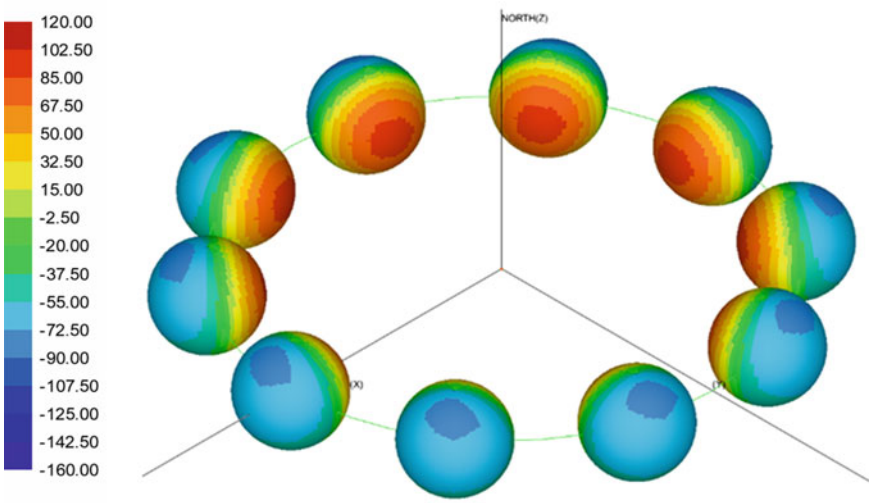


Fig. 16.16 Temperatures of perfectly black titanium sphere (thin shell with thickness 5 mm) circling the Sun in 1 AU distance. The sphere is discretized with 80 × 80 thermal nodes. Analysis and visualization by ESATAN-TMS

all the corresponding temperature gradients, a large number of thermal nodes must be employed.

Consider a black sphere (shell of 5 mm thickness) of titanium as a modest example of a satellite circling the Sun at distance of one astronomical unit (1 AU) as shown in Fig. 16.16. Thermal analysis with 80 × 80 thermal nodes reveals that it would experience temperatures in the range of +110 °C (facing the Sun) to −90 °C (facing space), while a global analytical heat balance analysis by a one-node model would suggest an average temperature of roughly +15 °C. This mean temperature would satisfy the typical permissible operational temperature range of +10 to +30 °C of a battery. However, in reality mounting the battery at any location within the S/C shell, would heavily violate the temperature limit. This simple example justifies why a certain minimum level of discretization is necessary even in the most global S/C thermal analysis. The discretization approach explained in Sects. 16.3.4 and 16.3.5 for the GMM and TMM provides the means to model such details.

Capturing Temporal Temperature Gradients

For most missions the S/C environment around a planet or on a planet's surface, that is the solar, albedo, and IR heat fluxes changes continuously. To capture all the transient temperature gradients of a S/C and to identify the most extreme thermal conditions, multiple points in time must be analyzed. Too few points may miss temperature extremes thus providing only approximate results. To properly capture transient temperature effects GMM and TMM must be set up with respective temporal granularity.

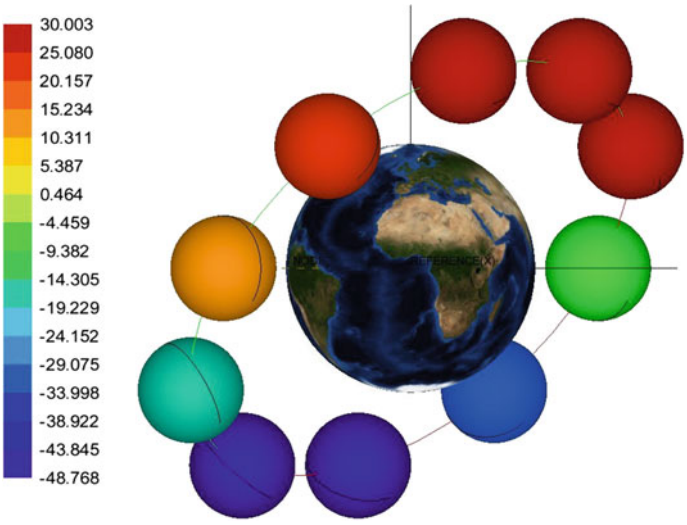


Fig. 16.17 Temperature evolution of perfectly black titanium sphere (shell with thickness 5 mm) in an ISS orbit (altitude 400 km, inclination 51.6°) around Earth. The sphere is modeled with one thermal node. Analysis and visualization by ESATAN-TMS. Sphere and orbit dimensions are not to scale

Therefore, in the GMM it is essential to ensure that not only critical transitions or attitude changes that strongly affect the heat fluxes (e.g. entry into eclipse) are covered in the analysis, but also that the gradual phases between those fast transitions are sampled with reasonable granularity. These rapid cases are then fed into the TMM to derive the temperature changes of a S/C over an orbit or along a trajectory. The TMM also requires the time history of the internal fluxes (e.g. heaters, electronic box heat dissipations) in order to cover the heat balance fluctuation caused by operational modes.

Consider again our battery with a temperature limit of +10 to +30 °C. If the local environment of the battery, that is for example the temperature of a dedicated compartment inside the S/C, changes evenly over an orbit between −20 and +60 °C, the average temperature of the battery would be +20 °C. This favorable temperature, however, would rarely occur. Most of the time the temperatures would violate the allowable upper and lower limits, which implies a failed thermal design.

Figure 16.17 shows the mean temperatures of a black sphere of titanium (shell of 5 mm thickness) circling Earth in an ISS-type orbit at different locations on its orbit. The temperature is maximal +30 °C on the dayside and minimal −48 °C on the nightside. Therefore, to capture such a temperature evolution, thermal analysis needs to be done at multiple points along the S/C trajectory.

In summary, for a reasonable temporal and spatial temperature profile a thermal model with multiple nodes and at multiple points along the trajectories is required as shown in Fig. 16.18.

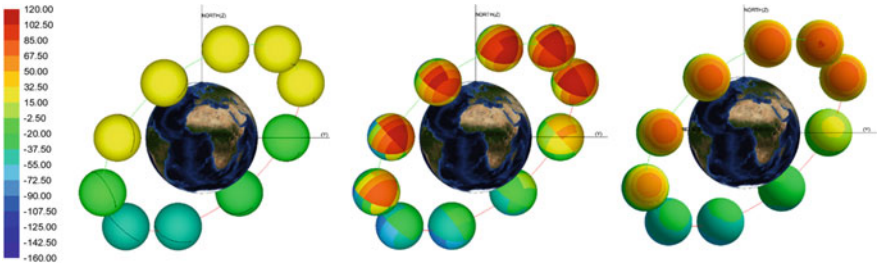


Fig. 16.18 Temperature evolution of a perfectly black titanium sphere (shell with thickness 5 mm) in an ISS orbit (altitude 400 km, inclination 51.6°) around Earth. The sphere is discretized with one (left), 6 × 6 (center) and 80 × 80 (right) thermal nodes, highlighting the impact of different orbital positions and of different spatial discretizations. Analysis and visualization by ESATAN-TMS. Sphere and orbit dimensions are not to scale

> **Note** *Most thermal analysis software packages offer the option to calculate time averaged temperatures in a so called "steady-state" mode. Caution is required when using this mode. It typically is used to establish initial temperature figures for subsequent transient analyses, and for this purpose it can be a suitable tool. For most missions it is useless for verifying the thermal design versus the thermal requirements because the steady-state hides all transient variations in the temperature.*

Spacecraft Lifetime Considerations

Space missions typically last several years. The degradation of the S/C and the change of the environment over such a long period must be considered in any thermal analysis. Most importantly, the S/C coatings degrade as discussed in Sect. 16.2.6. Also electronic components age in that they dissipate more heat as their electrical efficiency declines, primarily due to the total ionizing radiation dose received.

Also the environment changes over a mission, particularly for interplanetary missions. Consider for example a probe to Jupiter. At its final destination far away from the Sun the thermal design is all about keeping it warm, which often requires small heater units and MLI blankets on the S/C hull with outer layers having a high alpha/epsilon. Jupiter missions, however, initially often journey through the inner solar system to pick up speed by gravity assist maneuvers at inner planets, such as Venus. So close to the Sun the S/C needs to be shielded against the excessive solar heat flux. This is usually achieved by using a white coated high gain antenna as a solar shield.

Even in planetary orbits environmental changes are significant. In Earth orbit the variation of the solar declination from +23.5° to −23.5° over one year, the distance between Earth and Sun caused by the slightly elliptic orbit of the Earth, as well as the change in solar flux caused by the eleven year solar cycle lead to considerable variations in the environmental heat fluxes.

All these changes over a mission lifetime must be covered by thermal analyses in order to demonstrate that the thermal design is robust and that the thermal control system is able to meet the requirements at all times.

Worst-Case Analysis

Despite the need to provide the temperature information over an entire lifetime of a satellite, it is currently impracticable to simulate the entire mission duration. This would require numerical simulations yielding immense quantities of data, which subsequently have to be post-processed and interpreted. In order to limit the number of evaluated cases to a manageable number only so-called *worst case scenarios* are considered, which are the extreme cases of all possible environment and S/C combinations. Classically, the worst hot and worst cold case are established to predict extreme S/C temperatures. These cases are often associated with a certain phase of the mission.

In the worst cold case the lowest temperatures of a S/C are predicted and under nominal conditions the required heater power is derived. In addition, the worst cold case is often merged with the beginning of life (BOL) conditions. At the beginning of a mission the $\alpha_{sol}/\varepsilon_{IR}$ of the outer coatings is lowest (see Fig. 16.9) and hence the S/C coldest. The worst hot case predicts the highest S/C temperatures. It is merged with the end of life (EOL) conditions, as here the $\alpha_{sol}/\varepsilon_{IR}$ and the electrical components' heat loss are highest.

Take for example a classical thermal design of an electronic unit mounted on a radiator and equipped with heaters for temperature control. The worst-case analysis is as follows:

- The thermal design starts with the worst hot case, applying EOL conditions. This case is used to size the radiator for the highest heat loads.
- The worst cold case is established with the radiator set to the BOL conditions, to check the temperatures resulting from minimum heat loads and with the radiator sized to the maximum heat loads. This case determines if heating of the controlled item is necessary and if so, how much heater power is required.

The typical worst cases mentioned above are a minimum set of analyses, but it is not by any means sufficient to cover all contingency cases. In addition, S/C temperatures and/or heater power demands must also be determined for different operation modes (e.g. safe mode, communication mode, etc.) and mission scenarios (attitude changes, orbital maneuvers, launch and early operations phase, etc.).

Generally, the definition and description of a few example cases as a surrogate for the entire mission scenario is one of the most difficult tasks in thermal engineering and requires a significant level of experience.

Modeling Uncertainty

No mathematical model is absolutely exact. This is a result of all the simplifications that are part of the modeling process and the inherent inaccuracies of the many detailed properties of thermal hardware. So the question is: How precise is a thermal analysis and how much does it err? The answers to this question is key for

the reliability of thermal analyses as a part of the thermal design verification chain of a S/C. The reliability depends, of course, on the quality of the modeled S/C and of the modeled environment, in which the S/C operates or is tested. The simplifications and inaccuracies mentioned above are combined in the so-called *(thermal) modeling uncertainty*.

The so-called *calculated temperature range* (see Fig. 16.19) is the combination of the most extreme temperatures determined by all thermal analyses. The modeling uncertainty (i.e. the known or assumed inaccuracies of the model) is added to this range yielding the so-called *predicted temperature range*.

Modeling uncertainty is the upshot of uncertain bulk material properties or uncertain thermo-optical surface properties, interface conditions, geometrical details, and thermal control settings.

Uncertainties of bulk material properties result from deviations from assumed properties or from unknown or disregarded anisotropic material properties. Thermo-optical surface properties strongly vary with the substrate material or surface treatment (coating) applied to the surface. In addition, wavelength dependencies, diffuse and specular reflectivity, and the opacity of coatings usually are not or are only insufficiently modeled. Uncertain details about physical contacts between adjunct parts cause interface uncertainties. Applied geometrical model simplifications cause GMM uncertainties. In addition, uncertain heater locations, thermistor tolerances, and unknown PID control settings bring about thermal-control uncertainties. Another common source of uncertainty is the varying efficiency of MLI, which often is only known for flat samples. MLI edge effects and the folding round edges can significantly degrade blanket performance. In addition to S/C-related inaccuracies and uncertainties, environment uncertainties can also lead to variations in the model results.

The modeling uncertainty is typically obtained by running multiple sensitivity analyses for a specific thermal case. In each sensitivity analysis the S/C or the environmental parameters are varied within their inaccuracy ranges. In practice, sensitivity analyses are often performed only for the worst hot and cold cases or

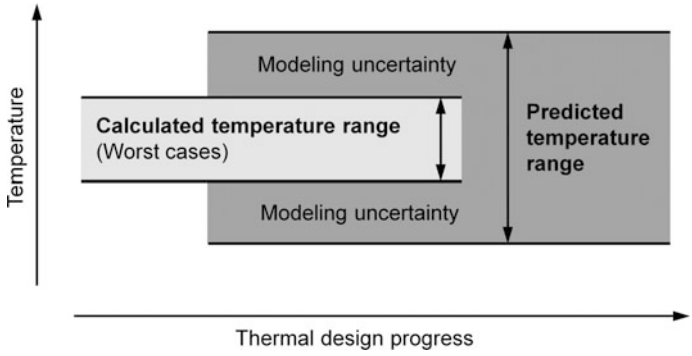


Fig. 16.19 Terminology in thermal analysis for temperature ranges and (thermal) modeling uncertainty, based on the uncertainty approach for thermal control system (TCS) used by the European Space Agency. Adapted from ECSS-E-ST-31C

Table 16.1 Exemplary values for thermal model parameter uncertainties

Class	Parameter	Inaccuracy
Environmental	Solar intensity	± 21 W/m^2
Physical	Absorptivity	± 0.1
	Emissivity	± 0.03
	Dissipation (for absolute value <10 W)	$\pm 10\%$
Geometrical	Shape (view) factors (simple geometry)	$\pm 10\%$

Credit ECSS

sometimes even only for one of these cases, while the resulting uncertainty then is applied to all cases.

A list of model quantities to be varied and recommendations on the variation ranges can be found for example in ECSS-E-HB-31-03A. The values given in this ESA reference are based on the experience of thermal engineers and analysts in the European space industry from numerous missions. In Table 16.1 a couple of typical model parameter uncertainties are shown. The full list can be found in ECSS-E-HB-31-03A, Sect. 6.5.

Due to the large number of nodes in thermal models, it is often impractical to derive the modeling uncertainty for each thermal node. Instead, for each subsystem and component representative nodes are identified, for which the modeling uncertainty is calculated. Those representative nodes can be, for example, the temperature reference points (TRP) or nodes representing a particularly sensitive piece of equipment. For each representative node the result of the sensitivity analysis is the root mean square (RMS) of all temperature deviations of the different parametric analysis runs (see ECSS-E-HB-31-03A).

In a last step, the modeling uncertainty, as given by the RMS, is applied to both sides of the *calculated temperature range* of each component: It is subtracted from the coldest temperatures and added to the hottest ones, thus widening the temperature range. This extended temperature range is the 'predicted' temperature range.

In early design phases the modeling uncertainty is often determined by experience or is based on standardized values. As the thermal design and thermal model matures, it is later replaced by modeling uncertainty values, derived by the aforementioned sensitivity analyses. As soon as the thermal design is tested and the measured temperatures become available, the thermal model is typically correlated against the test results, and the modeling uncertainty range can be further reduced.

The domain of thermal S/C modeling ends here, however not that of thermal design. The S/C and its subsystems and components still needs to be tested to qualify for orbital operations. In order to add reliability to the design, these tests are done at more extreme temperatures than those expected in orbit. Additional margins are applied in thermal tests to increase the reliability of the system. Yet, thermal testing and the system level verification process is beyond the scope of this book.

16.3.7 Case Studies

In order to illustrate how thermal modeling works, we present a simple example: a heated bar made of a metal is assumed to behave as a gray body in Low Earth Orbit. The heated bar is modeled by three diffusive thermal nodes along the x-axis (see Fig. 16.20) numbered 1, 2, and 3 and one boundary node 4.

Case 1
We make the following assumptions:

(a) The heated bar is assumed to circle Earth at 400 km altitude.
(b) All nodes are cubes and have equal edge lengths $l = 10$ cm.
(c) All nodes are made of aluminum with $\kappa_{alu} \approx 160$ W m^{-1} K^{-1}.
(d) Node 1 is exposed to the Sun on its $-x$-side, which has black paint properties $\varepsilon_{1,IR} = 0.9$ and $\alpha_{1,sol} = 0.9$ (see Fig. 16.9).
(e) Node 1 and 2 are covered with a perfectly isolating insulation on all $+/-y$ and $+/-z$ sides, i.e. these sides are adiabatic.
(f) Node 3 sides are not insulated, but covered by a SSM coating with properties $\varepsilon_{3,IR} = 0.9$ and $\alpha_{3,sol} = 0.1$ (see Fig. 16.9).

The boundary conditions for the computation are:

(I) The mean solar irradiance (solar constant) is $E_{sun} = 1361.5$ W m^{-2}.
(II) There is a constant internal heat source in node 2 of $\Phi_{2,\text{int}} = 2$ W.
(III) The boundary node 4 represents Earth, which is a black-body radiator with $T_{earth} = 255$ K.
(IV) The background temperature of space is $T_\infty = 2.7$ K.

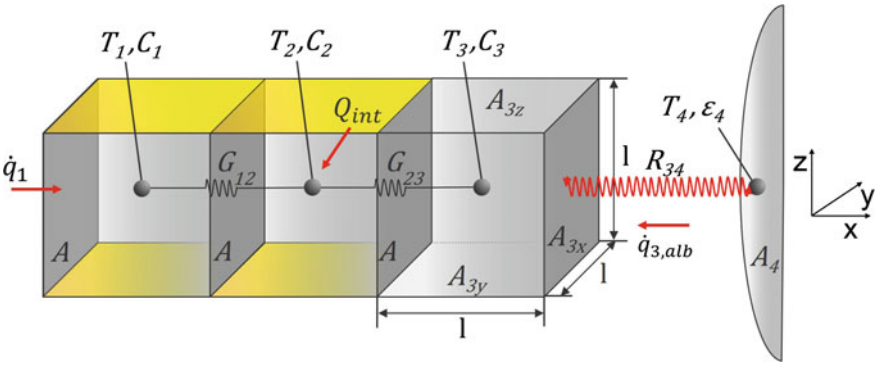


Fig. 16.20 A heated bar in LEO oriented to nadir. The bar is modeled by three cubes (nodes) with equal volumes. Shown is the conductive path through the solid material and the radiative exchange between two end surfaces

Geometry

$$A_{123,xyz} = l^2$$
$$V_{123} = l^3$$

Applying Eq. (16.2.10) for the view factor we have

$$F_{3x\to 4} = \mu^2 \cos\phi \quad \text{with } \phi = 0^\circ$$

and for the view factor of the surfaces A_{3y} and A_{3z} perpendicular to a sphere we can use (Howell 2011)

$$F_{3z\to 4} = F_{3y\to 4} = \frac{1}{\pi}\left(\arctan\frac{\mu}{\sqrt{1-\mu^2}} - \mu\sqrt{1-\mu^2}\right)$$

with $\mu = R/(R+h)$, where R is Earth's radius, and h orbit altitude above the Earth's surface. The view factor of the surfaces A_{3y} and A_{3z} to deep space (∞) can be derived from the summation rule Eq. (16.2.6) and the reciprocity relation, Eq. (16.2.7)

$$F_{3z\to\infty} = 1 - F_{3z\to 4}$$
$$F_{3y\to\infty} = 1 - F_{3y\to 4}$$

The view factor between the surface facing away from the Earth to deep space can be assumed to be

$$F_{1x\to\infty} = 1$$

From the geometry and the assumptions we can derive the following conductive and radiative CONDUCTORS

$$G_{12} = G_{23} = \frac{\kappa_{alu}A}{l} = \kappa_{alu}l$$
$$R_{3x\to 4} = \alpha_{4,IR}\varepsilon_{3,IR}A_{3x}F_{3x\to 4} = \varepsilon_{3,IR}A_{3x}F_{3x\to 4}$$
$$R_{3y\to 4} = R_{3z\to 4} = \alpha_{4,IR}\varepsilon_{3,IR}A_{3y}F_{3y\to 4} = \varepsilon_{3,IR}A_{3y}F_{3y\to 4}$$
$$R_{1x\to\infty} = \varepsilon_{1,IR}A_{1x}F_{1x\to\infty} = \varepsilon_{1,IR}A_{1x}$$

From the material properties we can derive the CONDUCTORS for transient calculations:

$$C_i = c_{p,i}\rho_i V_i, \quad i = 1,2,3$$

The boundary condition for node 1, i.e. the node exposed to the Sun, is

$$\Phi_{sun\to 1} = \alpha_{1,sol}A_1 E_{sun}$$

For node 3 there is an albedo heat flux from the Earth, which can be written as

$$\Phi_{4\to 3,\,alb} = \alpha_{3,\,sol} E_{sun} a \left(A_{3x} F_{3x\to 4} + 4 \cdot A_{3y} F_{3y\to 4}\right)$$

where $a = 0.3$ is Earth's albedo. Finally, we have the Earth's radiation at infrared wavelengths to node 3, which is addressed by the term

$$\Phi_{4\to 3,\,IR} = \left(R_{4\to 3x} + 4 \cdot F_{4\to 3y}\right)\sigma\left(T_4^4 - T_3^4\right)$$

where $R_{4\to 3x} = \alpha_{3x,y,IR}\varepsilon_{4,IR} A_4 F_{4\to 3x,y}$. Because we assume Earth to be a black body radiator, we have $\alpha_{4,IR} = \varepsilon_{4,IR} = 1$. Because we also assume the heated bar surface behaves as a gray body, we have from Eq. (16.1.35) $\varepsilon_{3y,\,IR} = \alpha_{3y,\,IR}$ and therefore

$$\alpha_{3y,\,IR}\varepsilon_{4,\,IR} = \alpha_{4,\,IR}\varepsilon_{3y,\,IR}$$

With the reciprocity relation (16.2.7) we therefore have

$$R_{4\to 3x} = \alpha_{3,\,IR}\varepsilon_{4,\,IR} A_4 F_{4\to 3x} = \alpha_{4,\,IR}\varepsilon_{3,\,IR} A_3 F_{3x\to 4} = R_{3x\to 4}$$
$$R_{4\to 3y} = \alpha_{3,\,IR}\varepsilon_{4,\,IR} A_4 F_{4\to 3y} = \alpha_{4,\,IR}\varepsilon_{3,\,IR} A_3 F_{3y\to 4} = R_{3y\to 4}$$

and therefore

$$\Phi_{4\to 3} = \left(R_{3x\to 4} + 4 \cdot R_{3y\to 4}\right)\sigma\left(T_4^4 - T_3^4\right)$$

In summary, we can write the transient node equation, Eq. (16.3.5) for each of the three thermal nodes:

$$\text{Node 1: } c_{p1}\rho_1 V_1 \frac{dT_1}{dt} = G_{12}(T_2 - T_1) + R_{1x\to\infty}\sigma\left(T_\infty^4 - T_1^4\right) + \Phi_{sun\to 1}$$

$$\text{Node 2: } c_{p2}\rho_2 V_2 \frac{dT_2}{dt} = G_{12}(T_1 - T_2) + G_{23}(T_3 - T_2) + \Phi_{2,\text{int}}$$

$$\text{Node 3: } c_{p3}\rho_3 V_3 \frac{dT_3}{dt} = G_{23}(T_2 - T_3) + \left(R_{3x\to 4} + 4 \cdot R_{3y\to 4}\right)\sigma\left(T_4^4 - T_3^4\right) + \Phi_{alb\to 3}$$

Case 2
We invert the optical surface properties, i.e. set the front of node 1 to black paint and the outer surfaces of node 3 to SSM.

Case 3
We vary Case 1 and assume that the internal cross section area between nodes 1, 2 and 3 is reduced to 10%.

$$G_{12} = G_{23} = \frac{\kappa_{alu} 0.1A}{l}$$

The area in radiative exchange with the Sun, space and Earth remains the same as in Case 1.

Table 16.2 Steady state temperatures for nodes 1, 2, and 3 for Cases 1–5

Case	Description	T_1 [°C]	T_2 [°C]	T_3 [°C]
1	Reference	11.99	12.06	11.95
2	Inverted optical surface properties	28.97	28.45	27.75
3	Reduced cross-section, reduced G_{ij}	81.73	82.04	80.55
4	Titanium material properties	12.73	14.52	11.83
5	Transient case with eclipse half or the orbit	−42.88	−42.81	−42.92

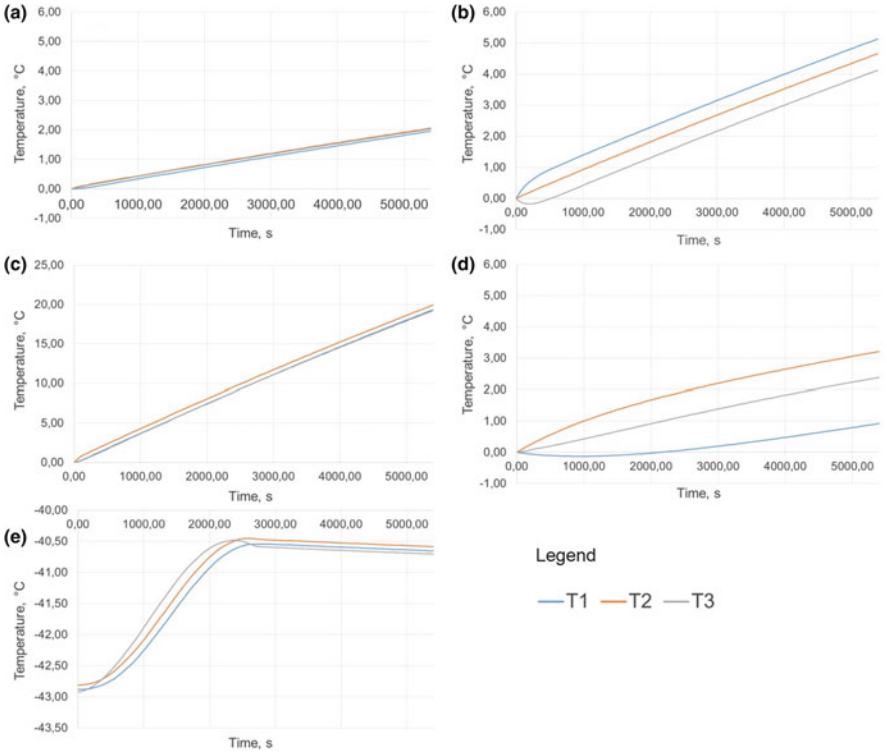


Fig. 16.21 Transient temperatures for nodes 1, 2 and 3 for cases 1–5. Initial temperature 0.0 °C except for figure **e**: **a** bar made of aluminum; **b** bar made of aluminum, but inverted optical surface properties; **c** bar made of aluminum with 10% cross section area; **d** bar made of titanium; **e** bar made of aluminum with an eclipse phase and a sinusoidal sun and albedo heat flux, initial temperature of −42.8 °C

Case 4

We vary Case 1 and use titanium material properties instead, $\rho_{Ti} = 4430$ kg m^{-3}, $c_{p,Ti} = 540$ J kg^{-1} K^{-1}, $\kappa_{Ti} = 6.7$ W m^{-1} K^{-1}.

Case 5
We look at the transient behavior of case 1 starting at an initial temperature of $T_{i,t=0} = 0.0\,°\text{C}$, $i = 1, 2, 3$. Furthermore we vary E_{sun} in a sinusoidal way, assuming an orbit period of 90 min and include an eclipse duration for half the orbit. We also assume that albedo follows a sinusoidal change in intensity, only on the dayside of the orbit.

The thermal results of cases 1-5 are given in Table 16.2 and Fig. 16.21.

16.4 Problems

Problem 1 *Wien's Displacement Law*
Prove Wien's displacement law Eq. (16.1.20) from Planck's law Eq. (16.1.19).

Problem 2 *Earth's IR Radiation*
According to Sect. 16.2.3 the solar irradiance at Earth on average is 1361.5 W m^{-2}. Show from applying Eq. (16.2.24) that the absorption of this radiation with an average albedo of $a = 0.31$ causes a radiated thermal energy that is equivalent to that of a black body radiator temperature of $T = 254$ K.

Problem 3 *Hot and Cold Cases of a Satellite in LEO*
A spherical satellite without any internal heat source is circling Earth in LEO at an altitude of 400 km. Consider the satellite as a body in full thermal equilibrium with its environment and Eq. (16.2.28) applies.

(a) Show that in the so-called *hot case*, where $\delta_{sun} = \delta_a = 1$, a change from a gray surface, $\alpha_{sol}/\varepsilon_{IR} = 1$, to a surface coated with zinc oxide, $\alpha_{sol}/\varepsilon_{IR} = 0.22$, yields a body temperature decrease of roughly 78 °C.
(b) Show that in the so-called *cold case* (nightside), where $\delta_{sun} = \delta_a = 0$, such a change of surface material does not have any effect on body temperature!
(c) Show that in the hot case the satellite with a gray surface would be cooler by 24 °C if Earth's albedo would be absent.

Appendix A
Planetary Parameters

Characteristic parameters of the planets in the solar system

	R [km]	μ [km^3/s^2]	a [10^6 km]	e	$\langle r_t \rangle$ [10^6 km]	$\langle v \rangle_t$ [km/s]
Sun	696,000	$1.327124400 \times 10^{11}$	–	–	–	–
Mercury	2440	2.2032×10^4	57.90923	0.20564	59.134	46.362
Venus	6052	3.24859×10^5	108.2096	0.00677	108.212	35.020
Earth	6378.136	3.98600442×10^5	149.5984	0.01670	149.619	29.783
(Moon)	1737.5	4902.801×10^3	0.384400	0.05540	0.38499	1.0175
Mars	3396.2	4.282837×10^4	227.9443	0.09341	228.938	24.076
Jupiter	71,492	1.2686534×10^8	778.338	0.04836	779.248	13.050
Saturn	60,268	3.7931187×10^7	1426.64	0.05378	1428.70	9.6379
Uranus	25,559	5.793939×10^6	2870.61	0.04725	2873.81	6.7955
Neptune	24,760	6.836529×10^6	4498.40	0.00860	4498.57	5.4315
Pluto	1151	871	5906.43	0.24883	6089.28	4.6659

R = equitorial planet radius, $\langle r \rangle_t$ = time-averaged orbit radius, $\langle v \rangle_t$ = time-averaged orbital velocity. The values hold for the year 2010. An underscore for $\boldsymbol{R}$ indicates uncertainties, for a, e, and $\langle r \rangle_t$ that this digit changes within a decade

A.1 Mean Orbit Radius

A.1.1 Titius–Bode Law

The orbit radii of the solar planets can be expressed empirically and approximately by the famous Titius–Bode law

$$\langle r \rangle = 0.4 + 0.3 \cdot 2^n \; [AU] \; \textbf{Titius–Bode law}$$

with

$$\begin{aligned} &\text{Mercury}: && n = -\infty \\ &\text{Venus}: && n = 0 \\ &\text{Earth}: && n = 1 \\ &\text{Mars}: && n = 2 \\ &\text{Asteroidbelt}: && n = 3 \\ &\text{Jupiter}: && n = 4 \\ &\text{Saturn}: && n = 5 \\ &\text{Uranus}: && n = 6 \end{aligned}$$

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$$1\,AU = 149.597870700 \times 10^6 \text{ km (exactly)}$$

A.1.2 Average over True Anomaly

$$\langle r\rangle_\theta = a\left(1-e^2\right)\frac{1}{\pi}\int_0^\pi \frac{d\theta}{1+e\,\cos\theta}$$

Since

$$\frac{1}{\pi}\int_0^\pi \frac{d\theta}{1+e\,\cos\theta} = \frac{2}{\pi\sqrt{1-e^2}}\arctan\left(\sqrt{\frac{1-e}{1+e}}\cdot\,\tan\frac{\theta}{2}\right)\Bigg|_0^\pi = \frac{2}{\pi\sqrt{1-e^2}}\left(\frac{\pi}{2}-0\right)$$
$$= \frac{1}{\sqrt{1-e^2}}$$

then

$$\langle r\rangle_\theta = a\sqrt{1-e^2} = b \quad (b = \text{semi-minor axis}) \tag{A.1}$$

A.1.3 Time Average

$$\langle r\rangle_t = a\left(1-e^2\right)\frac{2}{T}\int_0^{T/2}\frac{dt}{1+e\,\cos\theta} = a\left(1-e^2\right)\frac{2}{T}\int_0^\pi \frac{dt/d\theta}{1+e\,\cos\theta}d\theta$$

Since

$$\frac{dt}{d\theta} = \frac{r^2}{h} = \frac{a^2(1-e^2)^2}{(1+e\,\cos\theta)^2\sqrt{\mu a(1-e^2)}}$$

and

$$T = 2\pi\sqrt{\frac{a^3}{\mu}}$$

it follows that

$$\langle r\rangle_t = a\left(1-e^2\right)^{5/2}\frac{1}{\pi}\int_0^\pi \frac{d\theta}{(1+e\,\cos\theta)^3}$$

From Problem 7.6 we get

$$\left(1-e^2\right)^{5/2}\int\frac{d\theta}{\left(1+e\cdot\cos\theta\right)^3}=\int\left(1-e\cdot\cos E\right)^2 dE$$
$$=\left(1+\frac{e^2}{2}\right)E-2e\,\sin E+\frac{e^2}{4}\sin 2E$$

and hence finally

$$\langle r\rangle_t=a\left(1+\frac{e^2}{2}\right) \tag{A.2}$$

A.2 Mean Orbital Velocity

Denote by

$$K(x)=\int_0^{\pi/2}\frac{dt}{\sqrt{1-x^2\sin^2 t}}=\frac{\pi}{2}\left[1+\left(\frac{1}{2}\right)^2x^2+\left(\frac{1\cdot 3}{2\cdot 4}\right)^2x^4+\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2x^6+\ldots\right]$$

the *complete elliptic integral of the first kind* and by

$$E(x)=\int_0^{\pi/2}\sqrt{1-x^2\sin^2 t}\cdot dt$$
$$=\frac{\pi}{2}\left[1-\left(\frac{1}{2}\right)^2x^2-\left(\frac{1\cdot 3}{2\cdot 4}\right)^2\frac{x^4}{3}+\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2\frac{x^6}{5}+\ldots\right]$$

the *complete elliptic integral of the second kind.*

A.2.1 Average over True Anomaly

$$\langle v\rangle_\theta=\frac{\mu}{h}\frac{1}{\pi}\int_0^\pi\sqrt{1+2e\,\cos\theta+e^2}\cdot d\theta=\sqrt{\frac{\mu}{a}}\sqrt{\frac{1+e}{1-e}}\frac{2}{\pi}E\left(\frac{2\sqrt{e}}{1+e}\right)$$

From this it follows that

$$\langle v\rangle_\theta = \sqrt{\frac{\mu}{a}}\left[1+\frac{3}{4}e^2+\frac{33}{64}e^4+O(e^6)\right] \quad @\ \ e\to 0 \tag{A.3}$$

A.2.2 Time Average

$$\langle v\rangle_t = \frac{\mu}{h}\frac{2}{T}\int_0^{T/2}\sqrt{1+2e\,\cos\theta+e^2}\cdot dt = \frac{\mu}{h}\frac{2}{T}\int_0^{\pi}\sqrt{1+2e\,\cos\theta+e^2}\,\frac{dt}{d\theta}d\theta$$

From this it follows that

$$\langle v\rangle_t = \sqrt{\frac{\mu}{a}}\left(1-e^2\right)\cdot\frac{1}{\pi}\int_0^{\pi}\frac{\sqrt{1+2e\,\cos\theta+e^2}}{\left(1+e\,\cos\theta\right)^2}d\theta$$

$$\int_0^{\pi}\frac{\sqrt{1+2e\,\cos\theta+e^2}}{\left(1+e\,\cos\theta\right)^2}d\theta = \frac{1}{(1-e^2)}\left[(1-e)K\left(\frac{2\sqrt{e}}{1+e}\right)+(1+e)E\left(\frac{2\sqrt{e}}{1+e}\right)\right]$$

hence

$$\langle v\rangle_t = \sqrt{\frac{\mu}{a}}\cdot\frac{1}{\pi}\left[(1-e)K\left(\frac{2\sqrt{e}}{1+e}\right)+(1+e)E\left(\frac{2\sqrt{e}}{1+e}\right)\right]$$

and therefore

$$\langle v\rangle_t = \sqrt{\frac{\mu}{a}}\left[1-\frac{1}{4}e^2-\frac{3}{64}e^4-O(e^6)\right] \quad @\ \ e\to 0 \tag{A.4}$$

Appendix B
Approximate Analytical Solution for Uneven Staging

Let

$v_{*,i}$ arbitrary
ε_i arbitrary

We define

$$v_{*,i} = v_* + \Delta v_{*,i} =: v_*(1 + \delta_i)$$

and

$$\varepsilon_i = \varepsilon + \Delta\varepsilon_i =: \varepsilon(1 + \Delta_i)$$

with arithmetic means

$$v_* := \frac{1}{n}\sum_i v_{*,i}$$
$$\varepsilon := \frac{1}{n}\sum_i \varepsilon_i$$

implying

$$\sum_i \delta_i = 0, \quad \sum_i \Delta_i = 0$$

Assuming $\delta_i \ll 1$ and $\Delta_i \ll 1$ we derive from Eq. (3.2.4)

$$\begin{aligned}\frac{\Delta v}{v_*} &= \sum_{i=1}^{n}(1+\delta_i)\ln\left[\frac{1}{\varepsilon_i}\left(1-\frac{\alpha}{v_{*,i}}\right)\right] \\ &= -\sum_{i=1}^{n}(1+\delta_i)\ln\varepsilon_i + \sum_{i=1}^{n}(1+\delta_i)\ln\left(1-\frac{\alpha}{v_{*,i}}\right)\end{aligned}$$

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Expanding $\ln\left(1-\alpha/v_{*,i}\right)$ for $v_{*,i}=v_*(1+\delta_i)$ we get

$$\begin{aligned}\ln\left(1-\frac{\alpha}{v_{*,i}}\right) &\approx \ln\left[\left(1-\frac{\alpha}{v_*}\right)\left(1+\frac{\alpha\delta_i}{v_*-\alpha}\right)\right] \\ &\approx \ln\left(1-\frac{\alpha}{v_*}\right)+\frac{\alpha\delta_i}{v_*-\alpha}-\frac{1}{2}\left(\frac{\alpha\delta_i}{v_*-\alpha}\right)^2+O\left(\delta_i^3\right)\end{aligned} \qquad @\ \frac{\alpha\delta_i}{v_*-\alpha}\ll 1$$

With $\sum_i \delta_i = 0$ we get

$$\frac{\Delta v}{v_*}\approx\sum_i(1+\delta_i)\ln\left[\frac{1}{\varepsilon_i}\left(1-\frac{\alpha}{v_*}\right)\right]-\frac{1}{2}\left(\frac{\alpha}{v_*-\alpha}\right)^2\sum_{i=1}^{n}\delta_i^2+O\left(\sum_{i=1}^{n}\delta_i^3\right)$$

Due to

$$\sum_i(1+\delta_i)\ln\left[\frac{1}{\varepsilon_i}\left(1-\frac{\alpha}{v_*}\right)\right]=-\sum_i\delta_i\ln\varepsilon_i+\sum_i\ln\left[\frac{1}{\varepsilon_i}\left(1-\frac{\alpha}{v_*}\right)\right]$$

and

$$\sum_i\delta_i\ln\varepsilon_i=\sum_i\delta_i[\ln\varepsilon+\ln(1+\Delta_i)]=\sum_i\delta_i\Delta_i-\frac{1}{2}\sum_i\delta_i\Delta_i^2$$

and

$$\sum_i\ln\left[\frac{1}{\varepsilon_i}\left(1-\frac{\alpha}{v_*}\right)\right]=n\ln\frac{1}{\bar{\varepsilon}}\left(1-\frac{\alpha}{v_*}\right)$$

follows

$$\frac{\Delta v}{v_*}\approx n\ln\frac{1}{\bar{\varepsilon}}\left(1-\frac{\alpha}{v_*}\right)-nC$$

where

$$C:=\frac{1}{n}\sum_i\delta_i\left[\Delta_i+\frac{1}{2}\left(\frac{\alpha}{v_*-\alpha}\right)^2\delta_i+O\left(\delta_i^2\right)+O\left(\Delta_i^2\right)\right]$$

from which follows

$$\frac{\alpha}{v_*}\approx 1-\bar{\varepsilon}e^{\frac{\Delta v}{nv_*}}+C\approx 1-\bar{\varepsilon}e^{\frac{\Delta v}{nv_*}} \tag{B.1}$$

Because of $\frac{\alpha\delta_i}{v_*-\alpha} \ll 1$ we find from Eq. (B.1) the constraint $\exp\left(-\frac{\Delta v}{nv_*}\right) \ll \min_{i=1,n}\left(\frac{\bar{\varepsilon}}{\delta_i}\right)$

From Eq. (B.1) we determine the quantity C as

$$C = \frac{1}{n}\sum_i \frac{\Delta v_{*,i}}{v_*}\left[\frac{\Delta\varepsilon_i}{\varepsilon_i} + \frac{1}{2}\left(1 - \frac{1}{\bar{\varepsilon}}e^{-\frac{\Delta v}{nv_*}}\right)^2 \frac{\Delta v_{*,i}}{v_*} + O\left[(\Delta\varepsilon_i)^2\right] + O\left[\left(\Delta v_{*,i}\right)^2\right]\right] \tag{B.2}$$

We therefore find a solution equivalent to Eq. (3.3.7) with the substitution

$$\frac{\Delta v}{nv_*} \to \frac{\Delta v}{nv_*} + C$$

Applying this substitution rule we find the solutions equivalent to Eqs. (3.3.8) and (3.3.9)

$$\lambda_*^{1/n} = \frac{e^{-\Delta vn/v_* - C} - \bar{\varepsilon}}{\overline{1-\varepsilon}} \quad @ \quad \exp\left(-\frac{\Delta v}{nv_*}\right) \ll \min_{i=1,n}\left(\frac{\bar{\varepsilon}}{\delta_i}\right) \tag{B.3}$$

and

$$\Delta v = -nv_*\left\{\ln\left[\lambda_*^{1/n}\left(\overline{1-\varepsilon}\right) + \bar{\varepsilon}\right] + C'\right\} \tag{B.4}$$

From Eq. (B.3) follows

$$e^{-\Delta v/nv_*} \approx \lambda_*^{1/n}\left(\overline{1-\varepsilon}\right) + \bar{\varepsilon} \ll \min_{i=1,n}\left(\frac{\bar{\varepsilon}}{\delta_i}\right)$$

from which we obtain

$$C' = \frac{1}{n}\sum_i \frac{\Delta v_{*,i}}{v_*}\left[\frac{\Delta\varepsilon_i}{\varepsilon_i} + \frac{\lambda_*^{2/n}}{2}\left(\frac{\overline{1-\varepsilon}}{\bar{\varepsilon}}\right)^2 \frac{\Delta v_{*,i}}{v_*} + O\left[(\Delta\varepsilon_i)^2\right] + O\left[\left(\Delta v_{*,i}\right)^2\right]\right] \tag{B.5}$$

With Eq. (B.1) inserted into Eq. (3.2.3), and with $e^{-C/nv_*} \approx 1 - C/nv_*$ we find after some elementary calculations

$$\lambda_{i,opt} = \frac{e^{-\Delta v/nv_* - C'} - \bar{\varepsilon}}{\frac{\bar{\varepsilon}}{\varepsilon_i} - e^{-\Delta v/nv_* - C'}} = \frac{\lambda_*^{1/n} + K}{\frac{\bar{\varepsilon}}{\varepsilon_i}\frac{1-\varepsilon_i}{\overline{1-\varepsilon}} - \lambda_*^{1/n} - K} \tag{B.6}$$

with $K = \left(\lambda_*^{1/n} + \frac{\bar{\varepsilon}}{\overline{1-\varepsilon}}\right)C'$.

References

Anderson, J.D.: Hypersonic and High-Temperature Gas Dynamics. AIAA Education Series (2006). ISBN 1-56347-780-7

Anderson, J.D.: Fundamentals of Aerodynamics (5th SI unit edn). McGraw-Hill (2011). ISBN 13: 9780073398105

Barth, H.: Conrad Haas—Raketenpionier und Humanist. Johannis Reeg Verlag (2005). ISBN 3-937320-55-5

Bate, R.R., Mueller, D., White, J.E.: Fundamentals of Astrodynamics. Dover Publications (1971). ISBN 0-486-60061-0

Battin, R.H.: Astronautical Guidance. McGraw-Hill, New York (1964)

Battin, R.H.: An Introduction to the Mathematics and Methods of Astrodynamics. AIAA Education Series (1987). ISBN 0-930403-25-8

Berlin, P.: Satellite Platform Design. Department of Space Science of the Universities Luleå and Umeå (2005). ISBN 91-631-4917-6, orders by email to berlinspace@cs.com

Berlin, P.: Geostationary Applications Satellite. Cambridge University Press (1988). ISBN 0-521-33525-6

Bertin, J.J.: Hypersonic Aerothermodynamics. AIAA Education Series, Washington, DC (1994). ISBN 1-56347-036-5

Beutler, G.: Methods of Celestial Mechanics I: Physical Mathematical and Numerical Principles. Springer, Berlin (2005). ISBN 3-540-40749-9

Beutler, G.: Methods of Celestial Mechanics II: Application to Planetary System, Geodynamicsand Satellite Geodesy. Springer, Berlin (2005). ISBN 3-540-40750-2

Brandenberger, R.H., Vafa, C.: Superstrings in the Early Universe. Nucl. Phys. B **316**, 391 (1989)

Brankin, R., et al.: Algorithm 670: a Runge-Kutta-Nyström code. ACM Trans. Math. Softw. **15**(1), 31–40 (1989)

Brown, C.D.: Elements of Spacecraft Design. AIAA Education Series (2002). ISBN 1-56347-524-3

Campan, G., Alby, F., Gautier, H.: Station-keeping techniques for geostationary satellites. In: Carrou, J.-P. (ed.) Spaceflight Dynamics, Part II. CNES/CÉPADUÈS (1995). ISBN 2-85428-377-5

Chao, C.-C.: Applied Orbit Perturbation and Maintenance. Aerospace Press (2005). ISBN 1-884989-17-9

Chapman, D.R.: An approximate analytical method for studying entry into planetary atmospheres. NASA TR R-11 (1959)

Chapman, D.R.: Tables of Z Functions for Atmospheric Entry Analysis. NASA TR R-106 (1961)

Chobotov, V.A. (ed.) Orbital Mechanics, 3rd edn. AIAA Education Series (2002). ISBN 1-56347-537-5

Cook, G.E.: Perturbations of near-circular orbits by the gravitational potential. Planet. Space Sci. **14**(5), 433–444 (1966)

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Cornelisse, J.W., Schöyer, H.F.R., Wakker, K.F.: Rocket Propulsion and Spaceflight Dynamics. Pitman Publishing Ltd. (1979). ISBN 0-273-01141-3
Curtis, H.D.: Orbital Mechanics for Engineering Students. Elsevier (2005). ISBN 0-7506-6169-0
Danby, J.M.A.: Fundamentals of Celestial Mechanics, 2nd edn. Willmann-Bell (2003). ISBN 0-943396-20-4
Deser, S., Jackiw, R., T'Hooft, G.: Three-dimensional Einstein gravity: dynamics of flat space. Ann. Phys. **152**, 220–235 (1984)
Efroimsky, M.: Gauge freedom in astrodanamics. In: Gurfil, P. (ed.) Modern Astrodynamics. Elsevier (2006). ISBN 9780123735621
Esmaelzadeh, R., Ghadiri, H.: Appropriate starter for solving the Kepler's Equation. Int. J. Comput. Appl. **89**(7), 31–38 (2014)
Escobal, P.R.: Methods of Orbit Determination. Krieger Publishing (1965). ISBN 0-88275-319-3
Fehse, W.: Automated Rendezvous and Docking of Spacecraft. Cambridge University Press (2003). ISBN 0-521-82492-3
Fortescue, P., Stark, J., Swinerd, G.: Spacecraft Systems Engineering, 3rd edn. Wiley (2003). ISBN 0-470-85102-3
Goodman, J.L.: History of Space Shuttle Rendezvous and Proximity Operations. J. Spacecraft Rockets **43**(5), 944–959 (2006)
Griffin, M.D., French, J.R.: Space Vehicle Design, 2nd edn. AIAA Education Series (2004). ISBN 1-56347-539-1-90-8
Groves, G.V.: Motion of a satellite in the earth's gravitational field. Proc. Royal Soc. London **254** (1276), 48–65 (1960)
Gurfil, P.: Nonlinear feedback control of low-thrust orbital transfer in a central gravitational field. Acta Astronaut. **60**, 631–648 (2007)
Gurzadyan, G.A.: Theory of Interplanetary Flights. Gordon and Breach (1996). ISBN 2-919875-15-9
Guthmann, A.: Einführung in die Himmelsmechanik und Ephemeridenrechnung. In: Spektrum Akademischer Verlag, 2nd edn (2000). ISBN 3-8274-0574-2
Hale, F.J.: Introduction into Space Flight. Prentince-Hall (1994). ISBN 0-13-481912-8
Hankey, W.L.: Re-Entry Aerodynamics. AIAA Education Series (1988). ISBN 0-930403-33-9
Hastings, D., Garrett, H.: Spacecraft-Environment Interactions. Cambridge University Press (1996). ISBN 0-521-47128-1
Herrick, S.: Astrodynamics—Orbit Determination, Space Navigation, Celestial Dynamics, vol. 1. Van Nostrand Reinhold Company, London (1971). ISBN 978-0442033705
Herrick, S.: Astrodynamics—Orbit Correction, Perturbation Theory, Integration, vol. 2. Van Nostrand Reinhold Company, London (1972). ISBN 978-0442033712
Hill, P.G., Peterson, R.: Mechanics and Thermodynamics of Propulsion, 2nd edn. Addison-Wesley (1992). ISBN 0-201-14659-2
Hirschel, E.H.: Basics of Aerothermodynamics. Springer (2004). ISBN-10: 3540221328
Hughes, P.C.: Spacecraft Attitude Dynamics. Wiley (1986). ISBN 0-471-81842-9
Humble, R.W., Henry, G.N., Larson W.J.: Space Propulsion Analysis and Design. McGraw-Hill (1995). ISBN 0-07-031320-2)
Humble, R.W., Henry, G.N., Larson W.J.: Space Propulsion Analysis and Design. McGraw-Hill (1995). ISBN 0-07-031320-2)
Jochim, E.F.M.: Satellitenbewegung, Band IV (2013, vol. IV). ISSN 1434-8454
Johnson, N.L.: A new look at the GEO and near-GEO regimes: operations, disposals, and debris. Acta Astronaut. **80**, 82–88 (2012)
Kaplan, M.H.: Modern Spacecraft Dynamics and Control. Wiley (1976). ISBN 0-471-45703-5
Kaula, W.M.: Theory of Satellite Geodesy. Waltham (1966). ISBN 0-486-41465-5
Kemble, S.: Interplanetary Mission Analysis and Design. Springer (2006). ISBN 3-540-29913-0
King-Hele, D.: Theory of Satellite Orbits in an Atmosphere. Springer (1987). ISBN 0216922526
Koon, W.S., Lo, M.W., Marsden, J.E., Ross, S.D.: Heteroclinic connections between periodic orbits and resonance transitions in celestial mechanics. Chaos **10**(2), 427–469 (2000)

Larson, W.J., Wertz, J.R.: Space Mission Analysis and Design, 3rd edn. Microcosm Press (1999). ISBN 1-881883-10-8
Ledermann, W., Lloyd, E., Vaijda, S., Alexander, C.: Handbook of Applicable Mathematics, Supplement. Wiley (1990). ISBN 978-0471918257
Liu, J.J.F., Alford, R.L.: Semianalytic theory for a closed-earth artificial satellite. J. Guidance Control **3**(4), 304–311 (1980)
Loh, W.H.T.: Re-entry and Planetary Entry: Physics and Technology, vol. I: Dynamics, Physics, Radiation, Heat, Transfer and Ablation, and vol. II: Advanced Concepts, Experiments, Guidance-Control and Technology. Springer, New York (1968)
Martin, J.J.: Atmospheric Reentry—An Introduction to its Science and Engineering. Prentice-Hall (1966)
Messerschmid, E., Fasoulas, S.: Raumfahrtsysteme. Springer (2000). ISBN 3-540-66803-9
Micheau, P.: Orbit Control Techniques for Low Earth Orbiting (LEO) Satellites. In: Carrou, J.-P. Spaceflight Dynamics, Part I. CNES/CÉPADUÈS (1995). ISBN 2-85428-376-7
Montenbruck, O., Gill, E.: Satellite Orbits—Models, Methods, Applications. Springer, Berlin (2000). ISBN 3-540-67280-X
Murray, C.D., Dermott, S.F.: Solar Systems Dynamics. Cambridge University Press (1999). ISBN 0-521-57295-9
Noton, M.: Spacecraft Navigation and Guidance. Springer (1998). ISBN 3-540-76248-5
Perozzi, E., Ferraz-Mello, S.: Space Manifold Dynamics. Springer (2010). ISBN 978-1441903471
Pisacane, V.L.: Fundamentals of Space Systems, 2nd edn. Oxford University Press (2005). ISBN 0-19-516205-6
Pollard, H.: Celestial Mechanics, (The Carus Mathematical Monographs, no. 18). Mathematical Association of America (1976). ISBN-13: 978-0883850190
Prussing, J.E., Conway, B.A.: Orbital Mechanics. Oxford University Press (1993). ISBN 0-19-507834-9
Rao, G.V.R.: Exhaust nozzle contour for optimum thrust. Jet Propul. **28**, 377 (1958)
Rao, A.V.: Trajectory optimization. In: Blockley, R., Shyy, W. Encyclopedia of Aerospace Engineering. Wiley (2010). ISBN 978-0-470-68665-2
Reif, F.: Fundamentals of Statistical and Thermal Physics. McGraw-Hill (1965). ISBN 0070518009
Reimann, P.: Brownian motors: noisy transport far from equilibrium. Phys. R. **361**, 57–265 (2002)
Regan, F.J.: Re-Entry Vehicle Dynamics. AIAA Education Series (1984). ISBN 0-915928-78-7
Richardson, D.L.: Analytical construction of periodic orbits about the collinear points. Celest. Mech. **22**, 241–253 (1980)
Roy, A.E.: Orbital Motion, 4th edn. Institute of Physics Publishing (2005). ISBN 0-7503-1015-4
Ruppe, H.O.: Introduction to Astronautics, vol 1. Academic Press (1966)
Schaub, H., Junkins, J.L.: Analytical Mechanics of Space Systems. AIAA Education Series (2003). ISBN 1-56347-563-4
Sellers, J.J.: Understanding Space, 3rd edn. McGraw-Hill (2005). ISBN 0-07-340775-5
Shapiro, B.E.: Phase plane analysis and observed frozen orbit for the Topex/Poseidon mission. In: 6th International Space Conference of Pacific Basin Societies (1995)
Szebehely, V.: Theory of Orbits—The Restricted Problem of Three Bodies. Academic Press, New York (1967)
Sidi, Marcel J.: Spacecraft Dynamics and Control. Cambridge University Press (1997). ISDN 0-521-55072-6
Soop, E.M.: Handbook of Geostationary Orbits. Kluwer Academic Publisher (1994). ISDN 0-792-33054-4
Steiner, W., Schagerl, M.: Raumflugmechanik. Springer (2004). ISBN 3-540-20761-9
Sutton, G.P., Biblarz O.: Rocket Propulsion Elements, 7th edn. Wiley (2001). ISBN 0-471-32642-9
Tapley, B.C., Schulz, B.E., Born, G.H.: Statistical Orbit Determination. Academic Press, New York (2004). ISBN 0126836302

Tegmark, M.: On the dimensionality of spacetime. Class. Quantum Grav. **14**, L69–L75 (1997)
Thomson, W.T.: Introduction to Space Dynamics. Dover Books (1986). ISBN 0-486-65113-4
Tribble, A.C.: The Space Environment—Implications for Spacecraft Design. Princton University Press (2003). ISBN 0-691-10299-6
Turner, M.J.L.: Rocket and Spacecraft Propulsion, 2nd edn. Springer (2006). ISBN 3540221905
Vallado, D.A.: Fundamentals of Astrodynamics and Applications, 3nd edn. McGraw Hill (2007). ISBN 978-1-881883-14-2
Vinh, N.X., Busemann, A., Culp, R.D.: Hypersonic and Planetary Entry Flight Mechanics. Ann Arbor University of Michigan Press (1980). ISBN 0-472-09304-5
Volk, O.: Miscellania from the History of Celestial Mechanics. Celest. Mech. **14**, 365–382 (1976)
Walter, U.: Relativistic rocket and space flight. Acta Astronaut. **59**, 453–516 (2006)
Westmorelend, S.: A note on relativistic rocketry. Acta Astronaut. **67**, 1248–1251 (2010)
Wiesel, W.E.: Spaceflight Dynamics. McGraw Hill (1997). ISBN 0-07-070110-5
Zhong, W., Gurfil, P.: Mean orbital elements estimation for autonomous satellite guidance and orbit control. J. Guidance Control Dyn. **36**, 1624–1641 (2013)
Zoby, E.V.: Approximate heating analysis for the windward symmetry plane of shuttle-like bodies at angle of attack. In: Horton, T.E. (ed.) Thermodynamics of Atmospheric Entry, vol. 82. Progess in the Astronautics and Aeronautics, AIAA, pp. 521–528 (1982)

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Directions for Use

- Compound terms that are set expressions (Example: characteristic velocity (c^*)) are listed as such, otherwise they are listed by their superordinate term (Example: "departure velocity" is listed as "velocity, departure"). In case of doubt, cross references are provided.
- Only significant page references are listed.
- Page references that are definitions or are essential in terms of important results are printed in **bold**.
- Braces {…} mean a.k.a. …

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