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# Advances in Computer Algebra

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Carsten Schneider · Eugene Zima
Editors

# Advances in Computer Algebra

## In Honour of Sergei Abramov's 70th Birthday, WWCA 2016, Waterloo, Ontario, Canada

Springer

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*This book is devoted to the 70th birthday of Sergei Abramov, whose classical algorithms for symbolic summation and solving linear differential, difference, and q-difference equations inspired many.*

# Preface

The Waterloo Workshop on Computer Algebra (WWCA-2016) was held on July 23–24, 2016, at Wilfrid Laurier University (Waterloo, Ontario, Canada). The workshop provided a forum for researchers and practitioners to discuss recent advances in the area of Computer Algebra. WWCA-2016 was dedicated to the 70th birthday of Sergei Abramov (Computer Center of the Russian Academy of Sciences, Moscow, Russia) whose influential contributions to symbolic methods are highly acknowledged by the research community and adopted by the leading Computer Algebra systems. The workshop attracted world-renowned experts in Computer Algebra and symbolic computation. Presentations on original research topics or surveys of the state of the art within the research area of Computer Algebra were made by

- Moulay A. Barkatou, University of Limoges, France
- Shaoshi Chen, Chinese Academy of Sciences, China
- Mark van Hoeij, Florida State University, USA
- Manuel Kauers, Johannes Kepler University, Austria
- Christoph Koutschan, RICAM, Austria
- Ziming Li, Chinese Academy of Sciences, China
- Johannes Middeke, RISC, Johannes Kepler University Linz, Austria
- Mark Round, RISC, Johannes Kepler University Linz, Austria
- Evans Doe Ocansey, RISC, Johannes Kepler University Linz, Austria
- Carsten Schneider, RISC, Johannes Kepler University Linz, Austria
- Eric Schost, University of Waterloo, Canada
- Vo Ngoc Thieu, RISC, Johannes Kepler University Linz, Austria
- Eugene Zima, WLU, Canada

Success of the workshop was due to the generous support of the Office of the President, Research Office, and Department of Physics and Computer Science of the Wilfrid Laurier University.

This book presents a collection of formally refereed selected papers submitted after the workshop. Topics discussed in this book are the latest achievements in algorithms of symbolic summation, factorization, symbolic-numeric linear algebra, and linear functional equations, i.e., topics of symbolic computations that were extensively advanced due to Sergei's influential works.

In Chapter "On Strongly Non-singular Polynomial Matrices" (Sergei A. Abramov and Moulay A. Barkatou), an algorithm is worked out that decides whether or not a matrix with polynomial entries is a truncation of an invertible matrix with power series entries. Using this new insight, the computation of solutions of higher order linear differential systems in terms of truncated power series is pushed forward. In particular, a criterion is provided when a truncation of a power series solution can be computed.

In Chapter "On the Computation of Simple Forms and Regular Solutions of Linear Difference Systems" (Moulay A. Barkatou, Thomas Cluzeau and Carole El Bacha), first-order linear difference systems with factorial series coefficients are treated. Factorial series, which play an important role in the analysis of linear difference systems, are similar to power series, but instead of $z^n$ the "power" is 1 over the $n$th rising factorial of $z$. New reduction algorithms are presented to provide solutions for such systems in terms of factorial series.

In Chapter "Refined Holonomic Summation Algorithms in Particle Physics" (Johannes Blümlein, Mark Round and Carsten Schneider), the summation approach in the setting of difference rings is enhanced by tools from the holonomic system approach. It is now possible to deal efficiently with summation objects that are described by linear inhomogeneous recurrences whose coefficients depend on indefinite nested sums and products. The derived methods are tailored to challenging sums that arise in particle physics problems.

In Chapter "Bivariate Extensions of Abramov's Algorithm for Rational Summation" (Shaoshi Chen), a general framework is developed to decide algorithmically if a bivariate sequence/function can be summed/integrated by solving the bivariate anti-difference/anti-differential equation. The summation/integration problem for double sums/integrals is elaborated completely: Besides the rational case also its $q$-generalization is treated for both, the summation and integration setting.

In Chapter "A $q$-Analogue of the Modified Abramov-Petkovšek Reduction" (Hao Du, Hui Huang and Ziming Li), an algorithm is provided that simplifies a truncated $q$-hypergeometric sum to a summable part that can be expressed in terms of $q$-hypergeometric products and a non-summable sum whose summand satisfies certain minimality criteria. In case that the input sum is completely summable, this representation is computed (i.e., the non-summable part is zero). Experimental results demonstrate that this refined reduction of the Abramov-Petkovšek reduction gains substantial speedups.

In Chapter "Factorization of C-finite Sequences" (Manuel Kauers and Doron Zeilberger), a new algorithm is elaborated that factorizes a linear recurrence with constant coefficients into two non-trivial factors whenever this is possible. Instead of the usage of expensive Gröbner basis computation, the factorization task is reduced to a combinatorial assignment problem. Concrete examples demonstrate the practical relevance of these results.

In Chapter "Denominator Bounds for Systems of Recurrence Equations Using $\Pi\Sigma$-Extensions" (Johannes Middeke and Carsten Schneider), a general framework in the setting of difference fields is presented to tackle coupled systems of linear difference equations. Besides the rational and $q$-rational cases, the coefficients of the system and the solutions thereof might be given in terms of indefinite nested sums and products.

In Chapter "Representing ($q$–)Hypergeometric Products and Mixed Versions in Difference Rings" (Evans Doe Ocansey and Carsten Schneider), algorithms are presented that enable one to represent a finite set of hypergeometric products and more generally $q$-hypergeometric products and their mixed versions within the difference ring theory of $R\Pi\Sigma$-extensions. As a consequence, one can solve the zero-recognition problem for expressions in terms of such products and obtains expressions in terms of products whose sequences are algebraically independent among each other.

In Chapter "Linearly Satellite Unknowns in Linear Differential Systems" (Anton A. Panferov), an algorithm is worked out that determines for a given system of linear differential equations whether or not a component of a solution can be expressed in terms of the solutions of a fixed set of other components. The solutions of these components can be composed by taking their linear combination and by applying the differential operator to them. The derived knowledge turns out to be useful if one is only interested in parts of the solution.

In Chapter "Rogers-Ramanujan Functions, Modular Functions, and Computer Algebra" (Peter Paule and Silviu Radu), the connection of $q$-series in partition theory and modular functions, with the Rogers-Ramanujan functions as key players, is worked out. Special emphasis is put on Computer Algebra aspects dealing, e.g., with zero recognition of modular forms, $q$-holonomic approximations of modular forms, or projections of $q$-holonomic series. This algorithmic machinery is illuminated by the derivation of Felix Klein's classical icosahedral equation.

This book would not have been possible without the contributions and hard work of the anonymous referees, who supplied detailed referee reports and helped authors to improve their papers significantly.

Linz, Austria — Carsten Schneider
Waterloo, Canada — Eugene Zima
October 2017

# Contents

# On Strongly Non-singular Polynomial Matrices

**Sergei A. Abramov and Moulay A. Barkatou**

**Abstract** We consider matrices with infinite power series as entries and suppose that those matrices are represented in an "approximate" form, namely, in a truncated form. Thus, it is supposed that a polynomial matrix $P$ which is the $l$-truncation ($l$ is a non-negative integer, $\deg P = l$) of a power series matrix $M$ is given, and $P$ is non-singular, i.e., $\det P \neq 0$. We prove that the strong non-singularity testing, i.e., the testing whether $P$ is not a truncation of a singular matrix having power series entries, is algorithmically decidable. Supposing that a non-singular power series matrix $M$ (which is not known to us) is represented by a strongly non-singular polynomial matrix $P$, we give a tight lower bound for the number of initial terms of $M^{-1}$ which can be determined from $P^{-1}$. In addition, we report on possibility of applying the proposed approach to "approximate" linear differential systems.

**Keywords** Polynomial matrices · Strong non-singularity · Linear differential systems · Truncated series

## 1 Introduction

We discuss an "approximate" representation of infinite power series which appear as inputs for computer algebra algorithms. A well-known example is given in [10], it is related to the number of terms in $M$ that can influence some components of formal exponential-logarithmic solutions of a differential system $x^{s+1}y' = My$, where $s$ is a non-negative integer, $M$ is a matrix whose entries are power series; see also its generalization in [11] and our previous paper [4].

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In the present paper, we consider matrices with infinite power series (over a field $K$ of characteristic 0) as entries and suppose that those series are represented in a truncated form. Thus, it is assumed that a polynomial matrix $P$ which is the $l$-truncation $M^{\langle l \rangle}$ ($l$ is a non-negative integer, $\deg P = l$) of a power series matrix $M$ is given, and $P$ is non-singular, i.e., $\det P \neq 0$. We prove that the the question of strong non-singularity, i.e., the question whether $P$ is not the $l$-truncation of a singular matrix having power series entries, is algorithmically decidable.

Assuming that a non-singular power series matrix $M$ (which is not known to us) is represented by a strongly non-singular polynomial matrix $P$, we give a tight lower bound for the number of initial terms of $M^{-1}$ which can be determined from $P^{-1}$. As it turns out, for the answer to these questions, the number $h = \deg P + \operatorname{val} P^{-1}$ plays the key role, and $h \geq 0$ is a criterion of the impossibility of a prolongation of polynomials to series so that the determinant of the matrix vanishes. If this inequality holds then first, for any prolongation, the valuations of the determinant and the inverse of the approximate matrix and, resp., of the prolonged matrix coincide. Second, in the expansions of the determinants of the approximate and prolonged matrices the coefficients coincide for $x^{\operatorname{val} \det P}$, as well as $h$ subsequent coefficients (for larger degrees of $x$). The similar statement holds for the inverse matrices.

In addition, we prove that if $M$ is an $n \times n$-matrix having power series entries, $\det M \neq 0$ then there exists a non-negative integer $l$ such that $M^{\langle l \rangle}$ is a strongly non-singular polynomial matrix. If the entries of $M$ are represented algorithmically (for each power series that is an entry of $M$, an algorithm is specified that, given an integer $i$, finds the coefficient of $x^i$) then an upper bound for such $l$ can be computed.

In Sect. 7, we discuss the possibility of applying the proposed approach to approximate linear differential systems of arbitrary order with power series matrix coefficients: if a system $S$ is given in the approximate truncated form $\tilde{S}$, $\operatorname{ord} \tilde{S} = \operatorname{ord} S$, and the leading matrix of $\tilde{S}$ is strongly non-singular then one can guarantee, under some extra specific conditions, that Laurent series solutions of the truncated system $\tilde{S}$ coincide with Laurent series solutions of the system $S$ up to some degree of $x$ that can be estimated by the algorithm we proposed in [4].

In our paper we are considering a situation where a truncated system is initially given and we do not know the original system. We are trying to establish, whether it is possible to get from the solutions of this system an information on solutions of any system obtained from this system by a prolongation of the polynomial coefficients to series. In comparison with, e.g., [8, 10], this is a different task.

## 2 Preliminaries

Let $K$ be a field of characteristic 0. We denote by $K[[x]]$ the ring of *formal power series* and $K((x)) = K[[x]][x^{-1}]$ its quotient field (the field of *formal Laurent series*) with coefficients in $K$. For a nonzero element $a = \sum a_i x^i$ of $K((x))$ we denote by $\operatorname{val} a$ the *valuation* of $a$ defined by $\operatorname{val} a = \min \{i \text{ such that } a_i \neq 0\}$; by convention, $\operatorname{val} 0 = \infty$.

If $l \in \mathbb{Z}$, $a(x) \in K((x))$ then we define the *l-truncation* $a^{\langle l \rangle} \in K[x, x^{-1}]$ as the Laurent polynomial obtained by omitting all the terms of valuation larger than $l$ in $a$.

The ring of $n \times n$ matrices with entries belonging to a ring (a field) $R$ is denoted by $\mathrm{Mat}_n(R)$. The *identity* $n \times n$-matrix is denoted by $I_n$. The notation $M^T$ is used for the transpose of a matrix (vector) $M$.

For $M \in \mathrm{Mat}_n(K((x)))$ we define $\mathrm{val}\, M$ as the minimum of the valuations of the entries of $M$. We define the *leading coefficient* of a nonzero matrix $M \in \mathrm{Mat}_n(K((x)))$ as $\mathrm{lc}\, M = (x^{-\mathrm{val}\, M} M)|_{x=0}$. For $M \in \mathrm{Mat}_n(K[x])$ we define $\deg M$ as the maximum of the degrees of the entries of $M$.

A matrix $M \in \mathrm{Mat}_n(K((x)))$ is *non-singular* if $\det M \neq 0$, otherwise $M$ is *singular*.

For $M \in \mathrm{Mat}_n(K((x)))$ we denote by $M^*$ the adjugate matrix of $M$, i.e. the transpose of the cofactor matrix of $M$. One has

$$MM^* = M^*M = (\det M)I_n,$$

and, when $M$ is non-singular

$$M^{-1} = (\det M)^{-1}M^*, \tag{1}$$

Given $M \in \mathrm{Mat}_n(K((x)))$, we define $M^{\langle l \rangle} \in \mathrm{Mat}_n(K[x, x^{-1}])$ obtained by replacing the entries of $M$ by their $l$-truncations (if $M \in \mathrm{Mat}_n(K[[x]])$ then $M^{\langle l \rangle} \in \mathrm{Mat}_n(K[x])$).

## 3 Strongly Non-singular Polynomial Matrices

**Definition 1** Let $P \in \mathrm{Mat}_n(K[x])$ be a non-singular polynomial matrix and denote by $d$ its degree. We say that $P$ is *strongly non-singular* if there exists no singular matrix $M \in \mathrm{Mat}_n(K[[x]])$ such that $M^{\langle d \rangle} = P$.

*Remark 1* Clearly, a non-singular matrix $P \in \mathrm{Mat}_n(K[x])$ of degree $d$ is strongly non-singular if and only if there exists no $Q \in \mathrm{Mat}_n(K[[x]])$ such that $P + x^{d+1}Q$ is singular.

Now we prove a simple criterion for a polynomial matrix to be strongly non-singular.

**Proposition 1** *Let* $P \in \mathit{Mat}_n(K[x])$, $\det P \neq 0$. *Then* $P$ *is strongly non-singular if and only if*

$$\deg P + \mathrm{val}\, P^* \geq \mathrm{val} \det P. \tag{2}$$

*Proof* Let $d = \deg P$, $v = \mathrm{val} \det P$ and

$$\tilde{P} = (P^*)^T$$

be the cofactor matrix of $P$.

Necessity: Suppose that the condition (2) is not satisfied. Let $\tilde{P} = (\tilde{p}_{i,j})_{i,j=1,\ldots,n}$, and $\tilde{p}_{i_0,j_0}$ an entry of $\tilde{P}$ such that

$$d + \mathrm{val}\, \tilde{p}_{i_0,j_0} < v,$$

then $v - \mathrm{val}\, \tilde{p}_{i_0,j_0} \geq d + 1$. Divide $\det P$ by $\tilde{p}_{i_0,j_0}$, considering them as power series.

The quotient is a power series $q$, $\mathrm{val}\, q \geq d + 1$. For the matrix $Q = (q_{i,j})_{i,j=1,\ldots,n}$ such that

$$q_{i,j} = \begin{cases} -x^{-d-1}q, & \text{if } i = i_0 \text{ and } j = j_0, \\ 0, & \text{otherwise}, \end{cases} \tag{3}$$

we get $\det(P + x^{d+1}Q) = 0$ by the Laplace expansion along the $i_0$-th row. According to Remark 1 the matrix $P$ is not strongly non-singular.

Sufficiency: Suppose that the condition (2) is satisfied and let $Q \in \mathrm{Mat}_n(K[[x]])$. Since $\mathrm{val}\, x^{d+1}Q \geq d + 1$ we have

$$\mathrm{val}\det(P + x^{d+1}Q) = v,$$

and $\det(P + x^{d+1}Q) \neq 0$.

*Remark 2* Obviously, the inequality (2) can be rewritten in the equivalent form

$$\mathrm{val}\,(P^{-1}) + \deg P \geq 0, \tag{4}$$

since $\mathrm{val}\, P^* - \mathrm{val}\det P = \mathrm{val}\,(P^{-1})$ due to (1). Note also that

$$\deg P \geq \mathrm{val}\det P \tag{5}$$

is a sufficient condition for a matrix $P$ to be strongly non-singular, since (5) implies (2).

*Example 1* Every non-singular constant matrix is strongly non-singular. More generally, every polynomial matrix $P$ such that $\mathrm{val}\det P = 0$ is strongly non-singular. □

It follows from the given proof of Proposition 1 that if $P$ is not strongly non-singular, then one can *construct explicitly* a matrix $Q \in \mathrm{Mat}_n(K[[x]])$ such that $\det(P + x^{\deg P+1}Q) = 0$, and $Q$ has only one nonzero entry, which is factually a rational function of $x$ that can be expanded into a power series.

*Example 2* Consider the following matrix

$$P = \begin{pmatrix} x & 0 \\ 1 & x \end{pmatrix}. \tag{6}$$

One has $\deg P = 1$, $\operatorname{val} P^* = 0$, $\operatorname{val}\det P = 2$, so inequalities (2), (4) are not satisfied. Hence $P$ is not strongly non-singular. Its cofactor matrix $\tilde{P}$ is given by

$$\tilde{P} = \begin{pmatrix} x & -1 \\ 0 & x \end{pmatrix},$$

In accordance with (3), the corresponding matrix $Q$ is

$$Q = \begin{pmatrix} 0 & -x^{-2}\,\frac{x^2}{-1} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

The matrix

$$P + x^2 Q = \begin{pmatrix} x & x^2 \\ 1 & x \end{pmatrix}$$

is singular as expected. □

**Proposition 2** *Let $P$ be a strongly non-singular polynomial matrix of degree $d$. Let $v = \operatorname{val}\det P$ and $h = \operatorname{val} P^{-1} + \deg P$. Then for any $Q \in Mat_n(K[[x]])$ one has*

$$\det(P + x^{d+1}Q) - \det P = O(x^{v+h+1}). \tag{7}$$

*Proof* Put $\bar{P} = x^{-\operatorname{val} P^{-1}} P^{-1}$ so that $\operatorname{val}\bar{P} \geq 0$. For any $Q \in \operatorname{Mat}_n(K[[x]])$ one has

$$P + x^{d+1}Q = P(I_n + x^{d+1}P^{-1}Q) = P(I_n + x^{d+1+\operatorname{val} P^{-1}}\bar{P}Q)$$

Hence

$$\det(P + x^{d+1}Q) = \det P \det(I_n + x^{h+1}\bar{P}Q).$$

The matrix $P$ is strongly non-singular hence $h \geq 0$, and since $\operatorname{val}(\bar{P}Q) \geq 0$ it follows that

$$\det(I_n + x^{h+1}\bar{P}Q) = 1 + O(x^{h+1}),$$

and therefore $\det(P + x^{d+1}Q) = \det P + O(x^{v+h+1})$.

As a consequence, Proposition 2 states that $\det(P + x^{d+1}Q)$ and $\det P$ have the same valuation $v$ for any $Q \in \operatorname{Mat}_n(K[[x]])$. Moreover, the $h+1$ first terms in the power series expansion of $\det(P + x^{d+1}Q)$ coincide with the corresponding terms of $\det P$.

*Example 3* Let

$$P = \begin{pmatrix} 1+x & 0 \\ 1 & 1-x \end{pmatrix}.$$

Here $\det P = 1 - x^2$, $v = \operatorname{val}\det P = 0$ hence the matrix $P$ is strongly non-singular. Here $\deg P = 1$ and $h = \operatorname{val} P^{-1} + \deg P = 1$ Let

$$Q = \begin{pmatrix} 1 + x + x^2 + \cdots & 0 \\ 0 & 0 \end{pmatrix}$$

then

$$P + x^2 Q = \begin{pmatrix} 1 + x + x^2 + \cdots & 0 \\ 1 & 1 - x \end{pmatrix}.$$

We have

$$\det P = 1 - x^2, \quad \det (P + x^2 Q) = 1$$

and (7) holds (here $x^{v+h+1} = x^2$). □

## 4 Inverse Matrix

The following proposition states that if $P$ is strongly non-singular then for any $Q \in \mathrm{Mat}_n(K[[x]])$, the Laurent series expansions of the matrices $P^{-1}$ and $(P + x^{d+1}Q)^{-1}$ have the same valuation and their first $h + 1$ terms coincide where $h = \deg P + \mathrm{val}\, P^{-1}$.

**Proposition 3** *Let $P$ be a strongly non-singular polynomial $n \times n$-matrix of degree $d$ and let $h = \deg P + \mathrm{val}\, P^{-1}$. Then for any $Q \in Mat_n(K[[x]])$ the Laurent series expansions of $(P + x^{d+1}Q)^{-1}$ and $P^{-1}$ coincide up to order $\mathrm{val}\, P^{-1} + h$, i.e.,*

$$(P + x^{d+1}Q)^{-1} - P^{-1} = O(x^{\mathrm{val}\, P^{-1}+h+1}). \tag{8}$$

*In particular, one has*

$$\mathrm{val}\, (P + x^{d+1}Q)^{-1} = \mathrm{val}\, P^{-1} \quad \text{and} \quad \mathrm{lc}(P + x^{d+1}Q)^{-1} = \mathrm{lc}\, P^{-1} \tag{9}$$

*for any $Q \in Mat_n(K[[x]])$.*

*Proof* Let $\bar{P} = x^{-\mathrm{val}\, P^{-1}} P^{-1}$ so that $\mathrm{val}\, \bar{P} \geq 0$. For any $Q \in \mathrm{Mat}_n(K[[x]])$ one has

$$(P + x^{d+1}Q)^{-1} = (I_n + x^{d+\mathrm{val}\, P^{-1}+1} \bar{P} Q)^{-1} P^{-1}$$

It follows from (4) that $h \geq 0$, hence

$$(I_n + x^{h+1} \bar{P} Q)^{-1} = I_n + x^{h+1} C_1 + x^{h+2} C_2 + \cdots$$

where the $C_i$ are constant matrices and the dots denote terms of higher valuation. It follows that

$$(P + x^{d+1}Q)^{-1} = P^{-1} + O(x^{h+1}) \cdot P^{-1} = x^{\mathrm{val}\, P^{-1}} (\bar{P} + O(x^{h+1}) \cdot \bar{P})$$

Hence

$$(P + x^{d+1} Q)^{-1} = x^{\mathrm{val}\, P^{-1}} (\bar{P}^{<h+1>} + O(x^{h+1}))$$

and the claim follows.

*Example 4* Going back to the matrices $P$, $Q$ from Example 3, we see that

$$P + x^2 Q = \begin{pmatrix} \frac{1}{1-x} & 0 \\ 1 & 1-x \end{pmatrix}$$

and we can compute

$$(P + x^2 Q)^{-1} = \begin{pmatrix} 1-x & 0 \\ -1 & \frac{1}{1-x} \end{pmatrix} = \begin{pmatrix} 1-x & 0 \\ -1 & 1 + x + x^2 + \cdots \end{pmatrix}$$

while

$$P^{-1} = \begin{pmatrix} \frac{1}{1+x} & 0 \\ \frac{-1}{1-x^2} & \frac{1}{1-x} \end{pmatrix} = \begin{pmatrix} 1 - x + x^2 + \cdots & 0 \\ -1 - x^2 - \cdots & 1 + x + x^2 + \cdots \end{pmatrix}.$$

Taking into account that here $d = 1$, $h = 1$, we see that (8) and (9) hold. □

*Remark 3* Examples 3, 4 show that estimates (7), (8) are tight: $O(x^{\nu+h+1})$ and $O(x^{\mathrm{val}\, P^{-1}+h+1})$ cannot be replaced by $O(x^{\nu+h+2})$ and, resp., $O(x^{\mathrm{val}\, P^{-1}+h+2})$.

## 5 Product of Strongly Non-singular Matrices

The product of two strongly non-singular matrices is not in general a strongly non-singular matrix.

*Example 5* By Proposition 1, the matrices

$$P_1 = \begin{pmatrix} x & 1 \\ 1 & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 1 & x \\ 0 & -x^2 \end{pmatrix}$$

are both strongly non-singular, but their product

$$P_1 P_2 = \begin{pmatrix} x & 0 \\ 1 & x \end{pmatrix},$$

is not, as it has been shown in Example 2. □

However, the following proposition holds:

**Proposition 4** *Let $P_1, P_2 \in Mat_n(K[x])$ be strongly non-singular, and such that*

$$\deg P_1 P_2 = \deg P_1 + \deg P_2. \tag{10}$$

*Then* $P_1 P_2$ *is a strongly non-singular matrix.*

*Proof* The inequality

$$\text{val}\,(P_1 P_2)^{-1} \geq \text{val}\,P_1^{-1} + \text{val}\,P_2^{-1} \tag{11}$$

takes place (it holds for any matrices). Thus, it follows that if (10) is satisfied and (4) holds for both $P_1$ and $P_2$ then it holds for $P_1 P_2$ as well.

## 6 Width and s-Width of Non-singular Matrices with Power Series Entries

In [3], the *width* of a non-singular (full rank) matrix $M \in \text{Mat}_n(K[[x]])$ was defined as the minimal non-negative integer $w$ such that any truncation $M^{\langle l \rangle}$ of $M$, $l \geq w$, is non-singular. Besides the notion of the width we will consider a similar notion related to the strong non-singularity.

**Definition 2** The *s-width* (the *strong width*) of a non-singular (full rank) matrix $M \in \text{Mat}_n(K[[x]])$ is the minimal non-negative integer $w_s$ such that any $\hat{M} \in \text{Mat}_n(K[[x]])$ which satisfies $\hat{M}^{\langle w_s \rangle} = M^{\langle w_s \rangle}$ is a non-singular matrix.

We will use the notations $w(M)$, $w_s(M)$ when it is convenient.

It was shown in [3, Remark 3] that the width $w(M)$ is well defined for any non-singular matrix $M \in \text{Mat}_n(K[[x]])$. The following Proposition states that the s-width $w_s(M)$ is also well defined for any non-singular $M \in \text{Mat}_n(K[[x]])$ and it is bounded by $-\text{val}\,(M^{-1})$.

**Proposition 5** *Let* $M \in \text{Mat}_n(K[[x]])$ *with* $\det M \neq 0$ *and set* $l_0 = -\text{val}\,(M^{-1})$. *Then the matrix* $(M^{\langle l \rangle} + x^{l+1} Q)$ *is non-singular for any* $Q \in \text{Mat}_n(K[[x]])$ *and any* $l \geq l_0$.

*Proof* For any $Q \in \text{Mat}_n(K[[x]])$ and for any non-negative integer $l$ one has

$$M^{\langle l \rangle} + x^{l+1} Q = M + O(x^{l+1}) = M(I_n + x^{l+1+\text{val}\,M^{-1}} O(1)).$$

Hence

$$\det\,(M^{\langle l \rangle} + x^{l+1} Q) = (\det M)\det\,(I_n + x^{l+1+\text{val}\,M^{-1}} O(1))$$

If we take $l \geq -\text{val}\,(M^{-1})$ then

$$\text{val}\,(\det\,(M^{\langle l \rangle} + x^{l+1} Q)) = \text{val}\,(\det M)$$

for all $Q \in \text{Mat}_n(K[[x]])$. Thus the claim follows.

Evidently,

$$w_s(M) \geq w(M)$$

for any non-singular matrix $M \in \mathrm{Mat}_n(K[[x]])$. However, as it is shown by the following example, it may happen that $w_s(M) > w(M)$; in other words, $w_s(M) \neq w(M)$ in general.

*Example 6* Consider the matrix

$$M = \begin{pmatrix} x & x^3 \\ 1 & x \end{pmatrix}. \tag{12}$$

One has $\det M = x^2 - x^3 \neq 0$, $\det M^{\langle 0 \rangle} = 0$,

$$M^{\langle 1 \rangle} = M^{\langle 2 \rangle} = \begin{pmatrix} x & 0 \\ 1 & x \end{pmatrix}, \quad \det \begin{pmatrix} x & 0 \\ 1 & x \end{pmatrix} \neq 0,$$

and $M^{\langle l \rangle} = M$ for $l \geq 3$. Thus $w(M) = 1$. However, $w_s(M) > 1$, due to the fact that

$$\det \begin{pmatrix} x & x^2 \\ 1 & x \end{pmatrix} = 0.$$

It is easy to check that $w_s(M) = 2$. □

*Remark 4* The above example shows that the matrix $M^{\langle w_s \rangle}$ is not necessarily a strongly non-singular matrix. In fact, the matrix $M^{\langle w_s \rangle}$ is strongly non-singular if, and only if, $\deg M^{\langle w_s \rangle} = w_s$.

**Proposition 6** *Let $M \in Mat_n(K[[x]])$ with* $\det M \neq 0$. *Then the set*

$$L(M) = \{l \in \mathbb{N} \mid M^{\langle l \rangle} \text{ is strongly non-singular}\} \tag{13}$$

*is non-empty if, and only if, $M$ is either an infinite power series matrix or a polynomial matrix which is strongly non-singular.*

*Proof* Let $d = \deg M$, with $d = +\infty$ when $M \in \mathrm{Mat}_n(K[[x]] \setminus K[x])$, and $l_0 = -\mathrm{val}\,(M^{-1})$. If $L(M) \neq \emptyset$ and $d < +\infty$ then $M^{\langle l \rangle}$ is strongly non-singular for some integer $l \geq 0$, and hence $M$ is strongly non-singular as well. Reciprocally, suppose that $d = +\infty$. Then there exists an $l \geq l_0$ such that $\deg M^{\langle l \rangle} = l$. Now, according to Proposition 5, the matrix $M^{\langle l \rangle}$ is strongly non-singular. Hence $L(M) \neq \emptyset$.

**Proposition 7** *Let $M \in Mat_n(K[[x]])$ with* $\det M \neq 0$. *Suppose that $L(M) \neq \emptyset$ (see (13)), and denote the smallest element of $L(M)$ by $\widetilde{w}_s(M)$. Then*

$$w_s(M) \leq -\mathrm{val}\, M^{-1} \leq \widetilde{w}_s(M).$$

*In particular, the three quantities coincide if, and only if, the matrix $M^{\langle w_s \rangle}$ is strongly non-singular.*

*Proof* We know that the first inequality $w_s(M) \leq -\text{val}\,(M^{-1})$ always holds. It remains to prove the second inequality. Let $l \in L(M)$ and set $P = M^{\langle l \rangle}$. One has $l \geq \deg P$ and $\deg P + \text{val}\, P^{-1} \geq 0$, since $P$ is strongly non-singular. On the other hand, by Proposition 3, one has $\text{val}\, P^{-1} = \text{val}\, M^{-1}$. It follows that

$$l + \text{val}\, M^{-1} \geq \deg P + \text{val}\, P^{-1} \geq 0.$$

The last part of the proposition follows from the fact that $M^{\langle w_s \rangle}$ is strongly non-singular if, and only if, $w_s \in L(M)$.

The matrix $M$ in Example 6 satisfies the inequalities

$$w_s(M) = -\text{val}\, M^{-1} = \text{val}\det M = 2 < \widetilde{w}_s(M) = 3.$$

This shows in particular that, in general, $\text{val}\det M$ is not an upper-bound of $\widetilde{w}_s$, while we always have

$$w_s(M) \leq -\text{val}\,(M^{-1}) \leq \text{val}\det M.$$

The following example shows that $w_s(M)$ is not always equal to $\text{val}\det M$.

*Example 7* Let

$$M = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}.$$

It is easy to check that $w_s(M) = 1 = \widetilde{w}_s(M)$. Indeed, $\det M^{\langle 0 \rangle} = 0$ and

$$\det M^{\langle 1 \rangle} = x^2, \quad \det\,(M^{\langle 1 \rangle} + x^2 Q) = \det \begin{pmatrix} x + O(x^2) & O(x^2) \\ O(x^2) & x + O(x^2) \end{pmatrix} = x^2 + O(x^3).$$

In the same time, $\text{val}\det M = 2$. □

**Proposition 8** *There exists an algorithm which, given a non-singular matrix $M \in Mat_n(K[[x]] \setminus K[x])$ that is represented algorithmically*[1] *computes $\widetilde{w}_s(M)$.*

*Proof* For $l = 0, 1, \ldots,$ we set step-by-step $P = M^{\langle l \rangle}$ and test wether condition (4) holds. Proposition 6 guarantees that this process terminates.

Note that the existence of an algorithm for computing $w_s(M)$ for a non-singular matrix $M$ represented algorithmically is still an open problem, although we can compute upper bounds $\widetilde{w}_s(M)$, $\text{val}\det M$ for it.

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[1]For each power series that is an entry of $M$, an algorithm is specified that, given an integer $i$, finds the coefficient of $x^i$—see [2].

# 7 Linear Differential Systems with Truncated Coefficients

We write $\vartheta$ for $x\frac{d}{dx}$ and consider linear differential systems with power series coefficients of the form

$$A_r(x)\vartheta^r y + A_{r-1}(x)\vartheta^{r-1} y + \cdots + A_0(x)y = 0 \tag{14}$$

where $y = (y_1, y_2, \ldots, y_m)^T$ is a column vector of unknown functions of $x$ and where the coefficient matrices

$$A_0(x), A_1(x), \ldots, A_r(x) \tag{15}$$

belong to $\mathrm{Mat}_m(K[[x]])$. We suppose that matrices $A_0(x)$, $A_r(x)$ are non-zero and $\min_i\{\mathrm{val}\, A_i\} = 0$. For a system $S$ of the form (14) we define the $l$-truncation $S^{\langle l\rangle}$ as the differential system with polynomial matrix coefficients obtained from $S$ by omitting all the terms of valuation larger than $l$ in the coefficients of $S$ (the $l$-truncation is with respect to $x$ only, not with respect to $\vartheta$).

## *7.1 Width and s-Width of Differential Systems of Full Rank*

**Definition 3** Let $S$ be a system of full rank over $K[[x]][\vartheta]$. The minimal integer $w$ such that $S^{\langle l\rangle}$ is of full rank for all $l \geq w$ is called the *width* of $S$; this notion was first introduced in [3]. The minimal integer $w_s$ such that any system $S_1$ satisfying the condition $S_1^{\langle w_s\rangle} = S^{\langle w_s\rangle}$, is of full rank, is called the *s-width* (the *strong width*) of $S$.

We will use the notations $w(S)$, $w_s(S)$ when it is convenient.

Any linear algebraic system can be considered as a linear differential system of zero order. This let us state using Example 6 that for an arbitrary differential system $S$ we have $w_s(M) \neq w(M)$ in general. However the inequality

$$w_s(S) \geq w(S)$$

holds.

It was proven in [3, Thm 2] that if a system $S$ of the form (14) is of full rank then the width $w$ of $S$ is well defined, and the value $w$ may be computed if the entries of $S$ are represented algorithmically.

Concerning the s-width, we get the following proposition:

**Proposition 9** *Let $S$ be a full rank system of the form (14). Then the s-width $w_s(S)$ is defined. If the power series coefficients of $S$ are represented algorithmically then we can compute algorithmically a non-negative integer $N$ such that $w_s(S) \leq N$.*

*Proof* The idea that was used to prove the mentioned Theorem 2 from [3] can be used here as well. For this, the induced recurrent system $R$ is considered (such $R$

is a specific recurrent system for the coefficients of Laurent series solutions of $S$). This system has polynomial coefficients of degree less than or equal to $r = \mathrm{ord} S$. The original system $S$ is of full rank if and only if $R$ is of full rank as a recurrent system. A recurrent system of this kind can be transformed by a special version of EG-eliminations ([3, Sect. 3]) into a recurrent system $\tilde{R}$ whose leading matrix is non-singular. It is important that only a finite number of the coefficients of $R$ are involved in the obtained leading matrix of $\tilde{R}$ (due to some characteristic properties of the used version of EG-eliminations). Each of polynomial coefficients of $R$ is determined from a finite number (bounded by a non-negative integer $N$) of the coefficients of the power series involved in $S$. This proves the existence of the width and of the s-width as well. The mentioned number $N$ can be computed algorithmically when all power series are represented algorithmically.

In conclusion of the proof, note that we can compute the width of $S$ since we can test [1, 5, 7] whether a finite order differential system with polynomial coefficients is of full rank or not. From this point we can consider step-by-step $S^{\langle N-1\rangle}, S^{\langle N-2\rangle}, \ldots, S^{\langle 1\rangle}, S^{\langle 0\rangle}$ until the first one of them is not of full rank. If all the truncated systems are of full rank then $w = 0$. However, it is not exactly clear how to find $w_s(S)$, using the upper bound $N$. Is this problem algorithmically solvable? The question is still open.

*Remark 5* If $A_r$, the leading matrix of $S$, is non-singular then $w_s(S) \leq w_s(A_r)$, since a system with non-singular leading matrix is necessarily of full rank.

## 7.2 When Only a Truncated System is Known

In this section we are interested in the following question (this is the main issue of the whole Sect. 7): suppose that for a system $S$ of the form (14) only a finite number of terms of the entries of $A_0(x), A_1(x), \ldots, A_r(x)$ is known, i.e., we know not the system $S$ itself but the system $S^{\langle l\rangle}$ for some non-negative integer $l$. Suppose that we also know that

(a) $\mathrm{ord} S^{\langle l\rangle} = \mathrm{ord} S$, and
(b) $A_r(x)$ is invertible.

Is it possible to check the existence of nonzero Laurent series of $S$ from the given approximate system $S^{\langle l\rangle}$ and if yes how many terms of these solutions of $S$ can be computed from the solutions of $S^{\langle l\rangle}$? We will show that under the condition that the leading (polynomial) matrix of $S^{\langle l\rangle}$ is strongly non-singular we can apply our approach from [4] to get a non-trivial answer to this question.

We first recall the following result that we proved in [4]:

**Proposition 10** *([4, Proposition 6]) Let $S$ be a system of the form (14) having a non-singular $A_r(x)$ and*

$$\gamma = \min_i \text{ val } \left( A_r^{-1}(x) A_i(x) \right), \quad q = \max\{-\gamma, 0\}.$$

*There exists an algorithm, that uses only the terms of valuation less than*

$$rmq + \gamma + \text{val} \det A_r(x) + 1 \tag{16}$$

*of the entries of the matrices $A_0(x), A_1(x), \ldots, A_r(x)$, and computes a nonzero polynomial $I(\lambda)$ (the so-called* indicial *polynomial [9, Chapter 4, §8], [6, Definition 2.1], [4, Sect. 3.2]) such that:*

- *if $I(\lambda)$ has no integer root then (14) has no solution in $K((x))^m \setminus \{0\}$,*
- *otherwise, there exist Laurent series solutions of S. Let $e_*, e^*$ be the minimal and maximal integer roots of $I(\lambda)$; then the sequence*

$$a_k = rmq + \gamma + \text{val} \det A_r(x) + \max\{e^* - e_* + 1,\ k + (rm-1)q\}, \tag{17}$$

*$k = 1, 2, \ldots,$ is such that the system S possesses a solution $y(x) \in K((x))^m$ if and only if, the system $S^{\langle a_k \rangle}$ possesses a solution $\tilde{y}(x) \in K((x))^m$ such that $\tilde{y}(x) - y(x) = O(x^{e+k})$.*

Let us now assume that we are given a truncated system $S^{\langle l \rangle}$ and denote by $\tilde{A}_i$ its coefficients so that $\tilde{A}_i = A_i^{\langle l \rangle}$ for $i = 0, \ldots, r$. Suppose that its leading coefficient $\tilde{A}_r$ is strongly non-singular and let $d = \deg \tilde{A}_r$, $p = -\text{val}\, \tilde{A}_r^{-1}$ and $h = d - p$. Since $h \geq 0$, we have that $p \leq d \leq l$. Moreover, using (7) and (8) we have that

$$\text{val}\,(\det A_r) = \text{val}\,(\det \tilde{A}_r), \quad \text{val}\,(\det A_r^{-1}) = \text{val}\,(\det \tilde{A}_r^{-1}),$$

and

$$A_r^{-1} = \tilde{A}_r^{-1} + O(x^{-p+h+1}).$$

Hence, for $i = 0, \ldots, r-1$, one has

$$A_r^{-1} A_i = \tilde{A}_r^{-1} \tilde{A}_i + O(x^{-p+h+1}).$$

Let

$$\tilde{\gamma} = \min_{0 \leq i \leq r-1} (\text{val}\,(\tilde{A}_r^{-1} \tilde{A}_i)), \quad \gamma = \min_{0 \leq i \leq r-1} (\text{val}\,(A_r^{-1} A_i)).$$

It follows that if $h - p \geq \tilde{\gamma}$ then $\gamma = \tilde{\gamma}$. We obtain using (16) that, under the conditions

$$h - p \geq \tilde{\gamma}, \quad l \geq mr \max(-\tilde{\gamma}, 0) + \tilde{\gamma} + \text{val}\,(\det \tilde{A}_r),$$

the indicial polynomial $I(\lambda)$ of $S$ coincides with the indicial polynomial of $S^{\langle l \rangle}$. Moreover, the sequence (17) is the same for the two systems $S$ and $S^{\langle l \rangle}$. We thus have proven the following

**Proposition 11** *Let $\tilde{S}$ be a system of the form*

$$\tilde{A}_r(x)\vartheta^r y + \tilde{A}_{r-1}(x)\vartheta^{r-1} y + \cdots + \tilde{A}_0(x)y = 0$$

*with polynomial matrices $\tilde{A}_0(x), \tilde{A}_1(x), \ldots, \tilde{A}_r(x)$. Let its leading matrix $\tilde{A}_r(x)$ be strongly non-singular. Let*

$$d = \deg \tilde{A}_r, \quad p = -\mathrm{val}\, \tilde{A}_r^{-1}, \quad h = d - p, \quad \gamma = \min_{0 \le i \le r-1} (\mathrm{val}(\tilde{A}_r^{-1}\tilde{A}_i)), \quad q = \max(-\gamma, 0)$$

*and*

$$h - p - \gamma \ge 0. \tag{18}$$

*Let $l$ be an integer such that*

$$l \ge mrq + \gamma + \mathrm{val}\,(\det \tilde{A}_r). \tag{19}$$

*Denote by $I(\lambda)$ the indicial polynomial of $\tilde{S}$. Let the set if integer roots of $I(\lambda)$ be non-empty, and $e_*, e^*$ be the minimal and maximal integer roots of $I(\lambda)$. Let $S$ be of the form (14) and $S^{\langle l \rangle} = \tilde{S}$. Let $k$ satisfies the equality*

$$\max\{e^* - e_* + 1,\ k + (rm-1)q\} = l - rmq - \gamma - \mathrm{val}\ \det A_r(x). \tag{20}$$

*Then for any $e \in \mathbb{Z}$ and column vectors $c_e, c_{e+1}, \ldots, c_{e+k-1} \in K^m$, the system $S$ possesses a solution*

$$y(x) = c_e x^e + c_{e+1}x^{e+1} + \cdots + c_{e+k-1}x^{e+k-1} + O(x^{e+k}),$$

*if and only if, the system $\tilde{S}$ possesses a solution $\tilde{y}(x) \in K((x))^m$ such that $\tilde{y}(x) - y(x) = O(x^{e+k})$.*

*Example 8* Let

$$\tilde{A}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1-x \end{pmatrix}, \quad \tilde{A}_0 = \begin{pmatrix} 0 & -1 \\ -x + 2x^2 + 2x^3 + 2x^4 & -2 + 4x \end{pmatrix}.$$

For the first-order differential system $\tilde{S}$

$$\tilde{A}_1(x)\vartheta y + \tilde{A}_0(x)y = 0$$

we have

$$d = 1, \quad p = 0, \quad h = 1, \quad \gamma = 0, \quad I(\lambda) = \lambda(\lambda - 2), \quad e^* - e_* + 1 = 3.$$

The conditions of Proposition 11 are satisfied.

The general solution of $\tilde{S}$ is

$$\tilde{y}_1 = C_1 - C_1x + C_2x^2 - C_2x^3 + 0x^4 + \frac{2C_1}{15}x^5 + \frac{C_1}{30}x^6 + \left(\frac{C_1}{210} + \frac{2C_2}{35}\right)x^7 + \cdots,$$

$$\tilde{y}_2 = \quad - C_1x + 2C_2x^2 - 3C_2x^3 + 0x^4 + \frac{2C_1}{3}x^5 + \frac{C_1}{5}x^6 + \left(\frac{C_1}{30} + \frac{2C_2}{5}\right)x^7 + \cdots,$$

where $C_1, C_2$ are arbitrary constants. We can put $l = 4$ in (20), because $\deg \tilde{A}_0 = 4$, $\deg \tilde{A}_1 < 4$, and (18) holds. Then (20) has the form $\max\{3, k\} = 4$, thus $k = 4$. This means that all Laurent series solutions of any system $S$ of the form

$$A_1(x)\vartheta y + A_0(x)y = 0 \tag{21}$$

with non-singular matrix $A_1$ and such that $S^{\langle 4\rangle} = \tilde{S}$ are power series solutions having the form

$$y_1 = C_1 - C_1x + C_2x^2 - C_2x^3 + O(x^4),$$

$$y_2 = -C_1x + 2C_2x^2 - 3C_2x^3 + O(x^4),$$

where $C_1, C_2$ are arbitrary constants. Consider, e.g., the first-order differential system $S$ of the form (21) with

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 - x \end{pmatrix},$$

$$A_0 = \begin{pmatrix} 0 & -1 \\ -x + 2x^2 + 2x^3 + 2x^4 + 2x^5 + 2x^6 + x^7 + x^8 + \cdots & -2 + 4x \end{pmatrix}.$$

Its general solution is

$$y_1 = C_1 - C_1x + C_2x^2 - C_2x^3 + 0x^4 + 0x^5 + 0x^6 + \frac{C_1}{35}x^7 + \cdots,$$

$$y_2 = \quad - C_1x + 2C_2x^2 - 3C_2x^3 + 0x^4 + 0x^5 + 0x^6 + \frac{C_1}{5}x^7 + \cdots,$$

which corresponds to the forecast and expectations. □

The following example shows that if the condition '*strong non-singularity of the leading matrix of the truncated system*' of Proposition 11 is not satisfied then it may happen that the correspondence between the Laurent solutions of $\tilde{S}$ and $S$ as described in that proposition do not occur.

*Example 9* Consider the first-order differential system $S$:

$$A_1(x)\vartheta y + A_0(x)y = 0,$$

where

$$A_1 = \begin{pmatrix} x & x^3 \\ 1 & x \end{pmatrix} \quad A_0 = \begin{pmatrix} 0 & -x^4 + 3\,x^3 \\ 0 & -x^3 + 3\,x \end{pmatrix}.$$

Its general solution is

$$y_1\,(x) = C_1 + C_2\,\ln\,(x)\,, \quad y_2\,(x) = \frac{C_2}{x^3}$$

where $C_1, C_2$ are arbitrary constants.

The truncated systems $S^{\langle l \rangle}$ for $l = 1, 2$ coincide and have the leading matrix

$$\begin{pmatrix} x & 0 \\ 1 & x \end{pmatrix}$$

which is non-singular but not strongly non-singular. The general solution of $S^{\langle 2 \rangle}$ is

$$y_1\,(x) = C_1, \quad y_2\,(x) = \frac{C_2}{x^3}.$$

The truncated system $S^{\langle 3 \rangle}$ has the leading matrix $A_1$ which is strongly non-singular. Note that $(d, p, h, \gamma, q) = (3, 2, 1, 0, 0)$ and the condition (11), i.e., $h - p - \gamma \geq 0$ of Proposition 11 is not satisfied. The general solution of $S^{\langle 3 \rangle}$ is

$$y_1\,(x) = C_1 + C_2 \int \frac{\mathrm{e}^{-x}}{(x-1)^2}\,\mathrm{d}x, \quad y_2\,(x) = C_2 \frac{\mathrm{e}^{-x}}{x^3\,(x-1)}.$$

The expansions of $y_1(x)$ and $y_2(x)$ at $x = 0$ are respectively given by

$$y_1\,(x) = C_1 + C_2(x + \frac{1}{2}x^2 + O\left(x^3\right)),$$

$$y_2\,(x) = C_2(-x^{-3} - \frac{1}{2}x^{-1} - \frac{1}{3} - \frac{3}{8}x - \frac{11}{30}x^2 + O\left(x^3\right)).$$

Let now consider, instead of $S$, the first-order system $R$ :

$$B_1(x)\vartheta y + B_0(x)y = 0$$

where $B_1 = A_1$ and

$$B_0 = \begin{pmatrix} 0 & -x^5 + 3\,x^3 \\ 0 & -x^3 + 3\,x \end{pmatrix},$$

so that $R^{\langle 3 \rangle} = S^{\langle 3 \rangle}$. We find that the general solution of $R$ is

$$y_1\,(x) = C_{\mathit{1}}, \quad y_2\,(x) = \frac{C_2\,\mathrm{e}^{\frac{1}{2}\,x^2}}{x^3}.$$

It has no logarithmic term, and the statement of Proposition 11 holds. □

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# On the Computation of Simple Forms and Regular Solutions of Linear Difference Systems

**Moulay A. Barkatou, Thomas Cluzeau and Carole El Bacha**

**Abstract** In our previous article Barkatou et al. (Proceedings of CASC 2015: 72–86, 2015 [10]), we have described a method for computing regular solutions of *simple* linear difference systems. In the present paper, we develop a new algorithm that transforms any first-order linear difference system with factorial series coefficients into a simple system. Such an algorithm can thus be seen as a first step toward the computation of regular solutions. Moreover, computing a simple form can also be used to recognize the nature of the singularity at infinity: if the singularity is regular, we are then reduced to a system of the first kind. For this, we also devote a particular study to systems of the first kind and provide a direct algorithm for computing a formal fundamental matrix of regular solutions of such systems. This yields an alternative to the algorithm in Barkatou et al. (Proceedings of CASC 2015: 72–86, 2015 [10]) for computing regular solutions in the case of a regular singularity. Finally, we note that the algorithms developed in the present paper have been implemented in MAPLE, thanks to our new package for handling factorial series. We give examples illustrating our different methods.

**Keywords** Linear difference systems · Factorial series
Regular/irregular singularity · Singularity of the first/second kind · Simple forms

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# 1 Introduction

Throughout this article, the variable $z$ stands for a complex variable and $\tau$ is the backward shift operator acting on a function $f$ as follows $\tau(f(z)) = f(z-1)$. The difference operator is denoted by $\Delta$ and its action on a function $f$ is defined by

$$\Delta(f(z)) = (z-1)(f(z) - f(z-1)).$$

For any two functions $f$ and $g$ of $z$, the operator $\Delta$ satisfies the rule

$$\Delta(fg) = \tau(f)\Delta(g) + \Delta(f)g.$$

We consider first-order linear difference systems having a singularity at $z = \infty$ of the form

$$\Delta(\mathbf{y}(z)) = B(z)\,\mathbf{y}(z), \tag{1}$$

where $\mathbf{y}(z)$ is the vector of unknowns and $B(z)$ is an $n \times n$ matrix with generalized factorial series entries, i.e.,

$$B(z) = z^q \left( B_0 + \sum_{k=1}^{\infty} \frac{B_k}{z(z+1)\dots(z+k-1)} \right),$$

with $q \in \mathbb{Z}$ and $B_k \in \mathbb{C}^{n\times n}$ such that $B_0 \neq 0$. When $q \geq 0$, the integer $q$ is called the *Poincaré rank* of the system. As in the analytic theory of linear differential systems, there exist two ways of classifying the singularity $z = \infty$ of System (1): the first kind/second kind classification and the regular/irregular classification. More precisely, if $q \leq 0$, then the infinity is called *singularity of the first kind* and, otherwise (i.e., $q \geq 1$), it is called *singularity of the second kind*. In the sequel, we shall shortly say that System (1) is of the first/second kind. However, the regular/irregular classification is not immediately apparent. Indeed, the singularity at infinity, resp. System (1), is said to be *regular*, resp. *regular singular*, if System (1) has a formal fundamental matrix of solutions of the form

$$S(z)z^{\Lambda},$$

where $\Lambda \in \mathbb{C}^{n\times n}$ and $S(z)$ is an $n \times n$ matrix with factorial series entries. Otherwise, the singularity at infinity, resp. System (1), is said to be *irregular*, resp. *irregular singular*. Any regular singular system of the form (1) admits a basis of $n$ linearly independent *formal regular solutions* which constitute the columns of a fundamental matrix of solutions. Any regular solution can be written in the form

$$\mathbf{y}(z) = z^{-\rho} \left( \mathbf{y}_0(z) + \mathbf{y}_1(z)\log(z) + \cdots + \mathbf{y}_s(z)\log^s(z) \right), \tag{2}$$

where $s < n$, $\rho \in \mathbb{C}$, and for $i = 1, \dots, s$, $\mathbf{y}_i(z)$ is a vector of factorial series.

Linear systems of difference equations having a singularity of the first kind at $z = \infty$ have been extensively studied and are known to be regular singular. There exists a method [3, 16] which broadly follows the Frobenius method in the differential case for computing a fundamental matrix of solutions of such systems. However, it may also happen that systems of the second kind are regular singular. Indeed, it has been shown that a necessary and sufficient condition for a linear difference system of the form (1) to be regular singular is to be *gauge equivalent* to a system of the first kind [3, 4]. In other words, System (1) is regular singular if and only if there exists a *gauge transformation* $\mathbf{y}(z) = T(z)\mathbf{w}(z)$ with invertible matrix $T(z)$ that transforms System (1) into the new system

$$\Delta(\mathbf{w}(z)) = \widetilde{B}(z)\mathbf{w}(z),$$

where

$$\widetilde{B}(z) = (\tau(T(z)))^{-1}\,(B(z)T(z) - \Delta(T(z)))$$

is a matrix with factorial series entries. Note that if $z = \infty$ is an irregular singularity of System (1), a basis of solutions is composed of regular solutions as well as irregular solutions involving exponential parts.

In order to compute the formal regular solutions of System (1) having a regular or irregular singularity at infinity, it happens to be more convenient to write System (1) in the form

$$D(z)\,\Delta(\mathbf{y}(z)) + A(z)\,\mathbf{y}(z) = 0, \tag{3}$$

where $D(z)$ and $A(z)$ are two $n \times n$ matrices of factorial series such that $D(z)$ is invertible. One may classify the singularity $z = \infty$ of systems of the form (3) by bringing them back to the form (1). Indeed, System (3) is said to be regular singular (resp., irregular singular, of the first kind, of the second kind) if, written back in the form (1), it is so. In particular, if $D(z) = I_n$ (the $n \times n$ identity matrix) or if $D(z)$ is invertible at $z = \infty$, then System (3) is of the first kind.

We shall say that a system of the form (3) is *simple* (or, in *simple form*) if $\det(D(\infty)\lambda - A(\infty)) \not\equiv 0$. The main contribution of the present paper is to provide an algorithm that transforms any non-simple system of the form (3) into an equivalent simple system. This algorithm can be seen as an alternative to Moser's algorithm for determining the minimal Poincaré rank and thus the nature of the singularity at infinity. The computation of simple forms is crucial for the following two tasks. First, it allows to detect the nature of the singularity. Indeed, as we shall see later, a simple system of the form (3) is regular singular if and only if the matrix $D(\infty)$ is invertible. Second, it is a first step toward the computation of regular solutions of System (3) (see [10]).

For the sake of completeness, we shall recall here the approach that we have proposed in [10] for computing a basis of the formal regular solutions of any simple system. As the factorial series have been proved to be very well suited for studying linear difference equations, we have defined in [10] a sequence of functions $(\phi_n)_{n\in\mathbb{N}}$ (see Definition 1 below) which is more adapted than $(\log^n)_{n\in\mathbb{N}}$ while working with the difference operator $\Delta$. Then the method of [10] searches for regular solutions written as

$$\mathbf{y}(z) = \sum_{m\geq 0} z^{-[\rho+m]}\mathbf{y}_m(z), \qquad z^{-[\rho]} = \frac{\Gamma(z)}{\Gamma(z+\rho)},$$

where $\Gamma$ stands for the usual Gamma function and $\mathbf{y}_m(z)$ is a finite linear combination of the functions $\phi_i$ with constant vector coefficients. It reduces the problem of computing such solutions to solving systems with constant coefficients.

This method can be applied to any simple system of the form (3) whatever is the nature of the singularity $z = \infty$. In particular, it can be applied to Systems (3) with $D(z) = I_n$ which are then of the first kind. Therefore, we get an alternative method to the Frobenius-like method described in [3, 16] for computing regular solutions. In the present paper, we shall also dedicate a particular study to systems of the form (3) with $D(z) = I_n$. As for the differential case, we shall show that they are gauge equivalent to systems with constant coefficients of the form

$$\Delta(\mathbf{w}(z)) + \Lambda\mathbf{w}(z) = 0,$$

whose fundamental matrix of solutions is given by

$$z^{-[\Lambda]} = \sum_{k\geq 0} \frac{(-1)^k}{k!}\phi_k(z)\Lambda^k.$$

The rest of the paper is organized as follows. In Sect. 2, we recall some background information on factorial series and define functions $\phi_n$ which have been introduced in our previous paper [10]. In Sect. 3, we treat linear difference systems of the first kind and give a method for computing a formal fundamental matrix of solutions. In Sect. 4, we recall the approach of [10] for computing regular solutions of simple linear difference systems. In Sect. 5, we develop the main contribution of this paper: an algorithm that transforms any linear difference system into an equivalent simple system. Finally, it is important to mention that we have implemented our algorithm in MAPLE so we give some remarks on the implementation and illustrate it with an example.

## 2 Preliminaries

In this section, we recall some definitions, notations and properties that are well-known in the literature or that have been introduced or proved in our paper [10], except Proposition 1 which is new. These notions are useful for the rest of the paper. This section is basically divided into two parts: the first part concerns the factorial series which have been shown to be well appropriate for studying linear difference systems, and the second part treats the functions $\phi_n$ which will replace the powers of the logarithm function in the computation of regular solutions.

For a complex variable $z$ and $\rho \in \mathbb{C}$, the notation $z^{-[\rho]}$ stands for

$$z^{-[\rho]} = \frac{\Gamma(z)}{\Gamma(z+\rho)},$$

where $\Gamma$ denotes the usual *Gamma function*. Note that $z^{-[\rho]}$ is not defined when $z \in \mathbb{Z}_{\leq 0}$, and if $\rho = n \in \mathbb{N}$, then we have

$$z^{-[0]} = 1, \quad \text{and} \quad \forall n \geq 1,\ z^{-[n]} = \frac{1}{z(z+1)\cdots(z+n-1)}.$$

A straightforward calculation shows that, for all $n \in \mathbb{N}$, $\Delta(z^{-[n]}) = -n\, z^{-[n]}$ so that the behavior of $z^{-[n]}$ with respect to $\Delta$ is the same as the one of $z^{-n}$ with respect to the Euler operator $\vartheta = z\,\frac{d}{dz}$. This is one of the reasons for replacing power series by factorial series when we deal with linear difference equations.

## 2.1 Factorial Series

A *factorial series* is a series of the form $\sum_{n\geq 0} a_n\, z^{-[n]}$ where $a_n \in \mathbb{C}$.

The domain of convergence of a factorial series is a half-plane $\mathrm{Re}(z) > \mu$, where $\mu$ is the abscissa of convergence whose formula is given in [20, Sect. 10.09, Theorem V]. Note that, in this paper, we only consider formal factorial series and we do not address the problem of convergence of the factorial series involved in the solutions.

The set $\mathscr{R}$ of factorial series can be equipped with a commutative unitary ring structure. The addition of two factorial series of the form $\sum_{n\geq 0} a_n\, z^{-[n]}$ and $\sum_{n\geq 0} b_n\, z^{-[n]}$ is defined by

$$\sum_{n\geq 0} a_n\, z^{-[n]} + \sum_{n\geq 0} b_n\, z^{-[n]} = \sum_{n\geq 0} (a_n + b_n)\, z^{-[n]}.$$

As for the multiplication, we need to introduce the binomial-like coefficient $\mathrm{C}^k_{x,y}$ which is defined for all $k \in \mathbb{N}$, $x, y \in \mathbb{C}$ by

$$\mathrm{C}^k_{x,y} = \frac{x^{[k]}\, y^{[k]}}{k!},$$

where $z^{[k]}$ denotes the usual rising factorial $z^{[k]} = \prod_{j=0}^{k-1}(z+j)$. The multiplication of $\sum_{n\geq 0} a_n\, z^{-[n]}$ and $\sum_{n\geq 0} b_n\, z^{-[n]}$ is then given by

$$\left(\sum_{n\geq 0} a_n\, z^{-[n]}\right)\left(\sum_{n\geq 0} b_n\, z^{-[n]}\right) = \sum_{s\geq 0}\left(\sum_{\substack{n,p,k\geq 0\\ n+p+k=s}} \mathrm{C}^k_{n,p}\, a_n\, b_p\right) z^{-[s]}.$$

The set $\mathscr{R}$ of factorial series endowed with the above addition and multiplication is a ring which is isomorphic to the ring $\mathbb{C}[[z^{-1}]]$ of formal power series in $z^{-1}$(see [12]). The isomorphism $\varphi : \mathbb{C}[[z^{-1}]] \to \mathscr{R}$ and its inverse $\varphi^{-1}$ are given by:

$$\varphi(1) = 1, \quad \varphi\left(\frac{1}{z}\right) = z^{-[1]}$$

$$\forall\, n \geq 1, \quad \varphi\left(\frac{1}{z^{n+1}}\right) = (-1)^n\,(n-1)!\sum_{k\geq 0}(-1)^k\, s(n+k,n)\, z^{-[n+k+1]},$$

$$\forall\, n \geq 1, \quad \varphi^{-1}(z^{-[n]}) = \sum_{k\geq 0}(-1)^k\, S(n+k-1,n-1)\,\frac{1}{z^{n+k}},$$

where the constants $s(n,k)$ (resp. $S(n,k)$) are the Stirling numbers of the first (resp. second) kind (see [2, Sect. 24]).

The fraction field $\mathscr{F}$ of the ring $\mathscr{R}$ of factorial series is isomorphic to the field $\mathbb{C}((z^{-1}))$ of meromorphic formal power series in $z^{-1}$ and any nonzero element $f$ of $\mathscr{F}$ can be written of the form $f = z^k g$ where $k \in \mathbb{Z}$ and $g \in \mathscr{R}$ such that $g(\infty) \neq 0$. We define the *valuation* $\mathrm{val}(f)$ of $f$ by $\mathrm{val}(f) = -k$.

We also recall the translation formula transforming any factorial series in $z$ into another factorial series in $z+\beta$ with $\beta \in \mathbb{C}$. We have

$$\forall\, \beta \in \mathbb{C}, \quad \sum_{n\geq 0} a_n\, z^{-[n]} = a_0 + \sum_{p\geq 1}\left(\sum_{k=1}^{p} \mathrm{C}^{p-k}_{\beta,k}\, a_k\right)(z+\beta)^{-[p]}, \tag{4}$$

and

$$\forall\, m \in \mathbb{N}^*, \quad \forall\, \beta \in \mathbb{C}, \quad \sum_{n\geq 0} a_n\, z^{-[n+m]} = \sum_{p\geq 0}\left(\sum_{k=0}^{p} \mathrm{C}^{p-k}_{\beta,k+m}\, a_k\right)(z+\beta)^{-[p+m]}.$$

Note that we have

$$\Delta\left(\sum_{n\geq 0} a_n z^{-[n]}\right) = \sum_{n\geq 1} -n a_n z^{-[n]},$$

which implies that $\mathscr{R}$ is stable by $\Delta$. The relation $\Delta(fg) = \tau(f)\Delta(g) + \Delta(f)g$ then shows that $\Delta$ is a $\tau$-derivation of $\mathscr{R}$.

We have implemented the arithmetic operations on $\mathscr{R}$ as well as the useful translation formula and the action of $\Delta$ on factorial series in a MAPLE package called FACTORIALSERIESTOOLS.[1] A truncated factorial series $\sum_{n=0}^{t} a_n\, z^{-[n]}$ of *precision* $t$ is represented by the list of its $t+1$ coefficients $[a_0, a_1, \ldots, a_t]$.

Note finally that, in the sequel, if $M(z)$ is a matrix with factorial series entries, we shall often denote by $M_0$ the matrix $M(z)$ evaluated at $z = \infty$, i.e., $M_0 = M(\infty)$.

## 2.2 Functions $\phi_n$ and Properties

In order to adapt the method of [9] to the setting of difference equations we shall replace the triple $(\vartheta = z\,\frac{d}{dz}, z^{-n}, \log^n)$ used in the differential case and satisfying

$$\forall\, n \in \mathbb{N}, \quad \vartheta(z^{-n}) = -n\, z^{-n}, \quad \vartheta(\log^n(z)) = n\, \log^{n-1}(z),$$

by a triple $(\Delta, z^{-[n]}, \phi_n)$ satisfying

$$\forall\, n \in \mathbb{N}, \quad \Delta(z^{-[n]}) = -n\, z^{-[n]}, \quad \Delta(\phi_n(z)) = n\, \phi_{n-1}(z). \tag{5}$$

The aim of this subsection is to introduce the sequence of functions $(\phi_n)_{n\in\mathbb{N}}$ which will play a role analogous to the one played by $(\log^n)_{n\in\mathbb{N}}$ in the differential case, i.e., which will satisfy the last equality of (5).

### 2.2.1 Definition

For $z \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$, we consider the function $\rho \in \mathbb{C} \mapsto z^{-[\rho]} = \frac{\Gamma(z)}{\Gamma(z+\rho)}$. As the reciprocal $1/\Gamma$ of the Gamma function is an entire function (see [2, Sect. 6] or [20, Chap. 9]), the function $\rho \mapsto z^{-[\rho]}$ is holomorphic at $\rho = 0$ and we define the functions $\phi_n$ from the coefficients of the Taylor series expansion of $\rho \mapsto z^{-[\rho]}$ at $\rho = 0$:

**Definition 1** For $n \in \mathbb{N}$ and $z \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$, we define

$$\phi_n(z) = (-1)^n\, \frac{\partial^n}{\partial \rho^n} \left(z^{-[\rho]}\right)\Big|_{\rho=0}, \tag{6}$$

and the sequence of functions $(\phi_n)_{n\in\mathbb{N}}$ where, for $n \in \mathbb{N}$, $\phi_n$ is the function of the complex variable $z$ defined by (6) for $z \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$.

From (6), we have $\phi_0(z) = 1$ and $\phi_1(z) = \Psi(z)$, where $\Psi$ is the *Digamma* or *Psi function* defined as the logarithmic derivative of the Gamma function $\Gamma$, i.e., $\Psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$ (see [2, Sect. 6] or [20, Chap. 9]).

[1] A beta version is available upon request from the authors.

The Taylor series expansion of the function $\rho \mapsto z^{-[\rho]}$ at $\rho = 0$ can thus be written as

$$z^{-[\rho]} = \frac{\Gamma(z)}{\Gamma(z+\rho)} = \sum_{n\geq 0} \frac{(-1)^n}{n!}\, \phi_n(z)\, \rho^n.$$

*Remark 1* The function $z \mapsto z^{-[\rho]}$ is asymptotically equivalent, as $z \to \infty$, to the function $z \mapsto z^{-\rho}$ which has the Taylor series expansion at $\rho = 0$

$$z^{-\rho} = e^{-\rho \log(z)} = \sum_{n\geq 0} \frac{(-1)^n}{n!}\, \log^n(z)\, \rho^n.$$

This illustrates the close relations between the functions $\phi_n$ and $\log^n$ which will be more developed in Sect. 2.2.2 below.

### 2.2.2 Properties

As we have seen above, the function $\phi_1$ coincides with the Digamma function $\Psi$ which satisfies the linear difference equation

$$\Delta(\Psi(z)) = 1,$$

and has the asymptotic expansion

$$\Psi(z) \sim \log(z) - \frac{1}{2\,z} - \sum_{n\geq 1} \frac{B_{2n}}{2\,n\,z^{2n}}, \quad z \to \infty,\ |\arg(z)| < \pi,$$

where $B_k$ represents the $k$th Bernoulli number ([2, Sect. 23]).

In our previous paper [10], we have extended the above two properties on $\Psi$ to handle any function $\phi_n$ for $n \geq 1$. Indeed, we have shown that

$$\Delta(\phi_n(z)) = n\,\phi_{n-1}(z), \tag{7}$$

$$\phi_n(z) = \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} \Psi^{(k)}(z)\,\phi_{n-k-1}(z), \tag{8}$$

$$\phi_n(z) = \Psi(z)\,\phi_{n-1}(z) - \frac{d}{dz}\left(\phi_{n-1}(z)\right), \tag{9}$$

$$\phi_n(z) = (-1)^n\,\Gamma(z)\,\frac{d^n}{dz^n}\left(\frac{1}{\Gamma(z)}\right), \tag{10}$$

$$\phi_n(z) \sim \log^n(z) + \sum_{i=0}^{n-1} P_{n,i}(z^{-1})\,\log^i(z), \qquad z \to \infty,\ |\arg(z)| < \pi,$$

where $\Psi^{(k)}$ denotes the *Polygamma function of order k* and for $i = 0, \dots, n-1$, $P_{n,i}(z^{-1}) \in \mathbb{C}[[z^{-1}]]$.

Note that one can also define the functions $\phi_n$ using the formula (10) (as it has been done in [13]) from which one can easily recover (7), (8) and (9). However, getting the expression (6) doesn't seem to be straightforward.

For later use, we present below a new formula giving the $n$th derivative of $z^{-[\rho]}$ with respect to $\rho$. Here also, we point out the resemblance with the formula of the $n$th derivative of the power function $z^{-\rho}$: $\frac{\partial^n}{\partial \rho^n}(z^{-\rho}) = (-1)^n z^{-\rho} \log^n(z)$.

**Proposition 1** *For any integer $n \in \mathbb{N}$, we have*

$$\frac{\partial^n}{\partial \rho^n}\left(z^{-[\rho]}\right) = (-1)^n z^{-[\rho]} \phi_n(z+\rho).$$

*Proof* We proceed by induction on $n$. The formula holds true for $n = 0$ since we have $\phi_0(z) = 1$. Assuming that the statement holds for all integers $m \le n$, we have

$$\begin{aligned}
\frac{\partial^{n+1}}{\partial \rho^{n+1}}(z^{-[\rho]}) &= \frac{\partial}{\partial \rho}\left(\frac{\partial^n}{\partial \rho^n}(z^{-[\rho]})\right) \\
&= \frac{\partial}{\partial \rho}\left((-1)^n z^{-[\rho]} \phi_n(z+\rho)\right) \\
&= (-1)^n \left(\frac{\partial}{\partial \rho}(z^{-[\rho]})\phi_n(z+\rho) + z^{-[\rho]} \frac{\partial}{\partial \rho}(\phi_n(z+\rho))\right) \\
&= (-1)^n \left(-z^{-[\rho]} \Psi(z+\rho)\phi_n(z+\rho) + z^{-[\rho]} \frac{\partial}{\partial \rho}(\phi_n(z+\rho))\right) \\
&= (-1)^{n+1} z^{-[\rho]} \left(\Psi(z+\rho)\phi_n(z+\rho) - \frac{\partial}{\partial \rho}(\phi_n(z+\rho))\right) \\
&= (-1)^{n+1} z^{-[\rho]} \phi_{n+1}(z+\rho),
\end{aligned}$$

using Property (9). This ends the proof.

## 3 Linear Difference Systems of the First Kind

We recall that a necessary and sufficient condition for a linear difference system to have a regular singularity at $z = \infty$ is to be gauge equivalent to a system of the first kind of the form

$$\Delta(\mathbf{y}(z)) + A(z)\mathbf{y}(z) = 0, \tag{11}$$

where $A(z) = \sum_{k \ge 0} A_k z^{-[k]}$ with $A_k \in \mathbb{C}^{n \times n}$. Therefore it is worth studying directly the latter linear difference systems. In the literature, the structure of a formal fundamental matrix of solutions of systems of the first kind has been investigated in details (see [16–18]). In particular, there exists a method, corresponding to the Frobenius method for linear differential systems, for computing a fundamental matrix of solutions of System (11) (see [3, 16]). This method produces a formal fundamental matrix of solutions of the form

$$S(z)z^{\Lambda}, \tag{12}$$

where $S(z)$ is an $n \times n$ matrix with factorial series entries and $\Lambda \in \mathbb{C}^{n\times n}$. Moreover, the factorial series matrix $S(z)$ converges whenever $A(z)$ does so.

In this section, we handle linear difference systems (11) of the first kind but in a different way than it has been done in [16]. We first consider the simplest case where the matrix $A(z)$ is a constant matrix, and give an expression of a formal fundamental matrix of solutions in terms of the functions $\phi_n$ defined in Sect. 2.2. Then, we show how the general case, where $A(z)$ is a matrix of factorial series, can be reduced to the first case using a gauge transformation, therefore a formal fundamental matrix of solutions can be determined.

### *3.1 Systems with Constant Coefficients*

In the theory of differential equations, it can be easily checked that a first-order linear differential system with singularity at infinity of the first kind of the form

$$z\frac{d}{dz}\mathbf{y}(z) + A\,\mathbf{y}(z) = 0,$$

where $A \in \mathbb{C}^{n\times n}$, has a formal fundamental matrix of solutions given by $z^{-A}$ which has the power series expansion

$$z^{-A} = \sum_{k\geq 0} \frac{(-1)^k}{k!} \log^k(z)\, A^k.$$

In an analogous manner, we shall define the matrix $z^{-[A]}$ by replacing $\log^k(z)$ by $\phi_k(z)$ in the latter power series expansion.

**Definition 2** Given a constant square matrix $A$, we define the matrix $z^{-[A]}$ as

$$z^{-[A]} = \sum_{k\geq 0} \frac{(-1)^k}{k!} \phi_k(z)\, A^k,$$

where the functions $\phi_k$ are defined in Eq. (6).

**Proposition 2** *A formal fundamental matrix of solutions of the linear difference system*

$$\Delta(\mathbf{y}(z)) + A\mathbf{y}(z) = 0,$$

*where $A \in \mathbb{C}^{n\times n}$, is given by $z^{-[A]}$.*

*Proof* A direct computation of $\Delta(z^{-[A]})$ using Definition 2 gives

$$\begin{aligned}\Delta(z^{-[A]}) &= \sum_{n\geq 0} \frac{(-1)^n}{n!} \Delta(\phi_n(z)) A^n \\ &= \sum_{n\geq 1} \frac{(-1)^n}{n!} n\phi_{n-1}(z) A^n \\ &= \sum_{n\geq 1} \frac{(-1)^n}{(n-1)!} \phi_{n-1}(z) A^n \\ &= -A \sum_{n\geq 1} \frac{(-1)^{n-1}}{(n-1)!} \phi_{n-1}(z) A^{n-1} \\ &= -Az^{-[A]},\end{aligned}$$

which ends the proof.

We shall now provide an efficient way to compute $z^{-[A]}$. From Definition 2, we remark that if the matrix $A$ is block-diagonal, i.e., $A = \mathrm{diag}(A_1, \ldots, A_s)$, then we have $z^{-[A]} = \mathrm{diag}(z^{-[A_1]}, \ldots, z^{-[A_s]})$. We also notice that if $P$ is an invertible constant matrix, then $z^{-[PAP^{-1}]} = Pz^{-[A]}P^{-1}$. This allows us to reduce the computation of $z^{-[A]}$ to the case where the matrix $A$ is a Jordan block.

**Proposition 3** *Let $J_{\rho,n} = \rho I_n + N$ be a Jordan block of size n with $\rho \in \mathbb{C}$, $I_n$ the identity matrix of size n, and N a nilpotent matrix. Then, we have*

$$z^{-[\rho I_n + N]} = z^{-[\rho]}(z+\rho)^{-[N]},$$

*where*

$$(z+\rho)^{-[N]} = \begin{bmatrix} 1 & -\phi_1(z+\rho) & \ldots & \ldots & \frac{(-1)^{n-1}}{(n-1)!}\phi_{n-1}(z+\rho) \\ 0 & 1 & \ldots & \ldots & \frac{(-1)^{n-2}}{(n-2)!}\phi_{n-2}(z+\rho) \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & -\phi_1(z+\rho) \\ 0 & 0 & \ldots & & 1 \end{bmatrix}.$$

*Proof* Using Definition 2, we have

$$\begin{aligned} z^{-[\rho I_n + N]} &= \sum_{k\geq 0} \frac{(-1)^k}{k!} \phi_k(z)(\rho I_n + N)^k \\ &= \sum_{k\geq 0} \frac{(-1)^k}{k!} \phi_k(z) \left( \sum_{j=0}^{k} \binom{k}{j} \rho^{k-j} N^j \right) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=0}^{n-1} N^j \left( \sum_{k \geq j} \frac{(-1)^k}{k!} \binom{k}{j} \phi_k(z) \rho^{k-j} \right) \\
&= \sum_{j=0}^{n-1} \frac{1}{j!} N^j \left( \sum_{k \geq j} \frac{(-1)^k}{k!} k(k-1)\ldots(k-j+1) \phi_k(z) \rho^{k-j} \right) \\
&= \sum_{j=0}^{n-1} \frac{1}{j!} N^j \frac{\partial^j}{\partial \rho^j} \left( \sum_{k \geq 0} \frac{(-1)^k}{k!} \phi_k(z) \rho^k \right) \\
&= \sum_{j=0}^{n-1} \frac{1}{j!} N^j \frac{\partial^j}{\partial \rho^j} \left( z^{-[\rho]} \right).
\end{aligned}
$$

Finally, using Proposition 1, we get

$$
z^{-[\rho I_n + N]} = z^{-[\rho]} \sum_{j=0}^{n-1} \frac{(-1)^j}{j!} \phi_j(z+\rho) N^j = z^{-[\rho]} (z+\rho)^{-[N]},
$$

which ends the proof.

### *3.2 Systems with Factorial Series Coefficients*

We consider now the general case where the matrix $A(z)$ in (11) is a factorial series matrix not necessarily reduced to the constant matrix $A_0$. The aim of this subsection is to show that any system of the form (11) is gauge equivalent to a system with constant coefficients. One can then apply Proposition 2 to obtain a formal fundamental matrix of solutions. To achieve this, we shall expose a method similar to the one used in the differential case where the study depends on whether the distinct eigenvalues of the matrix $A_0$ are congruent or not modulo 1 (see [11, 24]).

**Definition 3** A constant matrix $A \in \mathbb{C}^{n \times n}$ is said to be *non-resonant* if any two eigenvalues of $A$ do not differ by a positive integer, that is, for every $m \in \mathbb{N}^*$, the two matrices $A$ and $A - mI_n$ have no common eigenvalues. Otherwise, $A$ is said to be *resonant*.

**Proposition 4** *Given a linear difference system of the first kind of the form (11) with a non-resonant matrix $A_0$, one can compute a gauge transformation $\mathbf{y}(z) = T(z)\mathbf{w}(z)$, where $T(z) = \sum_{m \geq 0} T_m z^{-[m]}$, with $T_0 = I_n$ and $T_m \in \mathbb{C}^{n \times n}$, transforming System (11) into the equivalent system*

$$
\Delta(\mathbf{w}(z)) + A_0 \mathbf{w}(z) = 0.
$$

*Proof* We shall show how to compute the coefficient matrices $T_m$ of $T(z)$ in such a way that the transformation $\mathbf{y}(z) = T(z)\mathbf{w}(z)$ reduces System (11) to the gauge equivalent system $\Delta(\mathbf{w}(z)) + A_0\mathbf{w}(z) = 0$. It follows that the matrix $T(z)$ must satisfy

$$A(z)T(z) + \Delta(T(z)) = \tau(T(z))A_0. \tag{13}$$

Plugging $T(z) = \sum_{m\geq 0} T_m z^{-[m]}$ and $A(z) = \sum_{k\geq 0} A_k z^{-[k]}$ in (13) and using the product formula for factorial series, we obtain

$$\sum_{m\geq 0}\left(\sum_{\substack{p,q,k\geq 0\\ p+q+k=m}} C_{p,q}^{k}\, A_p T_q\right) z^{-[m]} - \sum_{m\geq 1} mT_m z^{-[m]} = T_0A_0 + \sum_{m\geq 1}\left(\sum_{k=1}^{m} C_{1,k}^{m-k} T_k A_0\right) z^{-[m]}.$$

Comparing the coefficients of $z^{-[m]}$ in each side of the latter equality yields

$$A_0T_0 = T_0A_0,$$

which implies that we can always choose $T_0 = I_n$, and for $m \geq 1$, we get

$$\sum_{\substack{p,q,k\geq 0\\ p+q+k=m}} C_{p,q}^{k}\, A_p T_q - mT_m = \sum_{k=1}^{m} C_{1,k}^{m-k} T_k A_0.$$

Collecting the terms in $T_m$ on the left-hand side, the latter equation can be written as

$$(A_0 - mI)T_m - T_mA_0 = -\sum_{\substack{p,k\geq 0\\ 0\leq q<m\\ p+q+k=m}} C_{p,q}^{k}\, A_p T_q + \sum_{k=1}^{m-1} C_{1,k}^{m-k} T_k A_0, \tag{14}$$

where the right-hand side depends only on $A_k$ and $T_k$ for $k < m$. Due to the assumption on the eigenvalues of $A_0$, the Sylvester equation (14) admits a unique solution $T_m$ (see [15]).

When $A_0$ is a resonant matrix, it is well known (see [3, 16]) that System (11) is gauge equivalent to a system satisfying the hypotheses of Proposition 4. For the sake of completeness, we recall this result below.

**Proposition 5** *Let (11) be a linear difference system of the first kind with a resonant matrix $A_0$. One can construct a gauge transformation $\mathbf{y}(z) = T(z)\mathbf{w}(z)$ with $T(z) \in \mathbb{C}[z^{-1}]^{n\times n}$, such that System (11) is equivalent to a linear difference system of the first kind of the form*

$$\Delta(\mathbf{w}(z)) + \left(\sum_{k\geq 0} B_k z^{-[k]}\right)\mathbf{w}(z) = 0,$$

*with a non-resonant matrix* $B_0$.

*Proof* First, gather the eigenvalues of $A_0$ into sets such that the elements of each set are pairwise congruent modulo 1. Consider one of these sets and denote by $\rho_1, \rho_2, \ldots, \rho_q \in \mathbb{C}$ its distinct elements with multiplicity $m_1, \ldots, m_q$ respectively. Assume that

$$\mathrm{Re}(\rho_1) > \mathrm{Re}(\rho_2) > \cdots > \mathrm{Re}(\rho_q),$$

and set

$$\rho_i - \rho_{i+1} = l_i \in \mathbb{N}^*, \quad \forall\, i = 1, \ldots, q-1.$$

By means of a constant gauge transformation, one can assume without loss of generality that $A_0$ is in the canonical form

$$A_0 = \begin{bmatrix} A_0^{11} & 0 \\ 0 & A_0^{22} \end{bmatrix},$$

where $A_0^{11}$ is an $m_1 \times m_1$ matrix in Jordan form with one eigenvalue $\rho_1$. The gauge transformation

$$\mathbf{y}(z) = U(z)\tilde{\mathbf{y}}(z),$$

where $U(z) = \mathrm{diag}(z^{-1} I_{m_1}, I_{n-m_1})$, transforms System (11) into

$$\Delta(\tilde{\mathbf{y}}(z)) + \widetilde{A}(z)\tilde{\mathbf{y}}(z) = 0,$$

with

$$\begin{aligned}
\widetilde{A}_0 &= \left[\tau(U^{-1}(z))\,(A(z)U(z) + \Delta(U(z)))\right]_{z=\infty},\\
&= \left[\tau(U^{-1}(z))\left(A_0 + \frac{A_1}{z}\right)U(z) + \tau(U^{-1}(z))\Delta(U(z))\right]_{z=\infty},\\
&= \left[\tau(U^{-1}(z))\left(A_0 + \frac{A_1}{z}\right)U(z)\right]_{z=\infty} + \mathrm{diag}\left(-I_{m_1}, 0_{n-m_1}\right).
\end{aligned}$$

Partitioning the matrix $A_1$ into blocks of the same size as those of $A_0$ as follows

$$A_1 = \begin{bmatrix} A_1^{11} & A_1^{12} \\ A_1^{21} & A_1^{22} \end{bmatrix},$$

the matrix $\widetilde{A}_0$ can be written as

$$\widetilde{A}_0 = \begin{bmatrix} A_0^{11} - I_{m_1} & A_1^{12} \\ 0 & A_0^{22} \end{bmatrix},$$

and its eigenvalues are $\rho_1 - 1, \rho_2, \ldots, \rho_q$ of multiplicity $m_1, \ldots, m_q$ respectively. Repeating this process $l_1$ times, the eigenvalues become

$$\rho_1 - l_1 = \rho_2, \rho_2, \ldots, \rho_q.$$

Thus after $l_1 + l_2 + \cdots + l_{q-1}$ iterations, one can make the eigenvalues of this set equal. Applying the same process to the other sets of eigenvalues, one can get a non-resonant matrix $B_0$ which ends the proof.

Combining Propositions 4 and 5, we get the following theorem:

**Theorem 1** *Given a linear difference system of the first kind of the form (11), one can compute a gauge transformation $\mathbf{y}(z) = T(z)\mathbf{w}(z)$, where $T(z)$ is an invertible matrix of factorial series, such that System (11) is equivalent to a linear difference system with constant coefficients of the form*

$$\Delta(\mathbf{w}(z)) + \Lambda\mathbf{w}(z) = 0,$$

*where $\Lambda \in \mathbb{C}^{n\times n}$.*

Consequently, using Proposition 2, we obtain:

**Corollary 1** *Any linear difference system (11) of the first kind admits a formal fundamental matrix of solutions given by*

$$T(z)z^{-[\Lambda]}, \tag{15}$$

*where $\Lambda \in \mathbb{C}^{n\times n}$ and $T(z) = \sum_{m\geq 0} T_m z^{-[m]}$ with $T_m \in \mathbb{C}^{n\times n}$.*

Here also, we can show that if the factorial series matrix $A(z)$ in (11) converges, then the factorial series matrix $T(z)$ of (15) does so. Therefore, every formal solution is an actual solution.

Assuming that the matrix $\Lambda$ in (15) is in Jordan form (see Proposition 3), each column vector of the formal fundamental matrix of solutions (15) can be written in the form

$$z^{-[\rho]} \sum_{k=0}^{m} \mathbf{f}_k(z)\phi_k(z+\rho), \tag{16}$$

where $\rho \in \mathbb{C}$, $m < n$ and for $k = 0, \ldots, m$, $\mathbf{f}_k$ is a vector of factorial series. The vector given in (16) is then a solution of System (11) and it is called a *formal regular solution*. Therefore, we have shown that any linear difference system of the first kind admits a full basis of $n$ $\mathbb{C}$-linearly independent formal regular solutions. In the following section, we shall consider a general class of systems which are not necessarily of the first kind and see how their regular solutions can also be computed.

We end this section with an example illustrating the computation of a formal fundamental matrix of solutions of linear difference systems of the first kind. We recall that in our MAPLE implementation a (truncated) factorial series $\sum_{n=0}^{t} a_n z^{-[n]}$ of precision $t$ is represented by the list of its $t+1$ coefficients $[a_0, a_1, \ldots, a_t]$.

*Example 1* We consider the first-order linear difference system of the first kind of the form (11) where matrix $A(z)$ is given by the truncated matrix of factorial series

$$A(z) = \begin{bmatrix} [1,-2,-1] & [-1,-1,2] & [0,1,1] \\ [1,0,-1] & [3,2,1] & [0,-1,2] \\ [-1,1,0] & [-3+i,-1,-1] & [i,0,-1] \end{bmatrix}$$

of precision $t = 2$ with $i$ denoting the complex number $\sqrt{-1}$. The matrix $A_0 = A(\infty)$ given by

$$A_0 = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 3 & 0 \\ -1 & -3+i & i \end{bmatrix}$$

has two distinct eigenvalues, namely $i$ of multiplicity 1 and 2 of multiplicity 2. So we are in the case where matrix $A_0$ is non-resonant. Applying Proposition 4 that we have implemented using our package FACTORIALSERIESTOOLS, we find the gauge transformation $\mathbf{y}(z) = T(z)\mathbf{w}(z)$, where the matrix $T(z)$ of factorial series is computed with precision $t = 2$

$$T(z) = \begin{bmatrix} [1,-2,-\frac{107}{8}-\frac{i}{8}] & [0,-\frac{53}{10}-\frac{i}{10},-\frac{16829}{680}-\frac{463\,i}{680}] & [0,-\frac{3}{10}-\frac{i}{10},-\frac{53}{85}-\frac{26\,i}{85}] \\ [0,1,\frac{47}{8}+\frac{i}{8}] & [1,\frac{13}{10}+\frac{i}{10},\frac{6357}{680}+\frac{259\,i}{680}] & [0,\frac{3}{10}+\frac{i}{10},-\frac{47}{170}+\frac{i}{170}] \\ [0,-\frac{1}{2},-\frac{79}{8}-\frac{37\,i}{8}] & [0,\frac{7}{10}+\frac{2}{5}\,i,-\frac{8159}{680}-\frac{6073\,i}{680}] & [1,\frac{7}{10}-\frac{i}{10},\frac{43}{340}+\frac{151\,i}{340}] \end{bmatrix}.$$

This transformation reduced the original system to the system with constant coefficients $\Delta(\mathbf{w}(z)) + A_0\mathbf{w}(z) = 0$. Therefore, by Proposition 2, a formal fundamental matrix of solutions (with precision $t = 2$) of the original system is given by $T(z)\,z^{-[A_0]}$. Now, computing the Jordan form of $A_0$, we find

$$J = P^{-1}A_0P = \begin{bmatrix} i & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \quad \text{with} \quad P = \begin{bmatrix} 0 & -1 & 1 \\ 0 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix},$$

so that Proposition 3 yields

$$z^{-[A_0]} = P z^{-[J]} P^{-1} = P \left[\begin{array}{c|cc} z^{-[i]} & 0 & 0 \\ \hline 0 & \multicolumn{2}{c}{\multirow{2}{*}{$z^{-[2]} \begin{pmatrix} 1 & -\phi_1(z+2) \\ 0 & 1 \end{pmatrix}$}} \\ 0 & & \end{array}\right] P^{-1}.$$

Finally, with the previous notation, a formal fundamental matrix of solutions (with precision $t = 2$) of the initial system of the first kind is given by

$$Y(z) = T(z)P \left[ \begin{array}{c|cc} z^{-[i]} & 0 & 0 \\ \hline 0 & \multicolumn{2}{c}{\multirow{2}{*}{$z^{-[2]} \begin{pmatrix} 1 & -\phi_1(z+2) \\ 0 & 1 \end{pmatrix}$}} \\ 0 & & \end{array} \right].$$

## 4 Computation of the Formal Regular Solutions

We consider now linear difference systems of the form

$$D(z)\Delta(\mathbf{y}(z)) + A(z)\mathbf{y}(z) = 0, \tag{17}$$

where $D(z)$ and $A(z)$ are two $n \times n$ matrices of factorial series such that $D(z)$ is invertible in $\mathscr{F}^{n\times n}$.

In this section, we shall recall the method presented in [10] for computing the formal regular solutions of Systems (17). This method is a generalization to the difference case of the one proposed in [9] for computing regular solutions of linear differential systems. It differs from the existing Frobenius-like methods given in [3, 16, 17] which are restricted to difference systems of the first kind.

Remark that when $D(z) = I_n$ in (17), we get systems of the first kind which have been studied in the previous section. Hence, the approach provided below for computing the regular solutions can be seen as an alternative way to the one that we have described previously in this paper.

We stress the fact that the method of [10] only deals with systems of the form (17) which are *simple*:

**Definition 4** A system of the form (17), or the associated linear difference operator $L$ defined by $L = D(z)\Delta + A(z)$, is said to be *simple* if the matrix pencil defined by $D_0\lambda - A_0$ is regular, that is, $\det(D_0\,\lambda - A_0) \not\equiv 0$.

Note however that the assumption that (17) is a simple system is not restrictive since in Sect. 5 below we shall provide an algorithm that transforms a non-simple difference system into a simple *equivalent* system (see Definition 5).

Instead of looking for regular solutions of the form (16), the idea behind the method in [10] is to search for regular solutions written as

$$\mathbf{y}(z) = \sum_{m\geq 0} z^{-[\rho+m]}\,\mathbf{y}_m(z), \quad \mathbf{y}_0(z) \neq 0, \tag{18}$$

where the $\mathbf{y}_m(z)$, for $m \geq 0$, are finite linear combinations with constant vector coefficients of the functions $\phi_i$ defined by (6), i.e., for all $m \geq 0$:

$$\mathbf{y}_m(z) = \mathbf{u}_{m,0}\,\phi_0(z) + \cdots + \mathbf{u}_{m,l_m}\,\phi_{l_m}(z); \quad l_m \in \mathbb{N}, \quad \mathbf{u}_{m,i} \in \mathbb{C}^n, \quad i = 0, \ldots, l_m.$$

**Theorem 2** ([10], Theorem 1) *With the above notation, the vector* $\mathbf{y}(z)$ *defined by (18) is a regular solution of the linear difference system (17) if and only if the complex number* $\rho$ *and the vector* $\mathbf{y}_0$ *are such that*

$$D_0\,\Delta(\mathbf{y}_0) - (D_0\,\rho - A_0)\,\mathbf{y}_0 = 0, \tag{19}$$

*and for* $m \geq 1$, $\mathbf{y}_m$ *satisfies*

$$D_0\,\Delta(\mathbf{y}_m) - (D_0\,(\rho + m) - A_0)\,\mathbf{y}_m = \mathbf{q}_m, \tag{20}$$

*where, for* $m \geq 1$, $\mathbf{q}_m$ *is a linear combination with constant coefficient matrices of the* $\mathbf{y}_i$ *and* $\Delta(\mathbf{y}_i)$, *for* $i = 0, \ldots, m-1$, *that can be effectively computed.*

Theorem 2 reduces the problem of computing regular solutions of linear difference systems (17) to the resolution of the linear difference systems with constant coefficient matrices given by (19) and (20).

We shall now state two results showing that System (19), resp. Systems (20) for $m \geq 1$, can always be solved for $\mathbf{y}_0$, resp. $\mathbf{y}_m$ for $m \geq 1$, of the desired form.

Proposition 6 gives a necessary and sufficient condition for System (19) to have a nontrivial solution $\mathbf{y}_0$ as a finite linear combination of the functions $\phi_i$ defined by (6) with constant vector coefficients.

**Proposition 6** ([10], Proposition 2) *With the above notation and assumptions, the linear difference system with constant coefficient matrices (19) has a solution of the form*

$$\mathbf{y}_0(z) = \sum_{i=0}^{k} \frac{(-1)^i}{i!}\,\mathbf{v}_i\,\phi_i(z), \tag{21}$$

*where* $\mathbf{v}_0, \ldots, \mathbf{v}_k \in \mathbb{C}^n$ *are constant vectors such that* $\mathbf{v}_k \neq 0$, *if and only if* $\rho$ *is an eigenvalue of the matrix pencil* $D_0\,\lambda - A_0$, *i.e.,* $\det(D_0\,\rho - A_0) = 0$ *and* $\mathbf{v}_k, \mathbf{v}_{k-1}, \ldots, \mathbf{v}_0$ *form a Jordan chain associated with* $\rho$ *(see [9, 15]).*

Assuming now that $\rho$ is an eigenvalue of $D_0\,\lambda - A_0$ and $\mathbf{y}_0$ given by (21) is a solution of (19), then, for $m \geq 1$, $\mathbf{y}_m$ satisfies a non-homogeneous linear difference system with constant coefficient matrices whose right-hand side is a finite linear combination of the $\mathbf{y}_i$ and $\Delta(\mathbf{y}_i)$, $i = 0, \ldots, m-1$, with constant coefficient matrices. The following proposition shows that such a system always admits a solution of the desired form.

**Proposition 7** ([10], Proposition 3) *With the above notation and assumptions, let further assume that the right-hand side* $\mathbf{q}_m$ *of (20) is a linear combination of the functions* $\phi_0, \phi_1, \ldots, \phi_d$ *with constant vector coefficients. Then System (20) has a solution* $\mathbf{y}_m$ *expressed as a linear combination of* $\phi_0, \phi_1, \ldots, \phi_p$ *with constant vector coefficients such that*

$$\begin{cases} d \leq p \leq d + \max\{\kappa_i, i = 1, \ldots, m_g(\rho+m)\} & \text{if } \det(D_0\,(\rho+m) - A_0) = 0, \\ p = d & \text{otherwise,} \end{cases}$$

*where $m_g(\rho + m)$ denotes the dimension of the kernel of the matrix $D_0\,(\rho + m) - A_0$ and the $\kappa_i$, $i = 1, \ldots, m_g(\rho + m)$, are the partial multiplicities of the eigenvalue $\rho + m$ of the matrix pencil $D_0\,\lambda - A_0$ (see [9, 15]).*

Theorem 2, Propositions 6 and 7 and their constructive proofs (see [10]) provide an algorithm for computing regular solutions of simple first-order linear difference systems of the form (17). In particular, it relies on the computation of eigenvalues and Jordan chains of the matrix pencil $D_0\,\lambda - A_0$ and determines a basis of the formal regular solutions space whose dimension equals the degree in $\lambda$ of $\det(D_0\,\lambda - A_0)$.

*Remark 2* It is worth noticing that this algorithm can be applied to any simple linear difference system (17) with either a regular or an irregular singularity at infinity. Indeed, a simple system has a regular singularity at $z = \infty$ if and only if $\det(D_0\,\lambda - A_0)$ is of degree $n$ in $\lambda$ which holds only if the constant matrix $D_0$ is invertible. However, the latter condition about $D_0$ is not necessary for the execution of the algorithm which only requires that $\det(D_0\,\lambda - A_0) \not\equiv 0$.

## 5 Reduction to Simple Forms

To System (17), we associate the linear difference operator $L$ defined by

$$L = D(z)\Delta + A(z). \tag{22}$$

Using the terminology of [9], we shall say that System (17), or Operator (22), is simple if the associated matrix pencil $L_0(\lambda)$ defined by

$$L_0(\lambda) = D_0\,\lambda + A_0$$

is regular, i.e., $\det(L_0(\lambda)) \not\equiv 0$. This definition of simple forms is equivalent to Definition 4 given in the previous section since $\det(D_0\lambda - A_0) = (-1)^n \det(L_0(-\lambda))$.

The aim of this section is to provide a procedure that transforms any operator of the form (22) into an equivalent simple operator where the terminology "equivalent operator" is defined as follows:

**Definition 5** A difference operator $L' = D'(z)\Delta + A'(z)$ is said to be *equivalent* to an operator $L = D(z)\Delta + A(z)$ if there exist two invertible matrices $S(z)$ and $T(z)$ such that

$$L' = S(z) L T(z),$$

that is,

$$D'(z) = S(z)D(z)\tau(T(z)), \quad \text{and} \quad A'(z) = S(z)D(z)\Delta(T(z)) + S(z)A(z)T(z).$$

In Definition 5, if the matrix $T(z)$ is a constant matrix $T$ (independent of $z$), then the coefficients of $L'$ are simply given by $D'(z) = S(z)D(z)T$ and $A'(z) = S(z)A(z)T$.

Computing simple forms of linear difference operators is not only useful for computing regular solutions (as seen in Sect. 4 above). It can also be used to determine the nature of the singularity $z = \infty$ in the regular/irregular classification (see Remark 3).

In [7], the authors describe an approach for computing simple forms (and more general forms called *k-simple forms*) of first-order linear differential operators. The approach presented in [7] requires the leading coefficient matrix $D(z)$ of the operator (or system) to be in Smith form. For linear difference operators, the method that we present in this section broadly follows the main idea of [7], except that the hypothesis considered on $D(z)$ is now finer. Indeed instead of assuming that $D(z)$ is in Smith form, we suppose that the singular matrix $D_0 = D(\infty)$ has the form

$$D_0 = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, \tag{23}$$

where $r$ stands for the rank of $D_0$.

Partitioning the matrix $A_0 = A(\infty)$ into blocks of the same size as those of $D_0$

$$A_0 = \begin{bmatrix} A_0^{11} & A_0^{12} \\ A_0^{21} & A_0^{22} \end{bmatrix},$$

the matrix pencil $L_0(\lambda) = D_0\lambda + A_0$ can be written in the form

$$L_0(\lambda) = \begin{bmatrix} I_r\lambda + A_0^{11} & A_0^{12} \\ A_0^{21} & A_0^{22} \end{bmatrix}. \tag{24}$$

In what follows and using the terminology of [7], the rows of $L_0(\lambda)$ of indices $r + 1$ to $n$, that is, the rows of the submatrix $\begin{bmatrix} A_0^{21} & A_0^{22} \end{bmatrix}$, will be referred to as the *$\lambda$-free rows* of $L_0(\lambda)$.

The singularity of the matrix pencil $L_0(\lambda)$ may be partly due to the fact that its $\lambda$-free rows are linearly dependent, that is, rank $\begin{bmatrix} A_0^{21} & A_0^{22} \end{bmatrix} < n - r$. Eliminating these dependency relations by means of performing row-operations on the operator, the matrix pencil may turn to be non-singular. However this is not sufficient because it may happen that the $\lambda$-free rows are linearly independent whereas the matrix $L_0(\lambda)$ is singular. Therefore, we shall first explain how given a non-simple operator, we can always manage to ensure that the $\lambda$-free rows of the matrix pencil are linearly dependent (Lemma 1 and Proposition 8). This can be done without altering the valuation of the determinant of the leading coefficient matrix of the operator. After that, we proceed to eliminate the dependency relations between the $\lambda$-free rows (Proposition 9). At this stage, the manipulations that we perform can only drop the valuation of the determinant of the leading coefficient matrix. Iterating this process a finite number of times, we are sure that we shall end up with a simple (equivalent) operator. Indeed, at worst case, the determinant of the leading coefficient matrix of the operator that we get has valuation zero which means that the leading coefficient matrix is invertible at $z = \infty$ so that the matrix pencil associated to the operator is necessarily regular.

## 5.1 Description of the Approach

As explained above, our first goal consists in making sure that the $\lambda$-free rows of the matrix pencil are linearly dependent. To achieve this, we shall first perform the following preliminary step in order to get an equivalent operator with a matrix pencil in an appropriate form.

**Lemma 1** *Let $L$ be a non-simple operator of the form (22) with a matrix pencil $L_0(\lambda)$ given by (24). One can compute two invertible constant matrices $S$ and $T$ in $\mathbb{C}^{n\times n}$ that transform $L$ into the equivalent operator $L' = S\,L\,T$ whose matrix pencil is of the form*

$$\begin{bmatrix} I_q\lambda + W^{11} & 0 & 0 \\ W^{21} & I_{r-q}\lambda + W^{22} & W^{23} \\ W^{31} & W^{32} & W^{33} \end{bmatrix}, \tag{25}$$

*where $0 \le q \le r$ ($r$ being the rank of $D_0$) and*

$$rank \begin{bmatrix} W^{32} & W^{33} \end{bmatrix} < n - r. \tag{26}$$

*Proof* Given the matrix pencil $L_0(\lambda)$ of the form (24), if rank$\begin{bmatrix} A_0^{21} & A_0^{22} \end{bmatrix} < n - r$, then $L_0(\lambda)$ is already of the form (25) with $q = 0$ and the condition (26) is satisfied so there is nothing to do. Otherwise, as the operator $L$ is non-simple, $L_0(\lambda)$ is then singular for all $\lambda \in \mathbb{C}$. In particular, for $\lambda = 0$, the matrix

$$L_0(0) = A_0 = \begin{bmatrix} A_0^{11} & A_0^{12} \\ A_0^{21} & A_0^{22} \end{bmatrix}$$

is singular as well. Swapping the rows and columns of $L$ of index 1 to $r$, we can suppose that there exists a nonzero row-vector of the form $\begin{pmatrix} 1 & \mathbf{u} & \mathbf{v} \end{pmatrix}$, where $\mathbf{u} \in \mathbb{C}^{r-1}$ and $\mathbf{v} \in \mathbb{C}^{n-r}$, in the left nullspace of $A_0$. Multiplying the operator $L$ on the left by

$$S_1 = \begin{bmatrix} 1 & \mathbf{u} & \mathbf{v} \\ 0 & I_{r-1} & 0 \\ 0 & 0 & I_{n-r} \end{bmatrix}$$

yields the operator $\overline{L} = S_1 L = \overline{D}(z)\Delta + \overline{A}(z)$, where $\overline{D}(z) = S_1 D(z)$ and $\overline{A}(z) = S_1 A(z)$. It follows that the first row of $\overline{A}_0$ is equal to $\begin{pmatrix} 1 & \mathbf{u} & \mathbf{v} \end{pmatrix} A_0 = 0$. Note here that the leading coefficient matrix $\overline{D}_0$ of the matrix pencil $\overline{L}_0(\lambda)$ may not be of the form (23) so we multiply $\overline{L}$ on the right by

$$T_1 = \begin{bmatrix} 1 & -\mathbf{u} & 0 \\ 0 & I_{r-1} & 0 \\ 0 & 0 & I_{n-r} \end{bmatrix}.$$

Denoting $\widetilde{L} = \overline{L}\, T_1 = \widetilde{D}(z)\Delta + \widetilde{A}(z)$, we have $\widetilde{D}_0 = D_0$ given by (23) and the first row of $\widetilde{A}_0$ is zero. The matrix pencil $\widetilde{L}_0(\lambda)$ associated with operator $\widetilde{L}$ is then of the form (25) with $q = 1$. We draw attention to the fact that $\widetilde{L}$ is also non-simple since $\widetilde{L}_0(\lambda) = S_1\, L_0(\lambda)\, T_1$. Now, if the condition (26) is satisfied, then we are done. Otherwise, we repeat the process on the submatrix of $\widetilde{L}(\lambda)$ composed of rows and columns of index 2 to $n$ (which is necessarily singular due to the particular structure of $\widetilde{L}(\lambda)$) and we increment the value of $q$. Thus, after at most $r$ successive iterations of this process, we obtain an equivalent non-simple operator $\widehat{L}$ whose associated matrix pencil is of the form (25) where either $q < r$ and Condition (26) holds, or $q = r$ which means that $\widehat{L}_0(\lambda)$ is of the form

$$\widehat{L}_0(\lambda) = \begin{bmatrix} I_r\,\lambda + W^{11} & 0 \\ W^{31} & W^{33} \end{bmatrix}.$$

In the latter case, the matrix $\widehat{L}_0(\lambda)$ is singular and the first diagonal block $I_r\,\lambda + W^{11}$ is regular. This implies that $W^{33}$ is necessarily singular which could be translated to Condition (26). Finally, the matrices $S$ and $T$ of Lemma 1 are the product of permutation matrices and upper triangular constant matrices.

It is important to mention that since the transformations $S$ and $T$ of Lemma 1 are constant, they surely do not alter the valuation of the determinant of the leading coefficient matrix of the operator.

We shall prove next that any non-simple difference operator with a matrix pencil of the form (25) is equivalent to an operator such that the $\lambda$-free rows of its associated matrix pencil are linearly dependent.

**Proposition 8** *Let $L = D(z)\Delta + A(z)$ be a non-simple operator having a matrix pencil $L_0(\lambda) = D_0\lambda + A_0$ of the form (25) and satisfying Condition (26). One can compute two invertible matrices $S(z)$ and $T(z)$ of the form*

$$S(z) = diag(zI_q, I_{r-q}, I_{n-r}) \quad and \quad T(z) = S^{-1}(z)T_1,$$

*where $T_1 \in \mathbb{C}^{n\times n}$ is an invertible constant matrix, that transform $L$ into the equivalent operator $L' = S(z)\, L\, T(z) = D'(z)\Delta + A'(z)$ such that the $\lambda$-free rows of the matrix pencil $L'_0(\lambda) = D'_0\lambda + A'_0$ are linearly dependent. Furthermore, we have $val(\det(D'(z))) = val(\det(D(z)))$.*

*Proof* In what follows, and for sake of brevity, we shall omit the explicit reference to the dependence of the matrices on $z$ in the notations. First, we partition the coefficient matrices $D$ and $A$ of $L$ into blocks of the same sizes as those of the matrix given in (25), that is,

$$D = \begin{bmatrix} D^{11} & D^{12} & D^{13} \\ D^{21} & D^{22} & D^{23} \\ D^{31} & D^{32} & D^{33} \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} A^{11} & A^{12} & A^{13} \\ A^{21} & A^{22} & A^{23} \\ A^{31} & A^{32} & A^{33} \end{bmatrix},$$

with $D^{11} = I_q + O(\frac{1}{z})$, $D^{22} = I_{r-q} + O(\frac{1}{z})$, and the other blocks $D^{ij}$, $A^{12}$ and $A^{13}$ are at least of valuation 1. Multiplying the operator $L$ on the left by the invertible matrix $S = \mathrm{diag}(zI_q, I_{r-q}, I_{n-r})$ and on the right by $S^{-1}$, we get an equivalent operator $\overline{L} = \overline{D}\Delta + \overline{A}$, where $\overline{D}$ and $\overline{A}$ are respectively given by

$$\overline{D} = SD\tau(S^{-1}) = \begin{bmatrix} \frac{z}{z-1}D^{11} & z\,D^{12} & z\,D^{13} \\ \frac{1}{z-1}D^{21} & D^{22} & D^{23} \\ \frac{1}{z-1}D^{31} & D^{32} & D^{33} \end{bmatrix} \tag{27}$$

and

$$\begin{aligned} \overline{A} &= SD\Delta(S^{-1}) + SAS^{-1} \\ &= SD\mathrm{diag}(-z^{-1}I_q, 0, 0) + SAS^{-1} \\ &= \begin{bmatrix} -D^{11} & 0 & 0 \\ -z^{-1}D^{21} & 0 & 0 \\ -z^{-1}D^{31} & 0 & 0 \end{bmatrix} + \begin{bmatrix} A^{11} & z\,A^{12} & z\,A^{13} \\ z^{-1}A^{21} & A^{22} & A^{23} \\ z^{-1}A^{31} & A^{32} & A^{33} \end{bmatrix} \\ &= \begin{bmatrix} A^{11} - D^{11} & z\,A^{12} & z\,A^{13} \\ z^{-1}(A^{21} - D^{21}) & A^{22} & A^{23} \\ z^{-1}(A^{31} - D^{31}) & A^{32} & A^{33} \end{bmatrix}. \end{aligned} \tag{28}$$

Notice that these manipulations do not affect the valuation of the determinant of the leading coefficient matrix of the operator, that is, $\mathrm{val}(\det(\overline{D})) = \mathrm{val}(\det(D))$, since $\mathrm{val}(\det(S)) = -q = -\mathrm{val}(\det(S^{-1})) = -\mathrm{val}(\det(\tau(S^{-1})))$. However, the matrix pencil associated to the operator is now of the form

$$\overline{L}_0(\lambda) = \begin{bmatrix} I_q\lambda + W^{11} - I_q & [z\,D^{12}]_{z=\infty}\lambda + [z\,A^{12}]_{z=\infty} & [z\,D^{13}]_{z=\infty}\lambda + [z\,A^{13}]_{z=\infty} \\ 0 & I_{r-q}\lambda + W^{22} & W^{23} \\ 0 & W^{32} & W^{33} \end{bmatrix}.$$

To get back the form (23) of the leading coefficient matrix of $\overline{L}_0(\lambda)$, we multiply $\overline{L}$ on the right by the invertible constant matrix

$$T_1 = \begin{bmatrix} I_q & -[z\,D^{12}]_{z=\infty} & -[z\,D^{13}]_{z=\infty} \\ 0 & I_{r-q} & 0 \\ 0 & 0 & I_{n-r} \end{bmatrix},$$

which transforms $\overline{L}$ into the equivalent operator $L' = \overline{L}T_1$ with matrix pencil

$$L_0'(\lambda) = \begin{bmatrix} I_q\lambda + W^{11} - I_q & * & * \\ 0 & I_{r-q}\lambda + W^{22} & W^{23} \\ 0 & W^{32} & W^{33} \end{bmatrix},$$

where the $*$ stands for constant matrices of suitable dimensions. As we have rank $\begin{bmatrix} W^{32} & W^{33} \end{bmatrix} < n - r$, the $\lambda$-free rows of $L_0'(\lambda)$ are then linearly dependent.

Now, the $\lambda$-free rows of the matrix pencil being linearly dependent, we proceed to cancel these dependence relations. This operation will decrease the valuation of the determinant of the leading coefficient matrix.

**Proposition 9** *Consider a non-simple operator $L = D(z)\Delta + A(z)$ having a matrix pencil $L_0(\lambda)$ of the form (24) with rank $\begin{bmatrix} A_0^{21} & A_0^{22} \end{bmatrix} < n - r$. One can compute two invertible matrices $T \in \mathbb{C}^{n\times n}$ and $S(z)$ of the form*

$$S(z) = S_3 diag(1, \ldots, 1, z^{[\gamma]}, 1, \ldots, 1)S_1,$$

*where $\gamma \in \mathbb{N}^*$, $z^{[\gamma]} = \prod_{j=0}^{\gamma-1}(z + j)$, and $S_3$, $S_1 \in \mathbb{C}^{n\times n}$ invertible constant matrices, that transform $L$ into the equivalent operator $L' = S(z)LT = D'(z)\Delta + A'(z)$ with $val(\det(D'(z))) < val(\det(D(z)))$.*

*Proof* Let $\mathbf{u}$ be a non zero row-vector in the left nullspace of the matrix $\begin{bmatrix} A_0^{21} & A_0^{22} \end{bmatrix}$. Let $i$ be the position of the first nonzero component of the vector $\mathbf{u}$. Let $S_1$ be the constant matrix obtained from the identity matrix $I_n$ by substituting its $(r + i)$th row by the $n$-dimensional row $\begin{pmatrix} 0 \ldots 0 \ \mathbf{u} \end{pmatrix}$. Multiplying the operator $L$ on the left by $S_1$ yields an operator $\overline{L} = S_1 L = \overline{D}(z)\Delta + \overline{A}(z)$, where $\overline{D} = S_1 D$ and $\overline{A} = S_1 A$. The $(r + i)$th row of the matrix $\overline{A}_0 = S_1 A_0$ is zero and that of $\overline{D}_0$ is zero as well due to the special forms of $D_0$ and $S_1$. Let $\gamma$ be the minimum value among the valuation of the $(r + i)$th row of $\overline{D}$ and that of the $(r + i)$th row of $\overline{A}$. Let

$$S_2 = \text{diag}(\underbrace{1, \ldots, 1}_{(r+i-1) \text{ times}}, z^{[\gamma]}, 1, \ldots, 1),$$

where $z^{[\gamma]} = \prod_{j=0}^{\gamma-1}(z + j)$, and multiply the operator $\overline{L}$ on the left by $S_2$. Notice that this operation may reveal in the entries of the $(r + i)$th row of the operator $S_2\overline{L}$ factorial series in $z + \gamma$ which can be easily transformed to factorial series in $z$ using the translation formula. At this stage, the valuation of the determinant of the leading coefficient matrix has been decreased since

$$\begin{aligned} \text{val}(\det(S_2\overline{D})) &= \text{val}(\det(S_2)) + \text{val}(\det(\overline{D})) \\ &= \text{val}(\det(\overline{D})) - \gamma \\ &< \text{val}(\det(\overline{D})) \\ &< \text{val}(\det(D)). \end{aligned}$$

Finally, we draw attention to the fact that if the integer $\gamma$ is equal to the valuation of the $(r+i)$th row of the matrix $\overline{D}$, then we may need to multiply the operator by two invertible constant matrices $S_3$ on the left and $T$ on the right in order to get the form (23) of the leading coefficient matrix at $z=\infty$.

Lemma 1 together with Propositions 8 and 9 give rise to an algorithm that takes as input (a truncation of) the coefficient matrices $D(z)$ and $A(z)$ of a non-simple linear difference operator $L = D(z)\Delta + A(z)$ and returns (a truncation of) the coefficient matrices $D'(z)$ and $A'(z)$ of a simple operator $L' = D'(z)\Delta + A'(z)$, together with two invertible matrices $S'(z)$ and $T'(z)$ such that $L' = S'(z)LT'(z)$. We summarize its steps below.

Algorithm `SimpleForm`

Input: The coefficient matrices $D$ and $A$ of operator $L$ given by (22).
Output: Four matrices $S'$, $T'$, $D'$ and $A'$ with $S' \in \mathbb{C}[z]^{n\times n}$ and $T' \in \mathbb{C}[z^{-1}]^{n\times n}$ invertible in $\mathbb{C}[z, z^{-1}]^{n\times n}$, $D' = S'\, D\, \tau(T')$ and $A' = S'\, D\, \Delta(T') + S'\, A\, T'$ such that the operator $D'\Delta + A'$ is simple.

Initialization: $S' \longleftarrow I_n,\ T' \longleftarrow I_n,\ D' \longleftarrow D,\ A' \longleftarrow A,\ L' \longleftarrow D_0\,\lambda + A_0$;
**While** $\det(L') = 0$ **do**

1. Compute two constant matrices $S$ and $T$ of $\mathbb{C}^{n\times n}$ as in Lemma 1;
2. Update $D' \longleftarrow S\,D'\,T,\ \ A' \longleftarrow S\,A'\,T,\ \ S' \longleftarrow S\,S'$ and $T' \longleftarrow T'\,T$;
3. Compute two matrices $S \in \mathbb{C}[z]^{n\times n}$ and $T \in \mathbb{C}[z^{-1}]^{n\times n}$ as in Proposition 8;
4. Update $D' \longleftarrow S\,D'\,\tau(T),\ A' \longleftarrow S\,D'\,\Delta(T) + S\,A'\,T,\ \ S' \longleftarrow S\,S'$ and $T' \longleftarrow T'\,T$;
5. Compute two matrices $S \in \mathbb{C}[z]^{n\times n}$ and $T \in \mathbb{C}^{n\times n}$ as in Proposition 9;
6. Update $D' \longleftarrow S\,D'\,T,\ \ A' \longleftarrow S\,A'\,T,\ \ S' \longleftarrow S\,S'$ and $T' \longleftarrow T'\,T$;
7. Let $L' \longleftarrow D'_0\,\lambda + A'_0$;

**end do**;
**Return** $S'$, $T'$, $D'$, $A'$;

*Remark 3* An important application of the `SimpleForm` algorithm above is that it allows to recognize the nature of the singularity $z=\infty$ in the regular/irregular classification. Indeed, as seen in Remark 2, the infinity is a regular singularity if and only if the matrix $D'(z)$ in the output of the algorithm is invertible at $z=\infty$. Therefore, this algorithm can be considered as an alternative way of Moser's algorithm [3, 4] which computes the minimal Poincaré rank $q$ of the system and therefore determines the nature of the singularity.

Once the `SimpleForm` algorithm has been applied, one can always use the method presented in Sect. 4 in order to compute a basis of the formal regular solutions space of a simple system. Moreover, if $z=\infty$ is a regular singularity, i.e., if $D'_0$ is invertible, there is an alternative method for computing regular solutions: one can write the system as $\Delta(\mathbf{y}(z)) + (D'(z))^{-1}A'(z)\mathbf{y}(z) = 0$ which is now a system of the first kind and then apply the method of Sect. 3 which is exclusively dedicated to first-order linear difference systems of the first kind.

### 5.2 Implementation and Example

The algorithm `SimpleForm` above has been implemented in MAPLE[2] based on our FACTORIALSERIESTOOLS package. In this subsection, we first give some remarks concerning the implementation of some steps of the algorithm. Then, we illustrate it on an example.

In the proof of Proposition 8, some block entries of the matrices $\overline{D}$ and $\overline{A}$ given respectively by (27) and (28), are obtained by multiplying factorial series by one of the following rational functions: $\frac{z}{z-1}$, $\frac{1}{z-1}$, $z$ or $z^{-1}$. We shall explain below how these operations are performed in our implementation.

Let us consider a factorial series

$$a_0 + \frac{a_1}{z} + \frac{a_2}{z(z+1)} + \cdots + \frac{a_t}{z(z+1)\cdots(z+t-1)}, \tag{29}$$

given up to the precision $t$. We recall that in our implementation such a factorial series is represented by the list of its coefficients $[a_0, a_1, a_2, \ldots, a_t]$. We must explain both how to multiply a factorial series $[0, a_1, a_2 \ldots, a_t]$ by $z + \alpha$ and how to multiply a factorial series $[a_0, a_1, a_2 \ldots, a_t]$ by $(z + \alpha)^{-1}$, where $\alpha \in \mathbb{C}$.

To multiply a factorial series $[0, a_1, a_2 \ldots, a_t]$ by $z + \alpha$ with $\alpha \in \mathbb{C}$, we first use the translation formula to transform it into a factorial series in $z + \alpha$ whose coefficients are given by a list $[0, b_1, b_2, \ldots, b_t]$, where $b_i$ can be explicitly computed using Formula (4). Then we multiply the latter factorial series by $z + \alpha$ which is equivalent to shifting the elements of the list to the left so that we get $[b_1, b_2, \ldots, b_t]$. Finally, we use again the translation Formula (4) to get a factorial series $[c_0, c_1, \ldots, c_{t-1}]$ in $z$. Note that the factorial series that we obtain is only known up to precision $t - 1$. So performing the multiplication by $(z + \alpha)$, we lose one term in the precision of the factorial series.

To multiply a factorial series $[a_0, a_1, a_2 \ldots, a_t]$ by $(z + \alpha)^{-1}$ with $\alpha \in \mathbb{C}$, we first make a translation in order to get a factorial series in $z + \alpha + 1$ whose coefficient list is given by $[b_0, b_1, b_2 \ldots, b_t]$, where $b_i$ can be explicitly computed using Formula (4). Then we divide the latter factorial series by $z + \alpha$ to get a factorial series in $z + \alpha$ whose coefficients $[0, b_0, b_1, \ldots, b_t]$ are obtained by shifting the elements of the list to the right. Finally, using again the translation formula (4), we end up with a factorial series $[0, c_1, c_2, \ldots, c_{t+1}]$ in $z$ which is the result of multiplying (29) by $(z + \alpha)^{-1}$. Note that here we obtain a factorial series of precision $t + 1$.

A useful consequence of the explanations above is that, to multiply a factorial series of precision $t$ by a rational fraction of the form $\frac{z+\alpha}{z+\beta}$ where $\alpha$, $\beta \in \mathbb{C}$, it is better to start by the multiplication by $(z + \beta)^{-1}$ followed by the multiplication by $(z + \alpha)$ since doing so the precision is preserved, i.e., the result is a factorial series of precision $t$. Moreover, from (27) and (28), we see that when we apply Proposition 8, some block entries of the resulting matrices are obtained by multiplying factorial series by $z$ so that we lose one term of precision. For the same reason, applying

[2] A beta version is available upon request from the authors.

Proposition 9 implies the loss of $\gamma$ terms (with the notation of Proposition 9) in the precision of the factorial series involved in some coefficients of the operator obtained.

*Example 2* We shall explain the different steps of the algorithm `SimpleForm` applied to the first-order linear difference operator $L = D(z)\Delta + A(z)$ with

$$D(z) = \begin{bmatrix} [1,2,2,0] & [0,2,0,0] & [0,-1,-1,1] \\ [0,0,0,-2] & [0,-1,-2,-1] & [0,0,1,1] \\ [0,2,-1,2] & [0,1,1,-2] & [0,-1,2,-2] \end{bmatrix}$$

and

$$A(z) = \begin{bmatrix} [0,1,1,-1] & [0,1,-1,-1] & [1,-2,-2,-2] \\ [0,1,0,-1] & [0,0,2,-1] & [1,1,2,1] \\ [1,1,1,0] & [0,-2,2,1] & [1,-1,2,0] \end{bmatrix}.$$

Note that the factorial series in the entries of the algorithm are then given up to precision 3. We first compute the matrix pencil $L_0(\lambda) = D_0\lambda + A_0$ associated to $L$

$$L_0(\lambda) = \begin{bmatrix} \lambda & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

The matrix $L_0(\lambda)$ is singular so $L$ is a non-simple operator, whereas the $\lambda$-free rows of $L_0(\lambda)$ are linearly independent. Consequently, our algorithm starts by applying Lemma 1 which implies multiplying the operator $L$ on the left by the constant invertible matrix

$$S^{(1)} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

This yields the operator $L^{(1)} = S^{(1)}L$ whose associated matrix pencil

$$L_0^{(1)}(\lambda) = S^{(1)}L_0(\lambda) = \left[\begin{array}{c|cc} \lambda & 0 & 0 \\ \hline 0 & 0 & 1 \\ 1 & 0 & 1 \end{array}\right]$$

is of the form (25) with $q = 1$ and Condition (26) holds. We then apply the following two transformations (see Proposition 8)

$$S^{(2)}(z) = \begin{bmatrix} z & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad T^{(2)}(z) = \begin{bmatrix} z^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

on $L^{(1)}$. This gives arise to a new operator $L^{(2)} = S^{(2)}(z)L^{(1)}T^{(2)}(z)$ whose coefficient matrices are given by

$$D^{(2)}(z) = \begin{bmatrix} [1, 3, 5] & [3, 2, -1] & [-1, -2, 2] \\ [0, 0, 0] & [0, -1, -2] & [0, 0, 1] \\ [0, 0, 2] & [0, 1, 1] & [0, -1, 2] \end{bmatrix}$$

and

$$A^{(2)}(z) = \begin{bmatrix} [-1, -2, -1] & [1, -3, 3] & [-3, -4, 1] \\ [0, 0, 1] & [0, 0, 2] & [1, 1, 2] \\ [0, 1, -1] & [0, -2, 2] & [1, -1, 2] \end{bmatrix}.$$

As we have seen before, at this stage, we lose one term in the precision of the factorial series matrices defining the operator $L^{(2)}$ compared to those defining $L^{(1)}$. Indeed, some entries of $D^{(2)}(z)$ and $A^{(2)}(z)$ are only known up to precision $3 - 1 = 2$ so that we have removed all the terms in $z^{-[3]}$ in $D^{(2)}(z)$ and $A^{(2)}(z)$. Now the matrix $D_0^{(2)} = D^{(2)}(\infty)$ is not of the required form (23) so we multiply the operator $L^{(2)}$ on the right by the constant invertible matrix

$$T^{(3)} = \begin{bmatrix} 1 & -3 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

to obtain the desired form. The matrix pencil associated to $L^{(3)} = L^{(2)}T^{(3)}$ is now given by

$$L_0^{(3)}(\lambda) = L_0^{(2)}(\lambda)T^{(3)} = \begin{bmatrix} \lambda - 1 & 4 & -4 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix},$$

and its $\lambda$-free rows are then linearly dependent. Consequently, we shall apply Proposition 9: we multiply the operator on the left by the constant invertible matrix

$$S^{(4)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix},$$

in order to eliminate the linear dependence relation. This operation yields a zero row in the matrix pencil of the new operator $L^{(4)} = S^{(4)}L^{(3)}$ whose coefficient matrices are given by

$$D^{(4)}(z) = \begin{bmatrix} [1, 3, 5] & [0, -7, -16] & [0, 1, 7] \\ [0, 0, 2] & [0, 2, -3] & [0, -1, 3] \\ [0, 0, 2] & [0, 1, -5] & [0, -1, 4] \end{bmatrix}$$

and

$$A^{(4)}(z) = \begin{bmatrix} [-1,-2,-1] & [4,3,6] & [-4,-6,0] \\ [0,1,-2] & [0,-5,6] & [0,-1,-2] \\ [0,1,-1] & [0,-5,5] & [1,0,1] \end{bmatrix}.$$

The second row of both $D^{(4)}(z)$ and $A^{(4)}(z)$ are of valuation $\gamma = 1$ in $z$ so we multiply $L^{(4)}$ on the left by

$$S^{(5)}(z) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & z & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

in order to decrease the valuation of the second rows and get rows of valuation 0. Doing so we obtain a new operator $L^{(5)} = S^{(5)}(z)L^{(4)}$ given by

$$D^{(5)}(z) = \begin{bmatrix} [1,3] & [0,-7] & [0,1] \\ [0,2] & [2,-3] & [-1,3] \\ [0.0] & [0,1] & [0,-1] \end{bmatrix}$$

and

$$A^{(5)}(z) = \begin{bmatrix} [-1,-2] & [4,3] & [-4,-6] \\ [1,-2] & [-5,6] & [-1,-2] \\ [0,1] & [0,-5] & [1,0] \end{bmatrix}.$$

The factorial series in the entries of $D^{(5)}(z)$ and $A^{(5)}(z)$ are now of precision $2 - \gamma = 2 - 1 = 1$ and thus, all the terms in $z^{-[2]}$ have been omitted. The matrix pencil associated to $L^{(5)}$ is

$$L_0^{(5)}(\lambda) = \begin{bmatrix} \lambda - 1 & 4 & -4 \\ 1 & 2\lambda - 5 & -\lambda - 1 \\ 0 & 0 & 1 \end{bmatrix},$$

and turns out to be regular. Therefore, we can either stop because we have reached our purpose of getting an equivalent simple operator or we can apply a final step in order to obtain the required form (23) for $D_0^{(5)}$ which could be provided by multiplying $L^{(5)}$ on the right by the constant invertible matrix

$$T^{(6)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}.$$

Finally, the matrices $S(z)$ and $T(z)$ that transform the original non-simple operator $L = D(z)\Delta + A(z)$ into the simple operator $L^{(6)} = L^{(5)}T^{(6)}$ are given by

$$S(z) = S^{(5)}(z)S^{(4)}S^{(2)}(z)S^{(1)} = \begin{bmatrix} z & -z & 0 \\ 0 & -z & z \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$T(z) = T^{(2)}(z)T^{(3)}T^{(6)} = \begin{bmatrix} z^{-1} & -\frac{3}{2}z^{-1} & -\frac{1}{2}z^{-1} \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}.$$

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# Refined Holonomic Summation Algorithms in Particle Physics

**Johannes Blümlein, Mark Round and Carsten Schneider**

*Dedicated to Sergei A. Abramov on the occasion of his 70th birthday*

**Abstract** An improved multi-summation approach is introduced and discussed that enables one to simultaneously handle sequences generated by indefinite nested sums and products in the setting of difference rings and holonomic sequences described by linear recurrence systems. Relevant mathematics is reviewed and the underlying advanced difference ring machinery is elaborated upon. The flexibility of this new toolbox contributed substantially to evaluating complicated multi-sums coming from particle physics. Illustrative examples of the functionality of the new software package `RhoSum` are given.

**Keywords** Symbolic summation · Parameterized telescoping · Difference rings Holonomic sequences · Particle physics

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# 1 Introduction

A standard approach to symbolic summation is that of telescoping. There are many individual variations and specific technologies, however one may summarize all technologies by stating the problem of parameterized telescoping. Given sequences $F_1(k), \dots, F_d(k)$ over an appropriate field $\mathbb{K}$, find $d$ constants (meaning free of $k$ and not all zero) $c_1, \dots, c_d \in \mathbb{K}$ and a sequence $G(k)$ such that

$$G(k+1) - G(k) = c_1 F_1(k) + \dots + c_d F_d(k). \tag{1}$$

If one does succeed then one can sum the relation to obtain

$$G(b+1) - G(a) = c_1 \sum_{k=a}^{b} F_1(k) + \dots + c_d \sum_{k=a}^{b} F_d(k). \tag{2}$$

for some properly chosen bounds $a, b$. Restricting to $d = 1$ gives the telescoping formula, which expresses the sum over $F_1$ as a difference. Alternatively, suppose that we started with the definite sum $S(n) = \sum_{k=l}^{L(n)} F(n, k)$ with a bivariate sequence $F(n, k)$, with $l \in \mathbb{N}$ and with some integer linear expression[1] $L(n)$. Then taking $F_i(k) := F(n+i-1, k)$ for $1 \leqslant i \leq d$, the parameterized telescoping Eq. (2) reduces to Zeilberger's creative telescoping. Namely, omitting some mild assumptions, Eq. (2) with $a = l$ and $b = L(n)$ yields a linear recurrence of the form

$$h(n) = c_1(n)\, S(n) + c_2(n)\, S(n+1) + \dots + c_d(n)\, S(n+d-1) \tag{3}$$

where $h(n)$ comes from $G(L(n)+1) - G(l)$ and extra terms taking care of the shifts in the boundaries. In the creative telescoping setting we assume that the field $\mathbb{K}$ contains the variable $n$ and thus the constants $c_i$ may depend on $n$.

This recurrence finding technology based on parameterized telescoping started for hypergeometric sums [37, 48, 50, 69] and has been extended to $q$-hypergeometric and their mixed versions [22, 47]. A generalization to multi-summation has also been performed [21, 66, 67]. Further, the input class has been widened significantly by the holonomic summation paradigm [68] and the efficient algorithms worked out in [34, 44]. Using these tools it is possible to solve the parameterized telescoping problem for (multivariate) sequences that are a solution of a system of linear difference (or differential) equations. In particular, applying this technology recursively enables one to treat multi-sum problems.

Another general approach was initiated by M. Karr's summation algorithm [40] in the setting of $\Pi\Sigma$-fields and has been generalized to the more general setting of $R\Pi\Sigma$-difference ring extensions [61–63]. Using the summation package `Sigma` [57] one can solve the parameterized telescoping equation in such difference rings and fields not only for the class of ($q$–)hypergeometric products and their mixed

[1] $L(n_1, \dots, n_l)$ stands for $z_0 + z_1 n_1 + \dots + z_l n_l$ for some integers $z_0, \dots, z_l$.

versions, but also for indefinite nested sums defined over such objects covering as special cases, e.g., the generalized harmonic sums [13]

$$S_{r_1,\ldots,r_m}(x_1,\ldots,x_m,n)=\sum_{k_1=1}^{n}\frac{x^{k_1}}{k_1^{r_1}}\sum_{k_2=1}^{k_1}\frac{x^{k_2}}{k_2^{r_2}}\cdots\sum_{k_m=1}^{k_{m-1}}\frac{x^{k_m}}{k_m^{r_m}} \tag{4}$$

with $x_1,\ldots,x_m\in\mathbb{K}\setminus\{0\}$ which contain as special case the so-called harmonic sums [31, 65] defined by $S_{x_1 r_1,\ldots,x_m r_m}(n)=S_{r_1,\ldots,r_m}(x_1,\ldots,x_m,n)$ with $x_1,\ldots,x_r\in\{-1,1\}$. Further, $R\Pi\Sigma$-extensions enable one to model cyclotomic sums [12] or nested binomial sums [10]. Using efficient recurrence solvers [17, 33, 49, 53, 56] that make use of d'Alembertian solutions [18, 19], a strong machinery has been developed to transform definite sums to expressions in terms of indefinite nested sums defined over ($q$–)hypergeometric products and their mixed versions. (d'Alembertian solutions are a subclass of Liouvillian solutions [38, 51].) In the last years this strong machinery [60] has been utilized heavily for problems in particle physics, see, e.g., [3, 5, 7, 27] and references therein. However, for several instances we were forced to push forward our existing summation technologies to be able to carry out our calculations.

More precisely, we utilized and refined the Sigma-approach that has been developed in [54, 57] to unite Karr's $\Pi\Sigma$-field setting with the holonomic system approach: one can solve the parameterized telescoping problem in terms of elements from a $\Pi\Sigma$-field together with summation objects which are solutions of inhomogeneous linear difference equations. In particular, a refined tactic has been worked out for the well-known holonomic approach [34] that finds recurrences without Gröbner basis computations or expensive uncoupling algorithms [32, 70]. This efficient and flexible approach has been applied to derive the first alternative proof [20] of Stembridge's TSPP theorem [64].

This article is the continuation of this work and explains new features that were necessary to compute highly non-trivial problems coming from particle physics [3, 5, 7, 27]. First, the ideas of [54] are generalized from the difference field to the ring setting: we consider a rather general class of difference rings that is built by the so-called $R\Pi\Sigma$-extensions [61, 63] and introduce on top a so-called higher-order linear difference extension. In this way, indefinite nested sums can be defined covering in addition a summation object that is a solution of an (inhomogeneous) linear difference equation defined over indefinite nested sums and products. In particular, our new techniques from [58, 59, 61] are applied to derive new and more flexible algorithms for the parameterized telescoping problem. Further, we push forward the theory of higher-order linear extensions in connection with $R\Pi\Sigma$-extension. We show that certain non-trivial constants, in case of existence, can be computed in such rings and that such constants can be utilized to design improved higher-order linear extension with smaller recurrence order. Finally, this machinery is applied recursively to multi-sums in order to produce linear recurrences. As it turns out, our refined difference ring algorithms in combination with the ideas from [54] introduce various new options as to how such recurrences can be calculated: using different telescoping strategies will lead to more or less complicated recurrence relations and the calculation time might vary heavily. In order to dispense the user from all these considerations, a new summation package `RhoSum` has been developed that analyzes the different possibilities by clever heuristics and performs the (hopefully) optimal calculation automatically.

The manuscript is organized as follows. In Sect. 2 we present our toolbox to solve the parameterized telescoping problem in our general setting built by $\Pi\Sigma$-fields, $R\Pi\Sigma$-extensions and a higher-order linear extension on top. In addition, new theoretical insight is provided that connects non-trivial constants in such extensions and the possibility to reduce the recurrence order of higher-order extensions. Then, in Sect. 3, our multi-sum approach based on our refined holonomic techniques is presented and specific technical aspects of the algorithm are explained. Such details are important for an efficient implementation. With the main results of the summation approach discussed, Sect. 4 gives an illustrative example arising from particle physics which shows some of the features of the algorithm in practice. Finally, in Sect. 5 there is a brief summary.

## 2 Parameterized Telescoping Algorithms in Difference Rings

In our difference ring approach the summation objects are represented by elements in a ring or field $\mathbb{A}$ and the shift operator acting on these objects is rephrased in terms of a ring or field automorphism $\sigma : \mathbb{A} \to \mathbb{A}$. In short, we call $(\mathbb{A}, \sigma)$ a difference ring or a difference field. The set of units of a ring $\mathbb{A}$ is denoted by $\mathbb{A}^*$ and the set of constants of $(\mathbb{A}, \sigma)$ is defined by

$$\mathrm{const}_\sigma \mathbb{A} = \{f \in \mathbb{A} \mid \sigma(f) = f\}.$$

In general $\mathbb{K} := \mathrm{const}_\sigma \mathbb{A}$ is a subring of $\mathbb{A}$ (or a subfield of $\mathbb{A}$ if $\mathbb{A}$ is field). In the following we will take care that $\mathbb{K}$ is always a field containing the rational numbers $\mathbb{Q}$ as subfield. $\mathbb{K}$ will be also called the constant field of $(\mathbb{A}, \sigma)$. For a vector $\mathbf{f} = (f_1, \ldots, f_d) \in \mathbb{A}^d$ we define $\sigma(\mathbf{f}) = (\sigma(f_1), \ldots, \sigma(f_d)) \in \mathbb{A}^d$. $\mathbb{N}$ denotes the non-negative integers.

Finally, we will heavily use the concept of difference ring extensions. A difference ring $(\mathbb{E}, \sigma')$ is a difference ring extension of a difference ring $(\mathbb{A}, \sigma)$ if $\mathbb{A}$ is a subring of $\mathbb{E}$ and $\sigma'(f) = \sigma(f)$ for all $f \in \mathbb{A}$. If it is clear from the context, we will not distinguish anymore between $\sigma$ and $\sigma'$.

Suppose that one succeeded in rephrasing the summation objects $F_1(k), \ldots, F_d(k)$ in a difference ring $(\mathbb{A}, \sigma)$ with constant field $\mathbb{K}$, i.e., $F_i(k)$ can be modeled by $f_i \in \mathbb{A}$ where the corresponding objects $F_i(k+1)$ correspond to the elements $\sigma(f_i)$. Then the problem of parameterized telescoping (1) can be rephrased as follows.

**Problem RPT in** $(\mathbb{A},\sigma)$**:** Refined Parameterized Telescoping.
*Given* a difference ring $(\mathbb{A},\sigma)$ with constant field $\mathbb{K}$ and $\mathbf{f}=(f_1,\dots,f_d)\in\mathbb{A}^d$.
*Find*, if possible, an[a] "optimal" difference ring extension $(\mathbb{E},\sigma)$ of $(\mathbb{A},\sigma)$ with $\text{const}_\sigma\mathbb{E}=\text{const}_\sigma\mathbb{A}$, $g\in\mathbb{E}$ and $c_1,\dots,c_d\in\mathbb{K}$ with $c_1\neq 0$ and

$$\sigma(g)-g=c_1 f_1+\dots+c_d f_d. \qquad (5)$$

[a]The optimality criterion will be specified later in the setting of $R\Pi\Sigma$-extensions.

Namely, suppose that we find $g\in\mathbb{E}$ and $c_1,\dots,c_d\in\mathbb{K}$ and we succeed in reinterpreting $g$ as $G(k)$ in terms of our class of summation objects where $\sigma(g)$ represents $G(k+1)$. Then this will lead to a solution of the parameterized telescoping Eq. (1). In Sect. 2.1 we will work out this machinery for difference rings that are built by $R\Pi\Sigma$-extensions [61–63] and which enables one to model summation objects, e.g., in terms of indefinite nested sums defined over ($q$–)hypergeometric products and their mixed versions. Afterwards, we will use this technology in Sect. 2.2.2 to tackle summation objects that can be represented in terms of recurrences whose coefficients are given over the earlier defined difference rings.

## 2.1 $\Pi\Sigma$-Fields and $R\Pi\Sigma$-Extensions

A central building block of our approach are Karr's $\Pi\Sigma$-fields [40, 41].

**Definition 2.1** A difference field $(\mathbb{F},\sigma)$ with constant field $\mathbb{K}$ is called a $\Pi\Sigma$-field if $\mathbb{F}=\mathbb{K}(t_1,\dots,t_e)$ where for all $1\leqslant i\leqslant e$ each $\mathbb{F}_i=\mathbb{K}(t_1,\dots,t_i)$ is a transcendental field extension of $\mathbb{F}_{i-1}=\mathbb{K}(t_1,\dots,t_{i-1})$ (we set $\mathbb{F}_0=\mathbb{K}$) and $\sigma$ has the property that $\sigma(t_i)=a\,t_i$ or $\sigma(t_i)=t_i+a$ for some $a\in\mathbb{F}^*_{i-1}$.

*Example 2.1* (1) The simplest $\Pi\Sigma$-field is the rational difference field $\mathbb{F}=\mathbb{K}(t)$ for some field $\mathbb{K}$ and the field automorphism $\sigma:\mathbb{F}\to\mathbb{F}$ defined by $\sigma(c)=c$ for all $c\in\mathbb{K}$ and $\sigma(t)=t+1$.
(2) Another $\Pi\Sigma$-field is the $q$-rational difference field. Here one takes a rational function field $\mathbb{K}=\mathbb{K}'(q)$ over a field $\mathbb{K}'$ and the rational function field $\mathbb{F}=\mathbb{K}(t)$ over $\mathbb{K}$. Finally, one defines the field automorphism $\sigma:\mathbb{F}\to\mathbb{F}$ by $\sigma(c)=c$ for all $c\in\mathbb{K}$ and $\sigma(t)=q\,t$.
(3) One can combine the two constructions (1) and (2) and arrives at the mixed $(q_1,\dots,q_e)$-multibasic rational difference field [22]. Here one considers the rational function field $\mathbb{K}=\mathbb{K}'(q_1,\dots,q_e)$ over the field $\mathbb{K}'$ and the rational function field $\mathbb{F}=\mathbb{K}(t,t_1,\dots,t_e)$ over $\mathbb{K}$. Finally, one takes the field automorphism $\sigma:\mathbb{F}\to\mathbb{F}$ determined by $\sigma(c)=c$ for all $c\in\mathbb{K}$, $\sigma(t)=t+1$ and $\sigma(t_i)=q_i\,t_i$ for all $1\leqslant i\leqslant e$. By [46, Corollary 5.1] $(\mathbb{F},\sigma)$ is again a $\Pi\Sigma$-field.
(4) Besides these base fields, one can model nested summation objects. E.g., one can define the $\Pi\Sigma$-field $(\mathbb{K}(t)(h),\sigma)$ with constant field $\mathbb{K}$ where $(\mathbb{K}(t),\sigma)$ is the rational difference field and $\sigma$ is extended from $\mathbb{K}(t)$ to the rational function field

$\mathbb{K}(t)(h)$ subject to the relation $\sigma(h) = h + \frac{1}{t+1}$. Here $h$ in $\mathbb{F}$ scopes the shift behavior of the harmonic numbers $S_1(k) = \sum_{i=1}^{k} \frac{1}{i}$ with $S_1(k+1) = S_1(k) + \frac{1}{k+1}$.

A drawback of Karr's very elegant $\Pi\Sigma$-field construction is the inability to treat the frequently used summation object $(-1)^k$. In order to overcome this situation, $R\Pi\Sigma$-extensions have been introduced [61–63].

**Definition 2.2** A difference ring $(\mathbb{E}, \sigma)$ is called an *APS-extension* of a difference ring $(\mathbb{A}, \sigma)$ if $\mathbb{A} = \mathbb{A}_0 \leqslant \mathbb{A}_1 \leqslant \cdots \leqslant \mathbb{A}_e = \mathbb{E}$ is a tower of ring extensions where for all $1 \leqslant i \leqslant e$ one of the following holds:

- $\mathbb{A}_i = \mathbb{A}_{i-1}[t_i]$ is a ring extension subject to the relation $t_i^n = 1$ for some $n > 1$ where $\frac{\sigma(t_i)}{t_i} \in (\mathbb{A}_{i-1})^*$ is a primitive $n$th root of unity ($t_i$ *is called an A-monomial, and n is called the order of the A-monomial*);
- $\mathbb{A}_i = \mathbb{A}_{i-1}[t_i, t_i^{-1}]$ is a Laurent polynomial ring extension with $\frac{\sigma(t_i)}{t_i} \in (\mathbb{A}_{i-1})^*$ ($t_i$ *is called a P-monomial*);
- $\mathbb{A}_i = \mathbb{A}_{i-1}[t_i]$ is a polynomial ring extension with $\sigma(t_i) - t_i \in \mathbb{A}_{i-1}$ ($t_i$ *is called an S-monomial*).

If all $t_i$ are $A$-monomials, $P$-monomials or $S$-monomials, we call $(\mathbb{E}, \sigma)$ also a (nested) $A$-extension, $P$-extension or $S$-extension. If in addition the constants remain unchanged, i.e., $\text{const}_\sigma\mathbb{A} = \text{const}_\sigma\mathbb{E}$ an $A$-monomial is also called $R$-monomial, a $P$-monomial is called a $\Pi$-monomial and an $S$-monomial is called a $\Sigma$-monomial. In particular, such an $APS$-extension (or $A$-extension or $P$-extension or $S$-extension) is called an $R\Pi\Sigma$-extension (or $R$-extension or $\Pi$-extension or $\Sigma$-extension).

For the $R\Pi\Sigma$-extension $(\mathbb{E}, \sigma)$ of $(\mathbb{A}, \sigma)$ we also will write $\mathbb{E} = \mathbb{A}\langle t_1\rangle \ldots \langle t_e\rangle$. Depending on the case whether $t_i$ with $1 \leqslant i \leqslant e$ is an $R$-monomial, $\Pi$-monomial or $\Sigma$-monomial, $\mathbb{G}\langle t_i\rangle$ with $\mathbb{G} = \mathbb{A}\langle t_1\rangle \ldots \langle t_{i-1}\rangle$ stands for the algebraic ring extension $\mathbb{G}[t_i]$ with $t_i^n$ for some $n > 1$, for the ring of Laurent polynomials $\mathbb{G}[t_1, t_1^{-1}]$ or for the polynomial ring $\mathbb{G}[t_i]$, respectively.

We will rely heavily on the following property of $R\Pi\Sigma$-extensions [61, Theorem 2.12] generalizing the $\Pi\Sigma$-field results given in [41, 53].

**Theorem 2.1** *Let $(\mathbb{A}, \sigma)$ be a difference ring. Then the following holds.*

1. *Let $(\mathbb{A}[t], \sigma)$ be an S-extension of $(\mathbb{A}, \sigma)$ with $\sigma(t) = t + \beta$ where $\beta \in \mathbb{A}$ such that $\text{const}_\sigma\mathbb{A}$ is a field. Then this is a $\Sigma$-extension (i.e., $\text{const}_\sigma\mathbb{A}[t] = \text{const}_\sigma\mathbb{A}$) iff there does not exist a $g \in \mathbb{A}$ with $\sigma(g) = g + \beta$.*
2. *Let $(\mathbb{A}[t, t^{-1}], \sigma)$ be a P-extension of $(\mathbb{A}, \sigma)$ with $\sigma(t) = \alpha\, t$ where $\alpha \in \mathbb{A}^*$. Then this is a $\Pi$-extension (i.e., $\text{const}_\sigma\mathbb{A}[t, t^{-1}] = \text{const}_\sigma\mathbb{A}$) iff there are no $g \in \mathbb{A} \setminus \{0\}$ and $m \in \mathbb{Z} \setminus \{0\}$ with $\sigma(g) = \alpha^m\, g$.*
3. *Let $(\mathbb{A}[t], \sigma)$ be an A-extension of $(\mathbb{A}, \sigma)$ of order $\lambda > 1$ with $\sigma(t) = \alpha\, t$ where $\alpha \in \mathbb{A}^*$. Then this is an R-extension (i.e., $\text{const}_\sigma\mathbb{A}[t] = \text{const}_\sigma\mathbb{A}$) iff there are no $g \in \mathbb{A} \setminus \{0\}$ and $m \in \{1, \ldots, \lambda - 1\}$ with $\sigma(g) = \alpha^m\, g$.*

In the following we will focus on the special class of simple $R\Pi\Sigma$-extensions.

**Definition 2.3** Let $(\mathbb{A}, \sigma)$ be a difference ring extension of $(\mathbb{G}, \sigma)$. Then an $R\Pi\Sigma$-extension $(\mathbb{A}\langle t_1\rangle \dots \langle t_e\rangle, \sigma)$ of $(\mathbb{A}, \sigma)$ is called $\mathbb{G}$-simple if for any $R$-monomial $t_i$ with $1 \leqslant i \leqslant e$ we have that $\frac{\sigma(t_i)}{t_i} \in (\text{const}_\sigma\mathbb{G})^*$, and for any $\Pi$-monomial $t_i$ with $1 \leqslant i \leqslant e$ we have that $\frac{\sigma(t_i)}{t_i} \in \mathbb{G}^*$. If $\mathbb{A} = \mathbb{G}$, we just say simple and not $\mathbb{G}$-simple.

Take a simple $R\Pi\Sigma$-extension $(\mathbb{A}\langle \mathsf{t}_1\rangle \dots \langle \mathsf{t}_e\rangle, \sigma)$ of $(\mathbb{A}, \sigma)$. If $\mathsf{t}_i$ is an $R$-monomial or $\Pi$-monomial, we can reorder the generator and obtain the difference ring $(\mathbb{E}, \sigma)$ with $\mathbb{E} = \mathbb{A}\langle \mathsf{t}_i\rangle\langle \mathsf{t}_1\rangle \dots \langle \mathsf{t}_{i-1}\rangle\langle \mathsf{t}_{i+1}\rangle \dots \langle \mathsf{t}_e\rangle$. Note that this rearrangement does not change the set of constants. Further note that the recursive nature of $\sigma$ is respected accordingly. Thus $(\mathbb{E}, \sigma)$ is again a simple $R\Pi\Sigma$-extension of $(\mathbb{A}, \sigma)$. Applying such permutations several times enables one to move all $R$-monomials and $\Pi$-monomials to the left and all the $\Sigma$-monomials to the right yielding a simple $R\Pi\Sigma$-extension of the form $(\mathbb{A}\langle t_1\rangle \dots \langle t_u\rangle\langle \tau_1\rangle \dots \langle \tau_v\rangle, \sigma)$ with $u + v = e$ where the $t_i$ with $1 \leqslant i \leqslant u$ are $R$- or $\Pi$-monomials and the $\tau_i$ with $1 \leqslant i \leqslant v$ are $\Sigma$-monomials.

We emphasize that the class of simple $R\Pi\Sigma$-extensions defined over the rational difference field, $q$-rational difference or mixed multibasic difference field (see Example 2.1) cover all the indefinite nested summation objects that the we have encountered so far in practical problem solving: this class enables one to treat $(-1)^k$, e.g., with the $R$-monomial $t_1$ with $\sigma(t_1) = -t_1$ and $t_1^2 = 1$ and more generally it allows one to formulate all hypergeometric/$q$-hypergeometric/mixed multibasic hypergeometric products [46, 55] and nested sums defined over such products [61, 63]. For the definition of the hypergeometric class see Definition 3.1 below. In particular, this class enables one to represent d'Alembertian solutions [18, 19], a subclass of Liouvillian solutions [38, 51].

For $\Pi\Sigma$-fields $(\mathbb{G}, \sigma)$ and more generally for simple $R\Pi\Sigma$-extensions $(\mathbb{A}, \sigma)$ of $(\mathbb{G}, \sigma)$ many variations of Problem RPT have been worked out [40, 61–63]. In this regard, the depth function $\delta : \mathbb{A} \to \mathbb{N}$ will be used. More precisely, let $(\mathbb{A}, \sigma)$ be a simple $R\Pi\Sigma$-extension of $(\mathbb{G}, \sigma)$ with $\mathbb{A} = \mathbb{G}\langle t_1\rangle \dots \langle t_e\rangle$. By definition we have $\sigma(t_i) = \alpha_i\, t_i + \beta_i$ for $1 \leqslant i \leqslant e$ where the $\alpha_i, \beta_i$ are taken from the ring below. Then the depth function $f : \mathbb{A} \to \mathbb{N}$ is defined iteratively as follows. For $f \in \mathbb{G}$ we set $\delta(f) = 0$. If $\delta$ has been defined for $\mathbb{E} = \mathbb{A}\langle t_1\rangle \dots \langle t_{i-1}\rangle$, then define[2] $\delta(t_i) = 1 + \max_{j\in\text{supp}(\{\alpha_i,\beta_i\})} \delta(\tau_j)$. Further, for $f \in \mathbb{E}\langle t_i\rangle$, define $\delta(f) = \max_{j\in\text{supp}(f)} \delta(t_j)$. In other words $\delta(t_i)$ for $1 \leqslant i \leqslant e$ gives the maximal nesting depth of an $R\Pi\Sigma$-monomial and $\delta(f)$ for $f \in \mathbb{A}$ measures the maximal nesting depth among all the arising $R\Pi\Sigma$-monomials $t_i$ in $f$.

In the following we emphasize the following four variants of Problem RPT that will play a role in this article.

*Remark 2.1* Let $(\mathbb{A}, \sigma)$ be a simple $R\Pi\Sigma$-extension of a $\Pi\Sigma$-field[3] $(\mathbb{G}, \sigma)$ with $\mathbb{A} = \mathbb{G}\langle t_1\rangle \dots \langle t_e\rangle$ and let $f_1, \dots, f_d \in \mathbb{A}$. Then the following strategies are proposed that can be executed within the summation package `Sigma`.

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[2]For a finite set $L \subseteq \mathbb{A}$ we define $\text{supp}(L) = \{1 \leqslant j \leqslant e \mid t_j \text{ occurs in } L\}$.

[3]In order to apply our summation algorithms, we must assume that the constant field $\mathbb{K} = \text{const}_\sigma\mathbb{G}$ has certain algorithmic properties [55]; this is guaranteed if we are given, e.g., a rational function field $\mathbb{K} = \mathbb{K}'(x_1, \dots, x_l)$ over an algebraic number field $\mathbb{K}'$.

RPT$_1$: Decide constructively, if Problem RPT is solvable with $\mathbb{E} = \mathbb{A}$; see [61].

RPT$_2$: Try to solve Problem RPT$_1$. If this is not possible, decide constructively if there is a $\Sigma$-extension $(\mathbb{E}, \sigma)$ of $(\mathbb{A}, \sigma)$ with $\mathbb{E} = \mathbb{A}[\tau_1]\dots[\tau_v]$ in which one finds the desired $c_i$ and $g \in \mathbb{E}$ with the extra property that $\delta(\tau_i) \leqslant \delta(c_1 f_1 + \dots + c_d f_d)$ holds for all $1 \leqslant i \leqslant v$; see [58] in combination with [61].

RPT$_3$: By the recursive nature, we may reorder the $R\Pi\Sigma$-monomials in $\mathbb{A}$ such that $\delta(t_1) \leqslant \delta(t_2) \leqslant \dots \leqslant \delta(t_e)$ holds. With this preparation step, try to solve Problem RPT$_2$. If this is not possible, decide if there is a $\Sigma$-extension $(\mathbb{E}, \sigma)$ of $(\mathbb{A}, \sigma)$ with $\mathbb{E} = \mathbb{A}[\tau]$ and $\sigma(\tau) - \tau \in \mathbb{G}\langle t_1\rangle \dots \langle t_i\rangle$ for some $0 \leqslant i < e$ where at least one of the $t_{i+1}, \dots, t_e$ occurs in $c_1 f_1 + \dots + c_d f_d$ and $i$ is minimal among all such possible choices; see [59] in combination with [61].

RPT$_4$: Try to solve Problem RPT$_3$. If this is not possible, it follows in particular that there is no $g \in \mathbb{A}$ with $\sigma(g) - g = f_1$. By part 1 of Theorem 2.1 we can construct the $\Sigma$-extension $(\mathbb{A}[\tau], \sigma)$ of $(\mathbb{A}, \sigma)$ with $\sigma(\tau) = \tau + f_1$ and return $g = \tau$ and $c_1 = 1$, $c_i = 0$ for $2 \leqslant i \leqslant d$.

*Example 2.2* Consider the definite sum $X(k) = \sum_{j=0}^{k} F(k, j)$ with the summand $F(k, j) = \binom{k}{j} S_1(j)^2$ and the shifted versions $F(k+i, j) = \prod_{l=1}^{i} \frac{k+l}{k-j+l} \binom{k}{j} S_1(j)^2$ for $i = 0, 1, 2, \dots$. We start with the difference field $(\mathbb{K}(t), \sigma)$ with $\sigma(t) = t + 1$ and constant field $\mathbb{K} = \mathbb{Q}(k)$. Further we rephrase $\binom{k}{j}$ with the shift behavior $\binom{k}{j+1} = \frac{k-j}{j+1}\binom{k}{j}$ with $b$ in the $\Pi$-extension $(\mathbb{K}(t)[b, b^{-1}], \sigma)$ of $(\mathbb{K}(t), \sigma)$ with $\sigma(b) = \frac{k-t}{t+1} b$. Further, we rephrase $S_1(k)$ with $h$ in the $\Sigma$-extension $(\mathbb{K}(t)[b, b^{-1}][h], \sigma)$ of $(\mathbb{K}(t)[b, b^{-1}], \sigma)$ with $\sigma(h) = h + \frac{1}{t+1}$. In this ring we are now in the position to represent $F_i(j) = F(k+i-1, j)$ with $f_i = \prod_{l=1}^{i-1} \frac{k+l}{k-j+l} b\, h^2$ for $i = 1, 2, \dots$. First we will solve Problem RPT with the simplest variant RPT$_1$. We start with $d = 0, 1, \dots$ and are successful with $d = 5$: `Sigma` computes $c_1 = -8(1+k)(3+k)$, $c_2 = 4\big(29 + 25k + 5k^2\big)$, $c_3 = -2(8+3k)(10+3k)$, $c_4 = 86 + 49k + 7k^2$, $c_5 = -(4+k)^2$ and

$$g = b\Big(\tfrac{(1+k)(2+k)(3+k)}{(1-t+k)(2-t+k)(3-t+k)} - \tfrac{2(1+k)(10t-6t^2+6tk-2t^2k+tk^2)}{(1-t+k)(2-t+k)(3-t+k)} h \\ + \tfrac{t^2(1+k)\big(56-56t+12t^2+42k-30tk+4t^2k+11k^2-4tk^2+k^3\big)}{(-4+t-k)(-3+t-k)(-2+t-k)(-1+t-k)} h^2\Big)$$

such that (5) holds. Reinterpreting $b$ and $h$ as $\binom{k}{j}$ and $S_1(j)$, yields a solution of (1) for $F_i(j)$. Finally, summing this relation over $j$ from 0 to $k$ and taking care of compensating terms yields the linear recurrence relation

$$X(4+k) = -\tfrac{8(1+k)(3+k)}{(4+k)^2} X(k) + \tfrac{4(29+25k+5k^2)}{(4+k)^2} X(1+k) \\ - \tfrac{2(8+3k)(10+3k)}{(4+k)^2} X(2+k) + \tfrac{86+49k+7k^2}{(4+k)^2} X(3+k) + \tfrac{1}{(4+k)^2}. \quad (6)$$

RPT$_2$ will not contribute to a shorter recurrence. However, applying RPT$_3$ `Sigma` finds for $d = 2$ the solution $c_1 = -4(1+k)$, $c_2 = 2(3+2k)$, $c_3 = -2-k$ and

$$g = -(1+k)(2+k)\tau + \Big(-\tfrac{2(1+k)}{-1+t-k} h + \tfrac{t(1+k)(-2+2t-k)}{(-2+t-k)(-1+t-k)} h^2\Big) b$$

within the $\Sigma$-extension $(\mathbb{K}(t)[b,b^{-1}][h][\tau],\sigma)$ of $(\mathbb{K}(t)[b,b^{-1}][h],\sigma)$ with $\sigma(\tau)=\tau+\frac{b}{(1+t)^2(1+k-t)}$. Interpreting $\tau$ as the sum $s(k,j)=\sum_{i=1}^{j}\frac{\binom{k}{i}}{i(1-i+k)(2-i+k)}$ and performing similar steps as above, Sigma produces the linear recurrence

$$-4(1+k)X(k)+2(3+2k)X(1+k)+(-2-k)X(2+k)\\ =\frac{-5-5k-k^2}{(1+k)(2+k)}-(1+k)(2+k)\sum_{i=1}^{k}\frac{\binom{k}{i}}{i(1-i+k)(2-i+k)}. \tag{7}$$

Note that the sum $s(k)=s(k,k)$ on the right hand side is given in a form that is not indefinite nested and thus cannot be represented automatically in terms of an $R\Pi\Sigma$-extension. However, the sum $s(k)$ can be simplified further. Applying $\text{RPT}_1$ Sigma computes the linear recurrence

$$-2(1+k)(2+k)s(k)+(2+k)(10+3k)s(1+k)-(4+k)^2s(2+k)=\frac{4-k-k^2}{(1+k)(2+k)(3+k)}$$

and solves the recurrence in terms of d'Alembertian solutions:

$$\{\tfrac{c_1}{(1+k)(2+k)}+c_2\Big[\tfrac{(4+3k)2^{-2+k}}{(1+k)^2(2+k)^2}+\tfrac{S_1(2,k)}{8(1+k)(2+k)}\Big]-\tfrac{2(3+2k)}{(1+k)^2(2+k)^2}-\tfrac{S_1(k)}{(1+k)(2+k)}\mid c_1,c_2\in\mathbb{K}\}.$$

Finally, taking the two initial values $s(1)=\frac{1}{2}$ and $s(2)=\frac{7}{12}$ determines $c_1=-1$ and $c_2=8$ so one gets

$$s(k)=-\tfrac{8+7k+k^2}{(1+k)^2(2+k)^2}+\tfrac{(4+3k)2^{1+k}}{(1+k)^2(2+k)^2}-\tfrac{1}{(1+k)(2+k)}S_1(k)+\tfrac{1}{(1+k)(2+k)}S_1(2,k). \tag{8}$$

Thus recurrence (7) can be simplified to

$$X(2+k)=-\tfrac{4(1+k)}{2+k}X(k)+\tfrac{2(3+2k)}{2+k}X(1+k)+\tfrac{-3-2k}{(1+k)(2+k)^2}+\tfrac{2^{1+k}(4+3k)}{(1+k)(2+k)^2}+\tfrac{S_1(k)}{-2-k}-\tfrac{S_1(2,k)}{-2-k}; \tag{9}$$

we emphasize that the sums in the inhomogeneous part of (9) are now all indefinite nested and can be rephrased in an $R\Pi\Sigma$-extension. We note that we can solve this recurrence (or alternatively the recurrence (6)) again by solving the recurrence in terms of d'Alembertian solutions using Sigma. This finally enables us to find the closed form representation

$$X(k)=2^k\Big(-2S_1(k)S_1\big(\tfrac{1}{2},k\big)-2S_2\big(\tfrac{1}{2},k\big)+3S_{1,1}\big(\tfrac{1}{2},1,k\big)\\ -S_{1,1}\big(\tfrac{1}{2},2,k\big)+S_1(k)^2+S_2(k)\Big). \tag{10}$$

Later we will consider the sum $S(n)=\sum_{k=0}^{n}\binom{n}{k}X(k)$ with $X(k)=\sum_{j=0}^{k}\binom{k}{j}S_1(j)^2$ and aim at computing a linear recurrence in $n$. One option is to take the representation (10) and to use one of the tactics from Remark 2.1—this is our usual strategy from [60] to tackle such sums. In Example 2.5 below we will follow an alternative strategy. Instead of working with the zero-order recurrence (10) we will work with the higher-order recurrences (6) or (9). The advantage will be to work in a smaller $R\Pi\Sigma$-extension and encoding parts of the expression (10) within the recurrence operator. In order to accomplish this new strategy, we introduce and explore higher-order extensions in the next subsection.

## 2.2 Higher Order Linear Extensions

So far we have considered indefinite nested sums of the form $S(k) = \sum_{i=l}^{k} F(i)$ and products of the form $P(k) = \prod_{i=l}^{k} F(i)$ with $l \in \mathbb{N}$ that can be encoded by the first order homogeneous recurrences $S(k+1) = S(k) + F(k+1)$ and $P(k+1) = F(k+1)\, P(k)$, respectively. However, many interesting summation objects can be only described by higher-order recurrences, like Legendre polynomials, Hermite polynomials, or Bessel functions. More precisely, we are interested in dealing with a sequence $X(k)$ which satisfies a linear recurrence

$$X(k+s+1) = A_0(k)\, X(k) + A_1(k)\, X(k+1) + \cdots + A_s(k)\, X(k+s) + A_{s+1}(k) \tag{11}$$

where the sequences $A_i(k)$ $(1 \leqslant i \leqslant s+1)$ are expressible in a difference ring $(\mathbb{A}, \sigma)$.

*Remark 2.2* Sequences that satisfy (11) are also called holonomic. Specializing to the case that the $A_i(k)$ with $0 \leqslant i \leqslant s+1$ are elements of the rational or $q$-rational difference field, many important properties have been elaborated [34, 35, 42, 43, 45, 52, 68].

Then the summation object $X(k)$ with the recurrence relation (11) can be represented in a higher order difference ring extension as follows [54].

**Definition 2.4** A *higher-order linear difference ring extension* (in short *h.o.l. extension*) $(\mathbb{E}, \sigma)$ of a difference ring $(\mathbb{A}, \sigma)$ is a polynomial ring extension $\mathbb{E} = \mathbb{A}[x_0, \ldots, x_s]$ with the variables $x_0, \ldots, x_s$ and the automorphism $\sigma : \mathbb{E} \to \mathbb{E}$ is extended from $\mathbb{A}$ to $\mathbb{E}$ subject to the relations $\sigma(x_i) = x_{i+1}$ for $0 \leqslant i < s$ and

$$\sigma(x_s) = a_0\, x_0 + a_1\, x_1 + \cdots + a_s\, x_s + a_{s+1} \tag{12}$$

for some $a_0, \ldots, a_{s+1} \in \mathbb{A}$. $s+1$ is also called the extension order or recurrence order.

Namely, if we rephrase $X(k)$ as $x_0$, then $X(k+1)$ corresponds to $\sigma(x_0) = x_1$, $X(k+2)$ corresponds to $\sigma(x_1) = x_2$, etc. Finally, $X(k+s)$ corresponds to $x_s$ and the relation (11) is encoded by (12).

Concerning concrete computations, we usually start with a $\Pi\Sigma$-field $(\mathbb{G}, \sigma)$ (in particular as defined in Example 2.1) in which the $A_i(k)$ are encoded by $a_i \in \mathbb{G}$.

Further, we assume that $A_{s+1}(k)$ can be rephrased as $a_{s+1}$ in an $R\Pi\Sigma$-extension $(\mathbb{A},\sigma)$ of $(\mathbb{G},\sigma)$. Then we construct the h.o.l. extension $(\mathbb{H},\sigma)$ of $(\mathbb{A},\sigma)$ with $\mathbb{H}=\mathbb{A}[x_0,\dots,x_s]$ and (12) with $a_0,\dots,a_s\in\mathbb{G}$ and $a_{s+1}\in\mathbb{A}$.

In the following we will work out summation algorithms that tackle Problem RPS in $(\mathbb{H},\sigma)$ with $f_i\in\mathbb{A}\,x_0+\cdots+\mathbb{A}\,x_s+\mathbb{A}$ for $1\leqslant i\leqslant d$ and $g\in\mathbb{E}\,x_0+\cdots+\mathbb{E}\,x_s+\mathbb{E}$ for an appropriate difference ring extension $(\mathbb{E},\sigma)$ of $(\mathbb{A},\sigma)$. To warm up we will first focus on the following basic telescoping problem.

*Given* $f\in\mathbb{H}$ with $f=f_0\,x_0+\cdots+f_s\,x_s+f_{s+1}$ where $f_i\in\mathbb{A}$ for $0\leqslant i\leqslant s+1$. *Find*, if possible, $g=g_0\,x_0+\cdots+g_s\,x_s+g_{s+1}$ with (13) where $g_i\in\mathbb{A}$ for $0\leqslant i\leqslant s+1$.

In order to tackle this problem (and more generally Problem RPT below) we rely on the following lemma that follows immediately by [54, Lemmas 1 and 2]; the proof is based on coefficient comparison.

**Lemma 2.1** *Let* $(\mathbb{A}[x_0,\dots,x_s],\sigma)$ *be a h.o.l. extension of* $(\mathbb{A},\sigma)$ *with* (12). *Let* $f=f_0\,x_0+\cdots+f_s\,x_s+f_{s+1}$ *with* $f_i\in\mathbb{A}$ *and* $g=g_0\,x_0+\cdots+x_s\,x_s+g_{s+1}$ *with* $g_i\in\mathbb{A}$. *Then*

$$\sigma(g)-g=f \tag{13}$$

*if and only if the following equations hold:*

$$\sum_{j=0}^{s}\sigma^{s-j}(a_j)\sigma^{s-j+1}(g_s)-g_s=\sum_{j=0}^{s}\sigma^{s-j}(f_j), \tag{14}$$

$$\sigma(g_{s+1})-g_{s+1}=f_{s+1}-a_{s+1}\,\sigma(g_s), \tag{15}$$

$$g_0=a_0\,\sigma(g_s)-f_0, \tag{16}$$

$$g_i=\sigma(g_{i-1})+a_i\,\sigma(g_s)-f_i,\quad (0\leqslant i<s). \tag{17}$$

Namely, suppose that we succeed in computing $g_s\in\mathbb{A}$ and $g_{s+1}\in\mathbb{A}$ with (14) and (15). Then we can compute $g_0,\dots,g_{s-1}\in\mathbb{A}$ using (16) and (17), and by Lemma 2.1 it follows that $g=g_0\,x_0+\cdots+x_s\,x_s+g_{s+1}$ is a solution of (13).

*Remark 2.3* If $s=0$, constraint (14) is nothing else than $\sigma(g_s)-g_s=0$ which gives the solution $g_s=1$. Hence what remains is constraint (15) which reduces to $\sigma(g_{s+1})-g_{s+1}=f_{s+1}$. In other words, in this special case Lemma 2.1 boils down to the telescoping problem in $(\mathbb{A},\sigma)$.

*Example 2.3* Consider the sum $S(n)=\sum_{k=0}^{n}F(k)$ with the summand $F(k)=\frac{X(k)}{2^k}$ where the sequence $X(k)$ is determined by the recurrence

$$X(2+k)=-\tfrac{4(1+k)}{2+k}X(k)+\tfrac{2(3+2k)}{2+k}X(1+k)-\tfrac{1}{2+k} \tag{18}$$

and the initial values $X(0)=0$, $X(1)=-1$. We take the rational difference field $(\mathbb{G},\sigma)$ with $\mathbb{G}=\mathbb{Q}(t)$ and $\sigma(t)=t+1$ and construct the $\Pi$-extension $(\mathbb{A},\sigma)$

of $(\mathbb{G},\sigma)$ with $\mathbb{A}=\mathbb{G}[p,p^{-1}]$ and $\sigma(p)=2\,p$. Finally, we construct the h.o.l. extension $(\mathbb{H},\sigma)$ of $(\mathbb{A},\sigma)$ with $\sigma(x_2)=-\frac{4(1+t)}{2+t}x_0+\frac{2(3+2t)}{2+t}x_1-\frac{1}{2+t}$ and search for $g=g_0\,x_0+g_1\,x_1+g_2$ with $g_i\in\mathbb{A}$ such that $\sigma(g)-g=\frac{x_0}{p}$ holds. The constraint (14) of Lemma 2.1 reads as $-\frac{4(2+t)}{3+t}\sigma^2(g_1)+\frac{2(3+2t)}{2+t}\sigma(g_1)-g_1=\frac{1}{2p}$. Using `Sigma` we compute $g_1=-\frac{(-1+t)(1+t)}{2p}\in\mathbb{A}$ and we get the constraint (compare (15)): $\sigma(g_2)-g_2=-\frac{t}{4p}$. Solving this telescoping equation gives $g_2=\frac{1+t}{2\,p}\in\mathbb{A}$. Further, using (16), we obtain $g_0=\frac{-1+t+t^2}{p}$. Reinterpreting $g$ in terms of our summation objects yields $G(k)=\frac{-1+k+k^2}{2^k}X(k)-\frac{(-1+k)(1+k)}{2\,2^k}X(k+1)+\frac{1+k}{2\,2^k}$ with $\frac{X(k)}{2^k}=G(k+1)-G(k)$. Finally, summing this equation over $k$ from 0 to $n$ produces

$$S(n)=\frac{1+n}{2^{1+n}}\Big(1+2\,n\,X(n)+(1-n)\,X(1+n)\Big).$$

*Remark 2.4* More generally, multivariate sequences are often described by a system of homogeneous linear recurrences with coefficients from the difference field $(\mathbb{K}(t),\sigma)$ with $\sigma(t)=t+1$ or with $\sigma(t)=q\,t$. Then the telescoping problem with $\mathbb{A}=\mathbb{K}(t)$ and more generally, the parameterized telescoping problem, can be treated in this setting using the holonomic system approach [68]. In this regard, Chyzak's fast algorithm [34] was a major breakthrough that has been improved further in [44]. Lemma 2.1 specializes to one linear recurrence (and does not treat a system in the multivariate sequence case). However, it dispenses the user to work with Gröbner bases and expensive uncoupling procedures [32, 70] that are needed in the standard approaches [34, 44]. In particular, the constraints (14) and (16) have been worked out explicitly which will be the basis for further explorations. An extra bonus is the treatment of inhomogeneous recurrences that will be utilized below.

We remark further that a special case of Lemma 2.1 can be also related to [14].

Suppose that the summand $F(k)$ can be rephrased by $f$ in a difference ring $(\mathbb{A},\sigma)$ as constructed above. In most applications, one will fail to find a telescoping solution for $f$ in $\mathbb{A}$. To gain more flexibility, we will consider two strategies.

(I) Try to extend the difference ring $(\mathbb{A},\sigma)$ with a simple $R\Pi\Sigma$-extension in which one finds a telescoping solution.

(II) In case that the summand $F(k)$ contains an extra parameter, say $F(k)=F(n,k)$, utilize the creative telescoping paradigm with $F_i(k)=F(n+i-1,k)$ for $1\leqslant i\leqslant d$.

As it turns out below, the successful application of strategy I can be connected to the problem of finding constants in a difference ring or equivalently to construct higher-order extensions with smaller recurrence order. In Sect. 2.2.1 we will provide a constructive theory that enables one compute such constants and thus to find improved higher order extensions in the setting of simple $R\Pi\Sigma$-extensions. Based on this insight, we will propose in Sect. 2.2.2 our algorithm to solve the parameterized telescoping problem in $(\mathbb{A},\sigma)$ or in a properly chosen simple $R\Pi\Sigma$-extension of it. In a nutshell, we will combine strategies (I) and (II) that will lead to efficient and flexible algorithms to tackle indefinite and definite summation problems.

### 2.2.1 Finding Constants or Finding Recurrences with Lower Order

We are interested in the following problem.

**Problem C for** $(\mathbb{A}, \sigma)$**:** Find a linear constant.
*Given* a h.o.l. extension $(\mathbb{A}[x_0, \dots, x_s], \sigma)$ of $(\mathbb{A}, \sigma)$ with (12) where $a_0, \dots, a_{s+1} \in \mathbb{A}$ .
*Find* a $g = g_0\, x_0 + \dots + g_s\, x_s + g_{s+1} \in \mathbb{A}[x_0, \dots, x_s] \setminus \mathbb{A}$ with $g_i \in \mathbb{A}$ (or in an appropriate extension of it) such that $\sigma(g) = g$ holds.

Setting $f_i = 0$ for $0 \leqslant i \leqslant s + 1$ in Lemma 2.1 yields a basic strategy for Problem C.

**Lemma 2.2** *Let* $(\mathbb{H}, \sigma)$ *be a h.o.l. extension of* $(\mathbb{A}, \sigma)$ *with* $\mathbb{H} = \mathbb{A}[x_0, \dots, x_s]$ *and* (12) *where* $a_i \in \mathbb{A}$*. Then there exists a* $g = g_0\, x_0 + \dots + g_s\, x_s + g_{s+1} \in \mathbb{H} \setminus \mathbb{A}$ *with* $g_i \in \mathbb{A}$ *such that* $\sigma(g) = g$ *if and only if there is a* $g \in \mathbb{A} \setminus \{0\}$ *with*

$$\sum_{j=0}^{s} \sigma^{s-j}(a_j)\sigma^{s-j+1}(h) - h = 0 \tag{19}$$

*and a* $\gamma \in \mathbb{A}$ *with*

$$\sigma(\gamma) - \gamma = -a_{s+1}\, \sigma(h). \tag{20}$$

*In this case we can set* $g_s = h$*,* $g_{s+1} = \gamma$*,* $g_0 = a_0\, \sigma(g_s)$ *and* $g_i = \sigma(g_{i-1}) + a_i\, \sigma(h)$ *for* $1 \leqslant i < s$*.*

In other words, if one finds an $h \in \mathbb{A} \setminus \{0\}$ with (19) and a $\gamma \in \mathbb{A}$ with (20), one can compute a $g \in \mathbb{H} \setminus \mathbb{A}$ with $\sigma(g) = g$.

**From constants to recurrences with smaller order**. Now suppose that we find such a $g \in \mathbb{H} \setminus \mathbb{A}$ with $\sigma(g) = g$ and reinterpret $g$ as

$$G(k) = G_0(k)X(k) + \dots + G_s(k)X(k+s) + G_{s+1}(k)$$

where we rephrase for $0 \leqslant i \leqslant s + 1$ the $g_i$ in terms of our summation objects yielding the expression $G_i(k)$. Suppose that $G(k + 1) = G(k)$ holds for all $k \in \mathbb{N}$ with $k \geqslant \lambda$ for some $\lambda$ chosen big enough. Evaluating $c := G(\lambda) \in \mathbb{K}$ with our given sequence $X(k)$ gives the identity $G(k) = c$. In other words, we find the new linear recurrence

$$G_0(k)X(k) + \dots + G_s(k)X(k+s) = c - G_{s+1}(k) \tag{21}$$

with order $s$; note that so far we used the recurrence (11) to model the object $X(k)$ which has order $s + 1$. Now suppose that $g_s \in \mathbb{A}^*$ holds. Then we can define the h.o.l. extension $(\mathbb{A}[y_0, \dots, y_{s-1}], \sigma)$ of $(\mathbb{A}, \sigma)$ with $\sigma(y_{s-1}) = g'_0\, y_0 + \dots + g'_{s-1}\, y_{s-1} + g'_s$ with $g'_i = -\frac{g_i}{g_s}$ for $0 \leqslant i < s$ and $g'_s = \frac{c-g_{s+1}}{g_s}$.

Summarizing, finding a constant indicates that the recurrence used to describe the object $X(k)$ is not optimal. But given such a constant also enables one to cure the situation. One can derive a recurrence that models $X(k)$ with a smaller order.

Before we look at a concrete application in Example 2.4 below, we will work out the different possible scenarios to hunt for constants. So far we considered

**Case 1.1**: there is an $h \in \mathbb{A} \setminus \{0\}$ with (19) and a $\gamma \in \mathbb{A}$ with (20). Then we activate Lemma 2.2 and find

$$g = g_0\, x_0 + \cdots + g_s\, x_s + g_{s+1} \tag{22}$$

with $g_i \in \mathbb{A}$ for all $0 \leqslant i \leqslant s+1$, $g_s \neq 0$, with $\sigma(g) = g$.

It might happen that one finds an $h \in \mathbb{A} \setminus \{0\}$ with (19) but one fails to find a $\gamma \in \mathbb{A}$ with (20). This situation can be covered as follows.

**Case 1.2**: There is no $\gamma \in \mathbb{A}$ with (20). By part (1) of Theorem 2.1 we can construct a $\Sigma$-extension $(\mathbb{A}[\tau], \sigma)$ of $(\mathbb{A}, \sigma)$ with $\sigma(\tau) = \tau - a_{s+1}\, \sigma(h)$ and can put on top our h.o.l. extension $(\mathbb{A}[\tau][x_0, \ldots, x_s], \sigma)$ of $(\mathbb{A}[\tau], \sigma)$ with (12). Then by Lemma 2.2 we get

$$g = g_0\, x_0 + \cdots + g_s\, x_s + \tau \tag{23}$$

with $g_i \in \mathbb{A}$ for $0 \leqslant i \leqslant s$ such that $\sigma(g) = g$ holds.

Now let us tackle the case that there is no $h \in \mathbb{A}$ with (19) but there is such an $h$ in a simple $R\Pi\Sigma$-extension. More precisely, we assume that $(\mathbb{A}, \sigma)$ itself is a simple $R\Pi\Sigma$-extension of $(\mathbb{G}, \sigma)$ and that for (12) we have that $a_0, \ldots, a_s \in \mathbb{G}$ and $a_{s+1} \in \mathbb{A}$. In this setting, suppose that there is a $\mathbb{G}$-simple $R\Pi\Sigma$-extension $(\mathbb{E}, \sigma)$ of $(\mathbb{A}, \sigma)$ in which one finds an $h \in \mathbb{E}$ with (19). Note that $(\mathbb{E}, \sigma)$ is a simple $R\Pi\Sigma$-extension of $(\mathbb{G}, \sigma)$. Then we can apply the following result. A simpler field version can be found in [53, Lemma 4.5.3]; for the Liouvillian case with $\mathbb{A} = \mathbb{K}(t)$ and $\sigma(t) = t + 1$ we refer to [38, Theorem 5.1].

**Proposition 2.1** *Let $(\mathbb{A}, \sigma)$ with $\mathbb{A} = \mathbb{G}\langle \tilde{t}_1 \rangle \ldots \langle \tilde{t}_{\tilde{u}} \rangle [\tilde{\tau}_1] \ldots [\tilde{\tau}_{\tilde{v}}]$ be a simple $R\Pi\Sigma$-extension of $(\mathbb{G}, \sigma)$, and let $(\mathbb{E}, \sigma)$ with $\mathbb{E} = \mathbb{A}\langle t_1 \rangle \ldots \langle t_u \rangle [\tau_1] \ldots [\tau_v]$ be a $\mathbb{G}$-simple $R\Pi\Sigma$-extension of $(\mathbb{A}, \sigma)$ where the $\tilde{t}_i, t_i$ are $R\Pi$-monomials and the $\tilde{\tau}_i, \tau_i$ are $\Sigma$-monomials. Let $f \in \mathbb{A}$ and $a_0, \ldots, a_s \in \mathbb{G}$. Suppose there is a $g \in \mathbb{E} \setminus \{0\}$ with*

$$a_s\, \sigma^s(g) + \cdots + a_0\, g = f. \tag{24}$$

1. *If $f = 0$ or $g \notin \mathbb{A}\langle t_1 \rangle \ldots \langle t_u \rangle$, then there are $l_i, \tilde{l}_i \in \mathbb{Z}$ and $w \in \mathbb{G}^*$ such that for $h = w\, \tilde{t}_1^{\tilde{l}_1} \ldots \tilde{t}_{\tilde{u}}^{\tilde{l}_{\tilde{u}}} t_1^{\tilde{l}_1} \ldots t_u^{l_u}$ we have*

$$a_s\, \sigma^s(h) + \cdots + a_0\, h = 0. \tag{25}$$

2. *Otherwise, if $f \neq 0$ and $g \in \mathbb{A}\langle t_1 \rangle \ldots \langle t_u \rangle$, there is also a solution of (24) in $\mathbb{A}$.*

*Proof* (1) Set $\mathbb{E}_j = \mathbb{A}\langle t_1 \rangle \ldots \langle t_u \rangle [\tau_1] \ldots [\tau_j]$ for $0 \leqslant j \leqslant v$. First we show that there is an $h \in \mathbb{E} \setminus \{0\}$ with (25). If $f = 0$, this holds by assumption. Otherwise, we can conclude that there is a $g \in \mathbb{E} \setminus \mathbb{E}_0$ with (24) again by assumption. Now take among all the possible $g$ with (24) an element $g \in \mathbb{E}_i \setminus \mathbb{E}_{i-1}$ where $i > 0$ is minimal. Then $g = h\, \tau_i^m + b$ for some $m > 0$ and $h, b \in \mathbb{E}_{i-1}$ with $h \neq 0$. By coefficient comparison w.r.t. $\tau_i$ in (24) and using the fact that $\sigma(\tau_i) - \tau_i \in \mathbb{E}_{i-1}$ and $a_i \in \mathbb{G}$, we conclude that $h$ is a solution of (25). Hence in any case there is an $h \in \mathbb{E} \setminus \{0\}$ with (25).

We can reorder $(\mathbb{E}, \sigma)$ to $\mathbb{E} = \mathbb{H}[s_1] \dots [s_e]$ with $\mathbb{H} = \mathbb{G}\langle \tilde{t}_1 \rangle \dots \langle \tilde{t}_{\tilde{u}} \rangle \langle t_1 \rangle \dots \langle t_u \rangle$ where $(s_1, \dots, s_{v+\tilde{v}}) = (\tilde{\tau}_1, \dots, \tilde{\tau}_{\tilde{v}}, \tau_1, \dots, \tau_v)$. Set $\mathbb{E}'_j = \mathbb{H}[s_1] \dots [s_j]$. Suppose there is no such $h$ with $h \in \mathbb{H} \setminus \{0\}$. Then we can choose among all the possible solutions $h$ with (25) an element $h \in \mathbb{E}'_k \setminus \mathbb{E}'_{k-1}$ with $k > 0$ being minimal. We can write $h = \alpha\, s_k^{\mu} + \beta$ with $\mu > 0$ and $\alpha, \beta \in \mathbb{E}'_{k-1}$ where $\alpha \neq 0$. Doing coefficient comparison w.r.t. $s_k^{\mu}$ in (25), using $\sigma(s_k) - s_k \in \mathbb{E}'_{k-1}$ and knowing that $a_0, \dots, a_s \in \mathbb{G}$, we conclude that $\alpha$ is a solution of (25); a contradiction to the minimality of $k$. Summarizing, we can find $h \in \mathbb{H} \setminus \{0\}$ with (25). Now write

$$h = \sum_{(\tilde{l}_1, \dots, \tilde{l}_{\tilde{u}}, l_1, \dots, l_u) \in S} h_{(\tilde{l}_1, \dots, \tilde{l}_{\tilde{u}}, l_1, \dots, l_u)} \tilde{t}_1^{\tilde{l}_1} \dots \tilde{t}_{\tilde{u}}^{\tilde{l}_u} t_1^{l_1} \dots t_u^{l_u}$$

for a finite set $S \subseteq \mathbb{Z}^{\tilde{u}+u}$ and $h_{(\tilde{l}_1, \dots, l_u)} \in \mathbb{G}$. Since $h \neq 0$, we can take $w = h_{(\tilde{l}_1, \dots, l_u)} \in \mathbb{G}^*$ for some $(\tilde{l}_1, \dots, l_u) \in S$.

By coefficient comparison it follows that $h' = w\, t_1^{\tilde{l}_1} \dots t_u^{l_u} \neq 0$ is a solution of (25).
(2) Let $f \in \mathbb{A} \setminus \{0\}$ and $g \in \mathbb{A}\langle t_1 \rangle \dots \langle t_u \rangle$ with (24). Note that we can write $g = \sum_{(l_1, \dots, l_u) \in S} h_{(l_1, \dots, l_u)} t_1^{l_1} \dots t_u^{l_u}$ for a finite set $S \subseteq \mathbb{Z}^u$ and $h_{(l_1, \dots, l_u)} \in \mathbb{A}$. By coefficient comparison w.r.t. $t_1^0 \dots t_u^0$ in (24) we conclude that for $h = h_{(0, \dots, 0)} \in \mathbb{A} \setminus \{0\}$ the equation $a_s\, \sigma^s(h) + \dots + a_0\, h = f$ holds. □

We will reformulate Proposition 2.1 for homogeneous difference equations to Corollary 2.1 by using the following lemma; for a simpler version see [55, Proposition 6.13].

**Lemma 2.3** *Let* $(\mathbb{E}, \sigma)$ *with* $\mathbb{E} = \mathbb{A}\langle t_1 \rangle \dots \langle t_e \rangle$ *be a simple* $R\Pi$*-extension of* $(\mathbb{A}, \sigma)$ *with* $\alpha_i := \frac{\sigma(t_i)}{t_i} \in \mathbb{A}^*$*, and let* $l_i \in \mathbb{Z}$ *such that* $t^{l_1} \dots t^{l_e} \notin \mathbb{A}$*. Then there exists an* $R\Pi$*-extension* $(\mathbb{A}\langle t \rangle, \sigma)$ *of* $(\mathbb{A}, \sigma)$ *with* $\sigma(t) = \alpha\, t$ *where* $\alpha = \prod_{i=1}^{e} \alpha_i^{l_i}$.

*Proof* Define $M = \{i \mid l_i \neq 0\}$ and set $h := t^{l_1} \dots t^{l_e}$. Note that $\sigma(h) = \alpha\, h$. First, suppose that $\alpha^n = 1$ for some $n > 0$. Then for all $i \in M$, $t_i$ is an $R$-monomial. In particular, since the $\alpha_i$ are roots of unity, $\alpha$ is a root of unity. Let $m > 0$ be minimal such that $\alpha^m$. If $m = 1$, then $\alpha = 1$ thus $\sigma(h) = h$, and consequently $h \in \mathrm{const}_{\sigma}\mathbb{A}$. Therefore $h = 1$, a contradiction. Thus $\alpha$ is a primitive $m$th root of unity with $m > 1$. Now construct the $A$-extension $(\mathbb{A}[t], \sigma)$ of $(\mathbb{A}, \sigma)$ with $\sigma(t) = \alpha\, t$ and suppose that there is a $k \in \mathbb{N}$ with $0 < k < m$ and $g \in \mathbb{A} \setminus \{0\}$ such that $\sigma(g) = \alpha^k\, g$. We can find an $r$ such that $t_r^{l_r\, k} \neq 1$ (otherwise $h^k = 1$, thus $1 = \sigma(h^k) = \alpha^k\, h^k = \alpha^k$ and thus $m$ is not minimal with $\alpha^m = 1$). But this implies that $\alpha_r^{l_r\, k} \neq 1$ (otherwise $t_r^{l_r\, k} \in \mathrm{const}_{\sigma}\mathbb{E} \setminus \mathbb{A}$, but $\mathrm{const}_{\sigma}\mathbb{E} = \mathrm{const}_{\sigma}\mathbb{A}$). Choose $r \geqslant 1$ to be maximal with this property, and let $u > 1$ be minimal such that $\alpha_r^u = 1$. Then we can find $k'$ with $1 \leqslant k' < u$ with $\alpha_r^{l_r\, k} = \alpha_r^{k'}$. Further, with $\tilde{h} = g/(t_1^{l_1} \dots t_{r-1}^{l_{r-1}})^k \in \mathbb{A}\langle t_1 \rangle \dots \langle t_{r-1} \rangle$ we get $\sigma(\tilde{h}) = \alpha_r^{l_r\, k}\, \tilde{h}$. Hence $t_r$ is not an $R$-monomial by part 3 of Theorem 2.1: a contradiction.

Otherwise, suppose that there is no $n > 0$ with $\alpha^n = 1$. Then there is at least one $i \in M$ such that $t_i$ is a $\Pi$-monomial. W.l.o.g. suppose that $t_r$ is a $\Pi$-monomial with $\max(M) = r$; otherwise we reorder the generators accordingly. Suppose that the $P$-extension $(\mathbb{A}\langle t\rangle, \sigma)$ of $(\mathbb{A}, \sigma)$ with $\sigma(t) = \alpha\, t$ is not a $\Pi$-extension. Then there is a $k \in \mathbb{Z} \setminus \{0\}$ and $g \in \mathbb{A}\langle t_1\rangle \dots \langle t_{r-1}\rangle \setminus \{0\}$ with $\sigma(g) = \alpha^k\, g$. Define $\tilde{h} = g/(t_1^{l_1} \dots t_{r-1}^{l_{r-1}})^k$. Then, as above, $\sigma(\tilde{h}) = \alpha_r^{l_r\,k}\, \tilde{h}$ with $l_r\, k \neq 0$ and consequently $t_r$ is not a $\Pi$-monomial by part (2) of Theorem 2.1, a contradiction. □

**Corollary 2.1** *Let $(\mathbb{A}, \sigma)$ be a simple $R\Pi\Sigma$-extension of $(\mathbb{G}, \sigma)$ with $a_0, \dots, a_s \in \mathbb{G}$. If there is a $\mathbb{G}$-simple $R\Pi\Sigma$-extension $(\mathbb{E}, \sigma)$ of $(\mathbb{A}, \sigma)$ in which one finds an $h \in \mathbb{E} \setminus \mathbb{A}$ with (25), then there is an $R\Pi$-extension $(\mathbb{A}\langle t\rangle, \sigma)$ of $(\mathbb{A}, \sigma)$ with $\frac{\sigma(t)}{t} \in \mathbb{G}^*$ in which one finds a solution of (25) with $h = w\, t^m$ where $m \in \mathbb{Z} \setminus \{0\}$ and $w \in \mathbb{G} \setminus \{0\}$.*

*Proof* Suppose there is a $\mathbb{G}$-simple $R\Pi\Sigma$-extension $(\mathbb{E}, \sigma)$ of $(\mathbb{A}, \sigma)$ as in Proposition 2.1 in which we find an $h \in \mathbb{E} \setminus \{0\}$ with (25). Then by part 1 of Proposition 2.1 we can find an $h' = w\, \tilde{t}_1^{\tilde{l}_1} \dots \tilde{t}_u^{\tilde{l}_u}\, t_1^{l_1} \dots t_u^{l_u} \notin \mathbb{A}$ with $\tilde{l}_i, l_i \in \mathbb{Z}$ and $w \in \mathbb{G} \setminus \{0\}$ with (25) (where $h$ is replaced by $h'$). Set $a = \tilde{t}_1^{\tilde{l}_1} \dots \tilde{t}_u^{\tilde{l}_u}\, t_1^{l_1} \dots t_u^{l_u}$ and define $\alpha := \frac{\sigma(a)}{a} \in \mathbb{G}^*$. Then we can construct the $R\Pi$-extension $(\mathbb{A}\langle t\rangle, \sigma)$ of $(\mathbb{A}, \sigma)$ with $\sigma(t) = \alpha\, t$ by Lemma 2.3. By construction it follows that $h'' = w\, t$ is also a solution of (25). □

With these new results in $R\Pi\Sigma$-theory, we can continue to tackle Problem C. Recall that we assume that there is no $h \in \mathbb{A}$ with (19), but there exists a $\mathbb{G}$-simple $R\Pi\Sigma$-extension in which we find such an $h$. Then by Corollary 2.1 there is also an $R\Pi$-extension $(\mathbb{A}\langle t\rangle, \sigma)$ of $(\mathbb{A}, \sigma)$ with $\frac{\sigma(t)}{t} \in \mathbb{G}^*$ and $h \in \mathbb{A}\langle t\rangle$ with $h = w\, t^m$ where $m \in \mathbb{Z} \setminus \{0\}$ and $w \in \mathbb{G} \setminus \{0\}$ such that (19) holds. As above, we can consider two cases.

**Case 2.1**: Suppose that we find a $\gamma \in \mathbb{A}\langle t\rangle$ with (20). Then with Lemma 2.2 we get $g = g_0\, x_0 + \dots + g_s\, x_s + g_{s+1}$ with $g_i \in \mathbb{A}\langle t\rangle$ such that $\sigma(g) = g$ holds. Even more, looking at the construction it follows for $0 \leqslant i \leqslant s$ that $g_i = g_i'\, t^m$ for some $g_i' \in \mathbb{G}$. Further we can use the following simple lemma.

**Lemma 2.4** *Let $(\mathbb{A}\langle t\rangle, \sigma)$ be an $R\Pi$-extension of $(\mathbb{A}, \sigma)$. Let $f = f'\, t^m$ with $f' \in \mathbb{A}$, $m \neq 0$, and $g \in \mathbb{A}\langle t\rangle$. If $\sigma(g) - g = f$, then $g = g'\, t^m + c$ with $g' \in \mathbb{A}$ and $c \in \mathrm{const}_\sigma \mathbb{A}$.*

*Proof* Let $\alpha = \frac{\sigma(t)}{t} \in \mathbb{A}^*$ and $g = \sum_i g_i\, t^i \in \mathbb{A}\langle t\rangle$. By coefficient comparison it follows that $\alpha^i\, \sigma(g_i) - g_i = 0$ for all $i$ with $i \neq m$. By part (2) of Theorem 2.1 it follows that $g_i = 0$ if $i \neq 0$. Further, $g_i \in \mathrm{const}_\sigma \mathbb{A}$ if $i = 0$. □

Applying this lemma to (20), we conclude that we can choose $g_{s+1} = g_{s+1}'\, t^m$ for some $g_{s+1}' \in \mathbb{A}$ and therefore

$$g = t^m (g_0'\, x_0 + \dots + g_s'\, x_s + g_{s+1}') \tag{26}$$

with $g_i' \in \mathbb{G}$ for $0 \leqslant i \leqslant s$, $g_{s+1}' \in \mathbb{A}$ and $m \in \mathbb{Z} \setminus \{0\}$.

**Case 2.2**: There is no $\gamma \in \mathbb{A}\langle t\rangle$. Then as in Case 1.2 we can construct the $\Sigma$-extension $(\mathbb{A}\langle t\rangle[\tau], \sigma)$ of $(\mathbb{A}\langle t\rangle, \sigma)$ with $\sigma(\tau) = \tau - a_{s+1}\, \sigma(h)$ and get

$$g = t^m(g'_0 x_0 + \cdots + g'_s x_s) + \tau \tag{27}$$

with $m \in \mathbb{Z} \setminus \{0\}$ and $g_i \in \mathbb{G}$ for $0 \leqslant i \leqslant s$.

Summarizing, we obtain the following result.

**Theorem 2.2** *Let $(\mathbb{A}, \sigma)$ be a simple $R\Pi\Sigma$-extension of $(\mathbb{G}, \sigma)$ and let $(\mathbb{H}, \sigma)$ be a h.o.l. extension of $(\mathbb{A}, \sigma)$ with $\mathbb{H} = \mathbb{A}[x_0, \ldots, x_s]$ and (12) where $a_0, \ldots, a_a \in \mathbb{G}$ and $a_{s+1} \in \mathbb{A}$. Suppose that there is an $h$ in a $\mathbb{G}$-simple $R\Pi\Sigma$-extension with (19). Then there is a $\mathbb{G}$-simple $R\Pi\Sigma$-extension $(\mathbb{E}, \sigma)$ of $(\mathbb{A}, \sigma)$ and a h.o.l. extension $(\mathbb{E}[x_1, \ldots, x_s], \sigma)$ of $(\mathbb{E}, \sigma)$ with (12) in which one gets $g \in \mathbb{E}[x_1, \ldots, x_s]$ with $\sigma(g) = g$. In particular, one of the four situations hold.*

*Case 1.1:* (22) *with $g_i \in \mathbb{A}$.*
*Case 1.2:* $(\mathbb{E}, \sigma)$ *is a $\Sigma$-extension of $(\mathbb{A}, \sigma)$ with $\mathbb{E} = \mathbb{A}[\tau]$;* (23) *with $g_i \in \mathbb{A}$.*
*Case 2.1:* $(\mathbb{E}, \sigma)$ *is an $R\Pi$-extension of $(\mathbb{A}, \sigma)$ with $\mathbb{E} = \mathbb{A}\langle t\rangle$ and $\frac{\sigma(t)}{t} \in \mathbb{G}^*$;* (26) *with $g'_i \in \mathbb{G}$ for $0 \leqslant i \leqslant s$ and $g'_{s+1} \in \mathbb{A}$.*
*Case 2.2:* $(\mathbb{E}, \sigma)$ *is an $R\Pi\Sigma$-extension $(\mathbb{A}, \sigma)$ with $\mathbb{E} = \mathbb{A}\langle t\rangle[\tau]$ where $\frac{\sigma(t)}{t} \in \mathbb{G}$ and $\sigma(\tau) - \tau \in \mathbb{A}$;* (27) *with $g'_i \in \mathbb{G}$ for $0 \leqslant i \leqslant s$.*

Let $(\mathbb{A}, \sigma)$ be a simple $R\Pi\Sigma$-extension of a $\Pi\Sigma$-field $(\mathbb{G}, \sigma)$ and suppose that $a_0, \ldots, a_s \in \mathbb{G}$ and $a_{s+1} \in \mathbb{A}$. Then we can tackle problem C as follows.

1. Decide constructively if there is an $h \in \mathbb{A} \setminus \{0\}$ with (19) using the algorithms from [17, 33, 53, 56, 61, 63]. If such an $h$ exists, continue with step 6.
2. Otherwise, decide constructively if there is an $R\Pi$-extension $(\mathbb{G}\langle t\rangle, \sigma)$ of $(\mathbb{G}, \sigma)$ such that we find $h \in \mathbb{G}\langle t\rangle$ with (19). Here one can utilize, e.g., the algorithms given in [22, 50] if $(\mathbb{G}, \sigma)$ is one of the instances (1–3) from Example 2.1. Otherwise, we can utilize the more general algorithms from [17].
3. Check if there is an $h' \in \mathbb{A}$ with $\frac{\sigma(h')}{h'} = \frac{\sigma(t)}{t}$ using the algorithms from [61]. If yes, $h'$ is a solution of (19). Go to step 6 where $h'$ takes over the role of $h$.
4. Check if the $AP$-extension $(\mathbb{A}\langle t\rangle, \sigma)$ of $(\mathbb{A}, \sigma)$ is an $R\Pi$-extension using Theorem 2.1 and applying the algorithms from [61]. If yes, we get the solution $h' = t \in \mathbb{A}\langle t\rangle$ of (19) and we go to step 6 where $h'$ takes over the role of $h$.
5. Try to redesign and extend the difference ring $(\mathbb{A}, \sigma)$ to $(\mathbb{A}', \sigma)$ such that one can find $h \in \mathbb{A}'$ with (19) and such that $(\mathbb{A}', \sigma)$ is an $R\Pi\Sigma$-extension of $(\mathbb{G}, \sigma)$. If $(\mathbb{G}, \sigma)$ is one of the instances (1–3) from Example 2.1, this can be accomplished with the algorithms from [46, 55] in combination with [61]. For a general $\Pi\Sigma$-field $(\mathbb{G}, \sigma)$ our method might fail. Otherwise replace $\mathbb{A}$ by $\mathbb{A}'$ and go to step 6.
6. Compute, if possible, a $\gamma \in \mathbb{A}$ with (20) using the algorithms from [61]. If this is not possible, construct the $\Sigma$-extension $(\mathbb{A}[\tau], \sigma)$ of $(\mathbb{A}, \sigma)$ with $\sigma(\tau) = \tau - a_{s+1}\,\sigma(h)$ and set $\gamma = \tau$.
7. Use Lemma 2.2 with the given $h, \gamma$ to compute $g$ with $\sigma(g) = g$.

*Remark 2.5* (1) If $(\mathbb{G}, \sigma)$ is one of the base difference fields (1–3) from Example 2.1, all steps can be carried out. However, if $(\mathbb{G}, \sigma)$ is a general $\Pi\Sigma$-field, one might fail in step 5 with the existing algorithms to redesign and extend the difference ring $(\mathbb{A}, \sigma)$ to $(\mathbb{A}', \sigma)$ such that it is an $R\Pi\Sigma$-extension of $(\mathbb{G}, \sigma)$ in which one gets

$h \in \mathbb{A}'$ with (19).
(2) Suppose that there exists a $g = g_0\, x_0 + \cdots + g_s + g_{s+1}$ with $g_i \in \mathbb{E}$ for $0 \leqslant i \leqslant s+1$ and $\sigma(g) = g$ for some $\mathbb{G}$-simple $R\Pi\Sigma$-extension $(\mathbb{E}, \sigma)$ of $(\mathbb{A}, \sigma)$. Then the above method will always find such a $g$ as predicted in Theorem 2.2. Namely, by Lemma 2.2 there is an $h \in \mathbb{E}$ with (19). Hence we may either assume that there is a solution of (19) in $\mathbb{A}$ or by Corollary 2.1 there is an $R\Pi$-extension $(\mathbb{A}\langle t\rangle, \sigma)$ of $(\mathbb{A}, \sigma)$ with $\frac{\sigma(t)}{t} \in \mathbb{G}^*$ in which we can find a solution of (19) in $\mathbb{G}\langle t\rangle$. Thus the above method can be executed without entering in step 5.

*Example 2.4* Consider the sequence (18) with the initial values $X(0) = 0$, $X(1) = -1$. We remark that this recurrence is completely solvable in terms of d'Alembertian solutions:

$$X(k) = 2^k(S_1(\tfrac{1}{2}, k) - S_1(k)). \tag{28}$$

We will compute this zero-order recurrence by iteratively computing constants. We start with (18) and set up the underlying h.o.l. extension. Since there is a recurrence with smaller order (order zero), there must exist a non-trivial constant. Our algorithm produces the constant $G(k) = 2^{-k} + (1+k)2^{1-k}X(k) + (-1-k)2^{-k}X(1+k)$. With $G(0) = 2$ we get a new recurrence of order 1: $X(1+k) = \frac{-1+2^{1+k}}{-1-k} + 2X(k)$. We use this recurrence and set up a new h.o.l. extension and search again for a constant. We get $G(k) = 2^{-k}X(k) - S_1(\frac{1}{2}, k) + S_1(k)$ and with $G(0) = 0$ we obtain (28).

In other words, computing stepwise constants (where in each step the constant has the shape as worked out in Theorem 2.2), we find the smallest possible recurrence that can be given in terms of simple $R\Pi\Sigma$-extensions. Note that this mechanism has been utilized already earlier to find an optimal recurrence in the context of finite element methods [23]. In particular, if there is a recurrence of order 0 where the inhomogeneous part is given in a simple $R\Pi\Sigma$-extension, such a recurrence will be eventually calculated with our method from above. Note that this strategy to find minimal recurrences (and to solve the recurrence in terms of d'Alembertian solutions if possible) is also related to the remarks given in [50, p. 163] that deals with the computation of left factors of a recurrence.

### 2.2.2 The Refined Parameterized Telescoping Problem

Suppose that we are given $F_i(k)$ for $1 \leqslant i \leqslant d$ and suppose that we can represent them in a difference ring as introduced above. Namely, suppose that we succeeded in constructing an $R\Pi\Sigma$-extension $(\mathbb{A}, \sigma)$ of a $\Pi\Sigma$-field $(\mathbb{G}, \sigma)$ with $\mathbb{K} = \text{const}_\sigma\mathbb{G}$, and on top of this, we designed a h.o.l. extension $(\mathbb{A}[x_1, \ldots, x_s], \sigma)$ with (12) where $a_0, \ldots, a_s \in \mathbb{G}$ and $a_{s+1} \in \mathbb{A}$ with the following property: the $F_i(k)$ can be rephrased as

$$f_i = f_{i,0}\, x_0 + \cdots + f_{i,s}\, x_s + f_{i,s+1} \tag{29}$$

for $1 \leqslant i \leqslant d$ with $f_{i,j} \in \mathbb{G}$ for $0 \leqslant j \leqslant s$ and $f_{i,s+1} \in \mathbb{A}$.

In this setting, we are interested in solving Problem RPT. Namely, we aim at finding, if possible, an appropriate $\mathbb{G}$-simple $R\Pi\Sigma$-extension $(\mathbb{E}, \sigma)$ of $(\mathbb{A}, \sigma)$ in which one can solve Problem RPT with $c_1, \ldots, c_d \in \mathbb{K}$ where $c_1 \neq 0$ and $g = g_0\, x_0 + \cdots + g_s\, x_s + g_{s+1}$ with $g_i \in \mathbb{E}$ for $0 \leqslant i \leqslant s+1$. Then given such a result and rephrasing $g$ as $G(k)$ yields (1) and enables one to compute the sum relation (2).

In our main application we set $F_i(k) = F(n+i-1, k)$ for a bivariate sequence. Then (2) can be turned to a linear recurrence of the form (3) for a definite sum, say $S(n) = \sum_{k=l}^{L(n)} F(n, k)$ for $l \in \mathbb{N}$ and for some integer linear function $L(n)$. During this construction, we should keep in mind that various optimality criteria might lead to different preferable recurrences.

1. Find a recurrence (3) with lowest order $d-1$
2. Find a recurrence such that the underlying difference ring is as simple as possible (e.g., the number of arising sums and products is as small as possible, the nesting depth of the sums is minimal, or the number of objects within the summands is as low as possible.)

Note that in most examples both criteria cannot be fulfilled simultaneously: in an appropriate $R\Pi\Sigma$-extension the number $d$ might be reduced, but the difference ring will be enlarged by further, most probably more complicated $R\Pi\Sigma$-monomials; in the extreme case one might find a zero-order recurrence formulated in a rather large $R\Pi\Sigma$-extension. Contrary, increasing $d$ might lead to simpler $R\Pi\Sigma$-extensions in which the recurrence can be formulated; ideally, one can even find a recurrence without introducing any further $R\Pi\Sigma$-monomials. In our experience a compromise between these extremes are preferable to reduce the underlying calculation time. On one side, we are interested in calculating a recurrence as efficiently as possible. On the other side, we might use the found recurrence as the new defining h.o.l. extension and to tackle another parameterized telescoping problem in a recursive fashion (see Sect. 3). Hence the derivation of a good recurrence (not to large in $d$ but also not too complicated objects in the inhomogeneous part) will be an important criterion.

Having this in mind, we will focus now on various tactics to tackle Problem RPT that give us reasonable flexibility for tackling definite multi-sums but will be not too involved concerning the complexity of the underlying algorithms. We start as follows. Set $f = c_1\, f_1 + \cdots + c_d\, f_d$ for unknown $c_1, \ldots, c_d$ and write $f = h_0\, x_0 + \cdots + h_s\, x_s + h_{s+1}$ with $h_i = c_1\, f_{i,1} + \cdots + c_d\, f_{i,d}$. By Lemma 2.1 it follows that (14) and (15) must hold (where $f_i$ is replaced by $h_i$). Note that (14) reads as

$$\sum_{j=0}^{s} \sigma^{s-j}(a_j)\sigma^{s-j+1}(g_s) - g_s = \sum_{j=0}^{s} \sigma^{s-j}(h_j) = c_1\, \tilde{f}_1 + \cdots + c_d\, \tilde{f}_d \tag{30}$$

with

$$\tilde{f}_i = \sum_{j=0}^{s} \sigma^{s-j}(f_{i,j}) \in \mathbb{G}. \tag{31}$$

Hence one could utilize the summation package `Sigma` as follows: (1) look for a $g_s$ in $\mathbb{G}$; (2) if this fails, try to find a solution in $\mathbb{A}$; (3) if there is no such solution, search for a solution in a $\mathbb{G}$-simple $R\Pi\Sigma$-extension.

*Remark 2.6* Assume there exists such a solution $g_s \neq 0$ with (30) either in $(\mathbb{A}, \sigma)$ or in a $\mathbb{G}$-simple $R\Pi\Sigma$-extension $(\mathbb{E}, \sigma)$ of $(\mathbb{G}, \sigma)$, but not in $\mathbb{G}$. This implies that there is an $h \in \mathbb{E}$ with (19): if the right hand side of (30) is 0, we can set $h := g_s$. Otherwise, we utilize Proposition 2.1 (by taking the special case $\mathbb{A} = \mathbb{G}$). Namely, part 2 implies that a solution of (30) must depend on a $\Sigma$-monomial that is introduced by the extension $(\mathbb{E}, \sigma)$ of $(\mathbb{A}, \sigma)$. Finally, part 1 implies that there is an $h \in \mathbb{E}$ with (19). Hence we can utilize Theorem 2.2 and it follows that we can construct a $\mathbb{G}$-simple $R\Pi\Sigma$-extension in which one can compute $g' = g'_0 x_0 + \cdots + g'_s x_s + g'_{s+1}$ with $\sigma(g') = g'$ and $g_s \neq 0$. Note that such an extension and $g'$ can be even computed; see part (2) of Remark 2.5. Hence following the recipe after Lemma 2.2 we can construct a recurrence (21) ($g'_i$ is rephrased as the summation object $G_i(k)$) for our summation object $X(k)$ which has a smaller recurrence order. Further, we can construct an improved h.o.l. extension with recurrence order $s$ that describes better the shift behavior of the sequence $X(k)$.

Summarizing, finding a solution $g_s$ in $\mathbb{A}$ or in a $\mathbb{G}$-simple $R\Pi\Sigma$-extension implies that one can also reduce the recurrence order of the h.o.l. extension that models $X(k)$. In this regard, we emphasize that having a recurrence with smaller order will also increase the chance to find a parameterized telescoping solution: the larger the recurrence order is, the more sequences are satisfied by the recurrence and thus a solution of Problem PRS is more general. Conversely, the smaller the recurrence order is, the better the problem description and thus the higher the chances are to find a solution if it exists. Hence instead of searching for a $g_s$ in $\mathbb{A}$ or in an appropriate $\mathbb{G}$-simple $R\Pi\Sigma$-extension, we opt for outsourcing this task to the user: if it seems appropriate, the user should try to hunt for a recurrence with lower order by either using other summation tactics (see Example 2.2) or applying the machinery mentioned in part (2) of Remark 2.5 as a preprocessing step to produce a h.o.l. extension with lower order.

*Remark 2.7* Note that the classical holonomic summation algorithms [34, 44] handle the case (12) with $a_{s+1} = 0$ and $a_i \in \mathbb{G}$ where $(\mathbb{G}, \sigma)$ is the rational or $q$-rational difference field (see instances (1) and (2) of Example 2.1). In most cases, the arising recurrences are optimal in the following sense: the recurrence orders cannot be reduced and the recurrence system is the defining relation. Together with the above inside (see Remark 2.6) this explains why standard holonomic approaches are optimal: they hunt for solutions $g = g_0 x_0 + \cdots + g_s x_s$ where the $g_i$ are in $\mathbb{G}$ and do not try to look for any simple $R\Pi\Sigma$-extension.

With this understanding, we will restrict[4] ourselves to the following

[4]If one is only interested in the telescoping problem with $d = 1$, it might be worthwhile to look for a solution of (14) in a $\mathbb{G}$-simple $R\Pi\Sigma$-extension; this particular case is neglected in the following.

**Strategy 1**: we will search for $c_1, \dots, c_d \in \mathbb{K}$ with $c_1 \neq 0$ and for $g_s$ with (30) and (31) only in $\mathbb{G}$, but not in $\mathbb{A}$ or in any other simple $R\Pi\Sigma$-extension. To obtain all such solutions, we will assume that we can solve the following subproblem.

**Problem PRS in** $(\mathbb{G}, \sigma)$: Parameterized recurrence solving.
*Given* a difference field $(\mathbb{G}, \sigma)$ with constant field $\mathbb{K} = \text{const}_\sigma \mathbb{G}$, $\mathbf{0} \neq (\tilde{a}_0, \dots, \tilde{a}_m) \in \mathbb{G}^{m+1}$ and $\tilde{\mathbf{f}} = (\tilde{f}_1, \dots, \tilde{f}_d) \in \mathbb{G}^d$. *Find* a basis of the $\mathbb{K}$-vector space[a]

$$V = \{(c_1, \dots, c_d, g) \in \mathbb{K}^d \times \mathbb{G} \mid \tilde{a}_0\, g + \dots + \tilde{a}_m\, \sigma^m(g) = c_1\, \tilde{f}_1 + \dots + c_d\, \tilde{f}_d\} \qquad (32).$$

[a]The dimension of $V$ is at most $d + m$; see [36, 56].

We remark that this strategy is also down to earth: searching $g_s$ in $\mathbb{G}$ is usually very efficient and does not need any fancy algorithms. In particular, we can solve Problem PRS if $(\mathbb{G}, \sigma)$ is a $\Pi\Sigma$-field[5]; see [17, 33, 53, 56].

We continue with our algorithm for Problem RPT. Namely, suppose that we can compute a non-empty basis of $V$ as posed in Problem PRS. Then by Lemma 2.1 we have to find a $(c_1, \dots, c_d, g_s) \in V$ with $c_1 \neq 0$ such that there is a $g_{s+1}$ with (15) where $f_{s+1}$ must be replaced by $c_1\, f_{1,s+1} + \dots + c_d\, f_{d,s+1}$. If we find such a $g_{s+1}$ in $\mathbb{A}$, we are done. Namely, following Lemma 2.1 we take

$$\begin{aligned} g_0 &= a_0\, g_s - (c_1\, f_{0,1} + \dots + c_d\, f_{0,d}), \\ g_i &= \sigma(g_{i-1}) + a_i\, \sigma(g_s) - (c_1\, f_{i,1} + \dots + c_d\, f_{i,d}), \quad 1 \leqslant i < s \end{aligned} \qquad (33)$$

and get the desired solution $g = g_0\, x_0 + \dots + g_s\, x_s + g_{s+1}$ of (5). Otherwise, one could take any $(c_1, \dots, c_d, g_s) \in V$ with $c_1 \neq 0$. Then by Theorem 2.1 we can construct the $\Sigma$-extension $(\mathbb{A}[\tau], \sigma)$ of $(\mathbb{A}, \sigma)$ with $\sigma(\tau) = \tau + \phi$ where

$$\phi = (c_1\, f_{1,s+1} + \dots + c_d\, f_{d,s+1}) - a_{s+1}\, \sigma(g_s) \qquad (34)$$

and can choose $g_{s+1} = \tau$. However, this might not be the best choice. Using strategies $\text{RPT}_r$ with $r = 2, 3, 4$ of Remark 2.1 might produce a better result. In any case, if one fails to find a solution with the proposed tactics or if the produced $R\Pi\Sigma$-extension is too involved for further processing (in particular for our application in Sect. 3), one can also enlarge $d$ to search for a recurrence with a higher order but with simpler summation objects involved. These considerations yield
**Strategy 2**: we will search $g_{s+1}$ in $\mathbb{A}$ ($\text{RPT}_1$) and if this fails provide the option to use our refined algorithms $\text{RPT}_2$, $\text{RPT}_3$ or $\text{RPT}_4$ of Remark 2.1 to look for an optimal $\Sigma$-extension $(\mathbb{E}, \sigma)$ of $(\mathbb{A}, \sigma)$ in which $g_{s+1}$ can be found.

Summarizing, we propose the following general summation tactic in higher-order extensions that enables one to incorporate our Strategies 1 and 2 from above.

[5]For the rational and $q$-rational difference fields see also [15, 16, 39].

**Algorithm 1** *Refined holonomic parameterized telescoping.*
`ParameterizedTelescoping(`$(\mathbb{A}[x_0,\dots,x_s],\sigma),\mathbf{f}$`)`

Input: *A difference ring extension* $(\mathbb{A},\sigma)$ *of a difference field* $(\mathbb{G},\sigma)$ *with constant field* $\mathbb{K}=\mathrm{const}_\sigma\mathbb{G}=\mathrm{const}_\sigma\mathbb{A}$ *where one can solve Problems RPT in* $(\mathbb{A},\sigma)$ *and PRS in* $(\mathbb{G},\sigma)$*. A h.o.l. extension* $(\mathbb{A}[x_0,\dots,x_s],\sigma)$ *of* $(\mathbb{A},\sigma)$ *with* (12) *where* $a_0,\dots,a_s\in\mathbb{G}$ *and* $a_{s+1}\in\mathbb{A}$*.* $\mathbf{f}=(f_1,\dots,f_d)$ *with* (29) *for* $1\leqslant i\leqslant d$ *where* $f_{i,j}\in\mathbb{G}$ *with* $1\leqslant j\leqslant s$ *and* $f_{i,s+1}\in\mathbb{A}$*.*

Output: *A h.o.l. extension* $(\mathbb{E}[x_0,\dots,x_s],\sigma)$ *of* $(\mathbb{E},\sigma)$ *with* (12) *where* $(\mathbb{E},\sigma)$ *is an "optimal"*[6] *extension of* $(\mathbb{A},\sigma)$ *with* $g\in\mathbb{G}\,x_0+\dots+\mathbb{G}\,x_s+\mathbb{E}$ *and* $c_1,\dots,c_d\in\mathbb{K}$ *s.t.* $c_1\neq 0$ *and (5). If such an optimal extension does not exist, the output is "No solution".*

*(1) Compute* $\tilde{f}_i$ *for* $1\leqslant i\leqslant d$ *as given in* (31)*,* $\tilde{a}_0=-1$*,* $\tilde{a}_i=\sigma^{i-1}(a_{s+1-i})\in\mathbb{G}$ *for* $1\leqslant i\leqslant s+1$*.*
*(2) Solve Problem PRS: compute a basis* $B=\{(c_{i,1},\dots,c_{i,d},\gamma_i)\}_{1\leqslant i\leqslant n}$ *of (32).*
*(3) If* $B=\{\}$ *or* $c_{1,1}=c_{1,2}=\dots=c_{1,d}=0$*, then Return "No solution".*
*(4) We assume that* $(c_{1,1},\dots,c_{1,n})$ *has at most one entry which is non-zero. Otherwise, take one row vector in* $B$ *where the first entry is non-zero and perform row operations over* $\mathbb{K}$ *with the other row vectors of* $B$ *such that the first entries are zero (note that the result will be again a basis of (32)).*
*(5) Define* $\mathbf{C}=(c_{i,j})\in\mathbb{K}^{n\times d}$ *and* $\gamma=(\gamma_1,\dots,\gamma_n)\in\mathbb{G}^n$*, and compute*

$$\phi=(\phi_1,\dots,\phi_n)=\mathbf{C}\,(f_{1,s+1},\dots,f_{d,s+1})^t-a_{s+1}\,\sigma(\gamma)^t\in\mathbb{A}^n. \tag{35}$$

*(6) Solve Problem RPT: find, if possible, an "optimal" difference ring extension* $(\mathbb{E},\sigma)$ *of* $(\mathbb{A},\sigma)$ *with* $g_{s+1}\in\mathbb{E}$ *and* $\kappa_1,\dots,\kappa_n\in\mathbb{K}$ *with* $\kappa_1\neq 0$ *and* $\sigma(g_{s+1})-g_{s+1}=\kappa_1\,\phi_1+\dots+\kappa_n\,\phi_n$*.*
*(7) If such an optimal extension does not exist, return "No solution".*
*(8) Otherwise, compute* $(c_1,\dots,c_d)=(\kappa_1,\dots,\kappa_n)\,\mathbf{C}\in\mathbb{K}^d$ *and* $g_s=(\kappa_1,\dots,\kappa_n)\,\gamma^t\in\mathbb{G}$*.*
*(9) Compute the* $g_i$ *with* $0\leqslant i<s$ *as given in* (33)*.*
*(10) Return* $(c_1,\dots,c_d)\in\mathbb{K}^d$ *and* $g=g_0\,x_0+\dots g_s\,x_s+g_{s+1}$*.*

**Proposition 2.2** *If Algorithm 1 returns* $(c_1,\dots,c_d)\in\mathbb{K}^d$ *and* $g=g_0\,x_0+\dots g_s\,x_s+g_{s+1}$*, then* $g_1,\dots,g_s\in\mathbb{G}$*,* $g_{s+1}\in\mathbb{E}$*,* $c_1\neq 0$ *and (5) holds.*

*Proof* Suppose that the algorithm outputs $(c_1,\dots,c_d)\in\mathbb{K}^d$ and $g=g_0\,x_0+\dots+g_s\,x_s+g_{s+1}$. By construction, $g_i\in\mathbb{G}$ for $0\leqslant i\leqslant s$ and $g_{s+1}\in\mathbb{E}$. Further, the matrix $\mathbf{C}$ has in the first column precisely one nonzero entry (see step (4)). Thus with $(c_1,\dots,c_d)=(\kappa_1,\dots,\kappa_n)\,\mathbf{C}$ in step (8) it follows that $c_1\neq 0$ if and only if $\kappa_1\neq 0$. But $\kappa_1\neq 0$ is guaranteed in step (6) due to the specification of Problem RPT. Hence $c_1\neq 0$. Finally, define $h=c_1\,f_1+\dots+c_d\,f_d$ and write $h=h_0\,x_0+\dots+h_s\,x_s+h_{s+1}$ with $h_i\in\mathbb{A}$ for $0\leqslant i\leqslant s$ and $h_{s+1}\in\mathbb{E}$. Then

$$\sum_{j=0}^{s}\sigma^{s-j}(a_j)\sigma^{s-j+1}(g_s)-g_s=(\kappa_1,\dots,\kappa_n)\Big(\sum_{j=0}^{s}\sigma^{s-j}(a_j)\sigma^{s-j+1}(\gamma^t)-\gamma^t\Big)$$
$$=(\kappa_1,\dots,\kappa_n)\,\mathbf{C}\,(\tilde{f}_1,\dots,\tilde{f}_d)^t=c_1\,\tilde{f}_1+\dots,+c_d\,\tilde{f}_d\overset{(31)}{=}\sum_{j=0}^{s}\sigma^{s-j}(h_j),$$

$$\begin{aligned}\sigma(g_{s+1})-g_{s+1}&=\kappa_1\,\phi_1+\dots+\kappa_n\,\phi_n\\&=(\kappa_1,\dots,\kappa_n)(\mathbf{C}(f_{1,s+1},\dots,f_{d,s+1})^t-a_{s+1}\,\sigma(\gamma)^t)\\&=(c_1,\dots,c_d)(f_{1,s+1},\dots,f_{d,s+1})^t-a_{s+1}\,(\kappa_1,\dots,\kappa_n)\sigma(\gamma)^t=h_{s+1}-a_{s+1}\,\sigma(g_s).\end{aligned}$$

[6] We will make this statement precise in Theorem 2.3 by choosing specific variants of Problem RPT.

Thus by Lemma 2.1 it follows that $\sigma(g) - g = f$. □

**Theorem 2.3** *Let* $(\mathbb{A}, \sigma)$ *be a simple* $R\Pi\Sigma$*-extension of a* $\Pi\Sigma$*-field* $(\mathbb{G}, \sigma)$ *with* $\mathbb{A} = \mathbb{G}\langle t_1\rangle \dots \langle t_e\rangle$*; let* $f_1, \dots, f_d \in \mathbb{A}$ *with* (29) *for* $1 \leqslant i \leqslant d$ *where* $f_{i,j} \in \mathbb{G}$ *with* $1 \leqslant j \leqslant s$ *and* $f_{i,s+1} \in \mathbb{A}$*. Execute Algorithm 1 where in step (6) Problem RPT is specialized by one of the versions* $RPT_r$ *with* $r = 1, 2, 3, 4$*. as given in Remark 2.1. If the output is* $(c_1, \dots, c_d) \in \mathbb{K}^d$ *and* $g = g_0\, x_0 + \dots g_s\, x_s + g_{s+1}$*, then the following holds for the corresponding specialization.*

*$RPT_1$.* $g_{s+1} \in \mathbb{A} = \mathbb{E}$.
*$RPT_2$.* $g_{s+1}$ *as given in* $RPT_1$ *if this is possible. Otherwise, one gets a* $\Sigma$*-extension* $(\mathbb{E}, \sigma)$ *of* $(\mathbb{A}, \sigma)$ *with* $g_{s+1} \in \mathbb{E} \setminus \mathbb{A}$ *and* $\delta(g_{s+1}) \leqslant \delta(c_1\, f_{1,s+1} + \dots + c_d\, f_{d,s+1})$.
*$RPT_3$.* $g_{s+1}$ *as given in* $RPT_2$ *if this is possible. Otherwise, one obtains a* $\Sigma$*-extension* $(\mathbb{A}[\tau], \sigma)$ *of* $(\mathbb{A}, \sigma)$ *with* $g_{s+1} \in \mathbb{A}[\tau] \setminus \mathbb{A}$ *and* $\sigma(\tau) - \tau \in \mathbb{G}\langle t_1\rangle \dots \langle t_i\rangle$ *with* $0 \leqslant i < e$ *where at least one of the* $t_{i+1}, \dots, t_e$ *occurs in* $c_1\, f_{1,s+1} + \dots + c_d\, f_{d,s+1}$ *and i is minimal among all such possible solutions.*
*$RPT_4$.* $g_{s+1}$ *as given in* $RPT_3$ *if this is possible. Otherwise,* $g_{s+1} = \tau$ *within the* $\Sigma$*-extension* $(\mathbb{A}[\tau], \sigma)$ *of* $(\mathbb{A}, \sigma)$ *with* $\sigma(\tau) = \tau + \phi$ *where* (34).

*If the output is "No solution", then there is no solution of (5) with* $(c_1, \dots, c_d) \in \mathbb{K}^d$ *where* $c_1 \neq 0$ *and* $g = g_0\, x_0 + \dots g_s\, x_s + g_{s+1}$ *with* $g_i \in \mathbb{G}$ *for* $1 \leqslant i \leqslant s$ *and where* $g_{s+1}$ *can be represented as formulated in* $RPT_r$ *with* $r = 1, 2, 3, 4$*, respectively.*

*Proof* Problem PRS can be solved in a $\Pi\Sigma$-field; see [17, 33, 53, 56]. Further, Problems $\text{RPT}_r$ with $r = 1, 2, 3, 4$ can be solved in this setting; see Remark 2.1. Now let $r \in \{1, 2, 3, 4\}$ and suppose that our algorithm is executed with variant $\text{RPT}_r$. If Algorithm 1 returns $(c_1, \dots, c_d) \in \mathbb{K}^d$ and $g = g_0\, x_0 + \dots g_s\, x_s + g_{s+1}$, then $g_1, \dots, g_s \in \mathbb{G}$, $g_{s+1} \in \mathbb{E}$, $c_1 \neq 0$ and (5) holds by Proposition 2.2. Further, by construction the $g_{s+1}$ is given as specified in $\text{RPT}_r$. This completes the first part. Now suppose that the algorithm returns "No solution" but there exists a solution $(c_1, \dots, c_d) \in \mathbb{K}^d$ with $c_1 \neq 0$ and $g_{s+1}$ as specified in $\text{RPT}_r$. By Lemma 2.1 we conclude that $(c_1, \dots, c_d, g_s)$ is an element of (32). Thus we get $B \neq \{\}$ in step (2) and we do not quit in step (3). Since $B$ is a $\mathbb{K}$-basis of (32), there is a $(\kappa_1, \dots, \kappa_n) \in \mathbb{K}^n$ with $(c_1, \dots, c_d) = (\kappa_1, \dots, \kappa_n)\, \mathbf{C}$. By Lemma 2.1 we conclude that $(\kappa_1, \dots, \kappa_n)\phi^t = \sigma(g_{s+1}) - g_{s+1}$. Thus the variant $\text{RPT}_r$ is solvable, and the algorithm cannot return "No solution" in step (7). Consequently, the output "No solution" is not possible, a contradiction. □

We conclude this section by a concrete example that demonstrates the full flexibility of our refined holonomic machinery to hunt for linear recurrences.

*Example 2.5* Given $S(n) = \sum_{k=0}^{n} \binom{n}{k} X(k)$ with $X(k) = \sum_{j=0}^{k} \binom{k}{j} S_1(j)^2$, we aim at computing a linear recurrence of the form (3). We start with the $\Pi\Sigma$-field $(\mathbb{G}, \sigma)$ with constant field $\mathbb{K} = \mathbb{Q}(n)$ and $\mathbb{G} = \mathbb{K}(t)(b)$ where $\sigma(t) = t + 1$ and $\sigma(b) = \frac{n-t}{t+1}\, b$.
(A) In a first round, we will exploit the recurrence (6) to set up our h.o.l. extension defined $(\mathbb{G}, \sigma)$ (here we can set $\mathbb{A} = \mathbb{G}$) and search for a solution $g = g_0\, x_0 + g_1\, x_1 + g_2\, x_2 + g_3\, x_3 + g_4$ of Problem RPT with $g_0, g_1, g_2, g_3 \in \mathbb{G}$ and $g_4$

in $\mathbb{G}$ or in a properly chosen $R\Pi\Sigma$-extension of $(\mathbb{G}, \sigma)$. First, we will activate Algorithm 1 with the telescoping strategy $\mathrm{RPT}_1$ for $d = 0, 1, 2, 3, \ldots$ until we find a recurrence. Following our algorithm we search for $g_3 \in \mathbb{G}$ and $c_1, \ldots, c_d \in \mathbb{K}$ by solving the following parameterized difference equation

$$-\tfrac{8(4+t)(6+t)}{(7+t)^2}\sigma^4(g_3) + \tfrac{4\left(99+45t+5t^2\right)}{(6+t)^2}\sigma^3(g_3) - \tfrac{2(11+3t)(13+3t)}{(5+t)^2}\sigma^2(g_3) \\ + \tfrac{86+49t+7t^2}{(4+t)^2}\sigma(g_3) - g_3 = c_1\,\tilde{f}_1 + \cdots + c_d\,\tilde{f}_d$$

where the first six $\tilde{f}_i$ are given by

$$\tilde{f}_1 = -\tfrac{b(-n+t)(1-n+t)(2-n+t)}{(1+t)(2+t)(3+t)}, \qquad \tilde{f}_2 = \tfrac{b(1+n)(n-t)(-1+n-t)}{(1+t)(2+t)(3+t)},$$
$$\tilde{f}_3 = \tfrac{b(1+n)(2+n)(n-t)}{(1+t)(2+t)(3+t)}, \qquad \tilde{f}_4 = \tfrac{b(1+n)(2+n)(3+n)}{(1+t)(2+t)(3+t)},$$
$$\tilde{f}_5 = \tfrac{b(1+n)(2+n)(3+n)(4+n)}{(1+t)(2+t)(3+t)(1+n-t)}, \qquad \tilde{f}_6 = \tfrac{b(1+n)(2+n)(3+n)(4+n)(5+n)}{(1+t)(2+t)(3+t)(1+n-t)(2+n-t)}.$$

We obtain the first non-trivial solution with $d = 5$: the basis $B_5$ of the $\mathbb{K}$-vector space (32) has dimension 1 and is given by $B = \{(c_{1,1}, c_{1,2}, c_{1,3}, c_{1,4}, c_{1,5}, \gamma_1)\}$ with $c_{1,1} = 27(5+2n)$, $c_{1,2} = -\tfrac{27(28+27n+6n^2)}{2(1+n)}$, $c_{1,3} = \tfrac{3(418+544n+225n^2+30n^3)}{2(1+n)(2+n)}$, $c_{1,4} = -\tfrac{414+504n+187n^2+22n^3}{2(1+n)(2+n)}$, $c_{1,5} = \tfrac{(4+n)^2(3+2n)}{2(1+n)(2+n)}$, and $\gamma_1 = -\tfrac{b(3+t)^2(4+5t+n(2+2t))}{2(1+t)(2+t)(1+n-t)}$. So our hope is that $(c_1, c_2, c_3, c_4, c_5, g_3)$ equals the element of $B_5$. Next, we check if we can determine $g_4 \in \mathbb{G}$ with

$$\sigma(g_4) - g_4 = \tfrac{b(9+4n+5t+2nt)}{2(1+t)(2+t)(3+t)} =: \phi. \tag{36}$$

Since there is no such solution, we restart our algorithm for $d = 6$. This time the $\mathbb{K}$-vector space (32) has the dimension 2, i.e., we obtain a basis $B_6$ with two elements (which we do not print here). So we have more flexibility to set up $g_4$. In order to determine $g_4 \in \mathbb{A}$, it must be a solution of $\sigma(g_4) - g_4 = \kappa_1\,\phi_1 + \kappa_2\,\phi_2$ with

$$\phi_1 = \tfrac{2b(63+46n+8n^2+35t+24nt+4n^2t)}{(5+2n)(1+t)(2+t)(3+t)}, \qquad \phi_2 = -\tfrac{b(13+4n+7t+2nt)}{(1+t)(2+t)(3+t)(-1-n+t)};$$

for their calculation see (35). We find $\kappa_1 = \tfrac{3+2n}{(5+2n)(7+2n)}$, $\kappa_2 = -\tfrac{(3+n)(1+2n)}{(5+2n)^2}$, $g_4 = -\tfrac{b(-2n-3t-2nt)}{(5+2n)(1+t)(2+t)(1+n-t)}$. Combining this solution with entries of $B$ delivers

$$108(1+n)(2+n)(3+2n)S(n) - 54(2+n)\big(21+30n+8n^2\big)S(1+n) \\ + 3\big(831+1634n+795n^2+114n^3\big)S(2+n) \\ + \big(-1227-2556n-1095n^2-134n^3\big)S(3+n) \\ + \big(283+632n+243n^2+26n^3\big)S(4+n) - (5+n)^2(1+2n)S(5+n) = 0. \tag{37}$$

Algorithm 1 with the tactics $\mathrm{RPT}_2$, $\mathrm{RPT}_3$ will deliver the same recurrence. Applying $\mathrm{RPT}_4$ we will obtain for $d = 5$ the basis $B_5$ from above and have to find a solution for (36). Since there is no solution $g_4 \in \mathbb{G}$ (and the tactics from $\text{RPT}_2$, $\text{RPT}_3$ fail), we continue and construct the $\Sigma$-extension $(\mathbb{G}[\tau], \sigma)$ of $(\mathbb{G}, \sigma)$ with $\sigma(\tau) = \tau + \phi$ and get the solution $g_4 = \tau$. Gluing all the building blocks together, `Sigma` delivers

$$\begin{aligned} &54(1+n)(2+n)(5+2n)S(n) - 27(2+n)\big(28+27n+6n^2\big)S(1+n)\\ &+3\big(418+544n+225n^2+30n^3\big)S(2+n) - \big(414+504n+187n^2+22n^3\big)S(3+n)\\ &\quad + (4+n)^2(3+2n)S(4+n) = 2n + (1+n)(2+n)\sum_{i=0}^{n} \frac{\binom{n}{i}(4+5i+2n+2in)}{(1+i)(2+i)(1-i+n)}. \end{aligned} \tag{38}$$

We remark that the found sum on the right hand side can be turned to an expression in terms of indefinite nested objects (like for the sum (8)) and the right hand side can be simplified to $(1+2n)2^{2+n}$.

(B) In the second round, we will exploit the recurrence (9) and set up a h.o.l. extension defined over a properly chosen $R\Pi\Sigma$-extension $(\mathbb{A}, \sigma)$ of $(\mathbb{G}, \sigma)$. In this setting we search for a solution $g = g_0\, x_0 + g_1\, x_1 + g_2$ of Problem RPT with $g_0, g_1 \in \mathbb{G}$ and $g_2$ in $\mathbb{A}$ or in a properly chosen $R\Pi\Sigma$-extension of $(\mathbb{A}, \sigma)$. If we apply tactic $\text{RPT}_1$ or $\text{RPT}_2$, we will get (37). However, if we apply $\text{RPT}_3$, we find

$$\begin{aligned} &-36(1+n)^2(2+n)(3+n)S(n) + 6(1+n)(2+n)(3+n)(12+7n)S(1+n)\\ &+2(-19-8n)(1+n)(2+n)(3+n)S(2+n) + 2(1+n)(2+n)(3+n)^2 S(3+n)\\ &\qquad = -2^{3+n}(1+n)^2 + 2\, 3^{2+n}(1+n)(3+2n). \end{aligned} \tag{39}$$

Finally, if we activate tactic $\text{RPT}_4$ in `Sigma`, we end up at the recurrence

$$\begin{aligned} &-9(1+n)S(n) + 3(3+2n)S(1+n) + (-2-n)F(2+n)\\ &\qquad = \tfrac{2^{1+n}(4+3n)}{(1+n)(2+n)} - \tfrac{3^{1+n}(5+4n)}{(1+n)(2+n)} + 2^{1+n}S_1(n) - 2^{1+n}S_1\big(\tfrac{3}{2}, n\big). \end{aligned}$$

Note that we obtained in both cases first a recurrence where on the right hand side definite sums pop up which afterwards are simplified to indefinite versions.

(C) In a third round, one can use the zero-order recurrence (10) following the standard `Sigma`-approach [60] and can apply purely the tools from Sect. 2.1 (see Remark 2.1). In contrast to the variants (A) and (B), these calculations are more involved since they have to be carried out within a much larger $R\Pi\Sigma$-extension.

Solving any of the found recurrences in terms of d'Alembertian solutions yields

$$\begin{aligned} S(n) = 3^n\Big( &-2S_1(n)S_1\big(\tfrac{2}{3}, n\big) - 2S_2\big(\tfrac{2}{3}, n\big) - S_{1,1}\big(\tfrac{2}{3}, \tfrac{3}{2}, n\big)\\ &+ 3S_{1,1}\big(\tfrac{2}{3}, 1, n\big) + S_1(n)^2 + S_2(n)\Big). \end{aligned}$$

Summary: we provided different holonomic summation tactics in the context of $R\Pi\Sigma$-extensions to find linear recurrences. The smaller the obtained recurrence order is, the more the underlying difference ring algorithms are challenged to handle many $R\Pi\Sigma$-extensions. Conversely, the higher the recurrence order is, the larger will be the computed coefficients of the recurrence and thus the underlying arithmetic operations get more involved.

## 3 A Multi-sum Method to Determine Recurrences

We aim at computing a recurrence of an $m$-fold definite nested multi-sum

$$S(n) = \sum_{k_1=\alpha_1}^{L_1(n)} h_1(n, k_1) \sum_{k_2=\alpha_2}^{L_2(n,k_1)} h_2(n, k_1, k_2) \cdots \sum_{k_m=\alpha_m}^{L_m(n,k_1,\ldots,k_{m-1})} h_m(n, k_1, \ldots, k_m) \tag{40}$$

where for $1 \leqslant i \leqslant m$ the following holds: $\alpha_i \in \mathbb{N}$, $L_i(n, k_1, \ldots, k_{i-1})$ stands for an integer linear expression or equals $\infty$, and $h_i(n, k_1, \ldots, k_i)$ is an expression in terms of indefinite nested sums over hypergeometric products w.r.t. the variable $k_i$.

**Definition 3.1** Let $f(k)$ be an expression that evaluates at non-negative integers (from a certain point on) to elements of a field $\mathbb{K}$. $f(k)$ is called an *expression in terms of indefinite nested sums over hypergeometric products w.r.t. k* if it is composed of elements from the rational function field $\mathbb{K}(k)$, by the three operations $(+, -, \cdot)$, by *hypergeometric products* of the form $\prod_{j=l}^{k} h(j)$ with $l \in \mathbb{N}$ and a rational function $h(t) \in \mathbb{K}(t) \setminus \{0\}$, and by sums of the form $\sum_{j=l}^{k} F(j)$ with $l \in \mathbb{N}$ and where $F(j)$, being free of $k$, is an expression in terms of indefinite nested sums over hypergeometric products w.r.t. $j$.

For this task we will improve substantially the multi-sum approach introduced in [54] by exploiting our new difference ring machinery from Sect. 2. More precisely, we will process the sums in (40) from inside to outside and will try to compute for each sub-sum $X(\mathbf{n}, k)$ a refined holonomic system[7] w.r.t. $k$.

**Definition 3.2** Consider a multivariate sequence $X(\mathbf{n}, k)$ with the distinguished index $k$ and further indices $\mathbf{n} = (n_1, \ldots, n_u)$ and let $\mathbf{e}_i$ be the $i$th unit vector of length $u$. A refined holonomic system for $X(\mathbf{n}, k)$ w.r.t. $k$ is a set of equations of the form

$$X(\mathbf{n}, k+s+1) = A_0(\mathbf{n}, k)\, X(\mathbf{n}, k) + \cdots + A_s(\mathbf{n}, k)\, X(\mathbf{n}, k+s) + A_{s+1}(\mathbf{n}, k), \tag{41}$$

$$X(\mathbf{n} + \mathbf{e}_i, k) = A_0^{(i)}(\mathbf{n}, k)\, X(\mathbf{n}, k) + \cdots + A_s^{(i)}(\mathbf{n}, k)\, X(\mathbf{n}, k+s) + A_{s+1}^{(i)}(\mathbf{n}, k) \tag{42}$$

with $1 \leqslant i \leqslant u$ which holds within a certain range of $k$ and $\mathbf{n}$ and where the $A_j(\mathbf{n}, k)$ and $A_j^{(i)}(\mathbf{n}, k)$ with $0 \leqslant j \leqslant s$ and $1 \leqslant i \leqslant u$ are rational functions in $K(\mathbf{n}, k)$ for some field $K$ and the $A_{s+1}(\mathbf{n}, k)$ and $A_{s+1}^{(i)}(\mathbf{n}, k)$ for $1 \leqslant i \leqslant u$ are indefinite nested sums over hypergeometric products w.r.t. $k$.

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[7] Also in [34] coupled systems are constructed to handle multi-sums. Here we restrict to a special form so that the full power of our tools from Sect. 2 can be applied without using any Gröber bases or uncoupling computations. In particular, the recurrences can have inhomogeneous parts which can be represented in $\Pi\Sigma$-fields and $R\Pi\Sigma$-extensions. Also the coefficients could be represented in general $\Pi\Sigma$-fields (see [54]), but we will skip this more exotic case.

**Base case**. We process the trivial sum $X(n, k_1, \ldots, k_m) = 1$ and can construct the refined holonomic system $X(n+1, k_1, \ldots, k_m) = X(n, k_1, \ldots, k_m)$ and $X(n, k_1, \ldots, k_i + 1, \ldots, k_m) = X(n, k_1, \ldots, k_i, \ldots, k_m)$ for all $1 \leqslant i \leqslant m$.
Now suppose that we succeeded in treating the sum

$$X(n, k_1, \ldots, k_u) = \sum_{k_{u-1}=\alpha_{u-1}}^{L_{u+1}(n,k_1,\ldots,k_u)} h_{u-1}(\ldots) \cdots \sum_{k_m=\alpha_m}^{L_m(n,k_1,\ldots,k_{m-1})} h_m(n, k_1, \ldots, k_m). \tag{43}$$

For convenience, set $\mathbf{n} = (n_1, \ldots, n_u) := (n, k_1, \ldots, k_{u-1})$ and $k = k_u$; further set $\tilde{\mathbf{n}} = (n_1, \ldots, n_{u-1})$. By assumption we computed a refined holonomic system for $X(\mathbf{n}, k) = X(\tilde{\mathbf{n}}, n_u, k)$ w.r.t. $k$ as given in Definition 3.2.
If $u = 0$, we are done. Otherwise we proceed as follows.
**Recursion step**. Consider the next sum

$$\tilde{X}(\mathbf{n}) = \tilde{X}(\tilde{\mathbf{n}}, n_u) = \sum_{k=\alpha_u}^{L_u(\mathbf{n})} F(\mathbf{n}, k)$$

with $F(\mathbf{n}, k) = h_u(\mathbf{n}, k) X(\mathbf{n}, k)$. Then we aim at computing a refined holonomic system for $\tilde{X}(\mathbf{n})$ w.r.t. $n_u$. Namely, set $F_i(k) = F(\mathbf{n} + (i-1)\,\mathbf{e}_u, k) = F(\tilde{\mathbf{n}}, n_u + i - 1, k)$. Then using the rewrite rules (41) and (42) we can write $F_i(k)$ as

$$F_i(k) = F(\mathbf{n} + (i-1)\,\mathbf{e}_u, k) = F_{i,0}(k)\,X(k) + \cdots + F_{i,s}(k)\,X(k+s) + F_{i,s+1}(k)$$

where for $1 \leqslant i \leqslant d$ and $0 \leqslant j \leqslant s+1$ the $F_{i,j}(k)$ are indefinite nested sums over hypergeometric products w.r.t. $k$. Given this form, we try to construct an $R\Pi\Sigma$-extension $(\mathbb{A}, \sigma)$ of a $\Pi\Sigma$-field $(\mathbb{G}, \sigma)$ with the following properties: we can rephrase the $A_j(k)$ from (41) with $1 \leqslant j \leqslant s$ by $a_j$ in $\mathbb{G}$ and $A_{s+1}$ by $a_{s+1}$ in $\mathbb{A}$, and simultaneously, we can rephrase the $F_{i,j}(k)$ with $1 \leqslant i \leqslant d$ and $0 \leqslant j \leqslant s$ by $f_{i,j}$ in $\mathbb{G}$ and the $F_{i,s+1}$ with $1 \leqslant i \leqslant d$ by $f_{i,s+1}$ in $\mathbb{A}$.

*Remark 3.1* (1) Consider the special case $u = m$. Looking at the base case, we get $s = 0$ with $a_0 = 1$ and $a_1 = 0$. Further, $F_1(\mathbf{n}, k) = h_m(\mathbf{n}, k)$ is given in terms of indefinite nested sums over hypergeometric products, and also the shifted versions $F_i(\mathbf{n}, k) = h_m(\mathbf{n} + (i-1)\mathbf{e}_u, k)$ for $i \geqslant 2$ are again from this class. All these objects can be rephrased in one common $R\Pi\Sigma$-extension $(\mathbb{A}, \sigma)$ of the rational difference field $(\mathbb{G}, \sigma)$ with $\mathbb{G} = \mathbb{K}(t)$ and $\sigma(t) = t + 1$ using the algorithms from [46, 55, 61, 63]. In a nutshell: for this special case the desired construction is always possible.
(2) If $h_u(\mathbf{n}, k) \in K(\mathbf{n}, k)$ (for some field $K$), then $F_{i,j}(k) \in K(\mathbf{n}, k)$ for all $1 \leqslant i \leqslant d$ and $0 \leqslant j \leqslant s$. In addition, $A_j(\mathbf{n}, k) \in K(\mathbf{n}, k)$ for all $0 \leqslant j \leqslant s$ by our recursive construction. Further, $F_{i,s+1}(\mathbf{n}, k)$ with $1 \leqslant i \leqslant d$ and $A_{s+1}(\mathbf{n}, k)$ are indefinite nested sums over hypergeometric products w.r.t. $k$. Hence by using our tools from [46, 55, 61, 63], we can accomplish this construction.

If $h_u(\mathbf{n}, k)$ $(1 \leqslant u < m)$ is more involved, we refer to part (3) of Remark 3.2.

If this rephrasing in $\mathbb{G}$ and $\mathbb{A}$ is possible, take the h.o.l. $(\mathbb{H}, \sigma)$ of $(\mathbb{A}, \sigma)$ with $\mathbb{H} = \mathbb{A}[x_0, \dots, x_s]$ with (12). In other words, we model $F_i(k)$ by (29). Now we activate our Algorithm 1 by choosing an appropriate tactic $\text{RPT}_r$ with $r \in \{1, 2, 3, 4\}$: for $d = 0, 1, 2, \dots$ we check with the input $\mathbf{f} = (f_1, \dots, f_d)$ if we find a solution for Problem $\text{RPT}_r$. If we succeed for $d$ ($d$ is minimal for a given tactic) and rephrase the found solution in terms of indefinite nested sums and products, we obtain the summand recurrence (1) and summing this equation over the summation range[8] yields a recurrence of the form (41) for the next sum $\tilde{X}(\tilde{\mathbf{n}}, n_u)$.
Similarly, choose $i$ with $1 \leqslant i < u$. Then we can set $F_0^{(i)} = F(\mathbf{n} + \mathbf{e}_i, k)$ and using the rewrite rules (41) and (42) we obtain

$$F_0^{(i)}(k) = F(\mathbf{n} + \mathbf{e}_i, k) = F_0^{(i)}(k)\, X(k) + \dots + F_s^{(i)}(k)\, X(k+s) + F_{s+1}^{(i)}(k)$$

where the $F_j^{(i)}(k)$ are indefinite nested sums over hypergeometric products w.r.t. $k$.

As above, we try to represent these elements by $f_0^{(i)} = f_0^{(i)}\, x_0 + \dots + f_s^{(i)}\, x_s + f_{s+1}^{(i)}$ with $f_j^{(i)} \in \mathbb{G}$ for $0 \leqslant j \leqslant s$ and $f_{s+1}^{(i)} \in \mathbb{A}$ in an $R\Pi\Sigma$-extension $(\mathbb{A}, \sigma)$ of a $\Pi\Sigma$-field $(\mathbb{G}, \sigma)$. Now we activate again Algorithm 1 with the input $\mathbf{f} = (f_0^{(i)}, f_1, \dots, f_\delta)$ for $\delta = 0, 1, \dots$. In all our applications we have been successful for a $\delta$ with $\delta < d$ by choosing one of the tactics $\text{RPT}_j$ with $j \in \{1, 2, 3, 4\}$ (usually, with the tactic $\text{RPT}_r$ that lead to (41)). In other words, reinterpreting this solution in terms of indefinite nested sums and products and summing the found equation over the summation range will produce a recurrence of the form (42) for $\tilde{X}(\tilde{\mathbf{n}}, n_u)$.

Performing this calculation for all $i$ with $1 \leqslant i < u$ yields a system of recurrences for $\tilde{X}(\tilde{\mathbf{n}}, n_u)$ of the form (41) and (42). To turn this to a refined holonomic system, one has to face an extra challenge. The inhomogeneous sides $A_{s+1}(k)$ and $A_{s+1}^{(i)}(k)$ often contain definite sums (see, e.g., the recurrences (7) and (38)). To rewrite them to indefinite nested versions (which are expressible in an $R\Pi\Sigma$-extension) further symbolic simplifications are necessary; see Sect. 3.2.2 below. If this is not possible, our method fails. Otherwise, this completes the recursion step of our method.

*Remark 3.2* (1) If a refined holonomic system with $s = 0$ arises in one of these recursion steps (this is in particular the case if we treat the first summation), Algorithm 1 boils down to solve Problem RPT in $(\mathbb{A}, \sigma)$; compare also Remark 2.3.
(2) If the expression $\tilde{X}(\mathbf{n})$ in the recursion step is free of $n_i$ $(1 \leqslant i \leqslant u)$, one gets trivially $\tilde{X}(\mathbf{n} + \mathbf{e}_i) = \tilde{X}(\mathbf{n})$.
(3) Given (40), the summands $h_i(n, k_1, \dots, k_i)$ with $i < m$ (i.e., not the innermost summand $h_m$) might introduce complications. The indefinite nested sums over hypergeometric products w.r.t. $k_i$ in the $h_i$ and their shifted versions in the parameters $(n, k_1, \dots . k_{i-1})$ must be encoded in a $\Pi\Sigma$-field $(\mathbb{G}, \sigma)$; see Remark 3.1. If this

[8] If there are exceptional points within the summation range, we refer to Sect. 3.2.3. Further, if the upper bound is $\infty$, limit computations are necessary. For wide classes of indefinite nested sums asymptotic expansions can be computed [1, 10, 12, 13] that can be used for this task.

is not possible, our method fails. If it works, Problem PRS has to be solved in $(\mathbb{G}, \sigma)$. Hence the $\Pi\Sigma$-field should be composed only by a reasonable sized set of generators to keep the algorithmic machinery efficient. Conversely, if one sets $h_1 = \cdots = h_{m-1} = 1$ in (40) and moves all summation objects into $h_m$, one can choose for $(\mathbb{G}, \sigma)$ the rational difference field; see Remark 3.1. However, in this case the inhomogeneous parts of the refined holonomic system will blow up. In our experiments we found out that choosing $h_i$ as a hypergeometric product (that can be formulated in a $\Pi\Sigma$-field) was a reasonable trade-off to gain speed up and to keep the $\Pi\Sigma$-field simple; see Example 2.5 for a typical application.

Our machinery works also for sums (40) where the $h_i$ depend on mixed multi-basic hypergeometric products. This means that in Definition 3.1 one also allows products of the form $\prod_{j=l}^{k} f(j, q_1^j, \ldots, q_e^j)$ where $f(t, t_1, \ldots, t_e)$ is a rational function. The only extra adaption is to take as ground field instance (3) of Example 2.2.

## 3.1 Illustrative Examples

Our working example (see Examples 2.2 and 2.5) follows precisely the above multi-sum method. Namely consider our sum $S(n) = \sum_{k=0}^{n} \binom{n}{k} X(n, k)$ with $X(n, k) = \sum_{j=0}^{k} \binom{k}{j} S_1(j)^2$. We worked from inside to outside and computed a refined holonomic system for each summand. First, we took the inner sum $X(k) = X(n, k)$ and computed a recurrence purely in $k$ demonstrating our different telescoping strategies. Since $X(k)$ is free of $n$, we get trivially the recurrence $X(n + 1, k) = X(n, k)$. Afterwards, we applied our multi-sum machinery to the second sum: namely, as worked out in Example 2.5 we computed a recurrence of $S(n)$ by exemplifying our different summation tactics.

Now let us turn to a `Mathematica`—implementation of the refined holonomic approach called `RhoSum`. It is built on top of `Sigma`, `HarmonicSums` [1] and `EvaluateMultiSums` [60]. The first step is to load these packages,

In[1]:= **<< Sigma.m**
Sigma - A summation package by Carsten Schneider © RISC

In[2]:= **<< HarmonicSums.m**
HarmonicSums by Jakob Ablinger – © RISC

In[3]:= **<< EvaluateMultiSums.m**
EvaluateMultiSums by Carsten Schneider – © RISC

In[4]:= **<< RhoSum.m**
RhoSum by Mark Round – © RISC

By loading these packages one obtains recurrence finding and solving tools from `Sigma`, special function algorithms for indefinite nested sums [1, 10, 12, 13] from `HarmonicSums`, and summation technologies from `EvaluateMultiSums` and finally the refined summation package itself, `RhoSum`. Then with a single command the above method is applied to our double sum to deliver a recurrence.

In[5]:= **FindRecurrence[** $\binom{n}{k}\binom{k}{j}S_1[j]^2$**, {{j, 0, k}, {k, 0, n}}, {n}, {0}, {∞}]**

Out[5]= $-36(1+n)^2(2+n)(3+n)\text{nSUM}[n]+6(1+n)(2+n)(3+n)(12+7n)\text{nSUM}[1+n]+2(-19-8n)(1+n)(2+n)(3+n)\text{nSUM}[2+n]+2(1+n)(2+n)(3+n)^2\text{nSUM}[3+n]==-2^{3+n}(1+n)^2+2\,3^{2+n}(1+n)(3+2n)$

Internally, `RhoSum` used up to a certain complexity the subroutines of `Sigma` with the tactic $\text{RPT}_3$ (see Theorem 2.3) and delivers the recurrence (39). If one wants to solve the recurrence in addition in terms of d'Alembertian solutions (in case this is possible), one can execute the command

In[6]:= **FindSum[** $\binom{n}{k}\binom{k}{j}S_1(j)^2$**, {{j, 0, k}, {k, 0, n}}, {n}, {0}, {∞}]**

Out[6]= $3^n\left(-2S_1[n]S_1\left[\tfrac{2}{3},n\right]-2S_2\left[\tfrac{2}{3},n\right]-S_{1,1}\left[\tfrac{2}{3},\tfrac{3}{2},n\right]+3S_{1,1}\left[\tfrac{2}{3},1,n\right]+S_1[n]^2+S_2[n]\right)$

We will concentrate on the slightly more involved triple sum

$$S(N)=\sum_{n=0}^{N}\overbrace{\sum_{k=0}^{n+N}\underbrace{\sum_{j=0}^{k}S_1(j)\binom{n+N}{j}^2}_{=:c_{N,n,k}}}^{=:b_{N,n}}$$

in order to outline all steps of our multi-sum method. Our aim is to compute a refined holonomic system for the complete multi-sum $S(N)$. This refined holonomic system is particularly simple, it consists of just one recurrence of shifts in $N$. To compute the recurrence our algorithm is to encode the summand $b_{N,n}$ in to a refined holonomic system too. This is a system of two recurrences involving shifts in $N$ and $n$. Again this will be done by encoding the summand $c_{N,n,k}$ in yet another refined holonomic system. This is the base case because we can compute with the summand explicitly. Using `Sigma` the refined holonomic system

$$\begin{aligned}-c_{N,n,k}+c_{N,n,k+1}&=\binom{n+N}{k}^2\left(\tfrac{(k-n-N)^2}{(1+k)^3}+\tfrac{(k-n-N)^2}{(1+k)^2}S_1(k)\right)\\ 2(1+2n+2N)c_{N,n,k}-(1+n+N)c_{N,n+1,k}&=E(N,n,k)\\ 2(1+2n+2N)c_{N,n,k}-(1+n+N)c_{N+1,n,k}&=E(N,n,k)\end{aligned}$$

can be computed with

$$E(N,n,k)=\left(\tfrac{1-3k+4n+4N}{1+n+N}+(1-2k+3n+3N)S_1(k)\right)\binom{n+N}{k}^2+\tfrac{(-1-4n-4N)}{1+n+N}(1+s(k))$$

which contains the extra sum $s(k) = \sum_{i=1}^{k} \binom{n+N}{i}^2$. Next we use this system to compute a refined holonomic system for the sequence $b_{N,n}$. Using Sigma we get a recurrence purely shifted in $n$:

$$\begin{aligned}2(-1+n+N)(1+2n+2N)b_{N,n} - (-2+n+N)(1+n+N)b_{N,n+1}\\ = 2(1+2n+2N)c_{N,n,0} + 4(1+2n+2N)c_{N,n,N+n}\\ + \tfrac{(5+20n-7n^2-4n^3+20N-14nN-12n^2N-7N^2-12nN^2-4N^3)}{2(1+n+N)}(1+s(n+N)).\end{aligned}$$

Notice that the middle line contains "telescoping points" (see also Sect. 3.2.2) and $s(n+N)$ turns to a definite sum (the integer parameters $N$ and $n$ arise inside the sum and at the upper bound). These give three new summation problems. $c_{N,n,0} = 0$ is trivial, while the remaining sums can be treated similar to the sum in Example 2.2. Namely, we get

$$c_{N,n,N+n} = \sum_{j=0}^{n+N} S_1(j)\binom{n+N}{j}^2 = \frac{3}{2}\frac{(2n+2N)!S_1(n+N)}{((n+N)!)^2} - \frac{(2n+2N)!}{((n+N)!)^2}\sum_{j=1}^{n+N}\frac{1}{2j-1},$$

$$s(n+N) = -1 + \frac{(2n+2N)!}{((n+N)!)^2}.$$

The result contains a cyclotomic harmonic sum [12]. Replacing these definite sums with their indefinite nested sum representations leads to the final recurrence for $b_{N,n}$ purely with shifts in $n$. There is also a recurrence shifted in $N$ (note that $n$ and $N$ are symmetric) and we end up at the refined holonomic system

$$\begin{aligned}2(-1+n+N)(1+2n+2N)b_{N,n} - (-2+n+N)(1+n+N)b_{N,n+1} = r(N,n),\\ 2(-1+n+N)(1+2n+2N)b_{N,n} - (-2+n+N)(1+n+N)b_{N+1,n} = r(N,n)\end{aligned}$$

with the same right hand side

$$\begin{aligned}r(N,n) =& 4(1+2n+2N)\Big(\frac{3(2n+2N)!S_1(n+N)}{2((n+N)!)^2} - \frac{(2n+2N)!\sum_{j=1}^{n+N}\frac{1}{2j-1}}{((n+N)!)^2}\Big)\\ &+ \frac{\big(5+20n-7n^2-4n^3+20N-14nN-12n^2N-7N^2-12nN^2-4N^3\big)}{2(1+n+N)}\frac{(2n+2N)!}{(n+N)!^2}.\end{aligned}$$

Finally, this system can be used to compute a recurrence for the entire multi-sum. Using Sigma one obtains

$$\begin{aligned}&\tfrac{3(1+4N)(7+9N)}{(-1+N)(1+N)(1+2N)}S_1(2N) + \tfrac{(43+378N+527N^2-312N^3-828N^4-288N^5)}{4(-1+N)(1+N)^2(1+2N)^2}\frac{(4N)!}{(2N)!^2} + b_{N,0}\\ &+ \tfrac{(1-3N-38N^2-40N^3)b_{N,N}}{(-1+N)(1+N)(1+2N)} - \tfrac{2(1+4N)(7+9N)}{(-1+N)(1+N)(1+2N)}\frac{(4N)!}{(2N)!^2}\sum_{j=1}^{2N}\tfrac{1}{2j-1} = S(N) - S(N+1).\end{aligned}$$

There are two telescoping points $b_{N,N}$ and $b_{N,0}$ to evaluate which turn out to be two double sums. Applying our machinery again to these sums gives recurrences in $N$ and solving them produces closed form solutions in terms of indefinite nested sums. Plugging these simplifications into the telescoping points provides the final result: a recurrence for $S(N)$ in terms of indefinite nested sums. We remark that this recurrence can be also solved in terms of indefinite nested sums over hypergeometric products, but the result is too big to be printed here.

## 3.2 Implementation Remarks

This section contains a discussion of various technical components, knowledge of which is required for an efficient implementation of the underlying machinery in `RhoSum`. Some remarks refer to how the recurrence should be computed and handled before being returned by the recurrence finding technology. As such, in terms of our multi-sum approach described in the beginning of Sect. 3, these comments fit inside the calls to the recurrence finding technology.

Usually recurrence finding technologies are very costly. In general, a multi-summation algorithm based on refined difference field theory will compute up to $m(m+1)/2$ recurrences in an $m$-fold sum. This makes controlling the individual recurrences very important because if any individual recurrence is too large in size or the underlying difference ring consists of too many $R\Pi\Sigma$-monomials then the inherently high cost of recurrence finding technologies may easily lead to the entire multi-sum problem becoming intractable. Notice also that because recurrences are computed from sums of lower depth, efficiency issues can become cumulative; a recurrence which is not expressed in a simple form is likely to lead to longer computation times when used as an input for another calculation. This further serves to highlight the importance of the technical details of recurrence computations. We will discuss some of the specific issues that fit into this central problem.

### 3.2.1 Managing Recurrence Computation Time

When computing a recurrence, one must define the desired type of recurrence. The definition corresponds to how one constructs $g_{s+1}$ in Theorem 2.3. A configuration that searches for minimal order recurrences translates to applying tactic $\mathrm{RPT}_4$ where $d$ is minimal. Such an approach is, relatively speaking, cheap to compute; the linear system one must solve for the homogeneous part is of minimal size and one takes the first available solution then extends the ring to get an inhomogeneous solution. The potential penalty is to adjoin a $\Sigma$-extension which might be rather complicated which afterwards has to be converted to an expression in terms of indefinite nested sums. At the other extreme, one can use tactic $\mathrm{RPT}_1$, i.e., to relax the condition of minimal order and try to compute $g_{s+1}$ in the difference ring that one uses to describe the input problem—and if this fails to increase the recurrence order $d$ of the parameterized telescoping problem. When working with harder sums the different approaches, including tactics $\text{RPT}_2$ and $\text{RPT}_3$, may have very different computation times. The different possibilities are carried out in Example 2.5.

There are several reasonable heuristics, one could choose a single methodology for all recurrence computations. In the case of particle physics sums this can be useful. In fact, if a minimal order approach is taken for sums originating from particle physics, then the computations are likely to be representative of an optimum balance of the two methods. This is a strong motivation for pursuing refined holonomic summation. It offers the best approach to particle physics sums as compared to other techniques. Always applying a non-minimal order approach is likely to require computations that are not feasible with modern computer power. One could also choose to switch between the two methods. For example a crude but simple approach would be to adopt some non-minimal order approach and if the search is yet to return a result after a given time limit one switches to a minimal order approach. Another option would be to limit the order the non-minimal search takes place over. When that order has been exceeded one would then switch to a minimal order approach. Implementing both a time and recurrence order limit is recommended to avoid the scenario that a low but non-minimal order recurrence is very slow to compute and so must be avoided but at the same time allow the implementation to find simple recurrences of relatively high order.

### 3.2.2 Definite Sums Inside of Recurrences

As already observed in part (3) of Remark 3.2, one has to deal with definite sums that arise within the calculation of recurrence relations. Namely, given a summand recurrence (1) for properly chosen summands $F_i(k)$, one obtains a recurrence relation (3) where $h(n)$ comes from $G(L(n)+1) - G(l)$ which either evaluates nicely or otherwise turns to definite sums. More precisely, one either obtains the so-called telescoping points $X(l)$ or $X(L(n)+i)$ with $i \in \mathbb{N}$ or one gets definite sums coming from certain $\Sigma$-extensions. In both cases, these sums are simpler summation problems: in the first case, they are simpler than the main multi-sum because they are at specific values, in the second case they come from our $R\Pi\Sigma$-extensions which can be formulated in a simpler $R\Pi\Sigma$-extension (the summand can be formulated in a smaller difference ring). In other words, we have to solve simpler summation problems and the resulting recursive calls of our algorithms (i.e., calling `FindSum`) will eventually terminate. We remark that higher-order recurrences lead to larger numbers of telescoping points (tactic $\text{RPT}_1$) but more involved $R\Pi\Sigma$-extensions (tactic $\text{RPT}_4$) might also lead to more complicated definite sums. Here tactics $\text{RPT}_2$ and $\text{RPT}_3$ can be an interesting alternative to reduce the calculation time concerning the treatment of extra definite sums and avoiding any slow down of our refined holonomic summation implementation. With modern computers it is likely however that the telescoping points can be computed simultaneously by using parallelization.

### 3.2.3 Exceptional Points

It may be that the function $G(k)$ computed in solving the parameterized telescoping problem (compare to (1)) is not defined for some values in the range of summation. Thus when one tries to sum over the expression to obtain (2), one encounters ill-defined expressions even though the original summation problem is well-defined at that point. Usually such exceptional points restrict the summation range by a difference of one or two, should they occur at all. E.g., consider the sum $b_{n,k} = \sum_{j=0}^{k} c_{n,k,j}$ for which we want to compute a refined holonomic system, and suppose that we find a refined holonomic system for $c_{n,k,j}$ which is only valid within the range $j = j_1, \ldots, k - j_2$ for some $j_1, j_2 \in \mathbb{N}$. If one wishes to continue with a refined approach, there are two options. One could accept the restricted ranges returned by the summation technology and continue without making any adjustments. Then the final expression will be valid for a different sum which is contained within the original sum and it is most likely that only significantly simpler sums are required to solve the entire problem. However by disturbing the structure of the multi-sum, many unwanted, and possibly hard, sums might not cancel and one is faced with extra work to treat these sums. Alternatively one can compute the values of the exceptional points and compensate in the problem. To do this consider the rewriting

$$b_{n,k} = \sum_{j=0}^{k} c_{n,k,j} = \sum_{j=0}^{j_1-1} c_{n,k,j} + \sum_{j=k-j_2+1}^{k} c_{n,k,j} + \sum_{j=j_1}^{k-j_2} c_{n,k,j} = K + \sum_{j=j_1}^{k-j_2} c_{n,k,j}.$$

The expression for $K$ is just given by two definite sums, which are often easy to handle. Within this approach there is a subtlety as to where one places $K$ in the multi-sum expression. The choices are

$$a_n = \sum_{k=0}^{n} \Big( K + \sum_{j=j_1}^{k-j_2} c_{n,k,j} \Big) \quad \text{or} \quad a_n = \sum_{k=0}^{n} \sum_{j=j_1}^{k-j_2} \Big( \tfrac{K}{k-j_2-j_1+1} + c_{n,k,j} \Big).$$

In general, our heavy calculations coming from particle physics gave the experience that the second strategy is more preferable: within the summand further cancellations arise and the processing of the summations turn out to be easier.

## 4 Examples from Elementary Particle Physics

Perturbative calculations in quantum field theory lead to various summation problems [30], and one of the challenges is to find recurrence relations of a certain order and polynomial degree, where the polynomials contain an integer variable $N$ and a series of parameters. One of which is the dimensional parameter $\varepsilon = D - 4$, which is required to handle divergences in the Feynman diagrams. This introduces a small

parameter $\varepsilon > 0$ inside of the sum (40). Some of our results reproduce calculations that have only recently entered the particle physics literature [3, 7, 27]. In some very rare cases one can apply directly our method `FindRecurrence` to compute a recurrence for $S(n)$ or to apply `FindSum` to compute a closed form in terms of indefinite nested sums. In such cases the derived sums usually depend on the $\varepsilon$ parameter. However, in most cases one will fail to solve the arising recurrences within this class. In particular, the definite sums inside of our method as outlined in Sect. 3.2.2 cannot be expressed in our $R\Pi\Sigma$-extensions.
In the following we use the fact that the Laurent expansion of the Feynman integrals (and the underlying summation problems) around $\varepsilon = 0$ to a finite order is of primary interest to the physics community. Consider (40); for simplicity, we will assume that $h_i = 1$ for $1 \leqslant i < m$ and we set $h = h_m$. Then a more flexible tactic is to focus on the Laurent expansion of

$$h(n, k_1, \ldots, k_m) = f_l(n, k_1, \ldots, k_m)\varepsilon^l + \cdots + f_r(n, k_1, \ldots, k_m)\varepsilon^r + O(\varepsilon^{r+1}) \tag{44}$$

w.r.t. $\varepsilon$ up to a certain order $r$ with $r \geqslant l$; in 3-loop calculations one expects $l = -3$.

If the sums in (40) are finite, one obtains the first coefficients

$$F_i(n) = \sum_{k_1=\alpha_1}^{L_1(n)} \cdots \sum_{k_m=\alpha_m}^{L_m(n,k_1,\ldots,k_{m-1})} f_i(n, k_1, \ldots, k_m) \tag{45}$$

of the desired $\varepsilon$-expansion

$$S(n) = F_l(n)\varepsilon^l + \cdots + F_r(n)\varepsilon^r + O(\varepsilon^{r+1}).$$

If also infinite sums are involved, extra care has to be taken into account. As it turns out the $f_i$ themselves can be again written in terms of hypergeometric products together with harmonic numbers and cyclotomic harmonic sums [12]. Hence one option is to apply our summation methods to (45) which is free of $\varepsilon$. Then in basically all our calculations the arising definite sums turn out to be solvable within our difference ring approach. However, the coefficients $f_i$ in (44) and thus the summands in (45) get more and more involved (in particular they depend more and more on the harmonic sums) which blow up the calculations.

More successfully one can apply our new algorithms in combination with the following clever $\varepsilon$-expansion technique [11] to our simple example (46). As an illustration of the refined approach for a particle physics sum consider the following

$$S(n) = \sum_{k=0}^{n-2} \frac{n-k-1}{1+k} \sum_{j=0}^{n-k-2} \frac{(-1)^j (k+j)! \left(1-\frac{\varepsilon}{2}\right)_k \left(2-\frac{\varepsilon}{2}\right)_j}{(3-\varepsilon)_{k+j} \left(3+\frac{\varepsilon}{2}\right)_{k+j}} \binom{n-k-2}{j}. \tag{46}$$

We start to compute a refined holonomic system for the inner sum denoted by $b_{n,k}$:

$$\begin{aligned}0 =&-(1+k)(2-\varepsilon+2k)b_{n,k}+\big(14+\varepsilon-\varepsilon^2+14k+\varepsilon k\\&+4k^2-\varepsilon n-2kn\big)b_{n,k+1}-2(2+\varepsilon+k)(3+k-n)b_{n,k+2},\\0 =&\big(2-\varepsilon^2+2k+\varepsilon k+2k^2+2n-\varepsilon n-2kn+2n^2\big)b_{n,k}\\&-2(1+\varepsilon+k)(2+k-n)b_{n,k+1}+(-1+\varepsilon-n)(2+\varepsilon+2n)b_{n+1,k}.\end{aligned} \tag{47}$$

The complete double sum can be written as, $S(n)=\sum_{k=0}^{n-2}\frac{n-k-1}{1+k}b_{n,k}$. Using Sigma once more the sequence obeys a recurrence only valid for the upper bound $n-4$. For this adjusted sum $S'(n)$ we get

$$\begin{aligned}&-2(1+n)^2(2+n)(2+\varepsilon+2n)S'(n)\\&\quad-(2+n)(2+\varepsilon+2n)\big(-8+2\varepsilon+\varepsilon^2-10n+\varepsilon n-4n^2\big)S'(1+n)\\&\quad+(1+n)(-2+\varepsilon-n)(2+\varepsilon+2n)(4+\varepsilon+2n)S'(2+n)=r(\varepsilon,n)\end{aligned} \tag{48}$$

where $r(\varepsilon,n)$ depends on $b_{n,0}$, $b_{n,1}$ and $b_{n,n-4}$, $b_{n,n-3}$. Rewriting these definite sums, that depend on $\varepsilon$, to an expression in terms of indefinite nested sums is not possible. Therefore, we compute the $\varepsilon$-expansion of the arising sums (e.g., by expanding the summands and applying the summation quantifiers to the coefficients of their expansion as proposed above). Solving these telescoping points, i.e., computing the first coefficients of their $\varepsilon$-expansion gives

$$\begin{aligned}r(\varepsilon,n)=&\frac{16\big(336+48n-248n^2-70n^3+186n^4-121n^5+138n^6-81n^7+26n^8+7n^9-6n^{10}+n^{11}\big)}{(-2+n)^3(-1+n)^3n^2(1+n)}\\&+\varepsilon\Big[-\frac{4\big(-4032+7104n+\ldots-7n^{15}+n^{16}\big)}{(-2+n)^4(-1+n)^4n^3(1+n)^2(3+n)}-\frac{16(-1+n)S_1(n)}{3+n}\Big]\ldots\end{aligned} \tag{49}$$

Finally, given the first initial values $F_i(j)$ with $i=0,1$ and $j=2,3$ in $S(2)=F_0(2)+F_1(2)\varepsilon+\ldots$ and $S(3)=F_0(3)+F_1(3)\varepsilon+\ldots$ one can activate Sigma's $\varepsilon$-expansion solver [30] to (48) and obtains the coefficient $F_0(n)$ and $F_1(n)$ of $S'(n)=F_0(n)+F_1(n)\varepsilon+O(\varepsilon^2)$. Taking care of the extra points $k=n-2,n-3$ one finally obtains the expansion of the input sum $S(n)$. With the implementation RhoSum a complete automation is possible with the function call

In[7]:= $\textbf{FindSum}\Big[\frac{(-1)^j(-1-k+n)\binom{-2-k+n}{j}(j+k)!\left(1-\frac{\varepsilon}{2}\right)_k\left(2-\frac{\varepsilon}{2}\right)_j}{(1+k)(3-\varepsilon)_{j+k}\left(3+\frac{\varepsilon}{2}\right)_{j+k}},\{\{j,0,n-k-2\},\{k,0,n-2\}\},\{n\},\{3\},\{\infty\},\text{ExpandIn}\to\{\varepsilon,0,1\}\Big]$

Out[7]= $\Big\{-4S_1(n)+8nS_3(n)-4nS_{2,1}(n),\ -\frac{(5+n)}{1+n}S_1(n)-S_1(n)^2+3S_2(n)+2nS_2(n)^2-2nS_3(n)+6nS_4(n)+nS_{2,1}(n)-4nS_{3,1}(n)-2nS_{2,1,1}(n)\Big\}$

Instead of calculating the expansions of the sums in (49) in the old-fashioned way (i.e., by expanding the summands and applying the summation quantifiers to the coefficients of their expansion), one can apply recursively our proposed technology to obtain the expansion (49).

More generally, following the strategy in Sect. 3, we will calculate stepwise from inside to outside a refined holonomic system given by the recurrences (41) and (42) but in each step we will expand the inhomogeneous parts in an $\varepsilon$-expansion whose coefficients can be expressed in an $R\Pi\Sigma$-extension. E.g., the inhomogeneous parts of the recurrences in (47) are 0 and the $\varepsilon$-expansion is trivial. Further, the recurrence (48) with (49) can be a component of a refined holonomic system that might be used to tackle another sum which is on top.

More practically, RhoSum has been applied to solve many such sums being with many more summation quantifiers that originate in particle physics. Here we would like to mention the calculation of the massive 3-loop contributions to the heavy quark effects for the structure functions in deep-inelastic scattering [29]. This has already contributed to a rich literature [2–4, 6–9, 24–28] which has made new physical insight possible [5]. When this project is completed, the strong coupling constant $\alpha_s(M_Z^2)$ and the charm quark mass $m_c$ can be measured from the deep-inelastic world data with unprecedented accuracy and a significant improvement of the gluon distribution function of the nucleon can be achieved. This has important consequences for all precision measurements at the LHC (Large Hadron Collider, CERN), because these quantities determine the QCD corrections to the corresponding production cross sections.

## 5 Summary

A new method of summation known as the refined holonomic approach has been introduced that extends significantly the ideas worked out in [54]. Its main features are the ability to work with inhomogeneous recurrences and to balance the difficulty of a telescoping problem with the number of ring extensions made to find a solution. These techniques have proved useful in particle physics where inhomogeneous recurrences are essential to modern computation. A future development may consist of the application of the algorithm to summation problems in which further real parameters, beyond the dimensional parameter $\varepsilon$, are present. This is of high relevance for multi-leg scattering processes at high energy colliders like the LHC and a planned future $e^+e^-$ collider. These parameters may be the masses and/or the virtualities of the external legs of the corresponding Feynman diagrams. RhoSum is expected to handle problems of this kind more efficiently than other implementations. The algorithm has been implemented in Mathematica to employ the technique in practical situations and problems from particle physics involving large numbers of sums have been successfully solved.

The earlier approach [54] served as the central tool to provide the first computer assisted proof [20] of Stembridge's celebrated TSPP theorem [64] in the context of plane partitions. In that time the computation steps have been carried out manually. First experiments show that our new package RhoSum in interaction with Sigma can support the user heavily: many steps can be carried out now mechanically and critical special cases are discovered automatically. In a nutshell, our tools can guide the user to a big extend through these complicated and subtle calculations. It is expected that this machinery will contribute further in difficult calculation combing from particle physics but will also assist in new challenging problems in the context of combinatorics.

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# Bivariate Extensions of Abramov's Algorithm for Rational Summation

**Shaoshi Chen**

*Dedicated to Professor Sergei A. Abramov on the occasion of his 70th birthday*

**Abstract** Abramov's algorithm enables us to decide whether a univariate rational function can be written as a difference of another rational function, which has been a fundamental algorithm for rational summation. In 2014, Chen and Singer have generalized Abramov's algorithm to the case of rational functions in two ($q$-)discrete variables. In this paper we solve the remaining three mixed cases, which completes our recent project on bivariate extensions of Abramov's algorithm for rational summation.

**Keywords** Abramov's algorithm · Discrete residues · Ostrogradsky–Hermite reduction · Symbolic integration · Symbolic summation

## 1 Introduction

Symbolic summation has been a powerful tool in combinatorics and mathematical physics, whose history is as long as that of symbolic computation. Abramov's

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algorithm [1] for rational summation is one of the first few fundamental algorithms in symbolic summation. The central problem in symbolic summationis whether the sum of a given sequence can be written in “closed form”. A given sequence $f(n)$ belonging to some domain $D$ is said to be *summable* if $f(n) = g(n+1) - g(n)$ for some sequence $g \in D$. The problem of deciding whether a given sequence is summable or not in $D$ is called the *summability problem* in $D$. For example, if $D$ is the field of rational functions, then for $f = 1/(n(n+1))$ we can find $g = 1/n$, while for $f = 1/n$ no suitable $g$ exists in $D$. When $f$ is not summable in $D$, there are several other questions we may ask. One possibility is to ask whether there is a pair $(g, r)$ in $D \times D$ such that $f(n) = g(n+1) - g(n) + r(n)$, where $r$ is minimal in some sense and $r = 0$ if $f$ is summable. This problem is called the *decomposition problem* in [3].

For univariate sequences, extensive work has been done to solve the summability and decomposition problems. In 1971, Abramov solved the summability problem for univariate rational functions in [1]. The Gosper algorithm [19] solves the summability problem for univariate hypergeometric terms. This was then used by Zeilberger [30] in 1990s to design his celebrated telescoping algorithm for hypergeometric terms. The Gosper algorithm was extended further to the $D$-finite case by Abramov and van Hoeij in [6, 7], and to a more general difference-field setting by Karr [22, 23] and Schneider [29]. The decomposition problem was first considered by Ostrogradsky [25] in 1845 and later by Hermite [20] in the continuous setting for rational functions. The discrete case was solved by Abramov in [2], with alternative methods later presented by Abramov himself in [3], and also by Paule [26] and Pirastu [28]. Abramov’s decomposition algorithm was later extended to the hypergeometric case in [4, 5], as well as to continuous extensions in [9, 13, 17].

In 1993, Andrews and Paule [8] raised the general question: is it possible to provide any algorithmic device for reducing multiple sums to single ones? This question is related to symbolic summation in the multivariate case. To make the problem more tractable, we will focus on the first non-trivial case, namely the bivariate rational functions. To this end, let us first introduce some notations. Throughout the paper, let $k$ be a field of characteristic zero and $k(x, y)$ be the field of rational functions in $x$ and $y$. For any $f \in k(x, y)$, we define the shift operators $\sigma_x, \sigma_y$ by

$$\sigma_x(f(x, y)) = f(x+1, y), \quad \sigma_y(f(x, y)) = f(x, y+1),$$

and the $q$-shift operators with $q \in k \setminus \{0\}$ by

$$\tau_{x,q}(f(x, y)) = f(qx, y), \quad \tau_{y,q}(f(x, y)) = f(x, qy).$$

Let $\Delta_v := \sigma_v - 1$ and $\Delta_{v,q} := \tau_{v,q} - 1$ be the difference and $q$-difference operators with respect to $v \in \{x, y\}$, respectively. On the field $k(x, y)$, we can also define the usual derivations $D_x := \partial/\partial_x$ and $D_y := \partial/\partial_y$.

**Definition 1** A rational function $f \in k(x, y)$ is said to be *exact* with respect to the pair $(\partial_x, \partial_y) \in \{D_x, \Delta_x, \Delta_{x,q}\} \times \{D_y, \Delta_y, \Delta_{y,q}\}$ in $k(x, y)$ if $f = \partial_x(g) + \partial_y(h)$ for some $g, h \in k(x, y)$.

We study the following problem, which is a bivariate extension of the summability problem for univariate rational functions.

> **Exactness Testing Problem**. Given a rational function $f \in k(x, y)$, decide whether or not $f$ is exact with respect to $(\partial_x, \partial_y)$ in $k(x, y)$.

According to different types of $(\partial_x, \partial_y)$, the above problem has six different cases up to the symmetry between $x$ and $y$. In the pure continuous case, the problem is also called *integrability problem*, which was first solved by Picard [27, vol 2, p. 220], and see [14] for a more up-to-date presentation. Chen and Singer [16] presented the first necessary and sufficient conditions for the exactness in the pure discrete and $q$-discrete cases. Based on the theoretical criteria in [16], Hou and Wang [21] then gave a practical algorithm for deciding the exactness in the corresponding cases. The goal of this paper is to solve the remaining three mixed cases of the exactness testing problem, which completes our recent project on bivariate extensions of Abramov's algorithm for rational summation.

## 2 Residues and Reduced Forms

In this section, we will prepare some basic tools for testing the exactness of bivariate rational functions. We first introduce the classical residues and their discrete analogue for univariate rational functions. After this we will define reduced forms for bivariate rational functions.

Let $K$ be a field of characteristic zero and $K(z)$ be the field of rational functions in $z$ over $K$. We first define residues with respect to the derivation $D_z$ on $K(z)$. By irreducible partial fraction decomposition, we can always uniquely write a rational function $f \in K(z)$ as

$$f = p + \sum_{i=1}^{n} \sum_{j=1}^{m_i} \frac{a_{i,j}}{d_i^j}, \tag{1}$$

where $p, a_{i,j}, d_i \in K[z]$, $\deg_z(a_{i,j}) < \deg_z(d_i)$ and all of the $d_i$'s are distinct irreducible polynomials. We call $a_{i,1}$ the $D_z$*-residue* of $f$ at $d_i$, denoted by $\operatorname{res}_{D_z}(f, d_i)$.

We now recall the discrete analogue of $D_z$-residues introduced in [15, 21]. Let $\phi$ be an automorphism of $K(z)$ that fixes $K$. For a polynomial $p \in K[z]$, we call the set $\{\phi^i(p) \mid i \in \mathbb{Z}\}$ the $\phi$*-orbit* of $p$, denoted by $[p]_\phi$. Two polynomials $p, q \in K[z]$ are said to be $\phi$-equivalent (denoted as $p \sim_\phi q$) if they are in the same $\phi$-orbit, i.e., $p = \phi^i(q)$ for some $i \in \mathbb{Z}$. When $\phi = \sigma_z$, we can uniquely decompose a rational function $f \in K(z)$ into the form

$$f = p(z) + \sum_{i=1}^{n}\sum_{j=1}^{m_i}\sum_{\ell=0}^{e_{i,j}} \frac{a_{i,j,\ell}}{\sigma_z^{\ell}(d_i)^j}, \tag{2}$$

where $p, a_{i,j,\ell}, d_i \in K[z]$, $\deg_z(a_{i,j,\ell}) < \deg_z(d_i)$ and all of the $d_i$'s are irreducible polynomials such that any two of them are not $\sigma_z$-equivalent. We call the sum $\sum_{\ell=0}^{e_{i,j}} \sigma_z^{-\ell}(a_{i,j,\ell})$ the *$\sigma_z$-residue* of $f$ at $d_i$ of multiplicity $j$, denoted by $\mathrm{res}_{\sigma_z}(f, d_i, j)$.

The following lemma shows some commutativity properties of the residues at some special irreducible polynomials.

**Lemma 1** *Let $f = a/b \in k(x, y)$ and $d \in k[y]$ be an irreducible factor of $b$. Then the following commutativity formulae hold:*

*(i)* $res_{D_y}(\sigma_x(f), d) = \sigma_x(res_{D_y}(f, d))$;
*(ii)* $res_{D_y}(\tau_{x,q}(f), d) = \tau_{x,q}(res_{D_y}(f, d))$;
*(iii)* $res_{\sigma_y}(\tau_{x,q}(f), d, j) = \tau_{x,q}(res_{\sigma_y}(f, d, j))$ *for all* $j \in \mathbb{N}$.

*Proof* To show the first formula, we decompose $f \in k(x, y)$ into the form

$$f = p + \sum_{i=1}^{n}\sum_{j=1}^{m_i} \frac{a_{i,j}}{d_i^j},$$

where $p, a_{i,j} \in k(x)[y]$, $d_i \in k[x, y]$ with $\deg_y(a_{i,j}) < \deg_y(d_i)$ and the $d_i$'s are distinct irreducible polynomials with $d_1 = d \in k[y]$. Since $\sigma_x$ is an automorphism of $k(x, y)$, we have that

$$\sigma_x(f) = \sigma_x(p) + \sum_{j=1}^{m_1} \frac{\sigma_x(a_{1,j})}{d_1^j} + \sum_{i=2}^{n}\sum_{j=1}^{m_i} \frac{\sigma_x(a_{i,j})}{\sigma_x(d_i)^j}$$

is the irreducible partial fraction decomposition of $\sigma_x(f)$ with respect to $y$ over $k(x)$. Then $\mathrm{res}_{D_y}(\sigma_x(f), d) = \sigma_x(a_{1,1}) = \sigma_x(\mathrm{res}_{D_y}(f, d))$. The second formula can be proved similarly. To show the third formula, we decompose $f$ into the form

$$f = p + \sum_{i=1}^{n}\sum_{j=1}^{m_i}\sum_{\ell=0}^{e_{i,j}} \frac{a_{i,j,\ell}}{\sigma_y^{\ell}(d_i)^j},$$

where $p, a_{i,j,\ell} \in k(x)[y]$, $d_i \in k[x, y]$ with $\deg_y(a_{i,j,\ell}) < \deg_y(d_i)$ and the $d_i$'s are irreducible polynomials in distinct $\sigma_y$-orbits with $d_1 = d \in k[y]$. Since $\sigma_y$ is an automorphism of $k(x, y)$, the polynomial $d \in k[y]$ is not $\sigma_y$-equivalent to any irreducible polynomial $d' \in k[x, y]$ with $\deg_x(d') \neq 0$. Then we can decompose $\tau_{x,q}(f)$ into the form

$$\tau_{x,q}(f) = \tau_{x,q}(p) + \sum_{j=1}^{m_1}\sum_{\ell=0}^{e_{1,j}} \frac{\tau_{x,q}(a_{1,j,\ell})}{\sigma_y^{\ell}(d)^j} + \frac{s}{t},$$

where $s \in k(x)[y]$ and $t \in k[x, y]$ satisfying that $\deg_y(s) < \deg_y(t)$ and any irreducible factor of $t$ is not $\sigma_y$-equivalent to $d$. Then for all $j \in \mathbb{N}$ we have

$$\operatorname{res}_{\sigma_y}(\tau_{x,q}(f), d, j) = \sum_{\ell=0}^{e_{1,j}} \sigma_y^{-\ell} \tau_{x,q}(a_{1,j,\ell}) = \tau_{x,q}\left(\sum_{\ell=0}^{e_{1,j}} \sigma_y^{-\ell}(a_{1,j,\ell})\right) = \tau_{x,q}(\operatorname{res}_{\sigma_y}(f, d, j)).$$

This completes the proof. □

Let $\phi$ be any automorphism of $k(x, y)$ that fixes $k(y)$ which will be taken as $\tau_{x,q}$ or $\sigma_x$ in the next section. Then $\phi$ commutes with $D_y$. To study the exactness testing problem with respect to the pair $(\phi, D_y)$, we define reduced forms for rational functions in $k(x, y)$ as follows.

**Definition 2** A rational function $r = \sum_{i=1}^{m} \frac{a_i}{d_i}$ with $a_i \in k(x)[y]$ and $d_i \in k[x, y]$ is said to be $(\phi, D_y)$-*reduced* if $\deg_y(a_i) < \deg_y(d_i)$ and the $d_i$'s are irreducible polynomials in distinct $\phi$-orbits. Let $f \in k(x, y)$. We call the decomposition $f = \phi(g) - g + D_y(h) + r$ with $g, h, r \in k(x, y)$ and $r$ being $(\phi, D_y)$-reduced a $(\phi, D_y)$-*reduced form* of $f$.

We next show that $(\phi, D_y)$-reduced forms always exist for rational functions in $k(x, y)$. For any rational function $f \in k(x, y)$, Ostrogradsky–Hermite reduction [20, 25] decomposes $f$ into the form

$$f = D_y(h) + \sum_{i=1}^{m} \frac{a_i}{d_i}, \tag{3}$$

where $h \in k(x, y), a_i \in k(x)[y], d_i \in k[x, y]$ satisfying that $\deg_y(a_i) < \deg_y(d_i)$ and the $d_i$'s are irreducible over $k(x)$. Let $\phi_1, \phi_2$ be two automorphisms of $k(x, y)$ such that $\phi_1(\phi_2(f)) = \phi_2(\phi_1(f))$ for all $f \in k(x, y)$. Then for any $a, d \in k(x)[y]$, $m, n \in \mathbb{N}$, we have the following reduction formula

$$\frac{a}{\phi_1^m \phi_2^n(d)} = \phi_1(u) - u + \phi_2(v) - v + \frac{\phi_1^{-m}\phi_2^{-n}(a)}{d} \tag{4}$$

where

$$u = \sum_{j=0}^{m-1} \frac{\phi_1^{j-m}(a)}{\phi_1^j \phi_2^n(d)} \quad \text{and} \quad v = \sum_{k=0}^{n-1} \frac{\phi_2^{k-n}\phi_1^{-m}(a)}{\phi_2^k(d)}.$$

By applying the above reduction formula to (3) with $\phi_1 = \phi$ and $\phi_2 = id$, we can further decompose $f$ as

$$f = \phi(g) - g + D_y(h) + \sum_{i=1}^{\tilde{m}} \frac{\tilde{a}_i}{\tilde{d}_i},$$

where $g \in k(x, y)$ and the $\tilde{d}_i$'s are in distinct $\phi$-orbits, which is a $(\phi, D_y)$-reduced form of $f$. The above process for obtaining such a $(\phi, D_y)$-reduced form of $f$ is called a $(\phi, D_y)$*-reduction*.

Next we will define reduced forms for rational functions in $k(x, y)$ with respect to the pair $(\tau_{x,q}, \sigma_y)$. Two polynomials $p, p' \in k[x, y]$ are said to be $(\tau_{x,q}, \sigma_y)$-equivalent if $p = \tau_{x,q}^m \sigma_y^n(p')$ for some $m, n \in \mathbb{Z}$. The set $\{\tau_{x,q}^i \sigma_y^j(p) \in k[x, y] \mid i, j \in \mathbb{Z}\}$ is called the $(\tau_{x,q}, \sigma_y)$-orbit of $p$, denoted by $[p]_{(\tau_{x,q}, \sigma_y)}$.

**Definition 3** A rational function $r = \sum_{i=1}^n \sum_{j=1}^{m_i} \frac{a_{i,j}}{d_i^j} \in k(x, y)$ with $a_{i,j} \in k(x)[y]$ and $d_i \in k[x, y]$ is said to be $(\tau_{x,q}, \sigma_y)$*-reduced* if $\deg_y(a_{i,j}) < \deg_y(d_i)$ and the $d_i$'s are irreducible polynomials in distinct $(\tau_{x,q}, \sigma_y)$-orbits. The decomposition $f = \Delta_{x,q}(g) + \Delta_y(h) + r$ with $g, h, r \in k(x, y)$ and $r$ being $(\tau_{x,q}, \sigma_y)$-reduced is called a $(\tau_{x,q}, \sigma_y)$*-reduced form* of $f$.

The existence of $(\tau_{x,q}, \sigma_y)$-reduced forms for rational functions relies on Abrramov's reduction [3] that decomposes a rational function $f \in k(x, y)$ into the form

$$f = \Delta_y(h) + \sum_{i=1}^{n} \sum_{j=1}^{m_i} \frac{a_{i,j}}{d_i^j},$$

where $h \in k(x, y)$, $a_{i,j} \in k(x)[y]$, $d_i \in k[x, y]$ satisfying that $\deg_y(a_{i,j}) < \deg_y(d_i)$ and the $d_i$'s are irreducible polynomials in distinct $\sigma_y$-orbits. Using the formula (4) with $\phi_1 = \tau_{x,q}$ and $\phi_2 = \sigma_y$, we can further decompose $f$ as

$$f = \Delta_{x,q}(g) + \Delta_y(h) + \sum_{i=1}^{\tilde{n}} \sum_{j=1}^{\tilde{m}_i} \frac{a_{i,j}}{d_i^j},$$

where $g \in k(x, y)$ and the $d_i$'s are in distinct $(\tau_{x,q}, \sigma_y)$-orbits, which is a $(\tau_{x,q}, \sigma_y)$-reduced form of $f$. The above process for obtaining such a $(\tau_{x,q}, \sigma_y)$-reduced form of $f$ is called a $(\tau_{x,q}, \sigma_y)$*-reduction*.

## 3 Exactness Criteria

We first solve the exactness testing problem for the case in which $q \in k$ is a root of unity. Assume that $m$ is the minimal positive integer such that $q^m = 1$ and $k$ contains all $m$th roots of unity. For any $f \in k(x, y)$, it is easy to show that $\tau_{x,q}(f) = f$ if and only if $f \in k(y)(x^m)$. Note that $k(x, y)$ is a finite algebraic extension of $k(y)(x^m)$ of degree $m$. We recall a lemma in [16] on reduced forms for rational functions with respect to $\tau_{x,q}$.

**Lemma 2** *Let $q$ be such that $q^m = 1$ with $m$ minimal and let $f \in k(x, y)$.*

*(a)* $f = \tau_{x,q}(g) - g$ *for some* $g \in k(x, y)$ *if and only if the trace* $\mathrm{Tr}_{k(x,y)/k(y)(x^m)}(f) = 0$.

(b) *Any rational function $f \in k(x, y)$ can be decomposed into*

$$f = \tau_{x,q}(g) - g + c, \quad \text{where } g \in k(x, y) \text{ and } c \in k(y)(x^m). \tag{5}$$

*Moreover, $f$ is $\tau_{x,q}$-summable in $k(x, y)$ if and only if $c = 0$. We call this decomposition a $\tau_{x,q}$*-reduced form *for $f$.*

**Theorem 1** *Let $q$ be such that $q^m = 1$ with $m$ minimal and let $f \in k(x, y)$. Assume that $f = \tau_{x,q}(g) - g + c$ with $g \in k(x, y)$ and $c \in k(y)(x^m)$ is a $\tau_{x,q}$-reduced form of $f$. Then $f$ is exact with respect to $(\tau_{x,q}, \partial_y)$ with $\partial_y \in \{\Delta_y, D_y\}$ if and only if $c = \partial_y(d)$ for some $d \in k(y)(x^m)$.*

*Proof* The sufficiency is clear. To show the necessity, we assume that $f$ is exact with respect to $(\tau_{x,q}, \partial_y)$ with $\partial_y \in \{\Delta_y, D_y\}$, so is $c$, i.e., $c = \Delta_{x,q}(u) + \partial_y(v)$ for some $u, v \in k(x, y)$. Write $u = \sum_{i=0}^{m-1} u_i x^i$ and $v = \sum_{i=0}^{m-1} v_i x^i$ with $u_i, v_i \in k(y, x^m)$. Then we have

$$c = u_1(q-1)x + \cdots + u_{m-1}(q^{m-1}-1)x^{m-1} + \sum_{i=0}^{m-1} \partial_y(v_i)x^i.$$

Since $1, x, \ldots, x^{m-1}$ are linearly independent in $k(x, y)$ over $k(y, x^m)$, we get that $c = \partial_y(v_0)$. □

From now on, we assume that $q \in k \setminus \{0\}$ is not a root of unity. For any $f \in k(x, y)$, we have $\tau_{x,q}(f) = f$ if and only if $f \in k(y)$. We next solve the exactness testing problem in the case when $\partial_x \in \{\Delta_x, \Delta_{x,q}\}$ and $\partial_y = D_y$.

**Theorem 2** *Let $\phi \in \{\sigma_x, \tau_{x,q}\}$ and $f \in k(x, y)$. Assume that $f = \phi(g) - g + D_y(h) + \sum_{i=1}^m a_i/d_i$ with $a_i \in k(x)[y]$ and $d_i \in k[x, y]$ be a $(\phi, D_y)$-reduced form of $f$. Then $f$ is exact with respect to $(\partial_x, D_y)$ with $\partial_x = \phi - 1$ if and only if for each $i \in \{1, \ldots, m\}$, $d_i \in k[y]$ and $a_i = \partial_x(b_i)$ for some $b_i \in k(x)[y]$.*

*Proof* The sufficiency is clear. To show the necessity, we assume that $f$ is exact with respect to $(\partial_x, D_y)$. This implies that $r = \sum_{i=1}^m a_i/d_i$ is also exact with respect to $(\partial_x, D_y)$, i.e., $r = \phi(u) - u + D_y(v)$ for some $u, v \in k(x, y)$. By the Ostrogradsky–Hermite reduction, we first decompose $u$ into the form

$$u = D_y(\tilde{u}) + \sum_{i=1}^{s} \frac{v_i}{w_i},$$

where $\tilde{u} \in k(x, y)$, $v_i \in k(x)[y]$, and the $w_i$'s are irreducible polynomials in $k[x, y]$. Then we have

$$r = \sum_{i=1}^{m} \frac{a_i}{d_i} = T + D_y(\tilde{v}) \quad \text{with } T = \sum_{i=1}^{s} \left( \frac{\phi(v_i)}{\phi(w_i)} - \frac{v_i}{w_i} \right) \text{ and } \tilde{v} = \phi(\tilde{u}) - \tilde{u} + v.$$

Since $\phi$ is an automorphism of $k[x, y]$, the polynomials $\phi(w_i)$ are also irreducible and all of the simple fractions in the irreducible partial fraction decomposition of $T$ have simple poles.

We first show that all of the $d_i$'s are in $k[y]$. Set $\mathscr{D} := \{d_1, \ldots, d_m\}$ and $\mathscr{W} := \{w_1, \ldots, w_s\}$. Note that all of the simple fractions in $D_y(\tilde{v})$ have at least double poles. This implies that $r = T$ and each simple fraction $a_i/d_i$ can only be cancelled with some simple fractions of $T$. Then for each $i \in \{1, \ldots, m\}$, $d_i$ is equal to $w_{j_1}$ or $\phi(w_{j_1})$ for some $j_1 \in \{1, \ldots, s\}$. Assume that $d_i = w_{j_1}$. If $\phi(w_{j_1}) = w_{j_1}$, then $w_{j_1} \in k[y]$ by [15, Lemma 3.4]. Otherwise, $\phi(w_{j_1}) = w_{j_2}$ for some $j_2 \in \{1, \ldots, s\} \setminus \{j_1\}$. Indeed, If $\phi(w_{j_1}) = d_j$ with $i \neq j$, then $d_i$ is $\phi$-equivalent to $d_j$, which contradicts with the assumption that the $d_i$'s are in distinct $\phi$-orbits. If $w_{j_2} = \phi(w_{j_2})$, we also get that $w_{j_2}$ is in $k[y]$ and so is $d_i$. Otherwise $\phi(w_{j_2}) = w_{j_3}$ for some $j_3 \in \{1, \ldots, s\} \setminus \{j_1, j_2\}$. Continuing this process, we either conclude that $d_i \in k[y]$ or get a series of equalities

$$d_i = w_{j_1}, \phi(w_{j_1}) = w_{j_2}, \phi(w_{j_2}) = w_{j_3}, \ldots.$$

Since the set $\mathscr{W}$ is finite, there exists $t$ with $1 \leq t \leq s$ such that $\phi(w_{j_t}) = w_{j_{\tilde{t}}}$ with $1 \leq \tilde{t} \leq t$. Then $w_{j_{\tilde{t}}} = \phi^{t-\tilde{t}+1}(w_{j_{\tilde{t}}})$, which implies that $w_{j_{\tilde{t}}}$ is in $k[y]$ and so is $d_i$. Similarly, we have $d_i \in k[y]$ when $d_i = \phi(w_{j_1})$.

Since $d_i \in k[y]$, applying the commutativity formulae in Lemma 1 yields

$$a_i = \operatorname{res}_{D_y}(r, d_i) = \operatorname{res}_{D_y}(\phi(u) - u + D_y(v), d_i) = \operatorname{res}_{D_y}(\phi(u) - u, d_i) = \phi(b_i) - b_i,$$

where $b_i = \operatorname{res}_{D_y}(u, d_i) \in k(x)[y]$. □

*Example 1* By Theorem 2, the rational function $1/(x + y)$ is not exact with respect to $\Delta_x$ and $D_y$ since $x + y$ is not in $k[y]$. So is the rational function $1/(xy)$ since $1/x \neq \Delta_x(g)$ for any $g \in k(x, y)$.

We now consider the exactness testing problem in the case when $\partial_x = \Delta_{x,q}$ and $\partial_y = \Delta_y$. To this end, we first recall a lemma which is a special case of Lemma 5.4 in [10].

**Lemma 3** *Let $p$ be an irreducible polynomial in $k[x, y]$. Assume that $\tau_{x,q}^i \sigma_y^j(p) = p$ for some $i, j \in \mathbb{Z}$ with $i \neq 0$. Then $p \in k[y]$.*

Let $f \in k(x, y)$. We assume that $f = \Delta_{x,q}(g) + \Delta_y(h) + r$ is a $(\tau_{x,q}, \sigma_y)$-reduced form of $f$. Write $r = \sum_{i=1}^{n} \sum_{j=1}^{m_i} \frac{a_{i,j}}{d_i^j}$, where $a_{i,j} \in k(x)[y]$ and $d_i \in k[x, y]$ satisfying that $\deg_y(a_{i,j}) < \deg_y(d_i)$ and the $d_i$'s are in distinct $(\tau_{x,q}, \sigma_y)$-orbits. Then $f$ is exact with respect to $(\Delta_{x,q}, \Delta_y)$ if and only if $r$ is exact with respect to $(\Delta_{x,q}, \Delta_y)$. Note that the operators $\tau_{x,q}$ and $\sigma_y$ preserve the multiplicities of irreducible factors in the denominators of rational functions. Therefore the rational function $r$ is exact with respect to $(\Delta_{x,q}, \Delta_y)$ if and only if for each $j$, the rational function

$$r_j = \sum_{i=1}^{m} \frac{a_{i,j}}{d_i^j} \tag{6}$$

is exact with respect to $(\Delta_{x,q}, \Delta_y)$. By the same argument in the proof of Lemma 3.2 in [21], $r_j$ is exact with respect to $(\Delta_{x,q}, \Delta_y)$ if and only if each simple fraction $a_{i,j}/d_i^j$ is exact with respect to $(\Delta_{x,q}, \Delta_y)$. We now give an exactness criterion for rational functions of the form $a/d^m$.

**Lemma 4** *Let $f = a/d^m$, where $m \in \mathbb{N}$, $d \in k[x, y]$ is an irreducible polynomial and $a \in k(x)[y]$ is nonzero and $\deg_y(a) < \deg_y(d)$. Then $f$ is exact with respect to $(\Delta_{x,q}, \Delta_y)$ if and only if $d \in k[y]$ and $a = \Delta_{x,q}(b)$ for some $b \in k(x)[y]$.*

*Proof* The sufficiency is clear. For the necessity, we will outline the same argument used in the proof of Theorem 3.7 in [16] or that of Proposition 3.4 in [21]. We assume that $f$ is exact with respect to $(\Delta_{x,q}, \Delta_y)$, i.e., there exist $g, h \in k(x, y)$ such that

$$f = \Delta_{x,q}(g) + \Delta_y(h). \tag{7}$$

We decompose the rational function $g$ into the form

$$g = \sigma_y(g_1) - g_1 + g_2 + \frac{\lambda_1}{\tau_{x,q}^{\mu_1} d^m} + \cdots + \frac{\lambda_s}{\tau_{x,q}^{\mu_s} d^m}, \tag{8}$$

where $\lambda_k \in k(x)[y]$, $\mu_k \in \mathbb{Z}$, $g_1, g_2 \in k(x, y)$ such that $g_2$ is a rational function having no terms of the form $\lambda/(\tau_{x,q}^{\mu} d^m)$ in its partial fraction decomposition with respect to $y$, and the $(\tau_{x,q}^{\mu_i} d^m)$'s are irreducible polynomials in distinct $\sigma_y$-orbits.

The following claim can be shown by the same argument as in [16, 21].

*Claim 1* Let

$$\Lambda := \{\tau_{x,q}^{\mu_1} d, \ldots, \tau_{x,q}^{\mu_s} d, \tau_{x,q}^{\mu_1+1} d, \ldots, \tau_{x,q}^{\mu_s+1} d\}.$$

Then: (1) at least one element of $\Lambda$ is in the same $\sigma_y$-orbit as $d$; (2) for each $\eta \in \Lambda$, there is one element of $(\Lambda \setminus \{\eta\}) \cup \{d\}$ that is $\sigma_y$-equivalent to $\eta$.

Claim 1 implies that either $d \sim_{\sigma_y} \tau_{x,q}^{\mu_1'} d$ or $d \sim_{\sigma_y} \tau_{x,q}^{\mu_1'+1} d$ for some $\mu_1' \in \{\mu_1, \ldots, \mu_s\}$. Assume that $d \sim_{\sigma_y} \tau_{x,q}^{\mu_1'} d$. By the same argument as in [16, 21], we can show that there exists a positive integer $t \leq s$ and $j \in \mathbb{Z}$ such that $\tau_{x,q}^t \sigma_y^j(d) = d$, which implies $d \in k[y]$ by Lemma 3. Similarly, if $d \sim_{\sigma_y} \tau_{x,q}^{\mu_1'+1} d$, then we also have $d \in k[y]$.

Since $d \in k[y]$, applying the commutativity formulae in Lemma 1 yields

$$a = \operatorname{res}_{\sigma_y}(f, d, m) = \operatorname{res}_{\sigma_y}(\Delta_{x,q}(g) + \Delta_y(h), d, m) = \operatorname{res}_{\sigma_y}(\Delta_{x,q}(g), d, m) = \Delta_{x,q}(b),$$

where $b = \operatorname{res}_{\sigma_y}(g, d, m) \in k(x)[y]$. □

We conclude the above discussions by the following theorem.

**Theorem 3** *Let $f \in k(x, y)$ and assume that*

$$f = \Delta_{x,q}(g) + \Delta_y(h) + \sum_{i=1}^{n} \sum_{j=1}^{m_i} \frac{a_{i,j}}{d_i^j}$$

*with* $a_{i,j} \in k(x)[y]$ *and* $d_i \in k[x, y]$ *is a* $(\tau_{x,q}, \sigma_y)$*-reduced form of* $f$. *Then* $f$ *is exact with respect to the pair* $(\Delta_{x,q}, \Delta_y)$ *if and only if for each* $i \in \{1, \ldots, n\}$, $d_i \in k[y]$ *and for each* $j \in \{1, \ldots, m_i\}$, $a_{i,j} = \Delta_{x,q}(b_{i,j})$ *for some* $b_{i,j} \in k(x)[y]$.

*Example 2* By Theorem 3, the rational function $1/(x + y)$ is not exact with respect to $\Delta_{x,q}$ and $\Delta_y$ since $x + y$ is not in $k[y]$. But the rational function $1/(xy)$ is exact with respect to $\Delta_{x,q}$ and $\Delta_y$. In fact, $\frac{1}{xy} = \Delta_{x,q}\left(\frac{q}{(1-q)xy}\right)$.

*Remark 1* The exactness criteria given above reduce the exactness testing problem in the bivariate case to two subproblems: one is testing whether an irreducible polynomial $p \in k[x, y]$ is free of $x$, the other is testing whether a rational function is $(q)$-summable or not with respect to $x$. The first subproblem is easy and the second one can be solved by Abramov's algorithm and its $q$-analogue for univariate rational summation.

## 4 Conclusion

We conclude this paper by recalling the following open problem proposed in [12]:

**Problem 1** Develop an algorithm which takes as input a multivariate hypergeometric term $h$ in $m$ discrete variables $k_1, \ldots, k_m$, and decides whether there exist hypergeometric terms $g_1, \ldots, g_m$ such that

$$h = \Delta_1(g_1) + \cdots + \Delta_m(g_m).$$

Here, $\Delta_i$ is the forward difference operator with respect to the variable $k_i$, i.e.,

$$\Delta_i f(k_1, \ldots, k_m) = f(k_1, \ldots, k_i + 1, \ldots, k_m) - f(k_1, \ldots, k_i, \ldots, k_m).$$

A solution of this problem would be an important step towards the development of a Zeilberger-like algorithm for multisums. Together with the results in [16, 21], the exactness criteria in previous section enable us to completely solve the above problem in the case of bivariate rational functions. The summability criteria in [16, 21] were used in [11] to derive some conditions on the existence of telescopers for trivariate rational functions. Hopefully, the results in this paper can be used to solve the corresponding existence problems for the three mixed cases. An answer to the above open problem may analogously allow for the formulation of existence criteria for telescopers in the multivariate setting. In the long run, we would hope that a multivariate Gosper algorithm serves as a starting point for the development of a reduction-based creative telescoping algorithm for the multivariate setting. A necessary condition for bivariate hypergeometric summability has been given in [18] with applications to proving congruences for double sums in [24] but the summability criterion in this case is still missing and further new ideas and tools are needed to be developed.

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# A $q$-Analogue of the Modified Abramov-Petkovšek Reduction

**Hao Du, Hui Huang and Ziming Li**

*Dedicated to Professor Sergei A. Abramov on the occasion of his 70th birthday*

**Abstract** We present an additive decomposition algorithm for $q$-hypergeometric terms. It decomposes a given term $T$ as the sum of two terms, in which the former is $q$-summable and the latter is minimal in some technical sense. Moreover, the latter is zero if and only if $T$ is $q$-summable. Although our additive decomposition is a $q$-analogue of the modified Abramov-Petkovšek reduction for usual hypergeometric terms, they differ in some subtle details. For instance, we need to reduce Laurent polynomials instead of polynomials in the $q$-case. The experimental results illustrate that the additive decomposition is more efficient than $q$-Gosper's algorithm for determining $q$-summability when some $q$-dispersion concerning the input term becomes large. Moreover, the additive decomposition may serve as a starting point to develop a reduction-based creative-telescoping method for $q$-hypergeometric terms.

**Keywords** Additive decomposition · $q$-hypergeometric term · Reduction Symbolic summation

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## 1 Introduction

$q$-Hypergeometric terms are basic objects in $q$-analysis. An important question concerning $q$-hypergeometric terms is to decide whether such a term is the $q$-difference of another term of the same kind. This question is referred to as the *$q$-summability problem*, which can be solved by a direct $q$-analogue of Gosper's algorithm developed by Koornwinder in [12], or an algebraically motivated $q$-analogue by Paule and Riese in [14]. Both analogues need to compute a polynomial solution of some auxiliary linear $q$-recurrence equation of first order.

An alternative approach to dealing with $q$-summability problem is to decompose a given $q$-hypergeometric term as the sum of two terms of the same kind such that the former is $q$-summable and the latter is minimal in some technical sense. Moreover, the latter is zero if and only if the given term is $q$-summable. How to compute such a decomposition is referred to as the *$q$-additive decomposition problem*.

A rational function over a field of constants can be viewed as a usual hypergeometric or a $q$-hypergeometric term. For the usual shift case, Abramov in [1] developed an algorithm to decompose a rational function into a summable rational function and a nonsummable one whose denominator is of least possible degree. Moreover, a rational function is summable if and only if the nonsummable one in its additive decomposition is equal to zero. Abramov's algorithm can be easily adapted for solving the additive decomposition problem in the $q$-shift case. Both Abramov's algorithm and its $q$-analogue do not require computing polynomial solutions of any auxiliary ($q$-)recurrence equation. Schneider in [15] worked out a general approach to decomposing a rational function over a nonconstant difference field under the assumption that some parametric linear recurrence equation of first order is solvable in the difference field.

Abramov and Petkovšek developed an algorithm for computing an additive decomposition for usual hypergeometric terms in [2, 3]. We call it the Abramov-Petkovšek reduction. Their algorithm needs to compute polynomial solutions of some auxiliary recurrence equation. Part of the Abramov-Petkovšek reduction is translated to the $q$-case by Chen et al. in [9] on the way to establish a criterion on the termination of the $q$-analogue of Zeilberger's algorithm. Chen et al. in [6] present a modified Abramov-Petkovšek reduction for usual hypergeometric terms to avoid computing polynomial solutions of any auxiliary recurrence equation. This feature is crucial for reduction-based creative-telescoping methods.

The goal of this paper is to further develop a $q$-analogue of the modified Abramov-Petkovšek reduction, which provides a solution to the $q$-additive decomposition problem. The analogue also avoids solving any auxiliary $q$-recurrence equation. Similar to the modified Abramov-Petkovšek reduction, it consists of two steps, namely, shell and polynomial reductions. In the usual shift case, the shell reduction was carried out by calculating dispersions and partial fraction decomposition. When implementing its $q$-analogue in Maple, we observe that a combination of $q$-shift homogeneous factorization [3, 13] with the above two calculations yields an overall better performance. This is because the partial fraction decomposition of $q$-rational functions tends to be faster when their denominators split into powers of irreducible factors, which is particularly true when $q$ is an indeterminate. So the step for shell reduction is described in terms of $q$-shift homogeneous factorization. Moreover, in order to obtain $q$-shift-free denominators, we need to allow some numerators to be Laurent polynomials, which complicates the step for polynomial reduction. Experimental results illustrate that the $q$-analogue of the modified Abramov-Petkovšek reduction outperforms $q$-Gosper's algorithm when the $q$-dispersions of the denominators of shells become large. Please see Sect. 6 for more details. Hopefully, this $q$-analogue may enable us to develop a reduction-based creative-telescoping method for $q$-hypergeometric terms in a similar way as in [4, 6].

## 2 Summability and Congruences

Throughout the paper, let $C$ be a field of characteristic zero, and $\sigma$ be an automorphism of $C[x]$ such that $C$ is the subfield of constants with respect to $\sigma$. Then $\sigma(x) = \lambda x + \mu$ with $\lambda \in C \setminus \{0\}$ and $\mu \in C$, where either $\mu \neq 0$ or $\lambda \neq 1$ (cf. [11]). We call $\sigma$ the usual shift operator if $(\lambda, \mu) = (1, 1)$; and call it a $q$-shift operator if $(\lambda, \mu) = (q, 0)$, where $q$ is not a root of unity. The automorphism $\sigma$ can be naturally extended to $C(x)$. Let $\Delta$ be the difference operator $\sigma - \mathbf{1}$ on $C(x)$, where $\mathbf{1}$ stands for the identity map from $C(x)$ to itself.

Let $R$ be a ring extension of $C(x)$. Assume that $\sigma$ can be extended to a monomorphism of $R$. An element $r \in R$ is called a *constant* if $\sigma(r) = r$. The subset of constants in $R$ forms a subring, which is denoted by $C_R$.

An invertible element $T$ of $R$ is said to be *hypergeometric* with respect to $\sigma$ or *$\sigma$-hypergeometric* for short if its $\sigma$-quotient $\sigma(T)/T$ belongs to $C(x)$. Every nonzero element of $C(x)$ is $\sigma$-hypergeometric. When $\sigma$ is a $q$-shift operator, $\sigma$-hypergeometric terms are also called *$q$-hypergeometric terms*. All conclusions in this section are valid for general $\sigma$-hypergeometric terms.

Two $\sigma$-hypergeometric terms are said to be *similar* if their ratio belongs to $C(x)$. A $\sigma$-hypergeometric term $T$ is said to be *summable* if there exists another $\sigma$-hypergeometric term $G$ such that $T = \Delta(G)$. It is straightforward to verify that two $\sigma$-hypergeometric terms $T$ and $G$ are similar if $T = \Delta(G)$. A key idea on determining summability of a given $\sigma$-hypergeometric term $T$ is to write $T = fH$, where $f$ is a nonzero element of $C(x)$, and $H$ is another $\sigma$-hypergeometric term whose $\sigma$-quotient satisfies certain properties (see [2]). With such a multiplicative decomposition at hand, we see that determining the summability of $T$ amounts to finding a rational function $g$ such that $fH = \Delta(gH)$. Assume that $K$ is the $\sigma$-quotient of $H$. Then $f\,H$ is summable if and only if $f = K\sigma(g) - g$ for some $g \in C(x)$. In other words, determining the summability of $f\,H$ amounts to finding a rational solution of the first-order linear recurrence equation $K\sigma(z) - z = f$.

Let us formulate the above deduction in a different way, which will be convenient to describe various congruences in the sequel. Let $K$ be a nonzero rational function in $C(x)$. Then $K\sigma$ is a $C$-linear automorphism of $C(x)$ that maps $f$ to $K\sigma(f)$.

We define a $C$-linear map $\Delta_K = K\sigma - \mathbf{1}$ from $C(x)$ to itself. Then, for any $\sigma$-hypergeometric term $H$ with $\sigma(H)/H = K$, we have $fH$ is summable if and only if $f \in \text{im}(\Delta_K)$. This image is a $C$-linear subspace contained in $C(x)$.

Our reduction in the sequel relies on four congruences modulo the image of $\Delta_K$. The first two congruences are given below.

**Lemma 2.5** *Let $K$ be a nonzero rational function in $C(x)$. Then, for every $f \in C(x)$,*

$$f \equiv K\sigma(f) \mod \text{im}(\Delta_K) \quad \text{and} \quad f \equiv (K\sigma)^{-1}(f) \mod \text{im}(\Delta_K).$$

*Proof* The first congruence follows immediately from the definition of $\Delta_K$. To prove the second one, we note that $K\sigma$ is a bijection. Therefore, there exists $g \in C(x)$ such that $f = K\sigma(g)$. By the first congruence, $g \equiv K\sigma(g) \mod \text{im}(\Delta_K)$. Replacing $g$ with $(K\sigma)^{-1}(f)$ yields the second congruence. □

**Corollary 2.1** *Let $K$ be a nonzero rational function of $C(x)$. Then, for every $f \in C(x)$ and $m \in \mathbb{N}$, we have*

$$f \equiv \sigma^m(f) \prod_{i=0}^{m-1} \sigma^i(K) \mod \text{im}(\Delta_K)$$

*and*

$$f \equiv \sigma^{-m}(f) \prod_{i=1}^{m} \sigma^{-i}\left(K^{-1}\right) \mod \text{im}(\Delta_K).$$

*Proof* By Lemma 2.5 and a straightforward induction, we see that

$$f \equiv (K\sigma)^m(f) \mod \text{im}(\Delta_K) \quad \text{and} \quad f \equiv (K\sigma)^{-m}(f) \mod \text{im}(\Delta_K).$$

The corollary follows from the definition of $K\sigma$ and its inverse. □

The two congruences in the above corollary will be called the *forward and backward congruences*, respectively.

*Remark 2.2* The two congruences in Lemma 2.5 can be translated into two equalities:

$$f = \Delta_K(-f) + K\sigma(f) \quad \text{and} \quad f = \Delta_K(g) + (K\sigma)^{-1}(f),$$

where $g = (K\sigma)^{-1}(f)$. It follows that both forward and backward congruences can be translated into equalities.

The notions of shift and $q$-shift reduced rational functions are introduced in [2] and [9], respectively. We extend them slightly, because the next two congruences hold in both shift and $q$-shift cases. Let $K \in C(x)$ be a nonzero rational function with numerator $u$ and denominator $v$. We say that $K$ is *reduced with respect to* $\sigma$ or *$\sigma$-reduced* for short if $u$ and $\sigma^i(v)$ are relatively prime for all $i \in \mathbb{Z}$.

**Lemma 2.6** *Let $K \in C(x)$ be a $\sigma$-reduced rational function with numerator $u$ and denominator $v$. Then, for every $a \in C[x]$ and $n \in \mathbb{N}$, there exist two polynomials $b_1$ and $b_2$ in $C[x]$ such that*

$$\frac{a}{\prod_{i=0}^{n}\sigma^i(v)} \equiv \frac{b_1}{v} \mod \operatorname{im}(\Delta_K) \quad \textit{and} \quad \frac{a}{\prod_{j=1}^{n}\sigma^{-j}(u)} \equiv \frac{b_2}{v} \mod \operatorname{im}(\Delta_K).$$

*Proof* We prove the first congruence by induction on $n$. Let $w_n = \prod_{i=0}^{n}\sigma^i(v)$. The congruence is trivial when $n = 0$. Assume that it holds for $n - 1$. Setting $f = a/w_n$ in the second congruence in Lemma 2.5, we see that

$$\frac{a}{w_n} \equiv \sigma^{-1}\left(\frac{a}{w_n}\frac{v}{u}\right) = \frac{\sigma^{-1}(a)}{w_{n-1}\sigma^{-1}(u)} \mod \operatorname{im}(\Delta_K).$$

Since $K$ is $\sigma$-reduced, $\gcd(w_{n-1}, \sigma^{-1}(u)) = 1$, there exist $e_1, e_2 \in C[x]$ such that

$$\frac{a}{w_n} \equiv \frac{\sigma^{-1}(a)}{w_{n-1}\sigma^{-1}(u)} = \frac{e_1}{w_{n-1}} + \frac{e_2}{\sigma^{-1}(u)} \mod \operatorname{im}(\Delta_K).$$

By the induction hypothesis, the first summand is congruent to $b_1'/v$ for some $b_1'$ in $C[x]$. Setting $f = e_2/\sigma^{-1}(u)$ in the first congruence in Lemma 2.5, we see that the second summand is congruent to $\sigma(e_2)/v$. Setting $b_1 = b_1' + \sigma(e_2)$ establishes the first congruence in this lemma.

To prove the second congruence, we notice that the product in the denominator equals one when $n = 0$ and then there is nothing to show in this case. For $n = 1$, we set $f = a/\sigma^{-1}(u)$ in the forward congruence in Lemma 1 to get

$$\frac{a}{\sigma^{-1}(u)} \equiv K\sigma\left(\frac{a}{\sigma^{-1}(u)}\right) = \frac{\sigma(a)}{v} \mod \operatorname{im}(\Delta_K),$$

which is exactly the second congruence with $n = 1$. The induction can be completed in a similar way as in the proof of the first congruence. □

*Remark 2.3* In the above proof, all congruences are obtained from the congruences in Lemma 2.5. So they can be translated into equalities by Remark 2.2.

## 3 Kernels, Shells and $\sigma$-Factorizations in the $q$-Case

From now on, we assume that $\sigma$ is an automorphism of $C(x)$ such that $\sigma(x) = q\,x$, where $q$ is neither zero nor any root of unity in $C$. According to [14], a polynomial $p$ in $C[x]$ is said to be *$q$-monic* if $p(0) = 1$. Assume that $p$ is $q$-monic. Then so is $\sigma^i(p)$ for all $i \in \mathbb{Z}$. If, moreover, $p$ is irreducible, then $\sigma^i(p)$ and $p$ are coprime for all $i \in \mathbb{Z}$ with $i \neq 0$. Let $f$ be a nonzero rational function in $C(x)$ with denominator $a$

and numerator $b$. By a factor of $f$, we mean a factor of either $a$ or $b$. We say that $f$ is *$q$-monic* if both $a$ and $b$ are $q$-monic.

For a nonzero rational function $f$ in $C(x)$, there exist a $\sigma$-reduced rational function $K$ and a nonzero rational function $S$ such that

$$f = K\frac{\sigma(S)}{S}.$$

We call $K$ a *kernel* and $S$ the corresponding *shell* of $f$. They can be computed by gcd-calculations (cf. [9]).

Recall that an element of $C(x)$ is proper if its numerator has degree lower than that of the denominator, and that it is a Laurent polynomial if the denominator is a power of $x$. All Laurent polynomials in $C(x)$ form a subring, which is denoted by $C[x, x^{-1}]$. Every nonzero rational function can be decomposed as the sum of a Laurent polynomial and a proper rational function whose denominator is $q$-monic. This decomposition enables us to deal with Laurent polynomials and proper rational functions with $q$-monic denominators separately.

A nonzero Laurent polynomial $f$ can be written in the form $\sum_{i=m}^{n} c_i x^i$, where $m, n \in \mathbb{Z}$ with $m \leqslant n$ and $c_m, c_{m+1}, \ldots, c_n \in C$ with $c_m c_n \neq 0$. We call $n$ the *head degree of $f$* and $m$ the *tail degree of $f$*. They are denoted by $\text{hdeg}(f)$ and $\text{tdeg}(f)$, respectively. Moreover, we define $\text{hdeg}(0) = -\infty$ and $\text{tdeg}(0) = +\infty$. Such a convention agrees with the inequalities: for all $f, g \in C[x, x^{-1}]$,

$$\text{hdeg}(f+g) \leqslant \max(\text{hdeg}(f), \text{hdeg}(g)) \quad \text{and} \quad \text{tdeg}(f+g) \geqslant \min(\text{tdeg}(f), \text{tdeg}(g)).$$

Furthermore, the ring of Laurent polynomials in $\sigma$ over $\mathbb{Z}$, denoted by $\mathbb{Z}[\sigma, \sigma^{-1}]$, is useful to describe a number of notions uniformly in the sequel.

Let $p$ be a nonzero polynomial and $\alpha = \sum_{i=m}^{n} k_i \sigma^i$ be in $\mathbb{Z}[\sigma, \sigma^{-1}]$. We define

$$p^{\alpha} := \prod_{i=m}^{n} \sigma^i(p)^{k_i}.$$

Clearly, $p^{\alpha}$ is a polynomial if and only if $\alpha$ belongs to $\mathbb{N}[\sigma, \sigma^{-1}]$.

According to [11, Definition 11] and [3, Definition 1], two irreducible polynomials $a, b \in C[x]$ are said to be equivalent with respect to $\sigma$ or $\sigma$-equivalent for short if $a \mid \sigma^i(b)$ for some $i \in \mathbb{Z}$. The $\sigma$-equivalence of two polynomials can be easily recognized by comparing coefficients. A rational function $f \in C(x)$ is said to be *$q$-shift homogeneous* if all nonconstant irreducible factors of the numerator and denominator of $f$ belong to the same $\sigma$-equivalence class.

For any nonzero rational function $f \in C(x)$, by grouping together $\sigma$-equivalent factors of its numerator and denominator, it can be written in the form

$$f = c\,x^m \prod_{i=1}^{s} p_i^{\alpha_i}, \tag{1}$$

where $c \in C \setminus \{0\}$, $m \in \mathbb{Z}$, $s \in \mathbb{N}$, $\alpha_i \in \mathbb{Z}[\sigma, \sigma^{-1}]$, $p_i \in C[x]$ is nonconstant, $q$-monic and irreducible for $i = 1, \ldots, s$, and the $p_i$'s are pairwise inequivalent with respect to $\sigma$. Each $p_i^{\alpha_i}$ is both $q$-monic and $q$-shift homogeneous. Note that there are many different ways to express $p_i^{\alpha_i}$ in (1), because

$$p_i^{\alpha_i} = \left(p_i^{\sigma^\ell}\right)^{\sigma^{-\ell}\alpha_i}$$

for all $\ell \in \mathbb{Z}$. Nonetheless, the $q$-monic and $q$-shift homogeneous components $p_i^{\alpha_i}$'s are uniquely determined by $f$, since $C[x]$ is a unique factorization domain. So we call (1) the *$q$-shift-homogeneous factorization* of $f$ or *$\sigma$-factorization* for short.

Let $f$ be a nonzero rational function in $C(x)$, and $p$ be a $q$-monic and irreducible polynomial of positive degree. Then there exists a unique element $\alpha \in \mathbb{Z}[\sigma, \sigma^{-1}]$ such that $f/p^\alpha$ has no factor $\sigma$-equivalent to $p$. We call $\alpha$ the *$\sigma$-exponent* of $p$ in $f$. In addition, the multiplicity of $x$ in $f$ is also called the $\sigma$-exponent of $x$ in $f$.

Note that a rational function $K$ is $\sigma$-reduced if and only if, for every nonconstant, $q$-monic and irreducible polynomial $p$, the nonzero coefficients of the $\sigma$-exponent of $p$ in $K$ have the same sign. The next proposition describes a special property of $\sigma$-reduced rational functions and will be used to distinguish rational and irrational $q$-hypergeometric terms.

**Proposition 3.1** *Let $r$ be a $\sigma$-reduced rational function in $C(x)$. If $r = \sigma^k(f)/f$ for some $f \in C(x)$ and $k \in \mathbb{Z}$, then $r$ is a power of $q$.*

*Proof* The conclusion clearly holds if $k = 0$. Assume that $k$ is nonzero and that the $\sigma$-factorization of $f$ is given in (1). Suppose that $s > 0$. Then

$$r = q^{km} p_1^{\beta_1} \cdots p_s^{\beta_s},$$

where $m \in \mathbb{Z}$ and $\beta_i = \sigma^k\alpha_i - \alpha_i \neq 0$ for all $i$ with $1 \leqslant i \leqslant s$. It follows that $\beta_i$ must have both positive and negative coefficients. On the other hand, the coefficients of $\beta_i$ are either all nonpositive or all nonnegative, because $r$ is $\sigma$-reduced. This contradiction implies that $s = 0$, i.e., $r = q^{km}$. □

**Corollary 3.2** *Let $T \in R$ be a $q$-hypergeometric term. Assume that $K$ is a kernel of $\sigma(T)/T$. Then $K$ is a power of $q$ if and only if $T$ is of the form $cf$ for some $c \in C_R$ and $f \in C(x)$.*

*Proof* Assume that $K = q^m$ for some integer $m$. Then $\sigma(T)/T = q^m\sigma(S)/S$, where $S$ is the corresponding shell of $\sigma(T)/T$ with respect to $K$. It follows from the equality $q^m = \sigma(x^m)/x^m$ that $T/(x^m S)$ is a constant, say $c$, of the ring $R$. Thus $T = cx^m S$. Taking $f = x^m S$ yields the assertion. Conversely, assume that $T = cf$ with $c \in C_R$ and $f \in C(x)$. Then $\sigma(T)/T = \sigma(f)/f = K\sigma(S)/S$. Thus, $K = \sigma(r)/r$ with $r = f/S$, which belongs to $C(x)$. By Proposition 3.1, $K$ is a power of $q$. □

Note that $R$ can be chosen so that $C_R$ coincides with the field $C$ if $C$ is further assumed to be algebraically closed. Indeed, with an algebraically closed field $C$, we are able to construct a Picard-Vessiot extension of $C(x)$ that having no new constants and containing all $\sigma$-hypergeometric terms that interest us (cf. [5, 10]).

## 4 Shell Reduction

Let $T$ be a $q$-hypergeometric term whose $\sigma$-quotient has a kernel $K$ and the corresponding shell $S$. Then there exists another $q$-hypergeometric term $H$ with $\sigma$-quotient $K$ such that $T = SH$, which is called a *multiplicative decomposition* of $T$. We are going to reduce the shell $S$ modulo $\operatorname{im}(\Delta_K)$ to a rational function $r$, which is minimal in some sense. The reduction leads to $T = \Delta(G) + rH$ for some $q$-hypergeometric term $G$. Some special properties of $r$ and $H$ will make it easy to decide the $q$-summability of $T$. We begin with a description on the properties that $r$ should satisfy.

**Definition 4.4** A nonzero and $q$-monic polynomial $f \in C[x]$ is called *$q$-shift-free* or *$\sigma$-free* for short if $\gcd(f, \sigma^i(f)) = 1$ for all nonzero integer $i$.

The reader may find a more general definition of $q$-shift free polynomials in [9].

*Remark 4.1* For a nonzero polynomial with the $\sigma$-factorization given in (1), the polynomial is $\sigma$-free if and only if $m = 0$ and every $\alpha_i$ is a monomial in $\mathbb{N}[\sigma, \sigma^{-1}]$.

**Definition 4.5** Let $K \in C(x)$ be a $\sigma$-reduced rational function whose numerator and denominator are $u$ and $v$, respectively. A nonzero polynomial $f \in C[x]$ is said to be *strongly coprime with* $K$ if $\gcd(\sigma^i(f), u) = \gcd(\sigma^{-i}(f), v) = 1$ for all $i \in \mathbb{N}$.

*Remark 4.2* Let the $\sigma$-factorization of a nonzero polynomial $f$ be given in (1). Assume that $\lambda_i$ and $\mu_i$ are the $\sigma$-exponents of $p_i$ in $u$ and $v$, respectively, $i = 1, \ldots, s$. Then $f$ is strongly coprime with $K$ if and only if

$$\operatorname{tdeg}(\alpha_i) > \operatorname{hdeg}(\lambda_i) \quad \text{and} \quad \operatorname{hdeg}(\alpha_i) < \operatorname{tdeg}(\mu_i) \quad \text{for all } i \text{ with } 1 \leqslant i \leqslant s.$$

The next lemma is used to verify the minimality of our additive decomposition in the sequel.

**Lemma 4.1** *Let $K \in C(x)$ be a $\sigma$-reduced rational function with numerator $u$ and denominator $v$. Let $g \in C(x)$ with denominator $d$, which is $\sigma$-free and strongly coprime with $K$. If there exist two rational functions $\tilde{g}$ and $r$ such that*

$$v(g - \tilde{g}) - (u\sigma(r) - vr) \in C[x, x^{-1}], \tag{2}$$

*then the degree of $d$ is no more than that of the denominator of $\tilde{g}$.*

*Proof* Let $\tilde{d}$ be the denominator of $\tilde{g}$. There is nothing to show if $d \in C$. Now assume that $d \notin C$ and consider a nontrivial irreducible factor $p \in C[x]$ of $d$ with multiplicity $k$. Since $d$ is strongly coprime with $K$, it is coprime with $v$. Since $d$ is $\sigma$-free, it suffices to prove that $\sigma^\ell(p)^k \mid \tilde{d}$ for some $\ell \in \mathbb{Z}$. To this end, we let $e$ be the denominator of $r$. Suppose that $p^k$ does not divide $\tilde{d}$, otherwise we have done. Then it follows from (2) that either $p^k \mid e$ or $p^k \mid \sigma(e)$.

If $p^k \mid e$, then there is an integer $\ell \geqslant 1$ such that $\sigma^{\ell-1}(p)^k \mid e$ but $\sigma^\ell(p)^k \nmid e$. Moreover, $\sigma^\ell(p)^k \mid \sigma(e)$. On the other hand, $\sigma^\ell(p) \nmid d$ because $d$ is $\sigma$-free; and $\sigma^\ell(p) \nmid u$ because $d$ is strongly coprime with $K$. Thus, (2) implies that $\sigma^\ell(p)^k \mid \tilde{d}$.

If $p^k \mid \sigma(e)$, then there is an integer $\ell \leqslant -1$ such that $\sigma^\ell(p)^k \mid e$ but $\sigma^{\ell-1}(p)^k \nmid e$, i.e., $\sigma^\ell(p)^k \nmid \sigma(e)$. Observe that $\sigma^\ell(p) \nmid d$ because $d$ is $\sigma$-free, and $\sigma^\ell(p) \nmid v$ because $d$ is strongly coprime with $K$. Thus, $\sigma^\ell(p)^k \mid \tilde{d}$ by (2). □

Now, we describe how to perform shell reduction "locally".

**Lemma 4.2** *Let $K \in C(x)$ be a $\sigma$-reduced rational function with numerator $u$ and denominator $v$. Let $f \in C(x)$ be a nonzero rational function with denominator $p^\alpha$, where $p \in C[x]$ is nonconstant, $q$-monic and irreducible and $\alpha \in \mathbb{N}[\sigma, \sigma^{-1}]$. Assume that $\lambda$ and $\mu$ are the $\sigma$-exponents of $p$ in $u$ and $v$, respectively. Then we have the following two assertions.*

*(i) If $\mu = 0$, then, for every integer $\ell$ with $\ell \geqslant \operatorname{hdeg}(\alpha)$, there exist $k$ in $\mathbb{N}$ and $a, b$ in $C[x]$ such that*

$$f \equiv \frac{a}{p^{k\sigma^\ell}} + \frac{b}{v} \mod \operatorname{im}(\Delta_K). \tag{3}$$

*(ii) If $\lambda = 0$, then, for every integer $\ell$ with $\ell \leqslant \operatorname{tdeg}(\alpha)$, there exist $k$ in $\mathbb{N}$ and $a, b$ in $C[x]$ such that (3) also holds.*

*Proof* If $\alpha = 0$, then $f \in C[x]$. So we just need to set $k = 0$, $a = f$ and $b = 0$, and assume that $\alpha$ is nonzero in the rest of the proof.

(i) Assume that $\alpha = \sum_{i=m}^n k_i \sigma^i$, where $m \leqslant n$, $k_i \in \mathbb{N}$ and $k_m k_n \neq 0$. Since $p$ is $q$-monic and irreducible, the polynomials $p^{\sigma^m}, p^{\sigma^{m+1}}, \ldots, p^{\sigma^n}$ are pairwise coprime. Then we have a partial fraction decomposition $f = \sum_{i=m}^n f_i$, where $f_i$ is either zero or has the denominator $p^{k_i\sigma^i}$ for all $i$ with $m \leqslant i \leqslant n$.

Assume that $f_i$ is nonzero. By the forward congruence in Corollary 2.1, for every integer $\ell$ with $\ell \geqslant n$, there exists $g_i \in C[x]$ such that

$$f_i \equiv \frac{g_i}{p^{k_i\sigma^\ell} \prod_{j=0}^{\ell-i-1} \sigma^j(v)} \mod \operatorname{im}(\Delta_K).$$

It follows from $\mu = 0$ that $p^{\sigma^\ell}$ is coprime with any $q$-shifts of $v$. Then there exist two polynomials $a_i, \tilde{a}_i$ in $C[x]$ such that

$$f_i \equiv \frac{a_i}{p^{k_i\sigma^\ell}} + \frac{\tilde{a}_i}{\prod_{j=0}^{\ell-i-1} \sigma^j(v)} \mod \operatorname{im}(\Delta_K). \tag{4}$$

Applying the first congruence in Lemma 2.6 to the second summand in the right-hand side of the above congruence, we find $b_i \in C[x]$ such that

$$f_i \equiv \frac{a_i}{p^{k_i\sigma^\ell}} + \frac{b_i}{v} \mod \operatorname{im}(\Delta_K).$$

Summing up all these congruences yields

$$f \equiv \frac{a}{p^{k\sigma^\ell}} + \frac{b}{v} \mod \operatorname{im}(\Delta_K),$$

where $a, b \in C[x]$ and $k \in \mathbb{N}$ with $k \leqslant \max(k_m, k_{m+1}, \ldots, k_n)$.

(ii) The congruence (3) can be proved by a similar argument, in which we use the backward congruence in Corollary 2.1 and the second congruence in Lemma 2.6. Moreover, $\lambda = 0$ implies that $p^{\sigma^\ell}$ is coprime with any $q$-shifts of $u$. Therefore, a partial fraction decomposition similar to (4) holds, in which $\sigma^j(v)$ is replaced with $\sigma^{-j}(u)$ and $j$ ranges from 1 to $i - \ell$. □

The above lemma leads to a key step for the shell reduction.

**Corollary 4.1** *Let $K$, $f$, $p$ and $\alpha$ be the same as those in Lemma 4.2. Then there exist two polynomials $a, b \in C[x]$ and a monomial $\beta \in \mathbb{N}[\sigma, \sigma^{-1}]$ such that*

$$f \equiv \frac{a}{p^\beta} + \frac{b}{v} \mod \operatorname{im}(\Delta_K). \tag{5}$$

*Moreover, $p^\beta$ is both $\sigma$-free and strongly coprime with $K$.*

*Proof* Let $\lambda$ and $\mu$ be the $\sigma$-exponents of $p$ in $u$ and $v$, respectively. Then either $\lambda$ or $\mu$ is zero since $K$ is $\sigma$-reduced.

First, assume that $p^\alpha$ is strongly coprime with $K$. Set $\ell = \operatorname{hdeg}(\alpha)$ when $\mu = 0$ or $\ell = \operatorname{tdeg}(\alpha)$ when $\lambda = 0$. By Lemma 4.2, the congruence (5) holds in which $\beta = k\sigma^\ell$ is a monomial. Hence, $p^\beta$ is $\sigma$-free and strongly coprime with $K$.

Second, assume that $p^\alpha$ is not strongly coprime with $K$. Then either $\operatorname{tdeg}(\alpha)$ is no greater than $\operatorname{hdeg}(\lambda)$ or $\operatorname{hdeg}(\alpha)$ is no smaller than $\operatorname{tdeg}(\mu)$.

If $\operatorname{tdeg}(\alpha) \leqslant \operatorname{hdeg}(\lambda)$. then neither $\alpha$ nor $\lambda$ equals zero. Thus, $\mu = 0$ because $K$ is $\sigma$-reduced. Set $\ell = \max(\operatorname{hdeg}(\alpha), \operatorname{hdeg}(\lambda) + 1)$. By Lemma 4.2 (i), the congruence (5) holds, in which $\beta$ is a monomial. Consequently, $p^\beta$ is $\sigma$-free. Moreover, it is strongly coprime with $K$, as $\operatorname{tdeg}(\beta) > \operatorname{hdeg}(\lambda)$ and $\operatorname{hdeg}(\beta) < \operatorname{tdeg}(\mu) = +\infty$.

If $\operatorname{hdeg}(\alpha) \geqslant \operatorname{tdeg}(\mu)$, then neither $\alpha$ nor $\mu$ is zero. So $\lambda = 0$. The congruence (5) holds by Lemma 4.2 (ii), in which $\ell$ is set to be $\min(\operatorname{tdeg}(\alpha), \operatorname{tdeg}(\mu) - 1)$. □

The main result of this section is given below.

**Theorem 4.4** *Let $K \in C(x)$ be a $\sigma$-reduced rational function whose numerator and denominator are $u$ and $v$, respectively. For every rational function $f \in C(x)$, there* *exists a proper rational function $g \in C(x)$ and a Laurent polynomial $h \in C[x, x^{-1}]$ such that*

$$f \equiv g + \frac{h}{v} \mod \operatorname{im}(\Delta_K) \tag{6}$$

*with the property that the denominator of $g$ is $\sigma$-free and strongly coprime with $K$. Moreover, the denominator of $g$ is of minimal degree in the sense that if there exists another pair $(\tilde{g}, \tilde{h})$ with $\tilde{g} \in C(x)$ and $\tilde{h} \in C[x, x^{-1}]$ such that*

$$f \equiv \tilde{g} + \frac{\tilde{h}}{v} \mod \operatorname{im}(\Delta_K) \tag{7}$$

*then the degree of the denominator of $g$ is no greater than that of $\tilde{g}$. In particular, $g = 0$ if $f \in \operatorname{im}(\Delta_K)$.*

*Proof* Let $cx^m \prod_{i=1}^{s} p_i^{\alpha_i}$ be the $\sigma$-factorization of the denominator of $f$, as described in (1). Then a partial fraction decomposition of $f$ is

$$f = a + \sum_{i=1}^{s} f_i,$$

where $a$ is a Laurent polynomial and $f_i$ is proper with denominator $p_i^{\alpha_i}$ for $i = 1, \ldots, s$. By Corollary 4.1, we have, for all $i$ with $1 \leqslant i \leqslant s$,

$$f_i \equiv \frac{a_i}{p_i^{\beta_i}} + \frac{b_i}{v} \mod \operatorname{im}(\Delta_K),$$

where $a_i, b_i \in C[x]$ and $p_i^{\beta_i}$ is $\sigma$-free and strongly coprime with $K$. Then (6) holds with $g = \sum_{i=1}^{s} a_i / p_i^{\beta_i}$ and $h = va + \sum_{i=1}^{s} b_i$. Note that the irreducible polynomials $p_1, \ldots, p_s$ are $q$-monic and mutually inequivalent with respect to $\sigma$. Thus, the denominator of $g$ is $\sigma$-free. It is clearly strongly coprime with $K$. Moreover, $g$ is proper since the forward and backward congruences do not change the degrees.

It remains to verify that the degree of the denominator of $d$ is minimal. Assume that there exist $\tilde{g} \in C(x)$ and $\tilde{h} \in C[x, x^{-1}]$ such that (7) holds. By (6) and (7), there exists a rational function $r \in C(x)$ such that

$$g + \frac{h}{v} = \frac{u}{v}\sigma(r) - r + \tilde{g} + \frac{\tilde{h}}{v}.$$

Clearing the denominators in this equality, we see that $\deg(d)$ is no greater than the degree of the denominator of $\tilde{g}$ by Lemma 4.1.

Assume that $f \in \operatorname{im}(\Delta_K)$. Then $f \equiv 0 \mod \operatorname{im}(\Delta_K)$. Taking $\tilde{g} = \tilde{h} = 0$ in (7) implies that $g \in C[x]$ by the minimality of $\deg(d)$. Since $g$ is proper, it is zero. □

*Remark 4.3* On the way to compute $g$ and $h$ in (6), we can obtain another rational function $r$ such that

$$f = \Delta_K(r) + g + \frac{h}{v},$$

because all the reductions are based on the forward and backward congruences, which can be easily transformed into equalities, as described in Remark 2.2.

Let us translate Theorem 4.4 into the $q$-hypergeometric setting. This leads to a $q$-analogue of Proposition 3.3 in [6].

**Corollary 4.2** *Let $T$ be a $q$-hypergeometric term whose $\sigma$-quotient has a kernel $K$ with the denominator $v$. Then we have the following assertions.*

*(i) There exist two rational functions $r, g \in C(x)$, a Laurent polynomial $h \in C[x, x^{-1}]$, and a $q$-hypergeometric term $H$ with $\sigma(H)/H = K$ such that*

$$T = \Delta(rH) + \left(g + \frac{h}{v}\right) H.$$

*Moreover, $g$ is proper, and its denominator is $\sigma$-free, strongly coprime with $K$.*

*(ii) If $T$ is $q$-summable, then $g = 0$.*

*Proof* (i) Let $S$ be the shell of $\sigma(T)/T$ corresponding to $K$. By Theorem 4.4,

$$S \equiv g + \frac{h}{v} \mod \operatorname{im}(\Delta_K), \tag{8}$$

where $g$ is a proper rational function whose denominator is $\sigma$-free and strongly coprime with $K$, and $h$ belongs to $C[x, x^{-1}]$. Consequently, there exists $r \in C(x)$ such that

$$S = K\sigma(r) - r + g + \frac{h}{v}.$$

Set $H = T/S$. Then $\sigma(H)/H = K$. It follows that

$$T = \Delta(rH) + \left(g + \frac{h}{v}\right) H.$$

(ii) Assume now that $T$ is $q$-summable, that is, $SH$ is $q$-summable, which is equivalent to the fact $S \in \operatorname{im}(\Delta_K)$. Therefore, $g = 0$ by Theorem 4.4 and (8). □

The shell reduction for $q$-hypergeometric terms renders us an additive decomposition for $q$-rational functions.

**Corollary 4.3** *For $T \in C(x)$, there exist $f, g \in C(x)$ and $c \in C$ such that*

$$T = \Delta(f) + g + c, \tag{9}$$

*where $g$ is a proper rational function with $\sigma$-free denominator $d$. Moreover, if there exist $\tilde{f}, \tilde{g}$ in $C(x)$ and $\tilde{c}$ in $C$ such that*

$$T = \Delta(\tilde{f}) + \tilde{g} + \tilde{c}, \tag{10}$$

*then* $\deg(d)$ *is no greater than the degree of the denominator of* $\tilde{g}$. *In particular,* $T$ *is* $q$*-summable if and only if* $g = c = 0$.

*Proof* Let $K$ be a kernel of $\sigma(T)/T$. By Corollary 3.2, $K = q^m$ for some $m \in \mathbb{Z}$. So we may take 1 as the denominator of $K$. By Corollary 4.2 (i), there exist a rational function $r \in C(x)$, a proper rational function $s \in C(x)$ with $\sigma$-free denominator $d$, a Laurent polynomial $t \in C[x, x^{-1}]$, and a $q$-hypergeometric term $H$ with $\sigma$-quotient $q^m$ such that $T = \Delta(rH) + (s+t)H$. Thus, $H$ belongs to $C(x)$, and, consequently, is equal to $c'x^m$ for some $c' \in C$. It follows that

$$T = \Delta(c'rx^m) + c'sx^m + c'tx^m,$$

Moreover, we can split $c'sx^m$ into the sum of a Laurent polynomial and a proper rational function $g$ whose denominator is equal to $d$. So $T - \Delta(c'rx^m) - g \in C[x, x^{-1}]$, which, together with the fact

$$c_i x^i = \Delta\left(\frac{c_i}{q^i - 1} x^i\right) \quad \text{for all } i \in \mathbb{Z} \text{ with } i \neq 0 \text{ and } c_i \in C,$$

implies that (9) holds.

It follows from (9) and (10) that

$$g - \tilde{g} - \left(\sigma(f - \tilde{f}) - (f - \tilde{f})\right) \in C[x, x^{-1}].$$

Setting $u = 1$ and $v = 1$ in Lemma 4.1, we see that $\deg(d)$ is no greater than the degree of the denominator of $\tilde{g}$.

If both $g$ and $c$ in (9) are equal to zero, then $T$ is clearly $q$-summable. Conversely, assume that $T$ is $q$-summable. Then one can choose both $\tilde{g}$ and $\tilde{c}$ to be zero in (10). It follows from the minimality of $\deg(d)$ that $g = 0$. Consequently, $c$ is $q$-summable, and, thus, $c = 0$. □

Corollary 4.3 is derived from the shell reduction. It may also be obtained by translating the results in [1] directly into the $q$-case.

At last, we turn the proof of Theorem 4.4 into an algorithm, named after *ShellReduction*. To this end, we need to assume that one can factor univariate polynomials over $C$ in the rest of this paper. For example, $C$ is an algebraic number field over $\mathbb{Q}$ or the field of rational functions in several variables other than $x$ over $\mathbb{Q}$.

**ShellReduction**. Given a $\sigma$-reduced rational function $K \in C(x)$ whose numerator and denominator are $u$ and $v$, respectively, and a nonzero rational function $f \in C(x)$, compute two rational functions $r, g \in C(x)$ and a Laurent polynomial $h \in C[x, x^{-1}]$ such that

$$f = \Delta_K(r) + g + \frac{h}{v},$$

and $g$ is proper whose denominator is $\sigma$-free and strongly coprime with $K$.

1. Compute the $\sigma$-factorization $cx^m p_1^{\alpha_1} \cdots p_s^{\alpha_s}$ of the denominator of $f$, where $c \in C \setminus \{0\}$, $m \in \mathbb{N}$, $p_1, \ldots, p_s$ are $q$-monic and irreducible in $C[x] \setminus C$, inequivalent to each other with respect to $\sigma$, and $\alpha_1, \ldots, \alpha_s$ belong to $\mathbb{N}[\sigma, \sigma^{-1}] \setminus \{0\}$.
2. Compute the partial fraction decomposition of $f$ to get

$$f = a + \sum_{i=1}^{s} f_i$$

where $a \in C[x, x^{-1}]$ and $f_i$ is proper with denominator $p_i^{\alpha_i}$ for $i = 1, \ldots, s$.
3. For $i$ from 1 to $s$ do the following. Apply Corollary 4.1 to $f_i$ and find a rational function $r_i$ in $C(x)$, $a_i, b_i$ in $C[x]$ and a monomial $\beta_i$ in $\mathbb{N}[\sigma, \sigma^{-1}]$ such that

$$f_i = \Delta_K(r_i) + \frac{a_i}{p_i^{\beta_i}} + \frac{b_i}{v}.$$

4. Set

$$r := \sum_{i=1}^{s} r_i; \quad g := \sum_{i=1}^{s} \frac{a_i}{p^{\beta_i}}; \quad h := va + \sum_{i=1}^{s} b_i$$

and return.

*Example 4.3* Let $(q; q)_n := \prod_{i=1}^{n} (1 - q^i)$ be a $q$-Pochhammer symbol and

$$T(n) = \frac{(q; q)_n}{1 + q^n},$$

which is a $q$-hypergeometric term with $\sigma(T(n)) = T(n + 1)$ and $q^n = x$. Then the $\sigma$-quotient of $T$ has a kernel $K = -qx + 1$ and the corresponding shell $S = 1/(x + 1)$. Shell reduction yields

$$S = \Delta_K(0) + \frac{1}{x + 1} + \frac{0}{v}$$

where $v = 1$ and the second summand is nonzero. By Corollary 4.2 (ii), $T$ is non-summable.

*Example 4.4* Let $T = -q^{n+1}(q; q)_n$. Then a kernel $K$ of the $\sigma$-quotient of $T$ is equal to $-q^2x + q$ and the corresponding shell $S$ is equal to 1. According to the shell reduction algorithm, $S = \Delta_K(0) + 0 + 1/v$, where $v = 1$. But $T$ is $q$-summable as $T = \Delta((q; q)_n)$.

The above example illustrates that the shell reduction cannot decide $q$-summability completely. One way to proceed is to find a Laurent polynomial solution of an auxiliary first-order linear $q$-recurrence equation, as in the usual shift case [2, 3]. We show how this can be avoided in the next section.

## 5 Reduction for Laurent Polynomials

Corollary 4.2 (ii) provides us with a necessary condition on the summability of $q$-hypergeometric terms. To obtain a necessary and sufficient condition, we confine the numerator $h$ in (6) into a finite-dimensional linear subspace over $C$. This idea was first presented in [4], and has been extended in various ways [6–8].

To deal with Laurent polynomials whose tail and head degrees are arbitrary, we shall first reduce negative powers and then positive ones. To guarantee the termination of our reduction, we introduce the notion of *reduction index*, abbreviated as rind. For a Laurent polynomial $f \in C[x, x^{-1}]$,

$$\operatorname{rind}(f) := \begin{cases} \operatorname{tdeg}(f) & \text{if } \operatorname{tdeg}(f) < 0 \\ \operatorname{hdeg}(f) & \text{if } \operatorname{tdeg}(f) \geqslant 0 \end{cases}$$

Note that rind(0) is equal to $-\infty$, and that nonzero Laurent polynomials with distinct reduction indices are linearly independent over $C$.

**Lemma 5.1** *Let $K \in C(x)$ be a $\sigma$-reduced rational function with numerator $u$ and denominator $v$. Define*

$$\begin{array}{rcl} \phi_K : C[x, x^{-1}] & \longrightarrow & C[x, x^{-1}] \\ f & \mapsto & u\sigma(f) - vf. \end{array}$$

*Then we have the following assertions.*

*(i) The $C$-linear map $\phi_K$ is injective if $K$ is not a power of $q$.*
*(ii) Define*

$$\operatorname{im}(\phi_K)^\top = span_C \left\{ x^d \mid d \neq rind(p) \, for \, all \; p \in \operatorname{im}(\phi_K) \right\}.$$

*Then $C[x, x^{-1}] = \operatorname{im}(\phi_K) \oplus \operatorname{im}(\phi_K)^\top$.*

*Proof* (i) Assume that $K$ is not a power of $q$. If $\phi_K(f) = 0$ for some $f \in C[x, x^{-1}]$, then either $f = 0$ or $v/u = \sigma(f)/f$. The latter implies that $K$ is a power of $q$ by Proposition 3.1, which is impossible. So $f = 0$, that is, $\phi_K$ is injective.

(ii) By the definition of $\operatorname{im}(\phi_K)^\top$, we have $\operatorname{im}(\phi_K) \cap \operatorname{im}(\phi_K)^\top = \{0\}$ and there is a Laurent polynomial

$$f_m \in \operatorname{im}(\phi_K) \cup \operatorname{im}(\phi_K)^\top$$

such that $\operatorname{rind}(f_m) = m$ for every integer $m \in \mathbb{Z}$. Set $B = \{f_m \mid m \in \mathbb{Z}\}$, which consists of linearly independent Laurent polynomials. It suffices to show that $B$ is a $C$-basis of $C[x, x^{-1}]$. Let $g$ be a nonzero Laurent polynomial whose reduction index equals $r$.

*Case 1*. Assume that $r \geqslant 0$. Then $g$ is a $C$-linear combination of $f_0, f_1, \ldots, f_r$.

*Case 2*. Assume that $r < 0$. Then there is a $C$-linear combination $h$ of $f_r$, $f_{r+1}$, ..., $f_{-1}$ such that $g - h$ is of nonnegative tail degree. It follows from case 1 that $g - h$ belongs to the span of $B$ over $C$, and so does $g$.

Hence, $B$ is a $C$-basis of $C[x, x^{-1}]$. □

The map $\phi_K$ defined in the above lemma is called the *reduction map for Laurent polynomials with respect to* $K$ or *the LP-reduction map* for short when $K$ is clear from the context, and $\operatorname{im}(\phi_K)^\top$ is called the *standard complement of* $\operatorname{im}(\phi_K)$. The LP-reduction map is equal to the restriction of $v\Delta_K$ on $C[x, x^{-1}]$, where $v$ is the denominator of $K$.

The importance of standard complements is described in the next lemma.

**Lemma 5.2** *Let $K$ be a $\sigma$-reduced rational function with denominator $v$, and $\phi_K$ be the LP-reduction map. If $g \in \operatorname{im}(\phi_K)^\top$ and $g/v \in \operatorname{im}(\Delta_K)$, then $g$ is equal to zero.*

*Proof* Assume that $g \in \operatorname{im}(\phi_K)^\top$ and $g/v \in \operatorname{im}(\Delta_K)$. It follows from $g/v \in \operatorname{im}(\Delta_K)$ that there exists $f \in C(x)$ such that

$$u\sigma(f) - vf = g, \tag{11}$$

where $u$ is the numerator of $K$. Suppose that $f$ is not a Laurent polynomial. Then its denominator $d$ has a nonconstant, irreducible and $q$-monic factor $p$. Let $\alpha$ be the $\sigma$-exponent of $p$ in $d$ with tail and head degrees $k$ and $\ell$, respectively. Then $\sigma^k(p)$ is not a divisor of $\sigma(d)$. It follows from (11) that $\sigma^k(p)$ divides $v$. Similarly, $\sigma^{\ell+1}(p)$ divides $u$, as it is a divisor of the denominator of $\sigma(f)$ but not a divisor of $d$. We have reached a contradiction with the assumption that $K$ is $\sigma$-reduced. Thus, $f$ is a Laurent polynomial. Hence, $g \in \operatorname{im}(\phi_K) \cap \operatorname{im}(\phi_K)^\top$, which implies that $g = 0$. □

Next, we develop an algorithm for projecting a Laurent polynomial onto the image of an LP-reduction map and its standard complement, respectively.

Let $K \in C(x)$ be $\sigma$-reduced but not a power of $q$. By Lemma 5.1 (i), im $(\phi_K)$ has a $C$-basis $\left\{\phi_K(x^k) \mid k \in \mathbb{Z}\right\}$. From this basis, we can compute another $C$-basis whose elements have distinct reduction indices, which will be referred to as an *echelon basis*. With such a basis, we can project a Laurent polynomial by linear elimination. To this end, we let $K = u/v$ with $u, v \in C[x]$ and $\gcd(u, v) = 1$. Set

$$u = \sum_{i=0}^{d} u_i x^i \quad \text{and} \quad v = \sum_{i=0}^{d} v_i x^i,$$

where the $u_i$ and $v_i$ belong to $C$ for all $i$ with $0 \leqslant i \leqslant d$ and $d = \max(\deg(u), \deg(v))$. So at least one of $u_d$ and $v_d$ is nonzero. Moreover, either $u_0$ or $v_0$ is nonzero because $\gcd(u, v) = 1$. For all $k \in \mathbb{Z}$,

$$\phi_K\left(x^k\right) = \left(u_0 q^k - v_0\right) x^k + \left(u_1 q^k - v_1\right) x^{k+1} + \cdots + \left(u_d q^k - v_d\right) x^{k+d}. \tag{12}$$

We make a case distinction to construct respective echelon bases of $\operatorname{im}(\phi_K)$ and $\operatorname{im}(\phi_K)^\top$. In what follows, $\mathbb{Z}^-$ stands for the set of all negative integers.

*Case 1*. Assume that, for all $k \in \mathbb{Z}^-$, $u_0 q^k - v_0$ is nonzero. Then the reduction index of $\phi_K\left(x^k\right)$ is equal to $k$ for all $k \in \mathbb{Z}^-$ by (12). To compute the reduction index of $\phi_K\left(x^k\right)$ for $k \in \mathbb{N}$, we need to consider two subcases.

*Case 1.1*. Assume further that $u_d q^k - v_d \neq 0$ for all $k \in \mathbb{N}$. Then the reduction index of $\phi_K\left(x^k\right)$ is equal to $d + k$ for all $k \in \mathbb{N}$ by (12). So the images of distinct powers of $x$ under $\phi_K$ have distinct reduction indices, and thus form an echelon basis of $\mathrm{im}(\phi_K)$. It follows that $\mathrm{im}(\phi_K)^\top$ has a basis $\left\{1, x, \ldots, x^{d-1}\right\}$.

*Case 1.2*. Assume that $u_d q^\ell - v_d = 0$ for some $\ell \in \mathbb{N}$. The integer $\ell$ is unique, because $q$ is not a root of unity. Similar to case 1.1, the reduction index of $\phi_K\left(x^k\right)$ is equal to $d + k$ for all $k \in \mathbb{N}$ with $k \neq \ell$. However, the reduction index of $\phi_K\left(x^\ell\right)$ is less than $d + \ell$. Eliminating $x^{d+\ell-1}$, $x^{d+\ell-2}$, …, $x^d$ from $\phi_K\left(x^\ell\right)$ successively by $\phi_K\left(x^{\ell-1}\right)$, $\phi_K\left(x^{\ell-2}\right)$, …, $\phi_K\left(x^0\right)$, we find $c_{\ell-1}, c_{\ell-2}, \ldots, c_0 \in C$ and $r \in C[x]$ with $\deg(r) < d$ such that

$$\phi_K\left(x^\ell\right) = \sum_{i=0}^{\ell-1} c_i \phi_K(x^i) + r.$$

Note that $r$ is a nonzero polynomial in $C[x]$, because $\phi_K\left(x^0\right)$, …, $\phi_K\left(x^{\ell-1}\right)$, $\phi_K\left(x^\ell\right)$ are linearly independent over $C$. Thus, an echelon basis of $\mathrm{im}(\phi_K)$ is

$$\{r\} \cup \left\{\phi_K(x^k) \mid k \in \mathbb{Z} \text{ with } k \neq \ell\right\}.$$

Consequently, $\mathrm{im}(\phi_K)^\top$ has a $C$-basis

$$\left\{1,\ x,\ \ldots, x^{\deg(r)-1},\ x^{\deg(r)+1},\ \ldots,\ x^{d-1},\ x^{d+\ell}\right\}.$$

*Case 2*. There exists a negative integer $\ell$ such that $u_0 q^\ell - v_0 = 0$. Then the integer $\ell$ is unique. By (12), the reduction index of $\phi_K\left(x^k\right)$ is equal to $k$ for all $k \in \mathbb{Z}^-$ with $k \neq \ell$, while the reduction index of $\phi_K\left(x^\ell\right)$ is greater than $\ell$ and less than $d$. Eliminating $x^{\ell+1}$, $x^{\ell+2}$, …, $x^{-1}$ from $\phi_K\left(x^\ell\right)$ successively, we find $c_{\ell+1}, c_{\ell+2}, \ldots, c_{-1} \in C$ and a nonzero polynomial $r_\ell \in C[x]$ with $\deg(r_\ell) < d$ such that

$$\phi_K\left(x^\ell\right) = \sum_{i=\ell+1}^{-1} c_i \phi_K\left(x^i\right) + r_\ell. \tag{13}$$

We also need to consider two subcases.

*Case 2.1*. Assume that $u_d q^k - v_d \neq 0$ for all $k \in \mathbb{N}$. Then the reduction index of $\phi_K\left(x^k\right)$ is equal to $d + k$ for all $k \in \mathbb{N}$. So $\mathrm{im}(\phi_K)$ has an echelon basis

$$\left\{\phi_K\left(x^k\right) \mid k \in \mathbb{Z}^-, k \neq \ell\right\} \cup \{r_\ell\} \cup \left\{\phi_K\left(x^k\right) \mid k \in \mathbb{N}\right\}.$$

Consequently, $\mathrm{im}(\phi_K)^\top$ has a $C$-basis

$$\left\{x^{\ell},\ 1,\ x,\ \ldots,\ x^{\deg(r_\ell)-1},\ x^{\deg(r_\ell)+1},\ \ldots,\ x^{d-1}\right\}.$$

*Case 2.2*. Assume that $u_d q^m - v_d = 0$ for some $m \in \mathbb{N}$. The integer $m$ is again unique. By (12), the reduction index of $\phi_K\left(x^k\right)$ is $d+k$ for all $k \in \mathbb{N}$ with $k \neq m$, and the reduction index of $\phi_K\left(x^m\right)$ is a nonnegative integer less than $d+m$. So there exist $c_0, c_1, \ldots, c_{m-1} \in C$ and a nonzero polynomial $r_m \in C[x]$ with degree less than $d$ such that

$$\phi_K\left(x^m\right) = \sum_{j=0}^{m-1} c_j \phi_K(x^j) + r_m. \tag{14}$$

Moreover, $r_\ell$ and $r_m$ are linearly independent over $C$, for otherwise, the images

$$\phi_K\left(x^{\ell}\right), \phi_K\left(x^{\ell+1}\right), \ldots, \phi_K\left(x^{-1}\right), \phi_K(x^0), \phi_K(x), \ldots, \phi_K\left(x^{m-1}\right)$$

would be linearly dependent, a contradiction to Lemma 5.1 (i). Set $p_\ell = r_\ell$ and

$$p_m = \begin{cases} r_m & \text{if } \deg(r_\ell) \neq \deg(r_m) \\ \mathrm{lc}(r_m) r_\ell - \mathrm{lc}(r_\ell) r_m & \text{otherwise.} \end{cases}$$

Then $p_\ell, p_m$ and $r_\ell, r_m$ span the same linear subspace over $C$, but the degrees of $p_\ell$ and $p_m$ are distinct elements in $\{0, 1, \ldots, d-1\}$. It follows that $\mathrm{im}(\phi_K)$ has an echelon basis

$$\left\{\phi_K\left(x^k\right) \mid k \in \mathbb{Z}^-, k \neq \ell\right\} \cup \{p_\ell, p_m\} \cup \left\{\phi_K\left(x^k\right) \mid k \in \mathbb{N}, k \neq m\right\}$$

and that $\mathrm{im}(\phi_K)^\top$ has a $C$-basis

$$\left\{x^{\ell}, 1, x, \ldots, x^{d-1}, x^{d+m}\right\} \setminus \left\{x^{\deg(p_\ell)}, x^{\deg(p_m)}\right\}.$$

The above detailed case distinction leads to two interesting consequences. The first one tells us that all elements in a standard complement are "sparse" Laurent polynomials, as their numbers of terms are bounded.

**Proposition 5.1** *Let $K = u/v$, where $u, v \in C[x]$ and* $\gcd(u, v) = 1$. *Assume that $K$ is $\sigma$-reduced and not a power of $q$. Then the standard complement of the LP-reduction map is of dimension* $\max(\deg(u), \deg(v))$.

*Proof* It is immediate from the last conclusions in cases 1.1, 1.2, 2.1 and 2.2. □

*Example 5.1* Let $u = x^3 + q^{11}$ and $v = q^{20}x^3 + 1$. Then $K = u/v$ is $\sigma$-reduced. Note that $u_0 q^{-11} - v_0 = 0$, and $\mathrm{lc}(u) q^{20} - \mathrm{lc}(v) = 0$, where $u_0$ and $v_0$ are the coefficients of $x^0$ in $u$ and $v$, respectively. It follows from (13) and (14) that $r_{-11}$ and $r_{20}$ are polynomials of degrees 1 and 2, respectively. As $\mathrm{rind}(r_{-11}) \neq \mathrm{rind}(r_{20})$, we set $p_{-11} = r_{-11}$ and $p_{20} = r_{20}$. Thus, $\mathrm{im}(\phi_K)^\top = \mathrm{span}_C\{x^{-11}, 1, x^{23}\}$ by the last conclusion made in case 2.2. Consequently, every element of $\mathrm{im}(\phi_K)^\top$ has at most three terms.

Second, the case distinction enables us to project a Laurent polynomial onto $\mathrm{im}(\phi_K)$ and $\mathrm{im}(\phi_K)^\top$, respectively.

**LPReduction**. Given a Laurent polynomial $h \in C[x, x^{-1}]$, compute $a \in C[x, x^{-1}]$ and $b \in \mathrm{im}(\phi_K)^\top$ such that $h = \phi_K(a) + b$.

1. If $h = 0$, then set $a = 0$ and $b = 0$; return.
2. Find the subset $\{f_1, \ldots, f_s\} \subseteq C[x, x^{-1}]$ consisting of the preimages of all polynomials in an echelon basis of $\mathrm{im}(\phi_K)$ whose reduction indices are no more than $\mathrm{hdeg}(h)$ and no less than $\mathrm{tdeg}(h)$.
3. Order the echelon basis such that

$$\mathrm{rind}(\phi_K(f_1)) < \cdots < \mathrm{rind}(\phi_K(f_t)) < \mathrm{rind}(\phi_K(f_{t+1})) < \ldots < \mathrm{rind}(\phi_K(f_s)),$$

with $\mathrm{rind}(\phi_K(f_t)) < 0$ and $\mathrm{rind}(\phi_K(f_{t+1})) \geqslant 0$.
4. For $i = 1, 2, \ldots, t$, perform linear elimination to find $c_i \in C$ such that

$$g := h - \sum_{i=1}^{t} c_i \phi_K(f_i) \in C[x] + \mathrm{im}(\phi_K)^\top.$$

5. For $i = s, s-1, \ldots, t+1$, perform linear elimination to find $c_i \in C$ such that

$$b := g - \sum_{i=t+1}^{s} c_i \phi_K(f_i) \in \mathrm{im}(\phi_K)^\top.$$

6. Set $a := \sum_{i=1}^{s} c_i f_i$ and return $a, b$.

The truncated echelon basis in step 2 can be easily constructed according to the above case distinction. In step 4, we eliminate all negative power of $x$ in $h$ except those appearing in $\mathrm{im}(\phi_K)^\top$. In step 5, we eliminate all positive powers of $x$ in $g$ except those appearing in $\mathrm{im}(\phi_K)^\top$. Then the resulting Laurent polynomial $b$ is the projection of $h$ on $\mathrm{im}(\phi_K)^\top$. Tracing back the two elimination processes, we obtain the preimage of the projection of $h$ on $\mathrm{im}(\phi_K)$.

In summary, we have the following additive decomposition for irrational $q$-hypergeometric terms.

**Theorem 5.1** *Let $T$ be an irrational $q$-hypergeometric term whose $\sigma$-quotient has a kernel $K$. Let $u$ and $v$ be the numerator and denominator of $K$, respectively, and $\phi_K$ be the LP-reduction map. Then the following four assertions hold.*

*(i) There is an algorithm to compute a $q$-hypergeometric term $H$, two rational functions $f, g \in C(x)$ and a Laurent polynomial $p \in \mathrm{im}(\phi_K)^\top$ such that*

$$T = \Delta(fH) + \left(g + \frac{p}{v}\right) H, \tag{15}$$

*where the $\sigma$-quotient of $H$ is equal to $K$, $g$ is proper, and its denominator is $\sigma$-free and strongly coprime with $K$.*

*(ii) $p$ has at most $\max(\deg(u), \deg(v))$ many nonzero terms.*

*(iii) If there exist $\tilde{f}, \tilde{g} \in C(x)$ and $\tilde{p} \in C[x, x^{-1}]$ such that*

$$T = \Delta(\tilde{f}H) + \left(\tilde{g} + \frac{\tilde{p}}{v}\right)H, \tag{16}$$

*then the degree of the denominator of $g$ is no greater than that of $\tilde{g}$.*

*(iv) $T$ is $q$-summable if and only if both $g$ and $p$ are equal to zero.*

*Proof* (i) Let $S$ be the shell of $\sigma(T)/T$ with respect to $K$ and $H = T/S$. Applying the shell reduction algorithm to $S$, we obtain $r, g \in C(x)$ and $h \in C[x, x^{-1}]$ such that

$$T = \Delta(rH) + \left(g + \frac{h}{v}\right)H, \tag{17}$$

where $g$ is a proper rational function whose denominator is $\sigma$-free and strongly coprime with $K$.

The LP-reduction algorithm computes two Laurent polynomials $a$ and $p$ such that $h = \phi_K(a) + p$ and $p \in \operatorname{im}(\phi_K)^\top$. Hence, $h = u\sigma(a) - va + p$. It follows that

$$\frac{h}{v} = K\sigma(a) - a + \frac{p}{v} = \Delta_K(a) + \frac{p}{v},$$

which, together with (17), implies that (15) holds (setting $f = r + a$).

(ii) It is immediate from Proposition 5.1.

(iii) Assume that both (15) and (16) hold. Then

$$S = \Delta_K(f) + g + \frac{p}{v} = \Delta_K(\tilde{f}) + \tilde{g} + \frac{\tilde{p}}{v}.$$

It follows from Theorem 4.4 that the degree of the denominator of $g$ is no greater than that of $\tilde{g}$.

(iv) If both $g$ and $p$ are equal to zero, then $T$ is clearly $q$-summable. Conversely, assume that $T$ is $q$-summable. By (17) and Theorem 4.4, $g = 0$. Hence, $(p/v)H$ is also $q$-summable. In other words, $p/v \in \operatorname{im}(\Delta_K)$. By Lemma 5.2, $p = 0$. □

We now present an algorithm to decompose a $q$-hypergeometric term into a $q$-summable term and a non-summable one, which determines $q$-summability without solving any auxiliary $q$-recurrence equation explicitly. The algorithm, named $q$-MAP, is a $q$-analogue of the modified Abramov-Petkovšek algorithm.

**$q$-MAP**. Given a $q$-hypergeometric term $T$, compute two $q$-hypergeometric terms $T_1$ and $T_2$ such that $T = \Delta(T_1) + T_2$ with the property that $T_2$ is minimal in the sense of Theorem 5.1 (iii) and $T$ is $q$-summable if and only if $T_2$ is zero.

1. Compute a kernel $K$ and the corresponding shell $S$ of $\sigma(T)/T$. Set $v$ to be the denominator of $K$. Set $H = T/S$.
2. Apply **ShellReduction** to $S$ to find $f, g \in C(x)$ and $h \in C[x, x^{-1}]$ such that

$$T = \Delta(fH) + \left(g + \frac{h}{v}\right) H.$$

3. If $K = q^m$ for some integer $m$, then compute $a \in C[x, x^{-1}]$ and $c \in C$ such that

$$hx^m = \Delta(a) + c,$$

according to the proof of Corollary 4.3 (In this case, $v = 1$ and $H = x^m$). Set

$$T_1 := fx^m + a \quad \text{and} \quad T_2 := gx^m + c;$$

and return.
4. If $K \neq q^m$ for any integer $m$, then apply **LPReduction** to $h$ and find $a \in C[x, x^{-1}]$ and $b \in \mathrm{im}(\phi_K)^\top$ such that

$$h = \phi_K(a) + b,$$

where $\phi_K$ is the LP-reduction map. Set

$$T_1 := (f + a)H \quad \text{and} \quad T_2 := \left(g + \frac{b}{v}\right) H;$$

and return.

*Example 5.2* Consider the same term in Example 4.4. By shell reduction,

$$S = \Delta_K(0) + 0 + \frac{1}{v},$$

where $v = 1$. Then apply the LP reduction on the numerator 1 to get

$$S = \Delta_K(-(qx)^{-1}) + 0 + 0,$$

which implies that $T$ is $q$-summable and $T = \Delta((q; q)_n)$.

## 6 Experimental Results

We have implemented our $q$-analogue of the modified Abramov-Petkovšek reduction in the computer algebra system MAPLE 18, and compared with two analogues of Gosper's algorithm in [12, 14], respectively.

The first analogue, named $q$-Gosper's algorithm,[1] has three steps:

1. Compute a $q$-Gosper form $(a, b, c)$ of the $\sigma$-quotient of the input term.
2. Estimate degree bounds for a Laurent polynomial solution of a $q$-analogue of Gosper's equation in the form

$$a\sigma(z) - \sigma^{-1}(b)z = c. \tag{18}$$

3. Compute the Laurent polynomial solution by solving a linear system over $C$.

It takes little time to compute $q$-Gosper forms and estimate the head and tail degree bounds of Laurent polynomial solutions of (18). So most of the time is spent on solving a linear system over $C$.

The other $q$-analogue is named after *qTelescope* by the authors of [14]. It uses greatest factorial factorization to compute three polynomials $P$, $Q$, $R$ such that $P$ is coprime with both $Q$ and $R$, and $Q$ is coprime with every positive $q$-shift of $R$, and then computes a polynomial solution of a variant of (18) in the form

$$Q\sigma(Z) - RZ = P \tag{19}$$

by solving a linear system over $C$. Our implementation of the $q$Telescope algorithm is based on the description given in [14].

The test suite was generated by

$$T := \frac{a}{p_1\, \sigma^{\ell_1}(p_1)\; p_2\, \sigma^{\ell_2}(p_2)} \prod_{k=1}^{n-1} \frac{u_1 u_2}{v_1 v_2}, \tag{20}$$

where

(i) $a, p_1, p_2 \in \mathbb{Q}(q)[q^n]$ are random with $\deg(a) = 30$, $\deg(p_1) = \deg(p_2) = d$, where $q$ is transcendental over $\mathbb{Q}$, and $n$ is an integral variable;
(ii) $u_1, u_2, v_1, v_2 \in \mathbb{Q}(q)[q^n]$ are random whose degrees are all equal to 1;
(iii) $\ell_1, \ell_2 \in \mathbb{N}$.

In all the examples given above, the $q$-dispersion of $p_1\sigma^{\ell_1}(p_1)p_2\sigma^{\ell_2}(p_2)$ is equal to $\max(\ell_1, \ell_2)$. All timings are measured in seconds on an OS X computer with 16 GB 1600 MHz DDR3 and 2.5 GHz Intel Core i7 processors.

Table 1 contains the timings of the $q$-Gosper's algorithm ($q$-Gosper), $q$Telescope algorithm ($q$Telescope) and the $q$-analogue of the modified Abramov-Petkovšek reduction ($q$-MAP) for input $T$ given in (20) with different choices of $\ell_1$ and $\ell_2$. In general, randomly-generated terms in the form (20) are non-summable. In this case, both $q$-Gosper's algorithm and $q$Telescope algorithm return a message "non-summable"; while the $q$-MAP algorithm not only determines the non-summability, but also presents an additive decomposition. The experimental results illustrate that the $q$-MAP algorithm outperforms the two $q$-analogues of Gosper's algorithm when

[1] We thank Dr. Haitao Jin for sending us his maple scripts on $q$-Gosper's algorithm.

**Table 1** Non-summable case: $d = 5$ and $(\ell_1, \ell_2)$ varies

| Algorithm | $(\ell_1, \ell_2)$ | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | (2, 2) | (2, 3) | (3, 4) | (4, 4) | (4, 5) | (5, 5) | (5, 10) | (10, 10) |
| $q$-Gosper | 18.615 | 45.718 | 99.296 | 125.136 | 271.063 | 328.437 | 3332.533 | 7707.963 |
| $q$Telescope | 23.413 | 46.952 | 120.891 | 120.495 | 173.247 | 405.994 | 2541.752 | 7574.879 |
| $q$-MAP | 3.355 | 4.626 | 8.182 | 10.181 | 12.829 | 15.611 | 47.104 | 90.532 |

**Table 2** Non-summable case: $(\ell_1, \ell_2) = (1, 1)$ and $d$ varies

| Algorithm | $d$ | | | | | | |
|---|---|---|---|---|---|---|---|
| | 1 | 5 | 10 | 15 | 20 | 25 | 30 |
| $q$-Gosper | 2.3 | 2.1 | 0.6 | 0.03 | 0.04 | 0.04 | 0.04 |
| $q$Telescope | 3.1 | 3.8 | 1.7 | 0.5 | 0.8 | 1.3 | 1.8 |
| $q$-MAP | 3.9 | 1.6 | 5.5 | 32.3 | 150.4 | 517.1 | 1523.1 |

**Table 3** Summable case: $d = 5$ and $(\ell_1, \ell_2)$ varies

| Algorithm | $(\ell_1, \ell_2)$ | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | (2,2) | (5,5) | (5, 10) | (10, 10) | (10, 20) | (20, 20) | (20, 30) | (30, 30) |
| $q$-Gosper | 0.992 | 3.288 | 8.637 | 13.481 | 76.261 | 104.880 | 383.131 | 479.860 |
| $q$Telescope | 4.719 | 13.464 | 29.991 | 42.078 | 147.516 | 256.348 | 923.929 | 1823.568 |
| $q$-MAP | 3.288 | 11.546 | 18.939 | 22.203 | 40.440 | 44.703 | 90.023 | 88.876 |

the degree $d$ is equal to five and the $q$-dispersion of $p_1\sigma^{\ell_1}(p_1)p_2\sigma^{\ell_2}(p_2)$ is greater than one.

Table 2 contains the timings of the three algorithms for input $T$ given in (20), in which the $q$-dispersion of $p_1\sigma^{\ell_1}(p_1)p_2\sigma^{\ell_2}(p_2)$ is equal to one. One can show that the estimation on the degree bound in the second step of $q$-Gosper's algorithm has already implies that (18) has no Laurent polynomial solution when $d \geqslant 15$. Thus, $q$-Gosper's algorithm determines the non-summability of $T$ instantly, and so does the algorithm $q$Telescope.

Table 3 contains the timings of the three algorithms for input $\Delta(T)$, where $T$ is the same as in Table 1. So all the input terms are $q$-summable. Both $q$-Gosper and $q$Telescoper are either faster than or comparable with the $q$-MAP reduction when $\ell_1$ and $\ell_2$ take small values. In this case, the $q$-dispersion of the denominator of the input rational function in the shell reduction is less than or equal to 10. When the $q$-dispersion is more than 10, the $q$-modified Abramov-Petkovšek reduction outperforms both of the $q$-analogues.

When the degree bound estimates are loose in $q$-Gosper's algorithm, the $q$-modified Abramov-Petkovšek reduction is markedly superior to $q$-Gosper's algorithm, as illustrated in the next example.

*Example 6.1* Let

$$f := \frac{(x^3+q^{11})(q^5x^5+q^3x^3+2)(x-1)(q^{10}x-1)(x-2)(q^{20}x-2)}{(q^{20}x^3+1)(x^5+x^3+2)(qx-1)(q^{11}x-1)(qx-2)(q^{21}x-2)}$$

be the $\sigma$-quotient of some $q$-hypergeometric term $T$.

Applying $q$-Gosper's algorithm to the term $\Delta(T)$, we compute a $q$-Gosper form $(a, b, c)$ of the $\sigma$-quotient of $\Delta(T)$, where

$$a = (x^3+q^{11})(x-1)(x-2) \quad \text{and} \quad b = (q^{23}x^3+1)(q^{12}x-1)(q^{22}x-2).$$

As the ratio of the leading coefficients, as well as that of the tailing coefficients, is a power of $q$, the estimates on both head and tail degrees are not sharp. Indeed, the bounds on head and tail degrees estimated in $q$-Gosper's algorithm are 52 and $-11$, respectively. But a Laurent polynomial solution of the $q$-Gosper equation is of head degree 33 and tail degree 0. It takes about 35 s to find the indefinite sum of $\Delta(T)$.

Similarly, 63 is the degree bound for a polynomial solution of (19) in the algorithm $q$Telescope, and a polynomial solution of (19) is equal to $x^{11}p$, where $p$ is a polynomial of degree 33. It takes about 9 s to find the indefinite sum of $\Delta(T)$.

On the other hand, $q$-MAP takes less than 0.3 s to find the indefinite sum, although the case 2.2 happens in the LP-reduction.

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# Factorization of C-Finite Sequences

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*Dedicated to Sergei A. Abramov on the occasion of his 70th birthday*

**Abstract** We discuss how to decide whether a given C-finite sequence can be written nontrivially as a product of two other C-finite sequences.

**Keywords** Recurrence equations · Computer algebra · Closure properties

## 1 Introduction

It is well known that when $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ are two sequences that satisfy some linear recurrences with constant coefficients, then the product sequence $(a_n b_n)_{n=0}^{\infty}$ also satisfies such a recurrence. Sequences satisfying linear recurrences with constant coefficients are called C-finite [8, 17, 19], and the fact just refered to is one of several closure properties that this class of sequences enjoys. In this paper, we will consider the inverse problem: given a C-finite sequence $(c_n)_{n=0}^{\infty}$, can we write it in a nontrivial way as the product of two other C-finite sequences? This question is of interest in its own right, but it is also useful in some applications in combinatorics. For example, the celebrated solution by Kasteleyn, and Temperley-Fisher, of the dimer problem [3, 7] as well as the even more celebrated Onsager solution of the two-dimensional Ising model [10] can be (re)discovered using an algorithm for factorization of C-finite sequences.

A C-finite sequence is uniquely determined by a recurrence and a choice of sufficiently many initial values. The prototypical example of a C-finite sequence is the

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Fibonacci sequence $(F_n)_{n=0}^{\infty}$ defined by

$$F_{n+2} - F_{n+1} - F_n = 0, \qquad F_0 = 0, F_1 = 1.$$

Whether a C-finite sequence $(c_n)_{n=0}^{\infty}$ admits a factorization depends in general on both the recurrence as well as the initial values. For example, the sequence $(3^n + 4^n + 6^n + 8^n)_{n=0}^{\infty}$, which satisfies the recurrence

$$c_{n+4} - 21c_{n+3} + 158c_{n+2} - 504c_{n+1} + 576c_n = 0,$$

can be factored as $3^n + 4^n + 6^n + 8^n = (1 + 2^n)(3^n + 4^n)$, while the sequence $3^n + 4^n + 6^n - 8^n$, which satisfies the same recurrence, cannot be factored.

We shall consider a variant of the factorization problem that does not depend on initial values but only on the recurrence equations. Linear recurrences may be viewed as polynomials $p = p_0 + p_1x + \cdots + p_dx^d \in k[x]$ acting on sequences $(a_n)_{n=0}^{\infty}$ via

$$p \cdot (a_n)_{n=0}^{\infty} := (p_0a_n + p_1a_{n+1} + \cdots + p_da_{n+d})_{n=0}^{\infty}.$$

For every fixed $p \in k[x]$, denote by $V(p)$ the set of all sequences $(a_n)_{n=0}^{\infty}$ with $p \cdot (a_n)_{n=0}^{\infty} = (0)_{n=0}^{\infty}$, i.e., the solution space of the recurrence equation encoded by $p$. This is a vector space of dimension $\deg(p)$. For any two operators $p, q \in k[x] \setminus \{0\}$ there exists a unique monic polynomial $r \in k[x]$ such that $V(r)$ is the vector space generated by all sequences $(a_nb_n)_{n=0}^{\infty}$ with $(a_n)_{n=0}^{\infty} \in V(p)$ and $(b_n)_{n=0}^{\infty} \in V(q)$, i.e., $V(r) = V(p) \otimes V(q)$. We write $r = p \otimes q$.

Our problem shall be to decide, for a given monic polynomial $r \in k[x]$, whether there exist $p, q \in k[x]$ such that $r = p \otimes q$. In principle, it is known how to do this. Singer [12] gives a general algorithm for the analogous problem for linear differential operators with rational function coefficients; the problem is further discussed in [6]. Because of their high cost, these algorithms are mainly of theoretical interest. For the special case of differential operators of order 3 or 4 (still with rational function coefficients), van Hoeij [15, 16] combines several observations to an algorithm which handle these cases efficiently. For the recurrence case, Cha [1] gives an algorithm for operators of order 3 with rational function coefficients.

Also the case of recurrence equations with constant coefficients has already been considered. Everest et al. give an algorithm [2] based on a structure theorem of Ritt [11]. This algorithm relies on Ge's algorithm [4], which is efficient in theory but according to our experience rather costly in concrete examples. An alternative algorithm for the case of constant coefficients and arbitrary order was recently sketched by the second author [19]. This description, however, only considers the "generic case". The present paper is a continuation of this work in which we give a complete algorithm which also handles "degenerate" cases. Our algorithm is efficient in the sense that it does not depend on Ge's algorithm or on Gröbner basis computations, but it is inefficient in the sense that it requires a search that may take exponential time in the worst case.

## 2 Preliminaries

To fix notation, let us recall the basic facts about C-finite sequences. Let $k$ be an algebraically closed field.

**Definition 1** 1. A sequence $(a_n)_{n=0}^\infty$ is called *C-finite,* if there exist $p_0, \ldots, p_d \in k$ with $p_0 \neq 0 \neq p_d$ such that for all $n \in \mathbb{N}$ we have $p_0 a_n + \cdots + p_d a_{n+d} = 0$.
2. In this case, the polynomial $p = p_0 + p_1 x + \cdots + p_d x^d$ is called a *characteristic polynomial* for $(a_n)_{n=0}^\infty$.
3. For $p \in k[x]$, the set $V(p)$ denotes the set of all C-finite sequences whose characteristic polynomial is $p$. It is called the *solution space* of $p$.

**Theorem 1** *[8, 13] Let $p = (x - \phi_1)^{e_1} \cdots (x - \phi_m)^{e_m} \in k[x]$ for pairwise distinct $\phi_1, \ldots, \phi_m \in k \setminus \{0\}$. Then $V(p)$ is the k-vector space generated by the sequences*

$$\begin{array}{l} \phi_1^n, \ \ldots, \ n^{e_1-1}\phi_1^n, \\ \phi_2^n, \ \ldots, \ n^{e_2-1}\phi_2^n, \\ \ldots\ldots\ldots, \\ \phi_m^n, \ \ldots, \ n^{e_m-1}\phi_m^n. \end{array}$$

It is an immediate consequence of this theorem that for any two polynomials $p, q \in k[x]$ we have $V(\gcd(p,q)) = V(p) \cap V(q)$ and $V(\operatorname{lcm}(p,q)) = V(p) + V(q)$. The latter says in particular that when $(a_n)_{n=0}^\infty$ and $(b_n)_{n=0}^\infty$ are C-finite, then so is their sum $(a_n + b_n)_{n=0}^\infty$. A similar result holds for the product: write $p = \prod_{i=1}^m (x - \phi_i)^{e_i}$ and $q = \prod_{j=1}^\ell (x - \psi_j)^{\varepsilon_j}$ and define

$$r := p \otimes q := \operatorname{lcm}_{i=1}^m \operatorname{lcm}_{j=1}^\ell (x - \phi_i \psi_j)^{e_i + \varepsilon_j - 1}. \tag{1}$$

Then $r$ is a characteristic polynomial for the product sequence $(a_n b_n)_{n=0}^\infty$. Note that $\deg(p) + \deg(q) \leq \deg(r) \leq \deg(p)\deg(q)$ for every $p, q \in k[x]$ of degree at least 2. Note also that $p \otimes q = q \otimes p$ for every $p, q \in k[x]$.

Our goal is to recover $p$ and $q$ from a given $r$. The problem is thus to decide whether the roots of a given polynomial $r$ are precisely the pairwise products of the roots of two other polynomials $p$ and $q$. Besides the interpretation as a factorization of C-finite sequences, this problem can also be viewed as the factorization of algebraic numbers: given some algebraic number $\alpha$, specified by its minimal polynomial $r$, can we write $\alpha = \beta\gamma$ where $\beta, \gamma$ are some other algebraic numbers with respective minimal polynomials $p$ and $q$.

Trivial decompositions are easy to find: For each $r$ we obviously have $r = r \otimes (x - 1)$. Moreover, for every nonzero $\phi$ we have $(x - \phi) \otimes (x - \phi^{-1}) = (x - 1)$, so we can "decompose" $r$ into $r \otimes (x - \phi)$ and $x - \phi^{-1}$. In order for a decomposition $r = p \otimes q$ to be interesting, we have to require that both $p$ and $q$ have at least degree 2.

Even so, a factorization is in general not unique. Obviously, if $r = p \otimes q$ is a factorization, then for any nonzero $\phi$ also $r = \bigl(p \otimes (x-\phi)\bigr) \otimes \bigl((x-\phi^{-1}) \otimes q\bigr)$. Translated to sequences, this ambiguity corresponds to the facts that for every $\phi \neq 0$, both $(\phi^n)_{n=0}^\infty$ and $(\phi^{-n})_{n=0}^\infty$ are C-finite, and that a sequence $(a_n)_{n=0}^\infty$ is C-finite iff for all $\phi \neq 0$ the sequence $(a_n\phi^n)_{n=0}^\infty$ is C-finite. But there is even more non-uniqueness: the polynomial

$$r = (x-2)(x+2)(x-3)(x+3)$$

admits the two distinct factorizations

$$\begin{aligned} r &= (x-1)(x+1) \otimes (x-2)(x+3) \\ &= (x-1)(x+1) \otimes (x-2)(x-3) \end{aligned}$$

which cannot be obtained from one another by introducing factors $(x-\phi)$ and $(x-\phi^{-1})$. Our goal will be to compute a finite list of factorizations from which all others can be obtained by introducing factors $(x-\phi) \otimes (x-\phi^{-1})$.

There is a naive but very expensive algorithm which does this job when $r$ is squarefree: For some choice $n, m$ of degrees, make an ansatz $p = (x-\phi_1)\cdots(x-\phi_n)$ and $q = (x-\psi_1)\cdots(x-\psi_m)$ with variables $\phi_1,\dots,\phi_n,\psi_1,\dots,\psi_m$. Equate the coefficients of $r - \prod_{i=1}^n \prod_{j=1}^m (x-\phi_i\psi_j)$ with respect to $x$ to zero and solve the resulting system of algebraic equations for $\phi_1,\dots,\phi_n,\psi_1,\dots,\psi_m$. After trying all possible degree combinations $n \geq m \geq 2$ with $n+m \leq \deg(r) \leq nm$, either a decomposition has been found, or there is none.

## 3 The Generic Case

Typically, when $p$ and $q$ are square-free polynomials and $\phi_1,\dots,\phi_n \neq 0$ are the roots of $p$ and $\psi_1,\dots,\psi_m \neq 0$ are the roots of $q$, then the products $\phi_i\psi_j$ for $i = 1,\dots,n$, $j = 1,\dots,m$ will all be pairwise distinct. In this case, $r = p \otimes q$ will have exactly $nm$ roots, and the factorization problem consists in recovering $\phi_1,\dots,\phi_n$ and $\psi_1,\dots,\psi_m$ from the (known) roots $\rho_1,\dots,\rho_{nm}$ of $r$.

As observed in [19], a necessary condition for $r$ to admit a factorization into two polynomials of respective degrees $n$ and $m$ is then that there is a bijection $\pi : \{1,\dots,n\} \times \{1,\dots,m\} \to \{1,\dots,nm\}$ such that for all $j_1, j_2$ we have

$$\frac{\rho_{\pi(1,j_1)}}{\rho_{\pi(1,j_2)}} = \frac{\rho_{\pi(2,j_1)}}{\rho_{\pi(2,j_2)}} = \cdots = \frac{\rho_{\pi(n,j_1)}}{\rho_{\pi(n,j_2)}}$$

and for all $i_1, i_2$ we have

$$\frac{\rho_{\pi(i_1,1)}}{\rho_{\pi(i_2,1)}} = \frac{\rho_{\pi(i_1,2)}}{\rho_{\pi(i_2,2)}} = \cdots = \frac{\rho_{\pi(i_1,m)}}{\rho_{\pi(i_2,m)}}.$$

The explanation is simply that when a factorization exists, then the roots $\rho_\ell$ of $r$ are precisely the products $\phi_i\psi_j$, and if we define $\pi$ so that it maps each pair $(i, j)$ to the corresponding root index $\ell$, then the quotients

$$\frac{\rho_{\pi(i,j_1)}}{\rho_{\pi(i,j_2)}} = \frac{\phi_i\psi_{j_1}}{\phi_i\psi_{j_2}} = \frac{\psi_{j_1}}{\psi_{j_2}}$$

do not depend on $i$ and the quotients

$$\frac{\rho_{\pi(i_1,j)}}{\rho_{\pi(i_2,j)}} = \frac{\phi_{i_1}\psi_j}{\phi_{i_2}\psi_j} = \frac{\phi_{i_1}}{\phi_{i_2}}$$

do not depend on $j$.

In fact, the existence of such a bijection $\pi$ is also sufficient for the existence of a factorization: choose $\phi_1 \neq 0$ arbitrarily and set $\psi_1 := \rho_{\pi(1,1)}/\phi_1$ and

$$\phi_i := \phi_1 \frac{\rho_{\pi(i,1)}}{\rho_{\pi(1,1)}} \qquad (i = 2, \ldots, n)$$

and

$$\psi_j := \psi_1 \frac{\rho_{\pi(1,j)}}{\rho_{\pi(1,1)}} \qquad (j = 2, \ldots, m).$$

Then we have $\rho_{\pi(i,j)} = \phi_i\psi_j$ for all $i, j$, and therefore for $p = (x - \phi_1)\cdots(x - \phi_n)$ and $q = (x - \psi_1)\cdots(x - \psi_m)$ we have $r = p \otimes q$. Note that $p$ and $q$ are square-free, because if we have, say, $\phi_{i_1} = \phi_{i_2}$ for some $i_1, i_2$, then $\rho_{\pi(i_1,1)} = \rho_{\pi(i_2,1)}$, and then $\pi(i_1, 1) = \pi(i_2, 1)$, and then $i_1 = i_2$.

*Example 1* 1. Consider $r = (x - 4)(x - 6)(x + 6)(x + 9)$, i.e., $\rho_1 = 4$, $\rho_2 = 6$, $\rho_3 = -6$, $\rho_4 = -9$. A possible choice for $\pi \colon \{1, 2\} \times \{1, 2\} \to \{1, 2, 3, 4\}$ is given by the table

| $\pi$ | 1 | 2 |
|---|---|---|
| 1 | 1 | 2 |
| 2 | 3 | 4 |

(to be read like, e.g., $\pi(2, 1) = 3$), because

$$\frac{\rho_{\pi(1,2)}}{\rho_{\pi(1,1)}} = \frac{\rho_2}{\rho_1} = \frac{6}{4} = \frac{-9}{-6} = \frac{\rho_4}{\rho_3} = \frac{\rho_{\pi(2,2)}}{\rho_{\pi(2,1)}}$$

and

$$\frac{\rho_{\pi(2,1)}}{\rho_{\pi(1,1)}} = \frac{\rho_3}{\rho_1} = \frac{-6}{4} = \frac{-9}{6} = \frac{\rho_4}{\rho_2} = \frac{\rho_{\pi(2,2)}}{\rho_{\pi(1,2)}}.$$

Take $\phi_1 = 15$ (for no particular reason), $\psi_1 = \frac{4}{15}$, $\phi_2 = 15\ \frac{6}{4} = \frac{45}{2}$, $\psi_2 = \frac{4}{15}\frac{(-6)}{4} = -\frac{2}{5}$. Then

$$\begin{aligned}&(x-15)(x-\tfrac{45}{2})\otimes(x-\tfrac{4}{15})(x+\tfrac{2}{5})\\&=(x-15\tfrac{4}{15})(x+15\tfrac{2}{5})(x-\tfrac{45}{2}\tfrac{4}{15})(x+\tfrac{45}{2}\tfrac{2}{5})\\&=(x-4)(x+6)(x-6)(x+9),\end{aligned}$$

as required.

In this example, no other factorizations exist except for those that are obtained by replacing $p$ and $q$ by $p \otimes (x-\xi)$ and $(x-\xi^{-1}) \otimes q$ for some $\xi \neq 0$. This degree of freedom is reflected by the arbitrary choice of $\phi_1$.

2. The polynomial $(x-1)(x-2)(x-3)(x-4)$ cannot be written as $p \otimes q$ for two quadratic polynomials $p$ and $q$, because $\frac{1}{2} \neq \frac{3}{4}$, $\frac{1}{2} \neq \frac{4}{3}$, $\frac{1}{3} \neq \frac{2}{4}$, $\frac{1}{3} \neq \frac{4}{2}$, $\frac{1}{4} \neq \frac{2}{3}$, $\frac{1}{4} \neq \frac{3}{2}$.
3. Consider $r = (x-2)(x+2)(x-3)(x+3)$, i.e., $\rho_1 = 2$, $\rho_2 = -2$, $\rho_3 = 3$, $\rho_4 = -3$. We have seen that in this case there are two distinct factorizations. They correspond to the two bijections $\pi, \pi' \colon \{1,2\} \times \{1,2\} \to \{1,2,3,4\}$ defined via

| | (1, 1) | (1, 2) | (2, 1) | (2, 2) |
|---|---|---|---|---|
| $\pi$ | 1 | 2 | 3 | 4 |
| $\pi'$ | 1 | 2 | 4 | 3 |

## 4 Product Clashes

Again let $p, q \in k[x]$ be two square-free polynomials, and write $\phi_1, \dots, \phi_n$ for the roots of $p$ and $\psi_1, \dots, \psi_m$ for the roots of $q$. Generically, the degree of $p \otimes q$ is equal to $\deg(p)\deg(q)$. It cannot be larger than this, and it is smaller if and only if there are two index pairs $(i, j) \neq (i', j')$ with $\phi_i \psi_j = \phi_{i'} \psi_{j'}$. In this case, we say that $p$ and $q$ have a product clash. Recall from Eq. (1) that $p \otimes q$ is formed as the least common multiple of the factors $x - \phi_i \psi_j$, not as their product.

Product clashes appear naturally in the computation of $p \otimes p$. For example, for $p = (x-\phi_1)(x-\phi_2)$ we have

$$\begin{aligned}p \otimes p &= \operatorname{lcm}(x-\phi_1\phi_1, x-\phi_1\phi_2, x-\phi_2\phi_1, x-\phi_2\phi_2)\\&=(x-\phi_1\phi_1)(x-\phi_1\phi_2)(x-\phi_2\phi_2),\end{aligned}$$

because $\phi_1\phi_2 = \phi_2\phi_1$ is a clash. More generally, if $p$ is a square-free polynomial of degree $d \geq 2$, then $\deg(p \otimes p) \leq \frac{1}{2}d(d+1) < d^2$.

As an example that does not come from a product of the form $p \otimes p$, consider $p = (x-1)(x-2)(x-4)$ and $q = (x-\frac{1}{2})(x-\frac{1}{4})$. Here we have the clashes $1 \cdot \frac{1}{2} = 2 \cdot \frac{1}{4}$ and $2 \cdot \frac{1}{2} = 4 \cdot \frac{1}{4}$, so that $p \otimes q = (x-\frac{1}{2})(x-\frac{1}{4})(x-1)(x-2)$ only has degree 4.

In order to include product clashes into the framework of the previous section, we need to relax the requirement that $\pi$ be injective. We still want it to be surjective, because every root of $r$ must be produced by the product $\phi\psi$ of some root $\phi$ of $p$ and some root $\psi$ of $q$. If the $\phi_i$ and the $\psi_j$ are defined according to the formulas above, it can now happen that $\phi_{i_1} = \phi_{i_2}$ for some $i_1 \neq i_2$. We therefore adjust the definition of $p$ and $q$ to $p = \operatorname{lcm}(x - \phi_1, \ldots, x - \phi_n)$, $q = \operatorname{lcm}(x - \psi_1, \ldots, x - \psi_m)$. Then $p$ and $q$ are squarefree and for the set of roots of $p \otimes q$ we obtain

$$\{\, \phi_i \psi_j : i = 1, \ldots, n;\ j = 1, \ldots, m \,\} = \{\rho_1, \ldots, \rho_\ell\},$$

as desired.

*Example 2* 1. To find the factorization $(x - \phi_1^2)(x - \phi_1\phi_2)(x - \phi_2^2) = (x - \phi_1)(x - \phi_2) \otimes (x - \phi_1)(x - \phi_2)$, set $\rho_1 = \phi_1^2, \rho_2 = \phi_1\phi_2, \rho_3 = \phi_2^2$. Then a suitable choice for $\pi : \{1, 2\} \times \{1, 2\} \to \{1, 2, 3\}$ is given by

| $\pi$ | 1 | 2 |
|---|---|---|
| 1 | 1 | 2 |
| 2 | 2 | 3 |

because

$$\frac{\rho_{\pi(1,1)}}{\rho_{\pi(1,2)}} = \frac{\rho_1}{\rho_2} = \frac{\phi_1}{\phi_2} = \frac{\rho_2}{\rho_3} = \frac{\rho_{\pi(2,1)}}{\rho_{\pi(2,2)}}$$

and

$$\frac{\rho_{\pi(1,1)}}{\rho_{\pi(2,1)}} = \frac{\rho_1}{\rho_2} = \frac{\phi_1}{\phi_2} = \frac{\rho_2}{\rho_3} = \frac{\rho_{\pi(1,2)}}{\rho_{\pi(2,2)}}.$$

2. Consider $r = (x - \frac{1}{2})(x - \frac{1}{4})(x - 1)(x - 2)$, i.e., $\rho_1 = \frac{1}{2}, \rho_2 = \frac{1}{4}, \rho_3 = 1, \rho_4 = 2$. A possible choice for $\pi : \{1, 2\} \times \{1, 2, 3\} \to \{1, 2, 3, 4\}$ is

| $\pi$ | 1 | 2 | 3 |
|---|---|---|---|
| 1 | 1 | 3 | 4 |
| 2 | 2 | 1 | 3 |

because

$$\left\{\frac{\rho_{\pi(1,1)}}{\rho_{\pi(1,2)}}, \frac{\rho_{\pi(2,1)}}{\rho_{\pi(2,2)}}\right\} = \left\{\frac{\rho_1}{\rho_3}, \frac{\rho_2}{\rho_1}\right\} = \left\{\frac{1}{2}\right\}$$
$$\left\{\frac{\rho_{\pi(1,1)}}{\rho_{\pi(1,3)}}, \frac{\rho_{\pi(2,1)}}{\rho_{\pi(2,3)}}\right\} = \left\{\frac{\rho_1}{\rho_4}, \frac{\rho_2}{\rho_3}\right\} = \left\{\frac{1}{4}\right\}$$
$$\left\{\frac{\rho_{\pi(1,2)}}{\rho_{\pi(1,3)}}, \frac{\rho_{\pi(2,2)}}{\rho_{\pi(2,3)}}\right\} = \left\{\frac{\rho_3}{\rho_4}, \frac{\rho_1}{\rho_3}\right\} = \left\{\frac{1}{2}\right\}$$

and

$$\left\{\frac{\rho_{\pi(1,1)}}{\rho_{\pi(2,1)}}, \frac{\rho_{\pi(1,2)}}{\rho_{\pi(2,2)}}, \frac{\rho_{\pi(1,3)}}{\rho_{\pi(2,3)}}\right\} = \left\{\frac{\rho_1}{\rho_2}, \frac{\rho_3}{\rho_1}, \frac{\rho_4}{\rho_3}\right\} = \left\{2\right\}$$

## 5 Searching for Assignments

We now turn to the question how for a given $r = (x - \rho_1) \cdots (x - \rho_\ell) \in k[x]$ we can find a map $\pi$ as required. Of course, since $\ell$ is finite, there are only finitely many possible choices for $n$ and $m$ such that $n + m \le \ell \le nm$, and for each choice $n, m$ there are only finitely many functions $\pi : \{1, \dots, n\} \times \{1, \dots, m\} \to \{1, \dots, \ell\}$. We can simply try them all. But going through all these $(nm)^\ell$ many functions one by one would take very long.

In order to improve the efficiency of the search, we can exploit the fact that for most partial functions $\pi$ it is easy to see that they cannot be extended to a total function with the required properties. We can further reduce the search space by taking into account that the order of the roots of the factors is irrelevant, i.e., we can restrict the search to functions $\pi$ with $\pi(1, 1) \le \pi(2, 1) \le \cdots \le \pi(n, 1)$ and $\pi(1, 1) \le \pi(1, 2) \le \cdots \le \pi(1, m)$. Furthermore, because of surjectivity, the root $\rho_1$ must be reached, and we can choose to set $\pi(1, 1) = 1$ without loss of generality. Next, discard all functions with $\pi(i, j_1) = \pi(i, j_2)$ for some $i, j_1, j_2$ with $j_1 \ne j_2$ or with $\pi(i_1, j) = \pi(i_2, j)$ for some $i_1, i_2, j$ with $i_1 \ne i_2$, because these just signal some roots of a factor of $r$ several times without providing any additional information. So we can in fact enforce $1 = \pi(1, 1) < \pi(2, 1) < \cdots < \pi(n, 1)$ and $\pi(1, 1) < \pi(1, 2) < \cdots < \pi(1, m)$. Next, $\pi$ is a solution iff $\pi^\top : \{1, \dots, m\} \times \{1, \dots, n\} \to \{1, \dots, \ell\}$ with $\pi^\top(i, j) = \pi(j, i)$ is a solution. We can therefore restrict the search to functions where $n \le m$.

The following algorithm takes these observations into account. It maintains an assignment table $M$ which encodes a function $\pi : \{1, \dots, n\} \times \{1, \dots, m\} \to \{1, \dots, \ell\}$ with

$$\frac{\rho_{\pi(1,j_1)}}{\rho_{\pi(1,j_2)}} = \frac{\rho_{\pi(2,j_1)}}{\rho_{\pi(2,j_2)}} = \cdots = \frac{\rho_{\pi(n,j_1)}}{\rho_{\pi(n,j_2)}}$$

for all $i, j_1, j_2$ and

$$\frac{\rho_{\pi(i_1,1)}}{\rho_{\pi(i_2,1)}} = \frac{\rho_{\pi(i_1,2)}}{\rho_{\pi(i_2,2)}} = \cdots = \frac{\rho_{\pi(i_1,m)}}{\rho_{\pi(i_2,m)}}.$$

for all $i_1, i_2, j$. At every recursion level, the candidate under consideration is extended to a function $\pi$ with $\pi(n + 1, 1) = p$ for some $p$. As soon as $p$ is chosen, there is for each $j = 2, \dots, m$ at most one choice $q \in \{1, \dots, \ell\}$ for the value of $\pi(n + 1, j)$. The matrix $M$ stores these values $q$ and marks the indices $j$ for which no $q$ exists with $q = 0$. The result is a function $\{1, \dots, n + 1\} \times \{1, \dots, \tilde{m}\} \to \{1, \dots, \ell\}$ for some $\tilde{m} \le m$. If this function is surjective, we have found a solution. Otherwise, we proceed recursively unless we already have $n + 1 = \tilde{m}$, because in this case any further extension could only produce transposes of solutions that will be found at some other stage of the search.

INPUT: The roots $\rho_1, \dots, \rho_\ell$ of some square-free polynomial $r \in k[x]$.
OUTPUT: A list of functions $\pi$ as required for solving the factorization problem.

1 let $M = ((M[i, j]))_{i,j=1}^{\ell}$ be a matrix with $M[1, j] = j$ for $j = 1, \dots, \ell$.
2 call the procedure addRow$(M, 2)$ as defined below.
3 stop.

4 procedure addRow$(M, n)$
5   for $p = M[n-1, 1] + 1, \dots, \ell$ do:
6     set the $n$th row of $M$ to $(p, 0, \dots, 0)$ and let $J$ be the empty list
7     for $j = 2, \dots, \ell$ do:
8       if $M[n-1, j] \neq 0$ and there exists $q \in \{1, \dots, \ell\}$ such that $\rho_1/\rho_p = \rho_j/\rho_q$ and $\rho_1/\rho_j = \rho_p/\rho_q$
9         set $M[n, j] = q$ and append $j$ to $J$
10     if $\{M[i, j] : i = 1, \dots, n; j \in J\} = \{1, \dots, \ell\}$ then:
11       report the assignment $\pi : \{1, \dots, n\} \times \{1, \dots, |J|\} \to \{1, \dots, \ell\}$ with $\pi(i, j) = M[i, J[j]]$ for all $i, j$.
12     else if $|\{j : M[n, j] \neq 0\}| < n$ then
13       recursively call the procedure addRow$(M, n+1)$

In the interest of readability, we have refrained from some obvious optimizations. For example, an actual implementation might perform some precomputation in order to improve the search for $q$ in Step 8.

It is not hard to implement the algorithm. A Mathematica implementation by the authors is available on the website of this paper,

http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/Cfac.html.

The relevant function is `CFiniteFactor`.

*Example 3* Let $r = (x - \rho_1) \cdots (x - \rho_6)$ where $\rho_1 = -8, \rho_2 = -6, \rho_3 = -4, \rho_4 = -3, \rho_5 = -2, \rho_6 = -1$.

After initialisation, at the first level of the recursion, there are five choices for the first entry in the second row of $M$. Each of them uniquely determines the rest of the row, as follows (writing $\cdot$ for 0):

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & \cdot & 4 & \cdot & \cdot & \cdot \end{pmatrix},$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & \cdot & 6 & \cdot \end{pmatrix},$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix},$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & \cdot & 6 & \cdot & \cdot & \cdot \end{pmatrix},$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}.$$

The second of these matrices corresponds to a solution

$$\pi : \{1, 2\} \times \{1, 2, 3, 4\} \to \{1, 2, 3, 4, 5, 6\},$$

which gives rise to the factorization

$$r = (x - 1)(x - \tfrac{1}{2}) \otimes (x + 8)(x + 6)(x + 4)(x + 2),$$

while the other partial solutions cannot be continued to further solutions.

## 6 Multiple Roots

Let us now drop the condition that $r \in k[x]$ is square free. Write $r^*$ for the square free part of $r$. It is clear from Eq. (1) that when $p, q \in k[x]$ are such that $r = p \otimes q$, then $r^* = p^* \otimes q^*$, where $p^*, q^*$ denote the square free parts of $p$ and $q$, respectively. It is therefore natural to first determine factorizations of the square free part $r^*$ of $r$ and in a second step obtain $p$ and $q$ from $p^*$ and $q^*$ (if possible) by assigning appropriate multiplicities to their roots. As the multiplicities in $p$ or $q$ cannot exceed those in $r$, there are again just finitely many candidates and we could simply try them all. And again, the search can be improved because many possibilities can be ruled out easily. In fact, the freedom for the multiplicities is so limited that we can compute them rather than search for them.

First consider the case when $p^*$ and $q^*$ were obtained from an injective map $\pi$, i.e., the case when there are no product clashes. In this case, each root $\rho_\ell$ of $r^*$ corresponds to exactly one product $\phi_i \psi_j$ of a root $\phi_i$ of $p^*$ and a root $\psi_j$ of $q^*$. The multiplicities $e_i$ of $\phi_i$ in $p$ and $\varepsilon_j$ of $\psi_j$ in $q$, respectively, must be such that $e_i + \varepsilon_j - 1$ equals the multiplicity of $\rho_\ell$ in $r$. This gives a linear system of equations. Every solution of this system in the positive integers gives rise to a factorization for $r$, and if there is no solution for the linear system of any of the factorizations of the square-free part $r^*$, then $r$ admits no factorization.

When there are product clashes, there are roots $\rho$ of $r$ which are obtained in several distinct ways as products of roots of $p$ and $q$, for instance $\rho = \phi_{i_1}\psi_{j_1} = \phi_{i_2}\psi_{j_2}$ for some $(i_1, j_1) \neq (i_2, j_2)$. If $m$ is the multiplicity of $\rho$ in $r$, then the requirement for the multiplicities $e_{i_1}, e_{i_2}, \varepsilon_{j_1}, \varepsilon_{j_2}$ of $\phi_{i_1}, \phi_{i_2}, \psi_{j_1}, \psi_{j_2}$ in $p$ and $q$, respectively, is that

$$\max(e_{i_1} + \varepsilon_{j_1} - 1, e_{i_2} + \varepsilon_{j_2} - 1) = m.$$

We obtain a system of such equations, one equation for reach root of $r$, and they can be solved by exhaustive search.

*Example 4* 1. Let $r = (x - 2)(x + 2)^2(x - 3)^2(x + 3)^3$. We have seen earlier that the square free part $r^*$ of $r$ admits two distinct factorizations

$$\begin{aligned} r^* &= (x-1)(x+1) \otimes (x-2)(x+3) \\ &= (x-1)(x+1) \otimes (x-2)(x-3). \end{aligned}$$

Assigning multiplicities to the first, we get

$$\begin{aligned} &(x-1)^{e_1}(x+1)^{e_2} \otimes (x-2)^{\varepsilon_1}(x+3)^{\varepsilon_2} \\ &= (x+2)^{e_1+\varepsilon_1-1}(x-3)^{e_1+\varepsilon_2-1}(x-2)^{e_2+\varepsilon_1-1}(x+3)^{e_2+\varepsilon_2-1}. \end{aligned}$$

Comparing the exponents to those of $r$ gives the linear system

$$\begin{aligned} e_1 + \varepsilon_1 - 1 &= 2, & e_1 + \varepsilon_2 - 1 &= 2, \\ e_2 + \varepsilon_1 - 1 &= 1, & e_2 + \varepsilon_2 - 1 &= 3, \end{aligned}$$

which has no solution. For the second factorization, we get

$$\begin{aligned} &(x-1)^{e_1}(x+1)^{e_2} \otimes (x-2)^{\varepsilon_1}(x-3)^{\varepsilon_2} \\ &= (x+2)^{e_1+\varepsilon_1-1}(x+3)^{e_1+\varepsilon_2-1}(x-2)^{e_2+\varepsilon_1-1}(x-3)^{e_2+\varepsilon_2-1}. \end{aligned}$$

Comparing the exponents to those of $r$ gives the linear system

$$\begin{aligned} e_1 + \varepsilon_1 - 1 &= 2, & e_1 + \varepsilon_2 - 1 &= 3, \\ e_2 + \varepsilon_1 - 1 &= 1, & e_2 + \varepsilon_2 - 1 &= 2, \end{aligned}$$

whose unique solution in the positive integers is $e_1 = 2, e_2 = 1, \varepsilon_1 = 1, \varepsilon_2 = 2$, thus

$$r = (x-1)^2(x+1) \otimes (x-2)(x-3)^2.$$

2. Let $r = (x - \frac{1}{2})^2(x - \frac{1}{4})(x-1)^2(x-2)^3$. We have seen earlier that the square free part $r^*$ of $r$ admits the factorization

$$r^* = (x - \tfrac{1}{2})(x - \tfrac{1}{4}) \otimes (x-1)(x-2)(x-4).$$

Assigning multiplicities to the factors, we get

$$\begin{aligned} &(x - \tfrac{1}{2})^{e_1}(x - \tfrac{1}{4})^{e_2} \otimes (x-1)^{\varepsilon_1}(x-2)^{\varepsilon_2}(x-4)^{\varepsilon_3} \\ &= (x - \tfrac{1}{2})^{\max(e_1+\varepsilon_1-1, e_2+\varepsilon_2-1)} \\ &\quad (x-1)^{\max(e_1+\varepsilon_2-1, e_2+\varepsilon_3-1)} \\ &\quad (x-2)^{e_1+\varepsilon_3-1}(x - \tfrac{1}{4})^{e_2+\varepsilon_1-1}. \end{aligned}$$

Comparing the exponents to the exponents of the factors of $r$ gives a tropical linear system in the unknowns $e_1, e_2, \varepsilon_1, \varepsilon_2, \varepsilon_3$, which turns out to have two solutions. They correspond to the two factorizations

$$\begin{aligned} r &= (x - \tfrac{1}{2})^2 (x - \tfrac{1}{4}) \otimes (x-1)(x-2)(x-4)^2 \\ &= (x - \tfrac{1}{2})^2 (x - \tfrac{1}{4}) \otimes (x-1)(x-2)^2(x-4)^2 \end{aligned}$$

## 7 When We Don't Want to Find the Roots

Sometimes our polynomials are with integer coefficients, and we prefer not to factorize them over the complex numbers. Of course, all the roots are algebraic numbers, by definition, and computer-algebra systems know how to compute with them (without "cheating" and using floating-point approximations), but it may be more convenient to find the tensor product (in the generic case: no product clashes and no repeated roots) of $p = p_0 + \cdots + p_m x^m$ and $q = q_0 + \cdots + q_n x^n$, a certain polynomial $r$ of degree $mn$, as follows. If the roots of $p$ are $\phi_1, \ldots, \phi_n$ and the roots of $q$ are $\psi_1, \ldots, \psi_m$, then the roots of $p \otimes q$ are, of course

$$\{\, \phi_i \psi_j \mid 1 \le i \le m,\ 1 \le j \le n \,\}.$$

Let $P_k(p) := \sum_{i=1}^{m} \phi_i^k$ be the *power-sum symmetric functions* [9], then of course

$$P_k(p \otimes q) = P_k(p) P_k(q), \quad 1 \le k \le nm.$$

Now using *Newton's relations* (e.g. [9], Eq. I.(2.11') p. 23), one can go back and forth from the elementary symmetric functions (essentially the coefficients of the polynomial up to sign) to the power-functions, and *back,* enabling us easily to compute the tensor product without factorizing.

If you define the reverse of a polynomial $p$, to be $\bar{p}(x) := x^d p(1/x)$, where $d$ is the degree of $p$, then $p \otimes \bar{p}$ has, of course, the factor $(x-1)^d$ but otherwise (generically) all distinct roots, unless it has good reasons not to. On the other hand, if $r = p \otimes q$ for some non-trivial polynomials $p$ and $q$ then $r \otimes \bar{r}$ has repeated roots other than $1 - x$, and the *repetition profile* can be easily predicted as above, or "experimentally". So using this approach it is easy to test quickly whether $r$ "factorizes" (with high probability), in the tensor-product sense. However, to actually find the factors would take more effort.

This is implemented in the Maple package accompanying this article, linked to from

http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/Cfac.html.

The tensor product operation is called `Mul` and the testing procedure is `TestFact`.

## 8 Linear Combinations of Factorizations

For almost all polynomials $r \in k[x]$ there does not exist a factorization. When no factorization exists, we may wonder whether $r$ admits a decomposition of a more general type. For example, we can ask whether there exist polynomials $p_1, p_2, q_1, q_2$ of degree at least two such that

$$r = \operatorname{lcm}(p_1 \otimes q_1,\ p_2 \otimes q_2).$$

Translated to the language of C-finite sequences, this means that we seek to write a given C-finite sequence $(a_n)_{n=0}^{\infty}$ as

$$a_n = b_n c_n + u_n v_n$$

for C-finite sequences $(b_n)_{n=0}^{\infty}$, $(c_n)_{n=0}^{\infty}$, $(u_n)_{n=0}^{\infty}$, $(v_n)_{n=0}^{\infty}$, none of which should satisfy a first-order recurrence in order to make the problem nontrivial.

It is not difficult to adapt the algorithm in Sect. 5 so that it can also discover such factorizations. Suppose that $r$ is squarefree. Then, instead of searching for a single surjective map

$$\pi : \{1, \ldots, n\} \times \{1, \ldots, m\} \to \{1, \ldots, \ell\},$$

it suffices to find two functions

$$\pi_1 : \{1, \ldots, n_1\} \times \{1, \ldots, m_1\} \to \{1, \ldots, \ell\}$$
$$\pi_2 : \{1, \ldots, n_2\} \times \{1, \ldots, m_2\} \to \{1, \ldots, \ell\}$$

satisfying the same conditions previously requested for $\pi$ but with surjectivity replaced by $\operatorname{im} \pi_1 \cup \operatorname{im} \pi_2 = \{1, \ldots, \ell\}$. Once two such maps $\pi_1, \pi_2$ have been found, we can construct $p_1, p_2, q_1, q_2$ by choosing $\phi_1^1$ and $\phi_1^2$ arbitrarily, setting $\psi_1^1 = \rho_{\pi_1(1,1)}/\phi_1^1$, $\psi_1^2 = \rho_{\pi_2(1,1)}/\phi_1^2$ and

$$\phi_i^1 = \phi_1^1 \frac{\rho_{\pi_1(i,1)}}{\rho_{\pi_1(1,1)}}, \qquad \psi_j^1 = \psi_1^1 \frac{\rho_{\pi_1(1,j)}}{\rho_{\pi_1(1,1)}},$$
$$\phi_i^2 = \phi_1^2 \frac{\rho_{\pi_2(i,1)}}{\rho_{\pi_2(1,1)}}, \qquad \psi_j^2 = \psi_1^2 \frac{\rho_{\pi_2(1,j)}}{\rho_{\pi_2(1,1)}}$$

for all $i, j$ in question. Then $p_1 := \prod_{i=1}^{n_1}(x - \phi_i^1)$, $q_1 := \prod_{i=1}^{m_1}(x - \psi_j^1)$, $p_2 := \prod_{i=1}^{n_2}(x - \phi_i^2)$, $q_2 := \prod_{i=1}^{m_2}(x - \psi_j^2)$, are such that $r = \operatorname{lcm}(p_1 \otimes q_1, p_2 \otimes q_2)$.

In order to search for a pair $\pi_1, \pi_2$, we can search for $\pi_1$ very much like we searched for $\pi$ before, and for each partial solution encountered during the recursion, initiate a search for another function $\pi_2$ which is required to hit all the indices $1, \ldots, \ell$ not hit by the partial solution $\pi_1$. Note that it is fine if some indices are hit by both $\pi_1$ and $\pi_2$. The suggested modification amounts to replacing lines 12 and 13 of the algorithm from Sect. 5 by the following:

12 else<br>
13 let $Q = \{M[i, j] : i = 1, \dots, n;\ j \in J\}$.<br>
14 let $M_2$ be an $\ell \times \ell$-matrix with $(1, \dots, \ell)$ as first row.<br>
15 call the procedure addRow$_2(M_2, 2, Q)$ defined below.<br>
16 for each function $\pi_2$ it reports, report $(\pi, \pi_2)$.<br>
17 if no $\pi_2$ is found and $|\{j : M[n, j] \neq 0\}| < n$ then<br>
18 recursively call addRow$(M, n + 1)$

19 procedure addRow$_2(M, n, Q)$<br>
20 [lines 5–9 literally as in the definition of addRow]<br>
21 if $\{1, \dots, \ell\} \setminus Q \subseteq \{M[i, j] : i = 1, \dots, n;\ j \in J\}$ then:<br>
22 [literally as line 11 in the definition of addRow]<br>
23 else if $|\{j : M[n, j] \neq 0\}| < n$ then<br>
24 recursively call addRow$_2(M, n + 1, Q)$.

This settles the case of square free input. The extension to arbitrary polynomials is like in the previous section. For every factorization of the square free part we can assign variables for the multiplicities of all the roots and compare the resulting multiplicities for $\mathrm{lcm}(p_1 \otimes q_1, p_2 \otimes q_2)$ to those of $r$. This gives again a tropical linear system of equations which can be solved with Grigoriev's algorithm [5].

*Example 5* The polynomial $r = (x - 1)(x - 2)(x - 3)(x - 4)(x - 6)(x - 12)$ cannot be written as $r = p \otimes q$ for some $p, q \in k[x]$. However, we have the representation

$$r = \mathrm{lcm}(p_1 \otimes q_1, p_2 \otimes q_2)$$

for

$$p_1 = (x - 1)(x - 2), \qquad p_2 = (x - 1)(x - 3),$$
$$q_1 = (x - 2)(x - 3), \qquad q_2 = (x - 1)(x - 4).$$

Note that the roots 3 and 4 of $r$ are produced by both $p_1 \otimes q_1$ and $p_2 \otimes q_2$.

## 9 Examples

Our main motivation for studying the factorization problem for C-finite sequences are two interesting identities that can be interpreted as such factorizations. They both originate from the transfer matrix method.

The first is a tiling problem studied in [3, 7], and more recently in [18]. Given a rectangle of size $m \times n$, the question is in how many different ways we can fill it using tiles of size $2 \times 1$ or $1 \times 2$. If $n$ and $m$ are even, it turns out that

$$T_{n,m} = 2^{nm/2} \prod_{i=1}^{m/2} \prod_{j=1}^{n/2} \Big( z_{\boxbar}^2 \cos^2\Big(\frac{i\pi}{m+1}\Big) + z_{\boxminus}^2 \cos^2\Big(\frac{j\pi}{n+1}\Big)\Big)$$

is a bivariate polynomial in the variables $z_{\boxbar}$, $z_{\boxminus}$ where the coefficient of a monomial $z_{\boxbar}^u z_{\boxminus}^v$ is exactly the number of tilings of the $m \times n$ rectangle that uses exactly $u$ tiles of size $2 \times 1$ and $v$ tiles of size $1 \times 2$. The transfer matrix method can be used to prove this result automatically for every fixed $m$ and arbitrary $n$ (or vice versa). For every fixed choice of $m$ (say), it delivers a polynomial $r$ which encodes a recurrence for $(T_{n,m})_{n=0}^{\infty}$. For every fixed $i \in \{1, \dots, m\}$, the sequence

$$\begin{aligned} &2^{n/2} \prod_{j=1}^{n} \Big( z_{\boxbar}^2 \cos^2\Big(\frac{i\pi}{m+1}\Big) + z_{\boxminus}^2 \cos^2\Big(\frac{j\pi}{n+1}\Big)\Big) \\ &= \frac{1}{w} z_{\boxminus}^n T_n(\sqrt{w}) + \Big(1 - \frac{1}{w}\Big) z_{\boxminus}^n U_n(\sqrt{w}) \end{aligned}$$

with $w = 1 + \big(\frac{z_{\boxbar}}{z_{\boxminus}} \cos(\frac{i\pi}{m+1})\big)^2$ and $T_n$ and $U_n$ the Chebyshev polynomials of the first and second kind, is C-finite with respect to $n$. An annihilating polynomial is

$$p_i = x^2 - 2\Big(z_{\boxminus}^2 + 2 z_{\boxbar}^2 \cos^2\big(\tfrac{i\pi}{2m+1}\big)\Big)x + z_{\boxminus}^4.$$

The formula for $T_{n,m}$ can be proven for each particular choice of $m$ and arbitrary $n$ by checking $r = p_1 \otimes \cdots \otimes p_m$ and comparing the first $2^m$ initial terms. While the standard algorithms can confirm the correctness of some conjectured factorization, the algorithm described in the present paper can help discover the factorization in the first place, taking only $r$ as input. Fisher, Temperly [3] or Kasteleyn [7] would probably have found it useful back in the 1960s to apply the algorithm to $m = 2, 4, 6, 8, 10$ and to detect the general pattern from the outputs.

The second identity which motivated our study has a similar nature. It describes the Ising model on an $n \times m$ grid wrapped around a torus [10, 14]. Starting from a certain model in statistical physics that we do not want to explain here, the transfer matrix method produces for every fixed $m \in \mathbb{N}$ an annihilating polynomial $r$ of degree $2^m$ for a certain C-finite sequence in $n$. The asymptotic behaviour of this sequence for $n \to \infty$ is of interest. In view of Theorem 1, it is goverend by the root of $r$ with the largest absolute value. Onsager discovered that this largest root of $r$ is equal to

$$(2\sinh(2\nu))^{m/2} \exp\big(\tfrac{1}{2}(\gamma_1 + \gamma_3 + \cdots + \gamma_{2m-1})\big)$$

where $\nu$ is some physical constant and $\gamma_k$ is defined as

$$\gamma_k = \operatorname{arccosh}\big(\cosh(2\nu)\coth(2\nu) - \cos(\tfrac{\pi k}{m})\big)$$

for $k = 1, 3, \dots, 2m - 1$ (compare eq. (V.5.1) (p. 131) in [14]).

Let us translate these formulas to a more familiar form. First note that because of periodicity and symmetry of the cosine, we have $\gamma_k = \gamma_{2m-k}$ for $k = 1, 3, \ldots$. Hence each of the $\gamma_k$ in the argument of exp appears twice, except the middle term $\gamma_m$, which only appears for odd $m$. Set $z = \exp(\nu)$ and $x_k = \exp(\gamma_k)$ for $k = 1, 3, \ldots, 2m - 1$. Then $2\sinh(2\nu) = z^2 - z^{-2}$, and Onsager's expression for the largest root of $r$ simplifies to

$$\begin{cases} (z^2 + z^{-2})^{m/2} x_1 x_3 \cdots x_{m-1} & \text{if } m \text{ is even} \\ (z^2 + z^{-2})^{(m-1)/2}(1 + z^2) x_1 x_3 \cdots x_{m-1} & \text{if } m \text{ is odd.} \end{cases}$$

For the second case we have used $\sqrt{(z^2 + z^{-2})x_m} = 1 + z^2$. The equation for $\gamma_k$ says that $x_k$ is a root of

$$p_k := x^2 + \Big(2\cos(\tfrac{\pi k}{m}) - \frac{(z^4+1)^2}{(z^4-1)z^2}\Big)x + 1.$$

Set $q = x - (z^2 - z^{-2})^{m/2}$ when $m$ is even and set $q = x - (z^2 - z^{-2})^{(m-1)/2}(1 + z^2)$ when $m$ is odd. Then Onsager's formula says that the largest root of $r$ is equal to the largest root of $q \otimes p_1 \otimes p_3 \otimes \cdots \otimes p_{m-1}$.

In fact, the polynomial $q \otimes p_1 \otimes p_3 \otimes \cdots \otimes p_{m-1} \in \mathbb{Q}(z)[x]$ happens to be exactly the irreducible factor of $r \in \mathbb{Q}(z)[x]$ corresponding to the largest root of $r$. Therefore, our algorithm applied to this irreducible factor of $r$ could have helped Onsager discover his formula.

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# Denominator Bounds for Systems of Recurrence Equations Using $\Pi\Sigma$-Extensions

**Johannes Middeke and Carsten Schneider**

*Dedicated to Sergei A. Abramov on the occasion of his 70th birthday.*

**Abstract** We consider linear systems of recurrence equations whose coefficients are given in terms of indefinite nested sums and products covering, e.g., the harmonic numbers, hypergeometric products, $q$-hypergeometric products or their mixed versions. These linear systems are formulated in the setting of $\Pi\Sigma$-extensions and our goal is to find a denominator bound (also known as universal denominator) for the solutions; i.e., a non-zero polynomial $d$ such that the denominator of every solution of the system divides $d$. This is the first step in computing all rational solutions of such a rather general recurrence system. Once the denominator bound is known, the problem of solving for rational solutions is reduced to the problem of solving for polynomial solutions.

**Keywords** Coupled systems · Difference equations · Denominator bounds Difference rings · Ore polynomials · Nested sums and products

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## 1 Introduction

Difference equations are one of the central tools within symbolic summation. In one of its simplest forms, the telescoping equation plays a key role: given a sequence $f(k)$, find a solution $g(k)$ of

$$f(k) = g(k+1) - g(k).$$

Finding such a $g(k)$ in a given ring/field or in an appropriate extension of it (in which the needed sequences are represented accordingly) yields a closed form of the indefinite sum $\sum_{k=a}^{b} f(k) = g(b+1) - g(a)$. Slightly more generally, solving the creative telescoping and more generally the parameterized telescoping equation enable one to search for linear difference equations of definite sums. Finally, linear recurrence solvers enhance the summation toolbox to find closed form solutions of definite sums. This interplay between the different algorithms to solve difference equations has been worked out in [1–6] for hypergeometric sums and has been improved, e.g., by difference ring algorithms [7–11] to the class of nested sums over hypergeometric products, $q$-hypergeometric products or their mixed versions, or by holonomic summation algorithms [12, 13] to the class of sequences/functions that can be described by linear difference/differential equations.

More generally, coupled systems of difference equations are heavily used to describe problems coming from practical problem solving. E.g., big classes of Feynman integrals in the context of particle physics can be described by coupled systems of linear difference equations; for details and further references see [14]. Here one ends up at $n$ Feynman integrals $I_1(k), \ldots, I_n(k)$ which are solutions of a coupled system. More precisely, we are given matrices $A_0(k), \ldots A_l(k) \in \mathbb{K}(k)^{m\times n}$ with entries from the rational function field $\mathbb{K}(k)$, $\mathbb{K}$ a field containing the rational numbers, and a vector $b(k)$ of length $m$ in terms of nested sums over hypergeometric products such that the following coupled system holds:

$$A_l(k)\begin{pmatrix} I_1(k+l) \\ \vdots \\ I_n(k+l) \end{pmatrix} + A_{l-1}(k)\begin{pmatrix} I_1(k+l-1) \\ \vdots \\ I_n(k+l-1) \end{pmatrix} + \cdots + A_0(k)\begin{pmatrix} I_1(k) \\ \vdots \\ I_n(k) \end{pmatrix} = b(k). \tag{1}$$

Then one of the main challenges is to solve such a system, e.g., in terms of d'Alembertian [15, 16] or Liouvillian solutions [17, 18]. Furthermore, solving coupled systems arises as crucial subproblem within holonomic summation algorithms [12]. In many situations, one proceeds as follows to get the solutions of such a coupled system: first decouple the system using any of the algorithms described in [16, 19–22] such that one obtains a scalar linear recurrence in only one of the unknown functions, say $I_1(k)$, and such that the remaining integrals $I_2(k), \ldots I_n(k)$ can be expressed as a linear combination of the shifted versions of $I_1(k)$ and the entries of $b(k)$ over $\mathbb{K}(k)$. Thus solving the system (1) reduces to the problem to solve the derived linear recurrence and, if this is possible, to combine the solutions such that $I_1(k)$ can be expressed by them. Then given this solution, one obtains for free also the solutions of the remaining integrals $I_2(k), \dots, I_n(k)$. This approach in general is often rather successful since one can rely on the very well explored solving algorithms [7, 15–18, 23–29] to determine, e.g., d'Alembertian and Liouvillian solutions for scalar linear recurrence relations and can heavily use summation algorithms [8, 10, 30, 31] to simplify the found solutions.

The main drawback of this rather general strategy of solving a decoupled system is efficiency. First, the decoupling algorithms themselves can be very costly; for further details see [20]. Second, the obtained scalar recurrences have high orders with rather huge coefficients and the existing solving algorithms might utterly fail to find the desired solutions in reasonable time. Thus it is highly desirable to attack the original system (1) directly and to avoid any expensive reduction mechanisms and possible blow-ups to a big scalar equation. Restricting to the first-order case ($m = n = 1$), this problem has been treated the first time in [32]. Given an invertible matrix $A(t)$ from $\mathbb{K}(t)^{n\times n}$, find all solutions $y(t) = (y_1(t), \dots, y_n(t)) \in \mathbb{K}(t)^n$ such that

$$y(t+1) - A\,y(t) = 0 \tag{2}$$

holds. As for many other symbolic summation approaches [1–4, 7, 8, 23, 25, 29, 31] one follows the following strategy (sometimes the first step is hidden in certain normal-form constructions or certain reduction strategies):

1. Compute a universal denominator bound, i.e., a $d(t) \in \mathbb{K}[t]\setminus\{0\}$ such that for any solution $y(t) \in \mathbb{K}(t)^n$ of (2) we have $d(t)\,y(t) \in \mathbb{K}[t]^n$.
2. Given such a $d(t)$, plugging $y(t) = \frac{y'(t)}{d(t)}$ into (2) yields an adapted system for the unknown polynomial $y'(t) \in \mathbb{K}[t]^n$. Now compute a degree bound, i.e., a $b \in \mathbb{N}$ such that the degrees of the entries in $y'$ are bounded by $b$.
3. Finally, inserting the potential solution $y' = y_0 + y_1\,t + \cdots + y_b\,t^b$ into the adapted system yields a linear system in the components $(y_{01}, \dots, y_{0n}, \dots, y_{b1}, \dots, y_{bn}) \in \mathbb{K}^{n(b+1)}$ of the unknown vectors $y_0, \dots, y_b \in \mathbb{K}^n$. Solving this system yields all $y_0, \dots, y_b \in \mathbb{K}^n$ and thus all solutions $y(t) \in \mathbb{K}(t)^n$ for the original system (2).

For an improved version exploiting also ideas from [26] see [33]. Similarly, the $q$-rational case (i.e., $t \mapsto q\,t$ instead of $t \mapsto t+1$) has been elaborated in [34, 35]. In addition, the higher order case $m = n \in \mathbb{N}$ has been considered in [36] for the rational case.

In this article, we will push further the calculation of universal denominators (see reduction step (1)) to the general difference field setting of $\Pi\Sigma$-fields [8] and more generally to the framework of $\Pi\Sigma$-extensions [8]. Here we will utilise similar as in [36, 37] algorithms from [38] to transform in a preprocessing step the coupled system to an appropriate form. Given this modified system, we succeed in generalising compact formulas of universal denominator bounds from [39, 40] to the general case of $\Pi\Sigma$-fields. In particular, we generalise the available denominator bounds in the setting of $\Pi\Sigma$-fields of [7, 28] from scalar difference equations to coupled systems. As consequence, the earlier work of the denominator bounding algorithms is covered in this general framework and big parts of the $q$-rational, multibasic and mixed multibasic case [41] for higher-order linear systems are elaborated. More generally, these denominator bound algorithms enable one to search for solutions of coupled systems (1) where the matrices $A_i(k)$ and the vector $b(k)$ might consist of expressions in terms of indefinite nested sums and products and the solutions might be derived in terms of such sums and products. Furthermore, these algorithms can be used to tackle matrices $A_i(k)$ which are not necessarily square. Solving such systems will play an important role for holonomic summation algorithms that work over such general difference fields [42]. In particular, the technologies described in the following can be seen as a first step towards new efficient solvers of coupled systems that arise frequently within the field of particle physics [14].

The outline of the article is as follows. In Sect. 2 we will present some basic properties of $\Pi\Sigma$-theory and will present our main result concerning the computation of the aperiodic part of a universal denominator of coupled systems in a $\Pi\Sigma$-extension. In Sect. 3 we present some basic facts on Ore polynomials which we use as an algebraic model for recurrence operators and introduce some basic definitions for matrices. With this set up, we will show in Sect. 4 how the aperiodic part of a universal denominator can be calculated under the assumption that the coupled system is brought into particular regularised form. This regularisation is carried out in Sect. 6 which relies on row reduction that will be introduced in Sect. 5. We present examples in Sect. 7 and conclude in Sect. 8 with a general method that enables one to search for solutions in the setting of $\Pi\Sigma$-fields.

## 2 Some $\Pi\Sigma$-Theory and the Main Result

In the following we model the objects in (1), i.e., in the entries of $A_0(k), \dots, A_l(k)$ and of $b(k)$ with elements from a field[1] $\mathbb{F}$. Further we describe the shift operation acting on these elements by a field automorphism $\sigma : \mathbb{F} \to \mathbb{F}$. In short, we call such a pair $(\mathbb{F}, \sigma)$ consisting of a field equipped with a field automorphism also a difference field.

*Example 2.1* 1. Consider the rational function field $\mathbb{F} = \mathbb{K}(t)$ for some field $\mathbb{K}$ and the field automorphism $\sigma : \mathbb{F} \to \mathbb{F}$ defined by $\sigma(c) = c$ for all $c \in \mathbb{K}$ and $\sigma(t) = t + 1$. $(\mathbb{F}, \sigma)$ is also called the rational difference field over $\mathbb{K}$.

2. Consider the rational function field $\mathbb{K} = \mathbb{K}'(q)$ over the field $\mathbb{K}'$ and the rational function field $\mathbb{F} = \mathbb{K}(t)$ over $\mathbb{K}$. Further define the field automorphism $\sigma : \mathbb{F} \to \mathbb{F}$ defined by $\sigma(c) = c$ for all $c \in \mathbb{K}$ and $\sigma(t) = q\,t$. $(\mathbb{F}, \sigma)$ is also called the $q$-rational difference field over $\mathbb{K}$.
3. Consider the rational function field $\mathbb{K} = \mathbb{K}'(q_1, \dots, q_e)$ over the field $\mathbb{K}'$ and the rational function field $\mathbb{F} = \mathbb{K}(t_1, \dots, t_e)$ over $\mathbb{K}$. Further define the field automorphism $\sigma : \mathbb{F} \to \mathbb{F}$ defined by $\sigma(c) = c$ for all $c \in \mathbb{K}$ and $\sigma(t_i) = q_i\,t_i$ for

[1] Throughout this article, all fields contain the rational numbers $\mathbb{Q}$ as subfield.

all $1 \leqslant i \leqslant e$. $(\mathbb{F}, \sigma)$ is also called the $(q_1, \dots, q_e)$-multibasic rational difference field over $\mathbb{K}$.

4. Consider the rational function field $\mathbb{K} = \mathbb{K}'(q_1, \dots, q_e)$ over the field $\mathbb{K}'$ and the rational function field $\mathbb{F} = \mathbb{K}(t_1, \dots, t_e, t)$ over $\mathbb{K}$. Further define the field automorphism $\sigma : \mathbb{F} \to \mathbb{F}$ defined by $\sigma(c) = c$ for all $c \in \mathbb{K}$, $\sigma(t) = t + 1$ and $\sigma(t_i) = q_i\, t_i$ for all $1 \leqslant i \leqslant e$. $(\mathbb{F}, \sigma)$ is also called the mixed $(q_1, \dots, q_e)$-multibasic rational difference field over $\mathbb{K}$.

More generally, we consider difference fields that are built by the following type of extensions. Let $(\mathbb{F}, \sigma)$ be a difference field; i.e., a field $\mathbb{F}$ together with an automorphism $\sigma : \mathbb{F} \to \mathbb{F}$. Elements of $\mathbb{F}$ which are left fixed by $\sigma$ are referred to as *constants*. We denote the set of all constants by

$$\operatorname{const} \mathbb{F} = \{c \in \mathbb{F} \mid \sigma(c) = c\}.$$

A *$\Pi\Sigma$-extension* $(\mathbb{F}(t), \sigma)$ of $(\mathbb{F}, \sigma)$ is given by the rational function field $\mathbb{F}(t)$ in the indeterminate $t$ over $\mathbb{F}$ and an extension of $\sigma$ to $\mathbb{F}(t)$ which can be built as follows: either

1. $\sigma(t) = t + \beta$ for some $\beta \in \mathbb{F}\backslash\{0\}$ (a $\Sigma$-monomial) or
2. $\sigma(t) = \alpha t$ for some $\alpha \in \mathbb{F}\backslash\{0\}$ (a $\Pi$-monomial)

where in both cases we require that $\operatorname{const} \mathbb{F}(t) = \operatorname{const} \mathbb{F}$; compare [8, 43]. More generally, we consider a tower $(\mathbb{F}(t_1) \dots (t_e), \sigma)$ of such extensions where the $t_i$ are either $\Pi$-monomials or $\Sigma$-monomials adjoined to the field $\mathbb{F}(t_1) \dots (t_{i-1})$ below. Such a construction is also called a $\Pi\Sigma$-extension $(\mathbb{F}(t_1) \dots (t_e), \sigma)$ of $(\mathbb{F}, \sigma)$. If $\mathbb{F}(t_1) \dots (t_e)$ consists only of $\Pi$-monomials or of $\Sigma$-monomials, it is also called a $\Pi$- or a $\Sigma$-extension. If $\mathbb{F} = \operatorname{const} \mathbb{F}$, $(\mathbb{F}(t_1) \dots (t_e), \sigma)$ is called a $\Pi\Sigma$-field over $\mathbb{F}$.

Note that all difference fields from Example 2.1 are $\Pi\Sigma$-fields over $\mathbb{K}$. Further note that $\Pi\Sigma$-extensions enable one to model indefinite nested sums and products that may arise as rational expressions in the numerator and denominator. See [8] or [9] for examples of how that modelling works.

Let $(\mathbb{F}, \sigma)$ be an arbitrary difference field and $(\mathbb{F}(t), \sigma)$ be a $\Pi\Sigma$-extension of $(\mathbb{F}, \sigma)$. In this work, we take a look at systems of the form

$$A_\ell \sigma^\ell y + \dots + A_1 \sigma y + A_0 y = b \tag{3}$$

where $A_0, \dots, A_\ell \in \mathbb{F}[t]^{m \times n}$ and $b \in \mathbb{F}[t]^m$. Our long-term goal is to find all rational solutions for such a system, i.e., rational vectors $y \in \mathbb{F}(t)^n$ which satisfy (3) following the three steps presented in the introduction. In this article we will look at the first step: compute a so-called *denominator bound* (also known as a *universal denominator*). This is a polynomial $d \in \mathbb{F}[t]\backslash\{0\}$ such that $dy \in \mathbb{F}[t]^n$ is polynomial for all solutions $y$ of (3). Once that is done, we can simply substitute the denominator bound into the system and then it will be sufficient to search for polynomial solutions. In future work, it will be a key challenge to derive such degree bounds; compare the existing results [8, 43, 44] for scalar equations. Degree bounds for the rational case ($l = 1$)

and the $q$-rational case ($l$ arbitrarily) applied to the system (3) can be found in [45, 46], respectively. Once a degree bound for the polynomial solutions is known, the latter problem translates to solving linear systems over $\mathbb{F}$ if $\mathbb{F} = \text{const}\,\mathbb{F}$. Otherwise, one can apply similar strategies as worked out in [7, 8, 29] to reduce the problem to find polynomial solutions to the problem to solve coupled systems in the smaller field $\mathbb{F}$. Further comments on this proposed machinery will be given in the conclusion.

In order to derive our denominator bounds for system (3), we rely heavily on the following concept [1, 7]. Let $a, b \in \mathbb{F}[t]\setminus\{0\}$ be two non-zero polynomials. We define the *spread* of $a$ and $b$ as

$$\text{spread}(a, b) = \{k \geqslant 0 \mid \gcd(a, \sigma^k(b)) \notin \mathbb{F}\}.$$

In this regard note that $\sigma^k(b) \in \mathbb{F}[t]$ for any $k \in \mathbb{Z}$ and $b \in \mathbb{F}[t]$. In particular, if $b$ is an irreducible polynomial, then also $\sigma^k(b)$ is an irreducible polynomial.

The *dispersion* of $a$ and $b$ is defined as the maximum of the spread, i.e., we declare $\text{disp}(a, b) = \max \text{spread}(a, b)$ where we use the conventions $\max \emptyset = -\infty$ and $\max S = \infty$ if $S$ is infinite. As an abbreviation we will sometimes use $\text{spread}(a) = \text{spread}(a, a)$ and similarly $\text{disp}(a) = \text{disp}(a, a)$. We call $a \in \mathbb{F}[t]$ periodic if $\text{disp}(a)$ is infinite and aperiodic if $\text{disp}(a)$ is finite.

It is shown in [7, 8] (see also [43, Theorem 2.5.5]) that in the case of $\Sigma$-extensions the spread of two polynomials will always be a finite set (possibly empty). For $\Pi$-extensions the spread will certainly be infinite if $t \mid a$ and $t \mid b$ as $\sigma^k(t) \mid t$ for all $k$. It can be shown in [7, 8] (see also [43, Theorem 2.5.5]), however, that this is the only problematic case. Summarising, the following property holds.

**Lemma 2.1** *Let* $(\mathbb{F}(t), \sigma)$ *be a* $\Pi\Sigma$*-extension of* $(\mathbb{F}, \sigma)$ *and* $a \in \mathbb{F}[t]\setminus\{0\}$. *Then* $a$ *is periodic if and only if* $t$ *is a* $\Pi$*-monomial and* $t \mid a$.

This motives the following definition.

**Definition 2.1** Let $(\mathbb{F}(t), \sigma)$ be a $\Pi\Sigma$-extension of $(\mathbb{F}, \sigma)$ and $a \in \mathbb{F}[t]\setminus\{0\}$. We define the periodic part of $a$ as

$$\text{per}(a) = \begin{cases} 1 & \text{if } t \text{ is a } \Sigma\text{-monomial}, \\ t^m & \text{if } t \text{ is a } \Pi\text{-monomial and } m \in \mathbb{N} \text{ is maximal s.t. } t^m \mid a \end{cases}.$$

and the aperiodic part as $\text{ap}(a) = \frac{a}{per(a)}$.

Note that $\text{ap}(a) = a$ if $t$ is a $\Sigma$-monomial. In this article we will focus on the problem to compute the aperiodic part of a denominator bound $d$ of the system (3). Before we state our main result, we will have to clarify what me mean by the denominator of a vector.

**Definition 2.2** Let $y \in \mathbb{F}(t_1, \ldots, t_e)^n$ be a rational column vector. We say that $y = d^{-1}z$ is a *reduced representation* for $y$ if $d \in \mathbb{F}[t_1, \ldots, t_e]\setminus\{0\}$ and $z = (z_1, \ldots, z_n) \in \mathbb{F}[t_1, \ldots, t_e]^n$ are such that[2] $\gcd(z, d) = \gcd(z_1, \ldots, z_n, d) = 1$.

[2] If $z$ is the zero vector, then the assumption $\gcd(z, d) = 1$ implies $d = 1$.

With all the necessary definitions in place, we are ready to state the main result. Its proof will take up the remainder of this paper.

**Theorem 2.1** *Let* $(\mathbb{F}(t), \sigma)$ *be a* $\Pi\Sigma$*-extension of* $(\mathbb{F}, \sigma)$ *and let* $A_0, \dots, A_l \in \mathbb{F}[t]^{m\times n}$, $b \in \mathbb{F}[t]^m$. *If one can compute the dispersion of polynomials in* $\mathbb{F}[t]$, *then one can compute the aperiodic part of a denominator bound of* (3). *This means that one can compute a* $d \in \mathbb{F}[t]\setminus\{0\}$ *such that for any solution* $q^{-1}p \in \mathbb{F}(t)^n$ *of* (3) *with* $q^{-1}p$ *being in reduced representation we have* $\mathrm{ap}(q) \mid d$.

Note that such a $d$ in Theorem 2.1 forms a complete denominator bound if $t$ is a $\Sigma$-monomial. Otherwise, if $t$ is a $\Pi$-monomial, there exists an $m \in \mathbb{N}$ such that $t^m\, d$ is a denominator bound. Finding such an $m$ algorithmically in the general $\Pi\Sigma$-case is so far an open problem. For the $q$-rational case we refer to [46].

In order to prove Theorem 2.1, we will perform a preprocessing step and regularise the system (3) to a more suitable form (see Theorem 6.1 in Sect. 6); for similar strategies to accomplish such a regularisation see [36, 37]. Afterwards, we will apply Theorem 4.1 in Sect. 4 which is a generalisation of [39, 40]. Namely, besides computing the dispersion in $\mathbb{F}[t]$ one only has to compute certain $\sigma$- and gcd-computations in $\mathbb{F}[t]$ in order to derive the desired aperiodic part of the universal denominator bound.

Summarising, our proposed denominator bound method is applicable if the dispersion can be computed. To this end, we will elaborate under which assumptions the dispersion can be computed in $\mathbb{F}[t]$. Define for $f \in \mathbb{F}\setminus\{0\}$ and $k \in \mathbb{Z}$ the following functions:

$$f_{(k,\sigma)} := \begin{cases} f\sigma(f)\dots\sigma^{k-1}(f) & \text{if } k > 0 \\ 1 & \text{if } k = 0 \\ \frac{1}{\sigma^{-1}(f)\dots\sigma^{-k}(f)} & \text{if } k < 0, \end{cases} \qquad f_{\{k,\sigma\}} := \begin{cases} f_{(0,\sigma)} + f_{(1,\sigma)} + \dots + f_{(k-1,\sigma)} & \text{if } k > 0 \\ 0 & \text{if } k = 0 \\ -(f_{(-1,\sigma)} + \dots + f_{(k,\sigma)}) & \text{if } k < 0. \end{cases}$$

Then analysing Karr's algorithm [8] one can extract the following (algorithmic) properties that are relevant to calculate the dispersion in $\Pi\Sigma$-extensions; compare [47].

**Definition 2.3** $(\mathbb{F}, \sigma)$ is weakly $\sigma^*$-computable if the following holds.

1. There is an algorithm that factors multivariate polynomials over $\mathbb{F}$ and that solves linear systems with multivariate rational functions over $\mathbb{F}$.
2. $(\mathbb{F}, \sigma^r)$ is torsion free for all $r \in \mathbb{Z}$, i.e., for all $r \in \mathbb{Z}$, for all $k \in \mathbb{Z}\setminus\{0\}$ and all $g \in \mathbb{F}\setminus\{0\}$ the equality $\left(\frac{\sigma^r(g)}{g}\right)^k = 1$ implies $\frac{\sigma^r(g)}{g} = 1$.
3. *$\Pi$-Regularity*. Given $f, g \in \mathbb{F}$ with $f$ not a root of unity, there is at most one $n \in \mathbb{Z}$ such that $f_{(n,\sigma)} = g$. There is an algorithm that finds, if possible, this $n$.
4. *$\Sigma$-Regularity*. Given $k \in \mathbb{Z}\setminus\{0\}$ and $f, g \in \mathbb{F}$ with $f = 1$ or $f$ not a root of unity, there is at most one $n \in \mathbb{Z}$ such that $f_{\{n,\sigma^k\}} = g$. There is an algorithm that finds, if possible, this $n$.

Namely, we get the following result based on Karr's reduction algorithms.

**Lemma 2.2** *Let $(\mathbb{F}(t), \sigma)$ be a $\Pi\Sigma$-extension of $(\mathbb{F}, \sigma)$. Then the following holds.*

1. *If $(\mathbb{F}, \sigma)$ is weakly $\sigma^*$-computable, one can compute the spread and dispersion of two polynomials $a, b \in \mathbb{F}[t]\backslash\mathbb{F}$.*
2. *If $(\mathbb{F}, \sigma)$ is weakly $\sigma^*$-computable, $(\mathbb{F}(t), \sigma)$ is weakly $\sigma^*$-computable.*

*Proof* (1) By Lemma 1 of [28] the spread is computable if the shift equivalence problem is solvable. This is possible if $(\mathbb{F}, \sigma)$ is weakly $\sigma^*$-computable; see Corollary 1 of [47] (using heavily results of [8]).
(2) holds by Theorem 1 of [47]. □

Thus by the iterative application of Lemma 2.2 we end up at the following result that supplements our Theorem 2.1.

**Corollary 2.1** *Let $(\mathbb{F}(t), \sigma)$ be a $\Pi\Sigma$-extension of $(\mathbb{F}, \sigma)$ where $(\mathbb{F}, \sigma)$ with $\mathbb{F} = \mathbb{G}(t_1)\dots(t_e)$ is a $\Pi\Sigma$-extension of a weakly $\sigma^*$-computable difference field $(\mathbb{G}, \sigma)$. Then the dispersion of two polynomials $a, b \in \mathbb{F}[t]\backslash\mathbb{F}$ is computable.*

Finally, we list some difference fields $(\mathbb{G}, \sigma)$ that one may choose for Corollary 2.1. Namely, the following ground fields $(\mathbb{G}, \sigma)$ are weakly $\sigma^*$-computable.

1. By [48] we may choose $\operatorname{const}\mathbb{G} = \mathbb{G}$ where $\mathbb{G}$ is a rational function field over an algebraic number field; note that $(\mathbb{F}, \sigma)$ is a $\Pi\Sigma$-field over $\mathbb{G}$.
2. By [47] $(\mathbb{G}, \sigma)$ can be a free difference field over a constant field that is weakly $\sigma^*$-computable (see item 1).
3. By [49] $(\mathbb{G}, \sigma)$ can be radical difference field over a constant field that is weakly $\sigma$-computable (see item 1).

Note that all the difference fields introduced in Example 2.1 are $\Pi\Sigma$-fields which are weakly $\sigma^*$-computable if the constant field $\mathbb{K}$ is a rational function field over an algebraic number field (see item 1 in the previous paragraph) and thus the dispersion can be computed in such fields. For the difference fields given in Example 2.1 one may also use the optimised algorithms worked out in [41].

Using Theorem 2.1 we obtain immediately the following multivariate case in the setting of $\Pi\Sigma$-extensions which can be applied for instance for the multibasic and mixed multibasic rational difference fields defined in Example 2.1.

**Corollary 2.2** *Let $(\mathbb{E}, \sigma)$ be a $\Pi\Sigma$-extension of $(\mathbb{F}, \sigma)$ with $\mathbb{E} = \mathbb{F}(t_1)(t_2)\dots(t_e)$ where $\sigma(t_i) = \alpha_i\, t_i + \beta_i$ ($\alpha_i \in \mathbb{F}^*$, $\beta_i \in \mathbb{F}$) for $1 \leqslant i \leqslant e$. Let $A_0, \dots, A_l \in \mathbb{E}^{m\times n}$, $b \in \mathbb{E}$. Then there is an algorithm that computes a $d \in \mathbb{F}[t_1, \dots, t_e, t]\backslash\{0\}$ such that $d' := t_1^{m_1} \cdots t_e^{m_e}\, d$ is a universal denominator bound of system (3) for some $m_1, \dots, m_e \in \mathbb{N}$ where $m_i = 0$ if $t_i$ is a $\Sigma$-monomial. That is, for any solution $y = q^{-1} p \in \mathbb{F}^n$ of (3) in reduced representation we have that $q \mid d'$.*

*Proof* Note that one can reorder the generators in $\mathbb{E} = \mathbb{F}(t_1, \dots, t_e)$ without changing the constant field $\operatorname{const}\mathbb{E} = \operatorname{const}\mathbb{F}$. Hence for any $i$ with $1 \leqslant i \leqslant e$, $(\mathbb{A}_i(t_i), \sigma)$ is a $\Pi\Sigma$-extension of $(\mathbb{A}_i, \sigma)$ with $\mathbb{A}_i = \mathbb{F}(t_1)\dots(t_{i-1})(t_{i+1})\dots(t_e)$. Thus for each $i$ with $1 \leqslant i \leqslant e$, we can apply Theorem 2.1 (more precisely, Theorems 6.1 and 4.1) to compute the aperiodic part $d_i \in \mathbb{A}_i[t_i]\setminus\{0\}$ of a denominator bound of (3). W.l.o.g. we may suppose that $d_1, \ldots, d_e \in \mathbb{A} := \mathbb{F}[t_1, \ldots, t_e]$; otherwise, one clears denominators: for $d_i$ one uses a factor of $\mathbb{A}_i$). Finally, compute $d := \operatorname{lcm}(d_1, \ldots, d_e)$ in $\mathbb{A}$. Suppose that $d\,t_1^{m_1} \cdots t_e^{m_e}$ is not a denominator bound for any choice $m_1, \ldots, m_e \in \mathbb{N}$ where for $1 \leqslant i \leqslant e, m_i = 0$ if $t_i$ is a $\Sigma$-monomial. Then we find a solution $y = q^{-1} p$ of (3) in reduced representation with $p \in \mathbb{A}^n$ and $q \in \mathbb{A}$ and an irreducible $h \in \mathbb{A}\setminus\mathbb{F}$ with $h \mid q$ and $h \nmid d$ where $h \neq t_i$ for all $i$ where $t_i$ is a $\Pi$-monomial. Let $j$ with $1 \leqslant j \leqslant e$ such that $h \in \mathbb{A}_j[t_j]\setminus\mathbb{A}_j$. Since $d_j$ is the aperiodic part of a denominator bound w.r.t. $t_j$, and the case $h = t_j$ is excluded if $t_j$ is a $\Pi$-monomial, it follows that $h\,w = d_j$ for some $w \in \mathbb{A}_j[t_j]$. Write $w = \frac{u}{v}$ with $u \in \mathbb{A}$ and $v \in \mathbb{A}_j$. Since $d_j \in \mathbb{A}$, $h\,w \in \mathbb{A}$ and thus the factor $v \in \mathbb{A}$ must be contained in $h \in \mathbb{A}$. But since $h$ is irreducible in $\mathbb{A}$, $v \in \mathbb{F}\setminus\{0\}$ and thus $w \in \mathbb{A}$. Hence $h$ divides $d_j$ and thus it divides also $d = \operatorname{lcm}(d_1, \ldots, d_e)$ in $\mathbb{A}$, a contradiction. □

## 3 Operators, Ore Polynomials, and Matrices

For this section, let $(\mathbb{F}, \sigma)$ be a fixed difference field. An alternative way of expressing the system (3) is to use operator notation. Formally, we consider the ring of *Ore polynomials* $\mathbb{F}(t)[\sigma]$ over the rational functions $\mathbb{F}(t)$ w.r.t. the automorphism $\sigma$ and the trivial $\sigma$-derivation.[3] Ore polynomials are named after Øystein Ore who first described them in his paper [50]. They provide a natural algebraic model for linear differential, difference, recurrence of $q$-difference operators (see, e.g., [50–53] and the references therein).

We briefly recall the definition of Ore polynomials and refer to the aforementioned papers for details: As a set they consist of all polynomial expressions

$$a_\nu \sigma^\nu + \cdots + a_1 \sigma + a_0$$

with coefficients in $\mathbb{F}(t)$ where we regard $\sigma$ as a variable.[4] Addition of Ore polynomials works just as for regular polynomials. Multiplication on the other hand is governed by the *commutation rule*

$$\sigma \cdot a = \sigma(a) \cdot \sigma \qquad \text{for all} \quad a \in \mathbb{F}(t).$$

(Note that in the above equation $\sigma$ appears in two different roles: as the Ore variable and as automorphism applied to $a$.) Using the associative and distributive law, this rule lets us compute products of arbitrary Ore polynomials. It is possible to show that this multiplication is well-defined and that $\mathbb{F}(t)[\sigma]$ is a (non-commutative) ring (with unity).

[3] Some authors would denote $\mathbb{F}(t)[\sigma]$ by the more precise $\mathbb{F}(t)[\sigma; \sigma, 0]$.

[4] A more rigorous way would be to introduce a new symbol for the variable. However, a lot of authors simply use the same symbol and we decided to join them.

For an operator $L = a_\nu\sigma^\nu + \cdots + a_0 \in \mathbb{F}(t)[\sigma]$ we declare the *application* of $L$ to a rational function $\alpha \in \mathbb{F}(t)$ to be

$$L(\alpha) = a_\nu\sigma^\nu(\alpha) + \cdots + a_1\sigma(\alpha) + a_0\alpha.$$

Note that this turns $\mathbb{F}(t)$ into a left $\mathbb{F}(t)[\sigma]$-module. We extend this to matrices of operators by setting $L(\alpha) = \left(\sum_{j=1}^n L_{ij}(\alpha_j)\right)_j$ for a matrix $L = (L_{ij})_{ij} \in \mathbb{F}(t)[\sigma]^{m\times n}$ and a vector of rational functions $\alpha = (\alpha_j)_j \in \mathbb{F}(t)[\sigma]^n$. It is easy to see that the action of $\mathbb{F}(t)[\sigma]^{m\times n}$ on $\mathbb{F}(t)^n$ is linear over const $\mathbb{F}$. With this notation, we can express the system (3) simply as $A(y) = b$ where

$$A = A_\ell\sigma^\ell + \cdots + A_1\sigma + A_0 \in \mathbb{F}(t)[\sigma]^{m\times n}.$$

The powers of $\sigma$ form a (left and right) Ore set within $\mathbb{F}(t)[\sigma]$ (see, e.g., [54, Chap. 5] for a definition and a brief description of localisation over non-commutative rings). Thus, we may localise by $\sigma$ obtaining the *Ore Laurent polynomials* $\mathbb{F}(t)[\sigma, \sigma^{-1}]$. We can extend the action of $\mathbb{F}(t)[\sigma]$ on $\mathbb{F}(t)$ to $\mathbb{F}(t)[\sigma, \sigma^{-1}]$ in the obvious way.

We need to introduce some notation and naming conventions. We denote the $n$-by-$n$ identity matrix by $\mathbf{1}_n$ (or simply $\mathbf{1}$ if the size is clear from the context). Similarly $\mathbf{0}_{m\times n}$ (or just $\mathbf{0}$) denotes the $m$-by-$n$ zero matrix. With $\operatorname{diag}(a_1, \ldots, a_n)$ we mean a diagonal $n$-by-$n$ matrix with the entries of the main diagonal being $a_1, \ldots, a_n$.

We say that a matrix or a vector is *polynomial* if all its entries are polynomials in $\mathbb{F}[t]$; we call it *rational* if its entries are fractions of polynomials; and we speak of *operator* matrices if its entries are Ore or Ore Laurent polynomials.

Let $M$ be a square matrix over $\mathbb{F}[t]$ (or $\mathbb{F}(t)[\sigma]$ or $\mathbb{F}(t)[\sigma, \sigma^{-1}]$). We say that $M$ is *unimodular* if $M$ possesses a (two-sided) inverse over $\mathbb{F}[t]$ (or $\mathbb{F}(t)[\sigma]$ or $\mathbb{F}(t)[\sigma, \sigma^{-1}]$, respectively). We call $M$ *regular*, if its rows are linearly independent over $\mathbb{F}[t]$ (or $\mathbb{F}(t)[\sigma]$ or $\mathbb{F}(t)[\sigma, \sigma^{-1}]$, respectively) and *singular* if they are not linearly independent. For the special case of a polynomial matrix $M \in \mathbb{F}[t]^{n\times n}$, we can characterise these concepts using determinants[5]: here, $M$ is singular if $\det M = 0$; regular if $\det M \neq 0$; and unimodular if $\det M \in \mathbb{F}\setminus\{0\}$. Another equivalent characterisation of regular polynomial matrices is that they have a rational inverse $M^{-1} \in \mathbb{F}(t)^{n\times n}$.

We denote the set of all unimodular polynomial matrices by $\mathrm{GL}_n(\mathbb{F}[t])$ and that of all unimodular operator matrices by $\mathrm{GL}_n(\mathbb{F}(t)[\sigma])$ or by $\mathrm{GL}_n(\mathbb{F}(t)[\sigma, \sigma^{-1}])$. We do not have a special notation for the set of regular matrices.

*Remark 3.1* Let $A \in \mathbb{F}(t)[\sigma]^{m\times n}$ and $b \in \mathbb{F}(t)^m$. Assume that we are given two unimodular operator matrices $P \in \mathrm{GL}_m(\mathbb{F}(t)[\sigma, \sigma^{-1}])$ and $Q \in \mathrm{GL}_n(\mathbb{F}(t)[\sigma, \sigma^{-1}])$. Then the system $A(y) = b$ has the solution $y$ if and only if $(PAQ)(\tilde{y}) = P(b)$ has the solution $\tilde{y} = Q^{-1}(y)$: Assume first that $A(y) = b$. Then also $P(A(y)) = (PA)(y) = P(b)$ and furthermore we have $P(b) = (PA)(y) = (PA)(QQ^{-1}(y)) =$

[5]The other two rings do not admit determinants since they lack commutativity.

$(PAQ)(Q^{-1}(y)) = (PAQ)(\tilde{y})$. Because $P$ and $Q$ are unimodular, we can easily go back as well. Thus, we can freely switch from one system to the other.

**Definition 3.1** We say that the systems $A(y) = b$ and $(PAQ)(\tilde{y}) = P(b)$ in Remark 3.1 are *related* to each other.

## 4 Denominator Bounds for Regularised Systems

Let $(\mathbb{F}(t), \sigma)$ be again a $\Pi\Sigma$-extension of $(\mathbb{F}, \sigma)$. Recall that we are considering the system (3) which has the form $A_\ell \sigma^\ell y + \cdots + A_1 \sigma y + A_0 y = b$ where $A_0, \ldots, A_\ell \in \mathbb{F}[t]^{m \times n}$ and $b \in \mathbb{F}[t]^m$. We start this section by identifying systems with good properties.

**Definition 4.1** We say that the system in Eq. (3) is *head regular* if $m = n$ (i.e., all the matrices are square) and $\det A_\ell \neq 0$.

**Definition 4.2** We say that the system in Eq. (3) is *tail regular* if $m = n$ and $\det A_0 \neq 0$.

**Definition 4.3** The system $A(y) = b$ in Eq. (3) is called *fully regular* if it is head regular and there exists a unimodular operator matrix $P \in \mathrm{GL}_n(\mathbb{F}(t)[\sigma, \sigma^{-1}])$ such that the related system $(PA)(\tilde{y}) = P(b)$ is tail regular.

We will show later in Sect. 6 that any head regular system is actually already fully regular and how the transformation matrix $P$ from Definition 4.3 can be computed.

Moreover, in Definition 4.3, we can always choose $P$ in such a way that the coefficient matrices $\tilde{A}_0, \ldots, \tilde{A}_{\tilde{\ell}}$ and the right hand side of the related system $(PA)(\tilde{y}) = P(b)$ are polynomial: simply multiplying by a common denominator will not change the unimodularity of $P$.

The preceding Definition 4.3 is very similar to *strongly row-reduced* matrices [37, Definition 4]. The main difference is that we allow an arbitrary transformation $P$ which translates between a head and tail regular system while [37] require their transformation (also called $P$) to be of the shape $\mathrm{diag}(\sigma^{m_1}, \ldots, \sigma^{m_n})$ for some specific exponents $m_1, \ldots, m_n \in \mathbb{Z}$. At this time, we do not know which of the two forms is more advantageous; it would be an interesting topic for future research to explore whether the added flexibility that our definition gives can be used to make the algorithm more efficient.

*Remark 4.1* In the situation of Definition 4.3, the denominators of the solutions of the original system $A(y) = b$ and the related system $\tilde{A}(\tilde{y}) = \tilde{b}$ are the same: By Remark 3.1, we know that $y$ solves the original system if and only if $\tilde{y}$ solves the related system. The matrix $Q$ of Remark 3.1 is just the identity in this case.

We are now ready to state the main result of this section. For the rational difference field this result appears in various specialised forms. E.g., the version $m = n = 1$ can be also found in [40] and gives an alternative description of Abramov's denominator bound for scalar recurrences [23]. Furthermore, the first order case $l = 1$ can be rediscovered also in [39].

**Theorem 4.1** *Let the system in Eq.* (3) *be fully regular, and let* $y = d^{-1}z \in \mathbb{F}(t)^n$ *be a solution in reduced form. Let* $(PA)(\tilde{y}) = P(b)$ *be a tail regular related system with trailing coefficient matrix* $\tilde{A}_0 \in \mathbb{F}[t]^{n\times n}$*. Let m be the common denominator of* $A_\ell^{-1}$ *and let p be the common denominator of* $\tilde{A}_0^{-1}$*. Then*

$$\operatorname{disp}(\operatorname{ap}(d)) \leqslant \operatorname{disp}(\sigma^{-\ell}(\operatorname{ap}(m)), \operatorname{ap}(p)) = D \tag{4}$$

*and*

$$\operatorname{ap}(d) \;\Big|\; \gcd\Big(\prod_{j=0}^{D} \sigma^{-\ell-j}(\operatorname{ap}(m)), \prod_{j=0}^{D} \sigma^{j}(\operatorname{ap}(p))\Big). \tag{5}$$

We will show in Sect. 6 that any coupled system of the form (3) can be brought to a system which is fully regular and which contains the same solutions as the original system. Note further that the denominator bound of the aperiodic part given on the right hand side of (5) can be computed if the dispersion of polynomials in $\mathbb{F}[t]$ (more precisely, if $D$) can be computed. Summarising, Theorem 2.1 is established if Theorem 4.1 is proven and if the transformation of system (5) to a fully regular version is worked out in Sect. 6.

*Remark 4.2* Let $(\mathbb{F}(t_1)\dots(t_e), \sigma)$ be a $\Pi\Sigma$-extension of $(\mathbb{F}, \sigma)$ with $\sigma(t_i) = \alpha_i\, t_i + \beta_i$ ($\alpha_i \in \mathbb{F}^*$, $\beta_i \in \mathbb{F}$) for $1 \leqslant i \leqslant e$. In this setting a multivariate aperiodic denominator bound $d \in \mathbb{F}[t_1, \dots, t_e]\setminus\{0\}$ has been provided for a coupled system in Corollary 2.2. Namely, within its proof we determine the aperiodic denominator bound $d$ by applying Theorem 2.1 (and thus internally Theorem 4.1) for each $\Pi\Sigma$-monomial $t_i$. Finally, we merge the different denominator bounds $d_i$ to the global aperiodic denominator bound $d = \operatorname{lcm}(d_1, \dots, t_e)$. In other words, the formula (5) is reused $e$ times (with possibly different $D$s). This observation gives rise to the following improvement: it suffices to compute for $1 \leqslant i \leqslant e$ the dispersions $D_i$ (using the formula (4) for the different $\Pi\Sigma$-monomials $t_i$), to set $D = \max(D_1, \dots, D_e)$ and to apply only once the formula (5). We will illustrate this strategy in an example of Sect. 7.

For the sake of clarity we split the proof into two lemmata.

**Lemma 4.1** *With the notations of Theorem 4.1, it is*

$$\operatorname{disp}(\operatorname{ap}(d)) \leqslant \operatorname{disp}(\sigma^{-\ell}(\operatorname{ap}(m)), \operatorname{ap}(p)) = D.$$

*Proof* For the ease of notation, we will simply write $\overline{p}$ instead of $\operatorname{ap}(p)$ and we will do the same with $\overline{m} = \operatorname{ap}(m)$ and $\overline{d} = \operatorname{ap}(d)$.

Assume that $\operatorname{disp}(\overline{d}) = \lambda > D$ for some $\lambda \in \mathbb{N}$. Then we can find two irreducible aperiodic factors $a, g \in \mathbb{F}[t]$ of $\overline{d}$ such that $\sigma^\lambda(a)/g \in \mathbb{F}$. In particular, due to Lemma 2.1 we can choose $a, g$ with this property such that $\lambda$ is maximal.

We distinguish two cases. First, assume that $a \mid \overline{p}$. We claim that in this case we have $\sigma^{\ell}(g) \nmid \overline{m}$. Otherwise, it was $g \mid \sigma^{-\ell}(\overline{m})$ which together with $g \mid \sigma^{\lambda}(a) \mid \sigma^{\lambda}(\overline{p})$ implied $\lambda \in \operatorname{spread}(\sigma^{-\ell}(\overline{m}), \overline{p})$ which contradicts $D < \lambda$. Moreover, $\sigma^{\ell}(g)$ cannot occur in $\sigma^{i}(\overline{d})$ for $0 \leqslant i < \ell$ because else $\sigma^{\ell}(g) \mid \sigma^{i}(\overline{d})$ and thus $\tilde{b} = \sigma^{\ell-i}(g) \mid \overline{d}$ implied that $a$ and $\tilde{g}$ are factors of $\overline{d}$. Now, since $\sigma^{\lambda+\ell-i}(a)/\tilde{g} = \sigma^{\ell-i}(\sigma^{\lambda}(a)/g) \in \mathbb{F}$, this contradicts the maximality of $\lambda$. Thus, $\sigma^{\ell}(g)$ must occur in the denominator of

$$A_{\ell}\sigma^{\ell}(y) + A_{\ell-1}\sigma^{\ell-1}(y) + \cdots + A_1\sigma(y) + A_0 y = b \in \mathbb{F}[t]^n \tag{6}$$

for at least one component: Let $A_{\ell}^{-1} = \overline{m}U$ for some $U \in \mathbb{F}[t]^{n\times n}$. Then $UA_{\ell} = \overline{m}\mathbf{1}_n$ and

$$\underbrace{UA_{\ell}}_{=\overline{m}\mathbf{1}}\sigma^{\ell}(y) + UA_{\ell-1}\sigma^{\ell-1}(y) + \cdots + UA_1\sigma(y) + UA_0 y$$
$$= \frac{\overline{m}\sigma^{\ell}(z)}{\alpha\sigma^{\ell}(g)} + \frac{\sum_{j\neq\ell}\Big(\prod_{k\neq j,\ell}\sigma^{j}(\overline{d})\Big)UA_j\sigma^{j}(z)}{\prod_{j\neq\ell}\sigma^{j}(\overline{d})} = Ub \in \mathbb{F}[t]^n$$

for some $\alpha \in \mathbb{F}[t]^n$ such that $\sigma^{\ell}(\overline{d}) = \alpha\sigma^{\ell}(g)$. The equation is equivalent to

$$\Big(\prod_{j\neq\ell}\sigma^{j}(\overline{d})\Big)\overline{m}\sigma^{\ell}(z) = \Big(\Big(\prod_{j\neq\ell}\sigma^{j}(\overline{d})\Big)Ub - \sum_{j\neq\ell}\Big(\prod_{k\neq j,\ell}\sigma^{j}(\overline{d})\Big)UA_j\sigma^{j}(z)\Big)\alpha\sigma^{\ell}(g).$$

Note that (every component of the vector on) the right hand side is divisible by $\sigma^{\ell}(g)$. For the left hand side, we have

$$\sigma^{\ell}(g) \nmid \overline{m}\prod_{j\neq\ell}\sigma^{j}(\overline{d}).$$

Also, we know that $g \nmid z_j$ for at least one $j$. Thus, $\sigma^{\ell}(g)$ does not divide (at least one component of) the left hand side. This is a contradiction.

We now turn our attention to the second case $a \nmid \overline{p}$. Here, we consider the related tail regular system $\tilde{A}_{\tilde{\ell}}\sigma^{\tilde{\ell}}(y) + \cdots + \tilde{A}_0 y = \tilde{b}$ instead of the original system. Recall that $y$ remains unchanged due to Remark 4.1. Similar to the first case, let $\tilde{A}_0^{-1} = \overline{p}V$, i.e., $V\tilde{A}_0 = \overline{p}\mathbf{1}_n$ for some $V \in \mathbb{F}[t]^{n\times n}$. Note that $a \nmid \sigma^{i}(\overline{d})$ for all $i \geqslant 1$; otherwise, $\sigma^{-i}(a)$ was a factor of $\overline{d}$ with $\sigma^{\lambda+i}(\sigma^{-i}(a))/b \in \mathbb{F}$ contradicting the maximality of $\lambda$. Let now

$$V\tilde{A}_{\tilde{\ell}}\sigma^{\tilde{\ell}}(y) + \cdots + V\tilde{A}_1\sigma(y) + \overline{p}\mathbf{1}_n y = \tilde{\xi} \in \mathbb{F}[t]^n.$$

We write again $y = \overline{d}^{-1}z$. Then, after multiplying with the common denominator $\overline{d}\sigma(\overline{d})\cdots\sigma^{\ell}(\overline{d})$ and rearranging the terms we obtain

$$\overline{p}\left(\prod_{k\neq 0}\sigma^k(\overline{d})\right)z = \Big(\prod_{j=0}^{\ell}\sigma^j(\overline{d})\Big)\tilde{\xi} - \sum_{j=1}^{\ell}\Big(\prod_{k\neq j}\sigma^k(\overline{d})\Big)V\tilde{A}_j\sigma^j(z)$$

where every term on the right hand side is divisible by $a$. However, on the left hand side $a$ does not divide the scalar factor $\overline{p}\prod_{k\neq 0}\sigma^k(\overline{d})$ and because of $\gcd(z, \overline{d}) = 1$ there is at least one component of $z$ which is not divisible by $a$. Thus, $a$ does not divide the left hand side which is a contradiction. □

**Lemma 4.2** *With the notations of Theorem 4.1, we have*

$$\operatorname{ap}(d) \;\Big|\; \gcd\Big(\prod_{j=0}^{D}\sigma^{-\ell-j}(m), \prod_{j=0}^{D}\sigma^j(p)\Big).$$

*Proof* Again, we will simply write $\overline{p}$, $\overline{m}$ and $\overline{d}$ instead of $\operatorname{ap}(p)$, $\operatorname{ap}(m)$ and $\operatorname{ap}(d)$, respectively. As in the proof of Lemma 4.1, we let $U \in \mathbb{F}[t]^{n\times n}$ be such that $UA_\ell = \overline{m}\mathbf{1}$. Multiplication by $U$ from the left and isolating the highest order term transforms the system (3) into

$$\sigma^\ell(y) = \frac{1}{\overline{m}}U\Big(b - \sum_{j=0}^{\ell-1}A_j\sigma^j(y)\Big). \tag{7}$$

Now, we apply $\sigma^{-1}$ to both sides of the equation in order to obtain an identity for $\sigma^{\ell-1}(y)$

$$\sigma^{\ell-1}(y) = \frac{1}{\sigma^{-1}(\overline{m})}\sigma(U)\Big(\sigma(b) - \sum_{j=0}^{\ell-1}\sigma(A_j)\sigma^{j-1}(y)\Big).$$

We can substitute this into (7) in order to obtain a representation

$$\begin{aligned}\sigma^\ell(y) = \frac{1}{\overline{m}}U\Big(b - A_{\ell-1}\frac{1}{\sigma^{-1}(\overline{m})}\sigma(U)\Big(\sigma(b) - \sum_{j=0}^{\ell-1}\sigma(A_j)\sigma^{j-1}(y)\Big) - \sum_{j=0}^{\ell-2}A_j\sigma^j(y)\Big)\\ = \frac{1}{\overline{m}\,\sigma^{-1}(\overline{m})}\tilde{U}\Big(\tilde{b} - \sum_{j=-1}^{\ell-2}\tilde{A}_j\sigma^j(y)\Big)\end{aligned}$$

for $\sigma^\ell(y)$ in terms of $\sigma^{\ell-2}(y), \dots, \sigma^{-1}(y)$ where $\tilde{b} \in \mathbb{F}[t]^n$ and $\tilde{A}_{\ell-2}, \dots, \tilde{A}_{-1}, \tilde{U} \in \mathbb{F}[t]^{n\times n}$.

We can continue this process shifting the terms on the right side further with each step. Eventually, after $D$ steps, we will arrive at a system of the form

$$\sigma^\ell(y) = \frac{1}{\overline{m}\,\sigma^{-1}(\overline{m})\cdots\sigma^{-D}(\overline{m})}U'\Big(b' - \sum_{j=-D}^{\ell-D-1}A'_j\sigma^j(y)\Big) \tag{8}$$

where $b' \in \mathbb{F}[t]^n$ and $A'_{-D}, \ldots, A'_{\ell-D-1}, U' \in \mathbb{F}[t]^{n\times n}$.

Assume now that $y = \overline{d}^{-1} z$ is a solution of (3) or thus of (8) which is in reduced representation for some $\overline{d} \in \mathbb{F}[t]$ and a vector $z \in \mathbb{F}[t]^n$. Substituting this in Eq. (8) yields

$$\frac{1}{\sigma^\ell(\overline{d})}\sigma^\ell(z) = \frac{1}{\overline{m}\sigma^{-1}(\overline{m})\cdots\sigma^{-D}(\overline{m})}U'\Big(b' - \sum_{j=-D}^{\ell-D-1} A'_j \frac{1}{\sigma^j(\overline{d})}\sigma^j(z)\Big)$$

$$= \frac{1}{\prod_{j=0}^{D}\sigma^{-j}(\overline{m})\cdot\prod_{j=-D}^{\ell-D-1}\sigma^j(\overline{d})}U'\Big(\prod_{j=-D}^{\ell-D-1}\sigma^j(\overline{d})\cdot b' - \sum_{j=-D}^{\ell-D-1}\prod_{k\neq j}\sigma^k(\overline{d})\cdot A'_j\sigma^j(z)\Big)$$

or, equivalently after clearing denominators,

$$\prod_{j=0}^{D}\sigma^{-j}(\overline{m})\cdot\prod_{j=-D}^{\ell-D-1}\sigma^j(\overline{d})\cdot\sigma^\ell(z)$$
$$= \sigma^\ell(\overline{d})\cdot U'\Big(\prod_{j=0}^{\ell-1}\sigma^{j-D}(\overline{d})\cdot b' - \sum_{j=-D}^{\ell-D-1}\prod_{k\neq j}\sigma^k(\overline{d})\cdot A'_j\sigma^j(z)\Big). \quad (9)$$

Let further $q \in \mathbb{F}[t]$ be an irreducible factor of the aperiodic part of $\overline{d}$. Then $\sigma^\ell(q)$ divides the right hand side of Eq. (9). Looking at the left hand side, we see that $\sigma^\ell(q)$ cannot divide $\prod_{j=0}^{\ell-1}\sigma^{j-D}(\overline{d})$ since $D = \operatorname{disp}(\overline{d})$ and there is at least one entry $z_k$ of $z$ with $1 \leqslant k \leqslant n$ such that $q \nmid z_k$ because $\overline{d}^{-1}z$ is in reduced representation. It follows that $\sigma^\ell(q) \mid \prod_{j=0}^{D}\sigma^{-j}(\overline{m})$, or, equivalently, $q \mid \prod_{j=0}^{D}\sigma^{-\ell-j}(\overline{m})$. We can thus cancel $q$ from the equation. Reasoning similarly for the other irreducible factors of the aperiodic part of $\overline{d}$ we obtain $\overline{d} \mid \prod_{j=0}^{D}\sigma^{-\ell-j}(\overline{m})$.

In order to prove $\overline{d} \mid \prod_{j=0}^{D}\sigma^j(\overline{p})$, we consider once more the related tail regular system $\tilde{A}_{\tilde{\ell}}\sigma^{\tilde{\ell}}(y) + \cdots + \tilde{A}_0 y = \tilde{b}$. Recall that by Remark 4.1 $y$ is both a solution of the original and the related. Let $V\tilde{A}_0 = \overline{p}\mathbf{1}$ for some $V \in \mathbb{F}[t]^{n\times n}$. Multiplying the related system by $V$ and isolating $y$ yields

$$y = \frac{1}{p}V\Big(\tilde{b} - \sum_{j=1}^{\tilde{\ell}}\tilde{A}_j\sigma^j(\tilde{y})\Big).$$

Now, an analogous computation allows us to shift the orders of the terms on the right hand side upwards. We obtain an equation

$$y = \frac{1}{\overline{p}\sigma(\overline{p})\cdots\sigma^D(\overline{p})}V'\Big(\tilde{b}' - \sum_{j=1}^{\tilde{\ell}}\tilde{A}'_j\sigma^{D+j}(y)\Big)$$

for suitable matrices $V', \tilde{A}'_1, \ldots, \tilde{A}'_{\tilde{\ell}} \in \mathbb{F}[t]^{n\times n}$ and $\tilde{b}' \in \mathbb{F}[t]^n$. Substituting again $y = \overline{d}^{-1} z$ and clearing denominators we arrive at an equation similar to (9) and using the same reasoning we can show that $\overline{d} \mid \prod_{j=0}^{D} \sigma^j(\overline{p})$. □

## 5 Row and Column Reduction

We will show in Sect. 6 below that it is actually possible to make any system of the form (3) fully regular. One of the key ingredients for this will be row (and column) reduction which we are going to introduce in this section. The whole exposition closely follows the one in [38]. We will concentrate on row reduction since column reduction works *mutatis mutandis* basically the same.

Consider an arbitrary operator matrix $A \in \mathbb{F}(t)[\sigma]^{m\times n}$. When we speak about the *degree* of $A$, we mean the maximum of the degrees (in $\sigma$) of all the entries of $A$. Similarly, the degree of a row of $A$ will be the maximum of the degrees in that row.

Let $\nu$ be the degree of $A$ and let $\nu_1, \ldots, \nu_m$ be the degrees of the rows of $A$. For simplicity, we first assume that none of the rows of $A$ is zero. When we multiply $A$ by the matrix $\Delta = \operatorname{diag}(\sigma^{\nu-\nu_1}, \ldots, \sigma^{\nu-\nu_m})$ from the left, then for each $i = 1, \ldots, m$ we multiply the $i$th row by $\sigma^{\nu-\nu_i}$. The resulting row will have degree $\nu$. That is, multiplication by $\Delta$ brings all rows to the same degree. We will write the product as

$$\Delta A = A_\nu \sigma^\nu + \cdots + A_1\sigma + A_0$$

where $A_0, \ldots, A_\nu \in \mathbb{F}(t)^{m\times n}$ are rational matrices. Since none of the rows of $A$ is zero, also none of the rows of $A_\nu$ is zero. We call $A_\nu$ the *leading row coefficient matrix* of $A$ and denote it by $A_\nu = \operatorname{LRCM}(A)$. In general, if some rows of $A$ are zero, then we simply define the corresponding rows in $\operatorname{LRCM}(A)$ to be zero, too.

**Definition 5.1** The matrix $A \in \mathbb{F}(t)[\sigma]^{m\times n}$ is *row reduced* (w.r.t. $\sigma$) if $\operatorname{LRCM}(A)$ has full row rank.

*Remark 5.1* If $A(y) = b$ is a head reduced system where $A = A_\ell\sigma^\ell + \cdots + A_1\sigma + A_0$ for $A_0, \ldots, A_\ell \in R^{n\times n}$, then $A$ is row-reduced. This is obvious since in this case $\operatorname{LRCM}(A) = A_\ell$ and $\det A_\ell \neq 0$. Conversely, if $A$ is row-reduced, then $\Delta A$ (with $\Delta$ as above) is head regular.

It can be shown that for any matrix $A \in \mathbb{F}(t)[\sigma]^{m\times n}$ there exists a unimodular operator matrix $P \in \operatorname{GL}_m(\mathbb{F}(t)[\sigma])$ such that

$$PA = \begin{pmatrix} \tilde{A} \\ \mathbf{0} \end{pmatrix}$$

for some row reduced $\tilde{A} \in \mathbb{F}(t)[\sigma]^{r\times n}$ where $r$ is the (right row) rank of $A$ over $\mathbb{F}(t)[\sigma]$. (For more details, see [38, Theorem 2.2] and [38, Theorem A.2].)

It is a simple exercise to derive an analogous column reduction of $A$. Moreover, it can easily be shown that it has similar properties. In particular, we can always compute $Q \in \mathrm{GL}_n(\mathbb{F}(t)[\sigma])$ such that the product will be

$$AQ = \begin{pmatrix} \hat{A} & \mathbf{0} \end{pmatrix}$$

for some column reduced $\hat{A} \in \mathbb{F}(t)[\sigma]^{m\times r}$ where $r$ is the (left column) rank of $A$.

We remark that in fact $r$ in both cases will be the same number since the left column rank of $A$ equals the right row rank by, e.g., [55, Theorem 8.1.1]. Therefore, in the following discussion we will simply refer to it as the *rank* of $A$.

## 6 Regularisation

In Theorem 4.1, we had assumed that we were dealing with a fully regular system. This section will explain how every arbitrary system can be transformed into a fully regular one with the same set of solutions.

Represent the system (3) by an operator matrix $A \in \mathbb{F}(t)[\sigma]^{m\times n}$. We first apply column reduction to $A$ which gives a unimodular operator matrix $Q \in \mathrm{GL}_n(\mathbb{F}(t)[\sigma])$ such that the non-zero columns of $AQ$ are column reduced. Next, we apply row reduction to $AQ$ obtaining $P \in \mathrm{GL}_m(\mathbb{F}(t)[\sigma])$ such that in total

$$PAQ = \begin{pmatrix} \tilde{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$$

where $\tilde{A} \in \mathbb{F}(t)[\sigma]^{r\times r}$ will now be a row reduced square matrix and $r$ is the rank of $A$.

If we define the matrix $\Delta$ as in the previous Sect. 5, then the leading coefficient matrix of $\Delta PAQ$ and that of $PAQ$ will be the same. Moreover, since $\Delta$ is unimodular over $\mathbb{F}(t)[\sigma, \sigma^{-1}]$, also $\Delta P$ is unimodular over $\mathbb{F}(t)[\sigma, \sigma^{-1}]$. Thus, if we define $\hat{A} \in \mathbb{F}(t)[\sigma]^{r\times r}$ by

$$\Delta PAQ = \begin{pmatrix} \hat{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix},$$

then we can write

$$\hat{A} = \hat{A}_\nu \sigma^\nu + \cdots + \hat{A}_1 \sigma + \hat{A}_0$$

where $\nu$ is the degree of $\hat{A}$ and where $\hat{A}_0, \ldots, \hat{A}_\nu \in \mathbb{F}(t)^{r\times r}$ are rational matrices. Since $\hat{A}$ is still row reduced, we obtain that its leading row coefficient matrix $\hat{A}_\nu$ has full row rank.

Assume now that we started with the system $A(y) = b$. Then $(\Delta PAQ)(y) = (\Delta P)(b)$ is a related system with the same solutions as per Remark 3.1. More concretely, let us write

$$y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \quad \text{and} \quad (\Delta P)(b) = \begin{pmatrix} \tilde{b}_1 \\ \tilde{b}_2 \end{pmatrix}$$

where $y_1$ and $\tilde{b}_1 \in \mathbb{F}(t)^r$ are vectors of length $r$, $y_2 \in \mathbb{F}(t)^{m-r}$ has length $m - r$, and $\tilde{b}_2 \in \mathbb{F}(t)^{n-r}$ has length $n - r$. Then $(\Delta PAQ)(y) = (\Delta P)(b)$ can be expressed as

$$\hat{A}(y_1) = \tilde{b}_1 \quad \text{and} \quad 0 = \tilde{b}_2.$$

The requirement that $\tilde{b}_2 = 0$ is a necessary condition for the system to be solvable. We usually refer to it as a *compatibility condition*. Moreover, we see that the variables in $y_2$ can be chosen freely.

If the compatibility condition does not hold, then the system does not have any solutions and we may abort the computation right here. Otherwise, $A(y) = b$ is equivalent to a system $\hat{A}(y_1) = \tilde{b}_1$ of (potentially) smaller size. Clearing denominators in the last system does not change its solvability nor the fact that $\hat{A}$ is row reduced. Thus, we have arrived at an equivalent head regular system.

It remains to explain how we can turn a head regular system into a fully regular one. Thus, as above we assume now that the first regularisation step is already done and that the operator matrix $A \in \mathbb{F}(t)[\sigma]^{n\times n}$ is such that $A(y) = b$ is head regular. That does in particular imply that $A$ is row reduced and hence that $n$ equals the rank of $A$ over $\mathbb{F}(t)[\sigma]$.

We claim that $n$ is also the rank of $A$ over $\mathbb{F}(t)[\sigma, \sigma^{-1}]$, i.e., that the rows of $A$ are linearly independent over $\mathbb{F}(t)[\sigma, \sigma^{-1}]$. Assume that $vA = 0$ for some $v \in \mathbb{F}(t)[\sigma, \sigma^{-1}]^n$. There is a power $\sigma^k$ of $\sigma$ such that $\sigma^k v \in \mathbb{F}(t)[\sigma]^n$. Since then $(\sigma^k v)A = 0$, we obtain that $A$ did not have full rank over $\mathbb{F}(t)[\sigma]$. The claim follows by contraposition. Note that also the other direction obviously holds.

Let $\ell$ be the degree of $A$ and write $A$ as

$$A = A_\ell \sigma^\ell + \cdots + A_1 \sigma + A_0$$

where $A_0, \ldots, A_\ell \in \mathbb{F}(t)^{n\times n}$. We multiply $A$ over $\mathbb{F}(t)[\sigma, \sigma^{-1}]$ by $\sigma^{-\ell}$ from the left. This does not change the rank. The resulting matrix $\sigma^{-\ell}A$ will be in $\mathbb{F}(t)[\sigma^{-1}]^{n\times n}$. Using a similar argument as above, we see that the rank of $\sigma^{-\ell}A$ over $\mathbb{F}(t)[\sigma^{-1}]$ is still $n$. We have

$$\sigma^{-\ell} A_\ell = \sigma^{-\ell}(A_0)\sigma^{-\ell} + \cdots + \sigma^{-\ell}(A_{\ell-1})\sigma^{-1} + \sigma^{-\ell}(A_\ell),$$

i.e., $\sigma^{-\ell}A$ is similar to $A$ with the coefficients in reverse order.

We can now apply row reduction to $\sigma^{-\ell}A$ w.r.t. the Ore variable[6] $\sigma^{-1}$. Just as before we may also shift all the rows afterwards to bring them to the same degree. Let the result be

$$W\sigma^{-\ell}A = \tilde{A}_0 \sigma^{-\tilde{\ell}} + \cdots + \tilde{A}_{\tilde{\ell}-1}\sigma^{-1} + \tilde{A}_{\tilde{\ell}}$$

[6]Note that the commutation rule $\sigma^{-1}a = \sigma^{-1}(a)\sigma^{-1}$ follows immediately from the rule $\sigma a = \sigma(a)\sigma$.

where $\tilde{\ell}$ is the new degree, $W \in \mathrm{GL}_n(\mathbb{F}(t)[\sigma, \sigma^{-1}])$ is a unimodular operator matrix, the matrices $\tilde{A}_0, \ldots, \tilde{A}_{\tilde{\ell}} \in \mathbb{F}(t)^{n\times n}$ are rational, and where the non-zero rows of $\tilde{A}_0$ are independent. However, since the rank of $\sigma^{-\ell} A$ is $n$ (over $\mathbb{F}(t)[\sigma^{-1}]$), we obtain that $\tilde{A}_0$ does in fact not possess any zero-rows at all. Thus, $\tilde{A}_0$ has full rank.

Multiplication by $\sigma^{\tilde{\ell}}$ from the left, brings everything back into $\mathbb{F}(t)[\sigma]^{n\times n}$; i.e., we have

$$\sigma^{\tilde{\ell}} W \sigma^{-\ell} A = \sigma^{\tilde{\ell}}(\tilde{A}_{\tilde{\ell}})\sigma^{\tilde{\ell}} + \cdots + \sigma^{\tilde{\ell}}(\tilde{A}_1)\sigma + \sigma^{\tilde{\ell}}(\tilde{A}_0)$$

where $\sigma^{\tilde{\ell}}(\tilde{A}_0)$ still has full rank and where the transformation matrix $\sigma^{\tilde{\ell}} W \sigma^{-\ell}$ is unimodular over $\mathbb{F}(t)[\sigma, \sigma^{-1}]$. In other words, we have found a related tail regular system. Since we started with a head regular system, we even found that it is fully reduced.

We can summarise the results of this section in the following way. An overview of the process is shown in Fig. 1.

**Theorem 6.1** *Any system of the form* (3) *can be transformed into an related fully regular system. Along the way we acquire some compatibility conditions indicating where the system may be solvable.*

We would like to once more compare our approach to the one taken in [37]. They show how to turn a system into strongly row-reduced form (their version of fully regular as explained after Definition 4.3 in the proof of their [37, Proposition 5]. Although they start out with an input of full rank, this is not a severe restriction as the same preprocessing step (from $A$ to $\hat{A}$) which we used could be applied in their case, too. Just like our approach, their method requires two applications of row reduction. They do, however, obtain full regularity in the opposite order: The first reduction makes the system tail regular while the second reduction works on the leading matrix. In our case, the first row reduction (removes unnecessary equations) and makes the system head regular while the second one works on the tail. The

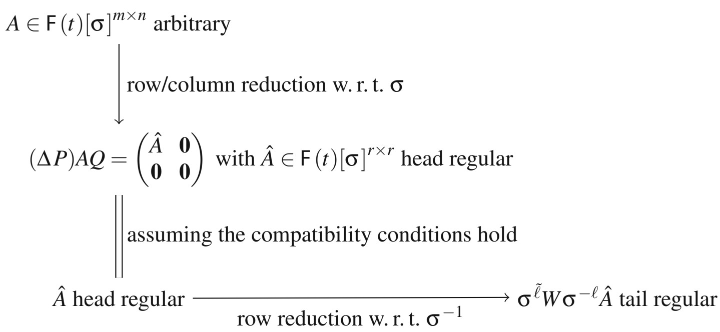


**Fig. 1** Outline of the regularisation

other big difference is that our second reduction is w.r.t. $\sigma^{-1}$ while [37] rewrites the system in terms of the difference operator $\Delta = \sigma - 1$ and then reduces w.r.t. $\Delta$. As mentioned after Definition 4.3, we cannot with certainty tell yet which of the two approaches is preferable. That will be a topic for future research.

## 7 Examples

As a first example, we consider the system

$$\begin{pmatrix} -2t^2 - t + 1 & 0 \\ -2t^5 - 9t^4 - 15t^3 - 8t + 3t + 3 & -t^7 - 2t^6 - 4t^5 - 6t^4 - 7t^3 - 8t^2 - 4t \end{pmatrix} y(t+1)$$
$$+ \begin{pmatrix} t^4 - t^3 + 2t^2 & t^4 - t^3 + 2t^2 \\ 0 & t^7 + 3t^6 + 4t^5 + 5t^4 + 9t^3 + 6t^2 \end{pmatrix} y(t)$$
$$= \begin{pmatrix} 0 \\ 2t^5 + 3t^4 + t^3 + 8t^2 + 4t \end{pmatrix}.$$

Here, we have $\mathbb{F} = \mathbb{Q}$ and we are in the $\Sigma$-extension case with $\sigma(t) = t + 1$. We can easily see that the leading and trailing matrices are both regular. Inverting them and computing common denominators, we arrive at

$$m = (2t - 1)t(t^2 + t + 2)(t^2 - t + 2)(t + 1)^2$$

and

$$p = t^2(t + 1)(t^2 - t + 2)(t^2 + 3t + 3).$$

We have $\operatorname{spread}(\sigma^{-1}m, p) = \{0\}$ and thus the dispersion is 0. We obtain the denominator bound

$$\gcd(\sigma^{-1}m, p) = t^2(t^2 - t + 2).$$

This does fit well with the actual solutions for which a $\mathbb{Q}$-basis is given by

$$\frac{1}{t^2(t^2 - t + 2)} \begin{pmatrix} -t(t^2 - t + 2) \\ t^3 - t^2 + 1 \end{pmatrix} \quad \text{and} \quad \frac{1}{t^2(t^2 - t + 2)} \begin{pmatrix} -t^3(t^2 - t + 2) \\ t^5 - t^4 - 3t^2 + 1 \end{pmatrix}.$$

(We can easily check that those are solutions; and according to [37, Theorem 6] the dimension of the solution space is 2).

For the second example, we consider a (2, 3)-multibasic rational difference field over $\mathbb{F} = \mathbb{Q}$; i.e., we consider $\mathbb{Q}(t_1, t_2)$ with $\sigma(t_1) = 2t_1$ and $\sigma(t_2) = 3t_2$. The system in this example is

$$\overbrace{\begin{pmatrix}(11t_1t_2-1)(36t_1t_2-1) & -(11t_1t_2-1)(36t_1t_2-1)\\ (4t_1-9t_2)(2t_1-3t_2) & (4t_1-9t_2)(2t_1-3t_2)\end{pmatrix}}^{=:A_2(t_1,t_2)}\begin{pmatrix}y_1(4t_1,9t_2)\\ y_2(4t_1,9t_2)\end{pmatrix}$$
$$+\begin{pmatrix}-(6t_1t_2-1)(143t_1t_2-3) & (6t_1t_2-1)(143t_1t_2-3)\\ -6(2t_1-3t_2)(t_1-2t_2) & -6(2t_1-3t_2)(t_1-2t_2)\end{pmatrix}\begin{pmatrix}y_1(2t_1,3t_2)\\ y_2(2t_1,3t_2)\end{pmatrix}$$
$$+\underbrace{\begin{pmatrix}2(t_1t_2-1)(66t_1t_2-1) & -2(t_1t_2-1)(66t_1t_2-1)\\ (4t_1-9t_2)(t_1-t_2) & (4t_1-9t_2)(t_1-t_2)\end{pmatrix}}_{=:A_0(t_1,t_2)}\begin{pmatrix}y_1(t_1,t_2)\\ y_2(t_1,t_2)\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}.$$

This is a 2-by-2 system of order 2 over $\mathbb{Q}(t_1,t_2)[\sigma]$. Both the $\sigma$-leading matrix $A_2$ and the $\sigma$-trailing matrix $A_0$ are invertible, which means that the system is both head and tail regular; hence it is fully regular. The denominator of $A_2^{-1}$ is

$$m=2(11t_1t_2-1)(36t_1t_2-1)(4t_1-9t_2)(2t_1-3t_2)$$

and the denominator of $A_0^{-1}$ is

$$p=4(t_1t_2-1)(66t_1t_2-1)(4t_1-9t_2)(t_1-t_2).$$

We have $\mathrm{ap}(m)=m$ and $\mathrm{ap}(p)=p$. Following the strategy/algorithm proposed in Remark 4.2 we compute the dispersions w.r.t. $t_1$ and $t_2$ (which turn out to be the same in this example); obtaining

$$D=\mathrm{disp}_{t_1,t_2}(\sigma^{-2}(\mathrm{ap}(m)),\mathrm{ap}(p))=0.$$

By Corollary 2.2 it follows that the denominator bound for this system is

$$d=\gcd(\sigma^{-2}(\mathrm{ap}(m)),\mathrm{ap}(d))=(t_1t_2-1)(t_1-t_2).$$

This fits perfectly with the actual $\mathbb{Q}$-basis of the solution space which is given by

$$\frac{1}{2(t_1t_2-1)(t_1-t_2)}\begin{pmatrix}(t_2+1)(t_1-1)\\(t_2-1)(t_1+1)\end{pmatrix},\quad \frac{1}{2(t_1-t_2)}\begin{pmatrix}t_1^2-t_1t_2+1\\-t_1^2+t_1t_2+1\end{pmatrix},$$
$$\frac{1}{4(t_1-t_2)}\begin{pmatrix}2t_1^2-2t_1t_2+4t_1-3t_2\\-2t_1^2+2t_1t_2+4t_1-3t_2\end{pmatrix},$$
$$\text{and}\quad\frac{1}{4(t_1t_2-1)(t_1-t_2)}\begin{pmatrix}4t_1^2t_2-3t_1t_2^2-2t_1+t_2\\4t_1^2t_2-3t_1t_2^2-6t_1+5t_2\end{pmatrix}.$$

(It is easy to check that these are solutions; and they are a basis of the solutions since the dimension of the solution space is 4 according to [37].)

## 8 Conclusion

Given a $\Pi\Sigma$-extension $(\mathbb{F}(t), \sigma)$ of $(\mathbb{F}, \sigma)$ and a coupled system of the form (3) whose coefficients are from $\mathbb{F}(t)$, we presented algorithms that compute an aperiodic denominator bound $d \in \mathbb{F}[t]$ for the solutions under the assumption that the dispersion can be computed in $\mathbb{F}[t]$ (see Theorem 2.1). If $t$ represents a sum, i.e., it has the shift behaviour $\sigma(t) = t + \beta$ for some $\beta \in \mathbb{F}$, this is the complete denominator bound. If $t$ represents a product, i.e., it has the shift behaviour $\sigma(t) = \alpha\, t$ for some $\alpha \in \mathbb{F}^*$, then $t^m\, d$ will be a complete denominator bound for a sufficiently large $m$. It is so far an open problem to determine this $m$ in the $\Pi$-monomial case by an algorithm; so far a solution is only given for the $q$-case with $\sigma(t) = q\, t$ in [56]. In the general case, one can still guess $m \in \mathbb{N}$, i.e., one can choose a possibly large enough $m$ ($m = 0$ if $t$ is a $\Sigma$-monomial) and continue. Namely, plugging $y = \frac{y'}{t^m\, d}$ with the unknown numerator $y' \in \mathbb{F}[t]^n$ into the system (3) yields a new system in $y'$ where one has to search for all polynomial solutions $y' \in \mathbb{F}[t]^n$. It is still an open problem to determine a degree bound $b \in \mathbb{N}$ that bounds the degrees of all entries of all solutions $y'$; for the rational case $\sigma(t) = t + 1$ see [32] and for the $q$-case $\sigma(t) = q\, t$ see [56]. In the general case, one can guess a degree bound $b$, i.e., one can choose a possibly large enough $b \in \mathbb{N}$ and continues to find all solutions $y'$ whose degrees of the components are at most $b$. This means that one has to determine the coefficients up to degree $b$ in the difference field $(\mathbb{F}, \sigma)$.

If $\mathbb{F} = \text{const}(\mathbb{F}, \sigma)$, this task can be accomplished by reducing the problem to a linear system and solving it. Otherwise, suppose that $\mathbb{F}$ itself is a $\Pi\Sigma$-field over a constant field $\mathbb{K}$. Note that in this case we can compute $d$ (see Lemma 2.1), i.e., we only supplemented a tuple $(m, b)$ of nonnegative integers to reach this point. Now one can use degree reduction strategies as worked out in [7, 8, 29] to determine the coefficients of the polynomial solutions by solving several coupled systems in the smaller field $\mathbb{F}$. In other words, we can apply our strategy again to solve these systems in $\mathbb{F} = \mathbb{F}'(\tau)$ where $\tau$ is again a $\Pi\Sigma$-monomial: compute the aperiodic denominator bound $d' \in \mathbb{F}'[\tau]$, guess an integer $m' \geqslant 0$ ($m' = 0$ if $\tau$ is a $\Sigma$-monomial) for a complete denominator bound $\tau^{m'}\, d'$, guess a degree bound $b' \geqslant 0$ and determine the coefficients of the polynomial solutions by solving coupled systems in the smaller field $\mathbb{F}'$. Eventually, we end up at the constant field and solve the problem there by linear algebra.

Summarising, we obtain a method that enables one to search for all solutions of a coupled system in a $\Pi\Sigma$-field where one has to adjust certain nonnegative integer tuples $(m, b)$ to guide our machinery. Restricting to scalar equations with coefficients from a $\Pi\Sigma$-field, the bounds of the period denominator part and the degree bounds has been determined only some years ago [57]. Till then we used the above strategy also for scalar equations [29] and could derive the solutions in concrete problems in a rather convincing way. It is thus expected that this approach will be also rather helpful for future calculations.

Influenced by Abramov's pioneering article [23] basically all existing difference equations solvers that hunt for rational solutions follow the same strategy: compute a denominator bound and reduce the problem to search for polynomial solutions. In Karr's work [8] there is another approach to solve first order difference equations. Inspired by that, one could derive denominator bounds for all factors which are shift equivalent. Further, exploiting partial fraction decomposition techniques from [8] one could split these subproblems further to deal only with denominators that are, up to multiplicities, irreducible. Summarizing, instead of finding one big solution for a coupled system, one could look for many small subparts of the solution that have to be combined accordingly. Exploring this Karr-type reduction mechanism further (or combining it in parts with Abramov's strategy) might lead to new and rather efficient algorithms.

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# Representing ($q$–)Hypergeometric Products and Mixed Versions in Difference Rings

**Evans Doe Ocansey and Carsten Schneider**

*Dedicated to Sergei A. Abramov on the occasion of his 70th birthday*

**Abstract** In recent years, Karr's difference field theory has been extended to the so-called R$\Pi\Sigma$-extensions in which one can represent not only indefinite nested sums and products that can be expressed by transcendental ring extensions, but one can also handle algebraic products of the form $\alpha^n$ where $\alpha$ is a root of unity. In this article we supplement this summation theory substantially by the following building block. We provide new algorithms that represent a finite number of hypergeometric or mixed $(q_1, \dots, q_e)$-multibasic hypergeometric products in such a difference ring. This new insight provides a complete summation machinery that enables one to formulate such products and indefinite nested sums defined over such products in R$\Pi\Sigma$-extensions fully automatically. As a side-product, one obtains compactified expressions where the products are algebraically independent among each other, and one can solve the zero-recognition problem for such products.

**Keywords** Symbolic summation · Hypergeometric products · Difference rings
Algebraic independence

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# 1 Introduction

Symbolic summation in difference fields has been introduced by Karr's groundbreaking work [12, 13]. He defined the so-called $\Pi\Sigma$-fields $(\mathbb{F}, \sigma)$ which are composed by a field $\mathbb{F}$ and a field automorphism $\sigma : \mathbb{F} \to \mathbb{F}$. Here the field $\mathbb{F}$ is built by a tower of transcendental field extensions whose generators either represent sums or products where the summands or multiplicands are elements from the field below. In particular, the following problem has been solved: given such a $\Pi\Sigma$-field $(\mathbb{F}, \sigma)$ and given $f \in \mathbb{F}$. Decide algorithmically, if there exists a $g \in \mathbb{F}$ with

$$f = \sigma(g) - g. \tag{1}$$

Hence if $f$ and $g$ can be rephrased to expressions $F(k)$ and $G(k)$ in terms of indefinite nested sums and products, one obtains the telescoping relation

$$F(k) = G(k+1) - G(k). \tag{2}$$

Then summing this telescoping equation over a valid range, say $a \leqslant k \leqslant b$, one gets the identity $\sum_{k=a}^{b} F(k) = G(b+1) - G(a)$.

In a nutshell, the following strategy can be applied: (I) construct an appropriate $\Pi\Sigma$-field $(\mathbb{F}, \sigma)$ in which a given summand $F(k)$ in terms of indefinite nested sums and products is rephrased by $f \in \mathbb{F}$; (II) compute $g \in \mathbb{F}$ such that (1) holds; (III) rephrase $g \in \mathbb{F}$ to an expression $G(k)$ such that (2) holds.

In the last years various new algorithms and improvements of Karr's difference field theory have been developed in order to obtain a fully automatic simplification machinery for nested sums. Here the key observation is that a sum can be either expressed in the existing difference field $(\mathbb{F}, \sigma)$ by solving the telescoping problem (2) or it can be adjoined as a new extension on top of the already constructed field $\mathbb{F}$; see Theorem 2.1(3) below. By a careful construction of $(\mathbb{F}, \sigma)$ one can simplify sum expressions such that the nesting depth is minimized [26], or the number [29] or the degree [24] of the objects arising in the summands are optimized.

In contrast to sums, representing products in $\Pi\Sigma$-fields is not possible in general. In particular, the alternating sign $(-1)^k$, which arises frequently in applications, can be represented properly only in a ring with zero divisors introducing relations such as $(1-(-1)^k)(1+(-1)^k) = 0$. In [23] and a streamlined version worked out in [28], this situation has been cured for the class of hypergeometric products of the form $\prod_{i=l}^{k} f(i)$ with $l \in \mathbb{N}$ and $f(k) \in \mathbb{Q}(k)$ being a rational function with coefficient from the rational numbers: namely, a finite number of such products can be always represented in a $\Pi\Sigma$-field adjoined with the element $(-1)^k$. In particular, nested sums defined over such products can be formulated automatically in difference rings built by the so-called $\mathrm{R}\Pi\Sigma$-extensions [30, 32]. This means in difference rings that are built by transcendental ring extensions and algebraic ring extensions of the form $\alpha^n$ where $\alpha$ is a primitive root of unity. Within this setting [30, 32], one can then solve the telescoping problem for indefinite sums (see Eq. (1)) and more generally the creative telescoping problem [19] to compute linear recurrences for definite sums. Furthermore one can simplify the so-called d'Alembertian [3, 5, 11, 18] or Liouvillian solutions [20, 21] of linear recurrences which are given in terms of nested sums defined over hypergeometric products. For many problems coming, e.g., from combinatorics or particle physics (for the newest applications see [33] or [1]) this difference ring machinery with more than 100 extension variables works fine. But in more general cases, one is faced with nested sums defined not only over hypergeometric but also over mixed multibasic products. Furthermore, these products might not be expressible in $\mathbb{Q}$ but only in an algebraic number field, i.e., in a finite algebraic field extension of $\mathbb{Q}$.

In this article we will generalize the existing product algorithms [23, 28] to cover also this more general class of products.

**Definition 1.1** Let $\mathbb{K} = K(q_1, \ldots, q_e)$ be a rational function field over a field $K$ and let $\mathbb{F} = \mathbb{K}(x, t_1, \ldots, t_e)$ be a rational function field over $\mathbb{K}$. $\prod_{k=l}^{n} f(k, q_1^k, \ldots, q_e^k)$ is a *mixed* $(q_1, \ldots, q_e)$*-multibasic hypergeometric product* in $n$, if $f(x, t_1, \ldots, t_e) \in \mathbb{F} \setminus \{0\}$ and $l \in \mathbb{N}$ is chosen big enough (see Example 2.9 below) such that $f(\ell, q_1^\ell, \ldots, q_e^\ell)$ has no pole and is non-zero for all $\ell \in \mathbb{N}$ with $\ell \geqslant l$. If $f(t_1, \ldots, t_e) \in \mathbb{F}$ which is free of $x$, then $\prod_{k=l}^{n} f(q_1^k, \ldots, q_e^k)$ is called a $(q_1, \ldots, q_e)$-*multibasic hypergeometric product* in $n$. If $e = 1$, then it is called a *basic* or $q$-*hypergeometric product* in $n$ where $q = q_1$. If $e = 0$ and $f \in \mathbb{K}(x)$, then $\prod_{k=l}^{n} f(k)$ is called a *hypergeometric product* in $n$. Finally, if $f \in \mathbb{K}$, it is called *constant or geometric product* in $n$.

Let $\boldsymbol{q}^n$ denote $q_1^n, \ldots, q_e^n$ and $\boldsymbol{t}$ denote $(t_1, \ldots, t_e)$. Further, we define the set of ground expressions[1] $\mathbb{K}(n) = \{f(n) \mid f(x) \in \mathbb{K}(x)\}$, $\mathbb{K}(\boldsymbol{q}^n) = \{f(\boldsymbol{q}^n) \mid f(\boldsymbol{t}) \in \mathbb{K}(\boldsymbol{t})\}$ and $\mathbb{K}(n, \boldsymbol{q}^n) = \{f(n, \boldsymbol{q}^n) \mid f(x, \boldsymbol{t}) \in \mathbb{K}(x, \boldsymbol{t})\}$. Moreover, we define $\mathrm{Prod}(\mathbb{X})$ with $\mathbb{X} \in \{\mathbb{K}, \mathbb{K}(n), \mathbb{K}(\boldsymbol{q}^n), \mathbb{K}(n, \boldsymbol{q}^n)\}$ as the set of all such products where the multiplicand is taken from $\mathbb{X}$. Finally, we introduce the set of product expressions $\mathrm{ProdE}(\mathbb{X})$ as the set of all elements

$$\sum_{(\nu_1,\ldots,\nu_m)\in S} a_{(\nu_1,\ldots,\nu_m)}(n)\; P_1(n)^{\nu_1} \cdots P_m(n)^{\nu_m} \tag{3}$$

with $m \in \mathbb{N}$, $S \subseteq \mathbb{Z}^m$ finite, $a_{(\nu_1,\ldots,\nu_m)}(n) \in \mathbb{X}$ and $P_1(n), \ldots, P_m(n) \in \mathrm{Prod}(\mathbb{X})$.

For this class where the subfield $K$ of $\mathbb{K}$ itself can be a rational function field over an algebraic number field, we will solve the following problem.

[1] Their elements are considered as expressions that can be evaluated for sufficiently large $n \in \mathbb{N}$.

**Problem RPE: Representation of Product Expressions.**

Let $\mathbb{X}_{\mathbb{K}} \in \{\mathbb{K}, \mathbb{K}(n), \mathbb{K}(\boldsymbol{q}^n), \mathbb{K}(n, \boldsymbol{q}^n)\}$. *Given* $P(n) \in \text{ProdE}(\mathbb{X}_{\mathbb{K}})$; *find* $Q(n) \in \text{ProdE}(\mathbb{X}_{\mathbb{K}'})$ with $\mathbb{K}'$ a finite algebraic field extension[a] of $\mathbb{K}$ and a natural number $\delta$ with the following properties:

1. $P(n) = Q(n)$ for all $n \in \mathbb{N}$ with $n \geqslant \delta$;
2. The product expressions in $Q(n)$ (apart from products over roots of unity) are algebraically independent among each other.
3. The zero-recognition property holds, i.e., $P(n) = 0$ holds for all $n$ from a certain point on if and only if $Q(n)$ is the zero-expression.

[a] If $\mathbb{K} = K(\kappa_1, \ldots, \kappa_u)(q_1, \ldots, q_e)$ is a rational function field over an algebraic number field $K$, then in worst case $\mathbb{K}$ is extended to $\mathbb{K}' = K'(\kappa_1, \ldots, \kappa_u)(q_1, \ldots, q_e)$ where $K'$ is an algebraic extension of $K$. Subsequently, all algebraic field extensions are finite.

Internally, the multiplicands of the products are factorized and the monic irreducible factors, which are shift-equivalent, are rewritten in terms of one of these factors; compare [2, 4, 8, 17, 23]. Then using results of [10, 27] we can conclude that products defined over these irreducible factors can be rephrased as transcendental difference ring extensions. Using similar strategies, one can treat the content coming from the monic irreducible polynomials, and obtains finally an RΠΣ-extension in which the products can be rephrased. We remark that the normal forms presented in [8] are closely related to this representation and enable one to check, e.g., if the given products are algebraically independent. Moreover, there is an algorithm [15] that can compute all algebraic relations for $c$-finite sequences, i.e., it finds certain ideals from ProdE($\mathbb{K}$) whose elements evaluate to zero. Our main focus is different. We will compute alternative products which are by construction algebraically independent among each other and which enable one to express the given products in terms of the algebraic independent products. In particular, we will make this algebraic independence statement (see property (2) of Problem RPE) very precise by embedding the constructed RΠΣ-extension explicitly into the ring of sequences [19] by using results from [32]. The derived algorithms implemented in Ocansey's Mathematica package `NestedProducts` supplement the summation package `Sigma` [25] and enable one to formulate nested sums over such general products in the setting of RΠΣ-extensions. As a consequence, it is now possible to apply completely automatically the summation toolbox [4, 7, 12, 23–32] for simplification of indefinite and definite nested sums defined over such products.

The outline of the article is as follows. In Sect. 2 we define the basic notions of RΠΣ-extensions and present the main results to embed a difference ring built by RΠΣ-extensions into the ring of sequences. In Sect. 3 our Problem RPE is reformulated to Theorem 3.1 in terms of these notions, and the basic strategy is presented how this problem will be tackled. In Sect. 4 the necessary properties of the constant field are worked out that enable one to execute our proposed algorithms. Finally, in Sects. 5 and 6 the hypergeometric case and afterwards the mixed multibasic case are treated. A conclusion is given in Sect. 7.

# 2 Ring of Sequences, Difference Rings and Difference Fields

In this section, we discuss the algebraic setting of difference rings (resp. fields) and the ring of sequences as they have been elaborated in [12, 30, 32]. In particular, we demonstrate how sequences generated by expressions in $\mathrm{ProdE}(\mathbb{K}(n))$ (resp. $\mathrm{ProdE}(\mathbb{K}(n, \boldsymbol{q}^n))$) can be modeled in this algebraic framework.

## *2.1 Difference Fields and Difference Rings*

A difference ring (resp. field) $(\mathbb{A}, \sigma)$ is a ring (resp. field) $\mathbb{A}$ together with a ring (resp. field) automorphism $\sigma : \mathbb{A} \to \mathbb{A}$. Subsequently, all rings (resp. fields) are commutative with unity; in addition they contain the set of rational numbers $\mathbb{Q}$, as a subring (resp. subfield). The multiplicative group of units of a ring (resp. field) $\mathbb{A}$ is denoted by $\mathbb{A}^*$. A ring (resp. field) is computable if all of it's operations are computable. A difference ring (resp. field) $(\mathbb{A}, \sigma)$ is computable if $\mathbb{A}$ and $\sigma$ are both computable. Thus, given a computable difference ring (resp. field), one can decide if $\sigma(c) = c$. The set of all such elements for a given difference ring (resp. field) denoted by

$$\mathrm{const}(\mathbb{A}, \sigma) = \{c \in \mathbb{A} \mid \sigma(c) = c\}$$

forms a subring (resp. subfield) of $\mathbb{A}$. In this article, $\mathrm{const}(\mathbb{A}, \sigma)$ will always be a field called the constant field of $(\mathbb{A}, \sigma)$. Note that it contains $\mathbb{Q}$ as a subfield. For any difference ring (resp. field) we shall denote the constant field by $\mathbb{K}$.

The construction of difference rings/fields will be accomplished by a tower of difference ring/field extensions. A difference ring $(\tilde{\mathbb{A}}, \tilde{\sigma})$ is said to be a *difference ring extension* of a difference ring $(\mathbb{A}, \sigma)$ if $\mathbb{A}$ is a subring of $\tilde{\mathbb{A}}$ and for all $a \in \mathbb{A}$, $\tilde{\sigma}(a) = \sigma(a)$ (i.e., $\tilde{\sigma}|_{\mathbb{A}} = \sigma$). The definition of a *difference field extension* is the same by only replacing the word ring with field. In the following we do not distinguish anymore between $\sigma$ and $\tilde{\sigma}$.

In the following we will consider two types of product extensions. Let $(\mathbb{A}, \sigma)$ be a difference ring (in which products have already been defined by previous extensions). Let $\alpha \in \mathbb{A}^*$ be a unit and consider the ring of Laurent polynomials $\mathbb{A}[t, t^{-1}]$ (i.e., $t$ is transcendental over $\mathbb{A}$). Then there is a unique difference ring extension $(\mathbb{A}[t, t^{-1}], \sigma)$ of $(\mathbb{A}, \sigma)$ with $\sigma(t) = \alpha\, t$ and $\sigma(t^{-1}) = \alpha^{-1}\, t^{-1}$. The extension here is called a *product-extension* (in short P-extension) and the generator is called a P-monomial. Suppose that $\mathbb{A}$ is a field and $\mathbb{A}(t)$ is a rational function field (i.e., $t$ is transcendental over $\mathbb{A}$). Let $\alpha \in \mathbb{A}^*$. Then there is a unique difference field extension $(\mathbb{A}(t), \sigma)$ of $(\mathbb{A}, \sigma)$ with $\sigma(t) = \alpha\, t$. We call the extension a P-*field extension* and $t$ a P-monomial. In addition, we get the chain of extensions $(\mathbb{A}, \sigma) \leqslant (\mathbb{A}[t, t^{-1}], \sigma) \leqslant (\mathbb{A}(t), \sigma)$.

Furthermore, we consider extensions which model algebraic objects like $\zeta^k$ where $\zeta$ is a $\lambda$-th root of unity for some $\lambda \in \mathbb{N}$ with $\lambda > 1$. Let $(\mathbb{A}, \sigma)$ be a difference ring and let $\zeta \in \mathbb{A}^*$ be a primitive $\lambda$-th root of unity, (i.e., $\zeta^\lambda = 1$ and $\lambda$ is minimal). Take the difference ring extension $(\mathbb{A}[y], \sigma)$ of $(\mathbb{A}, \sigma)$ with $y$ being transcendental over $\mathbb{A}$ and $\sigma(y) = \zeta\, y$. Note that this construction is also unique. Consider the ideal $I := \langle y^\lambda - 1 \rangle$ and the quotient ring $\mathbb{E} := \mathbb{A}[y]/I$. Since $I$ is closed under $\sigma$ and $\sigma^{-1}$ i.e., $I$ is a reflexive difference ideal, we have a ring automorphism $\sigma : \mathbb{E} \to \mathbb{E}$ defined by $\sigma(h + I) = \sigma(h) + I$. In other words, $(\mathbb{E}, \sigma)$ is a difference ring. Note that by this construction the ring $\mathbb{A}$ can naturally be embedded into the ring $\mathbb{E}$ by identifying $a \in \mathbb{A}$ with $a + I \in \mathbb{E}$, i.e., $a \mapsto a + I$. Now set $\vartheta := y + I$. Then $(\mathbb{A}[\vartheta], \sigma)$ is a difference ring extension of $(\mathbb{A}, \sigma)$ subject to the relations $\vartheta^\lambda = 1$ and $\sigma(\vartheta) = \zeta\,\vartheta$. This extension is called an algebraic extension (in short A-extension) of order $\lambda$. The generator, $\vartheta$ is called an A-monomial and we define $\lambda = \min\{n > 0 \mid \alpha^n = 1\}$ as its order. Note that the A-monomial $\vartheta$, with the relations $\vartheta^\lambda = 1$ and $\sigma(\vartheta) = \zeta\,\vartheta$ models $\zeta^k$ with the relations $(\zeta^k)^\lambda = 1$ and $\zeta^{k+1} = \zeta\,\zeta^k$. In addition, the ring $\mathbb{A}[\vartheta]$ is not an integral domain (i.e., it has zero-divisors) since $(\vartheta - 1)\,(\vartheta^{\lambda-1} + \cdots + \vartheta + 1) = 0$ but $(\vartheta - 1) \neq 0 \neq (\vartheta^{\lambda-1} + \cdots + \vartheta + 1)$.

We introduce the following notations for convenience. Let $(\mathbb{E}, \sigma)$ be a difference ring extension of $(\mathbb{A}, \sigma)$ with $t \in \mathbb{E}$. $\mathbb{A}\langle t\rangle$ denotes the ring of Laurent polynomials $\mathbb{A}[t, \frac{1}{t}]$ (i.e., $t$ is transcendental over $\mathbb{A}$) if $(\mathbb{A}[t, \frac{1}{t}], \sigma)$ is a P-extension of $(\mathbb{A}, \sigma)$. Lastly, $\mathbb{A}\langle t\rangle$ denotes the ring $\mathbb{A}[t]$ with $t \notin \mathbb{A}$ but subject to the relation $t^\lambda = 1$ if $(\mathbb{A}[t], \sigma)$ is an A-extension of $(\mathbb{A}, \sigma)$ of order $\lambda$. We say that the difference ring extension $(\mathbb{A}\langle t\rangle, \sigma)$ of $(\mathbb{A}, \sigma)$ is an AP-extension (and $t$ is an AP-monomial) if it is an A- or a P-extension. Finally, we call $(\mathbb{A}\langle t_1\rangle \cdots \langle t_e\rangle, \sigma)$ a (nested) AP-extension/P-extension of $\mathbb{A}, \sigma$ if it is built by a tower of such extensions.

Throughout this article, we will restrict ourselves to the following classes of extensions as our base field.

*Example 2.1* Let $\mathbb{K}(x)$ be a rational function field and define the field automorphism $\sigma : \mathbb{K}(x) \to \mathbb{K}(x)$ with $\sigma(f) = f|_{x \mapsto x+1}$. We call $(\mathbb{K}(x), \sigma)$ the *rational difference field* over $\mathbb{K}$.

*Example 2.2* Let $\mathbb{K} = K(q_1, \ldots, q_e)$ be a rational function field (i.e., the $q_i$ are transcendental among each other over the field $K$ and let $(\mathbb{K}(x), \sigma)$ be the rational difference field over $\mathbb{K}$. Consider a P-extension $(\mathbb{E}, \sigma)$ of $(\mathbb{K}(x), \sigma)$ with $\mathbb{E} = \mathbb{K}(x)[t_1, \frac{1}{t_1}] \cdots [t_e, \frac{1}{t_e}]$ and $\sigma(t_i) = q_i\, t_i$ for $1 \leqslant i \leqslant e$. Now consider the field of fractions $\mathbb{F} = Q(\mathbb{E}) = \mathbb{K}(x)(t_1) \cdots (t_e)$. We also use the shortcut $\boldsymbol{t} = (t_1, \ldots, t_e)$ and write $\mathbb{F} = \mathbb{K}(x)(\boldsymbol{t}) = \mathbb{K}(x, \boldsymbol{t})$. Then $(\mathbb{F}, \sigma)$ is a P-field extension of the difference field $(\mathbb{K}(x), \sigma)$. It is also called the *mixed* $\boldsymbol{q}$*-multibasic difference field* over $\mathbb{K}$. If $\mathbb{F} = \mathbb{K}(t_1) \cdots (t_e) = \mathbb{K}(\boldsymbol{t})$ which is free of $x$, then $(\mathbb{F}, \sigma)$ is called the $\boldsymbol{q}$*-multibasic difference field* over $\mathbb{K}$. Finally, if $e = 1$, then $\mathbb{F} = \mathbb{K}(t_1)$ and $(\mathbb{F}, \sigma)$ is called a $q$- or a *basic difference field* over $\mathbb{K}$.

Based on these ground fields we will define now our products. In the first sections we will restrict to the hypergeometric case.

*Example 2.3* Let $\mathbb{K} = \mathbb{Q}\big(\iota,\ (-1)^{\frac{1}{6}}\big)$ and let $(\mathbb{K}(x), \sigma)$ be a rational difference field. Then the product expressions

$$\prod_{k=1}^{n}(-1)^{\frac{1}{6}},\quad \prod_{k=1}^{n}(-1)^{\frac{1}{2}} \tag{4}$$

from $\mathrm{Prod}(\mathbb{K}(n))$ can be represented in an A-extension as follows. Here $\iota$ is the complex unit which we also write as $(-1)^{\frac{1}{2}}$. Now take the A-extension $(\mathbb{K}(x)[\vartheta_1],\sigma)$ of $(\mathbb{K}(x),\sigma)$ with $\sigma(\vartheta_1)=(-1)^{\frac{1}{6}}\,\vartheta_1$ of order 12. The A-monomial $\vartheta_1$ models $\big((-1)^{\frac{1}{6}}\big)^n$ with the shift-behavior $S_n\big((-1)^{\frac{1}{6}}\big)^n=\big((-1)^{\frac{1}{6}}\big)^{n+1}=(-1)^{\frac{1}{6}}\,\big((-1)^{\frac{1}{6}}\big)^n$. Further, $(\mathbb{K}(x)[\vartheta_1][\vartheta_2],\sigma)$ is also an A-extension of $(\mathbb{K}(x)[\vartheta_1],\sigma)$ with $\sigma(\vartheta_2)=\iota\,\vartheta_2$ of order 4. The generator $\vartheta_2$ models $(\iota)^n$ with $S_n(\iota)^n=\iota\,(\iota)^n$.

*Example 2.4* The product expressions

$$\prod_{k=1}^{n}\sqrt{13},\quad \prod_{k=1}^{n}7,\quad \prod_{k=1}^{n}169 \tag{5}$$

from $\mathrm{Prod}(\mathbb{K}(n))$ with $\mathbb{K}=\mathbb{Q}(\sqrt{13})$ are represented in a P-extension of the rational difference field $(\mathbb{K}(x),\sigma)$ with $\sigma(x)=x+1$ as follows.

1. Consider the P-extension $(\mathbb{A}_1,\sigma)$ of $(\mathbb{K}(x),\sigma)$ with $\mathbb{A}_1=\mathbb{K}(x)[y_1,\frac{1}{y_1}]$, $\sigma(y_1)=(\sqrt{13})\,y_1$ and $\sigma(\frac{1}{y_1})=\frac{1}{\sqrt{13}}\,\frac{1}{y_1}$. In this ring, we can model polynomial expressions in $\big(\sqrt{13}\big)^n$ and $\big(\sqrt{13}\big)^{-n}$ with the shift behavior $S_n\big(\sqrt{13}\big)^n=\sqrt{13}\,\big(\sqrt{13}\big)^n$ and $S_n\frac{1}{(\sqrt{13})^n}=\frac{1}{\sqrt{13}}\,\frac{1}{(\sqrt{13})^n}$. Here, $\big(\sqrt{13}\big)^n$ and $\frac{1}{(\sqrt{13})^n}$ are rephrased by $y_1$ and $\frac{1}{y_1}$, respectively.
2. Constructing the P-extension $(\mathbb{A}_2,\sigma)$ of $(\mathbb{A}_1,\sigma)$ with $\mathbb{A}_2=\mathbb{A}_1[y_2,\frac{1}{y_2}]$, $\sigma(y_2)=7\,y_2$ and $\sigma(\frac{1}{y_2})=\frac{1}{7}\,\frac{1}{y_2}$, we are able to model polynomial expressions in $7^n$ and $7^{-n}$ with the shift behavior $S_n7^n=7\,7^n$ and $S_n\frac{1}{7^n}=\frac{1}{7}\,\frac{1}{7^n}$ by rephrasing $7^n$ and $\frac{1}{7^n}$ with $y_2$ and $\frac{1}{y_2}$, respectively.
3. Introducing the P-extension $(\mathbb{A}_3,\sigma)$ of $(\mathbb{A}_2,\sigma)$ with $\mathbb{A}_3=\mathbb{A}_2[y_3,\frac{1}{y_3}]$, $\sigma(y_3)=169\,y_3$ and $\sigma(\frac{1}{y_3})=\frac{1}{169}\,\frac{1}{y_3}$, one can model polynomial expressions in $(169)^n$ and $(169)^{-n}$ with the shift behavior $S_n(169)^n=169\,(169)^n$ and $S_n\frac{1}{(169)^n}=\frac{1}{169}\,\frac{1}{(169)^n}$ by rephrasing $(169)^n$ and $(169)^{-n}$ by $y_3$ and $\frac{1}{y_3}$, respectively.

*Example 2.5* The hypergeometric product expressions

$$P_1(n)=\prod_{k=1}^{n}k,\quad P_2(n)=\prod_{k=1}^{n}\big(k+2\big) \tag{6}$$

from $\mathrm{Prod}(\mathbb{Q}(n))$ can be represented in a P-extension defined over the rational difference field $(\mathbb{Q}(x),\sigma)$ in the following way. Take the P-extension $(\mathbb{Q}(x)[z_1,\frac{1}{z_1}],\sigma)$ of $(\mathbb{Q}(x),\sigma)$ with $\sigma(z_1)=(x+1)\,z_1$ and $\sigma(\frac{1}{z_1})=\frac{1}{(x+1)}\,\frac{1}{z_1}$. In this extension, one can model polynomial expressions in the product expression $P_1(n)$ with the shift behavior $S_nP_1(n)=(n+1)\,P_1(n)$ and $S_n\frac{1}{P_1(n)}=\frac{1}{(n+1)}\,\frac{1}{P_1(n)}$ by rephrasing $P_1(n)$

and $\frac{1}{P_1(n)}$ by $z_1$ and $\frac{1}{z_1}$. Finally, taking the P-extension $(\mathbb{Q}(x)[z_1, \frac{1}{z_1}][z_2, \frac{1}{z_2}], \sigma)$ of $(\mathbb{Q}(x)[z_1, \frac{1}{z_1}], \sigma)$ with $\sigma(z_2) = (x + 3)\, z_2$ and $\sigma(\frac{1}{z_2}) = \frac{1}{(x+3)}\, \frac{1}{z_2}$, we can represent polynomial expressions in the product expression $P_2(n)$ with the shift-behavior $S_n P_2(n) = (n + 3)\, P_2(n)$ and $S_n \frac{1}{P_2(n)} = \frac{1}{(n+3)}\, \frac{1}{P_2(n)}$ by rephrasing $P_2(n)$ and $\frac{1}{P_2(n)}$ by $z_2$ and $\frac{1}{z_2}$, respectively.

In this article we will focus on the subclass of RΠ-extensions.

**Definition 2.1** An AP-extension (A- or P-extension) $(\mathbb{A}\langle t_1\rangle \cdots \langle t_e\rangle, \sigma)$ of $(\mathbb{A}, \sigma)$ is called an RΠ-extension (R- or Π-extension) if $\text{const}(\mathbb{A}\langle t_1\rangle \cdots \langle t_e\rangle, \sigma) = \text{const}(\mathbb{A}, \sigma)$. Depending on the type of extension, we call $t_i$ an R-/Π-/RΠ-monomial. Similarly, let $\mathbb{A}$ be a field. Then we call a tower of P-field extensions $(\mathbb{A}(t_1) \cdots (t_e), \sigma)$ of $(\mathbb{A}, \sigma)$ a Π-field extension if $\text{const}(\mathbb{A}(t_1) \cdots (t_e), \sigma) = \text{const}(\mathbb{A}, \sigma)$.

We concentrate mainly on product extensions and skip the sum part that has been mentioned in the introduction. Still, we need to handle the very special case of the rational difference field $(\mathbb{K}(x), \sigma)$ with $\sigma(x) = x + 1$ or the mixed $\boldsymbol{q}$-multibasic version. Thus it will be convenient to introduce also the field version of Σ-extensions [12, 22].

**Definition 2.2** Let $(\mathbb{F}(t), \sigma)$ be a difference field extension of $(\mathbb{F}, \sigma)$ with $t$ transcendental over $\mathbb{F}$ and $\sigma(t) = t + \beta$ with $\beta \in \mathbb{F}$. This extension is called a Σ-extension if $\text{const}(\mathbb{F}(t), \sigma) = \text{const}(\mathbb{F}, \sigma)$. In this case $t$ is also called a Σ-monomial. $(\mathbb{F}(t), \sigma)$ is called a ΠΣ-extension of $(\mathbb{F}, \sigma)$ if it is either a Π- or a Σ-extension. I.e., $t$ is transcendental over $\mathbb{F}$, $\text{const}(\mathbb{F}(t), \sigma) = \text{const}(\mathbb{F}, \sigma)$ and $t$ is a Π-monomial ($\sigma(t) = \alpha\, t$ for some $\alpha \in \mathbb{F}^*$) or $t$ is a Σ-monomial ($\sigma(t) = t + \beta$ for some $\beta \in \mathbb{F}$). $(\mathbb{F}(t_1) \cdots (t_e), \sigma)$ is a ΠΣ-extension of $(\mathbb{F}, \sigma)$ if it is a tower of ΠΣ-extensions.

Note that there exist criteria which can assist in the task to check if during the construction the constants remain unchanged. The reader should see [30, Proof 3.16, 3.22 and 3.9] for the proofs. For the field version, see also [12].

**Theorem 2.1** *Let* $(\mathbb{A}, \sigma)$ *be a difference ring. Then the following statements hold.*

1. *Let* $(\mathbb{A}[t, \frac{1}{t}], \sigma)$ *be a* P*-extension of* $(\mathbb{A}, \sigma)$ *with* $\sigma(t) = \alpha\, t$ *where* $\alpha \in \mathbb{A}^*$. *Then this is a* Π*-extension (i.e.,* $\text{const}(\mathbb{A}[t, \frac{1}{t}], \sigma) = \text{const}(\mathbb{A}, \sigma)$*) iff there are no* $g \in \mathbb{A} \setminus \{0\}$ *and* $v \in \mathbb{Z} \setminus \{0\}$ *with* $\sigma(g) = \alpha^v\, g$.
2. *Let* $(\mathbb{A}[\vartheta], \sigma)$ *be an* A*-extension of* $(\mathbb{A}, \sigma)$ *of order* $\lambda > 1$ *with* $\sigma(\vartheta) = \zeta\, \vartheta$ *where* $\zeta \in \mathbb{A}^*$. *Then this is an* R*-extension (i.e.,* $\text{const}(\mathbb{A}[\vartheta], \sigma) = \text{const}(\mathbb{A}, \sigma)$*) iff there are no* $g \in \mathbb{A} \setminus \{0\}$ *and* $v \in \{1, \ldots, \lambda - 1\}$ *with* $\sigma(g) = \zeta^v\, g$. *If it is an* R*-extension,* $\alpha$ *is a primitive* $\lambda$*th root of unity.*
3. *Let* $\mathbb{A}$ *be a field and let* $(\mathbb{A}(t), \sigma)$ *be a difference field extension of* $(\mathbb{F}, \sigma)$ *with* $t$ *transcendental over* $\mathbb{F}$ *and* $\sigma(t) = t + \beta$ *with* $\beta \in \mathbb{F}$. *Then this is a* Σ*-extension (i.e.,* $\text{const}(\mathbb{F}(t), \sigma) = \text{const}(\mathbb{F}, \sigma)$*) if there is no* $g \in \mathbb{F}$ *with* $\sigma(g) = g + \beta$.

Concerning our base case difference fields (see Examples 2.1 and 2.2) the following remarks are relevant. The rational difference field $(\mathbb{K}(x), \sigma)$ is a Σ-extension of $(\mathbb{K}, \sigma)$ by part (3) of Theorem 2.1 and using the fact that there is no

$g \in \mathbb{K}$ with $\sigma(g) = g + 1$. Thus $\text{const}(\mathbb{K}(x), \sigma) = \text{const}(\mathbb{K}, \sigma) = \mathbb{K}$. Furthermore, the mixed $\boldsymbol{q}$-multibasic difference field $(\mathbb{F}, \sigma)$ with $\mathbb{F} = \mathbb{K}(x)(t_1) \cdots (t_e)$ is a $\Pi$-field extension of $(\mathbb{K}(x), \sigma)$. See Corollary 5.1 below. As a consequence, we have that $\text{const}(\mathbb{F}, \sigma) = \text{const}(\mathbb{K}(x), \sigma) = \mathbb{K}$.

We give further examples and non-examples of R$\Pi$-extensions.

*Example 2.6* 1. In Example 2.3, the A-extension $(\mathbb{K}(x)[\vartheta_1], \sigma)$ is an R-extension of $(\mathbb{K}(x), \sigma)$ of order 12 since there are no $g \in \mathbb{K}(x)^*$ and $v \in \{1, \ldots, 11\}$ with $\sigma(g) = \left((-1)^{\frac{1}{6}}\right)^v g$. However, the A-extension $(\mathbb{K}(x)[\vartheta_1][\vartheta_2], \sigma)$ is not an R-extension of $(\mathbb{K}(x)[\vartheta_1], \sigma)$ since with $g = \vartheta_1^3 \in \mathbb{K}(x)[\vartheta_1]$ and $v = 1$, we have $\sigma(g) = \iota\, g$. In particular, we get $c = \vartheta_1^3\, \vartheta_2 \in \text{const}(\mathbb{K}(x)[\vartheta_1][\vartheta_2], \sigma) \setminus \text{const}(\mathbb{K}(x)[\vartheta_1], \sigma)$.

2. The P-extension $(\mathbb{A}_1, \sigma)$ of $(\mathbb{K}(x), \sigma)$ in Example 2.4(1) with $\sigma(y_1) = \sqrt{13}\, y_1$ is a $\Pi$-extension of $(\mathbb{K}(x), \sigma)$ as there are no $g \in \mathbb{K}(x)^*$ and $v \in \mathbb{Z} \setminus \{0\}$ with $\sigma(g) = \left(\sqrt{13}\right)^v g$. Similarly, the P-extension $(\mathbb{A}_2, \sigma)$ in Example 2.4(2) with $\sigma(y_2) = 7\, y_2$ is also a $\Pi$-extension of $(\mathbb{A}_1, \sigma)$ since there does not exist a $g \in \mathbb{A}_1 \setminus \{0\}$ and a $v \in \mathbb{Z} \setminus \{0\}$ with $\sigma(g) = 7^v\, g$. However, the P-extension $(\mathbb{A}_3, \sigma)$ in part (3) of Example 2.4 is not a $\Pi$-extension of $(\mathbb{A}_2, \sigma)$ since with $g = y_1^4 \in \mathbb{A}_2$ we have $\sigma(g) = (169)\, g$. In particular, $w = y_1^{-4}\, y_3 \in \text{const}(\mathbb{K}(x)[y_1, \frac{1}{y_1}][y_2, \frac{1}{y_2}][y_3, \frac{1}{y_3}], \sigma) \setminus \text{const}(\mathbb{K}(x)[y_1, \frac{1}{y_1}][y_2, \frac{1}{y_2}], \sigma)$.

3. Finally, in Example 2.5 the P-extension $(\mathbb{Q}(x)[z_1, \frac{1}{z_1}], \sigma)$ is a $\Pi$-extension of $(\mathbb{Q}(x), \sigma)$ with $\sigma(z_1) = (x + 1)\, z_1$ since there are no $g \in \mathbb{Q}(x)^*$ and $v \in \mathbb{Z} \setminus \{0\}$ with $\sigma(g) = (x + 1)^v\, g$. But the P-extension $(\mathbb{Q}(x)[z_1, \frac{1}{z_1}][z_2, \frac{1}{z_2}], \sigma)$ with $\sigma(z_2) = (x + 3)\, z_2$ is not a $\Pi$-extension of $(\mathbb{Q}(x)[z_1, \frac{1}{z_1}], \sigma)$ since with $g = (x + 2)\,(x + 1)\, z_1$ and $v = 1$ we have $\sigma(g) = (x + 3)\, g$. In particular, we get $c = \frac{g}{z_2} \in \text{const}(\mathbb{Q}(x)\, \langle z_1 \rangle\, \langle z_2 \rangle, \sigma) \setminus \text{const}(\mathbb{Q}(x)\, \langle z_1 \rangle, \sigma)$.

We remark that in [12, 30] algorithms have been developed that can carry out these checks if the already designed difference ring is built by properly chosen R$\Pi$-extensions. In this article we are more ambitious. We will construct AP-extensions carefully such that they are automatically R$\Pi$-extensions and such that the products under consideration can be rephrased within these extensions straightforwardly. In this regard, we will utilize the following lemma.

**Lemma 2.1** *Let* $(\mathbb{F}, \sigma)$ *be a* $\Pi\Sigma$*-extension of* $(\mathbb{K}, \sigma)$ *with* $\text{const}(\mathbb{K}, \sigma) = \mathbb{K}$. *Then the* A*-extension* $(\mathbb{F}[\vartheta], \sigma)$ *of* $(\mathbb{F}, \sigma)$ *with order* $\lambda > 1$ *is an* R*-extension.*

*Proof* By [13, Lemma 3.5] we have $\text{const}(\mathbb{F}, \sigma^k) = \text{const}(\mathbb{F}, \sigma)$ for all $k \in \mathbb{N} \setminus \{0\}$. Thus with [32, Proposition 2.20], $(\mathbb{F}[\vartheta], \sigma)$ is an R-extension of $(\mathbb{F}, \sigma)$.

## 2.2 Ring of Sequences

We will elaborate how RΠ-extensions can be embedded into the difference ring of sequences [32]; compare also [21]. Precisely this feature will enable us to handle condition (2) of Problem RPE.

Let $\mathbb{K}$ be a field containing $\mathbb{Q}$ as a subfield and let $\mathbb{N}$ be the set of non-negative integers. We denote by $\mathbb{K}^{\mathbb{N}}$ the set of all sequences $\langle a(n)\rangle_{n\geqslant 0} = \langle a(0),\ a(1),\ a(2),\ \cdots\rangle$ whose terms are in $\mathbb{K}$. With component-wise addition and multiplication, $\mathbb{K}^{\mathbb{N}}$ forms a commutative ring. The field $\mathbb{K}$ can be naturally embedded into $\mathbb{K}^{\mathbb{N}}$ as a subring, by identifying $c \in \mathbb{K}$ with the constant sequence $\langle c, c, c, \dots\rangle \in \mathbb{K}^{\mathbb{N}}$. Following the construction in [19, Sect. 8.2], we turn the shift operator

$$S : \begin{cases} \mathbb{K}^{\mathbb{N}} & \to \ \mathbb{K}^{\mathbb{N}} \\ \langle a(0),\ a(1),\ a(2),\ \dots\rangle & \mapsto \ \langle a(1),\ a(2),\ a(3),\ \dots\rangle \end{cases}$$

into a ring automorphism by introducing an equivalence relation $\sim$ on sequences in $\mathbb{K}^{\mathbb{N}}$. Two sequences $\boldsymbol{a} := \langle a(n)\rangle_{n\geqslant 0}$ and $\boldsymbol{b} := \langle b(n)\rangle_{n\geqslant 0}$ are said to be equivalent if and only if there exist a natural number $\delta$ such that $a(n) = b(n)$ for all $n \geqslant \delta$. The set of equivalence classes form a ring again with component-wise addition and multiplication which we will denote by $\mathscr{S}(\mathbb{K})$. For simplicity, we denote the elements of $\mathscr{S}(\mathbb{K})$ (also called germs) by the usual sequence notation as above. Now it is obvious that $S : \mathscr{S}(\mathbb{K}) \to \mathscr{S}(\mathbb{K})$ is a ring automorphism. Therefore, $(\mathscr{S}(\mathbb{K}), S)$ forms a difference ring called the (*difference*) *ring of sequences* (*over* $\mathbb{K}$).

*Example 2.7* The hypergeom. products in (4), (5) and (6) yield the sequences $\mathscr{M} = \{\langle ((-1)^{\frac{1}{6}})^n\rangle_{n\geqslant 0},\ \langle \iota^n\rangle_{n\geqslant 0},\ \langle (\sqrt{13})^n\rangle_{n\geqslant 0},\ \langle 7^n\rangle_{n\geqslant 0},\ \langle (169)^n\rangle_{n\geqslant 0},\ \langle P_1(n)\rangle_{n\geqslant 0},$ $\langle P_2(n)\rangle_{n\geqslant 0}\}$ with $S\langle a_n\rangle_{n\geqslant 0} := \langle S_n\, a_n\rangle_{n\geqslant 0}$ for $a_n \in \mathscr{M}$.

**Definition 2.3** Let $(\mathbb{A}, \sigma)$ and $(\mathbb{A}', \sigma')$ be two difference rings. We say that $\tau : \mathbb{A} \to \mathbb{A}'$ is a *difference ring homomorphism* between the difference rings $(\mathbb{A}, \sigma)$ and $(\mathbb{A}', \sigma')$ if $\tau$ is a ring homomorphism and for all $f \in \mathbb{A}$, $\tau(\sigma(f)) = \sigma'(\tau(f))$. If $\tau$ is injective, then it is called a *difference ring monomorphism* or a *difference ring embedding*. Consequently, $(\tau(\mathbb{A}), \sigma)$ is a sub-difference ring of $(\mathbb{A}', \sigma')$ where $(\mathbb{A}, \sigma)$ and $(\tau(\mathbb{A}), \sigma)$ are the same up to renaming with respect to $\tau$. If $\tau$ is a bijection, then it is a *difference ring isomorphism* and we say $(\mathbb{A}, \sigma)$ and $(\mathbb{A}', \sigma')$ are isomorphic.

Let $(\mathbb{A}, \sigma)$ be a difference ring with constant field $\mathbb{K}$. A difference ring homomorphism (resp. monomorphism) $\tau : \mathbb{A} \to \mathscr{S}(\mathbb{K})$ is called $\mathbb{K}$-*homomorphism* (resp. -*monomorphism*) if for all $c \in \mathbb{K}$ we have that $\tau(c) = \boldsymbol{c} := \langle c, c, c, \dots\rangle$.

The following lemma is the key tool to embed difference rings constructed by RΠ-extensions into the ring of sequences.

**Lemma 2.2** *Let* $(\mathbb{A}, \sigma)$ *be a difference ring with constant field* $\mathbb{K}$*. Then:*

1. *The map* $\tau : \mathbb{A} \to \mathscr{S}(\mathbb{K})$ *is a* $\mathbb{K}$*-homomorphism if and only if there is a map* $\mathrm{ev} : \mathbb{A} \times \mathbb{N} \to \mathbb{K}$ *with* $\tau(f) = \langle \mathrm{ev}(f, 0),\ \mathrm{ev}(f, 1), \ldots \rangle$ *for all* $f \in \mathbb{A}$ *satisfying the following properties:*
   a. *for all* $c \in \mathbb{K}$*, there is a natural number* $\delta \geqslant 0$ *such that*
   $$\forall n \geqslant \delta \ :\ \mathrm{ev}(c, n) = c;$$
   b. *for all* $f, g \in \mathbb{A}$ *there is a natural number* $\delta \geqslant 0$ *such that*
   $$\forall n \geqslant \delta \ :\ \mathrm{ev}(f\,g, n) = \mathrm{ev}(f, n)\ \mathrm{ev}(g, n),$$
   $$\forall n \geqslant \delta \ :\ \mathrm{ev}(f + g, n) = \mathrm{ev}(f, n) + \mathrm{ev}(g, n);$$
   c. *for all* $f \in \mathbb{A}$ *and* $i \in \mathbb{Z}$*, there is a natural number* $\delta \geqslant 0$ *such that*
   $$\forall n \geqslant \delta \ :\ \mathrm{ev}(\sigma^i(f), n) = \mathrm{ev}(f, n + i).$$
2. *Let* $(\mathbb{A}\langle t\rangle, \sigma)$ *be an* AP*-extension of* $(\mathbb{A}, \sigma)$ *with* $\sigma(t) = \alpha\, t$ *and suppose that* $\tau : \mathbb{A} \to \mathscr{S}(\mathbb{K})$ *as given in part (1) is a* $\mathbb{K}$*-homomorphism.*
   *Take some big enough* $\delta \in \mathbb{N}$ *such that* $\mathrm{ev}(\alpha, n) \neq 0$ *for all* $n \geqslant \delta$*. Further, take* $u \in \mathbb{K}^*$*; if* $t^\lambda = 1$ *for some* $\lambda > 1$*, we further assume that* $u^\lambda = 1$ *holds. Consider the map* $\tau' : \mathbb{A}\langle t\rangle \to \mathscr{S}(\mathbb{K})$ *with* $\tau(f) = \langle \mathrm{ev}(f, n)\rangle_{n \geqslant 0}$ *where the evaluation function* $\mathrm{ev}' : \mathbb{A}\langle t\rangle \times \mathbb{N} \to \mathbb{K}$ *is defined by*
   $$\mathrm{ev}'(\sum_i h_i\, t^i, n) = \sum_i \mathrm{ev}(h_i, n)\ \mathrm{ev}'(t, n)^i$$
   *with*
   $$\mathrm{ev}'(t, n) = u \prod_{k=\delta}^{n} \mathrm{ev}(\alpha, k - 1).$$
   *Then* $\tau$ *is a* $\mathbb{K}$*-homomorphism.*
3. *If* $(\mathbb{A}, \sigma)$ *is a field and* $(\mathbb{E}, \sigma)$ *is a (nested)* $\mathrm{R}\Pi$*-extension of* $(\mathbb{A}, \sigma)$*, then any* $\mathbb{K}$*-homomorphism* $\tau : \mathbb{E} \to \mathscr{S}(\mathbb{K})$ *is injective.*

*Proof* 1. The proof follows by [22, Lemma 2.5.1].
2. The proof follows by [32, Lemma 5.4(1)].
3. By [32, Theorem 3.3], $(\mathbb{E}, \sigma)$ is simple that is, any ideal of $\mathbb{E}$ which is closed under $\sigma$ is either $\mathbb{E}$ or $\{0\}$. Thus by [32, Lemma 5.8] $\tau'$ is injective. ∎

In this article, we will apply part (2) of Lemma 2.2 iteratively. As base case, we will use the following difference fields that can be embedded into the ring of sequences.

*Example 2.8* Take the rational difference field $(\mathbb{K}(x), \sigma)$ over $\mathbb{K}$ defined in Example 2.1 and consider the map $\tau : \mathbb{K}(x) \to \mathscr{S}(\mathbb{K})$ defined by $\tau(\frac{a}{b}) = \langle \text{ev}\left(\frac{a}{b}, n\right)\rangle_{n \geqslant 0}$ with $a, b \in \mathbb{K}[x]$ where

$$\text{ev}\left(\tfrac{a}{b}, n\right) := \begin{cases} 0, & \text{if } b(n) = 0 \\ \frac{a(n)}{b(n)}, & \text{if } b(n) \neq 0. \end{cases} \tag{7}$$

Then by Lemma 2.2(1) it follows that $\tau : \mathbb{K}(x) \to \mathscr{S}(\mathbb{K})$ is a $\mathbb{K}$-homomorphism. We can define the function:

$$Z(p) = \max\left(\{k \in \mathbb{N} \mid p(k) = 0\}\right) + 1 \quad \text{for any } p \in \mathbb{K}[x] \tag{8}$$

with $\max(\emptyset) = -1$. Now let $f = \frac{a}{b} \in \mathbb{K}(x)$ where $a, b \in \mathbb{K}[x]$, $b \neq 0$. Since $a(x)$, $b(x)$ have only finitely many roots, it follows that $\tau(\frac{a}{b}) = \mathbf{0}$ if and only if $\frac{a}{b} = 0$. Hence $\ker(\tau) = \{0\}$ and thus $\tau$ is injective. Summarizing, we have constructed a $\mathbb{K}$-embedding, $\tau : \mathbb{K}(x) \to \mathscr{S}(\mathbb{K})$ where the difference field $(\mathbb{K}(x), \sigma)$ is identified in the difference ring of $\mathbb{K}$-sequences $(\mathscr{S}(\mathbb{K}), S)$ as the sub-difference ring of $\mathbb{K}$-sequences $(\tau(\mathbb{K}(x)), S)$. We call $(\tau(\mathbb{K}(x)), S)$ the *difference field of rational sequences*.

*Example 2.9* Take the mixed $\boldsymbol{q}$-multibasic difference field $(\mathbb{F}, \sigma)$ with $\mathbb{F} = \mathbb{K}(x, \boldsymbol{t})$ defined in Example 2.2. Then, $\tau : \mathbb{F} \to \mathscr{S}(\mathbb{K})$ defined by $\tau(\frac{a}{b}) = \langle \text{ev}\left(\frac{a}{b}, n\right)\rangle_{n \geqslant 0}$ for $a, b \in \mathbb{K}[x, \boldsymbol{t}]$ with

$$\text{ev}\left(\tfrac{a}{b}, n\right) := \begin{cases} 0, & \text{if } b(n, \boldsymbol{q}^n) = 0 \\ \frac{a(n, \boldsymbol{q}^n)}{b(n, \boldsymbol{q}^n)}, & \text{if } b(n, \boldsymbol{q}^n) \neq 0 \end{cases} \tag{9}$$

is a $\mathbb{K}$-homomorphism. We define the function

$$Z(p) = \max\left(\{k \in \mathbb{N} \mid p(k, \boldsymbol{q}^k) = 0\}\right) + 1 \quad \text{for any } p \in \mathbb{K}[x, \boldsymbol{t}] \tag{10}$$

with $\max(\emptyset) = -1$. We will use the fact that this set of zeros is finite if $p \neq 0$ and that $Z(p)$ can be computed; see [6, Sect. 3.2]. For any rational function, $f = \frac{g}{h} \in \mathbb{F}$ with $g, h \in \mathbb{K}[x, \boldsymbol{t}]$ where $h \neq 0$ and $g, h$ are co-prime, let $\delta = \max(\{Z(g), Z(h)\})$. Then $f(n) \neq 0$ for all $n \geqslant \delta$. On the other hand, $\tau(\frac{g}{h}) = \mathbf{0}$ if and only if $\frac{g}{h} = 0$. Hence $\ker(\tau) = \{0\}$ and thus $\tau$ is injective. In summary, we have constructed a $\mathbb{K}$-embedding $\tau : \mathbb{F} \to \mathscr{S}(\mathbb{K})$ where the difference field $(\mathbb{F}, \sigma)$ is identified in $(\mathscr{S}(\mathbb{K}), S)$ as $(\tau(\mathbb{F}), S)$ which we call the *difference field of mixed* $\boldsymbol{q}$*-multibasic rational sequences*.

# 3 Main Result

We shall solve Problem RPE algorithmically by proving the following main result in Theorem 3.1. Here the specialization $e = 0$ covers the hypergeometric case. Similarly, taking $\boldsymbol{q}$-multibasic hypergeometric products in (11) and suppressing $x$ yield the multibasic case. Further, setting $e = 1$ provides the $q$-hypergeometric case.

**Theorem 3.1** *Let $\mathbb{K} = K(\kappa_1, \ldots, \kappa_u)(q_1, \ldots, q_e)$ be a rational function field over a field $K$ and consider the mixed $\boldsymbol{q}$-multibasic hypergeometric products*

$$P_1(n) = \prod_{k=\ell_1}^{n} h_1(k, \boldsymbol{q}^k), \quad \ldots, \quad P_m(n) = \prod_{k=\ell_1}^{n} h_m(k, \boldsymbol{q}^k) \in \mathrm{Prod}(\mathbb{K}(n, \boldsymbol{q}^n)) \tag{11}$$

*with $\ell_i \in \mathbb{N}$ and $h_i(x, \boldsymbol{t}) \in \mathbb{K}(x, \boldsymbol{t})$ s.t. $h_i(k, \boldsymbol{q}^k)$ has no pole and is non-zero for $k \geqslant \ell_i$.*
*Then there exist irreducible monic polynomials $f_1, \ldots, f_s \in \mathbb{K}[x, \boldsymbol{t}] \setminus \mathbb{K}$, nonnegative integers $\ell'_1, \ldots, \ell'_s$ and a finite algebraic field extension $K'$ of $K$ with a $\lambda$-th root of unity $\zeta \in K'$ and elements $\alpha_1, \ldots, \alpha_w \in K'^*$ which are not roots of unity with the following properties.*

*One can choose natural numbers $\mu_i, \delta_i \in \mathbb{N}$ for $1 \leqslant i \leqslant m$, integers $u_{i,j}$ with $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant w$, integers $v_{i,j}$ with $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant s$ and rational functions $r_i \in \mathbb{K}(x, \boldsymbol{t})^*$ for $1 \leqslant i \leqslant m$ such that the following holds:*

1. *For all $n \in \mathbb{N}$ with $n \geqslant \delta_i$,*

$$P_i(n) = (\zeta^n)^{\mu_i} (\alpha_1^n)^{u_{i,1}} \cdots (\alpha_w^n)^{u_{i,w}} r_i(n, \boldsymbol{q}^n) \left(\prod_{k=\ell'_1}^{n} f_1(k, \boldsymbol{q}^k)\right)^{v_{i,1}} \cdots \left(\prod_{k=\ell'_s}^{n} f_s(k, \boldsymbol{q}^k)\right)^{v_{i,s}}. \tag{12}$$

2. *The sequences with entries from the field $\mathbb{K}' = K'(\kappa_1, \ldots, \kappa_u)(q_1, \ldots, q_e)$,*

$$\langle \alpha_1^n \rangle_{n\geqslant 0}, \ldots, \langle \alpha_w^n \rangle_{n\geqslant 0}, \langle \prod_{k=\ell'_1}^{n} f_1(k, \boldsymbol{q}^k) \rangle_{n\geqslant 0}, \ldots, \langle \prod_{k=\ell'_s}^{n} f_s(k, \boldsymbol{q}^k) \rangle_{n\geqslant 0}, \tag{13}$$

*are among each other algebraically independent over $\tau\big(\mathbb{K}'(x, \boldsymbol{t})\big)\big[\langle \zeta^n \rangle_{n\geqslant 0}\big]$; here $\tau : \mathbb{K}'(x, \boldsymbol{t}) \to \mathcal{S}(\mathbb{K}')$ is a difference ring monomorphism where $\tau(\frac{a}{b}) = \langle \mathrm{ev}\left(\frac{a}{b}, n\right) \rangle_{n\geqslant 0}$ for $a, b \in \mathbb{K}'[x, \boldsymbol{t}]$ is defined by (9).*

*If $K$ is a strongly $\sigma$-computable field (see Definition 4.1 below), then the components in (12) are computable.*

Namely, Theorem 3.1 provides a solution to Problem RPE as follows. Let $P(n) \in \mathrm{ProdE}(\mathbb{K}(n, \boldsymbol{q}^n))$ be defined as in (3) with $S \subseteq \mathbb{Z}^m$ finite, $a_{(\nu_1,\ldots,\nu_m)}(n) \in \mathbb{K}(n, \boldsymbol{q}^n)$ and where the products $P_i(n)$ are given as in (11). Now assume that we have computed all the components as stated in Theorem 3.1. Then determine $\lambda \in \mathbb{N}$ such that all

$a_{(n_1,\dots,n_m)}(n)$ have no pole for $n \geqslant \lambda$, and set $\delta = \max(\lambda, \delta_1, \dots, \delta_m)$. Moreover, replace all $P_i$ with $1 \leqslant i \leqslant m$ by their right-hand sides of (12) in the expression $P(n)$ yielding the expression $Q(n) \in \mathrm{ProdE}(\mathbb{K}'(n, \boldsymbol{q}^n))$. Then by this construction we have $P(n) = Q(n)$ for all $n \geqslant \delta$. Furthermore, part (2) of Theorem 3.1 shows part (2) of Problem RPE.

Finally, we look at the zero-recognition statement of part (3) of Problem RPE. If $Q = 0$, then $P(n) = 0$ for all $n \geqslant \delta$ by part (1) of Problem RPE. Conversely, if $P(n) = 0$ for all $n$ from a certain point on, then also $Q(n) = 0$ holds for all $n$ from a certain point on by part (1). Since the sequences (13) are algebraically independent over $\tau(\mathbb{K}'(x, \boldsymbol{t}))[\langle \zeta^n \rangle_{n \geqslant 0}]$, the expression $Q(n)$ must be free of these products. Consider the mixed multibasic difference field $(\mathbb{K}'(x, \boldsymbol{t}), \sigma)$ and the A-extension $(\mathbb{K}'(x, \boldsymbol{t})[\vartheta], \sigma)$ of $(\mathbb{K}'(x, \boldsymbol{t}), \sigma)$ of order $\lambda$ with $\sigma(\vartheta) = \zeta\, \vartheta$. By Corollary 5.1 below it follows that the mixed multibasic difference field $(\mathbb{K}'(x, \boldsymbol{t}), \sigma)$ is a $\Pi\Sigma$-extension of $(\mathbb{K}', \sigma)$ with $\mathrm{const}(\mathbb{K}', \sigma) = \mathbb{K}'$. Thus by Lemma 2.1 it follows that the A-extension is an R-extension. In particular, it follows by Lemma 2.2 that the homomorphic extension of $\tau$ from $(\mathbb{K}'(x, \boldsymbol{t}), \sigma)$ to $(\mathbb{K}'(x, \boldsymbol{t})[\vartheta], \sigma)$ with $\tau(\vartheta) = \langle \zeta^n \rangle_{n \geqslant 0}$ is a $\mathbb{K}'$-embedding. Since $Q(n)$ is a polynomial expression in $\zeta^n$ with coefficients from $\mathbb{K}'(n, \boldsymbol{q}^n)$ ($\zeta^n$ comes from (12)), we can find an $h(x, \boldsymbol{t}, \vartheta) \in \mathbb{K}'(x, \boldsymbol{t})[\vartheta]$ such that the expression $Q(n)$ equals $h(n, \boldsymbol{q}^n, \zeta^n)$. Further observe that $\tau(h)$ and the produced sequence of $Q(n)$ agree from a certain point on. Thus $\tau(h) = \boldsymbol{0}$ and since $\tau$ is a $\mathbb{K}'$-embedding, $h = 0$. Consequently, $Q(n)$ must be the zero-expression.

We will provide a proof (and an underlying algorithm) for Theorem 3.1 by tackling the following subproblem formulated in the difference ring setting.

*Given* a mixed $\boldsymbol{q}$-multibasic difference field $(\mathbb{F}, \sigma)$ with $\mathbb{F} = \mathbb{K}(x)(t_1) \cdots (t_e)$ where $\sigma(x) = x + 1$ and $\sigma(t_\ell) = q_\ell\, t_\ell$ for $1 \leqslant \ell \leqslant e$; given $h_1, \dots, h_m \in \mathbb{F}^*$. *Find* an $R\Pi$-extension $(\mathbb{A}, \sigma)$ of $(\mathbb{K}'(x)(t_1) \cdots (t_e), \sigma)$ where $\mathbb{K}'$ is an algebraic field extension of $\mathbb{K}$ and $g_1, \dots, g_m \in \mathbb{A} \setminus \{0\}$ where $\sigma(g_i) = \sigma(h_i)\, g_i$ for $1 \leqslant i \leqslant m$.

Namely, taking the special case $\mathbb{F} = \mathbb{K}(x)$ with $\sigma(x) = x + 1$, we will tackle the above problem in Theorem 5.1, and we will derive the general case in Theorem 6.2. Then based on the particular choice of the $g_i$ this will lead us directly to Theorem 3.1. We will now give a concrete example of the above strategy for hypergeometric products. An example for the mixed $\boldsymbol{q}$-multibasic situation is given in Example 6.1.

*Example 3.1* Take the rational function field $\mathbb{K} = K(\kappa)$ defined over the algebraic number field $K = \mathbb{Q}\big((\iota + \sqrt{3}), \sqrt{-13}\big)$ and take the rational function field $\mathbb{K}(x)$ defined over $\mathbb{K}$. Now consider the hypergeometric product expressions

$$P(n) = \prod_{k=1}^{n} h_1(k) \; + \prod_{k=1}^{n} h_2(k) \; + \prod_{k=1}^{n} h_3(k) \in \mathrm{ProdE}(\mathbb{K}(n)) \tag{14}$$

with

$$h_1(x) = \frac{-13\sqrt{-13}\,\kappa}{x}, \quad h_2(x) = \frac{-784\,(\kappa+1)^2\,x}{13\sqrt{-13}\,(\iota+\sqrt{3})^4\,\kappa\,(x+2)^2}, \quad h_3 = \frac{-17210368\,(\kappa+1)^5\,x}{13\sqrt{-13}\,(\iota+\sqrt{3})^{10}\,\kappa\,(x+2)^5} \tag{15}$$

where $h_1, h_2, h_3 \in \mathbb{K}(x)$. With our algorithm (see Theorem 5.4 below) we construct the algebraic field extension $K' = \mathbb{Q}((-1)^{\frac{1}{6}}, \sqrt{13})$ of $K$, take the rational function field $\mathbb{K}' = K'(\kappa)$ and define on top the rational difference field $(\mathbb{K}'(x), \sigma)$ with $\sigma(x) = x + 1$. Based on this, we obtain the R$\Pi$-extension $(\mathbb{A}, \sigma)$ of $(\mathbb{K}'(x), \sigma)$ with

$$\mathbb{A} = \mathbb{K}'(x)[\vartheta][y_1, y_1^{-1}][y_2, y_2^{-1}][y_3, y_3^{-1}][y_4, y_4^{-1}][z, z^{-1}] \tag{16}$$

and the automorphism $\sigma(\vartheta) = (-1)^{\frac{1}{6}}\vartheta$, $\sigma(y_1) = \sqrt{13}\,y_1$, $\sigma(y_2) = 7\,y_2$, $\sigma(y_3) = \kappa\,y_3$, $\sigma(y_4) = (\kappa + 1)\,y_4$, $\sigma(z) = (x + 1)\,z$; note that $\operatorname{const}(\mathbb{A}, \sigma) = \mathbb{K}'$. Now consider the difference ring homomorphism $\tau : \mathbb{A} \to \mathscr{S}(\mathbb{K}')$ which we define as follows. For the base field $(\mathbb{K}'(x), \sigma)$ we take the difference ring embedding $\tau(\frac{a}{b}) = \langle \operatorname{ev}\left(\frac{a}{b}, n\right)\rangle_{n\geqslant 0}$ for $a, b \in \mathbb{K}'[x]$ where ev is defined in (7). Further, applying iteratively part (2) of Lemma 2.2 we obtain the difference ring homomorphism $\tau : \mathbb{A} \to \mathscr{S}(\mathbb{K}')$ determined by $\tau(\vartheta) = \langle\left((-1)^{\frac{1}{6}}\right)^n\rangle_{n\geqslant 0}$, $\tau(y_1) = \langle\left(\sqrt{13}\right)^n\rangle_{n\geqslant 0}$, $\tau(y_2) = \langle 7^n\rangle_{n\geqslant 0}$, $\tau(y_3) = \langle\kappa^n\rangle_{n\geqslant 0}$, $\tau(y_4) = \langle(\kappa + 1)^n\rangle_{n\geqslant 0}$ and $\tau(z) = \langle n!\rangle_{n\geqslant 0}$. In addition, since $(\mathbb{A}, \sigma)$ is an R$\Pi$-extension of $(\mathbb{K}'(x), \sigma)$, it follows by part (3) of Lemma 2.2 that $\tau$ is a $\mathbb{K}'$-embedding. Hence $\tau(\mathbb{K}'(x))[\tau(\vartheta)][\tau(y_1), \tau(y_1^{-1})]\cdots[\tau(y_4), \tau(y_4^{-1})]$ $[\tau(z), \tau(z^{-1})]$ is a Laurent polynomial ring over the ring $\tau(\mathbb{K}'(x))[\tau(\vartheta)]$ with $\tau(\vartheta) = \left\langle\left((-1)^{\frac{1}{6}}\right)^n\right\rangle_{n\geqslant\delta}$. In addition, we find

$$Q' = \underbrace{\frac{\vartheta^9\, y_1^3\, y_3}{z}}_{=:g_1} + 4\underbrace{\frac{\vartheta^{11}\, y_2^2\, y_4^2}{(x+1)^2\,(x+2)^2\, y_1^3\, y_3\, z}}_{=:g_2} + 32\underbrace{\frac{\vartheta^5\, y_2^5\, y_4^5}{(x+1)^5\,(x+2)^5\, y_1^3\, y_3\, z^4}}_{=:g_3} \tag{17}$$

where $\sigma(g_i) = \sigma(h_i)\, g_i$ for $i = 1, 2, 3$. Thus the $g_i$ model the shift behaviors of the hypergeometric products with the multiplicands $h_i \in \mathbb{K}(x)$. In particular, we have defined $Q'$ such that $\tau(Q') = \langle P(n)\rangle_{n\geqslant 0}$ holds. Rephrasing $x \leftrightarrow n$, $\vartheta \leftrightarrow \left((-1)^{\frac{1}{6}}\right)^n$, $y_1 \leftrightarrow \left(\sqrt{13}\right)^n$, $y_2 \leftrightarrow 7^n$, $y_3 \leftrightarrow \kappa^n$, $y_4 \leftrightarrow \left(\kappa + 1\right)^n$and $z \leftrightarrow n!$ in (17) we get

$$\begin{aligned} Q(n) =& \frac{\left(\left((-1)^{\frac{1}{6}}\right)^n\right)^9 \left(\left(\sqrt{13}\right)^n\right)^3 \kappa^n}{n!} + \frac{4\left(\left((-1)^{\frac{1}{6}}\right)^n\right)^{11} \left(7^n\right)^2 \left(\left(\kappa+1\right)^n\right)^2}{(n+1)^2\,(n+2)^2 \left(\left(\sqrt{13}\right)^n\right)^3 \kappa^n\, n!} \\ &+ \frac{32\left(\left((-1)^{\frac{1}{6}}\right)^n\right)^5 \left(7^n\right)^5 \left(\left(\kappa+1\right)^n\right)^5}{(n+1)^5\,(n+2)^5 \left(\left(\sqrt{13}\right)^n\right)^3 \kappa^n \left(n!\right)^4} \in \operatorname{ProdE}(\mathbb{K}(n)). \end{aligned} \tag{18}$$

Note: $n!$ and $a^n$ with $a \in \mathbb{K}'^*$ are just shortcuts for $\prod_{k=1}^n k$ and $\prod_{k=1}^n a$, respectively. Based on the corresponding proof of Theorem 3.1 at the end of Sect. 5.4 we can ensure that $P(n) = Q(n)$ holds for all $n \in \mathbb{N}$ with $n \geqslant 1$. Further details on the computation steps can be found in Examples 5.3 and 5.4.

## 4 Algorithmic Preliminaries: Strongly $\sigma$-Computable Fields

In Karr's algorithm [12] and all the improvements [4, 14, 24, 26, 29, 30, 32] one relies on certain algorithmic properties of the constant field $\mathbb{K}$. Among those, one needs to solve the following problem.

| **Problem GO for $\alpha_1, \ldots, \alpha_w \in K^*$** |
|---|
| Given a field $K$ and $\alpha_1, \ldots, \alpha_w \in K^*$. Compute a basis of the submodule $$\mathbb{V} := \Big\{(u_1, \ldots, u_w) \in \mathbb{Z}^w \,\Big|\, \prod_{i=1}^{w} {\alpha_i}^{u_i} = 1\Big\} \text{ of } \mathbb{Z}^w \text{ over } \mathbb{Z}.$$ |

In [23] it has been worked out that Problem GO is solvable in any rational function field $\mathbb{K} = K(\kappa_1, \ldots, \kappa_u)$ provided that one can solve Problem GO in $K$ and that one can factor multivariate polynomials over $K$. In this article we require the following stronger assumption: Problem GO can be solved not only in $K$ ($K$ with this property was called $\sigma$-computable in [14, 23]) but also in any algebraic extension of it.

**Definition 4.1** A field $K$ is strongly $\sigma$-computable if the standard operations in $K$ can be performed, multivariate polynomials can be factored over $K$ and Problem GO can be solved for $K$ and any finite algebraic field extension of $K$.

Note that Ge's algorithm [9] solves Problem GO over an arbitrary number field $K$. Since any finite algebraic extension of an algebraic number field is again an algebraic number field, it follows with Ge's algorithm, that any number field $K$ is $\sigma$-computable.

Summarizing, in this article we can turn our theoretical results to algorithmic versions, if we assume that $\mathbb{K} = K(\kappa_1, \ldots, \kappa_u)$ is a rational function field over a field $K$ which is strongly $\sigma$-computable. In particular, the underlying algorithms are implemented in the package `NestedProducts` for the case that $K$ is a finite algebraic field extension of $\mathbb{Q}$.

Besides these fundamental properties of the constant field, we rely on further (algorithmic) properties that can be ensured by difference ring theory. Let $(\mathbb{F}[t], \sigma)$ be a difference ring over the field $\mathbb{F}$ with $t$ transcendental over $\mathbb{F}$ and $\sigma(t) = \alpha\, t + \beta$ where $\alpha \in \mathbb{F}^*$ and $\beta \in \mathbb{F}$. Note that for any $h \in \mathbb{F}[t]$ and any $k \in \mathbb{Z}$ we have $\sigma^k(h) \in \mathbb{F}[t]$. Furthermore, if $h$ is irreducible, then also $\sigma^k(h)$ is irreducible.

Two polynomials $f, h \in \mathbb{F}[t] \setminus \{0\}$ are said to be *shift co-prime*, also denoted by $\gcd_\sigma(f, h) = 1$, if for all $k \in \mathbb{Z}$ we have that $\gcd(f, \sigma^k(h)) = 1$. Furthermore, we say that $f$ and $h$ are shift-equivalent, denoted by $f \sim_\sigma h$, if there is a $k \in \mathbb{Z}$ with $\frac{\sigma^k(f)}{h} \in \mathbb{F}$. If there is no such $k$, then we also write $f \nsim_\sigma h$.

It is immediate that $\sim_\sigma$ is an equivalence relation. In the following we will focus mainly on irreducible polynomials $f, h \in \mathbb{F}[t]$. Then observe that $f \sim_\sigma h$ holds if and only if $\gcd_\sigma(f, h) \neq 1$ holds. In the following it will be important to determine such a $k$. Here we utilize the following property of $\Pi\Sigma$-extensions whose proof can be found in [12, Theorem 4] ( [7, Corollary 1.2] or [22, Theorem 2.2.4]).

**Lemma 4.1** *Let $(\mathbb{F}(t), \sigma)$ be a $\Pi\Sigma$-extension of $(\mathbb{F}, \sigma)$ and $f \in \mathbb{F}(t)^*$. Then $\frac{\sigma^k(f)}{f} \in \mathbb{F}$ for some $k \neq 0$ iff $\frac{\sigma(t)}{t} \in \mathbb{F}$ and $f = u\,t^m$ with $u \in \mathbb{F}^*$ and $m \in \mathbb{Z}$.*

Namely, using this result one can deduce when such a $k$ is unique.

**Lemma 4.2** *Let $(\mathbb{F}(t), \sigma)$ be a $\Pi\Sigma$-extension of $(\mathbb{F}, \sigma)$ with $\sigma(t) = \alpha\,t + \beta$ for $\alpha \in \mathbb{F}^*$ and $\beta \in \mathbb{F}$. Let $f, h \in \mathbb{F}[t]$ be irreducible with $f \sim_\sigma h$. Then there is a unique $k \in \mathbb{Z}$ with $\frac{\sigma^k(f)}{h} \in \mathbb{F}^*$ iff $\frac{\sigma(t)}{t} \notin \mathbb{F}$ or $f = a\,t$ and $h = b\,t$ for some $a, b \in \mathbb{F}^*$.*

*Proof* Suppose on the contrary that $\beta = 0$ and $f = t = h$. Then $\frac{\sigma^k(f)}{h} \in \mathbb{F}^*$ for all $k \in \mathbb{Z}$ and thus $k$ is not unique. Conversely, suppose that $\sigma^{k_1}(f) = u\,h$ and $\sigma^{k_2}(f) = v\,h$ with $k_1 > k_2$. Then $\frac{\sigma^{k_1-k_2}(f)}{f} = \frac{u}{v} \in \mathbb{F}^*$. Thus by Lemma 4.1, $\frac{\sigma(t)}{t} \in \mathbb{F}$ and $f = a\,t$ for some $a \in \mathbb{F}^*$. Thus also $h = b\,t$ for some $b \in \mathbb{F}^*$. ■

Consider the rational difference field $(\mathbb{K}(x), \sigma)$ with $\sigma(x) = x + 1$. Note that $x$ is a $\Sigma$-monomial. Let $f, h \in \mathbb{K}[x] \setminus \mathbb{K}$ be irreducible polynomials. If $f \sim_\sigma h$, then there is a unique $k \in \mathbb{Z}$ with $\frac{\sigma^k(f)}{h} \in \mathbb{K}$. Similarly for the mixed $\boldsymbol{q}$-multibasic difference field $(\mathbb{K}(x)(t_1)\cdots(t_e), \sigma)$ with $\sigma(x) = x + 1$ and $\sigma(t_i) = q_i\,t_i$ for $1 \leqslant i \leqslant e$ we note that the $t_i$ are $\Pi$-monomials; see Corollary 5.1 below. For $1 \leqslant i \leqslant e$ and $\mathbb{E} = \mathbb{K}(x)(t_1)\cdots(t_{i-1})$, let $f, h \in \mathbb{E}[t_i]$ be monic irreducible polynomials. If $f \sim_\sigma h$, then there is a unique $k \in \mathbb{Z}$ with $\frac{\sigma^k(f)}{h} \in \mathbb{E}$ if and only $f \neq t_i \neq h$. In both cases, such a unique $k$ can be computed if one can perform the usual operations in $\mathbb{K}$; [14, Theorem 1]. Optimized algorithms for theses cases can be found in [6, Sect. 3]. In addition, the function $Z$ given in (8) or in (10) can be computed due to [6]. Summarizing, the following properties hold.

**Lemma 4.3** *Let $(\mathbb{F}, \sigma)$ be the rational or mixed $\boldsymbol{q}$-multibasic difference field over $\mathbb{K}$ as defined in Examples 2.1 and 2.2. Suppose that the usual operations[2] in $\mathbb{K}$ are computable. Then one compute*

1. *the $Z$-functions given in (8) or in (10);*
2. *one can compute for shift-equivalent irreducible polynomials $f, h$ in $\mathbb{K}[x]$ (or in $\mathbb{K}(x)(t_1, \ldots, t_{i-1})[t_i]$) a $k \in \mathbb{Z}$ with $\frac{\sigma^k(f)}{h} \in \mathbb{K}$ (or $\frac{\sigma^k(f)}{h} \in \mathbb{K}(x)(t_1, \ldots, t_{i-1})$).*

For further considerations, we introduce the following Lemma which gives a relation between two polynomials that are shift-equivalent.

[2]This is the case if $\mathbb{K}$ is strongly $\sigma$-computable, or if $\mathbb{K}$ is a rational function field over a strongly $\sigma$-computable field.

**Lemma 4.4** *Let* $(\mathbb{F}(t), \sigma)$ *be a difference field over a field* $\mathbb{F}$ *with* $t$ *transcendental over* $\mathbb{F}$ *and* $\sigma(t) = \alpha\, t + \beta$ *where* $\alpha \in \mathbb{F}^*$ *and* $\beta \in \mathbb{F}$. *Let* $f, h \in \mathbb{F}[t] \setminus \mathbb{F}$ *be monic and* $f \sim_\sigma h$. *Then there is a* $g \in \mathbb{F}(t)^*$ *with* $h = \frac{\sigma(g)}{g} f$.

*Proof* Since $f \sim_\sigma h$, there is a $k \in \mathbb{Z}$ and $u \in \mathbb{F}^*$ with $\sigma^k(f) = u\, h$. Note that $\deg(f) = \deg(h) = m$. By comparing coefficient of the leading terms and using that $f, h$ are monic, we get $u\, t^m = \sigma^k(t^m)$. If $k \geqslant 0$, set $g := \prod_{i=0}^{k-1} \sigma^i(t^{-m} f)$. Then $\frac{\sigma(g)}{g} = \frac{\sigma^k(t^{-m} f)}{t^{-m} f} = \frac{\sigma^k(f)\, t^m}{f\, \sigma^k(t^m)} = \frac{h\, u\, t^m}{f\, \sigma^k(t^m)} = \frac{h}{f}$. Thus, $h = \frac{\sigma(g)}{g} f$. If $k < 0$, set $g := \prod_{i=1}^{-k} \sigma^{-i}(\frac{t^m}{f})$. Then $\frac{\sigma(g)}{g} = \frac{\sigma^k(t^m f^{-1})}{t^m f^{-1}} = \frac{t^m \sigma^k(f)}{\sigma^k(t^m)\, f} = \frac{h\, u\, t^m}{f\, \sigma^k(t^m)} = \frac{h}{f}$. Hence again $h = \frac{\sigma(g)}{g} f$. ■

## 5 Algorithmic Construction of RΠ-Extensions for ProdE($\mathbb{K}(n)$)

In this section we will provide a proof for Theorem 3.1 for the case ProdE($\mathbb{K}(n)$). Afterwards, this proof strategy will be generalized for the case ProdE($\mathbb{K}(n, \boldsymbol{q}^n)$) in Sect. 6. In both cases, we will need the following set from [12, Definition 21].

**Definition 5.1** For a difference field $(\mathbb{F}, \sigma)$ and $\boldsymbol{f} = (f_1, \ldots, f_s) \in (\mathbb{F}^*)^s$ we define

$$\mathbf{M}\,(\boldsymbol{f}, \mathbb{F}) = \left\{(v_1, \ldots, v_s) \in \mathbb{Z}^s \;\middle|\; \frac{\sigma(g)}{g} = f_1^{v_1} \cdots f_s^{v_s} \text{ for some } g \in \mathbb{F}^*\right\}.$$

Note that $\mathbf{M}\,(\boldsymbol{f}, \mathbb{F})$ is a $\mathbb{Z}$-submodule of $\mathbb{Z}^s$ which has finite rank. We observe further that for the special case $\text{const}(\mathbb{A}, \sigma) = \mathbb{A}$ we have $\frac{\sigma(g)}{g} = 1$ for all $g \in \mathbb{A}^*$. Thus

$$\mathbf{M}\,(\boldsymbol{f}, \mathbb{A}) = \{(v_1, \ldots, v_s) \in \mathbb{Z}^s \mid f_1^{v_1} \cdots f_s^{v_s} = 1\}$$

which is nothing else but the set in Problem GO.

Finally, we will heavily rely on the following lemma that ensures if a P-extension forms a Π-extension; compare also [10].

**Lemma 5.1** *Let* $(\mathbb{F}, \sigma)$ *be a difference field and let* $\boldsymbol{f} = (f_1, \ldots, f_s) \in (\mathbb{F}^*)^s$. *Then the following statements are equivalent.*

1. *There are no* $(v_1, \ldots, v_s) \in \mathbb{Z}^s \setminus \{\mathbf{0}_s\}$ *and* $g \in \mathbb{F}^*$ *with* (31), *i.e.,* $\mathbf{M}\,(\boldsymbol{f}, \mathbb{F}) = \{\mathbf{0}_s\}$.
2. *One can construct a* Π*-field extension* $(\mathbb{F}(z_1) \cdots (z_s), \sigma)$ *of* $(\mathbb{F}, \sigma)$ *with* $\sigma(z_i) = f_i\, z_i$, *for* $1 \leqslant i \leqslant s$.
3. *One can construct a* Π*-extension* $(\mathbb{F}[z_1, z_1^{-1}] \cdots [z_s, z_s^{-1}], \sigma)$ *of* $(\mathbb{F}, \sigma)$ *with* $\sigma(z_i) = f_i\, z_i$, *for* $1 \leqslant i \leqslant s$.

*Proof* (1) $\Leftrightarrow$ (2) is established by [27, Theorem 9.1]. (2) $\Longrightarrow$ (3) is obvious while (3) $\Longrightarrow$ (2) follows by iterative application of [32, Corollary 2.6]. ■

Throughout this section, let $(\mathbb{K}(x), \sigma)$ be the rational difference field over a constant field $\mathbb{K}$, where $\mathbb{K} = K(\kappa_1, \ldots, \kappa_u)$ is a rational function field over a field $K$. For algorithmic reasons we will assume in addition that $K$ is strongly $\sigma$-computable (see Definition 4.1). In Sect. 5.1 we will treat Theorem 3.1 first for the special case $\text{ProdE}(K)$. Next, we treat the case $\text{ProdE}(\mathbb{K})$ in Sect. 5.2. In Sect. 5.3 we present simple criteria to check if a tower of $\Pi$-monomials $t_i$ with $\sigma(t_i)/t_i \in \mathbb{K}[x]$ forms a $\Pi$-extension. Finally, in Sect. 5.4 we will utilize this extra knowledge to construct $\Pi$-extensions for the full case $\text{ProdE}(\mathbb{K}(n))$.

## *5.1 Construction of* RΠ*-extensions for* ProdE($K$)

Our construction is based on the following theorem.

**Theorem 5.1** *Let $\gamma_1, \ldots, \gamma_s \in K^*$. Then there is an algebraic field extension $K'$ of $K$ together with a $\lambda$-th root of unity $\zeta \in K'$ and elements $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_w) \in K'^w$ with $\mathbf{M}\left(\boldsymbol{\alpha}, K'\right) = \{\mathbf{0}_w\}$ such that for all $i = 1, \ldots, s$,*

$$\gamma_i = \zeta^{\mu_i}\, \alpha_1^{u_{i,1}} \cdots \alpha_w^{u_{i,w}} \tag{19}$$

*holds for some $1 \leqslant \mu_i \leqslant \lambda$ and $(u_{i,1}, \ldots, u_{i,w}) \in \mathbb{Z}^w$.*

*If $K$ is strongly $\sigma$-computable, then $\zeta$, the $\alpha_i$ and the $\mu_i, u_{i,j}$ can be computed.*

*Proof* We prove the Theorem by induction on $s$. The base case $s = 0$ obviously holds. Now assume that there are a $\lambda$-th root of unity $\zeta$, elements $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_w) \in (K'^*)^w$ with $\mathbf{M}\left(\boldsymbol{\alpha}, K'\right) = \{\mathbf{0}_w\}$, $1 \leqslant \mu_i \leqslant \lambda$ and $(v_{i,1}, \ldots v_{i,w}) \in \mathbb{Z}^w$ such that $\gamma_i = \zeta^{\mu_i}\, \alpha_1^{v_{i,1}} \cdots \alpha_w^{v_{i,w}}$ holds for all $1 \leqslant i \leqslant s-1$.

Now consider in addition $\gamma_s \in K^*$. First suppose the case $\mathbf{M}\left((\alpha_1, \ldots, \alpha_w, \gamma_s), K'\right) = \{\mathbf{0}_{w+1}\}$. With $\alpha_{w+1} := \gamma_s$, we can write $\gamma_s$ as $\gamma_s = \zeta^{\lambda}\, \alpha_1^{\upsilon_1} \cdots \alpha_w^{\upsilon_w}\, \alpha_{w+1}$ with $\lambda = \upsilon_1 = \cdots = \upsilon_w = 0$. Further, with $v_{i,w+1} = 0$, we can write $\gamma_i = \zeta^{\mu_i}\, \alpha_1^{v_{i,1}} \cdots \alpha_w^{v_{i,w}}\, \alpha_{w+1}^{v_{i,w+1}}$ for all $1 \leqslant i \leqslant s-1$. This completes the proof for this case.

Otherwise, suppose that $\mathbf{M}\left((\alpha_1, \ldots, \alpha_w, \gamma_s), K'\right) \neq \{\mathbf{0}_{w+1}\}$ and take $(\upsilon_1, \ldots, \upsilon_w, u_s) \in \mathbf{M}\left((\alpha_1, \ldots, \alpha_w, \gamma_s), K'\right) \setminus \{\mathbf{0}_{w+1}\}$. Note that $u_s \neq 0$ since $\mathbf{M}\left(\boldsymbol{\alpha}, K'\right) = \{\mathbf{0}_w\}$. Then take all the non-zero integers in $(\upsilon_1, \ldots, \upsilon_w, u_s)$ and define $\delta$ to be their least common multiple. Define $\bar{\alpha}_j := \alpha_j^{\frac{1}{|u_s|}} \in K''$ for $1 \leqslant j \leqslant w$ where $K''$ is some algebraic field extension of $K'$ and let $\lambda' = \text{lcm}(\delta, \lambda)$. Take a primitive $\lambda'$-th root of unity $\zeta' := e^{\frac{2\pi\iota}{\lambda'}}$. Then we can express $\gamma_s$ in terms of $\bar{\alpha}_1, \ldots, \bar{\alpha}_w$ by

$$\gamma_s = (\zeta')^{\nu_s} \prod_{j=1}^{w} \alpha_j^{-\frac{\upsilon_j}{u_s}} = (\zeta')^{\nu_s} \prod_{j=1}^{w} (\bar{\alpha}_j)^{-\upsilon_j \cdot \text{sign}(u_s)} \tag{20}$$

with $1 \leqslant \nu_s \leqslant \lambda'$. Note that for each $j$, $-\upsilon_j \cdot \text{sign}(u_s) \in \mathbb{Z}$. Thus we have been able to represent $\gamma_s$ as a power product of $\zeta'$ and elements $\bar{\boldsymbol{\alpha}} = (\bar{\alpha}_1, \ldots, \bar{\alpha}_w) \in (K''^*)^w$ which are not roots of unity. Consequently, we can write $\gamma_i = (\zeta')^{\mu_i}\, \bar{\alpha}_1^{u_{i,1}} \cdots \bar{\alpha}_w^{u_{i,w}}$ for $1 \leqslant i \leqslant s-1$, where $u_{i,j} = |u_s|\, v_{i,j}$ for $1 \leqslant j \leqslant w$ and $1 \leqslant \mu_i \leqslant \lambda'$. Now suppose that $\mathbf{M}\left(\bar{\boldsymbol{\alpha}}, K''\right) \neq \{\mathbf{0}_w\}$. Then there is a $(m_1, \ldots, m_w) \in \mathbb{Z}^w \setminus \{\mathbf{0}_w\}$ such that

$$1 = \prod_{j=1}^{w} (\bar{\alpha}_j)^{m_j} = \prod_{j=1}^{w} \left(\alpha_j^{\frac{1}{|u_s|}}\right)^{m_j} \Longrightarrow \prod_{j=1}^{w} \left(\alpha_j^{\frac{1}{|u_s|}}\right)^{|u_s|\, m_j} = 1^{|u_s|} \iff \prod_{j=1}^{w} \alpha_j^{m_j} = 1$$

with $(m_1, \ldots, m_w) \neq \mathbf{0}_w$; contradicting the assumption that $\mathbf{M}\left(\boldsymbol{\alpha}, K'\right) = \{\mathbf{0}_w\}$ holds. Consequently, $\mathbf{M}\left(\bar{\boldsymbol{\alpha}}, K''\right) = \{\mathbf{0}_w\}$ which completes the induction step.

Suppose that $K$ is strongly $\sigma$-computable. Then one can decide if $\mathbf{M}\left(\boldsymbol{\alpha}, K'\right)$ is the zero-module, and if not one can compute a non-zero integer vector. All other operations in the proof rely on basic operations that can be carried out. ■

*Remark 5.1* Let $\gamma_1, \ldots, \gamma_s \in K^*$ and suppose that the ingredients $\zeta, \alpha_1, \ldots, \alpha_w$ and the $\mu_i$ and $u_{i,j}$ are given as stated in Theorem 5.1. Let $n \in \mathbb{N}$. Then by (19) we have that

$$\gamma_i^n = \prod_{k=1}^{n} \gamma_i = \prod_{k=1}^{n} \zeta^{\mu_i} \prod_{k=1}^{n} \alpha_1^{u_{i,1}} \cdots \prod_{k=1}^{n} \alpha_w^{u_{i,w}} = \left(\zeta^n\right)^{\mu_i} \left(\alpha_1^n\right)^{u_{i,1}} \cdots \left(\alpha_w^n\right)^{u_{i,w}}.$$

The following remarks are relevant.

1. Since $\mathbf{M}\left(\boldsymbol{\alpha}, K'\right) = \{\mathbf{0}_w\}$, we know that there are no $g \in K'^*$, and $(u_1, \ldots, u_w) \in \mathbb{Z}^w \setminus \{\mathbf{0}_w\}$ with $1 = \frac{\sigma(g)}{g} = \alpha_1^{u_1} \cdots \alpha_w^{u_w}$. In short we say that $\alpha_1, \ldots, \alpha_w$ satisfy no integer relation. Thus it follows by Lemma 5.1 that there is a $\Pi$-extension $(\mathbb{E}, \sigma)$ of $(K', \sigma)$ with $\mathbb{E} = K'[y_1, y_1^{-1}] \cdots [y_w, y_w^{-1}]$ and $\sigma(y_j) = \alpha_j\, y_j$ for $j = 1, \ldots, w$.
2. Consider the A-extension $(\mathbb{E}[\vartheta], \sigma)$ of $(\mathbb{E}, \sigma)$ with $\sigma(\vartheta) = \zeta\, \vartheta$ of order $\lambda$. By Lemma 2.1 this is an R-extension. (Take the quotient field of $\mathbb{E}$, apply Lemma 2.1, and then take the corresponding subring.)
3. Summarizing, the product expressions $\gamma_1^n, \ldots, \gamma_s^n$ can be rephrased in the R$\Pi$-extension $(K'[y_1, y_1^{-1}] \cdots [y_w, y_w^{-1}][\vartheta], \sigma)$ of $(K', \sigma)$. Namely, we can represent $\alpha_j^n$ by $y_j$ and $\zeta^n$ by $\vartheta$.
4. If $K = \mathbb{Q}$ (or if $K$ is the quotient field of a certain unique factorization domain), this result can be obtained without any extension, i.e., $K = K'$; see [23].

So far, Ocansey's Mathematica package `NestedProducts` contains the algorithmic part of Theorem 5.1 if $K$ is an algebraic number field, i.e., a finite algebraic field extension of the field of rational numbers $\mathbb{Q}$. More precisely, the field is given in the form $K = \mathbb{Q}(\theta)$ together with an irreducible polynomial $f(x) \in \mathbb{Q}[x]$ with $f(\theta) = 0$ such that the degree $n := \deg f$ is minimal ($f$ is also called the minimal polynomial of $\theta$). Let $\theta_1, \ldots, \theta_n \in \mathbb{C}$ be the roots of the minimal polynomial $f(x)$. Then the mappings $\varphi_i : \mathbb{Q}(\theta) \to \mathbb{C}$ defined as $\varphi_i(\sum_{j=0}^{n-1} \gamma_j\, \vartheta^j) = \sum_{j=0}^{n-1} \gamma_j\, \vartheta_i^j$ with

$\gamma_j \in \mathbb{Q}$ are the embeddings of $\mathbb{Q}(\theta)$ into the field of complex numbers $\mathbb{C}$ for all $i = 1, \dots, n$. Note that any finite algebraic extension $K'$ of $K$ can be also represented in a similar manner and can be embedded into $\mathbb{C}$. Subsequently, we consider algebraic numbers as elements in the subfield $\varphi_i(\mathbb{Q}(\theta))$ of $\mathbb{C}$ for some $i$.

Now let $K$ be such a number field. Applying the underlying algorithm of Theorem 5.1 to given $\gamma_1, \dots, \gamma_s \in K^*$ might lead to rather complicated algebraic field extensions in which the $\alpha_i$ are represented. It turned out that the following strategy improved this situation substantially. Namely, consider the map, $\| \ \| : K \to \mathbb{R}$ where $\mathbb{R}$ is the set of real numbers with $\gamma \mapsto \langle\gamma, \gamma\rangle^{\frac{1}{2}}$ where $\langle\gamma, \gamma\rangle$ denotes the product of $\gamma$ with its complex conjugate. In this setting, one can solve the following problem.

| **Problem RU for $\gamma \in K^*$.** |
|---|
| *Given* $\gamma \in K^*$. *Find*, if possible, a root of unity $\zeta$ such that $\gamma = \|\gamma\| \, \zeta$ holds. |

**Lemma 5.2** *If $K$ is an algebraic number field, then Problem RU for $\gamma \in K^*$ is solvable in $K$ or some finite algebraic extension $K'$ of $K$.*

*Proof* Let $\gamma \in K = \mathbb{Q}(\alpha)$ where $p(x)$ is the minimal polynomial of $\alpha$. We consider two cases. Suppose that $\|\gamma\| \notin K$. Then using the Primitive Element Theorem (see, e.g., [34, p. 145]) we can construct a new minimal polynomial which represents the algebraic field extension $K'$ of $K$ with $\|\gamma\| \in K'$. Define $\zeta := \frac{\gamma}{\|\gamma\|} \in K'$. Note that $\|\zeta\| = 1$. It remains to check if $\zeta$ is a root of unity,[3] i.e., if there is an $n \in \mathbb{N}$ with $\zeta^n = 1$. This is constructively decidable since $K'$ is strongly $\sigma$-computable. In the second case we have $\|\gamma\| \in K$, and thus $\zeta := \frac{\gamma}{\|\gamma\|} \in K$. Since $K$ is strongly $\sigma$-computable, one can decide again constructively if there is an $n \in \mathbb{N}$ with $\zeta^n = 1$.

As preprocessing step (before we actually apply Theorem 5.1) we check algorithmically if we can solve Problem RU for each of the algebraic numbers $\gamma_1, \dots, \gamma_s$. Extracting their roots of unity and applying Proposition 5.1, we can compute a common $\lambda$-th root of unity that will represent all the other roots of unity.

**Proposition 5.1** *Let $a$ and $b$ be distinct primitive roots of unity of order $\lambda_a$ and $\lambda_b$, respectively. Then there is a primitive $\lambda_c$-th root of unity $c$ such that for some $0 \leqslant m_a, m_b < \lambda_c$ we have $c^{m_a} = a$ and $c^{m_b} = b$.*

*Proof* Take primitive roots of unity of orders $\lambda_a$ and $\lambda_b$, say, $\alpha = e^{\frac{2\pi\iota}{\lambda_a}}$ and $\beta = e^{\frac{2\pi\iota}{\lambda_b}}$. Let $a = \alpha^u$ and $b = \beta^v$ for $0 \leqslant u < \lambda_a$ and $0 \leqslant v < \lambda_b$. Define $\lambda_c := \operatorname{lcm}(\lambda_a, \lambda_b)$ and take a primitive $\lambda_c$-th root of unity, $c = e^{\frac{2\pi\iota}{\lambda_c}}$. Then with $m_a = u \frac{\lambda_c}{\lambda_a} \mod \lambda_c$ and $m_b = v \frac{\lambda_c}{\lambda_b} \mod \lambda_c$ the Proposition is proven. ∎

[3] $\zeta$ lies on the unity circle. However, not every algebraic number on the unit circle is a root of unity: Take for instance $\frac{1-\sqrt{3}}{2} + \frac{3^{\frac{1}{4}}}{\sqrt{2}}\,\iota$ and its complex conjugate; they are on the unit circle, but they are roots of the polynomial $x^4 - 2x^3 - 2x + 1$ which is irreducible in $\mathbb{Q}[x]$ and which is not a cyclotomic polynomial. For details on number fields containing such numbers see [16].

*Example 5.1* With $K = \mathbb{Q}(\iota + \sqrt{3}, \sqrt{-13})$, we can extract the following products

$$\prod_{k=1}^{n} \underbrace{-13\sqrt{-13}}_{=:\gamma_1'}, \prod_{k=1}^{n} \underbrace{\frac{-784}{13\sqrt{-13}\,(\iota+\sqrt{3})^4}}_{=:\gamma_2'}, \prod_{k=1}^{n} \underbrace{\frac{-17210368}{13\sqrt{-13}\,(\iota+\sqrt{3})^{10}}}_{=:\gamma_3'} \tag{21}$$

from (14). Let $\gamma_1 = -13$, $\gamma_2 = \sqrt{-13}$, $\gamma_3 = -784$, $\gamma_4 = 13$, $\gamma_5 = (\iota + \sqrt{3})$ and $\gamma_6 = -17210368$. Applying Problem RU to each $\gamma_i$ we get the roots of unity $1, -1, \iota, \frac{\iota+\sqrt{3}}{2}$ with orders $1, 2, 4, 12$, respectively. By Proposition 5.1, the order of the common root of unity is 12. Among all the possible 12-th root of unity, we take $\zeta := e^{\frac{\pi\iota}{6}} = (-1)^{\frac{1}{6}}$. Note that we can express the other roots of unity with less order in terms of our chosen root of unity, $\zeta$. In particular, we can write $1, -1, \iota$ as $\zeta^{12}, \zeta^6, \zeta^3$, respectively. Consequently, (21) simplifies to

$$\left(\left((-1)^{\frac{1}{6}}\right)^n\right)^9 \prod_{k=1}^{n} 13\sqrt{13},\ \left(\left((-1)^{\frac{1}{6}}\right)^n\right)^{11} \prod_{k=1}^{n} \frac{49}{13\sqrt{13}},\ \left(\left((-1)^{\frac{1}{6}}\right)^n\right)^5 \prod_{k=1}^{n} \frac{16807}{13\sqrt{13}}. \tag{22}$$

The pre-processing step yields the numbers $\gamma_1^* = \sqrt{13}$, $\gamma_2^* = 13$, $\gamma_3^* = 49$ and $\gamma_4^* = 16807$ which are not roots of unity. Now we carry out the steps worked out in the proof of Theorem 5.1. `NestedProducts` uses Ge's algorithm [9] to given $\alpha_1 = \sqrt{13}$ and $\alpha_2' = 49$ and finds out that there is no integer relation, i.e., $\mathbf{M}\left((\alpha_1, \alpha_2'), K'\right) = \{\mathbf{0}_2\}$ with $K' = \mathbb{Q}\left((-1)^{\frac{1}{6}}, \sqrt{13}\right)$. For the purpose of working with primes whenever possible, we write $\alpha_2' = \alpha_2^2$ where $\alpha_2 = 7$. Note that, $\mathbf{M}\left((\alpha_1, \alpha_2), K'\right) = \{\mathbf{0}_2\}$. Now take the AP-extension $(K'[\vartheta][y_1, y_1^{-1}][y_2, y_2^{-1}], \sigma)$ of $(K', \sigma)$ with $\sigma(\vartheta) = (-1)^{\frac{1}{6}}\vartheta$, $\sigma(y_1) = \sqrt{13}\,y_1$ and $\sigma(y_2) = 7\,y_2$. By our construction and Remark 5.1 it follows that the AP-extension is an RΠ-extension. Further, with $\alpha_1$ and $\alpha_2$ we can write $13 = \left(\sqrt{13}\right)^2 \cdot 7^0$, $49 = \left(\sqrt{13}\right)^0 \cdot 7^2$ and $16807 = \left(\sqrt{13}\right)^0 \cdot 7^5$. Hence for $a_1' = \vartheta^9\, y_1^3$, $a_2' = \frac{\vartheta^{11}\, y_2^2}{y_1^3}$, $a_3' = \frac{\vartheta^5\, y_2^5}{y_1^3}$ we get $\sigma(a_i') = \gamma_i'\, a_i'$ for $i = 1, 2, 3$. Thus the shift behavior of the products in (21) is modeled by $a_1', a_2', a_3'$, respectively. In particular, the products in (21) can be rewritten to

$$\left(\left((-1)^{\frac{1}{6}}\right)^n\right)^9 \left(\left(\sqrt{13}\right)^n\right)^3,\ \left(\left((-1)^{\frac{1}{6}}\right)^n\right)^{11} \frac{\left(7^n\right)^2}{\left(\left(\sqrt{13}\right)^n\right)^3},\ \left(\left((-1)^{\frac{1}{6}}\right)^n\right)^5 \frac{\left(7^n\right)^5}{\left(\left(\sqrt{13}\right)^n\right)^3}. \tag{23}$$

## 5.2 *Construction of* RΠ*-extensions for* ProdE($\mathbb{K}$)

Next, we treat the case that $\mathbb{K} = K(\kappa_1, \ldots, \kappa_u)$ is a rational function field where we suppose that $K$ is strongly $\sigma$-computable.

**Theorem 5.2** *Let $\mathbb{K} = K(\kappa_1, \dots, \kappa_u)$ be a rational function field over a field $K$ and let $\gamma_1, \dots, \gamma_s \in \mathbb{K}^*$. Then there is an algebraic field extension $K'$ of $K$ together with a $\lambda$-th root of unity $\zeta \in K'$ and elements $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_w) \in K'(\kappa_1, \dots, \kappa_u)^w$ with $\mathbf{M}\left(\boldsymbol{\alpha}, K'(\kappa_1, \dots, \kappa_u)\right) = \{\mathbf{0}_w\}$ such that for all $i = 1, \dots, s$ we have (19) for some $1 \leqslant \mu_i \leqslant \lambda$ and $(u_{i,1}, \dots, u_{i,w}) \in \mathbb{Z}^w$.*

*If $K$ is strongly $\sigma$-computable, then $\zeta$, the $\alpha_i$ and the $\mu_i, u_{i,j}$ can be computed.*

*Proof* There are monic irreducible[4] pairwise different polynomials $f_1, \dots, f_s$ from $K[\kappa_1, \dots, \kappa_u]$ and elements $c_1, \dots, c_s \in K^*$ such that for all $i$ with $1 \leqslant i \leqslant s$ we have

$$\gamma_i = c_i\, f_1^{z_{i,1}}\, f_2^{z_{i,2}} \cdots f_s^{z_{i,s}} \tag{24}$$

with $z_{i,j} \in \mathbb{Z}$. By Theorem 5.1 there exist $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_w) \in (K'^*)^w$ in an algebraic field extension $K'$ of $K$ with $\mathbf{M}\left(\boldsymbol{\alpha}, K'\right) = \{\mathbf{0}_w\}$ and a root of unity $\zeta \in K'$ such that

$$c_i = \zeta^{\mu_i}\, \alpha_1^{u_{i,1}} \cdots \alpha_w^{u_{i,w}} \tag{25}$$

holds for some $m_i, u_{i,j} \in \mathbb{N}$. Hence $\gamma_i = \zeta^{\mu_i}\, \alpha_1^{u_{i,1}} \cdots \alpha_w^{u_{i,w}}\, f_1^{z_{i,1}}\, f_2^{z_{i,2}} \cdots f_s^{z_{i,s}}$. Now let $(\nu_1, \dots, \nu_w, \lambda_1, \dots, \lambda_s) \in \mathbb{Z}^{w+s}$ with $1 = \alpha_1^{\nu_1}\, \alpha_2^{\nu_2} \cdots \alpha_w^{\nu_w}\, f_1^{\lambda_1} f_2^{\lambda_2} \cdots f_s^{\lambda_s}$. Since the $f_i$ are all irreducible and the $\alpha_i$ are from $K' \setminus \{0\}$, it follows that $\lambda_1 = \dots = \lambda_s = 0$. Note that $\alpha_1^{\nu_1}\, \alpha_2^{\nu_2} \cdots \alpha_w^{\nu_w} = 1$ holds in $K'$ if and only if it holds in $K'(\kappa_1, \dots, \kappa_u)$. Thus by $\mathbf{M}\left(\boldsymbol{\alpha}, K'\right) = \{\mathbf{0}_w\}$ we conclude that $\nu_1 = \dots = \nu_w = 0$. Consequently, $\mathbf{M}\left((\alpha_1, \dots, \alpha_w, f_1, \dots, f_s), K'(\kappa_1, \dots, \kappa_u)\right) = \{\mathbf{0}_{w+s}\}$.

Now suppose that the computational aspects hold. Since one can factorize polynomials in $K[\kappa_1, \dots, \kappa_u]$, the representation (24) is computable. In particular, the representation (25) is computable by Theorem 5.1. This completes the proof. ■

Note that again Remark 5.1 is in place where $K'(\kappa_1, \dots, \kappa_u)$ takes over the role of $K'$: using Theorem 5.2 in combination with Lemma 5.1 we can construct a $\Pi$-extension in which we can rephrase products defined over $\mathbb{K}$. Further, we remark that the package `NestedProducts` implements this machinery for the case that $K$ is an algebraic number field. Summarizing, we allow products that depend on extra parameters. This will be used for the multibasic case with $\mathbb{K} = K(q_1, \dots, q_e)$ for a field $K$ ($K$ might be again, e.g., a rational function field defined over an algebraic number field). We remark further that for the field $\mathbb{K} = \mathbb{Q}(\kappa_1, \dots, \kappa_u)$ this result can be accomplished without any field extension, i.e., $\mathbb{K}' = \mathbb{K}$; see [23].

*Example 5.2* (Cont. 5.1) Let $\mathbb{K}' = K'(\kappa)$ with $K' = \mathbb{Q}\left((-1)^{\frac{1}{6}}, \sqrt{13}\right)$ and consider

[4]It would suffice to require that the $f_i \in K[\kappa_1, \dots, \kappa_u] \setminus K$ are monic and pairwise co-prime. For practical reasons we require in addition that the $f_i$ are irreducible. For instance, suppose we have to deal with $(\kappa(\kappa+1))^n$. Then we could take $f_1 = \kappa(\kappa+1)$ and can adjoin the $\Pi$-monomial $\sigma(t) = f_1\, t$ to model the product. However, if in a later step also the unforeseen products $\kappa^n$ and $(\kappa+1)^n$ arise, one has to split $t$ into two monomials, say $t_1, t_2$, with $\sigma(t_1) = \kappa\, t_1$ and $\sigma(t_2) = (\kappa+1)\, t_2$. Requiring that the $f_i$ are irreducible avoids such undesirable redesigns of an already constructed R$\Pi$-extension.

$$\prod_{k=1}^{n} \underbrace{-13\sqrt{-13}\,\kappa}_{=:\gamma_1}, \quad \prod_{k=1}^{n} \underbrace{\frac{-784\,(\kappa+1)^2}{13\,\sqrt{-13}\,(\iota+\sqrt{3})^4\,\kappa}}_{=:\gamma_2}, \quad \prod_{k=1}^{n} \underbrace{\frac{-17210368\,(\kappa+1)^5}{13\,\sqrt{-13}\,(\iota+\sqrt{3})^{10}\,\kappa}}_{=:\gamma_3} \tag{26}$$

which are instances of the products from (14). By Example 5.1 the products in (21) can be modeled in the RΠ-extension $(K'[\vartheta][y_1, y_1^{-1}][y_2, y_2^{-1}], \sigma)$ of $(K', \sigma)$. Note that $\kappa, (\kappa+1) \in K[\kappa] \setminus K$ are both irreducible over $K$. Thus $\mathbf{M}\Big((\sqrt{13}, 7, \kappa, \kappa+1), \mathbb{K}'\Big) = \{\mathbf{0}_4\}$ holds. Consequently by Remark 5.1, $(\mathbb{K}'[\vartheta]\,\langle y_1\rangle\,\langle y_2\rangle\,\langle y_3\rangle\,\langle y_4\rangle, \sigma)$ is an RΠ-extension of $(\mathbb{K}', \sigma)$ with $\sigma(y_3) = \kappa\, y_3$ and $\sigma(y_4) = (\kappa+1)\, y_4$. Here the Π-monomials $y_3$ and $y_4$ model $\kappa^n$ and $(\kappa+1)^n$, respectively. In particular, for $i = 1, 2, 3$ we get $\sigma(a_i) = \gamma_i\, a_i$ with

$$a_1 = \vartheta^9\, y_1^3\, y_3, \qquad a_2 = \frac{\vartheta^{11}\, y_2^2\, y_4^2}{y_1^3\, y_3}, \qquad a_3 = \frac{\vartheta^5\, y_2^5\, y_4^5}{y_1^3\, y_3}. \tag{27}$$

In short, $a_1, a_2, a_3$ model the shift behaviors of the products in (26), respectively.

## 5.3 *Structural Results for Single Nested* Π*-extensions*

Finally, we focus on products where non-constant polynomials are involved. Similar to Theorem 5.2 we will use irreducible factors as main building blocks to define our Π-extensions. The crucial refinement is that these factors are also shift co-prime; compare also [23, 28]. Here the following two lemmas will be utilized.

**Lemma 5.3** *Let* $(\mathbb{F}(t), \sigma)$ *be a* ΠΣ*-extension of* $(\mathbb{F}, \sigma)$ *with* $\sigma(t) = \alpha\, t + \beta$ *(*$\alpha \in \mathbb{F}^*$ *and* $\beta = 0$ *or* $\alpha = 1$ *and* $\beta \in \mathbb{F}$*). Let* $\boldsymbol{f} = (f_1, \dots, f_s) \in (\mathbb{F}[t] \setminus \mathbb{F})^s$*. Suppose that*

$$\forall\, i, j\ (1 \leqslant i < j \leqslant s) :\ \gcd_{\sigma}(f_i, f_j) = 1 \tag{28}$$

*holds and that for i with* $1 \leqslant i \leqslant s$ *we have that*[5]

$$\tfrac{\sigma(f_i)}{f_i} \in \mathbb{F}\ \vee\ \forall\, k \in \mathbb{Z} \setminus \{0\} : \gcd(f_i, \sigma^k(f_i)) = 1. \tag{29}$$

*Then for all* $h \in \mathbb{F}^*$ *there does not exist* $(v_1, \dots, v_s) \in \mathbb{Z}^s \setminus \{\mathbf{0}_s\}$ *and* $g \in \mathbb{F}(t)^*$ *with*

$$\frac{\sigma(g)}{g} = f_1^{v_1} \cdots f_s^{v_s}\, h. \tag{30}$$

*In particular,* $\mathbf{M}\,(\boldsymbol{f}, \mathbb{F}(t)) = \{\mathbf{0}_s\}$.

*Proof* Suppose that (28) and (29) hold. Now let $h \in \mathbb{F}^*$ and assume that there are a $g \in \mathbb{F}(t)^*$ and $(v_1, \dots, v_s) \in \mathbb{Z}^s \setminus \{\mathbf{0}_s\}$ with (30). Suppose that $\beta = 0$ and $g = u\, t^m$

[5] We note that (29) could be also rephrased in terms of Abramov's dispersion [2, 7].

for some $m \in \mathbb{Z}$ and some $u \in \mathbb{F}^*$. Then $\frac{\sigma(g)}{g} \in \mathbb{F}$. Hence $v_i = 0$ for $1 \leqslant i \leqslant s$ since the $f_i$ are pairwise co-prime by (28), a contradiction. Thus we can take a monic irreducible factor, say $p \in \mathbb{F}[t] \setminus \mathbb{F}$ of $g$ where $p \neq t$ if $\beta = 0$. In addition, among all these possible factors we can choose one with the property that for $k > 0$, $\sigma^k(p)$ is not a factor in $g$. Note that this is possible by Lemma 4.2. Then $\sigma(p)$ does not cancel in $\frac{\sigma(g)}{g}$. Thus $\sigma(p) \mid f_i$ for some $i$ with $1 \leqslant i \leqslant s$. On the other hand, let $r \leqslant 0$ be minimal such that $\sigma^r(p)$ is the irreducible factor in $g$ with the property that $\sigma^r(p)$ does not occur in $\sigma(g)$. Note that this is again possible by Lemma 4.2. Then $\sigma^r(p)$ does not cancel in $\frac{\sigma(g)}{g}$. Therefore, $\sigma^r(p) \mid f_j$ for some $j$ with $1 \leqslant j \leqslant s$. Consequently, $\gcd_\sigma(f_i, f_j) \neq 1$. By (28) it follows that $i = j$. In particular by (29) it follows that $\sigma(f_i)/f_i \in \mathbb{F}$. By Lemma 4.1 this implies $f_i = w\, t^m$ with $m \in \mathbb{Z}, w \in \mathbb{F}^*$ and $\beta = 0$. In particular, $p = t$, which we have already excluded. In any case, we arrive at a contradiction and conclude that $v_1 = \cdots = v_e = 0$. ■

Note that condition (28) implies that the $f_i$ are pairwise shift-coprime. In addition condition (29) implies that two different irreducible factors in $f_i$ are shift-coprime. The next lemma considers the other direction.

**Lemma 5.4** *Let* $(\mathbb{F}(t), \sigma)$ *be a difference field extension of* $(\mathbb{F}, \sigma)$ *with $t$ transcendental over* $\mathbb{F}$ *and* $\sigma(t) = \alpha\, t + \beta$ *where* $\alpha \in \mathbb{F}^*$ *and* $\beta \in \mathbb{F}$. *Let* $\boldsymbol{f} = (f_1, \ldots, f_s) \in (\mathbb{F}[t] \setminus \mathbb{F})^s$ *be irreducible monic polynomials. If there are no* $(v_1, \ldots, v_s) \in \mathbb{Z}^s \setminus \{\mathbf{0}_s\}$ *and* $g \in \mathbb{F}(t)^*$ *with*

$$\frac{\sigma(g)}{g} = f_1^{v_1} \cdots f_s^{v_s}, \tag{31}$$

*i.e., if* $\mathbf{M}(\boldsymbol{f}, \mathbb{F}(t)) = \{\mathbf{0}_s\}$, *then* (28) *holds.*

*Proof* Suppose there are $i, j$ with $1 \leqslant i < j \leqslant s$ and $\gcd_\sigma(f_i, f_j) \neq 1$. Since $f_i, f_j$ are irreducible, $f \sim g$. Thus by Lemma 4.4 there is a $g \in \mathbb{F}(t)^*$ with $f_i = \frac{\sigma(g)}{g} f_j$. Hence $\frac{\sigma(g)}{g} = f_i\, f_j^{-1}$ and thus we can find a $(v_1, \ldots, v_s) \in \mathbb{Z}^s \setminus \{\mathbf{0}_s\}$ with (30). ■

Summarizing, we arrive at the following result.

**Theorem 5.3** *Let* $(\mathbb{F}(t), \sigma)$ *be a* $\Pi\Sigma$*-extension of* $(\mathbb{F}, \sigma)$. *Let* $\boldsymbol{f} = (f_1, \ldots, f_s) \in (\mathbb{F}[t] \setminus \mathbb{F})^s$ *be irreducible monic polynomials. Then the following statements are equivalent.*

1. $\forall i, j : 1 \leqslant i < j \leqslant s,\ \gcd_\sigma(f_i, f_j) = 1.$
2. *There does not exist* $(v_1, \ldots, v_s) \in \mathbb{Z}^s \setminus \{\mathbf{0}_s\}$ *and* $g \in \mathbb{F}(t)^*$ *with* $\frac{\sigma(g)}{g} = f_1^{v_1} \cdots f_s^{v_s}$, *i.e.,* $\mathbf{M}(\boldsymbol{f}, \mathbb{F}(t)) = \{\mathbf{0}_s\}$.
3. *One can construct a* $\Pi$*-field extension* $(\mathbb{F}(t)(z_1)\cdots(z_s), \sigma)$ *of* $(\mathbb{F}(t), \sigma)$ *with* $\sigma(z_i) = f_i\, z_i$, *for* $1 \leqslant i \leqslant s$.
4. *One can construct a* $\Pi$*-extension* $(\mathbb{F}(t)[z_1, z_1^{-1}]\cdots[z_s, z_s^{-1}], \sigma)$ *of* $(\mathbb{F}(t), \sigma)$ *with* $\sigma(z_i) = f_i\, z_i$, *for* $1 \leqslant i \leqslant s$.

*Proof* Since the $f_i$ are irreducible, the condition (29) always holds. Therefore (1) $\Longrightarrow$ (2) follows from Lemma 5.3. Further, (2) $\Longrightarrow$ (1) follows from Lemma 5.4. The equivalences between (2), (3) and (4) follow by Lemma 5.1. ■

## *5.4 Construction of* **RΠ***-extensions for* **ProdE**$(\mathbb{K}(n))$

Finally, we combine Theorems 5.2 and 5.3 to obtain a Π-extension in which expressions from ProdE($\mathbb{K}(n)$) can be rephrased in general. In order to accomplish this task, we will show in Lemma 5.6 that the Π-monomials of the two constructions in the Sects. 5.2 and 5.3 can be merged to one RΠ-extension. Before we arrive at this result some preparation steps are needed.

**Lemma 5.5** *Let* $(\mathbb{F}(t), \sigma)$ *be a* Σ*-extension of* $(\mathbb{F}, \sigma)$ *with* $\sigma(t) = t + \beta$ *and let* $(\mathbb{E}, \sigma)$ *be a* Π*-extension of* $(\mathbb{F}, \sigma)$*. Then one can construct a* Σ*-extension* $(\mathbb{E}(t), \sigma)$ *of* $(\mathbb{E}, \sigma)$ *with* $\sigma(t) = t + \beta$*.*

*Proof* Let $(\mathbb{E}, \sigma)$ be a Π-extension of $(\mathbb{F}, \sigma)$ with $\mathbb{E} = \mathbb{F}(t_1)\cdots(t_e)$ and suppose that there is a $g \in \mathbb{E}$ with $\sigma(g) = g + \beta$. Let $i$ be minimal such that $g \in \mathbb{F}(t_1)\cdots(t_i)$. Since $\mathbb{F}(t)$ is a Σ-extension of $\mathbb{F}$, it follows by part (3) of Theorem 2.1 that there is no $g \in \mathbb{F}$ with $\sigma(g) = g + \beta$. Then [13, Lemma 4.1] implies that $g$ cannot depend on $t_i$, a contradiction. Thus there is no $g \in \mathbb{E}$ with $\sigma(g) = g + \beta$ and by part (3) of Theorem 2.1 we get the Σ-extension $(\mathbb{E}(t), \sigma)$ of $(\mathbb{E}, \sigma)$ with $\sigma(t) = t + \beta$. ■

As a by-product of the above lemma it follows that the mixed $\boldsymbol{q}$-multibasic difference field is built by Π-monomials and one Σ-monomial.

**Corollary 5.1** *The mixed* $\boldsymbol{q}$*-multibasic diff. ring* $(\mathbb{F}, \sigma)$ *with* $\mathbb{F} = \mathbb{K}(x)(t_1)\cdots(t_e)$ *from Example 2.2 is a* ΠΣ*-extension of* $(\mathbb{K}, \sigma)$*. In particular,* $\text{const}(\mathbb{F}, \sigma) = \mathbb{K}$.

*Proof* Since the elements $q_1, \ldots, q_e$ are algebraically independent among each other, there are no $g \in \mathbb{K}^*$ and $(v_1, \ldots, v_e) \in \mathbb{Z}^e \setminus \{\mathbf{0}_e\}$ with $1 = \frac{\sigma(g)}{g} = q_1^{v_1} \cdots q_e^{v_e}$. Therefore by Lemma 5.1, $(\mathbb{E}, \sigma)$ with $\mathbb{E} = \mathbb{K}(t_1)\cdots(t_e)$ is a Π-extension of $(\mathbb{K}, \sigma)$ with $\sigma(t_i) = q_i\, t_i$ for $1 \leqslant i \leqslant e$. Since $(\mathbb{K}(x), \sigma)$ is a Σ-extension of $(\mathbb{K}, \sigma)$, we can activate Lemma 5.5 and can construct the Σ-extension $(\mathbb{E}(x), \sigma)$ of $(\mathbb{E}, \sigma)$. Note that $\text{const}(\mathbb{E}(x), \sigma) = \text{const}(\mathbb{E}, \sigma)$ also implies that $\text{const}(\mathbb{K}(x)(t_1)\cdots(t_e), \sigma) = \text{const}(\mathbb{K}(x), \sigma) = \mathbb{K}$. In particular, the P-extension $(\mathbb{K}(x)(t_1)\cdots(t_e), \sigma)$ of $(\mathbb{K}(x), \sigma)$ is a Π-extension. Consequently, $(\mathbb{F}, \sigma)$ is a ΠΣ-extension of $(\mathbb{K}, \sigma)$. ■

**Proposition 5.2** *Let* $(\mathbb{F}(t_1)\cdots(t_e), \sigma)$ *be a* ΠΣ*-extension of* $(\mathbb{F}, \sigma)$ *with* $\sigma(t_i) = \alpha_i\, t_i + \beta_i$ *where* $\beta_i \neq 0$ *or* $\alpha_i = 1$*. Let* $f \in \mathbb{F}^*$*. If there is a* $g \in \mathbb{F}(t_1, \ldots, t_e)^*$ *with* $\frac{\sigma(g)}{g} = f$*, then* $g = \omega\, t_1^{v_1} \cdots t_e^{v_e}$ *where* $\omega \in \mathbb{F}^*$*. In particular,* $v_i = 0$*, if* $\beta_i \neq 0$ *(i.e.,* $t_i$ *is a* Σ*-monomial) or* $v_i \in \mathbb{Z}$*, if* $\beta_i = 0$ *(i.e.,* $t_i$ *is a* Π*-monomial).*

*Proof* See [22, Corollary 2.2.6, p. 76]. ■

**Lemma 5.6** *Let* $(\mathbb{K}(x), \sigma)$ *be the rational difference field with* $\sigma(x) = x + 1$ *and let* $(\mathbb{K}(x)[z_1, z_1^{-1}]\cdots[z_s, z_s^{-1}], \sigma)$ *be a* Π*-extension of* $(\mathbb{K}(x), \sigma)$ *as given in Theorem 5.3 (item (4)). Further, let* $\mathbb{K}'$ *be an algebraic field extension of* $\mathbb{K}$ *and let* $(\mathbb{K}'[y_1, y_1^{-1}]\cdots[y_w, y_w^{-1}], \sigma)$ *be a* Π*-extension of* $(\mathbb{K}', \sigma)$ *with* $\frac{\sigma(y_i)}{y_i} \in \mathbb{K}' \setminus \{0\}$*. Then the difference ring* $(\mathbb{E}, \sigma)$ *with* $\mathbb{E} = \mathbb{K}'(x)[y_1, y_1^{-1}]\cdots[y_w, y_w^{-1}][z_1, z_1^{-1}]\cdots[z_s, z_s^{-1}]$ *is a* Π*-extension of* $(\mathbb{K}'(x), \sigma)$*. Furthermore, the* A*-extension* $(\mathbb{E}[\vartheta], \sigma)$ *of* $(\mathbb{E}, \sigma)$ *with* $\sigma(\vartheta) = \zeta\, \vartheta$ *of order* $\lambda$ *is an R-extension.*

*Proof* By iterative application of [32, Corollary 2.6] it follows that $(\mathbb{F}, \sigma)$ is a $\Pi$-field extension of $(\mathbb{K}', \sigma)$ with $\mathbb{F} = \mathbb{K}'(y_1)\cdots(y_w)$. Note that $(\mathbb{K}'(x), \sigma)$ is a $\Sigma$-extension of $(\mathbb{K}', \sigma)$. Thus by Lemma 5.5 $(\mathbb{F}(x), \sigma)$ is a $\Sigma$-extension of $(\mathbb{F}, \sigma)$. We will show that $(\mathbb{E}, \sigma)$ with $\mathbb{E} = \mathbb{F}(x)(z_1)\cdots(z_s)$ forms a $\Pi$-extension of $(\mathbb{F}(x), \sigma)$. Since $(\mathbb{K}(x)[z_1, z_1^{-1}]\cdots[z_s, z_s^{-1}], \sigma)$ is a $\Pi$-extension of $(\mathbb{K}(x), \sigma)$ as given in Theorem 5.3 (item (4)), we conclude that also (item 2) of the theorem, i.e., condition (28) holds. Now suppose that there is a $g \in \mathbb{F}(x)^*$ and $(l_1, \ldots, l_s) \in \mathbb{Z}^s$ with $\frac{\sigma(g)}{g} = f_1^{l_1}\cdots f_s^{l_s} \in \mathbb{K}(x)$. By reordering of the generators in $(\mathbb{F}(x), \sigma)$ we get the $\Pi$-extension $(\mathbb{K}'(x)(y_1)\cdots(y_w), \sigma)$ of $(\mathbb{K}'(x), \sigma)$. By Proposition 5.2 we conclude that $g = q\, y_1^{n_1}\cdots y_w^{n_w}$ with $n_1, \ldots, n_w \in \mathbb{Z}$ and $q \in \mathbb{K}'(x)^*$. Thus $\frac{\sigma(g)}{g} = \frac{\sigma(q)}{q}\alpha_1^{n_1}\cdots\alpha_w^{n_w}$ and hence

$$\frac{\sigma(q)}{q} = u\, f_1^{l_1}\cdots f_s^{l_s} \tag{32}$$

for some $u \in \mathbb{K}'^*$. Now suppose that $f_i, f_j \in \mathbb{K}[x] \subset \mathbb{F}[x]$ with $i \neq j$ are not shift-coprime in $\mathbb{F}[x]$. Then there are a $k \in \mathbb{Z}$ and $v, \tilde{f}_i, \tilde{f}_j \in \mathbb{F}[x] \setminus \mathbb{F}$ with $\sigma^k(f_j) = v\,\tilde{f}_j$ and $f_i = v\,\tilde{f}_i$. But this implies that $f_i\,\frac{\tilde{f}_j}{\tilde{f}_i} = \sigma^k(f_j) \in \mathbb{K}[x]$. Since $f_i, \sigma(f_j) \in \mathbb{K}[x]$, this implies that $\frac{\tilde{f}_j}{\tilde{f}_i} \in \mathbb{K}(x)$. Since $f_i, \sigma(f_j)$ are both irreducible in $\mathbb{K}[x]$ we conclude that $\frac{\tilde{f}_j}{\tilde{f}_i} \in \mathbb{K}$. Consequently, $f_i$ and $f_j$ are also not shift-coprime in $\mathbb{K}[x]$, a contradiction. Thus the condition (28) holds not only in $\mathbb{K}[x]$ but also in $\mathbb{F}[x]$. Now suppose that $\gcd(f_i, \sigma^k(f_i)) \neq 1$ holds in $\mathbb{F}[x]$ for some $k \in \mathbb{Z} \setminus \{0\}$. By the same arguments as above, it follows that $\sigma^k(f_i) = u\, f_i$ for some $u \in \mathbb{K}$. By Lemma 4.1 we conclude that $f_i = t$ and $\sigma(t)/t \in \mathbb{F}$. Therefore also condition (29) holds. Consequently, we can activate Lemma 5.3 and it follows from (32) that $l_1 = \cdots = l_m = 0$. Consequently, we can apply Theorem 5.3 (equivalence (2) and (3)) and conclude that $(\mathbb{E}, \sigma)$ is a $\Pi$-extension of $(\mathbb{F}(x), \sigma)$. Finally, consider the A-extension $(\mathbb{E}[\vartheta], \sigma)$ of $(\mathbb{E}, \sigma)$ with $\sigma(\vartheta) = \zeta\,\vartheta$ of order $\lambda$. By Lemma 2.1 it is an R-extension. Finally, consider the sub-difference ring $(\mathbb{E}, \sigma)$ with $\mathbb{E} = \mathbb{K}'(x)[y_1, y_1^{-1}]\cdots[y_w, y_w^{-1}][z_1, z_1^{-1}]\cdots[z_s, z_s^{-1}][\vartheta]$ which is an AP-extension of $(\mathbb{K}'(x), \sigma)$. Since $\text{const}(\mathbb{E}, \sigma) = \text{const}(\mathbb{K}'(x), \sigma) = \mathbb{K}'$, it is an R$\Pi$-extension. ■

*Remark 5.2* Take $(\mathbb{E}, \sigma)$ with $\mathbb{E} = \mathbb{K}'(x)[y_1, y_1^{-1}]\cdots[y_w, y_w^{-1}][z_1, z_1^{-1}]\cdots[z_s, z_s^{-1}][\vartheta]$ as constructed in Lemma 5.6. Then one can rearrange the generators in $\mathbb{E}$ and gets the R$\Pi$-extension $(\mathbb{K}'(x)[\vartheta][y_1, y_1^{-1}]\cdots[y_w, y_w^{-1}][z_1, z_1^{-1}]\cdots[z_s, z_s^{-1}], \sigma)$ of $(\mathbb{K}'(x), \sigma)$.

With these considerations we can derive the following theorem that enables one to construct R$\Pi$-extension for $\text{ProdE}(\mathbb{K}(n))$.

**Theorem 5.4** *Let $(\mathbb{K}(x), \sigma)$ be the rational difference field with $\sigma(x) = x + 1$ where $\mathbb{K} = K(\kappa_1, \ldots, \kappa_u)$ is a rational function field over a field $K$. Let $h_1, \ldots, h_m \in \mathbb{K}(x)^*$. Then one can construct an* R$\Pi$*-extension $(\mathbb{A}, \sigma)$ of $(\mathbb{K}'(x), \sigma)$ with*

$$\mathbb{A} = \mathbb{K}'(x)[\vartheta][y_1, y_1^{-1}]\cdots[y_w, y_w^{-1}][z_1, z_1^{-1}]\cdots[z_s, z_s^{-1}]$$

*and* $\mathbb{K}' = K'(\kappa_1, \dots, \kappa_u)$ *where* $K'$ *is an algebraic field extension of* $K$ *such that*

- $\sigma(\vartheta) = \zeta\,\vartheta$ *where* $\zeta \in K'$ *is a* $\lambda$*-th root of unity;*
- $\dfrac{\sigma(y_j)}{y_j} = \alpha_j \in \mathbb{K}' \setminus \{0\}$ *for* $1 \leqslant j \leqslant w$ *where the* $\alpha_j$ *are not roots of unity;*
- $\dfrac{\sigma(z_\nu)}{z_\nu} = f_\nu \in \mathbb{K}[x] \setminus \mathbb{K}$ *are irreducible and shift co-prime for* $1 \leqslant \nu \leqslant s$*;*

*holds with the following property. For* $1 \leqslant i \leqslant m$ *one can define*

$$g_i = r_i\,\vartheta^{\mu_i}\,y_1^{u_{i,1}} \cdots y_w^{u_{i,w}}\,z_1^{v_{i,1}} \cdots z_s^{v_{i,s}} \in \mathbb{A} \tag{33}$$

*with* $0 \leqslant \mu_i \leqslant \lambda - 1$, $u_{i,1}, \dots, u_{i,w}, v_{i,1}, \dots, v_{i,s} \in \mathbb{Z}$ *and* $r_i \in \mathbb{K}(x)^*$ *such that*

$$\sigma(g_i) = \sigma(h_i)\,g_i. \tag{34}$$

*If* $K$ *is strongly* $\sigma$*-computable, the components of the theorem can be computed.*

*Proof* For $1 \leqslant i \leqslant m$ we can take pairwise different monic irreducible polynomials[6] $p_1, \dots, p_n \in \mathbb{K}[x] \setminus \mathbb{K}$ $\gamma_1, \dots, \gamma_m \in \mathbb{K}^*$ and $d_{i,1}, \dots, d_{i,n} \in \mathbb{Z}$ such that $\sigma(h_i) = \gamma_i\,p_1^{d_{i,1}} \cdots p_n^{d_{i,n}}$ holds. Note that this representation is computable if $K$ is strongly $\sigma$-computable. By Theorem 5.2 it follows that there are a $\lambda$-th root of unity $\zeta \in K'$, elements $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_w) \in (\mathbb{K}'^*)^w$ with $\mathbf{M}\left(\boldsymbol{\alpha}, \mathbb{K}'\right) = \{\mathbf{0}_w\}$ and integer vectors $(u_{i,1}, \dots, u_{i,w}) \in \mathbb{Z}^w$ and $\mu_i \in \mathbb{N}$ with $0 \leqslant \mu_i < \lambda$ such that $\gamma_i = \zeta^{\mu_i}\,\alpha_1^{u_{i,1}} \cdots \alpha_w^{u_{i,w}}$ holds for all $1 \leqslant i \leqslant m$. Obviously, the $\alpha_j$ with $1 \leqslant j \leqslant w$ are not roots of unity. By Lemma 5.1 we get the $\Pi$-extension $(\mathbb{K}'[y_1, y_1^{-1}] \cdots [y_w, y_w^{-1}], \sigma)$ of $(\mathbb{K}', \sigma)$ with $\sigma(y_j) = \alpha_j\,y_j$ for $1 \leqslant j \leqslant w$ and we obtain

$$a_i = \vartheta^{\mu_i}\,y_1^{u_{i,1}} \cdots y_w^{u_{i,w}} \tag{35}$$

with

$$\sigma(a_i) = \gamma_i\,a_i \tag{36}$$

for $1 \leqslant i \leqslant m$. Next we proceed with the non-constant polynomials in $\mathbb{K}[x] \setminus \mathbb{K}$. Set $\mathscr{I} = \{p_1, \dots, p_n\}$. Then there is a partition $\mathscr{P} = \{\mathscr{E}_1, \dots, \mathscr{E}_s\}$ of $\mathscr{I}$ with respect to $\sim_\sigma$, i.e., each $\mathscr{E}_i$ contains precisely the shift equivalent elements of $\mathscr{P}$. Take a representative from each equivalence class $\mathscr{E}_i$ in $\mathscr{P}$ and collect them in $\mathscr{F} := \{f_1, \dots, f_s\}$. Since each $f_i$ is shift equivalent with every element of $\mathscr{E}_i$, it follows by Lemma 4.4 that for all $h \in \mathscr{E}_i$, there is a rational function $r \in \mathbb{K}(x)^*$ with $h = \frac{\sigma(r)}{r}\,f_i$ for $1 \leqslant i \leqslant s$. Consequently, we get $r_i \in \mathbb{K}(x)^*$ and $v_{i,j} \in \mathbb{Z}$ with $p_1^{d_{i,1}} \cdots p_n^{d_{i,n}} = \frac{\sigma(r_i)}{r_i}\,f_1^{v_{i,1}} \cdots f_s^{v_{i,s}}$

[6] Instead of irreducibility it would suffice to require that the $p_i \in \mathbb{K}[x] \setminus \mathbb{K}$ satisfy property (29). However, suppose that one takes, e.g., $p_1 = x(2\,x + 1)$ leading to the $\Pi$-monomial $t$ with $\sigma(t) = x\,(2x + 1)$. Further, assume that later one has to introduce unexpectedly also $x$ and $2\,x + 1$. Then one has to split $t$ to the $\Pi$-monomials $t_1, t_2$ with $\sigma(t_1) = x\,t_1$ and $\sigma(t_2) = (2x + 1)\,t_2$, i.e., one has to redesign the already constructed R$\Pi$-extension. In short, irreducible polynomials provide an R$\Pi\Sigma$-extension which most probably need not be redesigned if other products have to be considered.

for all $1 \leqslant i \leqslant s$. Further, by this construction, we know that $\gcd_\sigma(f_i, f_j) = 1$ for $1 \leqslant i < j \leqslant s$. Therefore, it follows by Theorem 5.3 that we can construct the $\Pi$-extension $(\mathbb{K}(x)[z_1, z_1^{-1}] \cdots [z_s, z_s^{-1}], \sigma)$ of $(\mathbb{K}(x), \sigma)$ with $\sigma(z_i) = f_i\, z_i$. Now define $b_i = r_i\, t_1^{v_{i,1}} \cdots t_s^{v_{i,s}}$. Then we get

$$\sigma(b_i) = p_1^{d_{i,1}} \cdots p_n^{d_{i,n}}\, b_i. \tag{37}$$

Finally, by Lemma 5.6 and Remark 5.2 we end up at the R$\Pi$-extension $(\mathbb{A}, \sigma)$ of $(\mathbb{K}'(x), \sigma)$ with $\mathbb{A} = \mathbb{K}'(x)[\vartheta][y_1, y_1^{-1}] \cdots [y_w, y_w^{-1}][z_1, z_1^{-1}] \cdots [z_e, z_e^{-1}]$ with $\sigma(\vartheta) = \zeta\, \vartheta$, $\sigma(y_j) = \alpha_j\, y_j$ for $1 \leqslant j \leqslant w$ and $\sigma(z_i) = f_i\, z_i$ for $1 \leqslant i \leqslant s$.

Now let $g_i$ be as defined in (33). Since $g_i = a_i\, b_i$ with (36) and (37), we conclude that (34) holds. If $K$ is strongly $\sigma$-computable, all the ingredients delivered by Theorems 5.1 and 5.3 can be computed. This completes the proof. ■

*Example 5.3* Let $\mathbb{K} = K(\kappa)$ be the rational function field over the algebraic number field $K = \mathbb{Q}(\iota + \sqrt{3}, \sqrt{-13})$ and take the rational difference field $(\mathbb{K}(x), \sigma)$ with $\sigma(x) = x + 1$. Given (15), we can write

$$\sigma(h_1) = \gamma_1\, p_1^{-1}, \qquad \sigma(h_2) = \gamma_2\, p_1\, p_2^{-2}, \qquad \sigma(h_3) = \gamma_3\, p_1\, p_2^{-5}$$

where the $\gamma_1, \gamma_2, \gamma_3$ are given in (26) and where we set $p_1 = x + 1$, $p_2 = x + 3$ as our monic irreducible polynomials. Note that $p_1$ and $p_2$ are shift equivalent: $\gcd(p_2, \sigma^2(p_1)) = p_2$. Consequently both factors fall into the same equivalence class $\mathscr{E} = \{\sigma^k(x+1) \mid k \in \mathbb{Z}\} = \{\sigma^k(x+3) \mid k \in \mathbb{Z}\}$. Take $p_1 = x + 1$ as a representative of the equivalence class $\mathscr{E}$. Then by Lemma 4.4, it follows that there is a $g \in \mathbb{K}(x)^*$ that connects the representatives to all other elements in their respective equivalence classes. In particular with our example we have $x + 3 = \frac{\sigma(g)}{g}\,(x+1)$ where $g = (x + 1)\,(x + 2)$. By Theorem 5.3, it follows that $(\mathbb{K}(x)[z, z^{-1}], \sigma)$ is a $\Pi$-extension of the difference field $(\mathbb{K}(x), \sigma)$ with $\sigma(z) = (x + 1)\, z$. In this ring, the $\Pi$-monomial $z$ models $n!$. By Lemma 5.6 the constructed difference rings $(\mathbb{K}'[\vartheta]\,\langle y_1\rangle\,\langle y_2\rangle\,\langle y_3\rangle\,\langle y_4\rangle, \sigma)$ and $(\mathbb{K}(x)\,\langle z\rangle, \sigma)$ from Example 5.2 with $\mathbb{K}' = \mathbb{Q}\big((-1)^{\frac{1}{6}}, \sqrt{13}\big)(\kappa)$ can be merged into a single R$\Pi$-extension $(\mathbb{A}, \sigma)$ where $\mathbb{A}$ is (16) with the automorphism defined accordingly. Further note that for $b_1 = \frac{1}{z}$, $b_2 = \frac{1}{(x+1)^2(x+2)^2\, z}$, $b_3 = \frac{1}{(x+1)^5(x+2)^5 z^4}$ we have that $\sigma(b_1) = p_1^{-1}\, b_1$, $\sigma(b_2) = p_1\, p_2^{-2}\, b_2$ and $\sigma(b_3) = p_1\, p_2^{-5}\, b_3$. Thus together $a_1, a_2, a_3$ in (27) with $\sigma(\gamma_i) = a_i\, \gamma_i$ for $i = 1, 2, 3$, we define $g_i = a_i\, b_i$ for $i = 1, 2, 3$ and obtain $\sigma(g_i) = \sigma(h_i)\, g_i$. Note that the $g_i$ are precisely the elements given in (17).

Now we are ready to prove Theorem 3.1 for the special case ProdE$(\mathbb{K}(n)$. Namely, consider the products

$$P_1(n) = \prod_{k=\ell_1}^{n} h_1(k), \ldots, P_m(n) = \prod_{k=\ell_1}^{n} h_m(k) \ \in \mathrm{Prod}(\mathbb{K}(n))$$

with $\ell_i \in \mathbb{Z}$ where $\ell_i \geqslant Z(h_i)$. Further, suppose that we are given the components as claimed in Theorem 5.4.

• Now take the difference ring embedding $\tau(\frac{a}{b}) = \langle \text{ev}\left(\frac{a}{b}, n\right)\rangle_{n\geqslant 0}$ for $a, b \in \mathbb{K}[x]$ where ev is defined in (7). Then by iterative application of part (2) of Lemma 2.2 we can construct the $\mathbb{K}'$-homomorphism $\tau : \mathbb{A} \to \mathscr{S}(\mathbb{K}')$ determined by the homomorphic continuation of

- $\tau(\vartheta) = \langle \zeta^n \rangle_{n\geqslant 0}$,
- $\tau(y_i) = \langle \alpha_i^n \rangle_{n\geqslant 0}$ for $1 \leqslant i \leqslant w$ and
- $\tau(z_i) = \langle \prod_{k=\ell_i'}^{n} f_i(k-1)\rangle_{n\geqslant 0}$ with $\ell_i' = Z(f_i) + 1$ for $1 \leqslant i \leqslant s$.

In particular, since $(\mathbb{A}, \sigma)$ is an RΠ-extension of $(\mathbb{K}'(x), \sigma)$, it follows by part (3) of Lemma 2.2 that $\tau$ is a $\mathbb{K}'$-embedding.

• Finally, define for $1 \leqslant i \leqslant m$ the product expression

$$G_i(n) = r_i(n)\,(\zeta^n)^{\mu_i}\,(\alpha_1^n)^{u_{i,1}} \cdots (\alpha_w^n)^{u_{i,w}}\,(\prod_{k=\ell_1'}^{n} f_1(k-1))^{v_{i,1}} \cdots (\prod_{k=\ell_s'}^{n} f_s(k-1))^{v_{i,s}}$$

from $\text{Prod}(\mathbb{K}'(n))$ and define $\delta_i = \max(\ell_i, \ell_1', \ldots, \ell_s', Z(r_i))$. Observe that $\tau(g_i) = \langle G_i'(n)\rangle_{n\geqslant 0}$ with

$$G_i'(n) = \begin{cases} 0 & \text{if } 0 \leqslant n < \delta_i \\ G_i(n) & \text{if } n \geqslant \delta_i. \end{cases} \tag{38}$$

By (34) and the fact that $\tau$ is a $\mathbb{K}'$-embedding, it follows that $S(\tau(g_i)) = S(\tau(h_i))\,\tau(g_i)$. In particular, for $n \geqslant \delta_i$ we have that $G_i(n+1) = h_i(n+1)\,G_i(n)$. By definition, we have $P_i(n+1) = h_i(n+1)\,P_i(n)$ for $n \geqslant \delta_i \geqslant \ell_i$. Since $G_i(n)$ and $P_i(n)$ satisfy the same first order recurrence relation, they differ only by a multiplicative constant. Namely, setting $Q_i(n) = c\,G_i(n)$ with $c = \frac{P_i(\delta_i)}{G_i(\delta_i)} \in (\mathbb{K}')^*$ we have that $P_i(\delta_i) = Q_i(\delta_i)$ and thus $P_i(n) = Q_i(n)$ for all $n \geqslant \delta_i$. This proves part (1) of Theorem 3.1.

Since $\tau$ is a $\mathbb{K}'$-embedding, the sequences

$$\langle \alpha_1^n \rangle_{n\geqslant 0}, \ldots, \langle \alpha_w^n \rangle_{n\geqslant 0}, \langle \prod_{k=\ell_1'}^{n} f_1(k-1)\rangle_{n\geqslant 0}, \ldots, \langle \prod_{k=\ell_s'}^{n} f_s(k-1)\rangle_{n\geqslant 0}$$

are among each other algebraically independent over $\tau\big(\mathbb{K}'(x)\big)\big[\langle \zeta^n \rangle_{n\geqslant 0}\big]$ which proves property (2) of Theorem 3.1.

*Example 5.4* (Cont. Example 5.3) We have $\sigma(g_i) = \sigma(h_i)\,g_i$ for $i = 1, 2, 3$ where the $h_i$ and $g_i$ are given in (15) and (17), respectively. For the $\mathbb{K}'$-embedding defined in Example 3.1 we obtain $c_i\,\tau(g_i) = \langle P_i(n)\rangle_{n\geqslant 0}$ with $P_i(n) = \prod_{k=1}^{n} h_i(k)$ and $c_1 = 1$, $c_2 = 4$ and $c_3 = 32$. Since there are no poles in the $g_i$ we conclude that for

$$G_1(n) = \frac{\Big(\big((-1)^{\frac{1}{6}}\big)^n\Big)^9 \Big(\big(\sqrt{13}\big)^n\Big)^3 \kappa^n}{n!}, \qquad G_2(n) = \frac{4\Big(\big((-1)^{\frac{1}{6}}\big)^n\Big)^{11} \big(7^n\big)^2 \Big(\big(\kappa+1\big)^n\Big)^2}{(n+1)^2\,(n+2)^2 \Big(\big(\sqrt{13}\big)^n\Big)^3 \kappa^n\, n!},$$

$$G_3(n) = \frac{32\Big(\big((-1)^{\frac{1}{6}}\big)^n\Big)^5 \big(7^n\big)^5 \Big(\big(\kappa+1\big)^n\Big)^5}{(n+1)^5\,(n+2)^5 \Big(\big(\sqrt{13}\big)^n\Big)^3 \kappa^n \big(n!\big)^4}$$

we have $P_i(n) = G_i(n)$ for $n \geqslant 1$. With $P(n) = P_1(n) + P_2(n) + P_3(n)$ (see (14)) and $Q(n) = G_1(n) + G_2(n) + G_3(n)$ (see (18)) we get $P(n) = Q(n)$ for $n \geqslant 1$.

## 6 Construction of RΠ-extensions for ProdE($\mathbb{K}(n, q^n)$)

In this section we extend the results of Theorem 5.4 to the case ProdE($\mathbb{K}(n, q^n)$). As a consequence, we will also prove Theorem 3.1.

### *6.1 Structural Results for Nested Π-extensions*

In the following let $(\mathbb{F}_e, \sigma)$ be a $\Pi\Sigma$-extension of $(\mathbb{F}_0, \sigma)$ with $\mathbb{F}_e = \mathbb{F}_0(\mathsf{t}_1)\cdots(\mathsf{t}_e)$ with $\sigma(\mathsf{t}_i) = \alpha_i\,\mathsf{t}_i + \beta_i$ and $\alpha_i \in \mathbb{F}_0^*$, $\beta_i \in \mathbb{F}_0$ for $1 \leqslant i \leqslant e$. We set $\mathbb{F}_i = \mathbb{F}_0(\mathsf{t}_1)\cdots(\mathsf{t}_i)$ and thus $(\mathbb{F}_{i-1}(\mathsf{t}_i), \sigma)$ is a $\Pi\Sigma$-extension of $(\mathbb{F}_{i-1}, \sigma)$ for $1 \leqslant i \leqslant e$.

We will use the following notations. For $\boldsymbol{f} = (f_1, \ldots, f_s)$ and $h$ we write $\boldsymbol{f} \wedge h = (f_1, \ldots, f_s, h)$ for the concatenation of $\boldsymbol{f}$ and $h$. Moreover, the concatenation of $\boldsymbol{f}$ and $\boldsymbol{h} = (h_1, \ldots, h_u)$ is denoted by $\boldsymbol{f} \wedge \boldsymbol{h} = (f_1, \ldots, f_s, h_1, \ldots, h_u)$.

**Lemma 6.1** *Let $(\mathbb{F}_e, \sigma)$ be a $\Pi\Sigma$-extension of $(\mathbb{F}_0, \sigma)$ as above. If the polynomials in $\boldsymbol{f_i} \in (\mathbb{F}_{i-1}[\mathsf{t}_i] \setminus \mathbb{F}_{i-1})^{s_i}$ for $1 \leqslant i \leqslant e$ and $s_i \in \mathbb{N} \setminus \{0\}$ are irreducible and shift co-prime, then $\mathbf{M}(\boldsymbol{f_1} \wedge \cdots \wedge \boldsymbol{f_e}, \mathbb{F}_e) = \{\mathbf{0}_s\}$ where $s = s_1 + \cdots + s_e$.*

*Proof* Let $\boldsymbol{v_1} \in \mathbb{Z}^{s_1}, \ldots, \boldsymbol{v_e} \in \mathbb{Z}^{s_e}$ and $g \in \mathbb{F}_e^*$ with

$$\frac{\sigma(g)}{g} = \boldsymbol{f_1^{v_1}}\, \boldsymbol{f_2^{v_2}} \cdots \boldsymbol{f_e^{v_e}}. \tag{39}$$

Suppose that not all $\boldsymbol{v_i}$ with $1 \leqslant i \leqslant e$ are zero-vectors and let $r$ be maximal such that $\boldsymbol{v_r} \neq \mathbf{0}_{s_r}$. Thus the right hand side of (39) is in $\mathbb{F}_r$ and it follows by Proposition 5.2 that $g = \gamma\, t_{r+1}^{u_{r+1}} \cdots t_e^{u_e}$ with $\gamma \in \mathbb{F}_r^*$ and $u_i \in \mathbb{Z}$; if $\mathsf{t}_i$ is a $\Sigma$-monomial, then $u_i = 0$. Hence

$$\frac{\sigma(\gamma)}{\gamma} = \alpha_{r+1}^{-u_{r+1}} \cdots \alpha_e^{-u_e}\, \boldsymbol{f_1^{v_1}} \cdots \boldsymbol{f_{r-1}^{v_{r-1}}} \boldsymbol{f_r^{v_r}} = h\, \boldsymbol{f_r^{v_r}}$$

with $h = \alpha_{r+1}^{-u_{r+1}} \cdots \alpha_e^{-u_e}\, \boldsymbol{f_1^{v_1}} \cdots \boldsymbol{f_{r-1}^{v_{r-1}}} \in \mathbb{F}_{r-1}^*$. Since the entries in $\boldsymbol{f_r^{v_r}}$ are shift co-prime and irreducible, conditions (29) and (30) hold for these entries. Hence Lemma 5.3 is applicable and we get $\boldsymbol{v}_r = \mathbf{0}_{s_r}$, a contradiction. ■

We can now formulate a generalization of Theorem 5.3 for nested $\Pi\Sigma$-extensions.

**Theorem 6.1** *Let* $(\mathbb{F}_e, \sigma)$ *be the* $\Pi\Sigma$*-extension of* $(\mathbb{F}_0, \sigma)$ *from above. For* $1 \leqslant i \leqslant e$, *let* $\boldsymbol{f_i} = (f_{i,1}, \dots, f_{i,s_i}) \in (\mathbb{F}_{i-1}[\mathsf{t}_i] \setminus \mathbb{F}_{i-1})^{s_i}$ *with* $s_i \in \mathbb{N} \setminus \{0\}$ *containing irreducible monic polynomials. Then the following statements are equivalent.*

1. $\gcd_\sigma(f_{i,j}, f_{i,k}) = 1$ *for all* $1 \leqslant i \leqslant e$ *and* $1 \leqslant j < k \leqslant s_i$.
2. *There does not exist* $\boldsymbol{v_1} \in \mathbb{Z}^{s_1}, \dots, \boldsymbol{v_e} \in \mathbb{Z}^{s_e}$ *with* $\boldsymbol{v_1} \wedge \ldots \wedge \boldsymbol{v_e} \neq \boldsymbol{0}_s$ *and* $g \in \mathbb{F}_e^*$ *such that*
$$\frac{\sigma(g)}{g} = \boldsymbol{f_1}^{\boldsymbol{v_1}} \cdots \boldsymbol{f_e}^{\boldsymbol{v_e}}$$
*holds. That is,* $\mathbf{M}(\boldsymbol{f_1} \wedge \cdots \wedge \boldsymbol{f_e}, \mathbb{F}_e) = \{\boldsymbol{0}_s\}$ *where* $s = s_1 + \cdots + s_e$.
3. *One can construct a* $\Pi$*-field extension* $(\mathbb{F}_e(z_{1,1}) \cdots (z_{1,s_1}) \cdots (z_{e,1}) \cdots (z_{e,s_e}), \sigma)$ *of* $(\mathbb{F}_e, \sigma)$ *with* $\sigma(z_{i,k}) = f_{i,k}\, z_{i,k}$ *for* $1 \leqslant i \leqslant e$ *and* $1 \leqslant k \leqslant s_i$.
4. *One can construct a* $\Pi$*-extension* $(\mathbb{E}, \sigma)$ *of* $(\mathbb{F}_e, \sigma)$ *with the ring of Laurent polynomials* $\mathbb{E} = \mathbb{F}_e[z_{1,1}, z_{1,1}^{-1}] \cdots [z_{1,s_1}, z_{1,s_1}^{-1}] \cdots [z_{e,1}, z_{e,1}^{-1}] \cdots [z_{e,s_e}, z_{e,s_e}^{-1}]$ *and* $\sigma(z_{i,k}) = f_{i,k}\, z_{i,k}$ *for* $1 \leqslant i \leqslant e$ *and* $1 \leqslant k \leqslant s_i$.

*Proof* (1) $\Longrightarrow$ (2) follows by Lemma 6.1.

(2) $\Longrightarrow$ (3): We prove the statement by induction on the number of $\Pi\Sigma$-monomials $\mathsf{t}_1, \dots, \mathsf{t}_e$. For $e = 0$ nothing has to be shown. Now suppose that the implication has been shown for $\mathbb{F}_{e-1}$, $e \geqslant 0$ and set $\mathbb{E} = \mathbb{F}_{e-1}(z_{1,1}, \dots, z_{1,s_1}) \cdots (z_{e-1,1}, \dots, z_{e-1,s_{e-1}})$. Suppose that $(\mathbb{E}(z_{e,1}, \dots, z_{e,s_e}), \sigma)$ is not a $\Pi$-extension of $(\mathbb{E}, \sigma)$ and let $\ell$ be minimal with $s_\ell < s_e$ such that $(\mathbb{E}(z_{e,1}, \dots, z_{e,s_\ell}), \sigma)$ is a $\Pi$-extension of $(\mathbb{E}, \sigma)$. Then by Theorem 2.1(1) there are a $v_{e,s_\ell} \in \mathbb{Z} \setminus \{0\}$ and an $\omega \in \mathbb{E}(z_{e,1}, \dots, z_{e,s_j})^*$ with $j = \ell - 1$ such that $\sigma(\omega) = f_{e_{s_\ell}}^{v_{e,s_\ell}}\, \omega$ holds. By Proposition 5.2, $\omega = g\, z_{e,1}^{v_{e,1}} \cdots z_{e,s_j}^{v_{e,s_j}}$ with $(v_{e,1}, \dots, v_{e,s_j}) \in \mathbb{Z}^{s_j}$ and $g \in \mathbb{F}_{e-1}^*$. Thus $\frac{\sigma(g)}{g} = f_{e,1}^{-v_{e,1}} \cdots f_{e,s_j}^{-v_{e,s_j}}\, f_{e,s_\ell}^{v_{e,s_\ell}}$.

(3) $\Longrightarrow$ (2). We prove the statement by induction on the number of $\Pi\Sigma$-monomials $\mathsf{t}_1, \dots, \mathsf{t}_e$. For the base case $e = 0$ nothing has to be shown. Now suppose that the implication has been shown already for $e - 1$ $\Pi\Sigma$-monomials and set $\mathbb{E} = \mathbb{F}_e(z_{1,1}, \dots, z_{1,s_1}) \cdots (z_{e-1,1}, \dots, z_{e-1,s_{e-1}})$. Suppose that $(\mathbb{E}(z_{e,1}, \dots, z_{e,s_e}), \sigma)$ is a $\Pi$-extension of $(\mathbb{E}, \sigma)$ and assume on the contrary that there is a $g \in \mathbb{F}_e^*$ and $\boldsymbol{v_e} \in \mathbb{Z}^{s_e} \setminus \{\boldsymbol{0}_{s_e}\}$ such that $\frac{\sigma(g)}{g} = \boldsymbol{f_1}^{\boldsymbol{v_1}} \cdots \boldsymbol{f_{e-1}}^{\boldsymbol{v_{e-1}}}\, \boldsymbol{f_e}^{\boldsymbol{v_e}}$ holds. Let $j$ be maximal with $v_{e,j} \neq 0$ and define $\gamma := g\, \boldsymbol{z_1}^{-\boldsymbol{v_1}} \cdots \boldsymbol{z_{e-1}}^{-\boldsymbol{v_{e-1}}}\, z_{e,1}^{-v_{e,1}} \cdots z_{e,j-1}^{-v_{e,j-1}} \in \mathbb{E}(z_{e,1}, \dots, z_{e,j-1})^*$ where $\boldsymbol{z_i}^{-\boldsymbol{v_i}} = z_{i,1}^{-v_{i,1}} \cdots z_{i,s_i}^{-v_{i,s_i}}$ for $1 \leqslant i < e$ and $g \in \mathbb{F}_e^*$. Then $\frac{\sigma(\gamma)}{\gamma} = f_{e,j}^{v_{e,j}}$ with $v_{e_j} \neq 0$; a contradiction since $(\mathbb{E}(z_{e,1}, \dots, z_{e,j}), \sigma)$ is a $\Pi$-extension of $(\mathbb{E}(z_{e,1}, \dots, z_{e,j-1}), \sigma)$ by Theorem 2.1.

(2) $\Longrightarrow$ (1). We prove the statement by induction on the number of $\Pi\Sigma$-monomials $\mathsf{t}_1, \dots, \mathsf{t}_e$. For $e = 0$ nothing has to be shown. Now assume that the implication holds for the first $e - 1$ $\Pi\Sigma$-monomials. Now suppose that there are $k, \ell$ with $1 \leqslant k, \ell \leqslant s_e$ and $k \neq \ell$ such that $\gcd_\sigma(f_{e,k}, f_{e,\ell}) \neq 1$ holds. Since $\gcd_\sigma(f_{e,k}, f_{e,\ell}) \neq 1$ we know that they are shift equivalent and because $f_{e,k}, f_{e,\ell}$ are monic it follows by Lemma 4.4 that there is a $g \in \mathbb{F}_e^*$ with $\frac{\sigma(g)}{g} f_{e,k} = f_{e,\ell}$ and thus

$\frac{\sigma(g)}{g} = \boldsymbol{f}_1^{\boldsymbol{v}_1} \cdots \boldsymbol{f}_e^{\boldsymbol{v}_e}$ holds with $\boldsymbol{v}_1 = \cdots \boldsymbol{v}_{e-1} = 0$ and $\boldsymbol{v}_e = (0, \dots, 0, v_{e,k}, 0, \dots, 0, v_{e,\ell}, 0, \dots, 0) \in \mathbb{Z}^{s_e} \setminus \{\boldsymbol{0}_{s_e}\}$ where $v_{e,k} = -1$ and $v_{e,\ell} = 1$.

(3) $\Longrightarrow$ (4) is obvious and (4) $\Longrightarrow$ (3) follows by [32, Corollary 2.6]. ■

## 6.2 *Proof of the Main Result (Theorem 3.1)*

Using the structural results for nested $\Pi\Sigma$-extensions from the previous subsection, we are now in the position to handle the mixed $\boldsymbol{q}$-multibasic case. More precisely, we will generalize Theorem 5.4 from the rational difference field to the mixed $\boldsymbol{q}$-multibasic difference field $(\mathbb{F}, \sigma)$ with $\boldsymbol{q} = (q_1, \dots, q_{e-1})$. Here we assume that $\mathbb{K} = K(\kappa_1, \dots, \kappa_u)(q_1, \dots, q_{e-1})$ is a rational function field over a field $K$ where $K$ is strongly $\sigma$-computable. Following the notation from the previous subsection, we set $\mathbb{F}_0 := \mathbb{K}$ and $\mathbb{F}_i = \mathbb{F}_0(\mathsf{t}_1) \dots (\mathsf{t}_i)$ for $1 \leqslant i \leqslant e$. This means that $(\mathbb{F}_0(\mathsf{t}_1), \sigma)$ is the $\Sigma$-extension of $(\mathbb{F}_0, \sigma)$ with $\sigma(\mathsf{t}_1) = \mathsf{t}_1 + 1$ and $(\mathbb{F}_{i-1}(\mathsf{t}_i), \sigma)$ is the $\Pi$-extension of $(\mathbb{F}_{i-1}, \sigma)$ with $\sigma(\mathsf{t}_i) = q_{i-1}\, \mathsf{t}_i$ for $2 \leqslant i \leqslant e$.

As for the rational case we have to merge difference rings coming from different constructions. Using Theorem 6.1 instead of Theorem 5.3, Lemma 5.6 generalizes straightforwardly to Lemma 6.2. Thus the proof is omitted here.

**Lemma 6.2** *Let $(\mathbb{F}_e, \sigma)$ be the mixed $\boldsymbol{q}$-multi-basic difference field with $\mathbb{F}_0 = \mathbb{K}$ from above. Further, let $(\mathbb{K}[y_1, y_1^{-1}] \dots [y_w, y_w^{-1}], \sigma)$ be a $\Pi$-extension of $(\mathbb{K}, \sigma)$ with $\frac{\sigma(y_i)}{y_i} \in \mathbb{K}^*$ and $(\mathbb{F}_e[z_{1,1}, z_{1,1}^{-1}] \cdots [z_{1,s_1}, z_{1,s_1}^{-1}] \cdots [z_{e,1}, z_{e,1}^{-1}] \cdots [z_{e,s_e}, z_{e,s_e}^{-1}], \sigma)$ be a $\Pi$-extension of $(\mathbb{F}_0, \sigma)$ as given in Theorem 6.1 with item (4). Then $(\mathbb{E}, \sigma)$ with $\mathbb{E} = \mathbb{F}_e[y_1, y_1^{-1}] \cdots [y_w, y_w^{-1}][z_{1,1}, z_{1,1}^{-1}] \cdots [z_{1,s_1}, z_{1,s_1}^{-1}] \cdots [z_{e,1}, z_{e,1}^{-1}] \cdots [z_{e,s_e}, z_{e,s_e}^{-1}]$ is a $\Pi$-extension of $(\mathbb{F}_e, \sigma)$. Furthermore, the* A*-extension $(\mathbb{E}[\vartheta], \sigma)$ of $(\mathbb{E}, \sigma)$ with $\sigma(\vartheta) = \zeta\, \vartheta$ of order $\lambda$ is an* R*-extension.*

Gluing everything together, we obtain a generalization of Theorem 5.4. Namely, one obtains an algorithmic construction of an R$\Pi$-extension in which one can represent a finite set of hypergeometric, $q$-hypergeometric, $\boldsymbol{q}$-multibasic hypergeometric and mixed $\boldsymbol{q}$-multibasic hypergeometric products.

**Theorem 6.2** *Let $(\mathbb{F}_e, \sigma)$ be a mixed $\boldsymbol{q}$-multibasic difference field extension of $(\mathbb{F}_0, \sigma)$ with $\mathbb{F}_0 = \mathbb{K}$ where $\mathbb{K} = K(\kappa_1, \dots, \kappa_u)(q_1, \dots, q_{e-1})$ is a rational function field, $\sigma(\mathsf{t}_1) = \mathsf{t}_1 + 1$ and $\sigma(\mathsf{t}_\ell) = q_{\ell-1}\, \mathsf{t}_\ell$ for $2 \leqslant \ell \leqslant e$. Let $h_1, \dots, h_m \in \mathbb{F}_e^*$. Then one can define an* R$\Pi$*-extension $(\mathbb{A}, \sigma)$ of $(\mathbb{K}', \sigma)$ with*

$$\mathbb{A} = \mathbb{K}'(\mathsf{t}_1) \cdots (\mathsf{t}_e)[\vartheta][y_1, y_1^{-1}] \cdots [y_w, y_w^{-1}][z_{1,1}, z_{1,1}^{-1}] \cdots [z_{1,s_1}, z_{1,s_1}^{-1}] \cdots [z_{e,1}, z_{e,1}^{-1}] \cdots [z_{e,s_e}, z_{e,s_e}^{-1}] \quad (40)$$

*and $\mathbb{K}' = K'(\kappa_1, \dots, \kappa_u)(q_1, \dots, q_{e-1})$ where $K'$ is an algebraic field extension of $K$ such that*

- *$\sigma(\vartheta) = \zeta\, \vartheta$ where $\zeta \in K'$ is a $\lambda$-th root of unity.*

- $\frac{\sigma(y_j)}{y_j} = \alpha_j \in \mathbb{K}' \setminus \{0\}$ *for* $1 \leqslant j \leqslant w$ *where the* $\alpha_j$ *are not roots of unity;*
- $\frac{\sigma(z_{i,j})}{z_{i,j}} = f_{i,j} \in \mathbb{F}_{i-1}[\mathfrak{t}_i] \setminus \mathbb{F}_{i-1}$ *are monic, irreducible and shift co-prime;*

*holds with the following property. For* $1 \leqslant k \leqslant m$ *one can define*[7]

$$g_k = r_k\, \vartheta^{\mu_k}\, y_1^{u_{k,1}} \cdots y_1^{u_{k,w}}\, z_{1,1}^{\nu_{k,1,1}} \cdots z_{1,s_1}^{\nu_{k,1,s_1}}\, z_{2,1}^{\nu_{k,2,1}} \cdots z_{2,s_2}^{\nu_{k,2,s_2}} \cdots z_{e,1}^{\nu_{k,e,1}} \cdots z_{e,s_e}^{\nu_{k,e,se}} \tag{41}$$

*with* $0 \leqslant \mu_k \leqslant \lambda - 1$, $u_{k,i} \in \mathbb{Z}$, $\nu_{k,i,j} \in \mathbb{Z}$ *and* $r_k \in \mathbb{F}_e^*$ *such that*

$$\sigma(g_k) = \sigma(h_k)\, g_k.$$

*If* $K$ *is strongly* $\sigma$*-computable, the components of the theorem can be computed.*

*Proof* Take irreducible monic polynomials $\mathscr{B} = \{p_1, \ldots, p_n\} \subseteq \mathbb{F}_0[\mathfrak{t}_1, \mathfrak{t}_2, \ldots, \mathfrak{t}_e]$ and take $\gamma_1, \ldots, \gamma_m \in \mathbb{F}_0^*$ such that for each $k$ with $1 \leqslant k \leqslant m$ we get $d_{k,1}, \ldots, d_{k,n} \in \mathbb{Z}$ with $\sigma(h_k) = \gamma_i\, p_1^{d_{k,1}} \cdots p_n^{d_{k,n}}$. Following the proof of Theorem 5.4, we can construct an R$\Pi$-extension $\mathbb{F}'_0(x)[\vartheta][y_1, y_1^{-1}] \cdots [y_w, y_w^{-1}]$ of $(\mathbb{F}'_0(x), \sigma)$ with constant field $\mathbb{F}'_0 = K'(\kappa_1, \ldots, \kappa_u)(q_1, \ldots, q_{e-1})$ where $K'$ is an algebraic extension of $K$ and the automorphism is defined as stated in Theorem 6.2 with the following property: we can define $a_k$ of the form (35) in this ring with (36).

Set $\mathscr{I}_i = \{\omega \in \mathscr{B} \mid \omega \in \mathbb{K}[\mathfrak{t}_1, \mathfrak{t}_2, \ldots, \mathfrak{t}_i] \setminus \mathbb{K}[\mathfrak{t}_1, \mathfrak{t}_2, \ldots, \mathfrak{t}_{i-1}]\}$ for $1 \leqslant i \leqslant e$ and define $I = \{1 \leqslant i \leqslant e \mid \mathscr{I}_i \neq \{\}\}$. Then for each $i \in I$ there is a partition $\mathscr{P}_i = \{\mathscr{E}_{i,1}, \ldots, \mathscr{E}_{i,s_i}\}$ of $\mathscr{I}_i$ w.r.t. the shift-equivalence of the automorphism defined for each $\mathfrak{t}_i$, i.e., each $\mathscr{E}_{i,j}$ with $1 \leqslant j \leqslant s_i$ and $i \in I$ contains precisely the shift equivalent elements of $\mathscr{P}_i$. Take a representative from each equivalence class $\mathscr{E}_{i,j}$ in $\mathscr{P}_i$ and collect them in $\mathscr{F}_i := \{f_{i,1}, \ldots, f_{i,s_i}\}$. By construction it follows that property (1) in Theorem 6.1 holds; here we put all $\mathfrak{t}_i$ with $i \notin I$ in the ground field. Therefore by Theorem 6.1 we obtain the $\Pi$-extension $(\mathbb{F}_e(z_{1,1}) \cdots (z_{1,s_1}) \cdots (z_{e,1}) \cdots (z_{e,s_e}), \sigma)$ of $(\mathbb{F}_e, \sigma)$ with $\sigma(z_{i,k}) = f_{i,k}\, z_{i,k}$ for all $i \in I$ and $1 \leqslant k \leqslant s_i$ with $s_i \in \mathbb{N} \setminus \{0\}$; for $i \notin I$ we set $s_i = 0$. By Lemma 6.2 and Remark 5.2, $(\mathbb{A}, \sigma)$ with (40) is an R$\Pi$-extension of $(\mathbb{F}'_0(\mathfrak{t}_1) \cdots (\mathfrak{t}_e), \sigma)$. Let $i, j$ with $i \in I$ and $1 \leqslant j \leqslant s_i$. Since each $f_{i,j}$ is shift equivalent with every element of $\mathscr{E}_{i,j}$, it follows by Lemma 4.4 that for all $h \in \mathscr{E}_{i,j}$, there is a rational function $0 \neq r \in \mathbb{F}_i \setminus \mathbb{F}_{i-1}$ with $h = \frac{\sigma(r)}{r} f_{i,j}$. Putting everything together we obtain for each $k$ with $1 \leqslant k \leqslant m$, an $0 \neq r_k \in \mathbb{F}_e$ and $\boldsymbol{\nu}_{k,i} = (\nu_{k,i,1}, \ldots, \nu_{k,i,s_i}) \in \mathbb{Z}^{s_i}$ with $p_1^{d_{k,1}} \cdots p_n^{d_{k,n}} = \frac{\sigma(r_k)}{r_k} \boldsymbol{f}_{\boldsymbol{1}}^{\boldsymbol{\nu}_{\boldsymbol{1}}} \cdots \boldsymbol{f}_{\boldsymbol{e}}^{\boldsymbol{\nu}_{\boldsymbol{e}}}$. Note that for

$$b_k := r_k\, z_{1,1}^{\nu_{k,1,1}} \cdots z_{1,s_1}^{\nu_{k,1,s_1}}\, z_{2,1}^{\nu_{k,2,1}} \cdots z_{2,s_2}^{\nu_{k,2,s_2}} \cdots z_{e,1}^{\nu_{k,e,1}} \cdots z_{e,s_e}^{\nu_{k,e,se}} \in \mathbb{A}$$

we have that $\sigma(b_k) = p_1^{d_{k,1}} \cdots p_n^{d_{k,n}}\, b_k$. Now let $g_k \in \mathbb{A}$ be as defined in (41). Since $g_k = a_k\, b_k$ where $a_k$ equals (35) and has the property (36), we conclude that $\sigma(g_k) = \sigma(h_k)\, g_k$. The proof of the computational part is the same as that of Theorem 5.4. ■

[7] We remark that this representation is related to the normal form given in [8].

We are now ready to complete the proof for Theorem 3.1. To link to the notations used there, we set $\boldsymbol{q} = (q_1, \dots, q_{e-1})$ and set further $(x, t_1, \dots, t_{e-1}) = (\mathsf{t}_1, \dots, \mathsf{t}_e)$, in particular we use the shortcut $\boldsymbol{t} = (\mathsf{t}_2, \dots, \mathsf{t}_{e-1})$. Suppose we are given the products (11) and that we are given the components as stated in Theorem 6.2. Then we follow the strategy as in Sect. 5.4.

• Take the $\mathbb{K}'$-embedding $\tau : \mathbb{K}'(x, \boldsymbol{t}) \to \mathscr{S}(\mathbb{K}')$ where $\tau(\frac{a}{b}) = \langle \text{ev}\left(\frac{a}{b}, n\right)\rangle_{n\geqslant 0}$ for $a, b \in \mathbb{K}'[x, \boldsymbol{t}]$ is defined by (9). Then by iterative application of part (2) of Lemma 2.2 we can construct the $\mathbb{K}'$-homomorphism $\tau : \mathbb{A} \to \mathscr{S}(\mathbb{K}')$ determined by the homomorphic extension with

- $\tau(\vartheta) = \langle \zeta^n \rangle_{n\geqslant 0}$,
- $\tau(y_i) = \langle \alpha_i^n \rangle_{n\geqslant 0}$ for $1 \leqslant i \leqslant w$ and
- $\tau(z_{i,j}) = \langle \prod_{k=\ell'_{i,j}}^{n} f_{i,j}(k-1, \boldsymbol{q}^{k-1})\rangle_{n\geqslant 0}$ with $\ell'_{i,j} = Z(f_{i,j}) + 1$ for $1 \leqslant i \leqslant e$, $1 \leqslant j \leqslant s_i$.

In particular, since $(\mathbb{A}, \sigma)$ is an R$\Pi$-extension of $(\mathbb{K}'(x, \boldsymbol{t}), \sigma)$, it follows by part (3) of Lemma 2.2 that $\tau$ is a $\mathbb{K}'$-embedding.

• Finally, define for $1 \leqslant i \leqslant m$ the product expressions

$$\begin{aligned} G_i(n) =& r_i(n)\,(\zeta^n)^{\mu_i}\,(\alpha_1^n)^{u_{i,1}} \cdots (\alpha_w^n)^{u_{i,w}} \\ & \Big(\prod_{k=\ell'_{1,1}}^{n} f_{1,1}(k-1, \boldsymbol{q}^{k-1})\Big)^{v_{i,1,1}} \cdots \Big(\prod_{k=\ell'_{1,s_1}}^{n} f_{1,s_1}(k-1, \boldsymbol{q}^{k-1})\Big)^{v_{i,1,s_1}} \cdots \\ & \Big(\prod_{k=\ell'_{e,1}}^{n} f_{e,1}(k-1, \boldsymbol{q}^{k-1})\Big)^{v_{i,e,1}} \cdots \Big(\prod_{k=\ell'_{e,s_e}}^{n} f_{e,s_e}(k-1, \boldsymbol{q}^{k-1})\Big)^{v_{i,e,s_e}} \end{aligned}$$

and define $\delta_i = \max(\ell_i, \ell'_{1,1}, \dots, \ell'_{e,s_e}, Z(r_i))$. Then observe that $\tau(g_i) = \langle G'_i(n)\rangle_{n\geqslant 0}$ with (38). Now set $Q_i(n) := c\, G_i(n)$ with $c = \frac{P_i(\delta_i)}{G_i(\delta_i)} \in \mathbb{K}'$. Then as for the proof of the rational case we conclude that $P_i(n) = Q_i(n)$ for all $n \geqslant \delta_i$. This proves part (1) of Theorem 3.1. Since $\tau$ is a $\mathbb{K}'$-embedding, the sequences

$$\langle\alpha_1^n\rangle_{n\geqslant 0}, \dots, \langle\alpha_w^n\rangle_{n\geqslant 0}, \langle\prod_{k=\ell'_{1,1}}^{n} f_{1,1}(k-1, \boldsymbol{q}^{k-1})\rangle_{n\geqslant 0}, \dots, \langle\prod_{k=\ell'_{e,s_e}}^{n} f_{e,s_e}(k-1, \boldsymbol{q}^{k-1})\rangle_{n\geqslant 0}$$

are among each other algebraically independent over $\tau\big(\mathbb{K}'(x)\big)\big[\langle\zeta^n\rangle_{n\geqslant 0}\big]$ which proves property (2) of Theorem 3.1.

*Example 6.1* Let $\mathbb{K} = K(q_1, q_2)$ be the rational function field over the algebraic number field $K = \mathbb{Q}(\sqrt{-3}, \sqrt{-13})$, and consider the mixed $\boldsymbol{q} = (q_1, q_2)$-multibasic hypergeometric product expression

$$P(n) = \prod_{k=1}^{n} \frac{\sqrt{-13}\left(k\,q_1^k+1\right)}{k^2\left(q_1^{k+1}\,q_2^{k+1}+k+1\right)} + \prod_{k=1}^{n} \frac{k^2\left(k+q_1^k\,q_2^k\right)^2}{\sqrt{-3}\,(k+1)^2} + \prod_{k=1}^{n} \frac{169\left(k\,q_1^k\,q_2^k+q_2^k+k\,q_1^k+1\right)}{\left(k\,q_1^{k+2}+2\,q_1^{k+2}+1\right)k^2}. \quad (42)$$

Now take the mixed $\boldsymbol{q}$-multibasic difference field extension $(\mathbb{K}(x)(t_1)(t_2), \sigma)$ of $(\mathbb{K}, \sigma)$ with $\sigma(x) = x + 1$, $\sigma(t_1) = q_1\,t_1$ and $\sigma(t_2) = q_2\,t_2$. Note that $h_1(k, q_1^k, q_2^k)$, $h_2(k, q_1^k, q_2^k)$ and $h_3(k, q_1^k, q_2^k)$ with

$$h_1 = \frac{\sqrt{-13}\,(x\,t_1+1)}{x^2\,(q_1\,t_1\,q_2\,t_2+x+1)},\ h_2 = \frac{x^2\,(x+t_1\,t_2)^2}{\sqrt{-3}\,(x+1)^2},\ h_3 = \frac{169\,(x\,t_1\,t_2+t_2+x\,t_1+1}{(x\,q_1^2 t_1+2\,q_1^2 t_2+1)\,x^2} \in \mathbb{K}(x, t_1, t_2)$$

are the multiplicands of the above products, respectively. Applying Theorem 6.2 we construct the algebraic number field extension $\mathbb{K}' = \mathbb{Q}\big((-1)^{\frac{1}{2}}, \sqrt{3}, \sqrt{13}\big)$ of $\mathbb{K}$ and take the $\Pi\Sigma$-extension $(\mathbb{F}', \sigma)$ of $(\mathbb{K}', \sigma)$ with $\mathbb{F}' = \mathbb{K}'(x)(t_1)(t_2)$ where $\sigma(x) = x + 1, \sigma(t_1) = q_1\,t_1$ and $\sigma(t_2) = q_2\,t_2$. On top of this mixed multibasic difference field over $\mathbb{K}'$ we construct the R$\Pi$-extension $(\mathbb{A}, \sigma)$ with $\mathbb{A} = \mathbb{F}'[\vartheta]\,\langle y_1\rangle\,\langle y_2\rangle\,\langle z_1\rangle\,\langle z_2\rangle\,\langle z_3\rangle\,\langle z_4\rangle$ where the R-monomial $\vartheta$ with $\sigma(\vartheta) = (-1)^{\frac{1}{2}}\,\vartheta$ and the $\Pi$-monomials $y_1$, $y_2$ with $\sigma(y_1) = \sqrt{3}\,y_1$ and $\sigma(y_2) = \sqrt{13}\,y_2$ are used to scope the content of the polynomials in $h_1, h_2, h_3$. Furthermore, the $\Pi$-monomials $z_1, z_2, z_3, z_4$ with $\sigma(z_1) = (x + 1)\,z_1$, $\sigma(z_2) = \big((x + 1)\,q_1\,t_1 + 1\big)\,z_2$, $\sigma(z_3) = (q_2\,t_2 + 1)\,z_3$, $\sigma(z_4) = (q_2\,q_1\,t_2\,t_1 + x + 1)\,z_4$ are used to handle the monic polynomials in $h_1, h_2, h_3$. These $\Pi$-monomials are constructed in an iterative fashion as worked out in the proof of Theorem 6.2. In particular, within this construction we derive

$$Q = \underbrace{\frac{(q_2\,q_1 + 1)\,\vartheta\,y_2\,z_2}{(q_2\,q_1\,t_2\,t_1 + x + 1)\,z_1^2\,z_4}}_{=:g_1} + \underbrace{\frac{\vartheta^3\,z_4^2}{(x + 1)^2\,y_1}}_{=:g_2} + \underbrace{\frac{(q_1 + 1)\,(2\,q_1^2 + 1)\,y_2^4\,z_3}{\big((x + 1)\,q_1\,t_1 + 1\big)\,\big((x + 2)\,q_1^2\,t_1 + 1\big)\,z_1^2}}_{=:g_3}$$

such that $\sigma(g_i) = \sigma(h_i)\,g_i$ holds for $i = 1, 2, 3$.

Now take the $\mathbb{K}'$-embedding $\tau : \mathbb{K}'(x, \boldsymbol{t}) \to \mathcal{S}(\mathbb{K}')$ where $\tau(\frac{a}{b}) = \langle \mathrm{ev}\left(\frac{a}{b}, n\right)\rangle_{n\geqslant 0}$ for $a, b \in \mathbb{K}'[x, \boldsymbol{t}]$ is defined by (9). Then by iterative application of part (2) of Lemma 2.2 we can construct the $\mathbb{K}'$-embedding $\tau : \mathbb{A} \to \mathcal{S}(\mathbb{K}')$ determined by the homomorphic continuation of $\tau(\vartheta) = \langle(-1)^{\frac{1}{2}}\rangle_{n\geqslant 0}$, $\tau(y_1) = \langle\big(\sqrt{3}\big)^n\rangle_{n\geqslant 0}$, $\tau(y_2) = \langle\big(\sqrt{13}\big)^n\rangle_{n\geqslant 0}$, $\tau(z_1) = \langle n!\rangle_{n\geqslant 0}$, $\tau(z_2) = \langle\prod_{k=1}^{n}(k\,q_1^k + 1)\rangle_{n\geqslant 0}$, $\tau(z_3) = \langle\prod_{k=1}^{n}(q_2^k + 1)\rangle_{n\geqslant 0}$ and $\tau(z_4) = \langle\prod_{k=1}^{n}(q_2^k\,q_1^k + k)\rangle_{n\geqslant 0}$. By our construction we can conclude that $\tau(g_1)$, $\tau(g_2)$ and $\tau(g_3)$ equal the sequences produced by the three products in (42), respectively. In particular, $\tau(Q) = \langle P(n)\rangle_{n\geqslant 0}$. Furthermore, if we define

$$Q(n) = \frac{(q_2\,q_1+1)}{\left(q_2^{n+1}\,q_1^{n+1}+n+1\right)} \left((-1)^{\frac{1}{2}}\right)^n \left(\sqrt{13}\right)^n \frac{1}{(n!)^2} \prod_{k=1}^{n}\left(k\,q_1^k+1\right) \prod_{k=1}^{n} \frac{1}{\left(q_2^k\,q_1^k+k\right)}$$
$$+ \frac{1}{(n+1)^2} \left(\left((-1)^{\frac{1}{2}}\right)^n\right)^3 \left(\left(\sqrt{3}\right)^n\right)^{-1} \left(\prod_{k=1}^{n}\left(q_2^k\,q_1^k+k\right)\right)^2$$
$$+ \frac{(q_1+1)\left(2\,q_1^2+1\right)}{\left((n+1)\,q_1^{n+1}+1\right)\left((n+2)\,q_1^{n+2}+1\right)} \left(\left(\sqrt{13}\right)^n\right)^4 \frac{1}{(n!)^2} \prod_{k=1}^{n}\left(q_2^k+1\right)$$

then we can guarantee that $P(n) = Q(n)$ for all $n \geqslant 1$. The sequences generated by $\left(\sqrt{3}\right)^n$, $\left(\sqrt{13}\right)^n$, $n!$, $\prod_{k=1}^{n}\left(k\,q_1^k+1\right)$, $\prod_{k=1}^{n}\left(q_2^k+1\right)$, $\prod_{k=1}^{n}\left(q_2^k\,q_1^k+k\right)$ are algebraically independent among each other over $\tau(\mathbb{K}'(x,\boldsymbol{t}))[\langle((-1)^{\frac{1}{2}})^n\rangle_{n\geqslant 0}]$ by construction.

## 7 Conclusion

We extended the earlier work [23, 28] substantially and showed that any expression in terms of hypergeometric products $\mathrm{ProdE}(\mathbb{K}(n))$ can be formulated in an R$\Pi\Sigma$-extension if the original constant field $\mathbb{K}$ satisfies certain algorithmic properties. This is in particular the case if $\mathbb{K} = K(\kappa_1,\ldots,\kappa_u)$ is a rational function field over an algebraic number field $K$. In addition, we extended this machinery for the class of mixed $\boldsymbol{q}$-multibasic hypergeometric products. Internally, we rely on Ge's algorithm [9] that solves the orbit problem in $K$ and we utilize heavily results from difference ring theory [10, 27, 30, 32]. This product machinery implemented in Ocansey's package `NestedProducts` in combination with the summation machinery available in `Sigma` [25] yields a complete summation toolbox in which nested sums defined over $\mathrm{ProdE}(\mathbb{K}(n,\boldsymbol{q^n}))$ can be represented and simplified using the summation paradigms of telescoping, creative telescoping and recurrence solving [19, 25].

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# Linearly Satellite Unknowns in Linear Differential Systems

**Anton A. Panferov**

**Abstract** Let $K$ be a differential field of characteristic 0. Consider a linear differential system $S$ of the form $y' = Ay$, where $A \in K^{n\times n}$ and $y = (y_1, \dots, y_n)^T$ is a vector of unknowns. In the present work we introduce a concept of *linearly satellite* unknowns: for the nonempty set of *selected* unknowns $s = \{y_{i_1}, \dots, y_{i_k}\}$ an unselected unknown $y_j$ is called *linearly satellite* if the $j$-th component of any solution to $S$ can be linearly expressed over $K$ only via selected components of this solution and their derivatives. We present an algorithm for linearly satellite unknown recognition and its implementation in Maple. The ability to determine linearly satellite unknowns can be used for partial solving of differential systems.

**Keywords** Differential systems · Selected unknowns · Satellite unknowns
Computer algebra

## 1 Introduction

Let $K$ be a differential field of characteristic 0 with the derivation $'$. We assume that its field of constants $\mathrm{Const}(K) = \{c \in K \mid c' = 0\}$ is algebraically closed. Consider a linear differential system $S$ of the form

$$y' = Ay, \tag{1}$$

where $A \in K^{n\times n}$, $y = (y_1, \dots, y_n)^T$ is a vector of unknowns. Suppose that $s = \{y_{i_1}, \dots, y_{i_k}\}$ is a given nonempty set of *selected* unknowns (components of vector $y$) that does not contain all unknowns (i.e. $0 < k < n$).

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Earlier in [5, 8] we introduced the concept of *satellite* unknowns: an unselected unknown $y_j$ of a given system $S$ is called *satellite* for a set of selected unknowns $s$ if the $j$-th component of any solution to $S$ belongs to any differential field extension $F_s$ of $K$ containing all selected components of any solution to $S$. We recall this notion with some details in Sect. 2.1. We also recall the proposed satellite unknown testing algorithm in Sect. 2.3.

One of the problems that arises with differential systems is that of solving a linear differential system with respect to a part of the unknowns [1]. This is also related to the problem of partial stability of solutions with respect to a part of the variables and partial control ([10]). Satellite unknowns possess similar nice properties as the selected ones. Thus the possibility algorithmically determining satellite unknowns is useful in the context of these problems.

The Satellite testing algorithm that we proposed in [5, 8] allows one to construct the set of all satellite unknowns for a given system $S$ and a fixed set of selected unknowns $s$ for the case $K = \overline{\mathbb{Q}}(x)$. One of its steps checks the embedding of Picard–Vessiot extensions that can be done by means of the algorithm from [3, Sect. 5.3.3 (H)]. That algorithm is based on Hrushovski's algorithm [2] whose complexity is rather high. This fact makes it difficult to implement the Satellite testing algorithm in computer algebra systems and to use it in practice. That is why the partial satellite testing algorithms, which do not always solve the problem, were developed and implemented ([7]).

In the present work we introduce the concept of *linearly satellite* unknowns. Contrary to satellite unknowns, the definition of linearly satellite unknowns does not concern selected components of all solutions at once. An unselected unknown $y_j$ is said to be linearly satellite unknown if the $j$-th component of any solution to $S$ belongs to the $K$-linear span of the selected components of this solution and their derivatives. In other words, $y_j$ is said to be linearly satellite unknown for a set of selected unknowns $s$ if the $j$-th component of any solution to $S$ can be linearly expressed over $K$ only via selected components of this solution and their derivatives.

In this paper we also present the Linearly satellite testing algorithm, which has much lower complexity than that of the Satellite testing algorithm. So it can be easily implemented in computer algebra systems. One such implementation in Maple is presented in Sect. 3.3. In Sect. 4 we shall show how the presented algorithm can be generalized for the case of linear homogeneous differential systems of higher order and how this algorithm can be used for partial solution construction.

The work [1] of Sergei Abramov and Manuel Bronstein was the starting point of this research. The AB-algorithm, described there, is the basis for recognition algorithms, which will be discussed in this paper.

## 2 Preliminaries

Consider a differential system $S$ of the form (1) with a set of selected unknowns $s$.

**Definition 6** ([9, Prop. 1.22]) A *Picard–Vessiot extension* over $K$ for a normal differential system $y' = Ay$, with $A \in K^{n\times n}$, $y = (y_1, \ldots, y_n)^T$, is a differential extension $K_S \supseteq K$ satisfying:

- $Const(K_S) = Const(K)$.
- There is a fundamental matrix $B$ for $y' = Ay$ with coefficients in $K_S$, i.e., the invertible matrix satisfies $B' = AB$.
- $K_S$ is generated over $K$ by the entries of $B$.

It is known (see [9]) that a Picard–Vessiot extension exists for any differential system of the form (1) over $K$.

Denote by $V_S$ the solution space of $S$ in $K_S^n$: $V_S = \{y \in K_S^n \mid y' = Ay\}$, and denote by $\pi_s(V_S)$ the projection of $V_S$ onto the selected unknowns $s$.

The number of the equations of $S$ is called the *size* of the system and denoted by $|S|$. It is obvious that the size of $S$ is equal to the number of rows or columns of the matrix $A$, and also is equal to the length of the unknown vector $y$.

### 2.1 Satellite Unknowns

Let $F_s$ be a differential extension of $K$ satisfying:

1. $\pi_s(V_S) \subseteq F_s^k$, where $k$ is a cardinality of $s$ (that is the amount of selected unknowns).
2. For any $F \supseteq K$ if $\pi_s(V_S) \subseteq F^k$, then $F_s \subseteq F$.

The extension $F_s$ may be considered as the differential field generated over $K$ by all selected components of all solutions to $S$.

**Definition 7** ([8, Def. 2]) An unselected unknown $y_j$ is called *satellite* unknown for a set of selected unknowns $s$ in $S$ if the $y_j$ component of any solution to $S$ belongs to $F_s$.

*Example 13* Consider the following differential system:

$$y' = \begin{bmatrix} 1/x & 0 & 0 \\ 3 & -1/x & 0 \\ 0 & 0 & 1 \end{bmatrix} y, \tag{2}$$

where $y = (y_1, y_2, y_3)^T$. The general solution to (2) can be represented in the form

$$y = C_1 \begin{bmatrix} x \\ x^2 \\ 0 \end{bmatrix} + C_2 \begin{bmatrix} 0 \\ 1/x \\ 0 \end{bmatrix} + C_3 \begin{bmatrix} 0 \\ 0 \\ e^x \end{bmatrix}, \tag{3}$$

where $C_1, C_2, C_3$ are arbitrary constants. From (3) it follows that, for any solution to (2), its components $y_1$ and $y_2$ belong to $\overline{\mathbb{Q}}(x)$, but $y_3 \notin \overline{\mathbb{Q}}(x)$ in general. So, if $s = \{y_1\}$, then it is clear that $y_2$ is a satellite unknown, but $y_3$ is not.

Since $y_3 \in \overline{\mathbb{Q}}(x, e^x) \supset \overline{\mathbb{Q}}(x)$, we see that both $y_1$ and $y_2$ are satellite unknowns for $s = \{y_3\}$.

As it was noted in the introduction, all selected components of all solutions to $S$ are used in the definition of satellite unknowns. So these satellite unknowns may be called as satellite unknowns with respect to the system. In Sect. 3 we shall introduce the concept of linearly satellite unknowns, which are defined only by selected components of concrete solution.

## *2.2 AB-Algorithm*

Our Satellite testing algorithm is based on the AB-algorithm that was proposed by S. Abramov and M. Bronstein in [1]. We shall not present its formal description here but recall some facts about it.

The AB-algorithm for a given differential system $S$ of the form (1) and a set of selected unknowns $s$ produces a new differential system

$$z' = Bz, \tag{4}$$

where $B$ is a square matrix over $K$ whose size is less or equal to the size of $A$, and the components of the new unknown vector $z$ are the selected components from $s$ and, possibly, some of their derivatives.

Denote system (4) (i.e. the result of application of the AB-algorithm to the system $S$ with respect to the set of selected unknowns $s$) by $S_s^{\mathrm{AB}}$.

The AB-algorithm has the following properties (see [1, Prop. 1]):

1. For any differential field extension $L$ of $K$, if we let $V$ (resp., $V_s^{\mathrm{AB}}$) denote the projection of the solution space of $S$ (resp., $S_s^{\mathrm{AB}}$) over $L$, then the projections of $V$ and $V_s^{\mathrm{AB}}$ onto the subspaces generated by the components from $s$ are identical.
2. If a solution to the system $S_s^{\mathrm{AB}}$ is such that its selected components belong to some differential extension of $K$, then all the components of this solution belong to this extension.
3. If the size of $B$ is equal to the size of $A$ and the system $S$ has a solution whose selected components belong to some differential extension of $K$, then *all* the components of this solution belong to this extension.

*Remark 7* Property 3 can be reformulated in the following stronger form (see [1, Sect. 2]): if the size of $B$ is equal to the size of $A$ then the systems $S$ and $S_s^{\mathrm{AB}}$ are *equivalent* over $K$, i.e. there exists a nonsingular matrix $H \in K^{n\times n}$ such that $z = Hy$. In this case any unknown of $S$ can be linearly expressed over $K$ via the unknowns of $S_s^{\mathrm{AB}}$, which are the selected unknowns from $s$ and their derivatives.

The AB-algorithm was implemented in Maple as a part of the `OreTools` package (in the subpackage `Consequences`) in the form of the `ReducedSystem` procedure.

## 2.3 Satellite Testing Algorithm

Here we shortly recall our Satellite testing algorithm. We shall confine ourselves to present only its formal description, details and the proof of the correctness can be found in [8].

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ALGORITHM : **Satellite testing**

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**Input:**
- A differential system $S$ with a set of selected unknowns $s$.
- An unselected unknown $y_j \notin s$.

**Output:** YES if $y_j$ is a satellite unknown for $s$ in $S$ and NO otherwise.

1. Construct the system $S_s^{\mathrm{AB}}$.
2. If $|S_s^{\mathrm{AB}}| = |S|$, then **return** YES.
3. Construct the system $S_{\tilde{s}}^{\mathrm{AB}}$, where $\tilde{s} = s \cup \{y_j\}$.
4. If $|S_{\tilde{s}}^{\mathrm{AB}}| = |S_s^{\mathrm{AB}}|$, then **return** YES.
5. If $K_{S_s^{\mathrm{AB}}} \supseteq K_{S_{\tilde{s}}^{\mathrm{AB}}}$, then **return** YES.
6. **Return** NO.

---

The most complicated step of the algorithm is step 5, where Picard–Vessiot extensions are to be compared. The problem of testing whether the Picard–Vessiot extension of some differential system is a subfield of the Picard–Vessiot extension of another system can be solved algorithmically. Such an algorithm was proposed by A. Minchenko, A. Ovchinnikov and M. Singer in [3]. The algorithm from [3, Sect. 5.3.3(H)] uses Hrushovski's algorithm [2] for computing differential Galois groups. Hrushovski's algorithm has rather high complexity (in some cases, the degrees of the defining polynomials on the output of the algorithm are estimated to be from four-fold to six-fold exponential in the size of system matrix). So the complexity of the Satellite testing algorithm is very high.

## 3 Linearly Satellite Unknowns

The differential extension $F_s$ from Definition 7 depends on $V_S$ and includes selected components of all solutions to $S$. This yields that the notion of satellite unknown is determined by all solutions of the system. Here we propose the concept of *linearly satellite* unknowns. An unselected unknown $y_j$ is said to be linearly satellite unknown for a set of selected unknowns $s$ if the $j$-th component of any solution to $S$ belongs to the minimal differential extension of $K$ containing all nonzero selected components of this solution. If all selected components of the solution are zero, we require the linearly satellite component to be zero too.

*Example 14* Consider the following system

$$y' = \begin{bmatrix} 1/x & 1/(2x) & -1/x \\ 2/x & 3/x & -2/x \\ 1/x & 3/(2x) & -1/x \end{bmatrix} y, \tag{5}$$

where $y = (y_1, y_2, y_3)^T$. System (5) has only rational solutions (i.e. all components of any solution to (5) belong to $\overline{\mathbb{Q}}(x)$). This means that any unknown $y_j$ is satellite for any set $s$ such that $\varnothing \neq s \subset \{y_1, y_2, y_3\}$ and $y_j \notin s$. In particular, $y_2$ is a satellite unknown for $\{y_1\}$. At the same time $(0, x^2, x^2/2)^T$ is the solution to (5). Therefore $y_2$ is not linearly satellite for $\{y_1\}$. But $y_2$ is linearly satellite for $\{y_3\}$: the $y_2$ component of any solution to (5) with $y_3 = 0$ is also equal to zero.

### *3.1 Definition*

Consider $F_s$ defined in Sect. 2.1 as a differential extension generated by $K$ and all selected components of all solutions to $S$. $F_s$ also can be regarded as a linear space over $K$. Let $y = (y_1, \ldots, y_n)^T$ be a solution to $S$. Define $F_s(y)$ as the minimal $K$-subspace of $F_s$ that contains all selected components of $y$ and their derivatives. $F_s(y)$ also can be regarded as the linear space generated over $K$ by the selected components of the solution $y$ and their derivatives. It is important to note that $F_s(y)$ is not a differential field in general: if all selected components of the solution $y$ are zero, then $F_s(y) = \{0\}$.

*Example 15* Consider system (2) from Example 13. Suppose $s = \{y_1\}$. As it is easy to check, $y = (x, x^2, e^x)^T$ is a solution to (2). $F_s(y)$ for this solution is equal to $\overline{\mathbb{Q}}(x)$. At the same time $\tilde{y} = (0, -1/x, 0)^T$ is also a solution to (2). For this solution $F_s(\tilde{y}) = \{0\}$, which is not a differential extension of $K = \overline{\mathbb{Q}}(x)$.

**Definition 8** An unselected unknown $y_j$ is called *linearly satellite* unknown for a set of selected unknowns $s$ in $S$ if for any solution $y$ its $j$-th component belongs to $F_s(y)$.

It is clear that if $y_j$ is a linearly satellite unknown for $s$ then $y_j$ is a satellite unknown for $s$ ($y_j \in F_s(y) \subseteq F_s$ for any solution $y$).

Linearly satellite unknowns can be recognized algorithmically and the formal description of one such algorithm will be presented in the next section.

### *3.2 Algorithm*

Our "Linearly satellite testing" algorithm is based only on the AB-algorithm and repeats the Satellite testing algorithm with the exclusion of its step 5. As it will be shown, it is sufficient to test whether a given unselected unknown $y_j$ is a linearly satellite unknown for a set of selected unknowns $s$.

---

ALGORITHM : **Linearly satellite testing**

---

**Input:**
- A differential system $S$ with a set of selected unknowns $s$.
- An unselected unknown $y_j \notin s$.

**Output:** YES if $y_j$ is a linearly satellite unknown for $s$ in $S$ and NO otherwise.

1. Construct the system $S_s^{\mathrm{AB}}$.
2. If $|S_s^{\mathrm{AB}}| = |S|$, then **return** YES.
3. Construct the system $S_{\tilde{s}}^{\mathrm{AB}}$, where $\tilde{s} = s \cup \{y_j\}$.
4. If $|S_{\tilde{s}}^{\mathrm{AB}}| = |S_s^{\mathrm{AB}}|$, then **return** YES.
5. **Return** NO.

---

**Proposition 3** *The Linearly satellite testing algorithm is correct.*

To prove Proposition 3 we need the following lemma.

**Lemma 11** *Suppose that $s_1$ and $s_2$ are nonempty sets of the unknowns of $S$ such that $s_1 \neq s_2$, $s_1 \subset s_2$; then the following conditions are equivalent:*

*(i)* $|S_{s_1}^{\mathrm{AB}}| < |S_{s_2}^{\mathrm{AB}}|$;

*(ii)* $\dim \pi_{s_1}(V_S) < \dim \pi_{s_2}(V_S)$;

*(iii) There exists a solution to $S$ in $K_S^n$ such that all the components corresponding to the unknowns from $s_1$ are zero but at least one component corresponding to the unknown from $s_2 \setminus s_1$ is nonzero.*

*Proof* To prove the equivalence of (i) and (ii) we should note that for any nonempty set $s$ of unknowns $K_{S_s^{\mathrm{AB}}}$ is a subfield of $K_S$, since the unknowns of $S_s^{\mathrm{AB}}$ are only unknowns from $s$ and, possibly, some of their derivatives. Denote the solution spaces of $S$ and $S_s^{\mathrm{AB}}$ by $V_S$ and $V_{S_s^{\mathrm{AB}}}$ respectively. Then, from Property 1 of the AB-algorithm it follows that $\pi_s(V_S) = \pi_s(V_{S_s^{\mathrm{AB}}})$. Moreover, being derivatives of the unknowns from $s$, additional unknowns of $S_s^{\mathrm{AB}}$ do not affect the dimension of the solution space; thus $\dim \pi_s(V_{S_s^{\mathrm{AB}}}) = \dim V_{S_s^{\mathrm{AB}}}$. Note that $\dim V_{S_s^{\mathrm{AB}}} = |S_s^{\mathrm{AB}}|$; and this completes the proof.

To prove the equivalence of (ii) and (iii) consider subspace $P \subseteq V(S)$ that contains only solutions to $S$ whose components corresponding to unknowns from $s_1$ are zero:

$$P = \{z = (z_1, \ldots, z_n)^T \in V(S) \mid \forall i : y_i \in s_1 \Rightarrow z_i = 0\}.$$

Then (ii) becomes equivalent to condition $\dim \pi_{s_2}(P) > 0$, that is equivalent to (iii). ■

*Proof* (*of Proposition* 3) The correctness of step 2 follows immediately from Property 3 (Remark 7) of the AB-algorithm.

To prove the correctness of step 4 we should notice that $(S_{\tilde{s}}^{\mathrm{AB}})_s^{\mathrm{AB}} = S_s^{\mathrm{AB}}$. So, if we consider the system $S_{\tilde{s}}^{\mathrm{AB}}$ as the initial system with the set of selected unknowns $s \subset \tilde{s}$, then by Property 3 (Remark 7) of the AB-algorithm we get that all unselected unknowns in $S_{\tilde{s}}^{\mathrm{AB}}$ are linearly satellite for $s$; $y_j$ is an unselected unknown of $S_{\tilde{s}}^{\mathrm{AB}}$, so it is a linearly satellite unknown.

To conclude the proof, it remains to show the correctness of step 5. Indeed, in this case we have $|S_s^{\mathrm{AB}}| < |S_{\tilde{s}}^{\mathrm{AB}}|$. It follows from Lemma 11 that $S$ has a solution in which the selected components are zero but the component corresponding to $y_j$ is not zero. The existence of such a solution means that $y_j$ is not a linearly satellite unknown for $s$. ■

*Example 16* We continue with system (5) from Example 14. Denote this system by $S$. Suppose $s = \{y_1\}$, and let us test whether $y_2$ is linearly satellite for $s$. The AB-algorithm gives the following matrices

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & 2/x \end{bmatrix}$$

of systems $S_s^{\mathrm{AB}}$ and $S_{s\cup\{y_2\}}^{\mathrm{AB}}$ respectively. The different sizes ($2 \times 2$ versus $3 \times 3$) of these matrices show that $y_2$ is not linearly satellite for $\{y_1\}$ (as it was demonstrated in Example 14).

For $s = \{y_3\}$ the AB-algorithm produces the following $3 \times 3$ matrix of the system $S_s^{\mathrm{AB}}$:

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus $|S_s^{\mathrm{AB}}| = |S|$, and both $y_1$ and $y_2$ are linearly satellite for $\{y_3\}$.

### 3.3 Maple Implementation

Due to the existence of the AB-algorithm implementation in Maple (see procedure OreTools:-Consequences:-ReducedSystem), the implementation of the Linearly satellite testing algorithm is short enough. Instead of results YES and NO we use true and false respectively.

```
LinearlySatelliteTesting := proc(A::Matrix, s::set(posint), v::posint)
  local R, S1, S2;
  R := OreTools:-SetOreRing(x, 'differential');
  S1 := OreTools:-Consequences:-ReducedSystem(A, s, R)[1];
  if op(1, A) = op(1, S1) then return true end if;
  S2 := OreTools:-Consequences:-ReducedSystem(A, s union {v}, R)[1];
  if op(1, S1) = op(1, S2) then return true end if;
  return false
end proc
```

The LinearlySatelliteTesting procedure has the following parameters:

- A—the matrix of the differential system $S$;
- s—a set of positive integers—the indices of selected unknowns;
- v—the index of the testing unknown.

We assume that the matrix A is a square matrix, s is a nonempty set whose elements are correct indices of the unknowns (lie between 1 and $|S|$), v is a positive integer that is a correct index of the testing unknown and v $\notin$ s.

The ReducedSystem procedure is based on the functionality of the OreTools package and requires one to specify the Ore ring for computations. We use the local variable $R$ to specify the differential ring. The result of the ReducedSystem procedure is a tuple $[B, Z]$, where $B$ is the matrix of $S_s^{\mathrm{AB}}$, and $Z$ is a set of pairs, the first entry of each is the index of a selected unknown in $S$ and the second one is the index of the same unknown in $S_s^{\mathrm{AB}}$. We use only the first element of this tuple. To retrieve the sizes of the constructed system matrices we use the built-in Maple function op. Given 1 as the first parameter and a matrix as the second parameter, op returns the size of this matrix as a pair (rows, cols).

## 4 Differential Systems of Higher Order

The Linearly satellite testing algorithm can be extended to handle higher order linear homogeneous differential systems of the form

$$A_r y^{(r)} + A_{r-1} y^{(r-1)} + \cdots + A_0 y = 0, \tag{6}$$

where $r \in \mathbb{N}$ is called the *order* of the system, $A_i \in K^{n\times n}$ $(0 \le i \le r)$, $A_r \ne 0$, $y = (y_1, \dots, y_n)^T$ is a vector of unknowns. We assume that system (6) is of full rank, i.e. its equations are independent. We also assume that a subset of the unknowns is selected and denote the set of selected unknowns by $s$.

We define linearly satellite unknowns as previously (see Definition 8), considering $F_s(y)$ as the linear space generated over $K$ by the selected components of $y$ and all their derivatives.

It is known that any differential system of the form (6) of order $r > 1$ can be reduced to a differential system of the form (1) or to a first order differential-algebraic system of the form $\tilde{A}_1 y' + \tilde{A}_0 y = 0$ (see [4, 8] for details). Thus it remains to discuss the case of differential-algebraic systems.

## 4.1 *Differential-Algebraic Systems*

Consider a linear differential-algebraic system $S$ of the form

$$A_1 y' + A_0 y = 0, \tag{7}$$

where $A_1, A_0 \in K^{n\times n}$, $A_1 \neq 0$, $\det A_1 = 0$, $y = (y_1, \ldots, y_n)^T$ is a vector of unknowns; $s \subset \{y_1, \ldots, y_n\}$ is a nonempty set of selected unknowns.

To generalize the algorithm from Sect. 3.2 for systems of the form (7) we proceed similarly as when we generalized the Satellite testing algorithm in [8, Sect. 4] as an application of the Extract algorithm [4].

The details of the Extract algorithm can be found in [4] (see also [6]), and here we shall confine ourselves to present only an informal description. The Extract algorithm for a differential-algebraic system $S$ of the form (7) with a set of selected unknowns $s$, using a properly organized elimination process for unknowns, allows one to construct a partition $s = s_1 \cup s_2$ ($s_1 \cap s_2 = \varnothing$) and get a pair of systems $S_d$ and $S_a$ with the following properties. System $S_d$ is a differential system of the form $\tilde{y}' = A\tilde{y}$, where $A$ is a square matrix over $K$, and a new vector of unknowns $\tilde{y}$ contains only part of the unknowns of $S$ ($\tilde{y} \subset y$), but includes all unknowns from $s_1$. System $S_a$ is an algebraic system of the form $y_{s_2} = B y_{s_1}$, where $y_{s_i}$ ($i = 1, 2$) denotes a column vector whose entries are the unknowns from $s_i$, $B$ is a matrix (not necessary square) over $K$. So, system $S_a$ can be used to determine the part of the selected unknowns corresponding to $s_2$ from the part corresponding to $s_1$. From the form of $S_a$ it follows that every selected unknown from $s_2$ is linearly expressed over $K$ via selected unknowns from $s_1$. The part $s_1$ of the selected unknowns can be obtained from $S_d$. Systems $S_d$ and $S_a$ are *consistent* with the initial system $S$ and the set of selected unknowns $s$. This means that the selected components $s$ of solutions to $S$ in any differential extension of $K$ are uniquely determined by $S_d$ and $S_a$ (see [6] for details).

Applying the Extract algorithm to (7) with respect to the set of selected unknowns $\tilde{s} = s \cup \{y_j\}$, where $y_j \notin s$ is the unknown we are testing, we get two systems $S_d$

and $S_a$. If $y_j$ occurs in algebraic system $S_a$ (it can be an element of $s_2$ or it can appear in the right-hand side of some equation with a nonzero coefficient), then it can be linearly expressed via the selected unknowns from $s$; and this means that $y_j$ is linearly satellite for $s$ in (7). If $y_j$ does not occur in $S_a$, then it is an unknown in $S_d$, and the problem for the differential-algebraic system $S$ is reduced to the problem for the differential system $S_d$.

*Example 17* Consider the following differential system

$$\begin{cases} -y_1' + xy_3' - xy_4' + y_3 + y_5 + y_6 = 0 \\ (x+1)xy_4' + xy_6' - y_5 - (x+1)y_6 = 0 \\ xy_4' + y_5' + y_6' - y_5 - y_6 = 0 \\ y_4' - y_3 = 0 \\ y_1 - y_5 - y_6 = 0 \\ y_1 + xy_2 = 0 \end{cases} \tag{8}$$

System (8) can be presented in the form (7) where

$$A_1 = \begin{bmatrix} -1 & 0 & x & -x & 0 & 0 \\ 0 & 0 & 0 & (x+1)x & 0 & x \\ 0 & 0 & 0 & x & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad A_0 = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 & -(x+1) \\ 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 1 \\ 1 & x & 0 & 0 & 0 & 0 \end{bmatrix},$$

and $y = (y_1, y_2, y_3, y_4, y_5, y_6)^T$. Suppose that the set of selected unknowns contains only $y_1$, i.e. $s = \{y_1\}$. As it will be shown, the $y_1$-th component of any solution to (8) belongs to the extension $\overline{\mathbb{Q}}(x, e^x)$ and can be presented as a Laurent series. This is not so for all other unknowns. By determining the set of linearly satellite unknowns for $s$, we can build the partial solution of this system using the procedure to construct Laurent solutions (e.g., `LinearFunctionalSystems:-LaurentSolution` procedure from Maple).

As it follows from the last equation of (8), $y_2$ is linearly satellite for $\{y_1\}$. Let us show how this fact can be recognized algorithmically. The Extract algorithm being applied to (8) with respect to the set of selected unknowns $\{y_1, y_2\}$ gives the following systems:

$$S_d\colon \tilde{y}' = \begin{bmatrix} 1 - 1/x & 1 & 0 & 0 \\ 0 & -1/x & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & x+1 & 0 & 1 \end{bmatrix} \tilde{y}, \qquad S_a\colon y_1 = -xy_2,$$

where $\tilde{y} = (y_2, y_3, y_4, y_6)^T$. There is $y_2$ in $S_a$, so it is a linearly satellite unknown for $\{y_1\}$, and corresponding components of solutions to (8) also can be presented as a Laurent series in $x$.

Let us check that $y_3$ is also linearly satellite for $\{y_1\}$. The result of the Extract algorithm for $\tilde{s} = \{y_1, y_3\}$ contains only a differential system $S_d$ since all the unknowns from $\tilde{s}$ are in $\tilde{y}$ (system $S_a$ is empty in this case):

$$\tilde{y}' = \begin{bmatrix} 1 & -x & 0 & 0 \\ 0 & -1/x & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1/x & 1 & 0 & 1 \end{bmatrix} \tilde{y}, \quad \tilde{y} = (y_1, y_3, y_4, y_5)^T. \tag{9}$$

It remains to check if $y_3$ is linearly satellite for $\{y_1\}$ in (9). The AB-algorithm gives the following matrices

$$\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & -x \\ 0 & -1/x \end{bmatrix}$$

for systems $(S_d)_s^{\mathrm{AB}}$ and $(S_d)_{\tilde{s}}^{\mathrm{AB}}$ respectively. This implies that $y_3$ is also linearly satellite for $\{y_1\}$. Systems $(S_d)_{\{y_1\}}^{\mathrm{AB}}$ and $(S_d)_{\{y_1, y_3\}}^{\mathrm{AB}}$ can be easily solved and we get for components $y_1$, $y_2$, and $y_3$ the following general solutions:

$$y_1 = C_1 e^x + C_2, \quad y_2 = -(C_1 e^x + C_2)/x, \quad y_3 = C_2/x, \tag{10}$$

where $C_1$, $C_2$ are arbitrary constants. The application of the procedure for constructing series solutions directly to system (8) or (9) does not make it possible to find solutions (10) due to presence of the unknowns $y_4$, $y_5$, $y_6$, because the components of solutions corresponding to these unknowns cannot be presented as Laurent series in general.

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# Rogers-Ramanujan Functions, Modular Functions, and Computer Algebra

**Peter Paule and Silviu Radu**

*Dedicated to the symbolic summation pioneer Sergei Abramov who concretely passed milestone 70*

**Abstract** Many generating functions for partitions of numbers are strongly related to modular functions. This article introduces such connections using the Rogers-Ramanujan functions as key players. After exemplifying basic notions of partition theory and modular functions in tutorial manner, relations of modular functions to $q$-holonomic functions and sequences are discussed. Special emphasis is put on supplementing the ideas presented with concrete computer algebra. Despite intended as a tutorial, owing to the algorithmic focus the presentation might contain aspects of interest also to the expert. One major application concerns an algorithmic derivation of Felix Klein's classical icosahedral equation.

**Keywords** $q$-holonomic functions · $q$-series · $q$-products · Modular functions

## 1 Introduction

The main source of inspiration for this article was the truly wonderful paper [14] by William Duke. When reading Duke's masterly exposition, the first named author started to think of writing kind of a supplement which relates the beautiful ingredients of Duke's story to computer algebra. After starting, the necessity to connect to readers

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with diverse backgrounds soon became clear. As a consequence, this tutorial grew longer than originally intended. As a compensation for its length, we hope some readers will find it useful to find various things presented together at one place the first time. Owing to the algorithmic focus, some aspects might have a new appeal also to the expert.

Starting with partition generating functions and using the Omega package, in Sect. 2 the key players of this article are introduced, the Rogers-Ramanujan functions $F(1)$ and $F(q)$.

To prove non-holonomicity, in Sect. 3 the series presentations of $F(1)$ and $F(q)$ are converted into infinite products. Viewing things analytically, the Dedekind eta function, also defined via an infinite product on the upper half complex plane $\mathbb{H}$, is of fundamental importance, in particular, owing to its modular transformation properties.

Section 4 presents basic notions and definitions for modular functions associated to congruence subgroups $\Gamma$ of the modular group $\mathrm{SL}_2(\mathbb{Z})$. These groups are acting on $\mathbb{H}$ and, more generally, also on $\mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$. When restricting this extended action of $\Gamma$ to $\mathbb{Q} \cup \{\infty\}$ the resulting orbits are called cusps; cusps play a crucial role for zero recognition of modular functions.

Section 5 presents concrete examples of modular functions for congruence subgroups $\Gamma$ like the Klein $j$-invariant for $\Gamma := \mathrm{SL}_2(\mathbb{Z})$, the modular discriminant quotient $\Phi_2(\tau) := \Delta(2\tau)/\Delta(\tau)$ for $\Gamma := \Gamma_0(2)$, the (modified) Rogers-Ramanujan functions $G(\tau)$ and $H(\tau)$ being quasi-modular functions for $\Gamma := \Gamma_1(5)$, and the Rogers-Ramanujan quotient $r(\tau) := H(\tau)/G(\tau)$ for $\Gamma(5)$. To obtain information about congruence subgroups the computer algebra system SAGE [35] is used.

Section 6 introduces basic ideas of zero recognition of modular functions. To this end, one passes from modular functions $g$ defined on the upper half complex plane to induced functions $g^*$ defined on compact Riemann surfaces. By transforming problems into settings which involve modular functions with a pole at $\infty$ only, zero recognition turns into a finitary algorithmic procedure.

In Sect. 7 we present examples for zero recognition which despite being elementary should illustrate how to prove relations between $q$-series/$q$-products using modular function machinery. Among other tools, “valence formulas” are used which describe relations between orders of Laurent series expansions.

The example given in Sect. 8 shows that by transforming zero recognition problems into ones involving solely modular functions with a pole at $\infty$ only, one gets an “algorithmic bonus”: a method to derive identities algorithmically.

Many modular functions connected to partition generating functions are not holonomic. But there are strong connections to $q$-holonomic sequences and series which are briefly discussed in Sect. 9. Again the Rogers-Ramanujan functions serve as illustrating examples; here also $q$-hypergeometric summation theory comes into play.

Section 10 is devoted to another classical theme, the presentation of the Rogers-Ramanujan quotient $r(\tau)$ as a continued fraction. Evaluations at real or complex arguments are briefly discussed: most prominently, Ramanujan’s presentation of $r(i)$ in terms of nested radicals.

Finally, Sect. 11 returns to a main theme of Duke's beautiful exposition [14]. Namely, there is a stunning connection, first established by Felix Klein, between the fixed field of the icosahedral group and modular functions. In the latter context Ramanujan's evaluation of $r(i)$ finds a natural explanation as a root of Klein's icosohedral polynomial. An algorithmic derivation of this polynomial is given.

In Sect. 12 (Appendix 1) we briefly discuss general types of function families the Rogers-Ramanujan functions belong to. One such class are generalized Dedekind eta functions which were studied by Meyer [24], Dieter [12], and Schoeneberg [33, Chap. 8] in connection with work of Felix Klein. These functions form a subfamily of an even more general class, the theta functions studied extensively by Farkas and Kra [15].

In Sect. 7 "valence formulas" for $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ and $\Gamma = \Gamma_0(2)$ were used. The Rogers-Ramanujan function setting used in Sect. 11 connects to a "valence formula" for $\Gamma = \Gamma_1(5)$. For the sake of completeness, in Sect. 13 (Appendix 3) we present a "valence formula", Theorem 56, which contains all these instances as special cases. Being not relevant to the main themes of this article, we state this theorem without proof.

Concerning computer algebra packages: In addition to the SAGE examples and RISC packages used in our exposition, we want to point to Frank Garvan who has developed various software relevant to the themes discussed in this tutorial; see, for example, [16] and Garvan's web page for other packages.

## 2 Partition Generating Functions

Problem. Given $n, k \in \mathbb{Z}_{>0}$, determine

$$r_k(n) := \#\{(a_1, \dots, a_k) \in \mathbb{Z}^k_{>0} : a_1 + \cdots + a_k = n \text{ and } a_j - a_{j+1} \geqslant 2 \text{ for } 1 \leqslant j \leqslant k-1\}.$$

Example. $r_2(8) = 3$ because 8 equals $7 + 1, 6 + 2,$ and $5 + 3$. For convenience we define $r_k(0) := 1$ for $k \geqslant 0$.

To solve the problem we consider the generating function of such partitions,

$$R_k := \sum_{n \geqslant 0} r_k(n) q^n = \sum_{\substack{a_1, a_2, \dots, a_k \geqslant 1 \\ a_1 - a_2 \geqslant 2, a_2 - a_3 \geqslant 2, \dots, a_{k-1} - a_k \geqslant 2}} q^{a_1 + a_2 + \cdots + a_k}.$$

To compute such generating functions one can use the Omega package which implements MacMahon's method of partition analysis; see the references in [29].

In[8]:= << **RISC‘Omega‘**

Omega Package V2.49 written by Axel Riese
(in cooperation with George E. Andrews and Peter Paule) © RISC-JKU

To compute $R_4$ one calls

In[9]:= **OR[OSum[q^{a1+a2+a3+a4}, {a1 − a2 ⩾ 2, a2 − a3 ⩾ 2, a3 − a4 ⩾ 2, a4 ⩾ 1}, λ]]**

Out[9]= $$\frac{q^{16}}{(1-q)\left(1-q^2\right)\left(1-q^3\right)\left(1-q^4\right)}$$

In view of the instances for $k = 0, 1, 2, 3$:

$$\{R_0, R_1, R_2, R_3\} = \left\{1, \frac{q}{1-q}, \frac{q^4}{(1-q)(1-q^2)}, \frac{q^9}{(1-q)(1-q^2)(1-q^3)}\right\},$$

the general pattern

$$R_k = \frac{q^{k^2}}{(1-q)(1-q^2)\dots(1-q^k)}, \ k \geqslant 0,$$

becomes obvious. Its proof is by elementary partition reasoning.

Next we consider all such partitions with parts greater or equal to 2:

$$S_k := \sum_{n\geqslant 0} s_k(n)q^n = \sum_{\substack{a_1,a_2,\dots,a_k\geqslant 2 \\ a_1-a_2\geqslant 2, a_2-a_3\geqslant 2,\dots,a_{k-1}-a_k\geqslant 2}} q^{a_1+a_2+\dots+a_k}.$$

For $k = 4$ the Omega package gives:

In[10]:= **OR[OSum[q^{a1+a2+a3+a4}, {a1 − a2 ⩾ 2, a2 − a3 ⩾ 2, a3 − a4 ⩾ 2, a4 ⩾ 2}, λ]]**

Out[10]= $$\frac{q^{20}}{(1-q)\left(1-q^2\right)\left(1-q^3\right)\left(1-q^4\right)}$$

In view of the instances for $k = 0, 1, 2, 3$:

$$\{S_0, S_1, S_2, S_3\} = \left\{1, \frac{q^2}{1-q}, \frac{q^6}{(1-q)(1-q^2)}, \frac{q^{12}}{(1-q)(1-q^2)(1-q^3)}\right\},$$

the general pattern

$$S_k = \frac{q^{k^2+k}}{(1-q)(1-q^2)\dots(1-q^k)}, \ k \geqslant 0,$$

becomes obvious. Again the proof is by elementary partition reasoning.

We will use the standard $q$-notation

$$(q;q)_k := (1-q)(1-q^2)\dots(1-q^k),\, k \geqslant 1,\;\; (q;q)_0 := 1, \tag{1}$$

and

$$(q;q)_\infty := \prod_{k=1}^{\infty}\left(1-q^k\right). \tag{2}$$

For example, the generating function for $p(n)$, the number all partitions of $n$, is

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty}.$$

This infinite product representation implies that the sequence $(p(n))_{n\geqslant 0}$ is not holonomic,[1] because otherwise its generating function would have at most finitely many singularities; see, for instance, [19].

In connection with $R_k$ and $S_k$, the alternative notation

$$f_k(z) := \frac{q^{k^2} z^k}{(q;q)_k}$$

will be useful: $R_k = f_k(1)$ and $S_k = f_k(q)$. Defining

$$F(z) := \sum_{k=0}^{\infty} f_k(z) = \sum_{k=0}^{\infty} \frac{q^{k^2} z^k}{(q;q)_k}, \tag{3}$$

the key players of this article will be

$$F(1) = \sum_{k=0}^{\infty} \frac{q^{k^2}}{(q;q)_k} \text{ and } F(q) = \sum_{k=0}^{\infty} \frac{q^{k^2+k}}{(q;q)_k}, \tag{4}$$

called *Rogers-Ramanujan functions*; see, for instance, [7, Chap. 8].

From above it is clear that $F(1) = \sum_{n\geqslant 0} r(n)q^n$, resp. $F(q) = \sum_{n\geqslant 0} s(n)q^n$, are the generating functions for the number $r(n)$, resp. $s(n)$, of partitions into parts with minimal difference 2 with all parts greater than 0, respectively 1. For combinatorial purposes it is absolutely sufficient to view them as formal power series in the indeterminate $q$. But as we shall see, when interpreting them in the context of complex analysis—citing Zagier [39]—there is also a "hidden non-abelian symmetry" which can be used as the "magic principle of modular forms."

[1] i.e., it does not satisfy a linear recurrence with polynomial coefficients.

## 3 $q$-Products and Dedekind's eta Function

As with $(p(n))_{n\geqslant 0}$, to decide whether the sequences $(r(n))_{n\geqslant 0}$ and $(s(n))_{n\geqslant 0}$ are holonomic, one could try to convert their generating functions $F(1)$ and $F(q)$ into infinite product form. To do so, there is a tool already known to Euler and popularized by Andrews [4]; we state it in a (slightly modified) version taken from [22, Theorem 2.9].

**Theorem 3** *Let $\varphi(q)$ be an analytic function without zeros in the disk $|q| < R$ for some $R \leqslant 1$, and let $(\varepsilon_n)_{n\geqslant 1}$ be a sequence containing only the numbers 1 and $-1$. Then there exists a unique sequence $(a_n)_{n\geqslant 1}$ of complex numbers such that the product $\varphi(0)\prod_{n=1}^{\infty}(1+\varepsilon_n q^n)^{a_n}$ converges to $\varphi$ uniformly on compact subsets of the disk $|q| < R$. Moreover, if $\varphi(q) = 1 + \sum_{n=1}^{\infty} b_n q^n$ and $\varepsilon_n = -1$ for all $n \geqslant 1$, then*

$$-n\, a_n = n\, b_n + \sum_{\substack{d|n\\ d<n}} d\, a_d + \sum_{j=1}^{n-1}\sum_{d|j} d\, a_d\, b_{n-j}, \;\; n \geqslant 1. \tag{5}$$

Taking as input the Taylor series coefficients $b_n$, with recurrence (5) one can compute the exponents $a_n$. For example, for the truncated $F(1)$ series:

$$\begin{aligned}\varphi(q) := \sum_{k=0}^{30} f_k(1) = \sum_{k=0}^{30}\frac{q^{k^2}z^k}{(q;q)_k} = 1+\sum_{n=1}^{\infty} b_n q^n &= 1+q+q^2+q^3+2q^4+2q^5\\ &+3q^6+3q^7+4q^8+5q^9+6q^{10}+7q^{11}+9q^{12}+10q^{13}+12q^{14}+14q^{15}\\ &+17q^{16}+19q^{17}+23q^{18}+26q^{19}+31q^{20}+35q^{21}+41q^{22}+46q^{23}+54q^{24}\\ &+61q^{25}+70q^{26}+79q^{27}+91q^{28}+102q^{29}+117q^{30}+\dots,\end{aligned}$$

one obtains as output

$$\begin{aligned}(a_n)_{n\geqslant 1} = (&-1,0,0,-1,0,-1,0,0,-1,0,-1,0,0,-1,0,\\ &-1,0,0,-1,0,-1,0,0,-1,0,-1,0,0,-1,0,\dots).\end{aligned}$$

The pattern suggests that

$$F(1) = \sum_{n=0}^{\infty} r(n)q^n = \sum_{k=0}^{\infty}\frac{q^{k^2}}{(q;q)_k} = \prod_{m=0}^{\infty}\frac{1}{(1-q^{5m+1})(1-q^{5m+4})}, \tag{6}$$

and, after carrying out an analogous computation for $F(q)$,

$$F(q) = \sum_{n=0}^{\infty} s(n)q^n = \sum_{k=0}^{\infty}\frac{q^{k^2+k}}{(q;q)_k} = \prod_{m=0}^{\infty}\frac{1}{(1-q^{5m+2})(1-q^{5m+3})}. \tag{7}$$

As in the case of $p(n)$ the non-holonomicity of the partition number sequences $r(n)$ and $s(n)$ follows immediately from the infinite product representations (6) and (7). But in contrast to the simple derivation of the generating function for the $p(n)$, these product expansions are substantially more difficult to prove. In fact, (6) and (7) are the celebrated Rogers-Ramanujan identities, also called Rogers-Ramanujan-Schur identities owing to the fact that Issai Schur independently discovered and proved them. There is a vast literature on background, proofs, and history; [4] is a reference which also connects to computer algebra and to applications in the frame of Baxter's hard hexagon model in statistical mechanics.

In the general setting of Theorem 3, $q$ is interpreted as a complex variable. Nevertheless, to compute the exponent sequence $(a_n)$ we can apply the recurrence (5) in the case of a given *formal* power series $\varphi(q)$; i.e.; taking $q$ as an indeterminate. But, in order to consider the announced "magic principle of modular forms", one again turns to complex analysis by setting

$$q = q(\tau) := \exp(2\pi i \tau) = e^{2\pi i \tau},$$

where $\tau$ is taken from the upper half complex plane $\mathbb{H} := \{\tau \in \mathbb{C} : \mathrm{Im}(\tau) > 0\}$. In this setting, the analytic counterpart of the multiplicative inverse of the formal power series $\sum_{n=0}^{\infty} p(n) q^n$ is the Dedekind eta function,

$$\eta : \mathbb{H} \to \mathbb{C}, \tau \mapsto \eta(\tau) := q(\tau)^{\frac{1}{24}} \prod_{k=1}^{\infty} (1 - q(\tau))^k .$$

The above mentioned "hidden non-abelian symmetry" is with respect to modular transformations of $\tau$ under elements of the non-abelian modular group,

$$\mathrm{SL}_2(\mathbb{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{Z}^{2\times 2} : a\, d - b\, c = 1 \right\},$$

which acts on $\mathbb{H}$ by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \tau := \frac{a\tau+b}{c\tau+d}$, and which is generated by the matrices $S := \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $T := \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Under modular transformations $\eta$ behaves as follows [13, 23.18.5]:

$$\eta(\tau + 1) = e^{2\pi i/24} \eta(\tau) \text{ and } \eta\left(-\frac{1}{\tau}\right) = \sqrt{\frac{\tau}{i}} \cdot \eta(\tau), \tag{8}$$

and w.l.o.g. assuming that $c > 0$:

$$\eta\left(\frac{a\,\tau + b}{c\,\tau + d}\right) = e^{2\pi i\, \rho(a,b,c,d)/24} \cdot \sqrt{\frac{c\tau + d}{i}} \cdot \eta(\tau).$$

Here $\rho(a, b, c, d)$ is a complicated but integer-valued expression depending on $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$; the complex-valued square root is taken to have positive real part.

As a consequence, the modular discriminant[2]

$$\Delta(\tau) := \eta(\tau)^{24} = q\,(q;q)_\infty^{24},\ \ q = q(\tau), \tag{9}$$

behaves under modular transformation as

$$\Delta\left(\frac{a\,\tau + b}{c\,\tau + d}\right) = (c\tau + d)^{12} \cdot \Delta(\tau). \tag{10}$$

This, in view of $(c\tau + d)^{12}$ as the "factor of automorphy", makes $\Delta$ a modular form of weight 12 for $\mathrm{SL}_2(\mathbb{Z})$. However, in this article we will mostly deal with modular functions having 1 as the "factor of automorphy"; i.e., modular forms of weight 0.

For later we note that the Dedekind eta function $\eta(\tau)$ and hence the modular discriminant $\Delta(\tau)$ are non-zero analytic functions on $\mathbb{H}$. This is implied by

**Lemma 4** *Let* $f(\tau) := \prod_{m=0}^{\infty}\left(1 - e^{2\pi i \tau(am+b)}\right)$ *where* $\tau \in \mathbb{H}$ *and* $a, b \in \mathbb{Z}$ *such* $0 \leqslant b < a$. *Then* $f(\tau)$ *is an analytic function on* $\mathbb{H}$ *with* $f(\tau) \neq 0$ *for all* $\tau \in \mathbb{H}$.

*Proof* (*sketch*). The statement follows from convergence properties of the infinite product form of $f$; see, e.g., [5, Appendix A] for details.

## 4 Modular Functions: Definitions

We restrict our discussion to basic definitions and very few notions. For further details on modular forms and modular functions see, for instance, the classical monograph [11].

Besides the full modular group $\mathrm{SL}_2(\mathbb{Z})$ the following subgroups for $N \in \mathbb{Z}_{>0}$ will be relevant:

$$\Gamma_0(N) := \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod{N}\right\},$$
$$\Gamma_1(N) := \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod{N}\right\},$$
$$\Gamma(N) := \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N}\right\}.$$

Here $*$ serves as a placeholder for an integer; the congruence relation $\equiv$ between matrices has to be taken entrywise. Sometimes we write $I$ for the identity matrix; i.e., $I := \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. To indicate subgroup relations we use $\leqslant$; hence

[2] In our context it is convenient to normalize as in (9) instead of using the version $\Delta(\tau) := (2\pi)^{12}\eta(\tau)^{24}$.

$$\Gamma(N) \leqslant \Gamma_1(N) \leqslant \Gamma_0(N) \leqslant \mathrm{SL}_2(\mathbb{Z}) = \Gamma(1).$$

A subgroup $\Gamma \leqslant \mathrm{SL}_2(\mathbb{Z})$ with $\Gamma(N) \leqslant \Gamma$ for some fixed $N \in \mathbb{Z}_{>0}$ is called congruence subgroup. For the subgroup $\Gamma(N)$, called principal congruence subgroup of level $N$, one has

**Proposition 5** *The principal congruence subgroup $\Gamma(N)$ is normal in* $\mathrm{SL}_2(\mathbb{Z})$*; its index* $[\mathrm{SL}_2(\mathbb{Z}) : \Gamma(N)]$ *is finite for all $N$.*

*Proof* Considering the entries of $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ modulo $N$ induces a group isomorphism $\mathrm{SL}_2(\mathbb{Z})/\Gamma(N) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$. This implies the statement.

**Definition 6** An *analytic (resp. meromorphic) modular function* $g : \mathbb{H} \to \hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ for a congruence subgroup $\Gamma$ is defined by the following three properties:

- $g : \mathbb{H} \to \hat{\mathbb{C}}$ is analytic on $\{\tau \in \mathbb{H} : \mathrm{Im}(\tau) > M\}$ for some $M > 0$;
- for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$,

$$g\left(\frac{a\tau + b}{c\tau + d}\right) = g(\tau), \ \tau \in \mathbb{H}; \tag{11}$$

- for each $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ there exists a Laurent series expansion of $g(\gamma\tau)$ with *finite principal part*. This means, for all $\tau \in \mathbb{H}$ (resp. for all $\tau \in \mathbb{H}$ with $\mathrm{Im}(\tau) > M$ in case $g$ is meromorphic),

$$g\left(\frac{a\tau + b}{c\tau + d}\right) = g(\gamma\tau) = \sum_{n=-M_\gamma}^{\infty} c_n(\gamma) e^{2\pi i n\tau/w_\gamma^\Gamma} \tag{12}$$

where $M_\gamma \in \mathbb{Z}$ and

$$w_\gamma^\Gamma := \min\left\{h \in \mathbb{Z}_{>0} : \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \in \gamma^{-1}\Gamma\gamma \text{ or } \begin{pmatrix} -1 & h \\ 0 & -1 \end{pmatrix} \in \gamma^{-1}\Gamma\gamma\right\}. \tag{13}$$

*Example 7* It is possible that for $h \in \mathbb{Z}_{>0}$,

$$\begin{pmatrix} -1 & h \\ 0 & -1 \end{pmatrix} \in \gamma^{-1}\Gamma\gamma \text{ but } \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \notin \gamma^{-1}\Gamma\gamma;$$

take, for instance, $\Gamma = \Gamma_1(4)$, $h = 1$ and $\gamma = \left(\begin{smallmatrix} 1 & -1 \\ 2 & -1 \end{smallmatrix}\right)$.

The condition (12) has a strong technical flavour. Hence some background motivation is in place. The fundamental underlying observation is a basic fact concerning Fourier expansions:

**Lemma 8** *For $M > 0$ let $f : \mathbb{H} \to \hat{\mathbb{C}}$ be meromorphic such that* $\mathrm{Im}(p) \leqslant M$ *for each of its poles $p \in \mathbb{H}$. Suppose $f$ is periodic with period 1. Then there exists a unique analytic function*

$$h : \{z \in \mathbb{C} : 0 < |z| < R\} \to \mathbb{C} \textit{ for some } R \leqslant 1,$$

*such that*

$$f(\tau) = h(e^{2\pi i \tau}), \ \ \tau \in \mathbb{H}. \tag{14}$$

*Moreover, if $f$ has no poles, one can choose $R = 1$.*

Since $h$ is analytic on a punctured open disk, there exists a Laurent expansion $h(z) = \sum_{n=-\infty}^{\infty} h_n z^n$ about 0 with coefficients in $\mathbb{C}$; i.e., for $\mathrm{Im}(\tau)$ large enough,

$$f(\tau) = \sum_{n=-\infty}^{\infty} h_n \left(e^{2\pi i \tau}\right)^n. \tag{15}$$

Suppose $f$ is as in Lemma 8 but with period $w \in \mathbb{Z}_{>0}$ greater than 1. Then $F(\tau) := f(w\tau)$ has period 1, and $f$ has an expansion of the form

$$f(\tau) = F\left(\frac{\tau}{w}\right) = \sum_{n=-\infty}^{\infty} h_n \left(e^{2\pi i \tau}\right)^{\frac{n}{w}}. \tag{16}$$

Now let $g : \mathbb{H} \to \hat{\mathbb{C}}$ be a meromorphic function satisfying the same conditions as $f$ in Lemma 8 and, in addition, the modular invariance property (11). Then for any $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ the function $g \circ \gamma$ has period $w_\gamma^\Gamma$. Namely, according to the definition of $w_\gamma^\Gamma$,

$$\begin{pmatrix} 1 & w_\gamma^\Gamma \\ 0 & 1 \end{pmatrix} \in \gamma^{-1}\Gamma\gamma \text{ or } \begin{pmatrix} -1 & w_\gamma^\Gamma \\ 0 & -1 \end{pmatrix} \in \gamma^{-1}\Gamma\gamma.$$

Hence, in any case, there is a $\rho \in \Gamma$ such that $\tau + w_\gamma^\Gamma = \gamma^{-1}\rho\gamma\tau$, and thus

$$(g \circ \gamma)(\tau + w_\gamma^\Gamma) = g(\gamma(\tau + w_\gamma^\Gamma)) = g(\gamma\gamma^{-1}\rho\gamma\tau) = g(\rho\gamma\tau) = g(\gamma\tau) = (g \circ \gamma)(\tau).$$

As a consequence, $f := g \circ \gamma$ has the period $w_\gamma^\Gamma$, and an expansion as in (16) exists. Condition (12) now requires that this expansion has *finite principal part*. As we shall see this requirement is needed to extend $g \circ \gamma : \mathbb{H} \to \hat{\mathbb{C}}$ to a function $g \circ \gamma : \mathbb{H} \cup \mathbb{Q} \cup \{\infty\} \to \hat{\mathbb{C}}$.

We want to emphasize that representations of infinity as $\infty = \frac{a}{0}$, $a \in \mathbb{Z} \setminus \{0\}$, are explicitly included in our setting which formally is done by including the obvious arithmetical rules and by the natural extension of the group action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$ to an action on $\hat{\mathbb{H}} := \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$. Note that the extended action maps elements from $\mathbb{Q} \cup \{\infty\}$ to $\mathbb{Q} \cup \{\infty\}$.

Further remarks on Definition 6 are in place. In view of $q^{1/w_\gamma^\Gamma} = e^{2\pi i \tau / w_\gamma^\Gamma}$ and $\frac{a}{c} = \gamma\infty$ $(= \lim_{\mathrm{Im}(\tau)\to\infty} \gamma\tau)$, expansions as in (12) are called $q$-expansions of $g$ at $\frac{a}{c}$ for $\Gamma$. Taking $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, $\gamma' = \left(\begin{smallmatrix} a & b' \\ c & d' \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ such that $\frac{a}{c} = \gamma\infty = \gamma'\infty$, it is a natural question to ask in which way the corresponding $q$-expansions differ. The answer is given by the following fact that is straightforward to verify.

**Proposition 9** *Let g be a meromorphic modular function for a congruence subgroup $\Gamma$. Let $\gamma, \gamma' \in \mathrm{SL}_2(\mathbb{Z})$, $\rho \in \Gamma$, and $m \in \mathbb{Z}$ such that*

$$\gamma' = \rho\,\gamma \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \text{ or } \gamma' = \rho\,\gamma \begin{pmatrix} -1 & m \\ 0 & -1 \end{pmatrix} \tag{17}$$

*Suppose the q-expansion of g at $\frac{a}{c} := \gamma\infty$ is*

$$g(\gamma\tau) = \sum_{n=-M}^{\infty} c_n q^{\frac{n}{w}} \tag{18}$$

*with $w := w_\gamma^\Gamma$. Then*

$$w = w_{\gamma'}^{\Gamma} \tag{19}$$

*and*

$$g(\gamma'\tau) = g(\gamma(\tau + m)) = \sum_{n=-M}^{\infty} e^{2\pi i m n/w} c_n q^{\frac{n}{w}}. \tag{20}$$

This means, the $q$-expansions (18) and (20) at $\frac{a}{c}$ differ in their coefficients only by the factor $e^{2\pi i m n/w}$. This in particular holds if besides $\frac{a}{c} = \gamma\infty$ also $\frac{a}{c} = \gamma'\infty$. Because then $\gamma^{-1}\gamma'\infty = \infty$, which implies that $\gamma^{-1}\gamma' = \begin{pmatrix} \pm 1 & m \\ 0 & \pm 1 \end{pmatrix}$ for some $m \in \mathbb{Z}$ and we are in the case $\rho = I$.

This observation also enables us to extend the domain of the modular function $g$ to $\hat{\mathbb{H}}$. This extension is of particular relevance for the zero recognition of modular functions; see Sect. 6.

**Definition 10** Let $g : \mathbb{H} \to \hat{\mathbb{C}}$ be a meromorphic modular function for a congruence subgroup $\Gamma$ with $q$-expansion at $\frac{a}{c} := \gamma\infty$ as in (18). Then $g$ extends to $\hat{g} : \hat{\mathbb{H}} \to \hat{\mathbb{C}}$ as follows: $\hat{g}(\tau) := g(\tau)$ for $\tau \in \mathbb{H}$, and

$$\hat{g}\left(\frac{a}{c}\right) := \begin{cases} \infty, & \text{if } M > 0, \\ c_0, & \text{if } M = 0, \\ 0, & \text{if } M > 0. \end{cases}$$

Convention. Since each modular function has such an extension, we will also write $g$ for the extension $\hat{g}$.

Using Proposition 9 one can verify that the $\Gamma$-invariance (11) of $g$ on points $\tau \in \mathbb{H}$ carries over to the points $\frac{a}{c} \in \hat{\mathbb{Q}} := \mathbb{Q} \cup \{\infty\}$:

**Proposition 11** *Let g be a meromorphic modular function for a congruence subgroup $\Gamma$. Then for any $\frac{a}{c} \in \hat{\mathbb{Q}}$:*

$$g\left(\rho\frac{a}{c}\right) = g\left(\frac{a}{c}\right) \textit{ for all } \rho \in \Gamma. \tag{21}$$

Property (11) says that $g$ is invariant on the orbits of the $\Gamma$-action on $\mathbb{H}$; Proposition 11 says that $g$ is invariant on the orbits of the extended $\Gamma$-action on $\hat{\mathbb{Q}}$. The latter orbits got a special name.

**Definition 12** Let $\Gamma$ be a congruence subgroup. The $\Gamma$-orbits

$$\left[\frac{a}{c}\right]_\Gamma := \left\{\rho\frac{a}{c} : \rho \in \Gamma\right\}, \ \frac{a}{c} \in \hat{\mathbb{Q}},$$

of the action of $\Gamma$ on $\hat{\mathbb{Q}}$ are called cusps (of $\Gamma$).

*Convention.* If $\Gamma$ is clear from the context, we write $\left[\frac{a}{c}\right]$ instead of $\left[\frac{a}{c}\right]_\Gamma$.

**Proposition 13** *Let $\Gamma$ be a congruence subgroup. Then the number of cusps, this means, the number of orbits of the action of $\Gamma$ on $\hat{\mathbb{Q}}$, is finite.*

*Proof* The statement is true because congruence subgroups have finite index (Proposition 5) and any coset decomposition

$$\mathrm{SL}_2(\mathbb{Z}) = \Gamma\gamma_0 \cup \Gamma\gamma_1 \cup \ldots \cup \Gamma\gamma_k$$

implies

$$\hat{\mathbb{Q}} = \mathrm{SL}_2(\mathbb{Z})(\infty) = \Gamma(\gamma_0\infty) \cup \Gamma(\gamma_1\infty) \cup \cdots \cup \Gamma(\gamma_k\infty).$$

**Definition 14** Let $\Gamma$ be a congruence subgroup. Recalling (13), for $\frac{a}{c} \in \hat{\mathbb{Q}}$ define the width of the cusp $\left[\frac{a}{c}\right]_\Gamma$ as

$$w^\Gamma_{[a/c]} := w^\Gamma_\gamma \text{ with } \gamma \in \mathrm{SL}_2(\mathbb{Z}) \text{ such that } \gamma\infty = \frac{a}{c}. \tag{22}$$

The width is well-defined: Suppose $\frac{a'}{c'} = \gamma'\infty \in \left[\frac{a}{c}\right]_\Gamma$. Then $\frac{a'}{c'} = \rho\frac{a}{c}$ for some $\rho \in \Gamma$. Hence $\gamma'\infty = \rho\gamma\infty$; i.e., $(\gamma')^{-1}\rho\gamma = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$. The rest follows from (19).

*Convention.* If $\Gamma$ is clear from the context, we will write $w_{[a/c]}$ instead of $w^\Gamma_{[a/c]}$.

Another fact implied by Proposition 9 is that one has to consider only *finitely* many cases to check the finite principal-part-property (12). But more is true. Define the stabilizer subgroup

$$\mathrm{Stab}(\infty) := \{\gamma \in \mathrm{SL}_2(\mathbb{Z}) : \gamma\infty = \infty\} = \left\{\pm\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} : m \in \mathbb{Z}\right\} \leqslant \mathrm{SL}_2(\mathbb{Z}).$$

Given a coset decomposition $\mathrm{SL}_2(\mathbb{Z}) = \Gamma\gamma_0 \cup \cdots \cup \Gamma\gamma_k$, it is obvious that the set of all cusps of $\Gamma$ is formed by $\{[\gamma_j\infty]_\Gamma : j = 0, \ldots, k\}$.[3] The following lemma is important but straighforward to check.

[3]Despite the cosets being assumed to be pairwise different, it may well be that $[\gamma_i\infty]_\Gamma = [\gamma_j\infty]_\Gamma$ for $i \neq j$.

**Lemma 15**

$$[\gamma_i\infty]_\Gamma = [\gamma_j\infty]_\Gamma \quad \Leftrightarrow \quad \gamma_j = \rho\gamma_i\sigma \text{ for some } \rho \in \Gamma \text{ and } \sigma \in \mathrm{Stab}(\infty).$$

This lemma puts us into the position to verify that to establish the finite principal-part-property (12), it is sufficient to check (12) at the cusps:

Let $g : \mathbb{H} \to \hat{\mathbb{C}}$ be a meromorphic function which is analytic on $\{\tau \in \mathbb{H} : \mathrm{Im}(\tau) > M\}$ for some $M > 0$. Suppose that $g$ satisfies the modular transformation property (11) for a congruence subgroup $\Gamma$. Then by the same reasoning as to obtain (16) we know that for each $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ there exists a Laurent series expansion of $g(\gamma\tau)$. This means, for all $\tau \in \mathbb{H}$ with $\mathrm{Im}(\tau) > M$,

$$g\left(\frac{a\tau + b}{c\tau + d}\right) = g(\gamma\tau) = \sum_{n=-\infty}^{\infty} c_n(\gamma)e^{2\pi i n\tau/w_\gamma^\Gamma}. \tag{23}$$

**Lemma 16** *In the given setting, let* $\{[\delta_1\infty]_\Gamma, \dots, [\delta_m\infty]_\Gamma\}$ *with* $\delta_\ell \in \mathrm{SL}_2(\mathbb{Z})$ *be a complete set of different cusps of* $\Gamma$*. Suppose the* $q$*-expansions at all these cusps have a finite principal part; i.e.,*

$$g(\delta_\ell\tau) = \sum_{n=-M_{\delta_\ell}}^{\infty} c_n(\delta_\ell)e^{2\pi i n\tau/w_{\delta_\ell}^\Gamma}, \;\; \ell = 1, \dots, m.$$

*Then the* $q$*-expansions* (23) *have finite principal parts for all* $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$.

*Proof* Let $\mathrm{SL}_2(\mathbb{Z}) = \Gamma\gamma_0 \cup \dots \cup \Gamma\gamma_k$ be a coset decomposition. Then for a given $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ there is a $j \in \{0, \dots, k\}$ such that $\gamma \in \Gamma\gamma_j$. For the respective cusp we have $[\gamma_j\infty]_\Gamma = [\delta_\ell\infty]_\Gamma$ for some $\ell \in \{1, \dots, m\}$. By Lemma 15, $\gamma_j = \rho_1\delta_\ell\sigma$ for some $\rho_1 \in \Gamma$ and $\sigma \in \mathrm{Stab}(\infty)$. By assumption, $\gamma_j = \rho_2^{-1}\gamma$ for some $\rho_2 \in \Gamma$, and thus $\gamma = (\rho_2\rho_1)\delta_\ell\sigma$. But now Proposition 9 says: if the $q$-expansion of $g(\delta_\ell\tau)$ has finite principal part, this is also true for $g(\gamma\tau)$.

## 5 Modular Functions: Examples

In this section we present examples to illustrate the notions of Sect. 4 and which are of relevance for later sections.

*Example 17* Consider the $\Phi_2$ function which we will use also in Example 18,

$$\Phi_2 : \mathbb{H} \to \mathbb{C}, \; \tau \mapsto \Phi_2(\tau) := \left(\frac{\eta(2\tau)}{\eta(\tau)}\right)^{24} = \frac{\Delta(2\tau)}{\Delta(\tau)}. \tag{24}$$

It is an analytic modular function for $\Gamma_0(2)$: by Lemma 4 it is analytic on $\mathbb{H}$ and, as verified below, it satisfies for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(2)$,

$$\Phi_2\left(\frac{a\tau+b}{c\tau+d}\right) = \Phi_2(\tau). \tag{25}$$

The disjoint coset decomposition[4]

$$\mathrm{SL}_2(\mathbb{Z}) = \Gamma_0(2)\gamma_0 \dot\cup \Gamma_0(2)\gamma_1 \dot\cup \Gamma_0(2)\gamma_2 \text{ with } \gamma_0 = I, \gamma_1 = T, \gamma_2 = TS,$$

is straightforward to verify; hence $[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(2)] = 3$. Owing to $0 = \gamma_1\infty = \gamma_2\infty$,

$$\hat{\mathbb{Q}} = \mathrm{SL}_2(\mathbb{Z})(\infty) = \Gamma_0(2)(\infty) \cup \Gamma_0(2)(0);$$

i.e., $\Gamma_0(2)$ has the two cusps $[\infty]_{\Gamma_0(2)}$ and $[0]_{\Gamma_0(2)}$ with widths $w_{[\infty]}^{\Gamma_0(2)} = w_I^{\Gamma_0(2)} = 1$ and $w_{[0]}^{\Gamma_0(2)} = w_T^{\Gamma_0(2)} = 2$.[5] By Lemma 16, to check the finite principal part property (12) it is sufficient to inspect the $q$-expansions at $\frac{a}{c} = \infty = I\infty$ and $\frac{a}{c} = 0 = T\infty$:

$$\begin{aligned}\Phi_2(I\tau) = \Phi_2(\tau) &= \sum_{n=-M_I}^{\infty} c_n(I) e^{2\pi i n\tau/w_I^{\Gamma_0(2)}} = \sum_{n=-M_I}^{\infty} c_n(I) q^n \\ &= q + 24q^2 + 300q^3 + 2624q^4 + 18126q^5 + 105504q^6 + 538296q^7 + \dots, \end{aligned} \tag{26}$$

$$\begin{aligned}\Phi_2(T\tau) = \Phi_2\left(-\frac{1}{\tau}\right) &= \sum_{n=-M_T}^{\infty} c_n(T) e^{2\pi i n\tau/w_T^{\Gamma_0(2)}} = \sum_{n=-M_T}^{\infty} c_n(T) q^{\frac{n}{2}} \\ &= \frac{1}{2^{12}}\frac{1}{\Phi_2(\tau/2)} = \frac{1}{2^{12}}(q^{-1/2} - 24 + 276q^{1/2} - 2048q + 11202q^{3/2} - \dots). \end{aligned} \tag{27}$$

The first equality in (27) comes from (10), the second equality from (26).

*Proof (Proof of* (25)*)* The proof is a consequence of the following observation. For $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(2)$,

$$\Delta\left(2\frac{a\tau+b}{c\tau+d}\right) = \Delta\left(\frac{a(2\tau)+2b}{\frac{c}{2}(2\tau)+d}\right) = \Delta\left(\begin{pmatrix} a & 2b \\ \frac{c}{2} & d \end{pmatrix}(2\tau)\right) = \left(\frac{c}{2}(2\tau)+d\right)^{12}\Delta(2\tau).$$

---

[4]Recall, $T = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, $S = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$.

[5]$\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} a & b \\ 2c' & d \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} d & -2c' \\ -b & a \end{smallmatrix}\right)$ implies $\left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right) \in T^{-1}\Gamma_0(2)T$.

*Example 18* Consider the Klein $j$ function (also called: $j$-invariant),

$$\mathbb{H} \to \mathbb{C},\ \tau \mapsto j(\tau) := \frac{\left(1 + 2^8 \Phi_2(\tau)\right)^3}{\Phi_2(\tau)} \text{ with } \Phi_2(\tau) := \left(\frac{\eta(2\tau)}{\eta(\tau)}\right)^{24}. \tag{28}$$

By Lemma 4 it is analytic on $\mathbb{H}$. Hence, by Example 17, it is an analytic modular function for $\Gamma = \Gamma_0(2)$. But more is true: it is a well-known classical fact that $j$ is a modular function for the full modular group $\mathrm{SL}_2(\mathbb{Z})$. Nevertheless, this $\mathrm{SL}_2(\mathbb{Z})$-modularity cannot be directly deduced from the function presentation (28). To this end, one better uses one of the classical presentations like

$$j(\tau) = \frac{E_4(\tau)^3}{\Delta(\tau)} \text{ with } E_4(\tau) := 1 + 240 \sum_{n=1}^{\infty} \Bigg( \sum_{1 \leqslant d | n} d^3 \Bigg) q^n. \tag{29}$$

We will prove this identity in Example 6. Assuming the $\mathrm{SL}_2(\mathbb{Z})$-modularity of $j$ as proven, in view of (12), at *all* points $\frac{a}{c} = \gamma\infty \in \hat{\mathbb{Q}}$, $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, one can use one and the same $q$-expansion:

$$\begin{aligned} j\left(\frac{a\tau + b}{c\tau + d}\right) &= j(\gamma\tau) = j(\tau) = j(I\tau) = \sum_{n=-M_I}^{\infty} c_n(I) e^{2\pi i n \tau / w_I^{\mathrm{SL}_2(\mathbb{Z})}} \\ &= \frac{1}{q} + 744 + 196884\, q + 21493760\, q^2 + 864299970\, q^3 + \dots \end{aligned} \tag{30}$$

We point to the (elementary) fact that $\mathrm{SL}_2(\mathbb{Z})$ has only one cusp; namely, $\hat{\mathbb{Q}} = \mathrm{SL}_2(\mathbb{Z})(\infty) = [\infty] = [\gamma\infty]$ for any $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. Obviously, its width is $w_I^{\mathrm{SL}_2(\mathbb{Z})} = 1 = w_{[a/c]}^{\mathrm{SL}_2(\mathbb{Z})}$ for any $\frac{a}{c} \in \hat{\mathbb{Q}}$.

Concerning the presentation (28) of $j$ in terms of eta products, in Sect. 8 we shall not only give a proof but also *derive* it algorithmically. We also note that $E_4(\tau)$ belongs to the sequence of Eisenstein series defined as[6]

$$E_{2k}(\tau) := 1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \Bigg( \sum_{1 \leqslant d | n} d^{2k-1} \Bigg) q^n,\ \ k \geqslant 2,$$

and which under modular transformations behave similarly to $\Delta$:

$$E_{2k}\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^{2k} \cdot E_{2k}(\tau),\ \ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}). \tag{31}$$

[6] The $B_n$ are the Bernoulli numbers; as for $\Delta$, also for the Eisenstein series we prefer the normalized versions.

Now, this transformation property of $E_4$ together with that of $\Delta$ in a *direct fashion* yields that $j$ satisfies (11) for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. As for $j$ there are various presentations of the $E_{2k}$. For example, presenting $E_4$ in terms of the eta function as in [26, (1.28)] implies (28).

*Example 19* As we will see below, information about cusps, widths of cusps, etc. can be essential also for computational reasons. With computer algebra systems like SAGE such kind of data can be also obtained algorithmically; see, e.g., [34]. For example,

```
beagle:~> sage

SageMath version 7.6, Release Date: 2017-03-25
Type "notebook()" for the browser-based notebook interface.
Type "help()" for help.

sage: sage.modular.arithgroup.arithgroup_generic.ArithmeticSubgroup.coset_reps(Gamma0(2))

[
[1 0]  [ 0 -1]  [ 0 -1]
[0 1], [ 1  0], [ 1  1]
]
sage: Cusps=Gamma0(2).cusps()
sage: Cusps
[0, Infinity]
sage: [Gamma0(2).cusp_width(c) for c in Cusps]
[2, 1]
```

The first command loads SAGE; the second computes the coset representatives $I, T$ and $TS$; the third and fourth commands tell us that $\Gamma_0(2)$ has the two cusps [0] and [$\infty$] with widths 2 and 1, respectively.

*Example 20* Using again $q=q(\tau)=e^{2\pi i\tau}$, the slightly modified Rogers-Ramanujan functions,

$$G(\tau) := q^{-\frac{1}{60}} F(1) = q^{-\frac{1}{60}} \prod_{m=0}^{\infty} \frac{1}{(1-q^{5m+1})(1-q^{5m+4})} \tag{32}$$

and

$$H(\tau) := q^{\frac{11}{60}} F(q) = q^{\frac{11}{60}} \prod_{m=0}^{\infty} \frac{1}{(1-q^{5m+2})(1-q^{5m+3})} \tag{33}$$

behave well under the action of $\Gamma_1(5)$: for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1(5)$ with $\gcd(a, 6) = 1$:

$$G\left(\frac{a\tau+b}{c\tau+d}\right) = e^{2\pi i\,\alpha(a,b,c)/60}\, G(\tau) \tag{34}$$

and

$$H\left(\frac{a\tau+b}{c\tau+d}\right) = e^{2\pi i\,\beta(a,b,c)/60}\, H(\tau), \tag{35}$$

where

$$\alpha(a, b, c) = a(9 - b + c) - 9 \text{ and } \beta(a, b, c) = a(3 + 11b + c) - 3.$$

*Proof (Proof (sketch).)* As a general note, all known proofs of the modular transformation property (11) of $G(\tau)$ and $H(\tau)$, and of functions of similar type, rely on product representations like (32) and (33). In [32] Robins considered an important special case of a very general formula established by Schoeneberg [33]. For the crucial ingredient $\mu_{\delta,g}$ of this formula [32, (9)], Robins derived a very compact expression [32, Theorem 2] which for $\delta = 5$ and $g = 1, 2$ gives the stated versions of $\alpha(a, b, c)$ and $\beta(a, b, c)$. As a note, Robins' formula is correct provided $c > 0$. But in view of $(a\tau + b)/(c\tau + d) = (-a\tau - b)/(-c\tau - d)$ this is no problem; moreover, the special case $c = 0$ is a trivial check, since then we can assume $a = 1$.

In Sect. 12 we will discuss generalized Dedekind eta functions and their transformation behaviour (66) under elements $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. As stated in Corollary 53, this implies that the Rogers-Ramanujan functions $G(\tau)$ and $H(\tau)$ satisfy property (12) concerning the finiteness of the principal part. In addition we have (34) and (35), hence $G(\tau)^{60}$ and $H(\tau)^{60}$ are modular functions for $\Gamma_1(5)$. In view of Definiton 21 we say that $G$ and $H$ are *quasi-modular* functions for $\Gamma_1(5)$.

**Definition 21** Let $f : \mathbb{H} \to \hat{\mathbb{C}}$ be a meromorphic function. If there is an $\ell \in \mathbb{Z}_{>0}$ such that $f^\ell$ is a modular function for a congruence subgroup $\Gamma$, we say that $f$ is a quasi-modular function for $\Gamma$.

*Remark 22* Our notion of quasi-modular function should not be confused with some authors usage of the notion of quasi-modular form which applies for functions that are derivatives of modular forms, like for example the Eisenstein series of weight 2.

*Remark 23* One can show that the Rogers-Ramanujan functions $G(\tau)$ and $H(\tau)$ are modular functions for a subgroup $\Gamma_{\mathrm{RR}}$ of $\Gamma(5)$ of index $[\Gamma(5) : \Gamma_{\mathrm{RR}}] = 12$. Moreover, $\Gamma_{\mathrm{RR}}$ has $\Gamma(60)$ as a subgroup with index $[\Gamma_{\mathrm{RR}} : \Gamma(60)] = 96$. So one explicit way to present the Rogers-Ramanujan group $\Gamma_{\mathrm{RR}}$ is by its disjoint coset decomposition with respect to $\Gamma(60)$. This can be done without any effort using a computer algebra system.

*Example 24* The Rogers-Ramanujan quotient

$$r(\tau) := \frac{H(\tau)}{G(\tau)} = q^{\frac{1}{5}} \prod_{m=0}^{\infty} \frac{(1 - q^{5m+1})(1 - q^{5m+4})}{(1 - q^{5m+2})(1 - q^{5m+3})}$$

is an analytic[7] modular function for $\Gamma(5)$. Moreover, for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1(5)$,

$$r\left(\frac{a\,\tau + b}{c\,\tau + d}\right) = e^{2\pi i\, b/5} r(\tau). \tag{36}$$

[7] By Lemma 4.

*Proof* Corollary 53 implies that $r(\tau)$ satisfies property (12) concerning the finiteness of the principal part. Concerning the modularity property (11), we first prove this property for matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(5)$ with $\gcd(a, 6) = 1$. With this assumption we have $5|b$ and $a \equiv 1 \pmod 5$. The latter together with $\gcd(a, 6) = 1$ gives $a = 30m + 1$ or $a = 30m + 11$ for some $m \in \mathbb{Z}$. Hence the exponent resulting from Example 20, $(\beta(a, b, c) - \alpha(a, b, c))/6 = 2ab - a + 1$, reduces to $2b$ modulo 10. To extend the statement to arbitrary matrices in $\Gamma(5)$, apply Lemma 25(b). To prove the extended modular transformation property (36), apply Lemma 25(c).

**Lemma 25** *(a) Each matrix in* $\Gamma(10)$ *can be written as a product of matrices* $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(10)$ *with* $\gcd(a, 6) = 1$. *(b) Each matrix in* $\Gamma(5)$ *can be written as a product of matrices* $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(5)$ *with* $\gcd(a, 6) = 1$. *(c) Each matrix in* $\Gamma_1(5)$ *can be written as a product of matrices* $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1(5)$ *with* $\gcd(a, 6) = 1$.

*Proof* First of all, note that in the subgroup $\Gamma(30)$ of $\Gamma(5)$ all matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ have $\gcd(a, 6) = 1$. To prove (a), the first observation is that $\Gamma(30)$ is a normal subgroup of $\Gamma(10)$ with index 24. Hence there are $g_j \in \Gamma(10)$ which generate 24 (right) cosets such that

$$\Gamma(10)/\Gamma(30) = \{\Gamma(30)g_1, \ldots, \Gamma(30)g_{24}\}.$$

Consider the following 13 matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(10)$, all satisfying $\gcd(a, 6) = 1$. To save space we use $(a, b, c, d)$ instead of matrix notation $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$:

$$\begin{aligned}
&(1, 0, 0, 1), (1, 0, 10, 1), (1, 0, 20, 1), (11, 50, 20, 91), (11, 50, 130, 591),\\
&(11, 50, 240, 1091), (11, 100, 10, 91), (11, 100, 120, 1091), (11, 100, 230, 2091),\\
&(1121, 10200, 1020, 9281), (2231, 20190, 2030, 18371), (3421, 170, 1630, 81),\\
&(3631, 16520, 1730, 7871).
\end{aligned}$$

One can verify that these 13 matrices give rise to 13 pairwise disjoint (right) cosets in $\Gamma(10)/\Gamma(30)$. These cosets must generate the whole group $\Gamma(10)/\Gamma(30)$, because any proper subgroup would consist of maximally 12 elements. Hence, for any $j \in \{1, \ldots, 24\}$:

$$\Gamma(30)g_j = \Gamma(30)h_1\Gamma(30)h_2 \cdots = \Gamma(30)h_1h_2 \ldots$$

with particular $h_k$ chosen from the 13 matrices. This proves (a).

To prove (b), we apply the same strategy observing that $\Gamma(10)$ is a normal subgroup of $\Gamma(5)$ with index 6. Hence there are $G_j \in \Gamma(5)$ which generate 6 (right) cosets such that

$$\Gamma(5)/\Gamma(10) = \{\Gamma(10)G_1, \ldots, \Gamma(10)G_6\},$$

Consider the following 4 matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(5)$, all satisfying $\gcd(a, 6) = 1$:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 5 \\ 5 & 26 \end{pmatrix}, \begin{pmatrix} 31 & 285 \\ 160 & 2471 \end{pmatrix}, \begin{pmatrix} 281 & 1460 \\ 235 & 1221 \end{pmatrix}.$$

One can verify that these 4 matrices give rise to 4 pairwise disjoint (right) cosets which thus generate the full group $\Gamma(5)/\Gamma(10)$. Using the same argument as in (a), this proves (b).

To prove (c), we apply again the same strategy observing that $\Gamma(5)$ is a normal subgroup of $\Gamma_1(5)$ of index 5. One can verify that

$$\Gamma_1(5)/\Gamma(5) = \left\{\Gamma(5)\begin{pmatrix}1&0\\0&1\end{pmatrix}, \Gamma(5)\begin{pmatrix}1&1\\0&1\end{pmatrix}, \dots, \Gamma(5)\begin{pmatrix}1&4\\0&1\end{pmatrix}\right\}.$$

Hence for each element $\gamma \in \Gamma_1(5)$ we have $\gamma = \xi\left(\begin{smallmatrix}1&h\\0&1\end{smallmatrix}\right)$ with $\xi \in \Gamma(5)$. By (b), $\xi$ can be written as a product of matrices in $\Gamma(5)$ with $\gcd(a, 6) = 1$. Moreover, the matrices $\left(\begin{smallmatrix}1&h\\0&1\end{smallmatrix}\right)$ also have $\gcd(a, 6) = 1$. Consequently, every matrix in $\Gamma_1(5)$ can be written as a product of matrices in $\Gamma_1(5)$ with $\gcd(a, 6) = 1$.

*Example 26* For $\Gamma(5)$ SAGE computes 12 inequivalent cusps each of width 5:

```
sage: Cusps=Gamma(5).cusps(); Cusps
[0, 2/5, 1/2, 1, 3/2, 2, 5/2, 3, 7/2, 4, 9/2, Infinity]
sage: [Gamma(5).cusp_width(c) for c in Cusps]
[5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5]
```

*Example 27* The 5th power of the Rogers-Ramanujan quotient

$$R(\tau) := r(\tau)^5 = q\prod_{m=0}^{\infty}\frac{(1-q^{5m+1})^5(1-q^{5m+4})^5}{(1-q^{5m+2})^5(1-q^{5m+3})^5} \tag{37}$$

is an analytic[8] modular function for $\Gamma_1(5)$: for all $\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right) \in \Gamma_1(5)$,

$$r\left(\frac{a\tau+b}{c\tau+d}\right)^5 = e^{2\pi i b} r(\tau)^5 = r(\tau)^5.$$

*Proof* Immediate from Example 24.

*Remark 28* To prove property (12) of modular functions, i.e., the finiteness of the principal part of $q$-expansions, in Example 24 we were relying on the modular transformation property of generalized Dedekind eta functions; see Corollary 53 in connection with Proposition 52. Since this property is non-trivial to prove, we present a self-contained proof based on Abel's limit theorem.

[8] By Lemma 4.

*Example 29* For $\Gamma_1(5)$ SAGE computes 4 inequivalent cusps, two of them of width 1 and two of them of width 5:

```
sage: Cusps=Gamma1(5).cusps(); Cusps
[0, 2/5, 1/2, Infinity]
sage: [Gamma1(5).cusp_width(c) for c in Cusps]
[5, 1, 5, 1]
```

## 6 Zero Recognition of Modular Functions: Basic Ideas

Zagier's "magic principle" cited at the end of Sect. 2 enables algorithmic zero recognition of $q$-series/$q$-products which present meromorphic modular functions for congruence subgroups $\Gamma \leqslant \mathrm{SL}_2(\mathbb{Z})$. Obviously, for fixed $\Gamma$ such functions form a field. But for various (computational) reasons, in particular when working with the $q$-expansions, it can be useful to view these functions as elements from a $\mathbb{C}$-algebra.[9] In Sect. 8 we shall come back to this aspect.

We will denote such modular function fields, resp. $\mathbb{C}$-algebras, by $M(\Gamma)$; i.e.,

$$M(\Gamma) := \{g : \hat{\mathbb{H}} \to \hat{\mathbb{C}} \mid g \text{ is a meromorphic modular function for } \Gamma\}.$$

Recall that by extending the group action of $\Gamma$ on $\mathbb{H}$ to an action on $\hat{\mathbb{H}} = \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$, we extended modular functions $g : \mathbb{H} \to \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ to functions $g : \hat{\mathbb{H}} \to \hat{\mathbb{C}}$. For zero recognition we generalize further. Let

$$X(\Gamma) := \text{ set of orbits of } \Gamma \text{ on } \hat{\mathbb{H}} = \left\{[\tau]_\Gamma : \tau \in \hat{\mathbb{H}}\right\},$$

where we use the notation $[\tau]_\Gamma := \{\gamma\tau : \gamma \in \Gamma\}$ for orbits. Then to any meromorphic modular function $g \in M(\Gamma)$ we can assign a function $g^* : X(\Gamma) \to \hat{\mathbb{C}}$ (we say, $g^*$ is *induced* by $g$) defined by

$$g^*([\tau]_\Gamma) := g(\tau), \ \ \tau \in \hat{\mathbb{H}}.$$

Notice that the function values of $g^*$ are well-defined owing to Proposition 9 and the related discussions in Sect. 4.[10]

It is important to note that by defining a suitable topology on $\hat{\mathbb{H}}$ one can introduce a topology on $X(\Gamma)$ that makes $X(\Gamma)$ Hausdorff and compact. To this end, for any $M > 0$ one defines open neighborhoods of $\infty \in \hat{\mathbb{H}}$ as

$$U_M(\infty) := \{\tau \in \mathbb{H} : \mathrm{Im}(\tau) > M\} \cup \{\infty\}.$$

[9] i.e., a commutative ring with 1 which is also a vector space over $\mathbb{C}$.

[10] In fact one can use the observation (75) from Sect. 13.

The desired topology on $\hat{\mathbb{H}}$ then is defined to be generated by all finite intersections and all arbitrary unions built from the standard open sets in $\mathbb{H}$ and the sets $\gamma(U_M(\infty))$, $M > 0$, $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. Finally, the topology on $X(\Gamma)$ is defined as the quotient topology of the projection map $\pi : \hat{\mathbb{H}} \to X(\Gamma)$, $\tau \mapsto [\tau]_\Gamma$.

The final step towards building the framework for zero recognition is the observation that the connected and compact Hausdorff space $X(\Gamma)$ can be equipped with the structure of a Riemann surface as explained in detail in [11]. It is straightforward to verify that, given a meromorphic function $g \in M(\Gamma)$, the induced function $g^* : X(\Gamma) \to \hat{\mathbb{C}}$ turns into a meromorphic function on the compact Riemann surface $X(\Gamma)$.

In this setting, the expansions (12) correspond to the local (Laurent) series expansions for $g^*$ about the cusp $\left[\frac{a}{c}\right]_\Gamma$ with respect to local charts $\varphi$. To be more precise, suppose $\frac{a}{c} = \gamma\infty$ for $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, and consider $g^*$ in an open neighborhood of $\left[\frac{a}{c}\right]_\Gamma$ of the form $V_M := \{[t]_\Gamma : t = \gamma\tau \text{ for } \tau \in U_M(\infty)\}$.[11] If $M$ is sufficiently large, we have for $t \in \mathbb{H}$ such that $[t]_\Gamma \in V_M$,

$$g^*([t]_\Gamma) = g(t) = g(\gamma\tau) = h\left(e^{2\pi i \gamma^{-1} t / w_\gamma^\Gamma}\right),$$

where, according to (12), $h : \{z \in \mathbb{H} : |z| < m\} \to \hat{\mathbb{C}}$, $m > 0$ suitably chosen, is a meromorphic function with Laurent expansion of the form

$$h(z) = \sum_{n=-m_\gamma}^{\infty} c_n(\gamma) z^n.$$

In fact, one can verify that for suitably chosen $m, M > 0$,

$$\varphi : V_M \to \{z \in \mathbb{C} : |z| < m\},\ [t]_\Gamma \mapsto e^{2\pi i \gamma^{-1} t / w_\gamma^\Gamma} \tag{38}$$

is a coordinate chart at $\left[\frac{a}{c}\right]_\Gamma$[12]; i.e., a homeomorphism such that

$$g^*([t]_\Gamma) = h(\varphi([t]_\Gamma)) = \sum_{n=-m_\gamma}^{\infty} c_n(\gamma)\big(\varphi([t]_\Gamma) - \varphi([a/c]_\Gamma)\big)^n.$$

Notice that $\varphi([a/c])_\Gamma = e^{2\pi i \gamma^{-1} \frac{a}{c} / w_\gamma^\Gamma} = e^{-2\pi\infty / w_\gamma^\Gamma} = 0$.

**Definition 30** Let $g \in M(\Gamma)$, $\Gamma$ a congruence subgroup. Let $\frac{a}{c} = \gamma\infty \in \hat{\mathbb{Q}}$ for $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. Suppose the $q$-expansion of $g$ at $\frac{a}{c}$ is

---

[11] $\pi^{-1}(V_M)$ contains $\frac{a}{c}$ $(= \gamma\infty)$, and $\pi^{-1}(V_M) \setminus \{\frac{a}{c}\}$ is an open disc in $\mathbb{H}$ tangent to the real line at $\frac{a}{c}$.

[12] Apart from the requirement to be a homeomorphism, a second property one needs to verify is that such $\varphi$ also satisfy the Riemann surface compatibility conditions; see, e.g., [11, 25].

$$g(\gamma\tau) = \sum_{n=-M}^{\infty} c_n q^{n/w_\gamma^\Gamma} \tag{39}$$

with $c_{-M} \neq 0$. Then the order of $g$ at $\frac{a}{c}$ is defined as

$$\mathrm{ord}_{a/c}^{\Gamma}(g) := -M.$$

Moreover, we say that (39) is the "local $q$-expansion of the induced function $g^*$ at the cusp $\left[\frac{a}{c}\right]_\Gamma$" (with respect to the chart $\varphi$ defined as in (38)).

Because of Proposition 9, this order notion is well-defined and two local expansions at the same cusp differ in their coefficients only by an exponential factor.

A slightly more general implication of Proposition 9 is the fact that $\mathrm{ord}_{a/c}(g)$ for $g \in M(\Gamma)$ is invariant on the elements of cusps of $\Gamma$:

**Corollary 31** *Let $g \in M(\Gamma)$. Then for $\frac{a}{c} \in \hat{\mathbb{Q}}$ and $\rho \in \Gamma$:*

$$\mathrm{ord}_{\rho\frac{a}{c}}^{\Gamma}(g) = \mathrm{ord}_{\frac{a}{c}}^{\Gamma}(g). \tag{40}$$

*Proof* Let $\frac{a'}{c'} := \rho\frac{a}{c}$. There are $\gamma, \gamma' \in \mathrm{SL}_2(\mathbb{Z})$ such that $\gamma\infty = \frac{a}{c}$ and $\gamma'\infty = \frac{a'}{c'}$. Hence $\frac{a'}{c'} = \rho\frac{a}{c}$ translates into $\infty = \gamma^{-1}\rho^{-1}\gamma'\infty$; i.e., $\gamma^{-1}\rho^{-1}\gamma' = \left(\begin{smallmatrix}1 & m\\ 0 & 1\end{smallmatrix}\right)$ for some $m \in \mathbb{Z}$. The rest follows from Proposition 9.

In addition to having orders at cusps, we also need the usual notion of order for Laurent series with finite principal part:

**Definition 32** Let $f : U \to \hat{\mathbb{C}}$ be meromorphic in an open neighborhood $U \subseteq \mathbb{C}$ of $z_0$ containing no pole except possibly $z_0$ itself. Then, by assumption, $f$ is analytic in $U \setminus \{z_0\}$ and can be expanded in a Laurent series about $z_0$,

$$f(z) = \sum_{n=-M}^{\infty} c_n (z - z_0)^n.$$

Assuming that $c_{-M} \neq 0$, one defines $\mathrm{ord}_p(f) := -M$.

The following theorem is folklore, e.g. [25, Theorem 1.37], but of fundamental importance: it lies at the bottom of the "magic principle" for modular functions.

**Theorem 33** *Let $X$ be a compact Riemann surface. Suppose that $f : X \to \mathbb{C}$ is an analytic function on all of $X$. Then $f$ is a constant function.*

*Example 34* From Example 17 we know that $\Phi_2$ is an analytic modular function in $M(\Gamma_0(2))$ with no zeros in $\mathbb{H}$. Therefore $j(\tau) = \frac{(1+2^8\Phi_2(\tau))^3}{\Phi_2(\tau)}$ is an analytic modular function in $M(\Gamma_0(2))$. Since it is non-constant, its induced function $j^*$, which is a meromorphic function on $X(\Gamma_0(2))$, according to Theorem 33 we must have at least one pole. Indeed, it has a pole at $[\infty]_{\Gamma_0(2)}$ which is made explicit by the local $q$-expansion (Definition 30)

$$\begin{aligned} j(I\tau) &= \sum_{n=-M_I}^{\infty} c_n(I) e^{2\pi i n\tau/w_I^{\Gamma}} = \sum_{n=-M_I}^{\infty} c_n(I) e^{2\pi i n\tau} \\ &= j(\tau) = \frac{1}{q} + 744 + 196884\, q + 21493760\, q^2 + 864299970\, q^3 + \dots \end{aligned}$$

Recall from Example 17 that $X(\Gamma_0(2))$ has two cusps, $[\infty]_{\Gamma_0(2)}$ and $[0]_{\Gamma_0(2)}$, with widths $w_{[\infty]}^{\Gamma_0(2)} = w_I^{\Gamma_0(2)} = 1$ and $w_{[0]}^{\Gamma_0(2)} = w_T^{\Gamma_0(2)} = 2$. Accordingly, $j^*$ has another pole—of multiplicity 2—at $[0]_{\Gamma_0(2)}$, which is made explicit by the local $q$-expansion (Definition 30)

$$\begin{aligned} j(T\tau) &= \sum_{n=-M_T}^{\infty} c_n(T) e^{2\pi i n\tau/w_T^{\Gamma}} = \sum_{n=-M_T}^{\infty} c_n(T) e^{2\pi i n\tau/2} \\ &= j(\tau) = \left(\frac{1}{q^{1/2}}\right)^2 + 744 + 196884\, \left(q^{1/2}\right)^2 + 21493760\, \left(q^{1/2}\right)^4 + \dots \end{aligned}$$

Summarizing, $j^*$ is a meromorphic function on $X(\Gamma_0(2))$ having no other poles than a single pole at $[\infty]_{\Gamma_0(2)}$ and a double pole at $[0]_{\Gamma_0(2)}$; i.e.,

$$\mathrm{ord}_{\infty}^{\Gamma_0(2)}(j) = -1 \text{ and } \mathrm{ord}_{0}^{\Gamma_0(2)}(j) = -2.$$

*Proof of* (29). The properties of $j^*$ as a meromorphic function of $X(\Gamma_0(2))$ as exhibited in Example 34, put us into the position to prove the presentation of $j$ in (29),

$$j(\tau) = \frac{E_4(\tau)^3}{\Delta(\tau)}, \;\; \tau \in \mathbb{H};$$

as announced in Example 18. The definitions (9) and (29) tell us that $\Delta$ and $E_4$ are analytic functions on $\mathbb{H}$. Their modular symmetries (10) and (31) together with their $q$-expansions at $\infty$ imply that $g := E_4^3/\Delta \in M(\mathrm{SL}_2(\mathbb{Z})) \subseteq M(\Gamma_0(2))$ and also, by Lemma 4, that $g^*$ viewed as a meromorphic function on $X(\Gamma_0(2))$ has its only poles at the cusps $[\infty]_{\Gamma_0(2)}$ and $[0]_{\Gamma_0(2)}$, which are of orders 1 and 2, respectively. By taking the difference $h(\tau) := j(\tau) - g(\tau)$, one obtains a function $h \in M(\Gamma_0(2)$ with no poles, which can be verified by the $q$-expansions

$$h(\tau) = 0 \cdot \frac{1}{q} + 0 \cdot q^0 + \text{etc. and } h(T\tau) = 0 \cdot \left(\frac{1}{q^{1/2}}\right)^2 + \text{etc.}$$

Consequently, $h^*$ is an analytic function on $X(\Gamma_0(2))$, and Theorem 33 implies that $h^*$ is a constant function. From $h^*([\infty_{\Gamma_0(2)}]) = h(\infty) = 0$ we conclude $h^* = 0$, and thus $h = 0$. This means, we have proved (29). □

In view of Example 6, consider the following subalgebra of $M(\Gamma)$:

$$M^{\infty}(\Gamma) := \{g \in M(\Gamma) : g \text{ has no pole except at } [\infty]_{\Gamma}\}.$$

The $q$-expansions at the cusps $[\infty]_{\Gamma}$ give finitary normal form presentations for the modular functions in $M^{\infty}(\Gamma)$. More precisely, despite the analytic setting, to decide equality of two functions in $M^{\infty}(N)$ can be done in purely algebraic and finitary fashion:

**Lemma 35** *Let $g$ and $h$ be in $M^{\infty}(\Gamma)$ with $q$-expansions*

$$g(\tau) = \sum_{n=\mathrm{ord}_{\infty}^{\Gamma}(g)}^{\infty} a_n q^n \text{ and } h(\tau) = \sum_{n=\mathrm{ord}_{\infty}^{\Gamma}(h)}^{\infty} b_n q^n.$$

*Then $g = h$ if and only if* $\mathrm{ord}_{\infty}^{\Gamma}(g) = \mathrm{ord}_{\infty}^{\Gamma}(h) =: \ell$ *and*

$$(a_{\ell}, \dots, a_{-1}, a_0) = (b_{\ell}, \dots, b_{-1}, b_0).$$

*Proof* Apply Theorem 33.

In other words, if $g(\tau) = \sum_{n \geqslant \mathrm{ord}_{\infty}^{\Gamma}(g)} a_n q^n \in M^{\infty}(\Gamma)$, the coefficients $a_n, n \geqslant 1$, are uniquely determined by those of the principal part and $a_0$. Algebraically this corresponds to an isomorphic embedding of $\mathbb{C}$-algebras:

$$\begin{aligned} \varphi : M^{\infty}(N) &\to \mathbb{C}[z], \\ g = \sum_{n=\mathrm{ord}_{\infty}^{\Gamma}(g)}^{\infty} a_n q^n &\mapsto a_{\mathrm{ord}_{\infty}^{\Gamma}(g)} z^{-\mathrm{ord}_{\infty}^{\Gamma}(g)} + \cdots + a_{-1} z + a_0. \end{aligned} \tag{41}$$

In computationally feasible cases the zero test for $g - h \stackrel{?}{=} 0$ according to Lemma 35 trivializes the task of proving identities between modular functions.

In order to invoke this zero test for $G - H \stackrel{?}{=} 0$ with given $G, H \in M(\Gamma)$, in a preprocessing step one has to transform the problem into the form $g - h \stackrel{?}{=} 0$, where $g$ and $h$ are elements in $M^{\infty}(\Gamma)$. Computational examples of this strategy are given in the Sects. 8 and 11.2. As shown in these sections, when reducing things to $M^{\infty}(\Gamma)$ there is an “algorithmic bonus” which enables the algorithmic derivation of identities. For the single purpose of zero recognition, other variants of applying Theorem 33 can be used. This is illustrated by examples in Sect. 7.

## 7 Zero Recognition of Modular Functions: Examples

In this section we present various examples which despite being elementary should illustrate how to prove relations between $q$-series/$q$-products using modular function machinery.

For a better understanding of the "valence formulas" used in the concrete examples, we stress that the invariance property stated in Corollary 31 also holds for the usual order $\mathrm{ord}_\tau(g)$ when $\tau \in \mathbb{H}$:

**Corollary 36** *Let $g \in M(\Gamma)$. Then for $\tau \in \mathbb{H}$ and $\rho \in \Gamma$:*

$$\mathrm{ord}_{\rho\tau}(g) = \mathrm{ord}_\tau(g). \tag{42}$$

*Proof* (*Proof of Corollary* 31) The statement follows from a general fact which can be verified in a straighforward manner: Let $f$ be a meromorphic function on $\mathbb{H}$ and $\tau_0 \in \mathbb{H}$ then $\mathrm{ord}_{\gamma\tau_0}(f(\tau)) = \mathrm{ord}_{\tau_0}(f(\gamma\tau))$ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$.

In other words, $\mathrm{ord}_\tau(g)$ for $g \in M(\Gamma)$ is invariant on the elements of the orbits $[\tau]_\Gamma$ of $\Gamma$, and the "valence formulas" below should be read having this invariance property in mind.

For the first example in this section, we recall the "valence formula" for the full modular group $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ which can be found at many places in the literature; e.g. [26].

**Corollary 37** ("valence formula" for $\mathrm{SL}_2(\mathbb{Z})$) *If $g \in M(\mathrm{SL}_2(\mathbb{Z}))$ then*

$$\frac{\mathrm{ord}_i(g)}{2} + \frac{\mathrm{ord}_\omega(g)}{3} + \mathrm{ord}_\infty^{\mathrm{SL}_2(\mathbb{Z})}(g) + \sum_{\substack{\tau \in H(\mathrm{SL}_2(\mathbb{Z})) \\ [\tau]\neq[i],[\tau]\neq[\omega]}} \mathrm{ord}_\tau(g) = 0, \tag{43}$$

*where $H(\mathrm{SL}_2(\mathbb{Z})) \subseteq \mathbb{H}$ is a complete set of representatives of the orbits $[\tau]_{\mathrm{SL}_2(\mathbb{Z})}$ with $\tau \in \mathbb{H}$, $\mathrm{ord}_\tau(g)$ is the usual order as in Definition 32, and $\omega := e^{2\pi i/3}$.*

In Sect. 13 (Appendix 3) we state—without proof—Theorem 56 which presents a valence formula that holds for any congruence subgroup $\Gamma$ of $\mathrm{SL}_2(\mathbb{Z})$. Formula (43) is an immediate corollary setting $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ there.

*Example 38* We consider (43) for $g = E_4^3/\Delta = j \in M(\Gamma)$ where $\Gamma = \mathrm{SL}_2(\mathbb{Z})$.[13] Noting that $[\infty]_{\mathrm{SL}_2(\mathbb{Z})}$ is the only cusp of $X(\mathrm{SL}_2(\mathbb{Z}))$ and that $\mathrm{ord}_\infty^{\mathrm{SL}_2(\mathbb{Z})}(g) = -1$ (by inspection of the $q$-expansion at $\infty$), relation (43) turns into

$$\frac{\mathrm{ord}_i(g)}{2} + \frac{\mathrm{ord}_\omega(g)}{3} + \sum_{\substack{\tau \in H(\mathrm{SL}_2(\mathbb{Z})) \\ [\tau]\neq[i],[\tau]\neq[\omega]}} \mathrm{ord}_\tau(g) = 1.$$

[13] Cf. Example 6.

Therefore, and also in view of Corollary 36, there remain three possibilities for $g = j$ having a zero in $\mathbb{H}$[14]:

(a) $j$ has triple zeros at the points in $[\omega]$ and nowhere else;
(b) $j$ has double zeros at the points in $[i]$ and nowhere else;
(c) the only zeros of $j$ are single zeros at the points of some orbit $[\tau] \neq [i], [\omega]$.

By Lemma 4, $\Delta(\tau) \neq 0$ for all $\tau \in \mathbb{H}$. In view of this and of the third power of $E_4$, only alternative (a) can apply. In particular, we obtain:

$*$ $E_4(\tau_0) = 0$ for $\tau_0 \in [\omega] = \{\gamma\omega : \gamma \in \mathrm{SL}_2(\mathbb{Z})\}$;
$*$ the elements of $[\omega]$ are the only zeros of $E_4$;
$*$ each of these zeros has multiplicity 3.

*Example 39* Again $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. Consider (43) for

$$g = \frac{E_6^2}{\Delta} = \frac{1}{q} - 984 + 196884\,q + \cdots \in M(\mathrm{SL}_2(\mathbb{Z})).$$

With an argument analogous to Example 38 we conclude that

$$\frac{E_6^2}{\Delta} \text{ has double zeros at the points in } [i] \text{ and nowhere else.}$$

In particular:

$*$ $E_6(\tau_0) = 0$ for $\tau_0 \in [i] = \{\gamma i : \gamma \in \mathrm{SL}_2(\mathbb{Z})\}$;
$*$ the elements of $[i]$ are the only zeros of $E_6$;
$*$ each of these zeros has multiplicity 2.

*Example 40* The transformation property (31) implies

$$\frac{E_4^3 - E_6^2}{\Delta} = 1728 + \cdots \in M(\mathrm{SL}_2(\mathbb{Z})).$$

This quotient is an analytic modular function, hence Theorem 33 gives

$$\frac{E_4(\tau)^3 - E_6(\tau)^2}{\Delta(\tau)} = 1728, \quad \tau \in \mathbb{H}. \tag{44}$$

Hence (6) implies

[14]It is important to note that the orbit sets of modular transformations are discrete; i.e., they do not have a limit point.

$$j(\tau) = \frac{E_6(\tau)^2}{\Delta(\tau)} + 1728, \;\; \tau \in \mathbb{H}, \tag{45}$$

and by Example 39 we obtain the evaluation,

$$j(\tau_0) = 1728 \text{ for all } \tau_0 \in [i] = \{\gamma i : \gamma \in \mathrm{SL}_2(\mathbb{Z})\}. \tag{46}$$

The next example presents an alternative derivation of the equalities (45) and (46).

*Example 41* From Example 39 we know that

$$g(\tau) = \frac{E_6(\tau)^2}{\Delta(\tau)} = \frac{1}{q} - 984 + 196884\, q + \cdots \in M(\mathrm{SL}_2(\mathbb{Z}))$$

has a single pole at the points in $[\infty]$ and no pole elsewhere, and double zeros at the points in $[i]$ and nowhere else.[15] Applying the same reasoning as in Example 38 one obtains that also

$$f(\tau) := j(\tau) - j(i) = \frac{1}{q} + (744 - j(i)) + 196884\, q + \cdots \in M(\mathrm{SL}_2(\mathbb{Z}))$$

has a single pole at the points in $[\infty]$ and no pole elsewhere, and double zeros at the points in $[i]$ and nowhere else. Hence Theorem 33 implies that the induced function $(g/f)^*$ is constant and hence $g = c \cdot f$ for some $c \in \mathbb{C}$. Finally, the comparison of the $q$-expansions at $\infty$ of both sides gives:

$$\frac{1}{q} - 984 + 196884\, q + \cdots = \frac{c}{q} + c(744 - j(i)) + c\, 196884\, q + \ldots$$

Consequently, $c = 1$ and $j(i) = 744 + 984 = 1728$, which proves (45) and (46).

More generally, the reasoning used in Example 41 can be easily extended to prove the

**Theorem 42** *Suppose* $g \in M(\mathrm{SL}_2(\mathbb{Z}))$. *Then* $g \in \mathbb{C}(j)$*; i.e., g is a rational function in the Klein j function.*

For the next example we need

**Corollary 43** ("valence formula" for $\Gamma_0(2)$) *Let* $g \in M(\Gamma)$. *If* $\Gamma = \Gamma_0(2)$ *then*

$$\mathrm{ord}_i(g) + \frac{\mathrm{ord}_{TSi}(g)}{2} + \mathrm{ord}_\omega(g) + \mathrm{ord}_\infty^\Gamma(g) + \mathrm{ord}_0^\Gamma(g) + \sum_{\substack{\tau \in H(\mathrm{SL}_2(\mathbb{Z})) \\ [\tau] \neq [i], [\tau] \neq [\omega]}} \sum_{\gamma \in \{I, T, TS\}} \mathrm{ord}_{\gamma\tau}(g) = 0, \tag{47}$$

[15] Often one restricts to consider such functions only on a complete set of orbit representatives; for example, in the case of $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ to $\{\tau \in \mathbb{H} : -1/2 \leqslant \mathrm{Re}(\tau) \leqslant 0 \text{ and } \mathrm{Im}(\tau) \geqslant \mathrm{Im}(e^{i\tau})\} \cup \{0 < \mathrm{Re}(\tau) < 1/2 \text{ and } \mathrm{Im}(\tau) > \mathrm{Im}(e^{i\tau})\}$ .

*where $H(\mathrm{SL}_2(\mathbb{Z})) \subseteq \mathbb{H}$ is a complete set of representatives of the orbits $[\tau]_{\mathrm{SL}_2(\mathbb{Z})}$ with $\tau \in \mathbb{H}$,* $\mathrm{ord}_{\gamma\tau}(g)$ *is the usual order as in Definition 32, and* $\omega := e^{2\pi i/3}$.

As formula (43), also formula (47) is an immediate corollary of Theorem 56 setting $\Gamma = \Gamma_0(2)$ there.

*Example 44* By Example 17, $\Psi_2 := \frac{1}{\Phi_2}$ is an analytic modular function for $\Gamma_0(2)$. By Lemma 4, $\Psi_2 \in M(\Gamma_0(2))$ has no zero in $\mathbb{H}$. By (26) we have the $q$-expansion

$$\Psi_2(\tau) = \frac{\Delta(\tau)}{\Delta(2\tau)} = \frac{1}{\Phi_2(\tau)} = \frac{1}{q} - 24 + 276q - 2048q^2 + 11202q^3 - \dots,$$

which tells that $\Psi_2^*$ has a single pole at $[\infty]_{\Gamma_0(2)}$.[16] Hence Corollary 43 implies that $\Psi_2^*$ must also have a single zero which must sit at $[0]_{\Gamma_0(2)}$. Why? Because $\Psi_2$ has no zero in $\mathbb{H}$, so $[0]_{\Gamma_0(2)}$ is the only remaining option. This is in accordance with Example 17; namely, with the fact that $X(\Gamma_0(2))$ has two cusps $[\infty]_{\Gamma_0(2)}$ and $[0]_{\Gamma_0(2)}$ with widths $w_{[\infty]}^{\Gamma_0(2)} = 1$ and $w_{[0]}^{\Gamma_0(2)} = 2$. Consequently, the single zero of $\Psi_2^*$ must be at $[0]_{\Gamma_0(2)}$, which using (27) is confirmed by the expansion

$$\Psi_2\left(-\frac{1}{\tau}\right) = 2^{12}\Phi_2\left(\frac{\tau}{2}\right) = 2^{12}(q^{1/2} + 24q + 300q^{3/2} + 2624q^2 + 18126q^{5/2} + \dots).$$

The properties of $\Psi_2$ made explicit in Example 44 allow to apply the same argument as used in Example 41, resp. Theorem 42, to derive

**Theorem 45** *Suppose $g \in M(\Gamma_0(2))$. Then $g \in \mathbb{C}(\Psi_2)$; i.e., $g$ is a rational function in $\Psi_2 (= \frac{1}{\Phi_2})$.*

## 8 Zero Recognition: Computing Modular Function Relations

In view of Theorem 45 we consider the

*TASK.* Compute a rational function $\mathrm{rat}(x) \in \mathbb{C}(x)$ such that

$$j = \mathrm{rat}(\Psi_2).$$

From Examples 17 and 44 we know that $\Psi_2^*$ as a meromorphic function on $X(\Gamma_0(2))$

- has at $[\infty]_{\Gamma_0(2)}$ its only pole which is of order 1,
- and at $[0]_{\Gamma_0(2)}$ its only zero, also of order 1.

From Example 34 we know that $j^*$ as a meromorphic function on $X(\Gamma_0(2))$

- has at $[\infty]_{\Gamma_0(2)}$ a pole of order 1, at $[0]_{\Gamma_0(2)}$ a pole of order 2, and no pole elsewhere,

[16] Equivalently, $\Psi_2$ has single poles at all the elements of the orbit $[\infty]_{\Gamma_0(2)}$.

- has at $[\omega]_{\Gamma_0(2)}$ a triple zero and no zero elsewhere.

To solve our TASK the decisive observation is that for $F := j \cdot \Psi_2^2 \in M(\Gamma_0(2))$ the induced function $F^*$ as a meromorphic function on $X(\Gamma_0(2))$

- has a possible pole only at $[\infty]_{\Gamma_0(2)}$.

From our knowledge about poles and zeros of $j^*$ and $\Psi_2^*$ we expect a pole of order 3, which is confirmed by computing the $q$-expansion of $F$. We take as input the $q$-expansions (30) and (27):

In[11]:= $\mathbf{j = \frac{1}{q} + 744 + 196884q + 21493760q^2 + O[q]^3;}$

$\mathbf{\Psi = \frac{1}{q} - 24 + 276q - 2048q^2 + O[q]^3;}$

In[12]:= $\mathbf{F = j * \Psi^2}$

Out[12]= $\frac{1}{q^3} + \frac{696}{q^2} + \frac{162300}{q} + 12865216 + O[q]^1$

Since $F^*$ has the only pole at $[\infty]_{\Gamma_0(2)}$, we can successively reduce its local $q$-expansion (in the sense of Definition 30) using only powers of $\Psi_2$ ($= \Psi$ in the computation) until we reach a constant:

In[13]:= $\mathbf{F - \Psi^3}$

Out[13]= $\frac{768}{q^2} + \frac{159744}{q} + 12924928 + O[q]^1$

In[14]:= $\mathbf{F - \Psi^3 - 768\Psi^2}$

Out[14]= $\frac{196608}{q} + 12058624 + O[q]^1$

In[15]:= $\mathbf{F - \Psi^3 - 768\Psi^2 - 196608\Psi}$

Out[15]= $16777216 + O[q]^1$

In[16]:= $\mathbf{Factor[x^3 + 768x^2 + 196608x + 16777216]}$

Out[16]= $(256 + x)^3$

Consequently, $F = j \cdot \Psi_2^2 = (\Psi_2 + 2^8)^3$; i.e., we derived that

$$j = \Psi_2 \left(\frac{\Psi_2 + 2^8}{\Psi_2}\right)^3 = \frac{(1 + 2^8\Phi_2)^3}{\Phi_2},$$

which is (28).

Despite the simplicity of this example, the underlying idea of algorithmic reduction is quite powerful. For example, it is used in Radu's Ramanujan-Kolberg algorithm [31]. We want to stress that computationally one works with the coefficients of the principal parts of Laurent series as finitary representations of the elements in $M(\Gamma)$. Consequently, the underlying structural aspect relevant to such methods is that of a $\mathbb{C}$-algebra rather than a field. Other recent applications of this reduction strategy can be found in [17, 29, 30].

In Sect. 11.2 this algorithmic reduction is used to derive Felix Klein's icosahedral relation for the Rogers-Ramanujan quotient $r(\tau)$. Before returning to this theme in Sect. 10, in Sect. 9 we briefly discuss some connections between modular functions and holonomic functions.

## 9 Interlude: q-Holonomicity and Modular Functions

As mentioned in Sect. 2, the sequence $(p(n))_{n\geqslant 0}$ is not holonomic since its generating function $\prod_{k\geqslant 1}(1-q^k)^{-1}$ is not holonomic. The same applies to the other modular forms and functions we have presented so far. Nevertheless, there are several tight connections to *q-holonomic* sequences and series which we will briefly indicate in this section.

Let $\mathbb{F} = \mathbb{Q}(z_1, \ldots, z_\ell)$ be a rational function field over $\mathbb{Q}$ with parameters $z_1, \ldots, z_\ell$. Set $\mathbb{K} = \mathbb{F}(q)$ where $q$ is taken to be an indeterminate.[17] Let $\mathbb{K}[[x]]$ denote the ring of formal power series with coefficients in $\mathbb{K}$. The $q$-derivative $D_q$ on $\mathbb{K}[[x]]$ is defined as

$$D_q \sum_{n=0}^{\infty} a_n x^n := \sum_{n=0}^{\infty} a_n \frac{q^n - 1}{q - 1} x^{n-1}.$$

A sequence $(a_n)_{n\geqslant 0}$ with values in $\mathbb{K}$ is called $q$-holonomic (over $\mathbb{K}$), if there exist polynomials $p, p_0, \ldots, p_r$ in $\mathbb{K}[x]$, not all zero, such that

$$p_r(q^n)\, a_{n+r} + p_{r-1}(q^n)\, a_{n+r-1} + \cdots + p_0(q^n)\, a_n = p(q^n), \;\; n \geqslant 0. \tag{48}$$

If $r = 1$ and $p = 0$, the sequence $(a_n)_{n\geqslant 0}$ is called $q$-hypergeometric. A finite sum, resp. infinite series, over a $q$-hypergeometric summand sequence is said to be a $q$-hypergeometric sum, resp. series. For example, the Rogers-Ramanujan functions $F(1)$ and $F(q)$ are $q$-hypergeometric series.

A formal power series $f(x) \in \mathbb{K}[[x]]$ is called $q$-holonomic, if there exist polynomials $p, p_0, \ldots, p_r \in \mathbb{K}[x]$, not all zero, such that

$$p_r(x)\, D_q^r f(x) + p_{r-1}(x)\, D_q^{r-1} f(x) + \cdots + p_0(x)\, f(x) = p(x). \tag{49}$$

There are several variations of these definitions. We are following the setting of Kauers and Koutschan [18], who developed a computer algebra package for $q$-

[17] i.e., $q$ is transcendental over $\mathbb{F}$.

holonomic sequences and series which assists the manipulation of such objects, including the execution of closure properties. In our context we do not need to go into further $q$-holonomic details. We only remark that, as in the standard holonomic case "$q = 1$", a sequence $(a_n)_{n\geqslant 0}$ is $q$-holonomic if and only if its generating function $f(x) = \sum_{n=0}^{\infty} a_n x^n$ is $q$-holonomic. For example, a $q$-hypergeometric series is also a $q$-holonomic series since its summand sequence is $q$-hypergeometric, i.e., it satisfies a $q$-holonomic recurrence (48) of order 1.

## 9.1 q-*Holonomic Approximations of Modular Forms*

Despite being neither holonomic nor $q$-holonomic, modular forms and (quasi-)modular functions often find $q$-holonomic approximations; i.e., a presentation as a limit of a $q$-holonomic sequence.

There is theoretical and algorithmic framework for $q$-holonomic functions and sequences described in the literature; see [18, 37]. Nevertheles, to our knowledge no systematic account of $q$-holonomic approximation of modular forms or modular forms as projections of $q$-holonomic functions has been given so far. In this and in the next subsection we present illustrating examples for which we use the notation introduced in (1) and (2).

Set

$$a_n := \sum_{k=0}^{n} c(n,k) \text{ with } c(n,k) := \frac{q^{k^2}}{(q;q)_k (q;q)_{n-k}}, \;\; n \geqslant 0.$$

Using the notion of convergence in the formal power series ring $\mathbb{Q}[[q]]$,[18] or proceeding analytically with $|q| < 1$, it is straightforward to verify that

$$\lim_{n\to\infty} a_n = \prod_{\ell=1}^{\infty} \frac{1}{1-q^{\ell}} \sum_{k=0}^{\infty} \frac{q^{k^2}}{(q;q)_k} = \frac{F(1)}{(q;q)_\infty}. \tag{50}$$

The summand $c(n,k)$ of the definite $q$-hypergeometric sum $a_n$ is $q$-hypergeometric in both variables $n$ and $k$:

$$\frac{c(n+1,k)}{c(n,k)} = \frac{1}{1-q^{n+1-k}} \text{ and } \frac{c(n,k+1)}{c(n,k)} = q^{2k+1}\frac{1-q^{n-k}}{1-q^{k+1}}.$$

By applying a $q$-version of Zeilberger's algorithm one obtains a $q$-holonomic recurrence (48) for the sequence $a_n$. We use the implementation [28]. With respect to the input "`qZeil[f(n,k),{k,a(n),b(n)},n,order]`" the output symbol `SUM[n]` refers to the sum $\sum_{k=a(n)}^{b(n)} f(n,k)$; for instance, in `Out[12]` to $a_n$:

[18] See, for instance, [19].

```
In[17]:= << RISC`qZeil`
```

Package q-Zeilberger version 4.50
written by Axel Riese © RISC-JKU

```
In[18]:= qP = qPochhammer;
```

In[19]:= $\texttt{qZeil}\left[\frac{q^{k^2}}{\texttt{qP[q, q, k]qP[q, q, n - k]}}, \{k, 0, n\}, n, 2\right]$

Out[19]= $\text{SUM}[n] = \frac{(q^{2n} - q^{n+1} + q^2 + q)\,\text{SUM}[n-1]}{q(1-q^n)} - \frac{q\text{SUM}[n-2]}{1-q^n}$

Summarizing, the $q$-holonomic sequence[19] $((q;q)_n a_n)_{n\geqslant 0}$ is a $q$-holonomic approximation of $F(1)$ in the sense that

$$\lim_{n\to\infty}(q,q)_n a_n = \sum_{k=0}^{\infty}\frac{q^{k^2}}{(q;q)_k} = F(1).$$

Also the product side of (6) has a $q$-holonomic approximation: set

$$b_n = \sum_{k=-n}^{n}\frac{(-1)^k q^{(5k^2-k)/2}}{(q;q)_{n+k}(q;q)_{n-k}},\ \ n\geqslant 0.$$

Using the notion of convergence in the formal powers series ring $\mathbb{Q}[[q]]$,[20] or proceeding analytically with $|q|<1$, it is straightforward to verify that

$$\begin{aligned}\lim_{n\to\infty} b_n &= \frac{1}{(q;q)_\infty^2}\sum_{k=-\infty}^{\infty}(-1)^k q^{(5k^2-k)/2}\\ &= \frac{1}{(q;q)_\infty^2}\prod_{m=0}^{\infty}(1-q^{5m+2})(1-q^{5m+3})(1-q^{5m+5})\\ &= \frac{1}{(q;q)_\infty}\prod_{m=0}^{\infty}\frac{1}{(1-q^{5m+1})(1-q^{5m+4})}.\end{aligned}$$

The last equality is immediate, the second is by Jacobi's triple product identity [5],

$$\sum_{k=-\infty}^{\infty} q^{\binom{k}{2}}x^k = \prod_{m=0}^{\infty}(1+xq^m)(1+\frac{q}{x}q^m)(1-q^{m+1}). \tag{51}$$

Again one can use the $q$-version of Zeilberger's algorithm to derive a $q$-holonomic recurrence for $b_n$. Nevertheless, doing so results in a surprise:

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[19] That $((q,q)_n a_n)_{n\geqslant 0}$ is $q$-holonomic is immediate by $q$-holonomic closure properties.

[20] See, for instance, [19].

In[20]:= $\mathbf{qZeil}\left[\frac{(-1)^{k}q^{(5k^2-k)/2}}{qP[q,q,n+k]qP[q,q,n-k]}, \{k,-n,n\}, n, 5\right]$

Out[20]= $\mathrm{SUM}[n] = \frac{q^{10}\mathrm{SUM}[n-5]}{(1-q^{2n})(1-q^{2n-1})} + r_4\,\mathrm{SUM}[n-4] + r_3\,\mathrm{SUM}[n-3] + r_2\,\mathrm{SUM}[n-2] + r_1\,\mathrm{SUM}[n-1]$

When choosing instead of 5 the orders 1 to 4, the output will be empty. In other words, the first non-trivial recurrence the algorithm returns is `Out[13]` of order 5 (!), where the $r_j$ are rational functions in $q$ and $q^n$ —too big to be displayed here. But the $q$-holonomic approximations coincide; i.e.,

$$a_n = b_n,\ \ n \geqslant 0, \tag{52}$$

as proven by Andrews—inspired by Watson.[21] Hence, in order to prove (52) algorithmically, viewing the recurrences also as shift operators, one has to identify the order 2 recurrence for $a_n$ as a right factor of the order 5 recurrence for $b_n$. Alternatively, as described in [27], one can apply "symmetrization" which results in the following modification of the $b_n$ sum,

$$b_n = \frac{1}{2}\sum_{k=-n}^{n} \frac{(-1)^k(1+q^k)q^{(5k^2-k)/2}}{(q;q)_{n+k}(q;q)_{n-k}}.$$

Remarkably, for this version of $b_n$ the $q$-Zeilberger algorithm outputs the same recurrence as for $a_n$:

In[21]:= $\mathbf{qZeil}\left[\frac{(-1)^{k}(1+q^k)q^{(5k^2-k)/2}}{qP[q,q,n+k]qP[q,q,n-k]}, \{k,-n,n\}, n, 5\right]$

Out[21]= $\mathrm{SUM}[n] = \frac{(q^{2n}-q^{n+1}+q^2+q)\,\mathrm{SUM}[n-1]}{q(1-q^n)} - \frac{q\mathrm{SUM}[n-2]}{1-q^n}$

Note. There are also situations where for a given definite hypergeometric sum $S(n) := \sum_{k=0}^{n} f(n,k)$ the $q$-version of Zeilberger's algorithm returns a recurrence of order greater than one, but where, in fact, $S(n)$ is a $q$-hypergeometric sequence. Such situations can be resolved by applying a $q$-version q-version of Petkovšeks algorithm Hyper; see [1, 2].

For the second Rogers-Ramanujan identity (7) all these observations work completely the same.

[21] See [27] for more information about such finite versions of the Rogers-Ramanujan identities.

## *9.2 Modular Functions as Projections of* q*-Holonomic Series*

Despite being neither holonomic nor $q$-holonomic, modular forms and (quasi-)modular functions can arise as projections of $q$-holonomic series; i.e., can be obtained by specifying a parameter in a $q$-holonomic series. Instead of setting up a theoretical framework, we take again the Rogers-Ramanujan functions as an illustrating example. Recalling their common setting (3), they are the projections $z = 1$ and $z = q$ of

$$F(z) = \sum_{k=0}^{\infty} \frac{q^{k^2} z^k}{(q;q)_k}.$$

Using computer algebra, we now show that $F(z)$ is a $q$-holonomic series in $\mathbb{K}[[z]]$ with $\mathbb{K} = \mathbb{Q}(q)$.[22] Concretely, we derive a $q$-difference equation for $F(z)$; i.e., determine $r_j(z) \in \mathbb{K}[z]$ such that

$$r_0(z)\, F(z) + r_1(z)\, F(q\, z) + r_2(z)\, F(q^2\, z) = 0. \tag{53}$$

Owing to $D_q F(z) = \frac{F(q\,z)-F(z)}{(q-1)z}$ such $q$-shift equations are equivalent to $q$-differential equations (49).

The summand sequence of $F(z)$, $(f_k(z))_{k\geqslant 0} = \left(\frac{q^{k^2} z^k}{(q;q)_k}\right)_{k\geqslant 0}$ is $q$-hypergeometric:

$$\frac{f_{k+1}(z)}{f_k(z)} = \frac{q^{2k+1} z}{1 - q^{k+1}} \in \mathbb{K}(z)(q^k).$$

Consequently, one can apply *parametrized telescoping* to compute a $q$-hypergeometric sequence $(g_k(z))_{k\geqslant 0}$ and $r_j(z) \in \mathbb{K}[z]$ such that[23]

$$r_0(z)\, f_k(z) + r_1(z)\, f_k(qz) + r_2(z)\, f_k(q^2 z) = g_{k+1}(z) - g_k(z),\ \ k \geqslant 0. \tag{54}$$

Then summing (54) over $k$ from 0 to $\infty$ gives

$$r_0(z)\, F(z) + r_1(z)\, F(q\, z) + r_2(z)\, F(q^2\, z) = g_\infty(z) - g_0(z),$$

provided the limit $\lim_{k\to\infty} g_k(z) = g_\infty(z)$ exists. More precisely, the algorithm runs parameterized telescoping on the summand

$$f_k(z)\left(r_0(z) + r_1(z)\frac{f_k(qz)}{f_k(z)} + r_2(z)\frac{f_k(q^2 z)}{f_k(z)}\right) = f_k(z)\left(r_0(z) + r_1(z)\, q^k + r_2(z)\, q^{2k}\right)$$

with unknown $r_j(z)$. Using the RISC package `qZeil` this is executed as follows:

---

[22] $F(z)$ is also a $q$-hypergeometric series; its summand sequence $(f_k(z))_{k\geqslant 0}$ is $q$-hypergeometric over $\mathbb{K}$ with $\mathbb{K} = \mathbb{Q}(z)(q)$.

[23] In case no such order 2 equation exists, one proceeds with incrementing the order by one.

In[22]:= **qTelescope**$\left[\dfrac{q^{k^2} z^k}{qP[q, q, k]}, \{k, 0, N\}, \text{qParameterized} \to \{1, q^k, q^{2k}\}\right]$

Out[22]= $\text{Sum}\,[q\,z\,F_2[k] - F_0[k] + F_1[k], \{k, 0, N\}] = \dfrac{q^{N^2+2N+1} z^{N+1}}{(q;\,q)_N}$

The output corresponds to summing (54) over $k$ from 0 to $N$ where $F_j[k] = f_k(q^j z)$ and with the following ingredients computed in the steps of the algorithm:

$$r_0(z) = -1, r_1(z) = 1, r_2(z) = q\,z, \text{ and } g_N(z) = (1 - q^N)\frac{q^{N^2} z^N}{(q;q)_N}, N \geqslant 0.$$

Obviously, $g_0(z) = 0$ and $\lim_{N\to\infty} g_N(z) = 0$, and we thus obtained

$$F(qz) + q\,z\,F(q^2 z) = F(z). \tag{55}$$

In the next section we will see how (55) will be used to obtain a continued fraction representation of $r(\tau)$.

## 10 The Rogers-Ramanujan Continued Fraction

After deriving the functional relation (55) we unfold it as a continued fraction—following Ramanujan. Divide both sides of (55) by $F(z)$,

$$\frac{F(qz)}{F(z)} + qz\frac{F(q^2 z)}{F(z)} = 1,$$

and rewrite

$$\left(1 + qz\frac{F(q^2 z)}{F(qz)}\right)\frac{F(qz)}{F(z)} = 1$$

such that

$$\frac{F(qz)}{F(z)} = \frac{1}{1 + qz\dfrac{F(q^2 z)}{F(qz)}}.$$

Then iterate,

$$\frac{F(qz)}{F(z)} = \cfrac{1}{1 + \cfrac{qz}{1 + \cfrac{q^2 z}{1 + q^3 z\dfrac{F(q^4 z)}{F(q^3 z)}}}}. \tag{56}$$

This connects to the Rogers-Ramanujan quotient from Example 24,

$$r(\tau) = \frac{H(\tau)}{G(\tau)} = q^{\frac{1}{5}} \frac{F(q)}{F(1)} = q^{\frac{1}{5}} \prod_{m=0}^{\infty} \frac{(1-q^{5m+1})(1-q^{5m+4})}{(1-q^{5m+2})(1-q^{5m+3})},$$

which, as we noted, is a modular function for $\Gamma(5)$. Namely, taking $z = 1$ in (56) and iterating ad infinitum, one obtains

$$r(\tau) = q^{\frac{1}{5}} \cfrac{1}{1 + \cfrac{q}{1 + \cfrac{q^2}{1 + \cfrac{q^3}{1 + \cdots}}}}. \tag{57}$$

By Worpitzky's theorem the continued fraction (57) converges for $\tau \in \mathbb{H}$, i.e., for $|q| < 1$ when $q = q(\tau) = \exp(2\pi i \tau)$.[24] It converges also for some $\tau \in \mathbb{R}$. For example, for $\tau = 0$ one has $q = 1$ and thus

$$r(0) = \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}} \tag{58}$$

$$= \frac{1}{\phi}, \tag{59}$$

where

$$\frac{1}{\phi} = -\frac{1}{2} + \frac{\sqrt{5}}{2} \text{ and } \phi = \frac{1}{2} + \frac{\sqrt{5}}{2}.$$

This evaluation of $r(\tau)$ for $\tau = 0$ is made plausible by rewriting (58) as $r(0) = \frac{1}{1+r(0)}$. From a rigorous point of view, the situation is this: For $u_k, v_k \in \mathbb{C}$ the continued fraction

$$\cfrac{u_1}{v_1 + \cfrac{u_2}{v_2 + \cfrac{u_3}{v_3 + \dots}}}$$

converges to $c \in \mathbb{C}$, if there exists a $d \in \mathbb{Z}_{\geqslant 0}$ such $\lim_{n\to\infty} A_{n+d}(0) = c$, where the approximants $A_n$ are defined as

[24] An excellent account on convergence questions related to the Rogers-Ramanujan continued fraction is [10].

$$A_n(z) := (a_1 \circ a_2 \cdots \circ a_n)(z) \text{ with } a_k(z) := \frac{u_k}{v_k + z}.$$

With respect to (58) the approximants turn out to be quotients of successive Fibonacci numbers,

$$A_1(0) = \frac{1}{1}, A_2(0) = \frac{1}{2}, A_2(0) = \frac{2}{3}, A_3(0) = \frac{3}{5}, \text{ a.s.o.}$$

Similarly one obtains the evaluation of $r(\tau)$ for $\tau = \frac{1}{2}$:

$$\begin{aligned} r(1/2) &= e^{\pi i/5} \cfrac{1}{1 - \cfrac{1}{1 + \cfrac{1}{1 - \cfrac{1}{1 + \cdots}}}} \\ &= e^{\pi i/5} \phi. \end{aligned} \tag{60}$$

Here the approximants are of the form,

$$A_1(z) = \frac{1}{1+z}, A_3(z) = 2 + z, A_5(z) = \frac{3+z}{2+z}, A_7(z) = \frac{5+z}{3+z}, \text{ a.s.o.,}$$

and

$$A_2(z) = \frac{1+z}{z}, A_4(z) = \frac{1+2z}{1+z}, A_6(z) = \frac{2+3z}{1+2z}, A_8(z) = \frac{3+5z}{2+3z}, \text{ a.s.o.}$$

For $\tau \in \mathbb{H}$ Ramanujan gave several beautiful evaluations; for example,

$$\begin{aligned} r(i) &= e^{-\frac{2\pi}{5}} \cfrac{1}{1 + \cfrac{e^{-2\pi}}{1 + \cfrac{e^{-4\pi}}{1 + \cfrac{e^{-6\pi}}{1 + \cdots}}}} \\ &= \sqrt{\frac{5+\sqrt{5}}{2}} - \phi \end{aligned} \tag{61}$$

There is much history and literature connected to these evaluations of Ramanujan. Besides the pointers given in [14], see, for instance, the extensive survey [9] which presents many formulas related to $r(\tau)$ and discusses also analytic questions like convergence of the Rogers-Ramanujan continued fraction.

In the next section, using an algorithmic approach to Klein's icosahedral equation, we give a compact proof of the evaluation (61) using modular function machinery.

# 11 Klein's Icosahedron and Ramanujan's Evaluation

In this section we describe a beautiful connection, first established by Felix Klein [20] between the fixed field of the icosahedral group and modular functions. In the latter context Ramanujan's evaluation (61) finds a natural explanation.

## *11.1 Klein's Icosahedral Function and Ramanujan's Evaluation*

Consider the following subfield of $\mathbb{C}(z)$, the field of rational functions with complex coefficients,

$$\mathbb{K} = \left\{ f(z) \in \mathbb{C}(z) : f(z) = f\left(\frac{1}{z}\right) \right\}.$$

It is not too much a surprise that $\mathbb{K} = \mathbb{C}\left(z + \frac{1}{z}\right)$; this means, $\mathbb{K} = \mathbb{C}(f)$ is generated as a rational function field over $\mathbb{C}$ by one element, $f = z + \frac{1}{z} \in \mathbb{C}(z)$. By Lüroth's theorem this is true for all non-trivial subfields of $\mathbb{C}(z)$.

In order to produce such non-trivial subfields one can take fixed fields with respect to groups. For instance, $\mathbb{K}$ is the fixed field of the group $G = \left\{z \mapsto z, z \mapsto \frac{1}{z}\right\}$ acting on $\mathbb{C}(z)$. Felix Klein [20] considered finite subgroups $G$ of the three-dimensional rotation group which turn out to be the finite dihedral groups, and the symmetry groups of the Platonic solids up to conjugation by a rotation. For further details see [23] for Riemann surfaces aspects or [36] for the underlying geometry.

Of particular interest for our context is the case where $G$ is the group induced by the symmetry group of the icosahedron.[25] Defining the *icosahedral function* $I(z) \in \mathbb{C}(z)$ as

$$I(z) := -\frac{(z^{20} - 228z^{15} + 494z^{10} + 228z^5 + 1)^3}{z^5(z^{10} + 11z^5 - 1)^5}, \tag{62}$$

the subfield $\mathbb{K}$ of $\mathbb{C}(z)$ whose elements are invariant under the icosahedral mappings from $G$ is generated by $I(z)$, as computed by Klein [20]. This means, $\mathbb{K} = \mathbb{C}(I(z))$. As mentioned, such groups $G$ are determined up to conjugation by a rotation. Geometrically, the icosahedral function $I(z)$ emerges from inscribing an icosahedron into a sphere in a natural way; see [23, Sect. 1.7].

There is a beautiful connection between the icosahedral fixed field and modular functions which traces back to Felix Klein [20], namely

**Theorem 46** ("icosahedral key relation") *The Klein $j$ function and the Rogers-Ramanujan continued fraction $r$ are related via the icosahedral function as*

$$j(\tau) = I(r(\tau)), \quad \tau \in \mathbb{H}. \tag{63}$$

[25] By stereographic projection the rotations of the sphere turn into Möbius transformations $z \mapsto \frac{az+b}{cz+d}$ of the complex plane.

As a "by-product", for $\tau = i$ this gives Ramanujan's evaluation in a straightforward manner: by (46) we have

$$1728 = I(r(i));$$

this means, $r(i)$ is the root of a polynomial over the integers of degree 60. Despite the high degree it is a polynomial with underlying rich mathematical structure. For instance, its roots are related in geometrical fashion to Klein's icosahedron; see [14, 20, 23]. Computationally, using a computer algebra system like Mathematica gives

In[23]:= **Factor[(z^20 − 228z^15 + 494z^10 + 228z^5 + 1)^3 + 1728z^5(z^10 + 11z^5 − 1)^5]**

Out[23]= $(z^2+1)^2(z^4+2z^3-6z^2-2z+1)^2(z^8-z^6+z^4-z^2+1)^2$
$(z^8-6z^7+17z^6-18z^5+25z^4+18z^3+17z^2+6z+1)^2$
$(z^8+4z^7+17z^6+22z^5+5z^4-22z^3+17z^2-4z+1)^2$

In[24]:= **Solve[1 − 2z − 6z^2 + 2z^3 + z^4 == 0, z]**

Out[24]= $\left\{z \to -\frac{1}{2}+\frac{\sqrt{5}}{2}+\sqrt{\frac{1}{2}\left(5-\sqrt{5}\right)}\right\}, \left\{z \to \frac{1}{2}\left(-1+\sqrt{5}-\sqrt{2\left(5-\sqrt{5}\right)}\right)\right\},$
$\left\{z \to \frac{1}{2}\left(-1-\sqrt{5}-\sqrt{2\left(5+\sqrt{5}\right)}\right)\right\}, \left\{z \to \frac{1}{2}\left(-1-\sqrt{5}+\sqrt{2\left(5+\sqrt{5}\right)}\right)\right\}$

The fourth root is Ramanujan's evaluation for $r(i)$; it can be picked by the numerics of the continued fraction on the left side of (61).

Finally we show that modular function machinery not only enables to prove but also to *derive* the icosahedral key relation (63) in an algorithmic fashion.

## *11.2 Algorithmic Derivation of Klein's Icosahedral Key Relation*

Our task in this section is to derive the icosahedral key relation (63). To this end, notice that in (63) only powers of $r(\tau)^5$ arise. From Example 27 we know that $R(\tau) := r(\tau)^5$ is an analytic modular function for $\Gamma_1(5)$ which is non-zero on $\mathbb{H}$. By Example 29, $X(\Gamma_1(5))$ has 4 inequivalent cusps:

$[0]_{\Gamma_1(5)}$ and $[1/2]_{\Gamma_1(5)}$ of width 5, and $[2/5]_{\Gamma_1(5)}$ and $[\infty]_{\Gamma_1(5)}$ of width 1.

Analogous to the example treated in Sect. 8 we consider the

*TASK.* Compute a rational function $\mathrm{rat}(x) \in \mathbb{C}(x)$ such that

$$j = \mathrm{rat}(R).$$

Owing to the fact that $R(\tau) \in M(\Gamma_1(5))$ is an analytic modular function which, by Lemma 4, is non-zero on $\mathbb{H}$, $R^* : X(\Gamma_1(5)) \to \hat{\mathbb{C}}$ must have all its zeros and poles at the cusps. Indeed, formula (18) in [32, Theorem 4] gives that $R^*$

- has a zero of order 1 at $[\infty]_{\Gamma_1(5)}$[26];
- has a pole of order 1 at $[2/5]_{\Gamma_1(5)}$;
- has order zero at the cusps $[0]_{\Gamma_1(5)}$ and $[1/2]_{\Gamma_1(5)}$.

Analogous to Example 34 we have that $j^*$ as a meromorphic function on $X(\Gamma_1(5))$

- has poles of order 1 at the cusps $[2/5]_{\Gamma_1(5)}$ and $[\infty]_{\Gamma_1(5)}$;
- has poles of order 5 at the cusps $[0]_{\Gamma_1(5)}$ and $[1/2]_{\Gamma_1(5)}$.

*Example 47* Not being relevant to our derivation of (63), we only remark that as a consequence of the pole count and the "valence formula" (60),[27] $j^*$ must have 12 zeros (including multiplicities) at orbits $[\gamma\omega]_{\Gamma_1(5)}$ with $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. In fact, as sketched in Example 59, one can verify that $j^*$ has a zero of order 3 at each of the orbits

$$[\gamma_1\omega]_{\Gamma_1(5)}, [\gamma_4\omega]_{\Gamma_1(5)}, [\gamma_5\omega]_{\Gamma_1(5)}, [\gamma_7\omega]_{\Gamma_1(5)},$$

with $\gamma_j$ as in Example 59, and no other zero elsewhere.

Analogous to Sect. 8, to solve our TASK, the decisive observation is that for

$$F(\tau) := j(\tau)\frac{(R(\tau) - R(0))^5\,(R(\tau) - R(1/2))^5}{R(\tau)^{11}} \in M(\Gamma_1(5)) \tag{64}$$

the induced function $F^*$ as a meromorphic function on $X(\Gamma_1(5))$

- has a possible pole only at $[\infty]_{\Gamma_1(5)}$.

Namely, the factors $(R(\tau) - R(0))^5$ and $(R(\tau) - R(1/2))^5$ cancel the poles of $F^*$ at $[0]_{\Gamma_1(5)}$ and $[1/2]_{\Gamma_1(5)}$, but altogether introduce a pole of order 10 at $[2/5]_{\Gamma_1(5)}$. Since $j^*$ has a pole of order 1 also at $[2/5]_{\Gamma_1(5)}$, we cancel this pole by dividing with $R(\tau)^{11}$. Hence the only remaining pole of $F^*$ is located at $[\infty]_{\Gamma_1(5)}$. Counting the pole order on the right side of (64) gives $1 + 0 + 0 + 11 = 12$, since $j^*$ has a pole of order 1 and $(1/R)^*$ of order 11 at $[\infty]_{\Gamma_1(5)}$.

Set $S := 1/R$. As $F^*$, also $S^*$ has its only pole at $[\infty]_{\Gamma_1(5)}$. Owing to the fact that this only pole of $S^*$ is of order 1, we can proceed as in Sect. 8. This means, we will reduce $F$ successively using only powers of $S$ until we reach a constant.

To input $F$ we need to know the values $R(0)$ and $R(1/2)$. In general, to determine such specific values could be a serious problem. But in our case, the required evaluations are the limits of the Rogers-Ramanujan continued fraction, (59) and (60), which we found by using elementary means only:

[26] This is also immediate from the $q$-expansion (37) of $R(\tau)$ at $\infty$.

[27] And also taking into account the fact that $j$ has zeros of multiplicity 3 at each element of the orbit $[\omega]_{\mathrm{SL}_2(\mathbb{Z})}$, and no zero elsewhere; see Example 38.

$$R(0) = \frac{1}{\phi^5} = -\frac{11}{2} + \frac{5\sqrt{5}}{2} \text{ and } R(1/2) = e^{\pi i}\phi^5 = -\frac{11}{2} - \frac{5\sqrt{5}}{2}.$$

In view of the denominator in (62), we note explicitly that these values give

$$(R(\tau) - R(0))\,(R(\tau) - R(1/2)) = R(\tau)^2 + 11R(\tau) - 1.$$

We take as input the $q$-expansions (30) and (37), and compute the expansions for $F$ and $S = 1/R$:

In[25]:= $\mathbf{j = \frac{1}{q} + 744 + 196884q + 21493760q^2 + 864299970q^3 + \cdots + O[q]^{14};}$

$\mathbf{R = q - 5q^2 + 15q^3 - 30q^4 + 40q^5 - 26q^6 + \cdots + O[q]^{14};}$

In[26]:= $\mathbf{F = Series[j\frac{(R^2 + 11R - 1)^5}{R^{11}}, q, 0, 2]}$

Out[26]= $-\frac{1}{q^{12}} - \frac{744}{q^{11}} - \frac{196824}{q^{10}} - \frac{21449060}{q^9} - \frac{852444060}{q^8} - \frac{18945738096}{q^7} - \frac{280406147430}{q^6} - \frac{3024142415076}{q^5} - \frac{25050805181610}{q^4} - \frac{164605868039100}{q^3} - \frac{874299071995668}{q^2} - \frac{3783906304850712}{q} - 13295075401691261 + O[q]^1$

In[27]:= $\mathbf{S = Series[\frac{1}{R}, q, 0, 12]}$

Out[27]= $\frac{1}{q} + 5 + 10q + 5q^2 - 15q^3 - 24q^4 + 15q^5 + 70q^6 + 30q^7 - 125q^8 - 175q^9 + 95q^{10} + 420q^{11} + O[q]^{12}$

Then we reduce $F$ successively using powers of $S$ times a suitable constant, until the coefficient of $q^0$ vanishes:

In[28]:= $\mathbf{F + S^{12}}$

Out[28]= $-\frac{684}{q^{11}} - \frac{195054}{q^{10}} - \frac{21414900}{q^9} - \frac{851959965}{q^8} - \frac{18940379184}{q^7} - \frac{280358028740}{q^6} - \frac{3023783306916}{q^5} - \frac{25048541972115}{q^4} - \frac{164593704824480}{q^3} - \frac{874243078444152}{q^2} - \frac{3783685739309592}{q} - 13294338120698731 + O[q]^1$

In[29]:= $\mathbf{F + S^{12} + 684S^{11}}$

Out[29]= $-\frac{157434}{q^{10}}-\frac{20399160}{q^9}-\frac{834052845}{q^8}-\frac{18709129044}{q^7}-\frac{278032352816}{q^6}$
$-\frac{3004884410856}{q^5}-\frac{24921527108535}{q^4}-\frac{163877480555060}{q^3}$
$-\frac{870829109716752}{q^2}-\frac{3769918758959172}{q}-13247719680862951+O[q]^1$

In[30]:= . . .

In[31]:= **F + S$^{12}$ + 684S$^{11}$ + 157434S$^{10}$ + 12527460S$^9$ + 77460495S$^8$ + 130689144S$^7$ − 33211924S$^6$ − 130689144S$^5$ + 77460495S$^4$ − 12527460S$^3$ + 157434S$^2$ − 684S + 1**

Out[31]= $O[q]^1$

In[32]:= **Clear[S]**

In[33]:= **Factor[S$^{12}$ + 684S$^{11}$ + 157434S$^{10}$ + 12527460S$^9$ + 77460495S$^8$ + 130689144S$^7$ − 33211924S$^6$ − 130689144S$^5$ + 77460495S$^4$ − 12527460S$^3$ + 157434S$^2$ − 684S + 1]**

Out[33]= $\left(S^4+228S^3+494S^2-228S+1\right)^3$

This means, we obtained the relation

$$F(\tau)=j(\tau)\frac{(R(\tau)^2+11R(\tau)-1)^5}{R(\tau)^{11}}=-\left(\frac{1}{R(\tau)^4}+\frac{228}{R(\tau)^3}+\frac{494}{R(\tau)^2}-\frac{228}{R(\tau)}+1\right)^3$$

which completes the algorithmic derivation of the icosahedral key relation (63).

## 12 Appendix 1: Generalized Dedekind Eta-Functions

We give the definition of generalized Dedekind eta functions $\eta_{g,h}(\tau;N)$ following the notation of Berndt [8] and Schoeneberg [33, Chap. 8]. Again we put $q=e^{2\pi i\tau}$. For the Bernoulli polynomials $B_1(x)=x-\frac{1}{2}$ and $B_2(x)=x^2-x+\frac{1}{6}$ let

$$b_1(x):=B_1(\{x\})\text{ and }b_2(x):=B_2(\{x\}),\quad x\in\mathbb{R},$$

where $\{x\}:=x-\lfloor x\rfloor$ is the fractional part. Furthermore, for $g,h\in\mathbb{Z}$ define

$$\alpha(g,h):=\begin{cases}(1-e^{-2\pi ih})e^{\pi ib_1(h)}, & \text{if } g\in\mathbb{Z}\text{ and }h\notin\mathbb{Z},\\ 1, & \text{otherwise.}\end{cases}$$

**Definition 48** (*generalized Dedekind eta functions*) Let $g,h\in\mathbb{Z}$, $N\in\mathbb{Z}_{>0}$, and $\zeta_N:=e^{2\pi i/N}$. For $\tau\in\mathbb{H}$:

$$\eta_{g,h}(\tau;N):=\alpha(g/N,h/N)q^{b_2(g/N)/2}\prod_{\substack{m\geqslant 1\\ m\equiv g\,(\mathrm{mod}\,N)}}\left(1-\zeta_N^h q^{\frac{m}{N}}\right)\prod_{\substack{m\geqslant 1\\ m\equiv -g\,(\mathrm{mod}\,N)}}\left(1-\zeta_N^{-h} q^{\frac{m}{N}}\right).$$

If $g \not\equiv 0 \pmod N$ one can write this as

$$\eta_{g,h}(\tau;N) = \alpha(g/N,h/N)q^{b_2(g/N)/2}(\zeta_N^h q^{\frac{G}{N}};q)_\infty(\zeta_N^{-h}q^{\frac{N-G}{N}};q)_\infty, \tag{65}$$

where $G \in \{0,\dots,N-1\}$ such that $G \equiv g \pmod N$.

*Example 49* If $g = h = 0$: $\alpha(0,0) = 1$, $b_2(0)/2 = 1/12$, and

$$\eta_{0,0}(\tau;N) = q^{\frac{1}{12}} \prod_{k\geqslant 1} \left(1-q^k\right)^2 = \eta(\tau)^2.$$

This motivates to call the $\eta_{g,h}(\tau;N)$ *generalized Dedekind eta functions*.

The Rogers-Ramanujan functions $G(\tau)$ and $H(\tau)$ from Example 20 are obtained as follows.

*Example 50* According to (65),

$$\eta_{1,0}(\tau;5) = q^{\frac{1}{300}}(q^{\frac{1}{5}};q)_\infty(q^{\frac{4}{5}};q)_\infty \text{ and } \eta_{2,0}(\tau;5) = q^{-\frac{11}{300}}(q^{\frac{2}{5}};q)_\infty(q^{\frac{3}{5}};q)_\infty.$$

Hence

$$G(\tau) = q^{-\frac{1}{60}}F(1) = \frac{1}{\eta_{1,0}(5\tau;5)} \text{ and } H(\tau) = q^{\frac{11}{60}}F(q) = \frac{1}{\eta_{2,0}(5\tau;5)}.$$

*Example 51* If one expands the products in Definition 48 one obtains a Laurent series with *finite* principal part. The explicit expansion can be obtained with Jacobi's triple product identity (51); for example, if $g \not\equiv 0 \pmod N$,

$$\begin{aligned}\left(\frac{\alpha(g/N,h/N)q^{b_2(g/N)/2}}{(q;q)_\infty}\right)^{-1}\eta_{g,h}(\tau;N) &= \sum_{n=-\infty}^{\infty}(-1)^n q^{\binom{n}{2}}\left(\zeta_N^h q^{\frac{G}{N}}\right)^n \\ &= 1 + \sum_{n=1}^{\infty}(-1)^n q^{\binom{n}{2}}\zeta_N^{hn}q^{\frac{G}{N}n} + \sum_{n=1}^{\infty}(-1)^n q^{\binom{n}{2}}\zeta_N^{-hn}q^{\frac{N-G}{N}n},\end{aligned}$$

where $G \in \{0,\dots,N-1\}$ such that $G \equiv g \pmod N$.

The following transformation behaviour, respectively variants of it, has been studied and derived by Curt Meyer [24], Ulrich Dieter [12], and Bruno Schoeneberg [33, Chap. 8].

**Proposition 52** *Let* $N \in \mathbb{Z}_{>0}$ *and* $g,h \in \mathbb{Z}$ *such that* $g$ *and* $h$ *are not both* $\equiv 0 \pmod N$. *Then for any* $\gamma = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$*:*

$$\eta_{g,h}(\gamma\tau;N) = e^{\pi i\,\mu(\gamma,g',h';N)}\eta_{g',h'}(\tau;N) \tag{66}$$

*where*

$$\begin{pmatrix} g' \\ h' \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} g \\ h \end{pmatrix},$$

*and where the rational number* $\mu(\gamma, g', h'; N) \in \mathbb{Q}$ *is produced by a complicated expression.*[28]

In [8] Bruce Berndt succeeded to streamline work of Joseph Lewittes [21] and obtained (66) as a special case of his setting. For a different approach to transformation formulas for generalized Dedekind eta functions see Yifan Yang [38].

**Corollary 53** *The Rogers-Ramanujan functions* $G(\tau)$ *and* $H(\tau)$ *from Example 20 satisfy property* (12) *concerning the finiteness of the principal part.*

*Proof* This is a straightforward consequence of Example 50, Note 51, and the transformation formula (66).

We conclude Appendix 1 by mentioning some connections to theta functions.[29] Classically, there are four Jacobi theta functions $\theta_1, \ldots, \theta_4$, [5, (10.7.1)–(10.7.4)], but which, as stated in [5], "are really the same function." For example, noting that for $q = e^{2\pi i \tau}$, $\tau \in \mathbb{H}$, and $z \in \mathbb{C}$,

$$\theta_1(z, \tau) = (-i) q^{1/8} \sum_{n=-\infty}^{\infty} (-1)^n q^{\binom{n}{2}} e^{-(2n-1)iz}. \tag{67}$$

Using Jacobi's triple product identity (51) one obtains for the Rogers-Ramanujan functions from Example 20,

$$i q^{2/5} \theta_1(-2\pi\tau, 5\tau) = \eta(\tau) G(\tau) \text{ and } i q^{1/10} \theta_1(-\pi\tau, 5\tau) = \eta(\tau) H(\tau). \tag{68}$$

More generally, for $q = e^{2\pi i \tau}$, $\tau \in \mathbb{H}$, and $z = e^{2\pi i \zeta}$, $\zeta \in \mathbb{C}$, consider the theta function studied extensively by Farkas and Kra [15, (2.53)]:

$$\theta \begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix} (\zeta, \tau) = e^{\frac{\pi i \varepsilon \varepsilon'}{2}} z^{\frac{\varepsilon}{2}} q^{\frac{\varepsilon^2}{8}} (q; q)_\infty \left( -z e^{\pi i \varepsilon'} q^{\frac{1+\varepsilon}{2}}; q \right)_\infty \left( -z^{-1} e^{-\pi i \varepsilon'} q^{\frac{1-\varepsilon}{2}}; q \right)_\infty, \tag{69}$$

where $\varepsilon$ and $\varepsilon'$ are real parameters. One can verify, again by using the triple product identity (51), that generalized Dedekind eta functions are a subfamily of these functions. For instance, if $g, h \in \{1, \ldots, N-1\}$,

$$\theta \begin{bmatrix} 1 - 2g_N \\ 1 \end{bmatrix} (-h_N, \tau) = \frac{i e^{\pi i (2 g_N h_N - g_N - h_N)}}{2 \sin(h_N)} \eta(\tau) \eta_{g,h}(\tau; N), \tag{70}$$

[28] To obtain an explicit form of this expression set, for instance, $b_{g,h}(N) = 0$ on the right side of [8, (24)].

[29] Warning: in many texts on Jacobi theta functions $q = e^{\pi i \tau}$, in contrast to $q = e^{2\pi i \tau}$ as throughout this article.

where $g_N := g/N$ and $h_N := h/N$. For $h = 0$ and $g \in \{1, \dots, N-1\}$ one has,

$$\theta \begin{bmatrix} 1 - 2g_N \\ 1 \end{bmatrix} (0, \tau) = i e^{-\pi i g_N} \eta(\tau) \eta_{g,0}(\tau; N). \tag{71}$$

Theta functions with $z = 0$, i.e., $\theta \left[ \begin{smallmatrix} \varepsilon \\ \varepsilon' \end{smallmatrix} \right] (0, \tau)$, are called theta constants. As studied in detail in [15], already this subfamily satisfies a rich variety of transformation formulas. For example, Duke uses this tool-box to derive the following modular transformation [14, (4.10)] for the Rogers-Ramanujan quotient $r(\tau)$ from Example 24:

$$r\left(-\frac{1}{\tau}\right) = \frac{-(1+\sqrt{5})r(\tau) + 2}{2r(\tau) + 1 + \sqrt{5}}.$$

As revealed also by other applications in [14], the Farkas-Kra theta function calculus is providing computational alternatives to some of the methods presented in this tutorial.

## 13 Appendix 2: Valence Formula

For zero recognition of modular functions and, more generally, of modular forms "valence formulas" are often very useful. Such formulas describe relations between the orders at points $\tau \in \mathbb{H}$ corresponding to orbits $[\tau]_\Gamma$ in the sense of Corollary 36, and at points $\frac{a}{c} \in \hat{\mathbb{Q}}$ corresponding to cusps in the sense of Definition 30. In our context we only need to discuss "valence formulas" for modular functions which can be viewed as specializations of another "folklore theorem" from Riemann surfaces, e.g., [25, Proposition 4.12][30]:

**Theorem 54** *Let $f : X \to \hat{\mathbb{C}}$ be a non-constant meromorphic function on a compact Riemann surface $X$. Then*

$$\sum_{p \in X} \mathrm{Ord}_p(f) = 0. \tag{72}$$

For functions on Riemann surfaces the orders $\mathrm{Ord}_p(f)$ are defined via the orders of local (Laurent) series expansions about $p \in X$ with respect to charts $\varphi$. Concretely, let $U \subseteq X$ be an open neighborhood of $p$ containing no pole except possibly $p$ itself,

[30]The first such "folklore theorem" we considered was Theorem 33.

and let $\varphi : U \to V \subseteq \mathbb{C}$ be a homeomorphism.[31] Then, by assumption, $f \circ \varphi^{-1}$ is analytic in $V \setminus \{\varphi(p)\}$ and can be expanded in a Laurent series about $z_0 := \varphi(p)$,

$$f(\varphi^{-1}(z)) = \sum_{n=-M}^{\infty} c_n (z - z_0)^n.$$

Assuming that $c_{-M} \neq 0$, one defines $\mathrm{Ord}_p(f) := -M$.

*Note 55* Obviously, when taking the standard open sets as neighborhoods and as charts the identity maps, the complex plane can be turned into a Riemann surface. In this case, the order is the usual order $\mathrm{ord}_p(f)$ from Definition 32 for Laurent series with finite principal part; i.e., for $p \in X := \mathbb{C}$ and a function $f : U \to \hat{\mathbb{C}}$ being meromorphic in a neighborhood $U$ of $p \in U \subseteq \mathbb{C}$,

$$\mathrm{Ord}_p(f) = \mathrm{ord}_p(f). \tag{73}$$

Finally we connect (72) to our context; namely, where $X := X(\Gamma) = \{[\tau]_\Gamma : \tau \in \hat{\mathbb{H}}\}$ for some congruence subgroup $\Gamma$ of $\mathrm{SL}_2(\mathbb{Z})$ and where the $[\tau]_\Gamma = \{\gamma\tau : \gamma \in \Gamma\}$ are the orbits of the action of $\Gamma$ on $\hat{\mathbb{H}}$. Here as meromorphic functions $f : X \to \hat{\mathbb{C}}$ we have the induced functions $g^* : X(\Gamma) \to \hat{\mathbb{C}}$ of meromorphic $g \in M(\Gamma)$. If $p = \left[\frac{a}{c}\right]_\Gamma \in X(\Gamma)$ is a cusp, then in view of the remarks leading up to Definition 30 we have

$$\mathrm{Ord}_p(f) = \mathrm{Ord}_{[a/c]}(g^*) = \mathrm{ord}_{a/c}^{\Gamma}(g).$$

For orbits $p = [\tau]_\Gamma$ with $\tau \in \mathbb{H}$, the discussion of how to define $\mathrm{Ord}_p$ is more involved. Therefore we refrain from doing so, and state our modular function adaptation (77) of (72) without proof.

Nevertheless, we present a version of a "valence formula" which is sufficiently flexible for many (algorithmic) applications we have in mind.[32] We also note that our version is different from the many versions of "valence formulas" one finds in the literature in the following sense. The formula applied to a given group $\Gamma$ can be made explicit directly by knowing a complete set of representatives of the right cosets of $\Gamma$ in $\mathrm{SL}_2(\mathbb{Z})$. One basically lifts the formula valid form $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ to any $\Gamma$ in a natural way from our point of view. We view this as natural because we only need to consider how the orbit $[\tau]_{\mathrm{SL}_2(\mathbb{Z})}$ splits into smaller orbits under the action of $\Gamma$ for every $\tau$ going throw a complete set of representatives of the orbits of the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}^*$. So we can split our analysis into four cases: the orbits $[\tau]_{\mathrm{SL}_2(\mathbb{Z})}$ different from $[i]$, $[\omega]$ and $[\infty]$ and these remaining three orbits. This idea will be seen clearly from the examples where we apply the formula on the group $\Gamma_0(2)$ and $\Gamma(5)$. This gives, in particular, a more pragmatic flavour to our formula when compared to the classic versions that talk about elliptic points, parabolic points

[31] In addition, $\varphi$ is supposed to be compatible with the other charts; see e.g. [25].

[32] From modular *forms* point of view, (77) deals with the case of forms of weight zero only.

without making them more explicit. The transition from the formal statement to the concrete application can be tedious, at least from our experience.

Before stating it, we need some preparations.

Suppose $\gamma_1, \ldots, \gamma_m \in \mathrm{SL}_2(\mathbb{Z})$ is a complete set of right coset representatives of $\Gamma$ in $\mathrm{SL}_2(\mathbb{Z})$; i.e., as a disjoint union,

$$\mathrm{SL}_2(\mathbb{Z}) = \Gamma\gamma_1 \dot{\cup} \ldots \dot{\cup} \Gamma\gamma_m. \tag{74}$$

Then for any $\tau \in \mathbb{H}$ the $\mathrm{SL}_2(\mathbb{Z})$-orbit of $\tau$ splits into $\Gamma$-orbits accordingly,

$$[\tau]_{\mathrm{SL}_2(\mathbb{Z})} = [\gamma_1\tau]_\Gamma \cup \cdots \cup [\gamma_m\tau]_\Gamma.$$

We note explicitly that, in contrast to (74) it might well happen that $[\gamma_k\tau]_\Gamma = [\gamma_\ell\tau]_\Gamma$ for $k \neq \ell$. Actually it is true that

$$[\gamma_k\tau]_\Gamma = [\gamma_\ell\tau]_\Gamma \Leftrightarrow \gamma_\ell \in \Gamma\gamma_k/\mathrm{Stab}(\tau) \tag{75}$$

with

$$\mathrm{Stab}(\tau) := \{\gamma \in \mathrm{SL}_2(\mathbb{Z}) : \gamma\tau = \tau\},$$

and where $\Gamma\gamma_k/\mathrm{Stab}(\tau)$ is a particular subset of the right cosets of $\Gamma$ in $\mathrm{SL}_2(\mathbb{Z})$ defined as an orbit of an action of $\mathrm{Stab}(\tau)$ which permutes cosets:

$$\Gamma\gamma_j/\,\mathrm{Stab}(\tau) := \{\Gamma\gamma_j\gamma : \gamma \in \mathrm{Stab}(\tau)\}.$$

For fixed $\tau \in \mathbb{H}$, the set of different $\Gamma$-orbits is denoted by

$$S_\Gamma(\tau) := \{[\gamma_j\tau]_\Gamma : j = 1, \ldots, m\}.$$

Note that in general, $|S_\Gamma(\tau)| \leqslant m$. One can verify in a straightforward manner that for fixed $\tau \in \mathbb{H}$ the following map is bijective:

$$\phi : \{\Gamma\gamma_j/\mathrm{Stab}(\tau) : j = 1, \ldots, m\} \to S_\Gamma(\tau), \quad \phi\left(\Gamma\gamma_j/\mathrm{Stab}(\tau)\right) := [\gamma_j\tau]_\Gamma. \tag{76}$$

The stabilizer subgroup $\mathrm{Stab}(\tau)$ comes in because special care has to be taken of "elliptic" points; cf. [11, Sects. 2.3 and 2.4]. These are points $\tau_0 \in \mathbb{H}$, resp. orbits $[\tau_0]_\Gamma$, which are fixed by non-trivial elements from $\mathrm{SL}_2(\mathbb{Z})$. To handle this matter technically, it is convenient to introduce a special notation for the map induced by the action of $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$:

$$\overline{\gamma} : \hat{\mathbb{H}} \to \hat{\mathbb{H}}, \tau \mapsto \overline{\gamma}(\tau) := \gamma\tau = \frac{a\tau + b}{c\tau + d}.$$

For any subset $G \subseteq \mathrm{SL}_2(\mathbb{Z})$ we denote the image under this map by

$$\overline{G} := \{\overline{\rho} : \rho \in G\}\,.$$

We note that if $G$ is a subgroup of $\mathrm{SL}_2(\mathbb{Z})$, then $\overline{G}$ is a subgroup of $\overline{\mathrm{SL}_2(\mathbb{Z})} \cong \mathrm{SL}_2(\mathbb{Z})/\{\pm I\}$.

Collecting all these ingredients one can prove as a specialization of Thoerem 54:

**Theorem 56** ("valence formula") *Let $\Gamma$ be a congruence subgroup and $\mathrm{SL}_2(\mathbb{Z}) = \Gamma\gamma_1 \dot{\cup} \ldots \dot{\cup} \Gamma\gamma_m$ a disjoint coset decomposition. Then for any $g \in M(\Gamma)$:*

$$\sum_{\tau \in H(\mathrm{SL}_2(\mathbb{Z}))} \sum_{[\gamma_j\tau]_\Gamma \in S_\Gamma(\tau)} \frac{|\Gamma\gamma_j / \mathrm{Stab}(\tau)|}{w(\Gamma)|\overline{\mathrm{Stab}(\tau)}|} \,\mathrm{ord}_{\gamma_j\tau}(g) + \sum_{\substack{[a/c]_\Gamma \\ \text{cusp of } X(\Gamma)}} \mathrm{ord}^{\Gamma}_{a/c}(g) = 0, \tag{77}$$

*where $H(\mathrm{SL}_2(\mathbb{Z}))$ is a complete set of representatives of the orbits $[\tau]_{\mathrm{SL}_2(\mathbb{Z})}$ with $\tau \in \mathbb{H}$, $\mathrm{ord}_{\gamma_j\tau}(g)$ is the usual order as in Definition 32, and*

$$w(\Gamma) := \begin{cases} 1, \textit{if } -I = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in \Gamma, \\ 2, \textit{otherwise.} \end{cases}$$

It is well-known that the only points giving rise to non-trivial stabilizers are the elements in the orbits $[i]_{\mathrm{SL}_2(\mathbb{Z})}$ and $[\omega]_{\mathrm{SL}_2(\mathbb{Z})}$, where $\omega := e^{2\pi i/3}$. Indeed one has, for example,

$$\mathrm{Stab}(i) = \{I, -I, T, -T\},\ \mathrm{Stab}(\omega) = \{I, -I, TS, -TS, (TS)^2, -(TS)^2\}; \tag{78}$$

A detailed analysis of fixed points of modular transformations is given in [33, Sect. 1.3].

As examples we consider specializations of the "valence formula" (77) for three choices of $\Gamma$: $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, $\Gamma = \Gamma_0(2)$, and $\Gamma = \Gamma_1(5)$.

*Example 57* $\Gamma = \mathrm{SL}_2(\mathbb{Z})$: as coset decomposition we have $\mathrm{SL}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z})\gamma_1$ with $\gamma_1 = I$; $S_\Gamma(\tau) = \{[\tau]_{\mathrm{SL}_2(\mathbb{Z})}\}$; $\Gamma/\,\mathrm{Stab}(\tau) = \{\Gamma\}$; $w(\Gamma)$=1 since $-I\in\mathrm{SL}_2(\mathbb{Z})$. Finally, $X(\mathrm{SL}_2(\mathbb{Z}))$ has only one cusp $[\infty]_{\mathrm{SL}_2(\mathbb{Z})}$, hence (77) becomes

$$\sum_{\tau \in H(\mathrm{SL}_2(\mathbb{Z}))} \frac{1}{|\overline{\mathrm{Stab}(\tau)}|} \,\mathrm{ord}_\tau(g) + \mathrm{ord}^{\Gamma}_{\infty}(g) = 0. \tag{79}$$

Because of (78), the "valence formula" (79) turns into the version (43) of Corollary 37.

*Example 58* $\Gamma = \Gamma_0(2)$: $\mathrm{SL}_2(\mathbb{Z}) = \Gamma\gamma_1 \dot\cup \Gamma\gamma_2 \dot\cup \Gamma\gamma_3$ is the coset decomposition with $\gamma_1 = I$, $\gamma_2 = T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, and $\gamma_2 = TS = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$; $S_\Gamma(i) = \{[i]_\Gamma, [TSi]_\Gamma\}$, $S_\Gamma(\omega) = \{[\omega]_\Gamma\}$; $\Gamma/\operatorname{Stab}(i) = \{\Gamma, \Gamma T\}$, $\Gamma TS/\operatorname{Stab}(i) = \{\Gamma TS\}$; $\Gamma/\operatorname{Stab}(\omega) = \{\Gamma, \Gamma TS, \Gamma(TS)^2\}$; $w(\Gamma) = 1$ since $-I \in \Gamma_0(2)$. Finally, $\Gamma$ has two cusps, $[\infty]_\Gamma$ and $[0]_\Gamma$. Hence (77) turns into the version (47) of Corollary 43.

*Note 59* $\Gamma = \Gamma_1(5)$: To specify the elements $\gamma_j$ of the coset decomposition $\mathrm{SL}_2(\mathbb{Z}) = \dot\bigcup_{j=1}^{24} \Gamma\gamma_j$ we use $(a, b, c, d)$ instead of matrix notation $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$:

$$\begin{aligned}
&\gamma_1 := (1,0,0,1), \gamma_2 := (0,-1,1,0), \gamma_3 := (0,-1,1,1), \gamma_4 := (0,-1,1,2),\\
&\gamma_5 := (0,-1,1,3), \gamma_6 := (0,-1,1,4), \gamma_7 := (2,-1,5,-2), \gamma_8 := (-1,-2,-2,-5),\\
&\gamma_9 := (-1,-3,-2,-7), \gamma_{10} := (-1,-4,-2,-9), \gamma_{11} := (-1,-5,-2,-11),\\
&\gamma_{12} := (-1,-6,-2,-13), \gamma_{13} := (3,1,5,2), \gamma_{14} := (1,-3,2,-5),\\
&\gamma_{15} := (1,-2,2,-3), \gamma_{16} := (1,-1,2,-1), \gamma_{17} := (1,0,2,1), \gamma_{18} := (1,1,2,3),\\
&\gamma_{19} := (4,-1,5,-1), \gamma_{20} := (-1,-4,-1,-5), \gamma_{21} := (-1,-5,-1,-6),\\
&\gamma_{22} := (-1,-6,-1,-7), \gamma_{23} := (-1,-7,-1,-8), \gamma_{24} := (-1,-8,-1,-9).
\end{aligned}$$

The action of $\operatorname{Stab}(\omega)$ on the set $C := \{\Gamma\gamma_j : j = 1, \ldots, 24\}$ of cosets results in the disjoint orbit decomposition

$$C = \Gamma\gamma_1/\operatorname{Stab}(\omega) \,\dot\cup\, \Gamma\gamma_4/\operatorname{Stab}(\omega) \,\dot\cup\, \Gamma\gamma_5/\operatorname{Stab}(\omega) \,\dot\cup\, \Gamma\gamma_7/\operatorname{Stab}(\omega). \tag{80}$$

For each $j = 1, 4, 5, 7$ one has $|\Gamma\gamma_j/\operatorname{Stab}(\omega)| = 6$; for instance,

$$\Gamma\gamma_1/\operatorname{Stab}(\omega) = \{\Gamma\gamma_1, \Gamma\gamma_2, \Gamma\gamma_3, \Gamma\gamma_{19}, \Gamma\gamma_{20}, \Gamma\gamma_{21}\}.$$

Hence each of the six elements of $\operatorname{Stab}(\omega)$ gives rise to a different element of $\Gamma\gamma_j/\operatorname{Stab}(\omega)$. This is due to the fact that $-I = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \notin \Gamma$; for example,

$$\Gamma\gamma_{19} = \Gamma\begin{pmatrix} 4 & -1 \\ 5 & -1 \end{pmatrix} = \Gamma\begin{pmatrix} -4 & 1 \\ -5 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \Gamma\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

As another consequence of $-I \notin \Gamma = \Gamma_1(5)$, in the "valence formula" (77) we have to set $w(\Gamma) := 2$.

Finally, owing to the bijection $\phi$ from (76) we know that the orbit $[\omega]_{\mathrm{SL}_2(\mathbb{Z})}$ splits into four different $\Gamma$-orbits with the $\gamma_j$ as in (80); i.e.,

$$S_\Gamma(\omega) = \{[\gamma_1\omega]_\Gamma, [\gamma_4\omega]_\Gamma, [\gamma_5\omega]_\Gamma, [\gamma_7\omega]_\Gamma\}.$$

Proceeding along these lines one can establish the following "valence formula" for $\Gamma = \Gamma_1(5)$ as a consequence of Theorem 56:

**Corollary 60** ("valence formula" for $\Gamma_1(5)$) *Let $g \in M(\Gamma)$. If $\Gamma = \Gamma_1(5)$ then*

$$\sum_{j\in\{1,4,5,7\}} \frac{6}{2\times 3}\,\mathrm{ord}_{\gamma_j\omega}(g) + \sum_{j\in\{1,3,4,5,7,9\}} \frac{4}{2\times 2}\,\mathrm{ord}_{\gamma_j i}(g) + \sum_{\substack{[a/c]_\Gamma \\ \text{cusp of } X(\Gamma)}} \mathrm{ord}^{\Gamma}_{a/c}(g)$$
$$+ \sum_{\substack{\tau\in H(\mathrm{SL}_2(\mathbb{Z})) \\ [\tau]\neq[i],[\tau]\neq[\omega]}} \sum_{j=1}^{24} \mathrm{ord}_{\gamma_j\tau}(g) = 0, \tag{81}$$

*where $H(\mathrm{SL}_2(\mathbb{Z})) \subseteq \mathbb{H}$ is a complete set of representatives of the orbits $[\tau]_{\mathrm{SL}_2(\mathbb{Z})}$ with $\tau \in \mathbb{H}$, and where $\omega := e^{2\pi i/3}$.*

## 14 Conclusion

The Rogers-Ramanujan functions are embedded in a rich web of beautiful mathematics. So there are much more stories to tell. For example, as discussed in [14], one can ask for which evaluations the Rogers-Ramanujan continued fraction $r(\tau)$ gives an algebraic number and if so, in which situations such values can be expressed in terms of radicals over $\mathbb{Q}$. Finally we mention the fact that the Rogers-Ramanujan continued fraction is playing a prominent role in Ramanujan's "Lost" Notebook; see the first five chapters of [6].

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