Undergraduate Lecture Notes in Physics

Hafez A. Radi
John O. Rasmussen

Principles of Physics

For Scientists and Engineers

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Preface

This book on Principles of Physics is intended to serve fundamental college courses in scientific curricula.

Physics is one of the most important tools to aid undergraduates, graduates, and researchers in their technical fields of study. Without it many phenomena cannot be described, studied, or understood. The topics covered here will help students interpret such phenomena, ultimately allowing them to advance in the applied aspects of their fields.

The goal of this text is to present many key concepts in a clear and concise, yet interesting way, making use of practical examples and attractively colored illustrations whenever appropriate to satisfy the needs of today's science and engineering students.

Some of the examples, proofs, and subsections in this textbook have been identified as optional and are preceded with an asterisk *. For less intensive courses these optional portions may be omitted without significantly impacting the objectives of the chapter. Additional material may also be omitted depending on the course's requirements.

The first author taught the material of this book in many universities in the Middle East for almost four decades. Depending on the university, he leveraged different international textbooks, resources, and references. These used different approaches, but were mainly written in an expansive manner delivering a plethora of topics while targeting students who wanted to dive deeply into the subject matter. In this textbook, however, the authors introduce a large subset of these topics but in a more simplified manner, with the intent of delivering these topics and their key facts to students all over the world and in particular to students in the Middle East and neighboring regions where English may not be the native language. The second author went over the entire text with the background of study and/or teaching at Caltech, UC Berkeley, and Yale.

Instructors teaching from this textbook will be able to gain online access from the publisher to the solutions manual, which provides step-by-step solutions to all exercises contained in the book. The solutions manual also contains many tips, colored illustrations, and explanations on how the solutions were derived.

Acknowledgments from Prof. Hafez A. Radi

I owe special thanks to my wife and two sons Tarek and Rami for their ongoing support and encouragement. I also owe special thanks to my colleague and friend Prof. Rasmussen for his invaluable contributions to this book, and for everything that I learned from him over the years while carrying out scientific research at Lawrence Berkeley Lab. Additionally, I would like to express my gratitude to Prof. Ali Helmy Moussa, Prof. of Physics at Ain Shams University in Egypt, for his assistance, support, and guidance over the years. I also thank all my fellow professors and colleagues who provided me with valuable feedback pertaining to many aspects of this book, especially Dr. Sana'a Ismail, from Dar El Tarbiah School, IGCSE section and Dr. Hesham Othman from the Faculty of Engineering at Cairo University. I would also like to thank Professor Mike Guidry, Professor of Physics and Astronomy at the University of Tennessee Knoxville, for his valuable recommendations. I am also grateful to the CD Odessa LLC for their Concept-Draw software suite which was used to create almost all the figures in this book. I finally extend my thanks and appreciation to Professor Nawal El-Degwi, Professor Khayri Abdel-Hamid, Professor Said Ashour, and the staff members and teaching assistants at the faculty of Engineering at MSA University, Egypt, for all their support and input.

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I would like to thank Prof. Radi for the opportunity to join him as coauthor. I am grateful to the many teachers, students, and colleagues from whom I learned various aspects of the fascinating world of the physical sciences, notably the late Drs. Linus Pauling, Isadore Perlman, Stanley Thompson, Glenn Seaborg, Earl Hyde, Hilding Slätis, Aage Bohr, Gaja Alaga, and Hans-Järg Mang. There are many others, still living, too numerous to list here. I would also like to extend my special thanks to my wife for her support and encouragement.

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Contents

Fundamental Physical Constants

Quantity	Symbol	Approximate value
Speed of light in vacuum	c	3.00×10^8 m/s
Avogadro's number	N_A	$6.02 \times 10^{23}\ \mathrm{mol}^{-1} = 6.02 \times 10^{26}\ \mathrm{kmol}^{-1}$
Gas constant	R	$8.314\ \mathrm{J/mol \cdot K} = 8\,314\ \mathrm{J/kmol \cdot K}$
Boltzmann's constant	k	1.38×10^{-23} J/K
Gravitational constant	G	$6.67 \times 10^{-11}\ \mathrm{N \cdot m^2/kg^2}$
Planck's constant	h	$6.63 \times 10^{-34}\ \mathrm{J \cdot s}$
Permittivity of free space	ϵ_0	$8.85 \times 10^{-12}\ \mathrm{C^2/N \cdot m^2}$
Permeability of free space	$\mu_0 = 1/(c^2\epsilon_0)$	$4\pi \times 10^{-7}\ \mathrm{T \cdot m/A}$
Atomic mass unit	1u	$1.6605 \times 10^{-27}\ \mathrm{kg} = 931.49\ \mathrm{MeV}/c^2$
Electron charge	$-e$	-1.60×10^{-19} C
Electron rest mass	m_e	$9.11 \times 10^{-31}\ \mathrm{kg} = 0.000549\ \mathrm{u}$ $= 0.511\ \mathrm{MeV}/c^2$
Proton rest mass	m_p	$1.6726 \times 10^{-27}\ \mathrm{kg} = 1.00728\ \mathrm{u}$ $= 938.27\ \mathrm{MeV}/c^2$
Neutron rest mass	m_n	$1.6749 \times 10^{-27}\ \mathrm{kg} = 1.008665\ \mathrm{u}$ $= 939.57\ \mathrm{MeV}/c^2$

Other useful constants

Acceleration due to gravity at the Earth's surface (av.)	$g = 9.8\,\text{m/s}^2$
Absolute zero (0 K)	$-273.15\,°\text{C}$
Joule equivalent (1 kcal)	$4,186\,\text{J}$
Speed of sound in air (20 °C)	343 m/s
Density of air (dry)	$1.29\,\text{kg/m}^3$
Standard atmosphere	1.01×10^5 Pa
Electric breakdown strength	3×10^6 V/m
Earth: Mass Radius (av.)	5.98×10^{24} kg 6.38×10^3 km
Moon: Mass Radius (av.)	7.35×10^{22} kg 1.74×10^3 km
Sun: Mass Radius (av.)	1.99×10^{30} kg 6.96×10^5 km
Earth–Moon distance (av.)	3.84×10^5 km
Earth–Sun distance (av.)	1.5×10^8 km

The greek alphabet

Alpha	A	α	Nu	N	ν
Beta	B	β	Xi	Ξ	ξ
Gamma	Γ	γ	Omicron	O	o
Delta	Δ	δ	Pi	Π	π
Epsilon	E	ε	Rho	P	ρ
Zeta	Z	ζ	Sigma	Σ	σ
Eta	H	η	Tau	T	τ
Theta	Θ	θ	Upsilon	Y	υ
Iota	I	ι	Phi	Φ	ϕ
Kappa	K	κ	Chi	X	χ
Lambda	Λ	λ	Psi	Ψ	ψ
Mu	M	μ	Omega	Ω	ω

Some SI base units and derived units

Quantity	Unit name	Unit symbol	In terms of base units
Mass	kilogram	kg	Base SI units
Length	meter	m	
Time	second	s	
Electric current	ampere	A	
Force	newton	N	$kg \cdot m/s^2$
Energy and work	joule	J	$kg \cdot m^2/s^2$
Power	watt	W	$kg \cdot m^2/s^3$
Pressure	pascal	Pa	$kg/(m \cdot s^2)$
Frequency	hertz	Hz	s^{-1}
Electric charge	coulomb	C	$A \cdot s$
Electric potential	volt	V	$kg \cdot m^2/(A \cdot s^3)$
Electric resistance	ohm	Ω	$kg \cdot m^2/(A^2 \cdot s^3)$
Capacitance	farad	F	$A^2 \cdot s^4/(kg \cdot m^2)$
Magnetic field	tesla	T	$kg/(A \cdot s^2)$
Magnetic flux	weber	Wb	$kg \cdot m^2/(A \cdot s^2)$
Inductance	henry	H	$kg \cdot m^2/(s^2 \cdot A^2)$

SI multipliers

yotta	Y	10^{24}
zeta	Z	10^{21}
exa	E	10^{18}
peta	P	10^{15}
tera	T	10^{12}
giga	G	10^{9}
mega	M	10^{6}
kilo	k	10^{3}
hecto	h	10^{2}
deka	da	10^{1}
deci	d	10^{-1}
centi	c	10^{-2}
milli	m	10^{-3}
micro	μ	10^{-6}
nano	n	10^{-9}
pico	p	10^{-12}
femto	f	10^{-15}
atto	a	10^{-18}
zepto	z	10^{-21}
yocto	y	10^{-24}

Part I
Fundamental Basics

1 Dimensions and Units

The laws of physics are expressed in terms of basic quantities that require a clear definition for the purpose of measurements. Among these measured quantities are length, time, mass, temperature, etc.

In order to describe any physical quantity, we first have to define a **unit** of measurement (which was among the earliest tools invented by humans), i.e. a measure that is defined to be exactly 1.0. After that, we define a **standard** for this quantity, i.e. a reference to compare all other examples of the same physical quantity.

1.1 The International System of Units

Seven physical quantities have been selected as base quantities in the 14th General Conference on Weights and Measurements, held in France in 1971. These quantities form the basis of the International System of Units, abbreviated **SI** (from its French name *Système International*) and popularly known as the *metric system*. Table 1.1 depicts these quantities, their unit names, and their unit symbols.

Many SI derived units are defined in terms of the first three quantities of Table 1.1. For example, the SI unit for force, called the newton (abbreviated N), is defined in terms of the base units of mass, length, and time. Thus, as we will see from the study of Newton's second law, the unit of force is given by:

$$1\ \mathrm{N} = 1\ \mathrm{kg.m/s^2} \tag{1.1}$$

When dealing with very large or very small numbers in physics, we use the so-called scientific notation which employs powers of 10, such as:

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Undergraduate Lecture Notes in Physics, DOI: 10.1007/978-3-642-23026-4_1,

$$3\,210\,000\,000 \text{ m} = 3.21 \times 10^9 \text{ m} \tag{1.2}$$

$$0.000\,000\,789 \text{ s} = 7.89 \times 10^{-7} \text{ s} \tag{1.3}$$

Table 1.1 The seven independent SI base units

Quantity	Unit name	Unit symbol
Length	Meter	m
Time	Second	s
Mass	Kilogram	kg
Temperature	Kelvin	K
Electric current	Ampere	A
Amount of substance	Mole	mol
Luminous intensity	Candela	cd

An additional convenient way to deal with very large or very small numbers in physics is to use the prefixes listed in Table 1.2. Each one of these prefixes represents a certain power of 10.

Table 1.2 Prefixes for SI units[a]

Factor	Prefix	Symbol	Factor	Prefix	Symbol
10^{24}	yotta-	Y	10^{-24}	yocto-	y
10^{21}	zeta-	Z	10^{-21}	zepto-	z
10^{18}	exa-	E	10^{-18}	atto-	a
10^{15}	peta-	P	10^{-15}	**femto-**	**f**
10^{12}	**tera-**	**T**	10^{-12}	**pico-**	**p**
10^{9}	**giga-**	**G**	10^{-9}	**nano-**	**n**
10^{6}	**mega-**	**M**	10^{-6}	**micro-**	$\boldsymbol{\mu}$
10^{3}	**kilo-**	**k**	10^{-3}	**milli-**	**m**
10^{2}	hecta-	h	10^{-2}	**centi-**	**c**
10^{1}	deca-	da	10^{-1}	deci-	d

[a] The most commonly used prefixes are shown in bold face type

Accordingly, we can express a particular magnitude of force as:

$$\begin{aligned} 1.23 \times 10^6 \text{ N} &= 1.23 \text{ mega newtons} \\ &= 1.23 \text{ MN} \end{aligned} \tag{1.4}$$

or a particular time interval as:

$$\begin{aligned} 1.23 \times 10^{-9}\ \text{s} &= 1.23\ \text{nano seconds} \\ &= 1.23\ \text{ns} \end{aligned} \tag{1.5}$$

We often need to change units in which a physical quantity is expressed. We do that by using a method called *chain-link conversion*, in which we multiply by a conversion factor that equals unity. For example, because 1 minute and 60 seconds are identical time intervals, then we can write:

$$\boxed{\frac{1\ \text{min}}{60\ \text{s}} = 1 \quad \text{and} \quad \frac{60\ \text{s}}{1\ \text{min}} = 1} \tag{1.6}$$

This does not mean that $\frac{1}{60} = 1$ or $60 = 1$, because the number and its unit must be treated together.

Example 1.1

Convert the following: (a) 1 kilometer per hour to meter per second, (b) 1 mile per hour to meter per second, and (c) 1 mile per hour to kilometer per hour [to a good approximation 1mi = 1.609 km].

Solution: (a) To convert the speed from the kilometers per hour unit to meters per second unit, we write:

$$1\ \text{km/h} = \left(1\ \frac{\cancel{\text{km}}}{\cancel{\text{h}}}\right) \times \left(\frac{10^3\ \text{m}}{1\ \cancel{\text{km}}}\right) \times \left(\frac{1\ \cancel{\text{h}}}{60 \times 60\ \text{s}}\right) = 0.2777...\ \frac{\text{m}}{\text{s}} = 0.278\ \text{m/s}$$

(b) To convert from miles per hour to meters per second, we write:

$$1\ \text{mi/h} = \left(1\ \frac{\cancel{\text{mi}}}{\cancel{\text{h}}}\right) \times \left(\frac{1609\ \text{m}}{1\ \cancel{\text{mi}}}\right) \times \left(\frac{1\ \cancel{\text{h}}}{60 \times 60\ \text{s}}\right) = 0.447\ \frac{\text{m}}{\text{s}} = 0.447\ \text{m/s}$$

(c) To convert from miles per hour to kilometers per hour, we write:

$$1\text{mi/h} = \left(1\ \frac{\cancel{\text{mi}}}{\text{h}}\right) \times \left(\frac{1.609\ \text{km}}{1\ \cancel{\text{mi}}}\right) = 1.609\ \frac{\text{km}}{\text{h}} = 1.609\ \text{km/h}$$

1.2 Standards of Length, Time, and Mass

Definitions of the units of length, time, and mass are under constant review and are changed from time to time. We only present in this section the latest definitions of those quantities.

Length (L)

In 1983, the precision of the meter was redefined as the distance traveled by a light wave in vacuum in a specified time interval. The reason is that the measurement of the speed of light has become extremely precise, so it made sense to adopt the speed of light as a defined quantity and to use it to redefine the meter. In the words of the 17th General Conference on Weights and Measurements:

> One Meter
> One meter is the distance traveled by light in vacuum during the time interval of 1/299 792 458 of a second.

This time interval number was chosen so that the speed of light in vacuum c will be exactly given by:

$$\boxed{c = 299\ 792\ 458\ \text{m/s}} \tag{1.7}$$

For educational purposes we usually consider the value $c = 3 \times 10^8$ m/s.

Table 1.3 lists some approximate interesting lengths.

Table 1.3 Some approximate lengths

Length	Meters
Distance to farthest known galaxy	4×10^{25}
Distance to nearest star	4×10^{16}
Distance from Earth to Sun	1.5×10^{11}
Distance from Earth to Moon	4×10^{8}
Mean radius of Earth	6×10^{6}
Wave length of light	5×10^{-7}
Radius of hydrogen atom	5×10^{-11}
Radius of proton	1×10^{-15}

Time (T)

Recently, the standard of time was redefined to take advantage of the high-precision measurements that could be obtained by using a device known as an *atomic clock*. Cesium is most common element that is typically used in the construction of atomic clocks because it allows us to attain high accuracy.

Since 1967, the International System of Measurements has been basing its unit of time, the **second**, on the properties of the isotope cesium-133 ($^{133}_{55}Cs$). One of the transitions between two energy levels of the ground state of $^{133}_{55}Cs$ has an oscillation frequency of 9 192 631 770 Hz, which is used to define the second in SI units. Using this characteristic frequency, Fig. 1.1 shows the cesium clock at the National Institute of Standards and Technology. The uncertainty is about 5×10^{-16} (as of 2005). Or about 1 part in 2×10^{15}. This means that it would neither gain nor lose a second in 64 million years.

One Second

One second is the time taken for the cesium atom $^{133}_{55}Cs$ to perform 9 192 631 770 oscillations to emit radiation of a specific wavelength

Fig. 1.1 The cesium atomic clock at the National Institute of Standards and Technology (NIST) in Boulder, Colorado (photo with permission)

Table 1.4 lists some approximate interesting time intervals.

Table 1.4 Some approximate time intervals

Time intervals	Seconds
Lifetime of proton (predicted)	1×10^{39}
Age of the universe	5×10^{17}
Age of the Earth	1.3×10^{17}
Period of one year	3.2×10^{7}
Time between human heartbeats	8×10^{-1}
Period of audible sound waves	1×10^{-3}
Period of visible light waves	2×10^{-15}
Time for light to cross a proton	3.3×10^{-24}

Mass (M)

The Standard Kilogram

A cylindrical mass of 3.9 cm in diameter and of 3.9 cm in height and made of an unusually stable platinum-iridium alloy is kept at the International Bureau of Weights and Measures near Paris and assigned in the SI units a mass of 1 kilogram by international agreement, see Fig. 1.2.

One Kilogram
The SI unit of mass, 1 kilogram, is defined as the mass of a platinum-iridium alloy cylinder kept at the International Bureau of Weights and Measures in France.

Fig. 1.2 The standard 1 kilogram of mass is a platinum-iridium cylinder 3.9 cm in height and diameter and kept under a double bell jar at the International Bureau of Weights and Measures in France

Accurate copies of this standard 1 kilogram have been sent to standardizing laboratories in other countries. Table 1.5 lists some approximate mass values of various interesting objects.

A Second Standard Mass

Atomic masses can be compared with each other more precisely than the kilogram. By international agreement, the carbon-12 atom, ${}^{12}_{6}\mathrm{C}$, has been assigned a mass of 12 **atomic mass units** (u), where:

$$\boxed{1\ \mathrm{u} = (1.660\,540\,2 \pm 0.000\,001\,0) \times 10^{-27}\ \mathrm{kg}} \tag{1.8}$$

Experimentally, with reasonable precision, all masses of other atoms can be measured relative to the mass of carbon-12.

Table 1.5 Mass of various objects (approximate values)

Object	Kilogram
Known universe (predicted)	1×10^{53}
Our galaxy the milky way (predicted)	2×10^{41}
Sun	2×10^{30}
Earth	6×10^{24}
Moon	7×10^{22}
Small mountain	1×10^{12}
Elephant	5×10^{3}
Human	7×10^{1}
Mosquito	1×10^{-5}
Bacterium	1×10^{-15}
Uranium atom	4×10^{-25}
Proton	2×10^{-27}
Electron	9×10^{-31}

1.3 Dimensional Analysis

Throughout your experience, you have been exposed to a variety of units of length; the SI meter, kilometer, and millimeter; the English units of inches, feet, yards, and miles, etc. All of these derived units are said to have **dimensions** of length, symbolized by L. Likewise, all time units, such as seconds, minutes, hours, days, years, and centuries are said to have **dimensions** of time, symbolized by T. The kilogram and all other mass units have **dimensions** of mass, symbolized by M. In general, we may take the dimension (length, time, and mass) as the concept of the physical quantity.

From the three fundamental physical quantities of length L, time T, and mass M, we can derive a variety of useful quantities. Derived quantities have different dimensions from the fundamental quantities. For example, the area obtained by multiplying one length by another has the dimension L^2. Volume has the dimension L^3. Mass density is defined as mass per unit volume and has the dimension M/L^3. The SI unit of speed is meters per second (m/s) with the dimension L/T.

The concept of dimensionality is important in understanding physics and in solving physics problems. For example, the addition or subtraction of quantities with different dimensions makes no sense, i.e. 2 kg plus 8 s is meaningless. Actually, physical equations must be dimensionally consistent. For example, the equation giving the position of a freely falling body (see Chap. 3) is giving by:

$$x = v_\circ\, t + \frac{1}{2} g\, t^2 \tag{1.9}$$

where x is the position (length), $v_\circ$ the initial speed (length/time), g is the acceleration due to gravity (length/time^2), and t is time. If we analyze the equation dimensionally, we have:

$$\mathrm{L} = \frac{\mathrm{L}}{\cancel{\mathrm{T}}} \times \cancel{\mathrm{T}} + \frac{\mathrm{L}}{\cancel{\mathrm{T}^2}} \times \cancel{\mathrm{T}^2} = \mathrm{L} + \mathrm{L} \quad \text{(Dimensional analysis)}$$

Note that every term of this equation has the dimension of length L. Also note that numerical factors, such as $\frac{1}{2}$ in Eq. 1.9, are ignored in dimensional analysis because they have no dimension. Dimensional analysis is useful since it can be used to catch careless errors in any physical equation. On the other hand, Eq. 1.9 may be correct with respect to dimensional analysis, but could still be wrong with respect to dimensionless numerical factors.

If we had incorrectly written Eq. 1.9 as follows:

$$x = v_\circ\, t^2 + \frac{1}{2} g\, t \tag{1.10}$$

Then, by analyzing this equation dimensionally, we have:

$$\begin{aligned} \mathrm{L} &= \frac{\mathrm{L}}{\mathrm{T}} \times \mathrm{T}^2 + \frac{\mathrm{L}}{\mathrm{T}^2} \times \mathrm{T} \\ &= \frac{\mathrm{L}}{\cancel{\mathrm{T}}} \times \mathrm{T} \times \cancel{\mathrm{T}} + \frac{\mathrm{L}}{\mathrm{T} \times \cancel{\mathrm{T}}} \times \cancel{\mathrm{T}} \end{aligned} \quad \text{(Dimensional analysis)}$$

and finally we get: $\quad \xcancel{\mathrm{L} = \mathrm{LT} + \frac{\mathrm{L}}{\mathrm{T}}} \quad$ (Dimensional analysis)

Dimensionally, Eq 1.10 is meaningless, and thus cannot be correct.

Example 1.2

Use dimensional analysis to show that the expression $v = v_\circ + at$ is dimensionally correct, where v and $v_\circ$ represent velocities, a is acceleration, and t is a time interval.

Solution: Since L/T is the dimension of v and $v_\circ$, and the dimension of a is L/T^2, then when we analyze the equation $v = v_\circ + at$ dimensionally, we have:

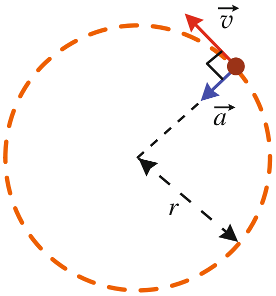
$$\begin{aligned} \frac{\mathrm{L}}{\mathrm{T}} &= \frac{\mathrm{L}}{\mathrm{T}} + \frac{\mathrm{L}}{\mathrm{T}^2} \times \mathrm{T} \\ &= \frac{\mathrm{L}}{\mathrm{T}} + \frac{\mathrm{L}}{\mathrm{T} \times \cancel{\mathrm{T}}} \times \cancel{\mathrm{T}} \end{aligned} \quad \text{(Dimensional analysis)}$$

and finally we get: $$\frac{\mathrm{L}}{\mathrm{T}} = \frac{\mathrm{L}}{\mathrm{T}} + \frac{\mathrm{L}}{\mathrm{T}} \qquad \text{(Dimensional analysis)}$$

Thus, the expression $v = v_\circ + at$ is dimensionally correct.

Example 1.3

A particle moves with a constant speed v in a circular orbit of radius r, see the figure below. Given that the magnitude of the acceleration a is proportional to some power of r, say r^m, and some power of v, say v^n, then determine the powers of r and v.



Solution: Assume that the variables of the problem can be expressed mathematically by the following relation:

$$a = k\, r^m v^n,$$

where k is a dimensionless proportionality constant. With the known dimensions of r, v, and a we analyze the dimensions of the above relation as follows:

$$\frac{\mathrm{L}}{\mathrm{T}^2} = \mathrm{L}^m \times \left(\frac{\mathrm{L}}{\mathrm{T}}\right)^n = \frac{\mathrm{L}^{m+n}}{\mathrm{T}^n} \qquad \text{(Dimensional analysis)}$$

This dimensional equation would be balanced, i.e. the dimensions of the right hand side equal the dimensions of the left hand side only when the following two conditions are satisfied:

$$m + n = 1,$$

and $$n = 2.$$

Thus: $$m = -1.$$

Therefore, we can rewrite the acceleration as follows:

$$a = k\, r^{-1} v^2 = k\, \frac{v^2}{r}$$

When we later introduce uniform circular motion in Chap. 4, we shall see that $k = 1$ if SI units are used. However, if for example we choose a to be in m/s^2 and v to be in km/h, then k would not be equal to one.

1.4 Exercises

Section 1.1 The International System of Units

(1) Use the prefixes introduced in Table 1.2 to express the following: (a) 10^3 lambs, (b) 10^6 bytes, (c) 10^9 cars, (d) 10^{12} stars, (e) 10^{-1} Kelvin, (f) 10^{-2} meter, (g) 10^{-3} ampere, (h) 10^{-6} newton, (i) 10^{-9} kilogram, (j) 10^{-15} second.

Section 1.2 Standards of Length, Time, and Mass

Length

(2) The original definition of the meter was based on distance from the North pole to the Earth's equator (measures along the surface) and was taken to be 10^7 m. (a) What is the circumference of the Earth in meters? (b) What is the radius of the Earth in meters, (c) Give your answer to part (a) and part (b) in miles. (d) What is the circumference of the Earth in meters assuming it to be a sphere of radius 6.4×10^6 m? Compare your answer to part (a)

(3) The time of flight of a laser pulse sent from the Earth to the Moon was measured in order to calculate the Earth-Moon distance, and it was found to be 3.8×10^5 km. (a) Express this distance in miles, meters, centimeters, and millimeters.

(4) A unit of area, often used in measuring land areas, is the *hectare*, defined as 10^4 m^2. An open-pit coal mine excavates 75 hectares of land, down to a depth of 26 m, each year. What volume of Earth, in cubic kilometers, is removed during this time?

(5) The units used by astronomers are appropriate for the quantities they usually measure. As an example, for planetary distances they use the astronomical

unit (AU), which is equal to the mean Earth-Sun distance (1.5×10^{11} m). For stellar distances they use the light-year (1 ly $= 9.461 \times 10^{12}$ km), which is the distance that light travels in 1 yr (1 yr $= 365.25$ days $= 3.156 \times 10^{7}$ s) with a speed of 299 792 458 m/s. They use also the parsec (pc), which is equal to 3.26 light-years. Intergalactic distances might be described with a more appropriate unit called the megaparsec. Convert the following to meters and express each with an appropriate metric prefix: (a) The astronomical unit, (b) The light-year, (c) The parsec, and (d) The megaparsecs.

(6) When you observe a total solar eclipse, your view of the Sun is obstructed by the Moon. Assume the distance from you to the Sun (d_s) is about 400 times the distance from you to the Moon (d_m). (a) Find the ratio of the Sun's radius to the Moon's radius. (b) What is the ratio of their volumes? (c) Hold up a small coin so that it would just eclipse the full Moon, and measure the distance between the coin and your eye. From this experimental result and the given distance between the Moon and the Earth (3.8×10^{5} km), estimate the diameter of the Moon.

(7) Assume a spherical atom with a spherical nucleus where the ratio of the radii is about 10^5. The Earth's radius is 6.4×10^{6} m. Suppose the ratio of the radius of the Moon's orbit to the Earth's radius (3.8×10^{5} km) were also 10^5. (a) How far would the Moon be from the Earth's surface? (b) How does this distance compare with the actual Earth-Moon distance given in exercise 6?

Time

(8) Using the day as a unit, express the following: (a) The predicted life time of proton, (b) The age of universe, (c) The age of the Earth, (d) The age of a 50-year-old tree.

(9) Compare the duration of the following: (a) A microyear and a 1-minute TV commercial, and (b) A microcentury and a 60-min TV program.

(10) Convert the following approximate *maximum* speeds from km/h to mi/h: (a) snail (5×10^{-2} km/h), (b) spider (2 km/h), (c) human (37 km/h),(d) car (220 km/h), and (e) airplane (1,000 km/h).

(11) A 12-hour-dial clock happens to gain 0.5 min each day. After setting the clock to the correct time at 12:00 noon, how many days must one wait until it again indicates the correct time?

(12) Is a cesium clock sufficiently precise to determine your age (assuming it is exactly 19 years, not a leap year) within 10^{-6} s? How about within 10^{-3} s?

(13) The slowing of the Earth's rotation is measured by observing the occurrences of solar eclipses during a specific period. Assume that the length of a day is increasing uniformly by 0.001 s per century. (a) Over a span of 10 centuries, compare the length of the last and first days, and find the average difference. (b) Find the cumulative difference on the measure of a day over this period.

Mass

(14) A person on a diet loses 2 kg per week. Find the average rate of mass loss in milligrams every: day, hour, minute, and second.

(15) Density is defined as mass per unit volume. If a crude estimation of the average density of the Earth was 5.5×10^3 kg/m^3 and the Earth is considered to be a sphere of radius 6.37×10^6 m, then calculate the mass of the Earth.

(16) A carbon-12 atom ($^{12}_{6}$C) is found to have a mass of $1.992\,64 \times 10^{-26}$ kg. How many atoms of $^{12}_{6}$C are there in: (a) 1 kg? (b) 12 kg? (This number is Avogadro's number in the SI units.)

(17) A water molecule (H_2O) contains two atoms of hydrogen ($^{1}_{1}$H), each of which has a mass of 1 u, and one atom of oxygen ($^{16}_{8}$O), that has a mass 16 u, approximately. (a) What is the mass of one molecule of water in units of kilograms? (b) Find how many molecules of water are there in the world's oceans, which have an estimated mass of 1.5×10^{21} kg?

(18) Density is defined as mass per unit volume. The density of iron is 7.87 kg/m^3, and the mass of an iron atom is 9.27×10^{-26} kg. If atoms are cubical and tightly packed, (a) What is the volume of an iron atom, and (b) What is the distance between the centers of two adjacent atoms.

Section 1.3 Dimensional Analysis

(19) A simple pendulum has periodic time T given by the relation:

$$T = 2\pi\sqrt{L/g}$$

where L is the length of the pendulum and g is the acceleration due to gravity in units of length divided by the square of time. Show that this equation is dimensionally correct.

(20) Suppose the displacement s of an object moving in a straight line under uniform acceleration a is giving as a function of time by the relation $s = ka^m t^n$, where k is a dimensionless constant. Use dimensional analysis to find the values of the powers m and n.

(21) Using dimensional analysis, determine if the following equations are dimensionally correct or incorrect: (a) $v^2 = v_\circ^2 + 2a\,s$, (b) $s = s_\circ + v_\circ\, t + \frac{1}{2} a\, t^2$, (c) $s = s_\circ \cos kt$, where k is a constant that has the dimension of the inverse of time.

(22) Newton's second law states that the acceleration of an object is directly proportional to the force applied and inversely proportional to the mass of the object. Find the dimensions of force and show that it has units of kg·m/s^2 in terms of SI units.

(23) Newton's law of universal gravitation is given by $F = Gm_1 m_2/r^2$, where F is the force of attraction of one mass, m_1, upon another mass, m_2, at a distance r. Find the SI units of the constant G.

Vectors

2

When a particle moves in a straight line, we can take its motion to be positive in one specific direction and negative in the other. However, when this particle moves in three dimensions, plus or minus signs are no longer enough to specify the direction of motion. Instead, we must use a vector.

2.1 Vectors and Scalars

A **vector** has **magnitude** and **direction**, examples being displacement (change of position), velocity, acceleration, etc. Actually, not all physical quantities involve direction, examples being temperature, mass, pressure, time, etc. These physical quantities are not vectors because they do not point in any direction, and we call them **scalars**.

A vector, such as a **displacement vector**, can be represented graphically by an arrow denoting the magnitude and direction of the vector. All arrows of the same direction and magnitude denote the same vector, as in Fig. 2.1a for the case of a displacement vector.

The displacement vector in Fig. 2.1b tells us nothing about the actual path taken from point A to B. Thus, displacement vectors represent only the overall effect of the motion, not the motion itself.

Another way to specify a vector is to determine its magnitude and the angle it makes with a reference direction, as in Example 2.1.

H. A. Radi and J. O. Rasmussen, *Principles of Physics*,
Undergraduate Lecture Notes in Physics, DOI: 10.1007/978-3-642-23026-4_2,

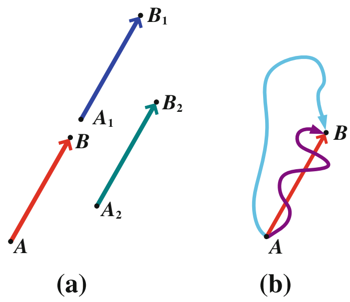


Fig. 2.1 (a) Three vectors of the same direction and magnitude represent the same displacement. (b) All three paths connecting the two points A and B correspond to the same displacement vector

Example 2.1

A person walks 3 km due east and then 2 km due north. What is his displacement vector?

Solution: We first make an overhead view of the person's movement as shown in Fig. 2.2. The magnitude of the displacement d is given by the Pythagorean theorem as follows:

$$d = \sqrt{(3\ \text{km})^2 + (2\ \text{km})^2} = 3.61\ \text{km}$$

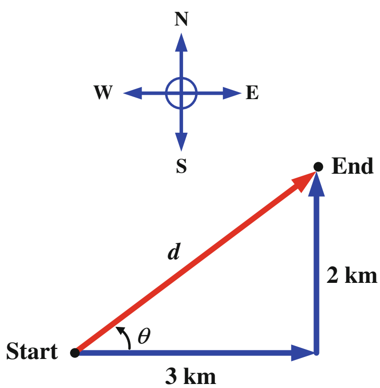
The angle that this displacement vector makes relative to east is given by:

$$\tan\,\theta = \frac{2\ \text{km}}{3\ \text{km}} = 0.666...$$

Then: $\theta = \tan^{-1}(0.666...) = 33.69°$

Thus, the person's displacement vector is 56.31° east of north.

Fig. 2.2



2.2 Properties of Vectors

In text books, it is common to use boldface symbols to identify vectors, such as $\boldsymbol{A}$, $\boldsymbol{B}$, etc., but in handwriting it is usual to place an arrow over the symbol, such as, $\vec{A}, \vec{B}$, etc. Throughout this text we shall use the handwriting style only and use the italic symbols A, B, etc. to indicate the magnitude of vectors.

Equality of Vectors

The two vectors $\vec{A}$ and $\vec{B}$ are said to be equal if they have the same magnitude, i.e. $A = B$, and point in the same direction; see for example the three equal vectors AB, A_1B_1, and A_2B_2 in Fig. 2.1a.

Addition of Vectors

Of course, all vectors involved in any addition process must have the same units. The rules for vector sums can be illustrated by using a graphical method. To add vector $\vec{B}$ to vector $\vec{A}$, we first draw vector $\vec{A}$ on graph paper with its magnitude represented by a convenient scale, and then draw vector $\vec{B}$ to the same scale with its tail coinciding with the arrow head of $\vec{A}$, see Fig. 2.3a. This is known as the triangle method of addition. Thus, the resultant vector $\vec{R}$ is the red vector drawn from the tail of $\vec{A}$ to the head of $\vec{B}$ and is shown in the vector addition equation:

$$\boxed{\vec{R} = \vec{A} + \vec{B}}\,, \tag{2.1}$$

which says that the vector $\vec{R}$ is the vector sum of vectors $\vec{A}$ and $\vec{B}$. The symbol + in Eq. 2.1 and the words "sum" and "add" have different meanings for vectors than they do in elementary algebra of scalar numbers.

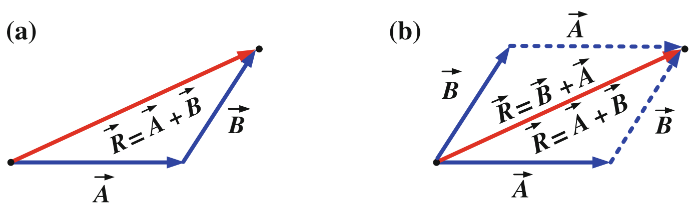


Fig. 2.3 (a) In the triangle method of addition, the resultant vector $\vec{R}$ is the red vector that runs from the tail of $\vec{A}$ to the head of $\vec{B}$. (b) In the parallelogram method of addition, the resultant vector $\vec{R}$ is the red diagonal vector that starts from the tails of both $\vec{A}$ and $\vec{B}$. This method shows that $\vec{A} + \vec{B} = \vec{B} + \vec{A}$

An alternative graphical method for adding two vectors is the parallelogram rule of addition. In this method, we superpose the tails of the two vectors $\vec{A}$ and $\vec{B}$; then the resultant $\vec{R}$ will be the diagonal of the parallelogram that starts from the tail of both $\vec{A}$ and $\vec{B}$ (which form the sides of that parallelogram), as shown in Fig. 2.3b.

Vector addition has two important properties. First, the order of addition does not matter, and this is known as the commutative law of addition, i.e.

$$\boxed{\vec{A}+\vec{B}=\vec{B}+\vec{A}} \qquad \text{(Commutative law)} \tag{2.2}$$

Second, if there are more than two vectors, their sum is independent of the way in which the individual vectors are grouped together. This is known as the associative law of addition, i.e.

$$\boxed{\vec{A}+(\vec{B}+\vec{C})=(\vec{A}+\vec{B})+\vec{C}} \qquad \text{(Associative law)} \tag{2.3}$$

The Negative of a Vector

The negative of a vector $\vec{B}$ is a vector with the same magnitude which points in the opposite direction, namely $-\vec{B}$, see Fig. 2.4a. Therefore, when we add a vector and its negative we will get zero, i.e.

$$\boxed{\vec{B}+(-\vec{B})=0} \tag{2.4}$$

Adding $-\vec{B}$ to $\vec{A}$ has the same effect of subtracting $\vec{B}$ from $\vec{A}$, see Fig. 2.4b, i.e.

$$\boxed{\begin{aligned}\vec{S}&=\vec{A}+(-\vec{B})\\&=\vec{A}-\vec{B}\end{aligned}} \tag{2.5}$$

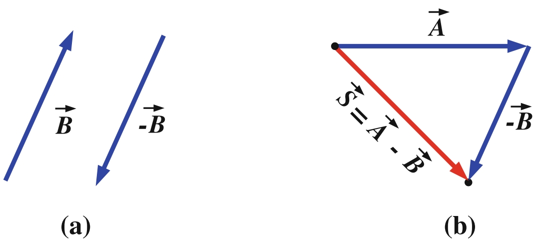


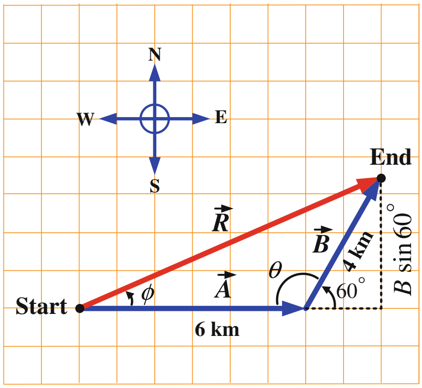
Fig. 2.4 (a) This part of the figure shows vector $\vec{B}$ and its corresponding negative vector $-\vec{B}$, both of which have the same magnitude but are opposite in direction. (b) To subtract vector $\vec{B}$ from vector $\vec{A}$, we add the vector $-\vec{B}$ to vector $\vec{A}$ to get $\vec{S}=\vec{A}-\vec{B}$

Example 2.2

A car travels 6 km due east and then 4 km in a direction 60° north of east. Find the magnitude and direction of the car's displacement vector by using: (a) the graphical method, and (b) the analytical method.

Solution: (a) Let $\vec{A}$ be a vector directed due east with magnitude $A = 6$ km and $\vec{B}$ be a vector directed 60° north of east with magnitude $B = 4$ km. Using graph paper with a reasonable scale and a protractor, we draw the two vectors $\vec{A}$ and $\vec{B}$; then we measure the length of the resultant vector $\vec{R}$. The measurements shown in Fig. 2.5 indicates that $R = 8.7$ km. Also, the angle ϕ that the resultant vector $\vec{R}$ makes with respect to the east direction can be measured and will give $\phi = 23.4°$.

Fig. 2.5



(b) The analytical solution for the magnitude of $\vec{R}$ can be obtained from geometry by using the law of cosines $R = \sqrt{A^2 + B^2 - 2AB\cos\theta}$ as applied to an obtuse triangle with angle $\theta = 180° - 60° = 120°$, see exercise (10b). Thus:

$$\begin{aligned} R &= \sqrt{A^2 + B^2 - 2\,AB\cos\theta} \\ &= \sqrt{(6\text{ km})^2 + (4\text{ km})^2 - 2(6\text{ km})(4\text{ km})\cos 120°} \\ &= \sqrt{(36 + 16 + 24)\ (\text{km})^2} = 8.72\text{ km} \end{aligned}$$

The angle that this displacement vector $\vec{R}$ makes relative to the east direction, see Fig. 2.5, is given by:

$$\sin\phi = \frac{B\sin 60°}{R} = \frac{4\,\text{km}\sin 60°}{8.72\,\text{km}} = 0.397$$

Then: $\phi = \sin^{-1}(0.397) = 23.41°$.

2.3 Vector Components and Unit Vectors

Vector Components

Adding vectors graphically is not recommended in situations where high precision is needed or in three-dimensional problems. A better way is to make use of the projections of a vector along the axes of a rectangular coordinate system.

Consider a vector $\vec{A}$ lying in the xy-plane and making an angle θ with the positive x-axis, see Fig. 2.6. This vector $\vec{A}$ can be expressed as the sum of two vectors $\vec{A}_x$ and $\vec{A}_y$ called the rectangular vector components of $\vec{A}$ along the x-axis and y-axis, respectively. Thus:

$$\boxed{\vec{A} = \vec{A}_x + \vec{A}_y} \tag{2.6}$$

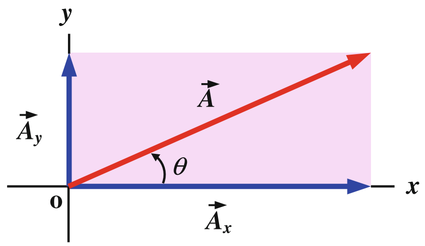


Fig. 2.6 A vector $\vec{A}$ in the xy-plane can be presented by its rectangular vector components $\vec{A}_x$ and $\vec{A}_y$, where $\vec{A} = \vec{A}_x + \vec{A}_y$

From the definitions of sine and cosine, the rectangular components of $\vec{A}$, namely A_x and A_y, will be given by:

$$\boxed{A_x = A\cos\theta} \quad \text{and} \quad \boxed{A_y = A\sin\theta}\,, \tag{2.7}$$

where the sign of the components A_x and A_y depends on the angle θ.

The magnitudes A_x and A_y form two sides of a right triangle that has a hypotenuse of magnitude A. Thus, from A_x and A_y we get:

$$\boxed{A = \sqrt{A_x^2 + A_y^2}} \quad \text{and} \quad \boxed{\theta = \tan^{-1}\left(\frac{A_y}{A_x}\right)} \tag{2.8}$$

The *inverse tan* obtained from your calculator is from $-90° < \theta < 90°$. This may lead to incorrect answer when $90° < \theta \le 360°$. A method used to achieve the correct answer is to calculate the angle ϕ such as:

$$\phi = \tan^{-1}\left(|A_y|/|A_x|\right) \tag{2.9}$$

Then, depending on the signs of A_x and A_y, we identify the quadrant where the vector $\vec{A}$ lies, as shown in Fig. 2.7.

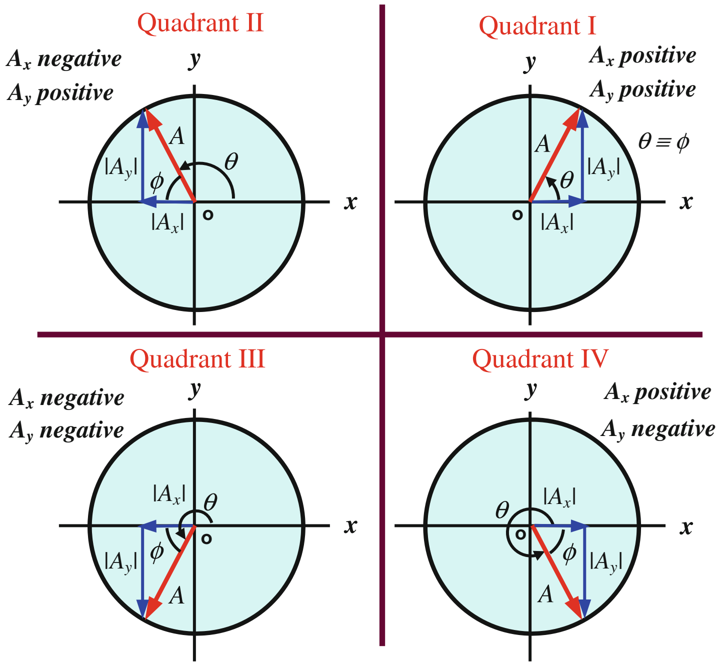


Fig. 2.7 The signs of A_x and A_y depend on the quadrant where the vector $\vec{A}$ is located

Once we determine the quadrant, we calculate θ using Table 2.1.

Table 2.1 Calculating θ from ϕ according to the signs of A_x and A_y

Sign of A_x	Sign of A_y	Quadrant	Angle θ
$+$	$+$	I	$\theta = \phi$
$-$	$+$	II	$\theta = 180^\circ - \phi$
$-$	$-$	III	$\theta = 180^\circ + \phi$
$+$	$-$	IV	$\theta = 360^\circ - \phi$

Unit Vectors

A unit vector is a dimensionless vector that has a magnitude of exactly one and points in a particular direction, and has no other physical significance. The unit vectors in the positive direction of the x, y, and z axes of a **right-handed coordinate system** are often labeled $\vec{\mathrm{i}}$, $\vec{\mathrm{j}}$, and $\vec{\mathrm{k}}$, respectively; see Fig. 2.8. The magnitude of each unit vector equals unity; that is:

$$|\vec{\mathrm{i}}| = |\vec{\mathrm{j}}| = |\vec{\mathrm{k}}| = 1 \tag{2.10}$$

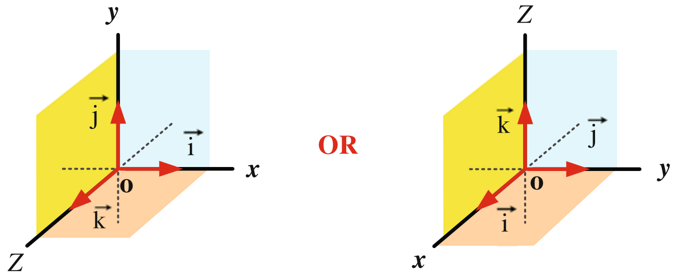
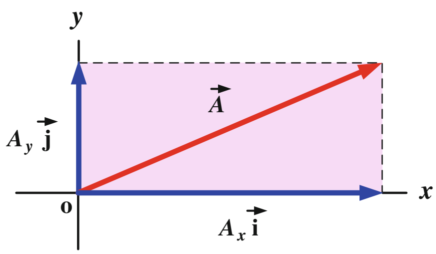


Fig. 2.8 Unit vectors $\vec{\mathrm{i}}$, $\vec{\mathrm{j}}$, and $\vec{\mathrm{k}}$ define the direction of the commonly-used right-handed coordinate system

Consider a vector $\vec{A}$ lying in the xy-plane as shown in Fig. 2.9. The product of the component A_x and the unit vector $\vec{\mathrm{i}}$ is the vector $\vec{A}_x = A_x\,\vec{\mathrm{i}}$, which is parallel to the x-axis and has a magnitude A_x. Similarly, $\vec{A}_y = A_y\,\vec{\mathrm{j}}$ is a vector parallel to the y-axis and has a magnitude A_y. Thus, in terms of unit vectors we write $\vec{A}$ as follows:

$$\vec{A} = A_x\,\vec{\mathrm{i}} + A_y\,\vec{\mathrm{j}} \tag{2.11}$$

Fig. 2.9 A vector $\vec{A}$ in the xy-plane can be represented by its rectangular components A_x and A_y and the unit vectors $\vec{\mathrm{i}}$ and $\vec{\mathrm{j}}$, and can be written as $\vec{A} = A_x\,\vec{\mathrm{i}} + A_y\,\vec{\mathrm{j}}$



This method can be generalized to three-dimensional vectors as:

$$\vec{A} = A_x\,\vec{\mathrm{i}} + A_y\,\vec{\mathrm{j}} + A_z\,\vec{\mathrm{k}} \tag{2.12}$$

We can define a unit vector $\vec{n}$ along any vector, say, $\vec{A}$, as follows:

$$\vec{n} = \frac{\vec{A}}{A}. \tag{2.13}$$

Adding Vectors by Components

Suppose we wish to add the two vectors $\vec{A} = A_x \vec{i} + A_y \vec{j}$ and $\vec{B} = B_x \vec{i} + B_y \vec{j}$ using the components method, such as:

$$\begin{aligned}\vec{R} &= \vec{A} + \vec{B} \\ &= (A_x \vec{i} + A_y \vec{j}) + (B_x \vec{i} + B_y \vec{j}) \\ &= (A_x + B_x)\vec{i} + (A_y + B_y)\vec{j}\end{aligned} \tag{2.14}$$

If the vector sum $\vec{R}$ is denoted by $\vec{R} = R_x \vec{i} + R_y \vec{j}$, then the components of the resultant vector will be given by:

$$\begin{aligned}R_x &= A_x + B_x \\ R_y &= A_y + B_y\end{aligned} \tag{2.15}$$

The magnitude of $\vec{R}$ can then be obtained from its components or the components of $\vec{A}$ and $\vec{B}$ using the following relationships:

$$\boxed{R = \sqrt{R_x^2 + R_y^2} = \sqrt{(A_x + B_x)^2 + (A_y + B_y)^2}} \tag{2.16}$$

and the angle that $\vec{R}$ makes with the x-axis can also be obtained by using the following relationships:

$$\boxed{\theta = \tan^{-1}\left(\frac{R_y}{R_x}\right) = \tan^{-1}\left(\frac{A_y + B_y}{A_x + B_x}\right)} \tag{2.17}$$

The components method can be verified using the geometrical method, as shown in Fig. 2.10.

If $\vec{A} = A_x \vec{i} + A_y \vec{j} + A_z \vec{k}$ and $\vec{B} = B_x \vec{i} + B_y \vec{j} + B_z \vec{k}$, then we can generalize the previous case to three dimensions as follows:

$$\boxed{\begin{aligned}\vec{R} &= \vec{A} + \vec{B} = (A_x + B_x)\vec{i} + (A_y + B_y)\vec{j} + (A_z + B_z)\vec{k} \\ &= R_x \vec{i} + R_y \vec{j} + R_z \vec{k}\end{aligned}} \tag{2.18}$$

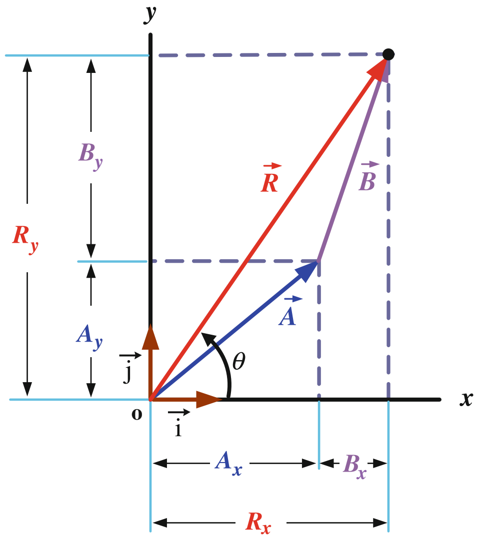


Fig. 2.10 Geometric representation of the sum of the two vectors $\vec{A}$ and $\vec{B}$, showing the relationship between the components of the resultant $\vec{R}$ and the components of $\vec{A}$ and $\vec{B}$

Example 2.3

Find the sum of the following two vectors:

$$\vec{A} = 8\,\vec{i} + 3\,\vec{j}$$

$$\vec{B} = -5\,\vec{i} - 7\,\vec{j}$$

For convenience, the units of the two vectors have been omitted, but for instance, you may take them to be kilometers.

Solution: The two vectors lie in the *xy*-plane, since there is no component in the *z*-axis. By comparing the two expressions of $\vec{A}$ and $\vec{B}$ with the general relations $\vec{A} = A_x\,\vec{i} + A_y\,\vec{j}$ and $\vec{B} = B_x\,\vec{i} + B_y\,\vec{j}$ we see that, $A_x = 8$, $A_y = 3$, $B_x = -5$, and $B_y = -7$. Therefore, the resultant vector R is obtained by using Eq. 2.14 as follows:

$$\begin{aligned}\vec{R} &= \vec{A} + \vec{B} = (A_x + B_x)\,\vec{i} + (A_y + B_y)\,\vec{j}\\ &= (8-5)\,\vec{i} + (3-7)\,\vec{j} = 3\,\vec{i} - 4\,\vec{j}\end{aligned}$$

That is: $R_x = 3$ and $R_y = -4$. The magnitude of $\vec{R}$ is given according to Eq. 2.16 as:

$$R = \sqrt{R_x^2 + R_y^2} = \sqrt{(3)^2 + (-4)^2} = \sqrt{9+16} = \sqrt{25} = 5$$

while the value of the angle θ that $\vec{R}$ makes with the positive x-axis is given according to Eq. 2.17 as:

$$\theta = \tan^{-1}\left(\frac{R_y}{R_x}\right) = \tan^{-1}\left(\frac{-4}{3}\right) = 360^\circ - \tan^{-1}\left(\frac{4}{3}\right) = 360^\circ - 53^\circ = 307^\circ$$

where we used Table 2.1 to calculate θ in case of negative R_y (Q IV).

2.4 Multiplying Vectors

Multiplying a Vector by a Scalar

If we multiply vector $\vec{A}$ by a scalar a we get a new vector $\vec{B}$, i.e.

$$\boxed{\vec{B} = a\vec{A}} \tag{2.19}$$

The vector $\vec{B}$ has the same direction as $\vec{A}$ if a is positive but has the opposite direction if a is negative. The magnitude of $\vec{B}$ is the product of the magnitude of $\vec{A}$ and the absolute value of a.

The Scalar Product (or the Dot Product)

The **scalar product** of the two vectors $\vec{A}$ and $\vec{B}$ is denoted by $\vec{A} \bullet \vec{B}$ and is defined as:

$$\boxed{\vec{A} \bullet \vec{B} = AB\cos\theta} \tag{2.20}$$

where A and B are the magnitudes of the two vectors $\vec{A}$ and $\vec{B}$, and θ is the angle between them, see Fig. 2.11. The two angles θ and $360^\circ - \theta$ could be used, since their cosines are the same. As we see from Eq. 2.20, the result of $\vec{A} \bullet \vec{B}$ is a scalar quantity, and is known as the **dot product** from its notation. Also, we get:

$$\boxed{\vec{A} \bullet \vec{B} = AB\cos\theta = \begin{cases} +AB & \text{if } \theta = 0^\circ \\ 0 & \text{if } \theta = 90^\circ \\ -AB & \text{if } \theta = 180^\circ \end{cases}} \tag{2.21}$$

According to Fig. 2.11, the dot product can be regarded as the product of the magnitude of one of the vectors with the scalar component of the second along the direction of the first. That is:

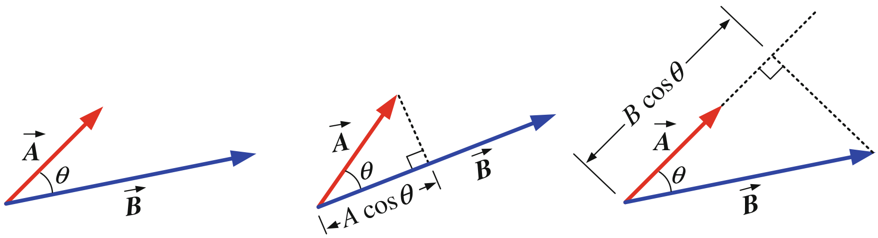


Fig. 2.11 The *left* part shows two vectors $\vec{A}$ and $\vec{B}$, with an angle θ between them. The middle and the *right* parts show the component of each vector along the other

$$\vec{A} \bullet \vec{B} = (A\cos\theta)B = A(B\cos\theta) \tag{2.22}$$

This indicates that scalar products obey the commutative and associative laws, so that:

$$\vec{A} \bullet \vec{B} = \vec{B} \bullet \vec{A} \quad \text{(Commutative law)} \tag{2.23}$$

$$\vec{A} \bullet (\vec{B} + \vec{C}) = \vec{A} \bullet \vec{B} + \vec{A} \bullet \vec{C} \quad \text{(Associative law)} \tag{2.24}$$

By applying the definition of dot product to the unit vectors $\vec{\mathrm{i}}$, $\vec{\mathrm{j}}$, and $\vec{\mathrm{k}}$, we get the following:

$$\begin{aligned} \vec{\mathrm{i}} \bullet \vec{\mathrm{i}} &= \vec{\mathrm{j}} \bullet \vec{\mathrm{j}} = \vec{\mathrm{k}} \bullet \vec{\mathrm{k}} = 1 \\ \vec{\mathrm{i}} \bullet \vec{\mathrm{j}} &= \vec{\mathrm{i}} \bullet \vec{\mathrm{k}} = \vec{\mathrm{j}} \bullet \vec{\mathrm{k}} = 0, \end{aligned} \tag{2.25}$$

where the angle between any two identical unit vectors is 0° and the angle between any two different unit vectors is 90°.

When two vectors are written in terms of the unit vectors $\vec{\mathrm{i}}$, $\vec{\mathrm{j}}$, and $\vec{\mathrm{k}}$, then to get their dot product, each component of the first vector is to be dotted into each component of the second vector. After that, we use Eq. 2.25 to get the following:

$$\begin{aligned} \vec{A} \bullet \vec{B} &= (A_x\,\vec{\mathrm{i}} + A_y\,\vec{\mathrm{j}} + A_z\,\vec{\mathrm{k}}) \bullet (B_x\,\vec{\mathrm{i}} + B_y\,\vec{\mathrm{j}} + B_z\,\vec{\mathrm{k}}) \\ &= A_xB_x + A_yB_y + A_zB_z \end{aligned} \tag{2.26}$$

Thus, from Eqs. 2.20 and 2.26, we can generally write the dot product as follows:

$$\vec{A} \bullet \vec{B} = A\ B\ \cos\ \theta = A_xB_x + A_yB_y + A_zB_z \tag{2.27}$$

Example 2.4

Find the angle between the vector $\vec{A} = 8\,\vec{\text{i}} + 3\,\vec{\text{j}}$ and the vector $B = -5\,\vec{\text{i}} - 7\,\vec{\text{j}}$.

Solution: Since $A = \sqrt{A_x^2 + A_y^2}$ and $B = \sqrt{B_x^2 + B_y^2}$, then using the dot product given by Eq. 2.20 we get:

$$\begin{aligned}\vec{A} \bullet \vec{B} &= AB\cos\theta = \sqrt{8^2 + 3^2} \times \sqrt{(-5)^2 + (-7)^2}\cos\theta \\ &= 8.544 \times 8.60\cos\theta \\ &= 73.5\cos\theta\end{aligned}$$

Keeping in mind that there is no component for $\vec{A}$ and $\vec{B}$ along the z-axis, we can find the dot product from Eq. 2.26 as follows:

$$\begin{aligned}\vec{A} \bullet \vec{B} &= A_xB_x + A_yB_y + A_zB_z \\ &= 8 \times (-5) + 3 \times (-7) + 0 \times 0 \\ &= -61\end{aligned}$$

Equating the results of the last two steps to each other, we find:

$$73.5\cos\theta = -61$$

Thus:

$$\theta = \cos^{-1}\left(\frac{-61}{73.5}\right) = 146.1°.$$

The Vector Product (or the Cross Product)

The **vector product** of the two vectors $\vec{A}$ and $\vec{B}$ is denoted by $\vec{A} \times \vec{B}$ and defined as a third vector $\vec{C}$ whose magnitude is:

$$\boxed{C = AB\sin\theta}\,, \tag{2.28}$$

where θ is the smaller angle between $\vec{A}$ and $\vec{B}$ (hence, $0 \leq \sin\theta \leq 1$). The direction of $\vec{C}$ is perpendicular to the plane that contains both $\vec{A}$ and $\vec{B}$, and can be determined by using the **right-hand rule**, see Fig. 2.12. To apply this rule, we allow the tail of $\vec{A}$ to coincide with the tail of $\vec{B}$, then the four fingers of the right hand are pointed along $\vec{A}$ and then "wrapped" into $\vec{B}$ through the angle θ. The direction of the erect right thumb is the direction of $\vec{C}$, i.e. the direction of $\vec{A} \times \vec{B}$. Also, the direction of $\vec{C}$ is determined by the direction of advance of a right-handed screw as shown in Fig. 2.12.

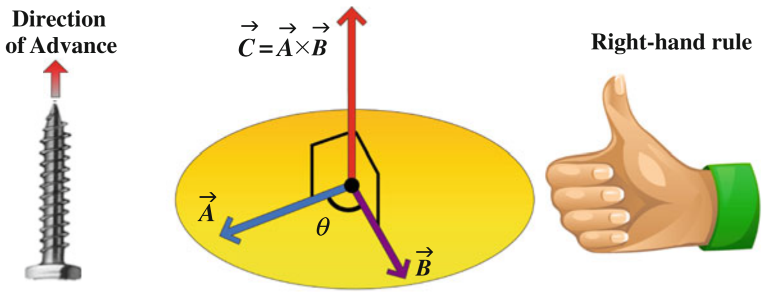


Fig. 2.12 The vector product $\vec{A} \times \vec{B}$ is a third vector $\vec{C}$ that has a magnitude of $AB \sin\theta$ and a direction perpendicular to the plane containing the vectors $\vec{A}$ and $\vec{B}$. Its sense is determined by the right-hand rule or the direction of advance of a right-handed screw

The vector product definition leads to the following properties:

1. The order of vector product multiplication is important; that is:

$$\vec{A} \times \vec{B} = -(\vec{B} \times \vec{A}) \tag{2.29}$$

which is unlike the scalar product and can be easily verified with the right-hand rule.

2. If $\vec{A}$ is parallel to $\vec{B}$ (that is, $\theta = 0°$) or $\vec{A}$ is antiparallel to $\vec{B}$ (that is, $\theta = 180°$), then:

$$\vec{A} \times \vec{B} = 0 \quad (\text{if } \vec{A} \text{ is parallel or antiparallel to } \vec{B}) \tag{2.30}$$

3. If $\vec{A}$ is perpendicular to $\vec{B}$, then:

$$|\vec{A} \times \vec{B}| = AB \quad (\text{if } \vec{A} \perp \vec{B}) \tag{2.31}$$

4. The vector product obeys the distributive law, that is:

$$\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C} \quad (\text{Distributive law}) \tag{2.32}$$

5. The derivative of $\vec{A} \times \vec{B}$ with respect to any variable such as t is:

$$\frac{d}{dt}(\vec{A} \times \vec{B}) = \vec{A} \times \frac{d\vec{B}}{dt} + \frac{d\vec{A}}{dt} \times \vec{B} \tag{2.33}$$

6. From the definition of the vector product and the unit vectors $\vec{i}$, $\vec{j}$, and $\vec{k}$, we get the following relationships:

$$\vec{i} \times \vec{i} = \vec{j} \times \vec{j} = \vec{k} \times \vec{k} = 0 \tag{2.34}$$

$$\vec{i} \times \vec{j} = \vec{k}, \quad \vec{j} \times \vec{k} = \vec{i}, \quad \vec{k} \times \vec{i} = \vec{j} \tag{2.35}$$

The last relations can be obtained by setting the unit vectors $\vec{i}$, $\vec{j}$, and $\vec{k}$ on a circle, see Fig. 2.13, and rotating in a clockwise direction to find the cross product of one unit vector with another. Rotating in a counterclockwise direction will involve a negative sign of the cross product of one unit vector with another, that is:

$$\vec{i} \times \vec{k} = -\vec{j}, \quad \vec{k} \times \vec{j} = -\vec{i}, \quad \vec{j} \times \vec{i} = -\vec{k} \tag{2.36}$$

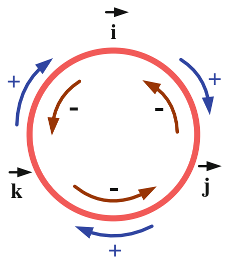


Fig. 2.13 The clockwise and counterclockwise cyclic order for finding the cross product of the unit vectors $\vec{i}$, $\vec{j}$, and $\vec{k}$

7. When two vectors $\vec{A}$ and $\vec{B}$ are written in terms of the unit vectors $\vec{i}$, $\vec{j}$, and $\vec{k}$, then the cross product will give the result:

$$\begin{aligned}\vec{A} \times \vec{B} &= (A_x \vec{i} + A_y \vec{j} + A_z \vec{k}) \times (B_x \vec{i} + B_y \vec{j} + B_z \vec{k}) \\ &= (A_y B_z - A_z B_y)\,\vec{i} + (A_z B_x - A_x B_z)\,\vec{j} \\ &\quad + (A_x B_y - A_y B_x)\,\vec{k}\end{aligned} \tag{2.37}$$

This result can be expressed in determinant form as follows:

$$\vec{A} \times \vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \vec{i} \begin{vmatrix} A_y & A_z \\ B_y & B_z \end{vmatrix} - \vec{j} \begin{vmatrix} A_x & A_z \\ B_x & B_z \end{vmatrix} + \vec{k} \begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix} \tag{2.38}$$

Example 2.5

(a) Find the cross product of the two vectors $\vec{A} = 8\,\vec{i} + 3\,\vec{j}$ and $\vec{B} = -5\,\vec{i} - 7\,\vec{j}$.
(b) Verify explicitly that $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$.

Solution: (a) Using Eq. 2.34 through Eq. 2.36 for the cross product of unit vectors, we will get the following for $\vec{A} \times \vec{B}$:

$$\begin{aligned}\vec{A} \times \vec{B} &= (8\,\vec{i} + 3\,\vec{j}) \times (-5\,\vec{i} - 7\,\vec{j})\\ &= -40\,\vec{i} \times \vec{i} - 56\,\vec{i} \times \vec{j} - 15\,\vec{j} \times \vec{i} - 21\,\vec{j} \times \vec{j}\\ &= 0 - 56\,\vec{k} + 15\,\vec{k} + 0 = -41\,\vec{k}\end{aligned}$$

As an alternative method for finding $\vec{A} \times \vec{B}$, we use Eq. 2.37, with $A_x = 8$, $A_y = 3$, $A_z = 0$, $B_x = -5$, $B_y = -7$, and $B_z = 0$:

$$\begin{aligned}\vec{A} \times \vec{B} &= (A_yB_z - A_zB_y)\,\vec{i} + (A_zB_x - A_xB_z)\,\vec{j} + (A_xB_y - A_yB_x)\,\vec{k}\\ &= (0)\,\vec{i} + (0)\,\vec{j} + (-56 - [-15])\,\vec{k} = -41\,\vec{k}\end{aligned}$$

(b) We can evaluate $\vec{B} \times \vec{A}$ as follows:

$$\begin{aligned}\vec{B} \times \vec{A} &= (-5\,\vec{i} - 7\,\vec{j}) \times (8\,\vec{i} + 3\,\vec{j})\\ &= -40\,\vec{i} \times \vec{i} - 15\,\vec{i} \times \vec{j} - 56\,\vec{j} \times \vec{i} - 21\,\vec{j} \times \vec{j}\\ &= 0 - 15\,\vec{k} + 56\,\vec{k} + 0 = +41\,\vec{k}\end{aligned}$$

Therefore, $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$.

Example 2.6

Is it possible to use the cross product to find the angle between the two vectors $\vec{A} = 8\,\vec{i} + 3\,\vec{j}$ and $\vec{B} = -5\,\vec{i} - 7\,\vec{j}$ of Example 2.5?

Solution: From Example 2.5 we found that:

$$\vec{A} \times \vec{B} = -41\,\vec{k}$$

If we let $\vec{C} = \vec{A} \times \vec{B}$, then according to Eq. 2.28 the magnitude of $\vec{C}$ is:

$$C = AB \sin\theta$$

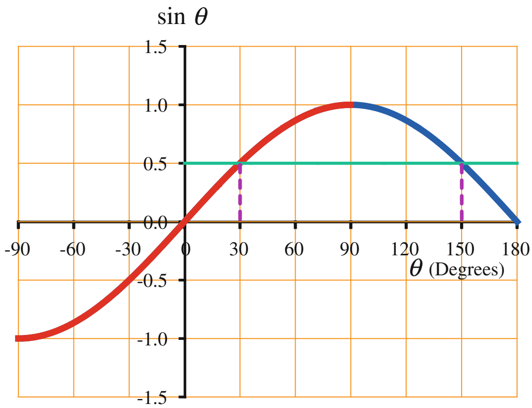
But $C = 41$. Therefore:

$$41 = \sqrt{8^2 + 3^2} \times \sqrt{(-5)^2 + (-7)^2}\,\sin\theta = 73.5\sin\theta$$

Thus, your calculator will give: $\theta = \sin^{-1}\left(\dfrac{41}{73.5}\right) = 33.91^\circ$

The calculator's range for $\sin^{-1}$ is only from -90° to 90°, (see the red part of the sine curve of Fig. 2.14.) So, when you calculate the inverse of a sine function, you must consider how reasonable your answer is, because there is usually another possible answer that the calculator does not display. For example, in Fig. 2.14, the horizontal line through 0.5 cuts the sine curve at 30° and 150°, i.e. the inverse sine of those two angles are equal to 0.5. But your calculator will give only the angle 30° (see the red part of the curve).

Fig. 2.14



Since $\sin\theta = \sin(180^\circ - \theta)$, then the angle between the two vectors could be either 33.91° or 146.1°. You can find the correct answer by using a graphical method or the dot product, as in Example 2.4, to prove that the correct answer is $\theta = 146.1^\circ$. Thus, the cross product is not the simplest method for determining the angle between any two vectors.

2.5 Exercises

Section 2.2 Properties of Vectors

(1) A car travels 10 km due north and then 5 km due west. Find graphically and analytically the magnitude and direction of the car's resultant displacement.

(2) A car travels 6 km due east and then 4 km in a direction 120° north of east. Use both the graphical and analytical methods to find the magnitude and direction of the car's displacement vector.

(3) Vector $\vec{A}$ has a magnitude of 10 units and makes 60° with the positive x-axis. Vector $\vec{B}$ has a magnitude of 5 units and is directed along the negative x-axis. Use geometry to find: (a) the vector sum $\vec{A} + \vec{B}$, and (b) the vector difference $\vec{A} - \vec{B}$.

(4) A car travels in a circular path of radius 10 m. (a) If the car traveled one half of the circle, find the magnitude of the displacement vector and find how far the car traveled. (b) Answer part (a) if the car makes one complete revolution.

Section 2.3 Vector Components and Unit Vectors

(5) Vector $\vec{A}$ has x and y components of 4 cm and −5 cm, respectively. Vector $\vec{B}$ has x and y components of −2 cm and 1 cm, respectively. If $\vec{A} - \vec{B} + 3\vec{C} = 0$, then what are the components of $\vec{C}$.

(6) Three vectors are oriented as shown in Fig. 2.15, where $A = 10$, $B = 20$, and $C = 15$ units. Find: (a) the x and y components of the resultant vector $\vec{D} = \vec{A} + \vec{B} + \vec{C}$, (b) the magnitude and direction of the resultant vector.

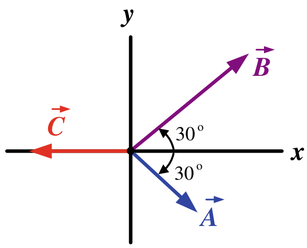


Fig. 2.15 See Exercise (6)

(7) The radar beam of a police car points at an angle of 30° away from the direction of a highway. The radar records the component of the car's speed along the beam as $v_{CR} = 120$ km/h, see Fig. 2.16. (a) What is the speed v_C of the car along the highway? (b) Can the radar beam be directed perpendicular to the direction of the highway? Why or why not?

(8) A radar device detects a rocket approaching directly from east due west. At one instant, the rocket was observed 10 km away and making an angle of 30° above the horizon. At another instant the rocket was observed at an angle of 150°

in the vertical east-west plane while the rocket was 8 km away, see Fig. 2.17. Find the displacement of the rocket during the period of observation.

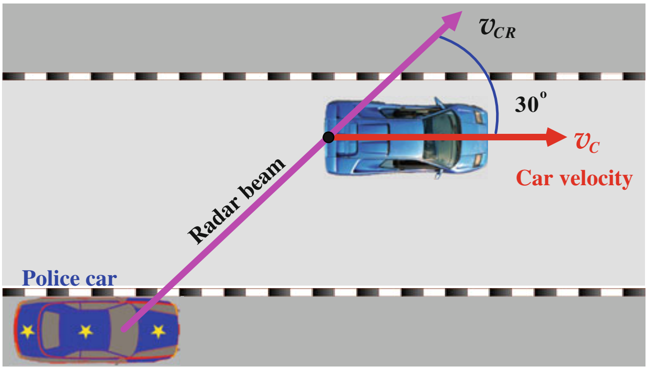


Fig. 2.16 See Exercise (7)

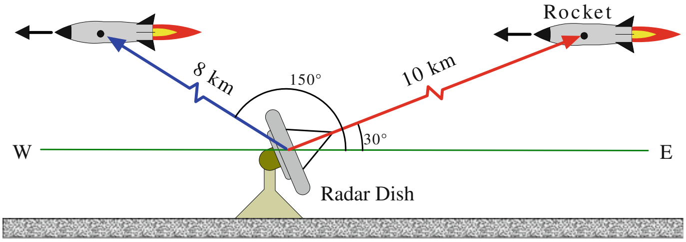


Fig. 2.17 See Exercise (8)

(9) Find the vector components of the sum $\vec{R}$ of the displacement vectors $\vec{A}$ and $\vec{B}$ whose components along three perpendicular directions are $A_x = 2$, $A_y = 1$, $A_z = 3$, $B_x = 1$, $B_y = 4$, and $B_z = 2$. Find the magnitude of $\vec{R}$.

(10) Two vectors $\vec{A}$ and $\vec{B}$ (of lengths A and B, respectively) make an angle θ with each other when they are placed tail to tail, see Fig. 2.18. (a) By taking components along two perpendicular axes, prove that the length of their vector sum $\vec{R} = \vec{A} + \vec{B}$ is:

$$R = \sqrt{A^2 + B^2 + 2AB\cos\theta}$$

(b) For the difference $\vec{C} = \vec{A} - \vec{B}$, where C is the length of the third side of a triangle formed from connecting the head of $\vec{B}$ to the head of $\vec{A}$ as in Fig. 2.19, use the same approach to prove that:

$$C = \sqrt{A^2 + B^2 - 2AB\cos\theta}$$

Fig. 2.18 See Exercise (10)

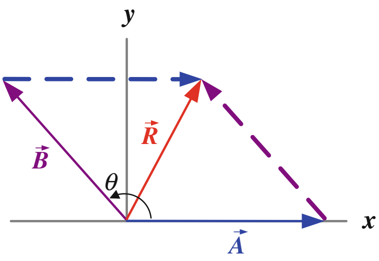
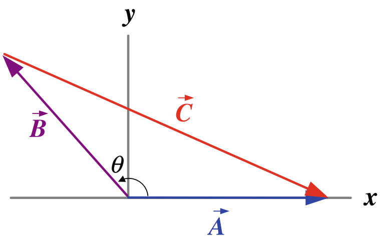
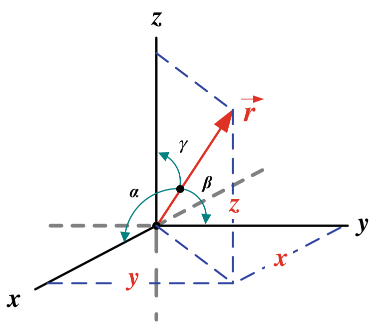


Fig. 2.19 See Exercise (10)



(11) A position vector $\vec{r} = x\,\vec{\text{i}} + y\,\vec{\text{j}} + z\,\vec{\text{k}}$ makes angles α, β, and γ with the x, y, and z axes of a perpendicular right-handed coordinate system as in Fig. 2.20. Show that the relation between what is known as the direction cosines $\cos\alpha$, $\cos\beta$, and $\cos\gamma$ are as follows: $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

Fig. 2.20 See Exercise (11)



(12) When vector $\vec{B}$ is added to vector $\vec{A}$ we get $5\,\vec{\text{i}} - \vec{\text{j}}$, and when $\vec{B}$ is subtracted from $\vec{A}$ we get $\vec{\text{i}} - 7\,\vec{\text{j}}$. What is the magnitude and direction of $\vec{A}$?

(13) Two vectors are given by $\vec{A} = 2\,\vec{i} + 3\,\vec{j}$ and $\vec{B} = 4\,\vec{i} - 3\,\vec{j}$. Find: (a) the magnitude and direction of the vector sum $\vec{R} = \vec{A} + \vec{B}$, (b) the magnitude and direction of the vector difference $\vec{S} = \vec{A} - \vec{B}$.

(14) Two vectors are given by $\vec{A} = -\vec{i} + \vec{j} + 4\,\vec{k}$ and $\vec{B} = 3\,\vec{i} - 4\,\vec{j} + \vec{k}$. Find: (a) $\vec{A} + \vec{B}$, (b) $\vec{A} - \vec{B}$, and (c) a vector $\vec{C}$ such that $\vec{A} + \vec{B} + \vec{C} = 0$.

Section 2.4 Multiplying Vectors

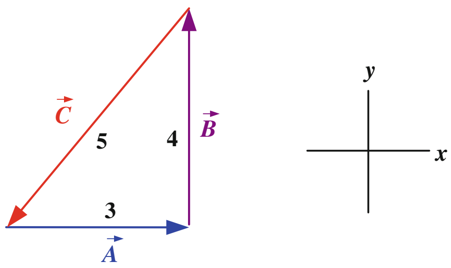
(15) Vector $\vec{A}$ has a magnitude of 3 units and lies along the negative x-axis. Vector $\vec{B}$ has a magnitude of 6 units and makes an angle 30° with the positive x-axis. (a) Find the scalar product $\vec{A} \bullet \vec{B}$ without using the concept of components. (b) Find $\vec{A} \bullet \vec{B}$ by using vector components.

(16) Show that for any vector $\vec{A}$: (a) $\vec{A} \bullet \vec{A} = A^2$ and (b) $\vec{A} \times \vec{A} = 0$.

(17) In Exercise 10, show that dotting vector $\vec{R}$ with itself and dotting vector $\vec{C}$ with itself leads directly to the results of both part (a) and part (b).

(18) For the vectors in Fig. 2.21, find the following: (a) $\vec{A} \bullet \vec{B}$, (b) $\vec{A} \bullet \vec{C}$, (c) $\vec{B} \bullet \vec{C}$, (d) $\vec{A} \times \vec{B}$, (e) $\vec{A} \times \vec{C}$, and (f) $\vec{B} \times \vec{C}$.

Fig. 2.21 See Exercise (18)



(19) (a) Show that $\vec{A} \bullet (\vec{A} \times \vec{B}) = 0$ for all vectors $\vec{A}$ and $\vec{B}$. (b) If θ is the angle between $\vec{A}$ and $\vec{B}$, then find the magnitude of $\vec{A} \times (\vec{A} \times \vec{B})$.

(20) Two vectors $\vec{A}$ and $\vec{B}$ make an acute angle θ with each other when they are placed tail to tail as shown in Fig. 2.22. (a) Prove that the area of the triangle that is contained by these two vectors is $\frac{1}{2}|\vec{A} \times \vec{B}|$. (b) Show that the area of the parallelogram formed by $\vec{A}$ and $\vec{B}$ is $|\vec{A} \times \vec{B}|$.

(21) Show that $\vec{A} \bullet (\vec{B} \times \vec{C})$ is equal in magnitude to the volume of the parallelepiped whose sides are formed from the three vectors $\vec{A}$, $\vec{B}$, and $\vec{C}$ as shown in Fig. 2.23.

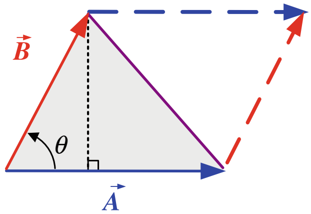


Fig. 2.22 See Exercise (20)

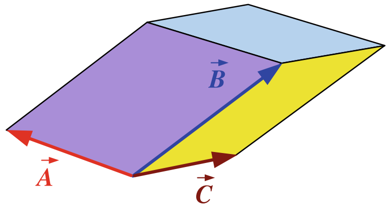


Fig. 2.23 See Exercise (21)

(22) In the xy plane, point P has coordinates (x_1, y_1) and is described by the position vector $\vec{r}_1 = x_1\,\vec{\mathrm{i}} + y_1\,\vec{\mathrm{j}}$. Similarly, point Q has coordinates (x_2, y_2) and is described by the position vector $\vec{r}_2 = x_2\,\vec{\mathrm{i}} + y_2\,\vec{\mathrm{j}}$, see Fig. 2.24. (a) Show that the displacement vector from P to Q is given by $\vec{d} = \vec{r}_2 - \vec{r}_1 = (x_2 - x_1)\,\vec{\mathrm{i}} + (y_2 - y_1)\,\vec{\mathrm{j}}$. (b) Find the magnitude and direction of $\vec{d}$.

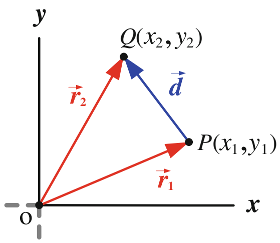


Fig. 2.24 See Exercise (22)

(23) The equation $\vec{F} = q(\vec{v} \times \vec{B})$ gives the force $\vec{F}$ on an electric point charge q moving with velocity $\vec{v}$ through a uniform magnetic field $\vec{B}$. Find the force on a proton of $q = 1.6 \times 10^{-19}$ coulomb moving with velocity $\vec{v} = (2\,\vec{\mathrm{i}} + 3\,\vec{\mathrm{j}} + 4\,\vec{\mathrm{k}}) \times 10^5$ m/s in a magnetic field of $0.5\,\vec{\mathrm{k}}$ tesla. (The given SI units yield a force in newtons.)

(24) The electromagnetic Poynting vector $\vec{S}$ is defined by $\vec{S} = \vec{E} \times \vec{H}$, where $\vec{E}$ and $\vec{H}$ are the electric and magnetic fields. Calculate $\vec{S}$ for $\vec{E} = \vec{\mathrm{i}} + 0.3\,\vec{\mathrm{j}} + 0.5\,\vec{\mathrm{k}}$ and $\vec{H} = -0.4\,\vec{\mathrm{i}} + \vec{\mathrm{j}} + 0.2\,\vec{\mathrm{k}}$. You can disregard units for this calculation.

Part II
Mechanics

Motion in One Dimension

3

Mechanics is the science that deals with motion of objects. It is **basic** to all other **branches of physics**. The branch of mechanics that *describes* the motion of objects is called **kinematics**. In this branch we answer questions like “Does the object speed up, slow down, stop, or reverse direction?” and “How is time involved in these situations?”

In this chapter, we only study motion along straight lines. The moving object of concern is either a **particle** (a point-like object) or an object that can be viewed to move like a particle.

3.1 Position and Displacement

To locate an object in one-dimensional space, we find its position with respect to some reference point, called the **origin** of an axis, such as the x-axis shown in Fig. 3.1. The **positive/negative direction** of this axis is the direction of increasing/decreasing numbers.

A change in the object’s position from an initial position x_i to a final position x_f is called displacement Δx (read delta x), where:

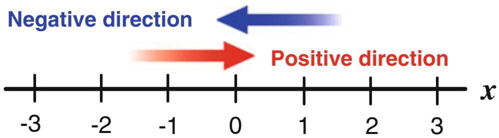


Fig. 3.1 The position of a particle that moves in one dimension is identified on an x-axis that is marked in units of length

H. A. Radi and J. O. Rasmussen, *Principles of Physics*,
Undergraduate Lecture Notes in Physics, DOI: 10.1007/978-3-642-23026-4_3,

$$\Delta x = x_f - x_i \tag{3.1}$$

The displacement is a **vector quantity** which has a magnitude and a direction. The magnitude is the distance between the initial and final positions and the direction is represented in Fig. 3.1 by a plus or minus sign for motion to the right or to the left, respectively.

3.2 Average Velocity and Average Speed

Average Velocity

Consider a particle moving along the x-axis, where its *position-time graph* is as shown in Fig. 3.2. At point P, let its position be x_i when the time was t_i and at point Q, let its position be x_f when the time was t_f (the indices i and f refer to the initial and final values for the variables under consideration). Accordingly, during the time interval $\Delta t = t_f - t_i$, the particle's displacement is $\Delta x = x_f - x_i$.

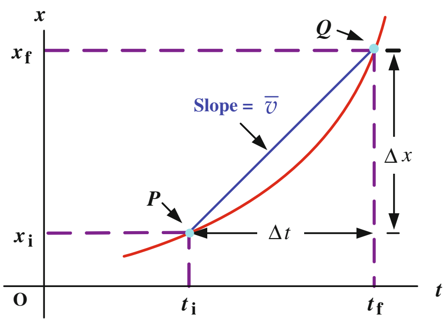


Fig. 3.2 The *position-time graph* for a particle moving along the x-axis. The slope of the line PQ measures the average velocity $\overline{v}$

One of several quantities associated with the phrase "how fast" a particle moves is the average velocity, $\overline{v}$, which is defined as follows:

> Average velocity
> The average velocity, $\overline{v}$, of a particle is defined as the ratio of its displacement, Δx, to the time interval, Δt. That is:

$$\overline{v} = \frac{\Delta x}{\Delta t} = \frac{x_{\mathrm{f}} - x_{\mathrm{i}}}{t_{\mathrm{f}} - t_{\mathrm{i}}} \tag{3.2}$$

From this definition, $\overline{v}$ has the dimension of length divided by time, that is m/s in SI units. The average velocity is a vector quantity which has a magnitude and direction represented by a plus or minus sign for motion to the right or to the left, respectively, see Fig. 3.1.

Average Speed

The average speed $\overline{s}$ is a different way of describing "how fast" a particle moves and it is defined as follows:

Average speed
The average speed, $\overline{s}$, of a particle is defined as the ratio of the total distance covered d to the time interval $\Delta t = t_{\mathrm{f}} - t_{\mathrm{i}}$. That is:

$$\overline{s} = \frac{\text{total distance}}{\Delta t} = \frac{d}{t_{\mathrm{f}} - t_{\mathrm{i}}} \tag{3.3}$$

So, $\overline{s}$ is different from $\overline{v}$ in that $\overline{s}$ does not depend on direction, and hence is always positive. In some cases $\overline{s}$ might be the same as $\overline{v}$.

Example 3.1

A car moving along the x-axis starts from the position $x_{\mathrm{i}} = 2$ m when $t_{\mathrm{i}} = 0$ and stops at $x_{\mathrm{f}} = -3$ m when $t_{\mathrm{f}} = 2$ s. (a) Find the displacement, the average velocity, and the average speed during this interval of time. (b) If the car goes backward and takes 3 s to reach the starting point, then repeat part (a) for the *whole* time interval.

Solution: (a) The car's displacement, see Fig. 3.3, is given by:

$$\Delta x = x_{\mathrm{f}} - x_{\mathrm{i}} = -3 \text{ m} - 2 \text{ m} = -5 \text{ m}$$

The average velocity is then given by:

$$\overline{v} = \frac{\Delta x}{\Delta t} = \frac{x_{\mathrm{f}} - x_{\mathrm{i}}}{t_{\mathrm{f}} - t_{\mathrm{i}}} = \frac{-3 \text{ m} - 2 \text{ m}}{2 \text{ s} - 0 \text{ s}} = \frac{-5 \text{ m}}{2 \text{ s}} = -2.5 \text{ m/s}$$

Since Δx and $\overline{v}$ are negative for this time interval, then the car has moved to the left, toward decreasing values of x, see Fig. 3.3. The total covered distance is $d = 5$ m and the average speed is thus:

$$\overline{s} = \frac{\text{total distance}}{\Delta t} = \frac{d}{t_f - t_i} = \frac{5\text{ m}}{2\text{ s} - 0\text{ s}} = \frac{5\text{ m}}{2\text{ s}} = 2.5\text{ m/s}$$

In this case, $\overline{s}$ is the same as $\overline{v}$ (except for a minus sign).

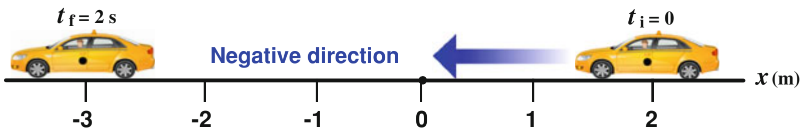


Fig. 3.3 Example 3.1

(b) After the backward movement, the final position and final time of the car are $x_f = 2$ m and $t_f = 2\text{ s} + 3\text{ s} = 5$ s, respectively, while the total distance covered by the car is $d = 5\text{ m} + 5\text{ m} = 10$ m. As we know, the displacement involves only the initial and final positions and will be:

$$\Delta x = x_f - x_i = 2\text{ m} - 2\text{ m} = 0$$

Then, the average velocity will be:

$$\overline{v} = \frac{\Delta x}{\Delta t} = \frac{x_f - x_i}{t_f - t_i} = \frac{0}{5\text{ s} - 0\text{ s}} = 0$$

Finally, the average speed for the whole movement of the car will be:

$$\overline{s} = \frac{\text{total distance}}{\Delta t} = \frac{d}{t_f - t_i} = \frac{10\text{ m}}{5\text{ s} - 0\text{ s}} = \frac{10\text{ m}}{5\text{ s}} = 2\text{ m/s}$$

As you can see, the average velocity is zero, while the average speed is 2 m/s, since the latter depends only on the total covered distance d.

3.3 Instantaneous Velocity and Speed

More commonly, we ask how fast a particle is moving at a given instant, which refers to its **instantaneous velocity** (or simply **velocity**). The velocity at any instant is obtained from the average velocity by allowing the time interval Δt to approach zero. Consider the motion of an object (for example a car). This object can be viewed as a particle for simplicity. The motion of that particle between two points P and Q on a position-time graph is shown in the right part of Fig. 3.4. As point Q is brought closer and closer to point P (through points Q_1, Q_2, ...), the time intervals (Δt_1, Δt_2, ...) get progressively smaller. The average velocity for each time interval is the slope of the dotted line in Fig. 3.4. As point Q approaches P, the time interval approaches zero, while the slope of the dotted line approaches the slope of the tangent to the curve at point P. This slope is defined to be the instantaneous velocity v at the time t_i. In short, we define:

Instantaneous velocity
The instantaneous velocity, v, of a particle is defined as the limiting value of the ratio $\Delta x/\Delta t$ as Δt approaches zero. Mathematically v can be expressed as:

$$v = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} \tag{3.4}$$

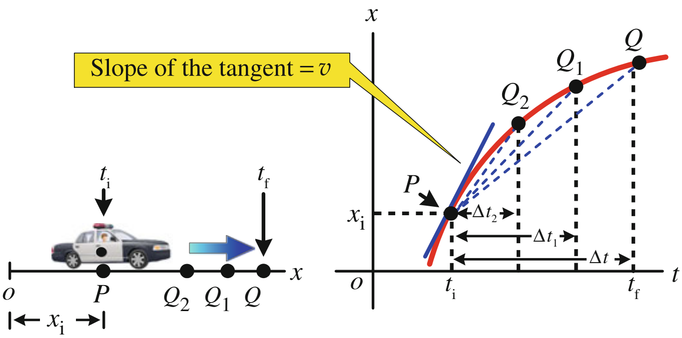


Fig. 3.4 The *left* part shows a police car (which can be considered as a particle) that moves along the x-axis. The *right* part shows the position-time graph for this motion. As Q approaches P, the average velocity $\overline{v}$ for the interval PQ approaches the slope of the tangent line at P, which is defined as the instantaneous velocity v at point P

In calculus notation, the above limit is called the derivative of x with respect to t, and written as dx/dt (abbreviated as $\dot{x}$). Thus:

$$v = \frac{dx}{dt} \quad \equiv \quad x_f - x_i = \int_{t_i}^{t_f} v\, dt \equiv \text{Area under } v\text{-}t \text{ graph} \tag{3.5}$$

The instantaneous velocity, v, can be positive, negative, or zero, depending on the slope of the position-time graph at the interval of interest in Fig. 3.5. In this figure, $v = 0$ represents the turning point, and occurs at any maximum or minimum of the x-t graph. From here on, we use the word *velocity* to denote *instantaneous velocity*.

The *speed* of a particle is defined as the magnitude of its velocity.

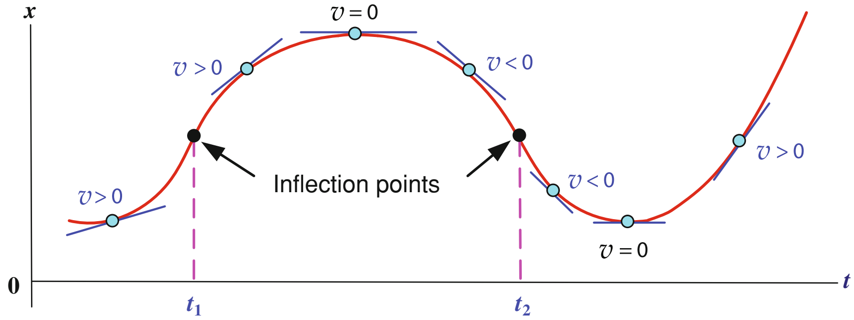


Fig. 3.5 The position-time graph for a particle moving along the x-axis. On this graph we display: (1) Positive velocities, where the slope of the tangent lines are positive, (2) Negative velocities, where the slope of the tangent lines are negative, (3) Zero velocities (turning points), where the slope of the tangent lines are zero, and (4) Inflection points at t_1 and t_2, where the increase/decrease of the velocity reaches a maximum/minimum

Example 3.2

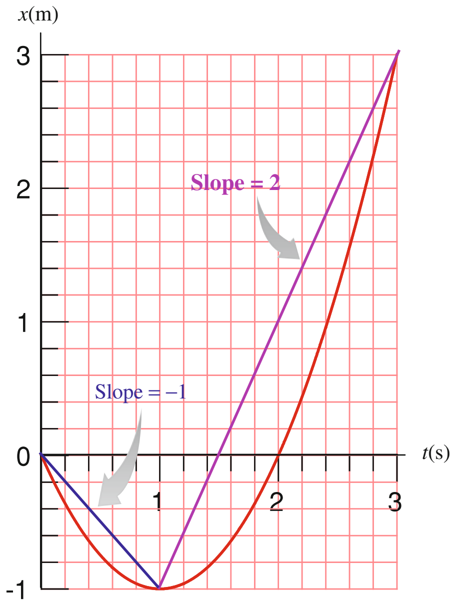
A particle moves along the x-axis and its coordinates vary with time according to the relation $x = t^2 - 2\,t$, where x is measured in meters and t is in seconds. The position-time graph for this motion is shown in Fig. 3.6. (a) Use this graph to comment about the particle's motion. (b) Find the displacement and the average velocity of the particle in the time intervals $0 \le t \le 1\,\text{s}$ and $1\,\text{s} \le t \le 3\,\text{s}$. (c) Find the velocity of the particle at $t = 2$ s.

Solution: (a) The particle starts from the origin of the x-axis and moves in the negative x direction for the first second. Its velocity is zero at $x = -1\,\text{m}$ when $t = 1\,\text{s}$ and then heads back in the positive x direction for $t > 1\,\text{s}$.

(b) In the interval $0 \le t \le 1\,\text{s}$ we have $t_i = 0$ and $t_f = 1$ s. Since $x = t^2 - 2\,t$, we get $x_i = t_i^2 - 2\,t_i = 0$ and $x_f = t_f^2 - 2\,t_f = -1$ m. Thus:

$$\Delta x = x_f - x_i = -1\text{ m} - 0\text{ m} = -1\text{ m}$$

Fig. 3.6



The average velocity is then:

$$\overline{v} = \frac{\Delta x}{\Delta t} = \frac{x_f - x_i}{t_f - t_i} = \frac{-1\ \text{m} - 0\ \text{m}}{1\ \text{s} - 0\ \text{s}} = \frac{-1\ \text{m}}{1\ \text{s}} = -1\ \text{m/s}$$

According to Fig. 3.6, this value equals the slope of the straight line drawn for this time interval.

In the interval $1\ \text{s} \le t \le 3\ \text{s}$ we have $t_i = 1$ s and $t_f = 3$ s. Again, from $x = t^2 - 2t$ we get $x_i = t_i^2 - 2t_i = -1$ m and $x_f = t_f^2 - 2t_f = 3$ m. Thus:

$$\Delta x = x_f - x_i = 3\ \text{m} - (-1\ \text{m}) = 4\ \text{m}$$

The average velocity is then:

$$\overline{v} = \frac{\Delta x}{\Delta t} = \frac{x_f - x_i}{t_f - t_i} = \frac{3\ \text{m} - (-1\ \text{m})}{3\ \text{s} - 1\ \text{s}} = \frac{4\ \text{m}}{2\ \text{s}} = 2\ \text{m/s}$$

According to Fig. 3.6, this value equals the slope of the straight line drawn for this time interval.

(c) To find the instantaneous velocity at any time t, we use Eq. 3.5 and apply the rules of differential calculus on the coordinate $x = t^2 - 2t$. That is:

$$v = \frac{dx}{dt} = \frac{d(t^2 - 2t)}{dt} = 2t - 2$$

Notice that this expression gives the velocity v at any time t and indicates that v is increasing linearly with time. It tells us that $v < 0$ during the interval $0 \le t < 1$ s (i.e. the particle is moving in the negative x direction), and that $v = 0$ at $t = 1$ s, and finally $v > 0$ for $t > 1$ s. When $t = 2$ s we use the above expression to get:

$$v = 2 \times 2 - 2 = 2 \text{ m/s}$$

3.4 Acceleration

When the velocity of a particle changes with time, the particle is said to be accelerating. Consider the motion of a particle along the x-axis. If the particle has a velocity v_i at time t_i and a velocity v_f at time t_f as in the velocity-time graph of Fig. 3.7, then we define the *average acceleration* as:

Average acceleration
The average acceleration, $\overline{a}$, of a particle is defined as the ratio of the change in velocity $\Delta v = v_f - v_i$ to the time interval $\Delta t = t_f - t_i$. That is:

$$\boxed{\overline{a} = \frac{\Delta v}{\Delta t} = \frac{v_f - v_i}{t_f - t_i}} \tag{3.6}$$

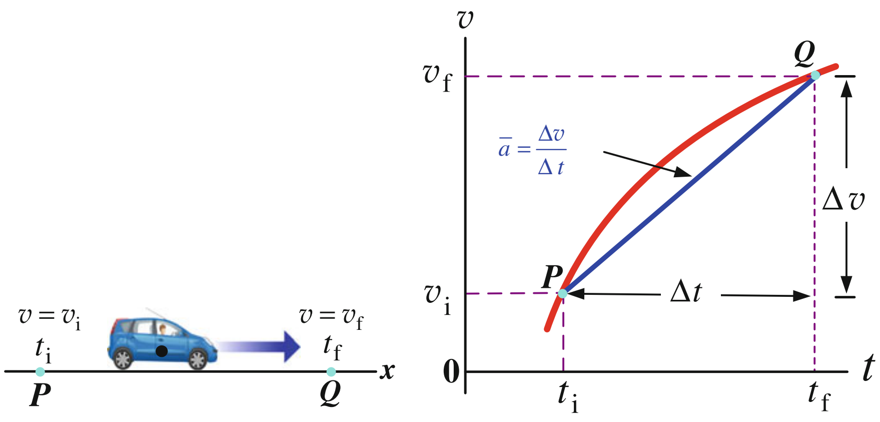


Fig. 3.7 The velocity-time graph for a car (or simply a particle) moving in a straight line. The slope of the straight line PQ is defined as the average acceleration in the time interval $\Delta t = t_f - t_i$

Acceleration is a vector quantity having dimensions of length divided by (time)^2, or L/T^2; that is m/s^2 in SI units.

It is useful to define the instantaneous acceleration as the limit of the average acceleration when Δt approaches zero. Consider the motion of a particle (for example a car that moves like a particle) between the two points P and Q on the velocity-time graph shown in the right part of Fig. 3.8. As point Q is brought closer and closer to point P (through points $Q_1, Q_2, \ldots$), the time intervals (Δt_1, $\Delta t_2, \ldots$) get progressively smaller. The average acceleration for each time interval is the slope of the dotted line in Fig. 3.8. As Q approaches P, the time interval approaches zero, while the slope of the dotted line approaches the slope of the tangent to the curve at point P. The slope of the tangent line to the curve at P is defined to be the instantaneous acceleration a at the time t_i. That is, we define the following:

Instantaneous acceleration

The instantaneous acceleration, a, of a particle is defined as the limiting value of the ratio $\Delta v/\Delta t$ when Δt approaches zero. Mathematically a can be expressed as:

$$a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} \tag{3.7}$$

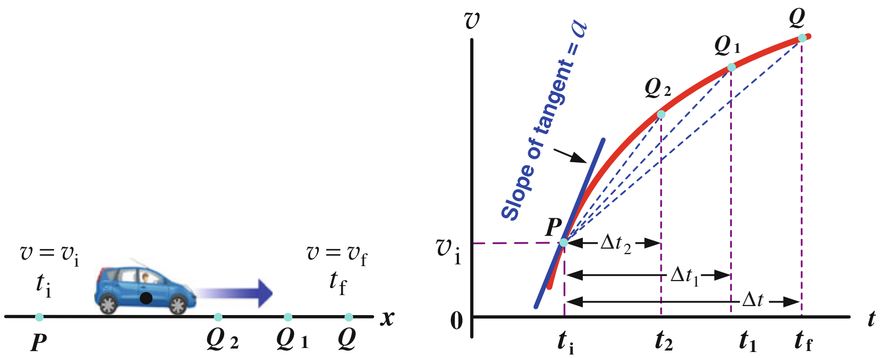


Fig. 3.8 The left part shows a car that moves along the x-axis. The right part shows the velocity-time graph that describes the car's motion. As Q approaches P, the average acceleration $\overline{a}$ for the interval PQ approaches the slope of the tangent line at P, which is defined as the instantaneous acceleration a at point P

In calculus notation, the above limit is called the *first derivative* of v with respect to t, and written as dv/dt (simplified sometimes as $\dot{v}$), or the *second derivative* of x with respect to t, and written as d^2x/dt^2 (simplified sometimes as $\ddot{x}$). Thus:

$$\boxed{a = \frac{dv}{dt} = \frac{d^2x}{dt^2}} \quad \equiv \quad \boxed{v_\mathrm{f} - v_\mathrm{i} = \int_{t_\mathrm{i}}^{t_\mathrm{f}} a\, dt \equiv \text{Area under } a\text{-}t \text{ graph}} \tag{3.8}$$

From here on, we use the word *acceleration* to designate *instantaneous acceleration*. Depending on the slope of the tangent to the velocity-time graph, acceleration a can be positive, negative (called deceleration), or zero. If $a = 0$ for a specific time interval in the $v - t$ graph, then the velocity must be a constant in this interval.

Example 3.3

The position of a particle moving along the x-axis varies with time t according to the relation $x = t^3 - 12\,t + 20$, where x is given in meters and t in seconds. (a) Find the velocity and the acceleration of the particle as a function of time. (b) Is there ever a time when $v = 0$? (c) Describe the particle's motion for $t \geq 0$.

Solution: (a) To get the velocity v as a function of time t, we differentiate the coordinate x with respect to t as follows:

$$v = \frac{dx}{dt} = \frac{d}{dt}\left(t^3 - 12\,t + 20\right) \quad \Rightarrow \quad v = 3\,t^2 - 12$$

To get the acceleration a as a function of time t, we differentiate the velocity v with respect to t as follows:

$$a = \frac{dv}{dt} = \frac{d}{dt}\left(3\,t^2 - 12\right) \quad \Rightarrow \quad a = 6\,t$$

(b) Setting $v = 0$ in the velocity relation yields:

$$0 = 3\,t^2 - 12,$$

which has the solution $t = \pm 2$ s. The negative answer has to be rejected, since time must be always positive. Thus at $t = 2$ s the velocity of the particle is zero.

(c) To describe the particle's motion for $t \geq 0$ we examine the expressions $x = t^3 - 12\,t + 20$, $v = 3\,t^2 - 12$, and $a = 6\,t$.

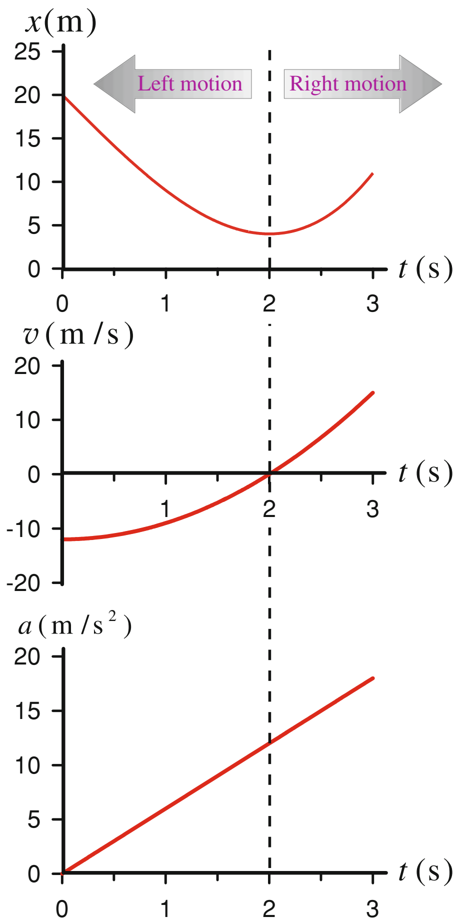
At $t = 0$, the particle is at $x = 20$ m from the origin and moving to the left with velocity $v = -12$ m/s and not accelerating since $a = 0$, see Fig. 3.9.

At $0 < t < 2$ s, the particle continues to move to the left (x decreases), but at a decreasing speed, because it is now accelerating to the right, $a =$ positive (Check the expressions of x, v, and a for $t = 1$ s and compare the results with Fig. 3.9).

At $t = 2$ s, the particle stops momentarily ($v = 0$) to reverse its direction of motion. At this moment $x = 4$ m, i.e. it will be as close as it will ever be to the origin. It will continue to accelerate to the right at an increasing rate, see Fig. 3.9.

For $t > 2$ s, the particle continues to accelerate and move to the right, and its velocity, which is now to the right, increases rapidly, see Fig. 3.9.

Fig. 3.9



3.5 Constant Acceleration

In many common types of one-dimensional motion, the acceleration is constant (or we say uniform). In this case, the average acceleration equals the instantaneous acceleration, i.e.

$$\boxed{\bar{a} = a = \text{constant}} \tag{3.9}$$

The shape of this relation can be displayed for positive a as shown in the left part of Fig. 3.10. Consequently, Eq. 3.6 becomes:

$$a = \frac{v_f - v_i}{t_f - t_i} \quad (\text{when } \bar{a} = a = \text{constant}) \tag{3.10}$$

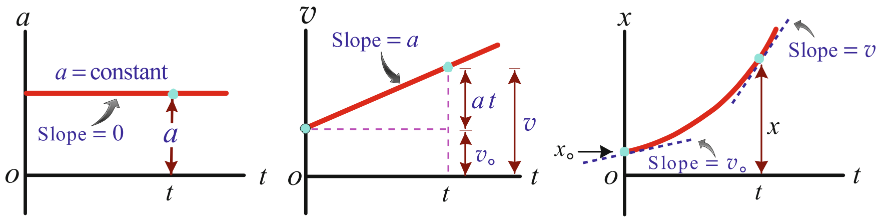


Fig. 3.10 (*Left part*) The acceleration-time graph of a particle moving along the x-axis with constant acceleration. (*Middle part*) the velocity-time graph of the particle's motion. (*Right part*) The position-time graph of the particle's motion

For convenience, we let $t_i = 0$ and $t_f = t$, where t is any arbitrary time. Also, we let $v_i = v_\circ$ (the initial velocity at time $t = 0$) and $v_f = v$ (the velocity at any time t). With this notation, we can express acceleration as:

$$a = \frac{v - v_\circ}{t}$$

Rearranging gives:

$$\boxed{v = v_\circ + a\,t} \quad (\text{for constant } a) \tag{3.11}$$

This linear relationship enables us to find the velocity at any time t; see the middle part of Fig. 3.10.

We can make use of the fact that when the acceleration is constant (i.e. when the velocity varies linearly with time according to Eq. 3.11 as in Fig. 3.10), the average velocity in any time interval is the arithmetic mean of the initial velocity, $v_\circ$, and the final velocity at the end of that interval, v. Thus:

$$\boxed{\overline{v} = \frac{v_\circ + v}{2}} \quad \text{(for constant } a\text{)} \tag{3.12}$$

To find the displacement as a function of time, we first let $x_i = x_\circ$ (the initial position at time $t = 0$) and $x_f = x$ (the position at any time t), and then use Eq. 3.2 and Eq. 3.12 to get:

$$\overline{v} = \frac{x - x_\circ}{t} = \frac{v_\circ + v}{2}$$

Rearranging gives:

$$\boxed{x - x_\circ = \tfrac{1}{2}(v_\circ + v)\ t} \quad \text{(for constant } a\text{)} \tag{3.13}$$

We can obtain another useful expression for the displacement by substituting Eq. 3.11 into Eq. 3.13 to get:

$$\boxed{x - x_\circ = v_\circ\ t + \tfrac{1}{2}\ a\ t^2} \quad \text{(for constant } a\text{)} \tag{3.14}$$

As a first check for Eq. 3.14, one can notice that substituting $t = 0$ yields $x = x_\circ$, as it must be. A further check, taking the derivative of Eq. 3.14 with respect to time, yields Eq. 3.11. The right part of Fig. 3.10 displays the position x as a function of time t for the parabolic Eq. 3.14.

We can use Eq. 3.11 to eliminate $v_\circ$ from Eq. 3.14 to obtain the following relation:

$$\boxed{x - x_\circ = v\ t - \tfrac{1}{2}\ a\ t^2} \quad \text{(for constant } a\text{)} \tag{3.15}$$

Finally, by replacing the value of t that was obtained from Eq. 3.11 into Eq. 3.13, we can obtain an expression that does not include the time variable as follows:

$$x - x_\circ = \tfrac{1}{2}(v_\circ + v)\ \frac{(v - v_\circ)}{a} = \frac{(v^2 - v_\circ^2)}{2\ a}$$

Rearranging gives:

$$\boxed{v^2 = v_\circ^2 + 2\ a\ (x - x_\circ)} \quad \text{(for constant } a\text{)} \tag{3.16}$$

Equations 3.11 through 3.16 are six *kinematic expressions* used to solve any one-dimensional problem with constant acceleration.

Table 3.1 lists the four kinematic equations that are used most often in solving problems for the case of constant acceleration.

Table 3.1 Equations for motion with constant acceleration

Equation	Missing quantity	Equation number
$v = v_\circ + a\,t$	$x - x_\circ$	Eq. 3.11
$x - x_\circ = \frac{1}{2}(v_\circ + v)\,t$	a	Eq. 3.13
$x - x_\circ = v_\circ\,t + \frac{1}{2}\,a\,t^2$	v	Eq. 3.14
$v^2 = v_\circ^2 + 2\,a\,(x - x_\circ)$	t	Eq. 3.16

Example 3.4

A car accelerates uniformly from rest to a speed of 100 km/h in 18 s.(a) Find the acceleration of the car. (b) Find the distance that the car travels. (c) If the car brakes to a full stop over a distance of 100 m, then find its uniform deceleration.

Solution: (a) In this problem we are given $v_\circ = 0$, $v = 100$ km/h, and $t = 18\text{ s} = 5 \times 10^{-3}$ h and we need to find a. So, we can use $v = v_\circ + a\,t$ to find the acceleration as follows:

$$a = \frac{v - v_\circ}{t} = \frac{100\text{ km/h} - 0}{5 \times 10^{-3}\text{ h}} = 2 \times 10^4\text{ km/h}^2 \equiv 2 \times 10^4\,\frac{1{,}000\text{ m}}{(60 \times 60\text{ s})^2} = 1.54\text{ m/s}^2$$

(b) If the car starts from the origin of the x-axis, i.e. $x_\circ = 0$, then we are given $v_\circ = 0$, $v = 100$ km/h, $x_\circ = 0$, and $t = 5 \times 10^{-3}$ h and we need to find x, which in this case equals the distance traveled by the car. So, we use $x - x_\circ = \frac{1}{2}(v_\circ + v)\,t$ to find the position x as follows:

$$x = x_\circ + \tfrac{1}{2}(v_\circ + v)\,t = 0 + \tfrac{1}{2}(0 + 100\text{ km/h}) \times 5 \times 10^{-3}\text{ h} = 0.25\text{ km} = 250\text{ m}$$

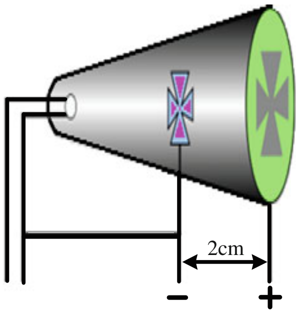
(c) We are given $v_\circ = 100$ km/h, $v = 0$, and $x - x_\circ = 0.1$ km and we need to find the deceleration a. We use $v^2 = v_\circ^2 + 2\,a\,(x - x_\circ)$ to get:

$$a = \frac{v^2 - v_\circ^2}{2\,(x - x_\circ)} = \frac{0 - (100\text{ km/h})^2}{2 \times 0.1\text{ km}} = -5 \times 10^4\text{ km/h}^2 = -3.86\text{ m/s}^2$$

Example 3.5

In a cathode ray tube of a TV set, an electron with initial velocity $v_\circ = 2 \times 10^4$ m/s enters a region 2 cm long (see Fig. 3.11) where it is electrically accelerated in a straight line. The electron emerges from this region with a velocity $v = 3 \times 10^5$ m/s. (a) What was its acceleration, assuming it was constant? (b) How long will the electron be in this region?

Fig. 3.11



Solution: (a) Taking the motion to be along the x-axis, and using $v_\circ = 2 \times 10^4$ m/s, $v = 3 \times 10^5$ m/s, and $x - x_\circ = 2$ cm $= 2 \times 10^{-2}$ m, we can find the acceleration a from the relation $v^2 = v_\circ^2 + 2\,a\,(x - x_\circ)$ as follows:

$$a = \frac{v^2 - v_\circ^2}{2\,(x - x_\circ)} = \frac{(3 \times 10^5 \text{ m/s})^2 - (2 \times 10^4 \text{ m/s})^2}{2 \ \times 2 \times 10^{-2} \text{ m}} = 2.24 \times 10^{12} \text{ m/s}^2$$

(b) Since the displacement and velocities are known, we can use $x - x_\circ = \frac{1}{2}(v_\circ + v)\,t$ to find the time t that the electron will be electrically accelerated as follows:

$$t = \frac{2(x - x_\circ)}{v_\circ + v} = \frac{2 \times 2 \times 10^{-2} \text{ m}}{2 \times 10^4 \text{ m/s} + 3 \times 10^5 \text{ m/s}} = 1.25 \times 10^{-7} \text{ s} = 0.125\,\mu\text{s}$$

Another way to find t is to use equation $v = v_\circ + a\,t$. In this case, $v = 3 \times 10^5$ m/s, and $a = 2.24 \times 10^{12}$ m/s^2. Thus:

$$t = \frac{v - v_\circ}{a} = \frac{3 \times 10^5 \text{ m/s} - 2 \times 10^4 \text{ m/s}}{2.24 \times 10^{12} \text{ m/s}^2} = 1.25 \times 10^{-7} \text{ s} = 0.125\,\mu\text{s}$$

Even though a is very high in this example, but such an acceleration occurs over a very short time interval which is a typical value for such an electrically accelerated charged particle.

Example 3.6

The remote-controlled truck shown in Fig. 3.12 moves along the x-axis with a constant acceleration of -2 m/s^2. As it passes the origin, i.e. $x_\circ = 0$, its initial velocity is 14 m/s. (a) At what time t' and position x' does $v' = 0$ (i.e. when the truck stops momentarily)? (b) At what times t_1 and t_2 is the truck at $x = 24$ m, and what is its velocity then?

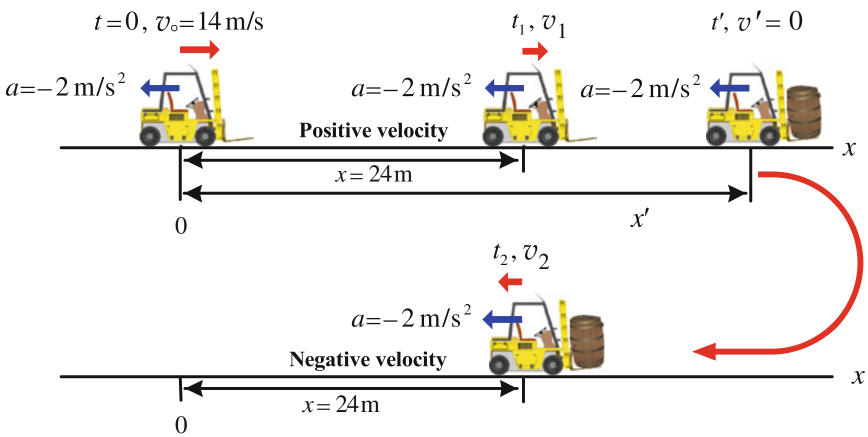


Fig. 3.12

Solution: (a) Given $v_\circ = 14$ m/s, $v' = 0$, and $a = -2$ m/s^2, we can find t' by using $v' = v_\circ + a\,t'$ as follows:

$$t' = \frac{v' - v_\circ}{a} = \frac{0 - 14 \text{ m/s}}{-2 \text{ m/s}^2} = 7 \text{ s}$$

To find the position x' we can use $v'^2 = v_\circ^2 + 2\,a\,(x' - x_\circ)$, since we are given $v_\circ = 14$ m/s, $v' = 0$, $x_\circ = 0$, and $a = -2$ m/s^2. Thus:

$$x' = x_\circ + \frac{v'^2 - v_\circ^2}{2\,a} = 0 + \frac{0 - (14 \text{ m/s})^2}{2 \times (-2 \text{ m/s}^2)} = 49 \text{ m}$$

(b) Using $x = 24$ m, $x_\circ = 0$, $v_\circ = 14$ m/s, and $a = -2$ m/s^2 in $x - x_\circ = v_\circ\,t + \frac{1}{2}\,a\,t^2$, we find, after omitting the units temporarily, that:

$$24 - 0 = 14\,t + \tfrac{1}{2}\,(-2)\,t^2 \quad \Rightarrow \quad t^2 - 14\,t + 24 = 0$$

Solving this quadratic equation yields:

$$t = \frac{14 \pm \sqrt{(-14)^2 - 4 \times 1 \times 24}}{2 \times 1} = \frac{14 \pm 10}{2} \quad \Rightarrow \quad t = \begin{cases} t_1 = 2 \text{ s} \\ t_2 = 12 \text{ s} \end{cases}$$

Thus, $t_1 = 2$ s is the time the truck takes from the origin to the position $x = 24$ m. Furthermore, $t_2 = 12$ s is the time the truck takes from O, passing the point $x = 24$ m, reaching the point $x' = 49$ m and returning back to $x = 24$ m.

For $x = 24$ m, $v_\circ = 14$ m/s, $a = -2$ m/s^2, and $t_1 = 2$ s, we use the formula $v = v_\circ + a\,t$ to get v_1 as follows:

$$v_1 = v_\circ + a\,t_1 = 14 \text{ m/s} + (-2 \text{ m/s}^2) \times (2 \text{ s}) = 10 \text{ m/s}$$

Also, for $x = 24$ m, $v_\circ = 14$ m/s, $a = -2$ m/s^2, and $t_2 = 12$ s, we use the formula $v = v_\circ + a\,t$ to get v_2 as follows:

$$v_2 = v_\circ + a\,t_2 = 14 \text{ m/s} + (-2 \text{ m/s}^2) \times (12 \text{ s}) = -10 \text{ m/s}$$

Observe that the two speeds are equal, i.e. $|v_1| = |v_2| = 10$ m/s.

In this example, we do not pay any attention to the cause of this constant acceleration, but this will be clarified later on when we study the dynamical aspect of mechanics.

3.6 Free Fall

Due to gravity, it is well known that all dropped objects near the Earth's surface will accelerate downward with a nearly constant acceleration when the effect of air resistance is very small and can be neglected. We use the term "free fall" for this motion and the same will be applied to objects that are either thrown up or down. We shall denote the magnitude of the *acceleration due to gravity* by the symbol g, which is very close to 9.8 m/s^2 near the Earth's surface.

Therefore, for free falls near the Earth's surface, the constant acceleration equations of motion Eqs. 3.11 through 3.16, and hence equations of Table 3.1, can be applied. However, we can make them simpler to use with the following minor changes:

(1) The motion is along the vertical y-axis.
(2) The free-fall acceleration is negative if the y-axis is chosen to be upward, and hence we replace the acceleration a with $-g$.
(3) The free-fall acceleration is positive if the y-axis is chosen to be downward, and hence we replace the acceleration a with $+g$.

Table 3.2 lists the four kinematic equations that are frequently used in solving free-fall problems with constant acceleration, where always $|a| = g = 9.8$ m/s^2 for motions near the Earth's surface.

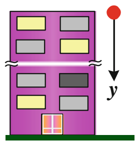
Table 3.2 Equations for free-fall motion with constant acceleration

y (up), $a = -g$	Equation	y (down), $a = g$	Equation
$y\uparrow$	$v = v_\circ - g\,t$	$y\downarrow$	$v = v_\circ + g\,t$
	$y - y_\circ = \frac{1}{2}(v_\circ + v)\,t$		$y - y_\circ = \frac{1}{2}(v_\circ + v)\,t$
	$y - y_\circ = v_\circ\,t - \frac{1}{2}\,g\,t^2$		$y - y_\circ = v_\circ\,t + \frac{1}{2}\,g\,t^2$
	$v^2 = v_\circ^2 - 2\,g(y - y_\circ)$		$v^2 = v_\circ^2 + 2\,g(y - y_\circ)$

Example 3.7

A ball is dropped from a tall building, as shown in Fig. 3.13. Choose the positive y to be downward with its origin at the top of the building, i.e. $y_\circ = 0$. Find the following for the ball's motion: (a) its acceleration, (b) the distance it falls in 2 s, (c) its velocity after falling 15 m, (d) the time it takes to fall 25 m, and (e) the time it takes to reach a velocity of 29.4 m/s.

Fig. 3.13



Solution: (a) Since the positive y is downward, then the ball's acceleration is positive (downward) and will be given by $a = g = 9.8$ m/s^2. Also, the ball's velocity will be always positive.

(b) We are given $v_\circ = 0$, $y_\circ = 0$, $a = g = 9.8$ m/s^2, and $t = 2$ s. To find y, we use $y - y_\circ = v_\circ\,t + \frac{1}{2}\,g\,t^2$ as follows:

$$y = 0 + \tfrac{1}{2}\,(9.8\text{ m/s}^2) \times (2\text{ s})^2 = 19.6\text{ m}$$

(c) We are given $v_\circ = 0$, $y_\circ = 0$, $a = g = 9.8$ m/s^2, and $y = 15$ m. To find v, we use $v^2 = v_\circ^2 + 2\,g(y - y_\circ)$ as follows:

$$v^2 = 0 + 2 \times (9.8\text{ m/s}^2) \times 15\text{ m} \quad\Rightarrow\quad v = \pm\sqrt{294}\text{ m/s} \quad\Rightarrow\quad v = 17.2\text{ m/s}$$

(d) We are given $v_\circ = 0$, $y_\circ = 0$, $y = 25$ m, and $a = g = 9.8$ m/s^2. To find t, we use $y - y_\circ = v_\circ\,t + \frac{1}{2}\,g\,t^2$ as follows:

$$25 = 0 + \tfrac{1}{2}\,(9.8\text{ m/s}^2) \times t^2 \quad\Rightarrow\quad t = \pm\sqrt{5.1}\text{ s} \quad\Rightarrow\quad t = 2.3\text{ s}$$

(e) We are given $v_\circ = 0$, $v = 29.4$ m/s, and $a = g = 9.8$ m/s^2. To find t, we use $v = v_\circ + g\,t$ as follows:

$$t = \frac{v - v_\circ}{g} = \frac{29.4\text{ m/s} - 0\text{ m/s}}{9.8\text{ m/s}^2} = 3\text{ s}$$

Example 3.8

A boy throws a ball upwards, giving it an initial speed $v_\circ = 15$ m/s. Neglect air resistance. (a) How long does the ball take to return to the boy's hand? (b) What will be its velocity then?

Solution: (a) We choose the positive y upward with its origin at the boy's hand, i.e. $y_\circ = 0$, see Fig. 3.14. Then, the ball's acceleration is negative (downward) during the ascending and descending motions, i.e. $a = -g = -9.8$ m/s^2. When the ball returns to the boy's hand its position y is zero. Since $v_\circ = 15$ m/s, $y_\circ = 0$, $y = 0$, and $a = -g$, then we can find t from $y - y_\circ = v_\circ\,t - \frac{1}{2}\,g\,t^2$ as follows:

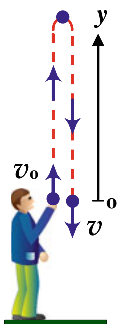
$$0 = (15\text{ m/s})\,t - \tfrac{1}{2}\,(9.8\text{ m/s}^2) \times t^2 \quad \Rightarrow \quad t = \frac{2 \times (15\text{m/s})}{9.8\text{ m/s}^2} = 3.1\text{ s}$$

(b) We are given $v_\circ = 15$ m/s, $y_\circ = 0$, $y = 0$, and $a = -g = -9.8$ m/s^2. To find v, we use $v^2 = v_\circ^2 - 2\,g(y - y_\circ)$ as follows:

$$v^2 = v_\circ^2 - 0 \quad \Rightarrow \quad v = \pm\sqrt{v_\circ^2} = \pm v_\circ = \pm 15\text{ m/s}$$

We should select the negative sign, because the ball is moving downward just before returning to the boy's hand, i.e. $v = -15$ m/s.

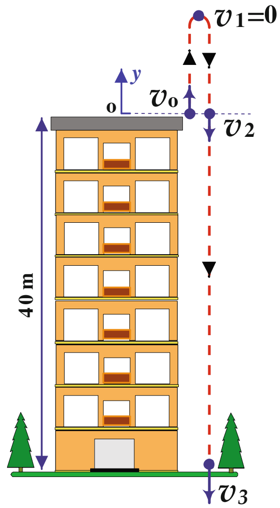
Fig. 3.14



Example 3.9

A ball is thrown upward from the top of a building with an initial velocity $v_\circ = 20$ m/s. The building is 40 m high and the ball just misses the edge of the building roof on its way down; see Fig. 3.15 and take $g = 10$ m/s^2. Neglecting air resistance, find: (a) the time t_1 for the ball to reach its highest point, (b) how high will it rise, (c) how long will it take to return to its starting point, (d) the velocity v_2 of the ball at this instant, and (e) the velocity v_3 and the total time of flight t_3 just before the ball hits the ground.

Fig. 3.15



Solution: (a) We choose upward as positive, i.e. $a = -g = -10$ m/s^2 during ascending and descending motions. Also, we choose the origin at the top of the building, i.e. $y_\circ = 0$, see Fig. 3.15. Since at the maximum height the ball stops momentarily, we use $v_\circ = 20$ m/s and $v_1 = 0$ in $v_1 = v_\circ - g\,t_1$ to find t_1 as follows:

$$0 = 20 \text{ m/s} - (10 \text{ m/s}^2)\ t_1$$

$$t_1 = \frac{20 \text{ m/s}}{10 \text{ m/s}^2} = 2 \text{ s}$$

(b) For the maximum height, we use the notation $y_1 \equiv y_{\max}$. To find the maximum height from the position of the thrower, we use the formula $y_{\max} - y_\circ = v_\circ\ t_1 - \frac{1}{2}\ g\ t_1^2$ as follows:

$$y_{\max} = (20 \text{ m/s}) \times (2 \text{ s}) - \tfrac{1}{2}(10 \text{ m/s}^2) \times (2\text{s})^2 = 20 \text{ m}$$

(c) When the ball returns to its starting point, the y coordinate is zero again, i.e. $y_2 = 0$. To find t_2 we use $y_2 - y_\circ = v_\circ\, t_2 - \frac{1}{2}\, g\, t_2^2$ as follows (after omitting the units temporarily, since they are consistent):

$$0 = 20\, t_2 - \tfrac{1}{2} \times 10 \times t_2^2$$

This equation can be factored to give:

$$t_2[20 - 5\, t_2] = 0$$

One solution is $t_2 = 0$, which corresponds to the time that the ball starts its motion. The other solution is $t_2 = 4\,\text{s}$, which is the solution we are after. Thus:

$$t_2 = 4\ \text{s}$$

(d) The value $t_2 = 4$ s found in part (c) can be inserted into the formula $v_2 = v_\circ - g\, t_2$ as follows:

$$v_2 = 20\ \text{m/s} - (10\ \text{m/s}^2) \times (4\ \text{s}) = -20\ \text{m/s}$$

Note that the velocity of the ball when it returns to its starting point is equal in magnitude to its initial velocity but opposite in direction. This indicates that the motion is symmetric, and generally we have:

$$v_2 = -v_\circ$$

(e) When the ball reaches the ground, its position is $y_3 = -40$ m. We can insert this value in $v_3^2 = v_\circ^2 - 2\, g(y_3 - y_\circ)$ to find v_3 as follows:

$$v_3^2 = (20\ \text{m/s})^2 - 2 \times (10\ \text{m/s}^2)[(-40\ \text{m}) - 0] = 1{,}200\,\text{m}^2/\text{s}^2$$

Thus:

$$v_3 = \pm\sqrt{1{,}200\,\text{m}^2/\text{s}^2} = \pm 34.64\ \text{m/s}$$

Since the ball is moving downward, we choose the negative value. Thus:

$$v_3 = -34.64\ \text{m/s}$$

To find the total time of flight t_3, we use $v_3 = v_\circ - g\, t_3$ as follows:

$$t_3 = \frac{v_\circ - v_3}{g} = \frac{(20\ \text{m/s}) - (-34.64\ \text{m/s})}{10\ \text{m/s}^2} = \frac{54.64\ \text{m/s}}{10\ \text{m/s}^2} = 5.5\ \text{s}$$

3.7 Exercises

Section 3.2 Average Velocity and Average Speed

(1) A runner on a straight track covers 1 km in 4 minutes. What is his average velocity in: (a) km/min, (b) km/s, and (c) km/h?

(2) A car travels in the positive x direction for 20 km at 40 km/h. It then continues in the same direction for another 20 km at 80 km/h. (a) What is the average velocity of the car during this 40 km trip? (b) What is its average speed?

(3) Suppose the motion of the particle in Fig. 3.2 is described by the equation $x = a + b\,t^2$, where $a = 10$ m and $b = 2$ m/s^2. (a) Find the displacement of the particle in the time interval between $t_i = 2$ s and $t_f = 4$ s. (b) Find the average velocity and the average speed during this interval of time.

(4) On an average, an eye blink lasts 100 ms. How far does a rocket moving with an average speed of $\bar{s} = 3{,}600$ km/h, see Fig. 3.16, travel during a pilot's blink?

Fig. 3.16 See Exercise (4)

(5) A graph of position (in meters) versus time (in seconds) for a boy traveling in the positive x direction is displayed in the Fig. 3.17. Find the average velocity for the following cases: (a) $t_i = 2$ s and $t_f = 4$ s, (b) $t_i = 2$ s and $t_f = 6$ s, and (c) $x_i = 12$ m and $x_f = 30$ m.

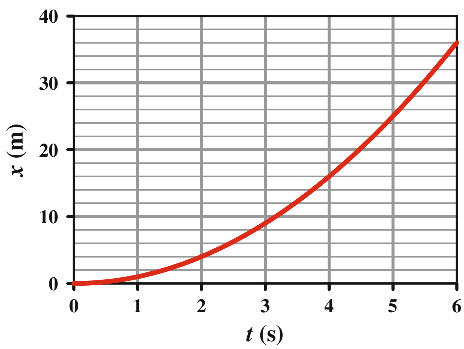


Fig. 3.17 See Exercise (5)

(6) A body moves along a straight line with position given by $x = 8\,t - 2\,t^2$, where x is in meters and t is in seconds. Find the average velocity and average speed of the body in the intervals: (a) from $t_i = 0$ to $t_f = 2$ s, and (b) from $t_i = 0$ to $t_f = 5$ s.

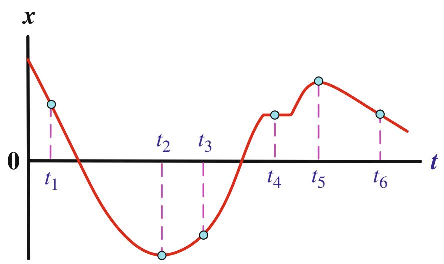
Section 3.3 Instantaneous Velocity and Speed

(7) The position of a plane during take-off along a straight runway is given by $x = k\,t^2$, where $k = 1.2\,\text{m/s}^2$,is measured in meters, and t is in seconds. (a) Find the displacement and the average velocity of the plane in the time intervals $0 \le t \le 4$ s and $4\text{ s} \le t \le 10$ s. (b) Find the velocity of the plane at $t = 4$ s and at $t = 10$ s.

(8) A particle moves along the x-axis according to the relation $x = 6 - 6\,t + t^2$, where x is measured in meters and t is measured in seconds. (a) Find the values of x for $t = 1, 2, 3, 4,$ and 5 s. (b) Find the values of the velocity v for $t = 1, 2, 3, 4,$ and 5 s. (c) For each value of t indicate whether the particle is moving toward an increasing or decreasing x. (d) Is there ever an instant when the velocity is zero? (e) Is there a time after $t = 5$ s when the particle is moving toward decreasing x?

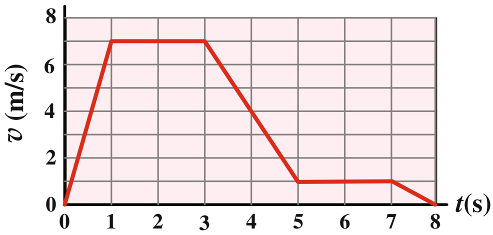
(9) The position-time graph for a particle moving along the x-axis is shown in the Fig. 3.18. Determine whether the velocity is positive, negative, or zero at the times t_1, t_2, t_3, t_4, t_5, and t_6.

Fig. 3.18 See Exercise (9)



(10) The graph of Fig. 3.19 shows the velocity of a runner plotted as a function of time. What is the interval where the velocity of the runner: (a) increases rapidly, (b) decreases rapidly, and (c) stays constant?

Fig. 3.19 See Exercise (10)



(11) How far does the runner whose $v - t$ graph is shown in the previous exercise travel in 8 s if at $t = 0$ the runner is at $x = 0$?

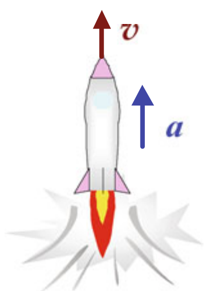
Section 3.4 Acceleration

(12) A particle is moving along the x-axis with velocity $v_{\mathrm{i}} = 50$ m/s at $t_{\mathrm{i}} = 0$. Its velocity decreases uniformly and reaches $v_{\mathrm{f}} = 0$ at $t_{\mathrm{f}} = 10$ s. What was the average acceleration during this 10 s interval?

(13) A car moving along the x-axis has a position given by the formula $x = 6 + 8\,t + 2\,t^2$, where x is measured in meters and t is in seconds. (a) Find the car's instantaneous velocity as a function of time. (b) Find its instantaneous acceleration as a function of time. (c) What will its velocity and acceleration be at $t = 5$ s?

(14) The velocity of a rocket during the first 6 s of its initial launch stage, see the Fig. 3.20, is given by $v = 20\,t - 0.4\,t^2$, where v is measured in meter/second and t is measured in seconds. (a) Find the average acceleration of the rocket from $t_{\mathrm{i}} = 0$ to $t_{\mathrm{f}} = 1$ s, and from $t_{\mathrm{i}} = 5$ s to $t_{\mathrm{f}} = 6$ s. (b) Find the acceleration a of the rocket at any time t during the interval $0 \le t \le 6$ s.

Fig. 3.20 See Exercise (14)



(15) Using the formula for the velocity given in the previous exercise, find the position x of the rocket at any time t during the interval $0 \leq t \leq 6$ s. Then, find the values of the position, velocity, and acceleration at $t = 0$, $t = 3$ s, and $t = 6$ s.

(16) A particle has $x = 0$ at $t = 0$ and its velocity as a function of time is shown in the Fig. 3.21. (a) Sketch the acceleration as a function of time. (b) Find the average acceleration of the particle in the time interval $t_i = 0$ to $t_f = 5$ s. (c) Find the acceleration of the particle at $t = 4$ s.

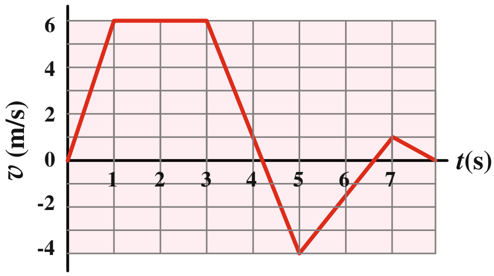


Fig. 3.21 See Exercise (16)

(17) A particle in one-dimensional motion has a velocity at any instant of time t given by $v = 6 + 4\,t + 3\,t^2$. (a) Find the initial velocity when $t = 0$. (b) Find the velocity when 2 s have passed. (c) Find the expression for the acceleration, and then its value when 2 s have elapsed. (d) Find the expression for the displacement $\Delta x = x - x_\circ$.

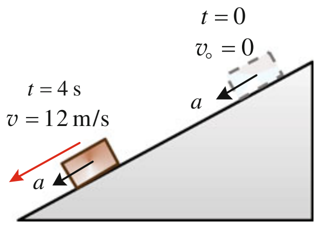
Section 3.5 Constant Acceleration

(18) An object starts from rest and moves with constant acceleration of 4 m/s^2. Find its speed and the distance it has traveled after 5 s have elapsed.

(19) A box slides down an incline with a uniform acceleration, see Fig. 3.22. It starts from rest and attains a speed of 12 m/s in 4 s. Find: (a) the acceleration, and (b) the distance moved in the first 4 s.

(20) A plane starts from rest and accelerates uniformly along a straight runway before takeoff. If the plane moves 1 km in 10 s, then find: (a) the acceleration, (b) the speed at the end of the 10 s period, (c) the distance moved in the first 20 s.

Fig. 3.22 See Exercise (19)



(21) A particle moving at 25 m/s in a straight line slows uniformly at a rate of 2 m/s every second. In an interval of 10 s, find: (a) the acceleration, (b) the final velocity, (c) the distance moved.

Section 3.6 Free Fall

(22) A stone strikes the ground with a speed of 25 m/s. (a) From what height was it released? (b) How long was it falling? (c) If the stone is thrown down with a speed of 10 m/s from the same height, then what will be its speed just before hitting the ground?

(23) A ball is thrown upward with a speed of 19.6 m/s. (a) How high does it go until its upward speed decreases to zero? (b) How long does the ball take in this upward trip? (c) How long does the ball take to return to the initial position? (d) What will be its velocity then?

(24) A bottle is dropped from a bridge and strikes the water after 5 s. (a) Find the speed of the bottle when it strikes the water. (b) Find how high the bridge is located above the water level.

(25) A sandbag dropped from a balloon reaches the ground in 5 s, see Fig. 3.23. Find the height of the balloon if: (a) it was at rest in the air, (b) it was ascending with a speed of 10 m/s when the sandbag was dropped, (c) it was descending with a speed of 10 m/s when the sandbag was dropped.

(26) A ball is thrown vertically *downward* from the edge of a cliff with an initial speed of 23 m/s. After a period of 1.4 s has elapsed, find: (a) how fast is it moving? and (b) how far has it moved?

(27) A ball is thrown vertically *upward* from the edge of a building with an initial velocity of 23 m/s. After a period of 1.4 s has elapsed, find: (a) how fast is it moving? and (b) how far has it moved?

(28) A ball is thrown vertically upward with a speed of 50 m/s from a building 20 m high, see Fig. 3.24. Find:(a) the time t_1 for the ball to reach the highest point,

(b) how high it will rise, (c) how long it will take to return to the starting point, (d) the velocity v_2 of the ball at this instant, (e) the velocity v_3 with which the ball strikes the ground, and (f) the total time of flight t_3.

Fig. 3.23 See Exercise (25)

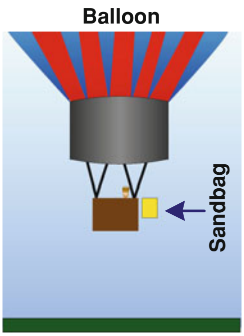
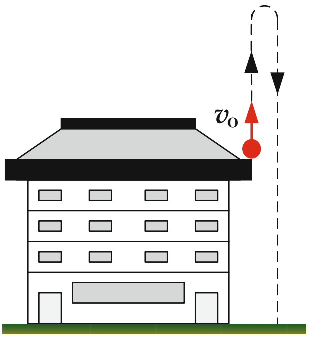


Fig. 3.24 See Exercise (28)



(29) A child drops balls from a bridge at regular intervals of 1 s, see Fig. 3.25. At the moment the fourth ball is released, the first strikes the water. (a) How high is the bridge? (b) How far above the water are each of the falling balls at this moment: (c) If the child decided to drop a ball once the previous one has reached the water surface, how long should he wait between every ball drop?

(30) A rubber ball is released from a height of 2 m above the floor, see the Fig. 3.26. The ball bounces repeatedly, always rising to 1/2 of the height through which it falls. Treat the ball as a particle that bounces an infinite number of times and take $g = 10\ \text{m/s}^2$. (a) What is the average speed of the ball during the first fall? (b) Show that its total distance traveled during an infinite number of bounces is 6 m? (c) Show that the total elapsed time for an infinite number of bounces

is $2[\sqrt{2}+1]^2/\sqrt{10}$ s. (d) Find the average speed from the time of release to the end of the infinite number of bounces.
[Hint: Use the binomial series $(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots,\ |x| < 1$]

Fig. 3.25 See Exercise (29)

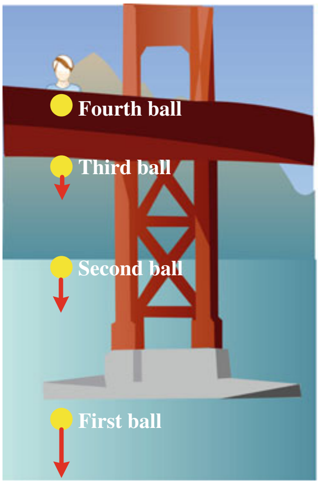
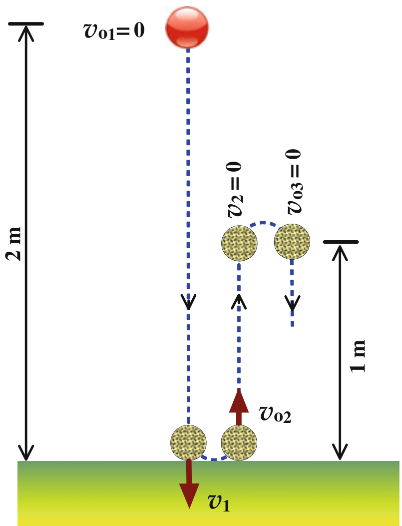


Fig. 3.26 See Exercise (30)



(31) An acrobat jumps straight up in the air and his center of mass (CM) took 0.2 s from the moment he just left the ground to the moment he just reached the

highest point, see the Fig. 3.27. Neglecting air resistance, (a) what was his initial vertical velocity just before his legs left the ground, (b) how high did his CM rise above the ground, and (c) what will be his velocity just before touching the ground in his way back?

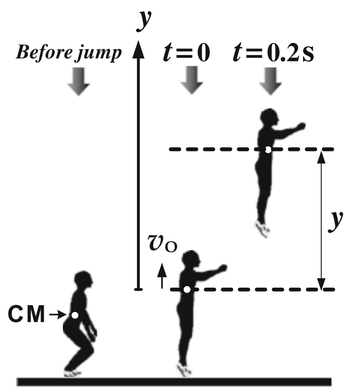


Fig. 3.27 See Exercise (31)

(32) Show that the vertical trajectory of a particle thrown upward is symmetric about its maximum when we neglect air resistance. That is, its height above the ground at time Δt before reaching its maximum equals its height above the ground after the same time interval Δt measured after reaching its maximum.

(33) A student drops a dartboard to the ground from a window, i.e. $v_{ob} = 0$. One second after dropping the board, he throws a dart at the board with initial speed of 20 m/s in order to score just before the board reaches the ground. See Fig. 3.28 and take $g = 10\ \text{m/s}^2$. (a) Find the time of flight T of the dartboard. (b) Find the height of the window. (c) Find the velocity of both the dart and the dartboard just before hitting the ground.

(34) The remote-controlled truck shown in Fig. 3.12 is used to pick up a package from a shelf in a factory. From rest and at $t = 0$, the truck accelerate at a_1 for a time interval t_1, then travels with constant speed for a time interval t_2, and finally decelerate at $-a_3$ for a time interval t_3. Show that the total distance traveled by the truck is $a_1 t_1 (t_2 + t_3) + \frac{1}{2}(a_1 t_1^2 - a_3 t_3^2)$.

(35) A diver drops his body from a diving board at a distance H above the water's surface into a deep swimming pool. The diver's motion stops at a distance h below the surface of the water. By choosing the downward direction to be

positive, see Fig. 3.29, prove that the average acceleration of the diver while he is under the water is $\overline{a} = -(H/h)g$.

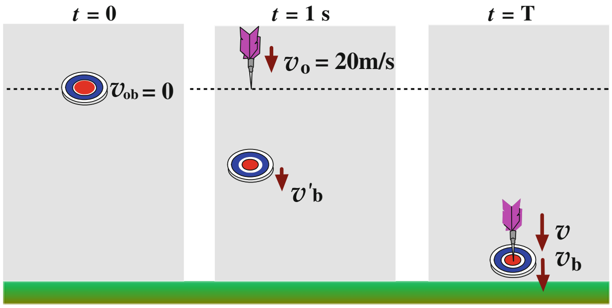


Fig. 3.28 See Exercise (33)

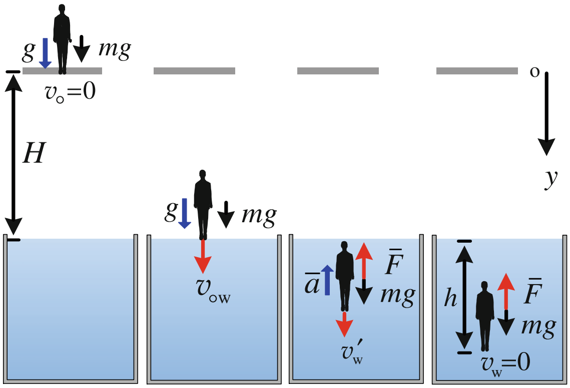


Fig. 3.29 See Exercise (35)

Motion in Two Dimensions

4

This chapter extends the study of the preceding chapter to two dimensions. We divide the study into two parts: motion of a particle in a plane, and circular motion of a particle in a plane.

4.1 Position, Displacement, Velocity, and Acceleration Vectors

The Position Vector

We describe the position of a particle with the position vector $\vec{r}$, which is a vector that extends from the origin of a certain coordinate system to the particle. Using the unit vector notation of Chap. 2, $\vec{r}$ can be written in two-dimensional form as:

$$\boxed{\vec{r} = x\,\vec{\mathrm{i}} + y\,\vec{\mathrm{j}}} \tag{4.1}$$

where $x\,\vec{\mathrm{i}}$ and $y\,\vec{\mathrm{j}}$ are the vector components of $\vec{r}$ along the x and y axes respectively, and the coefficients x and y are the scalar components, i.e., the particle has the rectangular coordinates (x, y). In three dimensions the position vector becomes $\vec{r} = x\,\vec{\mathrm{i}} + y\,\vec{\mathrm{j}} + z\,\vec{\mathrm{k}}$.

The Displacement Vector

Now, consider a particle moving in the xy plane as shown in Fig. 4.1. At point P, let its position be $\vec{r}_{\mathrm{i}}$ when the time was t_{i}. At point Q, let its position be $\vec{r}_{\mathrm{f}}$ when the time was t_{f} (the indices i and f refer to the initial and final values for our study). Accordingly, during the time interval $\Delta t = t_{\mathrm{f}} - t_{\mathrm{i}}$, the particle's displacement is:

$$\boxed{\Delta\vec{r} = \vec{r}_{\mathrm{f}} - \vec{r}_{\mathrm{i}}} \tag{4.2}$$

H. A. Radi and J. O. Rasmussen, *Principles of Physics*,
Undergraduate Lecture Notes in Physics, DOI: 10.1007/978-3-642-23026-4_4,

That is, the displacement vector $\Delta\vec{r}$ equals the difference between the final and initial position vectors. As seen from Fig. 4.1, the magnitude of the displacement vector is less than the distance traveled along the curved path, which was the particle's actual path of motion.

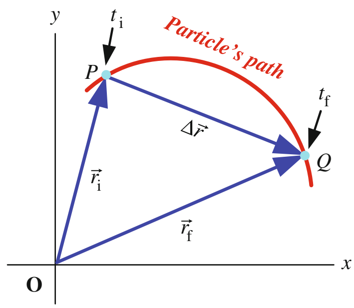


Fig. 4.1 The displacement $\Delta\vec{r} = \vec{r}_f - \vec{r}_i$ of a particle moving in a plane as it moves from P to Q during the time interval $\Delta t = t_f - t_i$

Average Velocity

One of several quantities associated with the phrase "how fast" a particle moves is the average velocity, $\bar{v}$, which is defined as follows:

Average velocity
The average velocity, $\overline{\vec{v}}$, of a particle is defined as the ratio of its displacement, $\Delta\vec{r}$, to the time interval, Δt. That is:

$$\overline{\vec{v}} = \frac{\Delta\vec{r}}{\Delta t} = \frac{\vec{r}_f - \vec{r}_i}{t_f - t_i} \tag{4.3}$$

From this definition, $\overline{\vec{v}}$, has the dimension of length divided by time, that is m/s in SI units. It is also a vector quantity that has a magnitude and direction along the displacement vector $\Delta\vec{r}$.

Instantaneous Velocity

Consider the motion of a particle between the two points P and Q in the xy plane, see Fig. 4.2. As the point Q is brought closer and closer to point P (through points $Q_1, Q_2, \ldots$), the time intervals $(\Delta t_1, \Delta t_2, \ldots)$ get progressively smaller. The average velocity for each time interval is directed along the displacement vector. As Q approaches P, the time interval approaches zero, and the direction of the instantaneous velocity $\vec{v}$, which is the direction of the displacement vector, approaches the direction of the tangent at P. We define $\vec{v}$ as follows:

Instantaneous velocity
The instantaneous velocity, $\vec{v}$, of a particle is defined as the limiting value of the ratio $\Delta\vec{r}/\Delta t$ as Δt approaches zero, i.e.

$$\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} \tag{4.4}$$

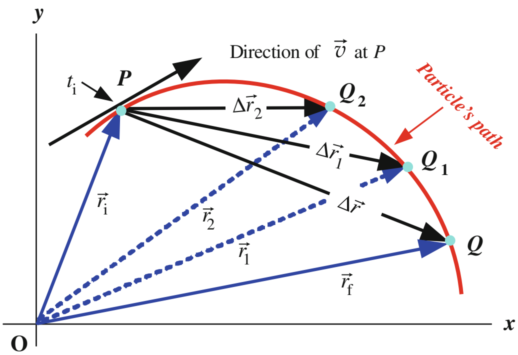


Fig. 4.2 The average velocity is in the direction of $\Delta\vec{r}$. As Q approaches P, the direction of $\Delta\vec{r}$ and hence the direction of the instantaneous velocity $\vec{v}$ approaches the tangent line to the curve at P

In calculus notation, the above limit is called the derivative of $\vec{r}$ with respect to t, and written as $d\vec{r}/dt$ (simplified as $\dot{\vec{r}}$). Thus:

$$\vec{v} = \frac{d\vec{r}}{dt} \quad \equiv \quad \vec{r}_f - \vec{r}_i = \int_{t_i}^{t_f} \vec{v}\; dt \tag{4.5}$$

From here on, we use the word *velocity* to designate *instantaneous velocity*, and *speed* is defined as the magnitude of that velocity.

In unit vector notation, the position vector can be written in the form $\vec{r} = x\,\vec{i} + y\,\vec{j}$, and hence we get:

$$\begin{aligned}\vec{v} &= \frac{d\vec{r}}{dt} = \frac{d(x\,\vec{i} + y\,\vec{j})}{dt} \\ &= \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j}\end{aligned} \tag{4.6}$$

or

$$\vec{v} = v_x \vec{i} + v_y \vec{j} \tag{4.7}$$

where the two components of the velocity vector are given by:

$$\begin{aligned} v_x &= \frac{dx}{dt}, \\ v_y &= \frac{dy}{dt} \end{aligned} \tag{4.8}$$

Figure 4.3 shows a velocity vector $\vec{v}$ and its scalar components for a particle moving in two dimensions.

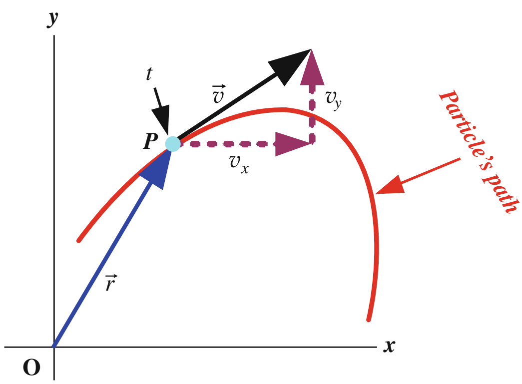


Fig. 4.3 The velocity $\vec{v}$ of a particle at point P along with its scalar components v_x and v_y

We should notice that, in a three dimensional study, the velocity vector can be written in the general form $\vec{v} = v_x \vec{i} + v_y \vec{j} + v_z \vec{k}$.

Average Acceleration

As the particle moves from P to Q along a certain path in the xy plane as in Fig. 4.4, its velocity changes from $\vec{v}_i$ at time t_i to $\vec{v}_f$ at time t_f.

We define the *average acceleration* as:

Average acceleration
The average acceleration, $\overline{\vec{a}}$, of a particle is defined as the ratio of the change in velocity $\Delta\vec{v} = \vec{v}_f - \vec{v}_i$ to the time interval $\Delta t = t_f - t_i$. That is:

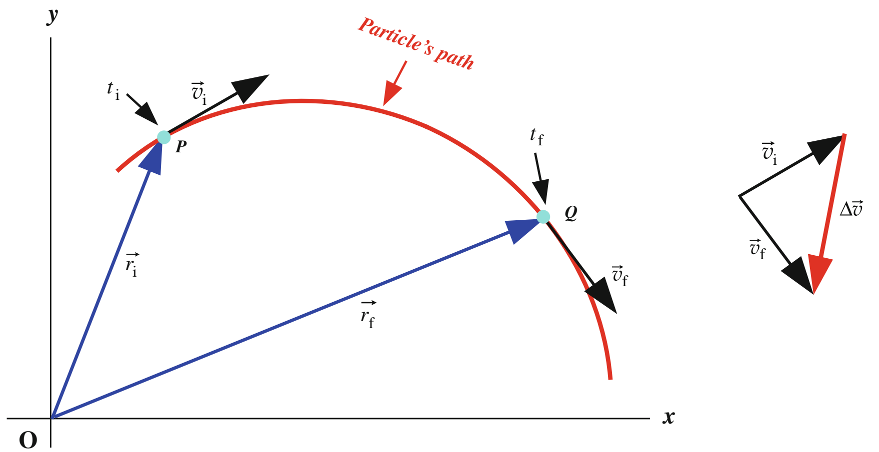


Fig. 4.4 The average acceleration $\overline{\vec{a}}$ for a particle moving from P to Q is in the direction of the change in velocity $\Delta\vec{v} = \vec{v}_f - \vec{v}_i$ shown in the right side of the figure

$$\overline{\vec{a}} = \frac{\Delta\vec{v}}{\Delta t} = \frac{\vec{v}_f - \vec{v}_i}{t_f - t_i} \tag{4.9}$$

Since Δt is a scalar quantity, the direction of $\overline{\vec{a}}$ is in the direction of the change in velocity $\Delta\vec{v} = \vec{v}_f - \vec{v}_i$.

Instantaneous Acceleration

It is useful to define instantaneous acceleration as the limit of the average acceleration when Δt approaches zero. When we consider the motion of the particle between the two points P and Q of the graph shown in Fig. 4.4, we see that as point Q approaches P, the time interval approaches zero, and we define the instantaneous acceleration $\vec{a}$ as follows:

> Instantaneous acceleration
> The instantaneous acceleration, $\vec{a}$, of a particle is defined as the limiting value of the ratio $\Delta\vec{v}/\Delta t$ when Δt approaches zero. Mathematically $\vec{a}$ can be expressed as:

$$\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta\vec{v}}{\Delta t} \tag{4.10}$$

In calculus notation, the above limit is called the *first derivative* of $\vec{v}$ with respect to t, and written as $d\vec{v}/dt$ (simplified sometimes as $\dot{\vec{v}}$), or the *second derivative* of $\vec{r}$ with respect to t, and written as $d^2\vec{r}/dt^2$ (simplified sometimes as $\ddot{\vec{r}}$). Thus:

$$\boxed{\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}} \equiv \boxed{\vec{v}_\mathrm{f} - \vec{v}_\mathrm{i} = \int_{t_\mathrm{i}}^{t_\mathrm{f}} \vec{a}\, dt} \tag{4.11}$$

From here on, we use the word *acceleration* to designate *instantaneous acceleration*.

In unit vector notation, we use $\vec{v} = v_x\vec{\mathrm{i}} + v_y\vec{\mathrm{j}}$, so that:

$$\boxed{\begin{aligned}\vec{a} &= \frac{d\vec{v}}{dt} = \frac{d(v_x\vec{\mathrm{i}} + v_y\vec{\mathrm{j}})}{dt}\\ &= \frac{dv_x}{dt}\vec{\mathrm{i}} + \frac{dv_y}{dt}\vec{\mathrm{j}}\end{aligned}} \tag{4.12}$$

or

$$\boxed{\vec{a} = a_x\vec{\mathrm{i}} + a_y\vec{\mathrm{j}}} \tag{4.13}$$

where the two components of the acceleration vector are given by:

$$\boxed{\begin{aligned}a_x &= \frac{dv_x}{dt},\\ a_y &= \frac{dv_y}{dt}\end{aligned}} \tag{4.14}$$

Figure 4.5 shows a general view of both the acceleration $\vec{a}$ and the velocity $\vec{v}$ for a particle moving in a plane. On the same figure, we display the scalar components of the acceleration vector $\vec{a}$.

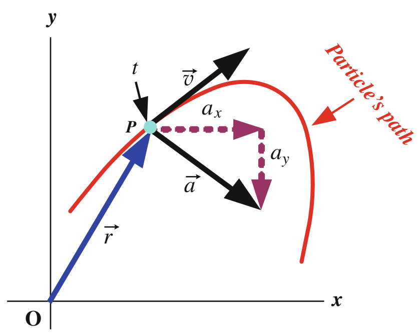


Fig. 4.5 A general view of the acceleration vector $\vec{a}$ of a particle at point P at a particular time t. The figure also displays the acceleration scalar components a_x and a_y, as well as the position vector $\vec{r}$ and the velocity vector $\vec{v}$

We should notice that in a three dimensional study, the acceleration vector will take the general form $\vec{a} = a_x \vec{\mathrm{i}} + a_y \vec{\mathrm{j}} + a_z \vec{\mathrm{k}}$.

A particle can accelerate for several reasons. One way is to change with time the magnitude of the velocity vector (called the speed) such as in one-dimensional motion. Another way is to change with time the direction of the velocity vector, as in circular motion. Finally, the acceleration may change due to a change in both the magnitude and the direction of the velocity vector.

Example 4.1

A particle moves over a path such that the components of its position with respect to an origin of coordinates are given as a function of time by:

$$x = -t^2 + 12\,t + 5$$
$$y = -2\,t^2 + 16\,t + 10$$

where t is in seconds and x and y are in meters. (a) Find the particle's position vector $\vec{r}$ as a function of time, and find its magnitude and direction at $t = 6$ s. (b) Find the particle's velocity vector $\vec{v}$ as a function of time, and find its magnitude and direction at $t = 6$ s. (c) Find the particle's acceleration vector $\vec{a}$ as a function of time, and find its magnitude and direction at $t = 6$ s.

Solution: (a) The position vector is given at time t by:

$$\vec{r} = x\vec{\mathrm{i}} + y\vec{\mathrm{j}} = (-t^2 + 12\,t + 5)\vec{\mathrm{i}} + (-2\,t^2 + 16\,t + 10)\vec{\mathrm{j}}$$

Figure 4.6 shows the variation of x and y as a function of time. At $t = 6$ s we have:

$$\vec{r} = x\vec{\mathrm{i}} + y\vec{\mathrm{j}} = 41\vec{\mathrm{i}} + 34\vec{\mathrm{j}}$$

The magnitude of $\vec{r}$ is:

$$r = \sqrt{x^2 + y^2} = \sqrt{41^2 + 34^2} = 53.26\,\mathrm{m}$$

The angle θ between $\vec{r}$ and the direction of increasing x is:

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{34\ \mathrm{m}}{41\ \mathrm{m}}\right) = \tan^{-1}(0.83) = 39.7^\circ$$

Figure 4.7 shows the path of the particle in the xy plane, and also shows its position vector $\vec{r}$ at $t = 6$ s.

Fig. 4.6

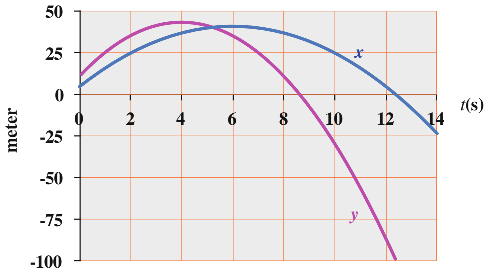
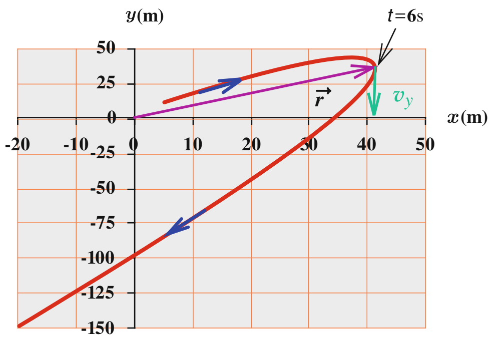


Fig. 4.7



(b) The velocity components along the x and y axes are:

$$v_x = \frac{dx}{dt} = \frac{d}{dt}(-t^2 + 12\,t + 5) = -2\,t + 12,$$

$$v_y = \frac{dy}{dt} = \frac{d}{dt}(-2\,t^2 + 16\,t + 10) = -4\,t + 16$$

At $t = 6$ s the components of $\vec{v}$ are:

$$v_x = 0, \quad v_y = -8\,\text{m/s}$$

That is, $\vec{v} = -8\,\vec{\mathrm{j}}$. The magnitude of $\vec{v}$ at this time is:

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{0 + (-8\,\text{m/s})^2} = 8\,\text{m/s}$$

Hence, the angle θ between $\vec{v}$ and the direction of increasing x is:

$$\theta = \tan^{-1}\left(\frac{v_y}{v_x}\right) = \tan^{-1}\left(\frac{-8\text{ m/s}}{0\text{ m/s}}\right) = \tan^{-1}(-\infty) = 270^\circ$$

where we used the fact that $\vec{v} = -8\,\vec{j}$ is a downward vector and its angle should be measured in a counterclockwise sense from the direction of increasing x, see the Fig. 4.7.

(c) The components of the acceleration along the x and y axes are:

$$a_x = \frac{dv_x}{dt} = \frac{d}{dt}(-2t + 12) = -2\,\text{m/s}^2,$$

$$a_y = \frac{dv_y}{dt} = \frac{d}{dt}(-4t + 16) = -4\,\text{m/s}^2$$

We see that the acceleration does not vary with time, i.e. it is a constant. We define the magnitude and direction of $\vec{a}$ as follows:

$$a = \sqrt{a_x^2 + a_y^2} = \sqrt{(-2\ \text{m/s}^2)^2 + (-4\ \text{m/s}^2)^2} = \sqrt{20\ (\text{m/s}^2)^2} = 4.47\,\text{m/s}^2$$

$$\theta = \tan^{-1}\left(\frac{a_y}{a_x}\right) = \tan^{-1}\left(\frac{-4\ \text{m/s}^2}{-2\ \text{m/s}^2}\right) = 180^\circ + \tan^{-1}(2)$$
$$= 180^\circ + 63.4^\circ = 243.4^\circ$$

where we used Table 2.1 to calculate θ for a negative a_x and a_y.

4.2 Projectile Motion

Any object that is thrown into the air is called a *projectile*. Near the Earth's surface, we assume that the downward acceleration due to gravity is constant and the effect of air resistance is negligible. Based on these two assumptions, we find that: (1) the horizontal motion and vertical motion are independent of each other and (2) the two dimensional path of a projectile (also called its *trajectory*) is always a parabola.

If we choose the axes of the xy plane such that the y axis is vertically upward, then $a_x = 0$ and $a_y = -g$ (as in one-dimensional free fall). Furthermore, let us assume that at $t = 0$ the projectile leaves the origin (i.e. $x_\circ = y_\circ = 0$) with initial velocity $\vec{v}_\circ$ that makes an angle $\theta_\circ$ with the positive x direction as in Fig. 4.8.

The projectile's initial velocity can be written as:

$$\boxed{\vec{v}_\circ = v_{x\circ}\,\vec{i} + v_{y\circ}\,\vec{j}} \tag{4.15}$$

The components $v_{x\circ}$ and $v_{y\circ}$ can then be found in terms of the initial speed $v_\circ$ and the launch angle $\theta_\circ$ as follows:

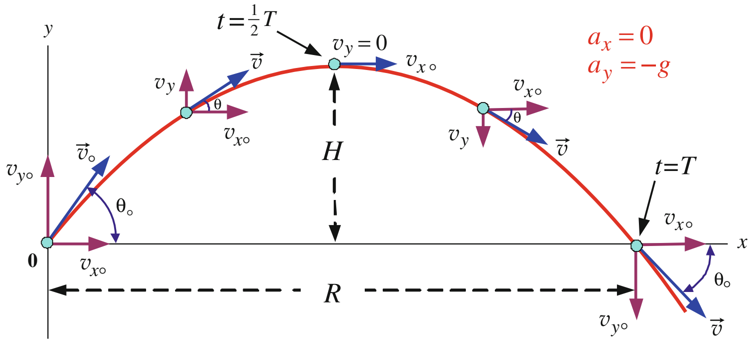


Fig. 4.8 The path of a projectile launched from the origin with initial velocity $\vec{v}_\circ$ makes an angle $\theta_\circ$ with the x axis. The horizontal velocity component $v_{x\circ}$ remains constant, but the vertical velocity component v_y changes continuously. At the peak of the parabolic trajectory $v_y = 0$ and y is maximum $(y = H)$. The horizontal range R is the distance traveled by the projectile when it returns to $y = 0$

$$\boxed{\begin{aligned} v_{x\circ} &= v_\circ \cos\theta_\circ, \\ v_{y\circ} &= v_\circ \sin\theta_\circ \end{aligned}} \tag{4.16}$$

Now, we decompose the horizontal motion and vertical motion as described below.

Horizontal Motion of a Projectile

Since $a_x = 0$, the horizontal velocity component $v_{x\circ}$ remains constant throughout the motion, as in Fig. 4.8. Thus, the horizontal velocity v_x and the horizontal position x are described as follows:

$$\boxed{v_x = v_{x\circ} = \text{constant}} \quad \Rightarrow \quad \boxed{v_x = v_\circ \cos\theta_\circ = \text{constant}} \tag{4.17}$$

$$\boxed{x = v_{x\circ}\, t} \quad \Rightarrow \quad \boxed{x = (v_\circ \cos\theta_\circ)t} \tag{4.18}$$

Vertical Motion of a Projectile

Since $a_y = -g$, then the vertical motion is the motion we discussed in Sect. 3.6 for a particle in free fall. Thus, the vertical velocity v_y and the vertical position y can be described as follows:

$$\boxed{v_y = v_{y\circ} - gt} \quad \Rightarrow \quad \boxed{v_y = v_\circ \sin\theta_\circ - gt} \tag{4.19}$$

$$\boxed{y = v_{y\circ}t - \tfrac{1}{2}gt^2} \quad \Rightarrow \quad \boxed{y = (v_\circ \sin\theta_\circ)t - \tfrac{1}{2}gt^2} \tag{4.20}$$

$$\boxed{v_y^2 = v_{y\circ}^2 - 2gy} \quad \Rightarrow \quad \boxed{v_y^2 = (v_\circ \sin\theta_\circ)^2 - 2gy} \tag{4.21}$$

As illustrated in Fig. 4.8, the vertical velocity component v_y and the coordinate y behave like those of an object that is thrown upwards. The initial upward velocity component steadily decreases reaching zero at the maximum height. The vertical component then reverses direction, and its magnitude becomes larger with time.

Horizontal Range of a Projectile

The horizontal range R is the distance traveled by the projectile when it returns to $y = 0$ after time $t = T$, as seen in Fig. 4.8. To find an expression for R we use Eq. 4.18 and set $x = R$ at time $t = T$. We also use Eq. 4.20 and set $y = 0$. Thus:

$$R = (v_\circ \cos\theta_\circ)T$$

$$0 = (v_\circ \sin\theta_\circ)T - \tfrac{1}{2}gT^2$$

From the last result, we get the following relation for T:

$$\boxed{T = \frac{2v_\circ \sin\theta_\circ}{g}} \tag{4.22}$$

Substituting with the expression of T in $R = (v_\circ \cos\theta_\circ)T$ and using the identity $\sin 2\theta_\circ = 2\,\sin\theta_\circ \cos\theta_\circ$, we get:

$$\boxed{R = \frac{v_\circ^2\,\sin 2\theta_\circ}{g}, \quad 0 \le \theta_\circ \le \pi/2} \tag{4.23}$$

The range is maximum when $\sin 2\theta_\circ = 1$, i.e. when $\theta_\circ = 45°$, and is the same for the two angles $\theta_\circ$ and $90° - \theta_\circ$ since $\sin 2\theta_\circ = \sin(180° - 2\theta_\circ)$.

Maximum Height of a Projectile

Since the trajectory is symmetric about the peak, then we can find the time $t = t_1 = T/2$ at the peak from Eq. 4.22. That is:

$$t_1 = \tfrac{1}{2}\,T = \frac{v_\circ \sin\theta_\circ}{g}$$

Note that we get the same answer when we set $v_y = 0$ in Eq. 4.19. Substituting with t_1 in Eq. 4.20 we get:

$$H = (v_\circ \sin\theta_\circ)\frac{v_\circ \sin\theta_\circ}{g} - \tfrac{1}{2}g\left(\frac{v_\circ \sin\theta_\circ}{g}\right)^2$$

Thus:

$$\boxed{H = \frac{v_\circ^2 \sin^2\theta_\circ}{2g}, \quad 0 \le \theta_\circ \le \pi/2} \tag{4.24}$$

Although $\theta_\circ$ and $90° - \theta_\circ$ are two angles that have the same range R, $\sin\theta_\circ$ is not equal to $\sin(90° - \theta_\circ)$. Therefore, the maximum height H is greater for the bigger angle. For the same initial speed $v_\circ$, Fig. 4.9 shows these properties by displaying three trajectories, where two of them are with the same range R when $\theta_\circ = 30°$ and $90° - \theta_\circ = 60°$, and the third trajectory has a maximum range $R_{\max}$ when $\theta_\circ = 45°$.

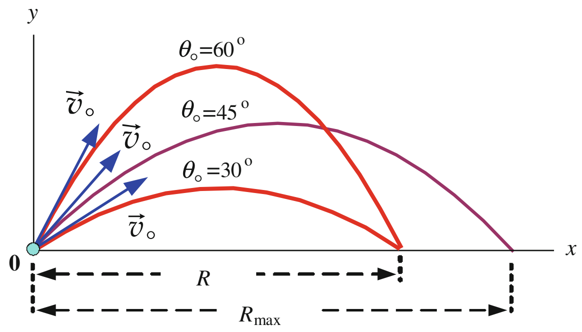


Fig. 4.9 The trajectories of a projectile fired from the origin with the same initial speed $v_\circ$. For the two angles $\theta_\circ = 30°$ and $\theta_\circ = 60°$, we get the same range R, but when $\theta_\circ = 45°$, we get the maximum range $R_{\max}$ for the projectile motion

Equation of the Trajectory

We can find the equation of the projectile's trajectory by solving Eq. 4.18 for t and substituting into Eq. 4.20. After performing a few manipulations, we get:

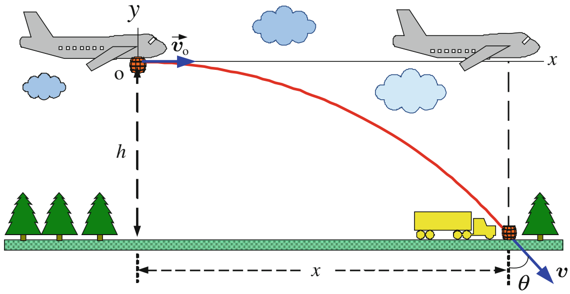
$$\boxed{y = \left(\frac{-g}{2v_\circ^2 \cos^2\theta_\circ}\right) x^2 + (\tan\theta_\circ)x, \quad 0 \le \theta_\circ < \pi/2} \tag{4.25}$$

This can be written in the form $y = ax^2 + bx$, which is the equation of a *parabola* that passes through the origin. The angle $\theta_\circ = 90°$ is excluded from Eq. 4.25 since it represents a projectile that is fired vertically up, i.e. the path of the projectile is a straight line.

Example 4.2

An airplane is flying horizontally with a constant speed $v_\circ = 400\,\text{km/h}$ at a constant elevation $h = 2\,\text{km}$ above the ground, see Fig. 4.10. (a) If the pilot decided to release a package of supplies very close to a truck on the ground, then what is the time of flight of the package? (b) What is the horizontal distance covered by the package in that time (which is the same horizontal distance covered by the plane)?

Fig. 4.10



Solution: (a) The initial velocity of the package is the same as the velocity of the plane. Therefore, the initial velocity of the package $\vec{v}_\circ$ is horizontal (i.e., $\theta_\circ = 0$) and has a magnitude of 400 km/h. Since we know the vertical distance that the package falls, then we find its time of flight from Eq. 4.20 as follows:

$$y = (v_\circ \sin\theta_\circ)\, t - \tfrac{1}{2} g t^2$$

Substituting with $y = -2{,}000\,\text{m}$ (we use the negative sign because the package is bellow the origin) and $\theta_\circ = 0$, we get:

$$-2{,}000\ \text{m} = 0 - \tfrac{1}{2}\,(9.8\,\text{m/s}^2) t^2$$

Solving for t and taking the positive root yields:

$$t = \sqrt{\frac{2{,}000\,\text{m}}{4.9\ \text{m/s}^2}} = 20.2\ \text{s}$$

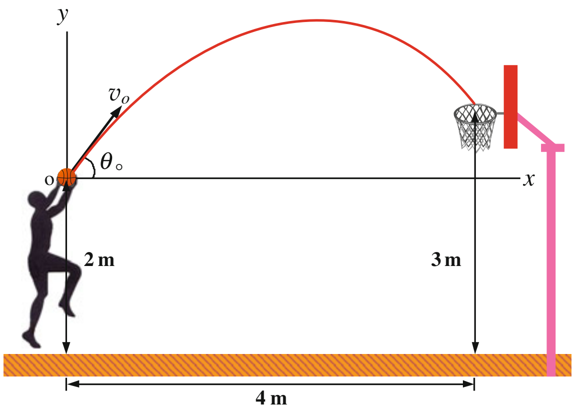
(b) The horizontal distance covered by the package in that time is:

$$\begin{aligned} x &= (v_\circ \cos\theta_\circ)\, t \\ &= (400\ \text{km/h})(1\ \text{h}/3600\ \text{s})(\cos 0^\circ)(20.2\ \text{s}) = 2.244\ \text{km} \\ &= 2{,}244\,\text{m} \end{aligned}$$

Example 4.3

A basketball player throws a ball at an angle $\theta_\circ = 60°$ above the horizontal, as shown in the Fig. 4.11. At what speed must he throw the ball to score?

Fig. 4.11



Solution: In terms of the unknown speed $v_\circ$, we must first use the horizontal Eq. 4.18 to find the time needed for the ball to reach the net after moving a horizontal distance of 4 m. Thus, after omitting the units temporarily since they are consistent, we get:

$$x = (v_\circ \cos\theta_\circ)t \quad \Rightarrow \quad 4 = v_\circ(\cos 60°)t = 0.5 v_\circ t \quad \Rightarrow \quad t = \frac{8}{v_\circ}$$

Since y is up, then $a = -g = -9.8\,\text{m/s}^2$ during ascending and descending motions. Also, choosing the origin of the xy plane to be at the player's hands makes $y_\circ = 0$. When the ball reaches the net, it gains vertical height $y = 1$ m. Then, by using Eq. 4.20 we get:

$$y = (v_\circ \sin\theta_\circ)t - \tfrac{1}{2}gt^2 \quad \Rightarrow \quad 1 = (v_\circ \sin 60°)\frac{8}{v_\circ} - \tfrac{1}{2} \times 9.8 \times \left(\frac{8}{v_\circ}\right)^2$$

or,

$$1 = 6.93 - \frac{313.6}{v_\circ^2}$$

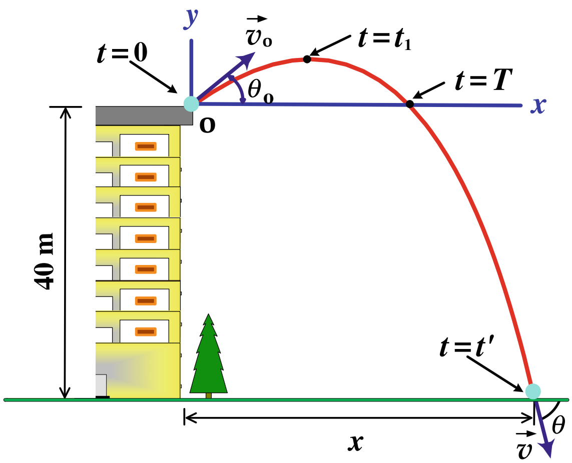
Thus, solving for $v_\circ$ and taking the positive root gives:

$$v_\circ = \sqrt{\frac{313.6}{6.93 - 1}} = 7.27\,\text{m/s}$$

Example 4.4

A ball thrown from the top of a building has an initial speed of 20 m/s at an angle of 30° above the horizontal. The building is 40 m high and the ball takes time t' before hitting the ground, see the Fig. 4.12. Take $g = 10\,\text{m/s}^2$. (a) Find the time t_1 for the ball to reach its highest point. (b) How high will it rise? (c) How long will it take to return to the level of the thrower? (d) Find the time of flight t'. (e) What is the horizontal distance covered by the ball during this time? (f) What is the velocity of the ball before striking the ground?

Fig. 4.12



Solution: (a) Since y is up, then $a = -g = -10\,\text{m/s}^2$ during ascending and descending motions. Also, since the origin is at the top of the building, then $y_\circ = 0$. The initial components of the velocity are:

$$v_{x\circ} = v_\circ \cos\theta_\circ = (20\,\text{m/s})\ (\cos 30°) = 17.32\,\text{m/s},$$

$$v_{y\circ} = v_\circ \sin\theta_\circ = (20\,\text{m/s})\ (\sin 30°) = 10\,\text{m/s}$$

Since at the maximum height the ball stops momentarily, we use $v_{y\circ} = 10\,\text{m/s}$ and $v_y = 0$ in $v_y = v_{y\circ} - g\ t$ to find t_1 as follows:

$$0 = 10\text{ m/s} - (10\text{ m/s}^2)\ t_1 \quad \Rightarrow \quad t_1 = \frac{10\text{ m/s}}{10\text{ m/s}^2} = 1\text{ s}$$

(b) To find the maximum height H from the position of the thrower, we use $t_1 = 1\,\text{s}$ and $y = v_{y\circ}t - \frac{1}{2}gt^2$, or Eq. 4.24, as follows:

$$H = (10 \text{ m/s}) \times (1 \text{ s}) - \tfrac{1}{2}(10 \text{ m/s}^2) \times (1 \text{ s})^2 = 5 \text{ m}$$

Thus, the maximum height of the ball from the ground is 45 m.

(c) When the ball returns to the level of the thrower, the y coordinate is zero again, i.e. $y = 0$, and $t = T$. To find the time T the ball takes to reach this location, we use $y = v_{y\circ}t - \frac{1}{2}gt^2$ as follows (after omitting the units temporarily since they are consistent):

$$0 = 10T - \tfrac{1}{2} \times 10 \times T^2 = (10 - 5T)T \quad \Rightarrow \quad T = 2\,\text{s}$$

We can also use $T = 2t_1 = 2\,\text{s}$ for the symmetric part of the path.

(d) To find t', we can use $y = v_{y\circ}t - \frac{1}{2}gt^2$ with $y = -40\,\text{m}$ and $v_{y\circ} = 10\,\text{m/s}$ so that (after omitting the units):

$$-40 = 10\,t' - 5\,t'^2 \quad \Rightarrow \quad 5\,t'^2 - 10\,t' - 40 = 0$$

Solving this quadratic equation yields:

$$t' = \frac{10 \pm \sqrt{(-10)^2 - 4 \times 5 \times (-40)}}{2 \times 5} = \frac{10 \pm 30}{10} \quad \Rightarrow \quad t' = \begin{cases} +4 \text{ s} \\ -2 \text{ s} \end{cases}$$

We reject the negative time and take only the positive root, i.e. $t' = 4\,\text{s}$.

(e) The horizontal distance x covered by the ball at $t' = 4\,\text{s}$ is:

$$\begin{aligned} x &= (v_\circ \cos\theta_\circ)\, t' = v_{x\circ}\, t' \\ &= (17.32 \text{ m/s})(4 \text{ s}) = 69.28 \text{ m} \end{aligned}$$

(f) The vertical component of the ball's velocity at $t' = 4\,\text{s}$ is given by:

$$v_y = v_{y\circ} - gt' = 10 \text{ m/s} - (10 \text{ m/s}^2)\ (4 \text{ s}) = -30 \text{ m/s}$$

The negative sign indicates that the vertical component is directed downwards. Since $v_x = v_{x\circ} = 17.32$ m/s, the required speed would be:

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{(17.32 \text{ m/s})^2 + (-30 \text{ m/s})^2} = 34.64 \text{ m/s}$$

The direction of $\vec{v}$ at $t' = 4\,\text{s}$ is indicated in the Fig. 4.12 by the angle θ. Thus, according to this figure we have:

$$\theta = \tan^{-1}\left(\frac{|v_y|}{v_x}\right) = \tan^{-1}\left(\frac{30 \text{ m/s}}{17.32 \text{ m/s}}\right) = \tan^{-1}(1.73) = 60^\circ$$

4.3 Uniform Circular Motion

A particle that moves around in a circle with a **constant speed**, like the car shown in Fig. 4.13a, is said to experience a *uniform circular motion*. In this case, the acceleration arises only from the change in the direction of the velocity vector.

We can use Fig. 4.13b to find the magnitude and direction of this acceleration. In this figure, the particle is seen first at point P with velocity $\vec{v}_i$ at time t_i and at point Q with velocity $\vec{v}_f$ at time t_f, where $\vec{v}_i$ and $\vec{v}_f$ are different only in direction, i.e. $v_i = v_f = v$. In order to calculate the acceleration we start with the average acceleration:

$$\overline{\vec{a}} = \frac{\Delta\vec{v}}{\Delta t} = \frac{\vec{v}_f - \vec{v}_i}{t_f - t_i} \tag{4.26}$$

where $\Delta\vec{v}$ can be accomplished graphically as shown in Fig. 4.13c.

The triangle OPQ in Fig. 4.13b, which has sides Δs and r, is similar to the triangle of Fig. 4.13c, which has sides Δv and v. This similarity enables us to write the following relationship:

$$\frac{\Delta v}{v} = \frac{\Delta s}{r} \tag{4.27}$$

Substituting with Δv from Eq. 4.27 into the magnitude form of Eq. 4.26 we get:

$$\overline{a} = \frac{\Delta v}{\Delta t} = \frac{v}{r}\frac{\Delta s}{\Delta t} \tag{4.28}$$

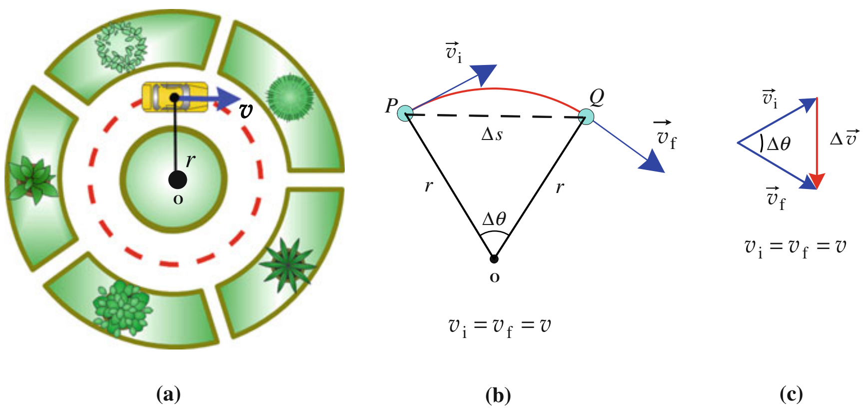


Fig. 4.13 (a) Circular motion with constant speed v. (b) Velocity vectors $\vec{v}_i$ and $\vec{v}_f$ at P and Q. (c) Graphical method to obtain $\Delta\vec{v}$

When Δt is very small, the two points P and Q of Fig. 4.8b becomes extremely close, and hence Δs and $\Delta\theta$ are very small too. In this limit, $\Delta\vec{v}$ would point toward the center of the circular path, and because the acceleration is in the direction of $\Delta\vec{v}$, it will also be toward the center. Consequently, in this limit the arc PQ ($PQ = r\,\Delta\theta$) will be equal to Δs and the ratio $\Delta s/\Delta t$ approaches the speed v. Thus, when $\Delta t \to 0$, the magnitude of the radial acceleration will be:

$$\boxed{a_\mathrm{r} = \frac{v^2}{r}}, \quad \text{(Radial acceleration)} \tag{4.29}$$

where the subscript "r" indicates that the acceleration of the particle is always toward the center of the circle. Because of this, the acceleration associated with uniform circular motion is called **centripetal acceleration** (meaning "center-seeking" acceleration).

Figure 4.14 shows the velocity and acceleration vectors at various stages of a body in circular motion, where both vectors have constant magnitude as the motion progresses. The velocity is always tangent to the circle and the acceleration is always directed toward the center.

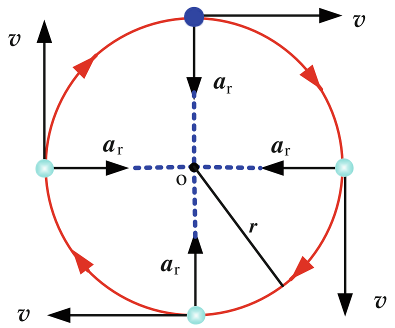


Fig. 4.14 Directional change of the velocity and acceleration vectors in uniform circular motion, where both have constant magnitude

In addition, during this circular motion with constant speed, the particle travels the circumference of the circle in a time T giving by:

$$\boxed{T = \frac{2\pi r}{v}} \tag{4.30}$$

where T is called the *period of revolution*, or simply the *period*.

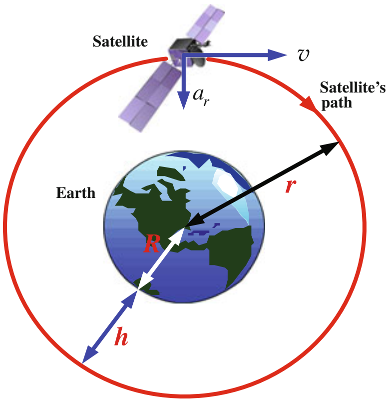
Example 4.5

A satellite is circulating the Earth at an altitude $h = 150$ km above its surface, where the free fall acceleration g is $9.4\,\text{m/s}^2$. The Earth's radius is 6.4×10^6 m. What is the orbital Speed and period of the satellite?

Solution: As shown in Fig. 4.15, the radius of the satellite's circular motion equals the sum of the Earth's radius R and the altitude h, i.e.

$$r = R + h$$

Fig. 4.15



By using the centripetal acceleration given by Eq. 4.29, we find that the magnitude of the satellite's acceleration can be written as:

$$a_\text{r} = \frac{v^2}{r} = \frac{v^2}{R + h}$$

For the uniform circular motion of the satellite around the Earth, the satellite's centripetal acceleration is then equal to the free fall acceleration g at this altitude. That is:

$$a_\text{r} = g = 9.4\,\text{m/s}^2$$

From the preceding two equations we have:

$$g = \frac{v^2}{R + h}$$

Solving for v and taking the positive root gives:

$$v = \sqrt{g(R+h)}$$
$$= \sqrt{(9.4\ \text{m/s}^2)(6.4 \times 10^6\ \text{m} + 150 \times 10^3\ \text{m})} = 7{,}847\ \text{m/s} \approx 28{,}000\ \text{km/h}$$

With this high speed, the satellite would take $T = 2\pi r/v = 1.46$ h to make one complete revolution around the Earth.

4.4 Tangential and Radial Acceleration

When the velocity of a particle changes in both direction and magnitude, the particle can move in a curved path as shown in Fig. 4.16. In this situation, the velocity $\vec{v}$ is always tangent to the path and usually the acceleration $\vec{a}$ makes an angle with the velocity. The vector $\vec{a}$ can be resolved into two component vectors: a tangential component vector, $\vec{a}_t$, and a radial component vector, $\vec{a}_r$. That is:

$$\boxed{\vec{a} = \vec{a}_r + \vec{a}_t} \tag{4.31}$$

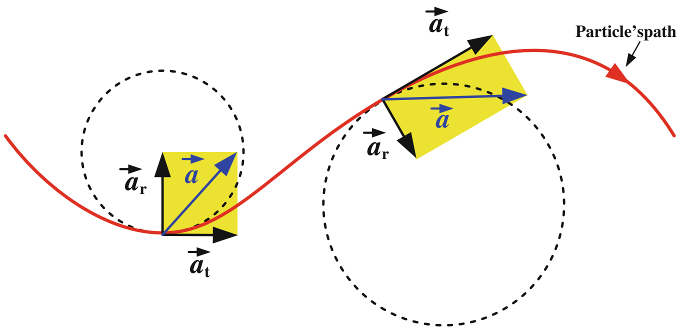


Fig. 4.16 When the velocity $\vec{v}$ of a particle changes in both direction and magnitude, the acceleration $\vec{a}$ can be decomposed to a radial component vector $\vec{a}_r$ and a tangential component vector $\vec{a}_t$

The *tangential acceleration* at a particular point arises from the time rate of the speed of the particle and has a magnitude given by:

$$\boxed{a_t = \frac{dv}{dt}} \quad \text{(Tangential acceleration)} \tag{4.32}$$

The *radial acceleration* at a particular point arises from the time rate of change in the direction of the velocity vector and has a magnitude:

$$a_r = \frac{v^2}{r} \quad \text{(Radial acceleration)} \tag{4.33}$$

where r is the particle's radius of curvature at the point in question.

Since $\vec{a}_t$ and $\vec{a}_r$ are perpendicular component vectors, then the *total* acceleration $\vec{a}$, its magnitude a, and its direction θ relative to the radius of curvature will be given by:

$$\vec{a} = \vec{a}_t + \vec{a}_r \qquad a = \sqrt{a_t^2 + a_r^2}\,, \quad \text{and} \quad \tan\theta = \frac{a_t}{a_r} \tag{4.34}$$

In the case of uniform circular motion, where v is constant, we have $a_t = dv/dt = 0$ and acceleration is always radial, i.e. $\vec{a} = \vec{a}_r$. Furthermore, if the direction of the velocity $\vec{v}$ does not change, then $a_r = 0$ and the motion will be in one dimension, i.e. $\vec{a} = \vec{a}_t$.

4.5 Non-uniform Circular Motion

A particle that moves around a circle with a **variable speed**, is said to experience a *non-uniform circular motion*. In this case, the total acceleration arises from the change in magnitude of $\vec{v}$ (represented by $\vec{a}_t$) and the change in direction of $\vec{v}$ (represented by $\vec{a}_r$), see Fig. 4.17a. Thus:

$$\vec{a} = \vec{a}_t + \vec{a}_r \tag{4.35}$$

To describe acceleration in terms of unit vectors, we consider the unit vectors $\hat{\vec{r}}$ and $\hat{\vec{\theta}}$ as in Fig. 4.17b, where $\hat{\vec{r}}$ is a unit vector directed outwards along the radius vector, and $\hat{\vec{\theta}}$ is a unit vector tangent to the circular path in the direction of increasing θ (measured in a counterclockwise sense from the positive x axis).

Using this notation, we can write the particle's total acceleration $\vec{a}$ as:

$$\vec{a} = \vec{a}_t + \vec{a}_r = \frac{dv}{dt}\,\hat{\vec{\theta}} - \frac{v^2}{r}\,\hat{\vec{r}} \tag{4.36}$$

These vectors are described in Fig. 4.17a. The negative sign of $\vec{a}_r$ indicates that it is always directed inward, opposite to $\hat{\vec{r}}$.

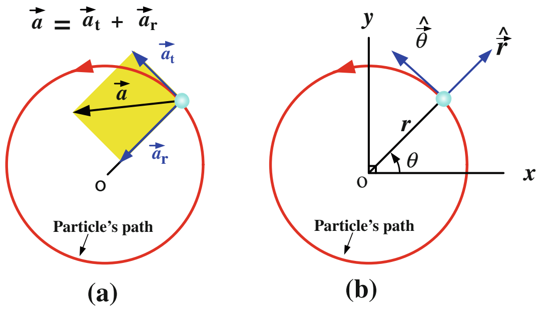
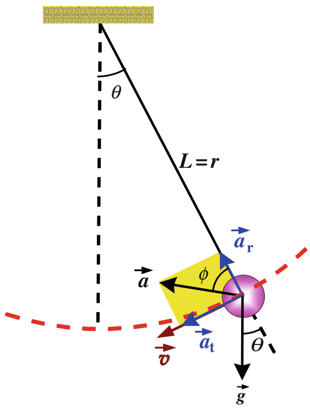


Fig. 4.17 (a) The acceleration of a particle moving in a circle with a tangential component $\vec{a}_t$ and a radial component $\vec{a}_r$ directed toward the center of the circle. (b) Definitions of the unit vectors $\hat{\vec{r}}$ and $\hat{\vec{\theta}}$

Example 4.6

A sphere attached to a cord of length $L = 1$ m swings in a vertical circle under the influence of gravity. The sphere has a speed of 2 m/s when the cord has an angle $\theta = 30°$ with the vertical, as shown in Fig. 4.18. At this instant, find its acceleration in terms of tangential and radial components.

Fig. 4.18



Solution: When the cord makes an angle θ to the vertical line, the component of the gravitational acceleration $\vec{g}$ that is tangent to the circular path has a magnitude $g \sin\theta$. Thus the magnitude of the tangential acceleration is:

$$a_t = g \sin\theta = (9.8\,\text{m/s}^2)(\sin 30°) = 4.9\,\text{m/s}^2$$

Since the speed of the sphere at this instant is $v = 2$ m/s and the radius of the circle that the sphere swings about equals the length of the cord, i.e. $r = L = 1$ m, then the magnitude of the radial acceleration is:

$$a_{\mathrm{r}}=\frac{v^2}{r}=\frac{(2\ \mathrm{m/s})^2}{1\ \mathrm{m}}=4\ \mathrm{m/s^2}$$

Therefore, $T-mg\cos\theta=ma_{\mathrm{r}}$, where T is the cord's tension. From the relation Eq. 4.34 we can find the magnitude of a at $\theta=30°$ as follows:

$$a=\sqrt{(4.9\ \mathrm{m/s^2})^2+(4\ \mathrm{m/s^2})^2}=6.32\ \mathrm{m/s^2}$$

The angle ϕ between the vector $\vec{a}$ and the cord will be:

$$\phi=\tan^{-1}\left(\frac{a_{\mathrm{t}}}{a_{\mathrm{r}}}\right)=\tan^{-1}\left(\frac{4.9\ \mathrm{m/s^2}}{4\ \mathrm{m/s^2}}\right)=\tan^{-1}(1.225)=50.77°$$

4.6 Exercises

Section 4.1 Position, Displacement, Velocity, and Acceleration Vectors

(1) The initial position vector of a butterfly can be described in unit vectors by $\vec{r}_{\mathrm{i}}=3\,\vec{\mathrm{i}}-7\,\vec{\mathrm{j}}+4\,\vec{\mathrm{k}}$, and five seconds later by $\vec{r}_{\mathrm{f}}=-2\,\vec{\mathrm{i}}+3\,\vec{\mathrm{j}}-\vec{\mathrm{k}}$ (all units in meters). (a) What is the butterfly's displacement vector? (b) What is the butterfly's average velocity?

(2) The position vector of a particle moving in two dimensions is given by $\vec{r}=x(t)\,\vec{\mathrm{i}}+y(t)\,\vec{\mathrm{j}}$, where $x(t)=2t+1$, $y(t)=2t^2$, t is the time in seconds, and all numerical coefficients have the proper units so that $\vec{r}$ is in meters. (a) Find the average velocity vector during the time interval from $t=0$ to $t=2$ s. (b) Find the particle's velocity vector $\vec{v}$ as a function of time, and find its magnitude and direction at $t=2$ s. (c) Find the particle's acceleration vector $\vec{a}$.

(3) The position vector of a particle moving in two dimensions is described by $\vec{r}=(4t^3-12t+9)\,\vec{\mathrm{i}}+(6t+4)\,\vec{\mathrm{j}}$, where t is the time in seconds and all numerical coefficients have the proper units so that $\vec{r}$ is in meters. (a) Find the average velocity vector between $t=1$ s and $t=2$ s. (b) Find the particle's instantaneous velocity vector $\vec{v}$ as a function of time, and then find its magnitude and direction at $t=1$ s. (c) Find the velocity and the speed of the particle at $t=3$ s. (d) Find the average acceleration vector between $t=1$ s and $t=2$ s. (e) Find the instantaneous acceleration vector $\vec{a}$ as a function of time, and find its magnitude and direction at $t=2$ s. (f) Find the time at which

the x component of the particle's displacement vector is at a relative maximum/minimum, showing how you can determine whether it is a maximum or a minimum.

(4) The position vector of a particle moving in three dimensions is given by $\vec{r} = 3t\,\vec{i} - 2t^2\,\vec{j} + 2\,\vec{k}$, where t is the time in seconds and all numerical coefficients have the proper units so that $\vec{r}$ is in meters. (a) Find the magnitude of the particle's position vector $\vec{r}$ as a function of time, and then find its value when $t = 2$ s. (b) Find the particle's velocity vector $\vec{v}$ as a function of time, and find its magnitude and direction at $t = 2$ s. (c) Find the particle's acceleration vector $\vec{a}$ as a function of time, and find its magnitude and direction at $t = 2$ s.

Section 4.2 Projectile Motion

(Neglect air resistance and take $g = 10\,\text{m/s}^2$ in all projectile Exercises)

(5) A small ball is projected horizontally from a tall building with a speed $v_\circ$ of 10 m/s, see Fig. 4.19. Find its position and its velocity components after $\frac{1}{2}$ s.

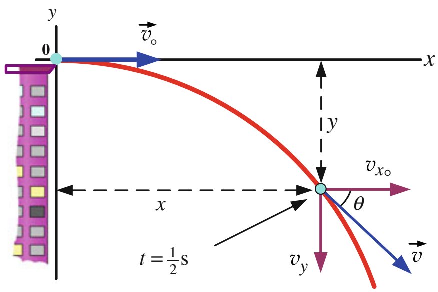


Fig. 4.19 See Exercise (5)

(6) A student running with a constant speed $v_\circ$ goes straight over the cliff shown in Fig. 4.20. The student is at a height $h = 45$ m above water level once he leaves the cliff. The student lands in the water at a point where $x = 39$ m. (a) How fast was the student running when he jumped over the cliff? (b) What is his speed and what is the angle of his impact with the water?

(7) A long jumper leaves a cliff at $\theta_\circ = 45°$ above the horizontal with an initial speed $v_\circ$ and lands 6 m away, see Fig. 4.21. The cliff is at a height $h = 2$ m above sea level. (a) What is the speed of the jumper? (b) How long will it take him to reach the water?

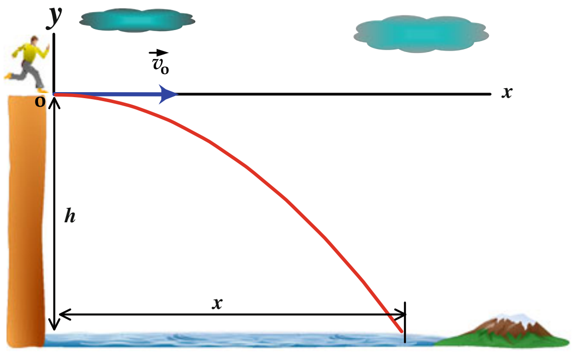


Fig. 4.20 See Exercise (6)

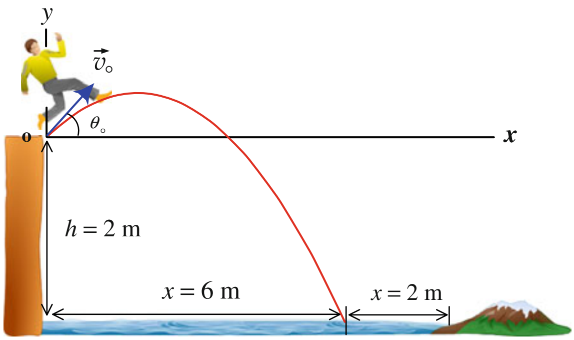


Fig. 4.21 See Exercise (7)

(8) After the long jumper in exercise 7 lands in water, he swims 2 m to the bank of the river, and then decides to go back to the cliff by making a long jump from the edge of the bank of the river at $\theta_\circ = 45°$ above the horizontal, see Fig. 4.22. (a) What must be the minimum speed of the jumper to reach the cliff? (b) How long will it take him to reach the cliff?

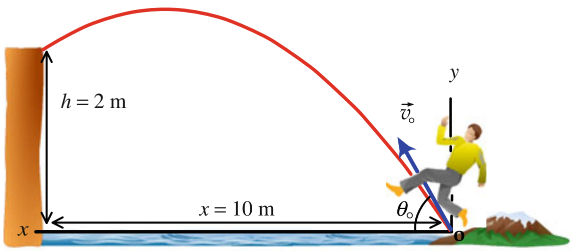
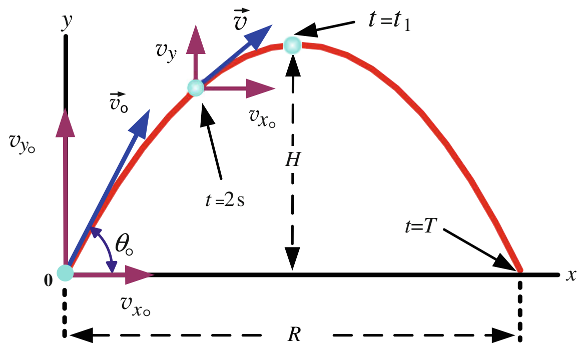


Fig. 4.22 See Exercise (8)

(9) A ball is thrown with an initial speed $v_\circ$ of 30 m/s at an angle $\theta_\circ$ above the horizontal, where $\sin\theta_\circ = 4/5$ and $\cos\theta_\circ = 3/5$, see Fig. 4.23. (a) Find the

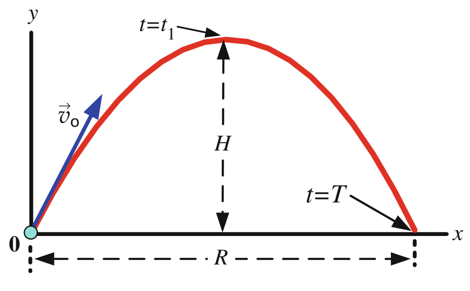
x and y components of $\vec{v}_\circ$. (b) When $t = 2$ s, find the position of the ball and the magnitude and direction of its velocity $\vec{v}$. (c) What is the value of the H (the highest point of the ball's trajectory) and how much time t_1 has elapsed for the ball to reach that point? (d) Calculate the values of the total time of the ball's flight T and the horizontal range R.

Fig. 4.23 See Exercise (9)



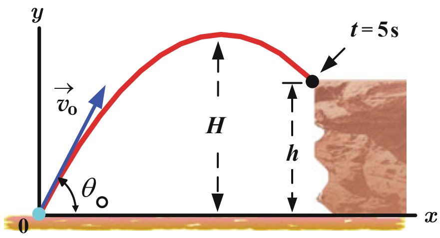
(10) A projectile is fired from the ground with an initial velocity $\vec{v}_\circ = (6\,\vec{i} + 12\,\vec{j})$ m/s, see Fig. 4.24. (a) What are the vertical and horizontal velocity components after time $t = 1$ s? (b) What are the time and the coordinates of the projectile when it reaches the highest point? (c) What are the values of the total time of the projectile's flight T and the horizontal range R?

Fig. 4.24 See Exercise (10)



(11) A ball is launched from the ground with an initial speed $v_\circ$ of 40 m/s at an angle $\theta_\circ = 60°$ towards a cliff of height h, see Fig. 4.25. The ball strikes the cliff after 5 s. Find: (a) the height h of the cliff, (b) the maximum height H, (c) the speed of impact, and (d) the horizontal distance between the cliff and the firing point.

Fig. 4.25 See Exercise (11)



(12) A stunt driver tries to jump with his car over 4 cars parked sequentially one after another, as shown in Fig. 4.26. The horizontal distance that he must cover is 15 m and the vertical height of the tip of the ramp is 1.5 m above the cars. (a) What is the minimum speed that he must drive off the horizontal ramp of Fig. 4.26a? (b) When the ramp is tilted upwards with angle $\theta_\circ = 8°$, as in Fig. 4.26b, what is the new minimum speed?

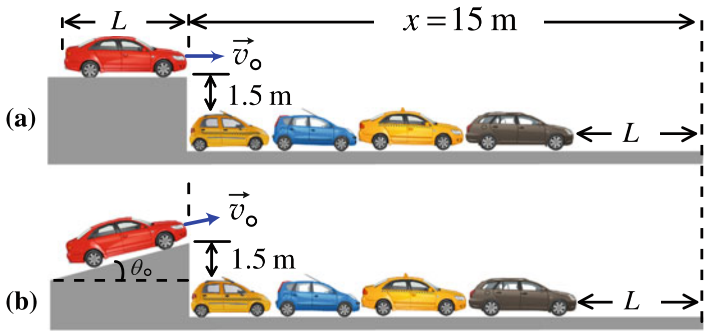


Fig. 4.26 See Exercise (12)

(13) Figure 4.27 shows a blasted chunk of a solid rock ejected from a volcano. It is safe to live at the foot of the volcano, but away from its center by 9 km (or more), i.e. $x = 9$ km when $\theta_\circ = 45°$ and the speed $v_\circ$is maximum. (a) At what maximum initial speed would a rock need to be ejected from the volcano's mouth and reach $x = 9$ km? (b) What would be its time of flight? (c) Does the maximum initial speed of part (a) increases or decreases when air resistance is taken into account?

(14) Figure 4.28 shows a fighter plane that has a speed $v_\circ$ of 300 km/h, flying at an angle $\theta_\circ = 15°$ below the horizontal when a decoy rocket is released. The horizontal distance between the release point and the point where the decoy

strikes the ground is $x = 600$ m. (a) How long was the decoy in the air? (b) How high was the plane when the decoy was released?

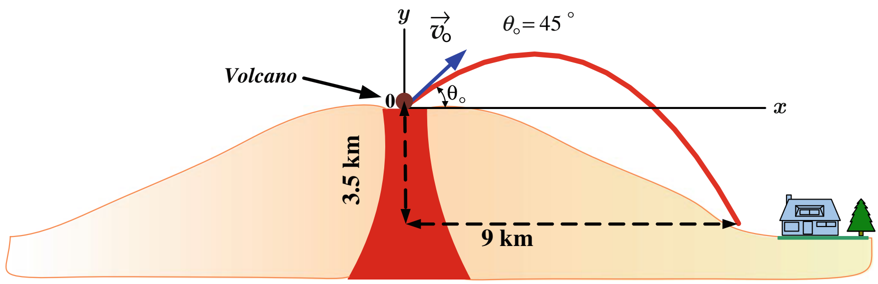
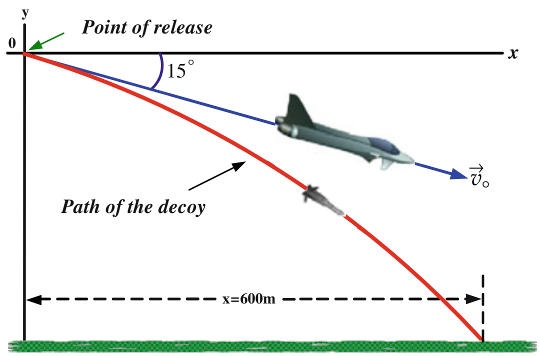


Fig. 4.27 See Exercise (13)

Fig. 4.28 See Exercise (14)



(15) At what initial angle $\theta_\circ$ will the maximum height attained by a projectile be equal to its horizontal range?

(16) A girl sitting on a hill aims her cork slingshot at a boy hanging from a tree at a horizontal distance L away and a vertical distance h up from the slingshot, see Fig. 4.29. At the instant the cork-projectile is fired, the boy releases himself from the tree, hoping to avoid being hit. Ignoring air resistance, show that the $y_{\text{cork}} = y_{\text{boy}}$ and thus the cork collides with the boy.

(17) An inclined plane makes an angle ϕ with the horizontal. At the lowest point of the incline, a projectile is fired with a speed $v_\circ$ that makes an angle $\theta_\circ$ above the horizontal, see Fig. 4.30. At what angle $\theta_\circ$ does the range R along the incline reach its maximum?

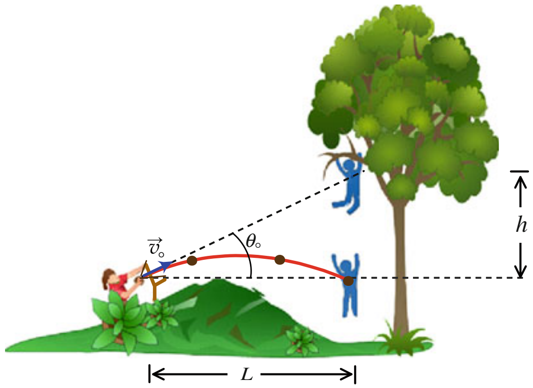


Fig. 4.29 See Exercise (16)

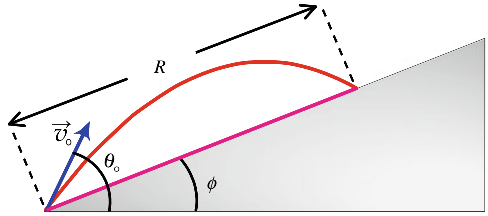


Fig. 4.30 See Exercise (17)

(18) A projectile is fired with a speed $v_\circ$ that makes an angle $\theta_\circ$ above the horizontal. Find the horizontal range R when the projectile is at a vertical position $y = +h$, see Fig. 4.31.

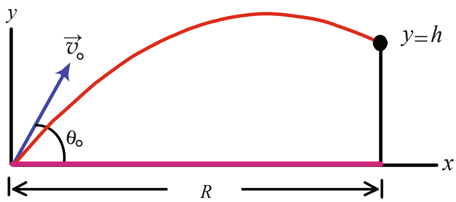


Fig. 4.31 See Exercise (18)

(19) What is the effect on the horizontal range R of Exercise 18 when the vertical position is $y = -h$.

Section 4.3 Uniform Circular Motion

(20) Calculate the magnitude of the acceleration of a particle moving in a circle of radius $r = 0.5$ m with a constant speed v of 10 m/s.

(21) A boy attaches a stone to the end of a rope of length $r = 0.25$ m, and rotates the stone at a constant speed in a circular fashion. Find the stone's radial acceleration when the period T is 2 s.

(22) As an approximation, assume the moon revolves about the Earth in a perfectly circular orbit with a radius $r = 3.85 \times 10^8$ m and takes 27.3 days (which is 2.36×10^6 s) to make a complete revolution, see Fig. 4.32. (a) What is the speed of the moon? (b) What is the magnitude of the radial acceleration of the moon toward the Earth's center?

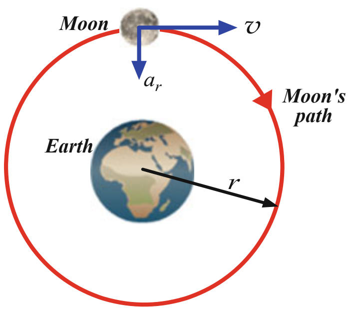


Fig. 4.32 See Exercise (22)

(23) A car moves in a circle of radius $r = 15$ m with a constant speed $v = 30$ m/s, see Fig. 4.33. (a) What is the change in velocity (magnitude and direction) when the car goes around an arc of $\Delta\theta = 60°$, as shown in the right part of Fig. 4.33? (b) What is the magnitude of the radial acceleration?

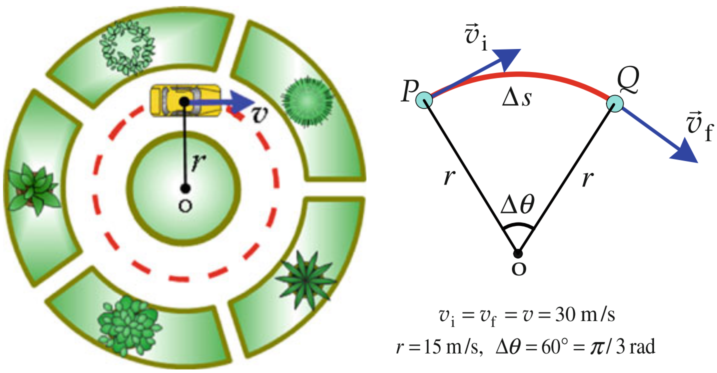
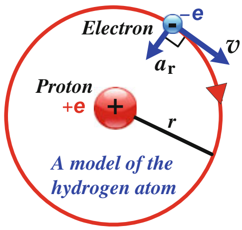


Fig. 4.33 See Exercise (23)

(24) In the model of the hydrogen atom proposed by Niels Bohr, an electron circulates a stationary proton in a circle of radius $r = 5.28 \times 10^{-11}$ m with a speed

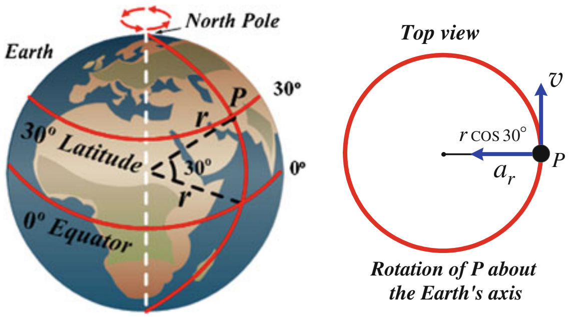
$v = 2.18 \times 10^6$ m/s, see Fig. 4.34. (a) Find the magnitude of the electron's radial acceleration in this model. (b) What is the period of the motion?

Fig. 4.34 See Exercise (24)



(25) A point P is located on the latitude that passes through Egypt's soil at exactly 30°N, and at a distance $r = 6.4 \times 10^6$ m away from Earth's center, see Fig. 4.35. As the Earth revolves about its axis, calculate the magnitude of the acceleration of the point P.

Fig. 4.35 See Exercise (25)



(26) The rotor of an ultracentrifuge has a radius $r = 6$ cm and rotates at 5×10^4 revolutions per minute. Find the centripetal acceleration of a particle at the circumference of the rotor in terms of the value of the acceleration due to gravity, g.

(27) A jet pilot performs a vertical loop when the speed of his aircraft is 1,200 km/h. Find the smallest radius of the circle when the centripetal acceleration at the lowest point does not exceed 6 g.

Sections 4.4 and 4.5 Tangential and Radial Acceleration, Non-uniform Circular Motion

(28) The speed of a particle moving in a circle of radius $r = 4.5$ m increases with a constant rate of 2 m/s. If at some instant, the magnitude of the total acceleration

$\vec{a}$ is $6\,\text{m/s}^2$, then find: (a) the magnitude of the tangential acceleration, (b) the magnitude of the radial acceleration, and (c) the speed of the particle.

(29) A particle has a non-uniform motion on a circular path of radius $r = 2\,\text{m}$. At a given instant of time, the magnitude of its total acceleration a was $10\,\text{m/s}^2$, see Fig. 4.36. At this instant, find: (a) the magnitude of both the centripetal and tangential accelerations, and (b) the speed v of the particle.

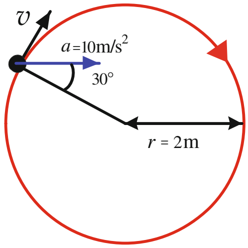


Fig. 4.36 See Exercise (29)

(30) A spaceship is in a circular orbit at an altitude of $h = 150\,\text{km}$ above the Earth's surface, see Fig. 4.37 (consider the Earth's radius to be $6.4 \times 10^6\,\text{m}$). The spaceship requires time $T = 5.25 \times 10^3\,\text{s}$ to complete one revolution around the Earth. In order to leave this orbit and head for the moon, an astronaut starts the engine of the spaceship, resulting in a tangential acceleration of magnitude $a_t = 25\,\text{m/s}^2$. (a) Find the spaceship's orbital speed v and the magnitude of its radial acceleration a_r before the engine is started. (b) What is the astronaut's total acceleration (magnitude and direction) just after starting the engine?

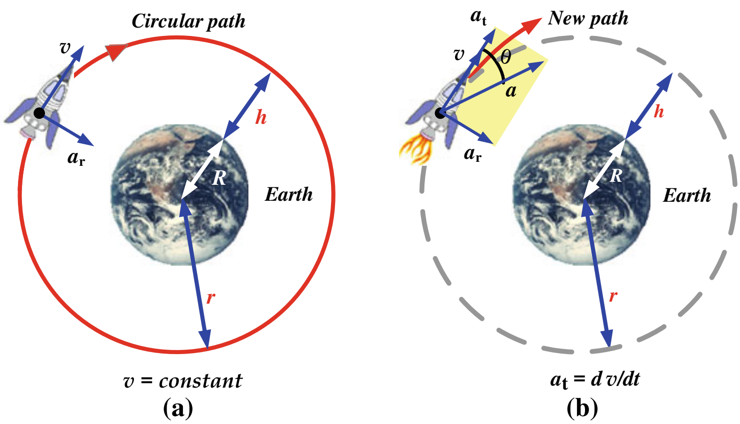


Fig. 4.37 See Exercise (30). (a) Before starting the engine, (b) just after starting the engine

Force and Motion

5

The kind of interaction that accelerates an object is called a **force**, which could be a push or pull. From now on, we shall use the capital letter $\vec{F}$ (with an arrow over it) to represent a general force vector. In addition, we shall use the symbol $\Sigma\vec{F}$ for the vector sum of several forces, which we call the **resultant force** or the **net force**.

5.1 The Cause of Acceleration and Newton's Laws

The relationship between forces and the produced acceleration is an aspect of **mechanics** called **dynamics**. Isaac Newton (1642–1727) first formulated this relationship in terms of laws known by his name.

Newton's First Law of Motion

Newton's original first law reads:

> Newton's First Law
> An object will remain at rest, or in motion with constant velocity, unless it experiences a net external force.

$$\text{If}\quad \Sigma\vec{F} = 0,\quad \text{then}\quad \begin{cases} \vec{v} = 0 \\ \quad \text{or} \\ \vec{v} = \text{constant} \end{cases} \qquad \text{(Newton's first law)} \tag{5.1}$$

H. A. Radi and J. O. Rasmussen, *Principles of Physics*,
Undergraduate Lecture Notes in Physics, DOI: 10.1007/978-3-642-23026-4_5,

Newton's first law is sometimes called the *law of inertia*, and the set of coordinates that are used to describe the object are called the *inertial reference frames* or alternatively *inertial frames*.

Inertial Frames
An inertial frame is one in which an object experiences zero net force.

Consequently, Newton's first law declares that if the net force on an object is zero, it must stay at rest or move with constant velocity with respect to any inertial frame.

Newton's Second Law of Motion

All observations reveal that the acceleration of an object is directly proportional to the net acting force. These observations are expressed in Newton's second law.

Newton's Second Law
The acceleration of an object, $\vec{a}$, is related to its mass, m, and the resultant force acting on it, $\Sigma\vec{F}$, by the relation:

$$\boxed{\Sigma\vec{F} = m\vec{a}} \quad \text{(Newton's second law)} \tag{5.2}$$

This equation is valid only when the speed of the object is much less than the speed of light. In SI units, we define the unit of force that accelerates a standard 1 kg by 1 m/s^2 as 1 newton (abbreviated to 1 N). Thus, according to Eq. 5.2, we have:

$$\boxed{1\,\text{N} = (1\,\text{kg})(1\,\text{m/s}^2) = 1\,\text{kg.m/s}^2,} \tag{5.3}$$

Although we shall use SI units only from now on, other systems like the CGS (centimeter-gram-second) system and the British system are still in use. Table 5.1 compares lists of all systems currently in use.

Table 5.1 Units in Newton's second law

System	Force[a]	Mass	Acceleration
SI	Newton (N)	Kilogram (kg)	m/s^2
CGS	dyne[b]	gram (g)	cm/s^2
British	Pound (lb)[c]	slug	ft/s^2

[a] 1 N = 10^5 dyne = 0.255 lb. [b] 1 dyne = 1 g.cm/s^2. [c] 1 lb = 1 slug.ft/s^2

Newton's Third Law of Motion

Forces come in pairs. For example, if you lean against a wall with a certain force, the wall reacts and pushes back on you with a force of equal magnitude. Another example of two interacting bodies is shown in Fig. 5.1, where body 1 exerts an action force $\vec{F}_{21}$ (a pull) on body 2. ($\vec{F}_{21}$ is read: force exerted on body 2 by body 1). Experiments show that body 2 would also exert a reaction force $\vec{F}_{12}$ on body 1. These two forces are equal in magnitude and opposite in direction. That is:

$$\boxed{\vec{F}_{12} = -\vec{F}_{21}} \quad \text{(Newton's third law)} \tag{5.4}$$

Equation 5.4 implies that $F_{12} = F_{21}$. Moreover, this equation holds true regardless of whether the two bodies move or remain stationary.

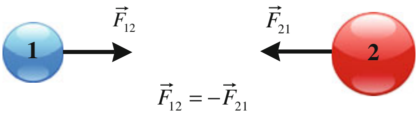


Fig. 5.1 The force exerted by body 1 on body 2 is equal in magnitude but opposite to the force exerted by body 2 on body 1

All observations similar to the previous two examples are summarized in Newton's third law, which states that:

> Newton's Third Law
> To every action there must be a reaction equal in magnitude and opposite in direction.

Forces of an action-reaction pair act on different bodies, i.e. they do not combine to give a net force. In Fig. 5.2, we display an orbiting satellite, where the only force that acts on it is $\vec{F}_{SE}$ (the gravitational force). The corresponding reaction force is $\vec{F}_{ES}$. This force causes the Earth to attain a very small yet undetectable acceleration.

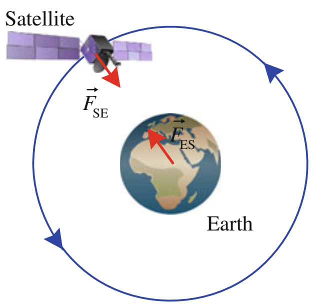


Fig. 5.2 Forces on a satellite and the Earth as action-reaction pair

5.2 Some Particular Forces

Weight ($\vec{W}$)

The **weight** $\vec{W}$ of a body is the force exerted by the Earth on the body. This force is directed toward the center of the Earth and is primarily due to an attraction (called a **gravitational attraction**) between the body and the Earth.

Since a freely falling body experiences an acceleration $\vec{g}$ acting toward the center of the Earth, then applying Newton's second law to a body of mass m, with $\vec{a} = \vec{g}$ and $\Sigma\vec{F} = \vec{W}$, gives the following:

$$\boxed{\vec{W} = m\vec{g}} \tag{5.5}$$

The magnitude of $\vec{W}$ in SI units is in newtons. We can weigh a body with a spring scale (see Fig. 5.3a). The body stretches the spring, moving its pointer along a scale that has been calibrated and marked in either mass or weight units. Alternatively, we can weigh a body by placing it on one pan of an equal-arm balance (Fig. 5.3b) and then adding reference masses on the other pan until we achieve a balance.

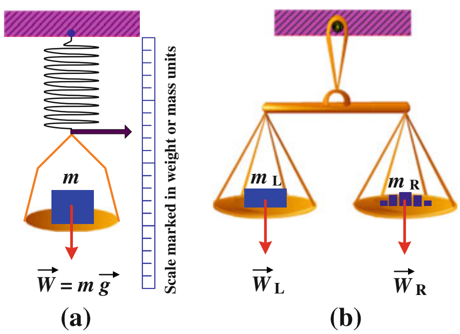


Fig. 5.3 (a) A spring scale. The reading gives the weight if marked in weight units. (b) An equal-arm balance. When balance is achieved, the masses on the *left* (L) and *right* (R) pans are equal

Normal Force ($\vec{N}$)

The reaction of a block of weight $\vec{W}$ is the force exerted on the Earth $\vec{W}'$, see Fig. 5.4a. When this block rests on a table, the table exerts an upward action force, $\vec{N}$, called the **normal force**; the name comes from the mathematical term *normal*, meaning “perpendicular”, see Fig. 5.4b. The normal force is the force that prevents the block from falling through the table, and can have any value up to the point of breaking the table. The reaction to $\vec{N}$ is the force that the block exerts on the table, $\vec{N}'$, see Fig. 5.4c. Therefore, we conclude that:

$$\vec{W} = -\vec{W}', \quad \vec{N} = -\vec{N}' \tag{5.6}$$

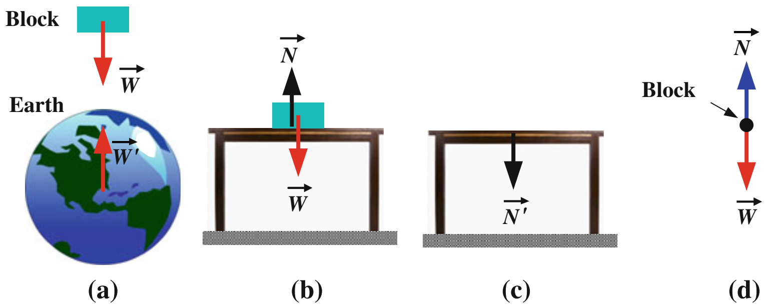


Fig. 5.4 (a) The reaction of a block of weight $\vec{W}$ is the force $\vec{W}'$. (b) A block resting on a table experiences a normal force $\vec{N}$ perpendicular to the table. (c) The reaction force $\vec{N}'$ exerted on the table. (d) The free-body diagram used to solve the block problem

The forces acting on the block are only $\vec{W} = m\vec{g}$ and $\vec{N}$, as seen in Fig. 5.4b. So, the normal force balances the weight of the block and provides equilibrium ($\vec{a} = 0$). To solve problems with Newton’s laws, we often draw a **free-body diagram**, representing the body by a dot (or a sketch of the body) and each external force by a vector with its tail on the dot, see Fig. 5.4d. With this figure $\Sigma\vec{F} = m\vec{a}$ becomes:

$$\Sigma\vec{F} = 0 \quad \Rightarrow \quad \vec{N} + \vec{W} = 0 \quad \Rightarrow \quad N - W = 0$$

Thus:

$$\boxed{N = W = mg} \tag{5.7}$$

Foces of Friction ($\vec{f}$)

When we attempt to slide a block over a surface, the intended motion is resisted by a bonding between the block and the surface. We represent this resistance by a force $\vec{f}$ called the **force of friction** or simply **friction**. This force is directed along the surface, opposite to the intended motion. Sometimes, we simplify situations by neglecting friction, and the surface is said to be *frictionless*.

Consider a block resting on a horizontal table, as in Fig. 5.5a, where its weight $\vec{W}$ is balanced by an equal but opposite normal force $\vec{N}$. In Fig. 5.5b, we apply a force $\vec{F}$ on the block, attempting to pull it to the right. The block will remain stationary if $\vec{F}$ is not large enough. The frictional force $\vec{f}$ acts to the left and keeps the block stationary, i.e. $F = f$. We call this frictional force the **force of static friction** $\vec{f}_s$. If we increase F, the static frictional force $\vec{f}_s$ increases, while the block remains at rest. When the applied force F reaches a certain value, the block will be on the verge of slipping and the frictional force will be maximum and denoted by $f_{s,max}$, see Fig. 5.5c. When F exceeds $f_{s,max}$, the block moves to the right, see Fig. 5.5d. When the block is in motion, the frictional force becomes less than $\vec{f}_{s,max}$ and is called the **force of kinetic friction** $\vec{f}_k$, see Fig. 5.5e. The horizontal net force $F - f_k$ accelerates the block to the right.

Experimentally, one finds that both f_s and f_k are proportional to the magnitude of the normal force N acting on the block through a *dimensionless constant* μ. These observations can be summarized as:

1. If the block is not moving, the force of static friction is opposite to the applied force and can have values given by:

$$f_s \le \mu_s N \tag{5.8}$$

   where the constant μ_s is called the **coefficient of static friction**.

2. When the block is on the **verge of slipping**, we have:

$$\boxed{f_{s,max} = \mu_s N} \tag{5.9}$$

3. If the block begins to move along the surface, the magnitude of the frictional force rapidly decreases to the value f_k given by:

$$\boxed{f_k = \mu_k N,} \quad \boxed{\mu_k < \mu_s} \tag{5.10}$$

where the constant μ_k is called the **coefficient of kinetic friction**.

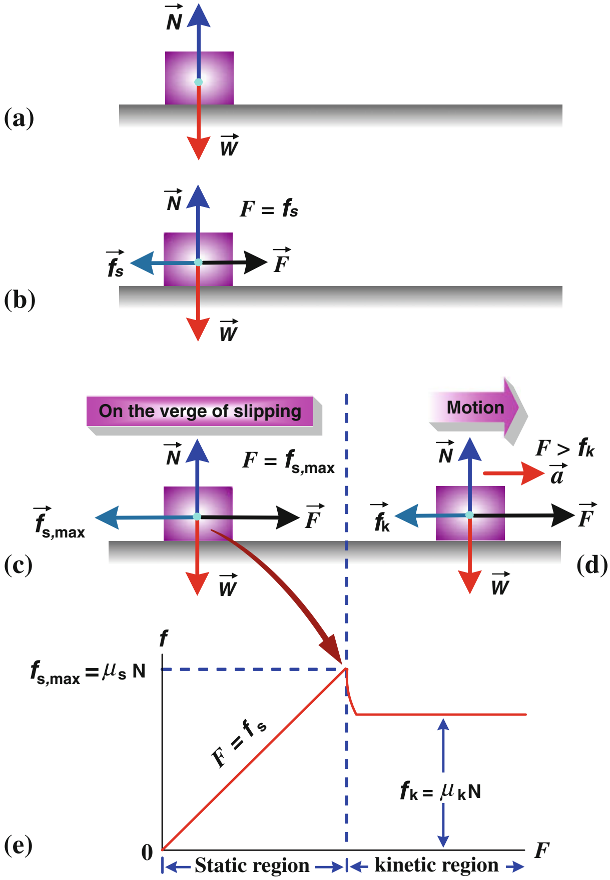


Fig. 5.5 (a) A block at rest on a horizontal table. The static frictions f_s and $f_{\mathrm{s,max}}$ are shown in parts (b) and (c). When the block moves, the kinetic friction f_k becomes less than $f_{\mathrm{s,max}}$ as in parts (d) and (e)

The values of the dimensionless coefficients μ_s and μ_k depend on the nature of the surfaces, not on their areas. Regardless, $\mu_\mathrm{k} < \mu_\mathrm{s}$ since $f_\mathrm{k} < f_{\mathrm{s,max}}$ as in Fig. 5.5e. Typical values of the coefficients lie in the range $0.05 \leq \mu \leq 1.5$. Table 5.2 lists some reported values.

A highly polished surface is far from being perfectly flat on the atomic scale. Moreover, the surfaces of everyday objects have layers of oxides and other contaminants. When two such surfaces are placed together, only high points touch each other, see Fig. 5.6. In addition, many contact points weld together, which is called *cold-welding*.

Table 5.2 Some approximate coefficients of friction

Material surfaces	μ_s	μ_k
Ice on ice	0.1	0.03
Wood on ice	0.08	0.06
Metal on metal (lubricated)	0.15	0.06
Wood on wood	0.25–0.5	0.2
Copper on steel	0.53	0.36
Glass on glass	0.94	0.4
Aluminum on steel	0.61	0.47
Steel on steel	0.74	0.57
Rubber on concrete	$\sim$ 0.9	$\sim$ 0.7

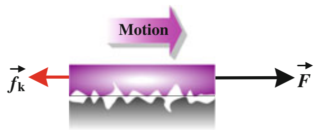


Fig. 5.6 A highly magnified cross section showing some welding at high spots. Force is required to break these welds to maintain motion

When two surfaces are pulled across each other, there is first a tearing of formed welds and then a continuous tearing of reforming welds as additional contacts are made. The kinetic friction $\vec{f}_k$ is the vector sum of all forces (against motion) at those many contacts.

Foces of Tension ($\vec{T}$)

When a rope (or a cord, cable, etc) is attached to a body and pulled, the rope is said to be in **tension**. The rope's function is to transfer force between two bodies. The tension in the rope is defined as the force that the rope exerts on the body. This force is denoted usually by the symbol $\vec{T}$, see Fig. 5.7a and b.

A rope is considered to be *massless* (i.e., its mass is negligible compared to the body's mass) and *non-stretchable*. It pulls on both bodies with a force of the magnitude T, even if the two bodies are accelerating, or the rope is run around a pulley as in Fig. 5.7c and d. Such a pulley is *massless* (has a negligible mass compared to the bodies) and *frictionless* (has negligible friction on its rotational axel).

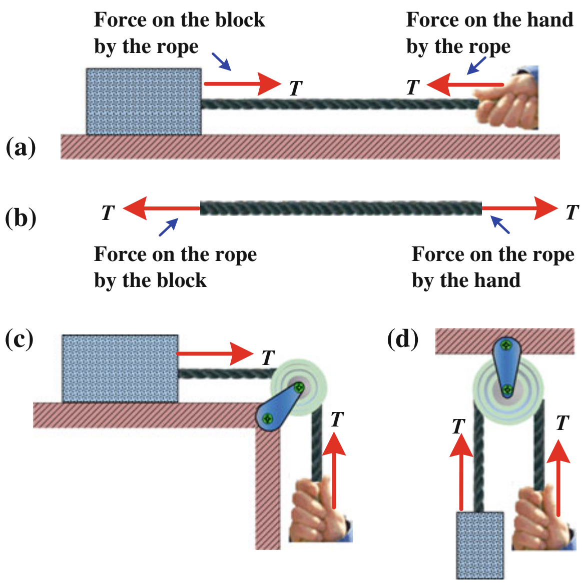


Fig. 5.7 When a rope is under tension, it pulls the block and the hand of parts (a), (c), and (d) with a force of magnitude T. According to Newton's third law, the block and the hand both exert a force on the rope of magnitude T, as shown in part (b) only

Drag Forces ($\vec{F}_D$)—Small Objects

When a small object moves at a low speed v through a viscous medium, it experiences a resistive **drag force** $\vec{F}_D$ that opposes its motion. In such situations, the force has a magnitude given by:

$$\boxed{F_D = bv} \tag{5.11}$$

where b is a proportionality constant that depends on the properties of the medium and on the shape of the object, and b has the units kg/s.

If we assume that a sphere of mass m and weight $W = mg$ is released from rest in a fluid as in Fig. 5.8, then application of Newton's second law $\Sigma\vec{F} = m\vec{a}$ in the vertical direction will give:

$$mg - bv = ma \quad \Rightarrow \quad mg - bv = m\frac{dv}{dt} \tag{5.12}$$

Note that when the initial speed $v = 0$, the resistive force is zero and the acceleration $a = dv/dt$ is g. As the time t increases, the speed increases and the resistive force increases, while the acceleration decreases. Finally, the acceleration becomes zero when the resistive force equals the weight mg. At this stage, the speed has reached its *terminal speed* v_t. The terminal speed can be obtained from Eq. 5.12 by setting $v = v_t$ and $a = dv/dt = 0$. Thus:

$$v_t = \frac{mg}{b} \tag{5.13}$$

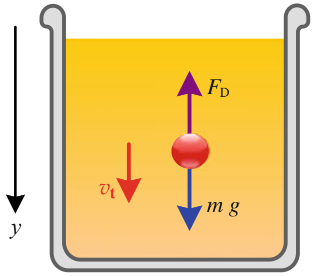


Fig. 5.8 A small sphere falling through a viscous fluid with a *low* speed. The resistive drag force $\vec{F}_D$ opposes the motion of the sphere

Drag Forces ($\vec{F}_D$)—Large Objects

When a large object (such as a baseball, skydiver, or an airplane) moves at a high speed v in a medium (gas or liquid) of density ρ (mass per unit volume), it experiences a **drag force** $\vec{F}_D$ that opposes the motion. From experiments, it was found that in these situations the magnitude F_D will be given by:

$$F_D = \tfrac{1}{2}\, C\rho A v^2 \tag{5.14}$$

where C is a dimensionless proportionality constant called the **drag coefficient**, and A is the effective cross sectional area of the object, taken to be perpendicular to its velocity $\vec{v}$. If v varies significantly, C can vary as well, but we ignore such complications.

If we assume a body of mass m and weight $W = mg$ falls from rest in air, as shown in Fig. 5.9, then application of Newton's second law $\Sigma\vec{F} = m\vec{a}$ in the vertical direction as in part (c) of the figure gives:

$$mg - \tfrac{1}{2}\, C\rho A v^2 = ma \quad \Rightarrow \quad mg - \tfrac{1}{2}\, C\rho A v^2 = m\frac{dv}{dt} \tag{5.15}$$

By setting $a = 0$ in this equation, the terminal speed is given by:

$$v_t = \sqrt{\frac{2mg}{C\rho A}} \tag{5.16}$$

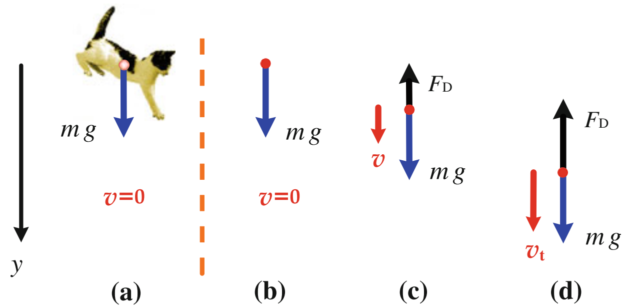


Fig. 5.9 Part (a) shows a body (cat) when it has just begun to fall through air and part (b) shows its corresponding free-body diagram. (c) Later, the drag force F_D has developed. (d) F_D has increased until it balances mg and the body falls with constant terminal speed v_t

5.3 Applications to Newton's Laws

This section is devoted to applications related to Newton's three laws of motion. The idea behind the examples is to let you know how to tackle a problem and how to translate a sketch of a situation to a free-body diagram with appropriate axes.

Example 5.1

(a) How much force is needed to give a 20,000 kg heavy loaded truck on a leveled track an acceleration of 1.5 m/s^2, and what is the force exerted by the track on the truck? (b) If the truck starts from rest, find its speed and position after 2 s.

Solution: Part (a) of Fig. 5.10 depicts the truck's travel. In part (b) we choose the coordinate axes and show the truck's free-body diagram. In this part we show the truck's weight $\vec{W}$ (acting downwards), the normal force $\vec{N}$ (acting perpendicularly to the track), and the driving force $\vec{F}$ (acting to the right).

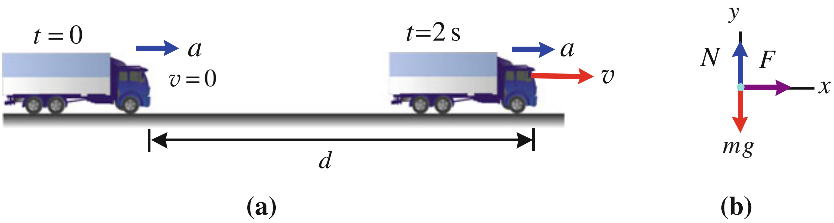


Fig. 5.10

(a) Applying Newton's second law in the component form, we find:

$$\Sigma F_x = F = ma \quad \text{and} \quad \Sigma F_y = N - mg = 0$$

From the first and second equations we find F and N as follows:

$$F = ma = (20{,}000 \text{ kg})(1.5 \text{ m/s}^2) = 30{,}000 \text{ N}$$

$$N = mg = (20{,}000 \text{ kg})(9.8 \text{ m/s}^2) = 196{,}000 \text{ N}$$

(b) Since a is constant, we can use $v = v_\circ + at$ and $x = v_\circ t + \frac{1}{2}at^2$ to find the speed v and the distance d after 2 s as follows:

$$v = v_\circ + at = 0 + (1.5 \text{ m/s}^2)(2 \text{ s}) = 3 \text{ m/s} = 10.8 \text{ km/h}$$

$$d = v_\circ t + \tfrac{1}{2}at^2 = 0 + \tfrac{1}{2}(1.5 \text{ m/s}^2)(2 \text{ s})^2 = 3 \text{ m}$$

Example 5.2

A block of mass $m = 21$ kg hangs from three cords as shown in part (a) of Fig. 5.11. Taking $\sin\theta = 4/5$, $\cos\theta = 3/5$, $\sin\phi = 5/13$, and $\cos\phi = 12/13$, find the tensions in the three cords.

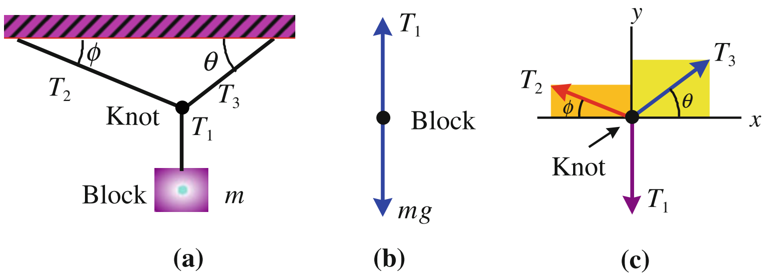


Fig. 5.11

Solution: We construct a free-body diagram for the block as shown in part (b) of Fig. 5.11. The tension in the vertical cord balances the weight of the block. Thus, by taking $g = 10 \text{ m/s}^2$, we get:

$$T_1 = mg = (21 \text{ kg})(10 \text{ m/s}^2) = 210 \text{ N}$$

In part (c) of Fig. 5.11, we first construct a free-body diagram of the *stationary* knot that holds the three cords together, and then we choose the coordinate axes. By applying Newton's second law in the x and y directions of part (c) of the figure, we find that:

$$\boxed{\Sigma F_x = T_3 \cos\theta - T_2 \cos\phi = 0} \quad \text{and} \quad \boxed{\Sigma F_y = T_3 \sin\theta + T_2 \sin\phi - T_1 = 0}$$

From the x-component equation we get the following relation:

$$T_3 = \frac{\cos\phi}{\cos\theta}\, T_2 = \frac{12/13}{3/5}\, T_2 = \frac{20}{13}\, T_2$$

When we substitute the result of T_3 into the y component equation, after putting $T_1 = mg = 210$ N, we get:

$$\frac{20}{13}\frac{4}{5}\, T_2 + \frac{5}{13} T_2 - 210 = 0 \quad \Rightarrow \quad \left(\frac{16}{13} + \frac{5}{13}\right) T_2 = 210 \quad \Rightarrow \quad T_2 = 130 \text{ N}$$

Consequently, one can find the value of the third tension to be:

$$T_3 = \frac{20}{13}\, T_2 = \frac{20}{13} \times 130\,\text{N} = 200 \text{ N}$$

Example 5.3

Two masses m_1 and m_2 ($m_2 > m_1$), are connected by a light cord that passes over a massless, frictionless pulley as shown in part (a) of Fig. 5.12. This arrangement is called Atwood's machine and sometimes is used to measure the acceleration due to gravity. Find the magnitude of acceleration of the two masses and the tension in the cord (consider $m_1 = 4$ kg and $m_2 = 6$ kg).

Solution: We construct a free-body diagram for the two masses as shown in parts (b) and (c) of Fig. 5.12. When Newton's second law is applied to m_1 in part (b) of the figure, we find:

$$\Sigma F_y = T - m_1 g = m_1 a$$

Also, we do the same for m_2 of part (c) of the figure, to get:

$$\Sigma F_y = m_2 g - T = m_2 a$$

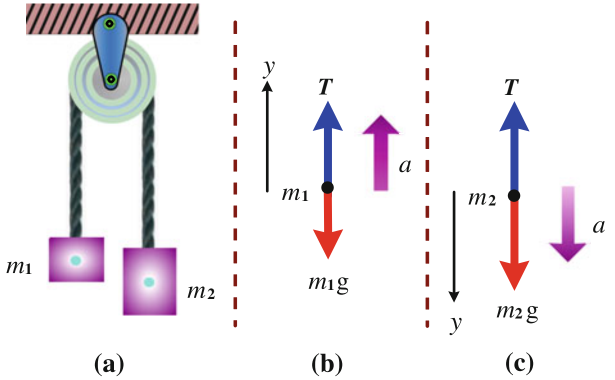


Fig. 5.12

When we add the last two equations, T will cancel out, and we get:

$$m_2 g - m_1 g = m_2 a + m_1 a$$

Thus: $\boxed{a = \dfrac{m_2 - m_1}{m_1 + m_2} g} \quad \Rightarrow \quad a = \dfrac{6\text{ kg} - 4\text{ kg}}{6\text{ kg} + 4\text{ kg}} \times 9.8\text{ m/s}^2 = 1.96\text{ m/s}^2$

If we substitute with a into the first equation we get:

$$\boxed{T = \frac{2m_1 m_2}{m_1 + m_2} g} \quad \Rightarrow \quad T = \frac{2(4\text{ kg})(6\text{ kg})}{6\text{ kg} + 4\text{ kg}} \times 9.8\text{ m/s}^2 = 47\text{ N}$$

Example 5.4

A car moving with an initial speed $v_\circ = 30$ m/s suddenly brakes, locking its wheels (i.e. it starts to skid). The car travels on the road a distance $d = 75$ m before it comes to a complete stop. Find the coefficient of kinetic friction between the tires of the car and the road.

Solution: Part (a) of Fig. 5.13 depicts the car's travel. In part (b) we choose the coordinate axes and show the car's free-body diagram during its skid. In this part we show the car's weight $\vec{W}$ (acting downwards), the normal force $\vec{N}$ (acting perpendicularly to the road), and kinetic frictional force $\vec{f}_k$ (acting to the left).

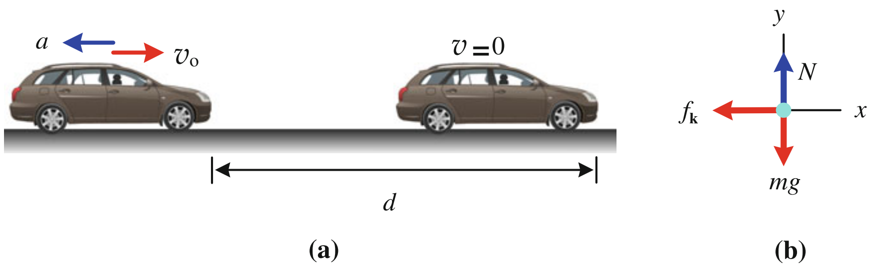


Fig. 5.13

Applying Newton's second law in the component form, we find that:

$$\Sigma F_x = -f_k = ma$$

$$\Sigma F_y = N - mg = 0$$

From the last equation we get $N = mg$. Since $f_k = \mu_k N = \mu_k mg$, then the first equation gives:

$$-\mu_k mg = ma$$

Thus: $$a = -\mu_k g$$

The negative sign means that the acceleration is to the left. Since a is constant, we can use $v^2 = v_\circ^2 + 2ax$, with $v = 0$ and $x = d$. This gives:

$$0 = v_\circ^2 + 2ad = v_\circ^2 - 2\mu_k\, gd$$

Thus: $$\mu_k = \frac{v_\circ^2}{2gd} = \frac{(30\ \text{m/s})^2}{2(9.8\ \text{m/s}^2)(75\text{m})} = 0.61$$

Example 5.5

A block of mass $m = 2$ kg is placed on an inclined plane of angle $\theta = 30°$, as shown in Fig. 5.14. The block is released from rest at the top of the plane, where the distance from the bottom is $d = 10$ m. (a) Find the magnitude of the acceleration of the block and the normal force exerted on the block. (b) How long does it take the block to reach the bottom, and what is its speed just as it gets there?

Solution: (a) We construct a free-body diagram to this example as shown in part (b) of Fig. 5.14. The only forces on the block are the weight $\vec{W}$ (acting downward) and the normal force $\vec{N}$ (acting perpendicular to the inclined plane). We choose a coordinate system with x axis parallel to the incline and y-axis perpendicular to it. With this choice, the angle between the weight vector and the negative direction of the y-axis equals the angle θ of the inclined plane. After that, we decompose the weight to a component of magnitude $mg\cos\theta$ along the negative y-axis and a component of magnitude $mg\sin\theta$ along the x axis. The block will slide along the inclined plane with acceleration a_x and will never leave the plane; i.e. $a_y = 0$. Applying Newton's second law to the x and y components gives:

$$\Sigma F_x = mg\sin\theta = m\,a_x$$

$$\Sigma F_y = N - mg\cos\theta = 0$$

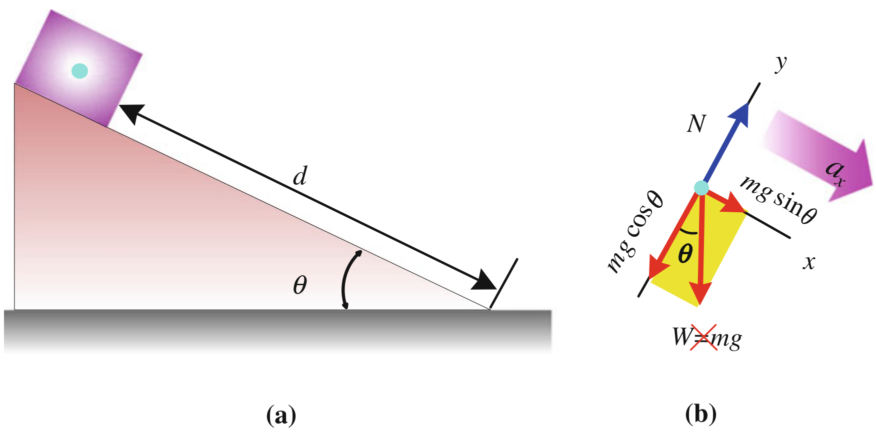


Fig. 5.14

From the x-component form, we see that the acceleration along the incline is provided by the component of the weight down the incline. By taking $g = 10$ m/s^2, we get:

$$a_x = g\sin\theta = (10\ \text{m/s}^2)(\sin 30^\circ) = 5\ \text{m/s}^2$$

Notice that when $\theta = 0$ (i.e. when the plane is horizontal) we have $a_x = 0$ (the minimum acceleration value). Also, we see that when $\theta = 90°$ (i.e. when the plane is vertical) the case resembles a free fall scenario, resulting in $a_x = g$ (the maximum acceleration value).

From the y component of Newton's second law we find N to be:

$$N = mg\cos\theta = (2\ \text{kg})(10\ \text{m/s}^2)(\cos 30°) = 17.32\ \text{N}$$

Also, notice that when $\theta = 0$, we have $N = mg = 20$ N (its maximum value), and when $\theta = 90°$, we have $N = 0$ (its minimum value).

(b) Since $a_x =$ constant, we apply the equation $x = v_{x\circ}\, t + \frac{1}{2} a_x\, t^2$ to the block with $v_{x\circ} = 0$ and $x = d = 10$ m to get the following relation:

$$d = \frac{1}{2} a_x\, t^2$$

Then, Solving for t and taking the positive root yields:

$$t = \sqrt{\frac{2d}{a_x}} = \sqrt{\frac{2 \times (10\ \text{m})}{5\ \text{m/s}^2}} = 2\ \text{s}$$

Also, we can apply the kinematics equation $v_x^2 = v_{x\circ}^2 + 2a_x\, x$ with $v_{x\circ} = 0$ and $x = d = 10$ m to get the following relation:

$$v_x^2 = 2a_x\, d$$

Then, solving for v_x and taking the positive root yields:

$$v_x = \sqrt{2a_x\, d} = \sqrt{2(5\ \text{m/s}^2)(10\ \text{m})} = 10\ \text{m/s}$$

Example 5.6

A block of mass $m_1 = 4$ kg lying on a rough horizontal surface is connected to a second block of mass $m_2 = 6$ kg by a light non-stretchable cord over a massless, frictionless pulley as shown in part (a) of Fig. 5.15. The coefficient of kinetic friction between the block and the surface is $\mu_k = 0.5$. (a) Find the magnitudes of the acceleration of the system and the tension in the cord. (b) Find the relation between m_1 and m_2 in the case when the system is on the verge of slipping.

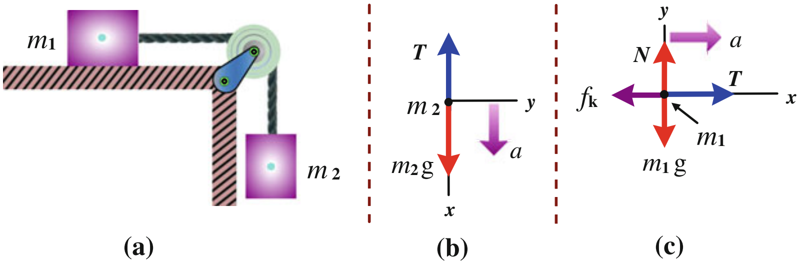


Fig. 5.15

Solution: (a) Since the cord is *non-stretchable*, the two masses have the same magnitude of acceleration. Consequently, we construct a free-body diagram for the two masses as shown in parts (b) and (c) of Fig. 5.15, where we take the x axis always along any of the body's motion. In this case a cannot take negative values. When Newton's second law is applied to m_2 in part (b) of the figure, we find:

$$(1) \qquad \Sigma F_x = m_2 g - T = m_2 a$$
$$\Sigma F_y = 0$$

From (1), we can find the magnitude of the tension in terms of g and a. That is:

$$(2) \qquad T = m_2 g - m_2 a$$

Doing the same for m_1 (see part (c) of Fig. 5.15) we get:

$$(3) \qquad \Sigma F_x = T - f_k = m_1 a$$
$$(4) \qquad \Sigma F_y = N - m_1 g = 0$$

Since $f_k = \mu_k N$, and from (4) we have $N = m_1 g$, then:

$$(5) \qquad f_k = \mu_k m_1 g$$

When this result is substituted into (3), we get:

$$(6) \qquad T = \mu_k m_1 g + m_1 a$$

Equating the magnitude of the tension in (2) and (6), we get:

$$\mu_k m_1 g + m_1 a = m_2 (g - a)$$

Solving for a we get:

$$\boxed{a = \frac{m_2 - \mu_k m_1}{m_1 + m_2} g}$$

Note that, when $m_2 > \mu_k m_1$ we have accelerated motion, and when $m_2 = \mu_k m_1$, we have motion with zero acceleration, i.e. the speed is constant. The value of a can then be evaluated as follows:

$$a = \frac{6\text{ kg} - 0.5(4\text{ kg})}{6\text{ kg} + 4\text{ kg}} \times 9.8\text{ m/s}^2 = 3.92\text{ m/s}^2$$

We can find T by substituting the expression of a into (6), to get:

$$\boxed{T = \frac{(\mu_k + 1)\, m_1\, m_2}{m_1 + m_2} g}$$

Thus: $$T = \frac{(0.5 + 1)(4\,\text{kg})(6\,\text{kg})}{6\,\text{kg} + 4\,\text{kg}} \times 9.8\,\text{m/s}^2 = 35.28\,\text{N}$$

(b) When the system is on the verge of slipping, the magnitude of the force T that acts on mass m_1 must equal the maximum static friction $f_{s,\max} = \mu_s N$, i.e. $T = \mu_s N = \mu_s m_1 g$. Also, the weight of the mass m_2 must equal the magnitude of the tension, i.e. $T = m_2 g$. Thus:

$$m_2 g = \mu_s m_1 g$$

Finally:

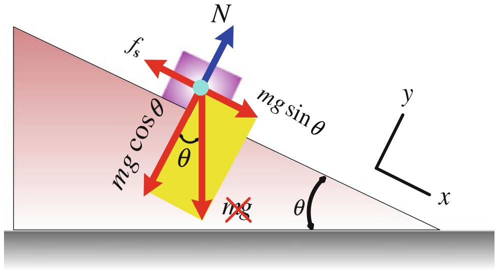
$$\boxed{m_2 = \mu_s\, m_1} \quad \text{(On the verge of slipping)}$$

Example 5.7

A block is at rest on a rough inclined plane of angle θ, as shown in Fig. 5.16. (a) Find the static frictional force f_s in terms of N and θ. (b) When the angle is increased until the block is on the verge of slipping at $\theta = \theta_c = 38.7°$, find the value of the coefficient of static friction μ_s. (c) After we increase θ further to allow the block to accelerate and then decrease θ again to the value $\theta = \theta' = 26.6°$

to allow the block to move with constant speed, find the coefficient of kinetic friction μ_k.

Fig. 5.16



Solution: (a) The block is balanced under its weight mg, the normal force N, and the static frictional force f_s. Taking x parallel to the plane and y perpendicular to it, then Newton's second law will give:

$$\Sigma F_x = mg\sin\theta - f_s = 0$$
$$\Sigma F_y = N - mg\cos\theta = 0$$

From the last equation we find $mg = N/\cos\theta$. Therefore, we can eliminate mg from the first equation to get:

$$f_s = mg\sin\theta = \frac{N}{\cos\theta}\sin\theta = N\tan\theta$$

(b) When the inclined plane is at the critical angle θ_c, the block is on the verge of slipping and $f_s = f_{s,\max} = \mu_s N$. So, at this angle the last equation becomes $\mu_s N = N\tan\theta_c$.

Thus: $$\boxed{\mu_s = \tan\theta_c} \xrightarrow[\text{when } \theta_c = 38.7^\circ]{} \mu_s = \tan 38.7^\circ = 0.8$$

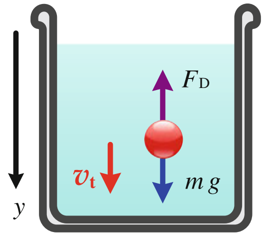
(c) When the block moves with constant speed at $\theta' = 26.6^\circ$, the kinetic friction $f_k = \mu_k N$ equals the weight component $mg\sin\theta'$.

Thus: $$\boxed{\mu_k = \tan\theta'} \xrightarrow[\text{when } \theta' = 26.6^\circ]{} \mu_k = \tan 26.6^\circ = 0.5$$

Example 5.8

A small sphere of mass $m = 1.5$ g is released from rest in a large vessel filled with liquid, see Fig. 5.17. The sphere reaches a terminal speed of $v_t = 2.45$ cm/s. Assume that the resistive drag force is given by Eq. 5.11.* (a) Solve Eq. 5.12 to find the speed of the sphere as a function of time. (b) Find the time t it takes the sphere to reach a speed of 0.9 v_t.

Fig. 5.17



Solution: * (a) To solve Eq. 5.12, we set $\tau = m/b$, which is called the **time constant**, and perform the following steps:

$$mg - bv = m\frac{dv}{dt} \quad \Rightarrow \quad \int_0^v \frac{dv}{v_t - v} = \tau^{-1}\int_0^t dt$$

The previous integration can be performed to get:

$$\boxed{v = v_t(1 - e^{-t/\tau})}$$

One can find from this result that the time $\tau = m/b$ is the time it takes the sphere to reach 63% of its terminal speed.

(b) Let us first determine the coefficient b in Eq. 5.11. Since the terminal speed is given by the relation $F_D = bv_t = mg$, i.e. $v_t = mg/b$, then the value of b will be given by:

$$b = \frac{mg}{v_t} = \frac{(1.5 \times 10^{-3}\,\text{kg})(9.8\,\text{m/s}^2)}{2.45 \times 10^{-2}\,\text{m/s}} = 0.6\,\text{kg/s}$$

Therefore, the value of the time τ is given by:

$$\tau = \frac{m}{b} = \frac{1.5 \times 10^{-3}\,\text{kg}}{0.6\,\text{kg/s}} = 2.5 \times 10^{-3}\,\text{s} = 2.5\,\text{ms}$$

We set $v = 0.9\, v_t$ in the resulting formula of part (a), and we perform the following steps, to find the corresponding time t:

$$0.9 = 1 - e^{-t/\tau} \quad \Rightarrow \quad e^{-t/\tau} = 0.1 \quad \Rightarrow \quad t = -\tau \ln(0.1)$$

Thus: $\quad t = -(2.5 \times 10^{-3}\,\text{s})(-2.303) = 5.76 \times 10^{-3}\,\text{s} = 5.76\ \text{ms}$

5.4 Exercises

Section 5.3 Applications to Newton's Laws

(Take $g = 10\ \text{m/s}^2$ in all the following exercises unless g is given)

(1) A 10 g bullet accelerates from rest to 500 m/s in a gun barrel of length 10 cm, see Fig. 5.18. Find the accelerating force (assuming it constant).

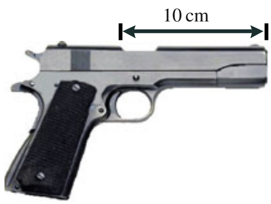


Fig. 5.18 See Exercise (1)

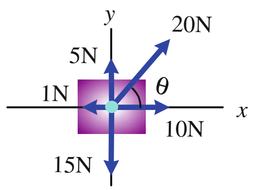
(2) A horizontal cable pulls a golf cart of mass 400 kg along a horizontal track. As in Fig. 5.19, the tension in the cable is 800 N. (a) Starting from rest, how long will it take the cart to reach a speed of 10 m/s? (b) Find the distance covered during this time.

Fig. 5.19 See Exercise (2)

(3) A block of mass 2 kg is accelerated by the two forces $\vec{F}_1 = 8\vec{i} + 3\vec{j}$ and $\vec{F}_2 = -5\vec{i} - 7\vec{j}$, (all units in newtons). (a) What is the net force on the block in unit vector notation, and what is its magnitude and direction? (b) What is the magnitude and direction of the acceleration?

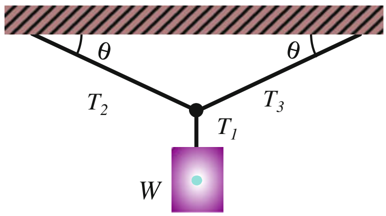
(4) Assume only five forces are acting in the xy plane on a block of mass 4 kg as shown in Fig. 5.20. (a) Taking $\sin\theta = 4/5$ and $\cos\theta = 3/5$, find the block's acceleration in unit-vector notation. (b) Find the acceleration's magnitude and direction.

Fig. 5.20 See Exercise (4)



(5) Consider a block of weight W hanging from three ropes as shown in Fig. 5.21. (a) At what angle θ will the magnitude of the tensions T_2 and T_3 each be equal to the weight W. (b) Is this angle independent of W?

Fig. 5.21 See Exercise (5)



(6) A traffic light of mass $m = 10$ kg is suspended over a road as shown in Fig. 5.22. The ropes are connected to the top of two vertical and identical posts at an angle $\theta = 65°$. Find the magnitude of the tension in all three cables.

(7) A block of mass 20 kg is suspended from the ceiling and the wall by three cords tied together as shown in Fig. 5.23. Find the magnitude of the tensions T_1, T_2, and T_3 in the ropes.

(8) After applying its brakes on a dry road, a 1,000 kg car moving at $v_\circ = 40$ m/s requires a minimum distance d of 50 m to come to a complete stop *without skidding*, see Fig. 5.24. (a) Find the car's acceleration. (b) Find the frictional force exerted on the car by the road. (c) Find the coefficient of static friction.

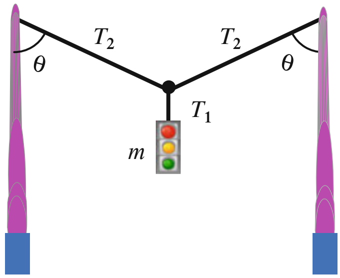


Fig. 5.22 See Exercise (6)

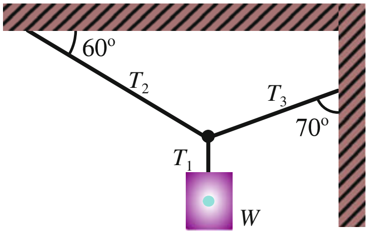


Fig. 5.23 See Exercise (7)

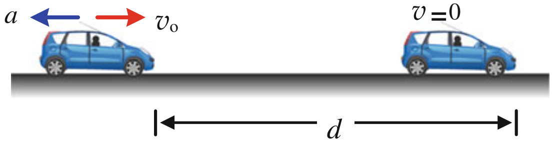


Fig. 5.24 See Exercise (8)

(9) A small sphere of mass m is attached to one end of a massless thread. The other end of that thread is fixed in the roof of a truck when it is at rest. Take $g = 9.8\,\text{m/s}^2$. (a) What angle θ does the thread make with the vertical when the truck has a constant acceleration $a = 1.5\,\text{m/s}^2$, see Fig. 5.25? (b) Find θ when the truck is moving with a constant velocity of magnitude $v = 100\,\text{km/h}$?

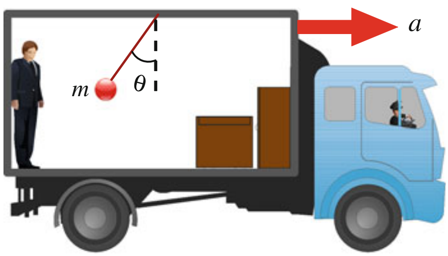
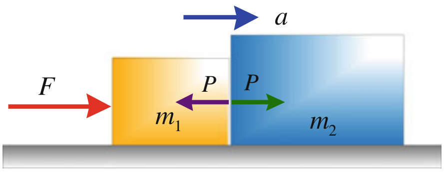


Fig. 5.25 See Exercise (9)

(10) A small object is hanging by a thread from the rearview mirror of a sports car. The car accelerates uniformly from rest to 90 km/h in 10 s. What angle θ does the thread make with the vertical?

(11) Two blocks of masses $m_1 = 4\,\text{kg}$ and $m_2 = 12\,\text{kg}$ are in contact on a smooth horizontal surface. A horizontal force of magnitude $F = 12$ N pushes them as shown in Fig. 5.26. (a) Find the magnitude of the acceleration of the system. (b) Find the magnitude of the force P that block m_1 exerts on block m_2.

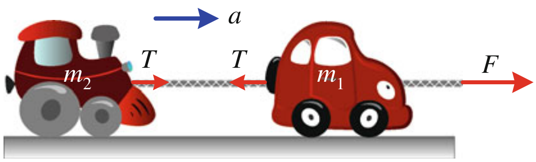
Fig. 5.26 See Exercise (11)



(12) Repeat exercise 11 if block m_2 is before block m_1 and the same force acts on m_2.

(13) Two toys of masses $m_1 = 40$ g and $m_2 = 120$ g are connected by a massless rope and lie on a horizontal frictionless surface as shown in Fig. 5.27. The toys are pulled to the right by a horizontal force of magnitude $F = 0.04$ N. (a) Find the acceleration of the system. (b) Find the magnitude of the tension force T in the connecting rope.

Fig. 5.27 See Exercise (13)

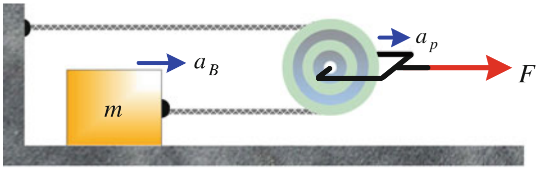


(14) When the surface of exercise 13 is rough, it is found that the toys move with a constant velocity. Find the common coefficient of kinetic friction μ_k between the toys and the surface.

(15) In Fig. 5.28, the pulley is assumed massless and frictionless and rotates freely about its axle. The block has a mass $m = 6$ kg and the pulley is pulled to the right by a horizontal force of magnitude $F = 24$ N. As the block moves a distance s_B in time t, the pulley moves half that distance in the same time t, i.e., $s_P = s_B/2$. (a) Find the acceleration ratio a_P/s_B. (b) Find the magnitudes of the tension in the cord and the acceleration of the block if there is no friction between the

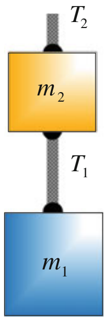
block and the surface. (c) Answer part (b) assuming that the kinetic friction between the block and the surface is $\mu_k = 0.1$.

Fig. 5.28 See Exercise (15)



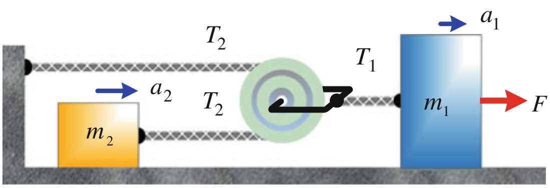
(16) A block of mass $m_1 = 6$ kg is hanging by a massless cord connected to another block of mass $m_2 = 4$ kg, which is also hanging by a massless cord, as shown in Fig. 5.29. (a) What is the tension in the cords when the system is at rest? (b) What is the tension in the cords when the two blocks are pulled up by the upper cord with an acceleration of 2 m/s^2?

Fig. 5.29 See Exercise (16)



(17) In Fig. 5.30, the pulley is assumed massless and frictionless and rotates freely about its axle. The blocks have masses $m_1 = 40$ g and $m_2 = 20$ g, and block m_1 is pulled to the right by a horizontal force of magnitude $F = 0.03$ N. Find the magnitude of the acceleration of block m_2 and the tension in the cord if the surface is frictionless.

Fig. 5.30 See Exercise (17)



(18) Figure 5.31 shows a bucket connected to a massless rope, which runs over a massless frictionless pulley. A man standing inside the bucket pulls the rope downwards in order to raise himself upwards. The mass of the man is 80 kg and the mass of the bucket is 20 kg. (a) How hard must the man pull the rope for him and the bucket to ascend at a constant speed? (b) Calculate the force that is needed for an upward acceleration of 1.2 m/s^2. (c) Using the man's free-body diagram, find the normal force exerted on the man by the bucket in parts (a) and (b).

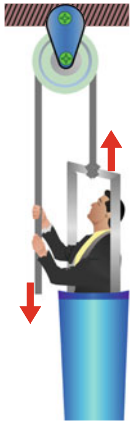


Fig. 5.31 See Exercise (18)

(19) A block of mass $m_1 = 2$ kg rests on the top of a second block of mass $m_2 = 8$ kg, as shown in Fig. 5.32. The left sides of the two blocks ate connected by a massless cord, which runs over a fixed massless frictionless pulley. The right side of block m_2 is pulled to the right by a horizontal force of magnitude F. How large must F such that block m_2 accelerates at 2 m/s^2?

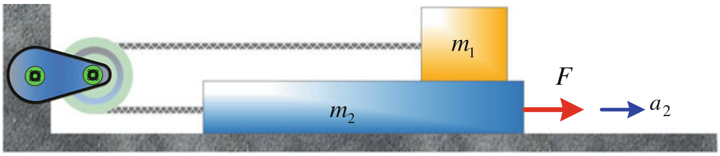
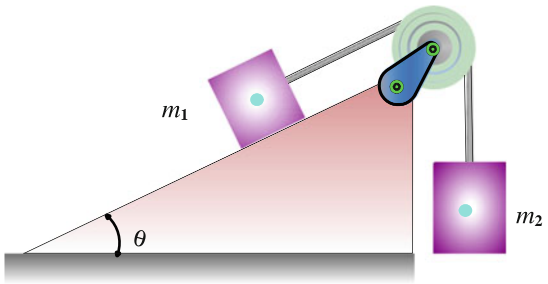


Fig. 5.32 See Exercise (19)

(20) Repeat exercise 19, this time assuming that the coefficient of kinetic friction between the surfaces at the top and bottom of the block m_2 is 0.5.

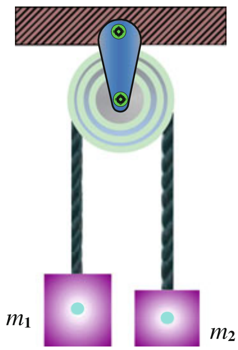
(21) A block of mass $m_1 = 4$ kg lies on a frictionless inclined plane of angle $\theta = 30°$. This block is connected by a cord over a massless, frictionless pulley to a second block of mass $m_2 = 6$ kg hanging vertically, as shown in Fig. 5.33. (a) For each block, find the magnitude and direction of its acceleration. (b) What is the magnitude of the tension in the cord? (c) Repeat parts (a) and (b) after replacing each block by the other.

Fig. 5.33 See Exercise (21)



(22) The Atwood's machine in Fig. 5.34 consists of two masses $m_1 = 6$ kg and $m_2 = 4$ kg that are connected by a light long cord that passes over a massless, frictionless pulley. Find the magnitude and direction of the acceleration of the two masses and the tension in the cord.

Fig. 5.34 See Exercise (22)



(23) If the two blocks in exercise 22 are initially at rest and are 2.25 m above the ground, find the speed of m_1 before hitting the ground.

(24) Two blocks having masses m_1 and m_2 ($m_2 > m_1$) are connected to each other by a light non-stretchable cord that passes over two identical massless frictionless pulleys, which rotate freely about their axles, as shown in Fig. 5.35. (a) Find the acceleration of each block and the tensions T_1, T_2, and T_3 in the cord.

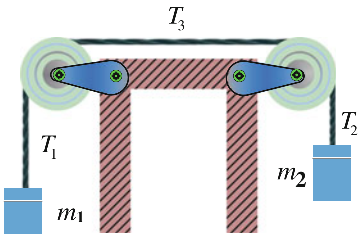


Fig. 5.35 See Exercise (24)

(25) A block of mass $m_1 = 6$ kg lying on a rough horizontal surface is connected to a second block of mass $m_2 = 3$ kg by a light non-stretchable cord over a massless, frictionless pulley as shown in Fig. 5.36. Assuming the two blocks move with constant speed, find the coefficient of kinetic friction between m_1 and the rough horizontal surface, and find the tension in the cord.

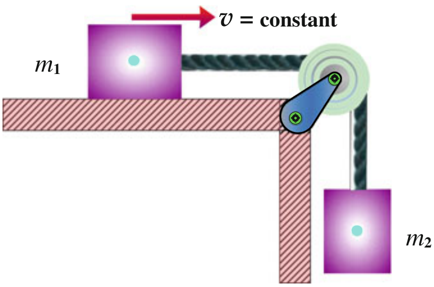


Fig. 5.36 See Exercise (25)

(26) A block of mass m_1 located on a horizontal frictionless surface is connected by a light non-stretchable cord that passes over a massless frictionless pulley to a second block of mass m_2, which is allowed to move on an inclined frictionless plane of angle θ, as shown in Fig. 5.37. Find the acceleration of the two blocks and the tension in the cord when $m_1 = 2$ kg, $m_2 = 6$ kg, $\sin\theta = 4/5$, and $\cos\theta = 3/5$.

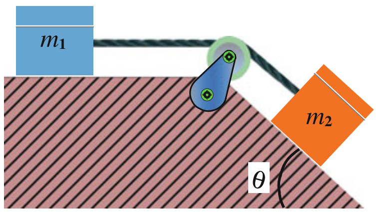


Fig. 5.37 See Exercise (26)

(27) Repeat exercise 26, this time assuming that the coefficient of kinetic friction for the two blocks on the horizontal and inclined planes are $\mu_{k1} = 0.3$ and $\mu_{k2} = 0.5$, respectively.

(28) A locomotive engine is pulling three cars behind it, see Fig. 5.38. Assume that the engine has mass m, and that each car also has mass m. If the driving force that is generated by the engine has a magnitude F, find the tension in the coupling between the cars in terms of F. Generalize this result when the number of the locomotive engine plus the cars is n.

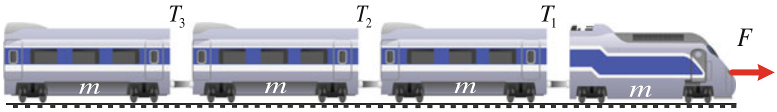
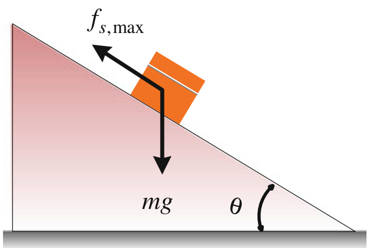


Fig. 5.38 See Exercise (28)

(29) A block is on the verge of skidding on a rough inclined plane of angle $\theta = \theta_c = 30°$, as shown in Fig. 5.39. (a) Find the value of the coefficient of static friction μ_s. (b) After we increase θ further to allow the block to accelerate and then decrease θ again to the value $\theta = \theta' = 20°$ to allow the block to move with constant speed, find the coefficient of kinetic friction μ_k.

Fig. 5.39 See Exercise (29)



(30) A box of mass $m = 200$ kg is pushed at a constant speed up an inclined frictionless ramp of angle $\theta = 30°$ by a horizontal force $\vec{F}$, as shown in Fig. 5.40. (a) What is the magnitude of the horizontal force? (b) What is the magnitude of the force exerted by the ramp on the box?

(31) A block of mass $m = 100$ kg is placed on an incline of angle $\theta = 25°$, see Fig. 5.41. What force is required to pull it up at a constant speed if the coefficient of kinetic friction between the block and the incline is 0.2?

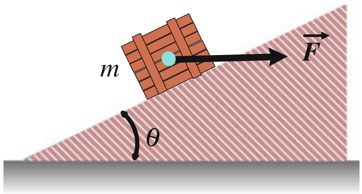


Fig. 5.40 See Exercise (30)

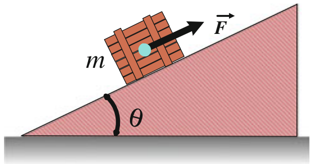


Fig. 5.41 See Exercise (31)

(32) Two blocks of masses $m_1 = 4\,\text{kg}$ and $m_2 = 8\,\text{kg}$ are connected by a massless rope and slide down an inclined plane of angle $\theta = 30°$, see Fig. 5.42. The coefficient of kinetic friction is $\mu_{k1} = 0.25$ between block m_1 and the plane, and $\mu_{k2} = 0.45$ between block m_2 and the plane. Find the acceleration of each block and the tension in the rope.

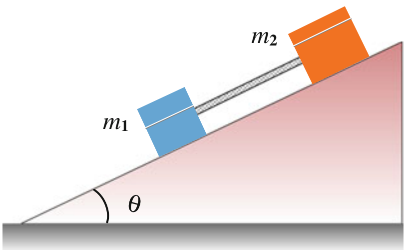


Fig. 5.42 See Exercise (32)

(33) What happens if the two blocks in exercise 31 are reversed such that block m_1 is located behind block m_2 on the plane?

(34) A ball of mass $m = 1.6\,\text{kg}$ hangs by a thread from the roof of an elevator. The thread can withstand only a tension force of 20 N. When the system accelerates upwards, see Fig. 5.43, the thread breaks. Within what range is the elevator accelerating?

(35) Assume that while ascending, an elevator has the same magnitude of acceleration a during starting (accelerating) and stopping (decelerating), see Fig. 5.44. A person stands on a scale in that elevator, and hence pushes downwards on the scale. The scale also reacts with an upward normal force. The maximum

and minimum scale readings are $N_{\mathrm{acc}} = 591$ N and $N_{\mathrm{dec}} = 391$ N, respectively. (a) Find the person's weight and mass. (b) Find the magnitude of the elevator's acceleration.

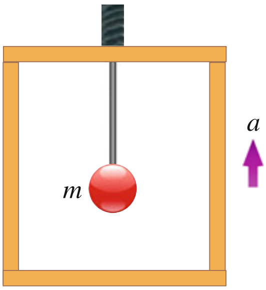


Fig. 5.43 See Exercise (34)

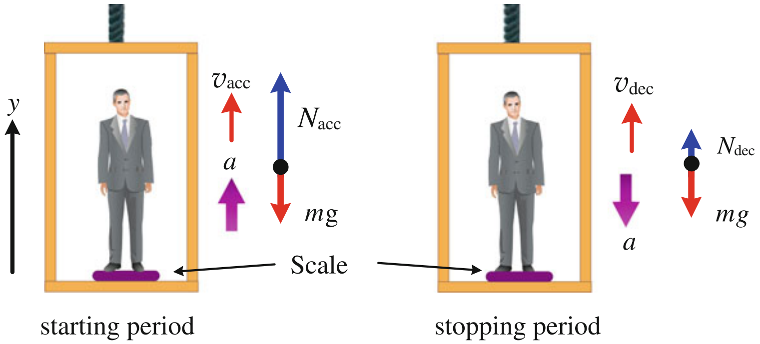


Fig. 5.44 See Exercise (35)

(36) Repeat exercise 35, this time assuming descending situation.

(37) A small sphere of mass $m = 2$ g is released from rest in a large vessel filled with oil. The sphere reaches a terminal speed of $v_{\mathrm{t}} = 3$ cm/s, see Fig. 5.45. Assume that the resistive drag force is given by Eq. 5.11. Find the time it takes the sphere to reach a speed of 0.99 v_{t}. [Use $v = v_{\mathrm{t}}(1 - e^{-t/\tau})$, where $\tau = m/b$ is the time constant]

(38) A spherical rain drop of radius $r = 2$ mm falls from a cloud at height $h = 1{,}000$ m above the ground. Assume that the drag coefficient C for the drop is 0.7, and that the density of water ρ_{w} is 1,000 kg/m^3, and that the density of air ρ_{a} is 1.2 kg/m^3. (a) What is the terminal speed of the drop? (b) If there is no drag force, what is the speed of the drop when it reaches the ground?

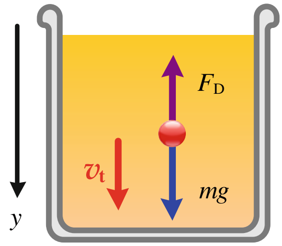


Fig. 5.45 See Exercise (37)

(39) A canonical pendulum consists of a bob of mass m attached to the end of a cord of length ℓ. The bob whirls around in a horizontal circle of radius r at a constant speed v while the cord always makes an angle θ with the vertical, see Fig. 5.46. Show that the bob's speed v and period T (the time for one complete revolution) are given by:

$$v = \sqrt{rg\tan\theta} = \sqrt{\ell g \sin\theta \tan\theta},$$

$$T = 2\pi\sqrt{\frac{\ell\cos\theta}{g}}$$

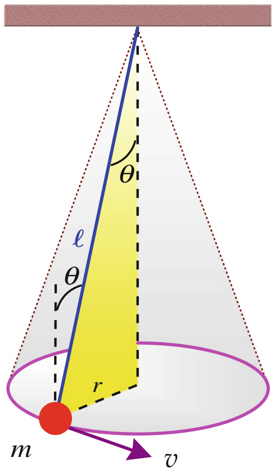


Fig. 5.46 See Exercise (39)

(40) What condition must be imposed on the relationship that governs the period of a canonical pendulum in order to reach to the period of a simple pendulum?

Work, Energy, and Power

6

Work, energy, and power are words that have different meanings in our everyday life. Nevertheless, physicists give them specific definitions, which we present in this chapter.

The *work-energy* power approach provides identical results to those obtained by Newtonian mechanics, but usually with simpler analysis, especially when dealing with complex situations where forces are not constant. Therefore, we will introduce two extremely important concepts: the *work-energy-theorem* and *conservation of energy*.

6.1 Work Done by a Constant Force

Consider a body that experiences a constant force $\vec{F}$ while undergoing a displacement $\vec{s}$ as it moves, see Fig. 6.1. We then define the work done by the constant force as follows:

> Work done by a constant force:
> Is defined as the product of the component of the force in the direction of the displacement and the magnitude of the displacement.

Thus:

$$W = (F\cos\theta)\, s = \vec{F} \bullet \vec{s} = \begin{cases} +Fs & \text{if } \theta = 0^\circ \\ 0 & \text{if } \theta = 90^\circ \\ -Fs & \text{if } \theta = 180^\circ \end{cases} \tag{6.1}$$

H. A. Radi and J. O. Rasmussen, *Principles of Physics*,
Undergraduate Lecture Notes in Physics, DOI: 10.1007/978-3-642-23026-4_6,

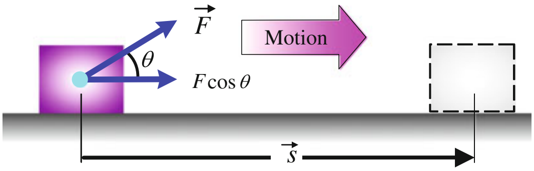


Fig. 6.1 The work done by a constant force $\vec{F}$ while undergoing a displacement $\vec{s}$ is $W = F\, s\cos\theta$

The unit of work in SI units is N.m [abbreviated by joule (J)], i.e., 1 J = 1 N.m, and in cgs units is dyne.cm (abbreviated by erg), i.e. 1 erg = 1 dyne.cm. Note that 1 J = 10^7 erg, see Table 6.1.

Table 6.1 Units of work

System	Unit of work	Name of combined unit
SI	N.m	joule (J)
cgs	dyne.cm	erg
British	ft.lb	ft.lb

Work Done by a Weight

Consider a block of mass m to be lifted up with almost zero acceleration (i.e., $a \simeq 0$) by a constant force $\vec{F}$ applied by a person, see Fig. 6.2. While in motion, the force $\vec{F}$ and the weight $m\vec{g}$ will be oppositely directed but equal in magnitude, i.e. $\vec{F} = -m\vec{g}$. That is:

$$F = mg \tag{6.2}$$

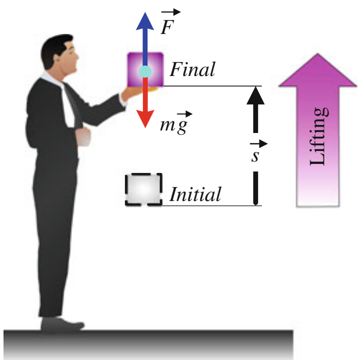


Fig. 6.2 *Lifting* a block with almost zero acceleration

If the upward displacement of the block is denoted by $\vec{s}$, as in Fig. 6.2, then we can calculate the work done by $\vec{F}$ as follows:

$$W_F = \vec{F} \bullet \vec{s} = Fs \cos 0^\circ = Fs = m\,g\,s \tag{6.3}$$

where we have used the fact that the angle between the two parallel vectors $\vec{F}$ and $\vec{s}$ is zero.

Also, we can calculate the work done by the gravitational force $m\,\vec{g}$ as follows:

$$W_g = m\,\vec{g} \bullet \vec{s} = m\,g\,s \cos 180^\circ = -m\,g\,s \tag{6.4}$$

Thus, we conclude that:

$$\boxed{W_F = m\,g\,s \quad \text{and} \quad W_g = -m\,g\,s \quad \text{(Lifting case)}} \tag{6.5}$$

where we have used the fact that the angle between the two antiparallel vectors $m\,\vec{g}$ and $\vec{s}$ is 180°. The net work $W_F + W_g$ done on the block is zero, as expected, because the net force on the block is zero. This is not, of course, to say that it takes no work to lift a block through a vertical height s. In such a context, we do not refer to the net work, but to the work done by the person.

When we lower the block vertically downward with almost zero acceleration for a displacement $\vec{s}$, see Fig. 6.3, the sign of the work done by $\vec{F}$ and $m\,\vec{g}$ will be reversed, since the sign of $\vec{s}$ has reversed.

Following similar steps, one can easily find:

$$W_F = \vec{F} \bullet \vec{s} = F\,s \cos 180^\circ = -F\,s = -m\,g\,s \tag{6.6}$$

$$W_g = m\,\vec{g} \bullet \vec{s} = m\,g\,s \cos 0^\circ = m\,g\,s \tag{6.7}$$

Thus, we conclude that:

$$\boxed{W_F = -m\,g\,s \quad \text{and} \quad W_g = m\,g\,s \quad \text{(Lowering case)}} \tag{6.8}$$

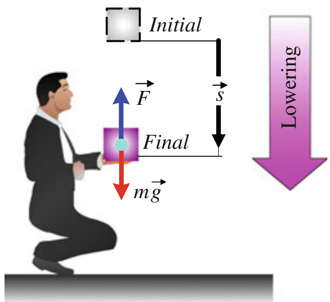


Fig. 6.3 *Lowering down* a block with almost zero acceleration

Work Done by Friction

A common example in which the work is always negative is the work done by friction. When a block slides over a rough surface due to an applied force $\vec{F}$, as shown in Fig. 6.4, the work done by the frictional force $\vec{f}_k$ while the block undergoes a displacement $\vec{s}$ is:

$$\begin{aligned} W_f &= \vec{f}_k \bullet \vec{s} = f_k\, s \cos 180^\circ \\ &= -f_k\, s \end{aligned} \tag{6.9}$$

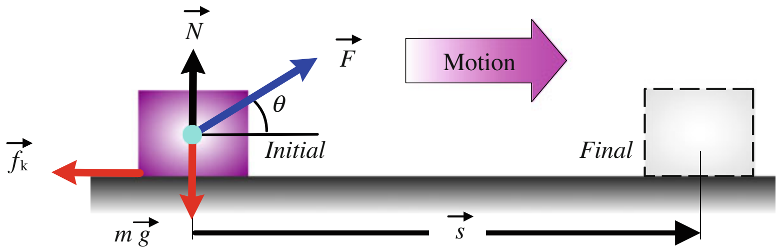


Fig. 6.4 The work done by the kinetic frictional force $\vec{f}_k$ while the block undergoes a displacement $\vec{s}$ is always negative and equals $W_f = -f_k\, s$

From Fig. 6.4, one can easily find the work done by gravity, the normal force, and the applied force as follows:

$$W_g = m\,\vec{g} \bullet \vec{s} = m\,g\,s\,\cos 90^\circ = 0 \tag{6.10}$$

$$W_N = \vec{N} \bullet \vec{s} = N\,s\,\cos 90^\circ = 0 \tag{6.11}$$

$$W_F = \vec{F} \bullet \vec{s} = F\,s \cos\theta \tag{6.12}$$

Example 6.1

A block of mass m is pushed up a rough inclined plane of angle θ by a constant force $\vec{F}$ parallel to the incline, as shown in Fig. 6.5. The displacement of the block up the incline is $\vec{d}$. (a) Find the work done by: the force $\vec{F}$, the kinetic friction $\vec{f}_k$, the force of gravity $m\,\vec{g}$, and the normal force $\vec{N}$. (b) Calculate the work done of part (a) for $m = 2$ kg, $\mu_k = 0.5$, $\theta = 30^\circ$, $F = 20$ N, and $d = 5$ m.

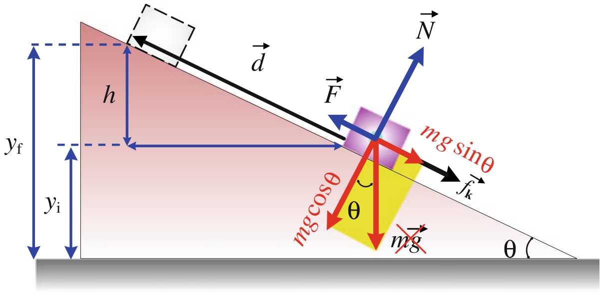


Fig. 6.5

Solution: (a) Since $\vec{F}$ is in the same direction as the displacement $\vec{d}$, we get:

$$W_F = \vec{F} \bullet \vec{d} = F\,d\cos 0° = F\,d$$

The work done by gravity is:

$$W_g = m\vec{g} \bullet \vec{d} = m\,g\,d\cos(90° + \theta) = -m\,g\,d\sin\theta = -m\,g\,h$$

where $h = y_f - y_i = d\sin\theta$ is the value of the vertical height. That is, the work done by gravity is negative and has a magnitude $m\,g$ multiplied by height h. This result and Eq. 6.4 proves that the work is independent of the path taken between any two points.

Since the force of friction $\vec{f}_k$ is opposite to the displacement $\vec{d}$, $f_k = \mu_k N$, and $N = mg\cos\theta$, the work done by friction will be:

$$W_f = \vec{f}_k \bullet \vec{d} = -f_k\,d = -\mu_k\,m\,g\,d\,\cos\theta$$

Since $\vec{N}$ is perpendicular to $\vec{d}$, we get:

$$W_N = \vec{N} \bullet \vec{d} = N\,d\,\cos 90° = 0$$

(b) Using the values given, the work done by each force will be:

$$W_F = F\,d = (20\,\text{N})(5\,\text{m}) = 100\,\text{J}$$

$$W_g = -m\,g\,d\sin\theta = -(2\,\text{kg})(9.8\,\text{m/s}^2)(5\,\text{m})(\sin 30°) = -49\,\text{J}$$

$$W_f = -\mu_k m\,g\,d\cos\theta = -0.5 \times (2\,\text{kg})(9.8\,\text{m/s}^2)(5\,\text{m})(\cos 30°) = -42.4\,\text{J}$$

Thus: $\quad W_{net} = W_F + W_g + W_f + W_N = 100 - 49 - 42.4 + 0 = 8.6\,\text{J}$

6.2 Work Done by a Variable Force

One-Dimensional Analysis

Consider an object that is being displaced along the x-axis from x_i to x_f due to the application of a varying positive force $F(x)$, as shown in Fig. 6.6a. To calculate the work done by this force, we imagine that the object undergoes a very small displacement Δx from x to $x + \Delta x$ due to the effect of an approximate constant force $F(x)$ as shown in Fig. 6.6b. For this very small displacement, we represent the amount of work done by the force by the expression:

$$\Delta W = F(x)\ \Delta x \tag{6.13}$$

which is just the area of the magnified rectangle shown in Fig. 6.6b. Then, the total work done from x_i to x_f by the variable force $F(x)$ is approximately equal to the sum of the large number of rectangles in Fig. 6.6b, i.e. the total area under the force curve. Thus:

$$W \simeq \sum_{x_\mathrm{i}}^{x_\mathrm{f}} F(x)\ \Delta x \tag{6.14}$$

In the limit where Δx approaches zero, the value of the sum in the last equation approaches the exact value of the area under the force curve, see Fig. 6.6c. As you probably know from calculus, the limit of that sum is called an integral and is represented by:

$$\lim_{\Delta x \to 0} \sum_{x_\mathrm{i}}^{x_\mathrm{f}} F(x)\ \Delta x = \int_{x_\mathrm{i}}^{x_\mathrm{f}} F(x)\ dx \tag{6.15}$$

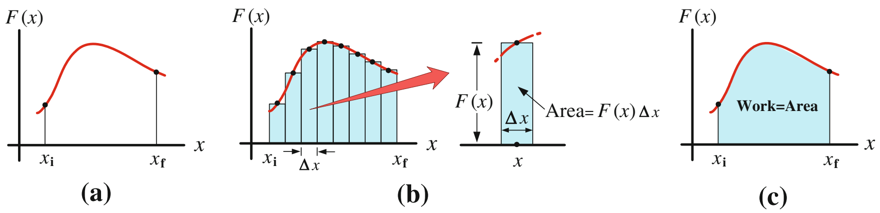


Fig. 6.6 (a) A variable force $F(x)$ displaces a body in the positive x direction from x_i to x_f. (b) The area under the curve is divided into narrow strips of thickness Δx, so that the approximate work done by the force $F(x)$ for the small displacement Δx is $\Delta W = F(x)\ \Delta x$. (c) In the limiting case, the work done by the force is the colored area under the force curve

Therefore, we can express the work done by a variable force $F(x)$ on an object that undergoes a displacement from x_i to x_f as follows:

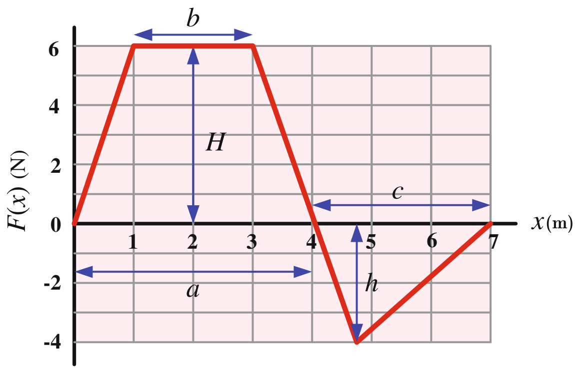
$$W = \int_{x_i}^{x_f} F(x)\, dx \tag{6.16}$$

If $F(x)$ is positive in some regions and negative in others, the last sum is called the *net signed area* and is equal to the area of the regions where $F(x) > 0$ minus the area of the regions where $F(x) > 0$.

Example 6.2

A force acting on an object varies with x as shown in Fig. 6.7. Find the work done by the force when the object undergoes a displacement from $x = 0$ to $x = 7$ m.

Fig. 6.7



Solution: The work done by the force equals the *net signed area* between the curve and the displacement from $x = 0$ to $x = 7$ m. That is, the area of the trapezoid minus the area of the triangle. Thus:

$$\begin{aligned} W &= \text{Area of the trapeziod} - \text{Area of the triangle} \\ &= \tfrac{1}{2}\,(a + b)\,H - \tfrac{1}{2}\,ch \\ &= \tfrac{1}{2} \times (4\,\text{m} + 2\,\text{m})\ \times (6\,\text{N}) - \tfrac{1}{2} \times (3\,\text{m})(4\,\text{N}) \\ &= 18\,\text{J} - 6\,\text{J} = 12\,\text{J} \end{aligned}$$

Work Done by a Spring

A spring is one type of common physical system in which the force (known as the spring force) varies with position. Figure 6.8a, shows a massless block on a horizontal frictionless surface attached to the free end of a **relaxed** spring. If the spring is stretched or compressed a small distance from equilibrium, the spring will exert a force on the block. This force is given by **Hooke's law** as follows:

$$\boxed{F = -\,k_{\mathrm{H}}\,x} \quad \text{(Hooke's law)} \tag{6.17}$$

where x is the displacement of the block from its equilibrium position ($x = 0$) and k_{H} is a positive constant known as the **spring constant** (or the **force constant**). The negative sign in Hooke's law indicates that the direction of the force is always opposite to the displacement. The spring force is positive (to the right) when $x < 0$, as in Fig. 6.8b, and is negative (to the left) when $x > 0$, as in Fig. 6.8c. This type of force always acts toward the equilibrium and is called a *restoring force*.

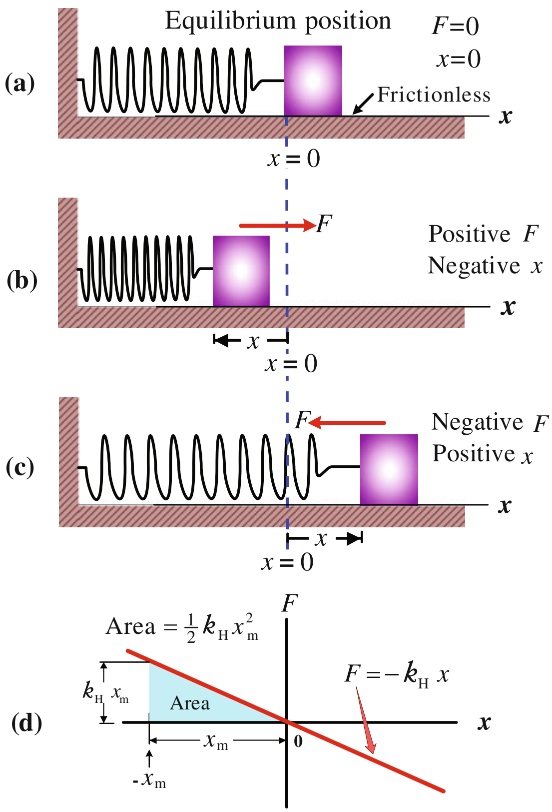


Fig. 6.8 The variation of the force of a spring on a block. (a) When $x = 0$, the force is zero (equilibrium position). (b) When x is negative, the force is positive (compressed spring). (c) When x is positive, the force is negative (stretched spring). (d) Graph of F versus x. The work done by the spring force as the block moves from $-x_{\mathrm{m}}$ to 0 is the colored triangular area which equals $\frac{1}{2}k_{\mathrm{H}}\,x_{\mathrm{m}}^2$

If we allow the block to compress the spring a distance of x_m from its equilibrium position and then release the block, it will move from $-x_m$ through the equilibrium position $x = 0$ to x_m. In the absence of friction, the block will oscillate indefinitely between $-x_m$ and x_m. In this case x_m is called the *amplitude* of the oscillations.

To calculate the work done by the spring force on the body as it moves from $x_i = -x_m$ to $x_f = 0$, we use Hooke's law in Eq. 6.16 as follows:

$$W_s = \int_{x_i}^{x_f} F(x)\,dx = \int_{-x_m}^{0} (-k_H\,x)\,dx = \left[-\tfrac{1}{2}k_H\,x^2\right]_{-x_m}^{0} = \tfrac{1}{2}k_H\,x_m^2 \tag{6.18}$$

Note that the work done by the spring force is positive because the spring force is in the same direction as the displacement. We can reach the same result of Eq. 6.18 if we plot F versus x, as shown in Fig. 6.8d, and then calculate the area of the colored triangle that has a base x_m and height $k_H\,x_m$. On the other hand, when $x_i = 0$ and $x_f = x_m$, we can find that $W_s = -\tfrac{1}{2}k_H\,x_m^2$. In this part of the motion, the spring force is to the left and the displacement is to the right, resulting in a negative work. Generally, if the block undergoes an arbitrary displacement from x_i to x_f, the work done by the spring force will be given by:

$$\boxed{W_s = \int_{x_i}^{x_f} F(x)\,dx = \int_{x_i}^{x_f} (-k_H\,x)\,dx = \tfrac{1}{2}k_H\,x_i^2 - \tfrac{1}{2}k_H\,x_f^2} \tag{6.19}$$

This shows that the work done is zero for any motion that has $x_i = x_f$.

Let us calculate the work done by the *applied force* $\vec{F}_{app}$ when the block moves *very slowly* from x_i to x_f, see Fig. 6.9a and b. To find this work we notice that $\vec{F}_{app}$ is equal and opposite to the spring force $\vec{F}$ at any displacement, i.e. $F_{app} = -F = -(-k_H\,x) = k_H\,x$. Thus:

$$\boxed{W_{F_{app}} = \int_{x_i}^{x_f} F_{app}\,dx = \int_{x_i}^{x_f} k_H\,x\,dx = \left[\tfrac{1}{2}k_H\,x^2\right]_{x_i}^{x_f} = \tfrac{1}{2}k_H\,x_f^2 - \tfrac{1}{2}k_H\,x_i^2} \tag{6.20}$$

Comparing Eq. 6.19 and Eq. 6.20 we find that $W_{F_{app}} = -W_s$, as expected. If we plot F_{app} versus x, as shown in Fig. 6.9c, then the work done by F in compressing the spring *very slowly* from $x_i = 0$ to $x_f = -x_m$ equals the area of the colored triangle that has a base x_m and height $k_H\,x_m$, i.e. $W_{app} = \tfrac{1}{2}k_H\,x_m^2$ (F_{app} *and the displacement are negative*).

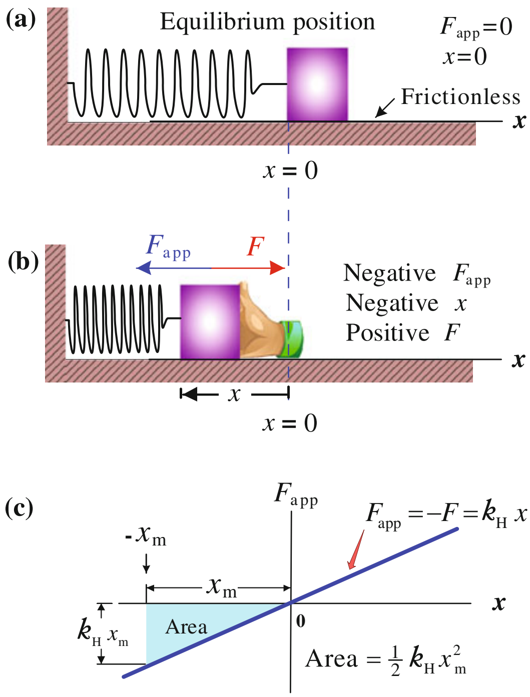


Fig. 6.9 (a) When $x = 0$, the applied force is zero (equilibrium position). (b) When x is negative, the applied force is negative (compressed spring). (c) Graph of the applied force versus x. The work done by the applied force as the block moves very slowly from $x = 0$ to $x = -x_m$ is the colored triangular area which equals $\frac{1}{2}k_H\,x_m^2$

Example 6.3

An applied force $F_{app} = -5\,\text{N}$ is exerted on the block that is attached to the free end of the spring of Fig. 6.9b. As a result of this force, the spring is compressed by 1 cm from its relaxed length. (a) What is the spring constant of the spring? (b) What force does the spring exert on the block if the spring is compressed by 2.5 cm? (c) How much work does the spring force do on the block as the spring is compressed from the relaxed state by 2.5 cm? (d) How much work does the spring force do on the block during a total displacement starting from a compression of 2.5 cm, passing through the equilibrium, and then to a stretch of 2.0 cm?

Solution: (a) The compressed spring pushes the block with a force $F = -F_{app} = +5\,\text{N}$. From $F = -k_H\,x$, with $x = -1$ cm, we have:

$$k_H = -\frac{F}{x} = -\frac{5\,\text{N}}{(-1 \times 10^{-2}\,\text{m})} = 500\,\text{N/m}$$

(b) Using Hooke's law, with $x = -2.5\,\text{cm} = -2.5 \times 10^{-2}$ m, we have:

$$F = -k_H x = -(500\,\text{N/m})(-2.5 \times 10^{-2}\,\text{m}) = 12.5\,\text{N}$$

(c) Since the spring is initially at its relaxed state, the work done by the spring force on the block from $x_{\mathrm{i}} = 0$ to $x_{\mathrm{f}} = -2.5 \times 10^{-2}$ m will be:

$$W_{\mathrm{s}} = \tfrac{1}{2} k_{\mathrm{H}}\, x_{\mathrm{i}}^2 - \tfrac{1}{2} k_{\mathrm{H}}\, x_{\mathrm{f}}^2 = 0 - \tfrac{1}{2}(500\,\mathrm{N/m})(-2.5 \times 10^{-2}\,\mathrm{m})^2 = -0.156\,\mathrm{J}$$

The work is negative because the spring force and the displacement are in opposite directions. Note that the amount of work done by the spring on the block would be the same when stretching by 2.5 cm.

(d) For this case, we have $x_{\mathrm{i}} = -2.5 \times 10^{-2}$ m (the spring is initially compressed) and $x_{\mathrm{f}} = +2.0 \times 10^{-2}$ m (the spring is finally stretched). Then Eq. 6.19 becomes:

$$\begin{aligned} W_{\mathrm{s}} &= \tfrac{1}{2} k_{\mathrm{H}}\, x_{\mathrm{i}}^2 - \tfrac{1}{2} k_{\mathrm{H}}\, x_{\mathrm{f}}^2 = \tfrac{1}{2} k_{\mathrm{H}} (x_{\mathrm{i}}^2 - x_{\mathrm{f}}^2) \\ &= \tfrac{1}{2}(500\,\mathrm{N/m})[(-2.5 \times 10^{-2}\,\mathrm{m})^2 - (+2.0 \times 10^{-2}\,\mathrm{m})^2] = 0.06\,\mathrm{J} \end{aligned}$$

Three-Dimensional Analysis

Consider a particle that is acted upon by a three-dimensional force of the following form:

$$\vec{F} = F_x\,\vec{\mathrm{i}} + F_y\,\vec{\mathrm{j}} + F_z\,\vec{\mathrm{k}} \tag{6.21}$$

where the components F_x, F_y, and F_z are generally a function of the position vector $\vec{r}$ of the particle. Furthermore, let the particle move through an incremental displacement $d\vec{r}$, i.e.

$$d\vec{r} = dx\,\vec{\mathrm{i}} + dy\,\vec{\mathrm{j}} + dz\,\vec{\mathrm{k}} \tag{6.22}$$

In this case, the increment of work dW done on the particle by the force $\vec{F}$ during the incremental displacement $d\vec{r}$ is giving by:

$$\begin{aligned} dW &= \vec{F} \bullet d\vec{r} = (F_x\,\vec{\mathrm{i}} + F_y\,\vec{\mathrm{j}} + F_z\,\vec{\mathrm{k}}) \bullet (dx\,\vec{\mathrm{i}} + dy\,\vec{\mathrm{j}} + dz\,\vec{\mathrm{k}}) \\ &= F_x\,dx + F_y\,dy + F_z\,dz \end{aligned} \tag{6.23}$$

The work W done by the force $\vec{F}$ on the particle when it moves from an initial position r_{i} of coordinates $(x_{\mathrm{i}}, y_{\mathrm{i}}, z_{\mathrm{i}})$ to a final position r_{f} of coordinates $(x_{\mathrm{f}}, y_{\mathrm{f}}, z_{\mathrm{f}})$ can be represented by:

$$W = \int_{r_i}^{r_f} dW = \int_{x_i}^{x_f} F_x\, dx + \int_{y_i}^{y_f} F_y\, dy + \int_{z_i}^{z_f} F_z\, dz \tag{6.24}$$

When $\vec{F}$ has only an x component, this equation reduces to Eq. 6.16.

6.3 Work-Energy Theorem

Consider a particle of mass m, moving with acceleration $a = a(x)$ along the x-axis under the effect of a net force $F(x)$ that points along this axis. Thus, according to Newton's second law of motion we have $F(x) = ma$. The work done by this net force on the particle as it moves from an initial position x_i to a final position x_f can be found using Eq. 6.16 as follows:

$$W = \int_{x_i}^{x_f} F(x)\, dx = \int_{x_i}^{x_f} m\, a\, dx \tag{6.25}$$

We can write the quantity $m\, a\, dx$ in the last equation as:

$$m\, a\, dx = m\frac{dv}{dt}\, dx \tag{6.26}$$

Since v is a function of time, then we can use the "chain rule" to have:

$$\frac{dv}{dt} = \frac{dv}{dx}\,\frac{dx}{dt} = \frac{dv}{dx}\, v \tag{6.27}$$

Then Eq. 6.26 becomes:

$$m\, a\, dx = m\frac{dv}{dx}\, v\, dx = m\, v\, dv \tag{6.28}$$

Substituting this result back into Eq. 6.25 yields:

$$W = \int_{v_i}^{v_f} m\, v\, dv = m\int_{v_i}^{v_f} v\, dv = m\left[\tfrac{1}{2}v^2\right]_{v_i}^{v_f} = \tfrac{1}{2}m\, v_f^2 - \tfrac{1}{2}m\, v_i^2 \tag{6.29}$$

Note that when we change the integration variable from x to v we are required to change the limits of the integration to the new variable.

For a particle of mass m that has a speed v well below the speed of light, we define its kinetic energy as:

Kinetic Energy
The kinetic energy K of a particle is defined as the product of one half of its mass and the square of its speed, i.e.

$$\boxed{K = \tfrac{1}{2} m\, v^2} \tag{6.30}$$

Kinetic energy is a scalar quantity and has the same units as work. In SI units we have:

$$1\,\mathrm{J} = 1\,\mathrm{kg.m^2/s^2} = 1\,\mathrm{N.m} \tag{6.31}$$

We can view kinetic energy as the energy associated with the motion of an object. It is more convenient to express Eq. 6.29 as:

$$\boxed{W = \Delta K = K_\mathrm{f} - K_\mathrm{i}} \tag{6.32}$$

where K_i is the particle's initial kinetic energy and K_f is its final kinetic energy after the work is done. That is, the work done by the *net force* in displacing a particle equals the change in its kinetic energy. If there are many forces, such as an applied force $\vec{F}$, a gravitational force $m\vec{g}$, a spring force $\vec{F}_\mathrm{s}$, a frictional force $\vec{f}$, etc, the work done by the net force in displacing a particle will be equal to the sum of the work done by all the forces acting on the particle. That is:

$$\boxed{W_\mathrm{net} = W_\mathrm{F} + W_\mathrm{g} + W_\mathrm{s} + W_\mathrm{f} + \ldots = K_\mathrm{f} - K_\mathrm{i} = \Delta K} \tag{6.33}$$

Equation 6.32 is known as the **work-energy theorem**. This theorem is valid even when the force varies in direction and magnitude while the particle (or the object) moves along an arbitrary curved path in three dimensions.

Example 6.4

A box of mass $m = 10\,\mathrm{kg}$ is initially at rest on a rough horizontal surface, where the coefficient of kinetic friction between the box and the surface is $\mu_\mathrm{k} = 0.2$. The box is then pulled horizontally by a force $F = 50\,\mathrm{N}$ that makes an angle $\theta = 60°$ with the horizontal, see Fig. 6.10. (a) Use the work-energy theorem to find the speed v_f of the box after it moves a distance s of 4 m. *(b) Repeat part (a) using Newtonian mechanics.

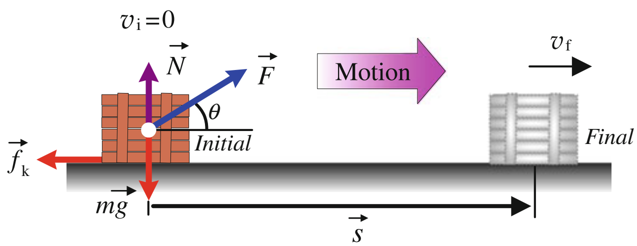


Fig. 6.10

Solution: (a) Both the weight-gravitational force $m\vec{g}$ and the normal force $\vec{N}$ do no work, since the displacement is horizontal, i.e. $W_g = W_N = 0$. The work done by the applied force is:

$$W_F = \vec{F} \bullet \vec{s} = F\, s \cos\theta = (50\,\text{N})(4\,\text{m})(\cos 60°) = 100\,\text{J}$$

The magnitude of the frictional force is $f_k = \mu_k N$, where in this case $N = mg - F\sin\theta$. Therefore, the work done by friction is:

$$\begin{aligned} W_f &= \vec{f}_k \bullet \vec{s} = -f_k\, s = -\mu_k(mg - F\sin\theta)\, s \\ &= -0.2 \times [(10\,\text{kg})(9.8\,\text{m/s}^2) - (50\,\text{N})(0.866)](4\,\text{m}) \\ &= -43.76\,\text{J} \end{aligned}$$

Thus, the net work done on the box is:

$$W_{\text{net}} = W_F + W_g + W_N + W_f = 100\,\text{J} + 0 + 0 + (-43.76\,\text{J}) = 56.24\,\text{J}$$

Applying the work-energy theorem with $v_i = 0$ gives:

$$W_{\text{net}} = K_f - K_i = \tfrac{1}{2} m\, v_f^2 \quad \Rightarrow \quad v_f = \sqrt{\frac{2\, W_{\text{net}}}{m}} = \sqrt{\frac{2 \times 56.24\,\text{J}}{10\,\text{kg}}} = 3.35\,\text{m/s}$$

*(b) Applying Newton's second law in the component form, then for the horizontal component, we find that:

$$\Sigma F_x = F\cos\theta - f_k = m\,a$$

Thus, the acceleration of the box will be given by:

$$\begin{aligned} a &= \frac{F\cos\theta - \mu_{\mathrm{k}}(mg - F\sin\theta)}{m} \\ &= \frac{(50\,\mathrm{N})(\cos 60^\circ) - 0.2\times[(10\,\mathrm{kg})(9.8\,\mathrm{m/s^2}) - (50\,\mathrm{N})(0.866)]}{10\,\mathrm{kg}} = 1.406\,\mathrm{m/s^2} \end{aligned}$$

To find the final speed, we use the kinematic equation $v_{\mathrm{f}}^2 = v_{\mathrm{i}}^2 + 2as$ when $v_{\mathrm{i}} = 0$ to get:

$$v_{\mathrm{f}} = \sqrt{2as} = \sqrt{2\times(1.406\,\mathrm{m/s^2})(4\,\mathrm{m})} = 3.35\,\mathrm{m/s}$$

Because the forces are constants in this example, the analysis used by Newtonian mechanics is easier than that of the work-energy theorem.

6.4 Conservative Forces and Potential Energy

In the previous section we introduced the concept of kinetic energy and found that it can change only if work is done on the object. In this section we introduce another form of energy, called *potential energy*, associated with the position or configuration of an object, and can be thought of as a stored energy that can be converted to kinetic energy or to work. We begin by defining the following:

(a) Conservative and Non-conservative Forces

Conservative Forces

In Example 6.1 we were able to see that the work done by gravity depends only on the initial and final vertical coordinates and hence is independent of the path taken between any two points. Also, we found the same holds true in the case of a spring. In addition, we can easily see from Sect. 6.2 that the net work done on the object by the gravitational force during a round trip is zero. When a force exhibits these properties, it is called a **conservative force**.

With reference to the arbitrary paths of Fig. 6.11a, we can write the first condition for a conservative force as:

$$W_{ab}(\text{path } 1) = W_{ab}(\text{path } 2) \tag{6.34}$$

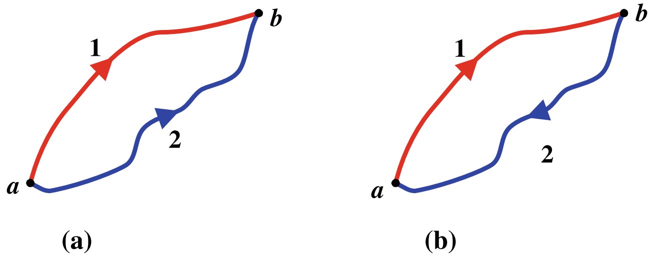


Fig. 6.11 (a) A conservative force acts on a particle moving from point a to point b by following either path 1 or path 2. (b) A conservative force acts on a particle moving in a round trip from point a to point b along path 1 and then back to point a along path 2

i.e., the work done by a conservative force on a particle moving from a to b along path 1 is the same as from a to b along path 2. In words:

> Spotlight
> The net work done by a conservative force on a particle moving between any two points does not depend on the path taken.

Also, with reference to the arbitrary paths of Fig. 6.11b, we can write the second condition for a conservative force as:

$$\left.\begin{array}{c} W_{ab}(\text{path 1}) = -W_{ba}(\text{path 2}) \\ \text{or} \\ W_{ab}(\text{path 1}) + W_{ba}(\text{path 2}) = 0 \end{array}\right\} \tag{6.35}$$

That is, the work done by a conservative force on a particle that moves in a round trip from a to b along path 1 and then from b to a along path 2 is zero. In other words:

> Soptlight
> The net work done by a conservative force on a particle that is moving around any closed path is zero.

From the work-energy theorem, $W = 0$ for a round trip, which means that the particle will return to its starting point with the same kinetic energy it had when it started its motion.

We recall from Example 6.1 that the work done by the gravitational force as a particle of mass m moves between two points of elevations y_{i} and y_{f} can be written as:

$$W_g = -mgh = -mg(y_{\mathrm{f}} - y_{\mathrm{i}}) \tag{6.36}$$

which satisfies the two conditions of a conservative force.

Non-conservative Forces

Not all forces are conservative. For example, let us allow a book to slide across a table that is not frictionless, see Fig. 6.12a. During the sliding, the kinetic frictional force does negative work on the book, slowing it by transferring energy from its kinetic energy to thermal energy of the book-table system. This energy transfer cannot be reversed. So, this force is not conservative. Therefore, all types of frictional forces are **non-conservative** forces. That is:

> Soptlight
> The work done by a non-conservative force on a particle that is moving between any two points depends on the path taken by the particle.

With reference to the arbitrary paths of Fig. 6.12a, we can write the first condition for a non-conservative force as:

$$W_{AB}(\text{path 1}) \neq W_{AB}(\text{path 2}) \quad \text{(Non-conservative forces)} \tag{6.37}$$

i.e., the work done by a non-conservative force on a particle moving from A to B along path 1 is always not the same along path 2.

Also, with reference to the arbitrary paths of Fig. 6.12b, we can write the second condition for a non-conservative force as:

$$\left.\begin{array}{c} W_{AB}(\text{path 1}) \neq -W_{BA}(\text{path 2}) \\ \text{or} \\ W_{AB}(\text{path 1}) + W_{BA}(\text{path 2}) \neq 0 \end{array}\right\} \tag{6.38}$$

That is, the work done by a non-conservative force on a particle that moves in a round trip from A to B along path 1 and then from B to A along path 2 is not zero.

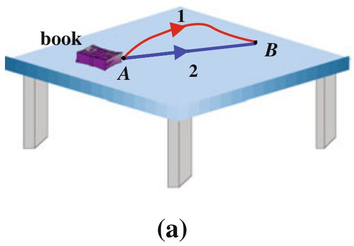


Fig. 6.12 (a) The work done by the force of friction depends on the path taken as the book is moved from A to B. (b) The work done by the force of friction in a round trip from point A to point B along path 1 and then back to point A along path 2 is not zero

(b) Potential Energy

We found that the work done by a conservative force is a function of the particle's initial and final coordinates and neither depends on the path taken nor depends on its velocity. Therefore, we can define a function U called the "*potential energy*" such that the work done by a conservative force equals the decrease of potential energy. That is:

$$\boxed{W_c = -\Delta U = U_\mathrm{i} - U_\mathrm{f}} \tag{6.39}$$

where the subscript "c" refers to a conservative force and the change in potential energy is defined as $\Delta U = U_\mathrm{f} - U_\mathrm{i}$. For a particle moving along the x axis under the effect of a conservative force $\vec{F}$ that has an x component F_x, we can express Eq. 6.39 as follows:

$$\boxed{\Delta U = U_\mathrm{f} - U_\mathrm{i} = -\int_\mathrm{i}^\mathrm{f} F_x\,dx} \quad \text{or} \quad \boxed{F_x = -\frac{dU}{dx}} \tag{6.40}$$

It is often convenient to choose some selected initial configuration that has a potential U_i as a **reference point** and measure all potential energy differences with respect to this point. Usually, we set $U_\mathrm{i} = 0$ at this point because it does not really matter what value we assign to U_i.

Gravitational Potential Energy

Consider a particle with mass m moving vertically along the y axis from point y_i to point y_f. Of course, the displacement will be an upward vector while the weight

$m\vec{g}$ will be a downward vector. To find the corresponding change in gravitational potential energy of the particle-Earth system, we change the integration in Eq. 6.40 to be along the y axis and substitute $-mg$ for the force F_x. Thus:

$$\Delta U = U_{\mathrm{f}} - U_{\mathrm{i}} = -\int_{y_{\mathrm{i}}}^{y_{\mathrm{f}}} (-mg)\, dy = mg \int_{y_{\mathrm{i}}}^{y_{\mathrm{f}}} dy = mg\left[y\right]_{y_{\mathrm{i}}}^{y_{\mathrm{f}}} = mg(y_{\mathrm{f}} - y_{\mathrm{i}})$$

That is:

$$\boxed{\Delta U = U_{\mathrm{f}} - U_{\mathrm{i}} = mg(y_{\mathrm{f}} - y_{\mathrm{i}}) = mg\Delta y} \tag{6.41}$$

Only the change in gravitational potential energy ΔU is physically important. So, according to the previous result, we can set $U_{\mathrm{i}} = 0$ when $y_{\mathrm{i}} = 0$. This gives:

$$U_{\mathrm{f}} - 0 = mg(y_{\mathrm{f}} - 0)$$

which is generally written as follows:

$$\boxed{U = mgy} \quad \text{(Gravitational potential energy)} \tag{6.42}$$

That is, the gravitational potential energy associated with the particle-Earth system depends on the vertical position y (or the height) of the particle relative to the reference position $y = 0$, and does not depend on the horizontal position. We can think of $U = mgy$ as the configuration energy stored in the particle-Earth system.

Elastic Potential Energy

Now Consider a block attached to a spring with a spring constant k_{H} as in Fig. 6.8. As the block moves from position x_{i} to position x_{f} the spring force $F = -k_{\mathrm{H}}\,x$ does work on the block. To find the corresponding change in elastic potential energy of the block-spring system, we substitute $-k_{\mathrm{H}}\,x$ for the force F_x in Eq. 6.40 to get:

$$\Delta U = U_{\mathrm{f}} - U_{\mathrm{i}} = -\int_{x_{\mathrm{i}}}^{x_{\mathrm{f}}} (-k_{\mathrm{H}}\,x)\, dx = k_{\mathrm{H}} \int_{x_{\mathrm{i}}}^{x_{\mathrm{f}}} x\, dx = k_{\mathrm{H}}\left[\tfrac{1}{2}x^2\right]_{x_{\mathrm{i}}}^{x_{\mathrm{f}}} = \tfrac{1}{2}k_{\mathrm{H}}\,x_{\mathrm{f}}^2 - \tfrac{1}{2}k_{\mathrm{H}}\,x_{\mathrm{i}}^2$$

That is:

$$\boxed{\Delta U = U_{\mathrm{f}} - U_{\mathrm{i}} = \tfrac{1}{2}k_{\mathrm{H}}\,x_{\mathrm{f}}^2 - \tfrac{1}{2}k_{\mathrm{H}}\,x_{\mathrm{i}}^2} \tag{6.43}$$

We set $U_{\mathrm{i}} = 0$ at the equilibrium position of the block, i.e. when $x_{\mathrm{i}} = 0$. This gives:

$$U_{\mathrm{f}} - 0 = \tfrac{1}{2}k_{\mathrm{H}}\,x_{\mathrm{f}}^2 - 0$$

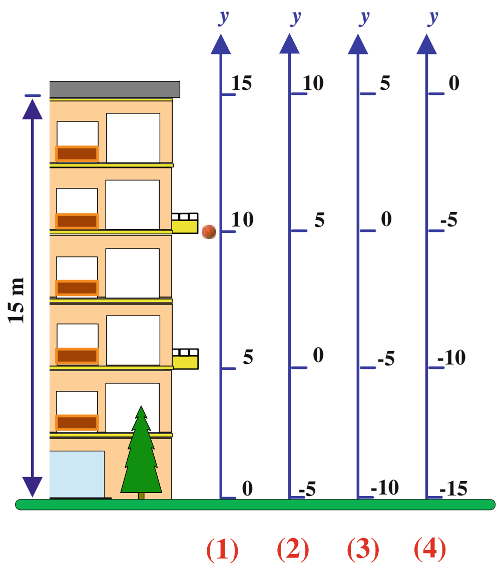
which is generally written as follows:

$$\boxed{U = \tfrac{1}{2}k_{\mathrm{H}}\,x^2} \quad \text{(Elastic potential energy)} \tag{6.44}$$

Example 6.5

A ball of mass $m = 0.2$ kg is at the level of a second balcony which is 10 m above the ground, see Fig. 6.13. (a) What is the gravitational potential energy of the ball if we take the reference point $y = 0$ to be: (1) at the ground, (2) at the first balcony, (3) at the second balcony, and (4) at the top of the building? (b) If the ball drops to the ground, for each of the reference points of part (a), what is the change of potential energy of the ball due to the fall?

Fig. 6.13 Example 6.5



Solution: (a) Using Eq. 6.42, we can calculate the potential energy U of the ball for each choice of $y = 0$ as follows:

coordinate choice (1): $U = mgy = (0.2\,\text{kg})(9.8\,\text{m/s}^2)(10\,\text{m}) = 19.6\,\text{J}$

coordinate choice (2): $U = mgy = (0.2\,\text{kg})(9.8\,\text{m/s}^2)(5\,\text{m}) = 9.8\,\text{J}$

coordinate choice (3): $U = mgy = (0.2\,\text{kg})(9.8\,\text{m/s}^2)(0\,\text{m}) = 0\,\text{J}$

coordinate choice (4): $U = mgy = (0.2\,\text{kg})(9.8\,\text{m/s}^2)(-5\,\text{m}) = -9.8\,\text{J}$

(b) For all the coordinate choices, we have $\Delta y = -10$ m. So Eq. 6.41 will give the same change in potential energy as follows:

$$\Delta U = mg\Delta y = (0.2\,\text{kg})(9.8\,\text{m/s}^2)(-10\,\text{m}) = -19.6\,\text{J}$$

Thus, although the value of U depends on the choice of where we let $y = 0$, the change in potential energy does not. In fact, only the change ΔU, not the value of U, in potential energy is physically important.

6.5 Conservation of Mechanical Energy

When a conservative force does work W_c on a particle, the work-energy theorem tells us that there will be a change in its kinetic energy given by Eq. 6.32, which can be rewritten as:

$$W_c = \Delta K \tag{6.45}$$

and a change in potential energy given by Eq. 6.39, rewritten:

$$W_c = -\Delta U \tag{6.46}$$

By equating the last two equations we get:

$$\Delta K = -\Delta U \tag{6.47}$$

or:

$$\Delta K + \Delta U = \Delta(K + U) = 0 \tag{6.48}$$

If we define the total mechanical energy E as the sum of the kinetic energy K and potential energy U, i.e.

$$\boxed{E = K + U}\,, \tag{6.49}$$

then Eq. 6.48 gives:

$$\boxed{\Delta E = 0} \tag{6.50}$$

which is called the **principle of conservation of mechanical energy**.

Conservation of Mechanical Energy:
When only a conservative force acts on a system, the kinetic energy and the potential energy can change. However, their sum, the mechanical energy E of the system, does not change. That is:

$$E_{\mathrm{i}} = E_{\mathrm{f}} \tag{6.51}$$

or:

$$K_{\mathrm{i}} + U_{\mathrm{i}} = K_{\mathrm{f}} + U_{\mathrm{f}} \tag{6.52}$$

If more than one conservative force acts on the system, where each one is associated with a potential energy, then the conservation of mechanical energy will take the form:

$$\boxed{K_{\mathrm{i}} + \sum U_{\mathrm{i}} = K_{\mathrm{f}} + \sum U_{\mathrm{f}}} \tag{6.53}$$

Example 6.6

A frictionless roller-coaster is given a maximum possible initial speed $v_{\circ} = 6\,\mathrm{m/s}$ when it is at height $y_{\circ} = 6\,\mathrm{m}$ above the ground and moves freely afterwards, see Fig. 6.14 and take $g = 10\,\mathrm{m/s^2}$. (a) What will be the roller-coaster's speed when it reaches the lowest point at $y_1 = 4\,\mathrm{m}$? (b) What will be its maximum height y_2?

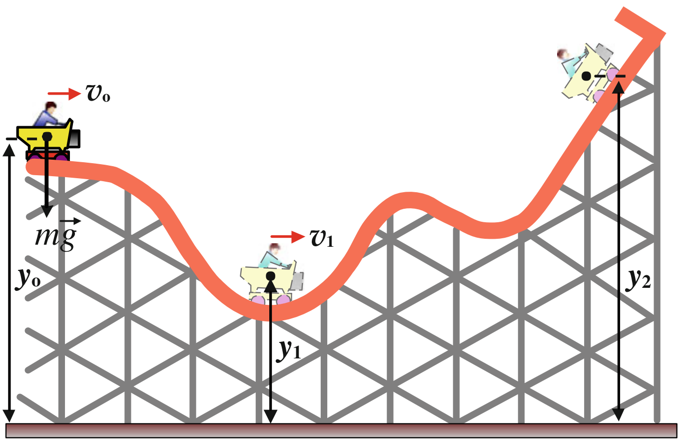


Fig. 6.14

Solution: (a) The only force that contributes to the work is the force of gravity. Therefore, we can use the law of conservation of mechanical energy. Initially, we have $K_\mathrm{i} = \frac{1}{2} m\, v_\circ^2$ and $U_\mathrm{i} = m\, g\, y_\circ$. Finally, at the lowest point we have $K_\mathrm{f} = \frac{1}{2} m\, v_1^2$ and $U_\mathrm{f} = m\, g\, y_1$. Thus, according to Eqs. 6.51 and 6.52, we get:

$$E_\mathrm{i} = E_\mathrm{f} \quad \Rightarrow \quad K_\mathrm{i} + U_\mathrm{i} = K_\mathrm{f} + U_\mathrm{f} \quad \Rightarrow \quad \frac{1}{2} m\, v_\circ^2 + m\, g\, y_\circ = \frac{1}{2} m\, v_1^2 + m\, g\, y_1$$

i.e.
$$v_1^2 = v_\circ^2 + 2g(y_\circ - y_1)$$

Then:
$$v_1 = \sqrt{(6\,\mathrm{m/s})^2 + 2(10\,\mathrm{m/s^2})(6\,\mathrm{m} - 4\,\mathrm{m})} = 8.7\,\mathrm{m/s}$$

(b) The roller-coaster will stop momentarily when it reaches the maximum height y_2, i.e. $v_2 = 0$. Accordingly, Eq. 6.52 gives:

$$E_\mathrm{i} = E_\mathrm{f} \quad \Rightarrow \quad K_\mathrm{i} + U_\mathrm{i} = K_\mathrm{f} + U_\mathrm{f} \quad \Rightarrow \quad \tfrac{1}{2} m\, v_\circ^2 + m\, g\, y_\circ = \tfrac{1}{2} m\, v_2^2 + m\, g\, y_2$$

Then:
$$y_2 = y_\circ + \tfrac{1}{2} v_\circ^2 / g \quad \Rightarrow \quad y_2 = 6\,\mathrm{m} + \tfrac{1}{2}(6\,\mathrm{m/s})^2/(10\,\mathrm{m/s^2}) = 7.8\,\mathrm{m}$$

6.6 Work Done by Non-conservative Forces

In real-life systems, the total mechanical energy is not constant due to the presence of non-conservative forces, such as friction or any applied forces. When the work done by all the non-conservative forces on a particle is W_{nc} and the work done by the conservative force is W_c, then the work-energy theorem tells us that there will be a change in the particle's kinetic energy given by Eq. 6.32 as:

$$W_{nc} + W_c = \Delta K \tag{6.54}$$

Since Eq. 6.39 gives $W_c = -\Delta U$, then this equation becomes:

$$W_{nc} = \Delta K + \Delta U = (K_\mathrm{f} - K_\mathrm{i}) + (U_\mathrm{f} - U_\mathrm{i}) \tag{6.55}$$

Since $E = K + U$ as we saw in Eq. 6.49, this equation becomes:

$$\boxed{W_{nc} = \Delta E = E_\mathrm{f} - E_\mathrm{i}} \tag{6.56}$$

which generally can be stated as:

Spotlight

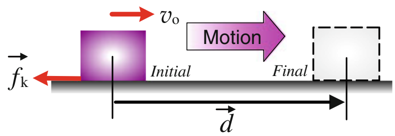
The work done by all non-conservative forces W (or $W_{\rm nc}$) equals the change in the total mechanical energy of the system.

When there are no non-conservative forces present, $W_{nc} = 0$ and hence $E_{\rm f} = E_{\rm i}$; that is, the total mechanical energy is conserved.

Example 6.7

A block of initial speed $v_\circ$ slides across a floor, see Fig. 6.15. A kinetic frictional force of magnitude $f_{\rm k} = 50\,{\rm N}$ does work on the block, stopping it over a displacement of magnitude $d = 2\,{\rm m}$. Find the dissipated mechanical energy.

Fig. 6.15



Solution: From Eq. 6.9, the work done by friction is given by:

$$W_{nc} \equiv W_f = -f_{\rm k}\, d = -(50\,{\rm N})(2\,{\rm m}) = -100\,{\rm J}$$

From Eq. 6.56 and the above result, the dissipated mechanical energy is:

$$\Delta E = W_{nc} = -100\,{\rm J}$$

Example 6.8

A boy of mass $m = 30\,{\rm kg}$ slides down a curved track of height $h = 3\,{\rm m}$, see Fig. 6.16. The boy starts at point i with a speed $v_{\rm i} = 0$ and reaches the bottom of the track at point f with a speed $v_{\rm f}$. (a) If the track is frictionless, i.e. $f_{\rm k} = 0$, then find the speed $v_{\rm f}$. (b) If the track is rough and $v_{\rm f} = 5\,{\rm m/s}$, then find the work done by friction.

Solution: (a) The normal force $\vec{N}$ does no work on the boy since it is always perpendicular to each displacement element on the curved track. The only force that has a change in potential energy is $m\vec{g}$. Therefore, we can use the law of conservation of mechanical energy. Initially, we have $K_{\rm i} = 0$ and $U_{\rm i} = m\,g\,h$. At the end we have $K_{\rm f} = \frac{1}{2}m\,v_{\rm f}^2$ and $U_{\rm f} = 0$. Thus, according to Eq. 6.51, we get:

$$E_{\rm i} = E_{\rm f} \quad \Rightarrow \quad K_{\rm i} + U_{\rm i} = K_{\rm f} + U_{\rm f} \quad \Rightarrow \quad 0 + m\,g\,h = \tfrac{1}{2}m\,v_{\rm f}^2 + 0$$

$$\text{i.e.,} \quad v_{\rm f} = \sqrt{2\,g\,h} = \sqrt{2(9.8\,\text{m/s}^2)(3\,\text{m})} = 7.67\,\text{m/s}$$

(b) In the presence of a non-conservative frictional force, i.e. $W_{nc} \neq 0$, then mechanical energy is not conserved and we can use Eq. 6.56 to find the work done by friction on the boy as follows:

$$\begin{aligned} W_{nc} \equiv W_f = E_{\rm f} - E_{\rm i} &= (K_{\rm f} + U_{\rm f}) - (K_{\rm i} + U_{\rm i}) = (\tfrac{1}{2}m\,v_{\rm f}^2 + 0) - (0 + m\,g\,h) \\ &= \tfrac{1}{2}(30\,\text{kg})(5\,\text{m/s})^2 - (30\,\text{kg})(9.8\,\text{m/s}^2)(3\,\text{m}) = -507\,\text{J} \end{aligned}$$

Note that W_f is negative, since the work done by friction is negative.

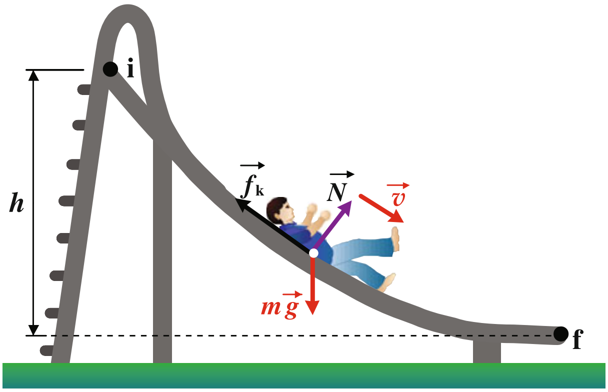


Fig. 6.16

Example 6.9

A block of mass $m = 2$ kg is placed on a rough horizontal surface against a compressed spring with a spring constant of $k_{\rm H} = 2{,}000$ N/m. The spring is compressed a distance of $x = 20$ cm, see Fig. 6.17. The block is released, and then it moves to the right until it stops completely after rising onto a rough track of height $h = 0.5$ m. Find the work done by friction.

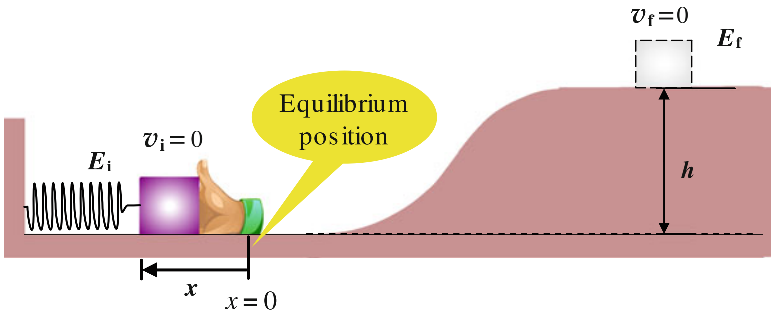


Fig. 6.17

Solution: As in Example 6.8, the normal force $\vec{N}$ does no work on the block since it is always perpendicular to each displacement element on the horizontal and curved parts of the track. The only force that has a change in potential energy is the force of gravity. Initially, we have $K_i = 0$ and only an elastic potential energy $\frac{1}{2}k\,x^2$, i.e. $\Sigma U_i = \frac{1}{2}k\,x^2$. At the end we have $K_f = 0$ and only a gravitational energy $m\,g\,h$, i.e. $\Sigma U_f = m\,g\,h$. In the presence of a non-conservative frictional force, $W_{nc} \neq 0$, the mechanical energy is not conserved. We then use Eq. 6.56 to find the work done by friction as follows:

$$
\begin{aligned}
W_{nc} &= E_f - E_i \\
&= (K_f + \Sigma U_f) - (K_i + \Sigma U_i) \\
&= (0 + m\,g\,h + 0) - (0 + 0 + \tfrac{1}{2}k_H\,x^2) \\
&= m\,g\,h - \tfrac{1}{2}k_H\,x^2 \\
&= (2\,\text{kg})(9.8\,\text{m/s}^2)(0.5\,\text{m}) - \tfrac{1}{2}(2{,}000\,\text{N/m})(0.2\,\text{m})^2 = -30.2\,\text{J}
\end{aligned}
$$

W_{nc} is negative since the work done by friction is always negative.

6.7 Conservation of Energy

When a block slides across a rough floor through a displacement $\vec{d}$, the frictional force $\vec{f}_k$ (which is an external force) does work on the block, and we found from Example 6.7 that $W_{nc} = W_f = -f_k\,d$. This allows us to write the dissipated mechanical energy given by Eq. 6.56 as follows:

$$\Delta E = \Delta K + \Delta U = -f_k\,d \tag{6.57}$$

In fact, this dissipated energy is transferred as thermal energy to the block and the floor. So, the energy of the block, which is considered to be our system, is not conserved.

When we expand our system to include both the block and the floor, the frictional force is no longer an external force, and the energy transfer will be within the system. So, again we have an *isolated system* within which energy is conserved.

To find this conservation principle, we look at the decrease ΔE in Eq. 6.57 as the total amount of energy transferred as thermal energy to the block and floor. If ΔE_{int} represents the change in the thermal energy (which is an *internal energy*) of the system consisting of the block and floor, then we get:

$$\Delta E_{\text{int}} = -\Delta E \tag{6.58}$$

which gives:

$$\Delta E + \Delta E_{\text{int}} = \Delta K + \Delta U + \Delta E_{\text{int}} = 0 \tag{6.59}$$

This means that, although the mechanical energy of the block is not conserved, the sum of the mechanical energy of the block and the thermal energy of the block and floor is conserved. This sum is called the total energy E_{tot} of the block-floor system. This conservation principle is called **the law of conservation of energy** and written as:

$$\boxed{\Delta E_{\text{tot}} = \Delta K + \Delta U + \Delta E_{\text{int}} = 0} \quad \left\{ \begin{matrix} \text{conservation of energy} \\ \text{for an isolated system} \end{matrix} \right\} \tag{6.60}$$

This law of conservation is not derived, but instead based on countless experiments done by scientists and engineers.

If the system is not isolated and applied external forces transfer energy to or from the system, then the work done *on* the system by external forces will be:

$$\boxed{W = \Delta E_{\text{tot}} = \Delta K + \Delta U + \Delta E_{\text{int}}} \quad \text{(non isolated system)} \tag{6.61}$$

For example, in Fig. 6.18, if we consider the rope to be external to the system, then the frictional force exerted by the rope on the metal rings of the system does an amount of work W on the system, transferring energy from the system to thermal energy in the rope while the values of K, U, and E_{int} change.

Fig. 6.18 A firewoman wrapping a rope around metal rings so that the rope rubs against the rings while she is descending from a helicopter. Doing so, she will transfer energy from the gravitational potential energy of a system consisting of her, her gear, and the Earth to thermal energy gained by the rope and the rings. While descending slowly, this allows most of the transferred energy to go to the rope and the rings rather than to her kinetic energy

Example 6.10

A steel ball of mass $m = 5\,\text{g}$ is projected vertically downward from a height $h = 14.8\,\text{m}$ with an initial speed $v_\circ = 10\,\text{m/s}$, see part a of Fig. 6.19. The ball penetrates itself in sand to a depth $d = 20\,\text{cm}$, see part c of the figure. Neglect air resistance and take g to be $10\,\text{m/s}^2$. (a) What is the change in the mechanical energy of the ball? (b) What is the change in the internal energy of the ball-Earth-sand system? (c) What is the magnitude of the average force exerted by the sand on the ball in part b of the figure?

Solution: (a) Let us take the reference point $y = 0$ to be at the point where the ball stops completely, as shown in part c of the figure. Therefore, at the stopping depth d, the kinetic energy and the potential energy are zero. Thus:

$$\begin{aligned}
\Delta E &= E_\text{f} - E_\text{i} \\
&= \Delta K + \Delta U \\
&= (K_\text{f} - K_\text{i}) + (U_\text{f} - U_\text{i}) \\
&= (0 - \tfrac{1}{2}mv_\circ^2) + (0 - mg[h + d]) \\
&= -\tfrac{1}{2}mv_\circ^2 - mg(h + d)
\end{aligned}$$

Inserting the given data into the final expression, we find:

$$\begin{aligned}\Delta E = &-\tfrac{1}{2}(5\times10^{-3}\,\text{kg})(10\,\text{m/s})^2\\ &-(5\times10^{-3}\,\text{kg})(10\,\text{m/s}^2)(14.8\,\text{m}+20\times10^{-2}\,\text{m})\\ =&-0.25-0.75=-1\,\text{J}\end{aligned}$$

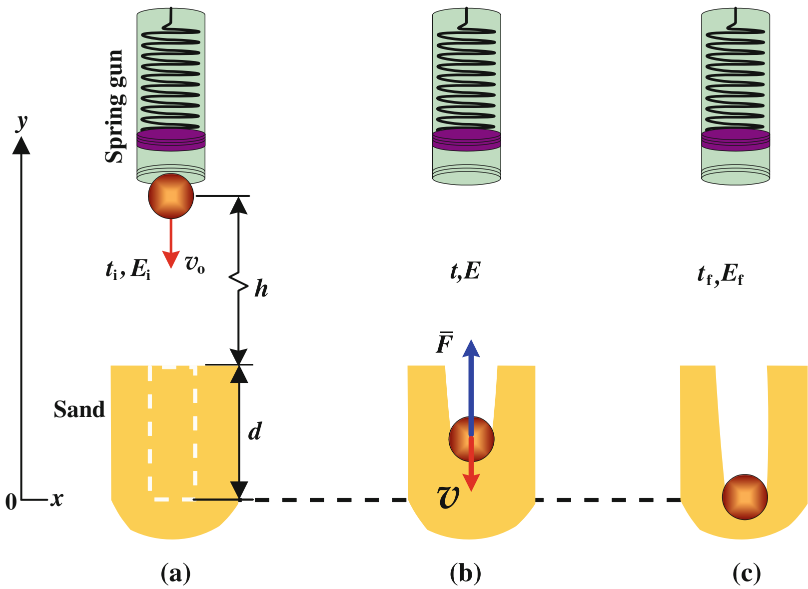


Fig. 6.19

(b) This system is isolated, and we can apply Eq. 6.59 as follows:

$$\Delta E + \Delta E_{\text{int}} = 0$$

or

$$\Delta E_{\text{int}} = -\Delta E = -(-1\,\text{J}) = 1\,\text{J}$$

That is to say, as the ball moves through the sand, the sand exerts an upward force on the ball and thus dissipates all the mechanical energy of the ball, transforming it to thermal energy of the sand and ball.

(c) When the ball reaches the surface of the sand, its mechanical energy will be the same as the initial mechanical energy E_i, since air resistance is neglected.

Then, as the ball moves through the sand, an average upward force $\overline{F}$ dissipates all its mechanical energy by the time the ball moves a distance d. Thus, the change in mechanical energy $\Delta E = E_{\rm f} - E_{\rm i}$ will be transferred to thermal energy of the sand and the ball. So, Eq. 6.57 can be written as:

$$\Delta E = E_{\rm f} - E_{\rm i} = -\overline{F}\, d$$

Solving this for $\overline{F}$, we find the following:

$$\overline{F} = -\frac{\Delta E}{d} = -\frac{(-1\,\text{J})}{20 \times 10^{-2}\,\text{m}} = 5\,\text{N}$$

We can arrive at this answer by using the techniques of Chap. 3 by finding the ball's speed at the surface of the sand and then its average deceleration within the sand. Then, using Newton's second law, we can find $\overline{F}$. Obviously, more algebraic steps would be required.

6.8 Power

It is more interesting to know not only the work done on an object, but also the time rate at which work is being done. This rate is defined as the **power**.

If ΔW is the work done by an applied force on an object during a time interval Δt, then the average power $\overline{P}$ during this time interval is defined as:

$$\overline{P} = \frac{\Delta W}{\Delta t} \tag{6.62}$$

The instantaneous power P is the limiting value of this average power as Δt approaches zero, i.e.

$$P = \lim_{\Delta t \to 0} \frac{\Delta W}{\Delta t} = \frac{dW}{dt} \tag{6.63}$$

The SI unit of power is joule per second (J/s), called a **watt** (W). In the British system, the unit of power is foot-pound per second (ft.lb/s). Often the term horsepower (hp) is used. These units relate as follows:

$$\left.\begin{array}{l} 1 \text{ watt} = 1\,\text{W} = 1\,\text{J/s} = 0.738\,\text{ft.lb/s} \\ 1 \text{ horsepower} = 1\,\text{hp} = 550\,\text{ft.lb/s} = 746\,\text{W} \end{array}\right\} \tag{6.64}$$

From Eq. 6.62, we see that the work can be expressed as power multiplied by time, as in the common unit, the kilowatt-hour, Thus:

$$\begin{aligned} 1\text{ kilowatt-hour} &= 1\text{ kW.h} = (10^3\text{ W})(3{,}600\text{ s}) \\ &= 3.6 \times 10^6\text{ J} = 3.6\text{ MJ} \end{aligned} \tag{6.65}$$

It is important to realize that a kW.h is a unit of energy, not power. For example, our electric bills are usually in kW.h, and this gives the consumed amount of energy, whereas an electric bulb rated at a power of 100 W means it would consume 3.6×10^5 J of energy in 1 h.

We can express the rate at which a force $\vec{F}$ does work on a particle (or a particle-like object) in terms of that force and the body's velocity $\vec{v}$. In Eq. 6.23, we were able to express the work done dW on the particle by a force $\vec{F}$ during a displacement $d\vec{r}$ as $dW = \vec{F} \bullet d\vec{r}$. Therefore, the instantaneous power can be written as:

$$P = \frac{dW}{dt} = \frac{\vec{F} \bullet d\vec{r}}{dt} = \vec{F} \bullet \frac{d\vec{r}}{dt}$$

Recognizing $d\vec{r}/dt$ as the instantaneous velocity $\vec{v}$, we get:

$$P = \vec{F} \bullet \vec{v} = F\,v\,\cos\theta = \begin{cases} +F\,v & \text{if } \theta = 0° \\ 0 & \text{if } \theta = 90° \\ -F\,v & \text{if } \theta = 180° \end{cases} \tag{6.66}$$

Positive power means that energy is transferred to the particle, while negative power means that energy is transferred from the particle.

Example 6.11

An elevator loaded fully with passengers has a mass $M = 2{,}000$ kg. When the elevator ascends, an almost constant frictional force $f = 5{,}000$ N acts against its motion, see Fig. 6.20. What power must be delivered by the motor (the tension T) to lift the elevator at: (I) a constant speed v of 4 m/s, see part (a) of the figure? (II) a constant acceleration a of 1.5 m/s^2 that produces a speed $v = at$, see part (b) of the figure?

Solution: (I) Let T be the force supplied by the elevator's motor to pull the elevator upward. From Newton's second law and from the fact that $a = 0$ (since v is a constant in part a of Fig. 6.20), we get:

$$T - f - M\,g = 0$$

Using M as the total mass of the elevator and the passengers and inserting the given data into this expression, we find:

$$
\begin{aligned}
T &= f + M\,g \\
&= 5{,}000\,\text{N} + (2{,}000\,\text{kg})(9.8\,\text{m/s}^2) \\
&= 24{,}600\,\text{N}
\end{aligned}
$$

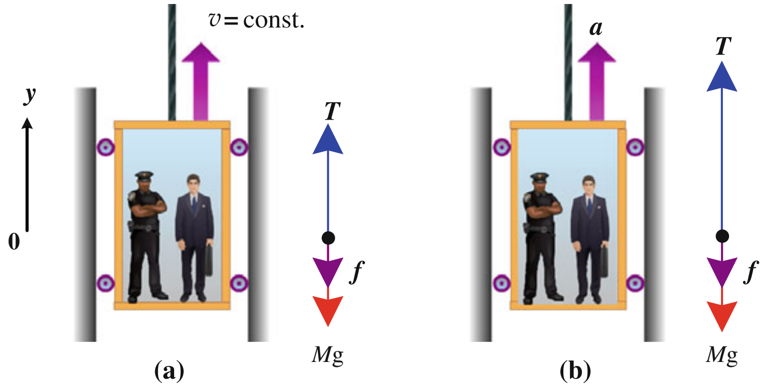


Fig. 6.20

Then, using Eq. 6.66 and the fact that $\vec{T}$ is in the same direction as $\vec{v}$ gives:

$$
\begin{aligned}
P = \vec{T} \bullet \vec{v} &= T v \cos 0^\circ = (24{,}600\,\text{N})(4\,\text{m/s}) \\
&= 98{,}400\ \text{W} = 98.4\,\text{kW} \simeq 132\,\text{hp}
\end{aligned}
$$

This means that to maintain a constant speed of 4 m/s, a force of magnitude 24,600 N is required to transfer energy *to* the elevator at a rate of 98,400 J/s.

(II) Applying Newton's second law to part (b) of the figure gives:

$$T - f - Mg = Ma$$

Inserting the given data into this expression, we find:

$$
\begin{aligned}
T &= f + M(g + a) \\
&= 5{,}000\,\text{N} + (2{,}000\,\text{kg})[9.8\,\text{m/s}^2 + 1.5\,\text{m/s}^2] \\
&= 27{,}600\,\text{N}
\end{aligned}
$$

Then, using Eq. 6.66 we get:

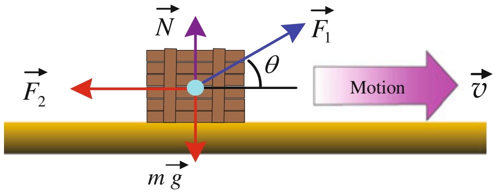
$$
\begin{aligned}
P = \vec{T} \bullet \vec{v} &= T v \cos 0^\circ = T\ at \\
&= (41{,}400\ t)\ \text{W}
\end{aligned}
$$

This indicates that the required power increases linearly with time t.

Example 6.12

Two forces $\vec{F}_1$ and $\vec{F}_2$ are acting on a box that slides horizontally to the right across a frictionless surface, see Fig. 6.21. Force $\vec{F}_1$ has a magnitude of 5 N and makes an angle $\theta = 60°$ with the horizontal. Force $\vec{F}_2$ is against the motion and has a magnitude of 2 N. The speed v of the box at a certain instant is 4 m/s. What is the power due to each force that acts on the box at that instant, and what is the net power? Is the net power changing with time?

Fig. 6.21



Solution: The weight $m\vec{g}$ and the normal force $\vec{N}$ are perpendicular to the velocity $\vec{v}$. Thus, their work done is zero, and hence the power due to each of them on the block is zero. We use Eq. 6.66 to find the power due to $\vec{F}_1$ and $\vec{F}_2$. First, for the force $\vec{F}_1$ that is applied at an angle $\theta = 60°$ to the velocity $\vec{v}$, we have:

$$P_1 = \vec{F}_1 \bullet \vec{v} = F_1 v \cos 60° = (5\,\mathrm{N})(4\,\mathrm{m/s})(0.5) = 10\,\mathrm{W}$$

which indicates that the force $\vec{F}_1$ is transferring energy *to* the box at a rate of 10 J/s. Similarly, for $\vec{F}_2$ we have:

$$P_2 = \vec{F}_2 \bullet \vec{v} = F_2 v \cos 180° = (2\,\mathrm{N})(4\,\mathrm{m/s})(-1) = -8\,\mathrm{W}$$

which indicates that the force $\vec{F}_2$ is transferring energy *from* the box at a rate of 8 J/s.

The net power is the sum of the individual powers. Thus:

$$P_{\mathrm{net}} = P_1 + P_2 = 10\,\mathrm{W} + (-8\,\mathrm{W}) = 2\,\mathrm{W}$$

This indicates that the net rate of energy transfer *to* the box is positive. So, the kinetic energy of the box will increase, and hence its speed. Consequently, the net power will increase with time.

6.9 Exercises

Section 6.1 Work Done by a Constant Force

(1) A 100 kg object moves in a straight line with a speed of 20 m/s. The object is to be stopped by a deceleration of 2 m/s^2. (a) What is the magnitude of the force required? (b) What distance does the object travel? (c) What work is done by the decelerating force? (d) Answer parts (a) to (c) for a deceleration of 4 m/s^2?

(2) How much work is done in moving a body of mass 2 kg vertically upward from an elevation of 1 m to an elevation of 3 m, (a) by gravity? (b) by an external agent that is slowly moving the body? (c) Answer parts (a) and (b) for a downward motion from an elevation of 3 m to an elevation of 2 m.

(3) Using Fig. 6.22, find the work done by the weight $m\vec{g}$ of a particle of mass m, as the particle is moved (by application of any other constant forces) from: (a) A to B, (b) B to A, (c) A to B to C, (d) A to C directly, and (e) A to B to C to A.

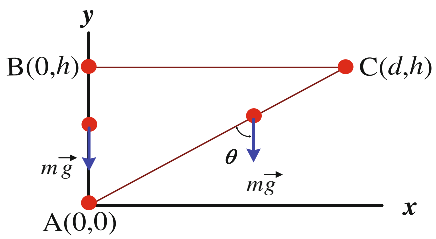


Fig. 6.22 See Exercise (3)

(4) A coin of mass $m = 0.5$ g slides a distance $d = 0.5$ m along a tabletop. If the coefficient of kinetic friction between the coin and the table is $\mu_k = 0.7$, find the work done on the coin by friction.

(5) A block of mass m is pushed along a rough horizontal surface by a constant horizontal force $\vec{F}$. The displacement of the block along the surface is $\vec{d}$. (a) Find the mathematical expression that represents the work done by: the force $\vec{F}$, the kinetic friction $\vec{f}_k$, the gravitational force $m\vec{g}$, and the normal force $\vec{N}$. (b) Calculate the work done when $m = 2$ kg, $\mu_k = 0.5$, $F = 20$ N, and $d = 5$ m.

(6) A block moves up an incline of angle $\theta = 30°$ under the action of the three forces shown in Fig. 6.23. Force $\vec{F}_1$ has a magnitude of 30 N and is parallel to the plane. Force $\vec{F}_2$ has a magnitude of 20 N and is normal to the plane. Force

$\vec{F}_3$ of 40 N is horizontal. Find the work done by each force as the block moves a distance $d = 2$ m up the incline.

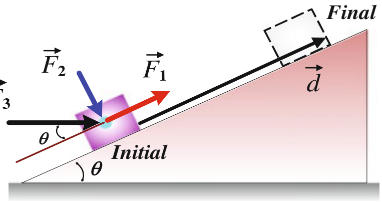


Fig. 6.23 See Exercise (6)

Section 6.2 Work Done by a Variable Force

(7) A force acting in the x direction on an object varies with x as shown in Fig. 6.24. Find the work done by the force in the intervals: (a) $0 \le x \le 1$ m, (b) 1 m $\le x \le 3$ m, (c) 3 m $\le x \le 4$ m, (d) 4 m $\le x \le 7$ m, and (e) $0 \le x \le 7$ m.

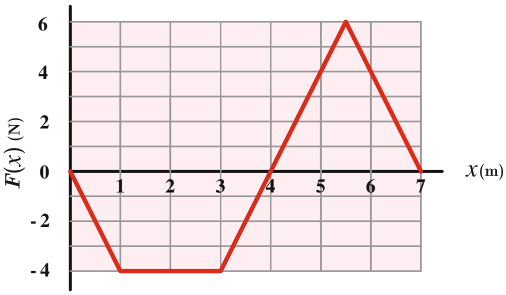


Fig. 6.24 See Exercise (7)

(8) A particle is subject to a force $f(x) = (2 + 0.5\,x)$ N. As the particle moves from $x = 0$ to $x = 8$ m, find the work done by the force using: (a) Equation 6.16, and (b) a graphical method.

(9) A smooth track in the form of a quarter of a circle of radius $r = 40$ cm lies in a vertical plane as shown in Fig. 6.25. A bead of mass 4 g moves from P_1 to P_2 under the effect of a force $\vec{F}(s)$ that is always acting tangentially to the track and of magnitude $F(s) = (10 - 2\,s)$ N, where the arc length s is measured in meters. (a) Find the work done by the applied force $\vec{F}$. (b) Find the work done by weight $m\vec{g}$.

(10) A force is used to compress a spring with a spring constant $k_H = 300$ N/m, see Fig. 6.26. (a) How much work does the applied force do when compressing

the spring a distance of 6 cm? (b) When the block is released, how much work does the spring force do on the block during a total displacement starting from a compression of 6 cm to a stretch of 4 cm?

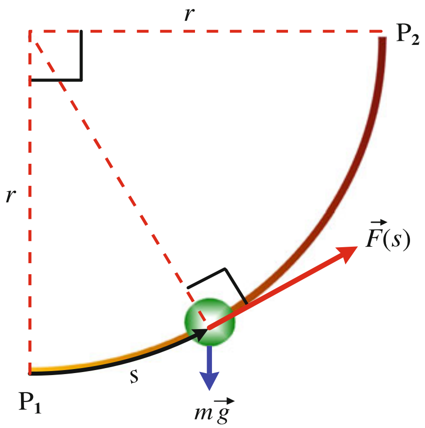


Fig. 6.25 See Exercise (9)

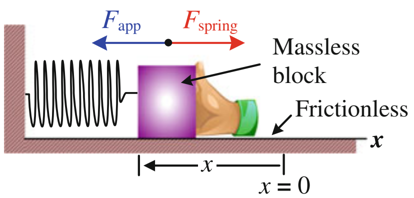


Fig. 6.26 See Exercise (10)

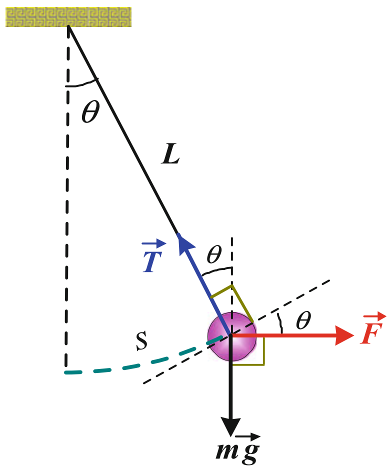
(11) A small sphere of weight $m\,g$ hangs from a string of length L, as shown in Fig. 6.27. A variable horizontal force $\vec{F}$, which starts from zero and gradually increases, is used to pull the sphere slowly (i.e., equilibrium exists at all the times) until the string makes an angle θ with the vertical. (a) Use Eq. 6.16 to show that the work done by the force $\vec{F}$ is $W_F = m\,g\,L(1 - \cos\theta)$. (b) Use the concept of equilibrium to reach the same answer without performing integration.

(12) The average resistive force against a nail penetrating a hard material is given by $\vec{F} = -kx^4\vec{i}$, where k is a constant and x is the penetration depth. Find the work done by this force when penetrating this material for a distance d.

(13) A bead is moving along the circumference of a circular hoop of radius R under a constant force of magnitude F. The force always makes an angle θ with

respect to the tangent to the circle. Find the work done by this force during one revolution.

Fig. 6.27 See Exercise (11)



Section 6.3 Work-Energy Theorem

(14) A car is moving at 100 km/h. If its mass is 1,000 kg, what is its kinetic energy?

(15) A 120 g mass has a velocity $\vec{v} = (3\,\vec{\mathrm{i}} + 4\,\vec{\mathrm{j}})$ m/s at a certain instant. What is its kinetic energy?

(16) Use the work-energy theorem to find the magnitude of the force required to accelerate a car of mass 1,300 kg from rest to 25 m/s in a distance of 100 m?

(17) The speed of a 10 kg object changes from 4 to 10 m/s. What is its change in kinetic energy?

(18) The velocity of a 0.4 kg object changes from $\vec{v}_{\mathrm{i}} = (4\,\vec{\mathrm{i}} + 3\,\vec{\mathrm{j}})$ to $\vec{v}_{\mathrm{f}} = (12\,\vec{\mathrm{i}} - 9\,\vec{\mathrm{j}})$ m/s. What is its change in kinetic energy?

(19) A force acting on a body that moves along the x-axis produces a velocity-time graph as shown in Fig. 6.28. If the body has a mass $m = 2$ kg, then find the change in kinetic energy in the intervals: (a) $0 \le t \le 1$ s, (b) $1\text{ s} \le t \le 3$ s, (c) $3\text{ s} \le t \le 5$ s, (d) $5\text{ s} \le t \le 7$ s, and (e) $0 \le t \le 7$ s.

(20) A force acts on a body of mass $m = 2$ kg that moves along the x-axis. The force varies with x as shown in Fig. 6.29. If the body was initially at rest, then find the change in kinetic energy in the intervals: (a) $0 \le x \le 1$ m, (b) $0 \le t \le 2$ m, and (c) $0 \le x \le 3$ m.

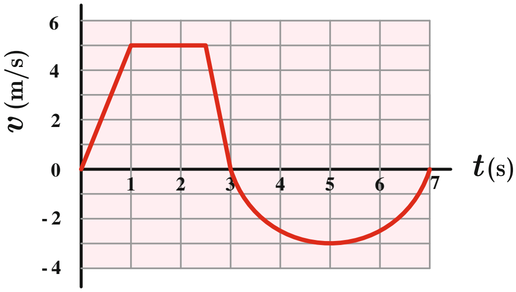


Fig. 6.28 See Exercise (19)

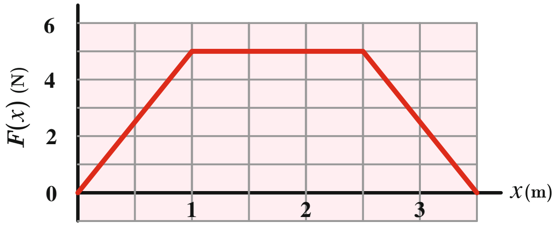


Fig. 6.29 See Exercise (20)

(21) A block of mass $m = 15$ kg slides from rest down a frictionless incline of inclination angle $\theta = 30°$ and is stopped by a spring that has a spring constant $k_H = 5{,}000$ N/m, see Fig. 6.30. The block moves a total distance $d = 1.5$ m from the point of release to the point where it stops momentarily as the spring reaches its maximum compression. Use the work-energy theorem to find the maximum compression of the spring.

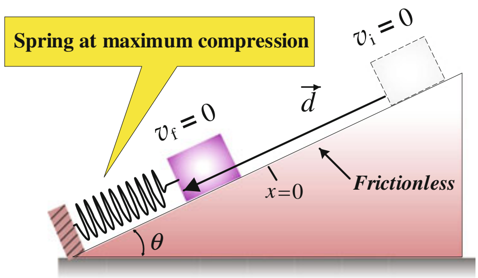


Fig. 6.30 See Exercise (21)

(22) A force acts on a particle of mass $m = 5$ kg and changes its velocity from $\vec{v}_i = (3\,\vec{i} + 4\,\vec{j})$ to $\vec{v}_f = (6\,\vec{i} + 8\,\vec{j})$ m/s. How much work is applied to this particle by this force?

Section 6.5 Conservation of Mechanical Energy

(23) A body of mass $m = 5$ kg is released from rest from a height of 2 m above the ground. (a) What is the kinetic energy of the body just before hitting the ground? (b) At that point, what is its speed?

(24) A freely falling ball of mass $m = 0.5$ kg passes a window 1.5 m high. (a) How much did the kinetic energy of the ball increase as it fell past the window? (b) If its speed at the top of the window was 2 m/s, what will its speed be at the bottom of the window?

(25) A pendulum bob has a mass $m = 0.5$ kg. It is suspended by a cord of length $L = 2$ m which is pulled back through an angle of 90° and released, see Fig. 6.31. (a) What is its maximum potential energy relative to its lowest position? (b) What is its maximum speed at point B? (c) What is its speed at point C when the cord makes an angle $\theta = 60°$ with the vertical?

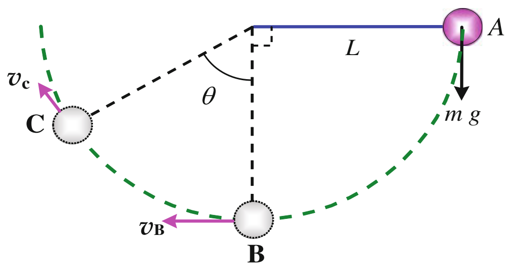


Fig. 6.31 See Exercise (25)

(26) In the track shown in Fig. 6.32, section AB is a quadrant of a circle of radius $r = 1$ m. A block is released at A and slides without friction until it reaches point B, then moves a distance $d = 4$ m on a horizontal rough plane before stopping at point C. (a) Haw fast is the block moving at point B? (b) What is the coefficient of kinetic friction between the block and the plane?

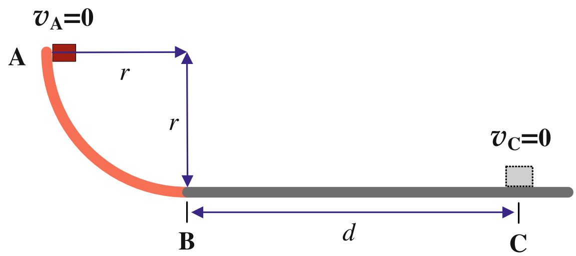
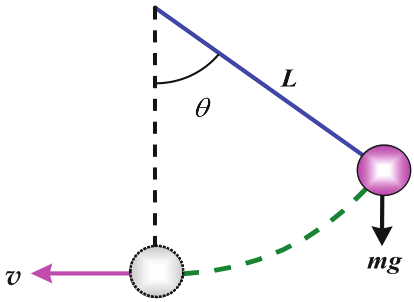


Fig. 6.32 See Exercise (26)

(27) A pendulum bob is pulled aside from its equilibrium position through an angle θ and then released, see Fig. 6.33. Show that the pendulum bob will pass through the equilibrium position with a speed $v = \sqrt{2gL(1 - \cos\theta)}$, where L is the length of the pendulum. When $\theta = 90°$, show that the relation of v will give an identical result to the result obtained in part (b) of exercise 25.

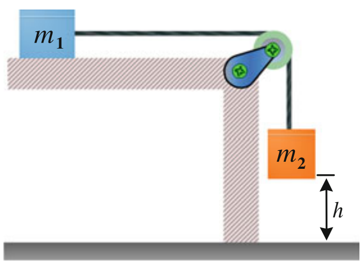
Fig. 6.33 See Exercise (27)



(28) A spring has one of its ends fixed and the other attached to a block of mass m that rests on a frictionless horizontal surface. The application of a horizontal force F on the block causes the spring to stretch a distance d from its equilibrium. The spring is held at this position momentarily and then the block is released. Find the speed of the block when the spring returns: (a) to half its original extension $(d/2)$, and (b) to its natural length.

(29) Two blocks of masses $m_1 = 4$ kg and $m_2 = 5$ kg are connected by a massless string that passes over a massless frictionless pulley as shown in Fig. 6.34. Block m_1 is initially at rest on a smooth horizontal plane while block m_2 is at a height $h = 0.75$ m above the ground. Use conservation of mechanical energy to find the speed of the masses just before m_2 hits the ground.

Fig. 6.34 See Exercise (29)



(30) Figure 6.35 shows a proposed roller-coaster track. Each car starts from rest at point A, where $y_A = 21$ m and it will roll freely without friction along the track. It is important that there be at least some small normal force exerted by the track on the car at all points; otherwise, the car will leave the track. What is the minimum safe value for the radius of the curvature at point B?

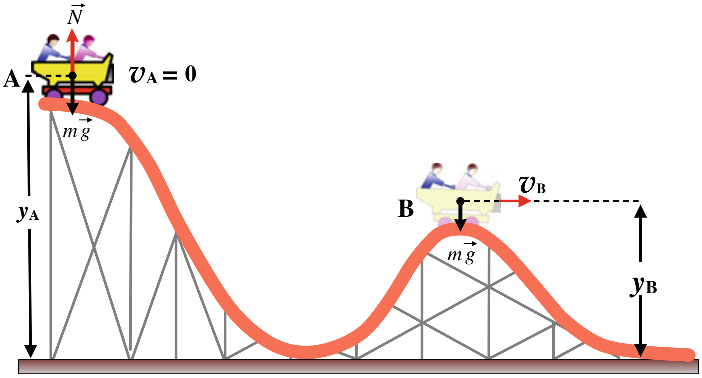
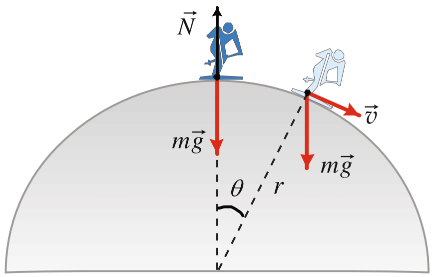


Fig. 6.35 See Exercise (30)

(31) A skier of mass m starts sliding from rest at the top of a solid frictionless hemisphere of radius r, see Fig. 6.36. At what angle θ will the skier leave the sphere?

Fig. 6.36 See Exercise (31)



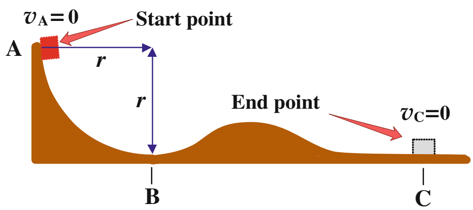
Sections 6.6 and 6.7 Work Done by Non-conservative Forces—Conservation of Energy

(32) If the mass of the block in Example 6.7 is 0.5 kg, then find the value of the speed $v_\circ$.

(33) If the mass of the boy in Example 6.8 is 50 kg, then redo parts (a) and (b) of the example and comment on the obtained results.

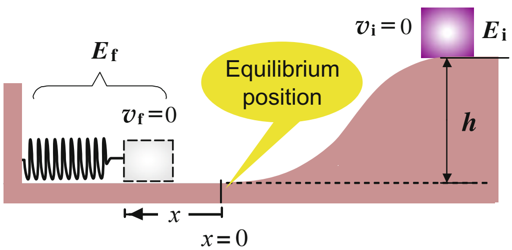
(34) In the rough track shown in Fig. 6.37, section AB is a quadrant of a circle of radius $r = 2$ m. A block of mass $m = 5$ kg is released at A and slides until it stops completely at point C. (a) Find the work done by friction. (b) What is the effect of having a more/less rough track on the block?

Fig. 6.37 See Exercise (34)



(35) A block of mass $m = 5$ kg is placed on the edge of a rough surface of height $h =$ 0.5 m, see Fig. 6.38. The block is released and moves until it stops momentarily after compressing a horizontal spring (with a spring constant $k_H = 2{,}000$ N/m) by a compression distance $x = 10$ cm. Find the work done by friction. Will the block ever be able to go back to its original location and why?

Fig. 6.38 See Exercise (35)



(36) (a) If the block in Exercise 35 traveled a total distance of 50 cm before coming to a momentary stop, estimate the average force of friction (assume it is roughly constant) on the block. (b) After the maximum compression of the spring is reached, the block starts its journey back on the surface. If the block reaches a

second momentary stop after moving a distance of 20 cm on the surface, what is the maximum height that the block can reach?

(37) A roller-coaster car of mass $m = 750$ kg starts from rest at the top of a hill 30 m high, see Fig. 6.39. The roller-coaster travels a total distance of 250 m without leaving the track and reaches a vertical height of only 25 m on the second hill before coming to a momentary stop. Find the thermal energy produced during the free motion and estimate the average frictional force on the car.

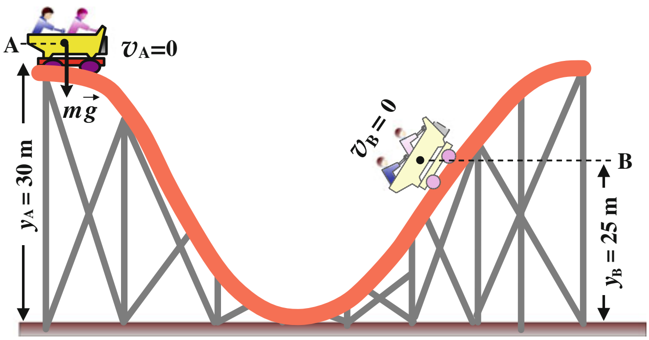


Fig. 6.39 See Exercise (37)

(38) A steel ball of mass $m = 0.5$ kg is projected horizontally with an initial speed $v_\circ = 10$ m/s, see Fig. 6.40a. The ball penetrates into a wall of clay until it stops at a depth $d = 20$ cm, see Fig. 6.40b–c. (a) What is the change in the mechanical energy of the ball ? (b) What is the change in the internal energy of the ball-Earth-wall system? (c) What is the magnitude of the average force exerted by the wall on the ball during the penetration process?

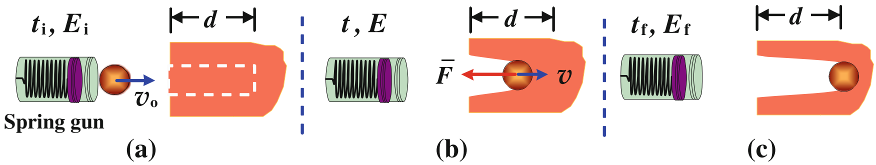


Fig. 6.40 See Exercise (38)

Section 6.8 Power

(39) How much average power in kilowatts and horsepower is required to lift a block of 100 kg to a height of 10 m in 30 s?

(40) At 30 piasters (Egyptian pound = 100 Piaster) per kilowatt-hour of electricity, what is the cost of operating a 5-hp motor for 2 h?

(41) An elevator fully loaded with passengers has a mass $M = 2{,}000$ kg. As the elevator descends, an almost constant frictional force $f = 4{,}000$ N acts against its motion. What power must be delivered by the motor to descend the elevator at: (a) a constant speed v of 4 m/s, and (b) a constant acceleration a of 1.5 m/s^2 that produces a speed $v = at$?

(42) A constant horizontal force $F = 20$ N acts on a block of mass $m = 4$ kg resting on a horizontal plane. The block starts from rest at $t = 0$. Show that the instantaneous power delivered by the force at any time t is given by $P = F^2 t/m$, and find its value at $t = 5$ s.

(43) A car generates 20 hp when traveling at a constant speed of 100 km/h. What is the total resistive force that acts on the car?

(44) A car of mass $m = 1{,}500$ kg accelerates from rest to 100 km/h in 8 s. What is the average power delivered by its engine?

(45) A car of mass m accelerates with acceleration a up an inclined plane of angle θ as in Fig. 6.41. The drag force f_D consists of rolling friction α (N) and air drag βv^2(N), i.e. $f_D = \alpha + \beta v^2$, where α and β are constants and v is the speed of the car. (a) Find the force F that propels the car. (b) Show that $P = mva + mvg \sin\theta + \alpha\, v + \beta\, v^3$ is the power delivered to the wheels by the engine, where mva is the power delivered to accelerate the car, $mvg \sin\theta$ is the power to overcome gravity, $\alpha\, v$ is the power to overcome rolling friction, and $\beta\, v^3$ is the power to overcome air drag. (c) Calculate the various components of P and hence the total P if we take $m = 1{,}000$ kg, $a = 2$ m/s^2, $v = 20$ m/s, $\alpha = 200$ N, $\beta = 0.5$ kg/m, and $\theta = 15°$.

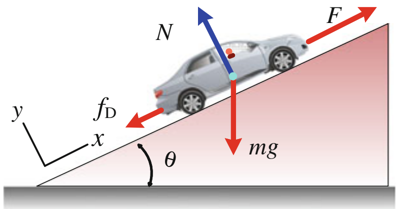


Fig. 6.41 See Exercise (45)

Linear Momentum, Collisions, and Center of Mass

7

In this chapter, we introduce the linear momentum of a particle and the law of *conservation of linear momentum* of a system of particles under certain conditions. We use this law and the conservation of energy to analyze translational motion when particles collide. For a system of isolated particles, or an extended object, we introduce the concept of *center of mass* to show that conservation of linear momentum applies under certain conditions, as it does for isolated particles. At the end of this chapter, we treat systems with variable mass. We first consider cases where the mass increases with time and then we consider cases where the mass decreases with time.

7.1 Linear Momentum and Impulse

First, let us consider Newton's second law, when a net force $\vec{F}$ acts on a particle of mass m, $\vec{F} = m\vec{a}$. After replacing $\vec{a}$ with $d\vec{v}/dt$, we get:

$$\vec{F} = m\frac{d\vec{v}}{dt} = \frac{d(m\vec{v})}{dt} \tag{7.1}$$

According to this equation, the net force $\vec{F}$ (abbreviation of $\sum\vec{F}$) acting on a particle is equal to the change in the product $m\vec{v}$ per unit time. This product is called the **linear momentum** (or the *momentum*) of a particle having a mass m and velocity $\vec{v}$, and it is assigned the symbol $\vec{p}$, that is:

$$\boxed{\vec{p} = m\vec{v}} \tag{7.2}$$

In the SI system, $\vec{p}$ has the units kg.m/s. In Cartesian coordinates, this equation is equivalent to the following component equations:

$$p_x = mv_x, \quad p_y = mv_y, \quad p_z = mv_z. \tag{7.3}$$

H. A. Radi and J. O. Rasmussen, *Principles of Physics*,
Undergraduate Lecture Notes in Physics, DOI: 10.1007/978-3-642-23026-4_7,

We can therefore rewrite Eq. 7.1 in a new form as follows:

$$\boxed{\vec{F} = \frac{d\vec{p}}{dt}} \quad \text{(Newton's second law)} \tag{7.4}$$

The two forms of Newton's second law $\vec{F} = m\vec{a}$ and $\vec{F} = d\vec{p}/dt$ are equivalent if the mass m is constant.

Next, to derive the linear impulse-momentum theorem, we rewrite Eq. 7.4 in a differential form as follows:

$$d\vec{p} = \vec{F}\,dt \tag{7.5}$$

If the momentum of the particle changes from $\vec{p}_\mathrm{i}$ at time t_i to $\vec{p}_\mathrm{f}$ at time t_f, we can then integrate this expression to find the change in momentum as follows:

$$\int_{\vec{p}_\mathrm{i}}^{\vec{p}_\mathrm{f}} d\vec{p} = \int_{t_\mathrm{i}}^{t_\mathrm{f}} \vec{F}\,dt \tag{7.6}$$

or

$$\Delta\vec{p} = \vec{p}_\mathrm{f} - \vec{p}_\mathrm{i} = \int_{t_\mathrm{i}}^{t_\mathrm{f}} \vec{F}\,dt \tag{7.7}$$

The right-hand side of this equation is called the impulse $\vec{J}$ (kg.m/s or N.s) of the net force $\vec{F}$ for the time interval $\Delta t = t_\mathrm{f} - t_\mathrm{i}$. Thus:

$$\boxed{\vec{J} = \int_{t_\mathrm{i}}^{t_\mathrm{f}} \vec{F}\,dt = \Delta\vec{p}} \tag{7.8}$$

This is known as the impulse-momentum theorem. During collisions, $\vec{F}$ jumps from zero to a large value and abruptly returns to zero again, all in a very short time interval $\Delta t = t_\mathrm{f} - t_\mathrm{i}$, see Fig. 7.1a. The integral in Eq. 7.8 can be represented by $\overline{\vec{F}}\,\Delta t$, where $\overline{\vec{F}}$ is the average force exerted on the particle during the time interval Δt, see Fig. 7.1b. Therefore, the impulse-momentum theorem reduces to the following:

$$\boxed{\vec{J} = \int_{t_\mathrm{i}}^{t_\mathrm{f}} \vec{F}\,dt = \overline{\vec{F}}\,\Delta t = \Delta\vec{p}} \quad \text{and} \quad \boxed{\overline{\vec{F}} = \frac{\Delta\vec{p}}{\Delta t}} \tag{7.9}$$

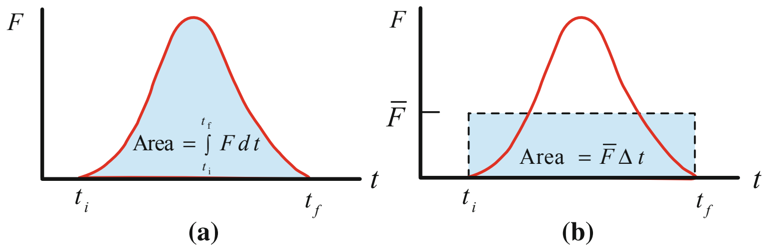
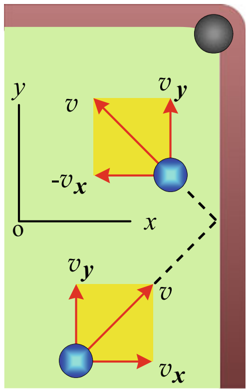


Fig. 7.1 (a) Variation of the force F with time t during a collision. (b) The average force $\overline{F}$ acting over a time interval $\Delta t = t_\mathrm{f} - t_\mathrm{i}$ gives the same impulse as the actual force F during the same time interval Δt

Example 7.1

A billiard ball of mass $m = 170\,\mathrm{g}$ has velocity components $v_x = v_y = 4\,\mathrm{m/s}$, see Fig. 7.2. The ball bounces back from a table's edge with the same speed and angle after being in contact with the edge for 0.2 s. Assume that friction and rotational motion are negligible. (a) What is the change in the horizontal and vertical components of the ball's momentum? (b) What is the average force exerted on the ball by the wall?

Fig. 7.2



Solution: (a) Bouncing with the same speed and angle means that the x component of the velocity is reversed, while the y component remains unchanged (this is known as an elastic collision). Since the x component of the ball's momentum is mv_x before the collision and $-mv_x$ afterward, the change in the ball's momentum will be:

$$\Delta p_x = (p_x)_\mathrm{f} - (p_x)_\mathrm{i} = -mv_x - mv_x$$
$$= -2mv_x = -2(0.17\,\mathrm{kg})(4\,\mathrm{m/s}) = -1.36\,\mathrm{kg.m/s}$$

Because of the unchanged y component of the velocity, we get

$$\Delta p_y = (p_y)_\mathrm{f} - (p_y)_\mathrm{i} = mv_y - mv_y = 0$$

(b) According to part (a), we have $\Delta\vec{p} = \Delta p_x \vec{\mathrm{i}} = -2mv_x \vec{\mathrm{i}}$, which by Eq. 7.9 means that the force exerted by the wall on the ball will be in the negative x direction. Thus:

$$\overline{\vec{F}} = \Delta\vec{p}/\Delta t = -2mv_x/\Delta t\,\vec{\mathrm{i}} = (-1.36\,\mathrm{kg.m/s})/(0.2\,\mathrm{s})\,\vec{\mathrm{i}} = -(6.8\,\mathrm{N})\,\vec{\mathrm{i}}$$

7.2 Conservation of Linear Momentum

Consider a system of n particles with linear momenta $\vec{p}_1, \vec{p}_2, \ldots,$ and $\vec{p}_n$. Some forces on these particles are external to the system, and others are internal. These forces may be of any type, including gravitational, electric, or magnetic.

Let $\vec{P}$ be the total linear momentum of the system, which is the vector sum of all individual momenta. Thus:

$$\vec{p}_1 + \vec{p}_2 + \cdots + \vec{p}_n = \sum \vec{p}_i = \vec{P} \tag{7.10}$$

When differentiating this equation with respect to time, we get:

$$\sum \frac{d\vec{p}_i}{dt} = \sum \vec{F}_i = \frac{d\vec{P}}{dt} \tag{7.11}$$

where $\sum \vec{F}_i$ represents the sum of all *forces* (internal plus external) exerted on the particles of the system. Then we can write the sum $\sum \vec{F}_i$ as follows:

$$\sum \vec{F}_i = \sum \vec{F}_\mathrm{ext} + \sum \vec{F}_\mathrm{int} \tag{7.12}$$

where $\sum \vec{F}_\mathrm{ext}$ is the vector sum of all external forces acting on the particles of the system. By Newton's third law, the internal forces form action-reaction pairs and their sum cancel each other out, i.e., $\sum \vec{F}_\mathrm{int} = 0$. Therefore, Eq. 7.11 reduces to:

$$\sum \vec{F}_\mathrm{ext} = \frac{d\vec{P}}{dt} \quad \text{(System of particles)} \tag{7.13}$$

This equation represents a generalization of the single-particle equation $\sum \vec{F} = d\vec{p}/dt$ that is deduced for a single particle.

For an isolated system, the sum of the external forces is zero. Setting $\sum \vec{F}_{\text{ext}} = 0$ in Eq. 7.13 yields $d\vec{P}/dt = 0$, or:

$$\boxed{\vec{P} = \text{constant}} \qquad \text{(Isolated system)} \tag{7.14}$$

Spotlight

Thus, the total linear momentum of an isolated system of particles remains constant

This is the **law of conservation of momentum**, which can be written as:

$$\boxed{\vec{P}_i = \vec{P}_f} \qquad \text{(Isolated system)} \tag{7.15}$$

where the subscripts refer to the total momentum of the system at initial time i and final time f.

Example 7.2

Two trams, 1 and 2, have an equal mass of $m = 5{,}000$ kg each. Tram 1 is traveling with a speed $v_1 = 15$ m/s before striking tram 2, which was at rest. If the two trams lock together as the result of the collision as shown in Fig. 7.3, what is their common speed immediately after collision?

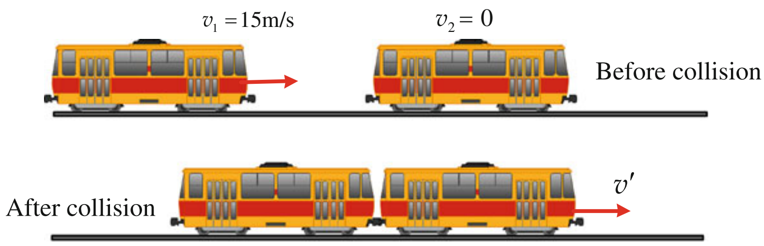


Fig. 7.3

Solution: We consider a short time interval after the collision so that heat and external forces such as friction can be ignored. Then we can apply the conservation of the total horizontal momentum:

$$P_i = P_f$$

The initial total momentum of the two trams before collision is:

$$P_i = m_1 v_1 + m_2 v_2 = m_1 v_1 + 0 = (5{,}000\,\text{kg})(15\,\text{m/s}) = 75{,}000\,\text{kg. m/s}$$

The final total momentum of the two trams after collision is:

$$P_f = m_1 v' + m_2 v' = (m_1 + m_2)v' = (5{,}000\,\text{kg} + 5{,}000\,\text{kg})v' = (10{,}000\,\text{kg})v'$$

Applying the conservation of total momentum $P_i = P_f$, we get:

$$75{,}000\,\text{kg.m/s} = (10{,}000\,\text{kg})v' \quad \Rightarrow \quad v' = \frac{75{,}000\,\text{kg.m/s}}{10{,}000\,\text{kg}} = 7.5\,\text{m/s}$$

Example 7.3

A cannon of mass $M = 1{,}500\,\text{kg}$ shoots a projectile of mass $m = 100\,\text{kg}$ with a horizontal speed $v = 30\,\text{m/s}$, as shown in Fig. 7.4. If the cannon can *recoil freely* on a horizontal ground, what is its recoil speed V just after shooting the projectile?

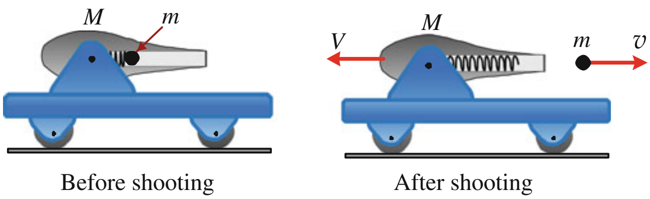


Fig. 7.4

Solution: We take our system to be the cannon and the projectile, which both are at rest initially before shooting. When the trigger is pulled, the forces involved in the shooting are internal and hence cancel. During the very short time of shooting, we can assume that the external forces such as friction are very small compared to the forces exerted by the shooting. In addition, the external gravitational forces acting on the system have no components in the horizontal direction. Then the momentum conservation along the horizontal direction is:

$$P_i = P_f$$

The initial total horizontal momentum before the shooting is:

$$P_i = m \times 0 + M \times 0 = 0$$

The final total horizontal momentum after the shooting is:

$$P_f = mv + MV$$

Applying the conservation of total momentum $P_i = P_f$, we get:

$$V = -\frac{mv}{M} = -\frac{(100\,\text{kg})(30\,\text{m/s})}{1{,}500\,\text{kg}} = -2\,\text{m/s}$$

The minus sign indicates that the velocity and momentum of the cannon is opposite to that of the projectile. Since the cannon has a much larger mass than the projectile, its recoil speed is much less than that of the projectile.

7.3 Conservation of Momentum and Energy in Collisions

During most types of collisions, forces are usually unknown. Nevertheless, by using the conservation laws of momentum and energy we can determine much information about the motion after collision in terms of information before collision. When objects are very hard, so that no heat or other forms of energy are produced during collisions, the kinetic energy is conserved before and after collision. Such a collision is referred to as an **elastic collision**. Thus, in elastic collisions we have the following for a system of particles:

$$\left\{\begin{matrix}\text{Total kinetic energy before} = \text{Total kinetic energy after}\\ \sum \frac{1}{2}mv^2_{before} = \sum \frac{1}{2}mv^2_{after}\end{matrix}\right\}\left\{\begin{matrix}\text{Elastic}\\ \text{collision}\end{matrix}\right\} \tag{7.16}$$

Collisions in which kinetic energy is not conserved are said to be **inelastic collisions**. However, we should remember that the total energy is conserved even if kinetic energy is not. Thus:

$$\left\{\begin{matrix}\text{Total energy before} = \text{Total energy after}\\ \sum \frac{1}{2}mv^2_{before} = \sum \frac{1}{2}mv^2_{after} + \text{other forms of energy}\end{matrix}\right\}\left\{\begin{matrix}\text{Inelastic}\\ \text{collision}\end{matrix}\right\} \tag{7.17}$$

7.3.1 Elastic Collisions in One and Two Dimensions

First, we apply the conservation laws of momentum and kinetic energy in an elastic collision of two small objects that collide head-on. Figure 7.5 shows two objects of masses m_1 and m_2 (treated as particles) moving along the x-axis with velocities v_1 and v_2, respectively. Usually the object of mass m_1 is called the projectile while the object of mass m_2 is called the target. After collision their velocities are v_1' and v_2', respectively. If the sign of any velocity is positive, then the object is moving in the direction of increasing x, whereas if the sign of the velocity is negative, then the object is moving in the direction of decreasing x.

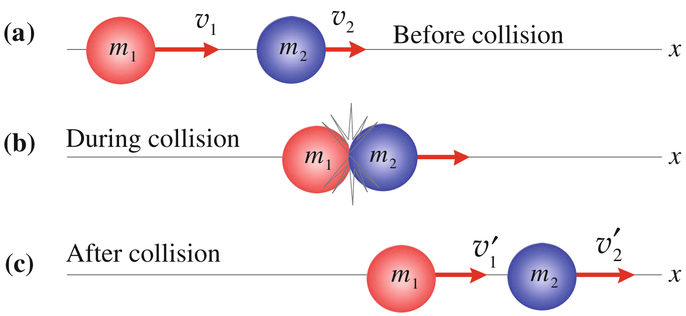


Fig. 7.5 Two small objects of masses m_1 and m_2, (a) approaching each other before collision, (b) colliding head-on, and (c) moving away from each other after collision

From the conservation of momentum, $P_i = P_f$, we have:

$$m_1 v_1 + m_2 v_2 = m_1 v_1' + m_2 v_2'$$

From the conservation of kinetic energy of elastic collisions, we have:

$$\tfrac{1}{2} m_1 v_1^2 + \tfrac{1}{2} m_2 v_2^2 = \tfrac{1}{2} m_1 v'^2_1 + \tfrac{1}{2} m_2 v'^2_2$$

If we know the masses and the velocities before collision, we can solve the above two equations for the two unknowns v'_1 and v'_2. We rewrite the momentum and kinetic-energy equations as follows:

$$m_1(v_1 - v_1') = m_2(v_2' - v_2) \tag{7.18}$$

$$m_1(v_1^2 - v'^2_1) = m_2(v'^2_2 - v_2^2) \tag{7.19}$$

Using the identity $a^2 - b^2 = (a - b)(a + b)$, we write the last equation as:

$$m_1(v_1 - v_1')(v_1 + v_1') = m_2(v_2' - v_2)(v_2' + v_2) \tag{7.20}$$

When dividing Eq. 7.20 by Eq. 7.18, we get:

$$v_1 + v_1' = v_2' + v_2 \tag{7.21}$$

We can rewrite this equation as:

$$v_1 - v_2 = -(v_1' - v_2') \tag{7.22}$$

This shows that for any elastic head-on collisions, the *relative velocity* of two objects before collision equals the negative of their *relative velocity* after collision, regardless of the masses of the objects.

In addition, Eqs. 7.18 and 7.21 can be used to find the final velocities (normally the unknown quantities) in terms of the initial velocities (normally the known quantities) as follows:

$$v_1' = \frac{m_1 - m_2}{m_1 + m_2} v_1 + \frac{2m_2}{m_1 + m_2} v_2 \tag{7.23}$$

$$v_2' = \frac{2m_1}{m_1 + m_2} v_1 + \frac{m_2 - m_1}{m_1 + m_2} v_2 \tag{7.24}$$

We can apply these equations to some very important special cases:

- Equal masses ($m_1 = m_2$). Equations 7.23 and 7.24 show that:

$$\boxed{v_1' = v_2} \quad \text{and} \quad \boxed{v_2' = v_1} \quad \text{(The objects exchange velocities)}$$

- Object 2 (the target) is initially at rest ($v_2 = 0$). Equations 7.23 and 7.24 becomes:

$$\boxed{v_1' = \frac{m_1 - m_2}{m_1 - m_2} v_1} \quad \text{and} \quad \boxed{v_2' = \frac{2m_1}{m_1 + m_2} v_1} \tag{7.25}$$

  (a) If $m_1 \gg m_2$, i.e., the projectile is *heavier* than the target, then:

$$\boxed{v_1' \approx v_1} \quad \text{and} \quad \boxed{v_2' \approx 2v_1}$$

  The much heavier object (projectile) continues with unaltered velocity, while the light object (target) takes off with twice the velocity of the heavy object

  (b) If $m_1 \ll m_2$, i.e., the projectile is much *lighter* than the target, then:

$$\boxed{v_1' \approx -v_1} \quad \text{and} \quad \boxed{v_2' \approx 0}$$

  The light object (projectile) has its velocity reversed while the heavy object (target) remains approximately at rest

The general Eqs. 7.23 and 7.24 should not be memorized. In each different problem we can easily start from scratch by applying the conservation of momentum and kinetic energy to solve questions in any elastic head-on collision.

Example 7.4

A tennis ball of mass $m_1 = 0.04\,\text{kg}$, moving with a speed of 5 m/s, has an elastic head-on collision with a target ball of mass $m_2 = 0.06\,\text{kg}$ that was moving at a speed of 3 m/s. What is the velocity of each ball after the collision if the two balls are moving: (a) in the same direction as shown in Fig. 7.6a? (b) in opposite direction as shown in Fig. 7.6b?

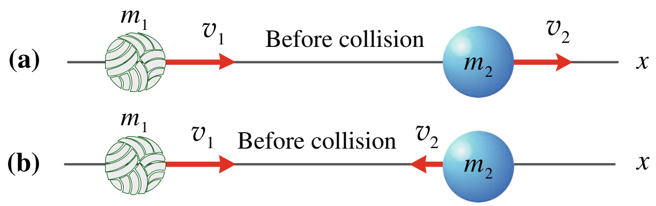


Fig. 7.6

Solution: (a) In Fig. 7.6a, we have $v_1 = +5$ m/s and $v_2 = +3$ m/s. Using Eq. 7.22, we find a relationship between the velocities as:

$$v_1 - v_2 = -(v_1' - v_2') \quad \Rightarrow \quad 5\,\text{m/s} - 3\,\text{m/s} = v_2' - v_1' \quad \Rightarrow \quad v_2' = 2\,\text{m/s} + v_1'$$

Using this result in the conservation of momentum, we have:

$$m_1 v_1 + m_2 v_2 = m_1 v_1' + m_2 v_2'$$

$$m_1 v_1 + m_2 v_2 = m_1 v_1' + m_2 (2\,\text{m/s} + v_1')$$

$$\begin{aligned} v_1' &= \frac{m_1 v_1 + m_2 (v_2 - 2\,\text{m/s})}{m_1 + m_2} = \frac{(0.04\,\text{kg})(5\,\text{m/s}) + (0.06\,\text{kg})(3\,\text{m/s} - 2\,\text{m/s})}{0.04\,\text{kg} + 0.06\,\text{kg}} \\ &= \frac{(0.2\,\text{kg.m/s}) + (0.06\,\text{kg.m/s})}{0.1\,\text{kg}} = \frac{0.26\,\text{kg.m/s}}{0.1\,\text{kg}} = 2.6\,\text{m/s} \end{aligned}$$

The other unknown velocity is v_2', which can now be obtained from:

$$v_2' = 2\,\text{m/s} + v_1' = 2\,\text{m/s} + 2.6\,\text{m/s} = 4.6\,\text{m/s}$$

After the collision, the plus signs of v_1' and v_2' tell us that the tennis ball and the target will move in the same positive x direction, but the tennis ball will slow down, while the target will speed up; see Fig. 7.7.

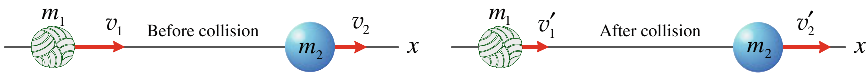


Fig. 7.7

(b) In Fig. 7.6b, we have $v_1 = +5$ m/s and $v_2 = -3$ m/s. Using Eq. 7.22, we find the relationship between the velocities as:

$$v_1 - v_2 = -(v_1' - v_2') \quad \Rightarrow \quad 5\,\text{m/s} - (-3\,\text{m/s}) = v_2' - v_1' \quad \Rightarrow \quad v_2' = 8\,\text{m/s} + v_1'$$

Similarly, using this result in the conservation of momentum, we get:

$$m_1 v_1 + m_2 v_2 = m_1 v_1' + m_2 v_2'$$

$$m_1 v_1 + m_2 v_2 = m_1 v_1' + m_2 (8\,\text{m/s} + v_1')$$

$$\begin{aligned} v_1' &= \frac{m_1 v_1 + m_2 (v_2 - 8\,\text{m/s})}{m_1 + m_2} = \frac{(0.04\,\text{kg})(5\,\text{m/s}) + (0.06\,\text{kg})(-3\,\text{m/s} - 8\,\text{m/s})}{0.04\,\text{kg} + 0.06\,\text{kg}} \\ &= \frac{(0.2\,\text{kg.m/s}) - (0.66\,\text{kg.m/s})}{0.1\,\text{kg}} = -\frac{0.46\,\text{kg.m/s}}{0.1\,\text{kg}} = -4.6\,\text{m/s} \end{aligned}$$

The other unknown velocity can now be obtained from:

$$v_2' = 8\,\text{m/s} + v_1' = 8\,\text{m/s} - 4.6\,\text{m/s} = 3.4\,\text{m/s}$$

After the collision the minus sign of v_1' tells us that the tennis ball reverses its motion and moves in the negative x direction, while the positive sign of v_2' tells us that the target also reverses its motion and moves in the positive x direction, see Fig. 7.8 with proper arrows.

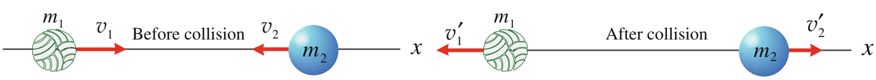


Fig. 7.8

Now, let us apply the conservation laws of momentum and kinetic energy to an elastic collision of two objects that are not colliding head-on. Figure 7.9 shows one common type of non-head-on collision at which one object (the "projectile") of mass m_1 moves along the x-axis with a speed v_1 and strikes a second stationary object (the "target") of mass m_2. After the collision, the two masses m_1 and m_2 go off at the angles θ_1 and θ_2, respectively, which are measured relative to the projectile's initial direction. We see this type of collision in nuclear experiments, or more commonly in billiard games.

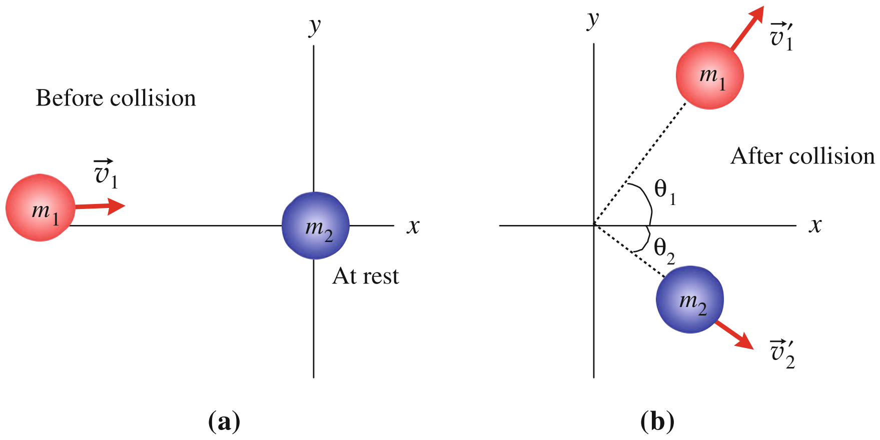


Fig. 7.9 (a) A projectile of mass m_1 moving in the x direction with velocity $\vec{v}_1$ toward a stationary target of mass m_2. (b) After collision, the projectile and target move away with velocities $\vec{v}'_1$ and $\vec{v}'_2$, respectively

We apply the law of conservation of momentum along the x and y axes, and in cases of elastic collisions we also apply the law of conservation of kinetic energy as follows:

$$\text{Momentum long } x\text{-axis:} \quad m_1 v_1 = m_1 v'_1 \cos\theta_1 + m_2 v'_2 \cos\theta_2 \tag{7.26}$$

$$\text{Momentum long } y\text{-axis:} \quad 0 = m_1 v'_1 \sin\theta_1 - m_2 v'_2 \sin\theta_2 \tag{7.27}$$

$$\text{Kinetic energy:} \quad \tfrac{1}{2} m_1 v_1^2 = \tfrac{1}{2} m_1 v_1'^2 + \tfrac{1}{2} m_2 v_2'^2 \tag{7.28}$$

If m_1, m_2, and v_1 are known quantities, then we are left with the four unknowns v'_1, θ_1, v'_2, and θ_2. Since we only have three equations, one of the four unknowns must be provided; otherwise, we cannot solve the problem.

Example 7.5

A projectile of mass $m_1 = m$ moving along the x direction with a speed $v_1 = 10\sqrt{3}$ m/s collides elastically with a stationary target of mass $m_2 = 2m$. After the collision, the projectile is deflected at an angle of 90°, as shown in Fig. 7.10. (a) What is the speed and angle of the target after collision? (b) What is the final speed of the projectile and the fraction of kinetic energy transferred to the target?

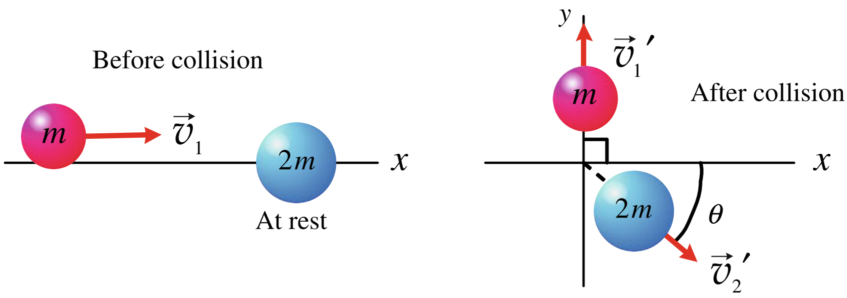


Fig. 7.10

Solution: (a) From the conservation of momentum in two dimensions and conservation of kinetic energy, we get the following relationships:

$$\begin{aligned}
&\text{Momentum along } x: && m v_1 = 2m v_2' \cos\theta && \Rightarrow && v_1 = 2v_2' \cos\theta \\
&\text{Momentum along } y: && 0 = m v_1' - 2m v_2' \sin\theta && \Rightarrow && v_1' = 2v_2' \sin\theta \\
&\text{Kinetic energy}: && \tfrac{1}{2} m v_1^2 = \tfrac{1}{2} m v_1'^2 + \tfrac{1}{2} 2m v_2'^2 && \Rightarrow && v_1^2 - v_1'^2 = 2v_2'^2
\end{aligned}$$

Squaring and adding the two momentum equations together, we get:

$$v_1^2 + v_1'^2 = 4v_2'^2$$

Adding this result to the one obtained from the conservation of kinetic energy, we get:

$$2v_1^2 = 6v_2'^2 \quad \Rightarrow \quad v_2'^2 = \frac{1}{3} v_1^2 \quad \Rightarrow \quad v_2' = \frac{1}{\sqrt{3}} v_1 = \frac{1}{\sqrt{3}} (10\sqrt{3}\,\text{m/s}) = 10\,\text{m/s}$$

Using this result in the x-momentum component, we find the angle:

$$v_1 = 2v_2' \cos\theta \quad \Rightarrow \quad v_1 = \frac{2}{\sqrt{3}} v_1 \cos\theta \quad \Rightarrow \quad \cos\theta = \frac{\sqrt{3}}{2} \quad \Rightarrow \quad \theta = 30^\circ$$

(b) We can substitute $v_2' = 10$ m/s and $\theta = 30°$ in the y-momentum component to find the speed v_1' as follows:

$$v_1' = 2(10\,\text{m/s})(\sin 30°) = 10\,\text{m/s}$$

The fraction transferred is the final energy of the target divided by the initial kinetic energy of the projectile.

$$\frac{K_{\text{target}}}{K_{\text{projectile}}} = \frac{\frac{1}{2}(2m)v_2'^2}{\frac{1}{2}mv_1^2} = \frac{\frac{1}{2}(2m)\left(v_1^2/3\right)}{\frac{1}{2}mv_1^2} = \frac{2}{3} \equiv 66.67\%$$

7.3.2 Inelastic Collisions

In some collisions, part of the initial kinetic energy is transferred to other types of energy (such as thermal or potential energy), or part of the internal energy (such as chemical or nuclear) is released as a form of kinetic energy. These types of collisions are called **inelastic collisions** because the total final kinetic energy can be less than or greater than the total initial kinetic energy (i.e., the kinetic energy is not conserved). If two objects stick together after collision, the collision is called a **completely inelastic collision**. Even though kinetic energy is not conserved in those collisions, total energy is conserved.

Example 7.6

A bullet of mass $m = 10$ g is fired horizontally with a speed v into a large wooden stationary block of mass $M = 2$ kg that is suspended vertically by two cords. This arrangement is called the ballistic pendulum, see Fig. 7.11. In a very short time, the bullet penetrates the pendulum and remains embedded. The entire system starts to swing through a maximum height $h = 10$ cm. Find the relation that gives the speed v in terms of the height h, and then find its value.

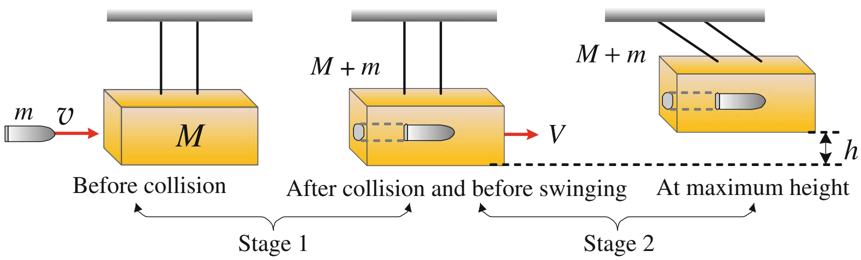


Fig. 7.11

Solution: In stage 1, momentum is conserved. Thus:

$$mv = (M + m)V \quad \Rightarrow \quad V = \frac{m}{M + m}v$$

In stage 2, the mechanical energy, $K + U$, is conserved. Thus:

$$\tfrac{1}{2}(M + m)V^2 + 0 = 0 + (M + m)gh \quad \Rightarrow \quad V^2 = 2gh \quad \Rightarrow \quad V = \sqrt{2gh}$$

Inserting this result into the previous relation gives v in terms of h as:

$$v = \frac{M + m}{m}\sqrt{2gh} \quad \Rightarrow \quad v = \frac{2.01\,\text{kg}}{0.01\,\text{kg}}\sqrt{2(10\,\text{m/s}^2)(0.1\,\text{m})} = 284.3\,\text{m/s}$$

7.4 Center of Mass (CM)

Until now, we have dealt with translation motion of an object that can be approximated by a point particle. In fact, real objects can undergo both translational and rotational motions. From general practical observations, it is found that when an applied resultant force $\Sigma\vec{F}_{\text{ext}}$ acts on an extended object (or a system of particles) of total mass M, the translation motion of the object moves as if the resultant force were applied on a single point at which the mass of the object were concentrated. This behavior is independent of other motion, such as rotational or vibrational motion. This special point is called the **center of mass** (abbreviated by CM) of the object.

As an example, consider the motion of the center of mass of the wrench over a horizontal surface shown in Fig. 7.12a. The CM follows a straight line under a zero net force. In Fig. 7.12b, the CM follows a straight line even when the wrench rotates about the CM.

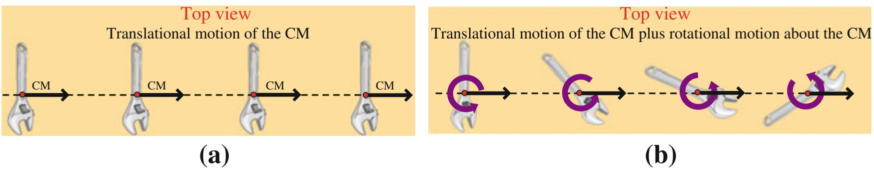


Fig. 7.12 (a) A *top view* of the translational motion of the CM of a wrench over a horizontal surface (the *red dot* represents the wrench's CM at different moments). (b) A *top view* of the translational motion of the CM plus the rotational motion about the CM

Figure 7.13 depicts a system of two masses m_1 and m_2 located on the x-axis at positions x_1 and x_2, respectively. The center of mass of this system of particles is at the position x_{CM} and defined as follows:

$$x_{CM} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} \tag{7.29}$$

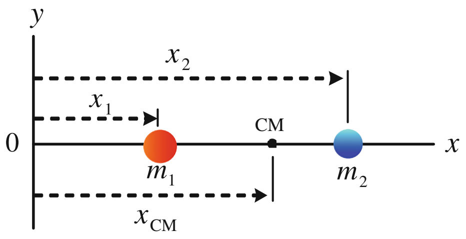


Fig. 7.13 The coordinate of the center of mass (x_{CM}) of a system of two particles is a point located between the particles

For a system consisting of n particles, where n could be very large, Eq. 7.29 becomes:

$$x_{CM} = \frac{m_1 x_1 + m_2 x_2 + \cdots + m_n x_n}{m_1 + m_2 + \cdots + m_n} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i} = \frac{\sum_i m_i x_i}{M} \tag{7.30}$$

The symbol $\sum_{i=1}^{n}$ indicates the sum over all particles, where i takes an integer values from 1 to n. Often the symbol $\sum_{i=1}^{n}$ is replaced by the symbol $\sum_i$ (or even $\sum$). The total mass of the system is $M = \sum m_i$.

If the particles are spread out in three dimensions and x_i, y_i, and z_i are the coordinates of the i^{th} particle of mass m_i and position vector $\vec{r}_i = x_i \vec{\text{i}} + y_i \vec{\text{j}} + z_i \vec{\text{k}}$, then we define the coordinates of the CM as:

$$x_{CM} = \frac{\sum m_i x_i}{M}, \quad y_{CM} = \frac{\sum m_i y_i}{M}, \quad z_{CM} = \frac{\sum m_i z_i}{M}, \tag{7.31}$$

where $M = \sum m_i$ is the total mass of the system. The position vector of the CM is thus:

$$\vec{r}_{CM} = x_{CM} \vec{\text{i}} + y_{CM} \vec{\text{j}} + z_{CM} \vec{\text{k}} = \frac{\sum m_i x_i \vec{\text{i}} + \sum m_i y_i \vec{\text{j}} + \sum m_i z_i \vec{\text{k}}}{M} \tag{7.32}$$

The position vector of the CM can be simplified as:

$$\boxed{\vec{r}_{CM} = \frac{\sum m_i \vec{r}_i}{M}} \tag{7.33}$$

For an extended object, we divide the object into tiny elements, each of mass Δm_i around a point with coordinates x_i, y_i, and z_i. When we take the limit as $n \to \infty$, then

Δm_i becomes an infinitesimal mass dm with coordinates x, y, and z. The summations in Eq. 7.31 become integrals and we get:

$$\boxed{x_{\text{CM}} = \frac{1}{M}\int x\,dm}, \quad \boxed{y_{\text{CM}} = \frac{1}{M}\int y\,dm}, \quad \boxed{z_{\text{CM}} = \frac{1}{M}\int z\,dm} \tag{7.34}$$

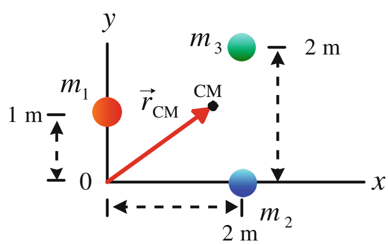
where $M = \int dm$ is the total mass of the system, and in vector notation, Eq. 7.33 becomes:

$$\vec{r}_{\text{CM}} = \frac{1}{M}\int \vec{r}\,dm \tag{7.35}$$

Example 7.7

A system of three particles of masses $m_1 = 0.5\,\text{kg}$, $m_2 = 1\,\text{kg}$, and $m_3 = 1.5\,\text{kg}$ are spread out in two dimensions and located as shown in Fig. 7.14. Find the center of mass of the system.

Fig. 7.14



Solution: According to Fig. 7.14, m_1, m_1, and m_1 have coordinates $(0, 1\,\text{m})$, $(2\,\text{m}, 0)$, and $(2\,\text{m}, 2\,\text{m})$, respectively. Thus, we use the x and y components of Eq. 7.31 with only three terms as follows:

$$\begin{aligned} x_{\text{CM}} &= \frac{\sum_{i=1}^{3} m_i x_i}{\sum_{i=1}^{3} m_i} = \frac{(0.5\,\text{kg})(0) + (1\,\text{kg})(2\,\text{m}) + (1.5\,\text{kg})(2\,\text{m})}{0.5\,\text{kg} + 1.0\,\text{kg} + 1.5\,\text{kg}} \\ &= \frac{5\,\text{kg.m}}{3\,\text{kg}} = 1.67\,\text{m} \end{aligned}$$

$$\begin{aligned} y_{\text{CM}} &= \frac{\sum_{i=1}^{3} m_i x_i}{\sum_{i=1}^{3} m_i} = \frac{(0.5\,\text{kg})(1\,\text{m}) + (1\,\text{kg})(0) + (1.5\,\text{kg})(2\,\text{m})}{0.5\,\text{kg} + 1.0\,\text{kg} + 1.5\,\text{kg}} \\ &= \frac{3.5\,\text{kg.m}}{3\,\text{kg}} = 1.17\,\text{m} \end{aligned}$$

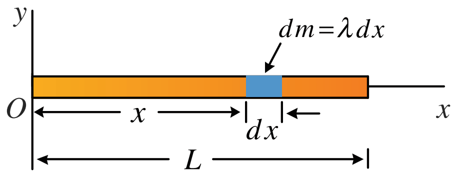
The center-of-mass position vector is thus $\vec{r}_{\text{CM}} = (1.67\,\text{m})\,\vec{i} + (1.17\,\text{m})\,\vec{j}$.

Example 7.8

A horizontal rod has a mass M and length L. Find the location of its center of mass from its left end: (a) if the rod has a uniform mass per unit length λ, and (b) if the rod has a mass per unit length λ that increases linearly from its left end according to the relation $\lambda = \alpha x$, where α is a constant.

Solution: (a) According to the geometry of Fig. 7.15, $y_{CM} = z_{CM} = 0$. For a uniform rod $\lambda = M/L$. If we divide the rod into infinitesimal elements of length dx, then the mass of each element is $dm = \lambda\, dx$.

Fig. 7.15



Accordingly, Eq. 7.34 gives:

$$x_{CM} = \frac{1}{M}\int x\, dm = \frac{1}{M}\int_0^L x\,\lambda\, dx = \frac{\lambda}{M}\int_0^L x\, dx = \frac{1}{L}\,\frac{x^2}{2}\bigg|_0^L = \frac{1}{L}\frac{L^2}{2} = \frac{L}{2}$$

where we used $\lambda = M/L$. Thus, as expected, the center of mass of a uniform rod is at its center.

(b) In this case, λ is not a constant. Therefore, Eq. 7.34 gives:

$$x_{CM} = \frac{1}{M}\int x\, dm = \frac{1}{M}\int_0^L x\,\lambda\, dx = \frac{\alpha}{M}\int_0^L x^2\, dx = \frac{\alpha}{M}\,\frac{x^3}{3}\bigg|_0^L = \frac{\alpha L^3}{3M}$$

We can eliminate α by writing M in terms of α and L as follows:

$$M = \int dm = \int_0^L \lambda\, dx = \int_0^L \alpha\, x\, dx = \alpha\,\frac{x^2}{2}\bigg|_0^L = \alpha\frac{L^2}{2}$$

Substituting this result into the expression of x_{CM}, we get:

$$x_{CM} = \frac{\alpha L^3}{3M} = \frac{\alpha L^3}{3\alpha L^2/2} = \frac{2}{3}L$$

7.5 Dynamics of the Center of Mass

In some cases, it is desirable to ignore rotational and vibrational motion in a system. In these cases, the center-of-mass concept greatly simplifies the analysis of the motion because the system of many-particles or an extended object can be treated as a single particle located at the CM of the system. To do this, we examine the motion of a system of n particles when the *total mass M of the system remains constant*. We begin by rewriting Eq. 7.33 as follows:

$$M\,\vec{r}_{\mathrm{CM}} = \sum m_i \vec{r}_i \tag{7.36}$$

Differentiating this equation with respect to time gives:

$$M\frac{d\vec{r}_{\mathrm{CM}}}{dt} = \sum m_i \frac{d\vec{r}_i}{dt}$$

or

$$M\,\vec{v}_{\mathrm{CM}} = \sum m_i \vec{v}_i \tag{7.37}$$

where $\vec{v}_{\mathrm{CM}}$ is the velocity of the center of mass and $\vec{v}_i$ is the velocity of the i^{th} particle that has a mass m_i. We differentiate again with respect to time to obtain:

$$M\frac{d\vec{v}_{\mathrm{CM}}}{dt} = \sum m_i \frac{d\vec{v}_i}{dt}$$

or

$$M\,\vec{a}_{\mathrm{CM}} = \sum m_i \vec{a}_i \tag{7.38}$$

where now $\vec{a}_{\mathrm{CM}}$ is the acceleration of the center of mass and $\vec{a}_i$ is the acceleration of the i^{th} particle. Although the center of mass is just a geometrical point in space, it has a position vector $\vec{r}_{\mathrm{CM}}$, a velocity $\vec{v}_{\mathrm{CM}}$, and an acceleration $\vec{a}_{\mathrm{CM}}$.

From Newton's second law, $m_i\vec{a}_i$ must equal the *net force* $\vec{F}_i$ that acts on the i^{th} particle of the system. Therefore, Eq. 7.38 takes the form:

$$M\,\vec{a}_{\mathrm{CM}} = \sum m_i \vec{a}_i = \sum \vec{F}_i \tag{7.39}$$

The sum of the net forces, $\sum \vec{F}_i$, that are exerted on the particles of the system can be divided into *external forces* (exerted on the particles from outside the system) and *internal forces* (exerted on the particles from within the system). By Newton's third law, as in Sect. 7.2, the internal forces cancel out in the sum $\sum \vec{F}_i$. Consequently, Eq. 7.39 can be written as follows:

$$\sum \vec{F}_{\text{ext}} = M\,\vec{a}_{CM} \tag{7.40}$$

Thus, for a system composed of a *group of particles* or formed out of an *extended object*, we conclude that:

> Spotlight
> The net external force on a system equals the total mass of the system times the acceleration of its center of mass.

If we compare Eq. 7.40 with Newton's second law for a single particle [see Eq. 5.2], we see that the point-particle model that has been used for all problems can be described in terms of the center of mass. Thus, we conclude that:

> Spotlight
> For a system of particles (or an extended object) of a total mass M, the center of mass point exists as if all the mass M were concentrated at that point and all the external forces acted on the same point.

Thus, the translational motion of any object or system of particles is known from the motion of the center of mass, as in Figs. 7.12 and 7.16.

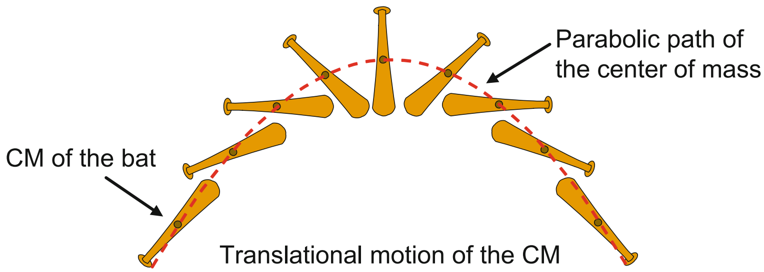


Fig. 7.16 When a bat is thrown into the air, the center of mass of the bat follows a parabolic path, but all other points of the bat follow complicated paths

Since $m_i\vec{v}_i$ is the linear momentum $\vec{p}_i$ of the i^{th} particle and $\sum \vec{p}_i = \vec{P}$ is the total linear momentum of the system, then we can rewrite Eq. 7.37 as follows:

$$M\,\vec{v}_{\text{CM}} = \vec{P} \tag{7.41}$$

Therefore, we conclude that:

For a system of particles:
The total linear momentum of a system of particles equals the total mass multiplied by the velocity of the center of mass.

For an extended object:
The linear momentum of an extended object equals its total mass multiplied by the velocity of its center of mass.

Now we differentiate Eq. 7.41 with respect to time to get:

$$M\frac{d\vec{v}_{\text{CM}}}{dt} = \frac{d\vec{P}}{dt} \quad \text{(System of particles or objects)} \tag{7.42}$$

We can use Eq. 7.40, $\sum \vec{F}_{\text{ext}} = M\,\vec{a}_{\text{CM}} = M\,d\vec{v}_{\text{CM}}/dt$, to get:

$$\sum \vec{F}_{\text{ext}} = \frac{d\vec{P}}{dt} \quad \text{(System of particles or objects)} \tag{7.43}$$

Equations 7.43 and 7.42 lead to the following conclusion:

$$\text{If} \sum \vec{F}_{\text{ext}} = 0, \text{ then} \begin{cases} \vec{P} = \text{constant} \\ \quad \text{and} \\ \vec{v}_{\text{CM}} = \text{constant} \end{cases} \tag{7.44}$$

That is, if the net force acting on a system is zero (which is true for any isolated system), then the total linear momentum as well as the velocity of the center of mass are both conserved. This is a generalization to the law of conservation of momentum discussed in Sect. 7.2. In fact, this result greatly simplifies the analysis of the motion of complex systems and extended objects.

Example 7.9

Two particles of masses $m_1 = 30$ g and $m_2 = 70$ g undergo an elastic head-on collision. Particle m_1 has an initial velocity of 2 m/s along the positive x-direction, while m_2 is initially at rest. (a) What are the velocities of the particles after the collision? (b) What is the velocity of the center of mass? Sketch the velocities of m_1, m_2, and CM at different times before and after the collision.

Solution: (a) From Eq. 7.23 we have:

$$v_1' = \frac{m_1 - m_2}{m_1 + m_2} v_1 = \frac{30\,\text{g} - 70\,\text{g}}{30\,\text{g} + 70\,\text{g}} (2\,\text{m/s}) = -0.8\,\text{m/s}$$

The negative sign indicates that m_1 rebounds after the collision and moves along the negative x-direction. From Eq. 7.24, we have:

$$v_2' = \frac{2m_1}{m_1 + m_2} v_1 = \frac{(2)(30\,\text{g})}{30\,\text{g} + 70\,\text{g}} (2\,\text{m/s}) = +1.2\,\text{m/s}$$

Thus, the relatively heavy target m_2 moves along the positive x-direction, but with a slower speed than the incoming particle m_1.

(b) Since $\sum \vec{F}_{\text{ext}} = 0$, then $P_{\text{before}} = P_{\text{after}}$ and Eq. 7.41 gives:

$$v_{\text{CM}} = \frac{P}{M} = \frac{m_1 v_1 + 0}{m_1 + m_2} = \frac{m_1}{m_1 + m_2} v_1 = \frac{30\,\text{g}}{30\,\text{g} + 70\,\text{g}} (2\,\text{m/s}) = +0.6\,\text{m/s}$$

Figure 7.17 displays v_1, v_2, v_1', v_2', and v_{CM} at different times. Notice that the velocity of the center of mass is unaffected by the collision.

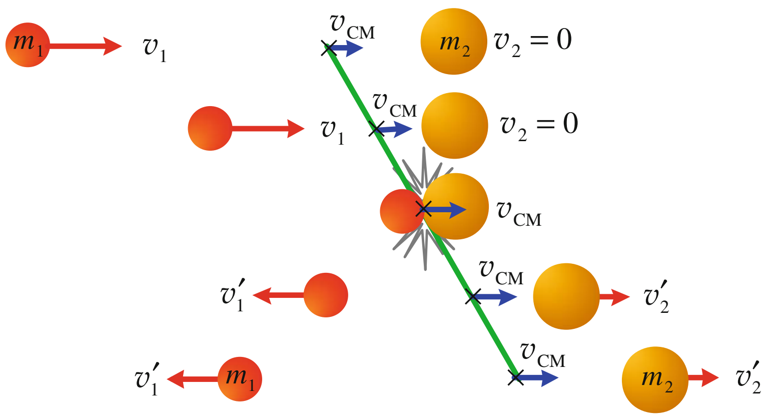


Fig. 7.17

Example 7.10

After the rocket of Fig. 7.18a is fired, the CM of the system continues to follow a parabolic trajectory from a constant downward gravitational force. When the system has a total mass M and speed $v_1 = 216\,\text{m/s}$, a prearranged explosion separates the system into two parts, a space capsule of mass $m_1 = M/4$ and a rocket of mass $m_2 = 3M/4$. The velocities of the two parts are perpendicular and the capsule has an upward initial speed $v'_1 = 571$ m/s, see Fig. 7.18b. Describe the motion of the CM and find the initial speed v'_2 of the rocket just after the separation of the space capsule and the rocket.

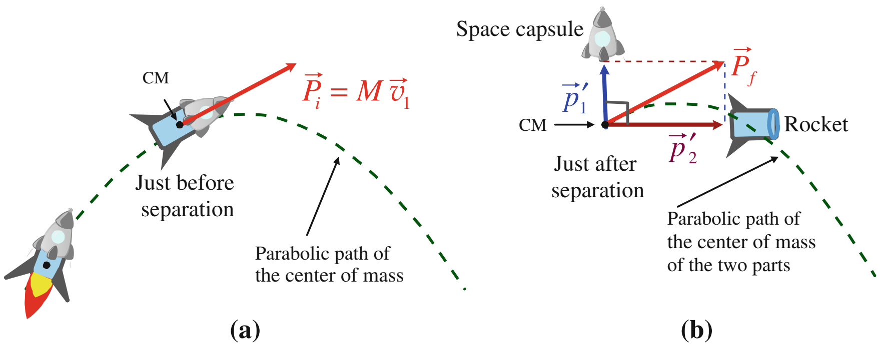


Fig. 7.18

Solution: Since the forces of the explosion are internal to the system composed of the rocket and the capsule, the initial momentum $\vec{P}_i$ just before the separation must equal the final total momentum $\vec{P}_f$ right after the separation. In addition, the center of mass of the two parts continues to follow the original parabolic path, until the rocket hits the ground. Conservation of total momentum gives:

$$\vec{P}_i = \vec{P}_f \quad\Rightarrow\quad \vec{P}_i = \vec{p}\,'_1 + \vec{p}\,'_2 \quad\Rightarrow\quad (Mv_1)^2 = \left(\frac{M}{4}v'_1\right)^2 + \left(\frac{3M}{4}v'_2\right)^2$$

Eliminating M from the last result and solving for v'_2, we find:

$$v'_2 = \sqrt{\frac{16v_1^2 - v'^2_1}{9}} = \sqrt{\frac{16(216\,\text{m/s})^2 - (571\,\text{m/s})^2}{9}} = 216\,\text{m/s}$$

7.6 Systems of Variable Mass

For systems with a variable mass M, we can use Eq. 7.43, $\sum \vec{F}_{\text{ext}} = d\vec{P}/dt$, whether the mass M increases (as in dropping material onto a conveyer belt, where $dM/dt > 0$) or the mass M decreases (as in rockets, where $dM/dt < 0$).

7.6.1 Systems of Increasing Mass

For the general treatment of systems of increasing mass, we use Fig. 7.19 that depicts the following:

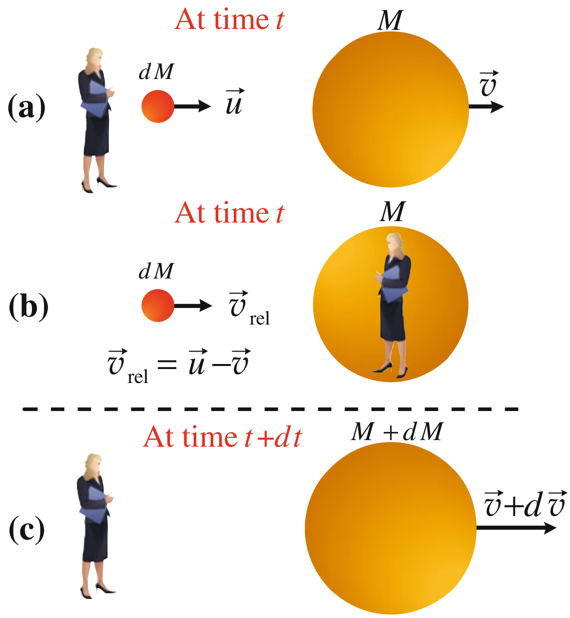


Fig. 7.19 (a) At time t, the differential mass dM is about to combine with the mass M. (b) The velocity of dM as seen by an observer on M at the same time t. (c) At time $t + dt$, the mass dM has combined with M

- At time t
  We have a system consisting of mass M moving with velocity $\vec{v}$ and momentum $M\,\vec{v}$. Also, we have an infinitesimal mass dM moving with velocity $\vec{u}$ and momentum $dM\,\vec{u}$, see Fig. 7.19a. The initial total momentum of the system can be expressed as:

$$\vec{P}_i = M\,\vec{v} + dM\,\vec{u}$$

  Relative to an observer sitting on the mass M, see Fig. 7.19b, the observer will view the infinitesimal mass dM moving with a relative velocity $\vec{v}_{\text{rel}}$ where:

$$\vec{v}_{\text{rel}} = \vec{u} - \vec{v}$$

- At time $t + dt$
  The infinitesimal mass dM combines with the mass M forming a system of mass $M + dM$ moving with velocity $\vec{v} + d\vec{v}$, see Fig. 7.19c. Then, the final total momentum of the system is:

$$\vec{P}_f = (M + dM)(\vec{v} + d\vec{v})$$

Note that dM can be positive (when momentum is being transferred *into* the mass M) or negative (when momentum is being transferred *out* of the mass M). The change in momentum of the system is thus:

$$\begin{aligned} d\vec{P} &= \vec{P}_f - \vec{P}_i = [(M + dM)(\vec{v} + d\vec{v})] - [M\,\vec{v} + dM\,\vec{u}] \\ &= M\,d\vec{v} - dM(\vec{u} - \vec{v}) \end{aligned} \tag{7.45}$$

where the term $dM\,d\vec{v}$ is dropped because it is the product of two differential quantities.

When we substitute Eq. 7.45 into $\sum \vec{F}_{\text{ext}} = d\vec{P}/dt$, we get:

$$\boxed{\sum \vec{F}_{\text{ext}} + (\vec{u} - \vec{v})\frac{dM}{dt} = M\frac{d\vec{v}}{dt}} \tag{7.46}$$

This can be simplified by using the relative velocity $\vec{v}_{\text{rel}} = \vec{u} - \vec{v}$, such as:

$$\boxed{\sum \vec{F}_{\text{ext}} + \vec{v}_{\text{rel}}\frac{dM}{dt} = M\frac{d\vec{v}}{dt}} \quad \Rightarrow \quad \boxed{\vec{F}_{\text{net}} = M\frac{d\vec{v}}{dt}} \tag{7.47}$$

The right-hand side of this equation, $M\,d\vec{v}/dt$, refers to the mass times the acceleration. The first term on the left-hand side of the equation, $\sum \vec{F}_{\text{ext}}$, refers to the external force on the mass M. The second term on the left-hand side, $\vec{v}_{\text{rel}}\,dM/dt$, refers to the force exerted on M, in terms of the rate at which the momentum is being transferred into M (due to the addition of mass).

7.6.2 Systems of Decreasing Mass; Rocket Propulsion

Now we treat systems with decreasing mass by considering the case of rocket propulsion. Figure 7.20a represents the following:

- At time t
  We have a system boundary consisting of a rocket of mass M moving with velocity $\vec{v}$ and momentum $M\,\vec{v}$, see Fig. 7.20a. The initial total momentum of the system can be expressed as: $\vec{P}_i = M\vec{v}$

- At time $t + dt$

  We have a system boundary consisting of a rocket of mass $M - dM$ moving with velocity $\vec{v} + d\vec{v}$ and an ejected exhaust of mass dM moving with velocity $\vec{u}$, see Fig. 7.20b. The final total momentum of the system boundary is:

$$\vec{P}_f = (M - dM)(\vec{v} + d\vec{v}) + dM\,\vec{u} \tag{7.48}$$

Relative to an observer sitting on the rocket, see Fig. 7.20c, that observer will view the exhaust of mass dM moving with a relative velocity $\vec{v}_{\rm rel}$ where: $\vec{v}_{\rm rel} = (\vec{v} + d\vec{v}) - \vec{u}$.

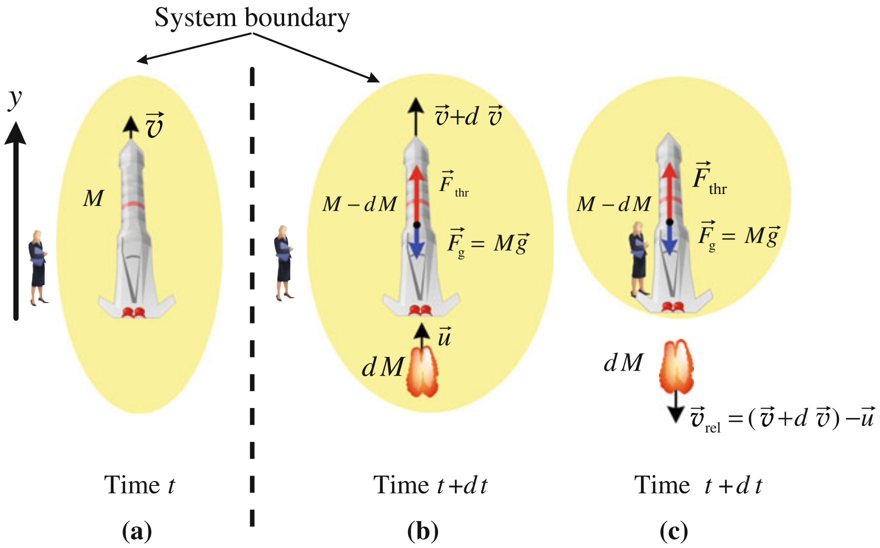


Fig. 7.20 (a) At time t, the rocket has a mass M. (b) At time $t + dt$, the mass of the exhaust dM has been ejected from M. (c) The velocity of the exhaust dM as seen by an observer on the rocket at time $t + dt$

The change in momentum between the system boundaries is thus:

$$\begin{aligned} d\vec{P} &= \vec{P}_f - \vec{P}_i = [(M - dM)(\vec{v} + d\vec{v}) + dM(\vec{v} + d\vec{v}) - \vec{v}_{\rm rel}] - M\vec{v} \\ &= M\,d\vec{v} - dM\,\vec{v}_{\rm rel} \end{aligned} \tag{7.49}$$

When we substitute with Eq. 7.49 into $\sum \vec{F}_{\rm ext} = d\vec{P}/dt$, we get:

$$\boxed{\sum \vec{F}_{\rm ext} + \vec{v}_{\rm rel}\frac{dM}{dt} = M\frac{d\vec{v}}{dt}} \tag{7.50}$$

This is identical to Eq. 7.47 except that $\vec{v}_{\mathrm{rel}}$ is against $\vec{v}$ and dM/dt is negative. The term $\vec{v}_{\mathrm{rel}}\, dM/dt$ refers to the force exerted on M in terms of the rate at which the momentum is being transferred out of M (due to the ejection of mass). For rockets, this term is positive since dM/dt is negative and $\vec{v}_{\mathrm{rel}}$ is negative (opposite to $\vec{v}$). This term is called the **thrust**, $\vec{F}_{\mathrm{thr}}$, and represents the force exerted on the rocket by the ejected gasses. Thrust is defined as follows:

$$\boxed{\vec{F}_{\mathrm{thr}} = \vec{v}_{\mathrm{rel}}\frac{dM}{dt}} \tag{7.51}$$

In one-dimensional vertical motion under a constant gravitational force, where $\sum F_{\mathrm{ext}} = -Mg$, we can find the speed of the rocket at any time t, by rewriting Eq. 7.50 as:

$$dv = -g dt + v_{\mathrm{rel}}\frac{dM}{M} \tag{7.52}$$

Since v_{rel} is constant, we can integrate this equation from an initial speed $v_{\circ}$ (when the mass was $M_{\circ}$) to a any speed v (when the mass becomes M). This gives:

$$\int_{v_{\circ}}^{v} dv = -g\int_{0}^{t} dt + v_{\mathrm{rel}}\int_{M_{\circ}}^{M}\frac{dM}{M}$$

or

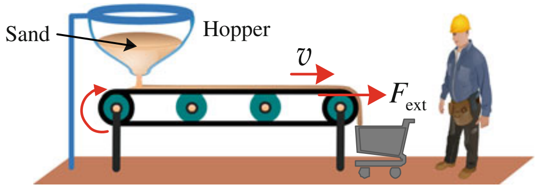
$$v - v_{\circ} = -gt + v_{\mathrm{rel}}\ln\frac{M}{M_{\circ}} \tag{7.53}$$

Note that v_{rel} is negative because it is opposite to the rocket's motion and $\ln M/M_{\circ}$ is also negative because $M_{\circ} > M$.

Example 7.11

Figure 7.21 shows a stationary hopper that drops sand at a rate $dM/dt = 80$ kg/s onto a conveyer belt. The belt is supported by frictionless rollers and moves at a constant speed $v = 1.5$ m/s under the action of a constant external force $\vec{F}_{\mathrm{ext}}$. (a) Find the value of the external force $\vec{F}_{\mathrm{ext}}$ that is needed to keep the belt moving with a constant speed. (b) Find the power delivered by the external force $\vec{F}_{\mathrm{ext}}$. (c) Find the rate of the kinetic energy acquired by the falling sand due to the change in its horizontal motion.

Fig. 7.21



Solution: (a) We use the one-dimensional form of Eq. 7.46 by considering $u = 0$ to represent the stationary hopper. We also take $dv/dt = 0$ because the belt is moving with constant speed. Thus:

$$F_{ext} + (u - v)\frac{dM}{dt} = M\frac{dv}{dt} \quad \Rightarrow \quad F_{ext} + (0 - v)\frac{dM}{dt} = 0 \quad \Rightarrow \quad F_{ext} = v\frac{dM}{dt}$$

$$\therefore \qquad F_{ext} = (1.2\,\text{m/s})(5\,\text{kg/s}) = 6\,\text{N}$$

The only horizontal force on the sand is the friction of the belt f_s. Thus, $f_s = F_{ext}$.

(b) The power delivered by $\vec{F}_{ext}$ is work done by this force in 1s. Thus:

$$P = \frac{dW}{dt} = \vec{F}_{ext} \bullet \vec{v} = F_{ext}\, v = v^2\frac{dM}{dt} = (1.2\,\text{m/s})^2(5\,\text{kg/s}) = 7.2\,\text{W}$$

This work per unit time is the power output required by the motor.

(c) The rate of the kinetic energy acquired by the falling sand is:

$$\frac{dK}{dt} = \frac{d}{dt}\left(\frac{1}{2}Mv^2\right) = \frac{1}{2}\frac{dM}{dt}v^2 = \frac{1}{2}(5\,\text{kg/s})(1.2\,\text{m/s})^2 = 3.6\,\text{W}$$

This is only half the power delivered by $\vec{F}_{ext}$. The other half goes into thermal energy produced by friction between the sand and the belt.

Example 7.12

A rocket has a mass 2×10^4 kg of which 10^4 kg is fuel. When the rocket is lunched vertically from the ground, it consumes fuel from its rear at a rate of 1.5×10^3 kg/s with an exhaust speed of 2.5×10^3 m/s relative to the rocket. Neglect air resistance and take the acceleration due to gravity to be $g = 9.8\,\text{m/s}^2$. (a) Find the thrust on the rocket. (b) Find the net force on the rocket, once when it is full of fuel and once when it is empty. (c) Find the final speed of the rocket when the fuel burns completely.

Solution: (a) Since the motion is in one dimension and we can take upward as positive, then v_{rel} is negative because it is downward and dM/dt is negative because the rocket's mass is decreasing. Therefore, the thrust is:

$$F_{\text{thr}} = v_{\text{rel}} \frac{dM}{dt} = (-2.5 \times 10^3 \text{ m/s})(-1.5 \times 10^3 \text{ kg/s}) = 3.75 \times 10^6 \text{ N}$$

(b) Initially the net force on the rocket is:

$$F_{\text{net}} = F_{\text{thr}} - M_\circ\, g = 3.75 \times 10^6 \text{ N} - (2 \times 10^4 \text{ kg})(9.8 \text{ m/s}^2) = 3.554 \times 10^6 \text{ N}$$

The net force just before the rocket is out of fuel is:

$$F_{\text{net}} = F_{\text{thr}} - M_\circ\, g = 3.75 \times 10^7 \text{ N} - (1 \times 10^4 \text{ kg})(9.8 \text{ m/s}^2) = 3.652 \times 10^6 \text{ N}$$

(c) The time required to reach fuel burnout is the time needed to use all the fuel (10^4 kg) at rate of 1.5×10^3 kg/s. Thus:

$$t = \frac{10^4 \text{ kg}}{1.5 \times 10^3 \text{ kg/s}} = 6.67 \text{ s}$$

By taking $v_\circ = 0$ and using Eq. 7.53, we find that:

$$v - v_\circ = -gt + v_{\text{rel}} \ln \frac{M}{M_\circ}$$

$$v = -(9.8 \text{ m/s}^2)(6.67 \text{ s}) + (-2.5 \times 10^3 \text{ m/s})$$

$$\times \left(\ln \frac{1 \times 10^4 \text{ kg}}{2 \times 10^4 \text{ kg}} \right) = 1667.5 \text{ m/s}$$

7.7 Exercises

Section 7.1 Linear Momentum and Impulse

(1) What is the momentum of an electron of speed $v = 0.99\,c$, if the rest mass of the electron is $m = 9.11 \times 10^{-31}$ kg and the speed of light is $c = 3 \times 10^8$ m/s?

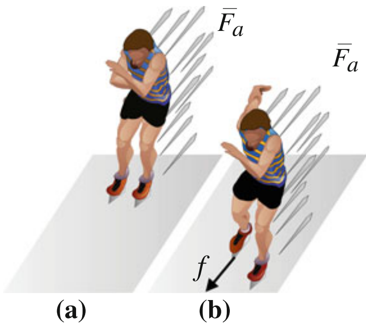
(2) (a) What is the momentum of an 8,000-kg truck when its speed is 20 m/s? What speed must a 2,000-kg car attain in order to have: (b) the same momentum as the truck, (c) the same kinetic energy as the truck?

(3) A ball of mass $m = 0.4$ kg is moving horizontally with a speed 6 m/s when it strikes a vertical obstacle. The ball rebounds with a speed 2 m/s. What is the change in momentum of the ball?

(4) A baseball has a mass of 0.2-kg and a speed of 30 m/s. After the baseball is struck by the batter, its velocity changed to 50 m/s in the opposite direction. (a) Find the change in momentum of the ball and the impulse of the strike. (b) Find the average force exerted by the bat on the ball if remains in contact for 0.002 s.

(5) A 70-kg ice skater experiences a constant air frictional force of magnitude $\bar{F}_a = 30$ N for 7 s, see Fig. 7.22a. (a) What is the change in the velocity of the skier? (b) What constant forward frictional force f must the skater apply in order to reduce the velocity of part (a) by half, see Fig. 7.22b?

Fig. 7.22 See Exercise (5)



(6) A 4-kg particle has a velocity $\vec{v} = (4\vec{i} - 3\vec{j})$ m/s. (a) What are the x and y components of its momentum? (b) Find the magnitude and direction of the momentum.

(7) Rain is falling on an object at time t with a force of $\vec{F} = (8\,t\,\vec{i} - 3\,t^2\vec{j}\,)$N. Find the change in the object's momentum between $t_i = 0$ and $t_f = 2$ s.

(8) In a training session, water with a horizontal speed of 25 m/s leaves a fireman's hose at a rate of 12 kg/s and comes to rest after striking a firewall, see Fig. 7.23. Ignoring the water splashes, what is the average force exerted by the water on the wall?

(9) The force-time graph for a ball struck by a bat is approximated as shown in Fig. 7.24. From this graph, find (a) the impulse delivered by the ball, (b) the average force exerted on the ball, and (c) the maximum force exerted on the ball.

(10) A mass m undergoes a free fall with a constant acceleration g. What is its momentum after it has been dropped (i.e., released from rest) and falls a distance h?

Fig. 7.23 See Exercise (8)

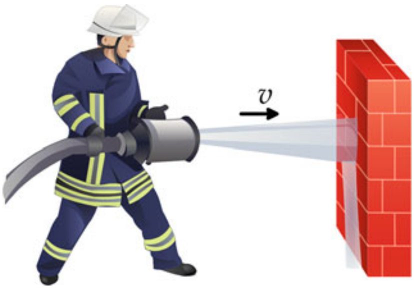
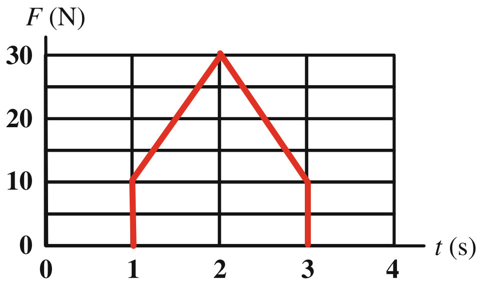
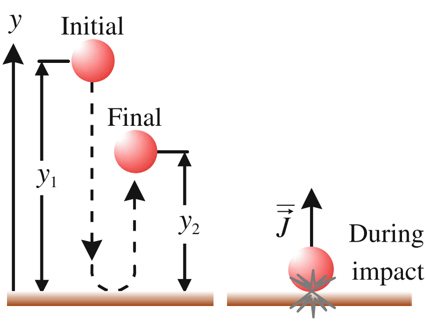


Fig. 7.24 See Exercise (9)



(11) A ball of mass 0.4 kg is dropped from a height $y_1 = 0.8$ m. The ball rebounds from the floor and reaches a maximum height $y_2 = 0.2$ m, see Fig. 7.25. Ignore air resistance and take $g = 10\,\text{m/s}^2$. (a) What is the impulse exerted by the floor on the ball? (b) What fraction of the ball's kinetic energy is lost in the Impact?

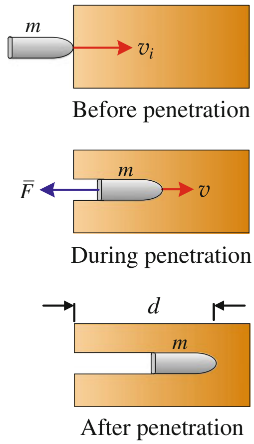
Fig. 7.25 See Exercise (11)



(12) A bullet of mass $m = 6$ g moving with $v_i = 80$ m/s strikes a wooden block and stops after penetrating a distance $d = 5$ cm, see Fig. 7.26. Assume that the bullet undergoes a constant deceleration due to an average resistive force $\overline{F}$.

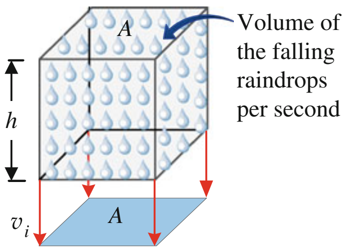
Find: (a) the penetration time, (b) the impulse on the wooden block, and (c) the average force $\bar{F}$ exerted on the bullet.

Fig. 7.26 See Exercise (12)



(13) Rain is falling vertically with a speed $v_i = 6\,\text{m/s}$ and can fill a container to a height of 18 cm in one hour. Water has a mass density $\rho = 10^3\,\text{kg/m}^3$. (a) Find the height h that the rain will fill the container in one second (neglect air volume between raindrops). (b) Estimate the mass of water that falls per unit time on a flat surface of an area $A = 2\,\text{m}^2$, see Fig. 7.27. (b) If the raindrops do not rebound, find the average force exerted by the rain on that surface.

Fig. 7.27 See Exercise (13)



(14) A 5-kg steel ball strikes a wall with a speed of 10 m/s at an angle $\theta = 60°$ with the wall's surface, see Fig. 7.28. The ball bounces off with the same speed and angle and is in contact with the wall for 0.01 s. Choose the x-axis to be toward the wall. (a) What is the change in momentum of the ball? (b) What is the average force exerted on the ball by the wall?

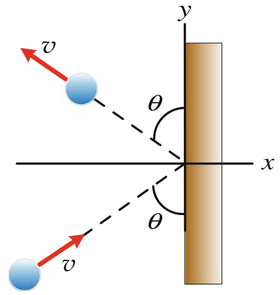


Fig. 7.28 See Exercise (14)

(15) Redo Exercise (14) with a value of θ that produce: (a) the smallest change in momentum and the average force, (b) the largest change in momentum and the average force.

Section 7.2 Conservation of Linear Momentum

(16) A locomotive of mass $m_1 = 40{,}000$ kg rolls at the speed $v_1 = 2$ m/s along a level track. It collides and couples with a stationary fully loaded freight car of mass $m_2 = 60{,}000$ kg, see Fig. 7.29. (a) What is the speed after the collision? (b) Find the decrease in kinetic energy that results from the collision. (c) With what velocity should the freight be moving toward the locomotive in order for both objects to stay at rest after the collision?

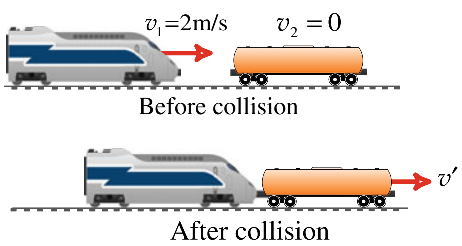


Fig. 7.29 See Exercise (16)

(17) An object at rest explodes into two fragments. One fragment of mass m_1 acquires twice the kinetic energy of the second fragment of mass m_2, see Fig. 7.30. What is the ratio of their masses?

(18) A parent atomic nucleus at rest decays radioactively into an alpha particle of mass m_1 and a residual nucleus of mass $m_2 = 232m_1$. What will be the speed of this recoiling nucleus if the speed of the alpha particle is $v_1' = 1.5 \times 10^5$ m/s?

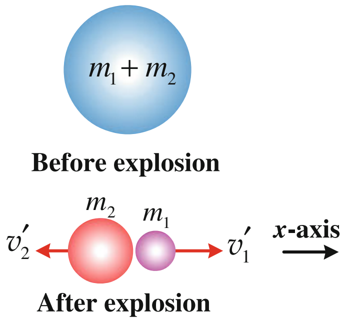
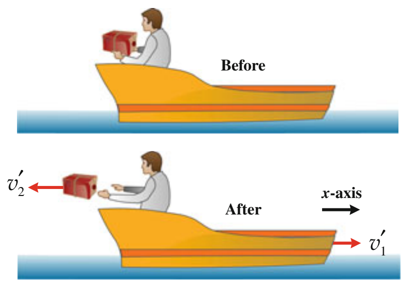


Fig. 7.30 See Exercise (17)

(19) A 60-kg boy holding a 4-kg package is sitting on a stationary boat of 100-kg mass, see Fig. 7.31. The boy throws the package horizontally with a velocity $v_2' = -5$ m/s. The boy and boat move together after the package is thrown and the boat moves without friction on the water surface. What is the speed of the boat?

Fig. 7.31 See Exercise (19)



(20) A railroad flatcar of mass M can roll without friction along a horizontal track. Initially, a man of mass m is standing on the car when it is at rest. The man starts to run on the car with a constant speed v, as measured with respect to an observer on the ground, see Fig. 7.32. (a) Find the speed V of the car with respect to the ground. (b) What is the relative speed v_{rel} of the man with respect to the car?

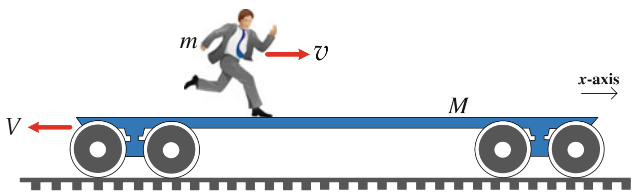


Fig. 7.32 See Exercise (20)

(21) A bullet of mass $m = 10$ g travels with velocity $v_1 = +250$ m/s toward a stationary wooden block of mass $M = 2$ kg that is resting on a horizontal frictionless surface, see Fig. 7.33. The bullet penetrates the block and emerges from the other side with velocity $v'_1 = +150$ m/s. Neglect the mass removed from the block by the bullet. (a) How fast does the block move after the bullet emerges from the other side of the block? (b) What fraction of the bullet's kinetic energy is lost in the penetration? (c) What fraction of the bullet's energy goes to heat?

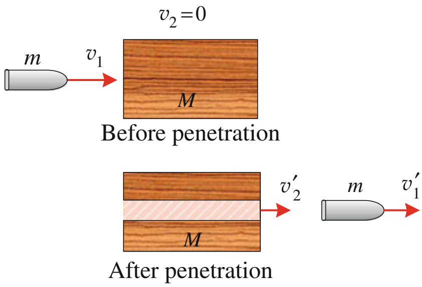


Fig. 7.33 See Exercise (21)

(22) A spaceship of mass M is traveling along the x-axis with a speed $v_i = 580$ m/s with respect to an observer on the Earth. The ship ejects a cargo module of mass 0.1 M and then travels relative to the cargo with a speed $v_{\text{rel}} = 140$ m/s, see Fig. 7.34. What is the velocity v_f of the ship with respect to the observer?

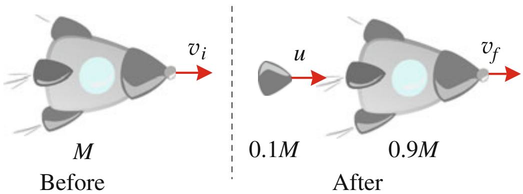


Fig. 7.34 See Exercise (22)

Section 7.3 Conservation of Momentum and Energy in Collisions

Subsection 7.3.1 Elastic Collisions in One and Two Dimensions

(23) A tennis ball of mass $m_1 = 0.06$ kg, moving with a speed of 4 m/s, has an elastic head-on collision with a target ball of mass $m_2 = 0.09$ kg initially moving in the same direction at a speed of 3 m/s. What is the velocity of each ball after the collision?

(24) A ball of mass $m_1 = 0.5\,\text{kg}$, moving along the x-axis with a speed of 5 m/s, has an elastic head-on collision with a target ball of mass $m_2 = 1\,\text{kg}$ initially at rest. What is the velocity of each ball after the collision?

(25) A ball of mass m_1 and velocity v_1 undergoes an elastic head-on collision with a second ball of mass m_2 initially at rest. Then m_1 rebounds with a speed $v_1' = -0.5v_1$. Find the value of m_2 in terms of m_1.

(26) A croquet ball of mass $m_1 = 1\,\text{kg}$ and velocity v_1 undergoes an elastic head-on collision with a second ball of mass m_2 that is initially at rest. Then m_1 moves with a velocity v_1' and m_2 moves with a velocity $v_2' = (4/5)v_1$. (a) Find the value of m_2. (b) Find the relation between v_1' and v_1. (c) What fraction of the original kinetic energy goes to the second ball?

(27) Find the fraction of kinetic energy lost by a neutron of mass $m_1 = 1.01\text{u}$ when it undergoes an elastic head-on collision with a stationary nucleus of: (a) a hydrogen atom ($^1_1$H) of mass $m_2 = 1.01\,\text{u}$, (b) a heavy hydrogen atom ($^2_1$H) of mass $m_2 = 2.01\,\text{u}$, (c) a carbon ($^{12}_6$C) atom of mass $m_2 = 12.00\,\text{u}$, and (d) a lead atom ($^{208}_{82}$Pb) of mass $m_2 = 208\,\text{u}$.

(28) A block of mass $m_1 = 1\,\text{kg}$ slides along a frictionless horizontal surface with a speed $v_1 = 4\,\text{m/s}$ toward a stationary second block of mass $m_2 = 0.5\,\text{kg}$. The second block is connected to an elastic spring that is not stretched and has a spring constant $k_H = 100\,\text{N/m}$. The other end of that spring is fixed to a wall, see Fig. 7.35. (a) Is the collision elastic? Explain your answer. (b) What will be the maximum compression of the spring?

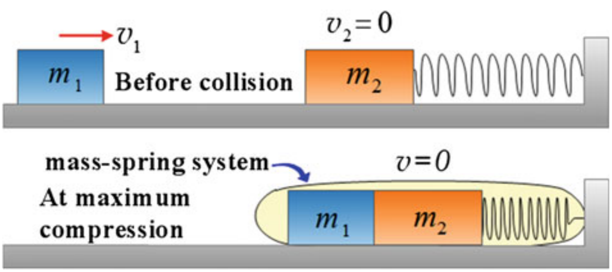


Fig. 7.35 See Exercise (28)

(29) A block of mass $m_1 = 2.5\,\text{kg}$ slides along a frictionless horizontal surface with a speed $v_1 = 8\,\text{m/s}$ toward a stationary second block of mass $m_2 = 7.5\,\text{kg}$. A massless spring with spring constant $k_H = 1{,}920\,\text{N/m}$ is attached to the near

side of m_2, as shown in Fig. 7.36. (a) Is the collision elastic? Explain. (b) What is the speed of the mass-spring system at the maximum compression? (c) What will be the maximum compression of the spring? (d) What will be the final velocities of the two blocks?

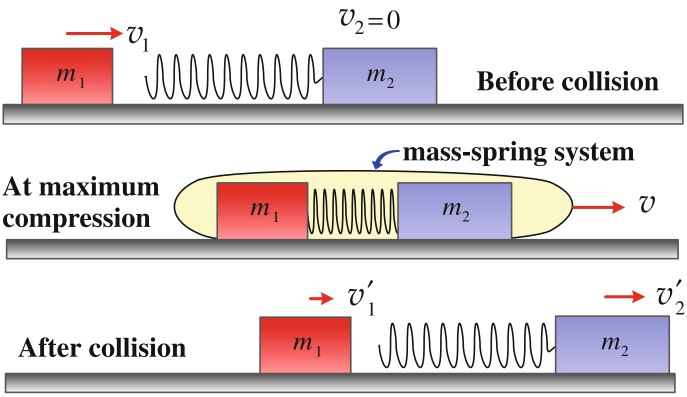


Fig. 7.36 See Exercise (29)

(30) Repeat Exercise (29), this time with $m_1 = 7.5$ kg and $m_2 = 2.5$ kg.

(31) Show that the fraction of kinetic energy transferred to the target in Example 7.5 is independent of the value of the speed of the projectile, v_1. Then, redo this example when $m_2 = 3m_1$.

(32) A hockey puck traveling at 30 m/s on a smooth ice surface is deflected by $\theta_1 = 30°$ from its original direction when it collides elastically with a second stationary identical puck. The second puck acquires a velocity at $\theta_2 = 60°$ from the original velocity of the first puck, see Fig. 7.37. Find the speed of the pucks after the collision.

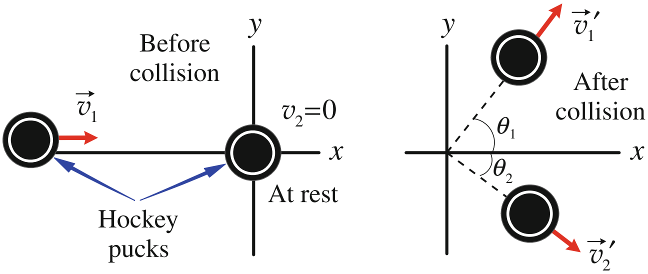


Fig. 7.37 See Exercise (32)

(33) If each angle in Exercise (32) is equal to 45°, then show that only the application of conservation of momentum is enough to find the speed of the pucks after the collision. Also, show that any other two equal angles (say 30°) are physically unacceptable.

(34) A ball of momentum $\vec{p}_1$ collides with an identical stationary ball. The first ball deflects by an angle θ_1 from its original direction with a momentum $\vec{p}\,'_1$ while the second ball deflects by an angle θ_2 from the original direction of the first ball with a momentum $\vec{p}\,'_2$. If the two balls have an elastic collision, then use the conservation of momentum vector diagram of Fig. 7.38 and conservation of kinetic energy to show that the two balls will always move off perpendicular to each other, i.e. $\theta_1 + \theta_2 = 90°$.

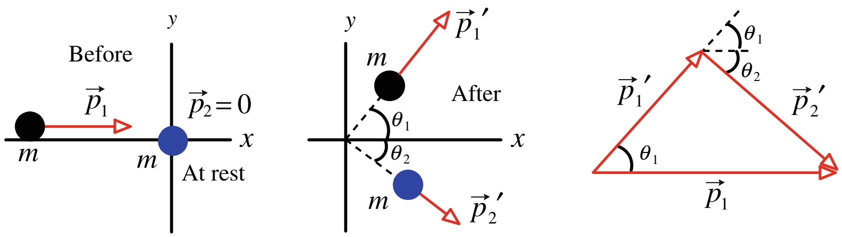


Fig. 7.38 See Exercise (34)

Subsection 7.3.2 Inelastic Collisions

(35) In a ballistic experiment like the one shown in Example 7.6, a bullet causes the pendulum to rise to a maximum height $h_1 = 1.3$ cm. A second bullet of the same mass causes the pendulum to rise to a maximum height $h_2 = 5.2$ cm. Express the speed of the second bullet as a multiple of the first bullet.

(36) Find a formula that gives the fractional change of kinetic energy, $(K_f - K_i)/K_i$, in terms of m and M in the first stage of the ballistic pendulum of Example 7.6. Calculate this fraction for $m = 10$ g and $M = 0.5$ kg.

(37) A 15-kg mass is moving along the positive x-axis at 30 m/s, and a 5-kg mass is moving along the negative x-axis at 50 m/s. The two masses collide head-on and stick together. (a) Find their velocity after the collision. (b) Find the fractional change of kinetic energy.

(38) A car of mass $m_1 = 1{,}000$ kg moving with a speed v_1 collides with another stationary car of mass $m_2 = 2{,}200$ kg. The two cars stick together after the

collision. Both drivers had their brakes locked throughout the incident, see Fig. 7.39. The police officer measures the skidding distance d to be 2.25 m and estimated the coefficient of kinetic friction between the tires and the road to be 0.8. (a) What was the speed of the oncoming car? (b) Find the fractional change of kinetic energy lost during the impact period.

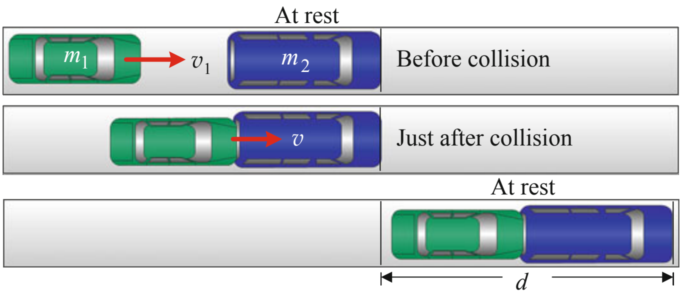


Fig. 7.39 See Exercise (38)

(39) A nucleus at rest spontaneously disintegrates into two nuclei, one of which has double the mass of the other. Assume that the total mass is conserved before and after disintegration. (a) Find the relation between the speeds of the two fragments. (b) If 6×10^{-17} J of energy is released in this disintegration, how much kinetic energy does each nucleus acquire?

(40) Two objects having the same mass $m = 5$ kg collide. Their velocities before collision are $\vec{v}_1 = (3\vec{i} + 6\vec{j})$(m/s) and $\vec{v}_2 = (-2\vec{i} + 1\vec{j})$(m/s). After collision, the first object acquires a velocity $\vec{v}'_1 = (-1\vec{i} + 4\vec{j})$(m/s). (a) What is the final velocity of the second object? (b) How much kinetic energy is lost or gained in this collision?

(41) A stationary radioactive nucleus decays into three fragments. Two of these fragments are emitted perpendicularly to each other and have momenta $|\vec{p}'_1| = 5 \times 10^{-23}$ kg.m/s and $|\vec{p}'_2| = 1.2 \times 10^{-22}$ kg.m/s. Find the magnitude and direction of the third fragment.

(42) A ball of mass $m_1 = 2.4$ kg moving horizontally with a speed of $v_1 = 3$ m/s collides (not head-on) with a second ball of $m_2 = 1.5$ kg moving in the opposite direction with a speed $v_2 = 5$ m/s, see Fig. 7.40. The first ball bounces off the second ball with an angle $\theta_1 = 60°$ and speed $v'_1 = 1.5$ m/s. (a) What is the

final velocity of the second ball? (b) How much kinetic energy is lost in this collision?

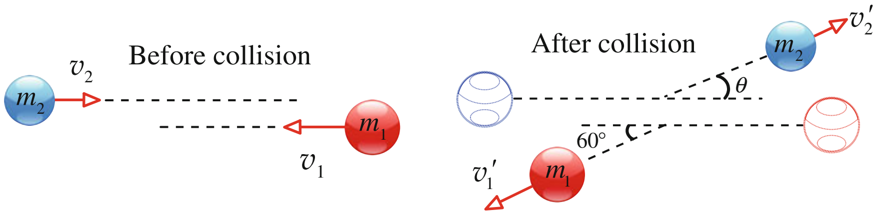


Fig. 7.40 See Exercise (42)

(43) Two identical putty balls move along a frictionless floor, as shown in Fig. 7.41. Their velocity vectors $\vec{v}_1$ and $\vec{v}_2$ make an angle θ and move with the same speed, i.e. $v_1 = v_2$. The two balls stick together after collision. (a) Use the momentum vector diagram shown in the figure to prove that the magnitude v and the direction ϕ of their common velocity $\vec{v}$ are given by:

$$v = \tfrac{1}{2}v_1\sqrt{2+2\cos\theta}, \quad \phi = \sin^{-1}\left[\sin\theta/\sqrt{2+2\cos\theta}\right]$$

(b) Taking $v_1 = v_2 = 20$ m/s and $\theta = 45°$ to calculate v, ϕ, and the fractional change in the kinetic energy.

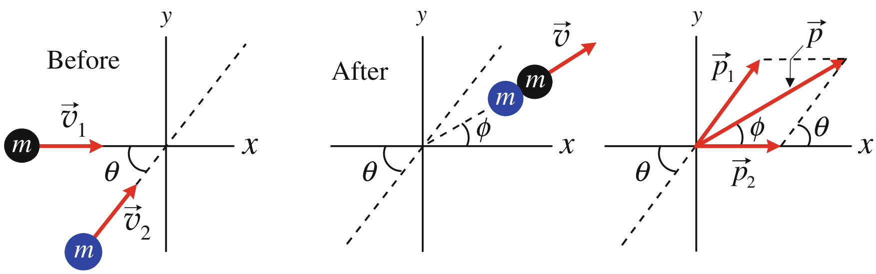


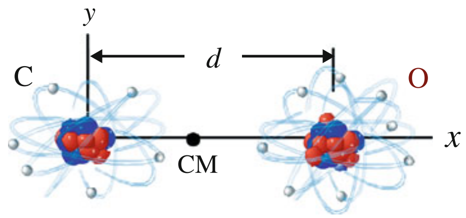
Fig. 7.41 See Exercise (43)

Section 7.4 Center of Mass (CM)

(44) An oxygen atom ($^{16}_{8}$O) has a mass $m_O = 16$ u, and a carbon atom ($^{12}_{6}$C) has a mass $m_C = 12$ u. The average distance between their nuclei in the CO molecule

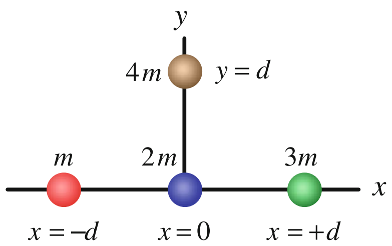
is $d = 0.113\,\text{nm}$, see Fig. 7.42. How far from the oxygen nucleus is the center of mass of the molecule?

Fig. 7.42 See Exercise (44)



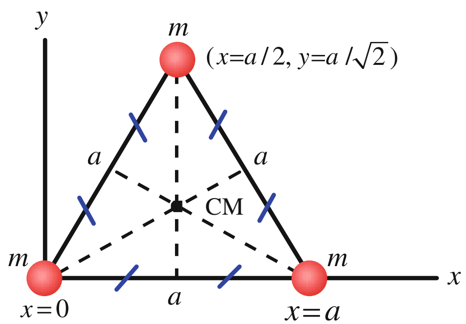
(45) For the system of particles shown in Fig. 7.43 and when $d = 1\,\text{m}$, find the location of the x and y components of the center of mass. Does your answer depend on the value of m? Explain.

Fig. 7.43 See Exercise (45)



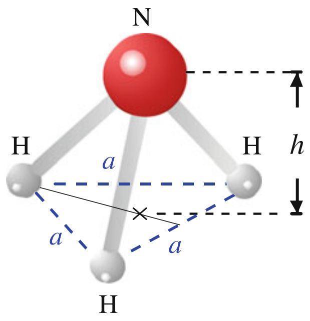
(46) Three particles, each of mass m, are located at the corners of an equilateral triangle of side a, as shown in the Fig. 7.44. Show that the center of mass of the system lies on a common point on the three lines that connect each vertex with the midpoint of the opposite side (the medians).

Fig. 7.44 See Exercise (46)



(47) In an ammonia molecule (NH_3), the three hydrogen (${}^1_1H$) atoms are at the corners of an equilateral triangle of side $a = 0.16$ nm that forms the base of a pyramid, with nitrogen atom (${}^{14}_6N$) at the apex above the center of this triangle by $h = 0.037$ nm, see Fig. 7.45. Find the distance of the center of mass of the ammonia molecule above the plane of the hydrogen atoms.

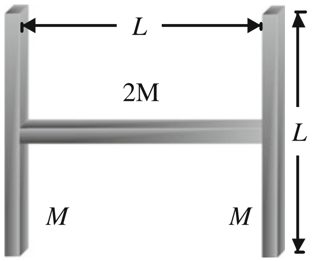
Fig. 7.45 See Exercise (47)



(48) A mass $m_1 = 2$ kg is connected to a mass $m_2 = 3$ kg by a massless rod. The location of m_1 and m_2 are given by the position vectors $\vec{r}_1 = (4\vec{i} + 5\vec{j})$(m) and $\vec{r}_2 = (2\vec{i} + 3\vec{j})$(m), respectively. Find the coordinates of the center of mass.

(49) Three uniform thin rods, each of length L, are arranged to form the shape shown in Fig. 7.46. The vertical arms have mass M and the horizontal arm has a mass $2M$. Find the center of mass of the assembly.

Fig. 7.46 See Exercise (49)



(50) Find the center of mass of a uniform cone of radius R and height h, see Fig. 7.47. (Hint: Divide the cone into an infinite number of disks, each of thickness dx.)

(51) A pyramid has a height H and square base area of side L, see Fig. 7.48. Find the center of mass of the pyramid above its base. Calculate z_{CM} for the Great

Pyramid of Khufu at Giza, Egypt, which has height $H = 138.8\,\text{m}$ and base square area of side $L = 230.4\,\text{m}$. (Hint: Divide the pyramid into an infinite number of squares, each of height dz.)

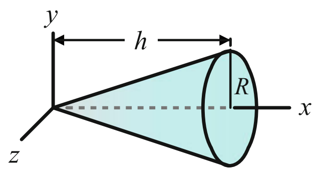


Fig. 7.47 See Exercise (50)

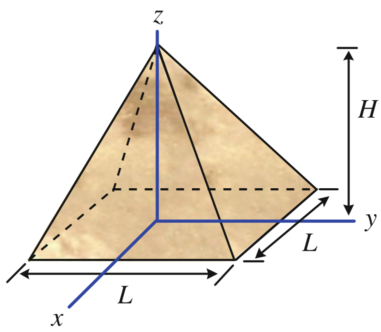


Fig. 7.48 See Exercise (51)

Section 7.5 Dynamics of the Center of Mass

(52) The velocities of two particles of masses $m_1 = 2\,\text{kg}$ and $m_2 = 3\,\text{kg}$ are given by the position vectors $\vec{v}_1 = (4\vec{i} + 5\vec{j})(\text{m/s})$ and $\vec{v}_2 = (2\vec{i} - 3\vec{j})(\text{m/s})$, respectively. Find the velocity of the center of mass of that system.

(53) A ball of mass $m_1 = 2\,\text{kg}$ traveling with velocity $\vec{v}_1 = 15\vec{i}$ m/s collides head-on and elastically with a second ball of mass $m_1 = 3\,\text{kg}$ traveling with velocity $\vec{v}_2 = -4\vec{i}$ m/s, see Fig. 7.49. (a) Find the velocities of the two balls after the collision. (b) Find the velocity of the center of mass before and after the collision.

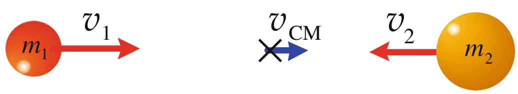


Fig. 7.49 See Exercise (53)

(54) Two particles of masses $m_1 = 0.2$ kg and $m_1 = 0.3$ kg are initially at rest 2 m apart. The two particles form an isolated system. Each particle starts to attract the other with an equal internal constant force of magnitude 0.12 N. (a) What is the speed of their center of mass before and after the start of the attractive force? (b) What distance does m_1 move before colliding with m_2? (c) What will be the speed of m_1 and m_2 just before the collision?

(55) A man of mass $m = 70$ kg stands on one end of a flat boat, which is always moving horizontally without friction at a speed of $v_\circ = 5$ m/s over water. The boat has a mass $M = 210$ kg and length $L = 20$ m, see Fig. 7.50. The man starts to walk to the other end in the direction of the boat's motion with a relative speed $v_{rel} = 2$ m/s. (a) What is the location and velocity of the center of mass before and after the motion of the man? (b) What is the velocity v_B of the boat once the man starts to move? (c) What time does the man take to reach the other end, and how far has the boat moved?

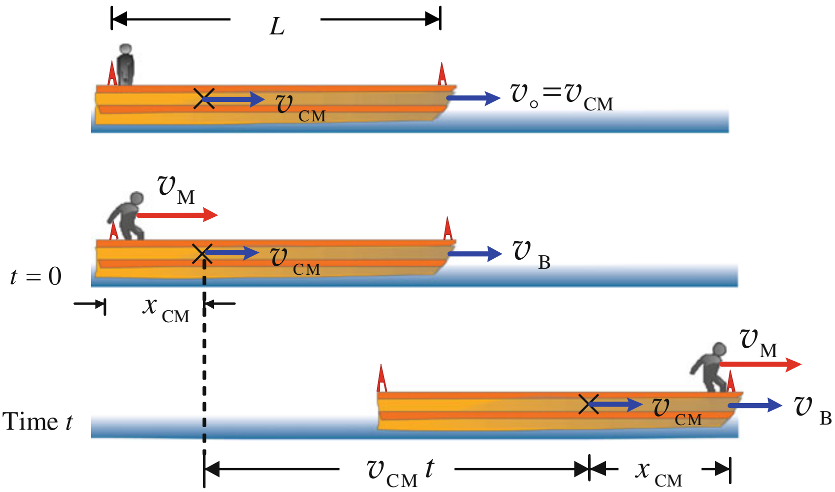


Fig. 7.50 See Exercise (55)

(56) A projectile is fired from the ground with an initial speed $v_\circ$ of 40 m/s at an angle $\theta_\circ$ of 15° m/s above the horizontal direction. At the maximum height, the projectile explodes into two fragments of equal mass, see Fig. 7.51. One fragment stops momentarily and falls vertically, while the second one flies

initially in a horizontal direction. How far from the ground do the center of mass and the second fragment land? Take $g = 10\,\text{m/s}^2$.

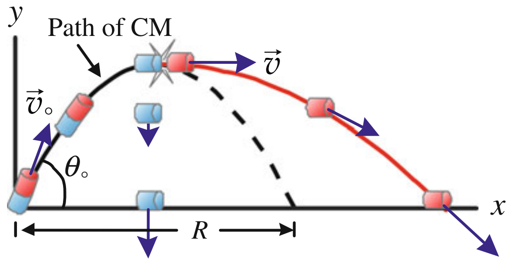


Fig. 7.51 See Exercise (56)

Section 7.6 Systems of Variable Mass

(57) A stationary grain funnel drops grain at a rate $dM/dt = 840\,\text{kg/min}$ onto a railroad car moving with a constant speed $v = 3.5\,\text{m/s}$, see Fig. 7.52. (a) What external force must be applied to the car to keep it moving at constant speed (in the absence of friction)? (b) Find the power delivered by this force. (c) Find the rate of the kinetic energy acquired by the falling grains.

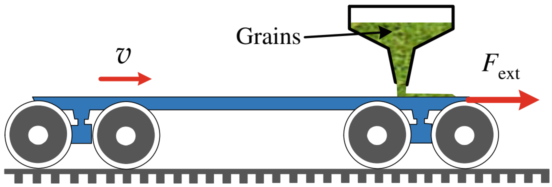


Fig. 7.52 See Exercise (57)

(58) During the first second of its flight in free space, a rocket ejects exhaust that is 1/50 of its mass with a relative speed $v_{\text{rel}} = 2.5 \times 10^3\,\text{m/s}$. What is the acceleration of the rocket?

(59) A rocket of mass $M = 3{,}000\,\text{kg}$ ejects fuel and oxidizer at a rate of 150 kg/s in order to acquire an acceleration $a = 4g$. What is the relative speed of the exhaust and the thrust on the rocket?

(60) A rocket of mass $M = 3{,}000\,\text{kg}$ is moving in free space with a speed $v = 3 \times 10^2\,\text{m/s}$ relative to the Earth. The rocket ejects fuel at a rate of 15 kg/s with a relative speed of $2.5 \times 10^3\,\text{m/s}$. (a) Find the thrust on the rocket. (b) Find the final speed of the rocket when its fuel burns completely after 20 s.

Rotational Motion

8

In this chapter, we first treat the rotation of an extended object about a fixed axis. This is commonly known as pure rotational motion. The analysis is greatly simplified when the object is rigid. To perform this analysis, we first ignore the cause of rotation and describe the rotational motion in terms of angular variables and time. This is known as ***rotational kinematics***. We then discuss the causes of rotation. This is known as ***rotational dynamics***, and through the study of this topic we introduce the concept of ***torque***. After that we treat some general cases where the axis of rotation is not fixed in space. In these cases, rigid bodies can undergo both rotational and translational motion, as in the rolling of objects.

8.1 Radian Measures

One **radian** (1 rad) is the angle subtended at the center of a circle of radius r by an arc of length s equal to the radius of the circle, i.e. $s = r$, see Fig. 8.1a. Since the circumference of a circle of radius r is $s = 2\pi r$, where $\pi \simeq 3.14$, then 360° (or one revolution) corresponds to an angle of $(2\pi r)/r = 2\pi$ rad, see also Appendix B. Thus:

$$1\,\text{rev} = 360^\circ = 2\pi\,\text{rad} \quad \Rightarrow \quad 180^\circ = \pi\,\text{rad} \tag{8.1}$$

Therefore:

$$1^\circ = (\pi/180)\,\text{rad} \simeq 0.02\,\text{rad}$$
$$1\,\text{rad} = 180^\circ/\pi \simeq 57.3^\circ.$$

H. A. Radi and J. O. Rasmussen, *Principles of Physics*,
Undergraduate Lecture Notes in Physics, DOI: 10.1007/978-3-642-23026-4_8,

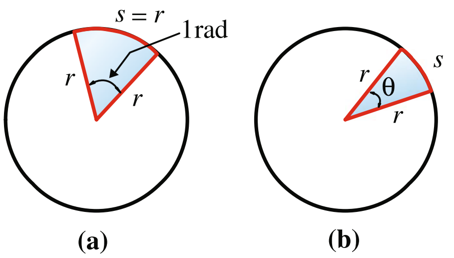


Fig. 8.1 (a) The definition of one radian (1 rad). (b) The definition of angle θ as the ratio of the arc length s to the radius r

Generally, if θ (in radians) represents any arbitrary angle subtended by an arc of length s on the circumference of a circle of radius r, see Fig. 8.1b, then the following relation must be satisfied:

$$\theta = \frac{s}{r} \quad \text{(Radian measure)} \tag{8.2}$$

8.2 Rotational Kinematics; Angular Quantities

Angular Position

The rotational motion of a rigid body (or a particle) about an axis is completely specified by an angle θ that a fixed line in the rigid body (or the particle) makes with some reference fixed line in the space, usually chosen as the x-axis. Additionally, the rotational motion is greatly simplified if θ is expressed in radians. This angle θ is defined as the **angular position** of the rigid body (or the particle).

Figure 8.2 represents a rigid body that is rotating about a fixed axis passing through point O, where that axis is perpendicular to the plane of the figure. Line OP is fixed in the body and completely specified at time t by the angular position θ which the line OP makes with respect to the x-axis. Therefore, the angular position of the rigid body, or the particle at point P which has polar coordinates (r, θ), is:

$$\boxed{\text{Angular position} = \theta} \tag{8.3}$$

Angular Displacement

When the rigid body rotates as shown in Fig. 8.3, the angular position of the line OP changes from θ_1 at time t_1 to θ_2 at a later time t_2. The quantity $\Delta\theta = \theta_2 - \theta_1$ is defined as the **angular displacement** of the a rigid body:

$$\text{Angular displacement} \equiv \Delta\theta = \theta_2 - \theta_1 \tag{8.4}$$

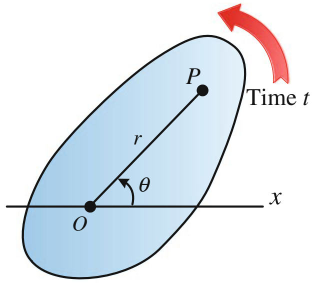


Fig. 8.2 The definition of the angular position θ of a rigid body, or a particle at point P with polar coordinates (r, θ), with respect to the x-axis

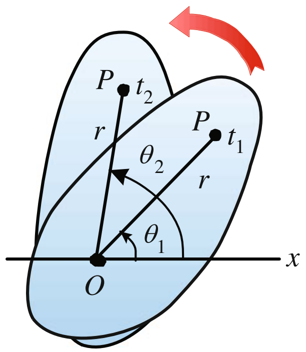


Fig. 8.3 The angular displacement $\Delta\theta = \theta_2 - \theta_1$ of a rigid body that occurs during the time interval $\Delta t = t_2 - t_1$

Angular velocity

In analogy to the average linear (translational) speed, we define the **average angular speed** $\overline{\omega}$ as the rate of change of angular displacement (ω is the lowercase Greek letter omega). That is:

$$\overline{\omega} = \frac{\Delta\theta}{\Delta t} = \frac{\theta_2 - \theta_1}{t_2 - t_1} \tag{8.5}$$

The **instantaneous angular velocity** ω is defined as the limiting value of the ratio $\Delta\theta/\Delta t$ when Δt approaches zero. Thus:

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt} \quad \equiv \quad \theta_{\mathrm{f}} - \theta_i = \int_{t_{\mathrm{i}}}^{t_{\mathrm{f}}} \omega \, dt \tag{8.6}$$

In SI units the angular velocity has the unit of radians per second (rad/s) and can be written as second^{-1} (s^{-1}) because radians are not dimensional quantities.

The last two equations hold for every point on the rigid body. That is, all points of the rigid body rotate through the same angular displacement in the same time. As in linear motion in one-dimension (where the linear velocity can be positive or negative), we take ω to be positive if θ increases (in a counterclockwise sense) and ω to be negative if θ decreases (in a clockwise sense).

Angular Acceleration

When the angular velocity of the rotating body is not constant, the body has an angular acceleration. Assume that ω_1 and ω_2 are the angular velocities at times t_1 and t_2, respectively, as shown in Fig. 8.4. Then we define the **average angular acceleration** $\overline{\alpha}$ (Greek alpha "α") as the rate of change of angular velocity as follows:

$$\boxed{\overline{\alpha} = \frac{\Delta\omega}{\Delta t} = \frac{\omega_2 - \omega_1}{t_2 - t_1}} \tag{8.7}$$

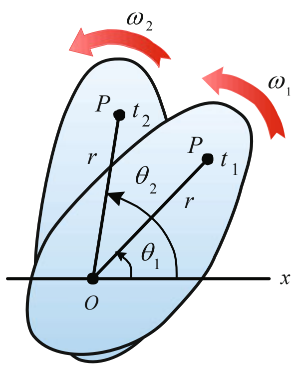


Fig. 8.4 The change in the angular velocity $\Delta\omega = \omega_2 - \omega_1$ of a rigid body which occurs during the time interval $\Delta t = t_2 - t_1$

Therefore, the **instantaneous angular acceleration** α is defined as the limiting value of the ratio $\Delta\omega/\Delta t$ when Δt approaches zero. Thus:

$$\boxed{\alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt}} \quad \equiv \quad \boxed{\omega_f - \omega_i = \int_{t_i}^{t_f} \alpha \; dt} \tag{8.8}$$

Angular acceleration has the unit of radians per second square (rad/s^2) and can be written as second^{-2} (s^{-2}).

Example 8.1

A reference line in a spinning disk has an angular position given by $\theta = 3\,t^2 - 12\,t + 9$, where θ is in radians and t is in seconds. (a) Find ω and α as a function of time. (b) Find the times when the angular position θ and the angular velocity ω become zero. (c) Describe the rotational motion of the disk for $t \geq 0$.

Solution: (a) To find ω, we differentiate θ with respect to time:

$$\omega = \frac{d\theta}{dt} = \frac{d}{dt}(3\,t^2 - 12\,t + 9) = (6\,t - 12)\,\text{rad/s}$$

Thus, ω could be negative or positive depending on t. To find the angular acceleration α, we differentiate ω with respect to time:

$$\alpha = \frac{d\omega}{dt} = \frac{d}{dt}(6\,t - 12) = 6\,\text{rad/s}^2$$

(b) Setting $\theta = 0$, we get:

$$3\,t^2 - 12\,t + 9 = 0 \quad \Rightarrow \quad t = \frac{12 \pm \sqrt{12^2 - 4 \times 3 \times 9}}{2 \times 3} \quad \Rightarrow \quad t = 1\,\text{s and } t = 3\,\text{s}$$

Thus, θ reaches zero at both $t = 1$ s and $t = 3$ s. Setting $\omega = 0$ gives:

$$6\,t - 12 = 0 \quad \Rightarrow \quad t = 2\,\text{s (when } \omega = 0)$$

(c) We can describe the rotation as follows:

- At $t = 0$ the reference line is at $\theta = 9$ rad and the disk's initial angular velocity is $\omega = -12$ rad/s.
- As time increases during the interval $0 < t < 2$ s, we have $\omega < 0$. That is, the disk is rotating in the clockwise sense, but with a decreasing angular speed, since $\alpha > 0$. In addition, θ reaches the value $\theta = 0$ at $t = 1$ s, and then attains negative values.
- At $t = 2$ s, the disk stops momentarily when $\theta = -3$ rad.
- As time increases during the interval $t > 2$ s, we have $\omega > 0$. In addition, θ goes back to zero again when $t = 3$ s. Afterward, both ω and θ will increase indefinitely.

8.3 Constant Angular Acceleration

The definitions of angular quantities are similar to those of linear quantities, except that θ, ω, and α replace the linear variables x, v, and a, respectively. Therefore, the angular equations for **constant angular acceleration** will be analogous to those presented in Chap. 3 and can be derived in exactly the same way, see Table 3.1. Table 8.1 summarizes the angular kinematic equations and their linear equivalents.

Table 8.1 Equations for motion with constant linear and angular accelerations

Linear equations		Angular equations
$v = v_\circ + a\,t$	$x \Leftrightarrow \theta$	$\omega = \omega_\circ + \alpha\,t$
$x - x_\circ = \frac{1}{2}(v_\circ + v)\,t$	$v \Leftrightarrow \omega$	$\theta - \theta_\circ = \frac{1}{2}(\omega_\circ + \omega)\,t$
$x - x_\circ = v_\circ\,t + \frac{1}{2}\,a\,t^2$	$a \Leftrightarrow \alpha$	$\theta - \theta_\circ = \omega_\circ\,t + \frac{1}{2}\,\alpha\,t^2$
$v^2 = v_\circ^2 + 2\,a\,(x - x_\circ)$		$\omega^2 = \omega_\circ^2 + 2\,\alpha\,(\theta - \theta_\circ)$

Example 8.2

A wheel accelerates uniformly from rest to an angular speed of 25 rad/s in 10 s. (a) Find the angular acceleration of the wheel. (b) Find the tangential and radial acceleration of a point 10 cm from the wheel's center. (c) How many revolutions has the wheel turned during this time interval? (d) Then, find the wheel's angular deceleration if it comes to a full stop after 5 rev.

Solution: (a) We are given $\omega_\circ = 0$, $\omega = 25$ rad/s, and $t = 10$ s. To find the angular acceleration a, we can use $\omega = \omega_\circ + \alpha\,t$ as follows:

$$\alpha = \frac{\omega - \omega_\circ}{t} = \frac{25\,\text{rad/s} - 0}{10\,\text{s}} = 2.5\,\text{rad/s}^2$$

(b) Using Eqs. 8.13 and 8.14 (See Sect. 8.5), we get:

$$a_\text{t} = r\,\alpha = (10 \times 10^{-2}\,\text{m})(2.5\,\text{rad/s}^2) = 0.25\,\text{m/s}^2$$

$$a_\text{r} = r\,\omega^2 = (10 \times 10^{-2}\,\text{m})(25\,\text{rad/s})^2 = 62.5\,\text{m/s}^2$$

(c) If we assume that the wheel starts from $\theta_\circ = 0$, then we are given $\omega_\circ = 0$, $\omega = 25$ rad/s, $\theta_\circ = 0$, and $t = 10$ s. To find θ, which in this case equals the angle traveled by a certain reference line in the wheel, we use $\theta - \theta_\circ = \frac{1}{2}(\omega_\circ + \omega)\,t$ as follows:

$$\theta = \theta_\circ + \tfrac{1}{2}(\omega_\circ + \omega)\, t = 0 + \tfrac{1}{2}(0 + 25\,\text{rad/s}) \times 10\,\text{s} = 125\,\text{rad}$$

Thus: $$\theta = 125\,\text{rad} \times \left(\frac{1\,\text{rev}}{2\pi\,\text{rad}}\right) \simeq 20\,\text{rev}$$

(d) We are given $\omega_\circ = 25$ rad/s, $\omega = 0$, and, $\theta - \theta_\circ = 5\,\text{rev} = 10\pi$ rad. To find the angular deceleration α, we use $\omega^2 = \omega_\circ^2 + 2\,\alpha\,(\theta - \theta_\circ)$ to get:

$$\alpha = \frac{\omega^2 - \omega_\circ^2}{2\,(\theta - \theta_\circ)} = \frac{0 - (25\,\text{rad/s})^2}{2\ \times 10\pi\,\text{rad}} = -9.95\,\text{rad/s}^2$$

8.4 Angular Vectors

We can treat angular velocity as a vector by choosing the axis of rotation to be the direction of the angular-velocity vector. By convention, the right-hand rule is used to determine $\vec{\omega}$. To apply this rule, we curl the four fingers of the right hand around the rotation axis and point in the direction of rotation; then the thumb would point in the direction of $\vec{\omega}$. The angular acceleration vector $\vec{\alpha} = d\vec{\omega}/dt$ will be along $\vec{\omega}$ if $|\omega|$ increases with time, and will be opposite to $\vec{\omega}$ if $|\omega|$ decreases with time, see Fig. 8.5.

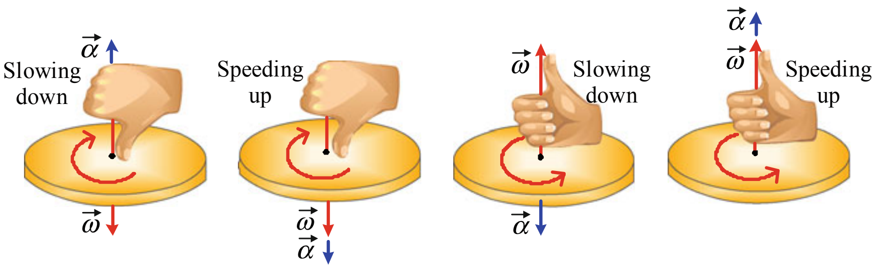


Fig. 8.5 Using the right-hand rule to obtain the direction of the vectors ω and α in cases of increasing and decreasing ω

8.5 Relating Angular and Linear Quantities

When a rigid body rotates with angular velocity ω, every point on the body moves in a circle with its center on the rotational axis, see Fig. 8.6. Because point P in the figure moves in a circle of radius r, this point defines a linear vector $\vec{v}$ whose direction is always tangent to its circular path. This tangential velocity has a magnitude defined by the tangential speed $v = ds/dt$, where s is the arc length traveled by point P along the circular path. Recalling from Eq. 8.2 that $s = r\,\theta$ and noting that r is constant, we find:

$$v = \frac{ds}{dt} = r\frac{d\theta}{dt} \tag{8.9}$$

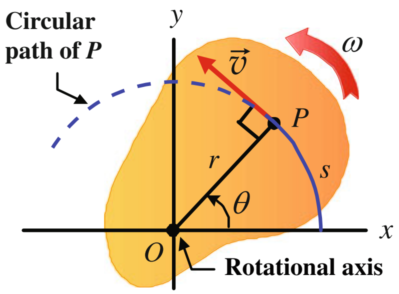


Fig. 8.6 The Point P on a rotating rigid body has a tangential velocity $\vec{v}$ which is always tangent to the circular path of this point

Using $\omega = d\theta/dt$, see Eq. 8.6, we get:

$$\boxed{v = r\,\omega} \quad \text{(Radian measure)} \tag{8.10}$$

This relation indicates that, although ω is the same for every point on the rigid body, points on the object with different radii have different tangential speeds v. In fact, the tangential speed v increases as one moves outward from the center of rotation.

When the angular speed ω is constant, then the linear speed v of any point on the rigid body is constant and hence undergoes uniform circular motion. The period of one revolution $T = 2\pi r/v$ and the frequency f (rev/s or Hz) can be written in terms of ω as follows:

$$T = \frac{2\pi}{\omega}, \quad f = \frac{1}{T} = \frac{\omega}{2\pi} \quad \text{(Radian measure)} \tag{8.11}$$

We can find the magnitude of the tangential acceleration a_t of point P by differentiating Eq. 8.10 with respect to time as follows:

$$a_t = \frac{dv}{dt} = r\frac{d\omega}{dt} \tag{8.12}$$

Using $\alpha = d\omega/dt$, see Eq. 8.8, we get:

$$\boxed{a_t = r\,\alpha} \quad \text{(Radian measure)} \tag{8.13}$$

In addition, we know that a point (or a particle) moving in a circular path of radius r with speed v undergoes a radial acceleration $\vec{a}_r$ of magnitude $a_r = v^2/r$ directed toward the axis of rotation. Thus, by using $v = r\,\omega$, the magnitude of the radial acceleration becomes:

$$\boxed{a_r = r\,\omega^2} \quad \text{(Radian measure)} \tag{8.14}$$

As shown in Fig. 8.7, the total linear acceleration $\vec{a}$ at point P is:

$$\vec{a} = \vec{a}_t + \vec{a}_r \tag{8.15}$$

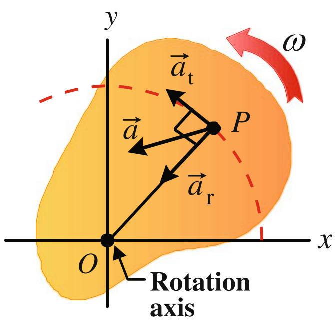


Fig. 8.7 The total linear acceleration $\vec{a}$ of point P on a rotating rigid body has two perpendicular components, the tangential component a_t and the radial component a_r

The magnitude of this acceleration is thus:

$$a = \sqrt{a_t^2 + a_r^2} = \sqrt{r^2\alpha^2 + r^2\omega^4} = r\sqrt{\alpha^2 + \omega^4} \tag{8.16}$$

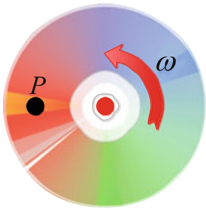
Example 8.3

A typical compact disk (CD) rotates at 300 rev/min. (a) What is the angular velocity of the disk? (b) What is the linear speed of a point P that lies 4 cm from its axis, see Fig. 8.8? (c) If one bit of data is represented at point P and has a length of $L = 0.5\ \mu\text{m}$, find the number of bits N that the reading head can read per second.

Solution: (a) First, we write the frequency f in SI unit as follows:

$$f = 300\left(\frac{\text{rev}}{\text{min}}\right)\left(\frac{\text{min}}{60\ \text{s}}\right) = 5\ \text{rev/s} = 5\ \text{Hz}$$

Fig. 8.8



Then, using Eq. 8.11, the angular speed ω is:

$$\omega = 2\pi f = 31.4\,\text{rad/s}$$

(b) The linear speed of a point 4 cm from the axis of the disk is:

$$v = r\,\omega = (4 \times 10^{-2}\,\text{m})(31.4\,\text{rad/s}) = 1.26\,\text{m/s}$$

(c) Using $NL = v$, we get the number per second N to be:

$$N = \frac{v}{L} = \frac{1.26\,\text{m/s}}{0.5\;\times\;10^{-6}\,\text{m}} = 2.5\,\times\,10^{6}\,\text{bits/s} = 2.5\,\text{Mbps}.$$

Example 8.4

A grindstone of radius $r = 2$ m rotates with an angular position $\theta = t^3 + 2\,t^2 - 2$, where θ is in radians and t is in seconds. (a) Find ω and α as a function of time and find their values at $t = 2$ s. (b) Find the speed v and the components of the acceleration a at $t = 2$ s for a point on the rim of the grindstone.

Solution: (a) To find ω, we differentiate θ with respect to time:

$$\omega = \frac{d\theta}{dt} = \frac{d}{dt}(t^3 + 2\,t^2 - 2) = (3\,t^2 + 4\,t)\,\text{rad/s} \quad \Rightarrow \quad \omega|_{t=2\,\text{s}} = 20\,\text{rad/s}$$

To find α, we differentiate ω with respect to time:

$$\alpha = \frac{d\omega}{dt} = \frac{d}{dt}(3\,t^2 + 4\,t) = (6\,t + 4)\,\text{rad/s}^2 \quad \Rightarrow \quad \alpha|_{t=2\,\text{s}} = 16\,\text{rad/s}^2$$

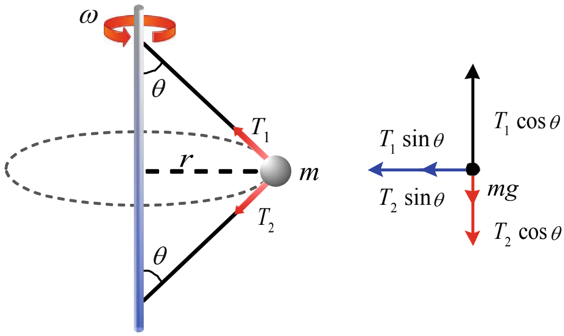
(b) Using Eqs. 8.10, 8.13, and 8.14, we get:

$$\begin{aligned}
v &= r\,\omega = r(3\,t^2 + 4\,t)\,\text{rad/s} &&\Rightarrow \quad v|_{t=2\,\text{s}} &&= 40\,\text{m/s}\\
a_\text{t} &= r\,\alpha = r(6\,t + 4)\,\text{m/s}^2 &&\Rightarrow \quad a_\text{t}|_{t=2\text{s}} &&= 32\,\text{m/s}^2\\
a_\text{r} &= r\,\omega^2 = r(3\,t^2 + 4\,t)^2\,\text{m/s}^2 &&\Rightarrow \quad a_\text{r}|_{t=2\,\text{s}} &&= 800\,\text{m/s}^2
\end{aligned}$$

Example 8.5

A ball of mass $m = 0.1$ kg rotates in a circular path of radius $r = 0.2$ m with an angular speed $\omega = 8$ rad/s while being attached to two strings of equal length, each making an angle $\theta = 30°$ with a vertical rod as shown in Fig. 8.9. Find the magnitude of the tension in the two strings.

Fig. 8.9



Solution: From the free-body diagram shown above, the vertical forces must balance. That is:

$$T_1 \cos\theta - T_2 \cos\theta = mg$$

According to Eq. 8.14, the magnitude of the radial acceleration is given in terms of the angular speed ω as $a_r = r\,\omega^2$. Therefore:

$$m\,r\,\omega^2 = T_1 \sin\theta + T_2 \sin\theta$$

Multiplying both sides of the first equation by $\sin\theta$ and both sides of the second equation by $\cos\theta$, then adding (or subtracting) the results, we can get the magnitude of the tension in the two strings as follows:

$$\begin{aligned} T_1 &= \frac{m}{2\sin\theta}(r\,\omega^2 + g\tan\theta) \\ &= \frac{0.1\,\text{kg}}{2\sin 30°}[(0.2\,\text{m})(8\,\text{rad/s})^2 + (10\,\text{m/s}^2)(\tan 30°)] = 1.86\,\text{N} \end{aligned}$$

$$\begin{aligned} T_2 &= \frac{m}{2\sin\theta}(r\,\omega^2 - g\tan\theta) \\ &= \frac{0.1\,\text{kg}}{2\sin 30°}[(0.2\,\text{m})(8\,\text{rad/s})^2 - (10\,\text{m/s}^2)(\tan 30°)] = 0.70\,\text{N} \end{aligned}$$

8.6 Rotational Dynamics; Torque

Rotational dynamics is the study of rotational motion and the causes of changes in motion. Just as linear motion is analogous to rotational motion from a kinematics perspective, we will see that this analogy applies also from a dynamics perspective.

We know from our everyday experience that, when an object rotates about an axis, the rate of this rotation depends on the magnitude and direction of the exerted force and how far this force is applied away from the rotation axis. This dependence is measured by a vector quantity called **torque (or moment)** $\vec{\tau}$ (Greek tau "τ").

Figure 8.10a depicts a cross section of a rigid body that is free to rotate about a fixed axis at O. A force $\vec{F}$ perpendicular to the axis of rotation acts on the body at point P whose position vector from O is $\vec{r}$. The smaller angle between the two vectors $\vec{F}$ and $\vec{r}$ is θ. The ability of $\vec{F}$ to rotate the body about O from point P depends on the torque $\vec{\tau}$ as follows:

$$\vec{\tau} = \vec{r} \times \vec{F} \tag{8.17}$$

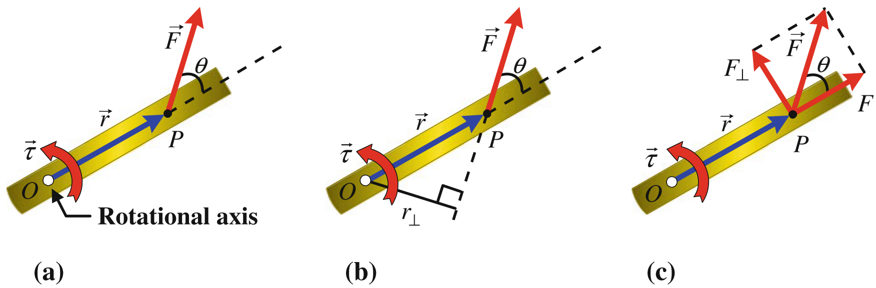


Fig. 8.10 (a) The torque $\vec{\tau}$ produced by a force $\vec{F}$ that acts at point P on a rigid body which can rotate freely about an axis passing through point O. (b) The torque can be written as $r_\perp F$, where $r_\perp$ is the moment arm of the force $\vec{F}$. (c) The torque can also be written as $r F_\perp$, where $F_\perp$ is the perpendicular component of the force to $\vec{r}$

Accordingly, its magnitude (see Chap. 2) is:

$$\tau = r F \sin\theta \tag{8.18}$$

The SI unit of the torque is m.N [not to be confused with the unit of energy (1 J = 1 N.m)]. By convention, torque is positive if the force has the tendency to rotate the object in a counterclockwise sense; and is negative if it has the tendency to rotate the object in a clockwise sense. Also, the reverse of this convention can be used.

Based on Fig. 8.10b and c, the magnitude τ can be written as:

$$\tau = r_{\perp} F \quad (\text{with } r_{\perp} = r \sin\theta) \tag{8.19}$$

$$\tau = r F_{\perp} \quad (\text{with } F_{\perp} = F \sin\theta) \tag{8.20}$$

where the distance $r_{\perp}$ is the perpendicular distance from the axis of rotation O to the line along which the force acts (also called the **lever arm**, or the **moment arm**). In addition, $F_{\perp}$ is the component of the force perpendicular to $\vec{r}$. This component is what causes that rotation. The other component, $F_{\parallel}$, is parallel to the position vector $\vec{r}$, passes through O and causes no rotation.

If two or more forces act on a rigid body, where each force tends to produce rotation about an axis passing through some point, the net torque on the body will be the sum of all torques:

$$\Sigma\vec{\tau} = \vec{\tau}_1 + \vec{\tau}_2 + \ldots \tag{8.21}$$

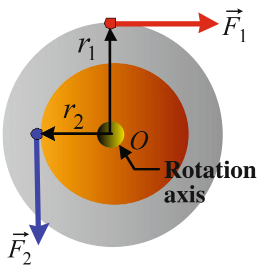
Using the sign convention introduced previously for torques, we can omit the vector notation and write the net torque as follows:

$$\Sigma\tau = \tau_1 + \tau_2 + \ldots \tag{8.22}$$

Example 8.6

Two wheels of radii $r_1 = 20\,\text{cm}$ and $r_2 = 30\,\text{cm}$ are fastened together as shown in Fig. 8.11. Together, they can rotate freely about an axle O perpendicular to the page. Two forces of magnitudes $F_1 = 20\,\text{N}$ and $F_2 = 40\,\text{N}$ are applied as shown in the figure. Find the net torque on the wheel.

Fig. 8.11



Solution: Designate counterclockwise torque as positive. The force $\vec{F}_1$ produces a torque $\vec{\tau}_1$ that tends to rotate the wheel in a clockwise sense. Thus, the sign of τ_1 is negative and equal to $-F_1 r_1$. The force $\vec{F}_2$ produces a torque $\vec{\tau}_2$ that tends to rotate the wheel in a counterclockwise sense. Thus, the sign of τ_2 is positive and equal to $+F_2 r_2$. By using Eq. 8.22, the net torque is:

$$\begin{aligned}\Sigma\tau &= \tau_1 + \tau_2 = -F_1 r_1 + F_2 r_2\\ &= -(20\text{ N})(20\times 10^{-2}\text{ m}) + (40\text{ N})(30\times 10^{-2}\text{ m})\\ &= 8\text{ m.N}\end{aligned}$$

The net torque acts to rotate the wheel in the counterclockwise sense.

8.7 Newton's Second Law for Rotation

We will show that Newton's second law $\Sigma F \propto a$ for translational motion corresponds to $\Sigma\tau \propto \alpha$ for rotational motion about a fixed axis.

First, we consider a particle of mass m attached to one end of a rod of negligible mass while the other end can rotate freely at point O. The mass rotates in a circle of radius r under the influence of a tangential force $\vec{F}_\text{t}$, as shown in Fig. 8.12. In this figure we do not display the radial force $\vec{F}_\text{r}$.

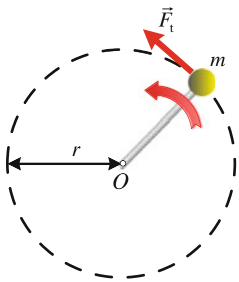


Fig. 8.12 A particle of mass m is rotating in a circle of radius r under the influence of a tangential force $\vec{F}_\text{t}$

According to Newton's second law, the tangential force $\vec{F}_\text{t}$ produces a tangential acceleration $\vec{a}_\text{t}$. Then:

$$F_\text{t} = m\, a_\text{t}$$

The tangential acceleration is related to the angular acceleration through the relationship $a_\text{t} = r\,\alpha$, see Eq. 8.13. Thus,

$$F_{\mathrm{t}} = m\,r\,\alpha \tag{8.23}$$

Since $\vec{F}_{\mathrm{t}}$ produces a torque $\vec{\tau}$ about the origin, this torque tends to rotate the particle in a counterclockwise sense. The magnitude of $\vec{\tau}$ is:

$$\tau = r\,F_{\mathrm{t}} \tag{8.24}$$

Substituting with Eq. 8.23 into Eq. 8.24, we get:

$$\tau = m\,r^2\,\alpha \tag{8.25}$$

which can be written as:

$$\boxed{\left.\begin{array}{l}\tau = I\,\alpha,\\ I = m\,r^2\end{array}\right\}} \qquad \text{(Single particle)} \tag{8.26}$$

That is, the applied torque is proportional to the angular acceleration, and represents the rotational equivalent of Newton's second law. The quantity $I = m\,r^2$ represents the *rotational inertia* of the particle about O and is called the **moment of inertia**. The SI units of I is kg.m^2.

We can apply this result to a system of particles located at various distances from a certain axis of rotation. For the i^{th} particle, we apply Eq. 8.25 to get $\tau_i = (m_i\,r_i^2)\alpha$. Then, the total torque about that axis will be $\sum\tau = (\sum m_i\,r_i^2)\,\alpha = I\alpha$. Thus:

$$\boxed{\left.\begin{array}{l}\Sigma\tau = I\alpha,\\ I = \sum m_i\,r_i^2\end{array}\right\}} \qquad \text{(System of particles)} \tag{8.27}$$

Notice the analogy between the translational relation $\Sigma F = m\ a$ and the rotational relation $\Sigma\tau = I\ \alpha$, where $F \Leftrightarrow \tau$ and $m \Leftrightarrow I$.

Now we consider a rigid body rotating about a fixed axis at O. We can think of this body as an infinite number of mass elements dm of infinitesimal size, see Fig. 8.13. Each mass element rotates in a circular path about the origin with an angular acceleration $\vec{a}_{\mathrm{t}}$ produced by an external tangential force $\vec{F}_{\mathrm{t}}$.

By applying Newton's second law to a given mass element, we get:

$$dF_{\mathrm{t}} = (dm)a_{\mathrm{t}}$$

All elements of the rigid body have the same angular acceleration α. Since $a_{\mathrm{t}} = r\,\alpha$ is the angular acceleration of each element, then:

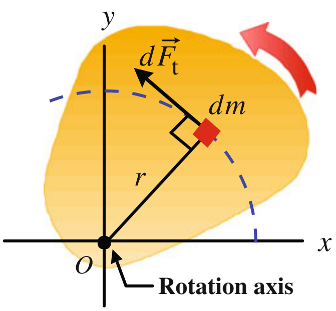


Fig. 8.13 Each element of mass dm is rotating about O in a circle of radius r under the influence of a tangential force $d\vec{F}_t$

$$dF_t = \alpha(dm)r \tag{8.28}$$

The magnitude of the differential torque $d\tau$ produced by dF_t is:

$$d\tau = r\,dF_t \tag{8.29}$$

Using Eq. 8.28, the expression for $d\tau$ becomes:

$$d\tau = \alpha r^2 dm \tag{8.30}$$

Now we can integrate both sides of this differential relation to find the net torque $\Sigma\tau$ about O due to external forces as follows:

$$\Sigma\tau = \alpha \int r^2 dm \tag{8.31}$$

which can be written as:

$$\boxed{\left.\begin{aligned} \Sigma\tau &= I\,\alpha, \\ I &= \int r^2 dm \end{aligned}\right\}} \qquad \text{(Rigid body)} \tag{8.32}$$

In this case, $I = \int r^2 dm$ is the moment of inertia of the rigid body about the rotation axis through O. All equations of the form $\Sigma\tau = I\,\alpha$ hold even if the external forces have radial components, since the action of these components passes through the axis of rotation.

Parallel-Axis Theorem

If we calculate the moment of inertia of a body about any axis that passes through its center of mass, then we can prove that the moment of inertia about any axis parallel to that axis is given by:

$$I = I_{CM} + M\,h^2 \tag{8.33}$$

where M is the total mass of the body and h is the perpendicular distance between the two parallel axes. Figure 8.15 shows this for the case of a rod.

Example 8.7

A horizontal rod of uniform mass per unit length λ has a mass M and length L. Use the relation $I = \int r^2 dm$ to calculate the moment of inertia of the rod about: (a) an axis passing through its center, and (b) an axis passing through its end. (c) Check your result by using the parallel-axis theorem.

Solution: (a) For a uniform rod, $\lambda = M/L$. If we divide the rod into infinitesimal elements of length dx, then the mass of each element is $dm = \lambda\, dx$. Figure 8.14 shows an axis through CM and the left end.

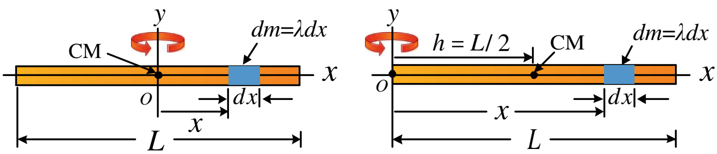


Fig. 8.14

For an axis passing through the CM, I in Eq. 8.32 leads to:

$$I_{CM} = \int r^2 dm = \int_{-L/2}^{+L/2} x^2 \lambda\, dx = \frac{M}{L}\int_{-L/2}^{+L/2} x^2 dx = \frac{M}{L}\int_{-L/2}^{+L/2} x^2 dx$$

$$= \frac{M}{L}\left[\frac{x^3}{3}\right]_{-L/2}^{+L/2} = \frac{1}{12} M\,L^2$$

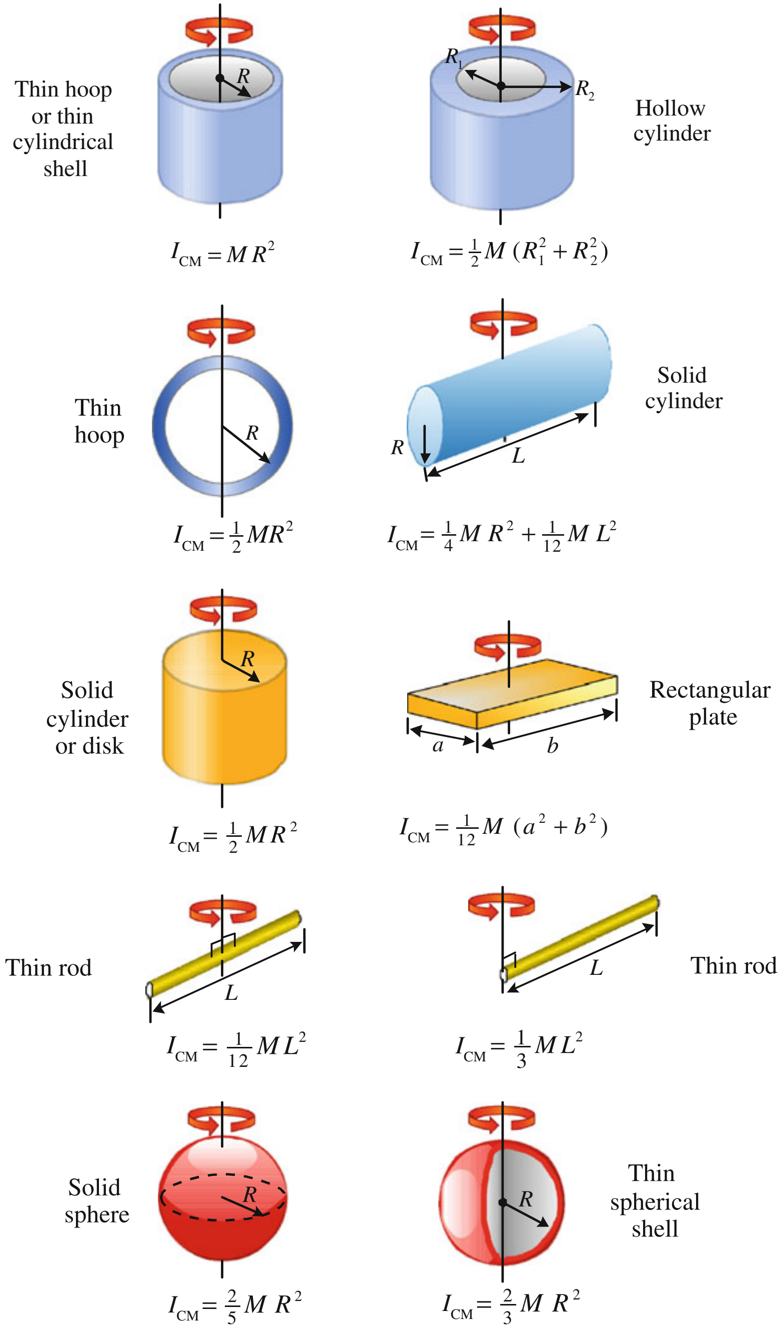


Fig. 8.15 Moments of inertia for some objects about specific axes

(b) For an axis passing through one end, I in Eq. 8.32 leads to:

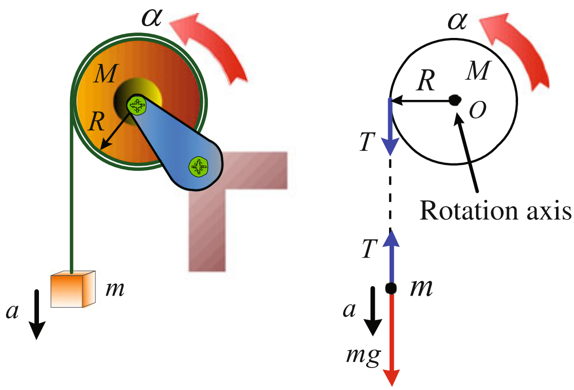
$$I = \int r^2 dm = \int_0^L x^2 \lambda\, dx = \frac{M}{L}\int_0^L x^2\, dx = \frac{M}{L}\int_0^L x^2\, dx = \frac{M}{L}\left[\frac{x^3}{3}\right]_0^L = \frac{1}{3} M L^2$$

(c) Applying the theorem $I = I_{\text{CM}} + M h^2$, one can obtain the same result.

Example 8.8

A pulley of mass $M = 6$ kg and radius $R = 20$ cm is mounted on a frictionless axis, as shown in Fig. 8.16. A massless cord is wrapped around the pulley while its other end supports a block of mass $m = 3$ kg. If the cord does not slip, find the linear acceleration of the block, the angular acceleration of the pulley, and the tension in the cord. Take $g = 10\,\text{m/s}^2$.

Fig. 8.16



Solution: For a downward motion of the block with acceleration a, the weight $m\,g$ must be greater than the tension T, see the free-body diagram of Fig. 8.16. Therefore, from Newton's second law of linear motion, we get:

$$(1) \quad m\,g - T = m\,a$$

From the free-body diagram of Fig. 8.16, we see that the torque τ that acts on the pulley is $R\,T$. Applying Newton's second law in angular form, Eq. 8.32, we obtain:

$$\Sigma\tau = I\,\alpha \quad \Rightarrow \quad R\,T = \left(\tfrac{1}{2} M R^2\right)\alpha \quad \Rightarrow \quad T = \tfrac{1}{2} M\,R\,\alpha$$

where the moment of inertia of the pulley $I = \frac{1}{2}M R^2$ is taken from Fig. 8.15. The linear acceleration of the block is equal to the tangential acceleration of the pulley, i.e., $a_t = a$. Since $a_t = R\,\alpha$, then the last equation reduces to:

$$(2) \quad T = \tfrac{1}{2}M\,a$$

Eliminating the tension from Eqs. (1) and (2), we get:

$$a = \frac{2m}{2m + M}g = \frac{2 \times (3\,\text{kg})}{2 \times (3\,\text{kg}) + 6\,\text{kg}} \times (10\,\text{m/s}^2) = 5\,\text{m/s}^2$$

The angular acceleration of the pulley is thus:

$$\alpha = \frac{a_t}{R} = \frac{a}{R} = \frac{5\text{m/s}^2}{0.2\,\text{m}} = 25\,\text{rad/s}^2$$

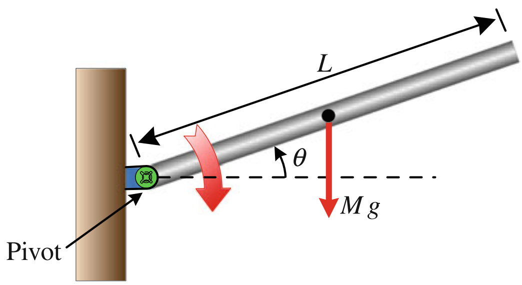
We use Eq. (2) to find the tension in the cord as follows:

$$T = \tfrac{1}{2}M\,a = \tfrac{1}{2} \times (6\,\text{kg})(5\ \text{m/s}^2) = 15\ \text{N}$$

Example 8.9

A uniform thin rod of mass $M = 2$ kg and length $L = 20$ cm is attached from one end to a frictionless pivot. The rod is free to rotate in a vertical plane. The rod is released when it is in the vertical position. Figure 8.17 shows the situation when the angle between the rod and the horizontal is θ. (a) Determine the angular acceleration of the rod as a function of θ for $-90° \leq \theta \leq +90°$ and find its maximum value. (b) Find the angle where the tangential acceleration of the free end of the rod equals g. Take $g = 10\,\text{m/s}^2$.

Fig. 8.17



Solution: (a) The moment arm of the force exerted by the pivot on the rod is zero. Therefore, the only force that contributes to the torque is the gravitational force $M\vec{g}$ with moment arm $\frac{1}{2}L\cos\theta$. Consequently, the angular acceleration is not constant because the torque exerted on the rod varies with θ. Call clockwise torques positive. Then the magnitude of this clockwise torque is:

$$\tau = \left(\tfrac{1}{2}L\cos\theta\right)M\,g$$

By applying Newton's second law in its angular form, $\sum\tau = I\,\alpha$, and taking $I = \frac{1}{3}M\,L^2$ from Fig. 8.15 for the axis of rotation at one end, we obtain:

$$\left(\tfrac{1}{2}L\cos\theta\right)M\,g = \left(\tfrac{1}{3}M\,L^2\right)\alpha$$

Thus:
$$\alpha = \frac{3g}{2L}\cos\theta$$

At any angle θ, all points on the rod have this angular acceleration and the maximum value of α occurs at $\theta = 0$. Thus:

$$\alpha_{\max} = \frac{3g}{2L}\cos 0^\circ = \frac{3(10\,\text{m/s}^2)}{2(20\times10^{-2}\,\text{m})} = 75\,\text{rad/s}^2$$

The dependence of α on the angle θ indicates that the angular acceleration starts from zero when $\theta = 90^\circ$, then increases with decreasing θ, becomes maximum of 75 rad/s at $\theta = 0$, then decreases for negative values of θ, and reaches zero again at $\theta = -90^\circ$.

(b) To find the tangential acceleration of the free end of the rod at any angle θ, we use the relation $a_t = L\,\alpha$ and substitute with α to get:

$$a_t = L\,\alpha = \tfrac{3}{2}g\cos\theta$$

Note that a_t does not depend on the length of the rod L. Now, setting $a_t = g$ in the previous relation, we find the value of θ to be:

$$\cos\theta = \tfrac{2}{3} \quad\Rightarrow\quad \theta = \cos^{-1}\tfrac{2}{3} = 48.2^\circ$$

8.8 Kinetic Energy, Work, and Power in Rotation

Rotational Kinetic Energy

Analogous to translational kinetic energy $\left(\frac{1}{2}mv^2\right)$, an object that rotates about an axis is said to have **rotational kinetic energy**. Using this analogy between translational and rotational motions, where $m \Leftrightarrow I$ and $v \Leftrightarrow \omega$, one would expect that the rotational kinetic energy will be given by the expression $\frac{1}{2}I\omega^2$. We can show that this expression is indeed true.

Consider the rigid body of Fig. 8.13 to be a collection of tiny particles rotating about a fixed axis with angular speed ω. If the i^{th} particle has a mass m_{i}, distance r_{i} from the axis of rotation, and tangential speed $v_{\mathrm{i}} = r_{\mathrm{i}}\omega$, then its kinetic energy is:

$$K_i = \tfrac{1}{2}m_{\mathrm{i}}v_{\mathrm{i}}^2 = \tfrac{1}{2}m_{\mathrm{i}}\,r_{\mathrm{i}}^2\omega^2 \tag{8.34}$$

The total kinetic energy of the rotating body will be:

$$K = \sum K_{\mathrm{i}} = \tfrac{1}{2}\left(\sum m_{\mathrm{i}}\,r_{\mathrm{i}}^2\right)\omega^2 \tag{8.35}$$

Since $\sum m_{\mathrm{i}}\,r_{\mathrm{i}}^2$ is the moment of inertial of the rigid body and tends to $\int r^2 dm$ for a continuous mass distribution, then as expected we get:

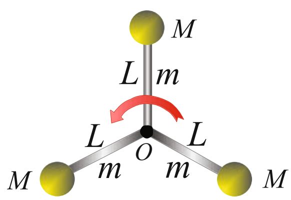
$$\boxed{K_{\mathrm{R}} = \tfrac{1}{2}I\,\omega^2} \quad \text{(Rotational kinetic energy)} \tag{8.36}$$

We refer to K_{R} as rotational kinetic energy, which has the units of energy.

Example 8.10

Figure 8.18 shows three tiny spheres, each of mass M, are fastened by three identical rods each of mass m and of length L. The system is allowed to rotate with an angular speed ω about an axis that is perpendicular to the page and passes through O. Find the moment of inertia and the rotational kinetic energy about this axis.

Fig. 8.18



Solution: Using I from Eq. 8.27 and taking $\frac{1}{3}m\,L^2$ as the moment of inertia of each rod about O, the system's moment of inertia will be:

$$I = 3(M\,L^2) + 3(\tfrac{1}{3}m\,L^2) = (3M + m)L^2$$

Therefore, the rotational kinetic energy of the system about O will be:

$$K_{\mathrm{R}} = \tfrac{1}{2}I\,\omega^2 = \tfrac{1}{2}(3M + m)L^2\omega^2$$

Example 8.11

A block of mass $m = 2$ kg rests on an inclined plane of angle $\theta = 30°$. The block is connected by a cord of negligible mass that is wrapped around a pulley of mass $M = 2.5$ kg and radius $R = 0.8$ m, see Fig. 8.19. The block slides on the incline against a frictional force f of 0.5 N, and the pulley rotates without friction about its axis. How fast will the block be moving after sliding a distance $d = 1.5$ m along the incline?

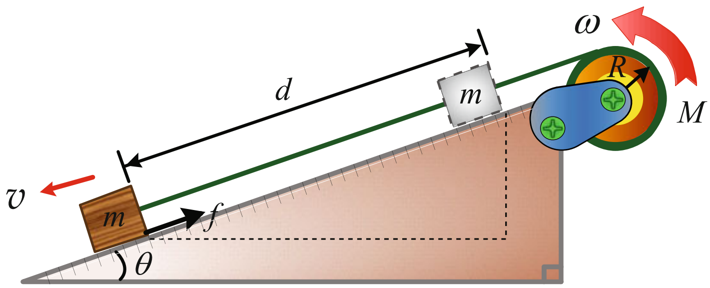


Fig. 8.19

Solution: The work done by the frictional force W should be equal to the change in the total energy ΔE of the block-pulley system. Thus:

$$W = \Delta E = E_{\mathrm{f}} - E_{\mathrm{i}}$$

where $E_{\mathrm{i}} = K_{\mathrm{i}} + (K_{\mathrm{R}})_{\mathrm{i}} + U_{\mathrm{i}}$ and $E_{\mathrm{f}} = K_{\mathrm{f}} + (K_{\mathrm{R}})_{\mathrm{f}} + U_{\mathrm{f}}$ are the initial and final total energies of the system, respectively. If we assign a zero value for the gravitational potential of the block at the final position, then $U_{\mathrm{i}} = m\,g(d\sin\theta)$ and $U_{\mathrm{f}} = 0$. Also, $(K_{\mathrm{R}})_{\mathrm{i}} = 0$ and $(K_{\mathrm{R}})_{\mathrm{f}} = \frac{1}{2}I\omega^2$, where $I = \frac{1}{2}M\,R^2$ for a disk rotating about its central axis. In addition, $K_{\mathrm{i}} = 0$ and $K_{\mathrm{f}} = \frac{1}{2}mv^2$. Using these relations and substituting with $W = -f\,d$ and $\omega = v/R$ into the last equation, we get:

$$-f\,d = [\tfrac{1}{2}m\,v^2 + \tfrac{1}{2}(\tfrac{1}{2}M\,R^2)(v/R)^2 + 0] - [0 + 0 + m\,g\,d\sin\theta]$$

By rearranging the terms, we have:

$$v = 2\sqrt{\frac{(m\,g\sin\theta - f)d}{2m + M}} = 2\sqrt{\frac{[(2\,\text{kg})(9.8\,\text{m/s}^2)\sin 30° - 0.5\,\text{N}](1.5\,\text{m})}{2\times(2\,\text{kg}) + (2.5\,\text{kg})}}$$
$$= 2.93\,\text{m/s}$$

Work done in Rotational Motion

We assume that the rotation of the rigid body in Fig. 8.20 is produced by an *external force* $\vec{F}$ that acts at a point P, which is at a distance r from the rotational axis through O. The radial component of $\vec{F}$ does not cause rotation, because it has a zero moment arm, while the tangential component $F_t = F\sin\phi$ does cause rotation. The differential work done by $\vec{F}$ on the rigid body as it rotates through an infinitesimal distance $ds = r\,d\theta$ about O is:

$$dW = \vec{F} \bullet d\,\vec{s} = F_t\,d\,s = F\sin\phi\,d\,s = F\sin\phi\,r\,d\theta \tag{8.37}$$

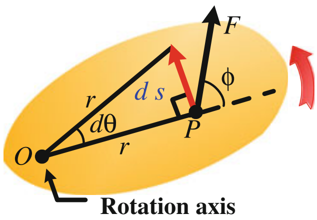


Fig. 8.20 A rigid body rotates about an axis through O under the action of a single external force $\vec{F}$ acting at point P

Since the magnitude of the torque due to $\vec{F}$ about O is $\tau = F_t\,r$, then:

$$dW = \tau\,d\theta \tag{8.38}$$

This is the rotational version of the one-dimensional relation $dW = F\,ds$, namely $F \Leftrightarrow \tau$ and $s \Leftrightarrow \theta$. For a single force, we use $\tau = I\,\alpha = I\,d\omega/dt$ and the chain rule of differentiation to get:

$$dW = \tau d\theta = I\frac{d\omega}{dt}d\theta = I\frac{d\omega}{d\theta}\frac{d\theta}{dt}d\theta = I\frac{d\omega}{d\theta}\omega\,d\theta = I\omega\,d\omega \tag{8.39}$$

By integrating Eq. 8.39, we obtain the total work as follows:

$$\boxed{W = \int_{\theta_i}^{\theta_f} \tau\, d\theta} \quad \text{or} \quad \boxed{W = \int_{\omega_i}^{\omega_f} I\omega\, d\omega = \tfrac{1}{2}I\omega_f^2 - \tfrac{1}{2}I\omega_i^2 = \Delta K_{\mathrm{R}}} \tag{8.40}$$

The relation $W = \Delta K_{\mathrm{R}}$ is the work-energy principle for rotational motion of a rigid body about a fixed axis.

Power in Rotational Motion

The rate of work done at time t, dW/dt, or the instantaneous power P, is obtained from Eq. 8.38 as follows:

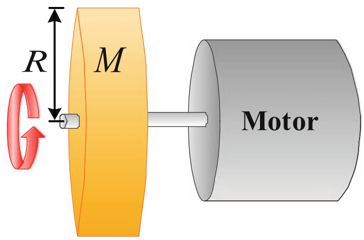
$$\boxed{P = \frac{dW}{dt} = \tau\omega} \tag{8.41}$$

The right-hand side of this expression is the rotational version of the linear-motion equation $P = Fv$, where $F \Leftrightarrow \tau$ and $v \Leftrightarrow \omega$.

Example 8.12

A disk of mass $M = 0.2$ kg and radius $R = 5$ cm is attached coaxially to the massless shaft of an electric motor, see Fig. 8.21. The motor runs steadily at 900 rpm and delivers 2 hp. (a) What is the angular speed of the disk in SI units? (b) What is the rotational kinetic energy of the disk? (c) How much torque does the motor deliver?

Fig. 8.21



Solution: (a) The angular speed of the motor of the disk is:

$$\omega = \left(900\,\frac{\text{rev}}{\text{min}}\right)\left(\frac{\text{min}}{60\,\text{s}}\right)\left(\frac{2\pi\,\text{rad}}{\text{rev}}\right) = 94.2\,\text{rad/s}$$

(b) The rotational kinetic energy of the disk is:

$$K_R = \tfrac{1}{2}I\omega^2 = \tfrac{1}{2}\left(\tfrac{1}{2}MR^2\right)\omega^2$$
$$= \tfrac{1}{4}(0.2\,\text{kg}) \times (0.05\,\text{m})^2(94.2\,\text{rad/s})^2 = 1.11\ \text{J}$$

This is the amount of energy needed to bring the disk from rest to the angular speed $\omega = 94.2\,\text{rad/s}$.

(c) The power delivered by the motor to maintain a constant angular speed $\omega = 94.2\,\text{rad/s}$ for the disk and to oppose all kinds of friction is:

$$P = 2 \times (746\ \text{W}) = 1{,}492\ \text{W}$$

Using Eq. 8.41, $P = \tau\omega$, we can find the torque as follows:

$$\tau = \frac{P}{\omega} = \frac{1{,}492\ \text{W}}{94.2\,\text{rad/s}} = 15.8\,\text{m.N}$$

8.9 Rolling Motion

Rolling as Rotation and Translation Combined

Assume that the wheel of Fig. 8.22 is rolling on a flat surface without slipping, and that its axes of rotation always remain parallel. In this figure, point Q on the rim of the wheel moves in a complex path called a cycloid while its center of mass moves in a straight line.

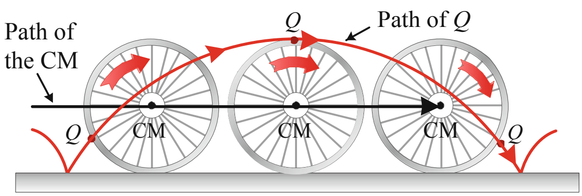


Fig. 8.22 When a wheel rolls without slipping on a flat surface, each point on the circumference (such as point Q) traces out a cycloid, while the center of mass (CM) traces out a straight line

Now, consider a wheel of a bicycle of radius R rolling without slipping on a horizontal surface at as shown in Fig. 8.23. Initially, the two points P and P' coincide, where P is the point of contact and $P'$is a point on the rim of the wheel.

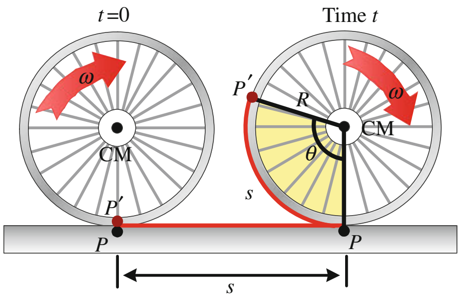


Fig. 8.23 When a wheel rolls through an angle θ due to a rotation about the center of mass CM, its CM moves a linear distance $s = R\theta$

During a time interval t, both the point of contact P and the center of mass CM move a linear distance s; while the point on the rim P' moves an arc length s that subtends an angle θ at CM. Thus:

$$s = r\,\theta$$

Consequently, the linear speed of the center of mass will be given by:

$$v_{\mathrm{CM}} = \frac{ds}{dt} = R\,\frac{d\theta}{dt} = R\,\omega \tag{8.42}$$

where ω is the angular speed of the wheel about its center of mass.

Rolling as Pure Rotation

To compare rolling-without-slipping motion with pure rotational motion, we consider Fig. 8.24. In this figure, the point of contact P is instantaneously at rest and the wheel rotates about an axis passing through this point. Since the point CM is at a distant R from P, and we proved that the CM has linear velocity $v_{\mathrm{CM}} = R\,\omega$, then, in order to preserve Eq. 8.42, the instantaneous angular velocity about P must be the same as the instantaneous angular velocity ω about CM. In addition, the linear speed of point Q must be $2v_{\mathrm{CM}}$.

As a result, rolling on a flat surface without slipping is equivalent to experiencing pure rotation about an axis through the point of contact P. Therefore, we can express the rolling kinetic energy of the wheel as:

$$K_{\mathrm{Roll}} = \tfrac{1}{2} I_P\,\omega^2 \tag{8.43}$$

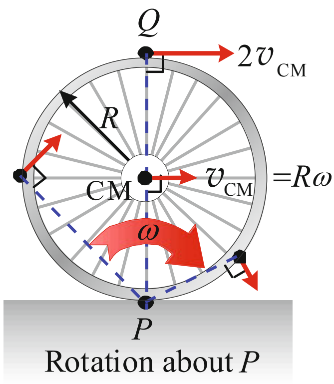


Fig. 8.24 Rotation about an axis through P with an angular velocity ω is equivalent to rotation about the CM with the same angular velocity

where I_P is the moment of inertia of the wheel about an instantaneous axis of rotation through P. By applying the parallel-axis theorem, we substitute $I_P = I_{CM} + M R^2$ into Eq. 8.43 to obtain:

$$K_{Roll} = \tfrac{1}{2} I_{CM}\,\omega^2 + \tfrac{1}{2} M\,R^2\omega^2$$

By using $v_{CM} = R\,\omega$, the relation leads to:

$$\boxed{K_{Roll} = \tfrac{1}{2} I_{CM}\,\omega^2 + \tfrac{1}{2} M v_{CM}^2} \tag{8.44}$$

Based on this relation, it seems natural to consider this type of rolling as a combination of rotational and translational motions. This consideration is explained graphically in Fig. 8.25.

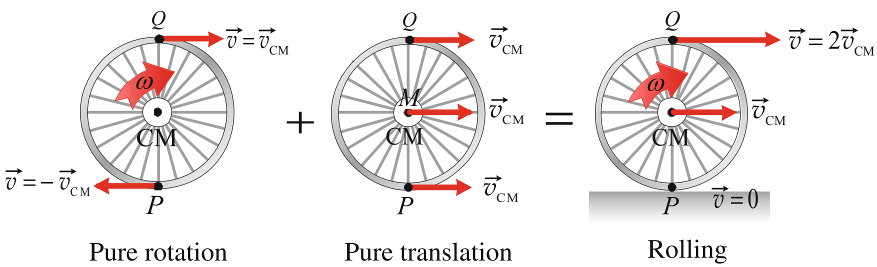


Fig. 8.25 Rolling without slipping can be considered as a combination of pure rotation and pure translation

Rolling with Friction

When the linear speed v_{CM} or the angular speed ω of a wheel changes, then a frictional force *tends to slide* the wheel at the point of contact P. Before sliding occurs, this frictional force is a static force f_s. Right on the verge of sliding, this frictional force is a maximum static force $f_{s,\max}$. When sliding occurs, this frictional force is a kinetic force f_k.

Figure 8.26a shows a wheel being rotated faster and faster (ω increases). The increase in ω tends to slide the point of contact P to the left. In Fig. 8.26b, the wheel tends to rotate more slowly, and the decrease in ω tends to slide the point of contact P to the right.

Figure 8.26c shows a wheel rolling down an incline without sliding. The weight $M\vec{g}$ at its center cannot cause rotation about the CM. Since $M\vec{g}$ tends to slide the wheel down the incline, a frictional force $\vec{f}_s$ must act at the point of contact P to oppose the sliding tendency; and this force has a moment arm about the center of mass.

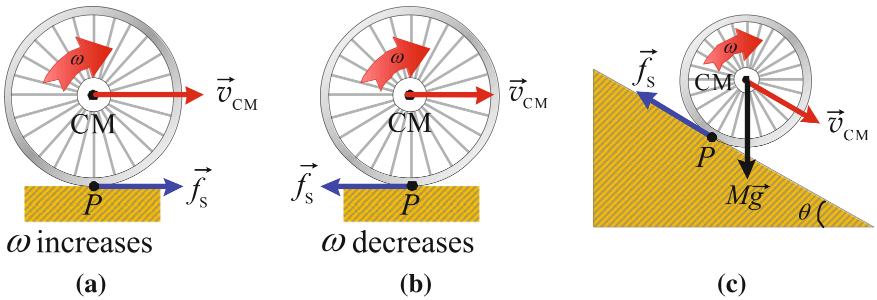


Fig. 8.26 (a) A wheel rolls horizontally without sliding while increasing its angular speed. The frictional force $\vec{f}_s$ acts at P to the right in order to oppose the sliding tendency. (b) Just like in (a) but with a decreasing angular speed. (c) A wheel rolls without sliding on an incline. The frictional force $\vec{f}_s$ acts at P to oppose the sliding tendency due to the wheel's weight $M\vec{g}$

Example 8.13

A disk of mass $M = 1.5$ kg and radius $R = 8$ cm rolls horizontally without sliding with a center-of-mass speed $v_{\mathrm{CM}} = 4$ m/s. (a) What is the angular speed of the disk? (b) What is the kinetic energy of the rolling disk?

Solution: (a) Using Eq. 8.42, we have:

$$\omega = \frac{v_{CM}}{R} = \frac{4\ \text{m/s}}{8 \times 10^{-2}\ \text{m}} = 50\ \text{rad/s} \simeq 8\ \text{rev/s}$$

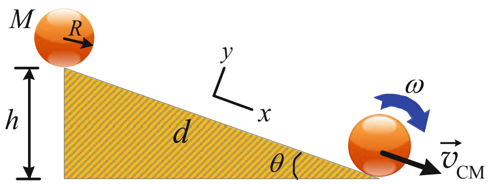
(b) The rolling kinetic energy of the disk is:

$$\begin{aligned} K_{\text{Roll}} &= K_R + K = \tfrac{1}{2} I_{CM}\,\omega^2 + \tfrac{1}{2} M v_{CM}^2 \\ &= \tfrac{1}{2}\left(\tfrac{1}{2} M R^2\right)\omega^2 + \tfrac{1}{2} M v_{CM}^2 \\ &= \tfrac{1}{4}(1.5\ \text{kg}) \times (0.08\ \text{m})^2 (50\ \text{rad/s})^2 + \tfrac{1}{2}(1.5\ \text{kg})(4\ \text{m/s})^2 = 18\ \text{J} \end{aligned}$$

Example 8.14

A solid sphere of mass M and radius R rolls without sliding when released from rest at the top of a frictional plane having a height h and inclination angle θ, see Fig. 8.27. The sphere starts at the top of the inclined plane and rolls to the bottom of the incline. Find the speed and acceleration of the sphere's center of mass when it reaches the bottom of the incline.

Fig. 8.27



Solution: Generally, the rolling kinetic energy of the sphere is:

$$K_{\text{Roll}} = K_R + K = \tfrac{1}{2} I_{CM}\,\omega^2 + \tfrac{1}{2} M v_{CM}^2$$

Using $v_{CM} = R\omega$ and $I_{CM} = \frac{2}{5} M R^2$ for a solid sphere, we can express K_{Roll} as a function of v_{CM} throughout the relation:

$$K_{\text{Roll}} = \tfrac{7}{10} M v_{CM}^2$$

We define the bottom of the incline to have zero gravitational potential energy. When rolling without sliding, the center of the sphere falls a vertical distance h, and the conservation of mechanical energy gives:

$$K_f + U_f = K_i + U_i$$

where $K_f = K_{Roll}$, $U_f = 0$, $K_i = 0$, and $U_i = M\,g\,h$. Thus:

$$\tfrac{7}{10} M v_{CM}^2 + 0 = 0 + M\,g\,h$$

Hence, we can express the dependence of v_{CM} on h as follows:

$$v_{CM} = \sqrt{\tfrac{10}{7} gh}$$

Notice that this is less than the speed $\sqrt{2gh}$ when an object slides on a frictionless incline without rolling (see Examples 5.5 and 6.8).

Using the kinematic equation $v^2 = v_\circ^2 + 2a(x - x_\circ)$ for the translational motion of the sphere along the incline, with $v \equiv v_{CM}$, $v_\circ = 0$, $a = a_{CM}$, and $(x - x_\circ) = d = h/\sin\theta$, we have:

$$\frac{10gh}{7} = 0 + 2\,a_{CM}\frac{h}{\sin\theta}$$
$$a_{CM} = \tfrac{5}{7} g \sin\theta$$

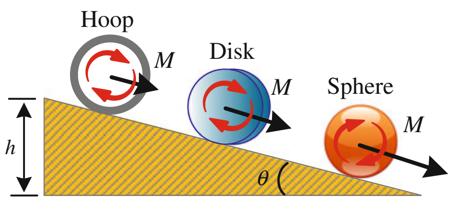
Notice also that this is less than the acceleration $g \sin\theta$ when an object slides down a frictionless incline without rolling (see Example 5.5).

The independence of v_{CM} and a_{CM} on R and M indicates that all homogeneous solid spheres experience the same speed and acceleration on a given incline.

Example 8.15

Three objects (a solid sphere, a disk, and a thin hoop) each having a mass M are at rest at the same height h. At the exact same instant, these objects start to roll without sliding down the incline of Fig. 8.28. In what order do they arrive at the bottom?

Fig. 8.28



Solution: For the given list of objects, we set $I_{\rm CM} = \beta M R^2$, where $\beta = 0.4$ for the sphere, $\beta = 0.5$ for the disk, and $\beta = 1$ for the thin hoop. Therefore, using $K_{\rm Roll} = \frac{1}{2} I_{\rm CM}\,\omega^2 = M\,g\,h$ and $v_{\rm CM} = R\omega$, the speed of the center of mass of any one of these objects at the bottom of the incline will be:

$$v_{\rm CM} = \sqrt{\frac{2gh}{\beta+1}}, \quad \beta = \begin{cases} 0.4 & \text{(for a sphere)} \\ 0.5 & \text{(for a disk)} \\ 1 & \text{(for a hoop)} \end{cases}$$

Note that $v_{\rm CM}$ does not depend on the object's mass M or radius R, but only depends on the shape (through the parameter β) and the height h. Moreover, according to the value of β, the sphere will attain the largest value of $v_{\rm CM}$, followed by the disk, and finally the hoop will attain lowest value of $v_{\rm CM}$, see Fig. 8.28.

In all cases, the acceleration of the center of mass is given by:

$$a_{\rm CM} = \frac{g\sin\theta}{(1+\beta)}$$

This is less than $g\sin\theta$ for the case of a box that slides down a frictionless incline of the same angle.

Table 8.2 summarizes the angular quantities and their linear analogs.

Table 8.2 Analogy between some linear and angular quantities and their connecting relations

Linear	Angular	Connecting relation
x	θ	$x = r\theta$
v	ω	$v = r\,\omega$
a	α	$a_{\rm t} = r\,\alpha$
m	I	$I = \sum m\,r^2$
F	τ	$\tau = r\,F\sin\theta$
$K = \frac{1}{2}mv^2$	$K_R = \frac{1}{2}I\omega^2$	
$W = Fd$	$W = \tau\theta$	
$P = Fv$	$P = \tau\omega$	
$\sum F = m\,a$	$\sum \tau = I\alpha$	

8.10 Exercises

Section 8.1 Radian Measures

(1) As fractions of π and as numerical values, express the following angles in radians: 30°, 45°, 60°, 90°, 180°, 270°, 360°.

(2) The Moon, which is 3.8×10^5 km away from the Earth, subtends an angle of about 0.4° to us. Estimate the radius of the Moon.

(3) A circle has a radius of 2 m. (a) What angle in radians and degrees is subtended by an arc that is 1.5 m in length? (b) What arc length is subtended by an angle of 1.2 rad between two radii in this circle?

(4) Through how many revolutions must a car wheel turn if the wheel has a radius of 0.5 m and the car travels 2 km?

Section 8.2 Rotational Kinematics; Angular Quantities

(5) A drill starts from rest and after 4.5 s reaches a rate of 4×10^4 rev/min. What is the drill's average angular acceleration?

(6) A motor rotates at a rate of 9×10^3 rpm. When the motor is turned off, it takes 5 s to stop rotating. What is the average angular acceleration during this period?

(7) A player throws a baseball in a straight line towards a target at a speed of 90 km/h. While traveling, the ball spins at a rate of 1,800 rev/min. If the target is 10 m away, how many revolutions does the ball make on its way to the target?

(8) A reference line in a rotating fan has an angular position given by $\theta = 4t^2 - 14t + 6$, where θ is in radians and t is in seconds. (a) Find ω and α as a function of time. (b) Find the times when the angular position θ and the angular velocity ω become zero.

(9) A wheel rotates with an angular acceleration $\alpha = 6at - 2b$. At $t = 0$, the wheel has an angular speed $\omega_\circ$ and angular position $\theta_\circ$. Write down the equations for the angular speed and angular position as a function of time t.

(10) A wheel with six spokes is rotating at an angular speed $\omega = 240$ rev/min about an axle passing through its central axis at O, see Fig. 8.29. A dart of length $L = 10$ cm is shot parallel to the wheel's axle towards the wheel. Assume the dart and the spokes are very thin. (a) What is the minimum speed that the dart must have in order to miss any one of the spokes? (b) Does it matter where between the axle and the rim of the wheel you must aim the dart?

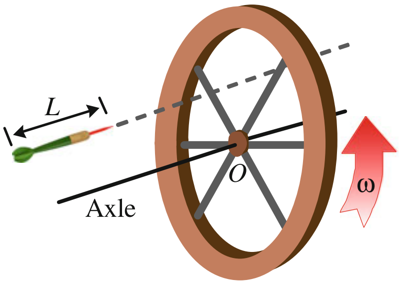


Fig. 8.29 See Exercise (10)

Section 8.3 Constant Angular Acceleration

(11) If the angular accelerations in Exercises 5 and 6 are constant, then find the change in angle through the corresponding rotational period. Provide your answer in radians, fractions of π, revolutions, and degrees.

(12) A wheel turning at an angular speed of 20 rev/s is brought to rest after 40 rev under a constant angular deceleration. (a) What is its angular deceleration? (b) How long does it take to stop?

(13) A car motor slows down from 5×10^3 rpm to 2×10^3 rpm in 2 s under a constant angular deceleration. (a) What is its angular deceleration? (b) Find the total number of revolutions of the motor in this period.

(14) A fan originally turning at 15 rev/s decelerates with $\alpha = -4\,\text{rad/s}^2$. (a) How long does the fan take to stop? (b) How many revolutions does it turn during this time period?

(15) A centrifuge rotates at an angular speed of 3.6×10^3 rev/min. When the centrifuge is turned off, it rotates 60 rev before coming to test. What is its angular deceleration? Assume it to be constant.

Section 8.4 Angular Vectors

(16) A wheel is mounted on fixed supports that are on a turntable that rotates about its axle with an angular speed $\omega_1 = 3$ rad/s, see Fig. 8.30. The turntable is rotating horizontally at an angular speed $\omega_2 = 4$ rad/s. Take the unit vectors along x, y, and z as $\vec{i}$, $\vec{j}$, and $\vec{k}$, respectively. (a) What are the directions of $\vec{\omega}_1$ and $\vec{\omega}_2$ at the instant shown in the figure? (b) Find the magnitude and direction of the resultant angular velocity $\vec{\omega}$ at the instant shown in the figure. (c) Find the magnitude and direction of the angular acceleration of the wheel $\vec{\alpha}_1$ at any time and then at the instant shown in the figure.

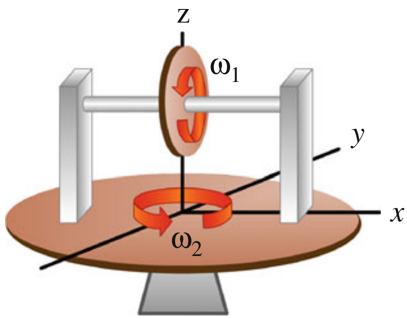


Fig. 8.30 See Exercise (16)

Section 8.5 Relating Angular and Linear Quantities

(17) A wheel 0.4 m in diameter rotates uniformly at an angular speed of 3.6×10^2 rev/min. (a) What is its angular speed in rad/s? (b) Find the linear speed and acceleration of a point on its rim.

(18) Figure 8.31 shows a synchronized analog 12-hour clock. Find the angular velocity of: (a) the second hand, (b) the minute hand, and (c) the hour hand. (d) Find the angular acceleration of each hand.

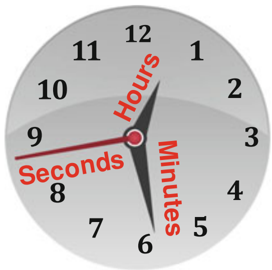


Fig. 8.31 See Exercise (18)

(19) In exercise 18, assume that the radii of the second hand, minute hand, and hour hand are 20, 15, and 10 cm, respectively. Find the linear speed of the tip of each hand.

(20) A merry-go-round completes one revolution in 1.5 s. (a) What is the linear speed of a child seated 3 m from the center? (b) Find the magnitude of the child's tangential and radial accelerations.

(21) Assume a point P is located at a latitude of exactly 30° N and is at a distance $r = 6.4 \times 10^6$ m away from Earth's center, see Fig. 8.32. As the Earth revolves about its axis, calculate: (a) the angular speed of the Earth, (b) the linear speed and magnitude of the acceleration of the point P, (c) the linear speed of a point on the equator.

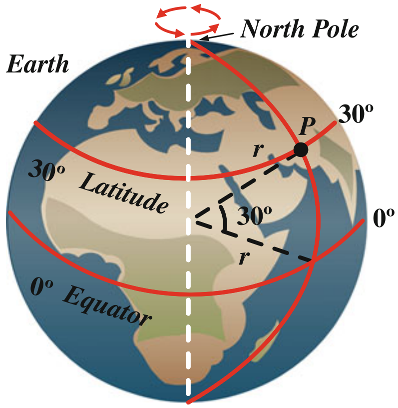


Fig. 8.32 See Exercise (21)

(22) If the lower string in Example 8.5 is removed, then find the proper angular speed ω that allows the ball to rotate with the same radius $r = 0.2$ m and angle $\theta = 30°$.

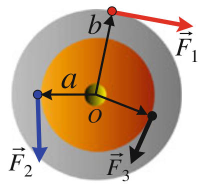
(23) A car accelerates uniformly from rest to a speed of 20 m/s during a 20 s time interval. The radius of the wheels of the car is 0.4 m. What is: (a) the angular acceleration of each wheel, and (b) the number of revolutions turned by each wheel during this time?

Section 8.6 Rotational Dynamics; Torque

(24) The pedals of a bike have a circular radius $r = 15$ cm. Find the maximum torque that can be exerted by the weight of a 70-kg person riding this bike?

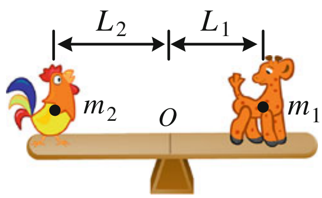
(25) The wheels in Fig. 8.33 have radii $a = 10$ cm and $b = 15$ cm. A frictional torque of 1.5 m.N opposes the motion when it rotates about an axle O perpendicular to the page. Find the net torque on the wheel when three forces of magnitudes $F_1 = 19$ N, $F_2 = 38$ N, and $F_3 = 45$ N are applied.

Fig. 8.33 See Exercise (25)



(26) A child wants to horizontally balance two toys of masses $m_1 = 0.1$ kg and $m_2 = 0.2$ kg by placing them at distances L_1 and L_2, respectively, from the central pivot of a seesaw of a massless board, see Fig. 8.34. (a) What is the ratio L_1/L_2 required to accomplish this balance? (b) If the child sets the toys 8 cm from the pivot, what is the magnitude and direction of the net torque?

Fig. 8.34 See Exercise (26)



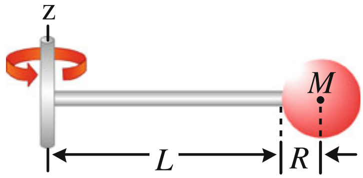
Section 8.7 Newton's Second Law for Rotation

(27) A rod of length 2 L is composed of an aluminum part with uniform length L and mass m_A and a brass part with uniform length L and mass m_B. Find the moment of inertia of the rod about an axis perpendicular to it yet passing through its center.

(28) A sphere of mass M and radius R is attached to one end of a massless rod of length L. The system rotates about the z-axis as shown in Fig. 8.35. (a) Use the parallel-axis-theorem to find the moment of inertia of the system about the z-axis. (b) Consider the sphere as a point particle and calculate its moment of inertia about the z-axis. (c) Find the percentage error introduced by the point approximation if $L = 0.5$ m and $R = 0.1$ m.

(29) A 1-kg wheel has a moment of inertia $I = 0.02\,\text{kg.m}^2$. The angular speed of the wheel reduces uniformly from 30 rev/s to zero after 150 rev. Find the torque used to slow down the wheel's rotation.

Fig. 8.35 See Exercise (28)

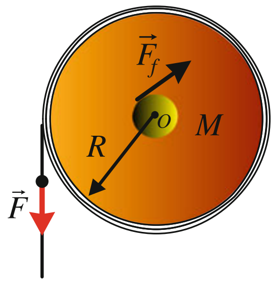


(30) Redo Example 8.8, this time assuming that a frictional torque $\vec{\tau}_f$ of magnitude 1.2 m.N exists at the axle.

(31) Redo Example 8.9, this time assuming that a frictional torque τ_f of magnitude 0.4 m.N exists at the pivot.

(32) A cord is wrapped around a pulley of mass $M = 2.5$ kg and radius $R = 0.2$ m. A constant force $\vec{F}$ of magnitude 30 N is applied to the cord as shown in Fig. 8.36. With the presence of a frictional torque $\vec{\tau}_f$ at the axle of magnitude 1.5 m.N, the pulley accelerates uniformly from rest to 21 rev/s in 2.8 s. (a) Find the moment of inertia of the pulley. (b) Does this moment of inertia equal the one obtained from the formula presented in Fig. 8.15? Explain.

Fig. 8.36 See Exercise (32)



(33) A pendulum of mass m with a string of length L is pulled aside to make an angle θ with the vertical. At the instant when the pendulum is released, find the torque on the mass m about the suspension point and its angular acceleration.

(34) A disk of mass M and radius R is attached to one end of a uniform rod of mass m and length L, as shown in Fig. 8.37. The other end of the rod is pivoted at P and the system is allowed to rotate freely. The system is released when the rod makes an angle θ with the vertical. Find the angular acceleration just after the system is released.

(35) An Atwood's machine consists of two boxes of masses $m_2 = 6$ kg and $m_1 = 4$ kg, which are connected by a massless cord that passes over a pulley, see

Fig. 8.38. The pulley has a moment of inertia $I = 5 \times 10^{-3}\,\text{kg.m}^2$ and radius $R = 5\,\text{cm}$. The cord does not slip over the pulley. Find the acceleration of the system and the tension in each cord. Take $g = 10\,\text{m/s}^2$.

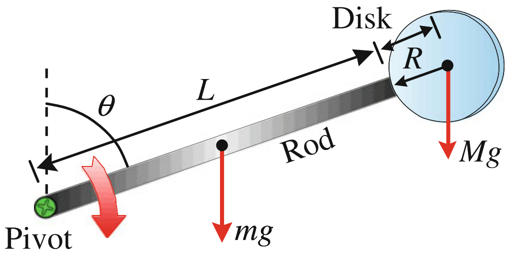


Fig. 8.37 See Exercise (34)

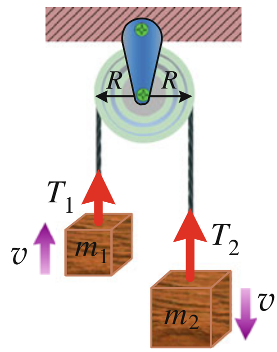


Fig. 8.38 See Exercise (35)

Section 8.8 Kinetic Energy, Work, and Power in Rotation

(36) What is the energy of an engine that has a moment of inertia $I = 5 \times 10^{-2}\,\text{kg.m}^2$ and is rotating at 1,500 rpm?

(37) Two small balls of masses $M = 4\,\text{kg}$ and $m = 2\,\text{kg}$ are connected by a horizontal massless rod of length $L = 3\,\text{m}$. The system is rotating with an angular speed $\omega = 8\,\text{rad/s}$ about an axle at a distance x from the mass M, see Fig. 8.39. Find the kinetic energy of the system and the net force on each mass: (a) when $x = L/2$, (b) when $x_{CM} = mL/(M + m)$; which is the position of the center of mass of the system.

(38) A horizontal massless rod is pivoted at one end. Three equal point masses are attached to this rod and are equidistant from each other and the pivot, see

Fig. 8.40. The system is released from its horizontal position. How fast will the bottom mass be moving when the rod becomes vertical?

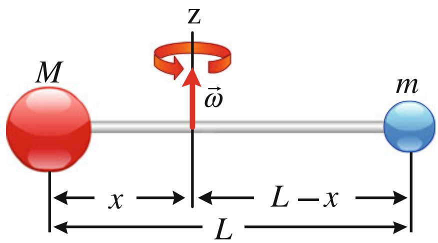


Fig. 8.39 See Exercise (37)

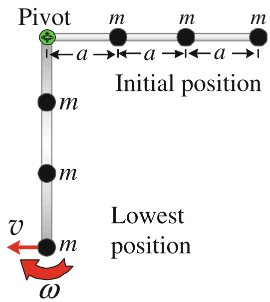


Fig. 8.40 See Exercise (38)

(39) The angular speed of a wheel increases from 60 to 180 rev/min when 125 J of work is added. What is its moment of inertia?

(40) Assume that the disk of Fig. 8.21 has a mass $M = 12$ kg and radius $R = 30$ cm. As in the figure, the disk is attached coaxially to the massless shaft of an electric motor. When the driving motor is disconnected, the motor slows down from 580 rpm to rest in 5 s. (a) What is the required power output of the motor to maintain a steady angular speed of 580 rpm? (b) How much torque does the motor deliver to maintain this steady angular speed?

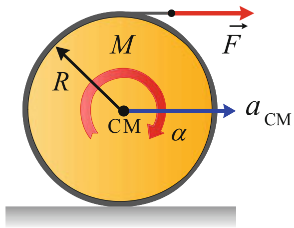
Section 8.9 Rolling Motion

(41) A cylinder of mass $M = 2$ kg and radius $R = 5$ cm rolls horizontally over the floor without sliding with a center of mass speed $v_{CM} = 0.8$ m/s. (a) What is the angular speed of the cylinder about its axis? (b) What are the magnitude

and direction of the speed and acceleration of a point on the top of the cylinder? (c) What is the kinetic energy of the rolling cylinder?

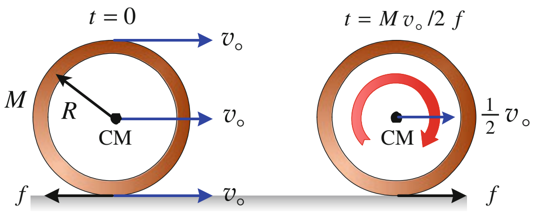
(42) A thread of negligible mass is wound around a cylinder of mass $M = 4$ kg and radius $R = 2$ cm. If the thread is unwound under a constant force of magnitude $F = 30$ N and the cylinder rolls without sliding, as shown in Fig. 8.41. (a) What is the direction of the frictional force? (b) What is acceleration of the center of mass? (c) What is the value of the frictional force?

Fig. 8.41 See Exercise (42)



(43) A hoop slides when projected horizontally at time $t = 0$ with an initial speed $v_\circ$. The frictional force causes the hoop to slow down and acquire an angular speed. Show that the hoop stops sliding and starts rolling when it has a speed $v_\circ/2$ at time $t = Mv_\circ/2f$, see Fig. 8.42.

Fig. 8.42 See Exercise (43)



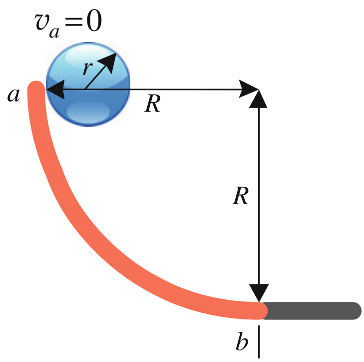
(44) A ball of radius r and mass m starts from rest and rolls without slipping on a track in the shape of a quarter circle of radius R, as shown in Fig. 8.43. Use conservation of mechanical energy to show that the ball's speed at the lowest point b is $v_b = \sqrt{10g\,(R - r)\,/7}$.

(45) A yo-yo of mass M and moment of inertia I has an axle of radius R. One end of a light string, assumed with negligible thickness, is tied to the axle and then wound several times around it. For idealized yo-yo, the thickness of wounded string can be neglected. While holding the other end of the string, the yo-yo

is released from rest, dropping as the string unwinds. Show that the linear acceleration, angular acceleration, and tension in the string of the yo-yo are given by:

$$a = \frac{g}{1 + I/MR^2}, \quad \alpha = \frac{g}{R + I/MR}, \quad T = \frac{Mg}{1 + MR^2/I}$$

Fig. 8.43 See Exercise (44)



Angular Momentum

9

In the previous chapter, we dealt with the kinematics and dynamics of the rotation of an extended object about a fixed axis. The rotational motion was analyzed in terms of Newton's second law for rotation as well as rotational kinetic energy.

In this chapter, we introduce the concept of ***angular momentum***, a quantity that plays a key role in *rotational dynamics*. Using classical physics, we saw how linear momentum was conserved. Similarly, we will see how the ***conservation of angular momentum*** is a fundamental law in rotational dynamics, and in further studies (not introduced in this book) can be proved to be equally valid for relativistic and quantum physics.

9.1 Angular Momentum of Rotating Systems

9.1.1 Angular Momentum of a Particle

Figure 9.1 depicts a particle of mass m that has a momentum $\vec{p} = m\vec{v}$ and a position vector $\vec{r}$ that is measured with respect to an origin O of an inertial frame. The angular momentum $\vec{L}$ of this particle about the origin O is defined by the vector product:

$$\vec{L} = \vec{r} \times \vec{p} \tag{9.1}$$

Following the right-hand rule introduced in Chap. 2, the direction of $\vec{L}$ is perpendicular to the plane containing the two vectors $\vec{r}$ and $\vec{p}$ as shown in Fig. 9.1 and its magnitude is given by:

$$L = r\,p\sin\theta = r\,p_{\perp} = r_{\perp}\,p \tag{9.2}$$

H. A. Radi and J. O. Rasmussen, *Principles of Physics*,
Undergraduate Lecture Notes in Physics, DOI: 10.1007/978-3-642-23026-4_9,

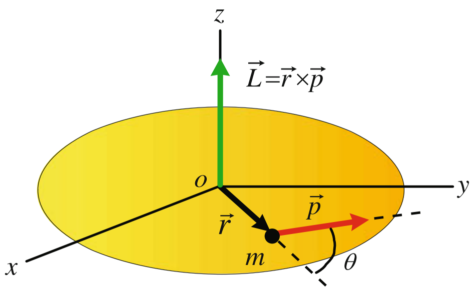


Fig. 9.1 The angular momentum $\vec{L}$ of a particle of mass m and momentum $\vec{p}$ located at position $\vec{r}$ is defined by $\vec{L} = \vec{r} \times \vec{p}$

where θ is the angle between $\vec{r}$ and $\vec{p}$. In addition, $p_\perp = p \sin\theta$ and $r_\perp = r \sin\theta$ are the components of $\vec{p}$ and $\vec{r}$ perpendicular to $\vec{r}$ and $\vec{p}$, respectively. It follows that $L = 0$ when $\vec{r}$ is parallel or antiparallel to $\vec{p}$ ($\theta = 0$ or $\theta = 180°$). The SI unit of L is kg.m^2/s (or J.s).

To find the relation between angular momentum and torque, we differentiate Eq. 9.1 with respect to time as follows:

$$\frac{d\vec{L}}{dt} = \frac{d\,(\vec{r} \times \vec{p}\,)}{dt} = \vec{r} \times \frac{d\vec{p}}{dt} + \frac{d\vec{r}}{dt} \times \vec{p}$$

Using Newton's second law $\Sigma\vec{F} = d\vec{p}/dt$ and the definition of net torque $\Sigma\vec{\tau} = \vec{r} \times \Sigma\vec{F}$, we see that the first term is just $\Sigma\vec{\tau}$. The second term is zero since $d\vec{r}/dt \times \vec{p} = \vec{v} \times (m\vec{v}\,) = m\vec{v} \times \vec{v} = 0$. Therefore:

$$\boxed{\Sigma\vec{\tau} = \frac{d\vec{L}}{dt}} \quad \text{(Single particle)} \tag{9.3}$$

That is, the net torque acting on a particle is equal to the time rate of change of the particle's angular momentum. The expression 9.3 is the rotational analogous to $\Sigma\vec{F} = d\vec{p}/dt$ in the case of linear motion, where $\Sigma\vec{F} \Leftrightarrow \Sigma\vec{\tau}$ and $\vec{p} \Leftrightarrow \vec{L}$.

For a particle of mass m moving with a constant speed v in a circular path of radius r, i.e. $v = r\omega$, the magnitude of the orbital angular momentum $\vec{L}$ is constant and given by:

$$L = rmv \sin 90° = mvr \quad \Rightarrow \quad L = I\omega \tag{9.4}$$

where $I = mr^2$ is the moment of inertia of the particle. Application of the right-hand rule shows that the direction of $\vec{L}$ is also constant and perpendicular to the plane of the circle, although the direction of $\vec{p} = m\vec{v}$ keeps changing.

9.1.2 Angular Momentum of a System of Particles

Consider a system made of n particles having angular momenta $\vec{L}_1, \vec{L}_2, \ldots, \vec{L}_n$. Regardless of whether these particles are loosely bound, or tightly bound together (as in a rigid body), or free, the total angular momentum $\vec{L}$ is always:

$$\vec{L} = \Sigma \vec{L}_i, \quad (i = 1, 2, \ldots, n) \tag{9.5}$$

If we differentiate this equation with respect to time, we get:

$$\frac{d\vec{L}}{dt} = \sum \frac{d\vec{L}_i}{dt} = \sum \vec{\tau}_i, \quad (i = 1, 2, \ldots, n) \tag{9.6}$$

Based on Newton's third law, the sum of all internal torques must add to zero due to the cancelation effect of all internal forces on the system. Therefore, the net torque on the system is only due to all external torques, and Eq. 9.6 reduces to:

$$\boxed{\Sigma \vec{\tau}_{\text{ext}} = \frac{d\vec{L}}{dt}} \quad \text{(System of particles)} \tag{9.7}$$

where $\vec{\tau}_{\text{ext}}$ and $\vec{L}$ are calculated with respect to a fixed point in an inertial frame. This is the rotational analogue to $\Sigma \vec{F}_{\text{ext}} = d\vec{P}/\,dt$ in the case of linear motion, where $\vec{F}_{\text{ext}} \Leftrightarrow \vec{\tau}_{\text{ext}}$ and $\vec{p} \Leftrightarrow \vec{L}$.

We can prove that Eq. 9.7 is valid for a fixed point on the center of mass of the system, even if the CM is accelerating. Thus:

$$\boxed{\Sigma\, \vec{\tau}_{\text{CM}} = \frac{d\vec{L}_{\text{CM}}}{dt}} \quad \text{(Even if CM is accelerating)} \tag{9.8}$$

9.1.3 Angular Momentum of a Rotating Rigid Body

Consider a rigid body rotating with an angular speed ω about a fixed axis; say the z-axis is as shown in Fig. 9.2. A typical mass element Δm_i of the rigid body moves with a speed v_i around the z-axis in a circular path of radius r_i, i.e. $v_i = r_i\ \omega$. If the position of this element is measured with respect to an origin O, then with the use of r_i we will be able to find the component of the angular momentum about the rotational axis, which is the z-axis in this case.

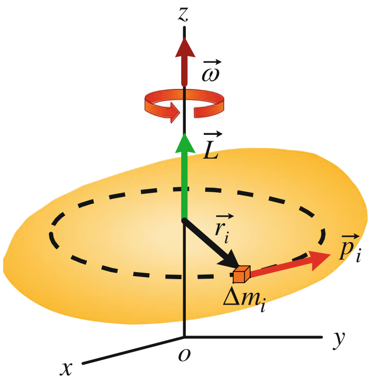


Fig. 9.2 A rigid body rotates about the z-axis with an angular speed ω. The component of the angular momentum will be along the z-axis

The z-component of the angular momentum of this element is:

$$L_i = r_i \Delta m_i v_i = \Delta m_i r_i^2\, \omega$$

where the vector $\vec{L_i}$ is directed along the z-axis just like the vector $\vec{\omega}$. The total component of the angular momentum about the rotational axis is the sum of all $\vec{L_i}$ and denoted by L_z. Thus:

$$L_z = \Sigma L_i = \Sigma\, \Delta m_i r_i^2 \omega = \left(\Sigma\, \Delta m_i r_i^2\right) \omega$$

Since $\sum \Delta m_i r_i^2 \to \int r^2 dm$, which is the moment of inertia of the body about the z-axis (see Chap. 8), then the above relation reduces to:

$$\boxed{L_z = I\omega} \quad \text{(Rigid body)} \tag{9.9}$$

Note that choosing any point on the z-axis and using that point as the origin O would yield the same Eq. 9.9. Accordingly, Eq. 9.7 will take on the following form for any rigid body:

$$\boxed{\Sigma \tau_{\text{ext}} = \frac{dL_z}{dt}} \quad \text{(Rigid body)} \tag{9.10}$$

If we differentiate Eq. 9.9 with respect to time, we get:

$$\frac{dL_z}{dt} = I \frac{d\omega}{dt} = I\, \alpha$$

Substituting this result into Eq. 9.10, we get:

$$\boxed{\Sigma\tau_{ext} = I\alpha} \quad \text{(Rigid body)} \tag{9.11}$$

This result is the same as in Eq. 8.32, which was derived using an approach that was based on the study of forces.

If the rigid body in Fig. 9.2 rotates about an axis of symmetry that passes through its center of mass, then L_z becomes the total angular momentum $\vec{L}$ of the body and Eqs. 9.9, 9.10, and 9.11 can be written in vector form as follows:

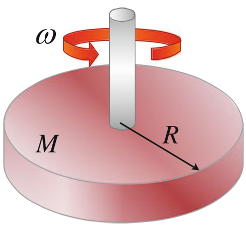
$$\boxed{\begin{aligned} &\vec{L} = I\vec{\omega} \\ &\sum\vec{\tau}_{ext} = \frac{d\vec{L}}{dt} \\ &\sum\vec{\tau}_{ext} = I\vec{\alpha} \end{aligned}} \quad \left(\begin{array}{l}\text{Rotation of rigid body}\\ \text{about its symmetry axis}\end{array}\right) \tag{9.12}$$

If the rigid object is *not symmetric*, then $\vec{L}$ and $\vec{\omega}$ may point in different directions and in this case $\vec{L}$ represents the component of the angular momentum along the axis of rotation.

Example 9.1

A disk of mass $M = 8$ kg and radius $R = 0.5$ m accelerates about its massless axle from rest to an angular speed $\omega = 8.5$ rad/s in a time $\Delta t = 2$ s, see Fig. 9.3. Find the angular momentum of the disk and the required constant torque used for this acceleration.

Fig. 9.3



Solution: According to Eq. 9.12, the angular momentum of the disk about its symmetry axis will be:

$$L = I\omega = \tfrac{1}{2}MR^2\,\omega = \tfrac{1}{2}(8\text{ kg})(0.5\text{ m})^2(8.5\text{ rad/s}) = 8.5\text{ J.s}$$

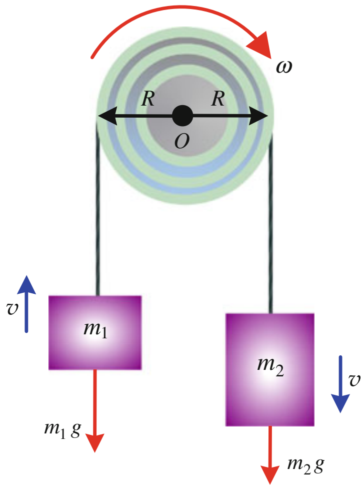
According to Eq. 9.12, the required constant torque that accelerates the disk from rest to 8.5 rad/s in 2 s is:

$$\tau_{\text{ext}} = \frac{\Delta L}{\Delta t} = \frac{L_{\text{f}} - L_{\text{i}}}{\Delta t} = \frac{8.5 \text{ J.s} - 0}{2 \text{ s}} = 4.25 \text{ m·N}$$

Example 9.2

An Atwood machine consists of two masses m_1 and m_2 ($m_2 > m_1$), which are connected by a light cord that passes over a freely rotating pulley, see Fig. 9.4. The pulley has a radius R and moment of inertia I about its axle. Find the acceleration of the two masses (consider $m_1 = 4$ kg, $m_2 = 6$ kg, $I = 2 \times 10^{-4}$ kg.m^2, and $R = 2$ cm).

Fig. 9.4



Solution: We can solve this problem by finding $\Sigma\vec{\tau}_{\text{ext}}$ and $\vec{L}_{\text{net}}$, and then by using Eq. 9.7, $\Sigma\vec{\tau}_{\text{ext}} = d\vec{L}_{\text{net}}/dt$, to find the acceleration $a = dv/dt$. Since the tensions in the two parts of the cord are internal forces, the net external torque of all external forces about the pulley's axel O (taking clockwise as positive since $m_2 > m_1$) is:

$$\begin{aligned}\Sigma\tau_{\text{ext}} &= m_2 gR - m_1 gR \\ &= (m_2 - m_1)gR\end{aligned}$$

At a given instant, when the speed of the two masses is v, the angular momenta of m_2 and m_1 are Rm_2v and Rm_1v, respectively. In addition, the angular momentum of the pulley is $I\omega$, where $v = R\omega$. Thus, the total clockwise angular momentum about O is:

$$L = Rm_1 v + Rm_2 v + I\frac{v}{R} = \left(m_1 + m_2 + \frac{I}{R^2}\right) Rv$$

By applying $\Sigma\tau_{\text{ext}} = dL/dt$, we get:

$$(m_2 - m_1)gR = \left(m_1 + m_2 + \frac{I}{R^2}\right) R\frac{dv}{dt}$$

Solving for $a = dv/dt$, we get:

$$a = \frac{m_2 - m_1}{m_1 + m_2 + \dfrac{I}{R^2}} g = \frac{6\text{ kg} - 4\text{ kg}}{6\text{ kg} + 4\text{ kg} + \dfrac{2\times 10^{-4}\text{ kg.m}^2}{(2\times 10^{-2}\text{ m})^2}} (9.8\text{ m/s}^2) = 1.87\text{ m/s}^2$$

If I is ignored, we get $a = (m_2 - m_1)g/(m_1 + m_2) = 1.96\text{ m/s}^2$ as proved in Example 5.3 of Chap. 5. Since this value is larger than 1.87 m/s^2, then the moment of inertia actually slows down the system.

Example 9.3

A rod having a mass $M = 3$ kg and length $d = 2$ m is pivoted (without friction) at its center O. Then, two masses $m_1 = 4$ kg and $m_2 = 7$ kg are treated as points and placed on the ends of that rod such that they are equidistant from O. At a particular moment in time the rod makes angle θ with the horizontal and the system is rotating in a vertical plane with an angular speed ω, see Fig. 9.5. (a) Find the system's angular momentum L and angular acceleration α. (b) How far away from the pivot O should m_2 be placed in order to acquire a balanced system having zero angular acceleration? Take $g = 10$ m/s^2.

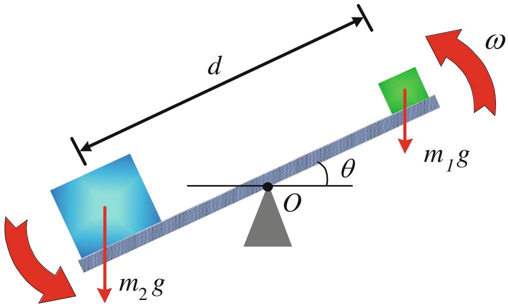
Solution: (a) The moment of inertia of the system about O is:

$$\begin{aligned} I &= m_1(\tfrac{1}{2}d)^2 + m_2(\tfrac{1}{2}d)^2 + \tfrac{1}{12}Md^2 = \tfrac{1}{4}(m_1 + m_2 + \tfrac{1}{3}M)d^2 \\ &= \tfrac{1}{4}(4\text{ kg} + 7\text{ kg} + \tfrac{1}{3}\times 3\text{ kg})(2\text{ m})^2 = 12\text{ kg.m}^2 \end{aligned}$$

Then, the magnitude of the total angular momentum of the system is:

$$L = I\omega = 12\,\omega \quad \text{(Out of the page with units of J.s)}$$

Fig. 9.5



To find the angular acceleration at any angle θ, we use $\Sigma\tau_{ext} = I\alpha$. To achieve this, we first find the magnitude of the two torques about O due to the forces m_1g and m_2g as follows:

$$\tau_1 = (d/2)m_1g\cos(90^\circ + \theta) = \tfrac{1}{2}m_1gd\cos\theta \quad \text{(Into page)}$$
$$\tau_2 = (d/2)m_2g\cos(90^\circ - \theta) = \tfrac{1}{2}m_2gd\cos\theta \quad \text{(Out of the page)}$$

Since $m_2 > m_1$, then the net external torque on the system about O is:

$$\begin{aligned}\sum \tau_{ext} &= \tau_2 - \tau_1 = \tfrac{1}{2}(m_2 - m_1)gd\cos\theta \\ &= \tfrac{1}{2}(7\text{ kg} - 4\text{ kg})(10\text{ m/s}^2)(2\text{ m})\cos\theta \\ &= 30\cos\theta\ \text{(m.N)} \quad \text{(Out of the page)}\end{aligned}$$

The angular acceleration at the instant shown in Fig. 9.5 is thus:

$$\alpha = \frac{\Sigma\tau_{ext}}{I} = \frac{\frac{1}{2}(m_2 - m_1)gd\cos\theta}{\frac{1}{4}(m_1 + m_2 + \frac{1}{3}M)d^2} = \frac{30\cos\theta\ \text{(m.N)}}{12\text{ kg.m}^2} = 2.5\cos\theta\text{ rad/s}^2$$

(b) Notice that as m_2 slides towards the pivot, the value of $\Sigma\tau_{ext}$ decreases and the system tends to be more balanced. In the case where the system is balanced we have $\Sigma\tau_{ext} = 0$ and $\alpha = 0$. If we assume the balance occurs when the distance between m_2 and the pivot is x, then:

$$\Sigma\tau_{ext} = \tau_2 - \tau_1 = m_2gx\cos\theta - \tfrac{1}{2}m_1gd\cos\theta = 0$$

Thus: $$x = \frac{1}{2}\frac{m_1}{m_2}d = \frac{1}{2}\frac{4\text{ kg}}{7\text{ kg}} \times (2\text{ m}) = \frac{4}{7}\text{ m}$$

9.2 Conservation of Angular Momentum

In Chap. 7, we found that the general form of Newton's second law for the translational motion, Eq. 7.43, is given by:

$$\sum \vec{F}_{\text{ext}} = \frac{d\vec{P}}{dt}$$

where $\sum \vec{F}_{\text{ext}}$ is the net external force acting on a system of particles (including rigid objects) and $\vec{P}$ is the total linear momentum of the system. If the system has a total mass M and its CM is moving with velocity $\vec{v}_{\text{CM}}$, then $\vec{P} = M\,\vec{v}_{\text{CM}}$. In addition, if the net external force is zero, then the total momentum $\vec{P}$ is conserved (which is the law of conservation of momentum) and $\vec{v}_{\text{CM}} = \text{constant}$.

In this chapter, we found an analogous relationship, Eq. 9.7, which describes the general rotational motion of a system of particles (including rigid objects). This was given by:

$$\Sigma \vec{\tau}_{\text{ext}} = \frac{d\vec{L}}{dt}$$

where $\Sigma \vec{\tau}_{\text{ext}}$ is the net external torque acting on a system of particles (including rigid objects) and $\vec{L}$ is the total angular momentum of the system. This relation is valid when $\Sigma \vec{\tau}_{\text{ext}}$ and $\vec{L}$ are evaluated either about a point fixed in an inertial reference frame, or about the CM of the system (even if the CM is accelerating). In addition, for isolated systems, the last relation leads to the following conclusion:

$$\text{If } \Sigma \vec{\tau}_{\text{ext}} = 0, \text{ then} \frac{d\vec{L}}{dt} = 0 \quad \text{and} \quad \vec{L} = \text{constant}$$

Therefore:

$$\boxed{\vec{L}_i = \vec{L}_f} \quad \text{(For an isolated system)} \tag{9.13}$$

This is the law of **conservation of angular momentum**, where i refers to some initial time, and f refers to a later time. In other words:

> Conservation of angular momentum:
> If the net external torque acting on a system is zero (i.e. an isolated system), the total angular momentum of the system remains constant in both magnitude and direction.

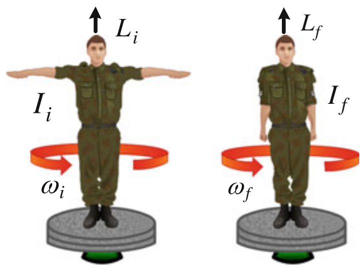
We can now state that the total energy, total linear momentum, and total angular momentum of an isolated system all remain constant:

$$\boxed{\begin{aligned} E_f &= E_i \\ \vec{P}_f &= \vec{P}_i \\ \vec{L}_f &= \vec{L}_i \end{aligned}} \quad \text{(For an isolated system)} \tag{9.14}$$

Example 9.4

A soldier stands with his arms stretched out at the center of a platform that rotates without friction with an angular speed $\omega_i = 1.8$ rev/s, see Fig. 9.6a. The rotational inertia of the soldier and platform is $I_i = 6\,\text{kg.m}^2$. When the soldier pulls his arms close to his body, as shown in Fig. 9.6b, he decreases the rotational inertia of the system to $I_f = 4\,\text{kg.m}^2$. (a) What is the resulting final angular speed of the system? (b) Is there a gain or a loss in the rotational kinetic energy of the system; and which of the objects, soldier or platform, gained or lost this energy? (c) If the platform is a disk of mass $M = 10$ kg and radius $R = 40$ cm, what is the moment of inertia of the soldier when his arms are close to his body?

Fig. 9.6



Solution: (a) Because there is no net external torque acting on the system about the axis of rotation, we can apply the law of conservation of angular momentum as follows:

$$L_f = L_i \quad \Rightarrow \quad I_f\omega_f = I_i\omega_i \quad \Rightarrow \quad \omega_f = \frac{I_i}{I_f}\omega_i = \frac{6\,\text{kg.m}^2}{4\,\text{kg.m}^2}1.8\,\text{rev/s} = 2.7\,\text{rev/s}$$

(b) The ratio of the final to the initial rotational kinetic energy is:

$$\frac{K_f}{K_i} = \frac{\frac{1}{2}I_f\omega_f^2}{\frac{1}{2}I_i\omega_i^2} = \frac{(4\,\text{kg.m}^2)(2.7\,\text{rev/s})^2}{(6\,\text{kg.m}^2)(1.8\,\text{rev/s})^2} = 1.5$$

This gain in rotational kinetic energy to both the solider and platform is due to the work done by the soldier by moving his arms inwards.

(c) Since $I_f = I_s + I_{\text{disk}}$, then the moment of inertia of the soldier I_s is:

$$I_s = I_f - \tfrac{1}{2}MR^2 = 4\,\text{kg.m} - \tfrac{1}{2}(10\,\text{kg})(0.4\,\text{m})^2 = 3.2\,\text{kg.m}^2$$

Example 9.5

A small mass m attached to one end of a light cord is constrained to rotate in a circular path over a frictionless table. The other end of the cord passes through a small hole O in the table, see Fig. 9.7. For an initial tension T_i and radius R_i, the initial angular speed of the mass is $\omega_i = 0.5$ rad/s, see Fig. 9.7a. The tension is then increased gradually to T_f when the cord is pulled until the radius is reduced to $R_f = R_i/3$, see Fig. 9.7b. (a) Find the final angular speed of the mass. (b) Find the ratio of the tensions T_f/T_i.

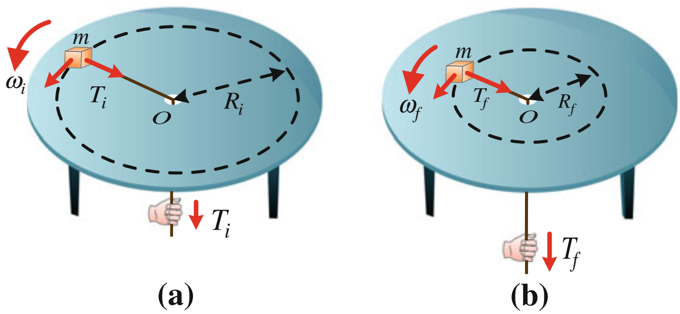


Fig. 9.7

Solution: (a) There is no torque about O since the force is central. Therefore, angular momentum is conserved. Thus:

$$L_f = L_i \quad \Rightarrow \quad I_f\omega_f = I_i\omega_i$$

If we treat the small mass as a particle with a moment of inertia $I = mr^2$, then we have:

$$mR_f^2\omega_f = mR_i^2\omega_i$$

Thus: $\omega_f = \left(\dfrac{R_i}{R_f}\right)^2 \omega_i = \left(\dfrac{R_i}{R_i/3}\right)^2 \omega_i = 9\omega_i = 9 \times 0.5\,\text{rad/s} = 4.5\,\text{rad/s}$

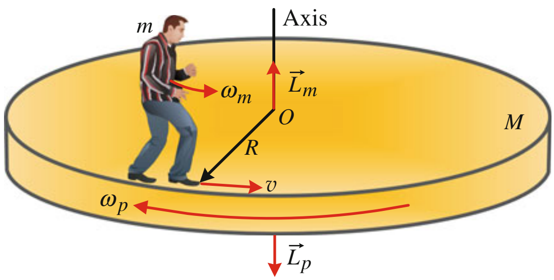
(b) The tension supplies the centripetal force which is needed to constrain the mass to move in a circle. So, $T = ma_\text{r} = mr\omega^2$ and we have:

$$\frac{T_f}{T_i} = \frac{mR_f\omega_f^2}{mR_i\omega_i^2} = \left(\frac{R_f}{R_i}\right)\left(\frac{\omega_f}{\omega_i}\right)^2 = (\tfrac{1}{3})(9)^2 = 27$$

Example 9.6

A man of mass $m = 60\,\text{kg}$ stands at the edge of a stationary circular platform of mass $M = 400\,\text{kg}$ and radius $R = 3\,\text{m}$. The platform is mounted on a frictionless bearing. When the man begins running at a speed $v = 4\,\text{m/s}$ around the platform's edge, the platform begins to rotate in the opposite direction as shown in Fig. 9.8. What is the angular speed and the period of the platform?

Fig. 9.8



Solution: Initially, the total angular momentum is zero, i.e. $\vec{L} = 0$. Since there is no net external torque on the system while the man is running on the platform, $\vec{L}_f$ of the system will remain zero. Thus:

$$\vec{L}_f = \vec{L}_i \quad \Rightarrow \quad \vec{L}_m + \vec{L}_p = 0 \quad \Rightarrow \quad L_m - L_p = 0 \quad \Rightarrow \quad L_p = L_m$$

where L_m and L_p are the magnitudes of the man's and platform's angular momentum, respectively. Modeling the man as a particle, we can write his moment of inertia as $I_m = mR^2$ and his angular speed about the axis of rotation as $\omega_m = v/R$. Then, treating the platform as a disk with a moment of inertia $I_p = \frac{1}{2}MR^2$, we can use the previous result of conservation of angular momentum $I_p\omega_p = I_m\omega_m$ to find:

$$\omega_p = \frac{I_m}{I_p}\omega_m = \frac{mR^2}{\frac{1}{2}MR^2}\frac{v}{R} = \frac{2mv}{MR} = \frac{2(60\,\text{kg})(4\,\text{m/s})}{(400\,\text{kg})(3\,\text{m})} = 0.4\,\text{rad/s}$$

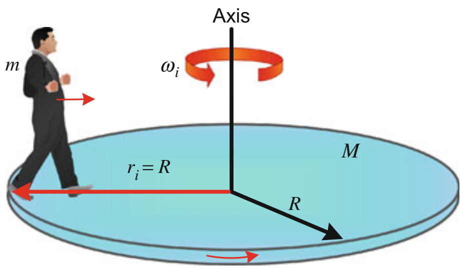
The rotational period of the platform is thus:

$$T_p = \frac{2\pi}{\omega_p} = \frac{2\pi}{0.4\,\text{rad/s}} = 15.7\,\text{s per revolution}$$

Example 9.7

A man of mass $m = 60\,\text{kg}$ stands at the edge of a rotating circular platform of mass $M = 220\,\text{kg}$ and radius R. The platform is mounted on a frictionless bearing. Initially, the angular speed of the system is $\omega_i = 0.5$ rad/s. The man starts to walk slowly and radially towards the center from the edge at $r_i = R$, see Fig. 9.9. What is the angular speed of the system when the man reaches a radius of $r_f = R/2$?

Fig. 9.9



Solution: The angular speed changes due to the change in the moment of inertia of the system during the walk. We model the man as a particle in this example. Since there is no net external torque on the system while the man is walking on the platform, the angular momentum of the system will remain constant. Thus:

$$L_f = L_i \quad \Rightarrow \quad I_f\ \omega_f = I_i\ \omega_i \quad \Rightarrow \quad (I_p + I_{mf})\ \omega_f = (I_p + I_{mi})\ \omega_i$$

where the moment of inertia of the platform about the rotational axis is constant during the man's walk and given by $I_p = \frac{1}{2}MR^2$. In addition, the initial and final moment of inertia of the man about this axis are $I_{mi} = mr_i^2 = mR^2$ and $I_{mf} = mr_f^2 = mR^2/4$, respectively. Therefore:

$$(\tfrac{1}{2}MR^2 + \tfrac{1}{4}mR^2)\omega_f = (\tfrac{1}{2}MR^2 + mR^2)\omega_i$$

$$(\tfrac{1}{2}M + \tfrac{1}{4}m)\omega_f = (\tfrac{1}{2}M + m)\omega_i$$

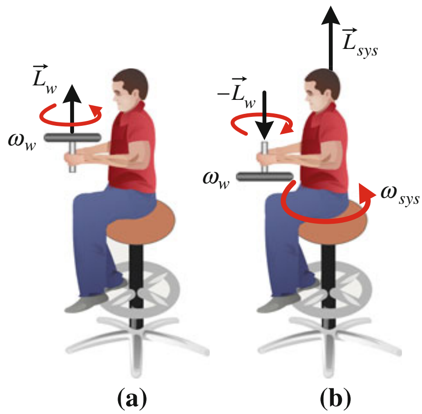
$$\omega_f = \frac{\frac{1}{2}M + m}{\frac{1}{2}M + \frac{1}{4}m}\omega_i = \frac{\frac{1}{2} \times 220\,\text{kg} + 60\,\text{kg}}{\frac{1}{2} \times 220\,\text{kg} + \frac{1}{4} \times 60\,\text{kg}} 0.5\,\text{rad/s} = 0.68\,\text{rad/s}$$

Note that ω_f is independent of R and that $\omega_f > \omega_i$ as expected.

Example 9.8

A student is sitting on a stationary stool that can rotate freely. This student is holding the axle of a rotating wheel whose moment of inertia about its axle is $I_w = 1.5\,\text{kg.m}^2$, see Fig. 9.10a. The rotating wheel has an angular speed $\omega_i = 4\,\text{rev/s}$ and its angular momentum $\vec{L}_w$ points upward. When the student inverts the wheel, its angular momentum becomes $-\vec{L}_w$, and the system (student+stool+wheel) starts rotating about the stool's axle, see Fig. 9.10b. The moment of inertia of the system about the stool's axle is $I_{sys} = 7.5\,\text{kg.m}^2$. What is the angular speed of the system after the inversion?

Fig. 9.10



Solution: The torque applied by the student to invert the wheel is internal to the system. Since there is no net external torque on the system, the angular momentum about any vertical axis is conserved. Initially, the total angular momentum of the system $\vec{L}_i$ comes entirely from the wheel. Thus:

$$\vec{L_i} = \vec{L_w}$$

After inverting the wheel, its angular momentum becomes $-\vec{L_w}$. For the total angular momentum to be conserved, the system must start rotating in the opposite direction with an angular momentum $\vec{L}_{sys}$, so:

$$\vec{L_f} = \vec{L}_{sys} + (-\vec{L_w})$$

Conservation of the angular momentum before and after the inversions of the wheel gives:

$$\vec{L_f} = \vec{L_i}$$
$$\vec{L}_{sys} - \vec{L_w} = \vec{L_w}$$

$$\vec{L}_{sys} = 2\vec{L_w} \quad \Rightarrow \quad L_{sys} = 2L_w \quad \Rightarrow \quad I_{sys}\,\omega_{sys} = 2I_w\omega_w$$

This yields: $$\omega_{sys} = \frac{2I_w}{I_{sys}}\omega_w = \frac{2 \times 1.5\,\text{kg.m}^2}{7.5\,\text{kg.m}^2} 4\,\text{rev/s} = 1.6\,\text{rev/s}$$

Example 9.9

Figure 9.11 shows a simple clutch which consists of two cylindrical disks that can be pressed together to connect two sections of an axle in a machine. The two disks have masses $M_1 = 5$ kg and $M_2 = 7$ kg, and have equal radii $R = 0.5$ m. Disk M_1 is accelerated from rest to an angular speed $\omega_1 = 6$ rad/s in a time interval $\Delta t = 2.5$ s. (a) Find the angular momentum of disk M_1. (b) Find the average torque required to accelerate M_1 to $\omega_1 = 6$ rad/s. (c) When disk M_2 (initially at rest) is coupled to disk M_1 such that they rotate as one unit, what is their angular speed after coupling?

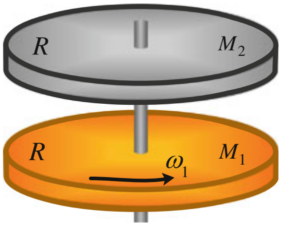
Solution: (a) The angular momentum of disk M_1 is:

$$L_1 = I_1\omega_1 = \tfrac{1}{2}M_1R^2\omega_1 = \tfrac{1}{2}(5\,\text{kg})(0.5\,\text{m})^2(6\,\text{rad/s}) = 3.75\,\text{J.s}$$

(b) The average torque required to accelerate M_1 is:

$$\overline{\tau}_{\text{ext}} = \frac{\Delta L}{\Delta t} = \frac{L_{\text{f}} - L_{\text{i}}}{\Delta t} = \frac{3.75\,\text{J.s} - 0}{2.5\,\text{s}} = 1.5\,\text{m.N}$$

Fig. 9.11



(c) When the stationary disk M_2 is coupled with M_1, each exerts a torque on the other, and there are no external torques in effect. Thus, conservation of angular momentum leads to:

$$\vec{L}_f = \vec{L}_i \quad \Rightarrow \quad (I_1 + I_2)\omega_2 = I_1\omega_1$$

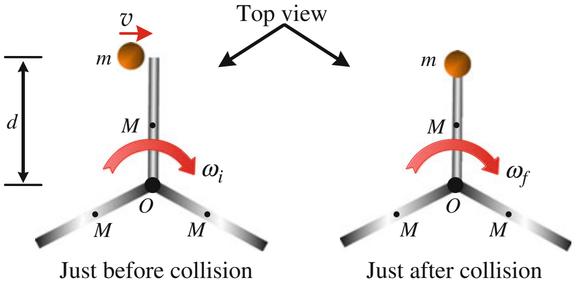
Thus:

$$\omega_2 = \frac{I_1}{I_1 + I_2}\omega_1 = \frac{M_1}{M_1 + M_2}\omega_1 = \frac{5\,\text{kg}}{5\,\text{kg} + 7\,\text{kg}} 6\,\text{rad/s} = 2.5\,\text{rad/s}$$

Example 9.10

Figure 9.12 shows a top view of three identical rods that are rigidly connected at one end at O and make an angle of 120° with each other. Each rod has a mass M and a length d, and the entire assembly is rotating horizontally with an initial angular speed ω_i about a vertical axle passing through O. A ball of clay of mass m moving horizontally with a speed v collides perpendicularly with the tip of one of the rods and sticks to it (i.e. the collision is completely inelastic). What is the final angular speed of the system?

Fig. 9.12



Solution: Just before the collision, the initial angular momentum of the clay about O is clockwise with magnitude $L_{c,i} = mvd$ and the initial angular momentum of the rod assembly is also clockwise with magnitude $L_{r,i} = I_r \omega_i$. We have $I_r = 3(Md^2/3) = Md^2$ as obtained from Fig 9.12. Thus, $L_{r,i} = Md^2\omega_i$ and the total initial angular momentum of the system about the axle is:

$$L_i = L_{c,i} + L_{r,i} = mvd + Md^2\omega_i$$

Just after collision, the system is composed of the clay with moment of inertia md^2 attached to the assembly having moment of inertia $I_r = Md^2$. Thus, the system has moment of inertia $I_{sys} = md^2 + Md^2$ and the total angular momentum of the system about the axle is:

$$L_f = I_{sys}\omega_f = (m + M)d^2\omega_f$$

During the impact (internal forces cancel), no external forces acting on the system have a torque about the rotational axis. Thus, conservation of angular momentum before and after the collision gives:

$$L_f = L_i \quad \Rightarrow \quad \omega_f = \frac{mvd + Md^2\omega_i}{(m + M)d^2}$$

9.3 The Spinning Top and Gyroscope

In all our previous studies, the axis of rotation either stayed fixed or was moving and kept moving in the same direction. However, a variety of new physical phenomena can occur when the axis of rotation changes its direction.

It is quite natural to wonder why a top spinning rapidly about its axis of symmetry does not fall over, even when its center of mass is not directly above its tip, see Fig. 9.13. In this figure, the top rotates rapidly about its axis of symmetry with angular speed ω. At the same time, this axis rotates slowly about the vertical direction (the z-axis) with angular speed Ω (capital Greek omega), where usually $\Omega \ll \omega$. The rotation of the top's axis about the vertical is called *precession*.

The essential features of the two rotations can be understood by examining the effect of the net torque $\vec{\tau}_{\text{ext}}$ on the top's angular momentum $\vec{L}$. During the rotational processes, the only two effective forces on the top are the weight $M\vec{g}$ acting at the CM and the normal force $\vec{N}$ acting upward on the tip O. The normal force produces zero torque about the tip, while the weight produced a torque $\vec{\tau}_{\text{ext}} = \vec{r} \times M\vec{g}$ about O. The direction of $\vec{\tau}_{\text{ext}}$ is perpendicular to the plane containing $\vec{r}$ and $M\vec{g}$. Also, $\vec{\tau}_{\text{ext}}$ is perpendicular to $\vec{L}$, since $\vec{r}$ and $\vec{L}$ are pointing in the same direction. In addition, $\vec{\tau}_{\text{ext}}$ always lies in the $x\,y$-plane.

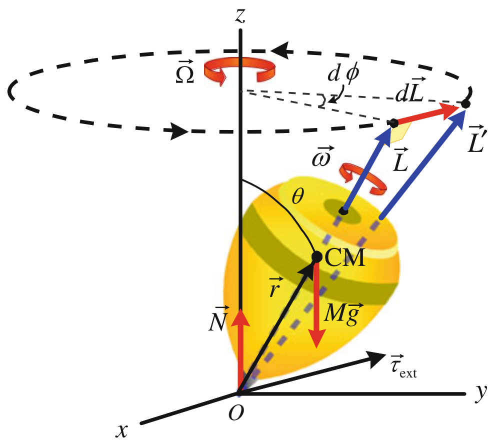


Fig. 9.13 A top rotating with angular velocity $\vec{\omega}$ about its symmetry axis and experiencing precession about the vertical axis with angular velocity $\vec{\Omega}$

Based on Eq. 9.7, the applied torque and angular momentum on the top are related through $\vec{\tau}_{\text{ext}} = d\vec{L}/dt$. Accordingly, during time interval dt, the change in angular momentum $d\vec{L}$ will be as follows:

$$d\vec{L} = \vec{L}' - \vec{L} = \vec{\tau}_{\text{ext}}\, dt \tag{9.15}$$

This relation indicates that the change in momentum $d\vec{L}$ has the same direction as $\vec{\tau}_{\text{ext}}$. But since $\vec{\tau}_{\text{ext}}$ is perpendicular to $\vec{L}$, then $d\vec{L}$ is also perpendicular to $\vec{L}$. Therefore, the magnitude of $\vec{L}$ does not change ($|\vec{L}'| = |\vec{L}|$) but only its direction changes perpendicular to $d\vec{L}$, as shown in Fig. 9.13. That is, the upper end of the top's axis moves in a horizontal circle. In other words, $\vec{\tau}_{\text{ext}}$ and $d\vec{L}$ rotate so as to be horizontal and perpendicular to $\vec{L}$.

To determine the angular velocity of precession, Ω, we notice that dL in Fig. 9.13 is related to the angle $d\phi$ by the relation:

$$dL = L \sin\theta\, d\phi \tag{9.16}$$

Substituting with $d\phi$ from this relation into the angular velocity of precession $\Omega = d\phi/dt$ and using $\tau_{\text{ext}} = dL/dt$, we get:

$$\Omega = \frac{d\phi}{dt} = \frac{1}{L\sin\theta}\frac{dL}{dt} = \frac{\tau_{\text{ext}}}{L\sin\theta} \tag{9.17}$$

But $|\vec{\tau}_{\text{ext}}| = |\vec{r} \times M\vec{g}\,| = rMg\sin(180° - \theta) = rMg\sin\theta$, then Ω becomes:

$$\Omega = \frac{Mgr}{L} \tag{9.18}$$

Using Eq. 9.12, we can write $L = I\omega$, where I and ω are the moment of inertia and angular speed of the spinning top about its axis of symmetry. Then the top's precessional angular speed becomes:

$$\boxed{\Omega = \frac{Mgr}{I\omega}} \tag{9.19}$$

This relation is valid only when $\Omega \ll \omega$, and this condition is satisfied if ω is large. If this condition is not fulfilled, the motion of the top becomes much more complicated.

Using $\Omega = 2\pi/T_p$ and $\omega = 2\pi/T_s$, where T_p is the precession period and T_s is the spinning period, we find that the period of precession T_p is given by:

$$\boxed{T_p = \frac{4\pi^2 I}{MgrT_s}} \tag{9.20}$$

In fact, Eqs. 9.19 and 9.20 also apply to gyroscopes. A gyroscope is a device for measuring or maintaining orientation, based on the concept of conservation of angular momentum.

The toy gyroscope shown in Fig. 9.14 has one end of its axle resting on a support (assumed to be a frictionless pivot), while the other end is free and precessing horizontally with angular speed Ω. A symmetric wheel attached to this axle spins rapidly about its axis with a large angular speed ω (like the top of Fig. 9.13).

Based on our findings for the spinning top, let us analyze the behavior of the toy gyroscope of Fig. 9.14. The wheel is rotating about its axis of symmetry with an angular speed ω and has an initial angular momentum $\vec{L}$ along the x-axis. Since $\vec{\tau}_{\text{ext}}$ and $d\vec{L}$ are along the y-axis and perpendicular to $\vec{L}$, this causes the direction of $\vec{L}$ to change, but not its magnitude. Therefore, the changes $d\vec{L}$ are always in the horizontal xy-plane. Consequently, the angular momentum $\vec{L}$ and the wheel axis with which the wheel moves are always horizontal. This means that the axis of the wheel does not fall, but will precess with the angular speed Ω given by Eq. 9.19.

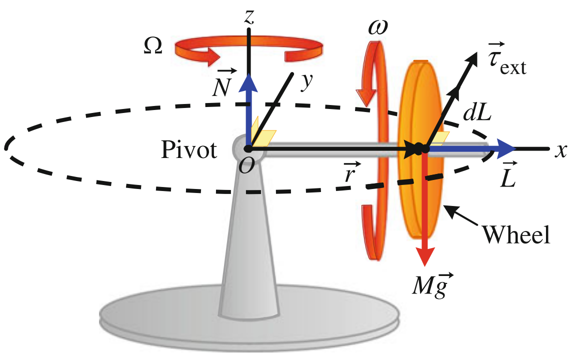


Fig. 9.14 A toy gyroscope is a wheel rotating with an angular speed ω about an axis supported at one end while the other is free. During time dt, the torque $\vec{\tau}_{ext}$ and the change in angular momentum $d\vec{L}$ are perpendicular to $\vec{L}$, which rotates in the xy-plane with a precessional angular speed Ω

Example 9.11

Assume that the cylindrical wheel of the gyroscope of Fig. 9.14 has a radius $R = 4$ cm and a center of mass located 3 cm from the pivot O. If the gyroscope takes 5 s for completing one revolution of precession, what is the spinning angular speed of the wheel and its period?

Solution: The precessional angular speed about the z-axis is:

$$\Omega = \frac{1\text{ rev}}{5\text{ s}} = \frac{2\pi\text{ rad}}{5\text{ s}} = 0.4\,\pi\text{ rad/s} = 1.257\text{ rad/s}$$

The moment of inertia of a cylindrical wheel about its axis of symmetry is $I = \frac{1}{2}MR^2$ and its weight is Mg. From Eq. 9.19:

$$\omega = \frac{Mgr}{I\Omega} = \frac{Mgr}{(\frac{1}{2}MR^2)\Omega} = \frac{2gr}{R^2\Omega}$$
$$= \frac{2(9.8\text{ m/s}^2)(0.03\text{ m})}{(0.04\text{ m})^2(1.257\text{ rad/s})} = 292.4\text{ rad/s} = 46.5\text{ rev/s}$$

Thus, the spinning period is: $T_s = \dfrac{2\pi}{\omega} = \dfrac{2\pi\text{ rad}}{292.4\text{ rad/s}} = 2.15 \times 10^{-2}\text{ s}$

9.4 Exercises

Section 9.1 Angular Momentum of Rotating Systems

(1) Calculate the angular momentum of a particle of mass $m = 2\,\text{kg}$ that has a velocity $\vec{v} = (2\vec{i} + 3\vec{j})$ m/s when its position vector is $\vec{r} = (3\vec{i} - 4\vec{j})$ m.

(2) Two cars, each having a mass $m = 1{,}500\,\text{kg}$, are moving in a horizontal circle of radius $r = 10\,\text{m}$ with the same speed $v = 10\,\text{m/s}$. The circle is centered at the origin O in the xy-plane, and the positive z-axis is directed upwards. If one of them is moving clockwise and the other counterclockwise, see Fig. 9.15, find the angular momentum of each car about O.

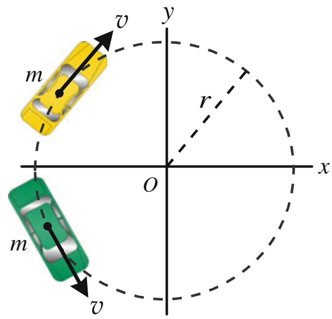


Fig. 9.15 See Exercise (2)

(3) A particle of mass $m = 2\,\text{kg}$ has a position vector that depends on time t and is given by $\vec{r} = (3t\vec{i} - 4t^2\vec{j})$ m. Find the angular momentum of the particle as a function of time.

(4) A particle of mass m is moving horizontally with constant velocity $\vec{v}$ as shown in Fig. 9.16. Find the magnitude and direction of the angular momentum of the particle, $\vec{L}_i$, $(i = 1, 2, \ldots, 8)$, respectively about the eight points O_i, $(i = 1, 2, \ldots, 8)$.

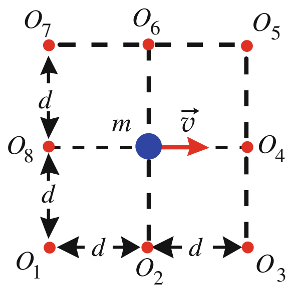


Fig. 9.16 See Exercise (4)

(5) A ball of mass $m = 0.5\,\text{kg}$ is moving horizontally with a speed $v = 10\,\text{m/s}$ at the instant when its position is identified in Fig. 9.17. (a) What is the angular momentum of the ball about O at this instant? (b) Neglecting air resistance, find the rate of change of its angular momentum about O at this instant.

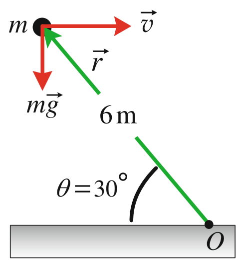


Fig. 9.17 See Exercise (5)

(6) By definition, kinetic energy $K = \frac{1}{2}mv^2$, where m and v are the mass and speed of a particle, respectively. Show that the kinetic energy of a particle moving in a circular path is $K = L^2/2I$, where L and I are, respectively, the angular momentum and moment of inertia of the particle about the center of the circle.

(7) A canonical pendulum consists of a bob of mass m attached to the end of a cord of length ℓ. The bob whirls around in a horizontal circle of radius r at a constant speed v while the cord always makes an angle θ with the vertical, see Fig. 9.18. Show that the magnitude of the angular momentum of the bob about its point of support O is given by:

$$L = \sqrt{m^2 g \ell^3 \sin\theta \tan\theta}$$

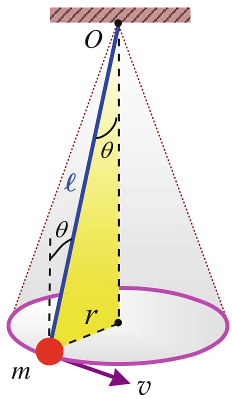


Fig. 9.18 See Exercise (7)

(8) Two identical particles 1 and 2 have respective position vectors $\vec{r}_1$ and $\vec{r}_2$ with respect to an arbitrary origin O. The two particles have equal and opposite linear momenta $\vec{p}$ and $-\vec{p}$ as shown in Fig. 9.19. Show that the total angular momentum of this system is independent of the choice of the origin and independent of where the traveling particles are located.

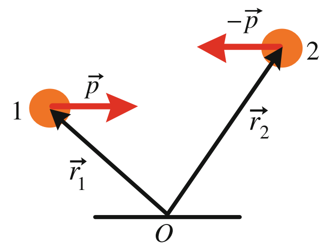


Fig. 9.19 See Exercise (8)

(9) Two particles of masses $m_1 = 2$ kg and $m_2 = 3$ kg are joined by a rod of mass $M = 0.5$ kg and length $d = 0.75$ m. The assembly rotates freely in the xy-plane about a pivot through the center of the rod, as shown in Fig. 9.20. Find the angular momentum of the system when the speed of each particle is $v = 6$ m/s.

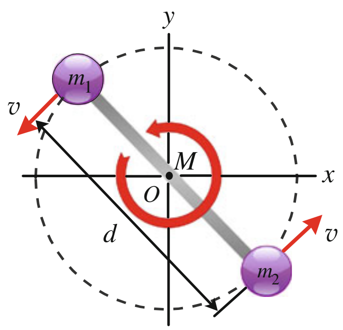


Fig. 9.20 See Exercise (9)

(10) Consider the seconds hand of a particular analog clock to be a thin rod of length $\ell = 14$ cm and mass $m = 5$ g, see Fig. 9.21. (a) If the seconds hand rotates constantly, what is its angular speed? (b) Find the magnitude of the angular momentum of the seconds hand about an axis perpendicular to the center of the clock's face.

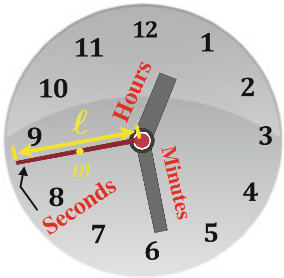


Fig. 9.21 See Exercise (10)

(11) Three identical particles, each of mass $m = 0.5$ kg, are attached at equal distances from one end of a rod of length $\ell = 2$ m and mass $M = 3$ kg, see Fig. 9.22. The system is rotating with angular speed $\omega = 2$ rad/s about an axis perpendicular to the rod through the free end at O. (a) What is the moment of inertia of the system about O? (b) What is the angular momentum of the system about O?

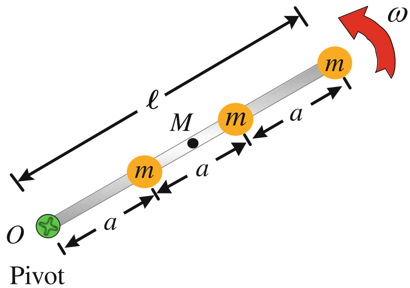


Fig. 9.22 See Exercise (11)

(12) Each of two identical particles of mass $m = 0.5$ kg attached at one end of two identical rods each of length $a = 0.2$ m and mass $M = 0.3$ kg. The other ends of the two rods are mounted perpendicular to a lightweight axle such that the distance between the rods is $d = 0.6$ m, see Fig. 9.23. The axle rotates at $\omega = 4$ rad/s. (a) What is the total angular momentum of the two particles about the CM of the system? (b) What is the total angular momentum of the two rods about the axle? (c) What angle does the total angular momentum of the whole system make with the axle?

(13) Three identical thin rods, each of mass m and length R, are fastened together to form the letter H. A circular hoop, of mass m and radius R, is fastened to the rods to form the rigid structure shown in Fig. 9.24. The rigid structure rotates

with a constant angular speed about a vertical axis with a period of rotation T. (a) Find an expression for the structure's moment of inertia and angular momentum about the axis of rotation. (b) Evaluate the two expressions of part (a) when $m = 0.5$ kg, $R = 0.1$ m, and $T = 2$ s.

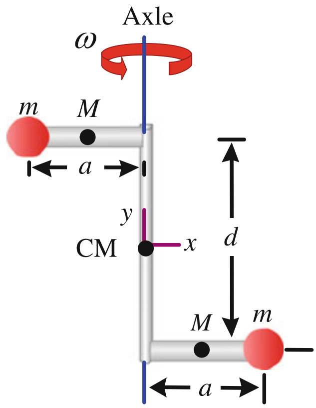


Fig. 9.23 See Exercise (12)

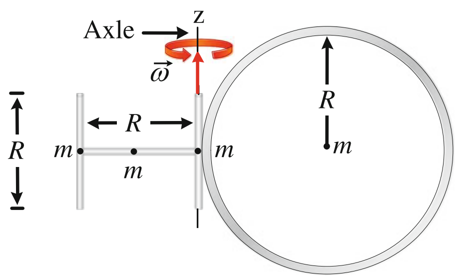


Fig. 9.24 See Exercise (13)

(14) A block of mass m_1 located on a smooth horizontal surface is connected by a light non-stretchable cord that passes over a pulley to a second block of mass m_2, which hangs vertically, see Fig. 9.25. The pulley is a uniform cylinder of mass M and radius R, and it rotates freely about its axle. (a) Find an expression for the net external torque about the pulley's axle. (b) Find an expression for the net angular momentum about the pulley's axle. (c) Find an expression for the magnitude of the acceleration of the two blocks and its value if $m_1 = 6$ kg, $m_2 = 3$ kg, $M = 2$ kg, $R = 0.1$ m, and $g = 10$ m/s^2.

(15) A disk has a moment of inertia $I = 2$ kg.m^2 about its axis of symmetry. The angular speed of the disk depends on the time t by $\omega = (12\text{ rad/s}^3)\, t^2$. (a) Find

the angular acceleration α and the angular momentum L of the disk as a function of time, and find their values at $t = 2$ s. (b) Show that using the expressions for α and L leads to the same expression for the net torque on the disk as a function of time, and find its value at $t = 2$ s.

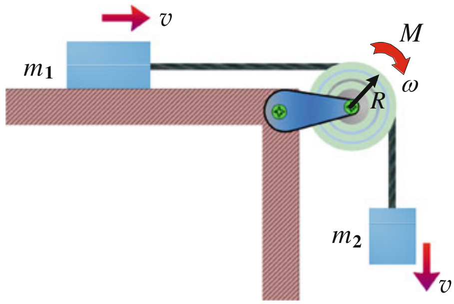


Fig. 9.25 See Exercise (14)

(16) A uniform solid sphere of mass $M = 10$ kg and radius $R = 10$ cm turns counterclockwise with an angular speed $\omega = 5$ rad/s about a vertical axis that touches its surface, see Fig. 9.26. What is the magnitude and direction of its angular momentum about this axis?

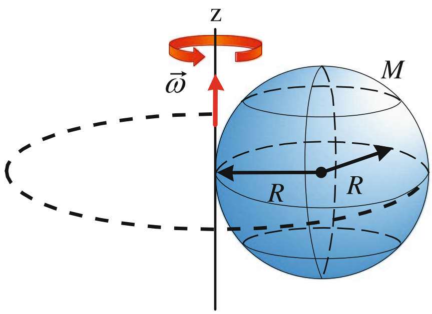


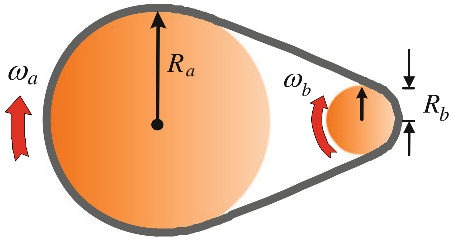
Fig. 9.26 See Exercise (16)

(17) A boy of mass $m = 40$ kg is standing on the rim of a merry-go-round that is rotating with angular speed $\omega = 0.5$ rev/s about an axis through its center. The merry-go-round is a uniform disk of mass $M = 120$ kg and radius $R = 3.5$ m. Find the total angular momentum of the boy-disk system by treating the boy as a point.

(18) Two wheels of radii R_a and R_b are connected by a non-stretchable belt that does not slip on their circumferences, see Fig. 9.27. The radius R_a is four times

the radius R_b. Find the ratio of the moment of inertia I_a/I_b and mass M_a/M_b if both wheels have: (a) the same angular momentum about their central axis, and (b) the same rotational kinetic energy.

Fig. 9.27 See Exercise (18)



(19) If an impulsive force $F(t)$ with moment arm R acts on a rigid body of moment of inertia I for a short time Δt, then show that the angular speed of the body will change from an initial value ω_i to a final value ω_f according to the *angular impulse* formula:

$$J_R = \int \tau\, dt = \bar{F} R \Delta t = I(\omega_f - \omega_i)$$

where $\bar{F}$ is the average value of the force during the time it acts on the body. [Hint: It is the rotational analogy of Eq. 7.9].

(20) A wheel of radius R_a and moment of inertia I_a is rotating about its central axle with angular speed ω_a. Another small wheel is stationary and has a radius R_b and moment of inertia I_b about its central axle. The smaller wheel is moved until it touches the larger wheel and rotates due to the friction between them, as in the upper part of Fig. 9.28. After the initial slipping period is over, the two wheels rotate at constant angular speeds ω'_a and ω'_b, see the lower part of Fig. 9.28. By applying the angular impulse relationship of Exercise 19, find the final angular speed ω'_b of the small wheel.

(21) A block of mass m_1 located on a rough horizontal surface is connected by a light non-stretchable cord that passes over a pulley to a second block of mass m_2, which is allowed to move on a rough inclined plane of angle θ, as shown in Fig. 9.29. The pulley is a uniform cylinder of mass M and radius R, and rotates freely about its axle. The coefficients of kinetic friction for the two blocks on the horizontal and inclined planes are $\mu_{k1} = 0.35$ and $\mu_{k2} = 0.5$, respectively. (a) Draw free-body diagrams of the two blocks and the pulley. (b) Find the acceleration of the two blocks and the tensions in the two sections

of the cord when $m_1 = 2\,\text{kg}$, $m_2 = 5\,\text{kg}$, $M = 10\,\text{kg}$, $R = 0.1\,\text{m}$, $\sin\theta = 4/5$, $\cos\theta = 3/5$, and $g = 10\,\text{m/s}^2$. (c) If the system starts from rest, find the angular momentum of the pulley about its axis as a function of time.

Fig. 9.28 See Exercise (20)

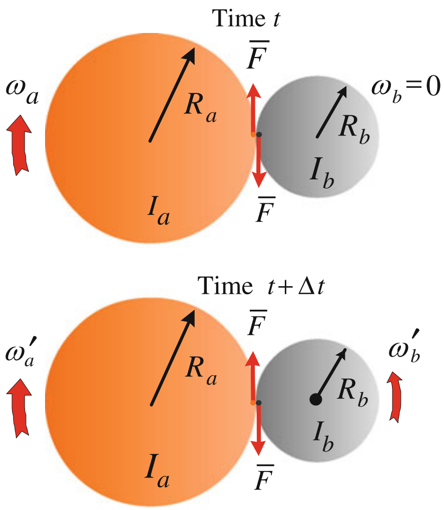
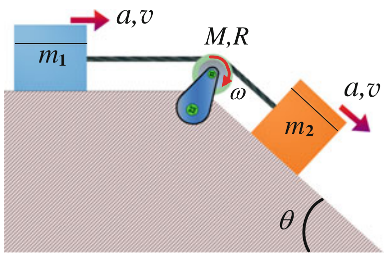


Fig. 9.29 See Exercise (21)



(22) Determine the angular momentum of the Earth: (a) about its rotational axis (assume that Earth is a uniform sphere of mass $M = 6.0 \times 10^{24}$ kg and radius $R = 6.4 \times 10^6$ m), and (b) about the Sun (assume Earth to be a particle at 1.5×10^{11} m from the Sun).

(23) Two blocks having masses m_1 and m_2 ($m_2 > m_1$) are connected to each other by a light non-stretchable cord that passes over two identical pulleys; each pulley is a uniform cylinder with a mass M and radius R, which rotates freely about its axle, as shown in Fig. 9.30. Assume no slipping happens between the cord and

the pulleys. (a) Find an expression for the net external torque of each pulley about its axle; then find the total net external torque of the system. (b) Find an expression for the net angular momentum of each pulley about its axle; and then find the total net angular momentum of the system. (c) Apply $\Sigma\tau_{\text{ext}} = dL/dt$ onto the whole system to find the acceleration of each block and the tensions T_1, T_2, and T_3 in the cord.

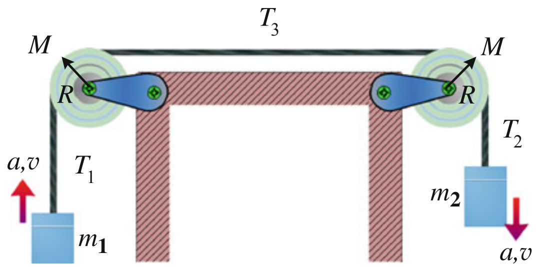


Fig. 9.30 See Exercise (23)

Section 9.2 Conservation of Angular Momentum

(24) A person is rotating on a frictionless surface at a rate of 1.5 rev/s with his arms at his sides. When he raises his arms to the horizontal position, the speed of rotation decreases to a rate of 0.75 rev/s. What is the percentage increase in moment of inertia of the person?

(25) A skater has a moment of inertia 4.5 kg.m^2 when rotating on a frictionless surface at a rate of 1 rev/s. What is her final moment of inertia if she increases her spin to the maximum value of 2.5 rev/s? How can she accomplish this change?

(26) A diver pushes a swimming pool board to jump into the air and acquires an initial angular momentum about her center of mass. Then she curls her body about her center of mass (by tucking in her arms and legs) to reduce her moment of inertia by a factor of 3.25. If she is able to make 3 revolutions in 2.25 s while she is in that tucked position, what was her initial angular speed?

(27) A uniform horizontal rod of mass M and length d rotates initially with angular speed ω_i about a vertical frictionless axle running through its center. Then two stationary small balls of clay, each of mass m, are made to stick to each end of the rod. What is the final angular speed of the system?

(28) A merry-go-round of radius $R = 2.5$ m and moment of inertia $I = 300$ kg.m^2 is rotating at 10 rev/min. A boy of mass $m = 40$ kg jumps onto the merry-go-round and manages to sit down quickly on its rim. What is the final angular speed of the system?

(29) A merry-go-round of a mass $M = 210$ kg and radius $R = 5.5$ m is mounted on a frictionless bearing. While a man of mass $m = 90$ kg is standing on its outer edge, the system is rotating with an angular speed $\omega_i = 0.2$ rev/s. Then, slowly, the man walks 3 m towards the center of the merry-go-round and stops. How fast will the merry-go-round be rotating after he stops?

(30) Rather than walking inwards, suppose the man in Exercise 29 decided to jump radially outwards relative to the merry-go-round. What will be the angular speed of the merry-go-round?

(31) A boy of mass $m = 30$ kg stands on the edge of a stationary small merry-go-round of moment of inertia $I_m = 150$ kg.m^2 and radius $R = 2$ m. The merry-go-round can rotate freely without friction about its axis. The boy jumps off the merry-go-round in a tangential direction with a linear speed $v = 2$ m/s. What is the angular speed of the merry-go-round after the boy leaves it?

(32) A person of mass $m = 80$ kg (treated as a point) stands at the center of a freely rotating cylindrical platform of a mass $M = 120$ kg and radius $R = 4$ m. The platform is mounted on a frictionless bearing and rotates with an angular speed $\omega_i = 1.5$ rad/s. The person walks radially and slowly to the edge of the platform and stops. (a) What is the final angular speed of the system? (b) Find the initial and final total rotational energy of the system.

(33) A uniform disk of radius R and a uniform rod of length $2R$ have the same mass M. The disk is rotating freely without friction about its axle with angular speed $\omega_i = 3$ rev/s while the rod is at rest and has its center coinciding with the disk's axle, see Fig. 9.31. The rod is dropped onto the disk and sticks to it such that their centers coincide (i.e. the collision is completely inelastic). What is the final angular speed of the system?

(34) Two disks have a common frictionless axle and moments of inertia $I_1 = 5$ kg.m^2 and $I_2 = 10$ kg.m^2. Initially, disk I_1 is rotating with an angular speed $\omega_i = 6$ rev/s about the axle, while disk I_2 is not rotating. Disk I_2 then drops onto disk I_1, see Fig. 9.32. Due to the friction between their surfaces, the two disks eventually reach the same angular speed ω_f. (a) Find the final angular frequency ω_f. (b) Find the percentage decrease in the rotational kinetic energy.

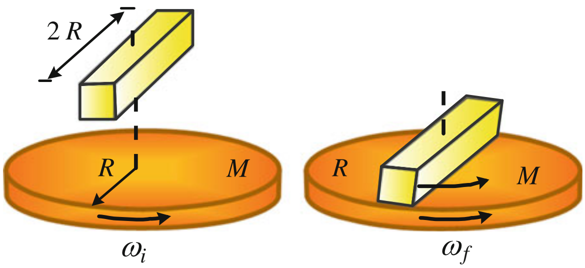


Fig. 9.31 See Exercise (33)

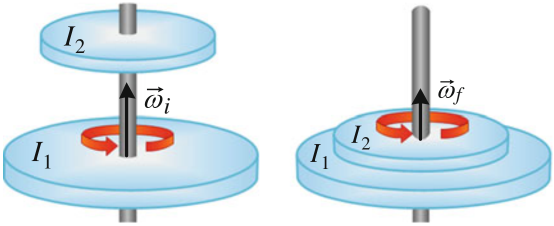


Fig. 9.32 See Exercise (34)

(35) An asteroid of mass $m = 10^6$ kg is traveling at a speed of $v = 4 \times 10^4$ m/s relative to the Earth. The asteroid hits the Earth tangentially at the equator in the direction opposite to its rotation and gets embedded at its surface, see Fig. 9.33. Assume that Earth is a uniform sphere of mass $M = 6.0 \times 10^{24}$ kg and radius $R = 6.4 \times 10^6$ m. Since no net external forces acting on this system can produce a torque about the axis of the Earth (all forces and torques are internal), then the angular momentum of the system is conserved about the axis of the Earth. As a result of this collision, find the percentage change in the Earth's angular speed.

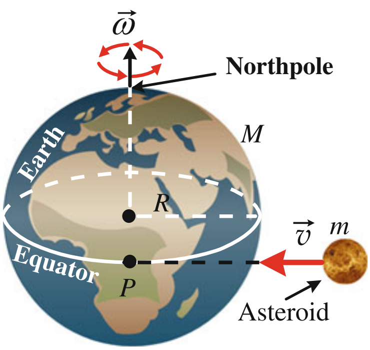
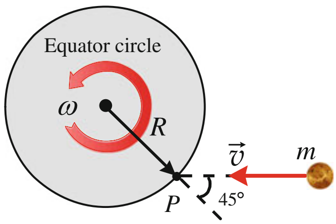


Fig. 9.33 See Exercise (35)

(36) Suppose the asteroid in Exercise 35 hits the equator circle with an incident angle $\theta = 45°$, see Fig. 9.34. By what factor does this completely inelastic collision affect the angular speed of the Earth?

Fig. 9.34 See Exercise (36)



(37) A thin vertical rod of mass M and length d can rotate about a frictionless pivot at its upper end, see Fig. 9.35. A small clay ball of mass m traveling horizontally with a speed v hits the rod at its center and sticks to it. (a) Find the angular speed of the system just after the collision. (b) How high does the lower end rise?

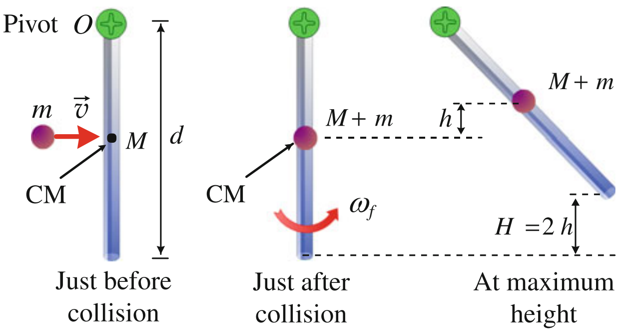


Fig. 9.35 See Exercise (37)

(38) A stationary thin horizontal rod of mass M and length d can rotate about a frictionless vertical axle through its center at O, see Fig. 9.36. A small clay ball of mass m traveling horizontally with a speed v hits the rod at one of its ends and sticks to it. (a) Find the angular speed of the system just after the collision. (b) What is the fractional loss in mechanical energy due to the collision?

(39) A stationary horizontal wooden stick of length $d = 75$ cm and mass $M = 0.4$ kg can rotate about a frictionless vertical axle through its center at O, see Fig. 9.37.

A bullet of mass $m = 5 \times 10^{-3}$ kg and horizontal speed $v_i = 200$ m/s is shot into the stick midway between the axle and one end. The bullet penetrates the stick in a very short time and leaves with a speed $v_f = 100$ m/s. (a) Find the angular speed of the stick after the collision. (b) Find the percentage decrease in total energy.

Fig. 9.36 See Exercise (38)

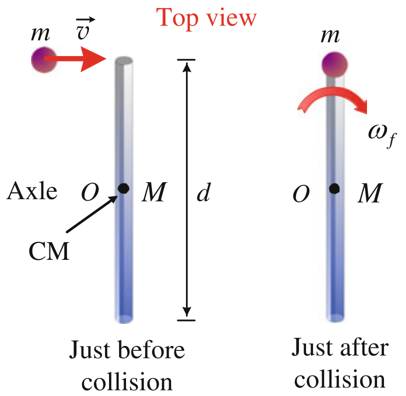
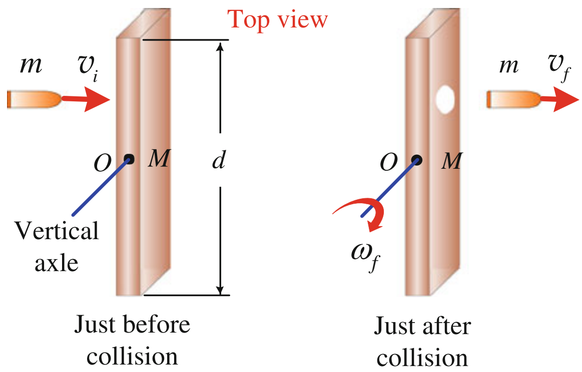


Fig. 9.37 See Exercise (39)

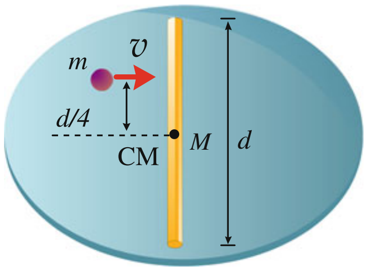


(40) A stationary thin rod of mass M and length d rests on a frictional table. A small clay ball of mass m traveling horizontally with a speed v hits the rod perpendicularly at a point $d/4$ from its center and sticks to it, see Fig. 9.38. Determine the translational and rotational motion of the rod after the collision.

(41) A student stands at the center of a turntable with his arms outstretched. In each hand, he holds a 10 kg-dumbbell at 1 m from the axis of the turntable. The turntable is rotating about a vertical frictionless axle with angular speed $\omega_i = 0.5$ rev/s. (a) Find his final angular speed if he pulls each dumbbell to

his stomach at 0.2 m from the axis of the turntable. The moment of inertia of the student with his arms outstretched is 4 kg.m^2, but it is 3.2 kg.m^2 with his hands at his stomach. (b) Find the initial and final kinetic energy of the system. Explain the meaning if they are different.

Fig. 9.38 See Exercise (40)



Section 9.3 The Spinning Top and Gyroscope

(42) To form a top, a uniform disk of mass $M = 50$ g and radius $R = 2$ cm is rigidly attached to an axial rod of negligible mass. The top spins on a frictionless surface about its axis of symmetry with angular speed $\omega = 6{,}000$ rev/min. How much work was done to get the top to spin at that rate?

(43) The center of the disk in exercise 42 is at $r = 3$ cm from the tip of the top at the surface of contact to the disk. What is the angular speed of precession of the top about the vertical axis?

(44) To form a toy gyroscope, a disk of mass $M = 150$ g and radius $R = 6$ cm is mounted at the center of a thin axle of 20 cm length. The disk spins at $\omega = 50$ rev/s when one end of the axle rests on a stand and the other end precesses horizontally. What is the angular speed of precession of the top about the vertical axis?

(45) A top of mass $M = 200$ g spins about its axis of symmetry with angular speed $\omega = 18$ rev/s and makes an angle $\theta = 25°$ with the vertical. It experiences precession at a rate of 1 rev every 5 s. The center of mass of the top is 4 cm from its tip. (a) What is the moment of inertia of the top? (b) Find the torque on the top.

Mechanical Properties of Matter

10

The physical states of matter can generally be divided into three broad classes: solids, liquids, and gases, see Fig. 10.1. A solid maintains its shape: it resists the action of external forces that tend to change its shape or volume. Liquids and gases are fluids. A fluid can easily change shape, and flows when subjected to a force. The three states of matter are distinguishable at the microscopic level as follows:

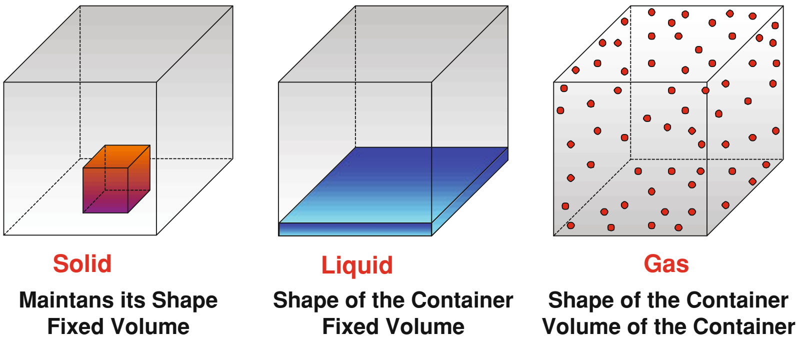


Fig. 10.1 The three states of matter: solid, liquid, and gas

1. A solid is a highly ordered array of atoms or molecules that are bound securely by mutual electrical forces.
2. A liquid is a crowded assembly of mobile atoms or molecules. Each atom or molecule is in contact with several neighbors, but is not bound securely to any of them. As an atom or molecule moves about in a liquid, it collides frequently with its neighbors.

H. A. Radi and J. O. Rasmussen, *Principles of Physics*,
Undergraduate Lecture Notes in Physics, DOI: 10.1007/978-3-642-23026-4_10,

3. A gas consists of atoms or molecules that are far apart and consequently move independently, with no forces keeping them together or pushing them apart. Collisions of atoms or molecules in gases are infrequent in comparison to those in liquids.

10.1 Density and Relative Density

The density (or mass density) of a material is defined as the mass per unit volume. If a mass m is distributed uniformly over a volume V, the density will be given by the following equation:

$$\rho = \frac{m}{V} \tag{10.1}$$

The SI unit of density is kg/m^3. If the mass is not uniformly distributed, then Eq. 10.1 defines the average density. The densities of several materials are listed in Table 10.1.

Table 10.1 Density and relative density comparison (approximates)

Material type	Material name	Density (kg/m^3)	Relative density
Gas	Helium	0.179	1.79×10^{-4}
	Air	1.29	1.29×10^{-3}
	Carbon dioxide	1.98	1.98×10^{-3}
Liquid	Alcohol	7.9×10^2	0.79
	Gasoline	8.6×10^2	0.86
	Water	1×10^3	1
	Mercury	13.6×10^3	13.6
Solid	Glass (common)	$2.4 - 2.8 \times 10^3$	2.5
	Aluminum	2.7×10^3	2.7
	Iron	7.86×10^3	7.86
	Copper	8.92×10^3	8.92
	Silver	10.5×10^3	10.5
	Lead	11.36×10^3	11.36
	Uranium	19.07×10^3	19.07
	Gold	19.3×10^3	19.3

The relative density of a substance tells us how many times more dense the substance is than pure water, see Table 10.1. Sometimes we refer to it as the specific gravity (SG). Thus:

$$\boxed{\mathrm{SG} = \frac{\rho}{\rho_{\text{water}}}} \tag{10.2}$$

Example 10.1

Calculate the average density of both the Earth and the Sun, given that the mass and radius of the Earth are $m_E = 5.98 \times 10^{24}$ kg and $R_E = 6.37 \times 10^6$ m, respectively, and the mass and radius of the Sun are $m_S = 1.99 \times 10^{30}$ kg and $R_S = 6.95 \times 10^8$ m, respectively. Compare between the resulting average densities.

Solution: Matter in the Earth and the Sun is not uniform. In spite of this fact, we can use Eq. 10.1 to calculate their average density. Using the given radius, we calculate the volume of the Earth to be:

$$V_E = \tfrac{4}{3}\pi R_E^3 = 1.08 \times 10^{21}\ \mathrm{m}^3$$

Thus, the average density of the Earth is:

$$\rho_E = \frac{m_E}{V_E} = \frac{5.98 \times 10^{24}\ \mathrm{kg}}{1.08 \times 10^{21}\ \mathrm{m}^3} = 5.54 \times 10^3\ \mathrm{kg/m}^3$$

In comparison to water, the average density of the Earth is 5.54 times more dense than water.

Similarly, we use the given radius of the Sun to calculate its volume:

$$V_S = \tfrac{4}{3}\pi R_S^3 = 1.41 \times 10^{27}\ \mathrm{m}^3$$

Thus, the average density of the Sun is:

$$\rho_S = \frac{m_S}{V_S} = \frac{1.99 \times 10^{30}\ \mathrm{kg}}{1.41 \times 10^{27}\ \mathrm{m}^3} = 1.41 \times 10^3\ \mathrm{kg/m}^3$$

In comparison to water, the average density of the Sun is 1.41 times more dense than water. Although the mass and volume of the Earth are much smaller than the mass and volume of the Sun, the average density of the Earth is nearly four times the average density of the Sun. That is:

$$\rho_E \simeq 4\rho_S$$

10.2 Elastic Properties of Solids

All solids are to some extent elastic. This means that we can change their dimensions slightly by pulling, pushing, twisting, and/or compressing them. We shall discuss the elastic properties of solids by introducing the concepts of *stress* and *strain*.

Stress:
Stress is the magnitude of the applied external force that acts perpendicularly on a unit area of the object.

Strain:
Strain is a measure of the degree of deformation of the object.

It is found that for small stresses, stress is proportional to strain. The *proportionality constant* is called the *elastic modulus* and it depends on the material being deformed, as well as on the nature of the deformation. Therefore:

$$\boxed{\text{Elastic modulus} = \frac{\text{Stress}}{\text{Strain}}} \tag{10.3}$$

This relation is equivalent to **Hooke's law** that states: Stress $\propto$ Strain. In this chapter, we introduce the three *most famous types of deformations* and their elastic moduli:

Young's Modulus:
Measures the resistance of a solid to a change in its length.

Shear Modulus:
Measures the resistance to motion of the planes of solids when sliding over each other.

Bulk Modulus:
Measures the resistance of a solid (or a liquid) to a change in its volume.

10.2.1 Young's Modulus: Elasticity in Length

Young's Modulus measures the resistance of a solid to a change in its length, which indicates its stiffness. Consider a metallic long rod of original length L and cross-sectional area A. When an external force $F_{\perp}$ is applied perpendicularly to the cross-sectional area A of a rod, its internal forces resist its distortion. As a final result, the rod attains equilibrium when its length increases to a new length $L + \Delta L$ and the magnitude of the perpendicular external force $F_{\perp}$ exactly balances the internal forces, see Fig. 10.2a. In light of this, we define the *tensile stress* and the *tensile strain* as follows:

$$\boxed{\text{Tensile stress} = \frac{F_{\perp}}{A}} \quad (\text{N/m}^2) \tag{10.4}$$

$$\boxed{\text{Tensile strain} = \frac{\Delta L}{L}} \tag{10.5}$$

The relation between the tensile stress and the tensile strain is linear when the rod is in its elastic range. When the stress exceeds what is called the *elastic limit*, the rod is permanently distorted and will not return to its original shape after the stress is removed. As the stress is increased even further, the rod will ultimately break, see Fig. 10.2b.

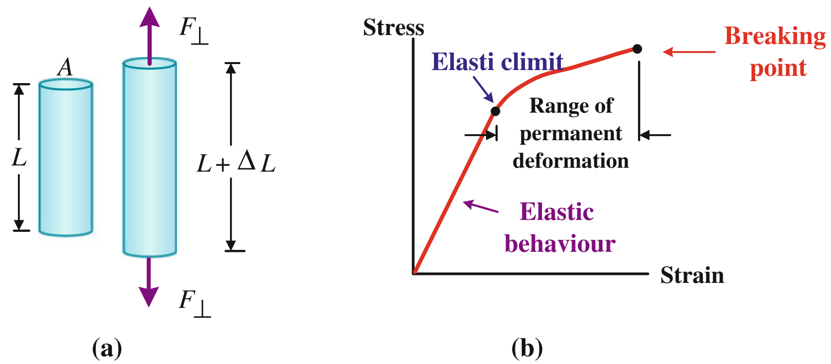


Fig. 10.2 (a) A rod of height L and cross-sectional area A. The rod stretches by an amount ΔL after application of a tensile stress. (b) The stress versus strain curve for an elastic solid

We use Eqs. 10.4 and 10.5 to define *Young's modulus*, Y, as:

$$Y = \frac{\text{Tensile stress}}{\text{Tensile strain}} = \frac{F_\perp/A}{\Delta L/L} \quad (\text{N/m}^2) \tag{10.6}$$

This quantity is used to characterize a solid that is stressed under either tension or compression. Table 10.2 depicts some values for Y.

Table 10.2 Young's modulus for different materials (approximates)

Material name	Young's modulus $Y \times (10^9\ \text{N/m}^2)$
Rubber	0.004
Lead	16
Glass	65–78
Aluminum	70
Brass	91
Copper	110
Steel	200
Tungsten	350

Example 10.2

A pendulum consists of a big sphere of mass $m = 30$ kg hung from the end of a steel wire that has a length $L = 15$ m, a cross-sectional area $A = 9 \times 10^{-6}$ m^2, and Young's modulus $Y = 200 \times 10^9$ N/m^2. Find the tensile stress on the wire and the increase in its length.

Solution: The applied force on the wire must equal to the weight of the sphere, i.e. $F_\perp = m\,g$. Thus, the tensile stress will be:

$$\text{Tensile stress} = \frac{F_\perp}{A} = \frac{m\,g}{A} = \frac{(30\ \text{kg}) \times (9.8\ \text{m/s}^2)}{9 \times 10^{-6}\ \text{m}^2} = 3.27 \times 10^7\ \text{N/m}^2$$

Using the value of the Young's modulus $Y = 20 \times 10^{10}$ N/m^2 and the length of the steel wire before stretching $L = 15$ m, we get:

$$Y = \frac{F_\perp/A}{\Delta L/L} \quad \Rightarrow \quad \Delta L = \frac{F_\perp/A}{Y}L = \frac{3.27 \times 10^7\ \text{N/m}^2}{200 \times 10^9\ \text{N/m}^2} \times 15\ \text{m} = 2.45 \times 10^{-3}\ \text{m}$$

Note that this large stress produces a relatively small change in L.

When we carefully study the deformation of the rod, we find that the rod's length L increases by ΔL in the direction of the force while it radius r decreases by $|\Delta r|$, where Δr is negative, in a direction perpendicular to the force, see Fig. 10.3. The tensile strain $\Delta L/L$ of the rod is called the linear strain. The strain $-\Delta r/r$ is called the lateral strain, and Poisson's ratio μ is defined as:

$$\boxed{\mu = \frac{\text{Lateral strain}}{\text{Linear strain}} = -\frac{\Delta r/r}{\Delta L/L} = -\frac{L}{r}\frac{\Delta r}{\Delta L}} \Rightarrow \boxed{\mu = -\frac{L}{r}\frac{dr}{dL}} \qquad (10.7)$$

The minus sign is inserted in this definition to make μ positive.

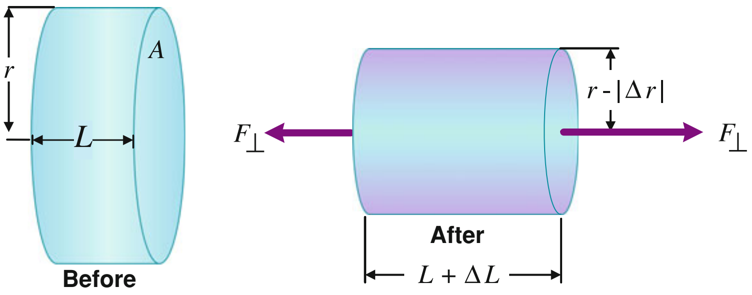


Fig. 10.3 The length of the rod will increase by ΔL and its radius will decrease by Δr (exaggerated scale) after applying a tensile stress $F_\perp/A$

Example 10.3

A cylindrical steel rod has a length of 2 m and a radius of 0.5 cm. A force of magnitude 2×10^4 N is acting normally on each of its ends. Find the change in its length and radius, if the Young's modulus Y is 200×10^9 N/m^2 and the Poisson's ratio μ is 0.25.

Solution: Using $F_\perp = 2 \times 10^4$ N, $A = \pi r^2 = \pi \times (0.5 \times 10^{-2}\text{ m})^2 = 7.9 \times 10^{-5}$ m^2, $L = 2$ m, and $Y = 200 \times 10^9$ N/m^2 in Eq. 10.6, we have:

$$\Delta L = \frac{F_\perp L}{YA} = \frac{(2 \times 10^4\text{ N})(2\text{ m})}{(200 \times 10^9\text{ N/m}^2)(7.9 \times 10^{-5})} = 2.5 \times 10^{-3}\text{ m} = 0.25\text{ cm}$$

From the definition of the Poisson's ratio μ, we have:

$$\Delta r = -\frac{\mu r \Delta L}{L} = -\frac{0.25 \times (0.5 \times 10^{-2}\text{ m})(2.5 \times 10^{-3}\text{ m})}{(2\text{ m})} = -1.56 \times 10^{-6}\text{ m}$$

Note that $\Delta L \approx 1{,}600\,|\Delta r|$, i.e. $|\Delta r|$ is extremely small compared to ΔL.

10.2.2 Shear Modulus: Elasticity of Shape

Another type of deformation occurs when a solid is subject to a force applied parallel to one of its surfaces while the opposite surface is kept fixed. Figure 10.4 shows a cylindrical rod subjected to a linear or torsional shear stress deforming it by an amount Δx due to a force $F_{\|}$ parallel to the surface area A. As a final result, the shape of the rod will attain equilibrium when the effect of the shear force $F_{\|}$ balances exactly the internal shear forces. For linear shearing, we define the *shearing stress* and the *shearing strain* as follows:

$$\boxed{\text{Shearing stress} = \frac{\text{Tangential acting force}}{\text{Area of surface being sheared}} = \frac{F_{\|}}{A}} \quad (\text{N/m}^2) \qquad (10.8)$$

$$\boxed{\text{Shearing strain} = \frac{\text{Distance sheared}}{\text{Distance between surfaces}} = \frac{\Delta x}{h} = \tan\theta \simeq \theta} \qquad (10.9)$$

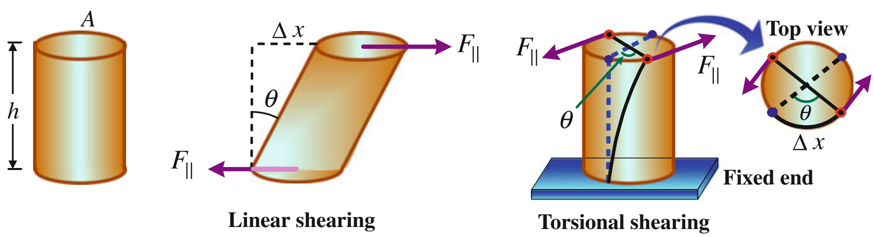


Fig. 10.4 The *left* part shows a cylindrical rod of height h. The *middle* part shows a linear shear where the rod is subject to a shearing force $F_{\|}$ parallel to each of its surface areas. The rod is deformed through an angle θ which is defined as the shearing strain. The *right* part shows a torsional shear when one end of the rod is kept fixed

The approximation $\tan\theta \simeq \theta$ is valid for small strains. We use Eqs. 10.8 and 10.9 to define the *shear modulus*, S, as follows:

$$\boxed{S = \frac{\text{Shearing stress}}{\text{Shearing strain}} = \frac{F_{\|}/A}{\Delta x/h}} \quad (\text{N/m}^2) \qquad (10.10)$$

S is also called the modulus of rigidity or the torsion modulus and is significant only for solids. Table 10.3 depicts some values for S.

Table 10.3 Shear modulus for different materials (approximates)

Material name	Shear modulus $S \times (10^9\ \mathrm{N/m^2})$
Rubber	0.001
Lead	6
Glass	23
Aluminum	23
Brass	36
Copper	42
Steel	80
Tungsten	120

Example 10.4

Assume that the rod in Fig. 10.4 has a cross-sectional area $A = 2 \times 10^{-3}\ \mathrm{m}^2$, length $h = 1$ m, and is made of brass with a shear modulus $S = 36 \times 10^9\ \mathrm{N/m^2}$. How large should the shear force $F_\parallel$ exerted on each edge of the rod be if the displacement Δx is 0.02 cm?

Solution: The shearing stress on each edge is:

$$\text{Shearing stress} = \frac{F_\parallel}{A} = \frac{F_\parallel}{2 \times 10^{-3}\ \mathrm{m}^2} = 500\ F_\parallel\ \mathrm{m}^{-2}$$

The shearing strain is:

$$\text{Shearing strain} = \frac{\Delta x}{h} = \frac{2 \times 10^{-4}\ \mathrm{m}}{1\ \mathrm{m}} = 2 \times 10^{-4}$$

From the definition of the shearing modulus Eq. 10.10 and the last two results, we have:

$$S = \frac{\text{Shearing stress}}{\text{Shearing strain}} = \frac{500 F_\parallel\ \mathrm{m}^{-2}}{2 \times 10^{-4}}$$

Using the given shear modulus value for brass, we get:

$$36 \times 10^9\ \mathrm{N/m^2} = \frac{500 F_\parallel\ \mathrm{m}^{-2}}{2 \times 10^{-4}}$$

Thus: $F_\parallel = (36 \times 10^9\ \mathrm{N/m^2})(2 \times 10^{-4})/(500\ \mathrm{m}^{-2}) = 14{,}400\ \mathrm{N} = 1.44 \times 10^4\ \mathrm{N}$

10.2.3 Bulk Modulus: Volume Elasticity

Another type of deformation occurs when an object is subject to an equal increase in normal forces acting on all its faces. For such a study, it is appropriate to define the pressure P as the force acting perpendicularly on a unit area of the object. That is:

$$\boxed{P = \frac{F_\perp}{A}} \quad (\mathrm{N/m^2}) \tag{10.11}$$

Hence, we can study the deformation of an object subject to an equal increase in pressure on all its faces. Figure 10.5 shows a cube of original volume V and face area A under uniform pressure P. When the force $F_\perp$ on each face increases to $F_\perp + \Delta F_\perp$, the pressure will increase to $P + \Delta P$ and consequently the volume V will decrease to $V' = V - |\Delta V|$, where ΔV is negative, see Fig. 10.5.

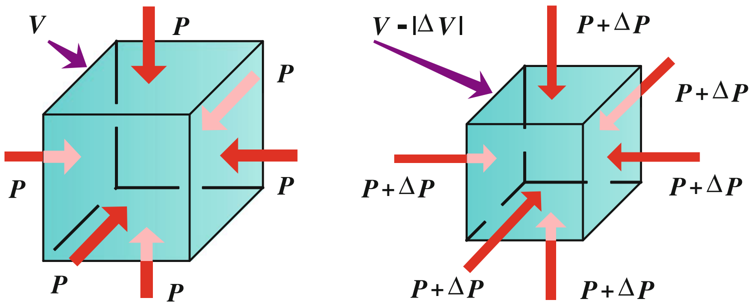


Fig. 10.5 When the uniform pressure P on each face of a cube of volume V increases to $P + \Delta P$, its volume will decrease to $V - |\Delta V|$.

In light of this, we define the *volume stress* and the *volume strain* as:

$$\boxed{\text{Volume stress} = \Delta P = \frac{\Delta F_\perp}{A}} \quad (\mathrm{N/m^2}) \tag{10.12}$$

$$\boxed{\text{Volume strain} = -\frac{\Delta V}{V}} \tag{10.13}$$

We use Eqs. 10.12 and 10.13 to define the *bulk modulus*, B, as:

$$\boxed{B = \frac{\text{Volume stress}}{\text{Volume strain}} = -\frac{\Delta F_\perp/A}{\Delta V/V} = -\frac{\Delta P}{\Delta V/V}} \Rightarrow \boxed{B = -V\frac{dP}{dV}} \tag{10.14}$$

The minus sign is inserted in Eq. 10.13 to make B a positive number, because an increase/decrease of pressure always causes a decrease/increase in volume.

Example 10.5

A sphere of lead has a volume $V = 0.5\,\text{m}^3$ when placed in atmospheric pressure ($P_a \simeq 10^5\,\text{N/m}^2$). The sphere is lowered to a particular depth in the ocean where the water pressure is $P = 10^8\,\text{N/m}^2 = 1{,}000\,P_a$. The bulk modulus B of lead is $8 \times 10^9\,\text{N/m}^2$. (a) What is the change in volume of the sphere? (b) What is the relative density change in lead?

Solution: (a) From the definition of the bulk modulus, we have:

$$B = -\frac{\Delta P}{\Delta V / V}$$

The change in pressure is:

$$\Delta P = P - P_a = 10^8\,\text{N/m}^2 - 10^5\,\text{N/m}^2 = 9.99 \times 10^7\,\text{N/m}^2$$

Using $V = 0.5\,\text{m}^3$, we can find the change in volume as follows:

$$\Delta V = -\frac{V\,\Delta P}{B} = -\frac{(0.5\,\text{m}^3)(9.99 \times 10^7\,\text{N/m}^2)}{8 \times 10^9\,\text{N/m}^2} = -6.2 \times 10^{-3}\,\text{m}^3$$

The negative sign indicates a decrease in volume.

(b) We can find the new volume V' of lead as follows:

$$V' = V - |\Delta V| = 0.5\,\text{m}^3 - |-6.2 \times 10^{-3}\,\text{m}^3| = 0.4938\,\text{m}^3 \quad \Rightarrow \quad V' = 0.9876\,V$$

If the original density of lead is denoted by $\rho = m/V$, then the new density ρ' will be:

$$\rho' = \frac{m}{V'} = \frac{m}{0.9876\,V} = 1.0126\,\frac{m}{V} = 1.0126\,\rho$$

Thus, a thousand times increase in pressure on the surfaces of a sphere of lead causes a *decrease* in its volume by about 1.3% and consequently an *increase* in density by the *same* percentage.

Table 10.4 depicts some values for the bulk modulus B.

Table 10.4 Bulk modulus for different materials (approximates)

Material name	Bulk modulus $B \times (10^9\ \text{N/m}^2)$
Rubber	3
Lead	8
Glass	37
Aluminum	70
Brass	61
Copper	140
Steel	160
Tungsten	200

10.3 Fluids

What is a Fluid?

Liquids and gases are fluids. The liquid state of any substance always exists at a higher temperature than its solid state. The reason for lumping liquids and gases together and calling them fluids is because neither liquids nor gases (such as liquid water and steam, for example) have a fairly rigid three-dimensional array of atoms/molecules as compared to solids (such as ice, for example).

In contrast to solids, fluids can flow and conform to the boundaries of any container in which they are placed. This is because a fluid cannot sustain a force that is tangent to its surface. In the language of the previous section, a fluid flows because it cannot withstand a shearing stress. On the other hand, a fluid can exert a force in a direction perpendicular to its surface.

Pressure in Fluids

Figure 10.6 shows a pressure device inside a fluid-filled vessel. The device consists of a light piston of area ΔA fitting in a vacuumed cylinder and resting on a light spring. As we insert the device into the fluid, the fluid will compress the piston due to the effect of a normal force of magnitude $\Delta F_\perp$. Using Eq. 10.11, after replacing $F_\perp$ by F, we define the average pressure exerted by the fluid on the piston by the following relation:

$$\boxed{\overline{P} = \frac{\Delta F}{\Delta A}} \quad (\text{N/m}^2) \tag{10.15}$$

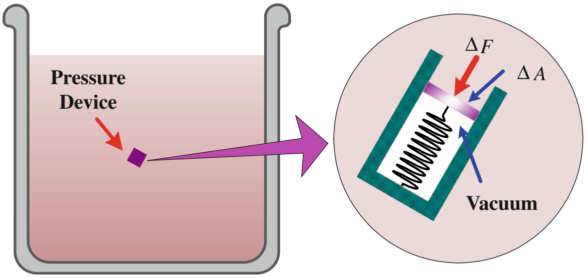


Fig. 10.6 A pressure device inside a fluid-filled vessel. The pressure is measured by the relative position of a movable piston in the device

The pressure at any point in the fluid is the limit of the above ratio as ΔA of the piston, centered on that point, approaches zero. That is:

$$\boxed{P = \lim_{\Delta A \to 0} \frac{\Delta F}{\Delta A} = \frac{dF}{dA}} \quad (\text{N/m}^2) \tag{10.16}$$

It was found by experiment that at a given point in a static fluid, the pressure P given by Eq. 10.16 has the same value no matter how the pressure device is oriented. Moreover, all points at the same depth from a liquid surface have the same value of pressure.

The SI unit of pressure is N/m^2, which is given the special name pascal (Pa). That is:

$$\boxed{1\,\text{Pa} = 1\,\text{N/m}^2} \tag{10.17}$$

Atmospheric pressure at sea level P_a (abbreviated by atm) is:

$$\boxed{1\,\text{atm} = 1.01 \times 10^5\,\text{Pa} \simeq 10^5\,\text{Pa}} \tag{10.18}$$

The pascal is also related to the torr, the bar, and the pound per square inch, which are other common (non-SI) pressure units. The torr unit (named after Evangelista Torricelli who invented the mercury barometer) was formerly called *millimeter of mercury* (mm Hg). The bar unit is usually used in meteorological sciences. Finally, the pound per square inch (lb/in^2) is often abbreviated by psi. Note that:

$$\boxed{1\,\text{atm} = \left\{\begin{array}{l} 760\,\text{torr} \\ \quad \text{or} \\ 760\,\text{mm Hg} \end{array}\right. = 14.7\,\text{psi}} \quad \text{and} \quad \boxed{1\,\text{bar} = 10^5\,\text{Pa (Exactly)}} \tag{10.19}$$

Table 10.5 depicts some approximate pressure values.

Table 10.5 Some approximate pressures

Locations	Pressure (Pa)	Pressure (atm)
Center of the Sun	2×10^{16}	2×10^{11}
Center of the Earth	4×10^{11}	4×10^{6}
Highest laboratory pressure	2×10^{10}	2×10^{5}
Deepest Ocean	1.1×10^{8}	1.1×10^{3}
Automobile tire (excess of 1 atm)	2×10^{5}	2
Atmosphere at sea level	1.0×10^{5}	1
Normal blood pressure (excess of 1 atm)[a]	0.16×10^{5}	0.16
Best laboratory vacuum	10^{-12}	10^{-17}

[a] The systolic pressure that corresponds to 120 mm Hg on the physician's pressure gauge.

To study the mechanics of fluids, we need to deal with:

1. Fluids at rest, or fluid statics (hydrostatics)
2. Fluids in motion, or fluid dynamics (hydrodynamics)

10.4 Fluid Statics

Variation of Pressure with Depth

As indicated in the previous section, all points at the same depth from a liquid surface have the same value of pressure. The variation of pressure P with depth h in a liquid of density ρ open to the atmosphere can be found by considering a small horizontal area dA at that depth, as shown in Fig. 10.7. The force dF that acts downwards on dA must be equal to the weight of the liquid column of height h plus the weight of the atmospheric air column. Accordingly, we have:

Volume of the liquid column $= h\,dA$
Mass of the liquid column $= h\,dA\rho$
Weight of the liquid column $= h\,dA\rho g$
Weight of the atmospheric air column $= P_a\,dA$
Total force dF on the horizontal area dA $= P_a\,dA + h\,dA\rho g$

Thus, from Eq. 10.16, the pressure $P = dF/dA$ at depth h gives:

$$\boxed{P = P_a + \rho g h} \quad \Rightarrow \quad \boxed{dP = P - P_a = \rho g h} \tag{10.20}$$

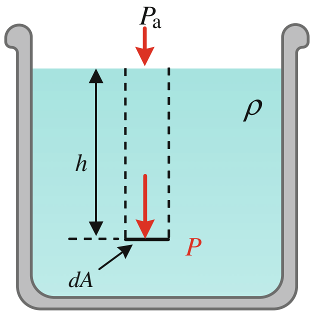


Fig. 10.7 The pressure P at a depth h below the surface of a liquid open to the atmosphere is given by $P = P_a + \rho g h$

This relation verifies that the pressure is the same at all points having the same depth from a liquid surface. Moreover, the pressure is not affected by the shape of the container, see Fig. 10.8.

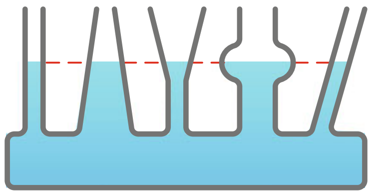


Fig. 10.8 The pressure in the liquid is the same at all points having the same depth. The shape of the vessel does not affect the pressure

Example 10.6

Find the pressure at depths of 10 m and 10 km in ocean water. Assume $P_a \equiv 1$ atm $\simeq 10^5$ Pa, $\rho \simeq 10^3\,\text{kg/m}^3$, and $g \simeq 10\,\text{m/s}^2$.

Solution: The pressure at a depth $h = 10$ m will be:

$$\begin{aligned} P &= P_a + \rho g h = 10^5\,\text{Pa} + (10^3\,\text{kg/m}^3)(10\,\text{m/s}^2)(10\,\text{m}) \\ &= 2 \times 10^5\,\text{Pa} = 2\,\text{atm} \end{aligned}$$

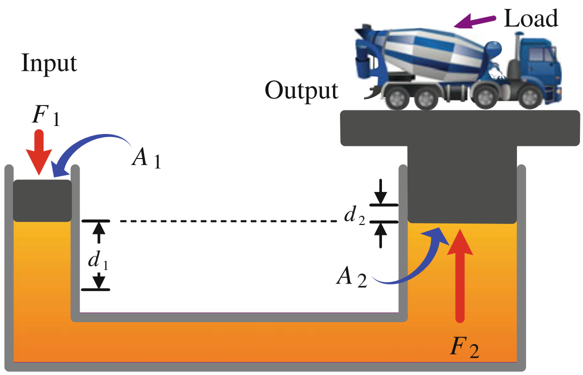
The pressure at a depth $h = 10\,\text{km} = 10^4$ m will be:

$$\begin{aligned} P &= P_a + \rho g h = 10^5\,\text{Pa} + (10^3\,\text{kg/m}^3)(10\,\text{m/s}^2)(10^4\,\text{m}) \\ &= 1{,}001 \times 10^5\,\text{Pa} \simeq 1{,}000\,\text{atm} \end{aligned}$$

The fact that the pressure in a fluid depends only on depth indicates that any increase in pressure at the liquid surface must be transmitted to every point in the liquid. This fact is known as **Pascal's law** or **Pascal's principle**.

An important application of Pascal's law is the *hydraulic lever* illustrated in Fig. 10.9. Let an external input force of magnitude F_1 be exerted downwards on a small piston of area A_1. The pressure will be transmitted through an incompressible fluid which then exerts an output force F_2 on a larger piston of area A_2, balancing the load.

Fig. 10.9 A hydraulic device used to magnify a force. However, for small strokes (small d_1), the input and output work done is the same



The pressure on both leveled pistons is the same. That is:

$$\boxed{P = \frac{F_1}{A_1} = \frac{F_2}{A_2}} \quad \Rightarrow \quad \boxed{F_2 = F_1 \frac{A_2}{A_1}} \quad \text{(Leveled pistons)} \tag{10.21}$$

Thus, the force F_2 is larger than F_1 by the multiplying factor A_2/A_1. Hydraulic brakes, car lifts, etc make use of this principle.

When we move the input piston downwards a distance d_1, the output piston moves upwards a distance d_2, such that the same volume V of the incompressible liquid is displaced at both pistons. Then, we get:

$$\boxed{V = A_1 d_1 = A_2 d_2} \quad \Rightarrow \quad \boxed{d_2 = d_1 \frac{A_1}{A_2}} \tag{10.22}$$

Thus, for $A_2 > A_1$, the output piston moves a smaller distance than the input piston. On the other hand, for *small values of* d_1, we can use Eq. 6.1 to find the following input/output relationship:

$$W_2 = F_2 d_2 = \left(F_1 \frac{A_2}{A_1}\right)\left(d_1 \frac{A_1}{A_2}\right) = F_1 d_1 = W_1 \quad \text{(for small } d_1 \text{ only)} \tag{10.23}$$

which shows that the work W_1 done on the input piston by the applied force equals the work W_2 done by the output piston in lifting the load.

Measuring Pressures

The Mercury Barometer

Figure 10.10 shows a very basic **mercury barometer** used to measure *atmospheric pressure*. Here, a long glass tube is first filled with mercury and then inverted with its open end in a container filled with mercury.

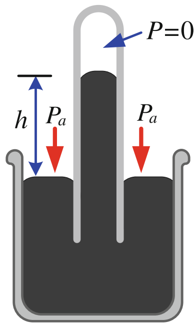


Fig. 10.10 A closed-end mercury barometer

The closed end of the tube is nearly in a state of vacuum, i.e. with $P \simeq 0$. Moreover, the pressure is the same at all points having the same horizontal level in mercury. Therefore, according to Figure 10.10, the atmospheric pressure P_a will be given by: $P_a = \rho g h$, where ρ is the density of mercury and h is the height of the mercury column.

The height of the mercury column is measured only if the barometer is set at a place where $g = 9.80665\,\text{m/s}^2$ and the temperature of the mercury is 0°C. At this temperature, the mercury has a density $\rho = 13.595 \times 10^3\,\text{kg/m}^3$ and the height of mercury is measured to be exactly 760 mm. Therefore:

$$\begin{aligned} P_a &= \rho g h \\ &= (13.595 \times 10^3\ \text{kg/m}^3)(9.80665\,\text{m/s}^2)(0.76\ \text{m}) \\ &= 1.013 \times 10^5\ \text{Pa} \end{aligned} \tag{10.24}$$

The Open-Tube Manometer

Figure 10.11 shows a very basic **open-tube manometer** used to measure the gauge *pressure of a gas*. It consists of a U-tube containing a liquid, with one end of the tube connected to a gas tank of pressure P, and the other end open to the atmosphere.

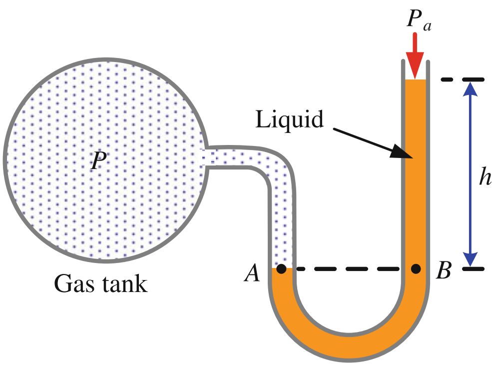


Fig. 10.11 An open-tube manometer for measuring gas pressure

From this figure, we see that the pressure at point A is the unknown pressure P of the gas in the tank. On the other hand, the pressure P_A at point A is equal to the pressure P_B at point B, which equals $P_{\mathrm{a}} + \rho g h$, where ρ is the density of the liquid and h is the height of the liquid column. Since $P_A = P_B$, we have:

$$\boxed{P = P_{\mathrm{a}} + \rho g h} \tag{10.25}$$

This relation gives us what we call the **absolute pressure** P. In general, the difference between an absolute pressure and an atmospheric pressure is called the **gauge pressure** P_g. That is:

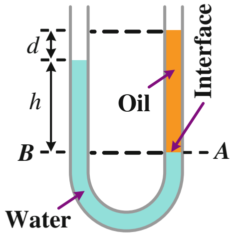
$$\boxed{P_g = P - P_{\mathrm{a}} = \rho g h} \tag{10.26}$$

The gauge pressure can be positive or negative, depending on whether $P > P_{\mathrm{a}}$ or $P < P_{\mathrm{a}}$. In inflated tires or in the human circulatory system, the absolute pressure is greater than the atmospheric pressure (i.e. $P > P_{\mathrm{a}}$), so the gauge pressure is positive (i.e. $P_g > 0$).

Example 10.7

The U shaped tube shown in Fig. 10.12 contains oil in the right arm and water in the left arm. In static equilibrium, the measurements give $h = 18$ cm and $d = 2$ cm. What is the value of the density of the oil ρ_o?

Fig. 10.12



Solution: The pressure P_A at the oil–water interface of the right arm must be equal to the pressure P_B in the left arm at the same level. In the right arm, we use Eq. 10.20 to get:

$$P_{\text{A}} = P_{\text{a}} + \rho_{\text{o}}\, g(h + d)$$

In the left arm, we use the same Eq. 10.20 to get:

$$P_B = P_{\text{a}} + \rho_{\text{w}} g h$$

Since $P_A = P_B$, we equate the last two equations to get:

$$\rho_{\text{o}} = \frac{h}{h + d}\,\rho_{\text{w}} = \frac{18\,\text{cm}}{18\,\text{cm} + 2\,\text{cm}} 10^3\ \text{kg/m}^3 = 900\,\text{kg/m}^3$$

Note that the answer does not depend on the atmospheric pressure.

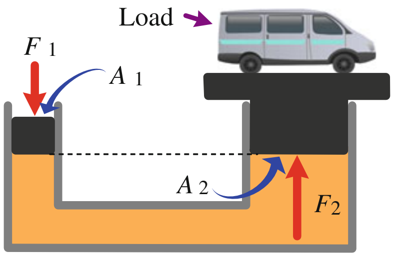
Example 10.8

For the car lift shown in the Fig. 10.13, the pistons on the left and right have areas $25\,\text{cm}^2$ and $750\,\text{cm}^2$ respectively. The car and the right piston have a total weight of 15,000 N. What force must be applied on the left piston (if it has negligible weight)? What pressure will produce this force?

Solution: From Eq. 10.21, we have:

$$F_1 = F_2 \frac{A_1}{A_2} = (15{,}000\,\text{N}) \frac{25\,\text{cm}^2}{750\,\text{cm}^2} = 500\,\text{N}$$

Fig. 10.13



The pressure that produce this force is given by:

$$P = \frac{F_1}{A_1} = \frac{500\,\text{N}}{25 \times 10^{-4}\,\text{m}^2} = 2 \times 10^5\,\text{Pa} \simeq 2\,\text{atm}$$

*** Example 10.9**

(a) A person dives to a depth $h = 50$ cm below the water surface without inhaling first. Find the pressure on his body and on his lungs. (b) Repeat part (a) when he dives to a depth $h = 5$ m. When the diver ignores diving rules and foolishly uses a snorkel tube at that depth, find the pressure on his lungs? Why he is in danger? Assume that $P_a = 1.01 \times 10^5$ Pa, $\rho = 10^3$ kg/m^3, and $g = 9.8$ m/s^2.

Solution: (a) The external pressure on the diver's body will be:

$$\begin{aligned} P_{\text{Body}} &= P_a + \rho g h \\ &= 1.01 \times 10^5\,\text{Pa} + (10^3\ \text{kg/m}^3)(9.8\ \text{m/s}^2)(0.5\ \text{m}) \\ &= 1.01 \times 10^5\,\text{Pa} + 4{,}900\,\text{Pa} = 1.059\ \times 10^5\ \text{Pa} \simeq 1.05\,\text{atm} \end{aligned}$$

The diver's body adjusts to that pressure by a very slight contraction until the internal pressure is in equilibrium with the external pressure. Consequently, his average blood pressure increases, and the average air pressure in his lungs increases until it balances the external pressure. Thus, his lung pressure will be at:

$$P_{\text{Lungs}} \simeq 1.05\,\text{atm}$$

(b) When $h = 5$ m, the external pressure on the diver's body will be:

$$\begin{aligned} P_{\text{Body}} &= P_a + \rho g h \\ &= 1.01 \times 10^5\,\text{Pa} + (10^3\ \text{kg/m}^3)(9.8\ \text{m/s}^2)(5\ \text{m}) \\ &= 1.01 \times 10^5\,\text{Pa} + 49{,}000\,\text{Pa} = 1.5\ \times 10^5\ \text{Pa} \simeq 1.5\,\text{atm} \end{aligned}$$

Again, as in part (a), the pressure inside his lungs will be:

$$P_{\text{Lungs}} \simeq 1.5\,\text{atm}$$

If the diver foolishly uses a 5 m snorkel tube, the pressurized air in his lungs will be expelled upwards through the tube to the atmosphere. Consequently, the air pressure in his lungs will drop rapidly from 1.5 to 1 atm. This 0.5 atm pressure difference is sufficient to collapse his lungs and force his still-pressurized blood into them.

Buoyant Forces and Archimedes' Principle

In a swimming pool, you may have noticed that it is relatively easy to carry an object that is totally or partially immersed in the water. This is because you must support only part of the object's weight, while the buoyant force supports the remainder.

This important property of fluids in hydrostatic equilibrium is summarized by Archimedes' principle, which can be stated as follows:

> Archimedes' Principle:
> A body fully or partially immersed in a fluid is buoyed up by a force equal to the weight of the fluid displaced by the body.

Let us show that the buoyant force is equal in magnitude to the weight of the displaced fluid. We can do this by considering a cube of fluid of height h (and hence volume $V_f = h^3$) as in Fig. 10.14a.

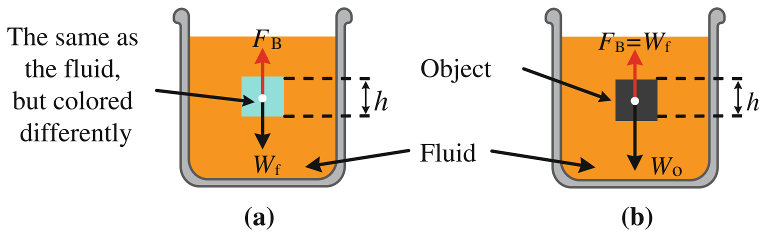


Fig. 10.14 (a) External forces acting on a cube of fluid (colored *blue*). Under equilibrium, the fluid's weight W_f is equal to the buoyant force F_B. (b) A cube of weight W_o is buoyed by a force $F_B = W_f$

The cube of this fluid is in equilibrium under the action of the forces on it. One of the forces is its own weight $\vec{W}_f$. Apparently, the rest of the fluid in the container is buoying up the cube and holding it in equilibrium. Therefore, the magnitude of this buoyant force, F_B, must be exactly equal in magnitude to the weight of the fluid. That is:

$$\boxed{F_B = W_f} \tag{10.27}$$

Now, imagine we replace the cube of fluid by a *cubical object* of the same dimensions. The fluid surrounding the cube will behave the same way, regardless whether the cube is a fluid or a solid. Therefore, *the buoyant force acting on an object of any density will be equal to the weight of the fluid displaced by the object*, i.e. $F_B = W_f$.

To show this result explicitly, we notice in Fig. 10.14b that the pressure at the bottom of the object is greater than the pressure at the top by $\Delta P = \rho_f g h$, where ρ_f is the density of the fluid. Since the pressure difference ΔP equals the buoyant force per unit area, then $\Delta P = F_B/A$, where $A = h^2$ is the area of one of the cube's faces. Therefore:

$$F_B = \Delta P\, A = \rho_f g h A = \rho_f V_f\, g = W_f \tag{10.28}$$

Consider the object of Fig. 10.14b to be of weight $W_o = \rho_o V_o g$, where ρ_o and V_o are its density and volume, respectively. If the object is totally immersed in a fluid of density ρ_f, the buoyant force will be $F_B = W_f = \rho_f V_f g$, where $V_f = V_o$. Thus, the net force on the object will depend only on ρ_o and ρ_f, see parts of Fig. 10.15a–c.

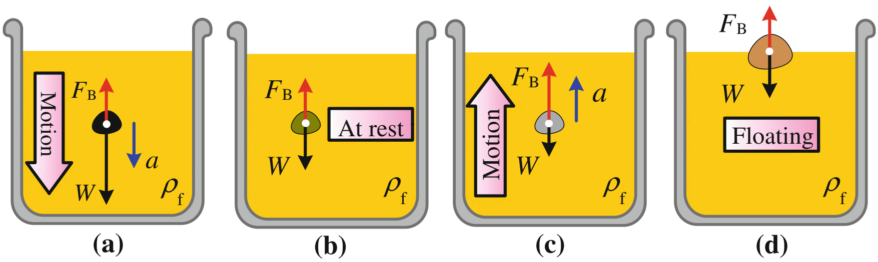


Fig. 10.15 An immersed object of density ρ_o when: (a) $\rho_o > \rho_f$, (b) $\rho_o = \rho_f$, (c) $\rho_o < \rho_f$, and (d) A floating object where $\rho_o V_o = \rho_f V_f$

Now, if that object floats; see Fig. 10.15d, the upward buoyant force will be $F_B = W_f = \rho_f V_f g$, where V_f is the volume of the displaced fluid and $V_f \neq V_o$.

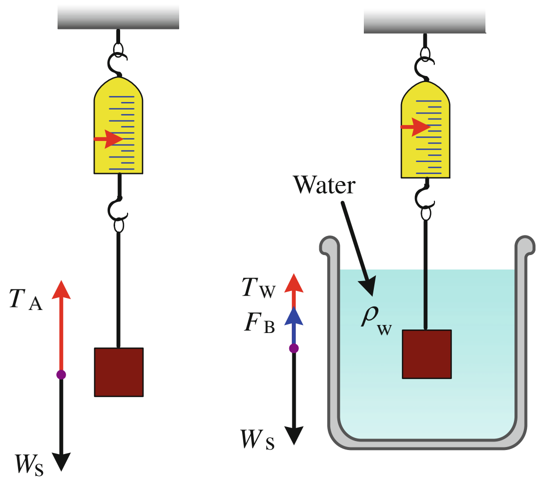
Equilibrium in this case gives:

$$\rho_o V_o = \rho_f V_f \tag{10.29}$$

Example 10.10

A piece of steel has a mass $m_s = 0.5\,\text{kg}$ and a density $\rho_s = 7.8 \times 10^3\,\text{kg/m}^3$. The steel is suspended in air by a string attached to a scale, see Fig. 10.16. After that, the steel is immersed in a container filled with water of density $\rho_w = 10^3\,\text{kg/m}^3$. Find the tension in the string before and after the steel is immersed.

Fig. 10.16



Solution: When the piece of steel is suspended in air, the tension in the string T_a equals the weight $m_s\,g$ of that piece of steel. That is:

$$T_a = m_s\,g = (0.5\,\text{kg})(9.8\,\text{m/s}^2) = 4.9\,\text{N}$$

When the steel is immersed in water, it experiences an upward buoyant force F_B. Thus, the tension in the string will be reduced to a new value T_w, see the figure. Equilibrium in this case gives:

$$T_w + F_B = m_s\,g \quad \Rightarrow \quad T_w = m_s\,g - F_B \quad \Rightarrow \quad T_w = 4.9\,\text{N} - F_B$$

To find F_B, we first calculate the volume of the steel as follows:

$$V_s = \frac{m_s}{\rho_s} = \frac{0.5\,\text{kg}}{7.8 \times 10^3\,\text{kg/m}^3} = 6.4 \times 10^{-5}\,\text{m}^3$$

This volume equals the volume of the displaced water. That is:

$$V_w = 6.4 \times 10^{-5}\,\text{m}^3$$

Since the buoyant force equals the weight of the displaced water, then:

$$F_B = m_w\, g = \rho_w V_w\, g = (10^3\,\text{kg/m}^3)(6.4 \times 10^{-5}\,\text{m}^3)(9.8\,\text{m/s}^2) = 0.63\,\text{N}$$

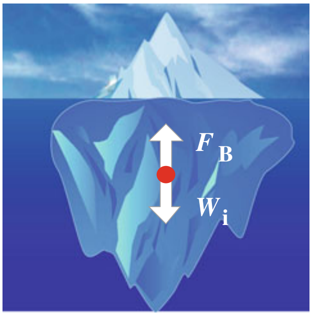
Therefore, the tension in the string T_w (the apparent weight) will be:

$$T_w = 4.9\,\text{N} - F_B = 4.9\,\text{N} - 0.63\,\text{N} = 4.3\,\text{N}$$

Example 10.11

The approximate density of ice is $\rho_i = 918\,\text{kg/m}^3$ and the approximate density of sea water in which an iceberg floats is $\rho_w = 1{,}020\,\text{kg/m}^3$, see Fig. 10.17. What fraction of the iceberg is beneath the water surface?

Fig. 10.17



Solution: The iceberg floats, as shown in the figure, due to the effect of an upward buoyant force given by:

$$F_B = W_w = \rho_w\, V_w\, g$$

where V_w is the volume of the displaced water or the volume of the iceberg beneath the water surface. The weight of the iceberg is:

$$W_i = \rho_i \, V_i \, g$$

where V_i is the volume of the iceberg. Equilibrium in this case gives $F_B = W_i$. That is:

$$\frac{V_w}{V_i} = \frac{\rho_i}{\rho_w} = \frac{918 \text{ kg/m}^3}{1{,}020 \text{ kg/m}^3} = 0.90 \quad \text{or} \quad 90\%$$

Thus, 90% of the iceberg lies below water level. This means that, only 10% of an iceberg—its tip is above the surface of the water.

Example 10.12

An object of a known density ρ_o floats three-fourths immersed in a liquid of unknown density ρ_f. Find the density of the liquid.

Solution: We use Eq. 10.29 when $V_f = \frac{3}{4} V_o$. Then:

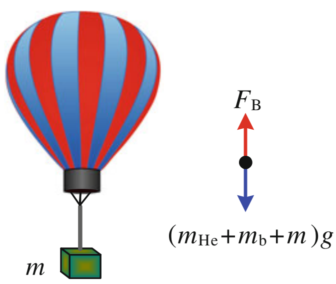
$$\rho_f = \rho_o \frac{V_o}{V_f} = \rho_o \frac{V_o}{\frac{3}{4} V_o} = \frac{4}{3} \rho_o$$

Note that ρ_f is larger than ρ_o by the reciprocal of the immersed ratio.

Example 10.13

A helium-filled balloon has a volume $V_b = 8 \times 10^3 \text{ m}^3$ and balloon mass $m_b = 200$ kg, see Fig. 10.18. What is the maximum mass m of a load that keeps the balloon in equilibrium? Neglect the air displaced by the load. Take $\rho_{He} = 0.18 \text{ kg/m}^3$ to be the density of helium and $\rho_{air} = 1.28 \text{ kg/m}^3$ to be the density of air.

Fig. 10.18



Solution: The volume of the displaced air equals the balloon's volume, i.e. $V_{air} = V_b$. According to Archimedes' principle, the buoyant force is the weight of the displaced air, i.e.:

$$F_B = W_{air} = \rho_{air} V_{air} g = \rho_{air} V_b\, g$$

Since the balloon's volume is approximately equal to the volume of helium, i.e. $V_{He} \simeq V_b$, then the weight of the helium is:

$$W_{He} = \rho_{He} V_{He}\, g = \rho_{He} V_b\, g$$

At equilibrium, see Fig. 10.18, we have:

$$F_B = W_{He} + W_b + mg \quad \Rightarrow \quad \rho_{air} V_b\, g = \rho_{He} V_b\, g + m_b g + m\, g$$

$$\text{Thus:} \qquad \begin{aligned} m &= (\rho_{air} - \rho_{He}) V_b - m_b \\ &= (1.28\,\text{kg/m}^3 - 0.18\,\text{kg/m}^3)(8 \times 10^3\,\text{m}^3) - 200\,\text{kg} = 8{,}600\,\text{kg} \end{aligned}$$

10.5 Fluid Dynamics

Ideal Fluids

The motion of a real fluid is very complicated. Instead, we shall discuss the motion of an **ideal fluid** that will obey the following four assumptions:

1. *Steady flow*: The velocity of the fluid at any specific point does not change with time. However, in general the velocity might vary from one point to another.
2. *Incompressible flow*: The density of the fluid does not change with time. That is, the density has a constant uniform value.
3. *Non-viscous flow*: A tiny object can move through the fluid without experiencing a viscous drag force; that is, there is no resistive force due to viscosity.
4. *Irrotational flow*: A tiny object can move through the fluid without rotating about an axis passing through its center of mass.

Streamlines

A **streamline** is the path traced out by a tiny fluid element, called a fluid "*particle*". As the fluid particle moves, its velocity may change in magnitude or in direction or both. However, the velocity of the fluid particle at any point is always tangent to the streamline at that point, see Fig. 10.19a. Streamlines never cross each other because if they do, a fluid particle could move either way at the cross over point, and the flow would not be steady.

When air flows around objects, the air particles must avoid the object, see Fig. 10.19b. Conservation of mass sets up the streamlines that the air particles must follow to avoid the object.

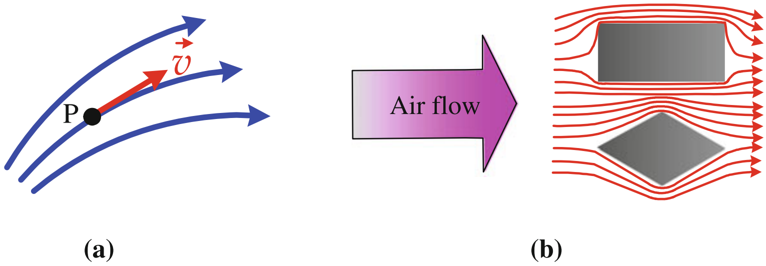


Fig. 10.19 (a) The diagram shows a set of streamlines. A fluid particle P traces out a streamline as it moves. The velocity vector of the fluid particle is tangent to the streamline at every point. (b) The streamlines of air flow near two different obstacles

Equation of Continuity

In flows like that of Fig. 10.19b, we consider a fluid flowing through a tube called a *stream tube*, or *tube of flow*, whose boundary is made up of streamlines, see Fig. 10.20. Such a tube acts like a pipe because any fluid particle entering it cannot escape through its walls.

Fig. 10.20 A stream tube formed by the streamlines. The flow rate of fluid at the cross sections A_1 and A_2 is the same

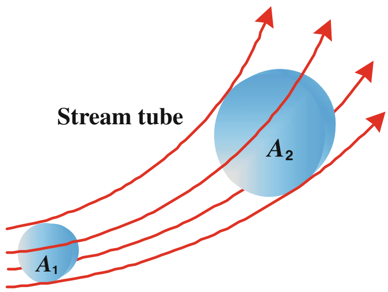


Figure 10.21 shows two cross-sectional areas A_1 and A_2 in a thin stream tube of fluid of varying cross-sectional areas. The fluid particles are moving steadily through this stream tube.

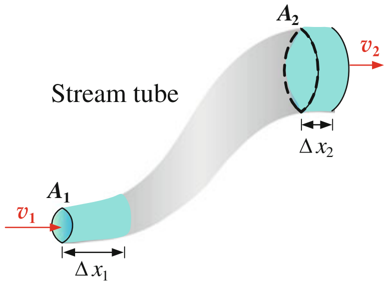


Fig. 10.21 A stream tube (streamlines are not shown) of fluid of varying cross sections with fluid particles moving steadily through it

In a small time interval Δt, the fluid at the area A_1 moves a small distance $\Delta x_1 = v_1 \Delta t$. Assuming uniform density over the area A_1, then the mass in the colored segment of Fig. 10.21 is:

$$\Delta m_1 = \rho_1(A_1 \Delta x_1) = \rho_1 A_1 v_1 \Delta t = \text{Mass into segment}$$

Similarly, the fluid that moves through the area A_2 in the same time interval will be:

$$\Delta m_2 = \rho_2(A_2 \Delta x_2) = \rho_2 A_2 v_2 \Delta t = \text{Mass out of segment}$$

Since **mass is conserved** and because the flow is steady, the mass that crosses A_1 in time interval Δt must be equal to the mass that crosses A_2 in the same time interval. Then, $\Delta m_1 = \Delta m_2$ and we get:

$$\boxed{\left\{\begin{array}{c} \rho_1 A_1 v_1 = \rho_2 A_2 v_2 \\ \text{or} \\ \rho A v = \text{constant} \end{array}\right.} \quad \text{(Steady flow)} \tag{10.30}$$

This is the *equation of continuity* for a steady flow. Since $\rho A v$ has the dimension of mass/time, it is called the *mass flow rate* R_m, i.e.:

$$\boxed{R_m = \rho A v = \text{constant}}\,. \tag{10.31}$$

If we assume the fluid is incompressible, then the density ρ is constant, and the continuity equation reduces to:

$$\boxed{\left\{\begin{array}{c} A_1 v_1 = A_2 v_2 \\ \text{or} \\ A v = \text{constant} \end{array}\right.} \quad \text{(Steady and Incompressible flow)} \tag{10.32}$$

This is another form of the *equation of continuity* for incompressible steady flow. Since Av has the dimension of volume/time, it is called the *volume flow rate* R_V, i.e.:

$$\boxed{R_V = Av = \text{constant}} \tag{10.33}$$

A constant-volume flow rate tells us that the flow is faster in narrower sections of a tube of flow, where the streamlines are close together.

Incompressible steady flow:
The product of the area and the fluid speed at all points along the pipe is constant.

Example 10.14

Water flows in a pipe from a large cross-sectional area $A_1 = 0.5\,\text{m}^2$ with a speed $v_1 = 15\,\text{m/s}$ to a smaller cross-sectional area $A_2 = 0.05\,\text{m}^2$. (a) What is the speed v_2 at which the water leaves the smaller cross section as in the left part of Fig. 10.22? (b) What is the effect of lowering A_2 by 10 m as in the right part of Fig. 10.22?

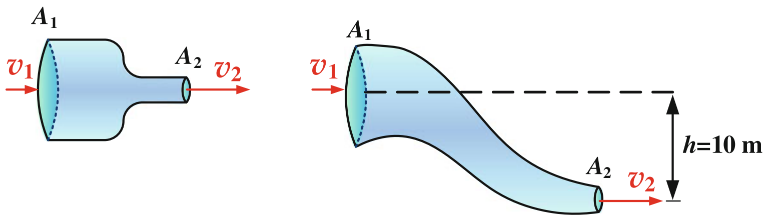


Fig. 10.22

Solution: (a) The flow of water through the pipe in the left part is governed by the continuity equation, or conservation of mass. That is:

$$\rho_1 A_1 v_1 = \rho_2 A_2 v_2$$

For most liquids, density is essentially constant. Then $A_1 v_1 = A_2 v_2$, and:

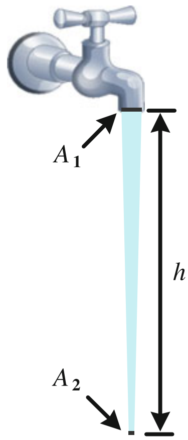
$$v_2 = v_1 \frac{A_1}{A_2} = (15\,\text{m/s}) \frac{0.5\,\text{m}^2}{0.05\,\text{m}^2} = 150\,\text{m/s}$$

(b) Since the continuity equation does not depend on altitude, then lowering A_2 by 10 m causes no change to the result.

Example 10.15

The fact that a water stream emerging from a faucet "necks down" as it falls is shown in Fig. 10.23, where $A_1 = 1.8\,\text{cm}^2$, $A_2 = 0.3\,\text{cm}^2$, and $h = 25\,\text{cm}$. What is the water flow rate from the faucet, assuming a steady flow?

Fig. 10.23



Solution: As water falls from a faucet, its speed increases due to gravity. Because the volume flow rate must be the same at all cross sections, the stream must "neck down".

The flow of water is governed by the continuity Eq. 10.32, that is:

$$A_1 v_1 = A_2 v_2$$

where v_1 and v_2 indicate the speed of the water at the marked levels shown in Fig. 10.23. Since water is falling freely, we must have:

$$v_2^2 = v_1^2 + 2gh$$

Eliminating v_2 from the last two equations, we get:

$$v_1 = \sqrt{\frac{2ghA_2^2}{A_1^2 - A_2^2}} = \sqrt{\frac{2(9.8\ \text{m/s}^2)(0.25\ \text{m})(0.3\ \text{cm}^2)^2}{(1.8\ \text{cm}^2)^2 - (0.3\ \text{cm}^2)^2}} = 0.374\,\text{m/s}$$

The volume flow rate R_V, given by Eq. 10.33, is thus:

$$R_V = A_1 v_1 = (1.8 \times 10^{-4}\ \text{m}^2)(0.374\ \text{m/s}) = 6.732 \times 10^{-5}\ \text{m}^3/\text{s}$$

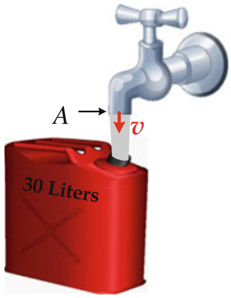
Finally, we find the mass-flow rate R_m using Eq. 10.31 to be:

$$R_m = \rho A_1 v_1 = \rho R_V = (10^3\ \text{kg/m}^3)(6.735 \times 10^{-5}\ \text{m}^3/\text{s}) = 0.067\,\text{kg/s} \simeq 70\,\text{g/s}$$

Example 10.16

Water flowing from a faucet of cross-sectional area $A = 2\,\text{cm}^2$ is used to fill a bucket of volume $V = 30\,\text{liters} = 30 \times 10^3\ \text{cm}^3$, see Fig. 10.24. What is the speed v at which the water leaves the faucet if it takes exactly 1 minute to fill the bucket?

Fig. 10.24



Solution: According to the given information, the volume flow rate is:

$$R_V = 30\ \text{liters/min} = 500\ \text{cm}^3/\text{s}$$

Using Eq. 10.33, $R_V = Av$, we can find the speed v as follows:

$$v = \frac{R_V}{A} = \frac{500\,\text{cm}^3/\text{s}}{2\,\text{cm}^2} = 250\,\text{cm/s} = 2.5\,\text{m/s}$$

Bernoulli's Equation

In static fluids, the pressure is the same at all points on the same horizontal level but increases with depth. This is not generally true when the fluid is in motion. In the year 1738, Bernoulli derived an expression for an **ideal fluid** (i.e. a fluid that is incompressible, non-viscous and flows in a non-rotational steady manner) that relates the pressure, speed, and elevation within different locations in the fluid.

* Consider a small portion of a tube of flow of an ideal fluid with density ρ through a non-uniform pipe as shown in Fig. 10.25. The width of the tube in this figure is exaggerated for clarity.

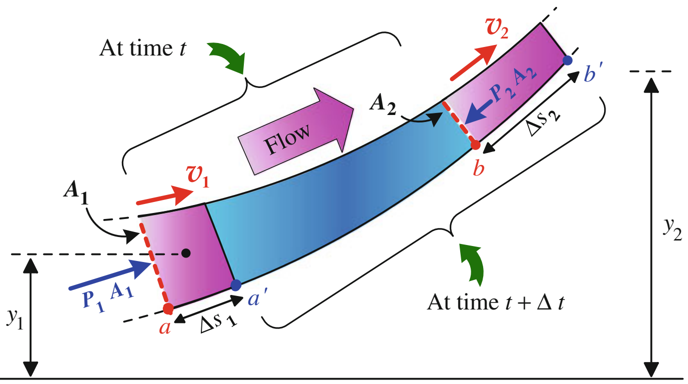


Fig. 10.25 The fluid in a section of length Δs_1 moves to a section of length Δs_2, while the volume of the two sections are the same

Using the information in Fig. 10.25, we perform the following steps:

1. At some initial time t, the fluid lies between two cross sections A_1 and A_2 at two points labeled a and b, respectively.
2. After a time interval Δt, the fluid's ends undergo displacements Δs_1 and Δs_2 to new points labeled a' and b', respectively.
3. The volume of fluid that passes from point a to point a' over a time Δt is equal to the volume of fluid that passes from point b to point b' in the same time interval, that is $\Delta V = A_1\Delta s_1 = A_2\Delta s_2$.
4. The force on the cross section A_1 is P_1A_1 and the force on the cross section A_2 is P_2A_2. Thus, the net work done on the fluid by these forces over the time Δt is:

$$W = P_1A_1\Delta s_1 - P_2A_2\Delta s_2 = (P_1 - P_2)\Delta V \tag{10.34}$$

where the negative sign in the second term is due to the fact that the fluid force P_2A_2 is opposite to the displacement Δs_2.

5. Part of this work goes into changing the kinetic energy of the fluid, and the other part goes into changing its gravitational potential energy. If Δm is the mass of the

fluid that passes through the tube during the time interval Δt, then $\Delta m = \rho \Delta V$ and the change in kinetic and potential energy will be given by:

$$\Delta K = \tfrac{1}{2}\Delta m v_2^2 - \tfrac{1}{2}\Delta m v_1^2 = \tfrac{1}{2}\rho \Delta V (v_2^2 - v_1^2) \tag{10.35}$$

$$\Delta U = \Delta m g y_2 - \Delta m g y_1 = \rho \Delta V g (y_2 - y_1) \tag{10.36}$$

6. We can now apply the work-energy theorem written in the form $W = \Delta K + \Delta U$, where W is the work done by all applied forces and is given by Eq. 10.34. Thus:

$$(P_1 - P_2)\Delta V = \tfrac{1}{2}\rho \Delta V (v_2^2 - v_1^2) + \rho \Delta V g (y_2 - y_1)$$

If we divide both sides by ΔV, we get:

$$P_1 - P_2 = \tfrac{1}{2}\rho (v_2^2 - v_1^2) + \rho g (y_2 - y_1) \tag{10.37}$$

7. Rearranging the terms, we get:

$$\boxed{P_1 + \tfrac{1}{2}\rho v_1^2 + \rho g y_1 = P_2 + \tfrac{1}{2}\rho v_2^2 + \rho g y_2} \tag{10.38}$$

This is the standard form of Bernoulli's equation for non-viscous, incompressible fluids experiencing steady flow. Since the subscripts 1 and 2 refer to any two points along the tube flow, Bernoulli's equation may also written as:

$$\boxed{P + \tfrac{1}{2}\rho v^2 + \rho g y = \text{constant}} \tag{10.39}$$

When the fluid is at rest, $v_1 = v_2 = 0$ and Bernoulli's equation becomes:

$$P_1 - P_2 = \rho g (y_2 - y_1) = \rho g h$$

which is of the same form as Eq. 10.20.

When we take y to be constant, say $y = 0$, so that the fluid does not change elevation as it flows, then Bernoulli's equation becomes:

$$\boxed{P_1 + \tfrac{1}{2}\rho v_1^2 = P_2 + \tfrac{1}{2}\rho v_2^2} \quad \text{(Horizontal flow)} \tag{10.40}$$

This tells us that if the speed of a fluid *increases* as it travels horizontally, then its pressure must *decrease*, and vice versa.

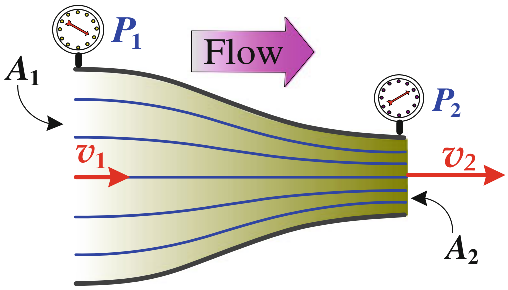
Example 10.17

Gasoline of density $\rho = 860\,\text{kg/m}^3$ flows steadily through a horizontal pipe that tapers in cross-sectional area from $A_1 = 1.5 \times 10^{-3}\,\text{m}^2$ to $A_2 = \frac{1}{2}A_1$, see Fig. 10.26. What is the volume flow rate when the pressure difference $P_1 - P_2$ is 5,160 Pa?

Solution: The flow of gasoline is governed by the continuity Eq. 10.32, i.e. $A_1 v_1 = A_2 v_2$. Then for $A_2 = \frac{1}{2}A_1$ we find:

$$A_1 v_1 = A_2 v_2 \quad \Rightarrow \quad A_1 v_1 = \tfrac{1}{2} A_1 v_2 \quad \Rightarrow \quad v_2 = 2 v_1$$

Fig. 10.26



That is, the gasoline speed at the narrower section is twice that at the wider section. Using this result in Bernoulli's equation of a fluid traveling horizontally, see Eq. 10.40, we get:

$$P_1 - P_2 = \tfrac{1}{2}\rho(v_2^2 - v_1^2) = \tfrac{1}{2}\rho(4v_1^2 - v_1^2) = \tfrac{3}{2}\rho v_1^2$$

Thus, we can find v_1 in terms of ρ and $P_1 - P_2$ as follows:

$$v_1 = \sqrt{\frac{2(P_1 - P_2)}{3\rho}} = \sqrt{\frac{2(5{,}160\,\text{Pa})}{3(860\,\text{kg/m}^3)}} = 2\ \text{m/s}$$

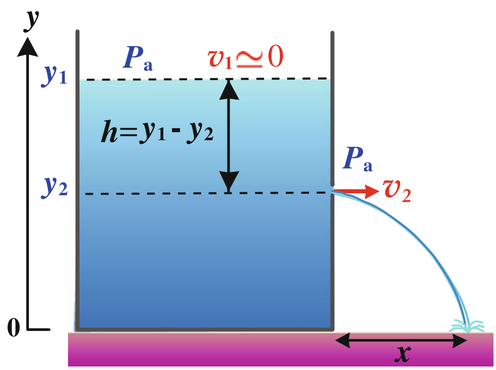
Finally, the volume flow rate R_V, given by Eq. 10.33, is thus:

$$R_V = A_1 v_1 = (1.5 \times 10^{-3}\ \text{m}^2)(2\ \text{m/s}) = 3 \times 10^{-3}\ \text{m}^3/\text{s}$$

Example 10.18

Flow Speed from a Reservoir (*Torricelli's Law*). A tank is filled with water to a height $y_1 = 3.5\,\text{m}$. The tank has a small hole in one of its walls at a height $y_2 = 1.5\,\text{m}$, see Fig. 10.27. (a) What is the speed v_2 of the water emerging from the hole? (b) What is the horizontal distance x from the base of the tank to the point at which the water stream strikes the floor?

Fig. 10.27



Solution: (a) The pressure at the top of reservoir and at the hole is the atmospheric pressure P_a, because both of them are exposed to the atmosphere. If we assume the tank has a large cross-sectional area A_1 compared to that of the hole's, i.e. $A_1 \gg A_2$, then water at the top of the reservoir will be almost stationary, i.e. $v_1 \simeq 0$. In addition to this, $h = y_1 - y_2$. Using Bernoulli's equation, see Eq. 10.38, we get:

$$P_a + \tfrac{1}{2}\rho \times (0) + \rho g y_1 = P_a + \tfrac{1}{2}\rho v_2^2 + \rho g y_2 \quad \Rightarrow \quad v_2^2 = 2gh$$

That is: $$v_2 = \sqrt{2gh}$$

Thus: $$v_2 = \sqrt{2gh} = \sqrt{2(9.8\ \text{m/s}^2)(2\ \text{m})} = 6.26\ \text{m/s}$$

The relation $v_2 = \sqrt{2gh}$ is the same as that for an object falling freely from rest through a height h. This is known as *Torricelli's law*.

(b) As in Sect. 4.3, the initial water velocity $\vec{v}_\circ \equiv \vec{v}_2$ is horizontal (i.e. $\theta_\circ = 0$) and the initial position is $y_\circ \equiv y_2 = 1.5$ m. Since water strikes the floor at $y = 0$, then we use $y - y_\circ = (v_\circ \sin\theta_\circ)\,t - \frac{1}{2}\,g\,t^2$ to first find the duration of descent for the water as follows:

$$0 - 1.5\text{ m} = 0 - \tfrac{1}{2}\,(9.8\,\text{m/s}^2)\,t^2 \quad \Rightarrow \quad t = 0.553\,\text{s}$$

The horizontal distance x covered by the water in that time is:

$$x = (v_\circ \cos\theta_\circ)\,t = (6.26\,\text{m/s})(\cos 0^\circ)(0.553\,\text{s}) = 3.46\,\text{m}$$

Example 10.19

In the Venturi meter of Fig. 10.28, air of density $\rho_{\text{air}} = 1.3\,\text{kg/m}^3$ flows from left to right through a horizontal pipe of radius $r_1 = 1.25\,\text{cm}$ that necks down to $r_2 = 0.5\,\text{cm}$. The U-shaped tube of the meter contains mercury of density $\rho_{\text{mer}} = 13.6 \times 10^3\,\text{kg/m}^3$. If the speed of the air entering the meter is $v_1 = 10\,\text{m/s}$, then find the mercury-level difference h between the two arms.

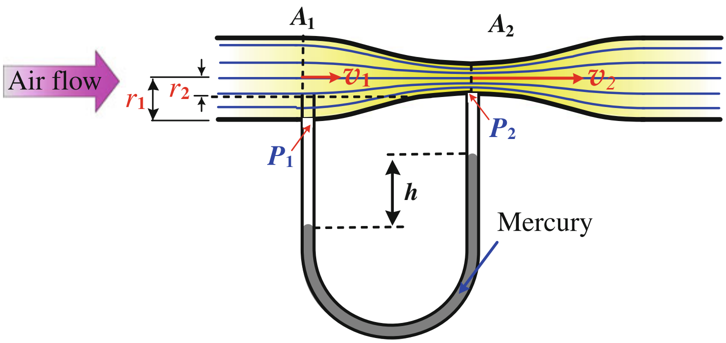


Fig. 10.28

Solution: Since the air moves horizontally, then Bernoulli's equation at the two openings of the U-shaped tube becomes:

$$P_1 - P_2 = \tfrac{1}{2}\rho_{\text{air}}(v_2^2 - v_1^2)$$

Also, from the continuity Eq. 10.32 we have:

$$A_1 v_1 = A_2 v_2 \quad \Rightarrow \quad v_2 = v_1 \frac{A_1}{A_2} = v_1 \frac{\pi r_1^2}{\pi r_2^2} = v_1 \left(\frac{r_1}{r_2}\right)^2$$

Substituting with the form of v_2 in the pressure difference, we get:

$$P_1 - P_2 = \tfrac{1}{2}\rho_{\text{air}} v_1^2 \left[\left(\frac{r_1}{r_2}\right)^4 - 1\right]$$

The pressure difference between the two openings of the U-shaped tube produces a mercury-level difference h given by $P_1 - P_2 = \rho_{mer} g h$. Thus, by combining the last result with this equation we get:

$$h = \frac{\rho_{air} v_1^2}{2\rho_{mer} g}\left[\left(\frac{r_1}{r_2}\right)^4 - 1\right] = \frac{(1.3\,\text{kg/m}^3)(10\text{ m/s})^2}{2(13.6\times 10^3\text{ kg/m}^3)(9.8\text{ m/s}^2)}\left[\left(\frac{1.25\text{ cm}}{0.5\text{ cm}}\right)^4 - 1\right]$$
$$= 0.019\text{ m} = 1.9\text{ cm}$$

* Example 10.8

The sketch in Fig. 10.29 shows a perfume atomizer before and after compressing its bulb. When the bulb is compressed gently, air with density ρ_{air} flows steadily through a narrow tube, reducing the pressure at the position of the vertical tube. Liquid of density ρ_L can rise in this vertical tube and enter the horizontal tube and be sprayed out. If the pressure in the bulb is $P_a + P$, where P_a is the atmospheric pressure, and v_2 is the speed of air in the horizontal tube, then find the pressure formula in the horizontal tube. What must the value of v_2 be in order to raise the liquid to the horizontal tube?

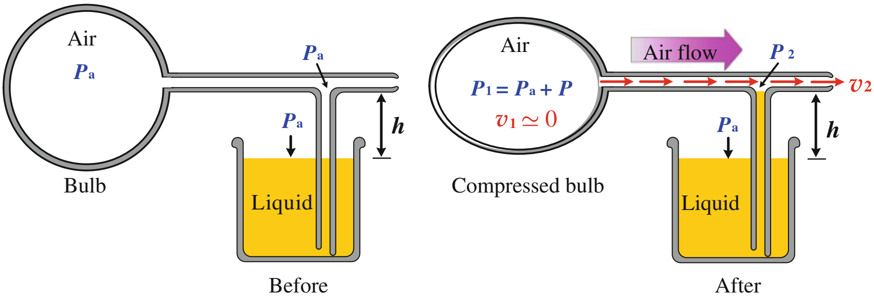


Fig. 10.29

Solution: By applying Bernoulli's equation on the bulb and the narrow horizontal tube we get the following:

$$(P_a + P) + \tfrac{1}{2}\rho \times (0) = P_2 + \tfrac{1}{2}\rho_{air} v_2^2$$

That is:

$$P_2 = P_a + P - \tfrac{1}{2}\rho_{air} v_2^2$$

So, the decrease in P_2 depends on the square of the speed v_2.

When the liquid rises a distance h to the horizontal tube, we have:

$$P_a = P_2 + \rho_L g h$$

By equating the expression of P_2 from the last two results we get:

$$P_2 = P_a + P - \tfrac{1}{2}\rho_{air} v_2^2 = P_a - \rho_L g h$$

Thus:

$$v_2 = \sqrt{\frac{2(P + \rho_L g h)}{\rho_{air}}}$$

Viscosity

Fluids cannot withstand a shearing stress. However, fluids show some degree of *resistance* to shearing motion, and this resistance is called **viscosity**.

The degree of viscosity can be understood by considering a fluid between two sheets of glass where the lower one is kept fixed; see the sketch in Fig. 10.30. It is easier to slide the upper glass if the fluid is oil as opposed to tar because tar has a higher viscosity than oil.

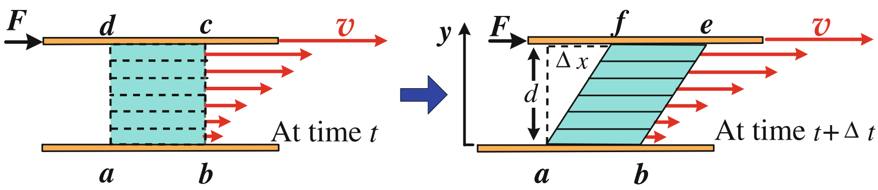


Fig. 10.30 A fluid between two sheets of glass where the lower one is kept fixed while the upper one moves to the right with a speed v under the action of an external force of magnitude F

We can think of fluids as a set of adjacent layers. Thus, a reasonable shearing stress produces smooth relative displacement of adjacent layers in fluids, called *laminar flow*. When we apply a force of magnitude F to the upper glass of area A, it will move to the right with a speed v. As a result of this motion, a portion of the fluid with shape *abcd* will take a new shape *abef* after a short time interval Δt.

According to Sect. 10.2, we can define the shearing stress and the shearing strain on the fluid of Fig. 10.30 as follows:

$$\text{Shearing stress} = \frac{F}{A}, \qquad \text{Shearing strain} = \frac{\Delta x}{d} \tag{10.41}$$

Since the upper sheet is moving with speed v, the fluid just beneath it will move with the same speed. Thus, in time Δt, the fluid just beneath the upper sheet moves a distance $\Delta x = v\Delta t$. Accordingly, we define the rate of shear strain as:

$$\text{Rate of shear strain} = \frac{\text{shear strain}}{\Delta t} = \frac{\Delta x/d}{\Delta t} = \frac{v}{d} \tag{10.42}$$

By analogy to the shear modulus in solids, we define in fluids, at a given temperature, the ratio of the shear stress to the *rate of shear strain*. This ratio, η, is known as the **coefficient of viscosity** or simply the **viscosity**:

$$\boxed{\eta = \frac{F/A}{v/d} = \frac{Fd}{Av}} \tag{10.43}$$

The SI unit of viscosity is N.s/m^2 = Pa.s which is called the **poiseuille** (abbreviated by Pl) while in cgs it is dyne.s/cm^2 which is called the **poise** (abbreviated by P). Thus, 1 N.s/m^2 = 1 Pa.s = 1 Pl = 10 P = 10^3 cP.

Table 10.6 depicts some viscosity values.

Table 10.6 The viscosity of some fluids at specific temperatures

Fluid	Temperature T (°C)	Viscosity η (N.s/m^2 = Pl)
Benzene	20	0.07×10^{-3}
Water	100	0.3×10^{-3}
Water	20	1×10^{-3}
Whole blood	37	2.7×10^{-3}
10-wt Motor oil	30	250×10^{-3}
Glycerin	20	830×10^{-3}

* Equation 10.43 is valid only when the velocity of the fluid varies linearly with the perpendicular distance to the fluid velocity. In this case, it is common to say that the velocity gradient is uniform. In case of non-uniform velocity gradient, the viscosity has the general form:

$$\boxed{\eta = \frac{F/A}{dv/dy}} \tag{10.44}$$

Stokes' law

The drag force of a small object that moves with a low speed v through a viscous medium was given in Sect. 5.2 of Chap. 5 by the relation $F_D = bv$, where b is a proportionality constant.

When the small object is a sphere of radius r and it moves with a terminal speed v_t through a viscous medium with viscosity η, it experiences a drag force F_{vis} which by **Stokes' law** has a magnitude:

$$\boxed{F_{vis} = 6\pi\eta r v_t} \tag{10.45}$$

As an application to Stokes' formula, Fig. 10.31 displays the fall of a small metallic spherical ball of volume $V_s = \frac{4}{3}\pi r^3$, density ρ_s, and mass $m_s = V_s\rho_s$ in a viscous liquid of density ρ. The forces that act on the sphere when it reaches its terminal (constant) speed v_t will be:

1. The sphere's weight $W = m_s\, g = \rho_s\, V_s\, g$ (downwards)
2. The buoyant force $F_B = \rho\, V g$ (upwards)
3. The viscous force $F_{vis} = 6\pi\eta r v_t$ (upwards)

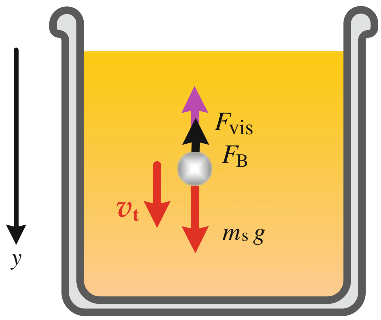


Fig. 10.31 A small sphere falling with terminal speed v_t in a liquid of density ρ and viscosity η

We must equate the volume of the liquid V that was displaced by the sphere with the volume of the falling sphere V_s. Thus:

$$m_s\, g = F_B + F_{vis} \quad \Rightarrow \quad \rho_s\, V_s\, g = \rho\, V_s\, g + 6\pi\eta r v_t$$

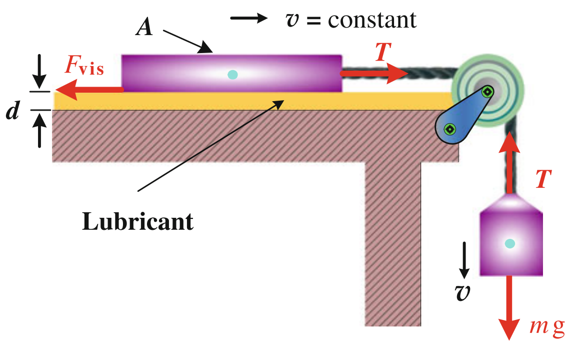
From this equation we get the following relation for the viscosity η:

$$\boxed{\eta = \frac{2}{9}\frac{(\rho_s - \rho)r^2 g}{v_t}} \tag{10.46}$$

Example 10.21

A steel plate of area $A = 0.2\,\text{m}^2$ is placed over a thin film of lubricant of thickness $d = 0.4\,\text{mm}$ sprayed over the flat horizontal surface of a table, see Fig. 10.32. When connected via a cord that passes over a massless and frictionless pulley to a mass $m = 10\,\text{g}$, the steel plate is found to move with a constant speed $v = 0.05\,\text{m/s}$. Find the viscosity of the lubricant oil.

Fig. 10.32



Solution: Since the steel plate moves with a constant speed, its resultant force must be zero. Thus, the magnitude of the tension force must equal the magnitude of viscous force exerted by the lubricant on the plate, i.e. $F_{\text{vis}} = T$. Also, the magnitude of the tension in the cord is equal to the magnitude of the suspended weight, i.e. $T = m\,g$. Thus:

$$F_{\text{vis}} = T = mg = (10 \times 10^{-3}\,\text{kg})(9.8\,\text{m/s}^2) = 9.8 \times 10^{-2}\,\text{N}$$

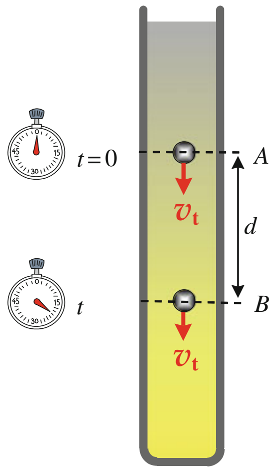
The layer of lubricant in contact with the horizontal surface of the table is at rest. The lubricant speed increases across the film, reaching a maximum, v, at the layer in contact with the steel plate which moves with speed v. If we assume that the rate of shear strain is constant, i.e. the velocity gradient is uniform, then we can use Eq. 10.43 to evaluate the viscosity as follows:

$$\eta = \frac{F_{\text{vis}}/A}{v/d} = \frac{F_{\text{vis}}\,d}{Av} = \frac{(9.8 \times 10^{-2}\,\text{N})(0.4 \times 10^{-3}\,\text{m})}{(0.2\,\text{m}^2)(0.05\,\text{m/s})}$$
$$= 3.92 \times 10^{-3}\,\text{N.s/m}^2 = 3.92\,\text{cP}$$

Example 10.22

A tiny glass sphere of density $\rho_s = 2.6 \times 10^3\,\mathrm{kg/m^3}$ falls with a terminal velocity v_t through oil which has a density $\rho = 950\,\mathrm{kg/m^3}$ and a viscosity coefficient $\eta = 0.2\,\mathrm{N.s/m^2}$. It is experimentally observed that the sphere drops a distance $d = 20\,\mathrm{cm}$ between the two points A and B in time $t = 50\,\mathrm{s}$, see Fig. 10.33. Find the radius r of the glass sphere.

Fig. 10.33



Solution: From experimental observations, the terminal speed of the sphere will be given by:

$$v_t = \frac{d}{t} = \frac{20 \times 10^{-2}\,\mathrm{m}}{50\,\mathrm{s}} = 4 \times 10^{-3}\,\mathrm{m/s}$$

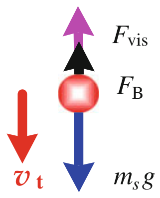
The forces that act on the sphere when it reaches its terminal (constant) speed v_t will be: the sphere's weight $W = m_s\, g = \rho_s\, V_s\, g$ (downwards), the buoyant force $F_B = \rho\, V g$ (upwards), and the viscous force $F_{vis} = 6\pi\,\eta r v_t$ (upwards), see Fig. 10.34. The volume of the liquid V that was displaced by the sphere equals the volume of the falling sphere, i.e. $V = V_s = \frac{4}{3}\pi r^3$. Since the sphere moves with constant speed v_t, its resultant force must be zero. Thus:

$$\rho_s\, V_s\, g = \rho\, V_s\, g + 6\pi\,\eta r v_t$$

Solving for the sphere's radius, we obtain:

$$r = \sqrt{\frac{9\eta v_t}{2(\rho_s - \rho)g}} = \sqrt{\frac{9(0.2\,\mathrm{N.s/m^2})(4 \times 10^{-3}\,\mathrm{m/s})}{2(2.6 \times 10^3\,\mathrm{kg/m^3} - 950\,\mathrm{kg/m^3})(9.8\,\mathrm{m/s^2})}}$$
$$= 4.7 \times 10^{-4}\,\mathrm{m} = 0.47\,\mathrm{mm}$$

Fig. 10.34



10.6 Exercises

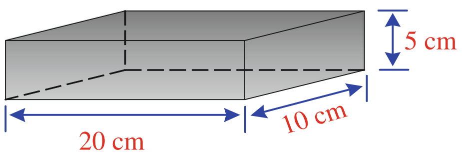
Section 10.1 Density and Relative Density

(1) A solid cube of mass of 0.04 kg has a side of length 1 cm. What is the density and the specific gravity of the cube?

(2) A solid sphere has a radius of 1 cm and a mass of 0.04 kg. What is the density and the specific gravity of the sphere?

(3) Lead bricks like the one in Fig. 10.35 are used to shield people from the hazards of radioactive materials. If the lead density is $\rho = 11.36 \times 10^3\ \text{kg/m}^3$, then find the mass and weight of such a brick.

Fig. 10.35 See Exercise (3)



Section 10.2 Elastic Properties of Solids

(4) A mass of 5 kg is suspended from the end of a copper wire that has a diameter of 1 mm. Find the tensile stress on the wire?

(5) A 4 m long structural steel rod with a cross-sectional area of 0.5 cm^2 stretches 1 mm when a mass of 250 kg is hung from its lower end. Find the value of Young's modulus for this steel.

(6) An iron rod 10 m long and 0.5 cm^2 in cross section, stretches 2.5 mm when a mass of 300 kg is hung from its lower end. Find Young's modulus for the iron rod.

(7) A wire has a length $L = 3$ m and a radius $r = 0.75$ cm, see Fig. 10.36. A force acting normally on each of its ends has a magnitude $F_\perp = 9 \times 10^4$ N.

Find the change in the wire's length and radius, when its Young's modulus Y is 190×10^9 N/m^2 and its Poisson's ratio μ is 0.25.

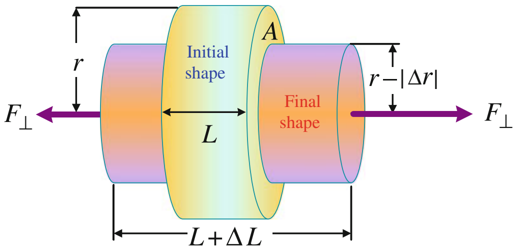


Fig. 10.36 See Exercise (7)

(8) A uniform platform is suspended by four wires, one on each corner. The wires are 2-m long and have a radius of 1 mm, and their material has a Young's modulus $Y = 190 \times 10^9$ N/m^2. How far will the platform drop if an 80 kg load is placed at its center?

(9) A block of gelatin resting on a rough dish has a length $L = 60$ cm, width $d = 40$ cm, and height $h = 20$ cm, see the vertical cross section abcd in Fig. 10.37. A force $F_{\parallel} = 0.6$ N is applied tangentially to the upper surface, leading to a new shape abef and hence a displacement $\Delta x = 5$ mm for the upper surface relative to the lower one. Find: (a) the shearing stress, (b) the shearing strain, and (c) the shear modulus.

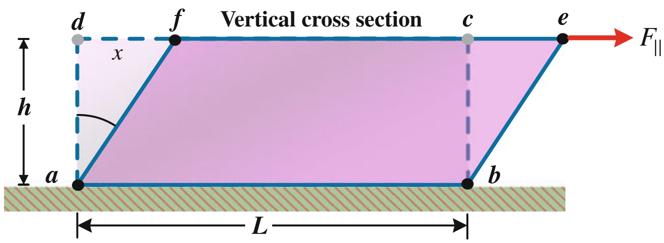


Fig. 10.37 See Exercise (9)

(10) Two parallel but opposite forces, each having a magnitude $F_{\parallel} = 4 \times 10^3$ N, are applied tangentially to the upper and lower faces of a cubical metal block of side $a = 25$ cm and shear modulus $S = 80 \times 10^9$ N/m^2, see Fig. 10.38. Find the displacement Δx of the upper surface relative to the lower one, and hence find the angle of shear θ.

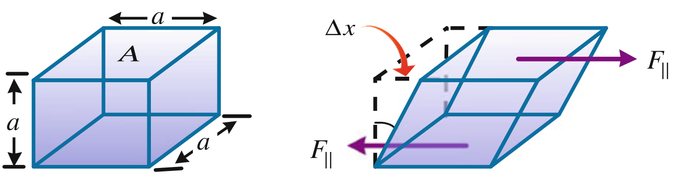


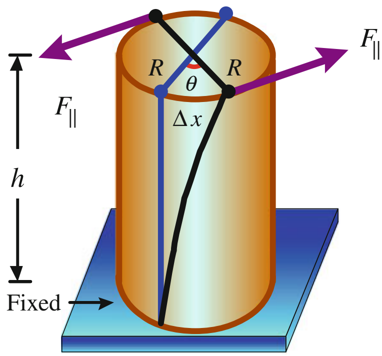
Fig. 10.38 See Exercise (10)

(11) Show that the angle of twist θ (in radians) for a torsional shearing caused by a tangential force $F_{\parallel}$ on the top of a cylindrical rod of height h, radius R, and shearing modulus S, see Fig. 10.39, is given by:

$$\theta = \frac{F_{\parallel} h}{\pi R^3 S}$$

Find θ when $F_{\parallel} = 500\,\text{N}$, $S = 80 \times 10^9\,\text{N/m}^2$, $R = 2.5\,\text{cm}$, and $h = 3\,\text{m}$.

Fig. 10.39 See Exercise (11)



(12) The pressure of the atmosphere around a metal block is reduced to almost zero when the block is placed in vacuum. Find the fractional change in volume if the bulk modulus of the metal is $B = 150 \times 10^9\,\text{N/m}^2$.

(13) The pressure around a cube of copper of side 40 mm is changed by $\Delta P = 2 \times 10^{10}\,\text{N/m}^2$. Find the change in volume if the bulk modulus for copper is $B = 125 \times 10^9\,\text{N/m}^2$.

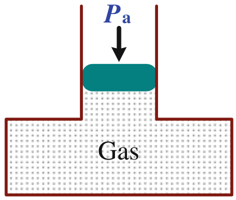
(14) What increase in pressure is required to decrease the volume of 200 liters of water by 0.004%? Take the bulk modulus of water B to be $2.1 \times 10^9\,\text{N/m}^2$.

(15) In an experiment, $750\,\text{cm}^3$ of water expands to $765\,\text{cm}^3$ when heated. What increase in pressure is required to squeeze the water back to its original volume? (Water bulk modulus is $2 \times 10^9\ \text{N/m}^2$)

Section 10.4 Fluid Statics

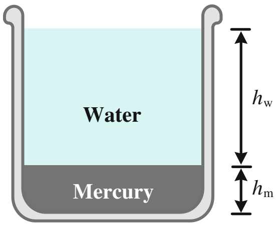
(16) A piston that has a cross-sectional area of $8\,\text{cm}^2$ and a mass of 20 kg holds a compressed gas in a tank as shown in Fig. 10.40. What is the total pressure of the gas in the tank? What would an ordinary pressure gauge inside the tank read?

Fig. 10.40 See Exercise (16)



(17) A vessel contains mercury of height $h_m = 10\,\text{cm}$ and water of height $h_w = 30\,\text{cm}$, see Fig. 10.41. The density of mercury is $\rho_m = 13.6 \times 10^3\,\text{kg/m}^3$ and the density of water is $\rho_w = 1.0 \times 10^3\,\text{kg/m}^3$. Find the pressure exerted by the two liquids on the bottom of the vessel.

Fig. 10.41 See Exercise (17)



(18) If a water gauge pressure at the ground floor of a building reads 280 kPa, how high will the water rise in the pipes of that building. Take density of water to be $\rho_w = 1.0 \times 10^3\,\text{kg/m}^3$.

(19) Two liquids are placed in a U-shaped tube as shown in Fig. 10.42. (a) Show that the heights of the liquids above their surface of separation are inversely proportional to their densities. (b) Assume that the two liquids are water and oil where $h_1 = 20\,\text{cm}$ and $h_2 = 25\,\text{cm}$. Find the density of oil if the density of water is $\rho_1 = 1.0 \times 10^3\,\text{kg/m}^3$.

(20) The special manometer shown in Fig. 10.43, uses mercury of density $\rho = 13.6 \times 10^3\,\text{kg/m}^3$. If the atmospheric pressure is 100 kPa and the height of mercury above the surface of separation at point B is $h = 10\,\text{cm}$, what is the pressure of the gas tank?

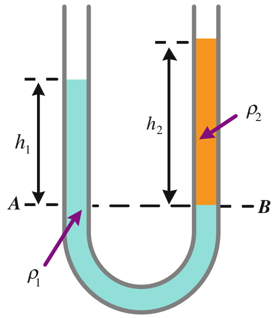


Fig. 10.42 See Exercise (19)

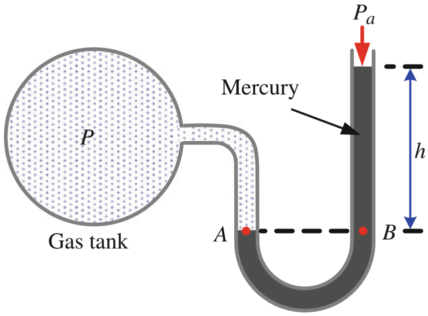


Fig. 10.43 See Exercise (20)

(21) Assume the value of the atmospheric pressure P_a is 1.01×10^5 Pa and water of density $\rho = 10^3$ kg/m^3 is used as a fluid in the barometer of Fig. 10.44. (a) What will be the height of the water column? (b) Repeat part (a) when water is replaced by alcohol of density $\rho = 7.9 \times 10^2$ kg/m^3. Comment on the practicality of your answers.

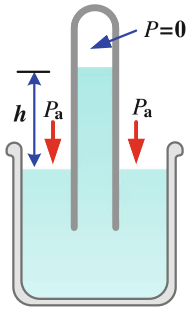
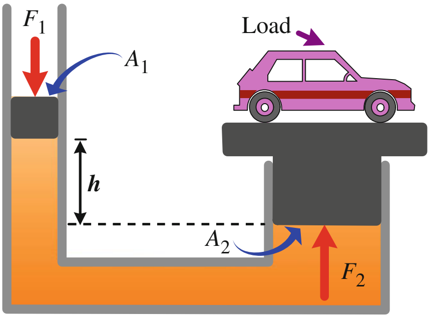


Fig. 10.44 See Exercise (21)

(22) The areas of the car lift pistons, shown in Fig. 10.45, are $A_1 = 25$ cm^2 and $A_2 = 500$ cm^2. The car and the right piston have a total weight of 10^4 N, while the left piston has a negligible weight and is at a height $h = 10$ m with respect

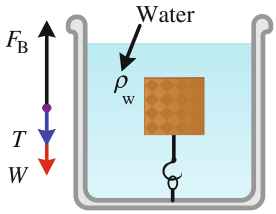
to the right one. The apparatus is filled with oil of density $\rho = 800\,\text{kg/m}^3$. What is the value of the force F_1 needed to keep the system in equilibrium?

Fig. 10.45 See Exercise (22)



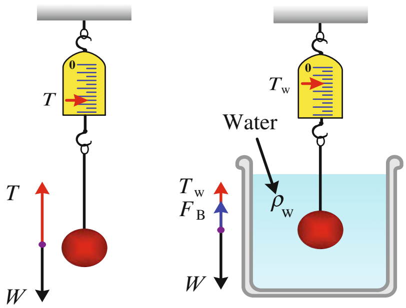
(23) A piece of wood has a mass $m = 0.25\,\text{kg}$ and a density $\rho = 750\,\text{kg/m}^3$. The wood is tied by a string to the bottom of a container of water in order to have the wooden piece fully immersed, see Fig. 10.46. Take the water density to be $\rho_\text{w} = 1.0 \times 10^3\,\text{kg/m}^3$. (a) What is the magnitude of the buoyant force F_B on the wood? (b) What is the magnitude of the tension T in the string?

Fig. 10.46 See Exercise (23)



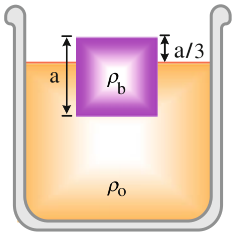
(24) Figure 10.47 shows a metal ball weighing $T = W = 9.5 \times 10^{-2}$ N in air. When the ball is immersed in water of density $\rho_\text{w} = 10^3\,\text{kg/m}^3$, it is found that it has an apparent weight $T_\text{w} = W_\text{w} = 7.0 \times 10^{-2}$ N. Find the density of the metal.

Fig. 10.47 See Exercise (24)



(25) A cubical block of side $a = 0.75$ cm floats in oil of density $\rho_o = 800\,\text{kg/m}^3$ with one-third of its side out the oil, see Fig. 10.48. (a) What is the magnitude of the buoyant force on the cube? (b) What is the density ρ_b of the block?

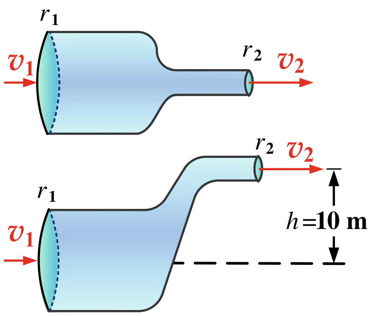
Fig. 10.48 See Exercise (25)



Section 10.5 Fluid Dynamics

(26) A fluid flows in a cylindrical pipe of radius r_1 with a speed v_1. (a) What would be the speed of this fluid at a point where the fluid is confined to a cylindrical part of radius $r_2 = r_1/5$, see the top part of Fig. 10.49. (b) What is the effect of elevating the constriction in the pipe by $h = 10$ m, see the lower part of Fig. 10.49?

Fig. 10.49 See Exercise (26)



(27) Water with a density of $10^3\,\text{kg/m}^3$ is flowing steadily through a closed-pipe system via the motor M, see Fig. 10.50. At height y_1, the water speed and pressure are $v_1 = 4$ m/s and $P_1 = 30$ kPa, respectively. At height y_2, which is a point higher than y_1 by a height $h = 2$ m, the water speed is $v_2 = 6$ m/s. (a) What is the pressure at y_2 ? (b) What would be the pressure at y_2 if the water in the closed system was to stop flowing and the pressure at y_1 were 25 kPa?

(28) A large tank is kept full at a height of $h = 4$ m as shown in Fig. 10.51. Take the water density to be $\rho = 1.0 \times 10^3\,\text{kg/m}^3$. (a) Find the speed v_2 of the jet

of water emerging from a small pipe at the bottom of the tank. (b) If $y_2 = 2$ m, then find the horizontal distance x (from the base of the tank) that the water stream travels before striking the floor.

Fig. 10.50 See Exercise (27)

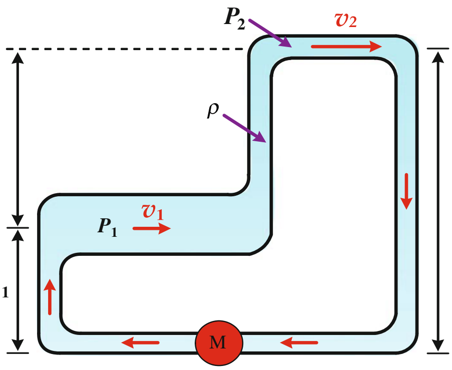
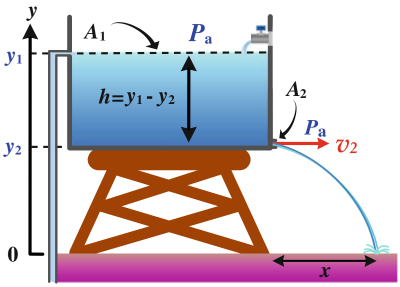


Fig. 10.51 See Exercise (28)



(29) Assume the faucet in the previous exercise is closed. In terms of the cross-sectional areas A_2 and A_1 of the small pipe and the tank, respectively, show that v_2 and x depend on the variable height h as follows:

$$v_2 = \sqrt{\frac{2gh}{[1 - (A_2/A_1)^2]}} \quad , \quad x = 2\sqrt{\frac{y_2 h}{[1 - (A_2/A_1)^2]}}$$

(30) Figure 10.52 shows a tube of uniform cross section that is filled with water to siphon water at a steady rate from a large vessel. (a) Use Torricelli's approach to find an expression for the speed at level C. (b) Use Bernoulli's equation to find an expression for the pressure at level B in terms of P_a, H, and h. (c) Find the maximum value H_{max} of H for which the siphon will work. (d) Calculate the answers to the previous parts when $\rho_w = 10^3$ kg/m^3, $P_a = 100$ kPa, $H = 2$ m, and $h = 3$ m.

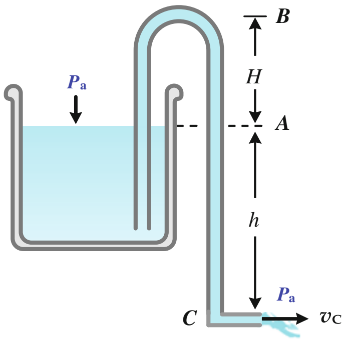


Fig. 10.52 See Exercise (30)

(31) A child tows a thin piece of wood of surface area $A = 200\ \text{cm}^2$ through a water puddle at a constant speed $v = 15$ cm/s, see Fig. 10.53. The depth of the water puddle is $d = 2$ mm and its viscosity is $\eta = 1$ cP. Assume that the velocity gradient is constant from the bottom to the surface of the water. Find the horizontal force component F exerted by the tow cord.

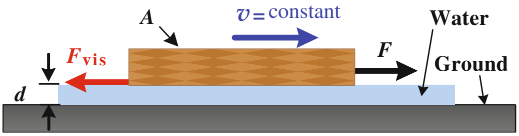


Fig. 10.53 See Exercise (31)

(32) A steel plate of weight $W = 0.5$ N and area $A = 0.2\ \text{m}^2$ is placed over a thin film of lubricant (oil) of thickness $d = 0.2$ mm sprayed over a flat surface inclined at an angle $\theta = 30°$, as shown in Fig. 10.54. What is the value of the constant speed of the plate v assuming that the rate of shear strain, v/d, is constant across the thickness of the film and the viscosity of the oil is $\eta = 0.05$ Pa.s.

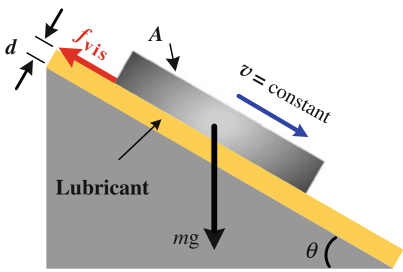
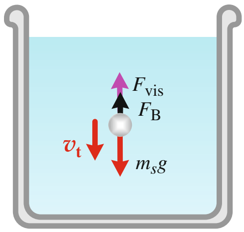


Fig. 10.54 See Exercise (32)

(33) How fast will an aluminum sphere of radius 1 mm fall through water at 20°C once its terminal speed has been reached, see Fig. 10.55? Assume that the

viscosity of water is $\eta = 8.5\,\mathrm{Pl}$ and that the aluminum sphere's density is $\rho_s = 2.7 \times 10^3\ \mathrm{kg/m^3}$.

Fig. 10.55 See Exercise (33)



(34) The viscous force on a liquid flowing steadily through a cylindrical pipe of length L is given by:

$$\boxed{F_{vis} = 4\pi\,\eta L v_m}$$

where η is the viscosity of the liquid and v_m is the maximum speed of the liquid which occurs along the central axis of the pipe, see Fig. 10.56. If the pressures in the rear and front horizontal segments of the pipe are respectively P_1 and P_2, where $P_1 > P_2$, then show that v_m will be given by:

$$v_m = (P_1 - P_2)\,r^2/4\eta L$$

where r is the radius of the cylinder.

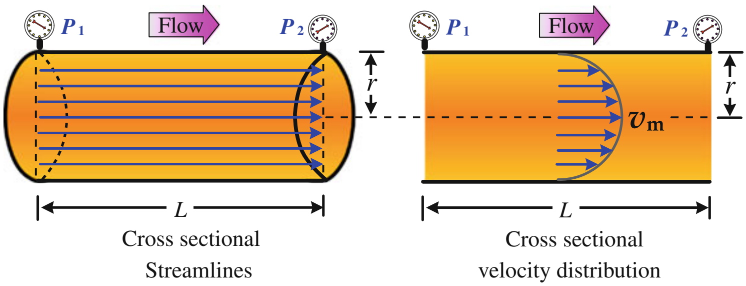


Fig. 10.56 See Exercise (34)

(35) Blood of viscosity $\eta = 4 \times 10^{-3}$ PI is passing through a capillary of length $L = 1$ mm and radius $r = 2\,\mu m$. If the speed of this blood as it travels through the center of this capillary is found to be $v_m = 0.66$ mm/s, calculate the blood pressure (in pascal and mm Hg) using the result from the previous exercise.

Part III
Introductory Thermodynamics

Thermal Properties of Matter

11

In this chapter, we introduce a physical quantity known as **temperature**, which is one of the seven SI base quantities. Temperature is associated with our sense of *hot* and *cold*. Physicists and engineers measure temperature more objectively using the **Kelvin scale**, which is independent of the properties of any substance. We will study the effect of temperature on matter; solid, liquid, and gas.

11.1 Temperature

The Kelvin Scale

The limiting temperature of a body is taken as the zero of the Kelvin scale, and called the *absolute zero*.

To set up the Kelvin temperature scale, we select a standard fixed point and give it a standard fixed point temperature. We select the **triple point of water**, where liquid water, solid ice, and water vapor can coexist in thermal equilibrium at only specific values of pressure and temperature. By international agreement, at water vapor pressure of 4.58 mm Hg, the temperature of this mixture has been assigned a value 273.16 Kelvins, written as 273.16 K. That is:

$$\boxed{T_3 = 273.16\ \text{K}} \qquad \text{(Triple-point temperature)} \tag{11.1}$$

where the subscript 3 denotes the triple point. This agreement sets the size of the kelvin as 1/273.16 of the difference between absolute zero and the triple-point temperature of water. This scale is used mostly in basic scientific calculations and studies.

H. A. Radi and J. O. Rasmussen, *Principles of Physics*,
Undergraduate Lecture Notes in Physics, DOI: 10.1007/978-3-642-23026-4_11,

On the Kelvin scale, measurements show that the lowest reached temperature is $\sim 10^{-10}$ K, and the freezing and boiling (at 1 atm. Pressure) temperature points are:

$$T_{\text{ice}} = 273.15\,\text{K} \quad \text{and} \quad T_{\text{steam}} = 373.15\,\text{K}$$

The Celsius Scale

The symbol °C stands for **degrees Celsius**. The *size* of 1 °C on the Celsius scale is the same as the *size* of 1 K on the Kelvin scale. However, the zero of the Celsius scale is shifted by 273.15° with respect to the absolute zero of the Kelvin scale. On the Celsius scale, any temperature value T_C is related to its Kelvin equivalent T by:

$$T_C = T - 273.15, \;\; (°\text{C}) \quad \text{or} \quad T = T_C + 273.15, \;\; (\text{K}) \tag{11.2}$$

For example, the Celsius temperature of the triple point of water is 0.01 °C, because $T_3 = 273.16$ K and the ice point (273.15 K) corresponds to 0.00 °C, and the steam point (373.15 K) corresponds to 100.00 °C. Note that we do not use a degree mark in reporting Kelvin temperatures.

The Fahrenheit Scale

The symbol °F stands for **degrees Fahrenheit**. The Fahrenheit scale has a smaller degree size than the Celsius and has a different zero (the freezing point of a certain concentration of salt water). The relation between the Celsius and Fahrenheit scales is:

$$T_F = \tfrac{9}{5} T_C + 32 \quad (°\text{F}) \tag{11.3}$$

Accordingly, 9 degrees on the Fahrenheit scale equals 5 degrees on the Celsius scale. Moreover, 0 °C = 32 °F and 100 °C = 212 °F. Table 11.1 shows some corresponding temperatures and Fig. 11.1 compares graphically the Kelvin, Celsius, and Fahrenheit scales.

Table 11.1 Some corresponding temperatures

Temperature	K	°C	°F
Boiling point of water (at 1 atm.)	373.15	100	212
Normal body temperature (average)	310.15	37	98.6
Triple point of water	273.16	0.01	32.02
Freezing point of water	273.15	0	32
Triple point of hydrogen	13.81	−259.34	−434.82
Absolute zero	0	−273.15	−459.67

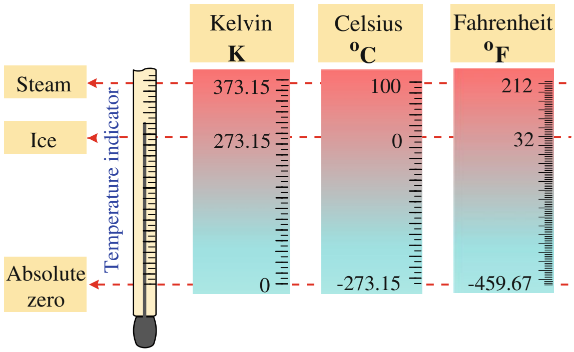


Fig. 11.1 The Kelvin, Celsius, and Fahrenheit temperature scales

Example 11.1

The normal boiling point of nitrogen is $-195.75\,°\mathrm{C}$. (a) What is this temperature in Kelvin and in Fahrenheit? (b) If the temperature changes from $-195.75\,°\mathrm{C}$ to $-100\,°\mathrm{C}$, find the change in the temperature on the Fahrenheit scale.

Solution: (a) Substituting $T_C = -195.75\,°\mathrm{C}$ into Eq. 11.2, we get:

$$T = T_C + 273.15 = -195.75 + 273.15 = 77.4\ \mathrm{K}$$

Also, from Eq. 11.3, we get:

$$T_F = \tfrac{9}{5} T_C + 32 = \tfrac{9}{5} \times (-195.75) + 32 = -320.35\,°\mathrm{F}$$

Thus, $-195.75\,°\mathrm{C}$, $77.4\,\mathrm{K}$, and $-320.35\,°\mathrm{F}$ are equivalent temperatures on different scales.

(b) For a change $\Delta T_C = [-100\,°\mathrm{C} - (-195.75\,°\mathrm{C})] = 95.75\,\mathrm{C}°$, we use Eq. 11.3 to find the change in temperature on the Fahrenheit scale as:

$$\Delta T_{\mathrm{F}} = \tfrac{9}{5}\Delta T_{\mathrm{C}} = \tfrac{9}{5}[-100 - (-195.75)] = 172.35\,\mathrm{F}^{\circ}$$

Thus, a change $95.75\,\mathrm{C}^{\circ} = 172.35\,\mathrm{F}^{\circ}$, where the notations C° and F° refer to temperature difference, not to be confused with actual temperatures, which are written in terms of symbols °C and °F.

11.2 Thermal Expansion of Solids and Liquids

Most bodies expand as their temperatures increase. This phenomenon plays an important role in numerous engineering applications, such as the joints in buildings, highways, railroad tracks, bridges . . . etc. Such thermal expansion is not always desirable.

Microscopically, thermal expansion arises from the change in the separation between the constituent atoms or molecules of the solid. To understand this, we consider a crystalline solid of a regular array of atoms or molecules held together by electrical forces. A mechanical model can be used to imagine the electrical interaction between the atoms or molecules, as shown in Fig. 11.2. At an ordinary temperature, the average spacing between the atoms is of the order of 10^{-10} m, and they vibrate about their equilibrium positions with an amplitude of about 10^{-11} m and a frequency of about 10^{13} Hz. As the temperature increases, the atoms vibrate with larger amplitudes and the average separation between them increases.

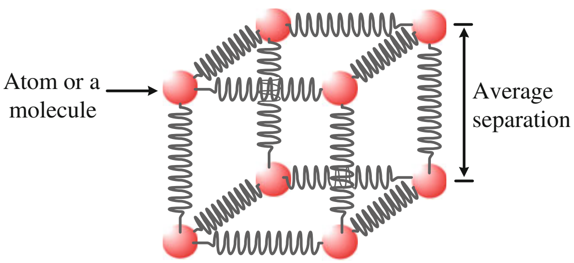


Fig. 11.2 A mechanical model representing the average spacing in a unit cell of crystalline solid at a given instant. Neighboring atoms or molecules (*red spheres*) are imagined to be attached to each other by elastic stiff springs, which represent the inter-atomic electric forces

If the thermal expansion of an object is sufficiently small compared to its initial dimensions, then the change in any dimension (length, width, or thickness) is, to a good approximation, a linear function of the temperature.

11.2.1 Linear Expansion

If a rod of length L and temperature T experiences a small change in temperature ΔT, its length changes by an amount ΔL, see Fig. 11.3. For a sufficiently small change ΔT, experiments show that ΔL is proportional to both L and ΔT. We introduce a proportionally coefficient α for the solid and write:

$$\boxed{\Delta L = \alpha L \Delta T} \tag{11.4}$$

where the proportionality constant α is called the **coefficient of linear expansion** for a given material. Note that Eq. 11.4 describes an expansion when ΔT is positive and a contraction when ΔT is negative. If we set[1] $\Delta T = 1\text{C}^\circ$, we see from Eq. 11.4 that α represents the fractional change in length $(\Delta L/L)$ per one degree change in temperature. Thus the unit of α is (degree^{-1}). An α value of $24 \times 10^{-6}\ (\text{C}^\circ)^{-1}$ means that the length L of an object changes by 24 parts per million for every Celsius degree change (C°) in temperature.

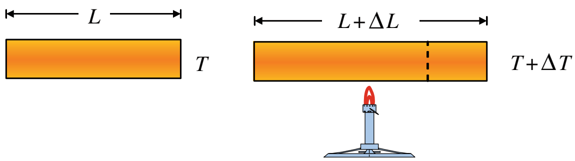


Fig. 11.3 The length L of the rod will increase by ΔL when its temperature changes from T to $T + \Delta T$. The expansion is exaggerated in the figure

Generally, the coefficient of linear expansion α varies with temperature, but this variation is negligible over the temperature range of most everyday measurements. Table 11.2 depicts some values of α.

Table 11.2 Coefficients of linear expansion α for some materials at room temperature (approximate)

Material Name	$\alpha\ (\text{C}^\circ)^{-1}$	Material Name	$\alpha\ (\text{C}^\circ)^{-1}$
Fused quartz	0.5×10^{-6}	Concrete	12×10^{-6}
Diamond	1.2×10^{-6}	Copper	17×10^{-6}
Glass (Pyrex)	3.2×10^{-6}	Brass & bronze	19×10^{-6}
Glass (ordinary)	9×10^{-6}	Aluminum	24×10^{-6}
Steel	11×10^{-6}	Lead	29×10^{-6}

[1] As in Example 11.1, temperature changes are expressed in units of Celsius degrees, abbreviated C° which should not to be confused with actual temperatures, written with the symbol °C and read degree Celsius.

Example 11.2

A steel rod has a length $L = 8\,\text{m}$ and radius $r = 1.5\,\text{cm}$ when the temperature is $20\,°\text{C}$. Take $\alpha = 11 \times 10^{-6}\,(\text{C}°)^{-1}$ and Young's modulus of the rod to be $Y = 200 \times 10^{9}\,\text{N/m}^2$ (a) What is its length on a hot day when the temperature is $50\,°\text{C}$? (b) If the rod's ends were originally fixed, then find the compression force on the rod?

Solution: (a) From Eq. 11.4, we can find the increase ΔL when the change in temperature is $\Delta T_C = 50\,°\text{C} - 20\,°\text{C} = 30\,\text{C}°$ as follows:

$$\Delta L = \alpha L \Delta T = [11 \times 10^{-6} (\text{C}°)^{-1}](8\text{ m})(30\text{ C}°) = 2.64 \times 10^{-3}\text{ m} = 2.64\text{ mm}$$

Therefore, the rod's new length at $50\,°\text{C}$ is 8.00264 m.

(b) If the rod is not allowed to expand, we then calculate what force would be required to compress the rod by the amount 2.64×10^{-3} m. From the definition of Young's modulus $Y = (F_\perp/A)/(\Delta L/L)$:

$$\begin{aligned} F_\perp &= \frac{AY\Delta L}{L} = \frac{\pi r^2 Y \Delta L}{L} \\ &= \frac{(3.14)(1.5 \times 10^{-2}\text{ m})^2(200 \times 10^9\text{ N/m}^2)(2.64 \times 10^{-3}\text{ m})}{(8\text{ m})} \\ &= 4.7 \times 10^4\text{ N} \end{aligned}$$

Note that this answer is independent of the length L.

11.2.2 Volume Expansion

Not only does the length of an object increase with temperature, but its area and volume change as well. The change in volume ΔV at a constant pressure is proportional to the original volume V and to the change in temperature ΔT according to the following relation:

$$\boxed{\Delta V = \beta\, V \Delta T} \tag{11.5}$$

where the proportionality constant β is called the **coefficient of volume expansion** for a given solid or liquid. Setting $\Delta T = 1\,\text{C}°$ in Eq. 11.5, we see that β is numerically equal to the fractional change in volume $(\Delta V/V)$ per one degree change in temperature. Thus, like α, the unit of β is (deg^{-1}). For example, a β value of

$24 \times 10^{-3}\ (\mathrm{C}^\circ)^{-1}$ means that the volume V of a solid or liquid changes by 24 parts in 10^3 for every Celsius degree change (C°) in temperature.

An **isotropic solid** is a solid that has a coefficient of linear expansion that is equal in all directions. Accordingly, for an isotropic solid, the coefficient of volume expansion is approximately three times the linear expansion coefficient, i.e. $\beta = 3\alpha$. Table 11.3 depicts some values of β.

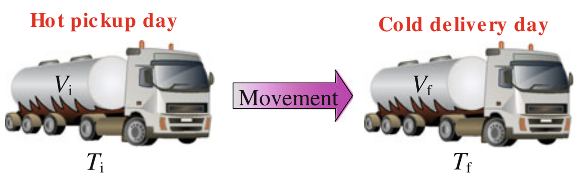
Table 11.3 Coefficients of volume expansion β for some materials at room temperature (approximate)

Material Name	$\beta\ (\mathrm{C}^\circ)^{-1}$	Material Name	$\beta\ (\mathrm{C}^\circ)^{-1}$
Alcohol, ethyl	1.12×10^{-4}	Water	6.3×10^{-4}
Benzene	1.24×10^{-4}	Turpentine	9×10^{-4}
Acetone	1.5×10^{-4}	Gasoline	9.6×10^{-4}
Mercury	1.82×10^{-4}	Air	3.67×10^{-3}
Glycerin	4.85×10^{-4}	Helium	3.665×10^{-3}

Example 11.3

On a hot day, when the temperature was $T_\mathrm{i} = 45\,^\circ\mathrm{C}$, an oil trucker fully loaded his truck from an oil station with 10,000 gal of gasoline (1 gallon $\simeq$ 3.8 liter). On his way to a delivery city, he encountered cold weather, where the temperature went down to $T_\mathrm{f} = 20\,^\circ\mathrm{C}$, see Fig. 11.4. The coefficients of volume expansion of gasoline and steel are $\beta = 9.6 \times 10^{-4}\ (\mathrm{C}^\circ)^{-1}$ and $\beta = 11 \times 10^{-6}\ (\mathrm{C}^\circ)^{-1}$, respectively. How many gallons did the trucker deliver?

Fig. 11.4



Solution: The change in temperature from the production city to the delivery city is $\Delta T_\mathrm{C} = T_\mathrm{f} - T_\mathrm{i} = 20\,^\circ\mathrm{C} - 45\,^\circ\mathrm{C} = -25\,\mathrm{C}^\circ$. From Eq. 11.5, we can find the change in the gasoline volume ΔV as follows:

$$\begin{aligned}\Delta V = \beta V \Delta T &= [9.6 \times 10^{-4}\ (\mathrm{C}^\circ)^{-1}](10{,}000\ \mathrm{gal})(-25\ \mathrm{C}^\circ)\\ &= -240\ \mathrm{gal}\end{aligned}$$

Thus, the amount of gasoline delivered was:

$$V_f = V_i + \Delta V = 10{,}000\,\text{gal} - 240\,\text{gal}$$
$$= 9{,}760\,\text{gal}$$

The thermal expansion of the volume of the steel tank can also be calculated as follows:

$$\Delta V = \beta V \Delta T = [11 \times 10^{-6}\,(\text{C}^\circ)^{-1}](10{,}000\,\text{gal})(-25\,\text{C}^\circ)$$
$$= -2.75\ \text{gal}$$

This change is very small and has nothing to do with the problem, since the decrease in the gasoline volume is much bigger than that of the steel. Question: Who paid for the missing gasoline?

The most common liquid, water, does not behave like other liquids, see Fig. 11.5a. Above 4 °C, water expands as its temperature rises, and thus its density decreases as shown in the Fig. 11.5b. Between 0 °C and 4 °C, however, water contracts as its temperature increases, and thus its density increases. Hence, the density of water reaches a maximum value of 1,000 kg/m^3 at 4 °C.

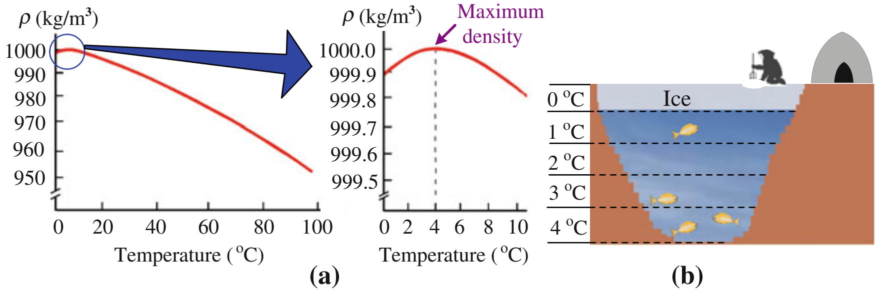


Fig. 11.5 (a) The density of water versus temperature at atmospheric pressure. The maximum density of 10^3 kg/m^3 occurs at 4 °C. (b) The warmer water from 1 to 4 degrees stays below the ice because it is more dense than the ice

This unusual thermal expansion behavior of water explains why a pond or lake frezes only at its surface. As the water on the surface is cooled towards the freezing point, it becomes denser (heavier) than the water below it and sinks to the bottom. Warmer, less dense (lighter) water rises upwards to take its place and this in turn is also cooled down. The water only stops circulating this way when it has all cooled to 4 °C (the maximum density). Further cooling below 4 °C makes the water on the surface less dense than the water below it, so it stays on the surface until it freezes. In time, ice continues to build up at the surface, and the denser warmer water at the bottom is unlikely to cool any further because it does not circulate, and water near the bottom remains at 4 °C. The water temperature stabilizes as shown in Fig. 11.5b. Fish can survive by staying in the warmer deeper water.

11.3 The Ideal Gas

Let us examine the basic thermal properties of gases from an elementary point of view. To do that, we will consider the properties of a gas of mass m confined within a container of volume V at absolute pressure P and temperature T. The relation that interrelates these quantities, the *equation of state,* is complicated. However, if the gas is maintained at a very low pressure (or density), this equation is found experimentally to be quite simple, and this keeps the mathematics relatively simple. This model is known as the **ideal gas model** and the low-pressure gas is commonly referred to as an **ideal gas**. Most gases at room temperature and atmospheric pressure behave as ideal gases.

Equation of State of an Ideal Gas

One mole (1 mol) of a substance is the amount of substance that contains as many particles as in exactly 12 g of the isotope carbon-12. Although the **mole** is one of the seven SI base units, it is convenient to introduce the **One kilomole** (1 kmol) of a substance as the amount of substance that contains as many particles as in exactly 12 kilograms of the isotope carbon-12. Thus:

> One kilomole (1 kmol):
> One kmol is the *number* of atoms in a 12 kg sample of pure carbon-12.

This number is called **Avogadro's number**, N_A, after A. Avogadro, who suggested that *all gases* contain the same number of particles (atoms or molecules) when they occupy the same volume under the same conditions of pressure and temperature. Avogadro's number is determined experimentally to be:

$$N_A = 6.022 \times 10^{26} \text{ particles/kmol} \tag{11.6}$$

Depending on the kind of study, atoms or molecules will replace the term "particles".

In addition, the **molar mass** of each chemical element is defined as:

Molar Mass (M):
The molar mass M of a chemical element is its atomic mass expressed in g/mol or equivalently in kg/kmol.

For example, the mass of one ^{12}C atom is 12 u (12 atomic mass units, see Chap. 1); then the *molar mass M* of ^{12}C is 12 kg/kmol and contains N_A atoms. For a molecular substance or chemical compound, we add up the molar mass from its molecular formula. Thus, the *molar mass M* of nitrogen gas (N_2) is 28 kg/kmol and it consists of N_A molecules; this is because the mass of one nitrogen atom is 14 u.

Now, for an ideal gas of mass m (in kg) confined to a container of volume V (in m^3) at a pressure P (in Pa) and temperature T (in K), it is convenient to express the amount of the gas in terms of the **number of kilomoles** n, see Fig. 11.6. This number is related to the gas mass m and its molar mass M through the expression:

$$\boxed{n = \frac{m}{M}} \qquad (m \text{ in kg and } M \text{ in kg/kmol}) \tag{11.7}$$

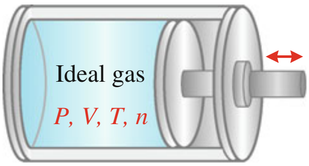


Fig. 11.6 An ideal gas defined by P, V, T, and n is contained in a cylinder with a movable piston to allow the volume to be varied

Moreover, we consider that the volume of the container can be varied, and hence the gas volume, by means of the movable piston of Fig. 11.6.

For this system, experiments show that at low densities, all gases tend to obey the following **equation of state** (which is known as the **ideal gas law**):

$$\boxed{PV = nRT} \qquad \text{(The ideal gas law)} \tag{11.8}$$

where n is the number of kilomoles of gas present and R is the **gas constant** which is determined from experiments to have the same value for all gases, namely:

$$\boxed{R = \begin{cases} 8.314 \times 10^3 \text{ J/kmol.K} & \text{if } n \text{ is the number of kmol} \\ 8.314 \text{ J/mol.K} & \text{if } n \text{ is the number of mol} \end{cases}} \tag{11.9}$$

Using this value of R and the equation of state, Eq. 11.8, we can find the volume occupied by 1 kmol (kilomolar volume) of any ideal gas at the *standard temperature and pressure* (**STP**), which means temperature of 0 °C (273.15 K) and atmospheric pressure (1 atm), as follows:

$$V = \frac{nRT}{P} = \frac{(1\text{ kmol})(8.314 \times 10^3\text{ J/kmol.K})(273.15\text{ K})}{(1.013 \times 10^5\text{ Pa})} = 22.42\text{ m}^3 = 22{,}420\text{ L}$$

where $1\text{ L} \equiv 1\text{ liter} = 10^3\text{ cm}^3 = 10^{-3}\text{ m}^3$. Thus:

> Volume of 1 kmol:
> One kmol of any ideal gas at atmospheric pressure and at 0 °C occupies a space of $22.42\text{ m}^3 = 22{,}420$ L.

We can express the ideal gas law in terms of the total number of molecules N by using the fact that N equals the product of the number of kmol n and Avogadro's number N_A, i.e. $N = n\,N_A$. Thus:

$$PV = nRT = \frac{N}{N_A}RT = N\frac{R}{N_A}T$$

or,

$$\boxed{PV = Nk_BT} \tag{11.10}$$

where k_B is called Boltzmann's constant, which has the value:

$$\boxed{k_B = \frac{R}{N_A} = \frac{8{,}314\text{ J/kmol.K}}{6.022 \times 10^{26}\text{ molecules/kmol}} = 1.38 \times 10^{-23}\text{ J/K}} \tag{11.11}$$

Equation 11.10 indicates that the pressure of a fixed volume of gas depends only on the temperature and the number of molecules in that volume.

Example 11.4

According to the periodic table of elements, see Appendix C, the molar mass of copper is $M(\text{Cu}) = 63.546$ kg/kmol. Use this information to find the mass of one atom.

Solution: The molar mass of $^{63.5}$Cu is $M(\text{Cu}) = 63.546$ kg/kmol and contains $N_A = 6.022 \times 10^{26}$ atoms/kmol. The mass of 1 atom is then:

$$\text{Mass of one Cu atom} = \frac{M(\text{Cu})}{N_A} = \frac{63.546\,\text{kg/kmol}}{6.022 \times 10^{26}\,\text{atoms/kmol}}$$
$$= 1.059 \times 10^{-25}\,\text{kg/atom}$$

Example 11.5

The main constituents of air are nitrogen molecules of molar mass $M(N_2) = 28\,\text{kg/kmol}$ and oxygen molecules of molar mass $M(O_2) = 32\,\text{kg/kmol}$ with approximate proportions of 80% and 20%, respectively. Using the ideal gas law, find the mass of air in a volume of 50 cm^3 at a pressure of 700 torr and temperature of 20 °C.

Solution: The molar mass of air can be obtained from the ratios of the two gases as follows:

$$M(\text{air}) = 0.8\,M(N_2) + 0.2\,M(O_2)$$
$$= 0.8\ (28\ \text{kg/kmol}) + 0.2\ (32\ \text{kg/kmol}) = 28.8\ \text{kg/kmol}$$

The volume, pressure, and temperature values can be written as:

$$V = 50\ \text{cm}^3 = 5 \times 10^{-5}\ \text{m}^3$$
$$P = 700\ \text{torr} = 700\ \text{torr} \times \frac{1\ \text{atm}}{760\ \text{torr}} \times \frac{1.01 \times 10^5\ \text{Pa}}{1\ \text{atm}} = 9.3 \times 10^4\ \text{Pa}$$
$$T = 20\,°\text{C} = 20 + 273 = 293\,\text{K} \quad \text{(From now on, we ignore the 0.15 K)}$$

We can use the ideal-gas equation $PV = nRT$ with $n = m/M(\text{air})$, where m is the mass of air under consideration, to find m as follows:

$$m = \frac{PVM(\text{air})}{RT} = \frac{(9.3 \times 10^4\ \text{Pa})(5 \times 10^{-5}\ \text{m}^3)(28.8\ \text{kg/kmol})}{(8.314 \times 10^3\ \text{J/kmol.K})(293\ \text{K})} = 5.5 \times 10^{-5}\ \text{kg}$$

Example 11.6

(a) How many molecules are there in 1 cm^3 of air at room temperature (27 °C)? (b) How many kilomoles of air are in that volume? (c) The best vacuum that can be produced corresponds to a pressure of about 10^{-16} atm. How many molecules remain in 1 cm^3?

Solution: The number of molecules in 1 cm^3 can be calculated from the ideal gas equation $PV = Nk_B T$. (a) Rewriting the quantities given, we have:

$$V = 1\ \text{cm}^3 = 10^{-6}\ \text{m}^3$$
$$P = 1\ \text{atm} \simeq 10^5\ \text{Pa}$$
$$T = 27\,^\circ\text{C} = 27 + 273 = 300\ \text{K}$$

$$\text{Thus:}\quad N = \frac{PV}{k_B T} = \frac{(10^5\ \text{Pa})(10^{-6}\ \text{m}^3)}{(1.38 \times 10^{-23}\ \text{J/K})(300\ \text{K})} = 2.4 \times 10^{19}\ \text{molecules}$$

(b) We use the ideal gas equation $PV = nRT$ to calculate the number of kilomoles as follows:

$$n = \frac{PV}{RT} = \frac{(10^5\ \text{Pa})(10^{-6}\ \text{m}^3)}{(8.314 \times 10^3\ \text{J/kmol.K})(300\ \text{K})} = 4 \times 10^{-8}\ \text{kmol}$$

Also, we can use the relation $N = n\,N_A$ to get n as follows:

$$n = \frac{N}{N_A} = \frac{2.4 \times 10^{19}\ \text{molecules}}{6.022 \times 10^{26}\ \text{molecules/kmol}} = 4 \times 10^{-8}\ \text{kmol}$$

(c) Rewriting the quantities given, we have:

$$V = 1\ \text{cm}^3 = 10^{-6}\ \text{m}^3$$
$$P = 10^{-16}\ \text{atm} \simeq 10^{-11}\ \text{Pa}$$
$$T = 27\,^\circ\text{C} = 27 + 273 = 300\ \text{K}$$

$$\text{Thus:}\qquad N = \frac{PV}{k_B T} = \frac{(10^{-11}\ \text{Pa})(10^{-6}\ \text{m}^3)}{(1.38 \times 10^{-23}\ \text{J/K})(300\ \text{K})} = 2{,}415\ \text{molecules}$$

There are still a large number of molecules left in this 1 cm^3 vacuum.

Example 11.7

A metal barrel is filled with air and is closed firmly when the pressure is 1 atm and the temperature is 20 °C. On a hot sunny day, the barrel's temperature rises to 60 °C while its volume remains almost the same. Find the final pressure inside the barrel.

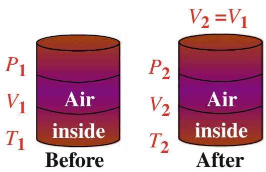
Solution: We mark the initial state of air with P_1, V_1, T_1 and final state with P_2, V_2, T_2, see Fig. 11.7. If no air escapes from the barrel, the number of moles of air n remains constant. Therefore, using the ideal gas law $PV = nRT$ in the initial and final states and the fact that $V_2 = V_1$, we get:

$$nR = \frac{P_1\,V_1}{T_1} = \frac{P_2\,V_2}{T_2} \quad \Rightarrow \quad \frac{P_1}{T_1} = \frac{P_2}{T_2} \quad \Rightarrow \quad P_2 = P_1\frac{T_2}{T_1}$$

The quantities given are: $\begin{cases} P_1 = 1 \text{ atm} = 1.01 \times 10^5 \text{ Pa} \\ T_1 = 20\,°\text{C} = 20 + 273 = 293 \text{ K} \\ T_2 = 60\,°\text{C} = 60 + 273 = 333 \text{ K} \end{cases}$

Thus: $$P_2 = (1\,\text{atm})\frac{333 \text{ K}}{293 \text{ K}} = 1.14 \text{ atm} = 1.15 \times 10^5 \text{ Pa}$$

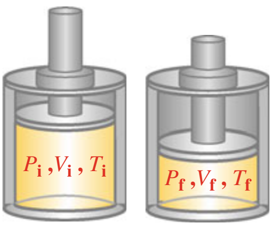
Fig. 11.7



Example 11.8

The initial volume, pressure, and temperature of helium gas trapped in a container with a movable piston are 2×10^{-3} m^3, 150 kPa, and 300 K, respectively; see Fig. 11.8. If the volume is decreased to 1.5×10^{-3} m^3 and the pressure increases to 300 kPa find the final temperature of the gas, assuming it behaves like an ideal gas.

Fig. 11.8



Solution: The initial state of helium is P_{i}, V_{i}, T_{i} and final state is P_{f}, V_{f}, T_{f}. With the use of the ideal gas law $PV = nRT$, we get:

$$\frac{P_{\mathrm{i}}\,V_{\mathrm{i}}}{T_{\mathrm{i}}} = \frac{P_{\mathrm{f}}\,V_{\mathrm{f}}}{T_{\mathrm{f}}} \quad \Rightarrow \quad T_{\mathrm{f}} = T_{\mathrm{i}}\frac{P_{\mathrm{f}}\,V_{\mathrm{f}}}{P_{\mathrm{i}}\,V_{\mathrm{i}}}$$

$$= (300\ \mathrm{K})\frac{(300\ \mathrm{kPa})(1.5\times10^{-3}\ \mathrm{m}^3)}{(150\ \mathrm{kPa})(2\times10^{-3}\ \mathrm{m}^3)} = 450\ \mathrm{K}$$

11.4 Exercises

Section 11.1 Temperature

(1) Convert the temperatures $-30\,°\mathrm{C}$, $10\,°\mathrm{C}$, and $50\,°\mathrm{C}$ to Kelvin and Fahrenheit.

(2) Express the normal human body temperature, $37\,°\mathrm{C}$, and the sun's surface temperature, $\sim 6000\,°\mathrm{C}$, in Fahrenheit and Kelvin.

(3) A Celsius thermometer indicates a temperature of $-40\,°\mathrm{C}$. (a) What Fahrenheit and Kelvin temperatures correspond to this Celsius temperature? (b) If the temperature changes from $-40\,°\mathrm{C}$ to $+10\,°\mathrm{C}$, find the change in temperature on the Fahrenheit scale.

(4) The normal melting point of gold is $1064.5\,°\mathrm{C}$ and its boiling point is $2660\,°\mathrm{C}$. (a) Convert these two values to the Fahrenheit and Kelvin scales. (b) Find the difference between those two values in Celsius. (c) Repeat (b) using the Kelvin scale.

(5) The height of an alcohol column in an alcohol thermometer has a length 12 cm at $0\,°\mathrm{C}$ and a length 22 cm at $100\,°\mathrm{C}$. Assume that the temperature and the length of the alcohol thermometer are linearly related. What is the temperature that the thermometer will measure if the alcohol column has a length 12.5 cm?

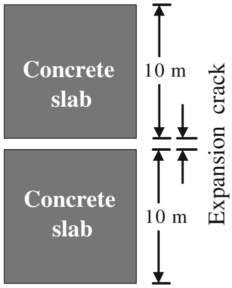
Section 11.2 Thermal Expansion of Solids and Liquids

(6) The Eiffel tower is built from iron and it is about 324 m high. Its coefficient of linear expansion is approximately $12\times10^{-6}\ (\mathrm{C}°)^{-1}$ and assumed constant. What is the increase in the tower's length when the temperature changes from $0\,°\mathrm{C}$ in winter to $30\,°\mathrm{C}$?

(7) A copper rod is 8 m long at 20 °C and has a coefficient of linear expansion $\alpha = 17 \times 10^{-6}\,(\mathrm{C}^\circ)^{-1}$. What is the increase in the rod's length when it is heated to 40 °C?

(8) A road is built from concrete slabs, each of 10 m long when formed at 10 °C, see Fig. 11.9. How wide should the expansion cracks between the slabs be at 10 °C to prevent road buckling if the range of temperature changes from −5 °C in winter to +40 °C in summer? The coefficient of linear expansion for concrete is $\alpha = 12 \times 10^{-6}\,(\mathrm{C}^\circ)^{-1}$.

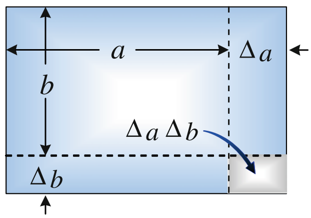
Fig. 11.9 See Exercise (8)



(9) An iron steam pipe is 100 m long at 0 °C and has a coefficient of linear expansion $\alpha = 10 \times 10^{-6}\,(\mathrm{C}^\circ)^{-1}$. What will be its length when heated to 100 °C?

(10) An ordinary glass window has a coefficient of linear expansion $\alpha = 9 \times 10^{-6}\,(\mathrm{C}^\circ)^{-1}$. At 20 °C the sides a and b have the values 1 m and 0.8 m respectively, see Fig. 11.10. By how much does the area increase when its temperature rises to 40 °C?

Fig. 11.10 See Exercise (10)



(11) A steel tape measure has a coefficient of linear expansion $\alpha = 12 \times 10^{-6}\,(\mathrm{C}^\circ)^{-1}$ and is calibrated at 20 °C. On a cold day when the temperature is −20 °C, what will be the percentage error for a reading made using this tape measure?

(12) A bar of length $L = 4$ m and linear expansion $\alpha = 25 \times 10^{-6}\,(\mathrm{C}^\circ)^{-1}$ has a crack at its center. The ends of the bars are fixed as shown in Fig. 11.11. As a result of a temperature rise of 40 C°, the bar buckles upwards, see Fig. 11.11. Find the vertical rise d of the bar's center.

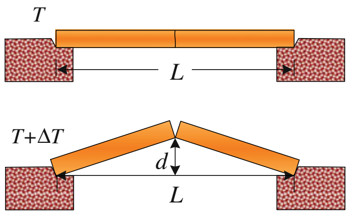


Fig. 11.11 See Exercise (12)

(13) A composite rod of length L is made from two different rods of lengths L_1 and L_2 with linear expansion coefficients of α_1 and α_2, respectively, see Fig. 11.12. (a) Show that the coefficient of linear expansion α for this composite rod is given by $\alpha = (\alpha_1 L_1 + \alpha_2 L_2)/L$. (b) Using the linear expansion coefficients of steel and brass given in Table 11.2, find L_1 and L_2 in the case where $L = 0.8$ m and $\alpha = 14 \times 10^{-6}\,(\mathrm{C}^\circ)^{-1}$.

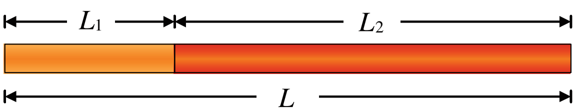


Fig. 11.12 See Exercise (13)

(14) A homogeneous metal ring of temperature T has inner and outer radii a and b, respectively. As the metal ring is heated to a temperature of $T + \Delta T$, its inner and outer radii increase linearly to $a + \Delta a$ and $b + \Delta b$ respectively, see Fig. 11.13. Show that the heating has no effect on the ratio between the inner and the outer radii.

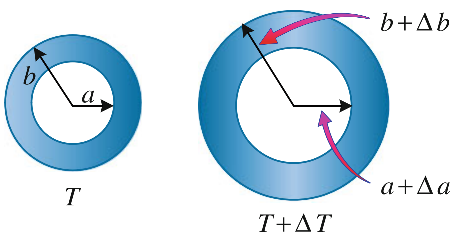


Fig. 11.13 See Exercise (14)

(15) A spherical brass plug has a diameter d of 10 cm at $T = 150\,\mathrm{C}^\circ$ and has a coefficient of linear expansion $\alpha = 19 \times 10^{-6}\,(\mathrm{C}^\circ)^{-1}$, see Fig. 11.14. At what temperature will its diameter be 9.950 cm?

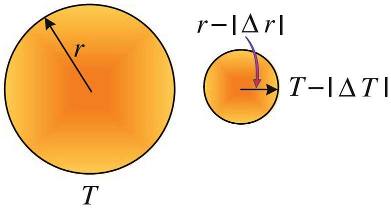


Fig. 11.14 See Exercise (15)

(16) Two rods of the same diameter, one made of brass of length $L_1 = 25$ cm, and the other rod made of steel of length $L_2 = 50$ cm, are placed end-to-end and pinned to two rigid supports, see Fig. 11.15. The Young's modulus for the brass and steel rods are $Y_1 = 100 \times 10^9\,\mathrm{N/m^2}$ and $Y_2 = 200 \times 10^9\,\mathrm{N/m^2}$ respectively, and their respective coefficients of linear expansion are $\alpha_1 = 18 \times 10^{-6}\,(\mathrm{C}^\circ)^{-1}$ and $\alpha_2 = 12 \times 10^{-6}\,(\mathrm{C}^\circ)^{-1}$. The two rods are heated until the rise in temperature becomes $\Delta T = 40\,\mathrm{C}^\circ$. What is the stress in each rod?

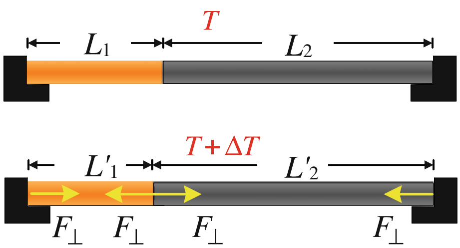


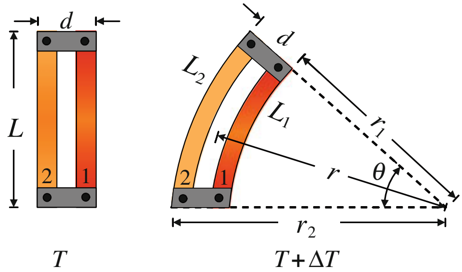
Fig. 11.15 See Exercise (16)

(17) Two parallel metal bars with the same length L and negligible width, but different linear expansion coefficients α_1 and α_2, are fixed at a distance d apart, see Fig. 11.16. When their temperature changes by ΔT, they will bend into two circular arcs intercepting at an angle θ as shown in Fig. 11.16. Find their mean radius of curvature r.

(18) Find the change in volume of an aluminum sphere that has a radius of 5 cm when it is heated from 0 °C to 300 °C. Assume that the coefficient of volume expansion is $\beta = 7.2 \times 10^{-5}\,(\mathrm{C}^\circ)^{-1}$.

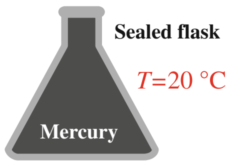
(19) A glass flask holds 50 cm^3 at a temperature of 20 °C. What is its capacity at 30 °C? Assume the coefficient of volume expansion of this glass flask is $2.7 \times 10^{-5}\,(\mathrm{C}^\circ)^{-1}$.

Fig. 11.16 See Exercise (17)



(20) A flask is completely filled with mercury at 20 °C and is sealed off, see Fig. 11.17. Ignore the expansion of the glass and assume that the bulk modulus of mercury is $B = 2.5 \times 10^9\ \mathrm{N/m^2}$ and its coefficient of volume expansion is $\beta = 1.82 \times 10^{-4}\ (\mathrm{C}^\circ)^{-1}$. Find the change in pressure inside the flask when it is heated to 100 °C.

Fig. 11.17 See Exercise (20)



(21) A glass flask of volume 200 cm^3 is filled with mercury when the temperature is $T = 20$ °C, see Fig. 11.18. The coefficient of volume expansion of the glass and mercury are $\beta = 1.2 \times 10^{-5}\ (\mathrm{C}^\circ)^{-1}$ and $\beta = 18 \times 10^{-5}\ (\mathrm{C}^\circ)^{-1}$ respectively. How much mercury will overflow when the temperature of the flask is raised to 100 °C?

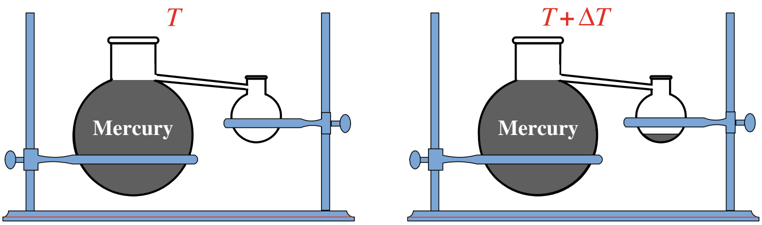


Fig. 11.18 See Exercise (21) (Take 1 atm $\simeq 10^5$ Pa unless specified)

Section 11.3 The Ideal Gas

(22) Find the density of nitrogen (N_2) and oxygen (O_2) at STP assuming they behave like an ideal gas.

(23) A tank contains $0.5\,\text{m}^3$ of nitrogen at a pressure of $1.5 \times 10^5\,\text{N/m}^2$ and a temperature of $27\,^\circ\text{C}$. (a) What will be the pressure if the volume is increased to $5.0\,\text{m}^3$ and the temperature is raised to $327\,^\circ\text{C}$? (b) Answer part (a) if the volume remains constant.

(24) A tank contains nitrogen N_2 at an absolute pressure of 2.5 atm. What will be the pressure of an equal mass of CO_2 that replaces the nitrogen at the same temperature?

(25) A tire is filled with air at $27\,^\circ\text{C}$ in a normal day to a gauge pressure of 2 atm. Then its temperature reaches $40\,^\circ\text{C}$ in a hot day. What fraction of the original air must be removed if the original pressure is to be restored?

(26) A 1,000 L container holds 50 kg of argon gas at $27\,^\circ\text{C}$. The molar mass of argon is $M = 40$ kg/kmol. What is the pressure of the gas?

(27) A bubble of air rises from the bottom of a lake, where the pressure is 3 atm and the temperature is $7\,^\circ\text{C}$, to the surface, where the pressure is 1 atm and the temperature is $27\,^\circ\text{C}$, see Fig. 11.19. What is the ratio of the volume of the bubble just as it reaches the surface to its volume at the bottom?

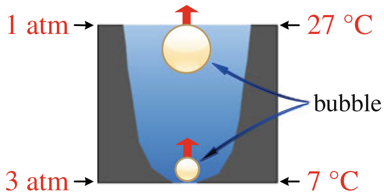
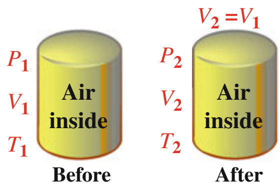


Fig. 11.19 See Exercise (27)

(28) (a) How many molecules are there in 1 L of air at a temperature of $27\,^\circ\text{C}$? (b) How many kilomoles of air are in that volume? (c) The best vacuum that can be produced corresponds to a pressure of about 10^{-16} atm. How many molecules remain in 1 L?

(29) A cylindrical metallic container is filled with air and is closed firmly when the pressure is $P_\text{i} = 1$ atm and the temperature is $T_\text{i} = 27\,^\circ\text{C}$. In a very hot sunny day, the container's temperature rises to $T_\text{f} = 70\,^\circ\text{C}$ while its volume remains almost the same, see Fig. 11.20. Find the final pressure inside the container.

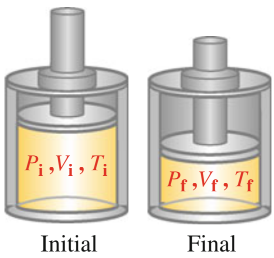
Fig. 11.20 See Exercise (29)



(30) The main constituents of air are nitrogen molecules of molar mass $M(N_2) = 28$ kg/kmol and oxygen molecules of molar mass $M(O_2) = 32$ kg/kmol with approximate ratios of 80 and 20%, respectively. Using the ideal gas law, find the mass of air in a volume of 1 L at atmospheric pressure and temperature of 27 °C.

(31) The initial volume, pressure, and temperature of helium gas trapped in a container with a movable piston are $V_i = 3$ L, $P_i = 150$ kPa, and $T_i = 300$ K, respectively, see Fig. 11.21. If the volume is decreased to $V_f = 2.5$ L and the pressure increases to $P_f = 300$ kPa, find the final temperature of the gas assuming that it behaves like an ideal gas.

Fig. 11.21 See Exercise (31)



(32) The volume of an oxygen tank is 50 L. As oxygen is withdrawn from the tank, the pressure of the remaining gas in the tank drops from 20 atm to 8 atm, and the temperature also drops from 30 to 10 °C. (a) How many kilograms of oxygen were originally in the tank? (b) How many kilograms of oxygen were withdrawn from the tank? (c) What volume would be occupied by the oxygen that withdrawn from the tank at a pressure of 1 atm and a temperature of 27 °C?

(33) A balloon filled with helium is left free on the surface of the ground when the temperature is 27 °C. When the balloon reaches an altitude of 3,000 m, where the temperature is 5 °C and the pressure is 0.65 atm, how will its volume compare to the original volume on the ground?

(34) The density of water vapor at exactly $100\,^{\circ}\mathrm{C}$ and $1\,\mathrm{atm} = 1.013 \times 10^5\,\mathrm{Pa}$ is $\rho = 0.598\,\mathrm{kg/m^3}$. Calculate the density of water vapor, with a molecular mass $M = 18\,\mathrm{kg/kmol}$, from the ideal gas law. Why would you expect a difference?

(35) An empty room of volume V contains air having a molar mass M. At atmospheric pressure P_a, the mass and temperature of the room are initially m_i and T_i, respectively. Assuming that the room is maintained at atmospheric pressure while its temperature is increased to T_f, show that the final mass of air left in the room, m_f, will be given by:

$$m_\mathrm{f} = m_\mathrm{i} - \frac{P_\mathrm{a}\,V\,M}{R}\left(\frac{1}{T_\mathrm{i}} - \frac{1}{T_\mathrm{f}}\right).$$

Heat and the First Law of Thermodynamics 12

Our focus in this chapter will be on the concept of internal energy, energy transfer, the first law of thermodynamics, and some applications of this law. The first law of thermodynamics expresses the general principle of conservation of energy. According to this law, an energy transfer to or from a system by either heat or work can change the internal energy of the system.

12.1 Heat and Thermal Energy

It is important to make a major distinction between *heat* and *internal energy* (*thermal energy*).

Internal energy is all the energy of a system that is associated with its microscopic constituents. Internal energy includes kinetic energy of random translational, rotational, and vibrational motion of molecules, potential energy of molecules and between molecules.

Heat is defined as the *transfer of energy* from one system to another due to a temperature difference between them.

12.1.1 Units of Heat, The Mechanical Equivalent of Heat

Previously, heat was measured in terms of its ability to raise the temperature of water. Thus, the **calorie** (cal), in cgs units, was defined as the amount of heat required to raise the temperature of 1 g of water from 14.5 to 15.5 °C. The

H. A. Radi and J. O. Rasmussen, *Principles of Physics*,
Undergraduate Lecture Notes in Physics, DOI: 10.1007/978-3-642-23026-4_12,

'Calorie' with a capital C, used by nutritionists, is a kilocalorie (1 Cal = 1 kcal = 10^3 cal).The **British Thermal Unit** (Btu) was also defined as the amount of heat required to raise the temperature of 1 lb of water from 63 to 64 °F. Since heat is now known as transferred energy, the SI unit for it is the **joule** (J).

In a famous experiment, see Fig. 12.1, Joule measured the calorie (cal) by converting mechanical energy into heat energy, expressed as an increase in water temperature.

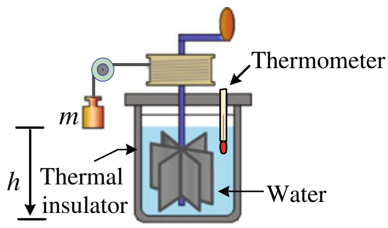


Fig. 12.1 Joule's experiment for measuring the mechanical equivalent of heat from the temperature rise in water

Joule found that the loss in mechanical energy is proportional to the increase in temperature of the water. The proportionality constant was found to be equal to 4,180 J/kg.C°. Hence, 4,180 J of mechanical energy will raise the temperature of 1 kg of water from 14.5 to 15.5 °C. One kilocalorie (1 kcal) is now defined to be exactly 4,186 J without reference to the heating of substance. Thus:

$$\boxed{1\,\text{kcal} = 4{,}186\,\text{J}} \tag{12.1}$$

The relations among the various heat units are as follows:

$$\begin{aligned} &1\,\text{J} = 2.389 \times 10^{-4}\,\text{kcal} = 9.478 \times 10^{-4}\,\text{Btu} \\ &\qquad\qquad\qquad\text{or} \\ &1\,\text{kcal} = 4{,}186\,\text{J} = 3.968\,\text{Btu} \end{aligned} \tag{12.2}$$

12.1.2 Heat Capacity and Specific Heat

The quantity of heat energy Q required to raise the temperature of an object by some amount ΔT varies from one substance to another.

The **heat capacity** C of an object is defined as:

The Heat Capacity C:
The heat capacity C of an object of a particular material is defined as the amount of heat energy needed to raise the object's temperature by one degree Celsius.

Accordingly, if Q units of heat energy are required to change the temperature by $\Delta T = T_{\mathrm{f}} - T_{\mathrm{i}}$, where T_{i} and T_{f} are the initial and final temperatures of the object, then:

$$\boxed{Q = C\,\Delta T}\,, \quad \text{where} \quad \boxed{\Delta T = T_{\mathrm{f}} - T_{\mathrm{i}}} \tag{12.3}$$

Heat capacity C has the unit J/C°(≡J/K) or kcal/C°(≡kcal/K).

The heat capacity for any object is proportional to its mass m. For this reason, we define the "heat capacity per unit mass" or the **specific heat** c which refers to a unit mass of the material of which the object is made. Thus, with $C = m\,c$, Eq. 12.3 becomes:

$$\boxed{Q = m\,c\,\Delta T}\,, \quad \text{where} \quad \boxed{\Delta T = T_{\mathrm{f}} - T_{\mathrm{i}}} \tag{12.4}$$

Specific heat c has the unit:

$$\boxed{\begin{array}{c} \text{J/kg.C}^\circ \equiv \text{J/kg.K} \\ \text{or} \\ \text{kcal/kg.C}^\circ \equiv \text{kcal/kg.K} \end{array}}$$

The specific heat of water at 15 °C and atmospheric pressure is:

$$\boxed{\mathrm{c}_{\text{water}} = 4{,}186\,\text{J/kg.K} = 1\,\text{kcal/kg.C}^\circ}$$

Note that, when heat energy is added to objects, Q and ΔT are both positive, i.e. the temperature increases. Likewise, when heat is removed from objects, Q and ΔT are both negative, i.e. the temperature decreases.

In general, specific heat c varies with temperature. However, if temperature intervals are not too big, the temperature variation can be ignored, and c can be treated as a constant. For example, the specific heat of water varies by about 1% from 0 to 100 °C at atmospheric pressure. Table 12.1 presents some specific heat values for various substances, measured at room temperature and atmospheric pressure.

Table 12.1 Specific heat c of some substances at atmospheric pressure and room temperature (20 °C) with few exceptions

Substance	Specific heat	
	J/kg.C°	kcal/kg.C°
Silver	230	0.0564
Copper	390	0.0923
Iron or steel	450	0.107
Aluminum	900	0.215
Brass	380	0.092
Granite	790	0.19
Glass	840	0.20
Ice (−5 °C)	2,100	0.50
Ice (−10 °C)	2,220	0.530
Mercury	140	0.033
Alcohol (Ethyl)	2,400	0.58
Seawater	3,900	0.93
Water (15 °C)	4,186	1
Steam (100 °C)	2,010	0.48

Measuring Specific Heat

Figure 12.2 shows an example of a calorimeter, which is a device used to determine the specific heat of a solid or liquid substance. The substance (represented by a circular object, having a specific heat c_x and mass m_x) is heated up to some known initial temperature T_x, and then placed in a perfectly insulated vessel containing water of specific heat c_w, mass m_w, and initial temperature T_w. If T_f is the final temperature after reaching equilibrium, then $T_w < T_f < T_x$. Using Eq. 12.4, we calculate the heat gained by the water to be $Q = m_w\, c_w\, (T_f - T_w)$, and calculate the heat energy lost by the object to be $-Q = m_x\, c_x\, (T_f - T_x)$.

Assuming that the entire system does not lose or gain any heat from its surrounding, then the heat gained by the water must equal the heat lost by the object. That is:

$$Q = m_w\, c_w\, (T_f - T_w) = -m_x\, c_x\, (T_f - T_x) \tag{12.5}$$

Solving for c_x gives:

$$c_x = c_w \frac{m_w}{m_x} \frac{(T_f - T_w)}{(T_x - T_f)}, \quad (T_w < T_f < T_x) \tag{12.6}$$

When calculating c_x, we neglected heat exchange with the vessel, which is acceptable when the mass of the water is considerably larger than that of the vessel, and when the vessel has a negligible specific heat.

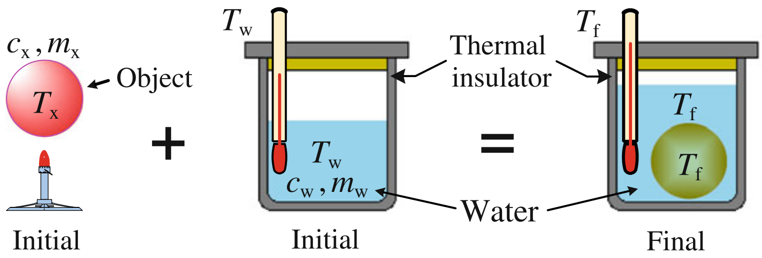


Fig. 12.2 In the method of mixtures, a calorimeter filled with water is used to find the specific heat of unknown heated objects

Example 12.1

The specific heat of zinc is 352 J/kg.C° for temperatures near 25 °C. Determine the amount of heat required to raise the temperature of 0.5 kg zinc from 20 to 30 °C. Take the specific heat to be constant in that temperature range.

Solution: The given values are $c = 352$ J/kg.C°, $m = 0.5$ kg, $T_i = 20$ °C, and $T_f = 30$ °C. The temperature change has the following magnitude:

$$\Delta T = T_f - T_i = 30\,°\text{C} - 20\,°\text{C} = 10\,\text{C}°$$

Using Eq. 12.4 we find the amount of heat required as follows:

$$Q = m\,c\,\Delta T = (0.5\,\text{kg})(352\,\text{J/kg.C}°)(10\,\text{C}°) = 1{,}760\,\text{J}$$

Example 12.2

A steel metal object of mass 0.05 kg is heated to 225 °C and then dropped into a vessel containing 0.55 kg of water initially at 18 °C. When equilibrium is reached, the temperature of the mixture is 20 °C. Find the specific heat of the metal.

Solution: For the steel metal object, we are given $m_x = 0.05$ kg and $T_x = 225$ °C, but its specific heat c_x is unknown. For water, the known values are $m_w = 0.55$ kg, $T_w = 18$ °C, and $c_w = 4{,}186$ J/kg.C° (Table 12.1). For the mixture, the equilibrium temperature occurs at $T_f = 20$ °C. Since the heat gained by the water is equal in magnitude to the heat lost by the steel, see Eq. 12.6 and Fig. 12.2, then we must have:

$$m_w\, c_w\, (T_f - T_w) = -m_x\, c_x\, (T_f - T_x), \quad (T_w < T_f < T_x)$$

Solving for c_x we get:

$$\begin{aligned} c_x &= c_w \frac{m_w}{m_x} \frac{(T_f - T_w)}{(T_x - T_f)} \\ &= (4{,}186\ \text{J/kg.C}^\circ) \frac{(0.55\ \text{kg})}{(0.05\ \text{kg})} \frac{(20\,^\circ\text{C} - 18\,^\circ\text{C})}{(225\,^\circ\text{C} - 20\,^\circ\text{C})} \\ &= 449\ \text{J/kg.C}^\circ \end{aligned}$$

12.1.3 Latent Heat

When heat energy is transferred from one substance to another, the temperature of the substance often changes. However, there are situations in which the transfer of energy does not change the temperature. Instead, the substance may change from one form to another. Such a change is commonly referred to as a **phase change or phase transition**, see Sect. 13.4 and especially Fig. 13.10.

We consider the following two main common phase changes:

1. A phase change from solid to liquid (as ice melting) and from liquid to gas (as water boiling), where heat energy is absorbed while the temperature remains constant.
2. A phase change from gas to liquid (as steam condensing) and from liquid to solid (as water freezing), where heat energy is released while the temperature remains constant.

The amount of heat energy per unit mass, L, that must be transferred when a substance completely undergoes a phase change *without changing temperature* is called the **latent heat** (literally, the "hidden" heat). If a quantity Q of heat energy transfer is required to change the phase of a pure substance of a mass m, then $L = Q/m$ characterizes an important thermal property of that substance. That is:

$$\boxed{Q = \pm m\,L} \tag{12.7}$$

A positive sign is used in this equation when energy enters the system, causing melting or vaporization of the substance, while a negative sign corresponds to energy leaving the system such that the substance condenses or solidifies.

When a substance experiences a phase change from solid to liquid by absorbing heat, the heat of transformation is called the **latent heat of fusion** L_F, see Fig. 12.3. When the substance releases heat and experiences a phase change from liquid back to solid, the heat of transformation is called the **latent heat of solidification** and is numerically equal to the latent heat of fusion, see Fig. 12.3. In the case of water at its normal melting or freezing temperature, we have:

$$L_F = 3.33 \times 10^5\ \text{J/kg} = 79.5\ \text{kcal/kg} = 6.01 \times 10^6\ \text{J/kmol} \tag{12.8}$$

When a substance experiences a phase change from liquid to gas by absorbing heat, the heat of transformation is called the **latent heat of vaporization** L_V, see Fig. 12.3. When the gas releases heat and experiences a phase change from gas back to liquid, the heat of transformation is called the **latent heat of condensation** and is numerically equal to the latent heat of vaporization, see Fig. 12.3. For water at its normal boiling and condensation temperatures, we have:

$$L_V = 2.256 \times 10^6\ \text{J/kg} = 539\ \text{kcal/kg} = 40.7 \times 10^3\ \text{J/kmol} \tag{12.9}$$

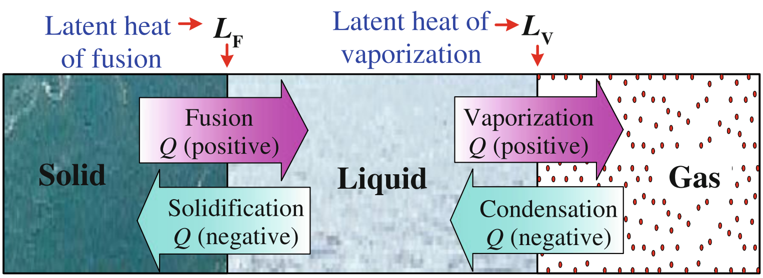


Fig. 12.3 A sketch showing heat of fusion/vaporization (positive Q) as well as heat of condensation/ solidification (negative Q)

Phase changes can be described in terms of a rearrangement of molecules when heat energy is added or removed from a substance. Consider, for example, the solid-to-liquid phase change. The molecules in the solid are strongly attracted to each other. As thermal energy is absorbed, the molecules usually move further apart and their potential energy increases. (Water-ice is an exception where there is shrinkage.) This leads to no change in the average kinetic energy of the molecules during the melting process, which involves molecules moving from fixed lattice positions to a random liquid state, the temperature stays constant. The latent heat of fusion is equal to the work done in separating the molecules during the melting process and hence breaking their bonds and transforming the substance from the ordered solid phase into the disordered liquid phase.

Now, we consider the liquid to gas phase change. The attractive forces between molecules in liquid form are stronger than in gas form because the average distance between molecules is smaller in the liquid state. As described in the solid-to-liquid phase transition, work must be done against these attractive forces. The latent heat of vaporization is the amount of energy added to the molecules in liquid form to accomplish this.

Table 12.2 gives some latent heats of various substances.

Table 12.2 Latent heats of fusion and vaporization (approximates)

Substance	Melting		Boiling	
	Melting point °C	Latent heat of fusion J/kg	Boiling point °C	Latent heat of vaporization J/kg
Helium	−270	5.23×10^3	−269	2.09×10^4
Nitrogen	−210	2.55×10^4	−196	2.01×10^5
Oxygen	−219	1.38×10^4	−183	2.13×10^5
Water	0	3.33×10^5	100	2.26×10^6
Sulfur	119	3.81×10^4	445	3.26×10^5
Lead	327	2.45×10^4	1,750	8.70×10^5
Aluminum	660	3.97×10^5	2,450	1.14×10^7
Silver	961	8.82×10^4	2,193	2.33×10^6
Gold	1,063	6.44×10^4	2,660	1.58×10^6
Copper	1,083	1.34×10^5	1,187	5.06×10^6
Silicon	1,410	1.65×10^6	2,447	1.06×10^7

To understand the role of latent heat in phase changes, we calculate the energy required to convert 1 g of ice at $-50\,°\mathrm{C}$ into steam at $150\,°\mathrm{C}$. Figure 12.4 shows the results obtained when energy is added gradually to 1 g of ice. The red curve of the figure is divided into the following five stages:

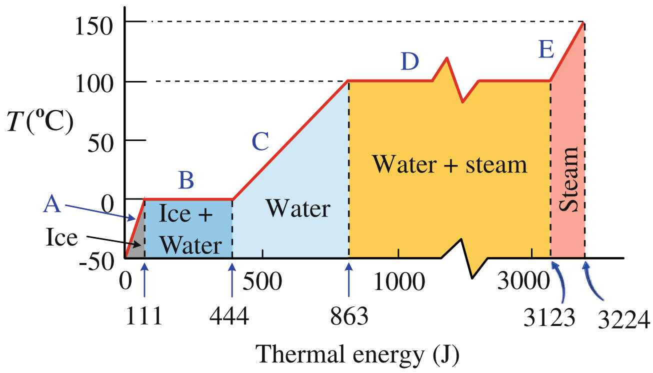


Fig. 12.4 Temperature as a function of the thermal energy added gradually to convert 1 g of ice at $-50\,°\mathrm{C}$ into steam at $150\,°\mathrm{C}$

Stage A—*Changing the temperature of ice from* -50 *to* $0\,°\mathrm{C}$:

With a specific heat of ice $c_i = 2{,}220\,\mathrm{J/kg.C°}$, the amount of heat added Q_A is:

$$Q_A = m_i c_i \Delta T = (1 \times 10^{-3}\,\mathrm{kg})(2{,}220\,\mathrm{J/kg.C°})(50\,\mathrm{C°}) = 111\,\mathrm{J}$$

Stage B—*Ice-water mixture remains at* $0\,°\mathrm{C}$ *(even heat is added)*:

With a latent heat of fusion $L_F = 3.33 \times 10^5\,\mathrm{J/kg}$, the amount of heatadded Q_B until all of the ice melts is:

$$Q_B = m\,L_F = (1 \times 10^{-3}\,\mathrm{kg})(3.33 \times 10^5\,\mathrm{J/kg}) = 3.33 \times 10^2\,\mathrm{J}$$

Stage C—*Changing the temperature of water from* 0 *to* $100\,°\mathrm{C}$:

With a specific heat of water $c_w = 4{,}186\,\mathrm{J/kg.C°}$, the amount of heat added Q_C is:

$$Q_C = m_w c_w \Delta T = (1 \times 10^{-3}\,\mathrm{kg})(4{,}186\,\mathrm{J/kg.C°})(100\,\mathrm{C°}) \simeq 419\,\mathrm{J}$$

Stage D—*Water-steam mixture remains at* 100 °C *(even heat is added)*:

With a latent heat of vaporization $L_V = 2.26 \times 10^6$ J/kg, the amount of heat added Q_D until all of the water evaporates is:

$$Q_D = m\,L_V = (1 \times 10^{-3}\,\text{kg})(2.26 \times 10^6\,\text{J/kg}) = 2.26 \times 10^3\,\text{J}$$

Stage E—*Changing the temperature of steam from* 100 *to* 150 °C:

With a specific heat of steam $c_s = 2{,}010$ J/kg.C°, the amount of heat added Q_E is:

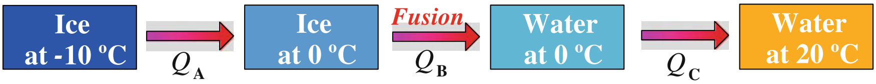
$$Q_E = m_s c_s \Delta T = (1 \times 10^{-3}\,\text{kg})(2{,}010\,\text{J/kg.C°})(50\,\text{C°}) \simeq 101\,\text{J}$$

The total heat added to change 1 g of ice at −50 °C to steam at 150 °C is $Q_{tot} = 3{,}224$ J. That is, if we cool 1 g of steam at 150 °C until we have ice at −50 °C, we must remove 3,224 J of heat.

Example 12.3

Find the quantity of heat required to convert ice of mass 500 g at −10 °C into water at 20 °C. The specific heat of ice is $c_i = 2{,}220$ J/kg.C°, the latent heat of fusion is $L_F = 3.33 \times 10^5$ J/kg, and the specific heat of water is $c_w = 4{,}186$ J/kg.C°.

Solution: The ice gains heat throughout the following three stages.



In stage A we raise the temperature of ice from −10 to 0 °C. Using Eq. 12.4 we get:

$$Q_A = m_i c_i\,\Delta T = (0.5\,\text{kg})(2{,}220\,\text{J/kg.C°})(10\,\text{C°}) = 11{,}100\,\text{J} = 11.1\,\text{kJ}$$

In stage B we melt the 500 g of ice at constant temperature (0 °C) by supplying the latent heat of fusion. Using Eq. 12.7 we get:

$$Q_B = m\,L_F = (0.5\,\text{kg})(3.33 \times 10^5\,\text{J/kg}) = 166{,}500\,\text{J} = 166.5\,\text{kJ}$$

In stage C we raise the temperature of water from 0 to 20 °C. Using Eq. 12.4 we get:

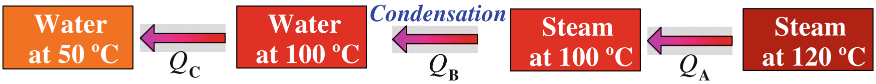
$$Q_C = m_w c_w \, \Delta T = (0.5\,\text{kg})(4{,}186\,\text{J/kg.C}^\circ)(20\,\text{C}^\circ) = 41{,}860\,\text{J} = 41.86\,\text{kJ}$$

Note that $Q_B > Q_C > Q_A$ and the total required heat is $Q_{tot} = 219.46\,\text{kJ}$.

Example 12.4

A glass beaker of water is at 20 °C. The beaker has a mass $m_g = 200$ g with specific heat $c_g = 840$ J/kg.C° and contains water of mass $m_w = 300$ g with specific heat $c_w = 4{,}186$ J/kg.C°. A quantity of steam initially at 120 °C is used to warm the system to 50 °C. If the specific heat of steam is $c_s = 2{,}010$ J/kg.C° and latent heat of vaporization is $L_V = 2.26 \times 10^6$ J/kg, what is the mass of the steam?

Solution: The heat lost by the steam equals the heat gained by both beaker and water. The steam loses heat over the stages shown below.



In the first stage, the steam is cooled from 120 to 100 °C, i.e. $\Delta T = T_f - T_i = 100\,°\text{C} - 120\,°\text{C} = -20\,\text{C}^\circ$. The heat liberated in this stage by the unknown mass m_s of steam is:

$$Q_A = m_s c_s \, \Delta T = m_s(2{,}010\,\text{J/kg.C}^\circ)(-20\,\text{C}^\circ) = -m_s(40{,}200\,\text{J/kg})$$

In the second stage, the steam is condensed to water at 100 °C. Since the latent heat of condensation equals the latent heat of vaporization, we use Eq. 12.7 to find the heat liberated as follows:

$$Q_B = -m_s \, L_V = -m_s(2.26 \times 10^6\,\text{J/kg})$$

In the last stage the temperature of water is reduced from 100 to 50 °C. This liberates an amount of heat given by:

$$Q_C = m_s c_w \, \Delta T = m_s(4{,}186\,\text{J/kg.C}^\circ)(-50\,\text{C}^\circ) = -m_s(209{,}300\,\text{J/kg})$$

The heat lost is thus $Q_{lost} = Q_A + Q_B + Q_C = -m_s(2{,}509{,}500\,\text{J/kg})$. The heat gained by the beaker and water system from 20 to 50 C° is:

$$\begin{aligned} Q_{gained} &= m_w c_w \, \Delta T + m_g c_g \, \Delta T = (m_w c_w + m_g c_g)\Delta T \\ &= [(0.3\,\text{kg})(4{,}186\,\text{J/kg.C}^\circ) + (0.2\,\text{kg})(840\,\text{J/kg.C}^\circ)](30\,\text{C}^\circ) \\ &= 42{,}714\,\text{J} \end{aligned}$$

If we equate the magnitude of heat lost by the steam, $|Q_{\text{lost}}|$, with the heat gained by the beaker and water system, Q_{gained}, we get:

$$m_s = 42{,}714\,\text{J}/(2{,}509{,}500\,\text{J/kg}) = 0.017\,\text{kg} = 17\,\text{g}$$

12.2 Heat and Work

In thermodynamics, when an isolated system is in thermal equilibrium internally, we describe its macroscopic state with the variables P, V, and T to represent pressure, volume and temperature. For such a system, we describe its microscopic state of internal energy with the variable E_{int} (some other textbooks use the symbol U).

Let us assume our system consists of gas confined to a cylinder with insulated walls and a movable frictionless piston of area A, as shown in Fig. 12.5. The cylinder rests on a heat reservoir whose temperature T is controlled by a knob. At equilibrium, the upward force on the piston due to the pressure of the confined gas is equal to the weight of the load on the top of the piston.

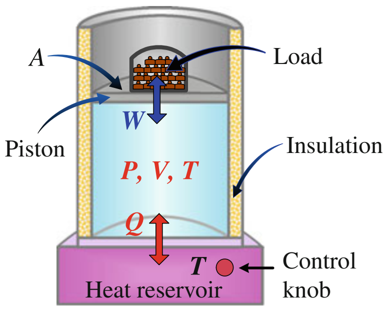


Fig. 12.5 Gas confined to a cylinder with a movable frictionless piston. The gas can do work W to raise or lower the piston. By regulating the temperature T of the thermal reservoir, by means of a control knob, a quantity of heat Q can be added or removed from the gas

Consider that we start at an *initial state* i, where the system is described to have pressure P_i, volume V_i, and temperature T_i. We then change the system to a *final state* f, described to have pressure P_f, volume V_f, and temperature T_f. The process of changing the system from the initial state to the final state is a **thermodynamic process**. During such a process, work is done *by* the system to raise the piston (positive work[1]) or lower it (negative work). In addition, heat may be transferred into the system from the thermal reservoir (positive heat) or vice versa. We assume that the state of the gas changes **quasi-statically**, i.e. slowly enough to allow the system to remain essentially in a thermodynamic equilibrium at all times.

Now, assume we reduce the load from the piston in such a way that the piston will move upward through a differential displacement $d\vec{s}$ with *almost* constant upward force $\vec{F}$, as shown in Fig. 12.6. From the definition of pressure, we have $F = PA$, where A is the area of the piston. The differential work dW done *by* the gas during the displacement is:

$$dW = \vec{F} \bullet d\vec{s} = F\,ds = P\,A\,ds$$

Since $A\,ds$ is the differential change in the volume of the gas dV (i.e. $dV = A\,ds$), we can express the work done by the gas as follows:

$$\boxed{dW = P\,dV} \tag{12.10}$$

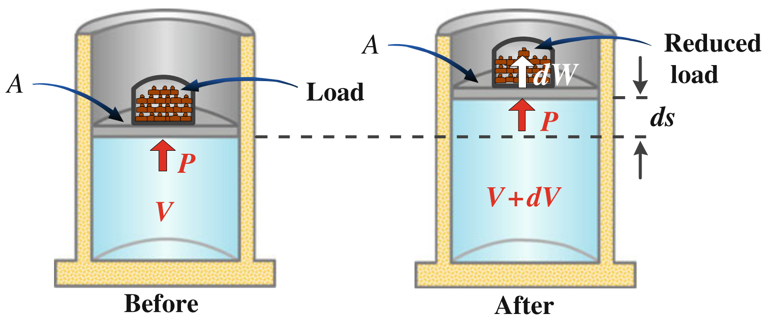


Fig. 12.6 A confined gas in a cylinder at pressure P does work dW on a free piston as the gas expands from volume V to volume $V + dV$ because of a decreased load

If the gas expands, as in Fig. 12.7, then dV is positive and the work done by the gas is positive, whereas if the gas is compressed, dV is negative, indicating that the work done by the gas is negative (which can be interpreted as work done *on* the gas). When we remove an appreciable amount of load from the piston, the volume of the gas changes from V_i to V_f, and the total work done by the gas is:

---

[1] For historical reasons, we choose W to represent the work done by the system. In other parts of the text, W is the work done on the system. This difference affects only the sign of W.

$$W = \int dW = \int_{V_i}^{V_f} P\,dV \tag{12.11}$$

During the change in volume of the gas, the pressure and temperature of the gas may also change. To evaluate the integral in the last equation, we need to know how the pressure varies with volume. For example, Fig. 12.7 indicates that the work done by the gas is represented by the area under the *PV* diagram of the figure.

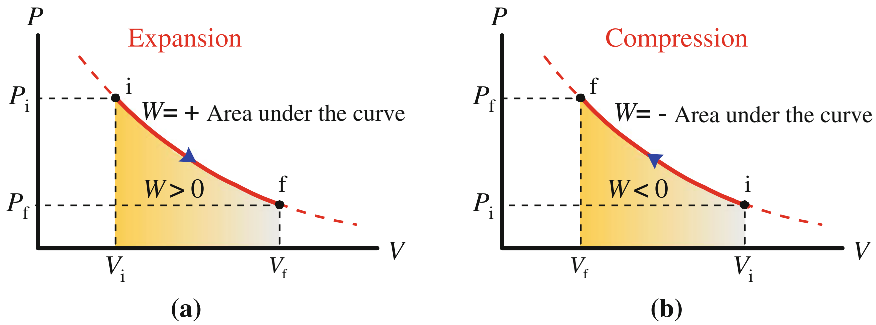


Fig. 12.7 The figure shows a gas that goes from an initial state i to a final state f by means of a thermodynamic process. (a) When the gas expands, the work done by the gas is positive and equals the area under the *PV* curve. (b) Similar to (a), except that the gas is compressed and the work done by the gas is negative

As seen from Fig. 12.7, the total work done during the expansion or compression of the gas depends on the specific path taken from the initial state i to the final state f.

In Fig. 12.8, we illustrate this important point further by considering several different paths for the gas along the *PV* curve, from state i to state f, regardless of how we achieve each path.

Path a—The gas expands from V_i to V_f while the pressure decreases from P_i to P_f. The work done by the gas along this path is positive and represented by the colored area under the curve between i and f.

Path b—The gas first expands from V_i to V_f at constant pressure P_i, and then its pressure is reduced to P_f at constant volume V_f. The work done along this path is $P_i\,(V_f - V_i)$.

Path c—The pressure of the gas is first reduced from P_i to P_f by cooling at a constant volume V_i, and then allowing the gas to expand from V_i to V_f at constant pressure P_f. The work done along this path is $P_f\,(V_f - V_i)$.

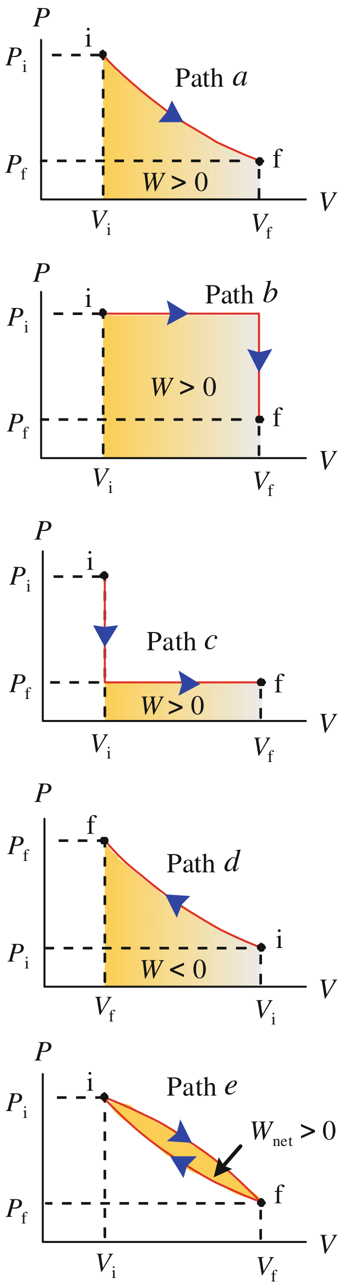


Fig. 12.8 The gas of Fig. 12.5 goes from an initial state i to a final state f by means of several different thermodynamic processes

Path d—The gas is compressed from V_i to V_f while the pressure increases from P_i to P_f. The work done along this path is the negative of the colored area under the curve.

Path e—The net work done by the system during a closed cycle is the sum of the positive work done during the expansion and the negative work done during the compression. Here, the net work done by the gas is positive and is represented by the enclosed area between the two curves.

From the graphs of Fig. 12.8, we see that W could be small or large depending on the thermodynamic path between i and f. Thus:

Spotlight

The net work done by a system W *depends* on the thermodynamic process (or the path) chosen between its initial and final states.

In a similar manner, we also find that the heat energy transfer Q into or out of a system depends on the thermodynamic process. This can be demonstrated for an ideal gas as shown in Fig. 12.9.

In Fig. 12.9a, the piston is held at a position where the gas is at its initial pressure P_i, volume V_i, and temperature T_i. When the force holding the piston is reduced slightly, the piston rises very slowly to a final pressure P_f and final volume V_f, i.e. the gas is doing work W on the piston. During this expansion process, heat energy Q is transferred from the reservoir to the gas to maintain a constant temperature T_i.

In Fig. 12.9b, the thermally insulated gas has the same initial state as in Fig. 12.9a, but with a membrane replacing the piston. When the membrane is broken, the gas expands *rapidly* into the vacuum until it acquires a pressure P_f and volume V_f. In this case, the gas does no work, i.e. $W = 0$, and no heat is transferred, i.e. $Q = 0$.

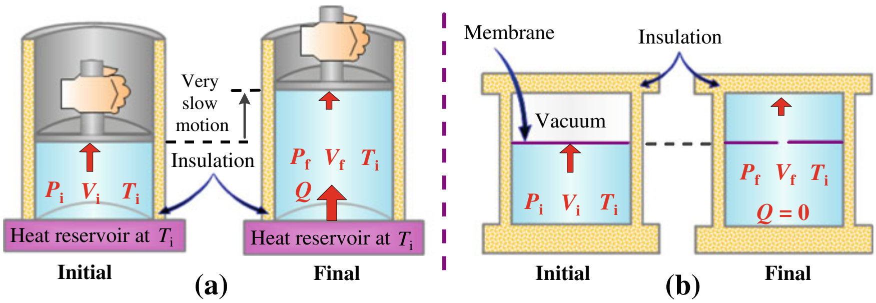


Fig. 12.9 (a) An ideal gas at temperature T_i expands slowly while absorbing heat energy Q from a reservoir in order to maintain its constant temperature T_i. (b) An ideal gas expands rapidly into an evacuated chamber after a membrane is broken

In both parts of Fig. 12.9, the initial and final states of the ideal gas are identical, although the path is different. In part (a) of the figure the gas does work W on the piston, and heat energy Q is transferred slowly to the gas from the reservoir. In part (b) of the figure the work done by the gas is zero and no heat energy is transferred. Thus:

Spotlight
The heat energy transfer Q *depends* on the thermodynamic process (or the path) chosen between the initial and final states of a system.

Finally, we conclude that neither the work done nor the heat energy are independently conserved during a thermodynamic process between the initial and final states of a system.

12.3 The First Law of Thermodynamics

In Chap. 6, we discussed the principle of conservation of energy as applied to systems that are *not isolated,* and we expressed this principle in Eq. 6.61, namely $W = \Delta E_{tot} = \Delta K + \Delta U + \Delta E_{int}$. In this chapter, we assume that there are no changes in kinetic energy and potential energy of the system as a whole; that is, $\Delta K = \Delta U = 0$ and hence $W = \Delta E_{tot} = \Delta E_{int}$. Moreover, before this chapter, the term *work* and the symbol W always meant the work done *on* a system. But starting from Eq. 12.10 and continuing to the rest of this chapter, we focus on the work done *by* a system. Thus, we replace the symbol W by $-W$ and Eq. 6.61 becomes $-W = \Delta E_{tot} = \Delta E_{int}$. If we need to account for the transfer of heat energy Q that is added (if Q positive) or taken (if Q negative) from the system, then we add Q to the left hand side of this equation and arrive at the following thermodynamic equation:

$$\boxed{\Delta E_{int} = Q - W} \quad \text{(The first law of thermodynamics)} \tag{12.12}$$

As we saw, W and Q are path-dependent, yet a surprising experimental discovery was found: *The quantity $Q - W$ is the same for all thermodynamic processes*. It depends only on the initial and final states of the system and is *path-independent*. Equation 12.12 is known as the **first law of thermodynamics**. This law states that a change in internal energy in a system can occur as a result of energy transfer by heat or by work, or by both. If the thermodynamic system undergoes only a differential change, we can write the first law as:

$$\boxed{dE_{\text{int}} = dQ - dW} \quad \text{(The first law of thermodynamics)} \tag{12.13}$$

Spotlight

The internal energy E_{int} of a system increases if energy is added via heat Q and decreases if energy is lost via work W done *by* the system.

Some special cases of the first law of thermodynamics are as follows

1. Isolated Systems

Consider a system that is not interacting with its surroundings. In this case, no energy transfer by heat takes place, i.e. $Q = 0$, and the value of the work done by the system is zero, i.e. $W = 0$. Then, from the first law we have $\Delta E_{\text{int}} = 0$. Thus, we conclude that the internal energy of an isolated system remains constant.

$$\boxed{E_{\text{int}} = \text{constant}} \quad \text{(Isolated system)} \tag{12.14}$$

2. Cyclic Processes

Consider a non-isolated system that is taken through a cyclic process, i.e. a process that starts and ends at the same state. In this case, the change in the internal energy must again be zero, i.e. $\Delta E_{\text{int}} = 0$. Then, from the first law we have:

$$\boxed{\Delta E_{\text{int}} = 0} \quad \text{and} \quad \boxed{Q = W} \quad \text{(Cyclic process)} \tag{12.15}$$

On the PV curve, a cyclic process appears as a closed curve as shown in path (e) of Fig. 12.8. For this clockwise cyclic path, the net work done by the system (and Q) equals the area enclosed by the path.

12.4 Applications of the First Law of Thermodynamics

The first law of thermodynamics relates the changes in internal energy of a system to transfers of energy by work W or heat Q, or both. In this section, we consider applications of the first law in processes in which certain restrictions are imposed.

1. Adiabatic Process

An adiabatic process is one that occurs so *rapidly* or occurs in *thermally insulated systems* during which no transfer of heat energy enters or leaves the system, i.e. $Q = 0$. With this restriction and the application of the first law of thermodynamics to an adiabatic process, we get:

$$\boxed{Q = 0} \quad \text{and} \quad \boxed{\Delta E_{\text{int}} = -W} \quad \text{(Adiabatic process)} \tag{12.16}$$

Figure 12.10 shows an idealized adiabatic process. Heat cannot enter or leave the system because of the insulation. The only way of transferring energy to the system is by work. We see in this figure that if a gas is compressed adiabatically such that W is negative, then ΔE_{int} is positive and hence the temperature of the gas increases. Conversely, if a gas expands adiabatically such that W is positive, then ΔE_{int} is negative, and hence the temperature of the gas decreases.

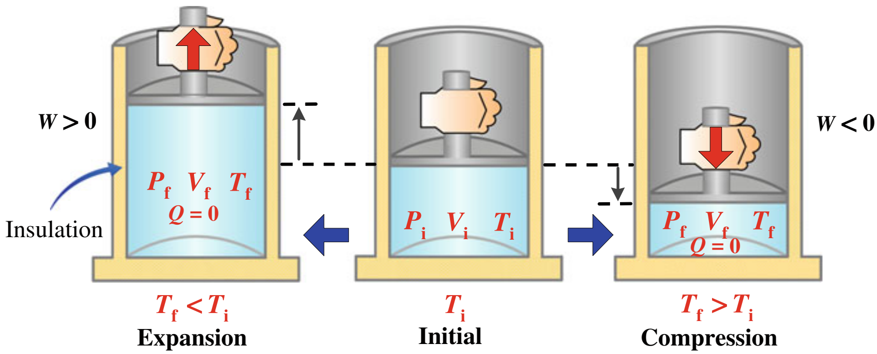


Fig. 12.10 An adiabatic compression/expansion is carried out for an ideal gas leading to an increase/decrease in internal energy

Adiabatic processes have a very important role in mechanical engineering. Some of the common examples include the approximately adiabatic compression/expansion of a mixture of gasoline vapor and air that takes place during operation of a combustion engine, leading to a temperature increase/decrease.

2. Adiabatic Free Expansion Process

The free expansion process is an adiabatic process, i.e. $Q = 0$, in which no work is done on or by the system, i.e. $W = 0$. Thus, with these restrictions and the application of the first law we have:

$$\boxed{Q = W = 0} \quad \text{and} \quad \boxed{\Delta E_{\text{int}} = 0} \quad \text{(Free expansion)} \tag{12.17}$$

Figure 12.11 shows how such an expansion can be carried out. An ideal gas in thermal equilibrium is initially confined by a closed valve to one-half of an insulated chamber; the other half is evacuated. When we open the valve, the gas expands freely to fill both halves of the chamber. No heat is transferred to or from the gas because of the insulation. No work is done by the gas because it rushes into vacuum, during which its motion is unopposed by any counteracting pressure.

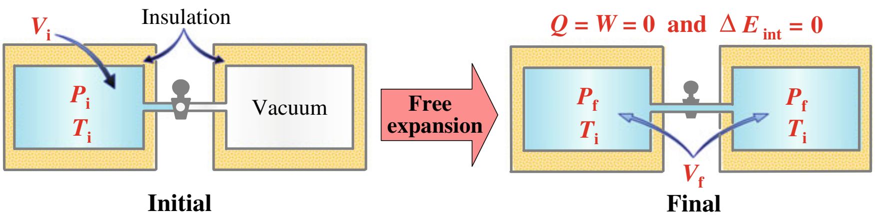


Fig. 12.11 In a free expansion process there will be no change in internal energy or temperature between the initial and final states

A free expansion differs from any other thermodynamic process since it cannot be performed slowly in a controlled way. As a result, at any given instant *during* the sudden expansion, the gas is not in thermal equilibrium and its pressure is not the same everywhere.

3. Isobaric Process

An isobaric process is one that takes place at constant pressure. In general, the first law of thermodynamics does not assume any special values for the isobaric process; that is, Q, W, and ΔE_{int} are all non-zero.

Assume the piston of Fig. 12.12 is free to move in such a way that it is always in equilibrium under the effect of the net force from a gas pushing upwards and the weight of the piston plus the force due to atmospheric pressure pushing downwards. Then, an isobaric process could be established by transferring heat energy Q to or from the gas by any mechanism. This transfer causes the gas to expand or contract depending on the sign of Q. In the PV diagram of Fig. 12.8, the first process in path (b) and the second process in path (c) are examples of isobaric processes.

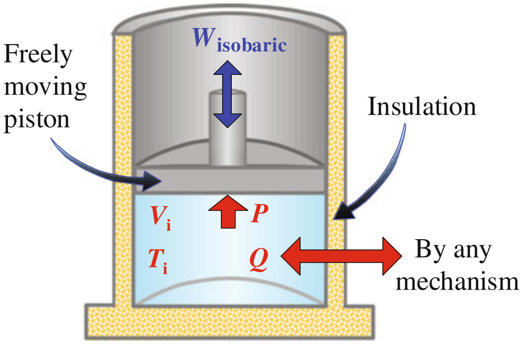


Fig. 12.12 An isobaric process could be achieved by transferring heat energy to a gas enclosed by a freely moving piston to attain a constant pressure

The work done by the gas as it expands or contracts in this isobaric process could be obtained from Eq. 12.11, after removing the constant pressure from the integral, as follows:

$$W_{\text{isobaric}} = P(V_f - V_i) \quad \text{(Isobaric process)} \tag{12.18}$$

4. Isovolumetric Process

An isovolumetric process is one that takes place at constant volume. In the PV diagram of Fig. 12.8, the second process in path (b) and the first process in path (c) are examples of isovolumetric processes.

Assume the piston of Fig. 12.13 is clamped to a fixed position to ensure an isovolumetric process. In such a process, the value of the work done by the gas is zero, i.e. $W = 0$, because the volume does not change. Thus, with this restriction and the application of the first law of thermodynamics to an isovolumetric process, we get:

$$W = 0 \quad \text{and} \quad \Delta E_{\text{int}} = Q \quad \text{(Isovolumetric process)} \tag{12.19}$$

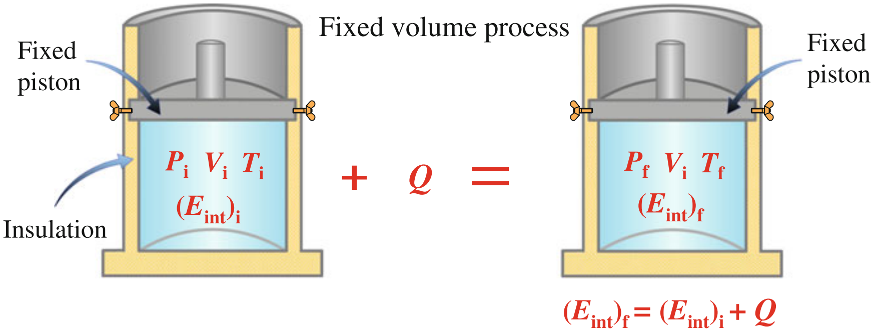


Fig. 12.13 An isovolumetric process could be achieved by fixing the piston's position. The pressure increases, and all the transferred heat energy remains in the system as an increase in its internal energy

This expression specifies that if energy is added by heat to a system kept at constant volume, then all of the transferred energy remains in the system as an increase in its internal energy, and hence, temperature. For example, when a closed metallic can is thrown into a fire, energy enters the gas in the can by the conduction of heat through the metal walls of the can. The temperature, and thus the pressure, in the can increases until the can possibly explodes, hence the warning label on such cans.

5. Isothermal Process

An isothermal process is one that takes place at constant temperature. This process can be established by putting a gas container in contact with a constant-temperature reservoir. If we plot P versus V at constant temperature for an ideal gas described by Eq. 11.8, the plot yields a hyperbolic curve called an *isotherm*. In Chap. 13, we will prove that the internal energy of an ideal gas is a function of temperature only. Consequently, in an isothermal process involving an ideal gas we must have $\Delta E_{int} = 0$. Therefore, for an isothermal process, we conclude from the first law that the energy transfer Q must be equal to the work done by the gas W. That is:

$$\boxed{\Delta\, E_{int} = 0} \quad \text{and} \quad \boxed{Q = W} \quad \text{(Isothermal process)} \tag{12.20}$$

Any energy that enters the system by heat is transferred out of the system by work; as a result, no change in the internal energy of the system occurs in an isothermal process.

Suppose that an ideal gas is allowed to expand at constant temperature as described by the PV diagram in Fig. 12.14. According to Eq. 11.10, the curve is a hyperbola with the equation $PV = \text{constant}$.

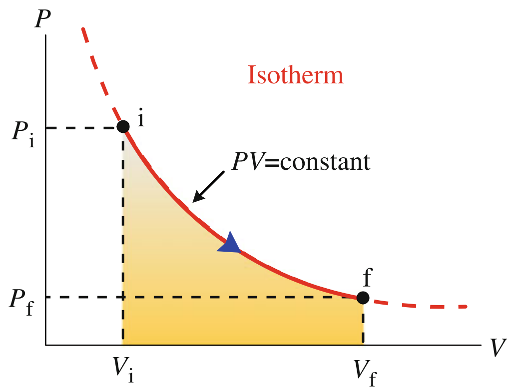


Fig. 12.14 The PV diagram for an isothermal expansion of an ideal gas from initial state i to a final state f

Let us calculate the work done by the gas in the isothermal expansion from state i to state f, as shown in Fig. 12.14. Because the gas is ideal and the process is quasi-static, we can use the expression $PV = nRT$ for each point on the path. Therefore, we have:

$$W = \int_{V_i}^{V_f} P\,dV = \int_{V_i}^{V_f} \frac{nRT}{V} dV \tag{12.21}$$

Since T is constant and also n and R are constants, then they can be moved from the integral sign. Thus:

$$W = nRT \int_{V_i}^{V_f} \frac{dV}{V} = nRT \left| \ln V \right|_{V_i}^{V_f} \tag{12.22}$$

where we used $\int dV/V = \ln V$ to evaluate the last integral. Thus:

$$W = nRT \ln\left(\frac{V_f}{V_i}\right) \tag{12.23}$$

If the gas expands, the work W equals the positive of the shaded area under the PV curve shown in Fig. 12.14; this is because $\ln(V_f/V_i) > 0$. If the gas is compressed, $V_f < V_i$, then $\ln(V_f/V_i) < 0$ and the work done is the negative of the area under the PV curve.

Table 12.3 summarizes the characteristics of the previous processes.

Table 12.3 The first law of thermodynamics in five special cases

Process	Restriction	Consequence
Adiabatic	$Q = 0$	$\Delta E_{\text{int}} = -W$
Free expansion	$Q = W = 0$	$\Delta E_{\text{int}} = 0$
Isobaric	P = constant	$W_{\text{isobaric}} = P(V_f - V_i)$
Isovolumetric	V = constant, $W = 0$	$\Delta E_{\text{int}} = Q$
Isothermal (ideal gas)	T = constant, $\Delta E_{\text{int}} = 0$	$Q = W = nRT\ln(V_f/\,V_i)$

Example 12.5

At a constant pressure of 1 atm and a temperature of 0 °C, the heat fusion of ice is $L_F = 3.33 \times 10^5$ J/kg, the density of ice is $\rho_i = 920\,\text{kg/m}^3$, and the density of liquid water is $\rho_w = 1{,}000\,\text{kg/m}^3$. (a) Find the work W done by 1 kg of ice that melts completely to water. (b) Find the change in internal energy of this process.

Solution: (a) The initial volume of ice is:

$$V_i = m/\rho_i = (1\,\text{kg})/(920\,\text{kg/m}^3) = 1.087 \times 10^{-3}\,\text{m}^3$$

The final volume of ice after it melts completely to water is:

$$V_w = m/\rho_w = (1\,\text{kg})/(1{,}000\,\text{kg/m}^3) = 10^{-3}\,\text{m}^3$$

The work done by 1 kg of ice that melts completely to water under constant pressure of 1 atm (1.01×10^5 Pa) and temperature of 0 °C, is:

$$\begin{aligned} W &= \int_{V_i}^{V_w} P\,dV = P(V_w - V_i) \\ &= (1.01 \times 10^5\,\text{Pa})(10^{-3}\,\text{m}^3 - 1.087 \times 10^{-3}\,\text{m}^3) = -8.787\,\text{J} \simeq -8.8\,\text{J} \end{aligned}$$

The minus sign appears because ice contracts when it melts.

(b) The heat energy transferred to change the phase of 1 kg of ice to water is:

$$Q = m\,L_F = (1\,\text{kg})(3.33 \times 10^5\,\text{J/kg}) = 3.33 \times 10^5\,\text{J}$$

Thus, from the first law of thermodynamics, we can find the change in internal energy of this process as follows:

$$\Delta E_{\text{int}} = Q - W = 3.33 \times 10^5\,\text{J} + 8.8\,\text{J} = 3.330088 \times 10^5\,\text{J}$$

We see from parts (a) and (b) that $|W|$ is less than 0.003% of Q in this process, i.e. $|W| \ll Q$. That is, the mechanical energy is negligible in comparison to the heat of fusion. So, all the added heat of fusion shows up as an increase in the internal energy.

Example 12.6

At a constant pressure of 1 atm, a movable piston encloses 1 kg of water with a volume of $10^{-3}\,\text{m}^3$ and a temperature of 100 °C, see Fig. 12.15. Heat is added from a reservoir until the liquid water changes completely into steam of volume $1.671\,\text{m}^3$, see the figure. (a) How much work is done by the system (water + steam) during the boiling process? (b) How much heat energy is added to the system? (c) What is the change in the internal energy of the system?

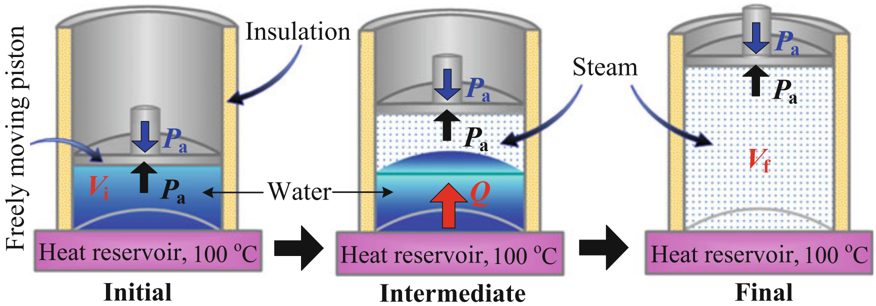


Fig. 12.15

Solution: (a) The work done by 1 kg of water that is converted completely into steam under a constant pressure of 1 atm (1.01×10^5 Pa) and a constant temperature of 100 °C, is:

$$W = \int_{V_i}^{V_f} P\,dV = P(V_f - V_i) = (1.01 \times 10^5\,\text{Pa})(1.671\,\text{m}^3 - 10^{-3}\,\text{m}^3) = 169\,\text{kJ}$$

(b) Since the heat of vaporization of water at atmospheric pressure is 2.26×10^6 J/kg, the heat energy required to change the phase of 1 kg of water to steam will be:

$$Q = m\,L_V = (1\,\text{kg})(2.26 \times 10^6\,\text{J/kg}) = 2{,}260\,\text{kJ}$$

(c) From the first law of thermodynamics, we can find the change in internal energy of this process as follows:

$$\Delta E_{\text{int}} = Q - W = 2.26 \times 10^6\,\text{J} - 1.69 \times 10^5\,\text{J} = 2{,}091\,\text{kJ}$$

We see that about 92.5% of the heat energy goes into internal energy while the remaining 7.5% goes into external work.

Example 12.7

An aluminum rod of mass 1 kg is heated from 25 to 55 °C at constant atmospheric pressure, see Fig. 12.16. The aluminum rod has a density ρ of $2.7 \times 10^3\,\text{kg/m}^3$, a coefficient of volume expansion β of $7.2 \times 10^{-5}\,(\text{C}°)^{-1}$ and a specific heat c of 900 J/kg.C°. (a) How much work is done by the rod? (b) How much heat is transferred to the rod? (c) Quantify the rod's internal energy change.

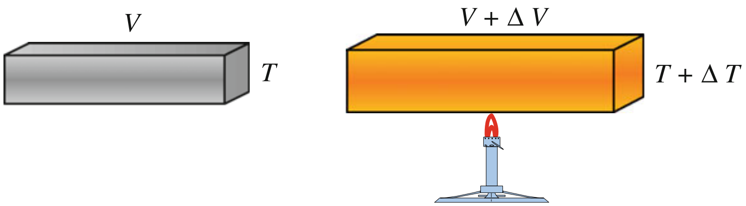


Fig. 12.16

Solution: (a) The initial volume of the aluminum rod is given by:

$$V = \frac{m}{\rho} = \frac{1\,\text{kg}}{2.7 \times 10^3\,\text{kg/m}^3} = 3.704 \times 10^{-4}\,\text{m}^3$$

Using the change in temperature $\Delta T = T_f - T_i = 55\,°\text{C} - 25\,°\text{C} = 30\,\text{C}°$, the change in the rod's volume can be obtained from Eq. 11.5 as follows:

$$\Delta V = \beta\, V \Delta T$$
$$= (7.2 \times 10^{-5}\,(\mathrm{C}^\circ)^{-1})(3.704 \times 10^{-4}\,\mathrm{m}^3)(30\,\mathrm{C}^\circ) = 8 \times 10^{-7}\,\mathrm{m}^3$$

Since the expansion is carried out at a constant pressure, the work done by the aluminum rod is:

$$W = \int_{V_\mathrm{i}}^{V_\mathrm{f}} P\,dV = P(V_\mathrm{f} - V_\mathrm{i}) = P\Delta V$$
$$= (1.01 \times 10^5\,\mathrm{Pa})(8 \times 10^{-7}\,\mathrm{m}^3) = 8.08 \times 10^{-2}\,\mathrm{J}$$

(b) We use the specific heat value in Eq. 12.4 to calculate the amount of heat transferred to the rod as follows:

$$Q = m\,c\,\Delta T = (1\,\mathrm{kg})(900\,\mathrm{J/kg.C}^\circ)(30\,\mathrm{C}^\circ) = 2.7 \times 10^4\,\mathrm{J}$$

(c) From the first law of thermodynamics, we can find the change in internal energy of this process as follows:

$$\Delta E_\mathrm{int} = Q - W = 2.7 \times 10^4\,\mathrm{J} - 8.09 \times 10^{-2}\,\mathrm{J} = 2.699 \times 10^4\,\mathrm{J}$$

We notice that almost all of the heat energy goes towards increasing the internal energy of the aluminum rod. The fraction of heat energy that is used as work against the atmospheric pressure is only about $4 \times 10^{-4}\%$. Therefore, in thermal expansion of solids, the amount of energy that goes into work is usually neglected.

Example 12.8

Find the work done by 1 kmol of an ideal gas that is kept at a constant temperature of 27 °C in an expansion process from 2 to 5 L.

Solution: Rewriting these values and the gas constant R, we have:

$$n = 1\,\mathrm{kmol}$$
$$R = 8.314 \times 10^3\,\mathrm{J/kmol.K}$$
$$T = 27\,^\circ\mathrm{C} = 27 + 273 = 300\,\mathrm{K}$$
$$V_\mathrm{i} = 2\,\mathrm{L} = 2{,}000\,\mathrm{cm}^3 = 2 \times 10^{-3}\,\mathrm{m}^3$$
$$V_\mathrm{f} = 5\,\mathrm{L} = 5{,}000\,\mathrm{cm}^3 = 5 \times 10^{-3}\,\mathrm{m}^3$$

Since this process is isothermal, the work done by the ideal gas is given by Eq. 12.23. Substitution in this equation results in:

$$\begin{aligned} W &= n\,RT\,\ln\left(\frac{V_\mathrm{f}}{V_\mathrm{i}}\right) \\ &= (1\,\mathrm{kmol})(8.314\times10^3\,\mathrm{J/kmol.K})(300\,\mathrm{K})\;\ln\left(\frac{5}{2}\right) = 2.29\times10^6\,\mathrm{J} \end{aligned}$$

This means that the heat energy Q that must be given to the ideal gas from the reservoir to keep its temperature $T = 27\,^\circ\mathrm{C}$ is also 2.29×10^6 J.

12.5 Heat Transfer

We discussed the transfer of heat energy between a system and its surroundings, but we did not describe *how that transfer takes place and at what rate*. The three common energy-transfer mechanisms that are responsible for changing the internal energy state of a system are:

1. Conduction:
   The flow of heat that reduces the temperature difference between two materials.
2. Convection:
   The flow of heat in liquids or gases that carries heat from one place to another if the liquids or gases are free to move.
3. Radiation:
   The transfer of energy in the form of electromagnetic waves from objects that have temperatures greater than absolute zero. The transfer of *heat* energy from one location to another is by infrared radiation.

In this section we focus only on the first mechanism, leaving the other two mechanisms for other thermodynamic studies.

Thermal Conduction in One Dimension (Plain Walls)

In thermal conduction, heat transfer can be represented on the atomic scale as an exchange of kinetic energy between microscopic particles (molecules, atoms, and electrons) in which less energetic particles gain energy in collisions with more energetic particles. By this method, heat energy is transferred from the hot parts of an object to its cold parts.

Consider the flow of heat along the x-axis between the faces of a slab of a material of thickness Δx and face area A, as shown in Fig. 12.17. Assume the opposite faces are maintained at different temperatures T_H and T_C, where $T_H > T_C$. Let $\Delta T = T_C - T_H$ denote the *change* in temperature that is maintained along the thickness Δx. The temperature *difference* $T_H - T_C = -\Delta T$ is what gives rise to heat flow.

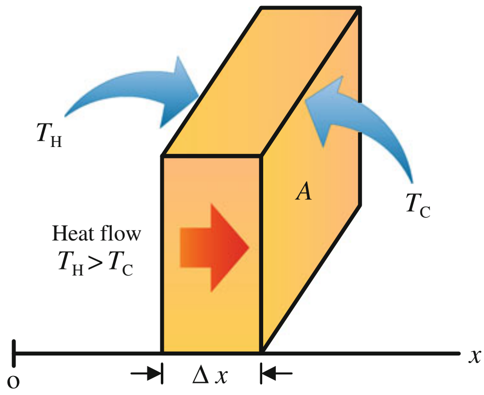


Fig. 12.17 Linear heat transfer through a conducting slab of face area A and thickness Δx, when the opposite faces are at different temperatures, T_H and T_C

Let ΔQ be the heat energy that is transferred through the slab from its hot face to its cold face, in a time interval Δt. Let $H = \Delta Q/\Delta t$ denote the rate of heat flow across the slab (H is measured in watts). Experiments show that H should be directly proportional to the face area A, the temperature difference $-\Delta T = T_H - T_C > 0$, and inversely proportional to the thickness Δx. That is:

$$H = \frac{\Delta Q}{\Delta t} \propto -A\frac{\Delta T}{\Delta x}$$

or

$$\boxed{H = \frac{\Delta Q}{\Delta t} = -kA\frac{\Delta T}{\Delta x}} \qquad (12.24)$$

where k is a proportionality constant that has the SI unit W/m.C° and is called the **thermal conductivity** of the material. For a slab of differential thickness dx and differential temperature difference dT, we can write what is called the **law of heat conduction** as follows:

$$\boxed{H = -kA\frac{dT}{dx}} \qquad (12.25)$$

where dT/dx is known as the temperature gradient. The minus sign in Eq. 12.25 is due to the fact that heat energy flows in the direction of decreasing temperature.

Now, consider a long uniform rod of length L, as shown in Fig. 12.18. The rod is insulated so that thermal energy cannot enter nor escape from its surface except at its ends, which are in thermal contact with heat reservoirs having temperatures T_H and T_C, where $T_H > T_C$.

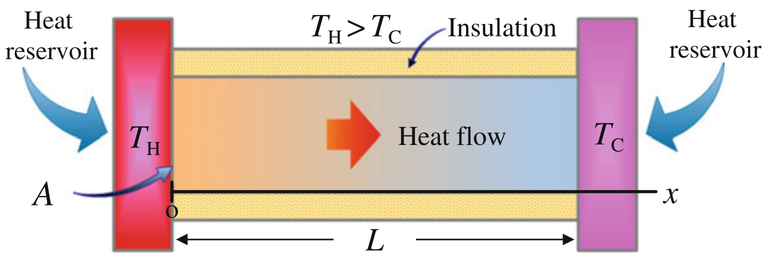


Fig. 12.18 Conduction of heat through a uniform conducting, insulated rod of length L and face area A, where the opposite faces are at different temperatures, T_H and T_C ($T_H > T_C$)

When a steady-state has been reached, the temperature at each point along the rod is constant in time. In such a case, the temperature gradient is the same everywhere along the rod and is given by:

$$\frac{dT}{dx} = \frac{T_C - T_H}{L} \tag{12.26}$$

Thus, the rate of heat flow becomes:

$$\boxed{H = kA\,\frac{T_H - T_C}{L}} \tag{12.27}$$

The thermal conductivity k is a constant that depends on the material of the rod. Large values of k indicate that a material is a good thermal conductor, and vice versa. Table 12.4 displays the thermal conductivities of some common metals, gases, and building materials.

Table 12.4 Thermal conductivity of some substances around normal room temperature.

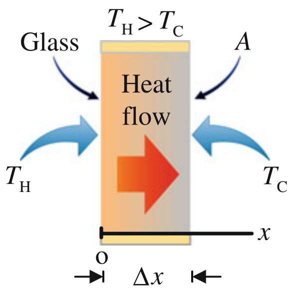
Substance	Thermal conductivity W/m.C°
Metals	
Stainless steel	14
Lead	35
Aluminum	238
Gold	314
Copper	401
Silver	427
Gases	
Air (dry)	0.026
Helium	0.15
Hydrogen	0.18
Building materials	
Foam	0.024
Rock wool	0.043
Fiberglass	0.048
Asbestos	0.08
Wood	0.08
Rubber	0.2
Glass	0.8
Concrete	0.8
Window glass	1.0
Steel	18

These values are approximate because k depends on the temperature

Example 12.9

A glass window measures $1\,\text{m} \times 1.5\,\text{m} \times 0.5\,\text{cm}$ and has a thermal conductivity of $0.8\,\text{W/m.C}^\circ$. The temperature of the inner surface of the glass is $T_{\text{H}} = 20\,^\circ\text{C}$, while the temperature for the outer surface is $T_{\text{C}} = -15\,^\circ\text{C}$, see Fig. 12.19. (a) Calculate the rate of heat flow by conduction through the window. (b) If the inner face of the window is taken to be at $x = 0$, see the figure, then find the temperature of the glass as a function of x.

Fig. 12.19



Solution: (a) The thickness of the glass is $\Delta x = 0.5\,\text{cm} = 5 \times 10^{-3}$ m and the change in temperature is $\Delta T = T_C - T_H = -15\,°\text{C} - 20\,°\text{C} = -35\,\text{C}°$. Using Eq. 12.24 we get:

$$H = -kA\frac{\Delta T}{\Delta x} = -(0.8\,\text{W/m.C}°)(1\,\text{m} \times 1.5\,\text{m})\frac{(-35\,\text{C}°)}{5 \times 10^{-3}\,\text{m}} = 8{,}400\,\text{W}$$

This enormous rate of heat flow by conduction shows that glass is not a very good insulator. The rate of heat flow through a glass window can be reduced substantially by using two layers of glass with a thin air layer between them. This is called *double glazing*.

(b) The temperature gradient for the window is given by:

$$\frac{dT}{dx} = \frac{\Delta T}{\Delta x} = \frac{(-35\,\text{C}°)}{5 \times 10^{-3}\,\text{m}} = -7{,}000\,\text{C}°/\text{m}$$

This equation can be integrated to give:

$$\int_{T_H}^{T} dT = (-7{,}000\,\text{C}°/\text{m})\int_0^x dx \quad \Rightarrow \quad T - T_H = (-7{,}000\,\text{C}°/\text{m})\,(x - 0)$$

Thus:

$$T = 20\,°\text{C} - (7{,}000\,\text{C}°/\text{m})\,x$$

We can check whether this gives the correct temperature of $-15\,°\text{C}$ for the outer surface as follows:

$$T = 20\,°\text{C} - (7{,}000\,\text{C}°/\text{m})\,(5 \times 10^{-3}\,\text{m}) = -15\,°\text{C}$$

Example 12.10

Figure 12.20 shows two slabs of thickness L_1 and L_2, thermal conductivities k_1 and k_2, and an equal surface area A. The temperatures at the outer faces of the slabs are T_H and T_C, where $T_H > T_C$. In a steady-state condition, find: (a) the interface temperature T, when $T_H = 25\,°C$, $T_C = -5\,°C$, $L_2 = 2L_1$, and $k_2 = 4k_1$, and (b) the rate of heat transfer by conduction through the slabs.

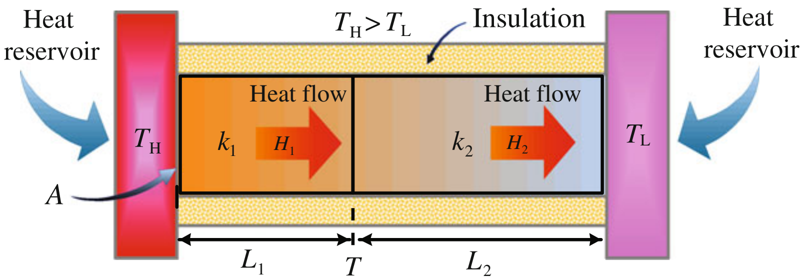


Fig. 12.20

Solution: (a) If T is the temperature at the interface, then the rate of heat flow through the two slabs is:

$$H_1 = k_1 A\,\frac{T_H - T}{L_1}, \quad \text{and} \quad H_2 = k_2 A\,\frac{T - T_C}{L_2}$$

When a steady-state is reached, these two rates must be equal, that is:

$$k_1 A\,\frac{T_H - T}{L_1} = k_2 A\,\frac{T - T_C}{L_2}$$

Solving for T gives: $$T = \frac{k_1 L_2 T_H + k_2 L_1 T_C}{k_1 L_2 + k_2 L_1}$$

Inserting the given relations and the known temperatures gives:

$$T = \frac{2k_1 L_1 T_H + 4k_1 L_1 T_C}{2k_1 L_1 + 4k_1 L_1} = \tfrac{1}{6}(2T_H + 4T_C) = \tfrac{1}{6}[2(25\,°C) + 4(-5\,°C)] = 5\,°C$$

(b) The expression of the rate of heat flow by conduction will be:

$$H = H_1 = H_2 = \frac{A\,(T_H - T_C)}{(L_1/k_1) + (L_2/k_2)} = \frac{10\,A\,k_1}{L_1}$$

Home Insulation

Insulation is important in building houses, since it helps limit heat loss and hence keeps homes at a comfortable temperature with less cost, see Fig. 12.21. Good insulation requires many insulation slabs.

Fig. 12.21 In houses, heat is conducted from the inside to the outside more rapidly where insulation is poor. Thus, houses should be well insulated especially in the attic to minimize heat loss

For a compound slab containing several materials of thicknesses $L_1,\ L_2, \ldots$ and thermal conductivities $k_1,\ k_2, \ldots$, we can perform similar steps as in Example 12.10 to show that the rate of heat transfer at a steady-state will take the form:

$$H = \frac{A\,(T_\mathrm{H} - T_\mathrm{C})}{\sum\limits_n L_n/k_n}, \qquad (n = 1, 2, \ldots) \tag{12.28}$$

In the engineering practice, the term L/k for a particular substance is referred to as the R value of the material, and Eq. 12.28 takes the following form:

$$H = \frac{A\,(T_\mathrm{H} - T_\mathrm{C})}{\sum\limits_n R_n}, \quad (n = 1, 2, \ldots) \tag{12.29}$$

where $R_n = L_n/k_n$. If a wall contains three slabs of insulation, then we can find the value of R for the wall by adding the values of R for each slab. Table 12.5 lists the R-values for common building materials.

Table 12.5 The *R*-values of some common building materials

Material	Thickness (cm)	R-value ($m^2 . C°/W$)
Hardwood siding	2	0.185
Wood shingles	1.3	0.111
Brick	10	0.704
Fiberglass batting	8	1.918
Fiberglass board	2.5	0.766
Cellulose fiber	2.5	0.651
Flat glass	0.3	0.151
Insulating glass	0.3	0.318
Air space	10	0.178
Drywall	1.5	0.095
Sheathing	1.5	0.233

Thermal Conduction in Two Dimensions (Cylindrical Shells)

We can apply the law of heat conduction to situations where heat flows in two dimensions by varying the area in consideration.

As an example, consider a steam pipe in which heat flows radially outwards. This type of heat flow is called *cylindrical heat flow* and is illustrated geometrically in Fig. 12.22.

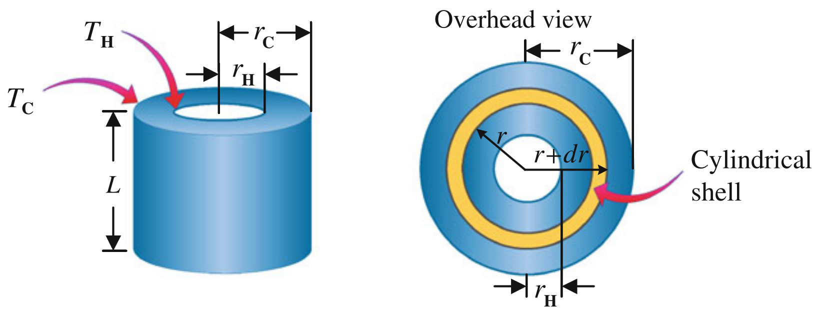


Fig. 12.22 Geometry for heat flow in a cylinder of length L. *Left* The inner and outer radii and temperatures are r_H, r_C, T_H, and T_C, respectively. *Right* A cylindrical shell has a radius r and thickness dr

Conceptually, we can divide a cylindrical pipe of length L into a series of thin concentric cylindrical shells. The rate of heat flow through a cylindrical shell of radius r and thickness dr is given by:

$$H = -kA\frac{dT}{dr} \tag{12.30}$$

where A is the surface area of the cylindrical shell and is given by:

$$A = 2\pi rL \tag{12.31}$$

Thus, for cylindrical heat flow, the law of heat conduction becomes:

$$H = -2\pi k\,Lr\frac{dT}{dr} \tag{12.32}$$

For steady-state conditions H remains constant, and we can find how T varies with r by rearranging Eq. 12.32 as follows:

$$dT = -\frac{H}{2\pi kL}\frac{dr}{r} \tag{12.33}$$

We can now integrate this equation from the initial radius r_H (where the temperature is T_H) to some arbitrary radius r where the temperature is $T \equiv T(r)$ as follows:

$$\int_{T_H}^{T} dT = -\frac{H}{2\pi kL}\int_{r_H}^{r}\frac{dr}{r} \tag{12.34}$$

Thus:

$$\boxed{T - T_H = -\frac{H}{2\pi kL}\,\ln\left(\frac{r}{r_H}\right)} \tag{12.35}$$

This result shows that for cylindrical heat flow the temperature decreases logarithmically with an increasing r.

The rate of heat flow through the pipe section that has inner and outer radii r_H and r_C, and inner and outer temperatures T_H and T_C, is given by letting $T = T_C$ and $r = r_C$ in Eq. 12.35. That is:

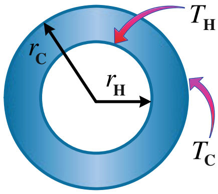
$$\boxed{H = \frac{2\pi kL\,(T_H - T_C)}{\ln\,(r_C/r_H)}} \tag{12.36}$$

Example 12.11

A stainless-steel pipe has inner and outer radii of 2 and 2.5 cm, respectively. The pipe carries hot water at a temperature of $T_H = 60\,°C$ and has a thermal conductivity of $19\,W/m.C°$. The pipe's outer surface temperature is $T_C = 56\,°C$, see Fig. 12.23. (a) What is the rate of heat flow per unit length of the pipe?

(b) When an additional cylindrical insulator of thermal conductivity of 0.2 W/m.C° is used, what is the thickness required to reduce heat loss by a factor of 10 and achieve an outer temperature of 37 °C?

Fig. 12.23



Solution: (a) The temperature difference is:

$$T_H - T_C = 60\,°\mathrm{C} - 56\,°\mathrm{C} = 4\,\mathrm{C}°$$

This value is used in Eq. 12.36 to get the value of the rate of heat flow per unit length, H/L, as:

$$\frac{H}{L} = \frac{2\pi k(T_H - T_C)}{\ln(r_C/r_H)} = \frac{2\pi(19\,\mathrm{W/m.C°})(4\,\mathrm{C°})}{\ln(2.5\,\mathrm{cm}/2\,\mathrm{cm})} = 2{,}140\,\mathrm{W/m}$$

This great rate of heat flow per unit length shows that stainless steel is not a very good material to use alone for an isolated hot-water pipe.

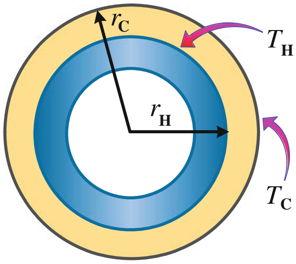
(b) As far as the stainless-steel pipe is concerned, a reduction in H/L by a factor of 10 requires that the temperature difference between the inner and outer surfaces be reduced by the same factor. Thus, the original 4 C° difference is reduced to 0.4 C°. Hence, the inner surface of the cylindrical insulator will be at $T_H = 59.6\,°\mathrm{C}$ and its outer surface temperature will be at $T_C = 37\,°\mathrm{C}$, i.e. $T_H - T_C = 59.6\,°\mathrm{C} - 37\,°\mathrm{C} = 22.6\,\mathrm{C}°$, see Fig. 12.24. In addition H/L will be reduced to 214 W/m. Solving Eq. 12.36 again for $\ln(r_C/r_H)$, where $r_H = 2.5$ cm, we get:

$$\ln(r_C/r_H) = \frac{2\pi k(T_H - T_C)}{H/L} = \frac{2\pi(0.2\,\mathrm{W/m.C°})(22.6\,\mathrm{C°})}{214\,\mathrm{W/m}} = 0.133$$

$$\text{Thus:}\quad r_C/r_H = \exp(0.133) = 1.142 \quad\Rightarrow\quad r_C = 1.142 \times r_H = 2.9\,\mathrm{cm}$$

The required insulation thickness is $r_C - r_H = 2.9\,\mathrm{cm} - 2.5\,\mathrm{cm} = 0.4\,\mathrm{cm}$.

Fig. 12.24



12.6 Exercises

Section 12.1 Heat and Thermal Energy

Subsection 12.1.1 Units of Heat, The Mechanical Equivalent of Heat

(1) A room is lighted by a 200 W light bulb. A 200 W of power is the rate at which the bulb converts electrical energy into heat and visible light. Assuming that 90% of the energy is converted into heat, how much heat is added to the room in 4 h?

(2) Suppose your mass is 70 kg and you ate a 250 kcal meal. To compensate, you decided to lose an equivalent amount of energy by climbing the stairs of a building. What is the total height that you must climb?

Subsection 12.1.2 Heat Capacity and Specific Heat

(3) 159.2 g of water is initially at 15 °C. To what temperature will this quantity of water rise when 1,000 J of energy is supplied?

(4) The brakes of a 1,500 kg car are used to decelerate its speed from 72 km/h to rest. How many joules and kilocalories are generated during the stopping process?

(5) How many calories of heat are required to raise the temperature of 4 kg of iron of specific heat 448 J/kg.C° from 20 to 40 °C?

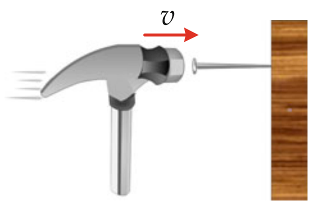
(6) The water cooling system (radiator) of a car holds 20 L of water. How much heat does the radiator absorb if its temperature rises from 20 to 95 °C?

(7) The specific heat of aluminum is 900 J/kg.C°. (a) What is the heat capacity of 5 kg of aluminum? (b) How much heat must be added to 5 kg of aluminum to raise its temperature from 27 to 37 °C?

(8) What is the specific heat of a 4 kg material when its temperature increases from 27 to 37 °C after 18 kJ of heat is added?

(9) A hammer head of mass 1.5 kg strikes an iron nail of mass 15 g that has a specific heat 450 J/kg.C°. The hammer has a speed $v = 6\,\mathrm{m/s}$ just before striking the nail and then comes to rest after the impact, see Fig. 12.25. Assume that all the energy of the hammer goes into heating the nail during the strike. What is the rise in temperature of the nail?

Fig. 12.25 See Exercise (9)



(10) What is the final equilibrium temperature when 20 g of milk at 10 °C is added to 200 g of tea at 100 °C? (Assume that the specific heat of milk, tea, and water are all the same, and neglect the heat capacity of the container).

(11) A 2 kg metallic object is heated to 500 °C and then dropped into a bucket containing 20 kg of water initially at 20 °C. When equilibrium is reached, the temperature of the mixture is 70 °C. What is the specific heat of the metal? (Neglect the heat capacity of the container).

(12) In an experiment where the specific heat of aluminum is measured using the method of mixtures, see Fig. 12.26, a student obtains the following data:

Mass of aluminum: $m_x = 0.2\,\mathrm{kg}$

Initial temperature of aluminum: $T_x = 27\,°\mathrm{C}$

Mass of water: $m_w = 0.4\,\mathrm{kg}$

Specific heat of water: $c_w = 4{,}186\,\mathrm{J/kg.C°}$

Mass of calorimeter: $m_c = 0.04\,\mathrm{kg}$

Initial temperature of water and calorimeter: $T_i = 70\,°\mathrm{C}$

Specific heat of the calorimeter: $c_c = 630\,\mathrm{J/kg.C°}$

Final temperature of the mixture: $T_f = 66.4\,°\mathrm{C}$

Use these data to determine the specific heat of aluminum c_x.

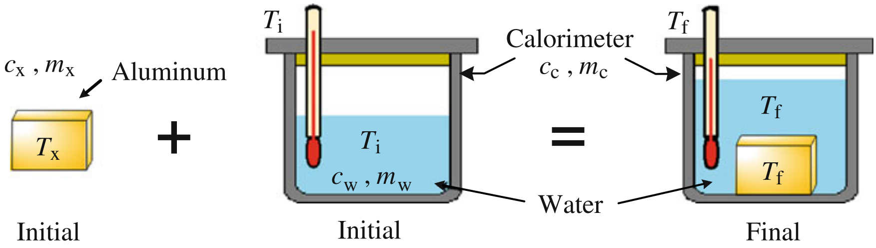


Fig. 12.26 See Exercise (12)

Subsection 12.1.3 Latent Heat

(13) Aluminum has a melting temperature of 660 °C, latent heat of fusion of 3.97×10^5 J/kg, and specific heat of 900 J/kg.C°. How much heat is needed to melt 15 kg of aluminum that is initially at 27 °C?

(14) A runner loses 150 kcal of heat in 15 min by evaporating water from his skin. The latent heat of vaporization of water at room temperature is 2.26×10^6 J/kg. How much water has been lost?

(15) Follow similar steps like the calculations done in Fig. 12.4 to find the energy required to change a 50 g ice cube from the ice state at −10 °C to the steam state at 110 °C.

(16) A 150 g of ice is enclosed in a thermally insulated container. What is the mass of steam at 100 °C that must be mixed with the ice to produce liquid water at 50 °C. (For the ice and steam, use the constants of Tables 12.1 and 12.2)

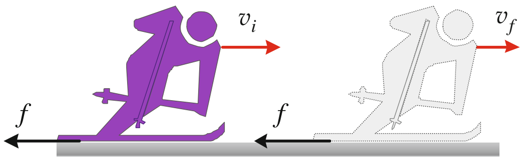
(17) A 100 g block of ice at 0 °C is added to 400 g of water at 30 °C. Assuming we have a perfectly insulated calorimeter for this mixture, what will be its final temperature when all of the ice has melted?

(18) A copper calorimeter has a mass $m_c = 100$ g. The calorimeter contains water of mass $m_w = 500$ g at a temperature of 20 °C. How much steam must be condensed into water if the final temperature of the mixture is to reach 50 °C? Assume the specific heat of copper is $c_c = 840$ J/kg.C°, the specific heat of water is $c_w = 4{,}186$ J/kg.C°, and the latent heat of condensation of steam is $L_V = 2.26 \times 10^6$ J/kg.

(19) A 60 kg ice skater glides from a speed $v_i = 10$ m/s to a speed $v_f = 4$ m/s on ice at 0 °C, see Fig. 12.27. Assume that 80% of the heat generated by friction is

absorbed by the ice and all of the melted ice stays at 0 °C. The latent heat of fusion of ice is 3.33×10^5 J/kg. How much ice melts?

Fig. 12.27 See Exercise (19)

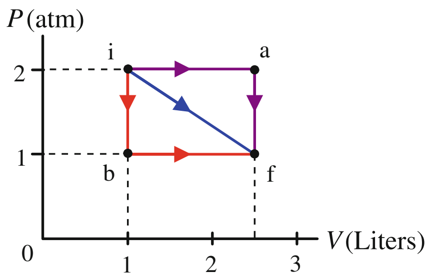


Section 12.2 Heat and Work (Take 1 atm $\simeq 10^5$ Pa)

(20) An ideal gas is enclosed in a container at a pressure of 2 atm and has a volume of 3 m^3. What is the work done by the gas if: (a) the gas expands at a constant pressure to three times its initial volume? (b) the gas is compressed at a constant pressure to one half of its initial volume?

(21) (a) An ideal gas is taken from an initial state i to a final state f, as shown in Fig. 12.28. Find the work done by the gas along the three paths iaf, if, and ibf. (b) Answer part (a) if the gas is taken from f to i.

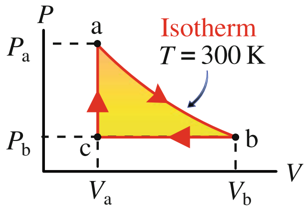
Fig. 12.28 See Exercise (21)



Section 12.4 Applications of the First Law of Thermodynamics

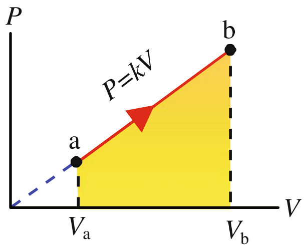
(22) An ideal gas of 2 kmol is carried around the thermodynamic cycle as shown in Fig. 12.29. The cycle consists of three parts; the isothermal expansion ab at $T = 300$ K an isobaric compression bc, and an isovolumetric increase in pressure ca. (a) When $P_a = 4$ atm and $P_b = 1$ atm, then find the work done by the gas per cycle. (b) Answer part (a) when the direction of the cycle is reversed.

Fig. 12.29 See Exercise (22)



(23) An ideal gas expands from an initial volume $V_a = 0.5\,\text{m}^3$ to a final volume $V_b = 1.5\,\text{m}^3$ in a quasi-static process for which $P = kV$, where $k = 2.5\,\text{atm/m}^3$, see Fig. 12.30. How much work was done by the expanding gas?

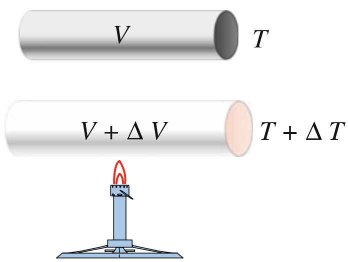
Fig. 12.30 See Exercise (23)



(24) An amount of work of 100 J is done on a system, and 100 cal of heat are extracted from it. In light of the first law of thermodynamics, what are the values (including algebraic signs) of: (a) W, (b) Q, and (c) ΔE_{int}?

(25) A cylindrical steel rod of mass 3.9 kg is heated from $T = 27\,°\text{C}$ to $T + \Delta T = 37\,°\text{C}$ at a constant atmospheric pressure, see Fig. 12.31. The rod has a density ρ of $7.8 \times 10^3\,\text{kg/m}^3$, a coefficient of volume expansion β of $3.3 \times 10^{-6}\,(\text{C°})^{-1}$, and a specific heat c of $450\,\text{J/kg.C°}$. (a) How much work is done by the rod? (b) How much heat is transferred to the rod? (c) What is the change in the rod's internal energy?

Fig. 12.31 See Exercise (25)



(26) An ideal helium gas of 1 kmol is carried around the thermodynamic cycle as shown in Fig. 12.32. The path ab is isothermal, with $P_a = 2$ atm, $P_b = 1$ atm, and $V_a = 22.4\,\text{m}^3$. (a) What are the values of T_a, V_b, and T_c? (b) How much work is done by the gas in this cycle?

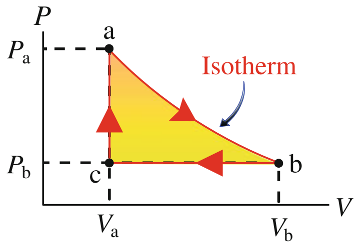


Fig. 12.32 See Exercise (26)

(27) An ideal gas of one kmol does 4,000 J of work as it expands isothermally to a volume of $12 \times 10^{-3}\,\text{m}^3$ that has a final pressure of 2 atm. (a) What is the temperature of the gas? (b) What is the initial volume of the gas?

(28) A fluid is carried through the cycle abcd as shown in Fig. 12.33. How much work (in kilojoules) is done by the fluid during: (a) the isobaric expansion ab, (b) the isovolumetric process bc, and (c) the isobaric compression cd? (d) What is the net amount of heat transferred to work during the cycle abcd?

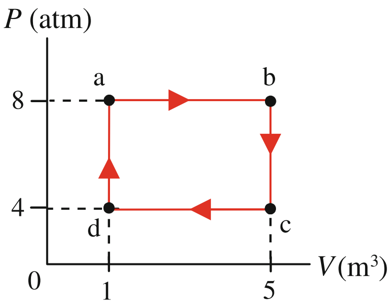


Fig. 12.33 See Exercise (28)

(29) At a constant pressure of 2 atm ($2\,P_a$), the boiling point of water is 120 °C, and its heat of vaporization is $L_V = 2.20 \times 10^6$ J/kg. Under these conditions, assume a movable piston encloses 1 kg of water with a volume of $V_i = 10^{-3}\,\text{m}^3$, see the left part of Fig. 12.34. Heat is added from a reservoir until the liquid water changes completely into steam of volume $V_f = 0.824\,\text{m}^3$, see the right part of Fig. 12.34. (a) How much work is done by the system (water + steam)

during the boiling process? (b) How much heat energy is added to the system? (c) What is the change in the internal energy of the system?

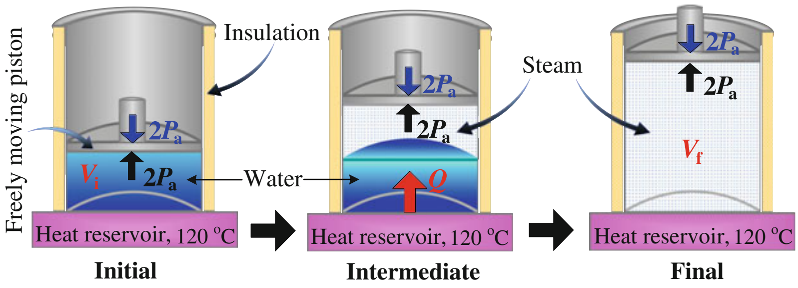
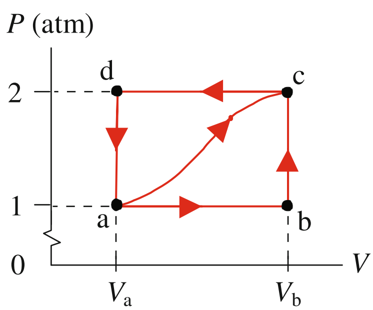


Fig. 12.34 See Exercise (29)

(30) An ideal gas has an initial temperature of 27 °C and an initial volume of 1 m^3. An isobaric expansion of the gas to a new volume of 3 m^3 is achieved by adding 9,500 J of heat at a constant pressure of 3,000 Pa. (a) Determine the work done by the gas during expansion. (b) What is the change in the internal energy of the gas? (c) Find the final temperature of the gas.

(31) An ideal gas is taken from a to c along the curved path in Fig. 12.35. Along this path, the work done by the gas is $W_{ac} = 15$ J and the heat added to the gas is $Q_{ac} = 43$ J. In addition, the work done along path abc is $W_{abc} = 34$ J. (a) What is the change in internal energy of the gas ΔE_{ac} for path ac? (b) What is Q_{abc} for path abc? (c) What is W_{cda} for path cda? (d) What is Q_{cda} for path cda?

Fig. 12.35 See Exercise (31)



(32) Helium with an initial volume of 10^{-3} m^3 and an initial pressure of 10 atm expands to a final volume of 1 m^3. The relationship between pressure and

volume during this expansion process is kept PV = constant by supplying heat at a constant temperature. (a) Calculate the value of the constant. (b) Find the final pressure. (c) Determine the work done by the helium during the expansion. (d) How much heat was absorbed by the expanding helium?

Section 12.5 Heat Transfer

(33) The thermal conductivity of a special type of Pyrex glass at 0 °C is 3×10^{-3} cal/cm.C°.s. (a) Express this quantity in W/m.C° and in Btu/ft.F°.h. (b) What is the R value of a 1 cm sheet of Pyrex?

(34) A slab of a thermal insulator has a cross section of 0.1 m^2, a thickness of 2 cm, and thermal conductivity of 0.1 J/m.s.C°. If the temperature difference between the opposite faces of the insulator is 100 C°, how much heat flows through the slab in 24 h?

(35) Consider the slab of copper shown in Fig. 12.36, where $A = 9 \times 10^{-3}$ m^2, $L = 0.25$ m, and the thermal conductivity is 400 W/m.C°. In the steady-state condition, the temperature of the hot surface is $T_H = 125$ °C, and the temperature for the cold surface is $T_C = 10$ °C. Find the rate of heat transfer through the slab.

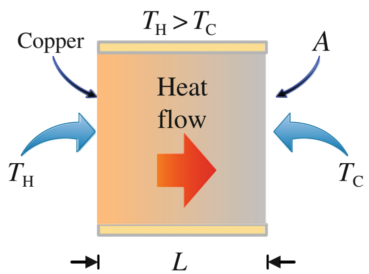


Fig. 12.36 See Exercise (35)

(36) A pair of metal plates having equal areas and equal thicknesses is in thermal contact, as shown in Fig. 12.37. One plate is made of aluminum, and the other is made of iron. Assume that the thermal conductivity of aluminum k_A is exactly three times that of iron k_F. The outer face of the iron plate is maintained at $T_C = 0$ °C, while the outer face of the aluminum plate is maintained at $T_H = 60$ °C. In a steady-state condition, find the interface temperature T and the relation that gives the rate of heat transfer by conduction through the slabs.

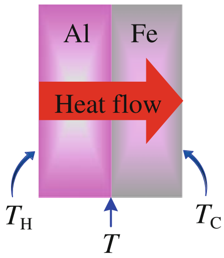


Fig. 12.37 See Exercise (36)

(37) Bricks and insulation are used to construct the walls of a house. The insulation has R_1-value of 0.095 m^2.C°/W. The bricks have an R_2-value of 0.704 m^2.C°/W, see Fig. 12.38. In the steady-state condition, the temperature inside the house is $T_H = 24\,°C$ and the outside temperature is $T_C = 10\,°C$. Find the rate of heat loss through such a wall, if its area is 20 m^2.

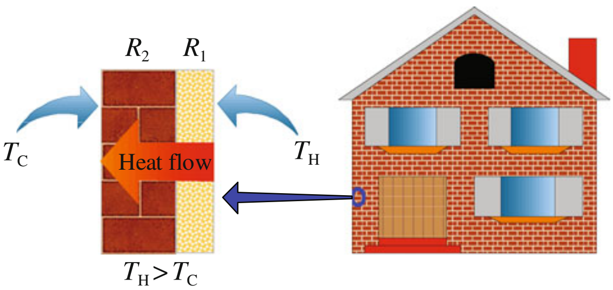


Fig. 12.38 See Exercise (37)

(38) A pipe made of steel has inner and outer radii of 2.5 and 3 cm respectively. The pipe carries hot water at a temperature $T_H = 70\,C°$ and has a thermal conductivity of 14 W/m.C°. The pipe's outer surface temperature is $T_C = 60\,°C$, see the left part of Fig. 12.39. (a) What is the rate of heat flow per unit length of the pipe? (b) When an additional cylindrical insulator of thermal conductivity of 0.2 W/m.C° is used, see the right part of Fig. 12.39, what is the thickness required to reduce heat loss by a factor of 10 and achieve an outer temperature of 30 °C?

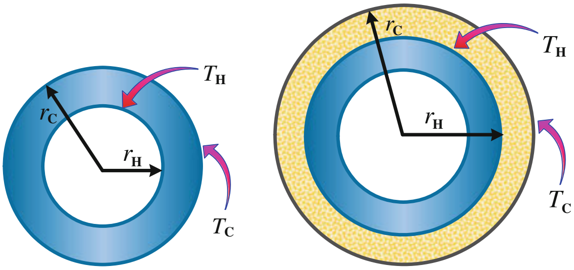
(39) Show that the rate of heat that flows radially outwards in a spherically symmetric system is governed by the equation:

$$H = -4\pi r^2 k \frac{dT}{dr}$$

where r is the distance from the center of the source to the point where the temperature is T. When the inner and outer radii and temperatures are r_H, r_C, T_H, and T_C, respectively, show that:

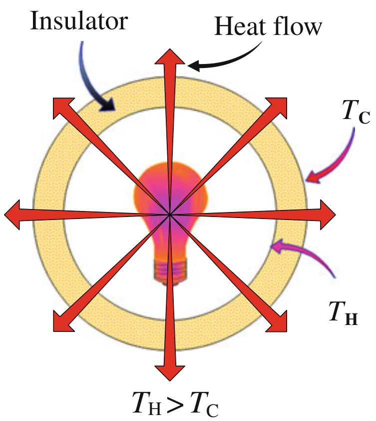
$$H = \frac{4\pi k(T_H - T_C)}{(1/r_H - 1/r_C)}$$

Fig. 12.39 See Exercise (38)



(40) An insulating spherical container has a total inner surface area of $1.5\,\text{m}^2$ and a thickness of 5 cm. A 60 W (assumed to be a point source) electric bulb inside the container is used to maintain a constant temperature difference $T_H - T_C = 100\,\text{C}^\circ$ between the inside and the outside of the container, see Fig. 12.40. What is the thermal conductivity k of the insulating material?

Fig. 12.40 See Exercise (40)



Kinetic Theory of Gases 13

In the simplest model of an ideal gas, which was presented in Chap. 11, we consider each atom/molecule to be a hard sphere that collides elastically with other atoms/molecules or with the walls of the container holding the gas.

In this chapter, our aim is to relate *macroscopic* parameters (such as volume, pressure, temperature, . . . etc.) to *microscopic* parameters (such as average kinetic energy per molecule, internal energy of the gas, . . . etc.). To keep the mathematics relatively simple, we develop a microscopic model that incorporates further justified assumptions.

13.1 Microscopic Model of an Ideal Gas

In this model, the pressure P that a gas exerts on the walls of its encompassing container, of volume V, is due to the collisions of n moles of the gas (or $N = nN_A$ molecules) with those walls. Moreover, in such a model we *assume* that gases have the following:

1. *A large number of molecules*
2. *Large separations between molecules compared to their average sizes*
3. *Molecules that move randomly and obey Newton's laws of motion*
4. *Negligible molecular collisions and experience only elastic collisions with the walls of the container*
5. *A thermal equilibrium at temperature T with the container's walls.*

Now consider the collision of the i$^{\text{th}}$ molecule of such a gas with the colored yz wall of a cubical container of side L, as shown in Fig. 13.1. In this figure, the gas

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Undergraduate Lecture Notes in Physics, DOI: 10.1007/978-3-642-23026-4_13,

molecule of mass m is moving with velocity $\vec{v}_i$ which has the velocity components v_{xi}, v_{yi}, and v_{zi}.

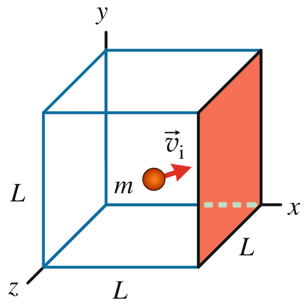


Fig. 13.1 A cubical container of side L, containing n moles or N molecules of an ideal gas. The figure shows an i$^{\text{th}}$ molecule of mass m and velocity $\vec{v}_i$ that is about to collide with the colored right yz wall

Because we assume elastic collisions, only the x component of the above molecule's velocity changes, while its y and z components remain unchanged. This is illustrated in Fig. 13.2, which captures only the motion in the xy plane. Using the definition of momentum (see Sect. 7.1), the only change in the molecule's momentum is along the x-axis. Its momentum before the collision is mv_{xi} and its momentum after the collision is $-mv_{xi}$. The change in momentum in one collision is:

$$\left.\begin{aligned} \Delta p_{xi} &= (p_{xi})_{\text{fin}} - (p_{xi})_{\text{ini}} = -mv_{xi} - mv_{xi} = -2mv_{xi} \\ \Delta p_{yi} &= 0 \\ \Delta p_{zi} &= 0 \end{aligned}\right\} \tag{13.1}$$

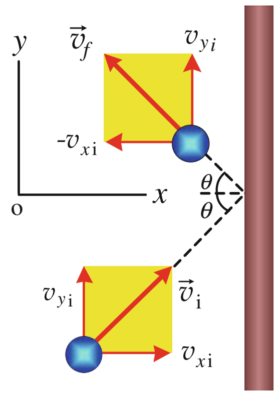


Fig. 13.2 A molecule moving in the xy plane undergoes an elastic collision with a wall perpendicular to that plane. The x component of the velocity is reversed, while the y component remains unchanged

Since the momentum of the system (molecule + wall) is conserved, the momentum delivered by the wall to the molecule for the i$^{\text{th}}$ molecule is $\Delta p_{xi} = -2mv_{xi}$. The molecule in Fig. 13.1 will travel to the opposite wall and back again. It will repeat this journey, hitting the colored wall repeatedly. The time between two successive collisions with this wall is Δt. This means that the molecule travels with a speed v_{xi} a distance $2L$ in time Δt. Thus:

$$\Delta t = \frac{2L}{v_{x\mathrm{i}}} \tag{13.2}$$

This time is very small and the molecule will make many collisions with the wall, each separated by time Δt. Therefore the number of collisions per unit time is large. Consequently, the average force exerted on the i$^{\text{th}}$ molecule over many collisions will be equal to the momentum change during one collision $\Delta p_{x\mathrm{i}}$ divided by the time Δt between collisions (Newton's second law).

If $\overline{F}_{x\mathrm{i}}$ is the average perpendicular force exerted by the wall on the molecule, then from Newton's third law, the average perpendicular force exerted on the wall by the molecule is $\overline{F}_{x\mathrm{i,on\,wall}} = -\overline{F}_{x\mathrm{i}}$. That is:

$$\begin{aligned} \overline{F}_{x\mathrm{i,on\ wall}} &= -\overline{F}_{x\mathrm{i}} = -\frac{\Delta p_{x\mathrm{i}}}{\Delta t} = -\frac{-2mv_{x\mathrm{i}}}{\Delta t} = \frac{2mv_{x\mathrm{i}}}{2L/v_{x\mathrm{i}}} \\ &= \frac{mv_{x\mathrm{i}}^2}{L} \end{aligned} \tag{13.3}$$

To find the total average force $\overline{F}_{x,\mathrm{on\ wall}}$ exerted on the wall we must add up the contributions of all molecules that strike the wall and then divide this total force by the area of the wall. This gives the average pressure P on the wall. Thus:

$$\begin{aligned} P &= \frac{\Sigma\overline{F}_{x\mathrm{i,on\,wall}}}{L^2} = \frac{mv_{x1}^2/L + mv_{x2}^2/L + \cdots + mv_{xN}^2/L}{L^2} \\ &= \frac{m}{L^3}(v_{x1}^2 + v_{x2}^2 + \cdots + v_{xN}^2) \\ &= \frac{m}{L^3}\sum_{\mathrm{i}=1}^{N} v_{x\mathrm{i}}^2 \end{aligned} \tag{13.4}$$

Since the average value of the square of the x component of all the molecular speeds is given by:

$$\begin{aligned} \overline{v_x^2} &= \frac{v_{x1}^2 + v_{x2}^2 + \cdots + v_{xN}^2}{N} \\ &= \frac{\sum_{\mathrm{i}=1}^{N} v_{x\mathrm{i}}^2}{N} \end{aligned} \tag{13.5}$$

and the volume of the container is given by $V = L^3$, then we can express the average pressure in the following form:

$$P = \frac{mN}{V}\,\overline{v_x^2} \tag{13.6}$$

Since $v_{\mathrm{i}}^2 = v_{x\mathrm{i}}^2 + v_{y\mathrm{i}}^2 + v_{z\mathrm{i}}^2$ for the i$^{\text{th}}$ molecule, then this result and Eq. 13.5 lead to $\overline{v^2} = \overline{v_x^2} + \overline{v_y^2} + \overline{v_z^2}$. In addition, because there is a large number of molecules moving randomly in all directions, the average values of the squares of their velocity components are equal, i.e. $\overline{v_x^2} = \overline{v_y^2} = \overline{v_z^2}$.

Thus: $$\overline{v_x^2} = \overline{v_y^2} = \overline{v_z^2} = \tfrac{1}{3}\overline{v^2} \tag{13.7}$$

Hence, Eq. 13.6 can be expressed as:

$$P = \tfrac{2}{3}\frac{N}{V}\left(\tfrac{1}{2}m\overline{v^2}\right) \tag{13.8}$$

The square root of $\overline{v^2}$ is called the *root mean square* (rms) *speed* of the molecules and is symbolically written as v_{rms}, i.e. $v_{\mathrm{rms}}^2 = \overline{v^2}$. Thus:

$$\boxed{v_{\mathrm{rms}} = \sqrt{\overline{v^2}}} \tag{13.9}$$

By substitution, Eq. 13.8 will take on the form:

$$\boxed{P = \tfrac{2}{3}\frac{N}{V}\left(\tfrac{1}{2}mv_{\mathrm{rms}}^2\right)} \tag{13.10}$$

where $\frac{1}{2}mv_{\mathrm{rms}}^2$ is the average translational kinetic energy per molecule. Equation 13.10 connects the macroscopic quantities P and V to the microscopic quantity representing the average molecular speed v_{rms}.

We should remember that we ignored inter-molecular collisions as we derived Eq. 13.10. Note that these collisions only affect the momenta of the molecules and have no net effect on the walls, so including such collisions will yield the same equation. This is consistent with the random motion assumption, which implies that the velocity distribution of the molecules does not change with time despite the collision between molecules. In addition, this equation is valid for any shaped container, although it was derived assuming a cubical container.

To get some insight into the meaning of temperature, we first rewrite Eq. 13.10 in the following form:

$$\boxed{PV = \tfrac{2}{3}N\left(\tfrac{1}{2}mv_{\mathrm{rms}}^2\right)} \tag{13.11}$$

Then we compare this with the ideal gas law Eq. 11.10:

$$\boxed{PV = Nk_B T} \tag{13.12}$$

where this equation is based on experimental facts concerning the macroscopic behavior of the ideal gas. Equating the right-hand sides of the last two equations, we find that:

$$\boxed{T = \frac{2}{3\, k_B} \left(\tfrac{1}{2} m v_{rms}^2\right)} \tag{13.13}$$

Since $K = \frac{1}{2} m v_{rms}^2$ is the average translational kinetic energy per molecule, we see that temperature is a direct measure of it. In addition, we can relate the average translational molecular kinetic energy per molecule to the temperature as follows:

$$\boxed{K = \tfrac{1}{2} m v_{rms}^2 = \tfrac{3}{2} k_B T} \tag{13.14}$$

With $\overline{v_x^2} = \overline{v_y^2} = \overline{v_z^2} = \frac{1}{3}\overline{v^2} = \frac{1}{3} v_{rms}^2$, we can find the average translational kinetic energy per molecule associated with the motion along the x, y, and z axes as follows:

$$\boxed{\begin{aligned} \tfrac{1}{2} m \overline{v_x^2} &= \tfrac{1}{2} k_B T \\ \tfrac{1}{2} m \overline{v_y^2} &= \tfrac{1}{2} k_B T \\ \tfrac{1}{2} m \overline{v_z^2} &= \tfrac{1}{2} k_B T \end{aligned}} \tag{13.15}$$

Thus, each translational degree of freedom[1] contributes an equal amount of energy to the gas, namely $\frac{1}{2} k_B T$. A generalization of that is known as the theory of equipartition of energy.

> Theory of equipartition of energy:
> The energy of a system experiencing thermal equilibrium is equally divided among all degrees of freedom. Each degree of freedom contributes $\frac{1}{2} k_B T$ to the energy of the system.

The total translational energy K_{tot} (*which is the internal energy in this model*) of N molecules of an ideal gas is the product of N with the average translational energy per molecule $K = \frac{1}{2} m v_{rms}^2$. That is:

---

[1] The translational degrees of freedom refer to the number of independent ways by which a molecule can possess energy when moving in a three-dimensional space.

$$\boxed{K_{\text{tot}} = N\left(\tfrac{1}{2}mv_{\text{rms}}^2\right) = \tfrac{3}{2}Nk_{\text{B}}T = \tfrac{3}{2}nRT} \tag{13.16}$$

where we have used $N = nN_{\text{A}}$ for the number n of kilomoles of the gas and $k_{\text{B}} = R/N_{\text{A}}$ for Boltzmann's constant.

Using the molar mass $M = mN_{\text{A}}$ in Eq. 13.16, where m here is the molecular mass and not to be confused with the mass of the gas as in Chap. 11, we can relate v_{rms} to the gas temperature T as follows:

$$v_{\text{rms}} = \sqrt{\frac{3k_{\text{B}}T}{m}} = \sqrt{\frac{3RT}{M}} \tag{13.17}$$

Table 13.1 shows some rms speeds calculated from Eq. 13.17.

Table 13.1 Some molecular speeds at room temperature ($T = 300$ K)

Gas	Molar mass (kg/kmol)	v_{rms} (m/s)
Hydrogen (H_2)	2.02	1,925
Helium (He)	4.0	1,368
Water vapor (H_2O)	18	645
Nitrogen (N_2) or Carbon monoxide (CO)	28	517
Nitrogen oxide (NO)	30	499
Oxygen (O_2)	32	484
Carbon dioxide (CO_2)	44	412
Sulfur dioxide (SO_2)	48	394

Example 13.1

Three moles of hydrogen gas are confined to a volume of 0.4 m^3 at a temperature of 24 °C. (a) What is the total translational kinetic energy of the gas molecules? (b) What is the average kinetic energy per molecule? (c) What is the rms speed of the molecules?

Solution: (a) Using Eq. 13.16 with $T = 24 + 273 = 297$ K, we get:

$$K_{\text{tot}} = \tfrac{3}{2}nRT = \tfrac{3}{2}\,(3 \times 10^{-3}\,\text{kmol})(8.314 \times 10^3\,\text{J/kmol.K})(297\,\text{K}) = 1.1 \times 10^4\,\text{J}$$

(b) Using Eq. 13.14 with $T = 297$ K, we get:

$$K = \tfrac{3}{2}k_{\text{B}}T = \tfrac{3}{2}\,(1.38 \times 10^{-23}\,\text{J/K})(297\,\text{K}) = 6.15 \times 10^{-21}\,\text{J}$$

(c) Using Eq. 13.17 with $T = 297$ K and the known hydrogen molar mass $M = 2.02$ kg/kmol, we get

$$v_{\text{rms}} = \sqrt{\frac{3RT}{M}} = \sqrt{\frac{3(8.314 \times 10^3\ \text{J/kmol.K})(297\ \text{K})}{2.02\ \text{kg/kmol}}} = 1{,}915\,\text{m/s}$$

Internal Energy of a Monatomic Ideal Gas

In the ideal-gas model presented in this section, we assumed each molecule behaved like a hard sphere and had no structure. We were then able to find its average translational kinetic energy in terms of the temperature of the gas. This average kinetic energy is associated with the motion of the center of mass of each molecule. This model does not include the energy associated with the internal motions of the gas, such as the vibrational and rotational motions of the molecules.

In light of this, consider an ideal monatomic gas, such as helium (He), where the gas molecules include only one atom. Essentially, all of the kinetic energy of such monatomic molecules are associated with the motion of each molecule's center of mass. Therefore, when we add energy to a monatomic gas, all the added energy goes into increasing the translational kinetic energy of the atoms. Thus, the total internal energy E_{int} of an ideal monatomic gas of N atoms (or n kmol) at pressure P, volume V, and temperature T will all be translational energy K_{tot}, i.e. $E_{\text{int}} = K_{\text{tot}}$. Using $K_{\text{tot}} = \frac{3}{2}Nk_{\text{B}}T = \frac{3}{2}nRT$ we find:

$$\begin{gathered} E_{\text{int}} = \begin{cases} \frac{3}{2}Nk_{\text{B}}T \\ \frac{3}{2}nRT \end{cases} \\ \text{or} \\ \Delta E_{\text{int}} = \begin{cases} \frac{3}{2}Nk_{\text{B}}\Delta T \\ \frac{3}{2}nR\Delta T \end{cases} \end{gathered} \qquad \text{(Monatomic ideal gas)} \qquad (13.18)$$

In general, *the internal energy of an ideal gas is a function of T only*, and the exact relationship depends on the type of gas.

> Internal energy of an ideal gas:
> The internal energy E_{int} of n kilomoles of an ideal gas is a function of the gas temperature only; it does not depend on any other variable.

With this result, we are going to find two expressions for the molar specific heat of an ideal gas. By convention, the symbol C_V will be used when the gas volume is constant while heat energy is added, whereas the symbol C_P will be used when the gas pressure remains constant while heat energy is added.

13.2 Molar Specific Heat Capacity of an Ideal Gas

Consider an ideal gas undergoing several processes such that the change in temperature is achieved by taking a variety of different paths from one isotherm at temperature T to another isotherm at temperature $T + \Delta T$, as shown in Fig. 13.3. Because ΔT is the same for each path, the change in internal energy ΔE_{int} is the same for all paths. However, from the first law of thermodynamics, $\Delta E_{\text{int}} = Q - W$, we know that the heat Q is different for each path because W is different. Thus, the heat associated with a given change in temperature ΔT does not have a unique value.

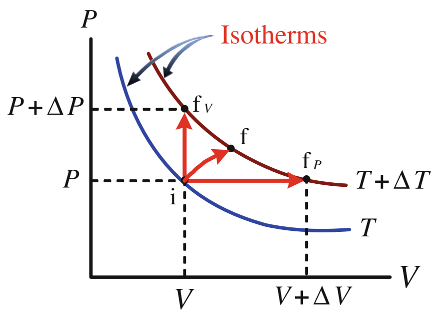


Fig. 13.3 An ideal gas is taken from initial state i of isotherm at temperature T to another at temperature $T + \Delta T$ along three different paths. The change in internal energy is the same for all paths

We can treat this situation by defining specific heats for two processes that frequently occur: changes at constant volume and changes at constant pressure. Because the number of *moles* is a convenient measure of the amount of gas, we define the *molar specific heats* associated with these processes with the following equations:

$$\boxed{Q_V = nC_V\Delta T} \quad \text{(Constant volume)} \tag{13.19}$$

$$\boxed{Q_P = nC_P\Delta T} \quad \text{(Constant pressure)} \tag{13.20}$$

where n ***now*** is the number of moles, C_V is the **molar specific heat at constant volume**, and C_P is the **molar specific heat at constant pressure**. When we deliver heat to the gas at constant pressure, the heat Q_P must account for both the increase in internal energy ΔE_{int} and the work W. But, when we deliver heat to the gas at constant volume, then the heat Q_V must account for only the same increase in internal energy

ΔE_{int}, since $W = 0$. For this reason, Q_P is greater than Q_V for all given values of n and ΔT. Thus, C_P is greater than C_V.

The molar specific heat capacities C_V and C_P are related to the specific heats c_V and c_P by the following two relations:

$$\boxed{\begin{aligned} C_V &= M\, c_V \\ C_P &= M\, c_P \end{aligned}} \tag{13.21}$$

where M is the molar mass of the gas.

13.2.1 Molar Specific Heat at Constant Volume

We consider n moles of an ideal gas at pressure P and temperature T, confined to a cylinder of fixed volume V, as shown in Fig. 13.4a. The *initial state* i of the gas is identified on the PV graph of Fig. 13.4b. When we add a small amount of heat Q_V to the gas, by slowly turning up the temperature of a heat reservoir, the gas temperature rises to $T + \Delta T$ and its pressure rises to $P + dP$, bringing the gas to the *final state* f that is identified in Fig. 13.4b.

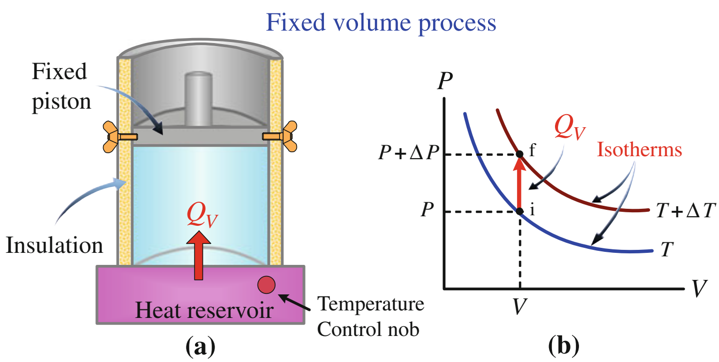


Fig. 13.4 (a) The temperature of an ideal gas is increased from T to $T + \Delta T$ in a constant *volume* process by adding heat Q_V. (b) The constant volume process i → f on a PV diagram

Since $W = \int P\, dV = 0$ for a *constant volume process*, then all of the transferred energy will be stored in the gas as an increase in its internal energy, and the change in the internal energy of the gas will be given by the first law of thermodynamics as:

$$\boxed{\Delta E_{\text{int}} = Q_V} \tag{13.22}$$

Substituting this expression into Eq. 13.19, we get:

$$\boxed{Q_V = \Delta E_{\text{int}} = nC_V \Delta T} \quad \text{(Ideal gas)} \tag{13.23}$$

If the molar specific heat C_V is constant, we can express the internal energy of an ideal gas as:

$$\boxed{E_{\text{int}} = nC_V T} \quad \text{(Ideal gas)} \tag{13.24}$$

This equation applies to all ideal gases (to gases having one or more than one atom per molecule). In the limit of infinitesimal changes, we can use Eq. 13.24 to express the molar specific heat at constant volume as follows:

$$\boxed{C_V = \frac{1}{n}\frac{dE_{\text{int}}}{dT}} \tag{13.25}$$

For monatomic gases, if we substitute the internal energy $E_{\text{int}} = \frac{3}{2}nRT$ from Eq. 13.18 into Eq. 13.25, we get:

$$\boxed{C_V = \tfrac{3}{2}R = \begin{cases} 12.5\ \text{J/mol.K} \\ 12.5 \times 10^3\ \text{J/kmol.K} \end{cases}} \quad \text{(Monatomic ideal gas)} \tag{13.26}$$

This result is in agreement with experimentally measured values for monatomic gases at a wide range of temperatures.

13.2.2 Molar Specific Heat at Constant Pressure

We now consider n moles of an ideal gas at pressure P and temperature T, confined to a volume V by a freely moving piston, as shown in Fig. 13.5a. The *initial state* i of the gas is identified on the PV diagram of Fig. 13.5b. Under a constant pressure, a small amount of heat Q_P is added to the gas, by slowly turning up the temperature of a heat reservoir; the gas temperature rises to $T + \Delta T$, bringing the gas to the *final state* f that is identified in Fig. 13.5b.

To relate C_P to C_V, we start with the first law of thermodynamics:

$$\Delta E_{\text{int}} = Q_P - W \tag{13.27}$$

and then replace each term. From Eq. 13.23, we have:

$$\Delta E_{\text{int}} = nC_V \Delta T \tag{13.28}$$

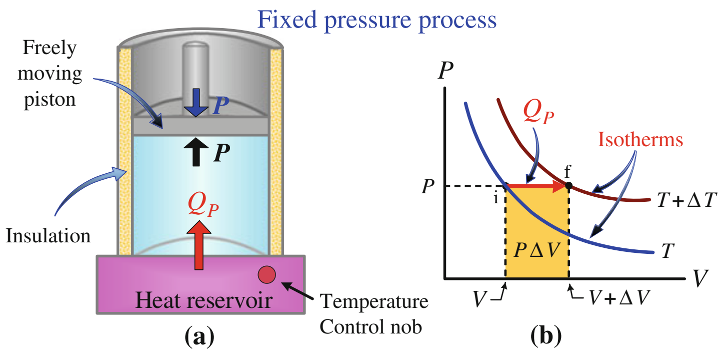


Fig. 13.5 (a) The temperature of an ideal gas is increased from T to $T + \Delta T$ in a constant *pressure* process by adding heat Q_P. (b) The work PdV is given by the colored area for the constant pressure process i $\rightarrow$ f on the PV diagram

Also, from Eq. 13.20, we have:

$$Q_P = nC_P \Delta T \tag{13.29}$$

The work done by the ideal gas in the *constant pressure process* of Fig. 13.5b is $W = P\Delta V$. Then we use the ideal-gas equation $PV = nRT$ to find W as follows:

$$W = P\Delta V = nR\Delta T \tag{13.30}$$

Substituting with Eq. 13.28, 13.29, and 13.30 into Eq. 13.27, and then dividing by $n\Delta T$, we find

$$\boxed{C_P - C_V = R} \tag{13.31}$$

This prediction of kinetic theory agrees well with experiments, not only for monatomic gases but for gases in general, as long as their density is low enough so that we may treat them as ideal.

For monatomic gases, we substitute with $C_V = 3R/2$ into Eq. 13.31 to find:

$$\boxed{C_P = \tfrac{5}{2}R = \begin{cases} 20.8\ \text{J/mol.K} \\ 20.8 \times 10^3\ \text{J/kmol.K} \end{cases}} \quad \text{(Monatomic ideal gas)} \tag{13.32}$$

The ratio of the molar specific heats C_P and C_V is a dimensionless quantity γ given by:

$$\boxed{\gamma = \frac{C_P}{C_V} = \frac{\frac{5}{2}R}{\frac{3}{2}R} = \frac{5}{3} = 1.67} \quad \text{(Monatomic ideal gas)} \tag{13.33}$$

Theoretical values of C_V, C_P, and γ are in excellent agreement with experimental values obtained for monatomic gases, but they are in serious disagreement with the values for more complex gases (those with multiple atoms per molecule). This is because their internal energy E_{int}, and their molar specific heats C_P and C_V include components from rotational and vibrational motions of the molecules.

Example 13.2

A cylinder contains 2 moles of helium at a temperature of 27 °C. Heat is added to the gas to increase its temperature to 227 °C. (a) Find the quantity of heat Q_V used if the gas was heated at constant volume. (b) Find the quantity of heat Q_P and the work done by the gas W if the gas was heated at constant pressure.

Solution: (a) Treating the helium gas as an ideal monatomic gas, the work done in this case is zero. We use $C_V = 3R/2 = 12.5$ J/mol.K and $\Delta T = 227\,°\text{C} - 27\,°\text{C} = 200\,\text{C}° = 200$ K in Eq. 13.23 to get:

$$Q_V = nC_V\,\Delta T = (2\text{ mol})(12.5\text{ J/mol.K})(200\text{ K}) = 5{,}000\text{ J}$$

(b) For a constant pressure process we use $C_P = 5R/2 = 20.8$ J/mol.K in Eq. 13.20 to get:

$$Q_P = nC_P\Delta T = (2\text{ mol})(20.8\text{ J/mol.K})(200\text{ K}) = 8{,}320\text{ J}$$

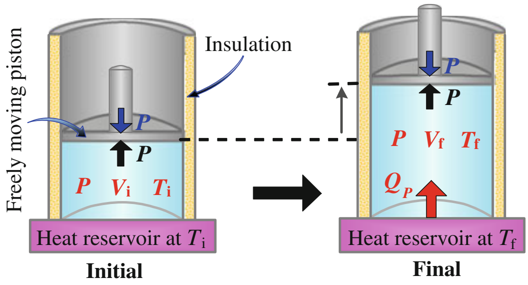
The work done by the gas in this process is:

$$W = Q_P - Q_V = 8{,}320\text{ J} - 5{,}000\text{ J} = 3{,}320\text{ J}$$

Example 13.3

A cylinder contains 5 moles of monatomic helium. At constant pressure, the helium gas undergoes a volume expansion and a temperature increase $\Delta T = T_f - T_i = 20\,\text{C}°$ due to the addition of heat Q_P, as shown in Fig. 13.6. (a) How much heat Q_P is added to the helium? (b) What is the change ΔE_{int} in the internal energy of the helium? (c) How much work W is done by the helium as it expands?

Fig. 13.6



Solution: (a) Treating the helium as an ideal monatomic gas undergoing a constant-pressure process, we use $C_P = 20.8\,\text{J/mol.K}$ and $\Delta T = 20\,\text{C}^\circ = 20\,\text{K}$ in Eq. 13.20 to get:

$$Q_P = nC_P\Delta T = n(5R/2)\Delta T = (5\text{ mol})(20.8\text{ J/mol.K})(20\text{ K}) = 2{,}080\,\text{J}$$

(b) Even though the temperature of the helium increases at a constant pressure (not at constant volume), we use Eq. 13.23 to calculate the change in internal energy when $C_V = 12.5\,\text{J/mol.K}$ as follows:

$$\Delta E_{\text{int}} = nC_V\Delta T = n(3R/2)\Delta T = (5\text{ mol})(12.5\text{ J/mol.K})(20\,\text{K}) = 1{,}250\,\text{J}$$

(c) From the first law of thermodynamics, $\Delta E_{\text{int}} = Q_P - W$, we calculate the work done by the gas in this process as follows:

$$W = Q_P - \Delta E_{\text{int}} = 2{,}080\,\text{J} - 1{,}250\,\text{J} = 830\text{ J}$$

Of all the heat energy $Q_P = 2{,}080\,\text{J}$ that is transferred to the helium during the increase in temperature, only 1,250 J goes to increasing the helium's internal energy and hence its temperature. The remaining 830 J is transferred out of the helium as work done during the expansion.

Internal Energy of a Diatomic Ideal Gas

In a diatomic ideal-gas model, a molecule can rotate about two different axes, while the rotation about the third axis passing through the two atoms gives very little energy because the moment of inertia about this axis is very small, see Fig. 13.7.

Therefore, a diatomic gas is said to have five energy degrees of freedom: three translational and two rotational. According to the principle of equipartition of energy, each active degree of freedom of a molecule has on average an energy equal to $\frac{1}{2}k_BT$. Thus, the average energy for a molecule in a diatomic gas is:

$$\boxed{E = \tfrac{5}{2}k_BT} \quad \text{(Diatomic ideal gas)} \tag{13.34}$$

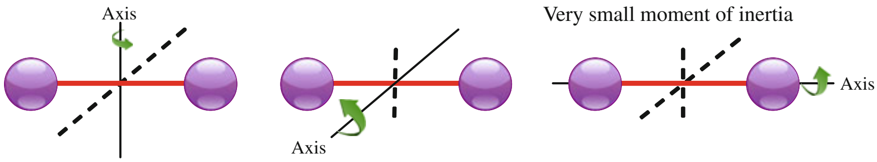


Fig. 13.7 A diatomic molecule can rotate about two perpendicular axes with appreciable rotational energy while the rotation about the third axis gives very little rotational energy (i.e. only two degrees of freedom)

Hence, the internal energy E_{int} of a diatomic ideal gas of N (or n kmol) at pressure P, volume V, and temperature T will be:

$$\boxed{E_{\text{int}} = \begin{cases} \tfrac{5}{2}Nk_BT \\ \tfrac{5}{2}nRT \end{cases}} \quad \text{(Diatomic ideal gas)} \tag{13.35}$$

In general, *the internal energy of an ideal gas is a function of T only*, and the exact relationship depends on the type of gas. The vibrational (kinetic and potential) degrees of freedom have only a tiny effect on Eqs. 13.34 and 13.35 unless the temperature is extremely high. Quantum mechanical study (which is not our aim) predicts discrete vibrational levels with spacing generally much larger than k_BT.

We can use the above results and Eq. 13.25 to find C_V and C_P as follows:

$$\boxed{\begin{aligned} C_V &= \frac{1}{n}\frac{dE_{\text{int}}}{dT} = \frac{1}{n}\frac{d}{dT}\left(\tfrac{5}{2}nRT\right) = \tfrac{5}{2}R \\ C_P &= C_V + R = \tfrac{7}{2}R \end{aligned}} \quad \text{(Diatomic ideal gas)} \tag{13.36}$$

Table 13.2 displays the measured molar specific heats of some gases. These results are in good agreement with the predicted C_V and C_P. The small deviations from the predicted values are due to the fact that real gases are not ideal gases. Real gases experience weak intermolecular interactions, which are not addressed in the presented ideal gas model.

Table 13.2 Some molar specific heats of various gases at 15 °C

Molar specific heat (J/mol C°)				
Gas	C_p	C_V	$C_P - C_V$	$\gamma = C_P/C_V$
Monatomic gases				
He	20.8	12.5	8.33	1.67
Ar	20.8	12.5	8.33	1.67
Ne	20.8	12.7	8.12	1.64
Kr	20.8	12.3	8.49	1.69
Diatomic gases				
H_2	20.8	20.4	8.33	1.41
N_2	29.1	20.8	8.33	1.40
O_2	29.4	21.1	8.33	1.40
CO	29.3	21.0	8.33	1.40
Cl_2	34.7	25.7	8.96	1.35
Triatomic gases				
CO_2	37.0	28.5	8.50	1.30
SO_2	40.4	31.4	9.00	1.29
H_2O	35.4	27.0	8.37	1.30

13.3 Distribution of Molecular Speeds

Molecules in a gas at thermal equilibrium are assumed to be in random motion, i.e. they have a wide range of molecular speeds. In 1859, James Clerk Maxwell derived an expression that describes the distribution of speeds in a gas containing N molecules in thermal equilibrium at temperature T. The number of molecules dN with speeds in the range v and $v + dv$ is defined by the distribution function $f(v)$ (known as Maxwell-Boltzmann distribution) through the following relation:

$$dN = f(v)\, dv = 4\pi N \left(\frac{m}{2\pi k_B T}\right)^{3/2} v^2\, e^{-\frac{1}{2}mv^2/k_B T}\, dv \tag{13.37}$$

where m is the mass of a gas molecule, k_B is Boltzmann's constant, and T is the absolute gas temperature. A sketch of the distribution function $f(v)$ is shown in Fig. 13.8 at a certain temperature T.

The average speed $\bar{v}$ can be obtained by integrating the product of the speed v with dN and dividing by the total number N. In addition, one can find v_{rms} and the most probable speed v_p as follows:

$$\bar{v} = \frac{1}{N}\int_0^\infty v f(v)\, dv = \sqrt{\frac{8}{\pi}\frac{k_B T}{m}} \tag{13.38}$$

$$v_{rms} = \sqrt{\overline{v^2}} = \sqrt{\frac{1}{N}\int_0^\infty v^2 f(v)\, dv} = \sqrt{\frac{3k_B T}{m}} \tag{13.39}$$

$$v_p = \sqrt{\frac{2k_B T}{m}} \tag{13.40}$$

From these results, we see that $v_{rms} > \bar{v} > v_p$ as displayed in Fig. 13.8.

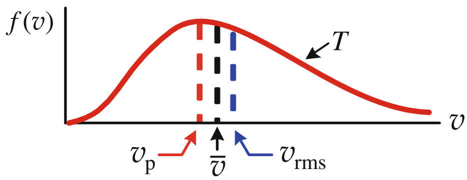


Fig. 13.8 Distribution of speeds of an ideal gas, $f(v)$. The area under the curve gives the total number of gas molecules. The speed at the peak v_p is less that $\bar{v}$ and v_{rms} because $f(v)$ is skewed to the right of the peak

13.4 Non-ideal Gases and Phases of Matter

The isothermal PV-diagram presented in Sect. 12.4 and Sect. 13.2 can help us grasp the overall behavior of a gas described by the gas law $PV = nRT$. The PV-isotherm for a constant amount of an ideal gas is displayed for temperatures $T_4 > T_3 > T_2 > T_1$ in Fig. 13.9a. Notice that the pressure P is inversely proportional to V and that the isotherms are hyperbolic curves.

Fig. 13.9b displays a PV-diagram for a substance that does not obey the ideal gas law for temperatures $T_4 > T_3 > T_c > T_2 > T_1$. The solid curve at T_4 represents the behavior of the non-ideal gas, while the dashed curve represents the behavior predicted by the ideal gas law at the same temperature. The curves at T_3 and T_c deviate more from the dashed curves predicted by the ideal gas law. At successively lower temperatures T_2 and T_1 (both below T_c), the behavior deviates even more from the curves of part a, and the isotherms develop flat regions in which the substance can be compressed without an increase in its pressure. Observation shows that the non-ideal gas is condensing from vapor (gas) to a **liquid state**. The colored region represents isotherms in their **liquid-vapor phase equilibrium**. A transition from one phase to another requires **phase equilibrium** between the two phases. This occurs at only one definite temperature for a given pressure value.

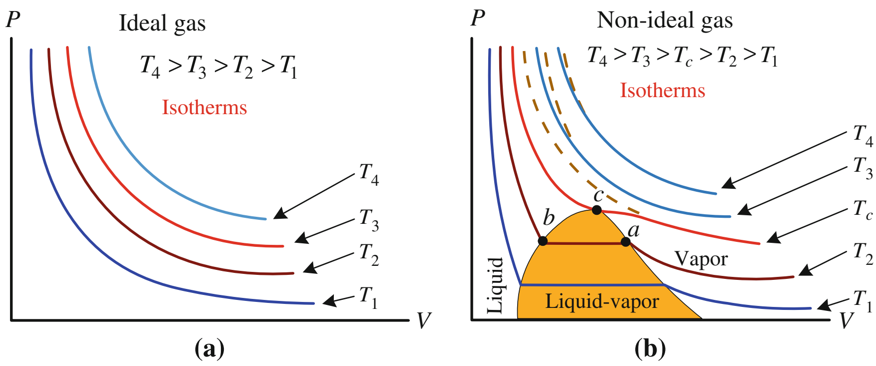


Fig. 13.9 (a) An isothermal P-V diagram of an ideal gas for temperatures $T_4 > T_3 > T_2 > T_1$. (b) An isothermal P-V diagram for a non-ideal gas for temperatures $T_4 > T_3 > T_c > T_2 > T_1$.

Kinetic theory can help us understand this behavior if we note that at higher pressures, and particularly at lower temperatures, the attractive potential energy due to attractive forces between molecules cannot be ignored as in the case of an ideal gas. These attractive forces at lower temperatures tend to pull the molecules and cause liquefaction.

The curve at T_c in Fig. 13.9b represents the substance at its **critical temperature**, and point c on the curve is called the **critical point**. When we compress the gas at constant temperature $T_2 (< T_c)$ it will be in the vapor state until point a is reached. As the volume decreases even more, it begins to liquefy and both the temperature and pressure remain constant. At point b, the substance is in the liquid state. Note that, a substance below T_c in the gaseous state is called a **vapor**, whereas above T_c, it is called a **gas**.

We can represent the condition of phase equilibrium on a PT-diagram such as that in Fig. 13.10. This diagram is referred to as a phase diagram and displays the following:

1. The curve labeled L-V represents points where liquid and vapor phases are in equilibrium (*boiling point* versus *pressure* curve)
2. The curve labeled S-L represents points where solid and liquid phases are in equilibrium (*freezing point* versus *pressure* curve)

3. The curve labeled S-V represents points where solid and vapor phases are in equilibrium (*sublimation point* versus *pressure* curve)

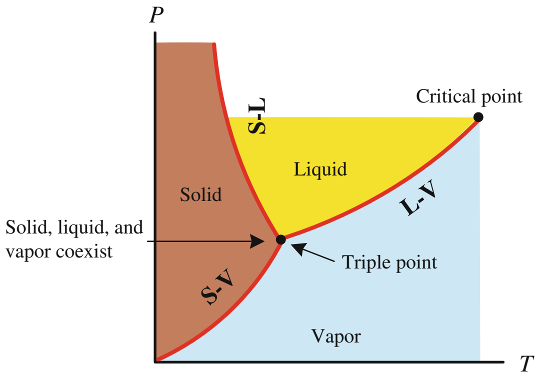


Fig. 13.10 A *P*-*T* diagram of a non-ideal substance showing regions of temperature and pressure at which the various phases exist

All three curves meet at the **triple point**, the only condition under which all three phases can coexist. Because the triple point corresponds to a unique value of temperature and pressure, it is precisely reproducible and is frequently used as a reference point. In Sect. 11.1, we used the triple-point temperature of water to define the Kelvin scale. This point occurs at $T = 273.16$ K and $P = 4.58$ mm Hg $= 6.026 \times 10^{-3}$ atm.

13.5 Exercises

Section 13.1 Microscopic Model of an Ideal Gas

(1) In an interval of 1 s, the total number of oxygen molecules that collide perpendicularly with a wall of an area $A = 10^{-3}\ \text{m}^2$ is $N = 5 \times 10^{23}$ molecules. Assume that each molecule has a mass $m = 5.3 \times 10^{-26}$ kg and moves with a perpendicular speed $v = 500$ m/s. What is the pressure exerted on the wall by these oxygen molecules?

(2) A 2-mol sample of oxygen gas is at STP [A *standard temperature and pressure* implies a temperature of 0 °C (273.15 K) and an atmospheric pressure of 1 atm]. (a) What is the average translational kinetic energy of an oxygen molecule under these conditions? (b) What is the total translational kinetic energy of the oxygen molecules?

(3) A gas is at 27 °C. Find the temperature at which the rms (*root mean square*) speed of its molecules is doubled.

(4) Bellatrix (or Amazon) star is one of the hottest stars that we can see with the naked eye and its surface temperature is about 3×10^4 K. (a) Find the rms speed of helium atoms near the surface of this star. (b) Compare the value of part (a) with the rms speed at 27 °C.

(5) A vessel of volume $V = 4 \times 10^{-2}$ m^3 filled with nitrogen gas contains $n = 4$ mol at a pressure $P = 2$ atm. (a) What is the temperature of this gas? (b) What is the average translational kinetic energy of a nitrogen molecule under these conditions?

(6) (a) Use the definition of Avogadro's number to find the mass of a helium atom, given that the molar mass of He is 4.00 kg/kmol. (b) How many atoms of helium are confined in a container of volume $V = 10^{-4}$ m^3 at 27 °C and 1 atm? (c) What is the rms speed of the helium atoms?

(7) A sample of 2-mol of monatomic argon is at 27 °C. The molar mass of argon (Ar) is 39.95 kg/kmol. (a) What is the average translational kinetic energy of an argon atom in this sample? (b) What is the total translational kinetic energy of all the argon atoms?

(8) At a temperature of 100 °C, a mixture of the monatomic helium and argon gases is in thermal equilibrium. (a) What is the average kinetic energy for each type of atom? (b) What is the rms speed for each type of atom? Assume this mixture displays the properties of an ideal gas, and that the molar mass of helium (He) is 4.00 kg/kmol, and the molar mass of argon (Ar) is 39.95 kg/kmol.

(9) (a) Find the change in internal energy of $n = 4$ mol of monatomic neon gas when its temperature is increased by 10 C°. (b) Will your answer change if the gas is diatomic? Explain your reasoning.

(10) Assume that oxygen at STP is an ideal gas, and that each molecule occupies the same cubical volume a^3, where a is the length of one side of the cube. (a) Find the volume of each molecule. (b) Estimate of the average distance between the oxygen molecules.

(11) Find a relation that gives the rms speed $v_{\text{rms}} = \sqrt{3RT/M}$ of gas molecules in terms of the pressure P of the gas and its density ρ.

(12) A sample of nitrogen gas is at 0 °C. (a) What is the rms speed of a nitrogen molecule at this temperature? (b) Assuming that each molecule has no preferred direction and does not collide with any other molecule, find the time that one molecule will cross a cubical container of side $a = 1.5$ m. (c) Estimate the number of times that a molecule would move back and forth on the average.

Section 13.2 Molar Specific Heat Capacity of an Ideal Gas

(13) A cylinder contains 3 kmol of helium at a temperature of 27 °C. Heat is added to the gas to increase its temperature to 127 °C, (a) Find the quantity of heat Q_V if the gas is heated at a constant volume. (b) As the gas is heated at a constant pressure, find the amount of added heat Q_P, the work done by the gas W, and the increase in internal energy.

(14) A cylinder contains 1 kmol of monatomic ideal gas at initial temperatures of 27 °C, see the Fig. 13.4. In a fixed volume process, $Q_V = 2 \times 10^5$ J of heat is transferred to the gas from a heat reservoir. Find (a) the increase in the internal energy of the gas, and (b) its final temperature.

(15) A monatomic ideal gas has a specific heat $c_V = 95.2069$ J/kg.C°, which is nearly constant over a wide range of temperatures. What is the molar mass of the gas? Use the periodic table to find the name of this gas.

(16) A room 4 m × 4.5 m × 2.8 m contains air at 17 °C and 1 atm. Assume that the air in the room is an ideal diatomic gas with a molar specific heat at constant volume $C_V = 20.79 \times 10^3$ J/kmol.K. Also assume that there is no air or heat loss to the outside. If a heater in the room supplies 10^6 J/h, by how much will the temperature of the room rise in one hour?

(17) A two mol of an ideal diatomic gas at 20 °C is heated to 70 °C at constant pressure of 1 atm. (a) What is the change in the internal energy? (b) What is the work done by the gas? (c) Determine the quantity of heat added to the gas?

(18) A four mol of an ideal diatomic gas is at −30 °C. Assume that all the active degrees of freedom are three translational, two rotational, and two vibrational. What is the internal energy of the gas?

(19) Assume that a molecule in an ideal gas has α active degrees of freedom. Show that for n moles of this gas, the principle of equipartition theory predicts that: (a) the total internal energy is $E_{\text{int}} = \frac{1}{2}\alpha nRT$; (b) the molar specific heat at constant volume is $C_V = \frac{1}{2}\alpha R$; (c) the molar specific heat at constant pressure is $C_P = \frac{1}{2}(\alpha+2)R$; (d) the molar specific heat ratio is $\gamma = C_P/C_V = (\alpha+2)/\alpha$.

(20) The diatomic molecule of chlorine (Cl_2) has a molar mass $M = 70$ kg/kmol. The distance between two chlorine atoms is $d = 2 \times 10^{-10}$ m. These two atoms rotate about their center of mass with an angular speed $\omega = 2 \times 10^{12}$ rad/s, see Fig. 13.11. What is the rotational kinetic energy of such a molecule?

(21) A cylinder contains 2 moles of air (a diatomic ideal gas) of volume $V = 6 \times 10^{-3}$ m^3 at 27 °C. At constant pressure, the air undergoes a volume

expansion due to the addition of heat $Q_P = 5\,\text{kJ}$, as shown in Fig. 13.12. (a) Use Table 13.2 to find the values of C_P and C_V for air, if air contains 78.08% N_2, 20.95% O_2, 0.93% Ar, and 0.03% CO_2. (b) Find the change in temperature. (c) What is the change ΔE_{int} of the internal energy of the air? (d) Find the final volume of the air.

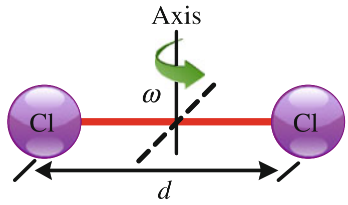


Fig. 13.11 See Exercise (20)

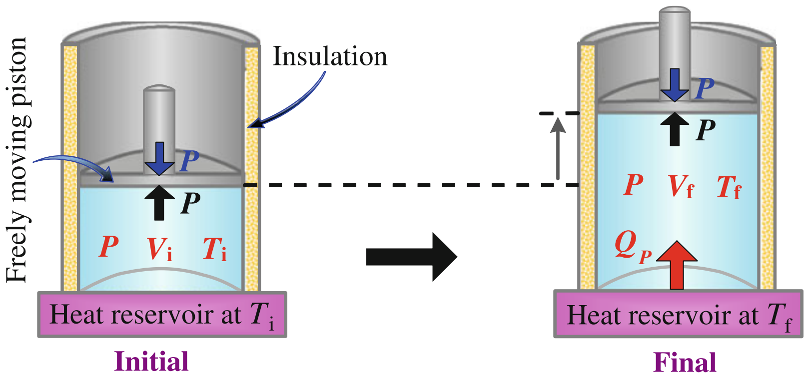


Fig. 13.12 See Exercise (21)

Section 13.3 Distribution of Molecular Speeds

(22) For the Maxwell-Boltzmann speed distribution given by Eq. 13.37 and displayed in Fig. 13.8, the most probable speed v_p of a gas molecule corresponds to a point on the curve at which the slope is zero, i.e. when the condition $df(v)/dv|_{v=v_p} = 0$ is fulfilled. Use this condition to show that v_p is given by Eq. 13.40.

(23) A container is filled with oxygen gas at $T = 300\,\text{K}$. (a) Find the rms speed v_{rms} of the oxygen molecule. (a) What is the average speed $\bar{v}$ of the oxygen molecule? (c) Find the value of the most probable speed v_p?

(24) The rms speed v_{rms} of a gas at temperature T_1 is equal to the most probable speed v_{p} at temperature T_2. Evaluate the ratio T_2/T_1.

(25) A container is composed of $N = 10^6$ oxygen molecules at $T = 300\,\mathrm{K}$. Assume that the gas obeys the Maxwell's speed distribution and the oxygen molecule has a mass $m = 5.31 \times 10^{-26}\,\mathrm{kg}$. Calculate the number of molecules with speeds between: (a) 300 m/s and 301 m/s; (b) 1,000 m/s and 1,001 m/s.

Part IV
Sound and Light Waves

Oscillations and Wave Motion

14

Any object that repeats its motion at regular time intervals is said to perform a **periodic** or **harmonic motion**. If the motion is a sinusoidal function of time, we call it **simple harmonic motion**.

14.1 Simple Harmonic Motion

Assume the motion of a particle moving back and forth about the origin o of the x-axis between the limits $x = +A$ and $x = -A$, as shown in Fig. 14.1.

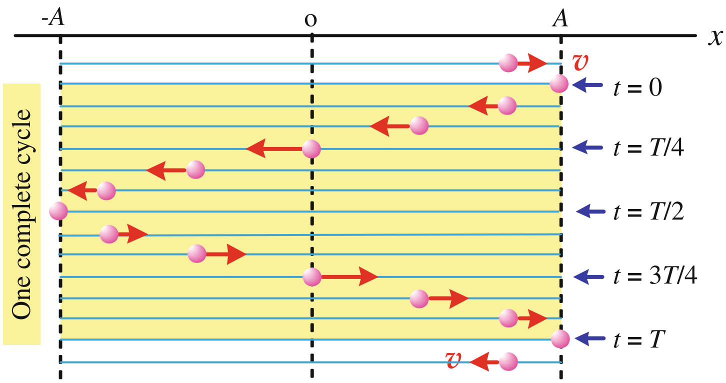


Fig. 14.1 Multiple snapshots of a particle oscillating about the origin of the x-axis between the two limits $x = +A$ and $x = -A$. If the time $t = 0$ is chosen to be when the particle is at $x = +A$, then the particle returns to $x = +A$ when $t = T$, where T is the period of the motion

In this figure, we chose $t = 0$ at the point where the particle is at $x = +A$ and $t = T$ when the particle returns to the same point $x = +A$ after one complete

H. A. Radi and J. O. Rasmussen, *Principles of Physics*,
Undergraduate Lecture Notes in Physics, DOI: 10.1007/978-3-642-23026-4_14,

cycle. In this case, T represents the **period** of the motion. The **frequency** f of the simple harmonic motion is equal to the number of oscillations per unit time. Therefore, the frequency is related to the period T by the following relation:

$$\boxed{f = \frac{1}{T}} \quad \text{(Harmonic motion)} \tag{14.1}$$

and has the SI unit s^{-1}, cycle/s, or hertz (Hz). Additionally, we define the **angular frequency** of the motion by the relation:

$$\boxed{\omega = \frac{2\pi}{T}} \quad \text{(Harmonic motion)} \tag{14.2}$$

Accordingly, this relation can be written in terms of the frequency f as follows:

$$\boxed{\omega = 2\pi f} \quad \text{(Harmonic motion)} \tag{14.3}$$

where the SI unit of ω is rad/s. For such a motion, the displacement x of the particle from o is given generally as a function of time as:

$$\boxed{x(t) = A\cos(\omega t + \phi)} \tag{14.4}$$

where A is the amplitude of the motion and ϕ is the **phase angle** or **phase constant** (ϕ is zero in Fig. 14.1). The cosine function in Eq. 14.4 varies between the limits ± 1, so the displacement $x(t)$ varies between the limits $\pm A$, as shown in Fig. 14.2.

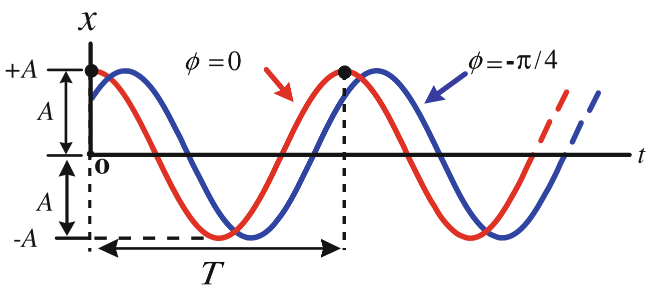


Fig. 14.2 A sketch of the relation $x(t) = A\cos(\omega t + \phi)$ for $\phi = 0$ and $\phi = -\pi/4$

14.1.1 Velocity and Acceleration of SHM

We can find an expression for the velocity v of a particle moving with a harmonic motion by differentiating Eq. 14.4 as follows:

$$v(t) = \frac{dx}{dt} = \frac{d[A\cos(\omega t + \phi)]}{dt}$$

Thus:

$$\boxed{v(t) = -\omega A \sin(\omega t + \phi) = -v_{\max} \sin(\omega t + \phi)}\,, \quad \boxed{v_{\max} = \omega A} \qquad (14.5)$$

By differentiating this expression, we generate the following expression for the acceleration of the oscillating particle:

$$a(t) = \frac{dv}{dt} = \frac{d[-\omega A \sin(\omega t + \phi)]}{dt}$$

Thus:

$$\boxed{a(t) = -\omega^2 A \cos(\omega t + \phi) = -a_{\max} \cos(\omega t + \phi)}\,, \quad \boxed{a_{\max} = \omega^2 A} \qquad (14.6)$$

Figure 14.3 shows a plot of Eqs. 14.4–14.6 for $\phi = 0$.

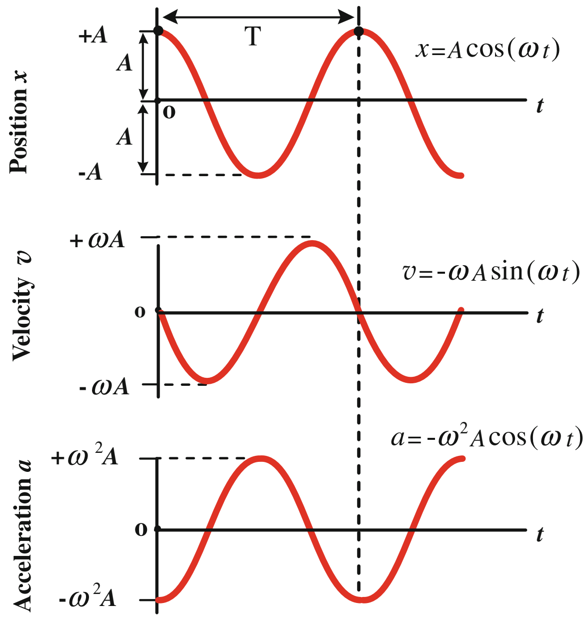


Fig. 14.3 The *upper part* of the figure shows the variation of the displacement $x(t)$ with time t of a particle oscillating with a SHM when the phase angle ϕ is equal to zero. The middle and lower parts display the variation of $v(t)$ and $a(t)$ with time. In all parts of the figure, the period T marks one complete oscillation

Example 14.1

A particle oscillates with a simple harmonic motion along the x axis. Its displacement from the origin varies with time according to the equation:

$$x = (2\text{ m})\cos(0.5\pi\, t + \pi/3)$$

where t is in seconds and the argument of the cosine is in radians. (a) Find the amplitude, frequency, and period of the motion. (b) Find the velocity and acceleration of the particle at any time. (c) Find both the maximum speed and acceleration of the particle. (d) Find the displacement of the particle between $t = 0$ and $t = 2$ s.

Solution: (a) By comparing the given equation to the general form $x(t) = A\cos(\omega t + \phi)$, we find that: the amplitude $A = 2$ m, the angular frequency $\omega = 0.5\pi$ rad/s, and the phase constant $\phi = \pi/3$ rad.

Therefore, the frequency will be:

$$f = \omega/2\pi = (0.5\pi\text{ rad/s})/(2\pi\text{ rad}) = 0.25\text{ s}^{-1} = 0.25\text{ Hz}$$

Hence the period will be given by:

$$T = 1/f = 1/0.25\text{ s}^{-1} = 4\text{ s}$$

(b) We differentiate x to find v, and then v to find a, as follows:

$$\begin{aligned}
v &= \frac{dx}{dt} = \frac{d[(2\text{ m})\cos(0.5\pi\, t + \pi/3)]}{dt} \\
&= (2\text{ m})[-\sin(0.5\pi\, t + \pi/3)] \times (0.5\pi\text{ rad/s}) \\
&= -(\pi\text{ m/s})\sin(0.5\pi\, t + \pi/3) \\
a &= \frac{dv}{dt} = \frac{d[(-\pi\text{ m/s})\sin(0.5\pi\, t + \pi/3)]}{dt} \\
&= (-\pi\text{ m/s})[\cos(0.5\pi\, t + \pi/3)] \times (0.5\pi\text{ rad/s}) \\
&= -(0.5\pi^2\text{ m/s}^2)\cos(0.5\pi\, t + \pi/3)
\end{aligned}$$

Remember, the rad is a dimensionless quantity and can be removed from or inserted into any calculation without altering the dimension of the result.

(c) Since the maximum values of the sine and cosine are unity, then v of part (b) varies between $\pm\pi$ m/s, and a of part (b) varies between $\pm 0.5\pi^2$ m/s^2. Thus, the maximum speed and maximum acceleration are as follows:

$$v_{max} = \pi \text{ m/s}$$

$$a_{max} = 0.5\pi^2 \text{ m/s}^2$$

We can also use Eqs. 14.5 and 14.6 to find v_{max} and a_{max} as follows:

$$v_{max} = \omega A = (0.5\pi \text{ rad/s})(2\text{ m}) = \pi \text{ m/s}$$

$$a_{max} = \omega^2 A = (0.5\pi \text{ rad/s})^2(2\text{ m}) = 0.5\pi^2 \text{ m/s}^2$$

(d) The position of the particle at $t = 0$ is denoted by x_i and is given by:

$$x_i = (2 \text{ m}) \cos(0 + \pi/3) = (2 \text{ m})(0.5) = 1 \text{ m}$$

The position of the particle at $t = 2$ s is denoted by x_f and is given by:

$$x_f = (2 \text{ m}) \cos(\pi + \pi/3) = (2 \text{ m})(-0.5) = -1 \text{ m}$$

Therefore, the displacement between $t = 0$ and $t = 2$ s is:

$$\Delta x = x_f - x_i = -1 \text{ m} - 1 \text{ m} = -2 \text{ m}$$

Figure 14.4a shows the plot of x versus t for the given function, while Fig. 14.4b depicts snapshots of the oscillating particle about the origin of the x-axis between the two limits $x = +2$ m and $x = -2$ m.

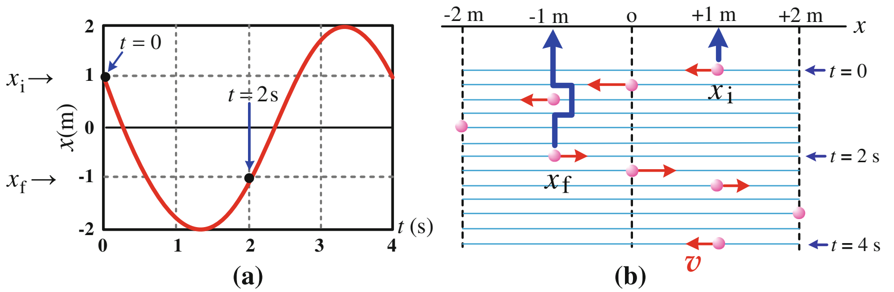


Fig. 14.4

14.1.2 The Force Law for SHM

We can combine Eqs. 14.4 and 14.6 to yield:

$$\boxed{a(t) = -\omega^2 x(t)} \tag{14.7}$$

This equation is the hallmark of simple harmonic motion. It states that the acceleration is proportional to the displacement but opposite in sign, and they are related by the square of the angular frequency, ω^2.

Once we know the acceleration as a function of time, we can use Newton's second law to find what force must act on the particle to produce this acceleration. Now, we combine Newton's second law with Eq. 14.7 as follows:

$$F = m\,a = -(m\,\omega^2)x \quad \text{or} \quad \boxed{\frac{d^2x}{dt^2} + \omega^2\,x = 0} \tag{14.8}$$

This form (a force proportional to the displacement but opposite in sign) is familiar to us. It is Hooke's law for a spring, which was introduced in Sect. 6.3. That is:

$$\boxed{F = -k_{\mathrm{H}}\,x} \qquad \text{(Hooke's law)} \tag{14.9}$$

By comparison, the equivalent spring constant in SHM is:

$$k_{\mathrm{H}} = m\,\omega^2 \tag{14.10}$$

We can take Eq. 14.9 as an alternative definition of SHM.

> Simple Harmonic Motion
> SHM is the motion executed by a particle of mass m subject to a force proportional to its displacement but opposite in sign.

The block-spring system of Fig. 14.5 forms a linear simple harmonic oscillator. The angular frequency ω of the simple harmonic motion is related to the spring constant k_{H} and the mass of the block by Eq. 14.10, which gives:

$$\boxed{\omega = \sqrt{\frac{k_{\mathrm{H}}}{m}}} \tag{14.11}$$

Therefore, using Eqs. 14.5, 14.6, and 14.11, we can find the maximum values for the velocity and acceleration of the oscillations as follows:

$$v_{\max} = \omega\,A = \sqrt{\frac{k_{\mathrm{H}}}{m}}\;A \tag{14.12}$$

$$a_{\max} = \omega^2 A = \frac{k_{\mathrm{H}}}{m}A \tag{14.13}$$

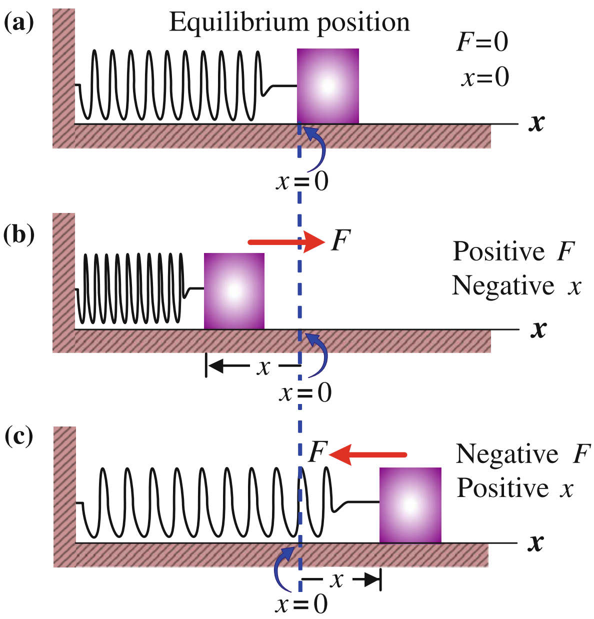


Fig. 14.5 The variation of the force of a spring on a block. (a) When $x = 0$, the force is zero (*equilibrium position*). (b) When x is negative, the force is positive (*compressed spring*). (c) When x is positive, the force is negative (*stretched spring*)

By combining Eqs. 14.2 and 14.11, we can find the period T of the oscillations as follows:

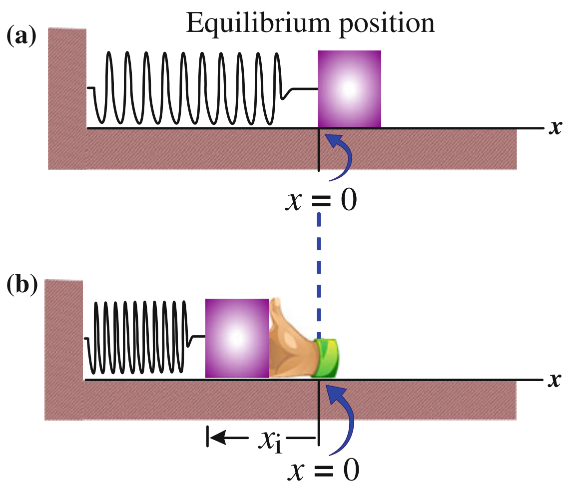
$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k_H}} \tag{14.14}$$

That is, the period T and hence the frequency $f = 1/T$ depend only on the mass of the particle m and the spring constant k_H, and not on the parameters of the motion, such as A or ϕ.

Example 14.2

A block of mass $m = 400\,\text{g}$ is attached to a light spring of force constant $k_H = 10\,\text{N/m}$, see Fig. 14.6a. The block is pushed against the spring from $x = 0$ to $x_i = -10\,\text{cm}$, see Fig. 14.6b, and then released to oscillate on a horizontal frictionless surface. (a) Find the angular frequency and the period of the block-spring system. (b) Find the maximum speed and maximum acceleration of the block. (c) Find the position, speed, and acceleration of the block at any time. (d) Repeat the above parts when the block is projected with initial velocity $v_i = -0.5$ m/s from another initial position $x_i = +10$ cm.

Fig. 14.6



Solution: (a) Using Eqs. 14.11 and 14.14 we find the angular frequency and the period of motion as follows:

$$\omega = \sqrt{\frac{k_H}{m}} = \sqrt{\frac{10 \text{ N/m}}{400 \times 10^{-3} \text{ kg}}} = 5 \text{ rad/s}$$

$$T = \frac{2\pi}{\omega} = \frac{2 \times 3.1416 \text{ rad}}{5 \text{ rad/s}} = 1.26 \text{ s}$$

(b) Since $A = |x_i| = 10$ cm, then Eqs. 14.12 and 14.13 will give:

$$v_{max} = \omega A = (5 \text{ rad/s})(0.1 \text{ m}) = 0.5 \text{ m/s}$$

$$a_{max} = \omega^2 A = (5 \text{ rad/s})^2 (0.1 \text{ m}) = 2.5 \text{ m/s}^2$$

(c) We can find the phase constant ϕ when we apply the initial condition $x(0) = -A$ at $t = 0$ to the form $x(t) = A\cos(\omega t + \phi)$. Thus:

$$x(t) = A\cos(\omega t + \phi) \quad \Rightarrow \quad x(0) = A\cos(\phi) \quad \Rightarrow \quad -A = A\cos(\phi) \quad \Rightarrow \quad \phi = \pi$$

Therefore, $x(t) = A\cos(\omega t + \pi)$. Using this expression and the results of parts (a) and (b), we get:

$$x(t) = A\cos(\omega t + \pi) = (0.1 \text{ m})\cos(5t + \pi)$$

$$v(t) = -\omega A\sin(\omega t + \pi) = -(0.5 \text{ m/s})\sin(5t + \pi)$$

$$a(t) = -\omega^2 A\cos(\omega t + \pi) = -(2.5 \text{ m/s}^2)\cos(5t + \pi)$$

(d) In this case, we start with the general form $x(t) = A\cos(\omega t + \phi)$, where A and ϕ are not known, but ω does not change because it is independent of how the oscillation is set into motion. Thus:

$$(1)\quad x(0) = A\cos(0+\phi) = x_i$$
$$(2)\quad v(0) = -\omega A\sin(0+\phi) = v_i$$

Dividing Eq. (2) by Eq. (1) gives a phase-constant relation:

$$\frac{-\omega A\sin(\phi)}{A\cos(\phi)} = \frac{v_i}{x_i} \quad\Rightarrow\quad \tan(\phi) = -\frac{v_i}{\omega x_i} = -\frac{-0.5\text{ m/s}}{(5\text{ rad/s})(0.1\text{ m})} = 1$$

Thus: $$\phi = \tan^{-1}(1) = 0.785\text{ rad} = 0.25\,\pi\text{ rad}$$

Now, Eq. (1) allows us to find the new amplitude A as follows:

$$A = \frac{x_i}{\cos(\phi)} = \frac{(0.1\text{ m})}{\cos(0.25\pi)} = 0.14\text{ m} = 14\text{ cm}$$

The new maximum speed and acceleration will be:

$$v_{max} = \omega A = (5\text{ rad/s})(0.14\text{ m}) = 0.7\text{ m/s}$$
$$a_{max} = \omega^2 A = (5\text{ rad/s})^2(0.14\text{ m}) = 3.5\text{ m/s}^2$$

Finally, the new expressions for x, v, and a are as follows:

$$x(t) = A\cos(\omega t+\phi) = (0.14\text{ m})\cos(5\,t + 0.25\pi)$$
$$v(t) = -\omega A\sin(\omega t+\phi) = -(0.7\text{ m/s})\sin(5\,t + 0.25\pi)$$
$$a(t) = -\omega^2 A\cos(\omega t+\phi) = -(3.5\text{ m/s}^2)\cos(5\,t + 0.25\pi)$$

14.1.3 Energy of the Simple Harmonic Oscillator

Consider the block-spring system shown in Fig. 14.6 when the spring is massless and the horizontal surface is frictionless (known as the linear oscillator). In such a situation, the kinetic energy of the system is associated entirely with the block. Its value depends only on the velocity v given by Eq. 14.5. Thus:

$$K = \tfrac{1}{2}mv^2 = \tfrac{1}{2}m\omega^2A^2\sin^2(\omega t+\phi) \tag{14.15}$$

When using Eq. 14.11 to substitute k_H/m for ω^2, we find:

$$\boxed{K = \tfrac{1}{2}mv^2 = \tfrac{1}{2}k_H A^2 \sin^2(\omega t + \phi)} \tag{14.16}$$

On the other hand, the potential energy of the linear oscillator of Fig. 14.6 is associated entirely with the spring. Its value depends only on the position x given by Eq. 14.4. Thus:

$$\boxed{U = \tfrac{1}{2}k_H x^2 = \tfrac{1}{2}k_H A^2 \cos^2(\omega t + \phi)} \tag{14.17}$$

The **mechanical energy** E of the simple harmonic oscillator is thus:

$$\begin{aligned} E = K + U &= \tfrac{1}{2}k_H A^2 \sin^2(\omega t + \phi) + \tfrac{1}{2}k_H A^2 \cos^2(\omega t + \phi) \\ &= \tfrac{1}{2}k_H A^2[\sin^2(\omega t + \phi) + \cos^2(\omega t + \phi)] \end{aligned} \tag{14.18}$$

When we use the identity $\sin^2\theta + \cos^2\theta = 1$, we find:

$$\boxed{E = K + U = \tfrac{1}{2}k_H A^2} \tag{14.19}$$

That is, the mechanical energy (or the **total energy**) of a simple harmonic oscillator is constant, independent of time, and is proportional to the square of the amplitude. Because $v = 0$ at $x = \pm A$, i.e. $K = 0$, the mechanical energy equals the maximum potential energy, i.e. $E = U_{max} = \frac{1}{2}k_H A^2$. At the equilibrium position $x = 0$ we have $U = 0$, and the mechanical energy equals the maximum kinetic energy, i.e. $E = K_{max} = \frac{1}{2}mv_{max}^2 = \frac{1}{2}k_H A^2$.

Since the potential energy U is expressed as a function of the position x through the relation $U = \frac{1}{2}k_H x^2$, then Eq. 14.19 allows us of to express the kinetic energy as a function of x as follows:

$$K = E - U = \tfrac{1}{2}k_H A^2 - \tfrac{1}{2}k_H x^2 = \tfrac{1}{2}k_H(A^2 - x^2) \tag{14.20}$$

Figure 14.7a displays both the kinetic energy K and potential energy U as a function of time t, while Fig. 14.7b displays the variation of K and U as a function of position x.

Finally, by using Eq. 14.20 we can find the velocity of the block at any arbitrary position x as follows:

$$K = \tfrac{1}{2}mv^2 = \tfrac{1}{2}k_H(A^2 - x^2)$$

$$\boxed{v = \pm\sqrt{\frac{k_H}{m}(A^2 - x^2)}} \tag{14.21}$$

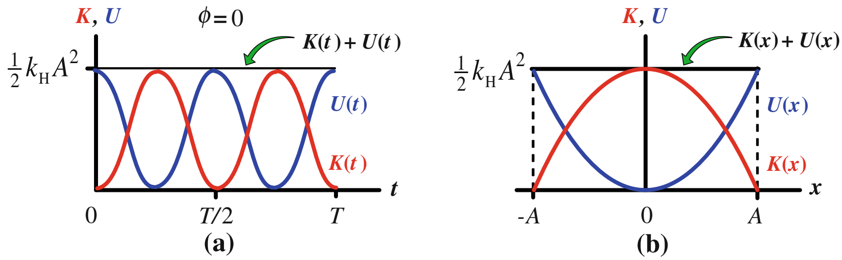


Fig. 14.7 (a) The kinetic energy $K(t)$ and the potential energy $U(t)$ as a function of time when $\phi = 0$ for a simple harmonic oscillator. Note that $K(t)$ and $U(t)$ peak twice during every period. (b) The kinetic energy $K(x)$ and the potential energy $U(x)$ as a function of x. For $x = 0$ the energy is entirely kinetic, and for $x = \pm A$ it is entirely potential

When using Eq. 14.11 to substitute k_H/m for ω^2, we find:

$$v = \pm\omega\sqrt{(A^2 - x^2)} \tag{14.22}$$

This relation verifies the fact that the speed is a maximum when $x = 0$ and is zero at both of the turning points $x = \pm A$.

Example 14.3

A block of mass $m = 320$ g is fastened to a light spring whose force constant k_H is 72 N/m, see Fig. 14.8a. The block is pulled a distance $x_i = 50$ cm from its equilibrium position at $x = 0$ on a horizontal frictionless surface, see Fig. 14.8b, and released at $t = 0$. (a) What is the mechanical energy of the oscillating block? (b) What is the maximum speed of the oscillating block? (c) Find the velocity, kinetic energy, and potential energy of the block when its position is 30 cm?

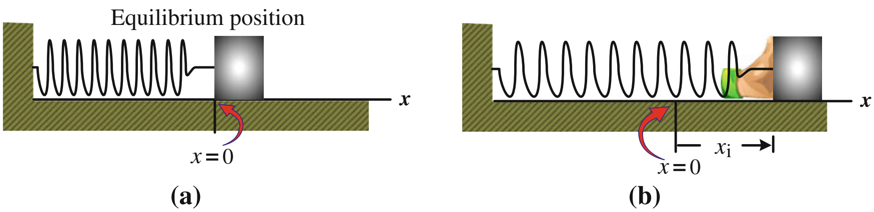


Fig. 14.8

Solution: (a) Since $A = x_i = 50\,\text{cm} = 0.5\,\text{m}$, then Eq. 14.19 gives:

$$E = \tfrac{1}{2}k_H A^2 = \tfrac{1}{2}(72\text{ N/m})(0.5\text{ m})^2 = 9\text{ J}$$

(b) At $x = 0$, we know that $U = 0$ and $E = \frac{1}{2}mv_{max}^2$; therefore:

$$\tfrac{1}{2}mv_{max}^2 = E = 9\text{ J}$$
$$v_{max} = \sqrt{\frac{2E}{m}} = \sqrt{\frac{2(9\text{ J})}{0.32\text{ kg}}} = 7.5\text{ m/s}$$

(c) We use Eq. 14.21 to find the velocity at $x = 30\,\text{cm}$ as follows:

$$v = \pm\sqrt{\frac{k_H}{m}(A^2 - x^2)} = \pm\sqrt{\frac{72\text{ N/m}}{0.32\text{ kg}}[(0.5\text{ m})^2 - (0.3\text{ m})^2]} = \pm 6\text{ m/s}$$

Now, we can find K and U when $x = 30\,\text{cm} = 0.3\,\text{m}$ as follows:

$$K = \tfrac{1}{2}mv^2 = \tfrac{1}{2}(0.32\text{ kg})(\pm 6\text{ m/s})^2 = 5.76\text{ J}$$
$$U = \tfrac{1}{2}k_H x^2 = \tfrac{1}{2}(72\text{ N/m})(0.3\text{ m})^2 = 3.24\text{ J}$$

14.2 * Damped Simple Harmonic Motion

When non-conservative forces, such as friction, oppose the motion of an oscillator, its mechanical energy diminishes with time, and the motion is said to be **damped**. One such system is a block of mass m attached to a spring and immersed in a viscous liquid, see Fig. 14.9a. Let us assume that the liquid exerts a damping force F_d that is proportional to the velocity v_x of the oscillator. If v_x is small, then:

$$\boxed{F_d = -bv_x} \tag{14.23}$$

where b is a damping constant. The total force acting on the block is:

$$\boxed{\Sigma F_x = -k_H\,x - bv_x} \tag{14.24}$$

If we set $v_x = dx/dt$ and substitute with ΣF_x in Newton's second law, $\Sigma F_x = m\,d^2x/dt^2$, we find the following differential equation:

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + k_H\,x = 0 \tag{14.25}$$

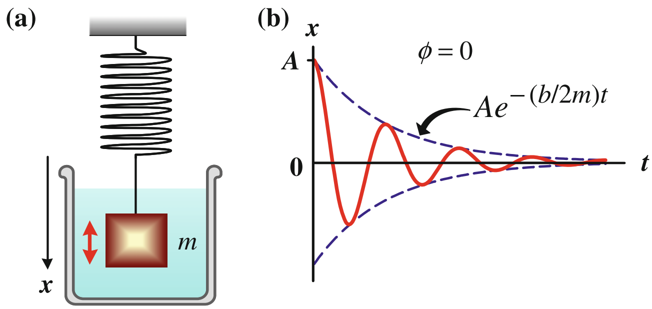


Fig. 14.9 (a) A damped oscillator consisting of a block immersed in a viscous liquid. (b) Graph of x versus t for a damped oscillator

This equation has a solution displayed in Fig. 14.9b, and is given by:

$$\boxed{x(t) = Ae^{-bt/2m}\cos(\omega_d\, t + \phi)} \tag{14.26}$$

where the angular frequency of the damped oscillator ω_d is given by:

$$\boxed{\omega_d = \sqrt{\frac{k_{\mathrm{H}}}{m} - \frac{b^2}{4m^2}}} \quad \left(\omega_d \xrightarrow[\text{or } b\to 0]{b\ll 2\sqrt{k_{\mathrm{H}}m}} \sqrt{\frac{k_{\mathrm{H}}}{m}} = \omega\right) \tag{14.27}$$

14.3 Sinusoidal Waves

Waves are of three types: mechanical, electromagnetic, and matter waves. This chapter deals only with mechanical waves. We encounter mechanical waves almost constantly every day in our lives. For such waves, information and energy move from one point to another, but matter does not. Common examples of such waves are sound, water, and seismic waves. These waves require the following:

(1) Some source of disturbance (or vibration),
(2) A medium that can be disturbed,
(3) Some physical mechanism responsible for allowing adjacent portions of the medium to influence each other.

14.3.1 Transverse and Longitudinal Waves

Figure 14.10 displays a single pulse wave sent from one end of a long stretched string toward the other fixed end. As the wave passes the point P on the string, the y

coordinate of this point will increase, reach a maximum, and then decrease to zero. In the case of an *ideal string* (that is when no frictional forces within the string cause the pulse to die out as it travels), the wave pulse in the string moves along the string with some constant velocity v.

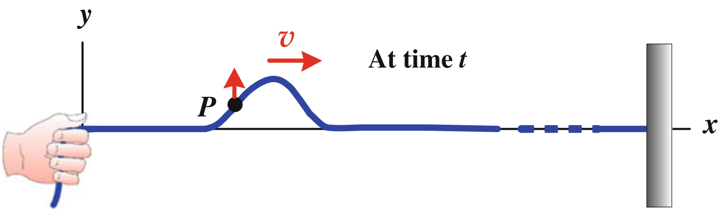


Fig. 14.10 Sending a single pulse through a long stretched string

Figure 14.11 displays a continuous sinusoidal wave sent from one end of a long stretched string toward the other fixed end. The wave has a sinusoidal shape at any time. That is, the wave has the shape of a sine curve (or a cosine curve) at any location x and time t. As the sinusoidal wave travels along the string with some constant velocity $\vec{v}$, the y coordinate of point P on the string will oscillate up and down continuously.

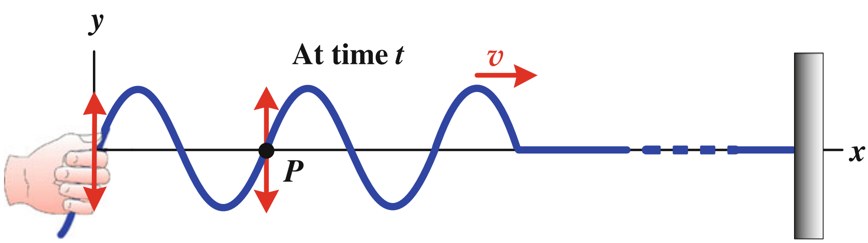


Fig. 14.11 Sending a continuous sinusoidal wave through a long stretched string. Any string element (represented by point P) oscillates perpendicular to the direction of the wave's velocity

From Figs. 1.10 and 1.11 we find that the displacement of every oscillating element on the string is perpendicular to the direction of the wave's velocity. This motion is called a **transverse** motion, and the generated wave is called a **transverse wave**.

Figure 14.12 shows how we can produce a sound wave by using a movable piston fitted in a long open pipe filled with air. A sound wave can be generated by pushing the piston toward the right or toward the left. The rightward motion of the piston compresses the air in the region next to it, i.e. increasing its pressure. Accordingly, the compressed air (or the change in pressure) travels as a pulse from one region to another toward the right along the pipe. If the push and pull of the piston is sinusoidal as in Fig. 14.12, a sinusoidal wave will travel along the pipe.

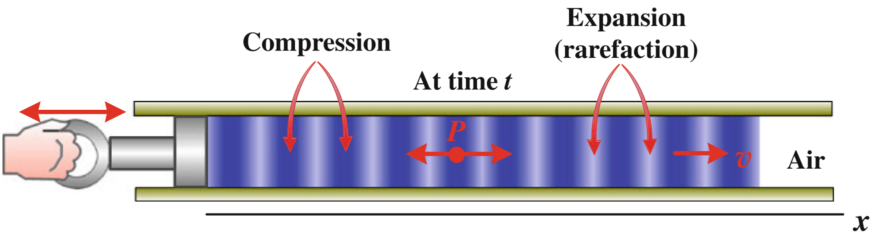


Fig. 14.12 Sending a continuous sinusoidal wave through an air-filled pipe by moving a piston back and forth in a sinusoidal manner. Any air element (represented by point P) oscillates back and forth parallel to the direction of the wave's velocity v

From Fig. 14.12, we find that the displacement of every air element in the pipe is parallel to the direction of the wave's velocity. This motion is **longitudinal**, and the generated wave is called a **longitudinal wave**.

Both transverse and longitudinal waves are **traveling waves**, because they travel from one point to another.

14.3.2 Wavelength and Frequency

Two physical characteristics are important in describing periodic waves: the **wavelength** (denoted by λ) and the **frequency** (denoted by f). Both are defined below:

Wavelength λ:
One wavelength λ is the *minimum distance between any two points on a wave where both points behave identically*.

Frequency f:
The frequency f of a wave is the *rate at which the disturbance repeats itself.*

A third important physical characteristic of waves is the **wave velocity** (denoted by v). In fact, mechanical waves travel, or propagate, with a specific velocity that is determined by the properties of the medium being disturbed.

14.3.3 Harmonic Waves: Simple Harmonic Motion

A harmonic wave that is traveling toward increasing x has a sinusoidal shape like the transverse wave of the string in Fig. 14.11. The displacement $y = y(x, t)$ of a harmonic wave can be written in terms of a sine (or a cosine) function of the position x at time t as follows:

$$\boxed{y = A\sin(kx - \omega t)} \quad \text{or} \quad \boxed{y = A\cos(kx - \omega t)}\,, \tag{14.28}$$

where A is the magnitude of the maximum displacement, known as the **amplitude** of the wave. The quantities k and ω are constants whose meanings will be discussed shortly. The quantity $kx - \omega t$ is called the **phase** of the wave. From now on, we will use only the sine form.

Figure 14.13 shows the transverse displacement y as a function of the position x at $t = 0$, i.e. the figure is a snapshot of the wave at that instant. With $t = 0$, the sine form of Eq. 14.28 becomes:

$$\boxed{y = A\sin kx} \quad (t = 0) \tag{14.29}$$

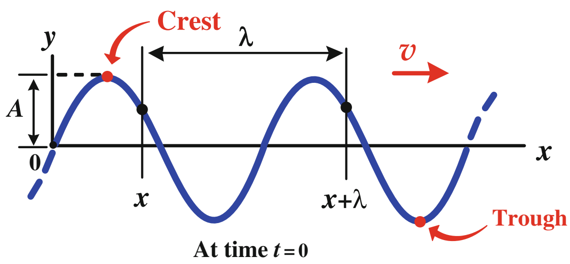


Fig. 14.13 A snapshot at $t = 0$ of a harmonic wave traveling to the right with a speed v in a taut string. A typical wavelength λ is shown, which is the minimum distance between any two points on the wave where both points behave identically

The **wavelength** λ of a wave is the distance between two successive crests, or identically behaving two points on the x axis having the same displacement y and slope dy/dx. If this occurs in Fig. 14.13 at x and $x + \lambda$, then Eq. 14.29 gives:

$$y = A \sin kx = A \sin k(x + \lambda) \qquad (14.30)$$

The sine function repeats itself when its angle is increased by 2π rad. Thus, Eq. 14.30 is satisfied only if $k\lambda = 2\pi$, i.e.:

$$\boxed{k = \frac{2\pi}{\lambda}} \quad \text{(Harmonic wave)} \qquad (14.31)$$

where k is called the **Angular wave number** (or simply the **wave number**) of the wave and has the SI unit rad/m (Not to be confused with the ***spring constant*** k_H in Hooke's law).

Figure 14.14 shows the transverse displacement y as a function of time t at an arbitrary location taken to be at $x = 0$. Thus, when monitoring the string, one would see point $x = 0$ moving up and down in a motion given by Eq. 14.28 with $x = 0$. That point is said to perform **simple harmonic motion**. So, Eq. 14.28 becomes:

$$\boxed{y = A \sin(-\omega t) = -A \sin(\omega t)} \quad (x = 0) \qquad (14.32)$$

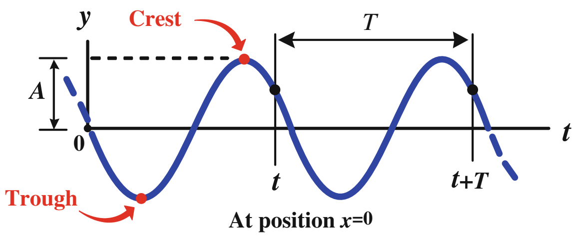


Fig. 14.14 A graph showing the displacement of the string at $x = 0$ as a function of t when the sinusoidal harmonic wave passes through that point. A typical period T is shown, which is the minimum time between any two points on the wave when both behave identically

The **period** T of the wave is the time between two successive points behaving identically on the t axis having the same displacement y. If this occurs in Fig. 14.14 at t and $t + T$, then Eq. 14.32 gives:

$$y = -A \sin \omega t = -A \sin \omega (t + T) \qquad (14.33)$$

Again, the sine function repeats itself when its angle is increased by 2π rad. Thus, Eq. 14.33 is satisfied only if $\omega T = 2\pi$, i.e.:

$$\boxed{\omega = \frac{2\pi}{T}} \quad \text{(Harmonic wave)} \qquad (14.34)$$

where ω is defined previously in Eq. 14.2 and called the **angular frequency** of the wave which has the SI unit rad/s.

The **frequency** f of a harmonic wave is equal to the number of crests (or troughs), or any point on the wave, that passes any point in a unit time interval. The following relation relates the frequency f to the period T and the angular frequency ω:

$$\boxed{f = \frac{1}{T} = \frac{\omega}{2\pi}} \quad \text{(Harmonic wave)} \tag{14.35}$$

As defined previously, it has the SI unit s^{-1}, or cycle/s, or hertz (Hz).

Figure 14.15 shows a snapshot of the wave at $t = 0$ and $t = \Delta t$. The ratio $\Delta x/\Delta t$ (or, in the differential limit dx/dt) is the **wave speed** v, i.e. $v = dx/dt$. As the wave moves, each point (such as point P) retains its displacement y. For each such point, although x and t are changing, Eq. 14.28 tells us that the argument of the sine function must be constant. That is:

$$k\,x - \omega\,t = \text{constant} \tag{14.36}$$

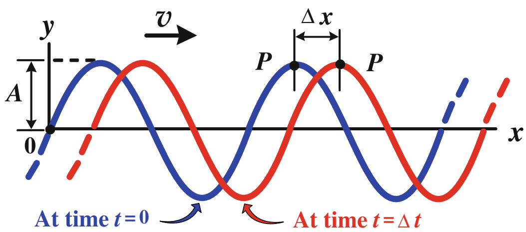


Fig. 14.15 Snapshots of a traveling wave at $t = 0$ and $t = \Delta t$. During this time the entire curve shifts a distance Δx with a speed $v = \Delta x/\Delta t$.

To find the speed of the wave v, we differentiate Eq. 14.36 with respect to t to get:

$$k\frac{dx}{dt} - \omega = 0$$

Thus:

$$\boxed{v = \frac{dx}{dt} = \frac{\omega}{k}} \tag{14.37}$$

Using $k = 2\pi/\lambda$ and $\omega = 2\pi/T$, we can rewrite the speed as:

$$\boxed{v = \frac{\omega}{k} = \frac{\lambda}{T}} \quad \text{or} \quad \boxed{v = \lambda\, f} \tag{14.38}$$

Therefore, we can rewrite Eq. 14.28 in different forms such as:

$$\boxed{y = A \sin\left[2\pi\left(\frac{x}{\lambda} - \frac{t}{T}\right)\right]} \quad \text{or} \quad \boxed{y = A \sin\left[\frac{2\pi}{\lambda}(x - vt)\right]} \tag{14.39}$$

The harmonic wave given by Eq. 14.28 assumes that the displacement y is zero at $x = 0$ and $t = 0$. If the transverse wave is not zero, we generally express the harmonic wave in the form:

$$\boxed{y = \begin{cases} A \sin(kx - \omega t + \phi) \\ \text{or} \\ A \cos(kx - \omega t + \phi) \end{cases}}, \quad \boxed{(k = 2\pi/\lambda,\ \omega = 2\pi f)} \tag{14.40}$$

where ϕ is a constant, called the **phase constant**, that can be determined from the wave's initial conditions.

Example 14.4

A harmonic wave traveling along a string in the direction of increasing x has the following form $y = 0.4 \sin(0.2\,x - 5\,t)$, where all the numerical constants are in SI units. (a) Find the amplitude, wave number, angular frequency, and speed of the wave. (b) Find the wavelength, period, and frequency of the wave.

Solution: (a) Comparing this wave with $y = A \sin(kx - \omega t + \phi)$, we find the amplitude, wave number, angular frequency, and phase to be:

$$A = 0.4\,\text{m}, \quad k = 0.2\,\text{rad/m}, \quad \omega = 5\,\text{rad/s}, \quad \text{and} \quad \phi = 0$$

From Eq. 14.37 we find the speed of the wave to be:

$$v = \frac{\omega}{k} = \frac{5\ \text{rad/s}}{0.2\ \text{rad/m}} = 25\ \text{m/s}$$

(b) Equation 14.31 gives the wavelength of the wave as follows:

$$\lambda = \frac{2\pi}{k} = \frac{2\pi\ \text{rad}}{0.2\ \text{rad/m}} = 31.4\ \text{m}$$

From Eq. 14.34 we can find the period of the wave as follows:

$$T = \frac{2\pi}{\omega} = \frac{2\pi\ \text{rad}}{5\ \text{rad/s}} = 1.26\ \text{s}$$

Equation 14.35 gives the frequency of the wave as follows:

$$f = \frac{1}{T} = \frac{1}{1.26\ \text{s}} = 0.8\ \text{s}^{-1} = 0.8\ \text{cycle/s} = 0.8\ \text{Hz}$$

Note that the quantities calculated are independent of the amplitude A.

14.4 The Speed of Waves on Strings

String waves are the most common examples of transverse waves. Let us consider a single symmetrical pulse traveling with a speed v in a stretched string that is under a tensional force of magnitude τ (we use the symbol τ to represent tension, which avoids confusion with the symbol T used to represent the period of oscillation), see Fig. 14.16. We assume that the string has a linear mass density $\mu = m/L$, where m is the mass of the string and L is its length.

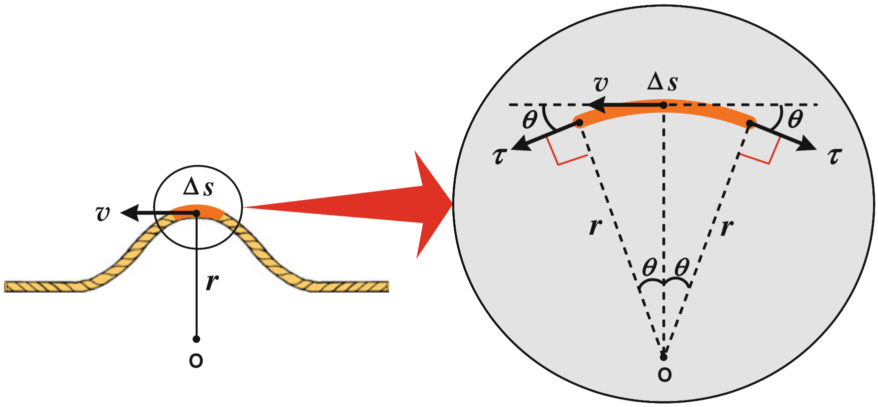


Fig. 14.16 A symmetrical pulse moving to the left on a stretched string with speed v. To find this speed we apply Newton's second law on a small segment of length Δs located at the top of the pulse.

*Consider a small segment at the top of the pulse, of length Δs, forming an arc of a circle of radius r, see Fig. 14.16. A force equal in magnitude to the string tension τ pulls tangentially on this segment at each end. The horizontal components of these forces cancel, but the vertical components add to form a radial restoring force of magnitude:

$$F_r = 2\tau \sin\theta \approx 2\tau\theta = \tau 2\theta = \tau \frac{\Delta s}{r} \tag{14.41}$$

where we have used the approximation $\sin\theta \approx \theta$ when Δs is very small and also used the relation $\Delta s = r \times (2\theta)$.

The mass Δm of the segment Δs is given by:

$$\Delta m = \mu \Delta s \tag{14.42}$$

According to Fig. 14.16, the string segment Δs is moving radially toward the center of a circle of radius r with a centripetal acceleration of magnitude given by:

$$a_r = \frac{v^2}{r} \tag{14.43}$$

When we apply Newton's second law *force* = *mass* × *acceleration*, i.e. $F_r = \Delta m\, a_r$, and also apply Eqs. 14.41–14.43, we get the following relation:

$$\tau \frac{\Delta s}{r} = \mu \Delta s \times \frac{v^2}{r}$$

Solving this equation for the speed v yields:

$$\boxed{v = \sqrt{\frac{\tau}{\mu}}} \tag{14.44}$$

This equation tells us that the speed of a wave along an ideal stretched string depends only on the characteristics of the string (the magnitude of the tension τ and the mass per unit length μ) and not on the frequency f of the wave. Actually, the frequency f is fixed by whatever generates the wave, while the wavelength is fixed by Eq. 14.38, i.e. by the relation $\lambda = v/f$.

Example 14.5

A uniform string has a linear mass density of 0.2 kg/m. The string passes over a massless frictionless pulley to a block of mass $m = 4$ kg, see Fig. 14.17. Find the speed of a single pulse sent from one end of the string toward the pulley.

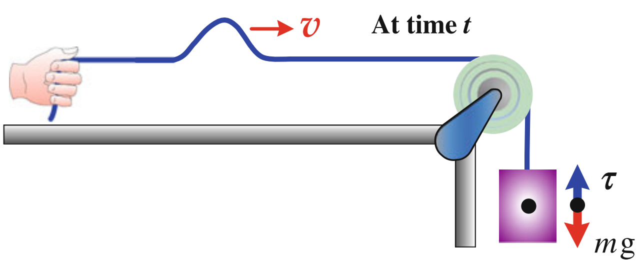


Fig. 14.17

Solution: The magnitude of the tension τ in the string is equal to the magnitude of the weight of the suspended block. Thus:

$$\tau = mg = (4 \text{ kg}) \times (9.8 \text{ m/s}^2)$$
$$= 39.2 \text{ N}$$

Using this result and the linear density $\mu = 0.2$ kg/m in Eq. 14.44, we find the value of the speed of the wave to be:

$$v = \sqrt{\frac{\tau}{\mu}}$$
$$= \sqrt{\frac{39.2 \text{ N}}{0.2 \text{ kg/m}}} = 14 \text{ m/s}$$

14.5 Energy Transfer by Sinusoidal Waves on Strings

Waves transport kinetic and potential energy when they propagate through a medium.

This can be easily demonstrated by hanging an object on a stretched string and then sending a pulse through it, see Fig. 14.18. As the pulse meets the object, the object will move up and hence acquire kinetic and potential energy.

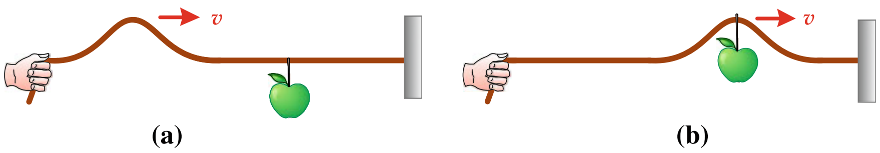


Fig. 14.18 (a) A pulse traveling on a stretched string over which an object is hung. (b) Kinetic energy and potential energy are transferred to the object when the pulse arrives

Consider a string of mass per unit length μ and tension of magnitude τ that is connected to a source of vibration as shown in Fig. 14.19a. When the source vibrates, it does work to produce a sinusoidal wave that travels to the right as shown in Fig. 14.19b. Now, let us focus our attention on an element of the string of mass Δm and length Δx located at a particular point x. This element will move up and down in a simple harmonic motion, see Fig. 14.19b.

Assume the oscillation of this element in the y direction has an amplitude A, wave number k, and angular frequency ω. Then, according to Eq. 14.28, the transverse velocity v_y (not to be confused with the wave velocity v) at a particular position x will be:

$$
\begin{aligned}
v_y = \left.\frac{dy}{dt}\right|_{x=\text{constant}} &= \frac{\partial y}{\partial t} = \frac{\partial\,[A\sin(kx-\omega t)]}{\partial t} \\
&= -\omega A\cos(kx-\omega t)
\end{aligned}
\tag{14.45}
$$

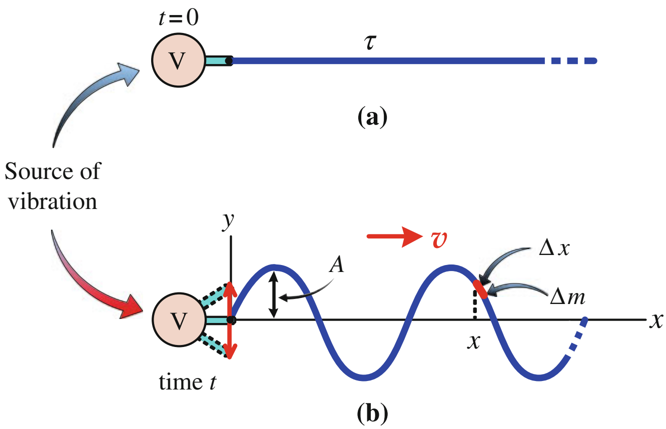


Fig. 14.19 (a) A source of vibration connected to a stretched string under tension τ. (b) A snapshot of a traveling harmonic wave on the string at a time chosen to be at time t

The kinetic energy ΔK associated with a string element of mass $\Delta m = \mu\Delta x$ will be given by:

$$
\Delta K = \tfrac{1}{2}\Delta m\, v_y^2 = \tfrac{1}{2}\mu\Delta x\, v_y^2 \tag{14.46}
$$

When allowing Δx to approach zero, this relation becomes a differential relationship and will take the following form:

$$
dK = \tfrac{1}{2}\mu\, dx\, v_y^2 = \tfrac{1}{2}\mu\omega^2A^2\cos^2(kx-\omega t)\,dx \tag{14.47}
$$

At a given instant, let us integrate this expression over all the string elements of a complete wavelength, which will give us the total kinetic energy K_λ in one wavelength:

$$
K_\lambda = \int dK = \tfrac{1}{2}\mu\omega^2A^2\int_0^\lambda \cos^2(kx-\omega t)\,dx \tag{14.48}
$$

*If we take a snapshot at time $t=0$, then we can evaluate the above integral by performing the following steps:

$$\begin{aligned}\int_{x=0}^{x=\lambda} \cos^2(k\,x)\;dx &= \tfrac{1}{k}\int_{z=0}^{z=k\lambda=2\pi} \cos^2 z\; dz\\ &= \tfrac{1}{k}\int_0^{2\pi} \tfrac{1}{2}[1+\cos 2z]\;dz\\ &= \tfrac{1}{2k}\left[z+\tfrac{1}{2}\sin 2z\right]_0^{2\pi}\\ &= \tfrac{1}{2k}\left[(2\pi+\tfrac{1}{2}\sin 4\pi)-0\right] = \frac{\lambda}{4\pi}2\pi\\ &= \frac{\lambda}{2}\end{aligned} \tag{14.49}$$

where we have used $z = kx$, $\cos^2 z = (1+\cos 2z)/2$ and $k = 2\pi/\lambda$ to arrive to the above result. Of course, we get the same answer if we perform the above steps at any other time different from zero. When we substitute the above result into Eq. 14.48, we get:

$$K_\lambda = \tfrac{1}{4}\mu\omega^2 A^2\,\lambda \tag{14.50}$$

A similar analysis to the total potential energy U_λ in one wavelength will give exactly the same result. Thus:

$$U_\lambda = \tfrac{1}{4}\mu\omega^2 A^2\lambda \tag{14.51}$$

The total energy in one wavelength of the wave is the sum of the obtained kinetic and potential energies:

$$E_\lambda = K_\lambda + U_\lambda = \tfrac{1}{2}\mu\omega^2 A^2\,\lambda \tag{14.52}$$

As the sinusoidal wave travels along the string, that amount of energy (E_λ) will cross any given point on the string during a time interval equal to one period of the oscillation. Thus, the rate of energy (or power) transferred by the wave through the string is:

$$P = \frac{\Delta E}{\Delta t} = \frac{E_\lambda}{T}$$

Therefore:

$$P = \tfrac{1}{2}\mu\omega^2 A^2\,\frac{\lambda}{T}$$

Using the relation $v = \lambda/T$ given by Eq. 14.38, we finally attain the following form:

$$\boxed{P = \tfrac{1}{2}\mu v \omega^2 A^2} \tag{14.53}$$

In this expression the factors μ and v depend on the material and tension of the string. On the other hand, the factors ω and A depend on the source that generates the sinusoidal wave. The dependence of the power of a wave on the square of its angular frequency and on the square of its amplitude is a general result, i.e. true for all wave types.

Example 14.6

A string that is taut under tension of magnitude $\tau = 40\,\text{N}$ has a linear density μ of 64 g/m. A wave is traveling along the string with a frequency f of 120 Hz and amplitude A of 8 mm. (a) Find the speed of the wave. (b) What is the rate of energy that must be supplied by a generator to produce this wave in the string? (c) If the string is to transfer energy at a rate of 500 W, what must be the required wave amplitude when all other parameters remain the same?

Solution: (a) Equation 14.44 gives the speed of the wave as follows:

$$v = \sqrt{\frac{\tau}{\mu}} = \sqrt{\frac{40\ \text{N}}{0.064\ \text{kg/m}}} = 25\ \text{m/s}$$

(b) First we calculate the angular frequency ω as follows:

$$\begin{aligned}\omega = 2\pi f &= 2 \times (3.1416\ \text{rad}) \times (120\ \text{s}^{-1})\\ &= 754\ \text{rad/s}\end{aligned}$$

The power supplied to the string is calculated by using the obtained values and the given information in Eq. 14.53 as follows:

$$\begin{aligned}P &= \tfrac{1}{2}\mu v \omega^2 A^2\\ &= \tfrac{1}{2}(0.064\ \text{kg/m})(25\ \text{m/s})(754\ \text{rad/s})^2(0.008\ \text{m})^2\\ &= 29.1\ \text{W}\end{aligned}$$

(c) The ratio between the new power P' and the old power P is:

$$\frac{P'}{P} = \frac{\tfrac{1}{2}\mu\, v\, \omega^2 A'^2}{\tfrac{1}{2}\mu\, v\, \omega^2 A^2} = \frac{A'^2}{A^2}$$

Thus: $$A' = A\sqrt{\frac{P'}{P}} = 0.008\ \text{m}\sqrt{\frac{500\ \text{W}}{29.1\ \text{W}}} = 0.033\ \text{m} = 3.3\ \text{cm}$$

14.6 The Linear Wave Equation

In Sect. 14.3.3 we introduced the wave function $y = y(x, t)$ to represent waves traveling on strings. Actually, all these wave functions represent solutions of a differential equation called the **linear wave equation**. This equation is basic to many forms of wave motions, such as waves on strings.

We consider a single symmetrical transverse pulse that is traveling with a speed v in a stretched ideal string under tensional force of magnitude τ and has a linear density μ, see Fig. 14.20.

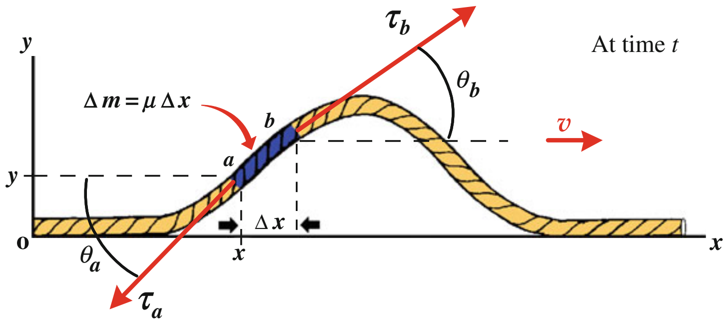


Fig. 14.20 A pulse traveling with a speed v in a string under tension τ. The figure shows an element of length Δx at the point (x, y)

In this figure we consider a small element $a\,b$ of length Δx with ends at angles θ_a and θ_b with the x axis. Also, for an ideal string we consider $\tau_b \cos\theta_b = \tau_a \cos\theta_a = \tau$. Thus, with the use of this result, the net vertical force acting on the string element can be written as:

$$\begin{aligned} \Sigma F_y &= \tau_b \sin\theta_b - \tau_a \sin\theta_a \\ &= \tau \tan\theta_b - \tau \tan\theta_a = \tau(\tan\theta_b - \tan\theta_a) \end{aligned} \tag{14.54}$$

The tangent of an angle is represented by dy/dx when y depends only on x. Since we are evaluating this tangent at a particular instant of time t, we need to express this tangent in partial form as $\partial y/\partial x$. Substituting this form of tangents into Eq. 14.54 gives:

$$\Sigma F_y = \tau \left[\left(\frac{\partial y}{\partial x} \right)_b - \left(\frac{\partial y}{\partial x} \right)_a \right] \tag{14.55}$$

When we apply Newton's second law to the vertical motion of an element of mass $\Delta m = \mu \; \Delta x$, we get:

$$\Sigma F_y = \Delta m \; a_y = \mu \; \Delta x \left(\frac{\partial^2 y}{\partial t^2} \right) \tag{14.56}$$

Combining Eqs. 14.55 with Eq. 14.56, we get:

$$\begin{aligned} \mu \; \Delta x \left(\frac{\partial^2 y}{\partial t^2} \right) &= \tau \left[\left(\frac{\partial y}{\partial x} \right)_b - \left(\frac{\partial y}{\partial x} \right)_a \right] \\ \frac{\mu}{\tau} \left(\frac{\partial^2 y}{\partial t^2} \right) &= \frac{\left(\frac{\partial y}{\partial x} \right)_b - \left(\frac{\partial y}{\partial x} \right)_a}{\Delta x} = \frac{\frac{\partial y(x + \Delta x, t)}{\partial x} - \frac{\partial y(x, t)}{\partial x}}{\Delta x} \end{aligned} \tag{14.57}$$

From the definition of partial differentiation, we know that:

$$\frac{\partial}{\partial x} f(x, t) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, t) - f(x, t)}{\Delta x}$$

Thus, if we associate $f(x + \Delta x, t)$ with $(\partial y / \partial x)_b$ and $f(x, t)$ with $(\partial y / \partial x)_a$, we see that, in the limit $\Delta x \to 0$, the right-hand side of Eq. 14.57 can be expressed as a partial derivative as follows:

$$\frac{\partial}{\partial x} \left(\frac{\partial y}{\partial x} \right) = \lim_{\Delta x \to 0} \frac{\left(\frac{\partial y}{\partial x} \right)_b - \left(\frac{\partial y}{\partial x} \right)_a}{\Delta x}$$

Then, with the use of this result and Eq. 14.44, namely $v = \sqrt{\tau / \mu}$, we can write Eq. 14.57 as a partial differential equation in the following general form:

$$\boxed{\frac{\partial^2 y}{\partial x^2} - \frac{1}{v^2} \frac{\partial^2 y}{\partial \; t^2} = 0} \tag{14.58}$$

This is the linear wave equation as it applies to waves on strings and generally applies to various types of traveling waves. We can prove that the sinusoidal wave $y(x, t) = A \sin(kx - \omega \, t)$ satisfies this equation.

14.7 Standing Waves

We consider two identical waves of the same wavelength and amplitude traveling simultaneously in opposite directions in a stretched string. The resultant wave in the string will be the algebraic sum of the two waves. This is one of the examples of a principle known as the **superposition principle**. Generally, this principle says that when several effects occur simultaneously, their net effect is the sum of the individual effects. The superposition principle will be introduced in more detail in Chap. 15 when we study the properties of standing sound waves.

To analyze this situation, we assume that the two string waves have the same frequency f (the same $\omega = 2\pi f$), wavelength λ (the same $k = 2\pi/\lambda$), and amplitude A but travel in opposite directions. Therefore, we can write these two waves in the following form:

$$\begin{aligned} y_1 &= A\sin(k\,x - \omega\,t), \\ y_2 &= A\sin(k\,x + \omega\,t) \end{aligned} \tag{14.59}$$

where y_1 represents a wave traveling in the positive x-direction and y_2 represents a wave traveling in the negative x-direction. The superposition of y_1 and y_2 gives the following resultant:

$$y = y_1 + y_2 = A\,[\sin(k\,x - \omega\,t) + \sin(k\,x + \omega\,t)] \tag{14.60}$$

To simplify this expression, we use the trigonometric identity:

$$\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b \tag{14.61}$$

If we substitute $a = kx$ and $b = \omega\,t$ in this identity, then the resultant wave y reduces to:

$$y = (2\,A\sin k\,x)\cos\omega\,t \tag{14.62}$$

The resultant wave y represented by Eq. 14.62 gives a special kind of simple harmonic motion. Here, every element of the medium oscillates in simple harmonic motion with the same angular frequency ω (through the factor $\cos\omega\,t$) with an amplitude (given by the factor $2\,A\sin k\,x$) that varies with position x. This wave is called a **standing wave** because there is no motion of the disturbance along the x-direction.

A standing wave is distinguished by stationary positions with *zero amplitudes* called *nodes* (see Fig. 14.21). This happens when x satisfies the condition $\sin kx = 0$, that is, when:

$$k\,x = 0,\ \pi,\ 2\pi,\ 3\pi, \ldots$$

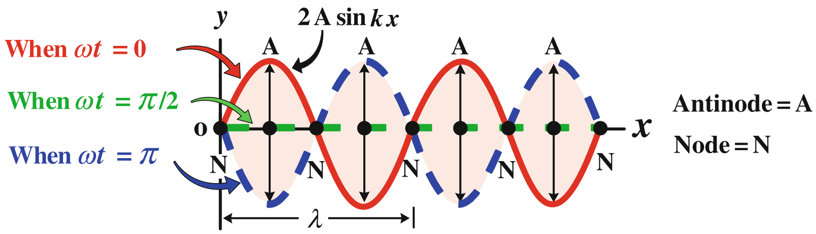


Fig. 14.21 The time dependence of the vertical displacement (from equilibrium) of any individual element in the standing wave y is governed by $\cos\omega t$. Each element vibrates within the confines of the envelope $2A\sin kx$. The nodes (N) are points of zero displacement, and the antinodes (A) are points of maximum displacement

When using $k = 2\pi/\lambda$, these values give $x = 0, \frac{\lambda}{2}, \lambda, \frac{3\lambda}{2}, \ldots$, that is:

$$x = 0,\ \frac{\lambda}{2},\ \lambda,\ \frac{3\lambda}{2}, \ldots = n\frac{\lambda}{2}, \quad (n = 0, 1, 2, \ldots) \quad \text{(Nodes)} \tag{14.63}$$

Also, a standing wave is distinguished by elements with the *greatest possible displacements* called *antinodes* (see Fig. 14.21). This happens when x satisfies the condition $\sin kx = \pm 1$, that is, when:

$$kx = \frac{\pi}{2},\ \frac{3\pi}{2},\ \frac{5\pi}{2}, \ldots$$

Also, using $k = 2\pi/\lambda$, these values give $x = \frac{\lambda}{4}, \frac{3\lambda}{4}, \frac{5\lambda}{4}, \ldots$, that is:

$$x = \frac{\lambda}{4},\ \frac{3\lambda}{4},\ \frac{5\lambda}{4}, \ldots = (n + \tfrac{1}{2})\frac{\lambda}{2}, \quad (n = 0, 1, 2, \ldots) \quad \text{(Antinodes)} \tag{14.64}$$

Equations 14.63 and 14.64 indicate the following general features of nodes and antinodes (see Fig. 14.21):

Spotlight
(1) The distance between adjacent nodes is $\lambda/2$.
(2) The distance between adjacent antinodes is $\lambda/2$.
(3) The distance between a node and adjacent antinode is $\lambda/4$.

At $t = 0$ $(\omega t = 0)$, the two oppositely traveling waves are in phase, producing a wave pattern in which each element of the medium is experiencing its maximum displacement from equilibrium, see Fig. 14.22a. At $t = T/4$, $(\omega t = \pi/2)$, the traveling waves have moved one quarter of a wavelength (one to the right and the other to the left). At this time, each element of the medium is passing through the equilibrium position in its simple harmonic motion. The result is zero displacement for each element at all values of x, see Fig. 14.22b. At $t = T/2$ ($\omega t = \pi$), the traveling waves are again in phase, producing a wave pattern that is inverted relative to the $t = 0$ pattern, see Fig. 14.22c. The pattern at $t = 3T/4$ (Fig. 14.22d) resembles that at $t = T/2$. Also, the pattern at $t = T$ (Fig. 14.22e) resembles that at $t = 0$.

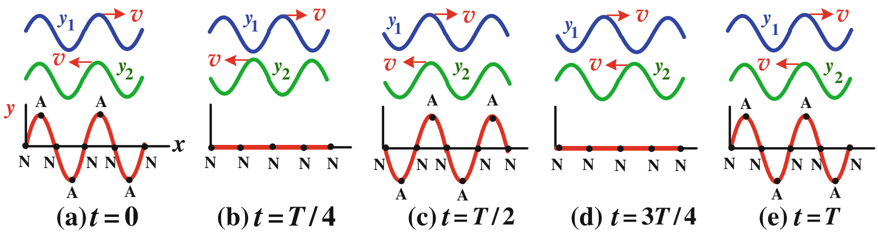


Fig. 14.22 Standing-wave patterns y at different times for the two oppositely traveling identical waves y_1 and y_2. Nodes (N) have no displacements while antinodes (A) have maximum displacements

Example 14.7

A standing wave is produced by two identical sinusoidal waves traveling in opposite directions in a taut string. The two waves are given by:

$$y_1 = (0.02\,\text{m})\sin(5x - 10t)$$
$$y_2 = (0.02\,\text{m})\sin(5x + 10t)$$

where x and y are in meters, t is in seconds, and the argument of the sine is in radians. (a) Find the amplitude of the simple harmonic motion of the element on the string located at $x = 10$ cm. (b) Find the positions of the nodes and antinodes in the string. (c) Find the maximum and minimum y values of the simple harmonic motion of a string element located at any antinode.

Solution: (a) Equation 14.62 gives the standing wave produced from y_1 and y_2 with $A = 0.02$ m, $k = 5$ rad/m, and $\omega = 10$ rad/s. Thus:

$$y = (2A\sin kx)\,\cos\omega t = [(0.04\,\text{m})\sin 5x]\cos 10\,t$$

The coefficient of the cosine at $x = 10\,\text{cm} = 0.1$ m will be:

$$y_{\max} = (0.04\,\text{m})\sin 5\,x|_{x=0.1} = (0.04\,\text{m})\sin(0.5\,\text{rad}) = 0.019\,\text{m} = 1.9\,\text{cm}$$

(b) When $k = 5\,\text{rad/m} = 2\pi/\lambda$, we find the wavelength to be $\lambda = 0.4\pi$ m. Therefore, from Eq. 14.63 we find the nodes to be located at:

$$x = n\frac{\lambda}{2} = (0.2\pi n)\ \text{m}, \quad (n = 0, 1, 2, \ldots)$$

From Eq. 14.64, the antinodes will be located at:

$$x = (n + \tfrac{1}{2})\frac{\lambda}{2} = [0.2\pi(n + \tfrac{1}{2})]\ \text{m}, \quad (n = 0, 1, 2, \ldots)$$

(c) The maximum and minimum y values of the simple harmonic motion of a string element located at any antinode are:

$$y_{\max/\min} = 2A\,(\sin 5\,x)|_{\max/\min} = 2A\,(\pm 1) = \pm\,0.04\,\text{m} = \pm\,0.04\,\text{m} = \pm\,4\,\text{cm}$$

14.7.1 Reflection at a Boundary

A wave moving along a stretched string can be reflected from one of its ends in two different ways, as shown in Fig. 14.23. The first way is to fix the far end of the string, and the second way is to allow the far end to move freely up and down.

When the incident pulse in Fig. 14.23a reaches the fixed end, it exerts an upward force on the wall through the support. By Newton's third law, the support at the wall exerts an opposite force on the string. This reaction force generates an inverted reflected pulse that travels in a direction opposite to the incident pulse. In a reflection of this kind, there must be no displacement of the string at the right end, which is referred to as a **node** at the support, because the string is fixed there.

In Fig. 14.23b, the right end of the string is tied to a weightless ring that is free to slide without friction along a vertical rod. When the incident pulse reaches the ring, the ring moves up along the rod. The ring rises as high as the incoming pulse, and then the downward component of the tension pulls the ring back down. This movement of the ring produces a non-inverted reflected pulse of the same amplitude as the incident pulse. In a reflection of this kind, there must be a maximum displacement of the string at the right end, which is referred to as an **antinode**, because the string is not fixed there.

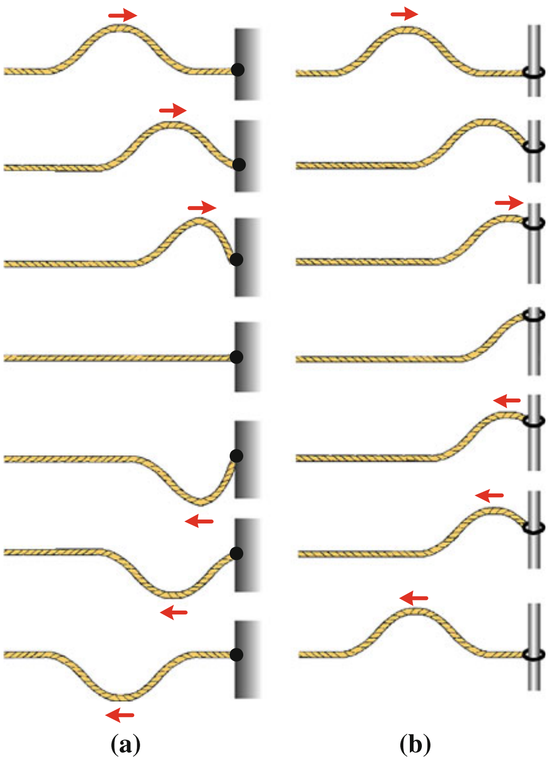


Fig. 14.23 (a) An incident pulse from the left is inversely reflected when the right side of the string is fixed to a wall. (b) The same incident pulse is reflected unchanged in sign when the right side of the string is tied to a ring that can slide without friction on a vertical rod

14.7.2 Standing Waves and Resonance

When one end of a stretched string is oscillating in a sinusoidal fashion while the other end is fixed, the incident wave and the reflected wave interfere with each other.

For certain frequencies, this interference produces a standing wave with nodes and antinodes like those shown in Figs. 14.21 and 14.22. Such a standing wave is said to be produced at **resonance**, and the string *resonates* at these **resonant frequencies**. If the string is oscillating at some other frequency, a standing wave is not set up.

Generally, an imposed boundary condition on a string sets up a number of natural patterns of oscillation called **normal modes**.

Consider a stretched string between two points separated by a distance *L*, see Fig. 14.24a. Visualize that the string is somehow made to oscillate at a resonance frequency to set up a specific standing wave pattern. Since both ends are fixed, then for this boundary condition there must be at least two nodes and one antinode for the standing wave pattern. The normal modes of oscillation for the string can be explained by considering the following three patterns:

(1) The first normal mode (the first harmonic, or the fundamental):
The simplest pattern that can meet the boundary condition of two fixed ends is shown in Fig. 14.24b. Note that there are two imposed nodes at both ends and only one antinode, which is at the center of the string. There is only half a wavelength in the length L. Thus, for this pattern, $\lambda_1/2 = L$, i.e. $\lambda_1 = 2\,L$.

(2) The second normal mode (the second harmonic):
The second pattern that can meet the boundary condition of two fixed ends is shown in Fig. 14.24c. This pattern has three nodes and two antinodes. This standing wave must have $\lambda_2 = L$.

(3) The third normal mode (the third harmonic):
The third pattern that can meet the boundary condition of two fixed ends is shown in Fig. 14.24d. This pattern has four nodes and three antinodes . This standing wave must have $\lambda_3 = 2L/3$.

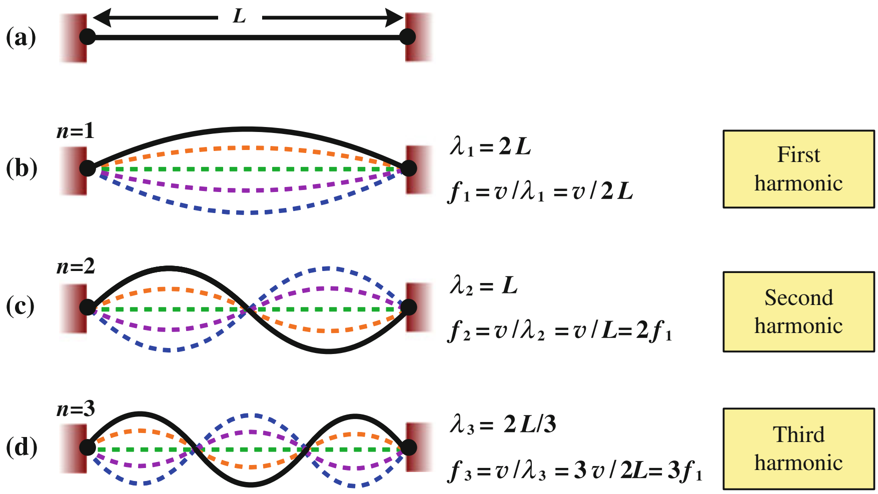


Fig. 14.24 (a) A string of length L that is fixed at both ends. The normal modes of vibration are shown for; (b) the first harmonic (or the fundamental), (c) the second harmonic, and (d) the third harmonic

In general, the relation between the wavelength λ of the various normal modes for a string of length L fixed at both ends is given by:

$$\lambda_n = \frac{2L}{n}, \quad (n = 1,\ 2,\ 3, \ldots) \quad \text{(String, fixed ends)} \tag{14.65}$$

where the index n refers to the n^{th} normal mode of the possible oscillation of the string (or the number of **loops** in the string).

The resonance frequencies associated with these modes are obtained from the relation $f = v/\lambda$, where the speed of the wave is the same for all the frequencies. Using Eq. 14.65, we find the resonance frequencies f_n of the normal modes to be (see Fig. 14.24):

$$f_n = \frac{v}{\lambda_n} = n\frac{v}{2L}, \quad (n = 1,\ 2,\ 3, \ldots) \quad \text{(String, fixed ends)} \tag{14.66}$$

According to Eq. 14.44, the speed of the wave v is related to the tension in the string τ and the linear mass density μ by the relation $v = \sqrt{\tau/\mu}$. Substituting with this relation into Eq. 14.66 we get:

$$f_n = \frac{n}{2L}\sqrt{\frac{\tau}{\mu}}, \quad (n = 1,\ 2,\ 3, \ldots) \quad \text{(String, fixed ends)} \tag{14.67}$$

The lowest resonance frequency f_1, which corresponds to $n = 1$, is called the **fundamental frequency** and is given by:

$$f_1 = \frac{1}{2L}\sqrt{\frac{\tau}{\mu}} \quad \text{(String, fixed ends)} \tag{14.68}$$

The resonance frequencies of the remaining normal modes are integer multiples of the fundamental frequency (Fig. 14.24), that is:

$$f_n = n f_1, \quad (n = 1,\ 2,\ 3, \ldots) \quad \text{(String, fixed ends)} \tag{14.69}$$

Example 14.8

The middle-C key on a piano (key No. 40) has a fundamental frequency of 262 Hz, and the A key above the middle C in frequency has a fundamental frequency of 440 Hz, see Fig. 14.25. (a) Find the frequencies of the next two harmonics of the C string. (b) The strings of the keys A and C have the same linear mass density but the length L_A of the string A is 65% of the length L_C of string C. What will be the ratio of the tensions τ_A/τ_C in the two strings?

Solution: (a) Equation 14.69 gives the higher harmonics in terms of the fundamental frequency. Thus, for $f_1 = 262$ Hz we get:

$$f_2 = 2f_1 = 2 \times 262 \text{ Hz} = 524 \text{ Hz}$$
$$f_3 = 3f_1 = 3 \times 262 \text{ Hz} = 786 \text{ Hz}$$

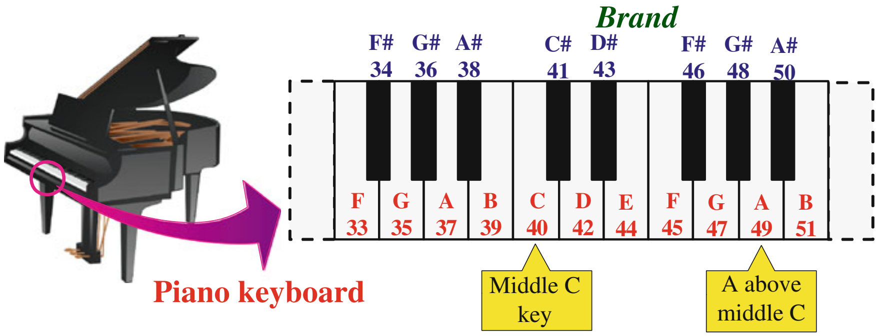


Fig. 14.25

(b) When the two strings vibrate at their fundamental frequencies, we can use Eq. 14.68 to write down the following relations:

$$f_{1A} = \frac{1}{2L_A}\sqrt{\frac{\tau_A}{\mu}} \quad \text{and} \quad f_{1C} = \frac{1}{2L_C}\sqrt{\frac{\tau_C}{\mu}}$$

Thus, the ratio of the two frequencies is $f_{1A}/f_{1C} = (L_C/L_A)\sqrt{\tau_A/\tau_C}$. When we square this relation, we get the ratio of the magnitude of the two tensions as follows:

$$\frac{\tau_A}{\tau_C} = \left(\frac{L_A}{L_C}\right)^2\left(\frac{f_{1A}}{f_{1C}}\right)^2 = \left(\frac{65}{100}\right)^2\left(\frac{440 \text{ Hz}}{262 \text{ Hz}}\right)^2 = 1.19$$

Example 14.9

The one end A of a string is attached to a vibrator of frequency 100 Hz, while the other end passes over a pulley at point B to a block of mass m, see Fig. 14.26. The separation L between A and B is 1.5 m and the linear mass density of the string is 1.5 g/m. (a) Find the mass m needed to allow the vibrator to set up the third harmonic on the string. (b) What standing-wave mode is set up if $m = 0.5$ kg?

Solution: (a) The tension τ in the string must equal to the weight of the mass m, i.e. $\tau = mg$. Substitution with this tension into Eq. 14.67 gives the resonance frequencies in a general form as follows:

$$f_n = \frac{n}{2L}\sqrt{\frac{mg}{\mu}}, \quad (n = 1,\ 2,\ 3, \ldots)$$

We need to set the tension in the string (by the mass m) so that the vibrator frequency is equal to the frequency of the third harmonic, i.e.:

$$f_3 = \frac{3}{2L}\sqrt{\frac{mg}{\mu}}$$

Thus:

$$m = \frac{4L^2\mu f_3^2}{9\,g} = \frac{4 \times (1.5\ \text{m})^2(1.5 \times 10^{-3}\ \text{kg/m})(100\ \text{Hz})^2}{9 \times (9.8\ \text{m/s}^2)} = 1.5306\ \text{kg}$$

(b) If we insert $m = 0.5$ kg and $f_n = 100$ Hz into the first equation, we get:

$$n = 2Lf_n\sqrt{\frac{\mu}{mg}} = 2 \times (1.5\ \text{m})(100\ \text{Hz})\sqrt{\frac{1.5 \times 10^{-3}\ \text{kg/m}}{(0.5\ \text{kg})(9.8\ \text{m/s}^2)}} = 5.25$$

With $m = 0.5$ kg, we get $n = 5.25$. Because n has to be an integer, then this vibrator cannot set up a standing wave on the string.

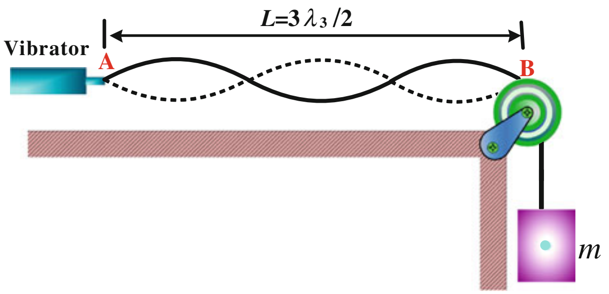


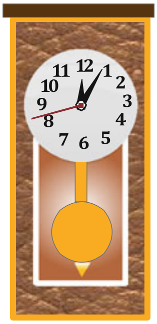
Fig. 14.26

14.8 Exercises

Section 14.1 Simple Harmonic Motion

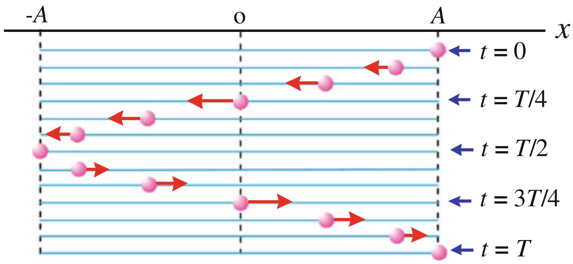
(1) Some clocks use a pendulum to keep time, see Fig. 14.27. The bob of a clock requires 1 s for a single small-amplitude swing. (a) What is the period of the pendulum? (b) What is frequency of the pendulum? (c) What is the angular frequency of the pendulum's oscillations?

Fig. 14.27 See Exercise (1)



(2) A particle executes a simple harmonic motion along the x-axis with amplitude A. The particle returns to its starting position every $T = 0.25$ s, see Fig. 14.28. (a) Find the period, frequency, and angular frequency of this motion. (b) Find the particle's displacement as a function of time.

Fig. 14.28 See Exercise (2)



(3) A particle oscillates with a simple harmonic motion along the x axis. Its displacement from the origin varies with time according to the equation $x = (1.5\text{ m})\cos(2\pi\, t + \phi)$, where $\phi = -\pi/4$ rad, t is in seconds and the argument of the cosine is in radians, see the blue curve of Fig. 14.29. (a) Find the value of the amplitude, frequency, and period of the motion. (b) Find the velocity and acceleration of the particle as a function of time. (c) Find both the maximum speed and acceleration of the particle. (d) Find the displacement of the particle between $t = 0$ and $t = 1$ s.

(4) When the mechanical energy of one oscillation of a spring-block system is doubled, what is the ratio of their amplitudes?

(5) A block of mass $m = 0.8$ kg oscillates freely with period $T = 0.9$ s when attached to a linear spring that obeys Hooke's law, see Fig. 14.30. An unknown mass M attached to the same spring is observed to have a period of oscillation

of 1.2 s. (a) Find the spring constant k_H of the spring. (b) Find the value of the unknown mass M.

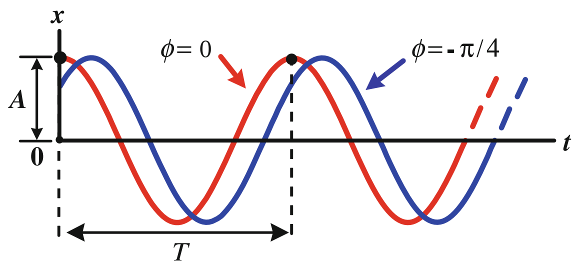


Fig. 14.29 See Exercise (3)

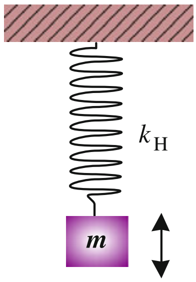


Fig. 14.30 See Exercise (5)

(6) A block of mass $m = 0.5$ kg rests on a horizontal frictionless surface and is connected to a spring, as shown in Fig. 14.31. When the system is set into motion with amplitude $A = 0.35$ m, it repeats its motion every 0.5 s. (a) Find the block's period, frequency, and angular frequency. (b) Find the spring constant, the maximum speed of the block, and the maximum force exerted by the spring on the block.

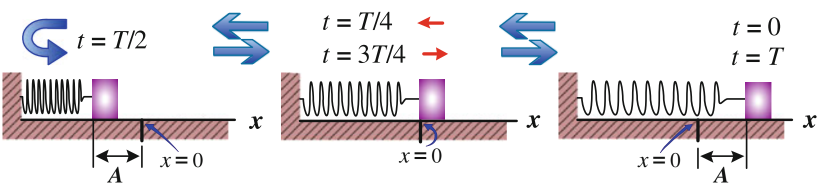


Fig. 14.31 See Exercise (6)

(7) Two springs 1 and 2 have the same un-stretched length but different force constants $k_{H_1} \equiv k_1$ and $k_{H_2} \equiv k_2$, respectively. The springs are connected to a

block of mass m that rests on a horizontal frictionless surface as shown in Fig. 14.32. Calculate the effective force constant k_{eff} in each of the three cases (a), (b), and (c) of the figure.

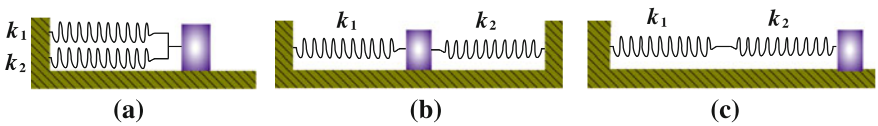


Fig. 14.32 See Exercise (7)

(8) When $k_1 = k_2 = k$ in Exercise 7, find the frequency of oscillation of the block in each of the three cases (a), (b), and (c).

(9) When a group of four persons, each of mass 60 kg, steps into a small car of mass 936 kg, the four springs of the car are compressed by 4 cm. Take $g = 10\,\text{m/s}^2$. (a) What is the effective spring constant k_H of the springs? (b) Find the period and frequency of the car after hitting a road bump that causes the car to oscillate up and down, assuming the oscillations of the four springs are in phase.

(10) In Exercise 7, show that the frequencies f of oscillation of the block in the two cases (b) and (c) are given respectively by:

$$f = \sqrt{f_1^2 + f_2^2}, \quad f = \sqrt{f_1^2 f_2^2/(f_1^2 + f_2^2)}$$

where f_1 and f_2 are the frequencies when the block is connected to only spring 1 or spring 2, respectively.

(11) The velocity of a 0.5 kg mass attached to the end of a spring is represented by $v = -(4\,\text{m/s})\sin(2\,t)$. Find the total energy E.

(12) A block of mass $m = 0.2$ kg is fastened to a light spring whose spring constant k_H is 5 N/m, see part (a) of Fig. 14.33. The block is pulled a distance $x_i = 5$ cm from its equilibrium position at $x = 0$ on a horizontal frictionless surface, see part (b) of Fig. 14.33, and then released at $t = 0$. (a) What is the mechanical energy of the oscillator? (b) What is the maximum speed of the oscillator? (c) Find the speed, kinetic energy, and potential energy of the block when its position is 2 cm.

(13) Assume that the mass in Exercise 12 is 0.025 kg, the force constant k_H is 0.4 N/m, and that the motion starts by imparting to the block at $x_i = 0.1$m a velocity toward the right of 0.4 m/s. (a) Find the period T, frequency f, and angular frequency ω of the oscillator. (b) Find the total energy E, amplitude A,

the phase angle ϕ, the maximum speed v_{max}, and the maximum acceleration a_{max}. (c) Write down the position, velocity, and acceleration in terms of time t, then substitute with $t = \pi/8$ s and find their values.

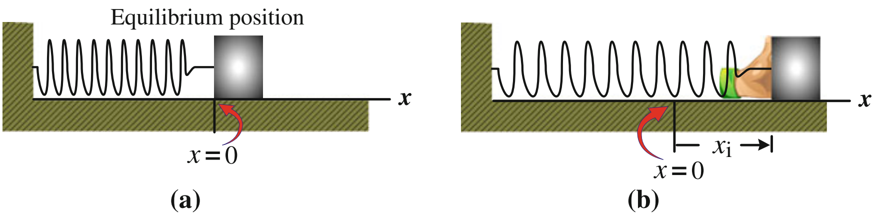


Fig. 14.33 See Exercise (12)

(14) A bullet of mass $m = 10$ g is fired horizontally with a speed v into a stationary wooden block of mass $M = 4$ kg. The block is resting on a horizontal smooth surface and attached to a massless spring with spring constant $k_H = 150$ N/m, where the other end of the spring is fixed through a wall, as shown in Fig. 14.34a. In a very short time, the bullet penetrates the block and remains embedded before compressing the spring, as shown in Fig. 14.34b. The maximum distance that the block compresses the spring is 8 cm, as shown in Fig. 14.34c. (a) What is the speed of the bullet? (b) Find the period T and frequency f of the oscillating system.

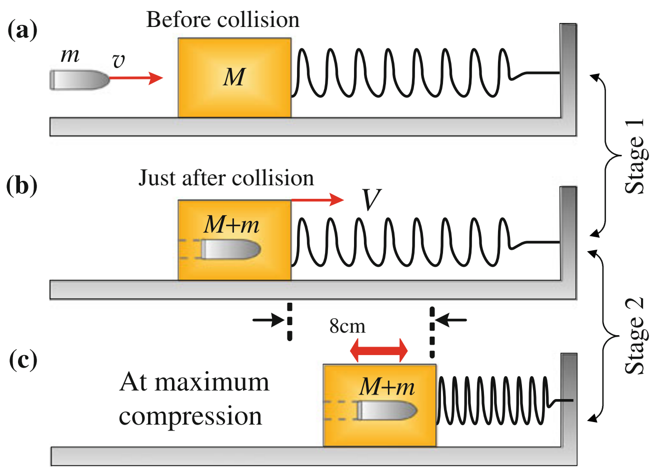


Fig. 14.34 See Exercise (14)

Section 14.2 Damped Simple Harmonic Motion

(15) An object of mass $m = 0.25$kg oscillates in a fluid at the end of a vertical spring of spring constant $k_H = 85$ N/m, see Fig. 14.35. The effect of the fluid resistance is governed by the damping constant $b = 0.07$kg/s. (a) Find the period of the damped oscillation. (b) By what percentage does the amplitude of the oscillation decrease in each cycle? (c) How long does it take for the amplitude of the damped oscillation to drop to half of its initial value?

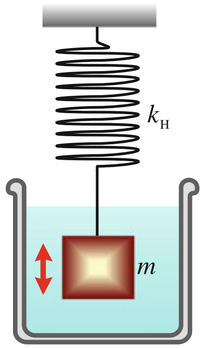


Fig. 14.35 See Exercise (15)

(16) A simple pendulum has a length L and a mass m. Let the arc length s and the angle θ measure the position of m at any time t, see Fig. 14.36. (a) When a damped force $F_d = -bv_s$ exists, show that the equation of motion of the pendulum is given for small angles by:

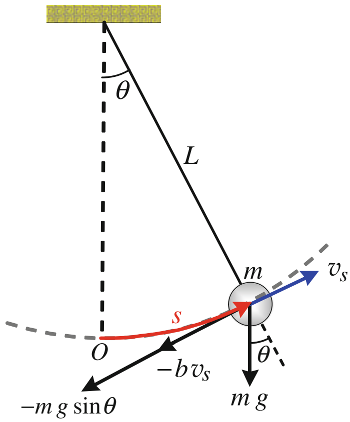
$$m\frac{d^2\theta}{dt^2} + b\frac{d\theta}{dt} + \frac{mg}{L}\theta = 0$$

(b) By comparison with Eq. 14.25, show that the above differential equation has a solution given by:

$$\theta = \theta_\circ\, e^{-bt/2m}\cos(\omega_d\; t), \quad \omega_d = \sqrt{\frac{g}{L} - \frac{b^2}{4m^2}}$$

where $\theta_\circ$ is initial angular amplitude at $t = 0$ and $\omega_d = 2\pi f_d$ is damped angular frequency, see Fig. 14.9b. (c) When the pendulum has $L = 1$ m, $m = 0.1$ kg, and the angular amplitude θ becomes 0.5 $\theta_\circ$ after 1 minute, find the damping constant b and the ratio $(f - f_d)/f$, where f is the undamped frequency.

Fig. 14.36 See Exercise (16)



Section 14.3 Sinusoidal Waves

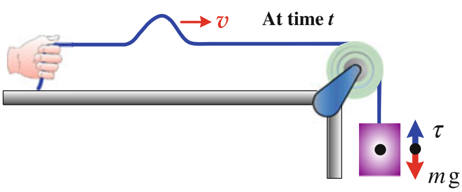
(17) Given a sinusoidal wave represented by $y = (0.2\,\mathrm{m})\sin(k\,x - \omega\,t)$, where $k = 4$ rad/m, and $\omega = 8$ rad/s, determine the amplitude, wavelength, frequency, and speed of this wave.

(18) A harmonic wave traveling along a string has the form $y = (0.25\,\mathrm{m})\sin(3\,x - 40\,t)$, where x is in meters and t is in seconds. (a) Find the amplitude, wave number, angular frequency, and speed of this wave. (b) Find the wavelength, period, and frequency of this wave?

Section 14.4 The Speed of Waves on Strings

(19) A uniform string has a mass per unit length of 5×10^{-3} kg/m. The string passes over a massless, frictionless pulley to a block of mass $m = 135$ kg, see Fig. 14.37, and take $g = 10\,\mathrm{m/s^2}$. Find the speed of a pulse that is sent from one end of the string toward the pulley. Does the value of the speed change when the pulse is replaced by a sinusoidal wave?

Fig. 14.37 See Exercise (19)



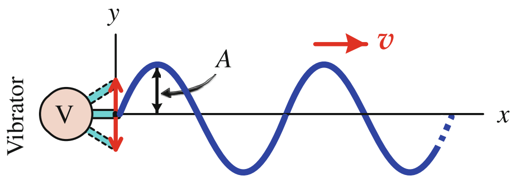
(20) Assume a transverse wave traveling on a uniform taut string of mass per unit length $\mu = 4 \times 10^{-3}$ kg/m. The wave has an amplitude of 5 cm, frequency of 50 Hz, and speed of 20 m/s. (a) Write an equation in SI units of the form $y = A\sin(kx - \omega t)$ for this wave. (b) Find the magnitude of the tension in the string.

Section 14.5 Energy Transfer by Sinusoidal Waves on Strings

(21) A sinusoidal wave of amplitude 0.05 m is transmitted along a string that has a linear density of 40 g/m and is under 100 N of tension. If the wave source has a maximum power of 300 W, what is the highest frequency at which the source can operate?

(22) A long string has a mass per unit length μ of 125 g/m and is taut under tension τ of 32 N. A wave is supplied by a generator as shown in Fig. 14.38. This wave travels along the string with a frequency f of 100 Hz and amplitude A of 2 cm. (a) Find the speed and the angular frequency of the wave. (b) What is the rate of energy that must be supplied by a generator to produce this wave in the string? (c) If the string is to transfer energy at a rate of 100 W, what must be the required wave amplitude when all other parameters remain the same?

Fig. 14.38 See Exercise (22)



(23) A sinusoidal wave is traveling along a string of linear mass density $\mu = 75$ g/m and is described by the equation:

$$y = (0.25 \text{ m})\sin(2x - 40t)$$

where x is in meters and t in seconds. (a) Find the speed, wavelength, and frequency of the wave. (b) Find the power transmitted by the wave.

Section 14.6 The Linear Wave Equation

(24) A one-dimensional wave traveling with velocity v is found to satisfy the partial differential equation [see Eq. 14.58]:

$$\frac{\partial^2 y}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} = 0$$

Show that the following functions are the solutions to this linear wave equation: (a) $y = A\sin(k\,x - \omega\,t)$. (b) $y = A\cos(k\,x - \omega\,t)$. (c) $y = \exp[b(x - v\,t)]$, where b is a constant. (d) $y = \ln[b(x - v\,t)]$, where b is a constant. (e) Any function y having the form $y = f(x - v\,t)$.

(25) If the linear wave functions $y_1 = f_1(x, t)$ and $y_2 = f_2(x, t)$ satisfy the wave Eq. 14.58, then show that the combination $y = C_1\,f_1(x, t) + C_2\,f_2(x, t)$ also satisfies the same equation, where C_1 and C_2 are constants.

Section 14.7 Standing Waves

(26) A standing wave having a frequency of 20 Hz is established on a rope 1.5 m long that has fixed ends. Its wavelength is observed to be twice the rope's length. Determine the wave's speed.

(27) A stretched string of length 0.6 m and mass 30 g is observed to vibrate with a fundamental frequency of 30 Hz. The amplitude of any antinodes in the standing wave is 0.04 m. (a) What is the amplitude of a transverse wave in the string? (b) What is the speed of a transverse wave in the string? (c) Find the magnitude of the tension in the string.

(28) A student wants to establish a standing wave with a speed 200 m/s on a string that is fixed at both ends and is 2.5 m long. (a) What is the minimum frequency that should be applied? (b) Find the next three frequencies that cause standing wave patterns on the string.

(29) Two identical waves traveling in opposite directions in a string interfere to produce a standing wave of the form:

$$y = [(2\,\mathrm{m})\sin(2\,x)]\cos(20\,t)$$

where x is in centimeters, t is in seconds, and the arguments of the sine and cosine are in radians. Find the amplitude, wavelength, frequency, and speed of the interfering waves.

(30) A standing wave is produced by two identical sinusoidal waves traveling in opposite directions in a taut string. The two waves are given by:

$$y_1 = (2\,\text{cm})\sin(2.3\,x - 4\,t) \quad \text{and} \quad y_2 = (2\,\text{cm})\sin(2.3\,x + 4\,t)$$

where x and y are in centimeters, t is in seconds, and the argument of the sine is in radians. (a) Find the amplitude of the simple harmonic motion of an element on the string located at $x = 3\,\text{cm}$. (b) Find the position of the nodes and antinodes on the string. (c) Find the maximum and minimum y values of the simple harmonic motion of a string element located at any antinode.

(31) A guitar string has a length $L = 64\,\text{cm}$ and fundamental frequency $f_1 = 330\,\text{Hz}$, see part (a) of Fig. 14.39. By pressing down with your finger on the string, it is found that the string is shortened in a way so that it plays an F note with a fundamental frequency $f_1' = 350\,\text{Hz}$, see part (b) of Fig. 14.39. [Assume the speed of the wave remains constant before and after pressing] How far is your finger from the near end of the string?

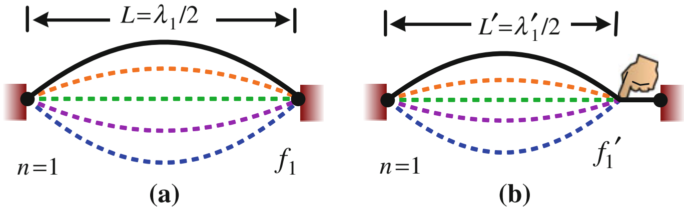


Fig. 14.39 See Exercise (31)

(32) A violin string oscillates at a fundamental frequency of 262 Hz when unfingered. At what frequency will it vibrate if it is fingered two-fifths of the length from its end?

(33) A string that has a length $L = 1\,\text{m}$, mass per unit length $\mu = 0.1\,\text{kg/m}$, and tension $\tau = 250\,\text{N}$ is vibrating at its fundamental frequency. What effect on the fundamental frequency occurs when only: (a) The length of the spring is doubled. (b) The mass per unit length of the spring is doubled. (c) The tension of the spring is doubled.

(34) Show that the resonance frequency f_n of standing waves on a string of length L and linear density μ, which is under a tensional force of magnitude τ, is given by $f_n = n\sqrt{\tau/\mu}/2L$, where n is an integer.

(35) Show by direct substitution that the standing wave given by Eq. 14.62,

$$y = (2\,A \sin k\,x)\ \cos \omega\,t$$

is a solution of the general linear wave Eq. 14.58:

$$\frac{\partial^2 y}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 y}{\partial\,t^2} = 0$$

(36) End A of a string is attached to a vibrator that vibrates with a constant frequency f, while the other end B passes over a pulley to a block of mass m, see Fig. 14.40. The separation L between points A and B is 2.5 m and the linear mass density of the string is 0.1 kg/m. When the mass m of the block is either 16 or 25 kg, standing waves are observed; however, standing waves are not observed for masses between these two values. Take $g = 10\,\text{m/s}^2$. (Hint: The greater the tension in the string, the smaller the number of nodes in the standing wave) (a) What is the frequency of the vibrator? (b) Find the largest m at which a standing wave could be observed.

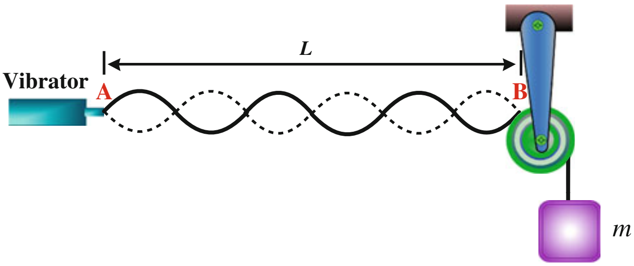


Fig. 14.40 See Exercise (36)

(37) Two identical sinusoidal waves traveling in opposite directions on a string of length $L = 3$ m interfere to produce a standing wave pattern of the form:

$$y = [(0.2\,\text{m})\sin(2\pi x)]\ \cos(20\pi\,t)$$

where x is in meters, t in seconds, and the arguments of the sine and cosine are in radians. (a) How many loops are there in this pattern? (b) What is the fundamental frequency of vibration of the string?

(38) Two strings 1 and 2, each of length $L = 0.5$ m, but different mass densities μ_1 and μ_2, are joined together with a knot and then stretched between two

fixed walls as shown in Fig. 14.41. For a particular frequency, a standing wave is established with a node at the knot, as shown in the figure. (a) What is the relation between the two mass densities? (b) Answer part (a) when the frequency is changed so that the next harmonic in each string is established.

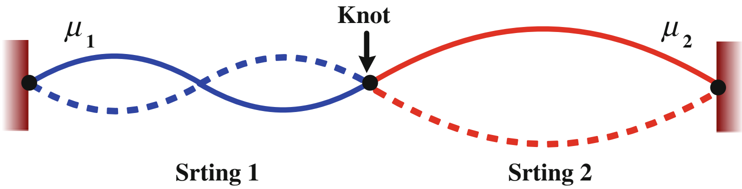


Fig. 14.41 See Exercise (38)

(39) The strings 1 and 2 of exercise 38 have $L_1 = 0.64$ m, $\mu_1 = 1.8$ g/m, $L_2 = 0.8$ m, and $\mu_2 = 7.2$ g/m, respectively, and both are held at a uniform tension $\tau = 115.2$ N. Find the smallest number of loops in each string and the corresponding standing wave frequency.

(40) In the case of the smallest number of loops in exercise 39, determine the total number of nodes and the position of the nodes measured from the left end of string 1.

Sound Waves

15

Sound waves are the most common examples of *longitudinal waves*. The speed of sound waves in a particular medium depends on the properties of that medium and the temperature. As discussed in Chap. 14, sound waves travel through air when air elements vibrate to produce changes in density and pressure along the direction of the wave's motion.

Sound waves can be classified into three frequency ranges:

(1) **Audible waves**: *within the range* of human ear sensitivity and can be generated by a variety of ways such as human vocal cords, etc.

(2) **Infrasonic waves**: *below the audible range* but perhaps within the range of elephant-ear sensitivity.

(3) **Ultrasonic waves**: *above the audible range* and lie partly within the range of dog-ear sensitivity.

15.1 Speed of Sound Waves

The motion of a one-dimensional, longitudinal pulse through a long tube containing undisturbed gas is shown in Fig. 15.1. When the piston is suddenly pushed to the right, the compressed gas (or the change in pressure) travels as a pulse from one region to another toward the right along the pipe with a speed v.

The speed of sound waves depends on the compressibility and density of the medium. We can apply equation $v = \sqrt{\tau/\mu}$, which gives the speed of a transverse wave along a stretched string, to the speed of longitudinal sound waves in fluids or

H. A. Radi and J. O. Rasmussen, *Principles of Physics*,
Undergraduate Lecture Notes in Physics, DOI: 10.1007/978-3-642-23026-4_15,

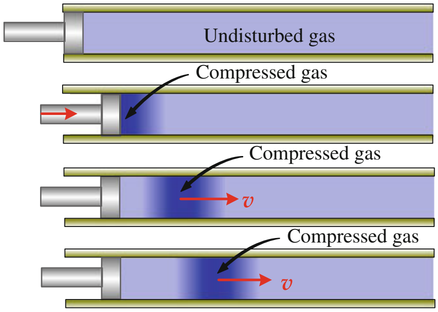


Fig. 15.1 Motion of a longitudinal sound pulse in a gas-filled tube

rods. In fluids we replace τ with the bulk modulus B, and in rods we replace τ with Young's modulus Y. In both, we replace μ with the density ρ. Then:

$$v = \sqrt{\frac{\text{elastic property}}{\text{medium property}}} = \begin{cases} \sqrt{B/\rho} & \text{(In fluids)} \\ \sqrt{Y/\rho} & \text{(In solid rods)} \end{cases} \tag{15.1}$$

Table 15.1 depicts the speed of sound in several different materials.

Table 15.1 The speed of sound in different materials

Medium	v(m/s)	Medium	v(m/s)
Gases		*Solids*	
Oxygen (0 °C)	317	Rubber	1,600
Air (0 °C)	331	Lead	1,960
Air (20 °C)	343	Lucite	2,680
Helium (0 °C)	972	Gold	3,240
Hydrogen (0 °C)	1,286	Brass	4,700
Liquids at (25 °C)		Copper	5,010
Kerosene	1,324	Pyrex	5,640
Mercury	1,450	Iron	5,950
Water	1,493	Granite	6,000
Sea water	1,533	Aluminum	6,420

For sound traveling through air, the relation between the speed and the temperature of the medium is given by the following relation:

$$v = (331\ \text{m/s})\sqrt{1 + \frac{T_C}{273\,°\text{C}}} \tag{15.2}$$

where 331 m/s is the speed of sound at 0 °C, and T_C is the temperature of the medium in degrees Celsius.

Example 15.1

Water at 20 °C has an approximate bulk modulus B of $2.1 \times 10^9\ \text{N/m}^2$ and density ρ of $10^3\ \text{kg/m}^3$. (a) Find the speed of sound in water. (b) Dolphins use sound waves to locate distant food targets by estimating the time Δt between the moment of emitting a sound pulse toward the food and the moment of receiving its reflection, see Fig. 15.2. Calculate such a Δt when the food is 100 m away from the dolphin.

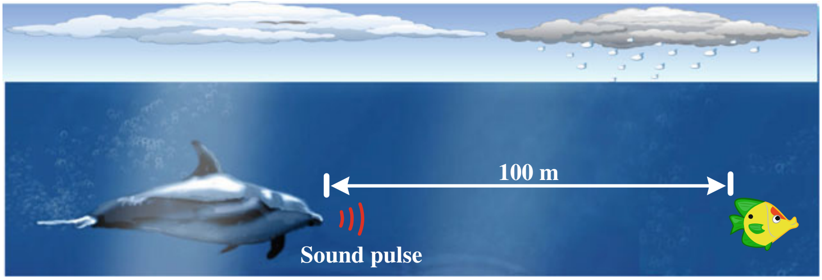


Fig. 15.2

Solution: (a) Using Eq. 15.1, we find that:

$$v = \sqrt{\frac{B}{\rho}} = \sqrt{\frac{2.1 \times 10^9\ \text{N/m}^2}{10^3\ \text{kg/m}^3}} = 1{,}449\ \text{m/s}$$

(b) The total distance traveled by the sound pulse from the dolphin to the food and back to the dolphin is $\Delta x = 2 \times 100\ \text{m} = 200\ \text{m}$. Thus:

$$\Delta t = \frac{\Delta x}{v} = \frac{200\ \text{m}}{1{,}449\ \text{m/s}} = 0.138\ \text{s}$$

15.2 Periodic Sound Waves

As a result of continuous push and pull of a piston in a gas tube, continuous regions of **compressions** and **expansions** (or called **rarefactions**) are generated, see Fig. 15.3a. The darker-colored areas in the figure represent regions where the gas is compressed, and thus the pressure and density are above their equilibrium values. The lighter-colored areas in the same figure represent regions of expansions, where the pressure and density are below their equilibrium values.

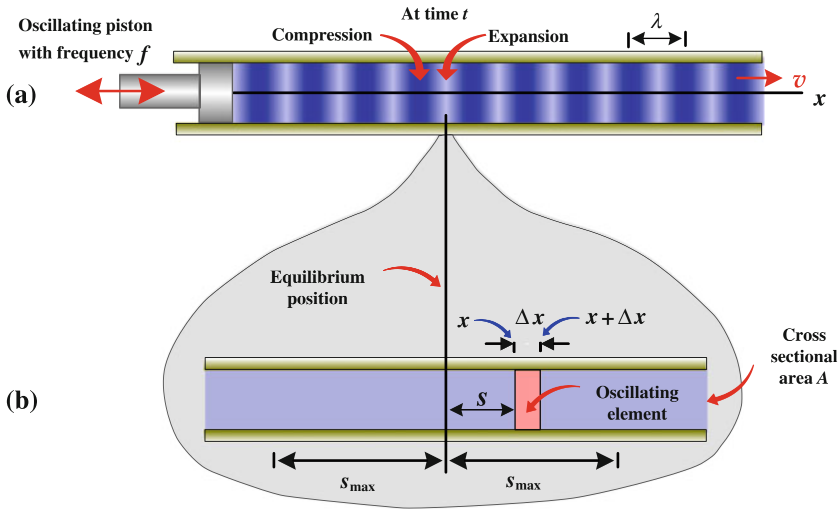


Fig. 15.3 (a) A longitudinal, sinusoidal sound wave is traveling through a long gas-filled tube with a speed v. The wave consists of a moving pattern of compressions and expansions. The wave is shown at an arbitrary time t. (b) An element of thickness Δx is displaced at a distance s to the right from its equilibrium position. Its maximum displacement, either right or left, is s_{max}, where $s_{max} \ll \lambda$

Consider a thin element of air of thickness Δx located at a position x along the tube. As the wave passes through the tube, this element oscillates back and forth in simple harmonic motion about its equilibrium position, see Fig. 15.3b. To describe this element from its equilibrium position, we can use either a sine function or a cosine function. In this book, we use a cosine function of the form:

$$s(x, t) = s_{max} \cos(kx - \omega t) \tag{15.3}$$

where s_{max} is the maximum displacement of the air element to either side of the equilibrium position, see Fig. 15.3b, and is called the **displacement amplitude** of the wave. For this longitudinal sound wave, the wave number k, wavelength λ, angular frequency ω, frequency f, speed v, and period T are all defined and interrelated exactly as for the transverse waves on strings in Sect. 14.3, except that λ is now along the direction of the wave.

For the sinusoidal longitudinal sound wave shown in Fig. 15.4a, the displacement $s(x, t)$ of Eq. 15.3 at $t = 0$ is displayed in Fig. 15.4b. Accordingly, the variation in the gas pressure ΔP about the equilibrium value must also be periodic, see Fig. 15.4c, and based on Eq. 15.3 it must be in the form:

$$\Delta P = \Delta P_{\max} \sin(kx - \omega t) \tag{15.4}$$

where $\Delta P_{\max}$ is the maximum change in pressure from the equilibrium value and is called the **pressure-variation amplitude**, as shown in Fig. 15.4c.

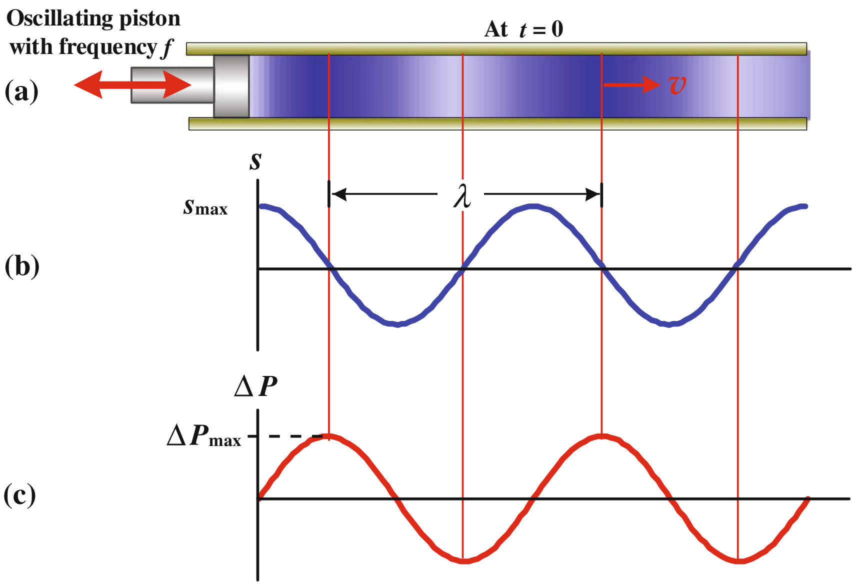


Fig. 15.4 (a) A snapshot at $t = 0$ of a longitudinal sinusoidal sound wave traveling through a long gas-filled tube with a speed v. The variation of both: (b) the displacement amplitude s and (c) the pressure difference ΔP as a function of position

* To find $\Delta P_{\max}$ in Eq. 15.4, we start with the definition of bulk modulus B, given by Eq. 10.14, and express the change in pressure at any time t as follows:

$$\Delta P = -B\frac{\Delta V}{V} \tag{15.5}$$

The quantity V is the volume element, given by:

$$V = A\ \Delta x \tag{15.6}$$

The quantity ΔV is the change in volume that arises from the difference Δs between the displacements of the two faces of the element in Fig. 15.3. That is, $\Delta s = s(x + \Delta x, t) - s(x, t)$. Thus:

$$\Delta V = A \Delta s \tag{15.7}$$

Substituting Eqs. 15.6 and 15.7 into Eq. 15.5 we get:

$$\Delta P = -B \frac{\Delta s}{\Delta x}$$

Allowing for the differential limit, $\Delta x \to 0$ at any time t, we get:

$$\Delta P = -B \frac{\partial s}{\partial x} \tag{15.8}$$

The partial derivative $\partial s/\partial x$ indicates how s changes with x at any time t. Using Eq. 15.3, and treating t as a constant, we find:

$$\frac{\partial s}{\partial x} = \frac{\partial}{\partial x}[s_{\max} \cos(kx - \omega t)] = -k s_{\max} \sin(kx - \omega t) \tag{15.9}$$

Thus:

$$\boxed{\Delta P = B\, k\, s_{\max} \sin(kx - \omega t)} \tag{15.10}$$

Comparing the two Eqs. 15.4 and 15.10, we find that:

$$\Delta P_{\max} = B\, k\, s_{\max} \tag{15.11}$$

Using Eq. 15.1 allows us to eliminate the bulk modulus B and get the following relation:

$$\Delta P_{\max} = \rho v^2 k\, s_{\max} \tag{15.12}$$

Also, we can eliminate k by using $v = \omega/k$, Eq. 14.38, to find:

$$\boxed{\Delta P_{\max} = \rho v \omega\, s_{\max}} \tag{15.13}$$

Example 15.2

The human ear can tolerate the loudest sound which has a pressure-variation amplitude $\Delta P_{max} = 28\,\text{Pa}$ (the *threshold of pain*), and can detect the faintest sound which has $\Delta P_{max} = 2.8 \times 10^{-5}\,\text{Pa}$ (the *threshold of hearing*). For a sound of frequency 1,000 Hz traveling with a speed $v = 343\,\text{m/s}$ in air of density $\rho = 1.21\,\text{kg/m}^3$, calculate the displacement amplitude s_{max} for the loudest and the faintest sounds.

Solution: From Eq. 15.13, we can find the displacement amplitude s_{max} for the loudest sound wave as follows:

$$\begin{aligned} s_{max} &= \frac{\Delta P_{max}}{\rho v \omega} = \frac{\Delta P_{max}}{\rho v (2\pi f)} \\ &= \frac{28\,\text{Pa}}{(1.21\,\text{kg/m}^3)(343\,\text{m/s})(2\pi \times 1{,}000\,\text{Hz})} \\ &= 1.1 \times 10^{-5}\,\text{m} \simeq 11\,\mu\text{m} \qquad \text{(Loudest; threshold of pain)} \end{aligned}$$

The displacement amplitude for the loudest sound that can be tolerated by the human ear is about one-tenth the thickness of this page.

Also, from Eq. 15.13, we find the following for the faintest sound wave:

$$\begin{aligned} s_{max} &= \frac{\Delta P_{max}}{\rho v \omega} = \frac{\Delta P_{max}}{\rho v (2\pi f)} \\ &= \frac{2.8 \times 10^{-5}\,\text{Pa}}{(1.21\,\text{kg/m}^3)(343\,\text{m/s})(2\pi \times 1{,}000\,\text{Hz})} \\ &= 1.1 \times 10^{-11}\,\text{m} \qquad \text{(Faintest; threshold of hearing)} \end{aligned}$$

This is a remarkably small number! This displacement amplitude is about one-tenth the size of a typical atom (diameter $\approx 10^{-10}\,\text{m}$). Indeed, the ear is an extremely sensitive detector for sound waves. On the other hand, the ear can detect a sound-wave pulse whose total energy is about the same as the total energy required to remove an outer electron from a single atom.

15.3 Energy, Power, and Intensity of Sound Waves

In Sect. 14.5, we showed that waves transport kinetic and potential energy when they propagate through a medium. The same concept applies to sound waves. Consider an element of air of mass Δm and length Δx in front of a piston oscillating with a frequency f in *one dimension*, as shown in Fig. 15.5.

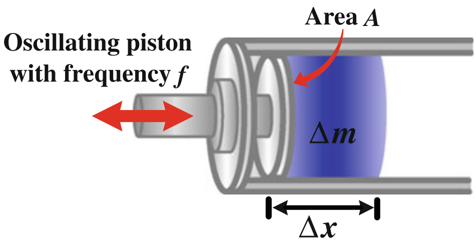


Fig. 15.5 A piston oscillates with frequency f in an air-filled tube. The piston transfers energy to an adjacent air element that has a mass Δm and length Δx, causing it to oscillate with an amplitude $s_{\max}$

The piston transfers energy to this element and hence the energy is propagated away through the tube by the sound wave. As the sound wave propagates away, the displacement of this element with respect to its equilibrium position will be given by Eq. 15.3, i.e.:

$$s(x,t) = s_{\max}\cos(k\,x - \omega\,t) \tag{15.14}$$

The speed of this element can be found by taking the partial time derivative of s as follows:

$$v(x,t) = \frac{\partial}{\partial t}s(x,t) = \frac{\partial}{\partial t}[s_{\max}\cos(k\,x - \omega\,t)] = +\,\omega\,s_{\max}\sin(k\,x - \omega\,t) \tag{15.15}$$

* The kinetic energy ΔK associated with the air element of mass $\Delta m = \rho A \Delta x$, where ρ is the air density, will be given by:

$$\Delta K = \tfrac{1}{2}\Delta m\,v^2 = \tfrac{1}{2}\rho\,A\Delta x\,\omega^2 s_{\max}^2\sin^2(k\,x - \omega\,t) \tag{15.16}$$

When we allow Δx to approach zero, this relation becomes a differential relationship and will take the following form:

$$dK = \tfrac{1}{2}\rho\,A\,\omega^2 s_{\max}^2\sin^2(k\,x - \omega\,t)dx \tag{15.17}$$

At a given instant, let us integrate this expression over all the elements in a complete sound wavelength, which will give us the total kinetic energy K_λ in one wavelength:

$$K_\lambda = \int dK = \tfrac{1}{2}\rho\,A\,\omega^2 s_{\max}^2\int_0^\lambda \sin^2(k\,x - \omega\,t)dx \tag{15.18}$$

If we take a snapshot at time $t = 0$, then we can evaluate the above integral by performing the following steps:

$$\begin{aligned}
\int_{x=0}^{x=\lambda} \sin^2(k\,x)dx &= \tfrac{1}{k}\int_{z=0}^{z=k\lambda=2\pi} \sin^2 z\,dz \\
&= \tfrac{1}{k}\int_0^{2\pi} \tfrac{1}{2}[1-\cos 2z]dz = \tfrac{1}{2k}\left[z-\tfrac{1}{2}\sin 2z\right]_0^{2\pi} \\
&= \tfrac{1}{2k}\left[(2\pi-\tfrac{1}{2}\sin 4\pi)-0\right] = \frac{\lambda}{4\pi}2\pi = \frac{\lambda}{2}
\end{aligned} \tag{15.19}$$

where we have used $z = k\,x$, $\sin^2 z = (1-\cos 2z)/2$ and $k = 2\pi/\lambda$ to arrive to the above result. Of course, we get the same answer if we perform the above steps at any other time different from zero. When we substitute the above result into Eq. 15.18, we get:

$$K_\lambda = \tfrac{1}{4}\rho A\omega^2 s_{max}^2\lambda \tag{15.20}$$

A similar analysis to the total potential energy U_λ in one wavelength will give exactly the same result. Thus:

$$U_\lambda = \tfrac{1}{4}\rho\,A\,\omega^2 s_{max}^2\,\lambda \tag{15.21}$$

The total energy in one wavelength of the sound wave (E_λ) is the sum of the obtained kinetic and potential energies. Thus:

$$E_\lambda = K_\lambda + U_\lambda = \tfrac{1}{2}\rho\,A\,\omega^2 s_{max}^2\,\lambda \tag{15.22}$$

As the sinusoidal sound wave travels along the tube, this amount of energy (E_λ) will cross any given point in the tube during a time interval equal to one period of the oscillation. Thus, the rate of energy (power) transferred by the sound wave through the air is:

$$\mathcal{P} = \frac{\Delta E}{\Delta t} = \frac{E_\lambda}{T}$$

where we used the symbol $\mathcal{P}$ for the power in this section to avoid confusion with the symbol P for pressure. Therefore:

$$\mathcal{P} = \tfrac{1}{2}\rho\,A\,\omega^2 s_{max}^2\,\frac{\lambda}{T}$$

Using the relation $v=\lambda/T$, given by Eq. 14.38, we finally reach the following power form:

$$\boxed{\mathcal{P}=\tfrac{1}{2}\rho\, A\, v\, \omega^2 s_{max}^2} \tag{15.23}$$

Thus, the power of a periodic sound wave is proportional to the square of the angular frequency and the square of the displacement amplitude (as in the case of periodic string waves).

For a wave crossing a particular surface, we define its **intensity** I as the power per unit area, or the rate of energy transfer (power $\mathcal{P}$) of the wave through a unit area perpendicular to the direction of the propagation of the wave, i.e. $I=\mathcal{P}/A$. Therefore:

$$\boxed{I=\frac{\mathcal{P}}{A}} \quad \Rightarrow \quad \boxed{I=\tfrac{1}{2}\rho v\omega^2 s_{max}^2} \tag{15.24}$$

By using Eq. 15.13, $\Delta P_{max}=\rho v\omega\, s_{max}$, the last relation can be written in terms of the pressure amplitude ΔP_{max} as:

$$I=\frac{\Delta P_{max}^2}{2\rho v} \tag{15.25}$$

On the other hand, we can express the pressure amplitude ΔP_{max} in terms of the measurable quantities ρ, v, and I as follows:

$$\boxed{\Delta P_{max}=\sqrt{2\rho v I}} \tag{15.26}$$

In three dimensions, we consider a point source S emitting sound waves uniformly in all directions as **spherical waves**, see Fig. 15.6. When we construct an imaginary sphere of radius r centered at the sound source, the power emitted by this source must be distributed uniformly over this spherical surface, which has an area $4\pi r^2$.

From the definition of the intensity, $I=\mathcal{P}/A$, given by Eq. 15.24, the intensity I at any point on the spherical surface will be given by:

$$\boxed{I=\frac{\mathcal{P}}{4\pi r^2}} \tag{15.27}$$

This equation is known as the **inverse square law** and tells us that the intensity of sound waves emitted from an isotropic point source decreases with the square of the distance r from the source, i.e. the intensity is inversely proportional to the square of the distance r.

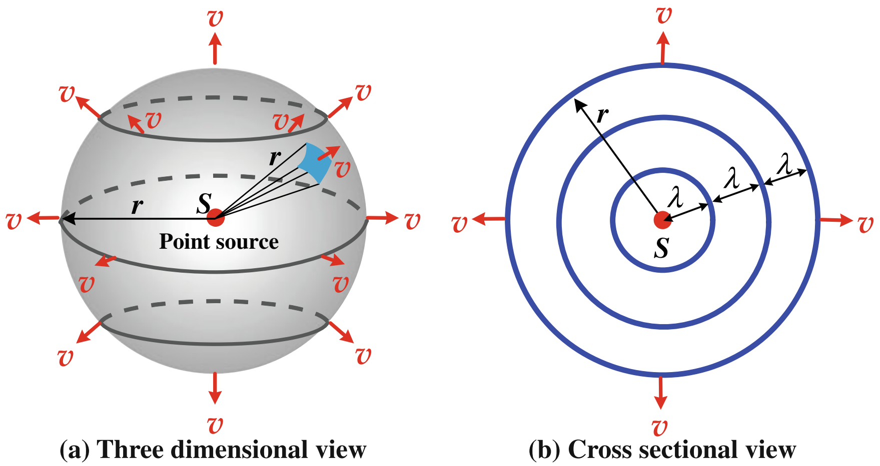


Fig. 15.6 (a) A point source S emitting sound waves uniformly in all directions with the waves passing through an imaginary sphere of radius r. (b) A cross-sectional view showing the wavelength λ between consecutive crests of the sound waves

Example 15.3

At a frequency of 1,000 Hz, the human ear can detect the loudest and faintest sounds with intensities of about $1.0\ \text{W/m}^2$ and $1.0 \times 10^{-12}\ \text{W/m}^2$, respectively. For sound waves traveling with a speed of $v = 343\ \text{m/s}$, find the pressure amplitude ΔP_{max} for the faintest and the loudest sound waves, assuming the air's density is $\rho = 1.21\ \text{kg/m}^3$.

Solution: From Eq. 15.26, we can find the pressure amplitude ΔP_{max} for the loudest sound waves as follows:

$$\begin{aligned}\Delta P_{max} &= \sqrt{2\rho v I} = \sqrt{2(1.21\ \text{kg/m}^3)(343\ \text{m/s})(1\ \text{W/m}^2)} \\ &= 28.8\ \text{N/m}^2 = 28.8\ \text{Pa} \qquad \text{(Loudest; threshold of pain)}\end{aligned}$$

Also, from Eq. 15.26, we find the pressure amplitude ΔP_{max} for the faintest sound waves as follows:

$$\begin{aligned}\Delta P_{max} &= \sqrt{2\rho v I} = \sqrt{2(1.21\ \text{kg/m}^3)(343\ \text{m/s})(1 \times 10^{-12}\ \text{W/m}^2)} \\ &= 2.88 \times 10^{-5}\ \text{N/m}^2 \\ &= 2.88 \times 10^{-5}\ \text{Pa} \qquad \text{(Faintest; threshold of hearing)}\end{aligned}$$

Example 15.4

A point source emits sound waves with a power of 50 W. (a) Find the intensity of the sound waves 2 m away from the source. (b) Find the distance at which the intensity of the sound is 10^{-6} W/m^2.

Solution: (a) The point source S shown in Fig. 15.7. emits energy in the form of spherical sound waves centered at the source. Thus, when using Eq. 15.27 we find that:

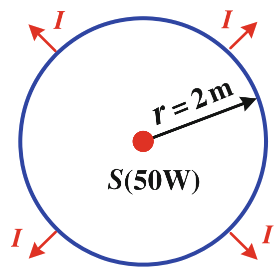
$$I = \frac{\mathcal{P}}{4\pi r^2} = \frac{50\,\text{W}}{4\pi(2\,\text{m})^2} = 0.995\,\text{W/m}^2$$

which is close to the intensity of the threshold of pain, see Example 15.3.

(b) Expressing r in Eq. 15.27 in terms of $\mathcal{P}$ and I, we obtain:

$$r = \sqrt{\frac{\mathcal{P}}{4\pi I}} = \sqrt{\frac{50\,\text{W}}{4\pi(10^{-6}\,\text{W/m}^2)}} = 1{,}995\,\text{m} \simeq 2\,\text{km}$$

Fig. 15.7



15.4 The Decibel Scale

According to Example 15.2, the displacement amplitude s_{max} for the human ear ranges from about 10^{-5} m for the loudest tolerable sound to about 10^{-11} m for faintest detectable sound, a ratio of 10^6. From Eq. 15.24, we see that the intensity I varies as the square of s_{max}, so the ratio of intensities at these two limits of the human audibility is 10^{12}. This goes to show that the human ear can accommodate an enormous range of intensities.

We can better represent large ranges of I by using logarithms. Now, consider the following logarithmic relation of the base 10:

$$y = \log x$$

It is usual to suppress explicit references to the base 10 (such as $\log_{10} x$) and instead write $\log x$. If x in this equation is multiplied by 10, then y increases by 1, i.e.:

$$y' = \log 10\,x = \log 10 + \log x = 1 + \log x = 1 + y$$

Similarly, if we multiply x by 10^{12}, y increases only by 12.

Consequently, instead of speaking of intensity I of a sound wave, it is much more convenient to speak of its **sound level** β (Greek beta), defined by:

$$\boxed{\beta = (10\,\text{dB})\log\frac{I}{I_\circ}} \tag{15.28}$$

Here dB is the abbreviation for **decibel**, the unit of sound level, a name chosen to recognize the work of Alexander Graham Bell. The constant $I_\circ$ in Eq. 15.28 is the *reference intensity*, taken to be near the threshold of hearing, i.e. $I_\circ = 1.0 \times 10^{-12}\ \text{W/m}^2$. The intensity I in the same equation is measured in watts per square meter.

On this scale, the threshold of hearing ($I = 1.0 \times 10^{-12}\ \text{W/m}^2$) corresponds to a sound level of:

$$\begin{aligned}\beta &= (10\,\text{dB})\log\frac{I}{I_\circ} = (10\,\text{dB})\log\frac{1.0\times 10^{-12}\ \text{W/m}^2}{1.0\times 10^{-12}\ \text{W/m}^2}\\ &= (10\,\text{dB})\log 1\\ &= 0\,\text{dB} \qquad \text{(Threshold of hearing)}\end{aligned}$$

So our threshold of hearing level corresponds to zero decibel. Also, the threshold of pain ($I = 1.0\ \text{W/m}^2$) corresponds to a sound level of:

$$\begin{aligned}\beta &= (10\,\text{dB})\log\frac{I}{I_\circ} = (10\,\text{dB})\log\frac{1.0\ \text{W/m}^2}{1.0\times 10^{-12}\ \text{W/m}^2}\\ &= (10\,\text{dB})\log 10^{12} = (10\,\text{dB}) \times 12\\ &= 120\,\text{dB} \qquad \text{(Threshold of pain)}\end{aligned}$$

In general, $\beta = 10 \times n\,\text{dB}$ corresponds to an intensity that is 10^n times the reference intensity, i.e. corresponds to $I = 10^n\, I_\circ = 10^{n-12}\ \text{W/m}^2$. Table 15.2 lists some sound-level values for some environments.

Table 15.2 Approximate sound levels (dB) for several sources

Source of sound	β(dB)	I(W/m^2)
Threshold of hearing in human auditory system	0	10^{-12}
Quiet rustling leaves, calm human breathing	10	10^{-11}
Very calm room	20	10^{-10}
Whispering	30	10^{-9}
Mosquito buzzing	40	10^{-8}
Normal talking (1 m distant)	50	10^{-7}
TV set—typical home level, 1 m distant	60	10^{-6}
Vacuum cleaner	70	10^{-5}
Traffic noise for a main road, 10 m distant	80	10^{-4}
Machine gun	90	10^{-3}
Jack hammer, 1 m distant	100	10^{-2}
Jet engine, 100 m distant	110	10^{-1}
Threshold of pain in human auditory system	120	1

Example 15.5

(a) Find the sound level in decibels for a sound wave of intensity 1.59×10^{-5} W/m^2.
(b) Find the sound intensity of a source rated at a 35 dB sound level.

Solution: (a) From Eq. 15.28, we find that:

$$\begin{aligned}\beta &= (10\,\text{dB})\log\frac{I}{I_\circ}\\ &= (10\,\text{dB})\log\frac{1.59\times10^{-5}\ \text{W/m}^2}{1.0\times10^{-12}\ \text{W/m}^2}\\ &= (10\,\text{dB})\log 1.59\times10^{7}\\ &= 72\,\text{dB}\end{aligned}$$

(b) Substituting in Eq. 15.28 with $\beta = 35$ dB, dividing both sides by 10, and taking the antilog of both sides, we can find I by performing the following steps:

$$\beta = (10\,\text{dB})\log\frac{I}{I_\circ}$$

$$35\,\text{dB} = (10\,\text{dB})\log\frac{I}{I_\circ}$$

$$3.5 = \log\frac{I}{I_\circ}$$

$$\text{antilog}(3.5) = \text{antilog}\left(\log\frac{I}{I_\circ}\right)$$

$$10^{3.5} = \frac{I}{I_\circ}$$

$$I = 10^{3.5} \times I_\circ$$

Thus, with the reference intensity $I_\circ = 1.0 \times 10^{-12}\ \text{W/m}^2$, we find that:

$$I = 10^{3.5} \times 1.0 \times 10^{-12}\ \text{W/m}^2$$
$$= 3.16 \times 10^{-9}\ \text{W/m}^2$$

Example 15.6

Two identical point sources, S_1 and S_2, have the same power and driven by one oscillator. The positions of the two sources relative to an observer is depicted in Fig. 15.8. The sound intensity at the observer's location from S_2 is found to be $I_2 = 3.0 \times 10^{-6}\ \text{W/m}^2$. (a) Find the total intensity of the combined sound waves that is received by the observer from the two sources. (b) Find the difference in sound level when the two sources operate simultaneously and when only the second source operates.

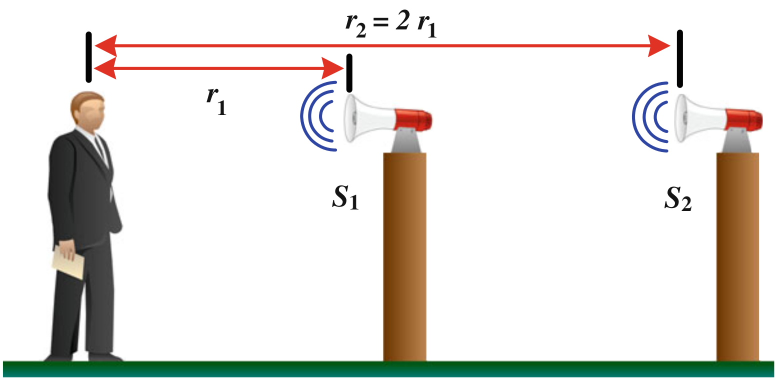


Fig. 15.8

Solution: (a) If I_1 and I_2 are the intensities received by the observer from the point sources S_1 and S_2, respectively, then their ratio will be:

$$\frac{I_1}{I_2} = \left[\frac{\mathcal{P}}{4\pi r_1^2}\right] \Big/ \left[\frac{\mathcal{P}}{4\pi r_2^2}\right] = \frac{r_2^2}{r_1^2} = \frac{(2r_1)^2}{r_1^2} = 4 \quad \Rightarrow \quad I_1 = 4\, I_2$$

This means that the intensity I_1 from S_1 is four times the intensity I_2 from S_2. Thus, the total intensity becomes:

$$I_{tot} = I_1 + I_2 = 4I_2 + I_2 = 5I_2 = 5(3.0 \times 10^{-6}\,\mathrm{W/m^2}) = 1.5 \times 10^{-5}\,\mathrm{W/m^2}$$

(b) If β_2 is the sound level when only the second source operates and β_{tot} is the sound level when both sources operate together, then:

$$\beta_2 = (10\,\mathrm{dB})\log\frac{I_2}{I_\circ} \quad \text{and} \quad \beta_{tot} = (10\,\mathrm{dB})\log\frac{I_{tot}}{I_\circ} = (10\,\mathrm{dB})\log\frac{5I_2}{I_\circ}$$

Then: $\Delta\beta = \beta_{tot} - \beta_2 = (10\,\mathrm{dB})\log\dfrac{5I_2}{I_\circ} - (10\,\mathrm{dB})\log\dfrac{I_2}{I_\circ} = (10\,\mathrm{dB})\log\dfrac{5I_2}{I_2} = 7\,\mathrm{dB}$

15.5 Hearing Response to Intensity and Frequency

The threshold of hearing in the human auditory system depends on the intensity of the sound (or the sound level in dB) and its frequency. We learned in the previous section that the threshold of hearing at 1,000 Hz requires an intensity of 10^{-12} W/m^2 and corresponds to a sound level of 0 dB. Conversely, at 100 Hz sound must have an intensity level of about 30 dB to be barely audible.

Figure 15.9 maps the sound regions that humans can respond to for a range of sound levels β (or intensity I) and sound frequencies f. Tentatively, the figure also overlays some sample sources. The lower blue curve of the white area shows the dependence of the threshold of hearing β on the frequency. This curve indicates that humans are sensitive to frequencies ranging from about 20 to 20,000 Hz. The upper bound to the white area is the threshold of pain, and does not depend much on frequency. The lower left region of the white area shows that our ears are particularly insensitive to low frequencies and low intensity levels.

15.6 The Doppler Effect

We move to a different phenomenon that applies to all kinds of waves, not only sound waves. You most probably have noticed that when a car *moves toward* you with a high speed and horns, you hear the horn with a *higher frequency* than when the car is at rest. Contrary wise, when the car *moves away*, you hear the horn with a *lower frequency*. This phenomenon is called the *Doppler effect*.

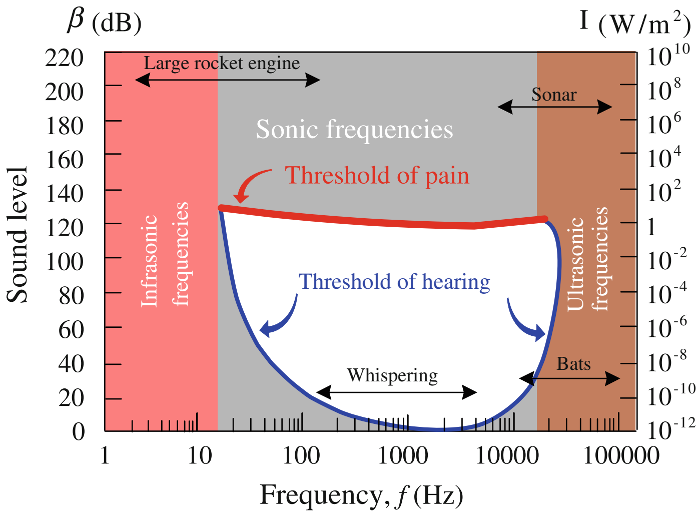


Fig. 15.9 The dependence of the sound level β on the frequency f for normal human hearing (the *white area*) and various sources

Let us now examine this phenomenon quantitatively. First, we consider a point source that emits sound waves radially in all directions in a uniform medium. It is useful to represent the emitted waves using a series of concentric spheres with the source located at their centers. Each sphere represents a wave **crest**, and it moves away from the source with the speed of sound. We call such a sphere of constant phase a **wave front**. Therefore, the distance between any two successive wave fronts equals the wavelength λ of the sound wave and has a frequency f and speed v. In our analyses that follow, we restrict ourselves to the motion of a sound source S and observer O along the line joining them.

Moving Observer and Stationary Source

Figure 15.10 shows an observer O (represented by an ear) moving with a speed v_o toward a stationary source S that emits spherical sound waves of speed $v(v > v_o)$, wavelength λ, and frequency f. The frequency detected by the observer O is the rate at which O intercepts successive wave fronts (or wavelengths).

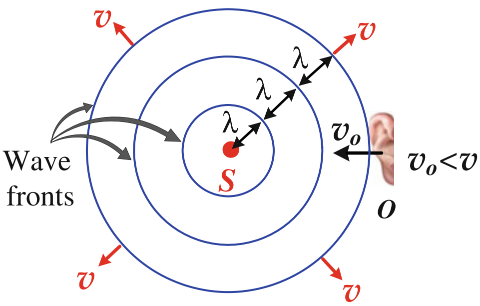


Fig. 15.10 A stationary sound source S emits spherical wave fronts (each is one wavelength λ from the next) with a speed v. An observer O (represented by an ear) moves with a speed v_O towards the source

If the observer O were stationary, the interception rate of wave fronts would be f. But if the observer O is moving toward the source S, then the interception rate f' is greater than f.

When the observer O moves with a speed v_O *toward* a stationary source S, the speed of the wave fronts *relative* to O is not v, but $v' = v + v_O$, while the wavelength λ is unchanged. When we apply the general relation $v = \lambda f$ to this case, i.e. $v' = \lambda f'$, we find that the frequency f heard by the observer has the following relation:

$$f' = \frac{v'}{\lambda} = \frac{v + v_O}{\lambda} = \frac{v + v_O}{v/f} \tag{15.29}$$

This relation can be rewritten as:

$$\boxed{f' = \left(1 + \frac{v_O}{v}\right) f} \qquad (O \text{ is moving towards } S) \tag{15.30}$$

When the observer O moves with a speed v_O *away* from a stationary source S, the speed of the wave fronts *relative* to O is not v, but $v' = v - v_O$, while the wavelength λ is unchanged. Steps similar to those above lead to the frequency heard by the observer as:

$$\boxed{f' = \left(1 - \frac{v_O}{v}\right) f} \qquad (O \text{ is moving away from } S) \tag{15.31}$$

Generally, for an observer O moving with a speed v_O relative to a stationary source S, a *positive sign* is used when O moves *toward* S and a *negative sign* is used when O moves *away from* S. Thus:

$$\boxed{f' = \left(1 \pm \frac{v_O}{v}\right) f} \qquad \left\{ \begin{array}{l} + \text{ when } O \text{ is moving towards } S \\ - \text{ when } O \text{ is moving away from } S \end{array} \right\} \tag{15.32}$$

Moving Source and Stationary Observer

Figure 15.11 shows a source S moving with a speed v_S toward an observer O while emitting spherical sound waves of speed v, wavelength λ, and frequency f. The figure indicates that the wave fronts detected by the observer O are closer together than they would be if the source S was not moving. Thus, the wavelength λ' measured by the observer O is shorter than the wavelength λ of the source S.

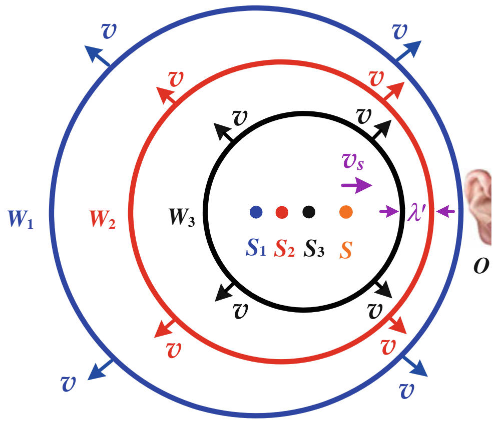


Fig. 15.11 A moving sound source S emits spherical wave fronts with a speed v and wavelength λ, while moving with a speed $v_S < v$ towards a stationary observer O. The wave front W_1 that arises from the source when it was at point S_1..., etc is shown

During a period T, the source S emits a wave front that moves a distance λ with a speed v, while the source itself moves a distance $v_S\, T$ before emitting the next wave front. Thus, the wavelength λ is shortened by $v_S\, T$. Then, the observed wavelength λ' will be:

$$\lambda' = \lambda - v_S\, T = \frac{v}{f} - \frac{v_S}{f} \tag{15.33}$$

Using $v = \lambda\, f$ in this case, i.e. $v = \lambda' f'$, we find that the frequency f' that is heard by the observer O is related to f as follows:

$$\frac{v}{f'} = \frac{v}{f} - \frac{v_S}{f} = \frac{1}{f}\,(v - v_S) \tag{15.34}$$

This relation can be rewritten as:

$$\boxed{f' = \left(\frac{1}{1 - v_S/v}\right) f} \qquad (S \text{ is moving towards } O) \tag{15.35}$$

When the source S is moving with a speed v_S *away* from a stationary observer O, the wavelength λ is increased by $v_S\, T$. Therefore, the observed wavelength λ' will be given by:

$$\lambda' = \lambda + v_S\, T = \frac{v}{f} + \frac{v_S}{f} \tag{15.36}$$

With similar steps to those of Eqs. 15.33–15.35, we get:

$$\boxed{f' = \left(\frac{1}{1 + v_S/v}\right) f} \qquad (S \text{ is moving away from } O) \tag{15.37}$$

Generally, for a source S moving with a speed v_S relative to a stationary observer O, a *negative sign* is used when S moves *toward* O and a *positive sign* is used when S moves *away from* O. Thus:

$$\boxed{f' = \left(\frac{1}{1 \mp v_S/v}\right) f} \qquad \left\{\begin{array}{l} - \text{ when } S \text{ is moving towards } O \\ + \text{ when S is moving away from } O \end{array}\right\} \tag{15.38}$$

One can find a generalized relation that includes all collinear motion of a source with speed v_S and an observer with speed v_o to be:

$$\boxed{f' = \left(\frac{1 \pm v_o/v}{1 \mp v_S/v}\right) f} \qquad (\text{General Doppler effect}) \tag{15.39}$$

The upper signs in the numerator and denominator ($+v_o/v$ and $-v_S/v$) refer to motion of one *toward* the other, while the lower signs ($-v_o/v$ and $+v_S/v$) refer to motion of one *away from* the other.

Spotlight
You can determine the signs in Eq. 15.39 by remembering that: the word *toward* is associated with an *increase* in observed frequency, while the word *away from* is associated with a *decrease* in observed frequency.

Example 15.7

A car moves on a straight road with a speed of 20 m/s. Its siren emits a sound with a frequency of 500 Hz. Find the frequencies heard by a stationary person on the sidewalk when the car approaches him (Fig. 15.12a) and then when it recedes from him (Fig. 15.12b). Assume collinear motion of the source and observer.

Solution: When the car *approaches* the observer, we use the upper signs in the numerator and denominator of the general formula of Doppler effect given by Eq. 15.39. In this formula, we take $v_o = 0$ for the stationary observer, $v_S = 20$ m/s for the speed of the car, $v = 343$ m/s for the speed of sound in air, and $f = 500$ Hz for the siren frequency. Thus, the frequency f' heard by the observer will be:

$$f' = \left(\frac{1 + v_O/v}{1 - v_S/v}\right) f = \left(\frac{1+0}{1-(20\,\text{m/s})/(343\,\text{m/s})}\right) \times 500\,\text{Hz}$$
$$= 531\,\text{Hz}$$

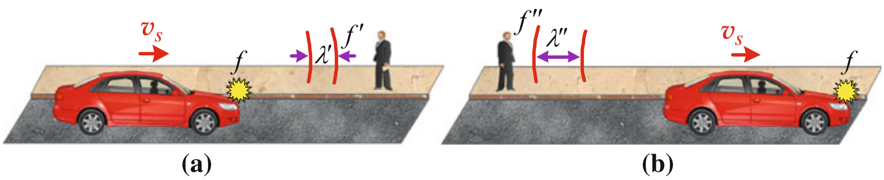


Fig. 15.12

When the car recedes from the observer, we use the lower signs in the numerator and denominator of Eq. 15.39. Thus:

$$f'' = \left(\frac{1 - v_O/v}{1 + v_S/v}\right) f = \left(\frac{1-0}{1+(20\,\text{m/s})/(343\,\text{m/s})}\right) \times 500\,\text{Hz}$$
$$= 473\,\text{Hz}$$

The change in frequency detected by the stationary observer is:

$$\Delta f = f' - f'' = 531\,\text{Hz} - 473\,\text{Hz} = 58\,\text{Hz}$$

This is about 8.6% of the actual frequency emitted from the siren.

Example 15.8

Submarines use sound propagation under water to navigate, communicate, or detect other objects; this technique is known as sonar (**SO**und **NA**vigation and **R**anging). A submarine 1 (sub 1) moves with a speed $v_1 = 10$ m/s and emits a sonar wave of frequency $f = 1{,}500$ Hz. A second submarine 2 (sub 2) moves directly towards the first one with a speed $v_2 = 8$ m/s. See Fig. 15.13, and take the speed of sound in water to be 1,533 m/s. (a) Find the frequency detected by an observer in sub 2. (b) Find the reflected frequency detected by an observer in sub 1. (c) Find the frequency detected by an observer in sub 2 when the two submarines miss each other and pass.

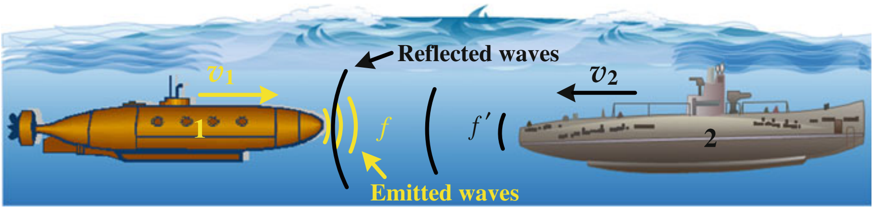


Fig. 15.13

Solution: (a) When the submarines move *toward* each other, we use the *upper signs* in the numerator and denominator of Eq. 15.39. Then we take:

$$\begin{aligned} v_o &= v_2 = 8\,\text{m/s for observer (sub 2)} \\ v_S &= v_1 = 10\,\text{m/s for the speed of the source (sub 1)} \\ v &= 1{,}533\,\text{m/s for the speed of sound in water,} \\ f &= 1{,}500\,\text{Hz for the emitted frequency from the source (sub 1)} \end{aligned}$$

Thus, the frequency f' received by an observer in sub 2 will be:

$$\begin{aligned} f' &= \left(\frac{1 + v_o/v}{1 - v_S/v}\right) f \\ &= \left(\frac{1 + (8\,\text{m/s})/(1{,}533\,\text{m/s})}{1 - (10\,\text{m/s})/(1{,}533\,\text{m/s})}\right) \times 1{,}500\,\text{Hz} = 1{,}518\,\text{Hz} \end{aligned}$$

(b) The frequency calculated in part (a) will be reflected from sub 2 (which acts as a moving source) and then be detected by sub 1 (the moving observer). In this case we take:

$$\begin{aligned} v_o &= v_1 = 10\,\text{m/s for observer (sub 1)} \\ v_S &= v_2 = 8\,\text{m/s for the speed of the source (sub 2)} \\ v &= 1{,}533\,\text{m/s for the speed of sound in water,} \\ f' &= 1{,}518\,\text{Hz for the emitted frequency from the source (sub 2)} \end{aligned}$$

Thus, the frequency f'' received by an observer in sub 1 will be:

$$f'' = \left(\frac{1 + v_o/v}{1 - v_S/v}\right) f' = \left(\frac{1 + (10\,\text{m/s})/(1{,}533\,\text{m/s})}{1 - (8\,\text{m/s})/(1{,}533\,\text{m/s})}\right) \times 1{,}518\,\text{Hz} = 1{,}536\,\text{Hz}$$

(c) When the submarines move *away from* each other, we use the *lower signs* in the numerator and denominator of Eq. 15.39. All the parameters used in this equation will be identical to the one in part (a). Thus, the frequency f' received by an observer in sub 2 will be:

$$f' = \left(\frac{1 - v_o/v}{1 + v_S/v}\right) f = \left(\frac{1 - (8\,\text{m/s})/(1{,}533\,\text{m/s})}{1 + (10\,\text{m/s})/(1{,}533\,\text{m/s})}\right) \times 1{,}500\,\text{Hz} = 1{,}483\,\text{Hz}$$

15.7 Supersonic Speeds and Shock Waves

When a source moves toward a stationary object with a speed equal to the speed of sound, i.e. when $v_o = 0$ and $v_S = v$, Eq. 15.39 predicts that $f' = (1+0)/(1-1)f = \infty$, which means that f' will be infinitely great. This also means that the source is moving as fast as its generated spherical wave fronts, as suggested by Fig. 15.14. Then the gas molecules pile up at what is called the shock front.

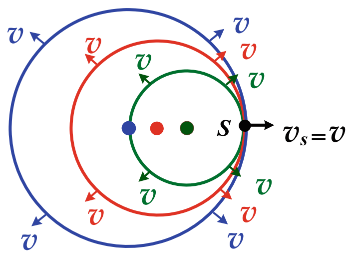


Fig. 15.14 A source of sound that moves at the speed of sound

Now what happens when v_S exceeds v? For such supersonic speeds, Eq. 15.39 predicts a negative f' and hence no longer applies. In such case, the speed of the source is faster than the speed of the wave fronts as shown in Fig. 15.15 for various source positions.

At $t = 0$, the source is at point S_0 and at a later time t, the source is at point S_t, see Fig. 15.15. At that instant, the radius of the wave front W_0 which originated when the source was at point S_0 is vt. In the same time interval, the source travels a greater distance $v_S\ t$ to the point S_t. The radius of any wave front is v multiplied by the elapsed time since the source emitted the wave front. The tangent line drawn from point S_t to the wave front centered at point S_0 is the tangent of all other wave fronts generated at intermediate times. The envelope to all of these wave fronts is a cone called the **Mach cone**. This conical wave front is known as a **shock wave** because it is the accumulation of all wave fronts and hence is causing an abrupt increase followed by a decrease of air pressure and then back to normal. The loud sound produced by this shock wave is known as a **sonic boom**.

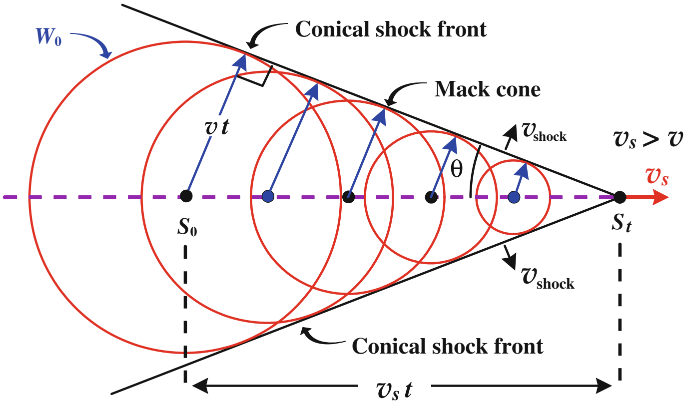


Fig. 15.15 A source of sound that moves with a speed v_S greater than the speed of sound v. All the spherical wave fronts expand at the speed of sound v and assemble along the surface of a cone called the Mach cone, forming a shock wave

The Mach cone has an apex half-angle θ (called the *Mach angle*):

$$\sin\theta = \frac{vt}{v_S\, t} = \frac{v}{v_S} \qquad \text{(Mach cone half-angle)} \qquad (15.40)$$

The ratio v_S/v is called the Mach number. When you hear that a jet plane has flown at Mach 3, it means that its speed v_S was 3 times the speed of sound ($v = 343$ m/s). With this supersonic speed, the jet plane generates a shock wave which produces a loud sound (*sonic boom*).

Example 15.9

A supersonic jet travels horizontally at Mach 2.5. At time $t = 0$, the jet is over a person's head at an altitude $h = 10$ km. (a) Where will the jet be before the ground observer hears the boom of the shock wave? (b) How long will the person wait before hearing that boom?

Solution: Figure 15.16 shows a sketch of the Mach cone at time $t = 0$, when the jet is just above the person's head. In addition, the figure shows the instant at time t when the person hears the *sonic boom*.

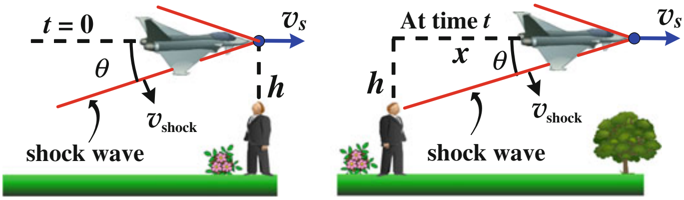


Fig. 15.16

(a) The half-angle of the shock wave cone can be obtained as follows:

$$\sin\theta = \frac{v}{v_S} = \frac{1}{2.5} = 0.4 \quad \Rightarrow \quad \theta = \sin^{-1} 0.4 \simeq 23.6^\circ$$

From the figure's geometry, we can find the distance x as follows:

$$\tan\theta = \frac{h}{x} \quad \Rightarrow \quad x = \frac{h}{\tan\theta} = \frac{10{,}000\,\text{m}}{\tan 23.6^\circ} = 22{,}889\,\text{m} = 22.9\,\text{km}$$

(b) The time the person will wait before hearing the sonic boom is:

$$t = \frac{x}{v_S} = \frac{x}{2.5v} = \frac{22{,}889\,\text{m}}{2.5 \times (343\,\text{m/s})} = 26.7\,\text{s}$$

15.8 Exercises

Section 15.1 Speed of Sound Waves

(1) Find the speed of sound in air when the temperature is 35 °C.

(2) The bulk modulus B and density ρ of mercury at 40 °C are 2.4×10^9 Pa and $13.45 \times 10^3\,\text{kg/m}^3$, respectively. Calculate the speed of sound in mercury at this temperature.

(3) Find the speed of sound in a steel rod that has a Yang's modulus $Y = 2 \times 10^{11}\,\text{N/m}^2$ and density $\rho = 7.8 \times 10^3\,\text{kg/m}^3$.

(4) A steel rod that has a Yang's modulus $Y = 2 \times 10^{11}\,\text{N/m}^2$, density $\rho = 7.8 \times 10^3\,\text{kg/m}^3$, and length $L = 100$ m is struck at one end. A person at the other end hears two sounds as a result of the propagation of two longitudinal waves, one that traveled through the rod and the other that traveled through the air at 20 °C. What is the time interval between the two sounds?

(5) The speed of a longitudinal wave in an adiabatic process is written as $v = \sqrt{B_{ad}/\rho}$, where $B_{ad} = -V\,dP/dV$ as given by Eq. 10.14. In the case of

an ideal gas, the relation between the pressure P and volume V during an adiabatic process is given by $PV^{\gamma} = \text{constant}$, where γ is the ratio of the heat capacity at constant pressure to the heat capacity at constant volume. (a) Show that $B_{\text{ad}} = \gamma P$ for an ideal gas. (b) Show that the speed of a longitudinal wave in the adiabatic process of an ideal gas is given by $v = \sqrt{\gamma RT/M}$, where R is the universal gas constant, T is the Kelvin temperature, and M is the molecular mass of the gas.

(6) Hydrogen is a diatomic gas with molecular mass $M = 2\,\text{kg/kmol}$ and $\gamma = 1.41$. Find the speed of sound in hydrogen gas at $27\,^{\circ}\text{C}$.

(7) The auto-focusing mechanism of old cameras used to depend on the camera sending a high frequency ultrasonic sound pulse toward the object being photographed. The camera would calculate the time that the pulse would take from the moment it left the camera to the moment it was detected by the camera's sensor. Based on the travel time of such a pulse, the camera would adjust its lens automatically. If the speed of sound in air is 343 m/s, find the travel time of a pulse for an object: (a) 1.5 m away, and (b) 5 m away.

(8) A fishing boat emits an ultrasonic pulse vertically toward the sea bed. Then pulse is received 1.5 s after being reflected from the ocean floor. If the speed of sound in sea water is 1,560 m/s, how far down is the ocean floor from the boat's location?

(9) On a warm summer day ($32.3\,^{\circ}\text{C}$), a boy drops a stone from the top of a cliff. Using his stopwatch, he finds that it took 20.9 s from the moment he dropped that stone until the moment he hears the sound of the splash that the stone makes with the surface of the water below. Take $g = 9.8\,\text{m/s}^2$. How high is the cliff?

Section 15.2 Periodic Sound Waves

(10) The pressure variation in a periodic sound wave is given by:

$$\Delta P = (2\,\text{Pa}) \sin \pi [(2\,\text{m}^{-1})x - (686\,\text{s}^{-1})t]$$

(a) Find the pressure-variation amplitude. (b) Find the wavelength and frequency of the pressure wave. (c) Find the speed of the pressure wave.

(11) A sinusoidal sound wave has the following displacement:

$$s(x, t) = (4\,\mu\text{m}) \cos[(20\,\text{m}^{-1})x - (6860\,\text{s}^{-1})t]$$

(a) Find the displacement amplitude, wavelength, frequency, and speed of the wave. (b) Find the value of the displacement of an element of air at the position $x = 2$ mm at time $t = 2$ ms. (c) Find the maximum speed of this oscillating element.

(12) In homogenous air of density $\rho = 1.21\,\text{kg/m}^3$ a sinusoidal periodic sound wave has a wavelength $\lambda = 0.2$ m, speed $v = 343$ m/s, and pressure-variation amplitude $\Delta P_{max} = 0.5$ Pa. (a) Show that the function that describes the pressure-variation depends on position x and time t according to the following expression:

$$\Delta P = (0.5\,\text{Pa}) \sin \pi [(10\,\text{m}^{-1})x - (3{,}430\,\text{s}^{-1})t]$$

(b) Show that the function that describes the displacement of an element of air is governed in position and time by the following expression:

$$s(x, t) = (0.112\,\mu\text{m}) \cos \pi [(10\,\text{m}^{-1})x - (3{,}430\,\text{s}^{-1})t]$$

(13) To generate a sound wave of speed $v = 343$ m/s and displacement amplitude $s_{max} = 5.5\,\mu$m in air of density $\rho = 1.2\,\text{kg/m}^3$, one finds that the pressure-variation amplitude ΔP_{max} has to be limited to a maximum value of 0.84 Pa. What is the minimum wavelength that the sound wave can have?

Section 15.3 Energy, Power, and Intensity of Sound Waves

(14) Figure 15.17 depicts a very long open tube of area $A = 5 \times 10^{-3}\,\text{m}^2$ that was filled at normal atmospheric pressure with air that has a density $\rho = 1.2\,\text{kg/m}^3$. When the piston is driven at a frequency of 500 Hz and amplitude of 0.15 cm, a sinusoidal sound wave with a speed $v = 343$ m/s is maintained in the tube. What power must be supplied by the piston to produce this sound wave?

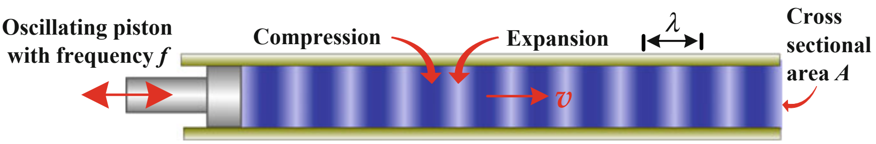


Fig. 15.17 See Exercise (14)

(15) A sound source vibrates at 1 kHz and produces sound waves of intensity $0.5\,\text{W/m}^2$ at a fixed point in space. (a) Find the intensity at this point if

the frequency is doubled while the displacement amplitude is kept constant. (b) Find the intensity at this point if the frequency is halved while the displacement amplitude is tripled.

(16) A loudspeaker emits a sound intensity of 100 μW/m^2 in a circular tube of radius $r = 7.5$ cm. How much power is being radiated as sound by the loudspeaker?

(17) Sound waves propagate with the same intensity I and angular frequency ω in: (1) air of density $\rho_a = 1.29$ kg/m^3 with a speed $v_a = 331$ m/s and, (2) water of density $\rho_w = 1{,}000$ kg/m^3 with a speed $v_w = 1{,}493$ m/s. Find the following for the two media: (a) the ratio of the values of the wavelength, (b) the ratio of the values of the displacement amplitude, and (c) the ratio of the values of the pressure-variation amplitude. (d) When $I = 10^{-6}$ W/m^2 and $\omega = 2{,}000\pi$ rad/s, evaluate the wavelength, displacement amplitude, and the pressure variation amplitude in each medium.

(18) The area of human eardrum is about $A = 5 \times 10^{-5}$ m^2. The intensity of sound at the threshold of hearing is $I = 10^{-12}$ W/m^2 and at the threshold of pain is $I = 1$ W/m^2. Find the sound power incident on the eardrum at both thresholds.

Section 15.4 The Decibel Scale

(19) When the human auditory system experiences a sound intensity of 1.2 W/m^2 it results in pain. Represent this amount in decibels.

(20) When a person speaks loudly, the sound level produced is 70 dB. When that person speaks normally, the sound level generated is at 40 dB. Find the ratio of the intensities of the two sounds.

(21) Two students argue loudly at sound levels of 80 dB and 78 dB. (a) Find the sound intensities for the individual students. (b) Find the combined sound level when the students argue simultaneously.

(22) (a) Show that doubling the intensity of sound will increase its level by 3 dB. (b) Show that halving the intensity of sound will decrease its level by 3 dB.

(23) One stereo amplifier is rated at 80 W and another is rated at 120 W. If the intensity of the sound produced at the maximum level of the first amplifier is taken as a reference, how much louder in dB will the second amplifier be at the maximum level?

(24) An engineer standing in front of an airplane with its four engines running experiences a sound level of 135 dB. What sound level would the engineer

experience if the pilot shut down: (a) only one engine, and (b) only two engines, and (c) only three engines?

(25) The amplitude of a sound wave is increased by a factor of 2.25. (a) By what factor will the intensity increase? (b) By how many dB will the sound level increase?

(26) Two identical point sources, S_1 and S_2, are located from an observer as shown in Fig. 15.18. They are emitting sound waves with the same power from the same oscillator. The sound intensity at the observer's location from S_2 is $I_2 = 4.0 \times 10^{-6}\,\text{W/m}^2$. (a) Find the total intensity of sound waves that is received by the observer from the two sources. (b) Find $\beta_{tot} - \beta_2$, which is the difference in the sound level when the two sources operate together and when the second source operates by itself. (c) Show that $\beta_1 - \beta_2 = (20\,\text{dB})\log(r_2/r_1) = 9.54\,\text{dB}$.

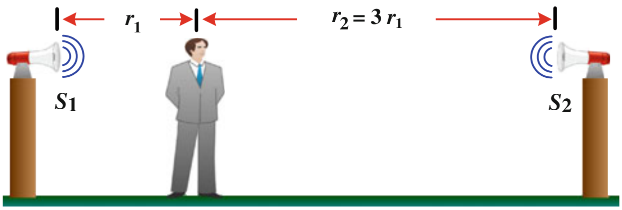


Fig. 15.18 See Exercise (26)

Section 15.5 Hearing Response to Intensity and Frequency

(27) What is the ratio of highest to lowest intensity that our auditory system can accommodate at: (a) 100 Hz, and (b) 1,000 Hz? (Use Fig. 15.9)

(28) What are the lowest and highest frequencies that our auditory system can detect if the sound level for normal talking is 50 dB? (Use Fig. 15.9)

Section 15.6 The Doppler Effect

(29) A source emits a 2.5 kHz sound wave. If this source moves toward you at 20 m/s while you stay still, will the observed frequency be the same as if you moved toward the source at 20 m/s while it stays still?

(30) While at rest, a bat sends out ultrasonic sound at 45 kHz. What is the bat's received sound frequency if that sound wave strikes a mouse running away with a speed of 20 m/s?

(31) While a bat is flying toward a wall at a speed of 5 m/s, it emits an ultrasonic sound of 35 kHz. What frequency does the bat receive from the reflected wave?

(32) A man holding an oscillating tuning fork with a frequency $f = 200$ Hz, runs toward a wall with a speed $v_m = 5$ m/s, see Fig. 15.19. The speed of sound in air is 343 m/s. (a) What frequency difference does he observe between the tuning fork and its echo? (b) How fast must he run away from the wall to observe a difference in frequency equal to 5 Hz?

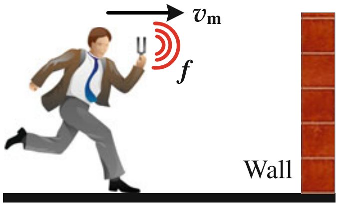


Fig. 15.19 See Exercise (32)

(33) An observer hears a frequency of 530 Hz from the siren of an approaching train; see part (a) of Fig. 15.20. After the train passes, the observer nearly in the path of the train hears a frequency of 470 Hz, see part (b) of Fig. 15.20. The speed of sound is 343 m/s. Find the train's speed.

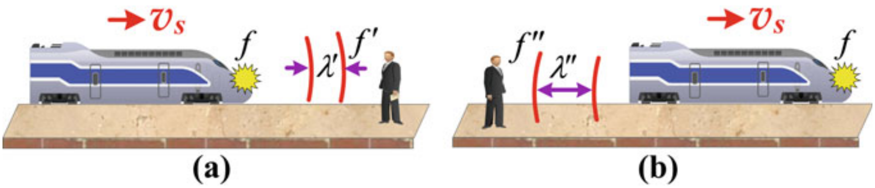


Fig. 15.20 See Exercise (33)

(34) A school bus moving with a speed $v_b = 15$ m/s generates a whistling sound at a frequency $f_b = 300$ Hz, see Fig. 15.21. A truck approaches the bus with a speed $v_t = 30$ m/s while its engine rumbles at a frequency $f_t = 500$ Hz. The speed of sound in air is 343 m/s. Assume approximately collinear paths. (a) What is the frequency detected by the driver in the truck? (b) What is the frequency detected by an observer in the bus? (c) After the truck passes the bus, what is the frequency detected by an observer in the bus?

(35) Two trams, A and B have identical sirens of frequency 500 Hz. Tram A is stationary and Tram B is moving towards the right, away from A at a speed of $v_B = 35$ m/s. An observer between the two sirens moves towards the right with a speed $v_o = 20$ m/s, see Fig. 15.22. Assume the speed of sound in air to be 340 m/s. (a) With what frequency does the observer hear the siren emitted from tram A? (b) With what frequency does the observer hear the siren emitted from tram B? (c) What is the difference in frequency heard by the observer?

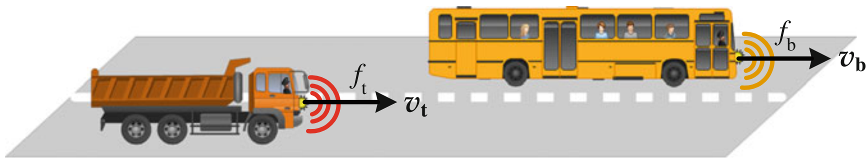


Fig. 15.21 See Exercise (34)

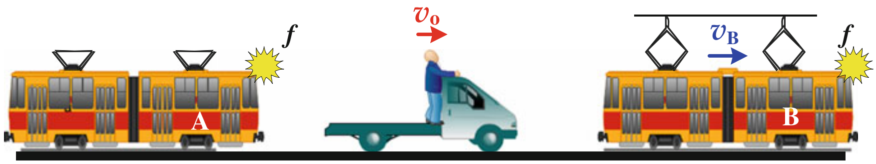


Fig. 15.22 See Exercise (35)

(36) A siren on the top of a stationary fire engine emits sound in all directions at a frequency $f = 900$ Hz. Assume that the speed of sound in calm air is 343 m/s and that a steady wind is blowing towards the East with a speed of 15 m/s. (a) Find the wavelength of the sound East of the siren. (b) Find the wavelength of the sound West of the siren. (c) Find the frequency of the sound heard when a firefighter approaches the siren with a speed of 15 m/s while walking *against* the wind. (d) Find the frequency of the sound heard when a firefighter approaches the siren with a speed of 15 m/s while walking *with* the wind.

Section 15.7 Supersonic Speeds and Shock Waves

(37) The Concorde could fly at Mach 1.5. The speed of sound is 340 m/s. (a) What does Mach 1.5 means? (b) What is the angle between the direction of the propagation of the shock wave front and the direction of the plane's velocity?

(38) A supersonic jet is traveling horizontally at Mach 3. At $t = 0$, the jet is over a person's head at an altitude $h = 15$ km, see the left part of the sketch in Fig. 15.23. (a) Where will the jet be before the person hears the boom of the shock wave, see the right part of the sketch in Fig. 15.23? (b) How long will the person wait before hearing that boom?

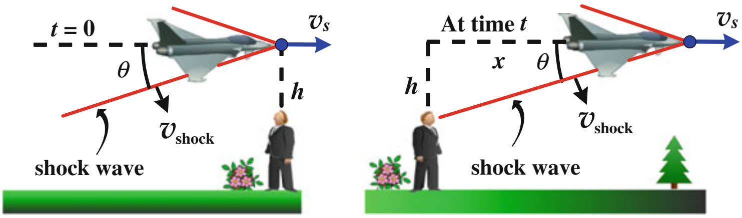


Fig. 15.23 See Exercise (38)

(39) A jet plane travels at Mach 2.5. The speed of sound is 320 m/s. (a) Find the angle of the shock wave compared to the direction of the jet's motion. (b) If the jet is flying $h = 6$ km vertically above a person on the ground, how long will it take for that person to hear the shock wave?

(40) A supersonic rocket travels at a constant speed of 1,190 m/s in a direction making an angle ϕ with the horizontal, see the sketch in Fig. 15.24. As the rocket gains altitude, an observer on the ground hears for the first time the boom of the shock wave when the rocket is directly above him. Assume the speed of sound in air to be 340 m/s. (a) Find the angle ϕ. (b) If the rocket is above the person at an altitude $h = 10$ km, find the time of flight. (c) Find the horizontal displacement of the rocket.

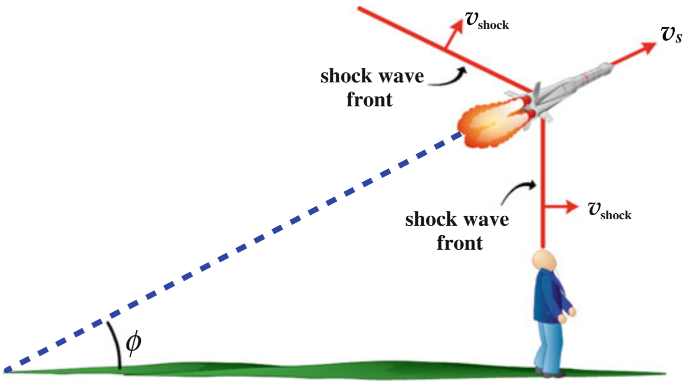


Fig. 15.24 See Exercise (40)

16 Superposition of Sound Waves

In this chapter, we explore the phenomenon that occurs when combining two or more waves at one point in the same medium. This phenomenon is known as **interference**. We first combine waves having the same frequencies. Then we combine waves that have slightly different frequencies. In both cases we only consider waves with small amplitudes so that we can use the **superposition principle**.

16.1 Superposition and Interference

To analyze complex combinations of traveling waves where each wave has a small amplitude, we use the superposition principle:

> The Superposition Principle
> If y_1 and y_2 are two traveling waves produced separately by two sources, then the resultant wave y at any point is the algebraic sum $y_1 + y_2$ when the two sources act together.

This principle is extremely important in all types of wave motion and applies not only to sound waves, but to string waves, light waves, and, in fact, to wave motion of any sort.

The general term **interference** is applied to the effect produced by two (or more) traveling waves when they are simultaneously passing through a given region. When the resultant wave has larger amplitude than that of either individual wave, we refer to their superposition as **constructive interference**. However, when the resultant

H. A. Radi and J. O. Rasmussen, *Principles of Physics*,
Undergraduate Lecture Notes in Physics, DOI: 10.1007/978-3-642-23026-4_16,

wave has smaller amplitude than that of either individual wave, we refer to their superposition as **destructive interference**.

Superposition of Sinusoidal Waves

Let us apply the principle of superposition to two sinusoidal waves traveling to the right in a homogeneous medium and having a different phase ϕ but the same frequency f, wavelength λ, and amplitude A. Accordingly, we write their individual waves as follows:

$$\begin{aligned} y_1 &= A\sin(kx - \omega t), \\ y_2 &= A\sin(kx - \omega t + \phi) \end{aligned} \tag{16.1}$$

where, as usual, $k = 2\pi/\lambda$, $\omega = 2\pi f$, and ϕ is the phase constant. The superposition of y_1 and y_2 gives the following resultant:

$$y = y_1 + y_2 = A\,[\sin(kx - \omega t) + \sin(kx - \omega t + \phi)] \tag{16.2}$$

To simplify the previous expression, we use the trigonometric identity:

$$\sin a + \sin b = 2\cos\left(\frac{a-b}{2}\right)\sin\left(\frac{a+b}{2}\right) \tag{16.3}$$

If we substitute in this identity with $a = kx - \omega t$ and $b = kx - \omega t + \phi$, then $a - b = -\phi$ and $a + b = 2kx - 2\omega t + \phi$. Accordingly, we will find that the resultant wave y is reduced to:

$$y = 2A\cos\left(\frac{\phi}{2}\right)\sin\left(kx - \omega t + \frac{\phi}{2}\right) \tag{16.4}$$

The resultant wave y has the following important characteristics:

(1) It is a sinusoidal wave and has the same frequency f and wavelength λ as any one of the contributing waves y_1 and y_2,

(2) It has an amplitude of $2A\cos(\phi/2)$,

(3) It has a phase of $\phi/2$.

Now, let us consider the following three cases:

(a) If $\phi = 0$, then $\cos(\phi/2) = +1$ and the amplitude of y is $+2A$, i.e. twice the amplitude of either one of the individual waves. In this case, the waves are said to be **in phase** and thus *interfere constructively*, see Fig. 16.1a. In general, constructive interference occurs if the phase ϕ is an *even* multiple of π, i.e. $\phi = 0,\ 2\pi,\ 4\pi, \ldots$ rad, then $\cos(\phi/2) = \pm 1$.

(b) If ϕ is an *odd* multiple of π, i.e. $\phi = \pi,\ 3\pi,\ 5\pi, \ldots$ rad, then $\cos(\phi/2) = 0$, and the crests of one wave occur at the same positions as the troughs of the second wave to produce a resultant amplitude of zero. In this case, the waves are canceling each other out and are said to be **out of phase** and thus *interfere destructively*, see Fig. 16.1b.

(c) If ϕ has an arbitrary value other than an odd or even multiple of π, then the resultant wave has an amplitude between 0 and 2 A, see Fig. 16.1c.

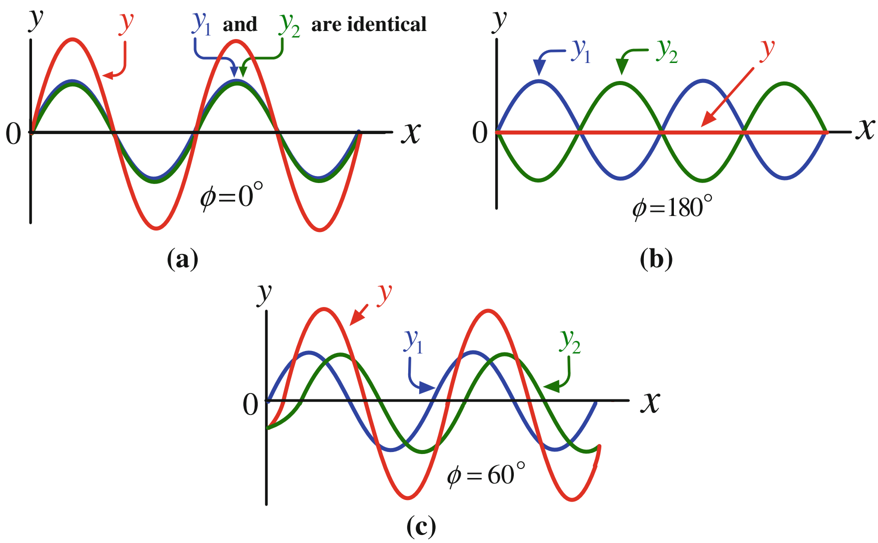


Fig. 16.1 Two identical waves, y_1 (*blue*) and y_2 (*green*), traveling in the same direction are added to each other at time $t = 0$ to give a resultant wave y (*red*). (a) When y_1 and y_2 are in phase ($\phi = 0$), they undergo constructive interference with a resultant wave $y = y_1 + y_2$ that has double the amplitude of either one of y_1 or y_2. (b) When y_1 and y_2 are out of phase ($\phi = \pi$ rad $= 180°$), they undergo destructive interference with a resultant wave $y = y_1 + y_2 = 0$, i.e. they cancel each other out. (c) When the phase is different from 0 or π rad, the resultant wave y falls somewhere between part (a) and part (b)

16.2 Spatial Interference of Sound Waves

Figure 16.2 depicts an acoustic interferometer device used to demonstrate sound interference. Sound energy from the source S is divided into two equal parts at the

T-shaped junction of the tube. This means that the sound wave that reached the receiver R traveled along either path A or path B. The distance along any path is called the **path length** L. The upper path length L_A is adjusted by a U-shaped tube, while the lower path length L_B is kept fixed.

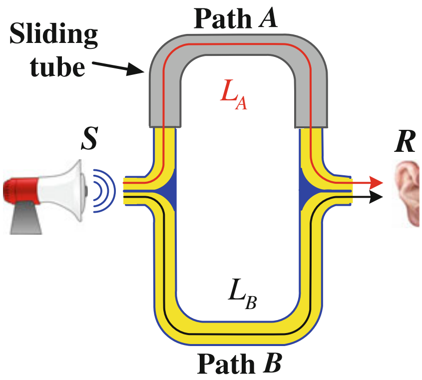


Fig. 16.2 A device used to demonstrate the interference of sound waves. Sound energy from the speaker (S) is divided into two parts at the T-shaped junction of the tube. Before reaching the receiver (R), half of the wave energy propagates through path A of length L_A, while the other half propagates through path B of length L_B. The *upper* path length L_A can be varied by sliding the U-tube *up* or *down*

Constructive interference occurs when the difference in the path length $\Delta L = |L_A - L_B|$ is given by:

$$\Delta L = |L_A - L_B| = (2n)\frac{\lambda}{2}, \quad n = 0, 1, 2, \ldots \quad \left\{\begin{matrix}\text{Constructive} \\ \text{interference}\end{matrix}\right\} \tag{16.5}$$

Therefore, the two waves reaching the receiver at any time are in phase ($\phi = 0$, 2π, 4π, ... rad), as shown in Fig. 16.1a, and hence a maximum sound intensity is detected at the receiver R.

Destructive interference occurs when the difference in the path length $\Delta L = |L_A - L_B|$ is given by:

$$\Delta L = |L_A - L_B| = (2n+1)\frac{\lambda}{2}, \quad n = 0, 1, 2, \ldots \quad \left\{\begin{matrix}\text{Destructive} \\ \text{interference}\end{matrix}\right\} \tag{16.6}$$

Therefore, the two waves reaching the receiver at any time are completely out of phase ($\phi = \pi$, 3π, 5π, ... rad), as shown in Fig. 16.1b, and hence no sound is detected at the receiver R.

Because a path difference of a complete wave length λ corresponds to a phase angle of 2π rad, one can relate path difference ΔL to the phase angle ϕ by the relation:

$$\Delta L = \frac{\phi}{2\pi}\,\lambda \tag{16.7}$$

Example 16.1

Two identical speakers, S_1 and S_2, are placed horizontally at a distance $d = 2$ m apart. Each emits sound waves of wavelength $\lambda = 80$ cm driven by the same oscillator, see Fig. 16.3. A listener is originally located at point O, which is midway between the two speakers. The listener walks to point P, which is a distance x from O, and reaches the *first minimum* in sound intensity. Find x.

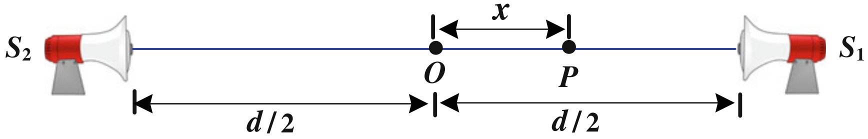


Fig. 16.3

Solution: If L_1 and L_2 are the distances from S_1 and S_2 to point P, respectively, then according to Fig. 16.3 we have:

$$L_1 = \frac{d}{2} - x, \quad L_2 = \frac{d}{2} + x$$

From these two relations and Eq. 16.6, the condition for the first destructive interference at point P leads to the following:

$$|L_2 - L_1| = \frac{\lambda}{2} \quad \Rightarrow \quad \left|\left[\frac{d}{2} + x\right] - \left[\frac{d}{2} - x\right]\right| = \frac{\lambda}{2} \quad \Rightarrow \quad x = \frac{\lambda}{4} = \frac{80\text{ cm}}{4} = 20\text{ cm}$$

Example 16.2

Two identical speakers, S_1 and S_2, are placed vertically at a distance $d = 2$ m apart and emit sound waves driven by the same oscillator, see Fig. 16.4. A listener is originally located at point O, which is a distance $R = 5$ m from the center of the line connecting the two speakers. The listener walks to point P, which is a distance $y = 0.5$ m above O, and thus reaches the first minimum in sound intensity. Find the wavelength λ of the sound wave.

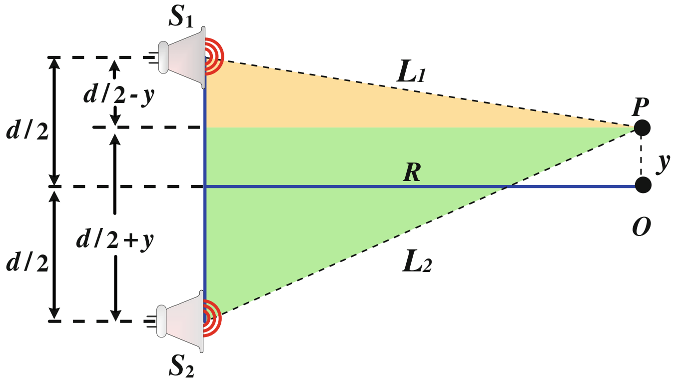


Fig. 16.4

Solution: The first minimum in sound intensity occurs when the two waves reaching the listener at point P are 180° out of phase. In other words, when their path difference equals $\lambda/2$. As per Fig. 16.4, we first calculate the path lengths L_1 and L_2 as follows:

$$L_1 = \sqrt{R^2 + (d/2 - y)^2} = \sqrt{(5\,\text{m})^2 + [(2\,\text{m})/2 - 0.5\,\text{m}]^2} = 5.0249 \text{ m}$$

and

$$L_2 = \sqrt{R^2 + (d/2 + y)^2} = \sqrt{(5\,\text{m})^2 + [(2\,\text{m})/2 + 0.5\,\text{m}]^2} = 5.2202 \text{ m}$$

Thus, from Eq. 16.6, the first destructive interference at point P leads to the following:

$$|L_2 - L_1| = \frac{\lambda}{2} \quad \Rightarrow \quad |5.2202 \text{ m} - 5.0249 \text{ m}| = \lambda/2 \quad \Rightarrow \quad 0.1953 \text{ m} = \lambda/2$$

Therefore:

$$\lambda = 0.3906 \text{ m} = 39.06 \text{ cm}$$

16.3 Standing Sound Waves

Assume we have two identical sound sources that face each other as shown in Fig. 16.5 and driven by the same oscillator. In this case, they produce two identical traveling waves each with a speed v. These waves would be moving in opposite directions in the same medium. Of course, these two waves combine according to the superposition principle.

Fig. 16.5 Two identical sound sources emitting traveling waves towards each other, each with a speed v. When the two waves overlap, they produce standing waves (not shown in the figure)

To analyze this situation, we assume that the two sound sources generate sound waves that have the same frequency f, wavelength λ, and amplitude A but differ by traveling in opposite directions. Therefore, we can write these two waves in the following form:

$$\boxed{\begin{aligned} y_1 &= A\sin(kx - \omega t), \\ y_2 &= A\sin(kx + \omega t) \end{aligned}} \tag{16.8}$$

where y_1 represents a wave traveling in the positive x-direction and y_2 represents a wave traveling in the negative x-direction. The superposition of y_1 and y_2 gives the following resultant:

$$y = y_1 + y_2 = A\,[\sin(kx - \omega t) + \sin(kx + \omega t)] \tag{16.9}$$

To simplify this expression, we use the trigonometric identity:

$$\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b \tag{16.10}$$

If we substitute in this identity with $a = k\,x$ and $b = \omega\,t$, then the resultant wave y reduces to:

$$\boxed{y = (2A\sin kx)\cos\omega t} \tag{16.11}$$

The resultant y represented by Eq. 16.11 gives a special kind of simple harmonic motion in which every element of the medium oscillates in simple harmonic motion with the same angular frequency ω (through the factor $\cos\omega t$) and an amplitude (given by the factor $2A\sin kx$) that varies with position x. This wave is called a *standing wave* because there is no motion of the disturbance along the x-direction.

A standing wave is distinguished by stationary positions with *zero amplitudes* called *nodes* (see Fig. 16.6). This happens when x satisfies the condition $\sin kx = 0$, that is, when:

$$kx = 0,\ \pi,\ 2\pi,\ 3\pi,\ \ldots$$

When using $k = 2\pi/\lambda$, these values give $x = 0,\ \dfrac{\lambda}{2},\ \lambda,\ \dfrac{3\lambda}{2},\ \ldots$, that is:

$$\boxed{x = 0,\ \frac{\lambda}{2},\ \lambda,\ \frac{3\lambda}{2},\ \ldots = n\frac{\lambda}{2}, \quad (n = 0, 1, 2, \ldots)} \quad \text{(Nodes)} \tag{16.12}$$

In addition, a standing wave is distinguished by elements with *greatest possible displacements* called *antinodes* (see Fig. 16.6). This happens when x satisfies the condition $\sin kx = \pm 1$, that is, when:

$$kx = \frac{\pi}{2},\ \frac{3\pi}{2},\ \frac{5\pi}{2},\ \ldots$$

Also, using $k = 2\pi/\lambda$, these values give $x = \dfrac{\lambda}{4},\ \dfrac{3\lambda}{4},\ \dfrac{5\lambda}{4},\ \ldots$, that is:

$$\boxed{x = \frac{\lambda}{4},\ \frac{3\lambda}{4},\ \frac{5\lambda}{4},\ \ldots = (n + \tfrac{1}{2})\frac{\lambda}{2}, \quad (n = 0, 1, 2, \ldots)} \quad \text{(Antinodes)} \tag{16.13}$$

Equations 16.12 and 16.13 indicate the following general features of nodes and antinodes (see Fig. 16.6):

Soptlight

(1) The distance between adjacent nodes is $\lambda/2$.

(2) The distance between adjacent antinodes is $\lambda/2$.

(3) The distance between a node and an adjacent antinode is $\lambda/4$.

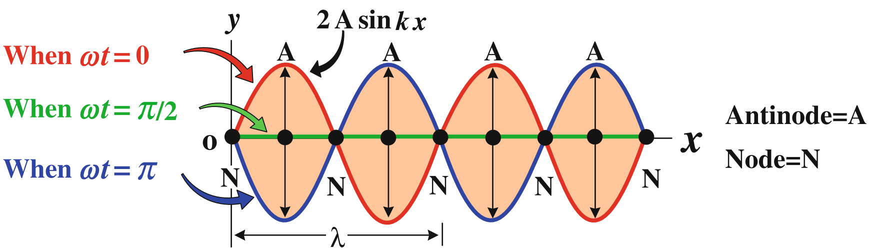


Fig. 16.6 The time dependence of the vertical displacement (from equilibrium) of any individual element in a standing wave y is governed by $\cos\omega t$. Each element vibrates within the confines of the envelope $2A\sin kx$. The *nodes* (N) are points of zero displacement, and the *antinodes* (A) are points of maximum displacement

In Fig. 16.7a, at $t = 0$ $(\omega t = 0)$, the two oppositely traveling waves are in phase, producing a wave pattern in which each element of the medium is experiencing its maximum displacement from equilibrium. In Fig. 16.7b, at $t = T/4$ $(\omega t = \pi/2)$, the traveling waves have moved one quarter of a wavelength (one to the right and the other to the left). At this time, each element of the medium is passing through the equilibrium position in its simple harmonic motion. The result is zero displacement for each element at all values of x. In Fig. 16.7c, at $t = T/2$ $(\omega t = \pi)$, the traveling waves are again in phase, producing a wave pattern that is inverted relative to the $t = 0$ pattern. The patterns at $t = 3T/4$ and $t = T$ are similar to $t = T/4$ and $t = 0$, respectively.

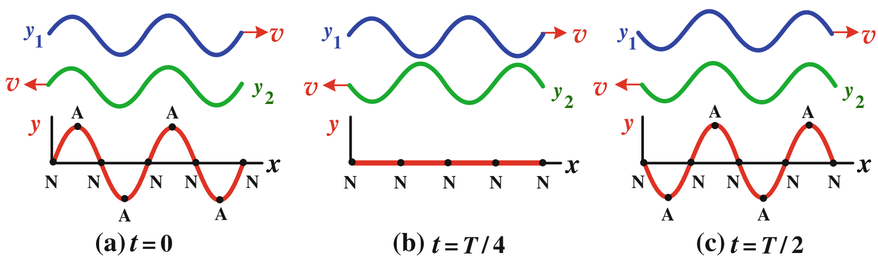


Fig. 16.7 Standing-wave patterns resulting from two oppositely traveling identical waves y_1 and y_2 at different phases. The displacement is zero for each node (N), and is maximum for each antinode (A)

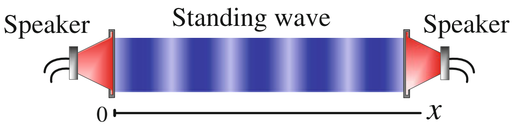
Example 16.3

Two opposing speakers are shown in Fig. 16.8. A standing wave is produced from two sound waves traveling in opposite directions; each can be described as follows:

$$y_1 = (5\text{ cm})\sin(4x - 2\,t),$$
$$y_2 = (5\text{ cm})\sin(4x + 2\,t).$$

where x and y, are in centimeters and t is in seconds. (a) What is the amplitude of the simple harmonic motion of a medium element lying between the two speakers at $x = 2.5$ cm? (b) Find the amplitude of the nodes and antinodes. (c) What is the maximum amplitude of an element at an antinode?

Fig. 16.8



Solution: (a) Using the general form of a standing wave given by Eq. 16.11, we find $A = 5$ cm, $k = 4$ rad/cm, and $\omega = 2$ rad/s. Thus:

$$y = (2\,A\sin kx)\cos\omega\,t = [(10\text{ cm})\sin(4x)]\cos(2t)$$

The amplitude of the simple harmonic motion of an element lying between the two speakers at $x = 2.5$ cm is the absolute value of the coefficient of $\cos(2t)$ evaluated at this point. Thus:

$$\begin{aligned}\text{Amplitude} &= |(10\text{ cm})\sin(4x)|_{x=2.5}| \\ &= |(10\text{ cm})\sin(10\text{ rad})| = |-5.4\text{ cm}| = 5.4\text{ cm}\end{aligned}$$

(b) With $k = 2\pi/\lambda = 4$ rad/cm, we have $\lambda = \pi/2$ cm. Then, from Eq. 16.12 we find that the nodes are located at:

$$x = n\frac{\lambda}{2} = n\frac{\pi}{4}\text{ cm}, \quad (n = 0, 1, 2, \ldots)$$

From Eq. 16.13, we find that the antinodes are located at:

$$x = (n + \tfrac{1}{2})\frac{\lambda}{2} = (n + \tfrac{1}{2})\frac{\pi}{4}, \quad (n = 0, 1, 2, \ldots)$$

(c) The maximum amplitude of antinodes will be $2\,A = 10$ cm

Example 16.4

Two sinusoidal sound waves, equal in amplitude and traveling in opposite directions along the x-axis, are superimposed on each other. The resultant wave is of the form:

$$y = (2\text{ m})\sin\left(\frac{\pi}{L}x\right)\cos\left(\frac{\pi}{T}t\right)$$

where x is in meters and t in seconds and the arguments of the sine and cosine functions are in radians. (a) What are the mathematical formulas of the two sinusoidal sound waves that are superimposed to give this resultant? (b) Find the values of the wavelength and the frequency of the two sinusoidal waves when $L = 2$ m and $T = 1$ s. (c) What are the velocities of the two sinusoidal waves?

Solution: (a) Using the general form of the standing waves given by Eq. 16.11, we find $A = 1$ m, $k = \pi/L$ rad/m, and $\omega = \pi/T$ rad/s. Using Eq. 16.8, we find the two sinusoidal waves as follows:

$$y_1 = (1\text{ m})\sin\left(\frac{\pi}{L}x - \frac{\pi}{T}t\right),$$
$$y_2 = (1\text{ m})\sin\left(\frac{\pi}{L}x + \frac{\pi}{T}t\right)$$

(b) Using $k = 2\pi/\lambda$, and $\omega = 2\pi f$ when $L = 2$ m and $T = 1$ s, we have:

$$k = \frac{2\pi}{\lambda} = \frac{\pi}{L} = \frac{\pi}{2\text{ m}} \quad \Rightarrow \quad \lambda = 4\text{ m}$$

$$\omega = 2\pi f = \frac{\pi}{T} = \frac{\pi}{1\text{ s}} \quad \Rightarrow \quad f = 0.5\text{ s}^{-1} = 0.5\text{ Hz}$$

(c) Using $v = \omega/k$, we find the speed of each of the sinusoidal waves as follows:

$$v = \frac{\omega}{k} = \frac{2\pi f}{2\pi/\lambda} = \lambda f = (4\text{ m})(0.5\text{ s}^{-1}) = 2\text{ m/s}$$

The velocity of y_1 is $v_1 = +2$ m/s (in the direction of increasing x) and the velocity of y_2 is $v_2 = -2$ m/s (in the direction of decreasing x).

16.4 Standing Sound Waves in Air Columns

In Chap. 14, we saw how a standing wave can be generated either on a stretched string with fixed ends or when one end is fixed and the other is left free to move. We learned that this happens when the wavelengths of the waves suitably match the length of the string, in which case the superposition of the traveling and reflecting waves produce a standing wave pattern. For such a match, the wavelength corresponds to the resonant frequency of the string.

We can set up standing sound waves in air-filled pipes in a way similar to that for strings. Here is how we can compare the two:

1. The closed end of a pipe is similar to the fixed end of a string in that it must be a **displacement node**. This is because the pipe's wall at this end does not allow longitudinal motion of the air and acts like a **pressure antinode** (point of maximum pressure variation).
2. The open end of a pipe acts like the end of a string that is free to move, so there must be a **displacement antinode** there[1]. This is because the pipe's open end allows longitudinal motion of the air and acts like a **pressure node** (point of no pressure variation, since the end must remain at atmospheric pressure).

It is interesting to know how sound waves reflect from the open end of a pipe. To get insight into this, we start with the fact that sound waves are in fact pressure waves. Next, we know that any compression region must be contained inside the pipe (between its two ends). Furthermore, any compression region that exists at an open end is free to expand into the atmosphere. This change in behavior of the air inside and outside the pipe is sufficient to allow some reflection.

With the boundary conditions of nodes and antinodes at the ends of air columns, we must set the normal modes of oscillations as we did in the case of stretched strings.

Air Columns of Two Open Ends

First, we consider a pipe of length L that is open at both ends. By representing the horizontal displacement of air elements on the vertical axis and applying the boundary condition that meets the case of two open ends, see Fig. 16.9, the normal modes of oscillations can be explained by considering the following first three patterns:

(1) The first normal mode (the first harmonic, or the fundamental):

The simplest pattern is shown in Fig. 16.9a. There are two imposed antinodes

[1] The antinode of an open end of a pipe is located slightly beyond the end because sound compression reaching an open end does not reflect until it passes the end. Therefore, the effective length of the air column is little greater than the true length L of the pipe.

at the two ends and only one node in the middle of the pipe. Also, there is only half a wavelength in the length L. Thus, this standing wave pattern has:

$$\lambda_1 = 2L \quad \text{and} \quad f_1 = \frac{v}{\lambda_1} = \frac{v}{2L}.$$

(2) The second normal mode (the second harmonic):
The second pattern is shown in Fig. 16.9b. This pattern has three antinodes and two nodes. This standing wave pattern has:

$$\lambda_2 = L \quad \text{and} \quad f_2 = \frac{v}{\lambda_2} = \frac{v}{L} = 2f_1$$

(3) The third normal mode (the third harmonic):
The third pattern is shown in Fig. 16.9c. This pattern has four antinodes and three nodes. This standing wave pattern has:

$$\lambda_3 = 2L/3 \quad \text{and} \quad f_3 = \frac{v}{\lambda_3} = \frac{3v}{2L} = 3f_1$$

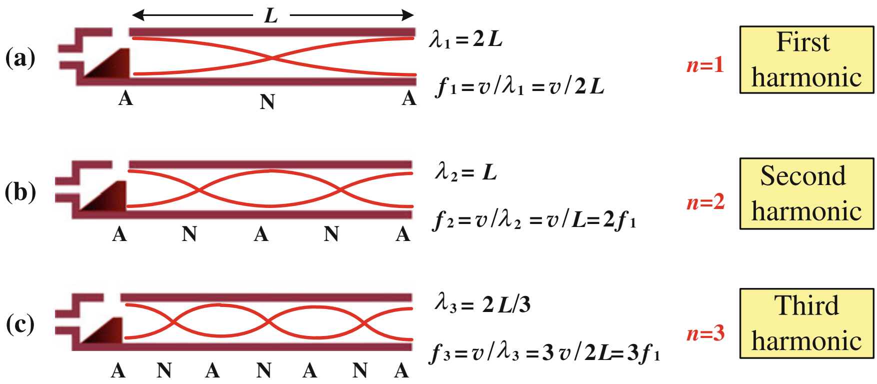


Fig. 16.9 The first three standing wave patterns (a), (b), and (c) of a longitudinal sound wave established in an organ pipe that is open to the atmosphere at both ends. The horizontal motion of air elements in the pipe is displayed vertically by using a *red* color. The difference between successive harmonics is the fundamental frequency f_1, and each harmonic is an integer multiple of the fundamental frequency f_1

Generally, the relation between the wavelength λ_n of the various normal modes and the length L of a pipe of two open ends is:

$$\boxed{\lambda_n = \frac{2L}{n}, \quad (n = 1,\ 2,\ 3, \ldots)} \quad \text{(Pipe, two open ends)} \tag{16.14}$$

Also, according to the relation $f = v/\lambda$, where the speed v of the sound wave is the same for all frequencies, the resonance frequencies f_n associated with these modes are (see Fig. 16.9):

$$\boxed{f_n = \frac{v}{\lambda_n} = n\frac{v}{2L}, \quad (n = 1,\ 2,\ 3, \ldots)} \quad \text{(Pipe, two open ends)} \tag{16.15}$$

The expressions of λ_n and f_n are the same as for the string, except that v is the speed of waves on the strings as in Eq. 14.66, whereas v in Eq. 16.15 is the speed of sound in air. The relation between the resonance frequencies and the fundamental frequency is:

$$\boxed{f_n = n f_1, \quad (n = 1,\ 2,\ 3, \ldots)} \quad \text{(Pipe, two open ends)} \tag{16.16}$$

Air Columns of One Closed End

Second, we consider a pipe of length L that is open at one end and closed at the other. By applying the boundary condition that meets this case, the normal modes of oscillations can be explained by considering the following first three patterns:

(1) The first normal mode (the first harmonic, or the fundamental):
Fig. 16.10a shows the simplest pattern. The standing wave extends from an antinode at the open end to the adjacent node at the closed end. The fundamental standing wave pattern has:

$$\lambda_1 = 4\,L \quad \text{and} \quad f_1 = \frac{v}{\lambda_1} = \frac{v}{4L}.$$

(2) The third normal mode (the third harmonic):
The next pattern is shown in Fig. 16.10b. This pattern has two antinodes and two nodes. Thus, this standing wave pattern has:

$$\lambda_3 = 4L/3 \quad \text{and} \quad f_3 = \frac{v}{\lambda_3} = \frac{3v}{4L} = 3f_1$$

(3) The fifth normal mode (the fifth harmonic):
The next pattern is shown in Fig. 16.10c. This pattern has four antinodes and four nodes. Thus:

$$\lambda_5 = 4L/5 \quad \text{and} \quad f_5 = \frac{v}{\lambda_5} = \frac{5v}{4L} = 5f_1$$

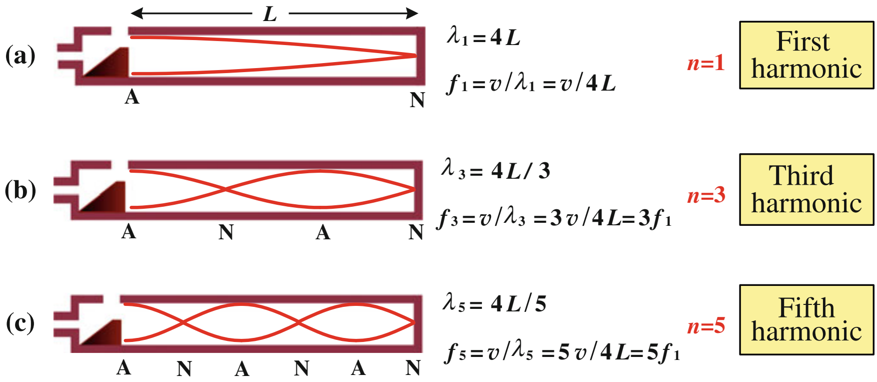


Fig. 16.10 The first three standing wave patterns (a), (b), and (c) of a longitudinal sound wave established in an organ pipe that is open to the atmosphere at only one end. The horizontal motion of air elements in the pipe is displayed vertically by using a *red* color. The harmonic frequencies are the odd-integer multiples of f_1, and the successive difference is $2f_1$

Generally, λ_n and f_n of the various normal modes for a pipe of length L with only one end open are given as (see Fig. 16.10):

$$\lambda_n = \frac{4L}{n}, \quad (n = 1,\ 3,\ 5, \ldots) \quad \text{(Pipe, one open end)} \tag{16.17}$$

$$f_n = \frac{v}{\lambda_n} = n\frac{v}{4L}, \quad (n = 1,\ 3,\ 5, \ldots) \quad \text{(Pipe, one open end)} \tag{16.18}$$

$$f_n = nf_1, \quad (n = 1,\ 3,\ 5, \ldots) \quad \text{(Pipe, one open end)} \tag{16.19}$$

Figure 16.11 shows a simple apparatus for demonstrating the resonance of sound waves in air columns. A tube that is open from both ends is immersed into a container filled with water, and a tuning fork of unknown frequency f and wavelength λ is placed at its top. The sound waves generated by the fork are reinforced when the length L corresponds to one of the resonance frequencies of the tube. Thus:

$$\lambda = \frac{4L_n}{n}, \quad f = \frac{v}{\lambda} = n\frac{v}{4L_n}, \quad (n = 1,\ 3,\ 5, \ldots) \tag{16.20}$$

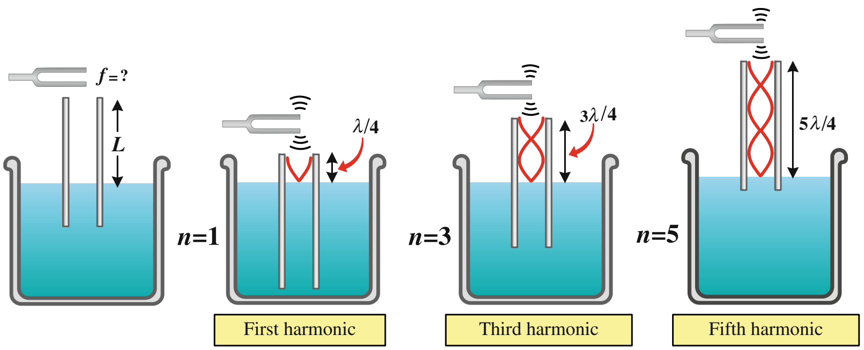


Fig. 16.11 An apparatus used to demonstrate the resonance of sound waves in a tube closed at one end. At resonance, L and λ are related

Example 16.5

When wind blows through a cylindrical drainage culvert of 2.5 m length, see Fig. 16.12, a howling noise is established. Take $v = 343$ m/s as the speed of sound in air. (a) Find the frequencies of the first three harmonics if the pipe is open at both ends. (b) How many of the harmonics fall within the normal human hearing range (from about 20 Hz $\rightarrow$ 20,000 Hz). (c) Answer part (a) if the pipe is blocked at the other end.

Fig. 16.12

Solution: (a) When the pipe is open at both ends, we use Eq. 16.15 with $n = 1$ to find the fundamental frequency as follows:

$$f_1 = 1 \times \frac{v}{2L} = \frac{343 \text{ m/s}}{2 \times 2.5 \text{ m}} = 68.6 \text{ Hz}$$

Also, all harmonics are available for a pipe open at both ends; thus:

$$f_2 = 2f_1 = 137.2 \text{ Hz} \quad \text{and} \quad f_3 = 3f_1 = 205.8 \text{ Hz}$$

(b) We can express the frequency of the highest harmonic heard as $f_n = nf_1$, where $f_n = 20{,}000$ Hz and n is the number of harmonics that can be heard. Therefore:

$$n = \frac{f_n}{f_1} = \frac{20{,}000\ \text{Hz}}{68.6\ \text{Hz}} = 292$$

Although we get $n = 292$, practically, only the first few harmonics have amplitudes that are sufficient to be heard.

(c) Using Eq. 16.18 and substituting with $n = 1$, the fundamental frequency of a pipe closed at one end will be given by:

$$f_1 = 1 \times \frac{v}{4L} = \frac{343\ \text{m/s}}{4 \times 2.5\ \text{m}} = 34.3\ \text{Hz}$$

In this case, only the odd harmonics can exist. Thus:

$$f_3 = 3f_1 = 102.9\ \text{Hz} \quad \text{and} \quad f_5 = 5f_1 = 171.5\ \text{Hz}$$

Example 16.6

A background noise in a hall sets up a fundamental standing wave frequency in a tube of length $L = 0.7$ m. What is the value of this fundamental frequency if your ear blocks one end of the tube (see Fig. 16.13a) and when your ear is far from the tube (see Fig. 16.13b)? Take $v = 343$ m/s as the speed of sound in air.

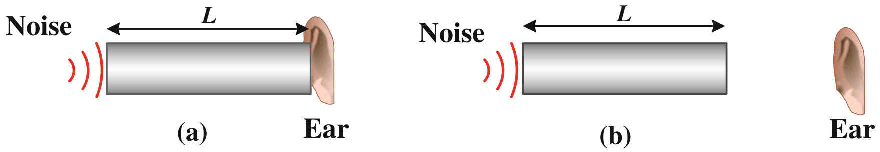


Fig. 16.13

Solution: When the tube is blocked by your ear (see Fig. 16.13a) the fundamental frequency is given by Eq. 16.18 with $n = 1$:

$$f_1 = 1 \times \frac{v}{4L} = \frac{343\ \text{m/s}}{4 \times 0.7\ \text{m}} = 122.5\ \text{Hz}$$

In addition, you can hear frequencies that are *odd integer multiples* of 122.5 Hz provided that the standing waves are formed with sufficient amplitudes.

When you move your head away enough (see Fig. 16.13b) the pipe becomes open at both ends and the fundamental frequency will be given by Eq. 16.15 with $n = 1$:

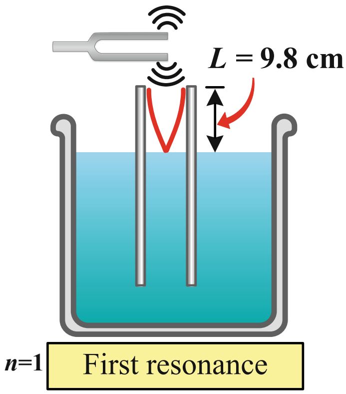
$$f_1 = 1 \times \frac{v}{2L} = \frac{343\ \text{m/s}}{2 \times 0.7\ \text{m}} = 245\ \text{Hz}$$

In addition, you can hear frequencies that are *multiples* of 245 Hz if the standing waves are formed with sufficient amplitudes.

Example 16.7

Resonance can occur in Fig. 16.14 when the smallest length of the air column is $L = 9.8$ cm. Take $v = 343$ m/s as the speed of sound in air. (a) What is the frequency f of the tuning fork? (b) What is the value of L for the next two resonances?

Fig. 16.14



Solution: (a) When the tube is blocked by the water's surface, it acts as if the tube is closed at one end. Thus, for the smallest air column L_1, the fundamental frequency is given by Eq. 16.20 with $n = 1$:

$$f = 1 \times \frac{v}{4L_1} = \frac{343\ \text{m/s}}{4 \times (0.098\ \text{m})} = 875\ \text{Hz} \quad \left\{ \begin{matrix} \text{First resonance} \\ \text{First harmonic} \end{matrix} \right\}$$

This frequency must be equal to the frequency f of the tuning fork.

(b) We know from Fig. 16.14 and Eq. 16.20 that the wavelength of the fundamental mode is four times the length of the air column. Thus:

$$\lambda = \frac{4L_1}{1} = 4(0.098\ \text{m}) = 0.392\ \text{m}$$

Because the frequency of the tuning fork is constant, then according to Fig. 16.11, the values of L for the next two normal modes are:

$$L_3 = \frac{3\,\lambda}{4} = \frac{3 \times (0.392\text{ m})}{4} = 0.294\text{ m} = 29.4\text{ cm} \quad \left\{\begin{array}{l}\text{Second resonance}\\ \text{Third harmonic}\end{array}\right\}$$

$$L_5 = \frac{5\,\lambda}{4} = \frac{5 \times (0.392\text{ m})}{4} = 0.49\text{ m} = 49\text{ cm} \quad \left\{\begin{array}{l}\text{Third resonance}\\ \text{Fifth harmonic}\end{array}\right\}$$

16.5 Temporal Interference of Sound Waves: Beats

Previously, we discussed the *spatial interference* of waves of same frequencies, where at fixed time the amplitude of the oscillating elements varies with the position in space. The standing waves in strings and air columns are good examples of this kind of interference.

Now, we consider another type of interference of waves having a slight difference in their frequencies, where at fixed position, the amplitude of the oscillating elements varies periodically with time. The standing wave produced by two tuning forks having a slight difference in their frequencies is a good example of this kind of interference. We refer to this *interference in time* by **temporal interference**, and this phenomenon is called **beating**:

Beating
Beating is defined as the periodic variation in amplitude at a fixed position due to the variation in the constructive and destructive interference between waves having slightly different frequencies.

Consider the time-dependent variations of the displacements of two sound waves of equal amplitude and slightly different frequencies f_1 and f_2 (angular frequencies $\omega_1 = 2\pi f_1$ and $\omega_2 = 2\pi f_2$) such that:

$$\begin{aligned} y_1 &= A\cos(k_1\,x - \omega_1\,t),\\ y_2 &= A\cos(k_2\,x - \omega_2\,t) \end{aligned} \tag{16.21}$$

At the fixed point $x = 0$ (chosen for convenience), the two wave functions become (see Fig. 16.15a):

$$\begin{aligned} y_1 &= A\cos\omega_1 t, \\ y_2 &= A\cos\omega_2 t \end{aligned} \tag{16.22}$$

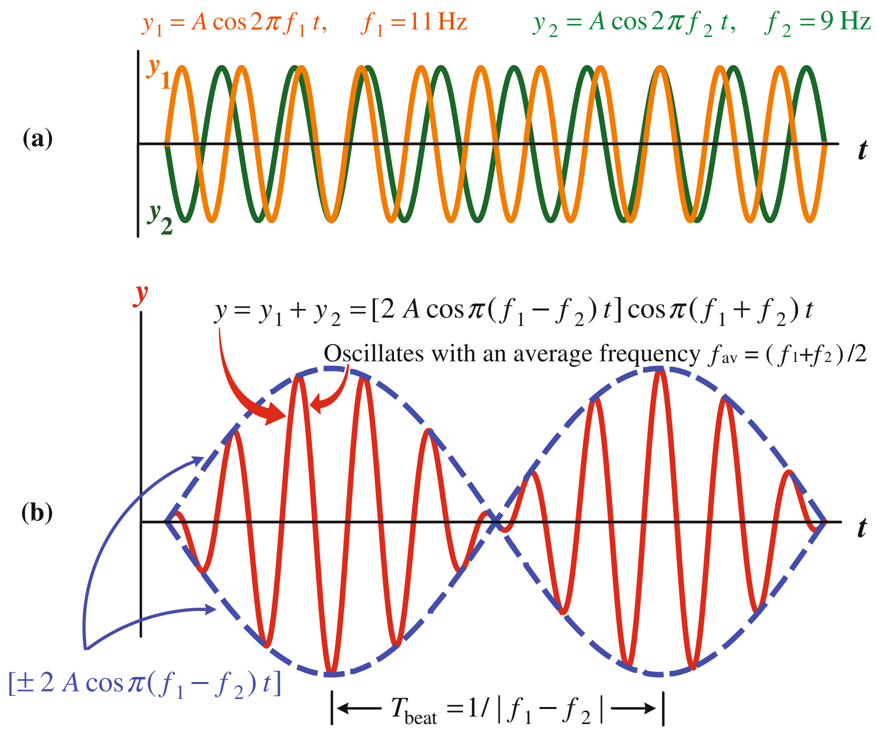


Fig. 16.15 (a) Formation of beats by combining two waves of slightly different frequencies f_1 and f_2 ($f_1 = 11$ Hz and $f_1 = 9$ Hz). (b) The slowly varying amplitude envelope $\pm 2A\cos\pi(f_1 - f_2)t$ limits the amplitude of the rapid sinusoidal function $\cos\pi(f_1 + f_2)t$, which proceeds with an average frequency $f_{av} = (f_1 + f_2)/2$

The superposition of y_1 and y_2 gives the following resultant:

$$y = y_1 + y_2 = A[\cos\omega_1 t + \cos\omega_2 t] \tag{16.23}$$

To simplify this expression, we use the trigonometric identity:

$$\cos a + \cos b = 2\cos\tfrac{1}{2}(a - b)\cos\tfrac{1}{2}(a + b) \tag{16.24}$$

If we substitute $a = \omega_1 t$ and $b = \omega_2 t$ in this identity, then the resultant wave y reduces to:

$$y = [2A\cos\tfrac{1}{2}(\omega_1 - \omega_2)\,t]\cos\tfrac{1}{2}(\omega_1 + \omega_2)\,t \tag{16.25}$$

When the difference in angular frequencies is small compared to the sum of angular frequencies, i.e.:

$$|\omega_1 - \omega_2| \ll \omega_1 + \omega_2 \quad \text{or} \quad |f_1 - f_2| \ll f_1 + f_2 \tag{16.26}$$

Then the time behavior of the factor $\cos\frac{1}{2}(\omega_1 + \omega_2)\,t$, is a *rapidly varying* sinusoidal oscillation, see Fig. 16.15b, with the average angular frequency $\frac{1}{2}(\omega_1 + \omega_2)$. Thus, the y equation indicates that the resultant sound wave at any given location has an effective angular frequency equal to the average angular frequency:

$$\omega_{\text{av}} = \frac{\omega_1 + \omega_2}{2} \quad \text{or} \quad f_{\text{av}} = \frac{f_1 + f_2}{2} \tag{16.27}$$

In addition, the oscillation is not precisely sinusoidal because the resultant amplitude varies with time according to the expression:

$$A_{\text{res}} = 2A\cos\tfrac{1}{2}(\omega_1 - \omega_2)\,t \tag{16.28}$$

This resultant amplitude is a *slowly varying envelope* in time, see Fig. 16.15b, that modulates the rapidly oscillating factor $\cos\frac{1}{2}(\omega_1 + \omega_2)\,t$. Moreover, this resultant amplitude A_{res} confirms the existence of a constructive interference when $\cos\frac{1}{2}(\omega_1 - \omega_2)\,t = \pm 1$. That is when:

$$\frac{1}{2}|\omega_1 - \omega_2|\,t = 0,\ \pi,\ 2\pi,\ \ldots \quad \text{(Constructive interference)} \tag{16.29}$$

Also, the resultant amplitude A_{res} confirms the existence of a destructive interference when $\cos\frac{1}{2}(\omega_1 - \omega_2)\,t = 0$. That is when:

$$\frac{1}{2}|\omega_1 - \omega_2|\,t = \frac{1}{2}\pi,\ \frac{3}{2}\pi,\ \frac{5}{2}\pi,\ \ldots \quad \text{(Destructive interference)} \tag{16.30}$$

The time between successive moments of constructive (or destructive) interferences is called the *beat period*, T_{beat}. During this time, the phase difference increases by π, i.e. $\frac{1}{2}|\omega_1 - \omega_2|\,T_{\text{beat}} = \pi$. Thus:

$$T_{\text{beat}} = \frac{2\pi}{|\omega_1 - \omega_2|} = \frac{1}{|f_1 - f_2|} \tag{16.31}$$

Hence, the number of beats per second, or the *beat frequency* f_{beat}, will be given by:

$$\boxed{f_{\text{beat}} = |f_1 - f_2|} \tag{16.32}$$

Musicians can use the beat phenomenon in tuning their instruments. If an instrument sounds different from how it is supposed to, it can be tuned by using a standard frequency until the beat disappears.

Example 16.8

Two identical violin A strings (see the left part of Fig. 16.16) of the same length and tension are tuned exactly to 440 Hz. The tension in one of them is increased by 2% (see the right part of Fig. 16.16). When both strings are struck, what will be the beat frequency between their fundamental frequencies?

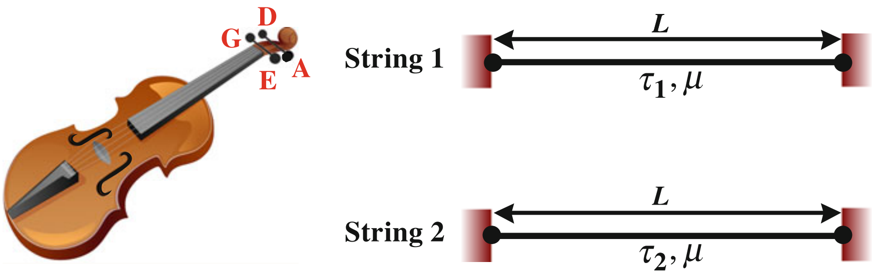


Fig. 16.16

Solution: The frequency of a string that is fixed at both ends is given by Eq. 14.68 as $f = \sqrt{\tau/\mu}/(2L)$, where L, τ, and μ are the length, tension, and mass per unit length of the string, respectively. Thus, the ratio of frequencies of the two strings after being struck is:

$$\frac{f_2}{f_1} = \frac{1}{2L}\sqrt{\frac{\tau_2}{\mu}} \bigg/ \frac{1}{2L}\sqrt{\frac{\tau_1}{\mu}} = \sqrt{\frac{\tau_2}{\tau_1}}$$

When tension τ_2 is 2% more than τ_1, we can find the frequency f_2 of string 2 as follows:

$$\frac{f_2}{f_1} = \sqrt{\frac{1.02\tau_1}{\tau_1}} = \sqrt{1.02} = 1.01 \quad \Rightarrow \quad f_2 = 1.01 f_1 = 1.01 \times (440\,\text{Hz}) = 444\,\text{Hz}$$

With the use of Eq. 16.32, the beat frequency will be:

$$f_{beat} = |f_1 - f_2| = |440\text{ Hz} - 444\text{ Hz}| = 4\text{ Hz} = 4\text{ beat/s}$$

Example 16.9

A musician wants to tune the A2 key (key No. 25) of a piano that has a *proper* fundamental frequency of 110 Hz, see Fig. 16.17. Assume he uses a fork of frequency $f_1 = 220$ Hz and was able to tune the A2 key after observing a beat frequency of 8 Hz. Explain the process of tuning and find the mistuned frequency.

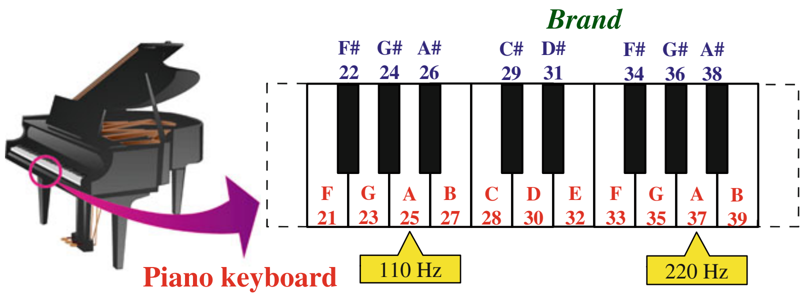


Fig. 16.17

Solution: Equation 16.25 leads to the beat phenomenon when the frequencies are close to each other, which is not the case for 110 and 220 Hz. However, based on Eq. 14.69 of fixed strings, the proper second harmonic of the string of the A2 key should be:

$$f_2 = 2 \times (110\text{ Hz}) = 220\text{ Hz} \quad \text{(Proper second harmonic)}$$

By listening to the beats of 8 Hz between the fundamental frequency $f_1 = 220$ Hz of the tuning fork and the unknown mistuned second harmonic frequency f_2 of the A2 key, he can adjust the tension in the string until the beat note disappears. From the beats Eq. 16.32, he can find the mistuned frequency of the A2 key as follows:

$$f_{beat} = |f_1 - f_2|$$
$$8\text{ Hz} = |220\text{ Hz} - f_2|$$

Hence,

$$f_2 = \begin{cases} 228 \text{ Hz} \\ \text{or} \\ 212 \text{ Hz} \end{cases}$$

Accordingly, the musician cannot tell whether the mistuned fundamental frequency of the string was 114 Hz or 106 Hz, because both frequencies produce the same beat frequency.

16.6 Exercises

Sections 16.1 and 16.2 Superposition and Interference and Spatial Interference of Sound Waves

(1) Two traveling waves are defined by the following relations:

$$y_1 = (2\,\text{cm})\sin(kx - \omega t),$$
$$y_2 = (2\,\text{cm})\sin(kx - \omega t + \phi)$$

Find the amplitude of the resultant wave $y = y_1 + y_2$ when $\phi = \pi/2$ and $\phi = \pi$.

(2) Two traveling waves are defined by the following relations:

$$y_1 = (1.5\text{ m})\sin(10x - 16t),$$
$$y_2 = (1.5\text{ m})\sin(14x - 20t)$$

where x is in meters, t is in seconds, and the arguments of the sine waves are in radians. (a) What is the phase difference between the two waves when $x = 4$ m and $t = 2$ s? (b) At $t = 4$ s, apply the condition of destructive interference (phase difference $= (2n+1)\pi,\ n = 0, 1, 2, \ldots$) to find the closest positive value of x to the origin.

(3) The two identical speakers shown in Fig. 16.18 are driven by one oscillator that has a frequency of 3,400 Hz. Take the speed of sound to be 343 m/s. (a) What are the values of x that correspond to a minimum sound intensity at point P? (b) What are the values of x that correspond to a maximum sound intensity at point P?

(4) A small speaker is placed in a circular pipe of radius $r = 1.35$ m, as shown in Fig. 16.19. Take the speed of sound to be 343 m/s and assume propagation

of one-dimensional waves for such a big radius. What are the three smallest frequencies that produce a maximum sound intensity in the tube?

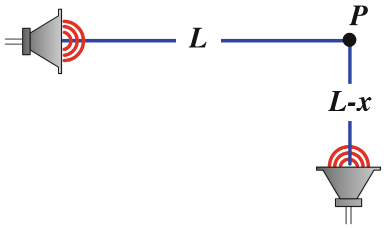


Fig. 16.18 See Exercise (3)

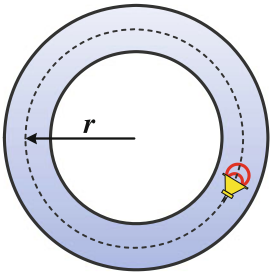


Fig. 16.19 See Exercise (4)

(5) Two identical speakers, S_1 and S_2, are placed vertically at a distance d apart. They emit sound waves driven by the same oscillator whose frequency is f. A listener at a distance R from the lower speaker walks straight towards it as shown in Fig. 16.20. If the speed of sound is v, show that the listener will hear a minimum sound when R satisfies the following relation:

$$R^2 = \frac{d^2 - (2n+1)^2(v/2f)^2}{2(2n+1)(v/2f)}, \quad (n = 0, 1, 2, \ldots) \quad \left\{ \begin{array}{l} \text{Destructive} \\ \text{interference} \end{array} \right\}$$

(6) In the previous example, assume that $d = 3$ m, $f = 350$ Hz, and $v = 343$ m/s. How many times will the listener hear a minimum in sound intensity while walking from a very far point to the nearest possible point in front of the lower speaker?

Section 16.3 Standing Sound Waves

(7) Two waves are traveling in opposite directions and are described by the following relations:

$$y_1 = A\sin(kx - \omega t),$$
$$y_2 = \tfrac{1}{2}A\sin(kx + \omega t)$$

Show that the resultant of these two waves can be written as a combination of a traveling wave and a standing wave of the following form:

$$y = y_1 + y_2 = \tfrac{1}{2}A\sin(kx - \omega t) + (A\sin kx)\cos\omega t$$

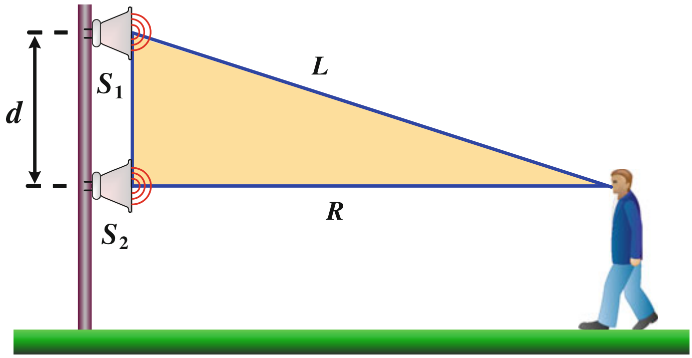


Fig. 16.20 See Exercise (5)

(8) Two identical speakers facing each other as shown in Fig. 16.21, establish a standing wave as a result of the production of the following two oppositely traveling sound waves:

$$y_1 = (2\,\text{cm})\sin(2.5x - 5t),$$
$$y_2 = (2\,\text{cm})\sin(2.5x + 5t).$$

where x and y are in centimeters and t is in seconds. (a) What is the amplitude of the simple harmonic motion of an element of the medium located at $x = 4$ cm? (b) Find the position of the nodes and antinodes. (c) What is the maximum amplitude of an element at an antinode?

(9) The two sources shown in the evacuated vessel of Fig. 16.22, are 1.2 m apart, and send sound waves of speed $v = 2$ m/s. Source S_1 vibrates according to the equation $(0.04\,\text{m})\sin 10\pi t$ while source S_2 vibrates according to the equation $(0.01\,\text{m})\sin 10\pi t$. (a) Show that S_1 sends sound in the positive x direction as:

$$y_1 = (0.04\,\text{m})\sin(5\pi x_1 - 10\pi\, t)$$

where x_1 is measured from an origin located at S_1. (b) Show that S_2 emits sound in the negative x-direction as:

$$y_2 = (0.01\text{ m})\sin(5\pi x_2 + 10\pi\, t)$$

where x_2 is measured from an origin located at S_2. (c) Show that the equation of motion of a particle at 0.8 m from S_1 and 0.4 m from S_2 is given by:

$$y = y_1 + y_2 = (-0.03\text{ m})\sin 10\pi\, t.$$

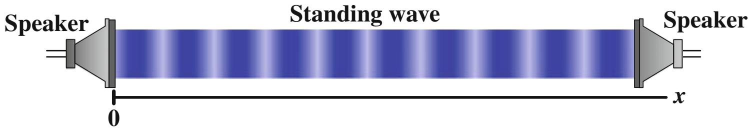


Fig. 16.21 See Exercise (8)

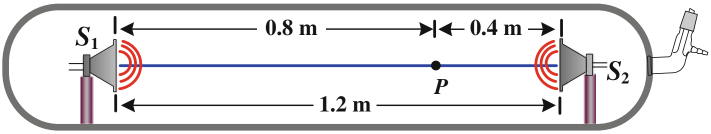


Fig. 16.22 See Exercise (9)

(10) Using direct substitution, show that the standing wave function:

$$y = 2\,A\cos\left(\frac{\phi}{2}\right)\sin\left(kx - \omega\, t + \frac{\phi}{2}\right)$$

is a solution of the general partial linear differential equation [see Eq. 14.58]:

$$\frac{\partial^2 y}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} = 0$$

Section 16.4 Standing Sound Waves in Air columns

Note: Unless otherwise specified, use the speed of sound in this section to be 343 m/s.

(11) An organ pipe of length 30 cm is open at both ends. What are the frequencies of the fundamental and the next two harmonics?

(12) If the organ pipe in the previous exercise has one end closed, what are the frequencies of the fundamental and the next two harmonics?

(13) The fundamental frequency of a pipe is found to be 110 Hz when the speed of sound is 330 m/s. (a) Find the pipe's length when it is closed at one end. (b) Find the pipe's length when it is open at both ends.

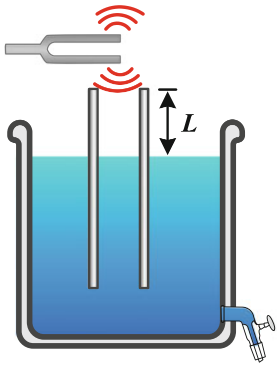
(14) The two adjacent harmonic frequencies of an organ pipe (with both ends open) are determined to be 540 Hz and 420 Hz. (a) Find the fundamental frequency of the pipe. (b) Find the pipe's length.

(15) Estimate the fundamental frequency that you would experience when blowing across the top of an empty cylindrical soft drink bottle that has a height of 10 cm. Assume that the bottle behaves like a tube with one end closed. Take the speed of sound to be 340 m/s. How would this frequency change if the bottle was only three quarters empty?

(16) What would be the range of an adjustable pipe length that has two open ends if its fundamental frequency spans the human hearing rang (form 20 Hz to 20 kHz)? Take the speed of sound to be 340 m/s.

(17) A tuning fork vibrating at a frequency of 384 Hz is held over the top end of a vertical tube while the other end is partially inserted in a water tank as shown in Fig. 16.23. The water level in the tube is lowered by opening a valve in the tank so that the length L of the air column slowly increases from an initial value of 30 cm. Determine the next two values of L that correspond to resonance.

Fig. 16.23 See Exercise (17)



(18) Assume that the speed of waves on a guitar string does not change when the string is fingered. If an unfingered string has a length $L = 0.75$ m and is tuned to play an F note (at 349 Hz). (a) How far from the end of this string must your finger be placed to play an A note (at 440 Hz). (b) What is the wavelength of the standing wave when this fingered string resonates at its fundamental frequency? (c) Find the frequency and wavelength of the sound waves that are produced by this string at that fundamental frequency.

(19) At a temperature of 25 °C, an open organ pipe produces the middle C note (262 Hz) with a fundamental standing wave. (a) What is the length of the pipe? (b) Find the frequency and wavelength of the fundamental standing wave in the pipe. (c) Find the frequency and wavelength of the sound produced in the air outside the pipe.

(20) An open organ pipe is tuned in a room where the temperature was set to 20 °C. If the temperature drops to 10 °C, what would be the percentage change in frequency generated by the pipe?

(21) In an air-filled tube closed at both ends, the distance between several nodes is 25 cm. When another gas replaces the air, the distance between that same number of nodes is 35 cm. If the speed of sound in air is 340 m/s, what is the speed in the gas?

(22) An organ pipe can resonate at the successive harmonics of frequencies 210, 350, and 490 Hz. (a) Is this pipe open at both ends or closed at one of its ends? Explain why. (b) What is the fundamental frequency of this pipe?

(23) A tube is open at both ends and has a length $L = 2$ m. It resonates at two successive harmonics of frequencies 355 and 440 Hz. (a) What is the fundamental frequency of this pipe? (b) What is the speed of sound in the air inside the tube?

(24) A tube has a length $L = 2.5$ m. How many harmonics are present in this tube within the human hearing range (from 20 Hz to 20 kHz) if: (a) the tube is open at both ends, and (b) is closed at one end?

(25) A pipe is open at one end and closed by a movable piston at the other end. A tuning fork of frequency 348 Hz is held at the open end. On a hot day a resonance occurs when the piston is at 0.25 m from the open end and again when it is at 0.75 m, see Fig. 16.24. (a) What is the speed of sound in the air inside the pipe? (b) How far from the open end will the piston be when the next resonance is experienced?

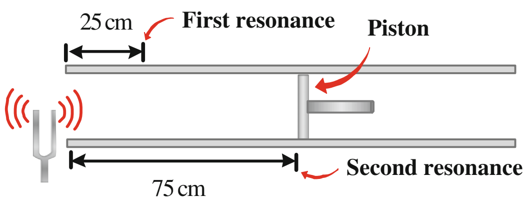


Fig. 16.24 See Exercise (25)

Section 16.5 Temporal Interference of Sound Waves: Beats

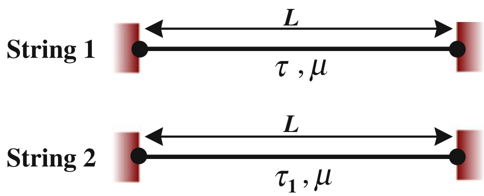
(26) Determine the beat frequency resulting from the superposition of the two sound waves given by:

$$y_1 = (1.5 \text{ cm}) \sin(3.5x - 1376\pi\, t),$$
$$y_2 = (1.5 \text{ cm}) \sin(3.5x - 1364\pi\, t).$$

where x and y are in centimeters and t is in seconds.

(27) Two identical violin strings have the same length L, tension τ, and exact fundamental frequency of 600 Hz. How much should we increase the tension of one of these strings to generate a sound beat of 6 Hz (see Fig. 16.25 for a new tension τ_1)?

Fig. 16.25 See Exercise (27)



(28) A standard tuning fork of frequency 512 Hz makes a beat frequency of 4 Hz with another fork of unknown frequency. The beat frequency disappears when the prongs of the second fork are waxed. What is the frequency of the unknown fork?

(29) A mistuned Middle C string in a piano (corresponds to key No. 40) has a *proper* fundamental frequency of 262 Hz, see Fig. 16.17. During the tuning trials, a musician hears 3 beats per second between the piano string and a standard oscillator of 262 Hz. (a) What are the possible frequencies of the string? (b) When the musician tightens the string slightly, he hears 4 beats per second. What is the frequency of the string now? (Hint: use the fact that tightening the string raises the wave speed and frequency) (c) By what percentage should the musician change the tension in the string to tune it?

(30) At a temperature of 30 °C, a source generates sound waves that propagate in the air with wavelengths $\lambda_1 = 1.62$ m and $\lambda_2 = 1.70$ m. (a) What beat frequency is heard? (b) How far in space is the distance between the maximum intensities? (Hint: see Fig. 16.15b)

Light Waves and Optics

17

Since ancient times, the nature and properties of light have been intensively investigated in an attempt to address many of our needs for a better life on Earth. Today, scientists view the behavior of light as **waves** (*electromagnetic waves*) in some situations and **particles** (*photons*) in other situations. In this chapter, we *briefly* introduce aspects of light that are understood best when using wave models, as applied to **geometrical** and **physical** optics. First, we study the reflection and refraction of light at the boundary between two media. Then we study formation of images when using the two types of mirrors and lenses.

17.1 Light Rays

It is useful to represent light waves with imaginary surfaces representing the **crests** of the electric field of the electromagnetic waves. These surfaces are called **wave fronts**, and the distance between any two successive wave fronts is referred to as the wavelength λ. While propagating in vacuum, light waves have a constant speed $c = \lambda f$, where $c = 2.9979 \times 10^8$ m/s $\simeq 3 \times 10^8$ m/s and f is the light's frequency. When we study light reflection from mirrors, refraction from a surface between two media, and propagation through lenses, we approximate light propagation by defining rays that travel in straight lines *perpendicular* to the wave fronts. This **ray approximation** technique is referred to as **geometrical optics**. On the other hand, when we study interference, diffraction, and polarization of light and need to get satisfactory descriptions of these phenomena, we treat light as waves. Such a study is referred to as **physical optics**.

H. A. Radi and J. O. Rasmussen, *Principles of Physics*,
Undergraduate Lecture Notes in Physics, DOI: 10.1007/978-3-642-23026-4_17,

In geometrical optics we first consider a point source S emitting light waves *isotropically* in all directions in a uniform medium. The emitted waves are a series of concentric spherical wave fronts with the source located at their common centers, and these waves can be approximated by straight-line rays perpendicular to the wave fronts, see Fig. 17.1a. Next, we consider the case when the source is very far and study the propagation of plane wave fronts. In this case, light rays propagate as straight lines perpendicular to the wave fronts in a given direction, see Fig. 17.1b.

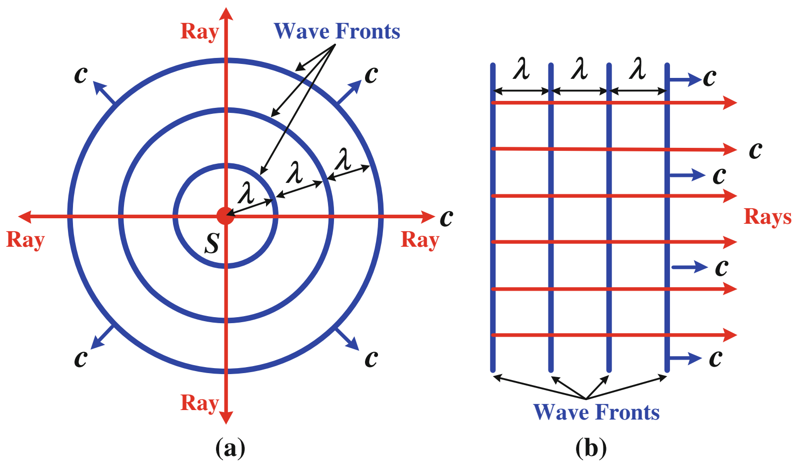


Fig. 17.1 Light waves of wavelength λ propagating with a speed c as: (a) spherical wave fronts and (b) plane wave fronts

> Spotlight
> We observe the following effects when plane wave fronts meet a barrier with a circular opening of diameter a:
>
> - If $\lambda \ll a$, the rays of the wave continue to move away from the opening in straight lines, see Fig. 17.2a.
> - If $\lambda \approx a$, the rays of the wave spread out from the opening in all directions, see Fig. 17.2b. This effect is called *diffraction*.
> - If $\lambda > a$ (or $\lambda \gg a$) the rays of the wave spread out more (*diffracted more*) in a way as if the opening is a point source, see Fig. 17.2c.

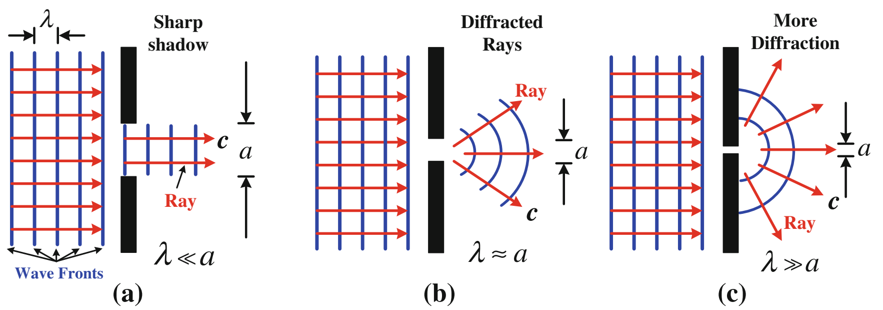


Fig. 17.2 A plane wave of light of wavelength λ is incident on a barrier that has a circular opening of diameter a. (a) When $\lambda \ll a$, the rays continue in a straight line and the ray approximation is valid. (b) When $\lambda \approx a$, the rays spread out from the opening in all directions. (c) When $\lambda > a$ (or $\lambda \gg a$) the circular opening behaves like a point source

17.2 Reflection and Refraction of Light

Figure 17.3 shows a beam of light of wavelength λ_1 and speed v_1 represented by a light ray traveling in a straight line in medium 1. The beam encounters the smooth boundary surface (or *interface*) of the transparent medium 2, which is more dense than medium 1. Part of the incident light is **reflected** by the surface and another part penetrates medium 2 with wavelength λ_2 and speed v_2. Unless the incident beam is perpendicular to the surface, the ray that enters medium 2 is bent at the boundary and is said to be **refracted**.

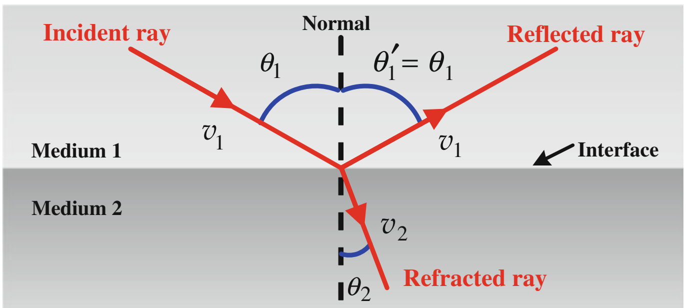


Fig. 17.3 An incident ray in medium 1 is reflected from the interface and maintains the same speed v_1, while the refracted ray is bent toward the normal and propagates in medium 2 with a speed $v_2 < v_1$

In Fig. 17.3, the incident, reflected, and refracted rays are all in a plane perpendicular to the boundary surface. In addition, the incident, reflected, and refracted rays make angles θ_1, θ_1', and θ_2, respectively, with the normal to the boundary surface. Moreover, v_1 and v_2 are the speeds of the light rays in media 1 and 2, respectively. Experiments and theory prove the following two laws:

Spotlight

- $\theta_1 = \theta_1'$ (Law of reflection) (17.1)
- $v_2 \sin\theta_1 = v_1 \sin\theta_2$ (Law of refraction) (17.2)

The speed of light v in any material is *less* than its speed in vacuum c. It is found that the value of v slightly depends on the wavelength λ. Also, it is convenient to define a dimensionless quantity known as the **index of refraction** n of a material as follows:

$$\boxed{n = \frac{c}{v}} \tag{17.3}$$

Since v is always less than c, then $n > 1$ for any material and $n = 1$ for vacuum. Table 17.1 lists the indices of refraction for various materials.

As light crosses an interface between two media, its speed v and wavelength λ change, but its frequency f remains the same. This can be understood by considering a normal incidence of light and treating light as photons, each with energy $E = hf$. If f changes, then energy will pile up at the interface, which is a mechanism that cannot take place under the laws of Physics. Since the relation $v = \lambda f$ must be satisfied in both media of Fig. 17.3, and since the frequency f of the incident and refracted rays must be the same, then:

$$v_1 = \lambda_1 f \quad \text{and} \quad v_2 = \lambda_2 f \tag{17.4}$$

If the media 1 and 2 have indices of refraction n_1 and n_2, respectively, then Eq. 17.3 leads to:

$$n_1 = c/v_1 \quad \text{and} \quad n_2 = c/v_2 \tag{17.5}$$

Table 17.1 Some indices of refraction[a]

Medium	Index of refraction n
Vacuum	Exactly 1
Air[b]	1.000 29
Carbon dioxide[b]	1.000 45
Water	1.333
Acetone	1.360
Ethyl alcohol	1.361
Sugar solution (30%)	1.38
Glycerin	1.473
Sugar solution (80%)	1.49
Benzene	1.501
Ice	1.309
Fused quartz	1.46
Polystyrene	1.49
Crown glass	1.52
Sodium chloride	1.544
Flint glass	1.66
Heaviest flint glass	1.89
Cubic zirconium	2.20
Diamond	2.419
Gallium	3.50

[a] For light with a wavelength of 589 nm traveling in a vacuum. [b] At 0 °C and 1 atm

Using Eq. 17.4 with Eq. 17.5 will give:

$$\frac{\lambda_1}{\lambda_2} = \frac{v_1}{v_2} = \frac{c/n_1}{c/n_2} = \frac{n_2}{n_1} \tag{17.6}$$

This gives:

$$\boxed{n_1\,\lambda_1 = n_2\,\lambda_2} \tag{17.7}$$

If medium 1 is vacuum (or air), then $n_1 = 1$ and $\lambda_1 \equiv \lambda$. In addition, if n is the index of refraction of medium 2, and λ_n is its refracted wavelength, then we find that:

$$\boxed{n = \frac{\lambda}{\lambda_n} = \frac{\text{Wavelength of the incident light in vacuum}}{\text{Wavelength of refracted light in the medium}}} \tag{17.8}$$

Since Eq. 17.3 leads to the ratio $v_1/v_2 = n_2/n_1$, then the law of refraction given by Eq. 17.2 can be written as:

$$n_1 \sin\theta_1 = n_2 \sin\theta_2 \quad \text{(Snell's Law)} \tag{17.9}$$

This form of the law of refraction is known as Snell's law of refraction, and we will use this form in tackling most of our examples.

To compare the refractive angle θ_2 with the incident angle θ_1 and the relative ratio n_1/n_2 for a light beam propagating from medium 1 to medium 2, we present the following results:

- If $n_2 = n_1$, then $\theta_2 = \theta_1$. In other words, the light beam will not be deflected (refracted) as it changes media, as in Fig. 17.4a.
- If $n_2 > n_1$, then $\theta_2 < \theta_1$. In other words, the light beam will refract and bend *toward* the normal as in Fig. 17.4b.
- If $n_2 < n_1$, then $\theta_2 > \theta_1$. In other words, the light beam will refract and bend *away* from the normal as in Fig. 17.4c.

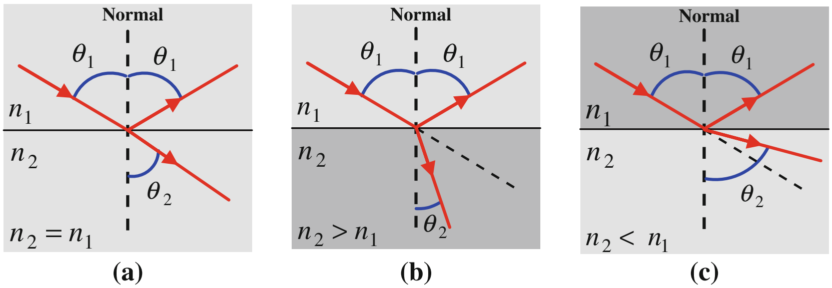


Fig. 17.4 Light propagating from a medium of index of refraction n_1 into a medium of index of refraction n_2. (a) When $n_2 = n_1$, the beam does not bend. (b) When $n_2 > n_1$, the beam bends toward the normal. (c) When $n_2 < n_1$, the beam bends away from the normal

Example 17.1

The wavelength of yellow light in vacuum is 600 nm. (a) What is the speed of this light in vacuum and water? (b) What is the frequency of this light in vacuum and water? (c) What is the wavelength of this light in water?

Solution: (a) The speed of the yellow light in vacuum ($n = 1$) and water ($n = 1.333$) can be obtained by using Eq. 17.3 as follows:

$$c = 3 \times 10^8 \text{ m/s} \quad \text{and} \quad v = \frac{c}{n} = \frac{3 \times 10^8 \text{ m/s}}{1.333} = 2.25 \times 10^8 \text{ m/s}$$

(b) We use the equation $v = \lambda f$ to prove that the frequency of the yellow light in vacuum and water is the same, as follows:

$$f = \frac{c}{\lambda} = \frac{3 \times 10^8 \text{ m/s}}{600 \times 10^{-9} \text{ m}} = 5 \times 10^{14} \text{ Hz}$$

$$f_n = \frac{v}{\lambda_n} = \frac{c/n}{\lambda/n} = \frac{c}{\lambda} = 5 \times 10^{14} \text{ Hz}$$

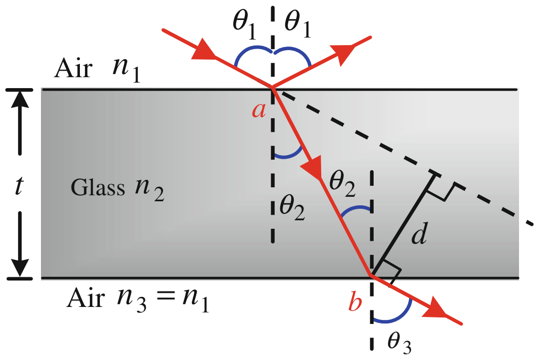
(c) By using Eq. 17.8, we can calculate the wavelength of the yellow light in water as follows:

$$\lambda_n = \frac{\lambda}{n} = \frac{600 \times 10^{-9} \text{ m}}{1.333} = 4.501 \times 10^{-7} \text{ m} = 450.1 \text{ nm}$$

Example 17.2

A beam of monochromatic light traveling through air strikes a slab of glass at an angle $\theta_1 = 60°$ to the normal, see Fig. 17.5. The glass has a thickness $t = 1$ cm and refractive index $n = 1.52$. (a) Find the angle of refraction θ_2. (b) Show that the emerging beam is parallel to the incident beam. (c) At what distance d does the beam shift from the original?

Fig. 17.5



Solution: (a) We apply Snell's law at point a on the upper surface:

$$(1) \quad n_1 \sin\theta_1 = n_2 \sin\theta_2$$

$$\sin\theta_2 = \frac{n_1}{n_2}\sin\theta_1 = \frac{1}{1.52}\sin 60° = 0.5698 \quad \Rightarrow \quad \theta_2 = 34.7°$$

(b) Applying Snell's law again at point b on the lower surface gives:

$$(2) \quad n_2 \sin\theta_2 = n_3 \sin\theta_3 = n_1 \sin\theta_3$$

Substituting $n_2 \sin\theta_2$ from Eq. (1) into (2) gives:

$$\sin\theta_1 = \sin\theta_3$$

Therefore,
$$\theta_3 = \theta_1$$

Thus, the slab does not alter the direction of the emerging beam, it only shifts the beam laterally by an offset distance of magnitude d.

(c) From the geometry of the figure, we find that:

$$d = ab\sin(\theta_1 - \theta_2) \quad \text{and} \quad \cos\theta_2 = t/ab$$

Thus:
$$d = \frac{\sin(\theta_1 - \theta_2)}{\cos\theta_2} t$$

Therefore, for a given incident angle θ_1, the refracted angle θ_2 is solely determined by n_2, and the shift d is directly proportional to the thickness of the slab, t.

Substituting the values of θ_1, θ_2, and t, into the above relation gives:

$$d = \frac{\sin(60° - 34.7°)}{\cos 34.7°} \times 1\,\text{cm} = 0.52\,\text{cm}$$

17.3 Total Internal Reflection and Optical Fibers

When light is directed from a medium having a higher index of refraction n_1 toward one having a lower index n_2, i.e. $n_1 > n_2$, the refracted ray is bent away from the normal. At some particular angle of incidence θ_c, called the **critical angle**, see Fig. 17.6, the refracted ray 4 moves parallel to the boundary, i.e. $\theta_2 = 90°$. In addition, all the incident light energy will be associated with the reflected ray $4'$.

All rays having angles of incidence θ_1 greater than θ_c are entirely reflected at the boundary, see ray 5 in Fig. 17.6. For those rays, the angle of incidence must be equal to the angle of reflection.

To find θ_c, we use Snell's law given by Eq. 17.9 and then substitute $\theta_1 = \theta_c$ and $\theta_2 = 90°$, to find that:

$$n_1 \sin\theta_c = n_2 \sin 90° = n_2 \tag{17.10}$$

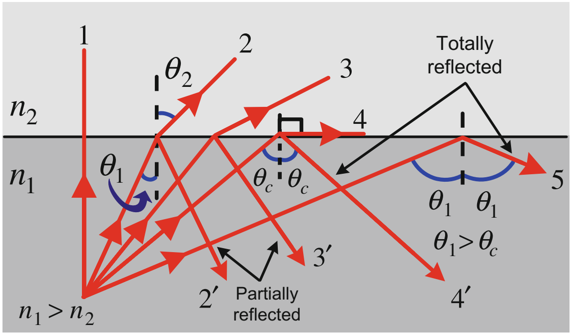


Fig. 17.6 When $n_1 > n_2$, the angle of refraction θ_2 will be greater than the angle of incidence θ_1. As θ_1 increases, θ_2 will increase until $\theta_2 = 90°$. For $\theta_1 > \theta_c$, all rays will be reflected without any refraction

This gives:

$$\boxed{\sin\theta_c = \frac{n_2}{n_1}} \qquad \boxed{(n_1 > n_2)} \tag{17.11}$$

When $n_1 \gg n_2$, Eq. 17.11 produces small values of θ_c.

Diamonds and Cubic zirconium crystals are good examples of media that have a high index of refraction. The critical angle for a diamond crystal in air is $\theta_c = \sin^{-1}(1/2.419) = 24.4°$. Any light ray inside the crystal that strikes its surfaces at an angle greater than the critical angle will be completely reflected back into the crystal. This ray might undergo repeated total internal reflections within the crystal, and this causes the crystal to sparkle.

Another important feature of internal reflection is the use of a thin flexible pipe made of glass or transparent plastic as a light transmitter. This kind of flexible light pipe is called an **optical fiber**. As shown in Fig. 17.7a, b, light is confined to travel within a thin curved fiber pipe because of successive total internal reflections. A bundle of fibers can be used to form an optical fiber cable, as in Fig. 17.7c. This cable can transmit light, images, and even telephone calls from one point to another with *little loss*. This technique is used extensively in modern industry and is known as **fiber optics**. A physician can explore or even perform surgery by inserting a bundle of optical fibers into the human body, avoiding the need to make large incisions. Optical fibers are also commonly used in fiber-optic communications, which permits data, voice, and video transmission over longer distances than other forms of communication media.

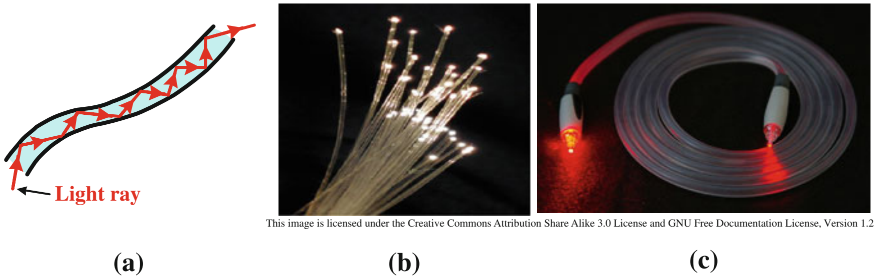


Fig. 17.7 (a) A ray of light traveling in a curved transparent pipe by multiple total internal reflections. (b) A bundle of optical fibers. (c) An illuminated fiber optic audio cable

Example 17.3

Part of a fish tank made of glass is shown in Fig. 17.8. A ray starting from the left passes through the glass and is totally internally reflected at the water-air interface. Take the index of refraction for the glass and water to be 1.5 and 1.33, respectively. (a) Find the critical angle θ_c for the total internal reflection at the water-air boundary. (b) Find the angle θ_2 between the light ray and the normal inside the glass wall. (c) Find the incident angle θ_1 between the light ray and the normal to the glass.

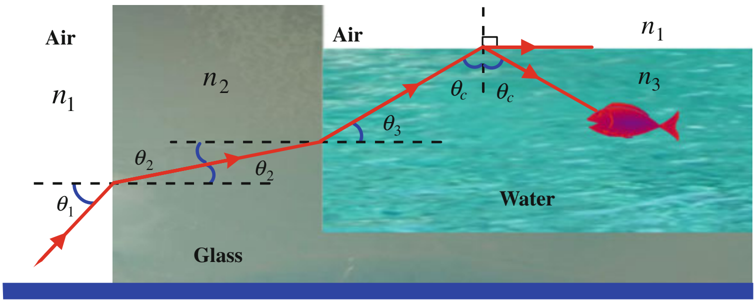


Fig. 17.8

Solution: (a) To find θ_c, we apply Snell's law at the water-air interface of the figure as follows:

$$n_3 \sin\theta_c = n_1 \sin 90^\circ$$

Thus: $$\theta_c = \sin^{-1}\frac{n_1}{n_3} = \sin^{-1}\frac{1}{1.33} = \sin^{-1} 0.752 = 48.8^\circ$$

(b) From the right-angle triangle at the glass-water interface we can find the refracted angle θ_3 in water to be:

$$\theta_3 = 90^\circ - \theta_c = 41.2^\circ$$

Using Snell's law again at the glass-water interface, we have:

$$n_2 \sin\theta_2 = n_3 \sin\theta_3$$

Thus: $$\sin\theta_2 = \frac{n_3 \sin\theta_3}{n_2} = \frac{1.33 \times \sin 41.2^\circ}{1.5} = 0.584$$

$$\theta_2 = \sin^{-1} 0.585 = 35.7^\circ$$

(c) Since the sides of the glass-walled fish tank are parallel, we can again apply Snell's law at the air-glass interface to calculate θ_1 as follows:

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

Thus: $$\sin\theta_1 = \frac{n_2 \sin\theta_2}{n_1} = \frac{1.5 \times \sin 35.7^\circ}{1} = 0.875$$

$$\theta_1 = \sin^{-1} 0.875 = 61^\circ$$

17.4 Chromatic Dispersion and Prisms

Except in vacuum, the index of refraction depends on the light's wavelength, i.e. its color, see Sect. 27.7. Therefore if a beam of light consists of rays of different wavelengths (as in the case of white light), each ray will refract by a different angle from a surface. This spread of light is called **chromatic dispersion**, or simply **dispersion**.

Generally, the index of refraction n decreases with increasing wavelengths. This means that the violet light (with wavelength $\lambda \simeq 425$ nm and index $n = 1.3435$) bends more than the red light (with wavelength $\lambda \simeq 700$ nm and index $n = 1.3318$) when passing through the interface between two materials. Figure 17.9a shows this for a glass block, and Fig. 17.9b shows this for a glass prism.

The prism of Fig. 17.9b is more commonly used to observe color separation of white light because the dispersion at the first surface is enhanced at the second interface. Thus, the violet ray in the white light of Fig. 17.9b will emerge from the right surface with an **angle of deviation** δ_V which is greater than the angle of deviation δ_R of the red ray. The difference $\delta_V - \delta_R$ is known as the **angular dispersion**, while δ_Y is the mean deviation of the yellow rays.

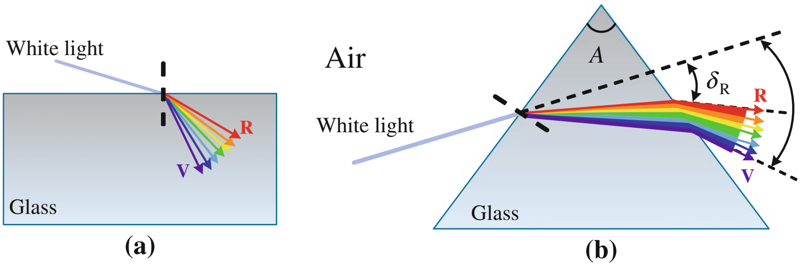


Fig. 17.9 A schematic representation of the dispersion of white light. The violet color is bent more than the red color. (a) Dispersion in a glass block. (b) Dispersion in a prism

The general expression of δ for any *color* turns out to be rather complicated. However, as the angle of incidence decreases from a large value, the angle of deviation δ is found to decrease at first and then increase. The angle of *minimum deviation* δ_m is found when the ray passes through the prism symmetrically. This angle is related to the angle of the prism A, and its index of refraction n by the relation:

$$n = \frac{\sin[(A+\delta_m)/2]}{\sin(A/2)} \xrightarrow[\text{When } A \text{ is small}]{} n = \frac{A+\delta_m}{A} \tag{17.12}$$

The most charming example of color dispersion is that of a rainbow. To understand the formation of a rainbow we consider a horizontal overhead white sunlight that is intercepted by spherical raindrops. Figure 17.10 shows refractions and reflection in two raindrops that explain how light rays from the Sun reach an observer's eye. The first refraction separates the sunlight into its color components. Each color is then reflected at the raindrop's inner surface. Finally, a second refraction increases the separation between colors, and these color rays finally make it to the observer's eye. Using Snell's law and geometry, we find that the maximum deviation angles of red and violet are about 42 and 40°, respectively. The rainbow that you can see is a personal one because different observers receive light from different raindrops.

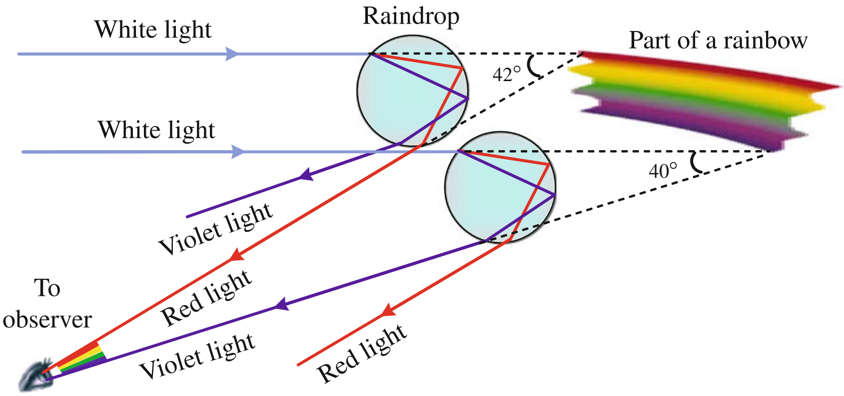
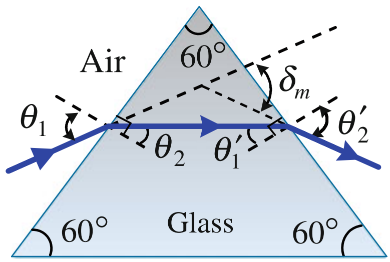


Fig. 17.10 A sketch of a rainbow formed by horizontal sunlight rays. Only two enlarged raindrops are used to explain the rainbow's formation for the case of the *red* and *violet* colors only

Example 17.4

A monochromatic light ray is incident from air (with index $n_1 = 1$) onto an equilateral glass prism (with index $n_2 = 1.5$) and is refracted parallel to one of its faces (i.e. we have a symmetric ray), see Fig. 17.11. (a) What is the angle of incidence θ_1 at the first face? (b) What is the subsequent angle of incidence at the second face? (c) Is the light ray totally reflected at the second face? If not, find the angle of minimum deviation of the light ray. Then check that Eq. 17.12 holds.

Fig. 17.11



Solution: (a) The path of a symmetric light ray going through the prism (of apex angle 60°) and back out again into the air is shown.

Using elementary geometry, this figure shows that the angle of refraction θ_2 can be found as follows:

$$\theta_2 + 60^\circ = 90^\circ$$

Thus:
$$\theta_2 = 30^\circ$$

Therefore, using Snell's law:

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

We get:

$$\theta_1 = \sin^{-1}\left(\frac{n_2 \sin\theta_2}{n_1}\right)$$
$$= \sin^{-1}\left(\frac{1.5 \times \sin 30^\circ}{1}\right)$$
$$= \sin^{-1}(0.75) = 48.59^\circ$$

(b) Again, by simple geometry the horizontal light ray inside the prism must be incident on the second face with an angle $\theta_1' = \theta_2 = 30^\circ$.

(c) We know that if the incident angle is greater than the critical angle, then total internal reflection must occur. Therefore, we first calculate the critical angle as follows:

$$\theta_c = \sin^{-1}\frac{n_1}{n_2}$$
$$= \sin^{-1}\frac{1}{1.5}$$
$$= \sin^{-1} 0.666$$
$$= 41.8^\circ$$

Since $\theta_1' < \theta_c$, then the light ray refracts at the second face, and total internal reflection will ***not*** occur.

Using the geometry shown in the figure, we can find for this special case that the angle of minimum deviation is given by the following relation:

$$\delta_m = 2(\theta_1 - \theta_2)$$
$$= 2(48.59^\circ - 30^\circ)$$
$$= 37.18^\circ$$

Substituting $A = 60^\circ$ and $\delta_m = 37.18^\circ$ in Eq. 17.12 gives:

$$n_2 = \frac{\sin[(A + \delta_m)/2]}{\sin(A/2)}$$
$$= \frac{\sin[(60^\circ + 37.18^\circ)/2]}{\sin(60^\circ/2)}$$
$$= \frac{0.75}{0.5} = 1.5$$

This value of n_2 obtained from Eq. 17.12 satisfies the given value of index of refraction of the prism.

17.5 Formation of Images by Reflection

Mirrors gather and redirect light rays to form images of objects by reflection. To explain this, we will use the ray approximation model in terms of geometric optics, in which light travels in straight lines.

17.5.1 Plane Mirrors

A **plane mirror** is a plane surface that can reflect a beam of light in one direction instead of either scattering it in many directions or absorbing it.

Figure 17.12a shows how a plane mirror can form an image of a point object O located at a distance p from the mirror. In this figure, we consider two diverging rays leaving O and strike the mirror and then are reflected to the eye of an observer. The rays appear to diverge from point I behind the mirror. Thus, point I is the **image** of point O. The geometry of the figure indicates that the image I is opposite to object O and is located at a distance as far behind the mirror as the object is in front of the mirror.

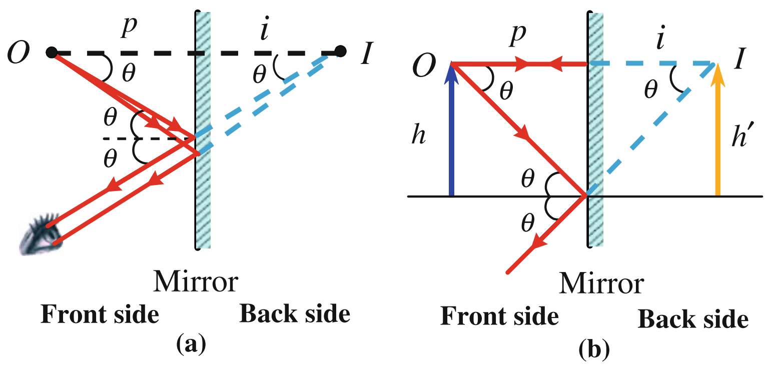


Fig. 17.12 A geometric sketch that is used to depict an image of an object placed in front of a plane mirror. (a) An image formed for a point object. (b) An image formed by an extended object, where the object is an upright arrow of height h

Figure 17.12b shows how a plane mirror can form an image of an extended object O. The object in this figure is an upright arrow of height h placed at a distance p from the mirror. The full image can be inferred by locating the images of selected points on the object. One of the two rays at the tip of the arrow follows a horizontal path to the mirror and reflects back on itself. The second ray follows an oblique path and reflects according to the laws of reflection, as shown in the figure. Using geometry we find that the image I is *upright*, opposite to the object, and located *behind* the mirror at a distance equal to the object's distance in front of the mirror. In addition, the height of the object and its image are equal. Also, the geometry of Fig. 17.12b indicates that $h'/h = i/p$.

The image I in both parts of Fig. 17.12 is called a **virtual image** because no light rays pass through it. In addition, the value of i is considered to be *negative* since the image is *behind the mirror* and the value of h' is considered to be *positive* since the image is *upright*.

We define the **lateral magnification** M of a horizontal overhead image as follows:

$$\boxed{M = \frac{\text{Image height}}{\text{Object height}} = \frac{h'}{h}} \tag{17.13}$$

We can use the relation $h'/h = i/p$ and the sign convention to write the lateral magnification M as follows:

$$\boxed{M = \frac{h'}{h} = -\frac{i}{p}} \tag{17.14}$$

For plane mirrors, $M = 1$, since h' is positive and equal to h, or i is negative and has a magnitude equal to p. The image formed by a plane mirror is upright but reversed. The reversal of right and left is the reason why the word **AMBULANCE** is printed as "ƎƆИA⅃UᗺMA" across the front of ambulance vehicles. People driving in front of such an ambulance can see the word "AMBULANCE" immediately evident when looking in their rear-view mirrors and make way.

17.5.2 Spherical Mirrors

A spherical mirror is simply a mirror in the shape of a small section of the surface of a sphere that has a center C and radius R. When light is reflected from the concave surface of the mirror, the mirror is called a **concave mirror**. However, when light is reflected from the convex surface of the mirror, the mirror is called a **convex mirror**.

Focal Point of a Spherical Mirror

The **principal axis** (or the **symmetry axis**) of a spherical mirror is defined as the axis that passes through its center of curvature C and the center of the mirror c, see Fig. 17.13. We consider the reflection of light coming from an infinitely far object O located on the principal axis of a concave or convex spherical mirror. Because of the great distance between the object and the mirror, the light rays reach the mirror *parallel* to its principal axis.

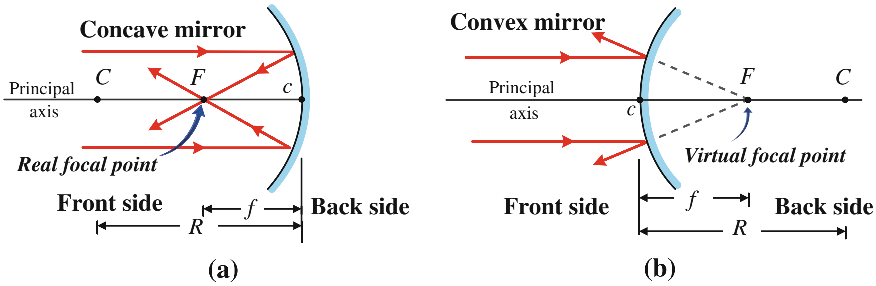


Fig. 17.13 (a) Two parallel light rays will meet at a real focal point after reflecting from a concave mirror. (b) The same rays will diverge from a convex mirror and appear to come from a virtual focal point

When parallel rays reach the surface of the **concave mirror** of Fig. 17.13a, they will reflect and pass through a common point F. If we place a card at F, a point image would appear at F. Therefore, this point is called the **real focal point**. However, in the case of the **convex mirror** of Fig. 17.13b, the parallel rays reflect from the mirror and appear to diverge from a common point F *behind* the mirror. If we could place a card at F, no image would appear on the card. Therefore, this point is called the **virtual focal point**. The distance f from the center of the mirror to the focal point (real or virtual) is called the **focal length** of the mirror.

For concave and convex mirrors, the following relation relates the focal length f to the radius of curvature R:

$$\boxed{f = \frac{R}{2}} \quad \text{(Spherical mirror)} \tag{17.15}$$

17.5.2.1 Concave Mirrors

Sharp and Blurred Images

Rays that diverge from any point on an object and make *small* angles with the principal axis (called **paraxial rays**) will reflect from the spherical concave mirror and intersect at *one* image point. See Fig. 17.14a for a point object on the principal axis. On the other hand, rays that diverge from the same point and make large angles with the principal axis will reflect and intersect at *different* image points, see Fig. 17.14b. This condition is called ***spherical aberration***.

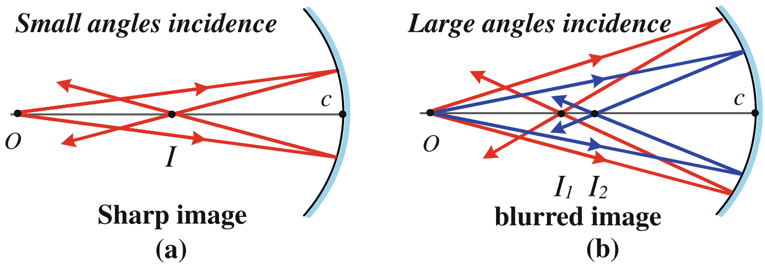


Fig. 17.14 (a) When rays diverge from point object O at small angles with the principal axis, they all reflect from the spherical concave mirror and meet at the same point image I. (b) When rays diverge from O at large angles with the principal axis, they reflect from the spherical concave mirror and meet at different points $I_1, I_2, \ldots$

The Mirror Equation

The relationship between an object's distance p, its image distance i, and the focal length f of a concave mirror can be found when light rays make *small* angles with the principal axis (paraxial rays). Figure 17.15a shows two rays (leaving an object O of height h) reflected to form an image I of height h'. The first ray strikes the mirror at its center c and is reflected. The second ray passes through the focal point F and reflects parallel to the principal axis.

From the purple triangles of Fig. 17.15a, we see that:

$$\tan\theta = \frac{h}{p} = \frac{h'}{i} \quad \Rightarrow \quad \frac{h'}{h} = \frac{i}{p} \tag{17.16}$$

From the yellow triangles of Fig. 17.15b, we see that:

$$\tan\alpha = \frac{h}{p-f} = \frac{h'}{f} \quad \Rightarrow \quad \frac{h'}{h} = \frac{f}{p-f} \tag{17.17}$$

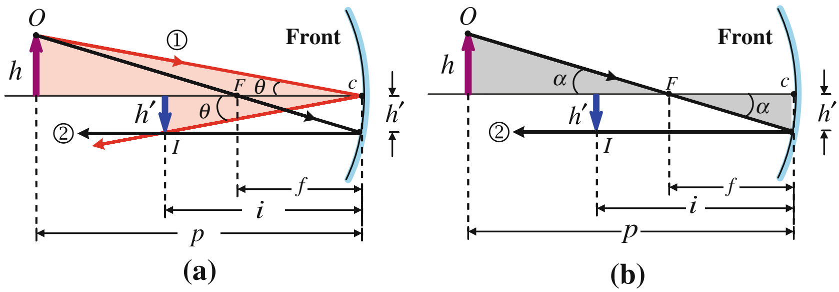


Fig. 17.15 (a) Intersection of two rays produced by a spherical concave mirror to form an image of the tip of an arrow. (b) Demonstration of the geometry produced by only the second ray

By comparing Eqs. 17.16 and 17.17, we find that:

$$\frac{i}{p} = \frac{f}{p - f} \quad \Rightarrow \quad ip - if = pf \tag{17.18}$$

Dividing both sides of this equation by *pif*, we get:

$$\boxed{\frac{1}{p} + \frac{1}{i} = \frac{1}{f}} \tag{17.19}$$

Equation 17.19 is known as the **mirror equation** for spherical mirrors, and this expression holds when we interchange p and i, i.e. when we can replace the object O by the image I and vice versa. For a given value of f, we notice the following for *concave* mirrors:

- When $p > f$, the image distance i is positive. A ***positive*** value of i means that the image is ***real*** and ***inverted***. See Fig. 17.16a,b for images ***smaller*** or ***larger*** than the object.
- When $p < f$, the mirror equation is satisfied by a negative value of the image distance i. The ***negative*** image distance means that the image is ***virtual***. When we extend two rays from the object we find that the virtual image is ***upright*** and ***enlarged***, see Fig. 17.16d.

If we use this sign convention in the lateral magnification Eq. 17.13, then we can also write M as follows:

$$\boxed{M = \frac{h'}{h} = -\frac{i}{p}} \tag{17.20}$$

We get an upright image for positive values of M and an inverted image for negative values of M as shown in Fig. 17.16.

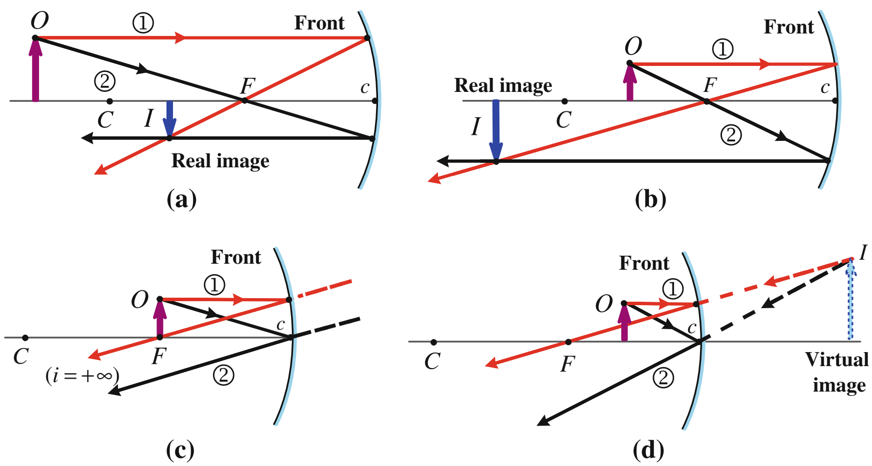


Fig. 17.16 (a) An object O outside the center of curvature C. (b) The object between the focal point F and C. (c) The object at F. (d) The object inside the focal point F and its virtual upright image I

17.5.2.2 Convex Mirrors

Convex mirrors like those shown in Fig. 17.17 are called **diverging mirrors**. The images formed by these types of mirrors are virtual because the reflected rays appear to originate from an image behind the mirror. Furthermore, the images are always ***upright*** and ***smaller*** than the object. Because of this feature, these types of mirrors are often used in stores to prevent shoplifting.

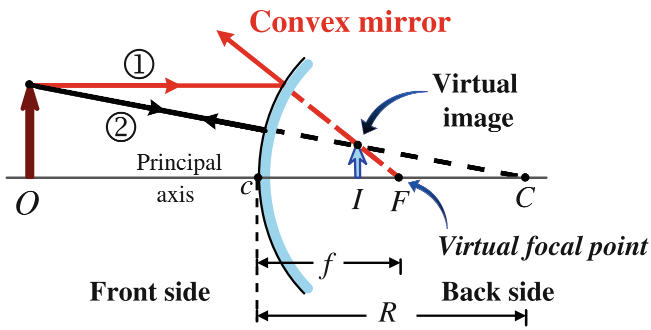


Fig. 17.17 When the object O is in front of a convex mirror, the image is virtual, upright, and smaller than the object

We can use Eqs. 17.19 and 17.20 for either concave or convex spherical mirrors if we stick to the sign conventions presented in Table 17.2. This table gives the sign conventions for the quantities f, i, h', and M.

Table 17.2 Sign conventions for spherical mirrors[a]

Quantity	Symbol	Positive values when	Negative values when
Focal length	f	The mirror is *concave*	The mirror is *convex*
Image location	i	The image is in *front* of mirror (*real image*)	The image is in *behind* the mirror (*virtual image*)
Image height	h'	The Image is *upright*	The Image is *inverted*
Magnification	M	The Image is *upright*	The Image is *inverted*

[a] The object location p and its height h are both positive

Example 17.5

A concave mirror has a focal length of 10 cm. Locate and describe the image formed by an object having distances: (a) $p = 25$ cm, (b) $p = 15$ cm, (c) $p = 10$ cm, and (d) $p = 5$ cm.

Solution: Concave mirrors have a positive focal length, i.e. $f = +10$ cm. (a) To find the image distance, we use Eq. 17.19 as follows:

$$\frac{1}{p} + \frac{1}{i} = \frac{1}{f} \quad \Rightarrow \quad \frac{1}{25\text{ cm}} + \frac{1}{i} = \frac{1}{10\text{ cm}} \quad \Rightarrow \quad \frac{1}{i} = \frac{1}{10\text{ cm}} - \frac{1}{25\text{ cm}}$$

$$\frac{1}{i} = \frac{25\text{ cm} - 10\text{ cm}}{250\text{ cm}^2} \quad \Rightarrow \quad \frac{1}{i} = \frac{15}{250\text{ cm}} \quad \Rightarrow \quad i = \frac{50}{3}\text{ cm}$$

The positive sign of i indicates that the image is *real* and located on the front side of the mirror. The magnification of the image can be determined using Eq. 17.20 as:

$$M = -\frac{i}{p} = -\frac{50/3\text{ cm}}{25\text{ cm}} \quad \Rightarrow \quad M = -\frac{2}{3}$$

The negative sign of M indicates that the image is *inverted*. In addition, the image is *reduced* (66.7% of the size of the object) because the absolute value of M is less than unity, see Fig. 17.16a.

(b) When $p = 15$ cm, the mirror and magnification equations give:

$$\frac{1}{15\text{ cm}} + \frac{1}{i} = \frac{1}{10\text{ cm}} \quad \Rightarrow \quad \frac{1}{i} = \frac{1}{10\text{ cm}} - \frac{1}{15\text{ cm}} \quad \Rightarrow \quad i = 30\text{ cm}$$

$$M = -\frac{i}{p} = -\frac{30 \text{ cm}}{15 \text{ cm}} \quad \Rightarrow \quad M = -2$$

The image is *real* when i is positive, *inverted* when M is negative, and *enlarged* when $|M| > 1$, see Fig. 17.16b.

(c) When $p = 10$ cm, the mirror equation gives:

$$\frac{1}{10 \text{ cm}} + \frac{1}{i} = \frac{1}{10 \text{ cm}} \quad \Rightarrow \quad i = \infty$$

This means that the reflected rays are parallel to one another and formed at an infinite distance from the mirror, see Fig. 17.16c.

(d) When $p = 5$ cm, the mirror and magnification equations give:

$$\frac{1}{5 \text{ cm}} + \frac{1}{i} = \frac{1}{10 \text{ cm}} \quad \Rightarrow \quad \frac{1}{i} = \frac{1}{10 \text{ cm}} - \frac{1}{5 \text{ cm}} \quad \Rightarrow \quad i = -10 \text{ cm}$$

$$M = -\frac{i}{p} = -\frac{(-10 \text{ cm})}{5 \text{ cm}} \quad \Rightarrow \quad M = +2$$

The image is *virtual* (or behind the mirror) because i is negative, *upright* because M is positive, and *enlarged* (*twice as large*) because M is greater than unity, see Fig. 17.16d.

Example 17.6

An anti-shoplifter convex spherical mirror has a radius of curvature of 0.4 m. Locate and describe the image formed by a man standing 3.8 m away from the mirror.

Solution: The focal length of a mirror is half of its radius of curvature, but for a convex mirror, the focal length that must be used in the mirror equation is:

$$f = -R/2 = -0.2 \text{ m}$$

When $f = -0.2$ m and $p = 3.8$ m, the mirror and magnification equations give:

$$\frac{1}{p} + \frac{1}{i} = \frac{1}{f} \quad \Rightarrow \quad \frac{1}{3.8 \text{ m}} + \frac{1}{i} = -\frac{1}{0.2 \text{ m}}$$

$$\frac{1}{i} = -\frac{1}{0.2 \text{ m}} - \frac{1}{3.8 \text{ m}} \quad \Rightarrow \quad i = -0.19 \text{ m}$$

$$M = -\frac{i}{p} = -\frac{(-0.19\ \text{m})}{3.8\ \text{m}} \quad \Rightarrow \quad M = +0.05$$

The image is *virtual* (or behind the mirror) because i is negative, *upright* because M is positive, and *reduced* (5% of the man's size) because M is less than unity.

17.6 Formation of Images by Refraction

Lenses gather and redirect light rays to form images of objects by refraction. Again, we will use the ray-approximation model of geometric optics in which light travels in straight lines to form images.

17.6.1 Spherical Refracting Surfaces

Consider two transparent media having indices of refraction n_1 and n_2 and the boundary between them is a spherical surface of radius R, see Fig. 17.18. Assume a point object O exists in the medium with an index of refraction n_1. In addition, assume that all rays make small angles with the principal axis (paraxial rays) when they leave O and focus at point I after being refracted at the spherical surface.

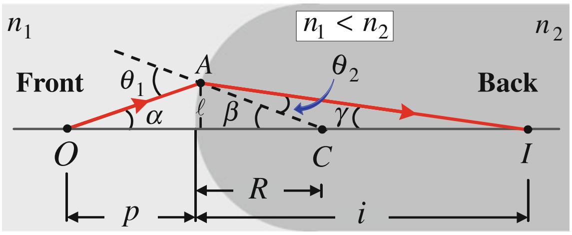


Fig. 17.18 Geometry used to derive Eq. 17.26 for $n_1 < n_2$

Applying Snell's law on the single ray of Fig. 17.18 gives:

$$n_1 \sin\theta_1 = n_2 \sin\theta_2 \tag{17.21}$$

Because θ_1 and θ_2 are assumed to be small angles, we use the small-angle approximation $\sin\theta \approx \theta$, where θ is measured in radians, to have:

$$n_1\,\theta_1 = n_2\,\theta_2 \tag{17.22}$$

Next, we use the rule that an exterior angle of any triangle equals the sum of the two opposite interior angles. Applying this rule to triangles *OAC* and *AIC* of Fig. 17.18, we get:

$$\theta_1 = \alpha + \beta \quad \text{and} \quad \beta = \theta_2 + \gamma \tag{17.23}$$

where α, β, and γ are also small angles. Eliminating θ_1 and θ_2 from the last two equations gives:

$$n_1\,\alpha + n_2\,\gamma = (n_2 - n_1)\beta \tag{17.24}$$

From the figure we find that the following holds true for paraxial rays:

$$\tan\alpha \approx \alpha \approx \frac{\ell}{p} \qquad \tan\beta \approx \beta \approx \frac{\ell}{R} \qquad \tan\gamma \approx \gamma \approx \frac{\ell}{i} \tag{17.25}$$

When substituting these expressions into Eq. 17.24 and eliminating h from the result, we get the following relation:

$$\boxed{\frac{n_1}{p} + \frac{n_2}{i} = \frac{n_2 - n_1}{R}} \tag{17.26}$$

which is valid regardless of which index of refraction is greater. We notice that for a fixed object distance p, the image distance i is independent of the small angle that the paraxial ray makes with the axis. Therefore, we conclude that all paraxial rays from point O focus at the same point I. The magnification is given by:

$$\boxed{M = \frac{h'}{h} = -\frac{n_1\ i}{n_2\ p}} \tag{17.27}$$

Again, we must use a sign convention if we want to apply Eq. 17.26 to a variety of cases; see Table 17.3. We define the side of the surface in which *light rays originate* as the *front side*. The other side is called the back side and is the side in which real images are formed.

17.6.2 Flat Refracting Surfaces

When the refracting surface is flat, its radius of curvature R is infinite (i.e. $R = \infty$) and Eq. 17.26 reduces to:

$$\boxed{i = -\frac{n_2}{n_1}p} \tag{17.28}$$

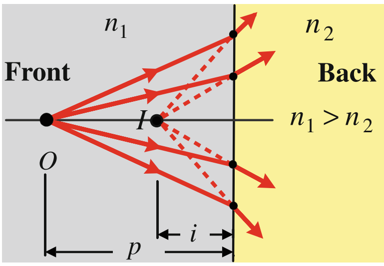
where the sign of i is opposite that of p. Thus, according to Table 17.3, the image formed by a flat refracting surface is on the same side of the surface as the object, see Fig. 17.19.

Table 17.3 Sign conventions for refracting surfaces[a]

Quantity	Symbol	Positive values when	Negative values when
Radius	R	The center of curvature is *behind* the surface	The center of curvature is in *front* of the surface
Image location	i	The image is in *behind* the surface (*real image*)	The image is in *front* of the surface (*virtual image*)
Image height	h'	The image is *upright*	The image is *inverted*
Magnification	M	The image is *upright*	The image is *inverted*

[a] When the object is in front of the surface, the object location p and its height h are positive.

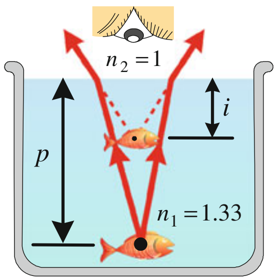
Fig. 17.19 A virtual image formed by a flat refracting surface when $n_1 > n_2$. All rays are assumed to be paraxial



Example 17.7

A small fish is at a distance p below the water surface, see Fig. 17.20. The index of refraction of water and air are $n_1 = 1.33$ and $n_2 = 1$, respectively. What is the apparent depth of the fish as viewed by an observer directly above the water?

Fig. 17.20



Solution: For flat refracting surfaces, we use Eq. 17.28 to find the location of the image. Thus:

$$i = -\frac{n_2}{n_1}p = -\frac{1}{1.33}p = -0.752\,p$$

The image of the fish is virtual because i is negative (both the object and image are in *front* of the flat surface in water). The apparent depth of the fish is approximately $3/4$ of the actual depth.

17.6.3 Thin Lenses

A lens is a transparent object with two refracting surfaces of different radii of curvature R_1 and R_2 but with a common principal axis, and when light rays bend across these surfaces we get the image of an object.

When a lens converges light rays parallel to the principal axis, we call it a **converging lens**, see Fig. 17.21a. If instead it causes such rays to diverge, we call it a **diverging lens**, see Fig. 17.21b.

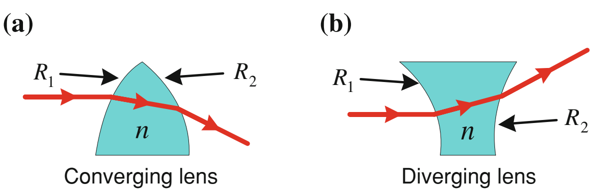


Fig. 17.21 (a) An enlargement of the top part of a converging lens. (b) An enlargement of the top part of a diverging lens

The Thin Lens Equation

First, we consider a thick glass lens bounded by two spherical surfaces, air-to-glass and glass-to-air. This lens is defined by the radii R_1 and R_2 of the two surfaces, its thickness Δ, and its index of refraction n, see Fig. 17.22.

Let us begin with an object O placed at a distance p in front of surface 1 of radius R_1. Using Eq. 17.26 with $n_1 = 1$ and $n_2 = n$, the position i_1 of image I_1 formed by surface 1 satisfies the equation:

$$\frac{1}{p} + \frac{n}{i_1} = \frac{n-1}{R_1} \tag{17.29}$$

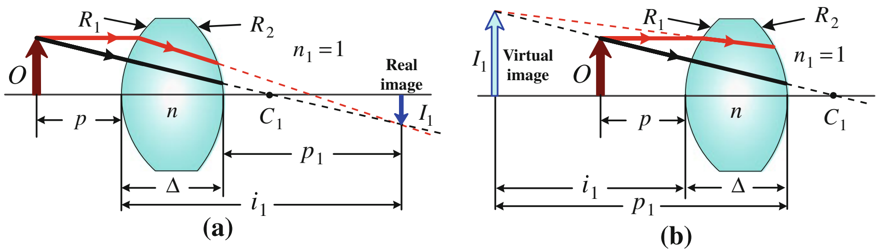


Fig. 17.22 When we ignore the existence of surface 2 (of radius R_2): (a) the first possibility is that an object O produces a real image I_1 by surface 1 (of radius R_1), and (b) The other possibility is that the image I_1 is virtual. Point C_1 is the center of curvature of surface 1

The position i_1 is positive in Fig. 17.18a when the image I_1 is real and negative in Fig. 17.18b when the image I_1 is virtual. In both cases, it seems as if I_1 is formed in the lens material with index n.

Next, we consider the image I_1 as a *virtual* object placed at a distance p_1 in front of surface 2 of radius R_2. Again, applying Eq. 17.26 with $n_1 = n$ and $n_2 = 1$, the position i of the final image I formed by surface 2 satisfies the equation:

$$\frac{n}{p_1} + \frac{1}{i} = \frac{1-n}{R_2} \tag{17.30}$$

We note from Fig. 17.22a, b that $p_1 = -i_1 + \Delta$, where i_1 is positive for *real images* and negative for *virtual objects*. For thin lenses, Δ is very small and therefore $p_1 \simeq -i_1$. Thus, the last equation becomes:

$$-\frac{n}{i_1} + \frac{1}{i} = \frac{1-n}{R_2} \quad \text{(For thin lenses)} \tag{17.31}$$

Adding Eqs. 17.29 and 17.31, we get:

$$\boxed{\frac{1}{p} + \frac{1}{i} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)} \tag{17.32}$$

The focal length f of a thin lens is obtained when $p \to \infty$ and $i \to f$ in this equation. Thus, the inverse of the focal length for a thin lens is:

$$\boxed{\frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)} \tag{17.33}$$

which is called the **lens-makers' equation** because it can be used to determine R_1 and R_2 for the desired values of n and f.

In conclusion, a thin lens of index n and two surfaces of radii R_1 and R_2 has an equation identical to the mirror equation, written as:

$$\boxed{\frac{1}{p} + \frac{1}{i} = \frac{1}{f}}, \quad \text{where} \quad \boxed{\frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)} \tag{17.34}$$

This is called the **thin-lens equation**. The sign conventions for R_1 and R_2 are presented in Table 17.3. Just as with mirrors, the thin lens lateral magnification is:

$$\boxed{M = \frac{h'}{h} = -\frac{i}{p}} \tag{17.35}$$

Since light rays can travel in both directions of a lens, then each lens has two focal points F_1 and F_2. Both focal points are at the same distance f (the focal length) from a thin lens. The focal length f is the same for light rays passing through a given lens in either direction. This is illustrated in Fig. 17.23 for a *biconvex lens* (converging lens) and a *biconcave lens* (diverging lens).

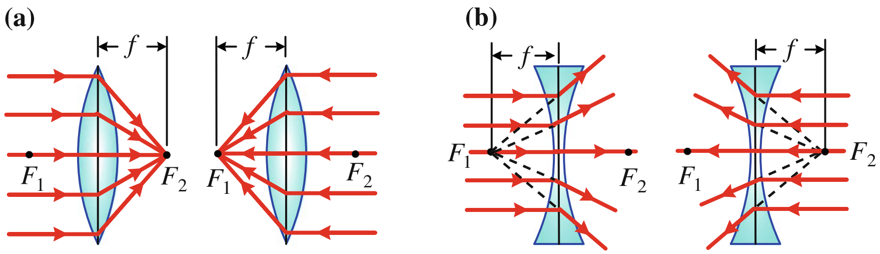


Fig. 17.23 Parallel rays passing through: (a) a converging lens, and (b) a diverging lens

Ray Diagrams for Thin Lenses

Ray diagrams are convenient tools that help us locate images formed by thin lenses. They also clarify our sign conventions. For the purpose of locating an image, we only use two special rays drawn from the top of the object to the top of the image as follows:

- Ray 1 starts parallel to the principal axis.
  - For a converging lens, the ray is refracted by the lens and passes through the focal point F_2 on the back side of the lens.
  - For a diverging lens, the ray is refracted by the lens and appears to originate from the focal point F_1 on the front side of the lens.

• Ray 2 passes through the center of the lens and continues in a straight line.

Figure 17.24 shows such ray diagrams for converging and diverging lenses.

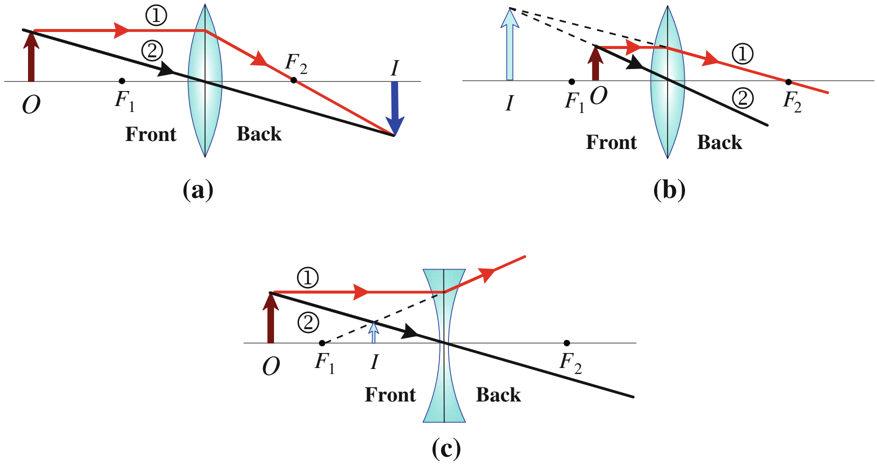


Fig. 17.24 Ray diagrams for locating the image formed by a thin lens. (a) An object in front of a converging lens (double convex lens). When the object is outside the focal point, the image is real, inverted, and on the back side of the lens. (b) When the object is between the focal point and the converging lens (double convex lens), the image is virtual, upright, larger than the object, and on the front side of the lens. (c) When an object is anywhere in front of a diverging lens (double concave lens), the image is virtual, upright, smaller than the object, and on the front side of the lens

When using Eq. 17.34, it is very important to use the proper sign conventions introduced in Table 17.4.

Table 17.4 Sign conventions for thin lenses

Quantity	Symbol	Positive values when	Negative values when
Radii	R_1 or R_2	The center of curvature is in *back* of lens	The center of curvature is in *front* of the lens
Object location	p	The object is in *front* of lens (*real object*)	The object is in *back* of lens (*virtual object*)
Image location	i	The image is in *back* of lens (*real image*)	The image is in *front* of lens (*virtual image*)
Image height	h'	The Image is *upright*	The Image is *inverted*
Magnification	M	The Image is *upright*	The Image is *inverted*

Table 17.5 shows a comparison of the image positions, magnifications, and types of images formed by convex and concave lenses when an object is placed at various

positions, p, relative to the lens. Notice that a converging (biconvex lens) can produce real images or virtual images, whereas a diverging (biconcave) lens only produces virtual images.

Table 17.5 Properties of a single spherical lens system

Type of lens	f	p	i	M	Image
Converging lens (Biconvex lens)	+	$p > 2f$	$2f > i > f$	Reduced inverted	Real
		$2f > p > f$	$i > 2f$	Enlarged inverted	Real
		$f > p > 0$	$\lvert i\rvert > p$ (Negative)	Enlarged upright	Virtual
Diverging lens (Biconcave lens)	−	$p > 0$	$\lvert f\rvert > \lvert i\rvert > 0$ (Negative)	Reduced upright	Virtual

* Combination of Thin Lenses

To understand and locate the image produced by two lenses, we follow two steps. The first image formed by the first lens is located as if the second lens were not present. Then this first image is treated as a virtual object and we use the second lens to find the final image. This procedure can be extended to three or more lenses.

Let us consider the case were two lenses of focal lengths f_1 and f_2 are in contact with each other. If p is the object distance from the system and i_1 the is image distance produced by the first lens, then:

$$\frac{1}{p} + \frac{1}{i_1} = \frac{1}{f_1} \tag{17.36}$$

and

$$M_1 = -\frac{i_1}{p} \tag{17.37}$$

This image is the object for the second lens. Since this image is behind the second lens, it serves as a virtual object and its distance for the second lens is negative, i.e. its distance to the second lens is $-i_1$ (see Table 17.4). Therefore, the distance i of the final image produced by the second lens satisfies:

$$-\frac{1}{i_1} + \frac{1}{i} = \frac{1}{f_2} \tag{17.38}$$

and

$$M_2 = -\frac{i}{(-i_1)} = \frac{i}{i_1} \tag{17.39}$$

Adding the two Eqs. 17.36 and 17.38 gives:

$$\boxed{\frac{1}{p}+\frac{1}{i}=\frac{1}{f}}\,, \quad \text{where} \quad \boxed{\frac{1}{f}=\frac{1}{f_1}+\frac{1}{f_2}} \tag{17.40}$$

Thus, two thin lenses in contact are equivalent to a single thin lens of focal length f given by $f^{-1}=f_1^{-1}+f_2^{-1}$. The overall magnification of the two lenses is:

$$M = M_1\,M_2 = -\frac{i_1}{p}\frac{i}{i_1} = -\frac{i}{p} \quad \text{(Thin lenses in contact)} \tag{17.41}$$

Example 17.8

A converging lens of focal length 20 cm forms an image of an object of height 30 cm located at a distance 40 cm from the lens. Locate and describe the image. Draw two rays to locate the image.

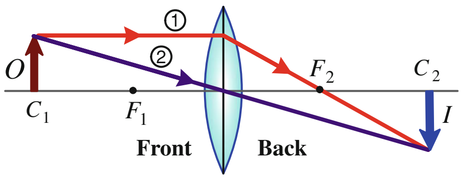
Solution: A converging lens has a positive value for its focal length, i.e. $f=+20$ cm. To find the image distance when $p=40$ cm and $f=+20$ cm, we use Eq. 17.34 as follows:

$$\frac{1}{p}+\frac{1}{i}=\frac{1}{f} \quad \Rightarrow \quad \frac{1}{40\text{ cm}}+\frac{1}{i}=\frac{1}{20\text{ cm}} \quad \Rightarrow \quad i=40\text{ cm}$$

$$\text{Consequently we have:} \quad M=-\frac{i}{p}=-\frac{40\text{ cm}}{40\text{ cm}} \quad \Rightarrow \quad M=-1$$

The image is *real* and on the back side because i is positive, *inverted* because M is negative, and *as large* as the object, see Fig. 17.25.

Fig. 17.25



Example 17.9

Repeat Example 17.8 using a diverging lens.

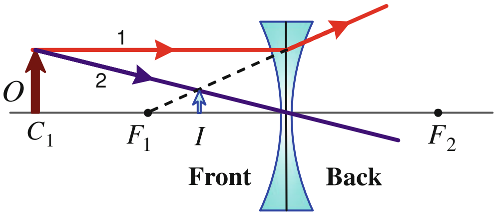
Solution: The diverging lens would have $f=-20$ cm. Thus:

$$\frac{1}{p}+\frac{1}{i}=\frac{1}{f} \quad \Rightarrow \quad \frac{1}{40\,\text{cm}}+\frac{1}{i}=-\frac{1}{20\,\text{cm}} \quad \Rightarrow \quad i=-40/3\,\text{cm}$$

Consequently, we have: $M = -\dfrac{i}{p} = -\dfrac{(-40/3 \text{ cm})}{40 \text{ cm}} \quad \Rightarrow \quad M = +1/3$

The image is *virtual* and on the front side because i is negative, *upright* because M is positive, and *reduced* because M is less than unity, see Fig. 17.26.

Fig. 17.26



Example 17.10

An object is placed 20 cm from a symmetrical lens that has an index of refraction $n = 1.65$. The lateral magnification of the object produced by the lens is $M = -1/4$. (a) Determine the type of the lens and describe the image. (b) What is the magnitude of the two radii of curvature of the lens?

Solution: (a) Using the lateral-magnification equation, we have:

$$M = -\frac{i}{p} = -\frac{1}{4} \quad \Rightarrow \quad i = \frac{p}{4} = \frac{20 \text{ cm}}{4} \quad \Rightarrow \quad i = +5 \text{ cm}$$

Because i is positive, the obtained image must be *real*. The only type of lens that can produce a real image is a *converging lens*. According to Fig. 17.24a, the object must be outside the focal point and the image must be *inverted* and on the back side of the lens.

(b) To find the focal length f of the lens when $p = 20$ cm and $i = +5$ cm, we use Eq. 17.34 as follows:

$$\frac{1}{p} + \frac{1}{i} = \frac{1}{f} \quad \Rightarrow \quad \frac{1}{20 \text{ cm}} + \frac{1}{5 \text{ cm}} = \frac{1}{f} \quad \Rightarrow \quad f = 4 \text{ cm}$$

From the general lens-makers' Eq. 17.33, f is related to the radii of curvatures R_1 and R_2 of the two surfaces of the lens and its index of refraction n by the relation:

$$\frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

For a symmetric lens, R_1 and R_2 have the same magnitude R. If R_1 is for the surface where the center of curvature is in the back of the lens, and R_2 is for the surface where the center of curvature is in the back of the lens, then using the sign convention of Table 17.4, we have $R_1 = +R$ and $R_2 = -R$. Thus:

$$\frac{1}{f} = (n-1)\left(\frac{1}{R} - \frac{1}{-R}\right) = \frac{2(n-1)}{R} \quad \Rightarrow \quad R = 2(n-1)f,$$

Hence, $\quad R = 2(n-1)f = 2(1.65-1) \times (4\text{ cm}) = 5.2\text{ cm}$

Example 17.11

Two thin coaxial lenses 1 and 2, with focal lengths $f_1 = +24$ cm and $f_2 = +9$ cm, respectively, are separated by a distance $L = 10$ cm; see part (a) of Fig. 17.27. An object is placed 6 cm in front of lens 1. Locate and describe the image. Draw the necessary sketches to show how you can reach to the answer.

Solution: We first ignore the presence of lens 2 and find the image I_1 produced by lens 1 alone, see part (b) Fig. 17.27. Equation 17.34 written for lens 1 leads to the following steps:

$$\frac{1}{p} + \frac{1}{i_1} = \frac{1}{f_1} \quad \Rightarrow \quad \frac{1}{6\text{ cm}} + \frac{1}{i_1} = \frac{1}{24\text{ cm}}$$

$$i_1 = -8\text{ cm}$$

Consequently, we have the following lateral magnification:

$$M_1 = -\frac{i_1}{p} = -\frac{(-8\text{ cm})}{6\text{ cm}} \quad \Rightarrow \quad M_1 = +4/3$$

This tells us that image I_1 is *virtual* (8 cm in front of lens 1), *upright* because M is positive, and *enlarged* because M is greater than unity, see part (b) of Fig. 17.27.

For the second step, we ignore lens 1 and treat the image I_1 as a virtual object O_1 in front of the second lens. The distance p_1 between the virtual object O_1 and lens 2 is:

$$p_1 = L - i_1 = 10\text{ cm} - (-8\text{ cm}) = 18\text{ cm}$$

Equation 17.34 written for lens 2 leads us to the following:

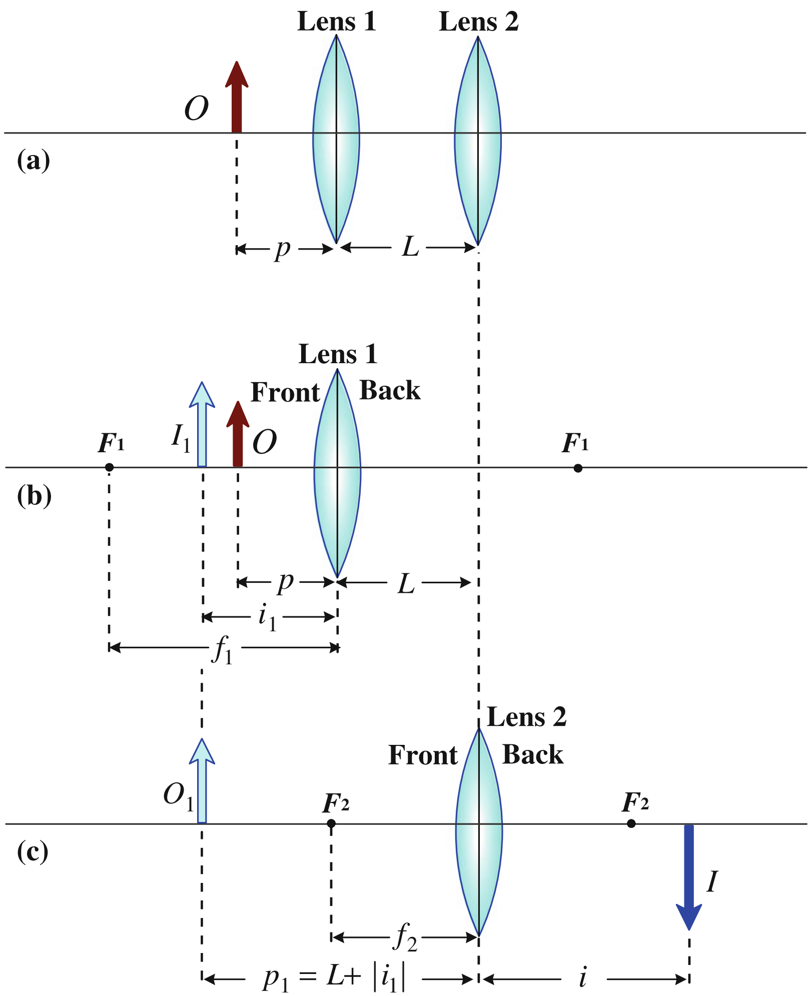


Fig. 17.27

$$\frac{1}{p_1} + \frac{1}{i} = \frac{1}{f_2} \quad \Rightarrow \quad \frac{1}{18\text{ cm}} + \frac{1}{i} = \frac{1}{9\text{ cm}}$$

$$i = +18\text{ cm}$$

Consequently, we have the following lateral magnification:

$$M_2 = -\frac{i}{p_1} = -\frac{18\text{ cm}}{18\text{ cm}} \quad \Rightarrow \quad M_2 = -1$$

The final image is *real* because i is positive and on the back side of lens 2, *inverted* because M is negative, and *as large* as the virtual object I_1, see part (c) of Fig. 17.27. The overall magnification of the two lenses is:

$$M = M_1\, M_2 = \frac{i_1}{p}\frac{i}{p_1} = \frac{(-8\text{ cm})}{(6\text{ cm})}\frac{(18\text{ cm})}{(18\text{ cm})}$$

$$M = -4/3$$

The final image is *enlarged* because $|M| > 1$. Notice that when $L = 0$, we get $M = -i/p$ as expected from Eq. 17.41.

17.7 Exercises

Section 17.2 Reflection and Refraction of Light

(1) A beam of light travels in vacuum and has a wavelength $\lambda = 500$ nm. The beam passes through a piece of diamond ($n = 2.4$). What is the wave's speed and wavelength in diamond?

(2) Assume that the wavelength of a yellow beam of light in vacuum is $\lambda = 600$ nm, and that the index of refraction of water is 1.33. (a) What is the speed of this light when it travels in vacuum? (b) What is the speed of this light when it travels in water? (c) What is the frequency of this light when it travels in vacuum? (d) What is the wavelength of this light when it travels in water? (e) What is the frequency of this light when it travels in water?

(3) A beam of light having a wavelength $\lambda = 600$ nm is incident perpendicular to a glass plate of thickness $d = 2$ cm and index of refraction $n = 1.5$. (a) How long does it take a point on the beam to pass through the plate? (b) Calculate the number of wavelengths in the glass plate.

(4) At what angle must a ray of light traveling in air be incident on acetone ($n = 1.38$) in order to be refracted at 30°?

(5) The index of refraction of alcohol is $n = 1.4$. (a) What is the speed of light in alcohol? (b) Find the angle of refraction in alcohol assuming light meets the air-alcohol boundary at an angle of incidence of 60°?

(6) A beam of light in air falls on a liquid surface at an angle of incidence of 55°. The liquid has an unknown index of refraction. (a) If the beam is deviated by 20°, what is the value of n? (b) What is the speed of light in this liquid?

(7) A beam of light in air strikes a glass plate at an angle of incidence of 53°. If the thickness of the glass plate is 2 cm and its index of refraction is 1.6, what will be the lateral displacement of the beam after it emerges from the glass?

(8) A beam of light in air falls on water at an angle of incidence of 45° and then passes through a glass block before it emerges out to air again. The surfaces of water and glass are parallel and their indexes of refraction are 1.33 and 1.63, respectively. (a) What is the angle of refraction in water? (b) What is the angle of refraction in glass? (c) Show that the incoming and outgoing beams are parallel. (d) At what distance does the beam shift from the original if the thickness of water and glass are both equal to 1 cm?

Section 17.3 Total Internal Reflection and Optical Fibers

(9) Diamond has a high index of refraction $n = 2.42$. To some extent, Diamond's "brilliance" is attributed to its total internal reflection. Find the critical angle for the diamond-air surface.

(10) A beam of light passes from glass to water. The index of refraction of glass and water are 1.52 and 1.333, respectively. (a) What is the critical angle of incidence in glass? (b) If the angle of incidence in glass is 45°, what is the angle of refraction in water?

(11) As it travels through ice, light has a speed of 2.307×10^8 m/s. (a) What is the index of refraction of ice? (b) What is the critical angle of incidence for light going from ice to air? (c) If the angle of incidence in ice is 45°, what is the angle of refraction in air?

(12) As the sun sets, its rays are nearly tangent to the surface of water, see Fig. 17.28. The index of refraction of water is 1.33. (a) At which angle from the normal would the fish in the figure see the sun? (b) Refraction at the water-air boundary changes the apparent position of the Sun. What is the apparent direction of the Sun with respect to the fish (measured above the horizontal)?

(13) Figure 17.29 shows a sketch of an Optical fiber cable that has a length $L = 1.51$ m, diameter of $D = 251$ μm, and index of refraction $n = 1.3$. A ray of light is incident on the left end of the cable at an angle of incidence $\theta_1 = 45°$. (a) What is the critical angle of incidence for light going from inside the cable to air? (b) Find the angle of refraction θ_2 and the length ℓ. Does the angle θ

fulfill the condition of total internal reflection? (c) How many reflections does the light ray make before emerging from the other end?

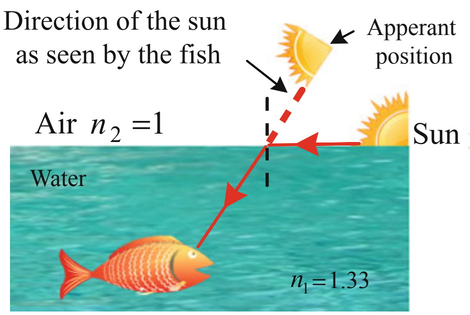


Fig. 17.28 See Exercise (12)

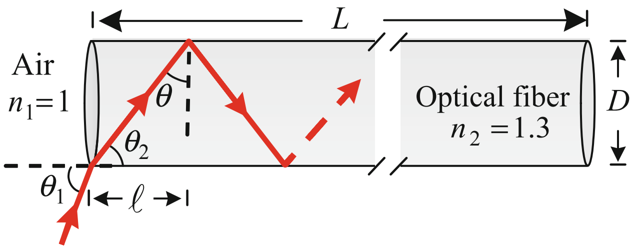


Fig. 17.29 See Exercise (13)

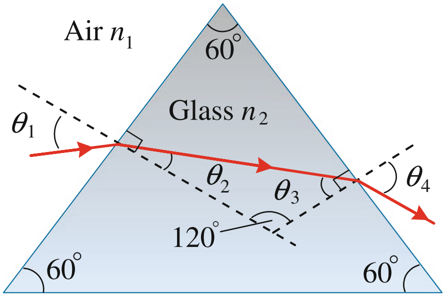
(14) Using the figure of Exercise 13, show that the largest angle of incidence θ_1 for which total internal reflection occuring at the top surface is given by the relation $\sin\theta_1 = \sqrt{(n_2/n_1)^2 - 1}$. Now find the value of this angle using the data of Exercise 13.

Section 17.4 Chromatic Dispersion and Prisms

(15) Find the difference in time needed for two short pulses of light to travel 12 km through a fiber optics cable, assuming that the cable's index of refraction for a pulse of wavelength 700 nm is 1.5 and 1.53 for a pulse of wavelength 400 nm.

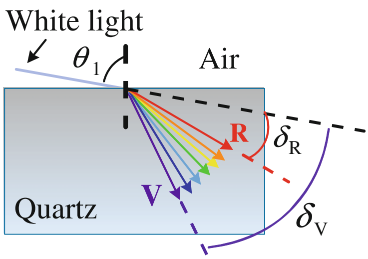
(16) A monochromatic light ray is incident from air ($n_1 = 1$) onto one of the faces of an equilateral prism that has an index of refraction $n_2 = 1.5$, see Fig. 17.30. If the angle of incidence θ_1 is 40°, then at what angle from the normal would this ray leave the prism?

Fig. 17.30 See Exercise (16)



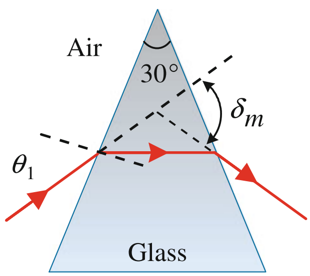
(17) A narrow beam of white light is incident from air onto a plate of fused quartz at an angle of incidence $\theta_1 = 60°$; see Fig. 17.31. The index of refraction of quartz for violet and red light is $n_V = 1.470$ and $n_R = 1.458$, respectively. Find the angular width $\delta_V - \delta_R$ between the violet and red light rays inside the quartz.

Fig. 17.31 See Exercise (17)



(18) A prism has an index of refraction $n = 1.5$ and an apex angle $A = 30°$. The prism is set for minimum deviation by allowing a ray of monochromatic light to pass through it symmetrically, as shown in Fig. 17.32. (a) Find the angle of minimum deviation δ_m. (b) Find the value of the angle of incidence θ_1.

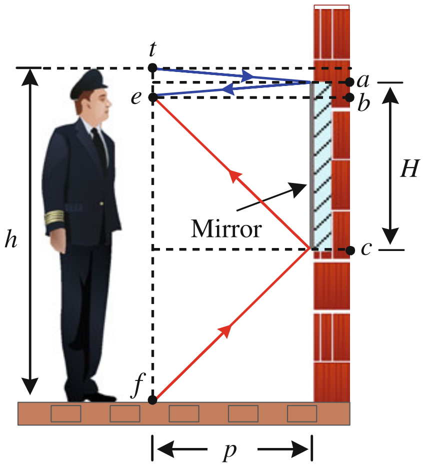
Fig. 17.32 See Exercise (18)



Section 17.5 Formation of Images by Reflection

(19) The height h of a man is 200 cm. The top of his hat t, his eyes e, and his feet f are marked by dots on Fig. 17.33. In order for the man to be able to see his entire length in a vertical plane mirror, he needs a mirror of height H, as shown. The figure also shows two paths, one for the light ray leaving his hat t and entering his eyes e, and another for the light ray leaving his feet f and again entering his eyes e. (a) Find the height H of the mirror. (b) Use two rays to make a geometric sketch for the location and the height of the man's image.

Fig. 17.33 See Exercise (19)



(20) A concave mirror has a radius of curvature of 1.5 m. Where is the focal point of this mirror?

(21) A concave mirror has a focal length $f = +0.2$ m. An object of height 3 cm is placed 0.1 m along its principal axis. Locate and describe the image formed by the mirror.

(22) Repeat Exercise 21 using a convex mirror.

(23) Assume a spherical concave mirror has a positive focal length $|f|$. Use the mirror equation $1/p + 1/i = 1/|f|$ to determine where an object must be placed if the image created has the same size as the object, i.e. when $M = |-i/p| = 1$.

(24) Assume a spherical convex mirror has a negative focal length $-|f|$. Use the mirror equation $1/p + 1/i = -1/|f|$ to show that the condition $M = |-i/p| = 1$ cannot not be satisfied.

(25) Six objects are located at the following positions from a spherical mirror: (i) $p = \infty$, (ii) $p = 15\,\text{cm}$, (iii) $p = 10\,\text{cm}$, (iv) $p = 7.5\,\text{cm}$, (v) $p = 5\,\text{cm}$, and (vi) $p = 2.5\,\text{cm}$. Locate and describe the image for each object when the spherical mirror is: (a) concave, with a focal length of 5 cm, (b) convex, with a focal length of 5 cm.

(26) Repeat Exercise 25, this time sketching the lateral magnification M for each object's location p.

Section 17.6 Formation of Images by Refraction

(27) (a) A cylindrical glass rod ($n_2 = 1.6$) has a hemispherical end of radius $R = 2\,\text{cm}$. An object of height $h = 0.2\,\text{cm}$ is placed in air ($n_1 = 1$) on the axis of the rod at a distance $p = 6\,\text{cm}$ from the spherical vertex, see Fig. 17.34. (a) Locate and describe the image. (b) Repeat part (a) when $p = 2\,\text{cm}$.

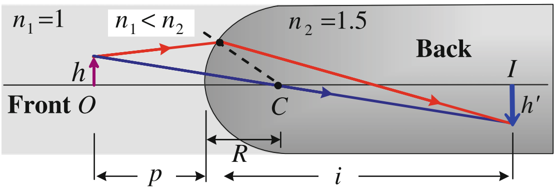


Fig. 17.34 See Exercise (27)

(28) A spherical fish bowl filled with water ($n_1 = 1.33$) has a radius of 15 cm. A small fish is located at a horizontal distance $p = 20\,\text{cm}$ from the left side of the bowl, see Fig. 17.35. Neglecting the effect of the glass walls of the bowl, where does an observer see the fish's image? What is the lateral magnification of the fish?

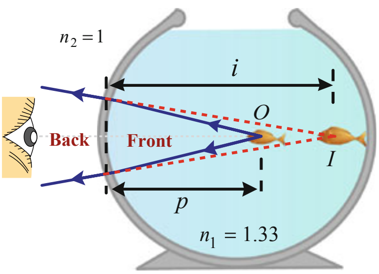


Fig. 17.35 See Exercise (28)

(29) (a) An object is placed 30 cm from a converging lens with a 10 cm focal length. Find the position of the image and its lateral magnification. Is the image real or virtual? Is it upright or inverted? (b) Repeat part (a) for an object placed 5 cm away.

(30) Repeat Exercise 29 with a diverting lens.

(31) Use a ray diagram to explain the results of Exercises 29 and 30.

(32) An object is placed 20 cm from a symmetrical lens that has an index of refraction $n = 1.6$. The lateral magnification of the object produced by the lens is $M = 1/4$. (a) Determine the type of the lens used and describe the image. (b) What are the values of the two radii of curvature of the lens?

(33) A thin converging lens of focal lens $f_1 = +15$ cm is placed in contact with a thin diverging lens of unknown focal length f_2. Find f_2 when incident Sunrays on the converging lens are focused by this combination at a point 25 cm behind the diverging lens.

(34) A converging lens of focal lens $f_1 = +2$ cm is placed at a distance $L = 4$ cm in front of a diverging lens of focal lens $f_2 = -14$ cm. An object is placed at infinite distance from the converging lens. Where will the object be focused?

(35) Repeat Exercise 34 with $f_1 = +10$ cm and $f_2 = -16$ cm.

(36) Two lenses with focal lenses $f_1 = +16$ cm and $f_2 = +20$ cm are at a distance $L = 64$ cm apart. An object is placed 48 cm in front of the first lens. Locate and describe the image formed by the system.

(37) Repeat Exercise 36 with $L = 19$ cm.

(38) A converging lens of $f_2 = +17$ cm is placed behind a diverging lens of unknown focal lens f_1 by a distance $L = 12$ cm. Find f_1 when parallel light rays strike the diverging lens and leave the converging lens parallel.

(39) Repeat Exercise 38 when the two lenses exchange positions.

(40) An object is moving with velocity $v = -dp/dt$ toward a converging lens of focal length f such that $p > f$. Find the image velocity di/dt as a function of p. Find p when $v = di/dt$.

Interference, Diffraction and Polarization of Light 18

In this chapter we treat light as waves to study interference, diffraction, and polarization. This study is known as ***wave optics*** or ***physical optics***.

We found in Chap. 16 that the superposition of two sound waves could be constructive or destructive. The same phenomena can be observed with light waves. When a resultant wave at a given position or time has an amplitude larger than the individual waves, we refer to their superposition as **constructive interference**. However, when a resultant wave has an amplitude smaller than the individual waves, we refer to their superposition as **destructive interference**.

18.1 Interference of Light Waves

If you have two ordinary light bulbs, incandescent or fluorescent, the light waves they emit have random phases. These phases change in time intervals that are less than a nanosecond apart; thus, the conditions for constructive and destructive interference are maintained for only very short time intervals, too short for our eyes to notice. Such light sources are said to be **incoherent**.

Spotlight
To observe detailed interference effects in light waves, the following conditions must be fulfilled:

- The sources must be ***coherent***, i.e., *they must maintain a constant phase with respect to each other.*
- The sources must be ***monochromatic***, i.e., *of a single wavelength.*

H. A. Radi and J. O. Rasmussen, *Principles of Physics*,
Undergraduate Lecture Notes in Physics, DOI: 10.1007/978-3-642-23026-4_18,

A common method for observing interference is to allow a single monochromatic light source to split to form two coherent light sources and then allow the light waves from the two sources to overlap. This can be achieved by using the **diffraction** of light waves from a small opening as introduced in Fig. 17.2. Figure 18.1 shows the overlap of monochromatic coherent light waves after being diffracted from two slits when $\lambda \approx a$, where a is the width of each slit.

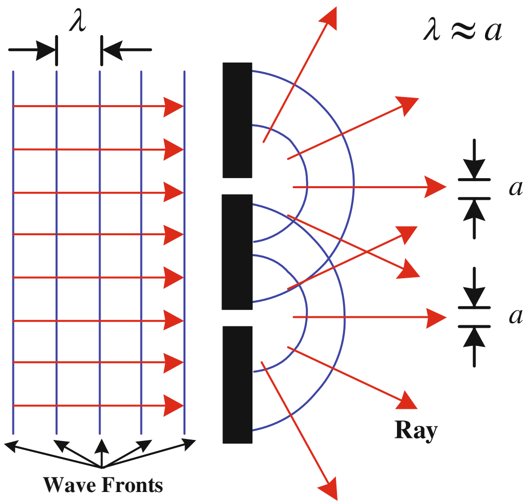


Fig. 18.1 Spreading of light waves from each slit (which is known as diffraction) ensures overlapping of waves, and hence interference effects can be observed when the light from the two slits arrive at a viewing screen (which is not shown in this figure)

18.2 Young's Double Slit Experiment

Figure 18.2 shows a schematic diagram of the apparatus used by Thomas Young in 1801 to demonstrate the interference of light waves. Plane monochromatic light waves arrive at a barrier that has two parallel slits S_1 and S_2. These two slits serve as a pair of coherent light sources because the emerging waves originate from the same wave front and hence have the same phase relationship.

Diffraction of the light by the two slits sends overlapping waves into the region between the barrier and the viewing screen. When light waves from the two slits combine constructively at any location on the screen, they produce a **bright band**. On the other hand, when light from the two slits combine destructively at any location on the screen they produce a **dark band**. These bands are called **fringes**, and the pattern of bright and dark fringes is called an **interference pattern**. Figure 18.2b shows a representation of the interference pattern observed on the screen.

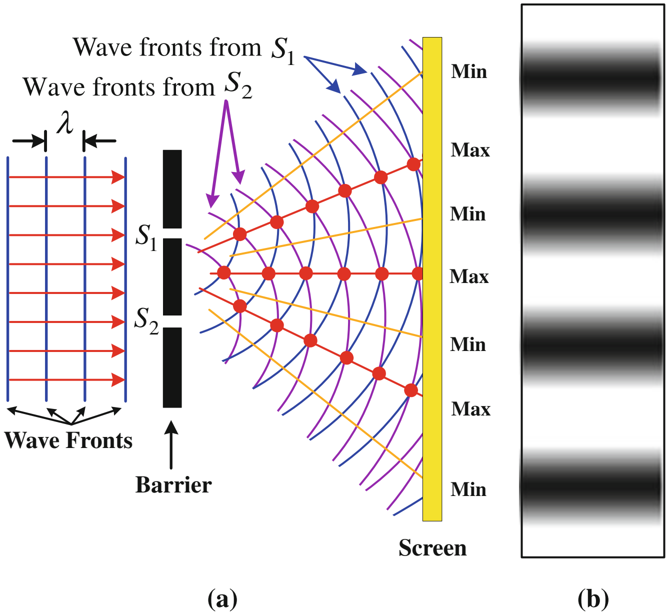


Fig. 18.2 (a) When light waves are diffracted from the two slits S_1 and S_2, waves overlap and undergo interference. Constructive interference in the region between the barrier and screen is represented by red circles on the red lines. Destructive interference is represented by the yellow lines. (b) Representation of the photograph that we get for the interference pattern in Young's double slit experiment

With the help of Fig. 18.3a, we can quantitatively specify the positions of bright and dark fringes in Young's experiment. In this figure, we show the following:

- Light waves of wavelength λ illuminating a barrier having two narrow slits
- The two slits S_1 and S_2 are separated by a distance d
- Point Q is half way between the two slits
- The central line QO between the barrier and the screen has a distance D (where $D \gg d$)
- Point P on the screen makes an angle θ above the central line QO. The central line QO will be taken as a reference line for measuring positive angles above or below the line
- The distance from P to S_1 is r_1, and the distance from P to S_2 is r_2.

The waves from S_2 must travel a longer distance to reach point P than the waves starting at S_1. This difference ΔL in distance is called the **path difference**. When

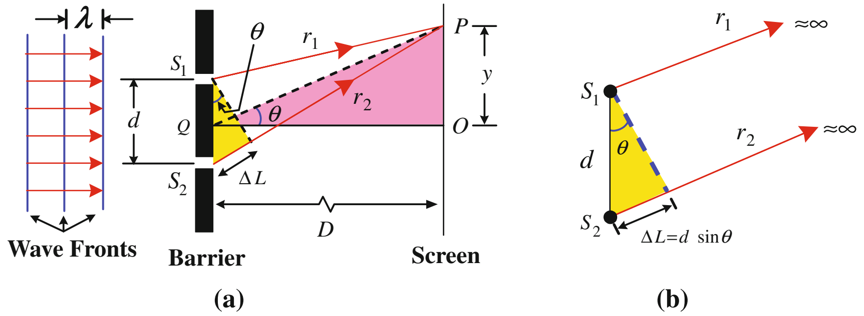


Fig. 18.3 (a) Locating the fringes for Young's double slit experiment geometrically (the figure is not to scale). (b) For the condition $D \gg d$, we can approximate rays r_1 and r_2 as being parallel, making an angle θ to the central line QO, and the path difference between the two rays is $r_2 - r_1 = d \sin\theta$

$D \gg d$, the rays r_1 and r_2 are approximately parallel. Using Fig. 18.3b the path difference will be:

$$\Delta L = |r_2 - r_1| \simeq d \sin\theta \tag{18.1}$$

Constructive interference (maximum light intensity) occurs at P when the two waves are in phase ($\phi = 0, \pm 2\pi, \pm 4\pi, \ldots$ rad), or when the path difference $d \sin\theta$ is an *integer multiple of the wavelength* λ. That is, when:

$$\boxed{d \sin\theta_m = m\,\lambda \qquad (m = 0, 1, 2, \ldots)} \quad \left\{\begin{array}{c}\text{Maxima}\\ \text{Bright fringes}\end{array}\right\} \tag{18.2}$$

The number m for a bright fringe is called the **fringe order number**. The central bright fringe at $\theta_m = 0$, where $m = 0$, is called the *zeroth-order maximum*. The first maximum on *either sides* of point O, where $m = 1$, is called the *first-order maximum*, and so forth.

Destructive interference occurs at P when the path difference $d \sin\theta$ is an *odd multiple of half the wavelength*. That is when:

$$\boxed{d \sin\theta_m = (m - \tfrac{1}{2})\,\lambda \qquad (m = 1, 2, 3, \ldots)} \quad \left\{\begin{array}{c}\text{Minima}\\ \text{Dark fringes}\end{array}\right\} \tag{18.3}$$

Similarly, the two waves reaching P at any time are completely out-of-phase ($\phi = \pm\pi, \pm 3\pi, \pm 5\pi, \ldots$ rad), and hence a minimum light intensity is detected. In this case, the first *minimum* on *either side* of point O, where $m = 1$, is called the *first-order minimum*, and so forth.

Using the triangle *OPQ* of Fig. 18.3a, we can find the location *y*, on *either side* of point *O*, of a fringe from the relation:

$$y = D\tan\theta \tag{18.4}$$

In addition to the conditions $\lambda \gg a$ (where a is the width of each slit) and $D \gg d$, we assume that $\lambda \ll d$. This assumption is valid only if θ is very small and hence $\tan\theta \simeq \sin\theta$. Therefore, $y = D\sin\theta$, or:

$$y_m = D\sin\theta_m \qquad \left\{ \begin{array}{c} \text{Bright or dark fringes} \\ \text{when } \theta_m \text{ is very small} \end{array} \right\} \tag{18.5}$$

Substituting with $\sin\theta_m$ into Eqs. 18.2 and 18.3, we get the following expressions for the locations of bright and dark fringes above or below the central point *O*:

$$\boxed{y_m = m\,\frac{\lambda\, D}{d} \quad (m = 0, 1, 2, \ldots)} \qquad \left\{ \begin{array}{c} \text{Bright fringes} \\ \text{for very small angles} \end{array} \right\} \tag{18.6}$$

$$\boxed{y_m = (m - \tfrac{1}{2})\,\frac{\lambda\, D}{d} \quad (m = 1, 2, 3, \ldots)} \qquad \left\{ \begin{array}{c} \text{Dark fringes} \\ \text{for very small angles} \end{array} \right\} \tag{18.7}$$

We can find the distance on the screen between the adjacent maxima or minima near the origin *O* by finding the difference:

$$\Delta y = y_{m+1} - y_m \qquad \left\{ \begin{array}{c} \text{Bright or dark} \\ \text{fringes} \end{array} \right\} \tag{18.8}$$

Using Eq. 18.6, for bright fringes, we find:

$$\Delta y = y_{m+1} - y_m = (m+1)\,\frac{\lambda\, D}{d} - m\,\frac{\lambda\, D}{d}$$

Therefore:

$$\boxed{\Delta y = \frac{\lambda\, D}{d}} \qquad \left\{ \begin{array}{c} \text{Above or below the central} \\ \text{point for very small angles} \end{array} \right\} \tag{18.9}$$

In other words, when the condition for small-angle approximation is valid, Δy does not depend on the order of the fringe m and the fringes are uniformly spaced. The same result is true for dark fringes.

Example 18.1

Two narrow slits are separated by 0.06 mm and are 1.2 m away from a screen. When the slits are illuminated by light of unknown wavelength λ, we obtain a fourth-order bright fringe 4.5 cm from the central line. Find the wavelength of the light.

Solution: Using Eq. 18.6, with $m = 4$ and $\lambda \ll d$, we find that:

$$y_m = m\,\frac{\lambda\, D}{d} \quad \Rightarrow \quad \lambda = \frac{d\, y_m}{m\, D} \quad \Rightarrow \quad \lambda = \frac{d\, y_4}{4\, D}$$

$$\text{Thus:} \quad \lambda = \frac{(0.06 \times 10^{-3}\,\text{m})(4.5 \times 10^{-2}\,\text{m})}{4 \times 1.2\,\text{m}} = 5.625 \times 10^{-7}\ \text{m} = 563\ \text{nm}$$

This wavelength is in the range of green light. The angle that this fringe makes with the central line is $\theta_4 = \tan^{-1}(y_4/D) = 2.15°$.

Example 18.2

Two narrow slits are separated by 1.5 mm and are 3 m away from a screen. The slits are illuminated by a yellow light of wavelength 589 nm from a sodium-vapor lamp. Find the spacing between the bright fringes.

Solution: Using Eq. 18.9 when $\lambda \ll d$, we have:

$$\Delta y = y_{m+1} - y_m = \frac{\lambda\, D}{d} = \frac{(589 \times 10^{-9}\,\text{m})(3\,\text{m})}{1.5 \times 10^{-3}\,\text{m}} = 1.178 \times 10^{-3}\ \text{m} = 1.178\ \text{mm}$$

Example 18.3

Two slits are separated by 0.4 mm and illuminated by light of wavelength 442 nm. How far must the screen be placed in order for the first dark fringes to appear directly opposite both slits?

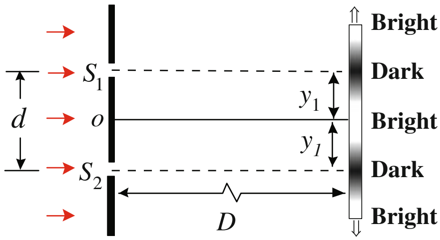
Solution: Taking $m = 1$, $d = 0.4$ mm, $y_1 = 0.2$ mm, and $\lambda = 442$ nm in Eq. 18.7, see Fig. 18.4 , we get:

$$y_m = (m - \tfrac{1}{2})\,\frac{\lambda\, D}{d} \quad \Rightarrow \quad D = \frac{2\, d\, y_1}{\lambda}$$

$$D = \frac{2(4 \times 10^{-4}\ \text{m})(2 \times 10^{-4}\ \text{m})}{442 \times 10^{-9}\ \text{m}} = 0.36\ \text{m} = 36\ \text{cm}$$

Geometric optics incorrectly predicts bright regions opposite the slits.

Fig. 18.4



Light Intensity in the Double-Slit Experiment

Let us assume that the waves emerging from the two slits of Fig. 18.3a are two sinusoidal electric fields having the same phase, wavelength λ, angular frequency $\omega = 2\pi f$, and amplitude $E_\circ$. When the two waves arrive at point P, their phase difference ϕ depends on the path difference $\Delta L = |r_2 - r_1| \simeq d \sin\theta$. We can write the magnitude of the electric field at point P due to each separate wave as:

$$\left.\begin{aligned} E_1 &= E_\circ \sin(\omega t), \\ E_2 &= E_\circ \sin(\omega t + \phi) \end{aligned}\right\} \tag{18.10}$$

The superposition of E_1 and E_2, $E = E_1 + E_2$, can be calculated in a similar way as in Sect. 16.6. Thus:

$$E = 2 E_\circ \cos\left(\frac{\phi}{2}\right) \sin\left(\omega t + \frac{\phi}{2}\right) \tag{18.11}$$

We can prove that the intensity I of light waves at P is proportional to the square of the resultant electric field averaged over one cycle. Thus:

$$\boxed{I = I_\circ \cos^2\left(\frac{\phi}{2}\right)}\ , \quad \boxed{I_\circ = 4 I_{\text{max}}} \tag{18.12}$$

where $I_\circ$ is the peak intensity and I_{max} is the maximum intensity of one slit when the second slit is closed.

Since a path difference of a complete wave length λ corresponds to a phase difference of 2π rad, then one can relate the path difference ΔL to the phase difference ϕ or vice-versa by the two relations:

$$\left.\begin{aligned} \Delta L &= \frac{\phi}{2\pi}\,\lambda \\ &\text{or} \\ \phi &= \frac{2\pi}{\lambda}\,\Delta L \end{aligned}\right\} \tag{18.13}$$

In the last form, when we replace ΔL by $d \sin\theta$, we get the following relation for the phase:

$$\phi = \frac{2\pi\, d}{\lambda}\,\sin\theta \tag{18.14}$$

In addition, when we use the condition $\sin\theta \simeq \tan\theta$, and replace $\tan\theta$ by y/D as shown in Fig. 18.3, we arrive at the following relation for the phase:

$$\phi \simeq \frac{2\pi\, d}{\lambda\, D}\,y \tag{18.15}$$

Substituting the expression of ϕ from Eqs. 18.14 and 18.15 into Eq. 18.12, we get:

$$\boxed{I = I_\circ \cos^2\left(\frac{\pi\, d}{\lambda}\,\sin\theta\right)} \tag{18.16}$$

$$\boxed{I \simeq I_\circ \cos^2\left(\frac{\pi\, d}{\lambda D}\,y\right)} \tag{18.17}$$

Constructive interference occurs when $\pi\, d\, y/\lambda\, D$ is an integer multiple of π, corresponding to $y_m = m\,\lambda\, D/d$, $(m = 0, 1, 2, \ldots)$. This is consistent with Eq. 18.6.

Figure 18.5 shows the variation of the intensity I against both $d \sin\theta$ and ϕ, when we satisfy both the conditions $D \gg d$ and the small observation angle.

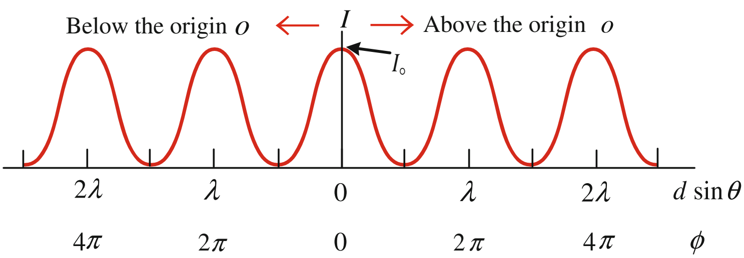


Fig. 18.5 A sketch showing intensity variations of a double-slit interference pattern as a function of the path difference $\Delta L = d \sin\theta$ or the phase difference ϕ. This variation limit is true only for very small values of θ

18.3 Thin Films—Change of Phase Due to Reflection

We saw that path differences can be used to generate a phase difference as given by Eq. 18.13. Reflection is another method that we can use to generate a phase difference for electromagnetic waves (especially light waves). Specifically, the reflection of light from surfaces has the following effects:

- When a light wave traveling in a homogenous medium meets a boundary of *higher* index of refraction, it reflects, undergoing *a phase change of* π rad $(= 180°)$.
- When a light wave traveling in a homogenous medium meets a boundary of *lower* index of refraction, it reflects, undergoing *no phase change*.

These two rules can be deduced from Maxwell's equations, but the treatment is beyond the scope of this text. Fig. 18.6 summarizes these two rules.

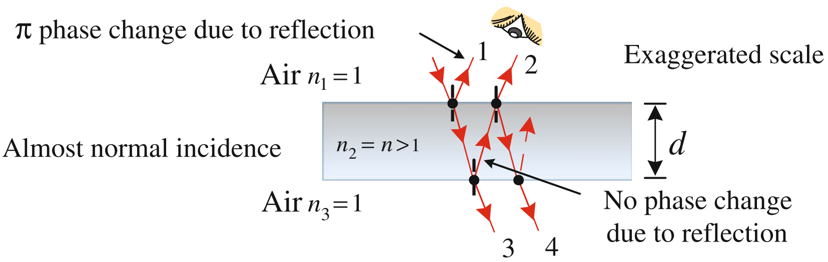


Fig. 18.6 When $n_1 < n_2$, light traveling in medium 1 will reflect from the surface between media 1 and 2 with 180° phase change. When $n_2 > n_3$, light traveling in medium 2 will reflect from the surface between media 2 and 3 with no phase change. Rays 1 and 2 lead to interference of the reflected light, while rays 3 and 4 lead to interference of the transmitted light. All rays are drawn not quite normal to the surface, so we can see each of them

The incoming ray in Fig. 18.6 is a light ray of wavelength λ that *almost normally* strikes a thin transparent film of thickness d. This ray is reflected from the upper surface of the film as ray 1 and has experienced a phase change $\phi_1 = \pi$ rad relative to the incident wave because $n_1 < n_2$. The transmitted ray has a wavelength $\lambda_n = \lambda/n$ and undergoes a second reflection at the lower surface without a phase change because $n_2 > n_3$. This ray is transmitted back to the air as ray 2 after traveling an extra distance $2d$ before recombining in the air with ray 1. Thus, it has a phase change

$\phi_2 = (2\pi/\lambda_n)(2d)$ due to the additional path length. Rays 1 and 2 have a net phase difference given by:

$$\phi_{\text{net}} = \phi_2 - \phi_1 = \frac{2\pi}{\lambda_n}(2d) - \pi \quad \Rightarrow \quad \phi_{\text{net}} = \frac{2\pi}{\lambda}(2\,n\,d) - \pi \tag{18.18}$$

where the first term is due to a $2\,d$ path difference for ray 2, while the second term is due to the reflection from the top surface for ray 1.

Rays 1 and 2 interfere **constructively** when $\phi_{\text{net}} = 0, 2\pi, 4\pi, 6\pi, \ldots$, or according to Eq. 18.18 when $2\,n\,d$ is $\lambda/2, 3\lambda/2, 5\lambda/2, \ldots$. Thus:

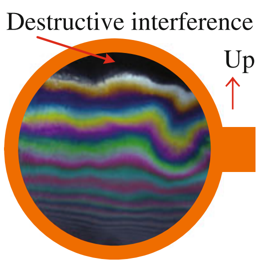
$$\boxed{2\,n\,d_m \;=\; (m - \tfrac{1}{2})\,\lambda \quad (m = 1, 2, 3, \ldots)} \quad \left\{ \begin{array}{c} \text{Maxima} \\ \text{Bright bands} \end{array} \right\} \tag{18.19}$$

Rays 1 and 2 interfere **destructively** (which indicates that they are **strongly transmitted** as rays 2 and 3), when $\phi_{\text{net}} = \pi,\ 3\pi,\ 5\pi, \ldots$, or according to Eq. 18.18 when $2\,n\,d$ is $0, \lambda, 2\lambda, 3\lambda, \ldots$. Thus:

$$\boxed{2\,n\,d_m = m\,\lambda \quad (m = 0, 1, 2, \ldots)} \quad \left\{ \begin{array}{c} \text{Minima} \\ \text{Dark bands} \end{array} \right\} \tag{18.20}$$

The last two equations explain what we occasionally notice as colored bands on a surface of oily water or in a thin film of soap, see Fig. 18.7. These colored bands arise from the interference of white light reflected from the top and bottom surfaces of the film. The different colors arise from the variations in thickness of the film, causing interference for different wavelengths at different points. When the top portion of the film is very thin, all reflected colors undergo destructive interference and produce dark colors.

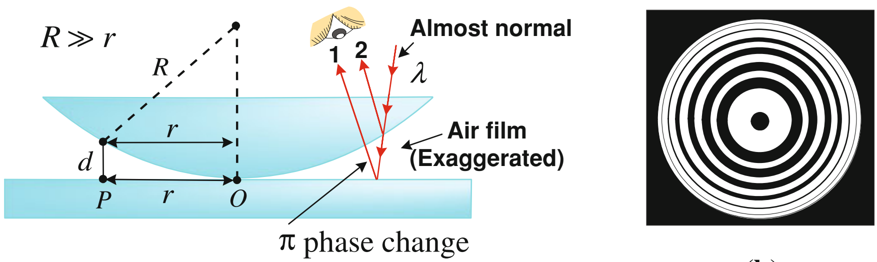
Fig. 18.7



Newton's Rings

Another method for observing light interference patterns from a thin film of varying width is shown in Fig. 18.8a. This figure shows a plano-convex lens of radius R on top of a flat glass surface. The thickness of the air film between the glass surfaces increases from zero at the point of contact O to some value d at point P, which is at a distance r from O. The loci of points of equal thickness d are circles concentric with the point of contact O.

Ray 1 is reflected from the lower surface of the air film and hence undergoes a π phase change (reflection from a medium of higher index of refraction). Ray 2 is reflected from the upper surface of the air film and undergoes no phase change (reflection from a medium of lower index of refraction). Therefore, if $R \gg r$, the conditions for constructive and destructive interference due to the combination of rays 1 and 2 are given by Eqs. 18.19 and 18.20, respectively, but with $n = 1$. The gap thickness changes by $\lambda/2$ as we move from one fringe to the next fringe of the same type. The observed interference pattern of bright and dark rings is shown in Fig. 18.8b.



(a) (b)

Fig. 18.8 (a) An air film of variable thickness between a convex surface and a plane surface. (b) Representation of Newton's rings, which are formed by interference in the air film. Near the center, the thickness of the film is negligible, and the interference is destructive because of the π phase change of ray 1 upon reflection from the lower air surface

Example 18.4

A soap film has an index of refraction $n = 1.33$. Light of wavelength $\lambda = 500$ nm is incident normally on the film. (a) What is the smallest thickness of the film that will give a maximum interference in the reflected light? (b) Would doubling the thickness calculated in part (a) produce maximum interference?

Solution: (a) For a maximum reflected interference, the minimum film thickness corresponds to $m = 1$ in Eq. 18.19. Thus:

$$2\,n\,d_m = (m-\tfrac{1}{2})\,\lambda \quad\Rightarrow\quad 2\,n\,d_1 = \tfrac{1}{2}\,\lambda \quad\Rightarrow\quad d_1 = \frac{\lambda}{4\,n} = \frac{500\,\text{nm}}{4\times 1.33} = 94\,\text{nm}$$

(b) With the new thickness $d' = 2d_1 = 2\lambda/(4n)$, Eq. 18.19 gives:

$$2\,n\,d' = (m-\tfrac{1}{2})\,\lambda \qquad (m = 1, 2, 3, \ldots) \quad\Rightarrow\quad 1 = m - \tfrac{1}{2}$$

The last relation cannot be satisfied, since m must be $1, 2, 3, \ldots$. Thus, maximum interference will *not* occur for a film with twice d. Only odd multiples of d give maximum interference in the reflected light.

Example 18.5

As in Fig.18.8, a plano-convex lens of radius R is placed on a flat sheet of glass. Red light of wavelength $\lambda = 670\,\text{nm}$ is incident normally on the lens. The radius r of the twentieth Newton's dark ring is 11 mm. Find the radius of curvature R of the lens.

Solution: The gap thickness changed by $\lambda/2$ as we move from fringe to the next of same type. The thickness of the twentieth dark ring is:

$$d_{20} = 20\,\frac{\lambda}{2} = 10 \times 670\ \text{nm} = 6{,}700\ \text{nm}$$

From the right triangle of Fig. 18.8a, we have:

$$R^2 = r^2 + (R-d)^2 = r^2 + R^2 - 2Rd + d^2 \quad\Rightarrow\quad 2Rd = r^2 + d^2$$

Neglecting d^2 compared to r^2, we get:

$$R = \frac{r^2}{2d} = \frac{(11 \times 10^6\ \text{nm})^2}{2 \times 6{,}700\ \text{nm}} = 9.03 \times 10^9\ \text{nm} = 9.03\ \text{m}$$

Example 18.6

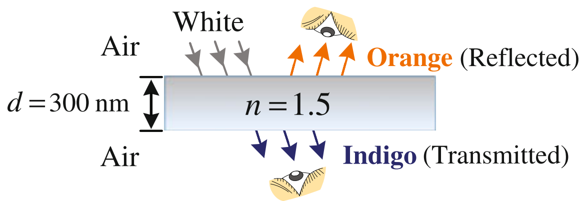
A film with thickness $d = 300\,\text{nm}$ and index of refraction $n = 1.5$ is exposed to white light from one side. Which colors of white light are strongly *reflected* and which are *transmitted*?

Solution: Interference is *constructive* for wavelengths that are most prominent in the *reflected* light. When using Eq. 18.19, $2\,n\,d_m = (m - 1/2)\,\lambda$, these wavelengths are:

$$\lambda = \frac{2\,n\,d_m}{m - 1/2} = \frac{2(1.5)(300\text{ nm})}{m - 1/2} = \frac{900\text{ nm}}{m - 1/2} \quad (m = 1, 2, 3, \ldots)$$

where d_m is fixed and always equal to $d = 300$ nm. For $m = 1, 2, 3,$ we get $\lambda = 1{,}800$ nm, 600 nm, and 360 nm. The first wavelength is in the infrared region (IR), the second is in the visible region (Orange color), and the third is in the ultra-violet region (UV). From these wavelengths that interfere constructively in reflection, only orange has a wavelength within the visible spectrum (400–700 nm) so the film will appear orange when viewed by reflection (see Fig. 18.9).

Fig. 18.9



Interference is *destructive* for wavelengths that are missing from reflected light and thus are strongly *transmitted*. Using Eq. 18.20, $2\,n\,d_m = m\,\lambda$, the transmitted rays have wavelengths:

$$\lambda = \frac{2\,n\,d_m}{m} = \frac{2(1.5)(300\text{ nm})}{m} = \frac{900\text{ nm}}{m} \quad (m = 0, 1, 2, \ldots)$$

For $m = 1, 2, 3,$ we get $\lambda = 900$ nm, 450 nm, and 300 nm. The first wavelength is in the infrared region (IR), the second is in the visible region (Indigo color), and the third is in the ultra violet region (UV). From these wavelengths that interfere destructively in reflection (and hence are transmitted), only indigo has a wavelength within the visible spectrum, so the film will appear indigo when viewed by transmission.

18.4 Diffraction of Light Waves

In Fig. 18.2, we introduced the fact that light waves of $\lambda \approx a$ or $\lambda > a$ spread out after passing through a single slit of width a, and this effect is called **diffraction**. We will see that this spread has interesting features. It has a diffraction pattern consisting of bright and dark areas somewhat similar to the interference pattern.

We have two models of diffractions, one observed when the viewing screen is placed close to the narrow slit (known as **Fresnel diffraction**), and another observed when the viewing screen is placed very far from the slit (known as **Fraunhofer diffraction**). We will consider only the second model, since it is easier to analyze. In this model, we need to focus the parallel rays by using a converging lens.

Figure 18.10a shows a light wave of wavelength λ entering a single slit of width a and diffracted towards a viewing screen. Fig. 18.10b shows a representation of a photograph obtained for a Fraunhofer diffraction pattern. Notice the existence of a wide bright central fringe followed by successive narrower dark fringes.

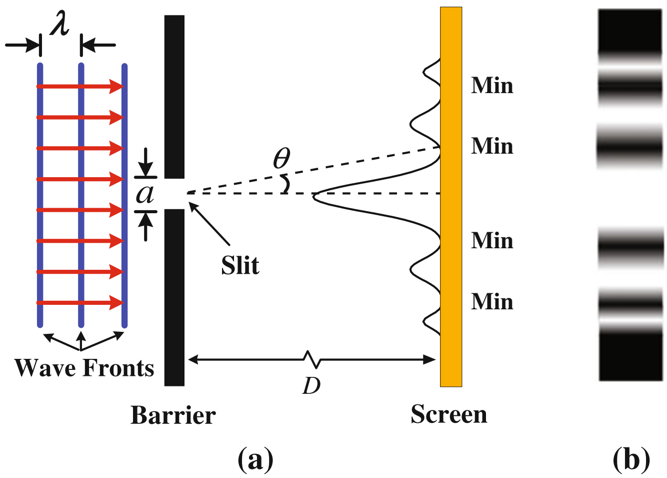


Fig. 18.10 (a) A Fraunhofer diffraction pattern for a single slit (not to scale). (b) Representation of a photograph showing this pattern with a wide central bright fringe followed by much weaker maxima

Figure 18.11 displays the geometry as viewed from above the slit. According to **Huygens' principle**, each point on the wave front within the slit acts like a **secondary wave** source. Waves reaching the screen from different portions of the slit *differ in phase* because they travel *different path lengths*. Differences in phase of the arrived secondary waves produce the diffraction pattern.

We can take advantage of the symmetry of path differences about the central axis by first adding the interference effect from two equal portions of the slit, each of width $a/2$, one above the central axis and one below it, see Fig. 18.11a. This means that the diffraction pattern is actually an interference pattern!

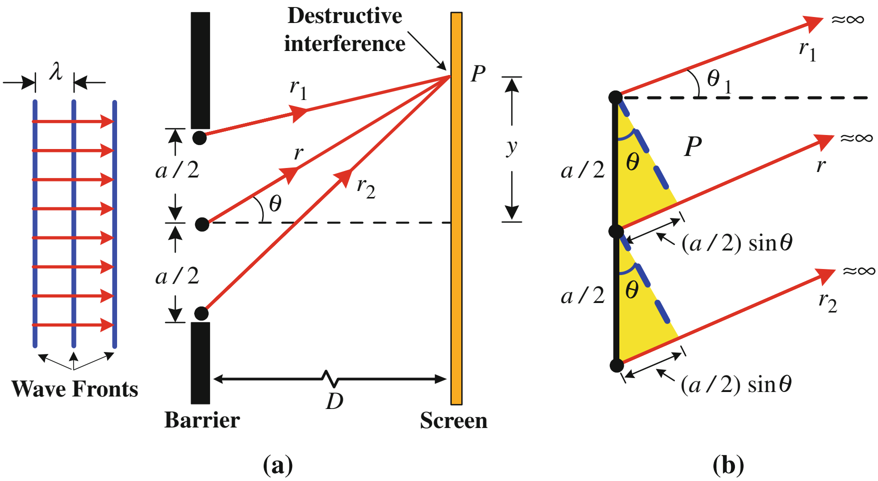


Fig. 18.11 (a) Waves from the two portions of the slit, each having a width $a/2$, undergo destructive interference at point P. (b) When $D \gg a$, we can approximate rays r_1, r, and r_2 as being parallel lines making an angle θ to the central axis. The path difference between rays r_1 and r, or rays r and r_2 is equal to $(a/2)\sin\theta$

The second step is to apply this strategy to locate the first dark fringe at point P, which makes an angle θ with the central line. If the screen is far away from the slit, $D \gg a$, the three rays of Fig. 18.11a are almost parallel, as shown in Fig. 18.11b . This figure indicates that the path difference between rays r_1 and r is $(a/2)\sin\theta$. Similarly, the path difference between rays r and r_2 is $(a/2)\sin\theta$. If this path difference is exactly $\lambda/2$, then the two waves at P cancel each other and produce a destructive interference. In other words, waves from the upper half interfere destructively with waves from the lower half. Consequently, we have:

$$\frac{a}{2}\sin\theta = \frac{\lambda}{2} \quad \xrightarrow{\text{or}} \quad a\sin\theta = \lambda \quad \text{(First minimum)} \tag{18.21}$$

To locate the second dark fringe at point P, we divide the slit into four equal portions, each of width $a/4$, two above the central axis and two below it. Then, the four waves corresponding to the four portions of the slit interfere destructively at P when the path difference is exactly $\lambda/2$. Thus:

$$\frac{a}{4}\sin\theta = \frac{\lambda}{2} \quad \xrightarrow{\text{or}} \quad a\sin\theta = 2\,\lambda \quad \text{(Second minima)} \tag{18.22}$$

We can extend this strategy to many portions of the slit of equal even numbers. Then the locations of the dark fringes can be generalized as:

$$\boxed{a \sin\theta_m = m\,\lambda \quad (m = 1, 2, 3, \ldots)} \quad \left\{ \begin{array}{c} \text{Minima} \\ \text{Dark fringes} \end{array} \right\} \tag{18.23}$$

Although this relation is derived for $D \gg a$, it is also applicable if we place a converging lens between the slit and the screen such that the lens focal plane coincides with the screen. Usually we are only interested in the first minimum because nearly all the light energy is contained in the wide central diffraction maximum.

In Fig. 18.11a, the distance y from the central maximum to the first diffraction minimum is related to the angle θ and the distance d from the slit to the screen by $y = D\tan\theta$. Generally, we have:

$$\boxed{y_m = D\tan\theta_m \quad (m = 1, 2, 3, \ldots)} \quad \text{(Minima)} \tag{18.24}$$

Intensity of Single-Slit Diffraction Patterns

The total path difference between r_1 and r_2 in Fig. 18.11a is $\Delta L = a\sin\theta$. Consequently, according to Eq. 18.13, the total phase difference δ between the two rays is:

$$\delta = \frac{2\pi}{\lambda} a \sin\theta \tag{18.25}$$

The intensity I of the diffraction pattern as a function of θ is given, without proof, in terms of δ as follows:

$$\boxed{I = I_\circ \left(\frac{\sin\delta/2}{\delta/2}\right)^2} \tag{18.26}$$

where $I_\circ$ is the intensity at the central maximum (when $\theta = 0°$).

Substituting the expression of δ into the last equation leads to:

$$I = I_\circ \left(\frac{\sin(\pi a \sin\theta/\lambda)}{\pi a \sin\theta/\lambda}\right)^2 \tag{18.27}$$

A minimum of I occurs when:

$$\frac{\pi a \sin\theta_m}{\lambda} = m\pi \quad (m = 1, 2, 3, \ldots) \tag{18.28}$$

or $a\sin\theta_m = m\lambda$, $(m = 1, 2, 3, \ldots)$, which agrees with Eq. 18.23. Part (a) of Fig. 18.12 displays the variation of I as a function of $\delta/2$. Part (b) is a representation of the obtained photograph.

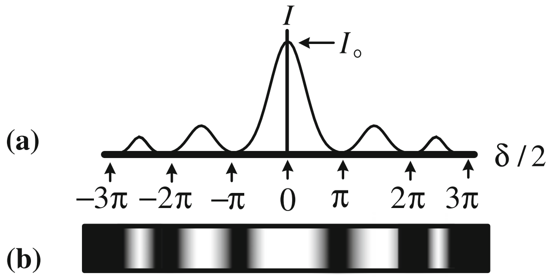


Fig. 18.12 (a) A sketch showing intensity variations as a function of half the total phase difference $\delta/2$. (b) Representation of a Fraunhofer diffraction pattern resulting from a single slit

Example 18.7

Parallel rays of light with wavelength $\lambda = 500$ nm are incident on a slit of width $a = 0.2$ mm. A diffraction pattern is formed on a screen at a distance $D = 2.5$ m from the slit. Find the position of the first minimum and the width of the central bright fringe.

Solution: We use Eq. 18.23, $a \sin\theta_m = m\,\lambda$, with $m = 1$ to find the angle of the first minimum as follows:

$$\sin\theta_1 = \frac{\lambda}{a} = \frac{500 \times 10^{-9}\ \text{m}}{0.2 \times 10^{-3}\ \text{m}} = 2.5 \times 10^{-3}$$

Because θ_1 is very small, we can approximate $\sin\theta_1 \approx \tan\theta_1$. Then we use Eq. 18.24, $y_m = D\tan\theta_m$, with $m = 1$ to calculate the position of the first minimum as follows:

$$y_1 = D\tan\theta_1 \simeq D\sin\theta_1 = (2.5\,\text{m})(2.5 \times 10^{-3}) = 6.25 \times 10^{-3}\ \text{m}$$

The width of the central bright fringe is twice y_1. Thus:

$$2y_1 = 2 \times (6.25 \times 10^{-3}\ \text{m}) = 0.0125\ \text{m} = 1.25\ \text{cm}$$

Example 18.8

Use Fig. 18.12a for a single-slit Fraunhofer diffraction pattern to estimate the ratio of the intensities of the first and second maxima to the central maximum. [Hint: use the intensity relation $I = I_\circ[(\sin\delta/2)/(\delta/2)]^2$, where δ is the total phase difference between the two rays r_1 and r_2 of Fig. 18.11a. See Exercise 32.]

Solution: From Fig. 18.12a, the first and second maxima occur at $\delta/2 = 3\pi/2$ and $\delta/2 = 5\pi/2$, respectively. Substituting these values in the intensity equation, we find the following for the first maximum:

$$\frac{I_1}{I_\circ} = \left(\frac{\sin 3\pi/2}{3\pi/2}\right)^2 = \frac{4}{9\pi^2} = 0.045 \quad \Rightarrow \quad I_1 \text{ is } 4.5\% \text{ of } I_\circ$$

Similarly, for the second maximum, we find that:

$$\frac{I_2}{I_\circ} = \left(\frac{\sin 5\pi/2}{5\pi/2}\right)^2 = \frac{4}{25\pi^2} = 0.016 \quad \Rightarrow \quad I_2 \text{ is } 1.6\% \text{ of } I_\circ$$

18.5 Diffraction Gratings

A **diffraction grating** is one of the most useful devices used to analyze light sources. This device is somewhat like the double-slit experiment of Fig. 18.2 but it has a much greater number of slits[1], perhaps as many as several thousands of slits per millimeter.

For example, a typical grating has about $N = 5{,}000$ grooves/cm, which means that the spacing between every two successive slits d is of the order of $(1/5{,}000)$ cm $= 2 \times 10^{-4}$ cm.

We deal with two types of diffraction gratings:

- *Transmission gratings*: can be obtained by cutting parallel *grooves* on a *glass* plate with a highly precise ruling machine. The spaces between the grooves act as separate slits and produce *transmitted interference fringes*.
- *Reflection gratings*: can be obtained by cutting parallel *grooves* on a *reflecting* surface with a highly precise ruling machine. The reflection of light from the spaces between the grooves form the *reflected interference fringes*.

When monochromatic light of wavelength λ is sent through any type of diffraction grating, it forms narrow interference fringes that can be analyzed to determine the wavelength of the light.

To study the effect of diffraction gratings, we first consider a small number of slits that produce an interference pattern on a distant viewing screen. Each slit produces diffraction, and the diffracted beams interfere with one another to produce the final pattern. Then we gradually increase the number of slits to a larger number N.

Figure 18.13a shows a small section of a diffraction grating containing only five slits. The grating is placed in front of a very distant screen. Plane light waves of

[1] For historical reasons, it is called *diffraction grating*, but it would be more correct to call it *interference grating* because it is somewhat like the double-slit experiment but with huge number of double slits.

wavelength λ are incident normally on the grating. Consider all rays leaving the slits in phase and traveling in an arbitrary direction θ measured from the central axis before reaching point P on the far screen.

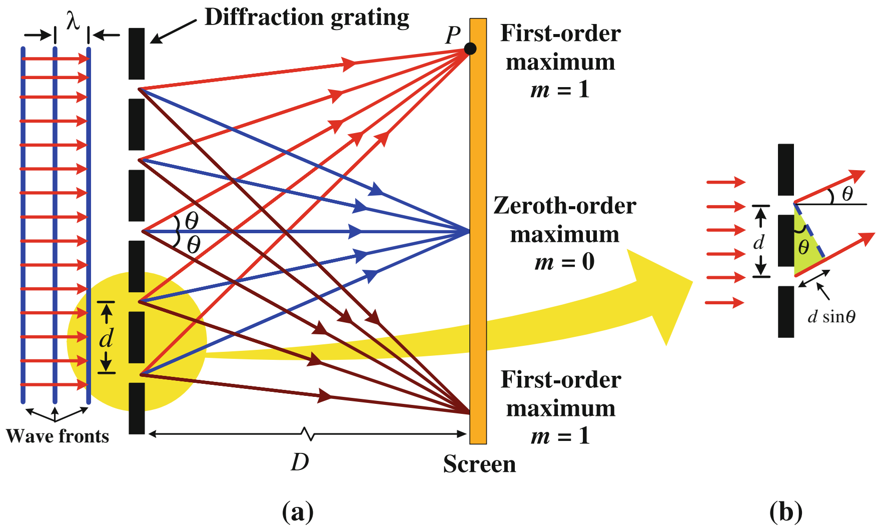


Fig. 18.13 (a) A small section of a diffraction grating having a slit-spacing d. (b) When $D \gg d$, the rays at point P, which make an angle θ to the central axis, are considered to be parallel. The path difference between adjacent slits is $d \sin\theta$

As shown in Fig. 18.13b , the path difference between rays from any two adjacent slits is $d \sin\theta$. If this path difference is equal to an *integral multiple of the wavelength*, then waves from all slits reach point P in phase, and a bright fringe is observed. The condition for a maximum to exist at P is thus:

$$\boxed{d \sin\theta_m = m\,\lambda \quad (m = 0, 1, 2, \ldots)} \quad \left\{ \begin{array}{c} \text{Maxima} \\ \text{Bright fringes} \end{array} \right\} \tag{18.29}$$

where m is the fringe *order number*. Generally, fringes are referred to as follows:

- The zeroth-order maximum, when $m = 0$. All waves must meet at $\theta = 0$,
- The first-order maximum, when $m = 1$. Each wavelength corresponds to an angle $\sin\theta_1 = \lambda/d$,
- The second-order maximum, when $m = 2$. Each wavelength corresponds to an angle $\sin\theta_2 = 2\,\lambda/d$, and so forth.

Therefore, Eq. 18.29 can be used to measure λ if the grating spacing d and θ_m are known.

For monochromatic light, Fig. 18.14 shows a sharp intensity distribution for the maxima and a broad distribution in the dark areas.

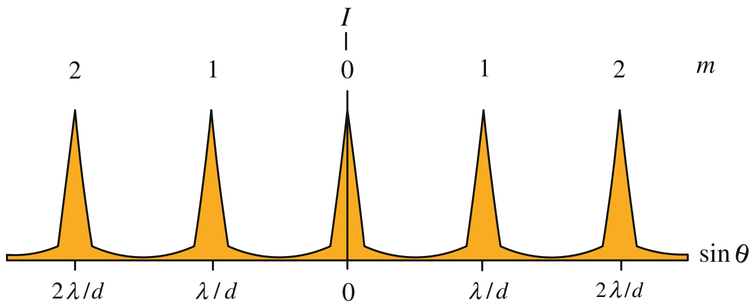


Fig. 18.14 A sketch of the intensity versus $\sin\theta$ for a diffraction grating. The zeroth-, first-, and second-order maxima are shown. The sharpness of the maxima and the broadness of the minima are shown

We can use this technique to distinguish and identify light of several unknown wavelengths. We cannot do that with the double-slit arrangement of Sect. 18.2, even though the same equation and wavelength dependencies apply there. In a double-slit interference, the bright fringes due to different wavelengths overlap too much to be distinguished.

Resolving Power of the Diffraction Gratings

Diffraction gratings are useful tools for accurately measuring wavelengths. To resolve two similar light sources with nearly equal wavelengths λ_1 and λ_2 (near a wavelength λ), the diffraction grating should have a high **resolving power** R, defined in terms of the average wavelength λ_{av} and the wavelength difference $\Delta\lambda$ as:

$$\boxed{R = \frac{\lambda_{av}}{\Delta\lambda}} \quad \boxed{\lambda_{av} = \frac{\lambda_1 + \lambda_2}{2} \simeq \lambda, \quad \Delta\lambda = \lambda_2 - \lambda_1} \tag{18.30}$$

If N is the number of illuminated slits in the grating, then it can be shown that the resolving power in the mth-order diffraction is:

$$\boxed{R = N\, m} \tag{18.31}$$

Thus, R increases as N and m increase. When $m = 0$, we know that all the wavelengths are indistinguishable and hence $R = 0$ as expected.

Example 18.9

A diffraction grating has 7,000 lines per centimeter. When the grating is illuminated normally with a monochromatic light, the second order spectral line is found at 62°.

(a) What is wavelength of the light? (b) Where can we observe the third order maximum?

Solution: (a) First, we must find the slit separation d as follows:

$$d = \frac{1\text{ cm}}{7{,}000} = 1.429 \times 10^{-4}\text{ cm} = 1{,}429\text{ nm}$$

Then, we use $d \sin\theta_m = m\,\lambda$ with $m = 2$ to find the wavelength:

$$\lambda = \frac{d \sin\theta_2}{2} = \frac{(1{,}429\text{ nm})\sin 62°}{2} = 631\text{ nm}$$

(b) For $m = 3$, we calculate $\sin\theta_3$ as follows:

$$\sin\theta_3 = \frac{3\,\lambda}{d} = \frac{3 \times 631\text{ nm}}{1{,}429\text{ nm}} = 1.33$$

Since $\sin\theta_3$ cannot exceed unity, then this order **cannot** be observed.

Example 18.10

A diffraction grating 1 cm wide has $N = 1{,}000$ equally spaced slits across its width. The diffraction grating is illuminated at normal incidence by a sodium-vapor yellow lamp. The yellow light (known as the sodium doublet) contains two colors, one with wavelengths $\lambda_1 = 589.0$ nm and the other with wavelength $\lambda_2 = 589.6$ nm. (a) What is the separation between the slits of the grating? (b) How many bright fringes are seen for both colors? (c) What must the resolving power of the grating be if the two colors are to be resolved (distinguished)? (d) How many slits of this grating must be illuminated in order to resolve these two colors in the fourth-order?

Solution: (a) The ruling separation distance d is:

$$d = \frac{1\text{ cm}}{1{,}000} = 10^{-3}\text{ cm} = 10^4\text{ nm}$$

(b) Maxima occur at $\sin\theta_m = m\,\lambda/d$ $(m = 0, 1, 2, \ldots)$. This condition is accepted only if $m\,\lambda/d < 1$, or $m < d/\lambda$. We select the larger wavelength $\lambda_2 = 589.6$ nm to find the possible values of m as follows:

$$m < \frac{d}{\lambda_2} = \frac{10^4\,\text{nm}}{589.6\,\text{nm}} = 16.96$$

Thus, the orders $m = 0, 1, 2, \ldots, 16$ are seen in the diffraction pattern.

(c) With $\lambda_{av} = (\lambda_1 + \lambda_2)/2 = 589.3\,\text{nm}$ and $\Delta\lambda = 0.6\,\text{nm}$, we use Eq. 18.30 to calculate the resolving power as follows:

$$R = \frac{\lambda_{av}}{\Delta\lambda} = \frac{589.3\text{ nm}}{0.6\text{ nm}} = 982$$

(d) Using Eq. 18.31 and $R = 982$, we find that:

$$N' = \frac{R}{m} = \frac{982}{4} = 246$$

Thus, in order to resolve the yellow sodium doublet up to a 4^{th}-order maximum, we must illuminate at least $N' = 246$ slits.

18.6 Polarization of Light Waves

Light propagates in vacuum with a speed $c = 2.9979 \times 10^8$ m/s, see Chap. 27. As shown in Fig. 18.15a, light waves have the properties of transverse electromagnetic waves, with an electric field vector $\vec{E}$ and a magnetic field vector $\vec{B}$ vibrating in two planes perpendicular to each other. In addition, $\vec{E}$ and $\vec{B}$ propagate with velocity $\vec{c}$ in the direction of the light wave, which is perpendicular to both.

The direction of vibration of $\vec{E}$ for an individual wave is defined as the direction of **polarization** of the wave. However, an ordinary beam of light contains a large number of electromagnetic waves emitted by atoms having random vibrational orientations, and hence the direction of the electric field vector $\vec{E}$ in the beam is random. In this case the beam of light is **unpolarized**.

An electromagnetic wave is said to be **polarized** if the electric field vector $\vec{E}$ at a given position vibrates in the same direction at all times. The **plane of polarization** is defined as the plane containing $\vec{E}$ and the direction of propagation $\vec{c}$. Fig. 18.15a displays a schematic diagram for a light wave polarized along the y axis. Fig. 18.15b represents an unpolarized light beam and Fig. 18.15c represents a polarized beam, both viewed along the direction of propagation.

A bean of unpolarized light can be polarized by reflection, refraction, scattering, or absorption.

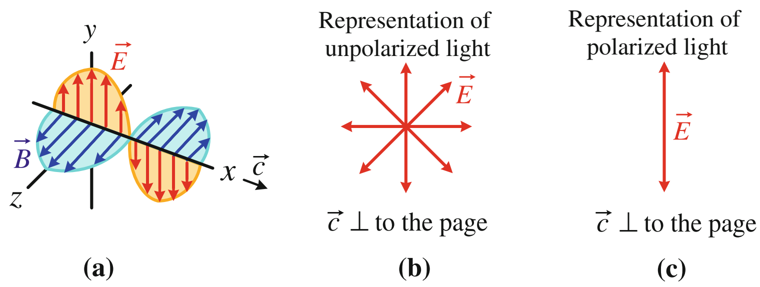


Fig. 18.15 (a) An electromagnetic wave propagating in the x direction with velocity $\vec{c}$, where $\vec{c}$ is perpendicular to both $\vec{E}$ and $\vec{B}$. (b) A sketch representing an unpolarized light beam. (c) A sketch representing a polarized light beam

In 1938, E. H. Land invented a polarizing sheet called *Polaroid*. This sheet transmits waves whose electric fields $\vec{E}$ vibrate in a certain direction (called the transmission axis) and absorbs waves whose electric fields vibrate in a perpendicular direction.

Figure 18.16 represents an unpolarized light beam incident first on a polarizing sheet, called the polarizer. Because the transmission axis is vertical in the figure, the light transmitted through this sheet is polarized vertically with an electric field vector denoted by $\vec{E}_\circ$. A second polarizing sheet, called the analyzer, with transmission axis making an angle θ with the polarizer, intercepts the beam. The only component that is allowed through by the analyzer is $\vec{E}\cos\theta$, while other components are absorbed.

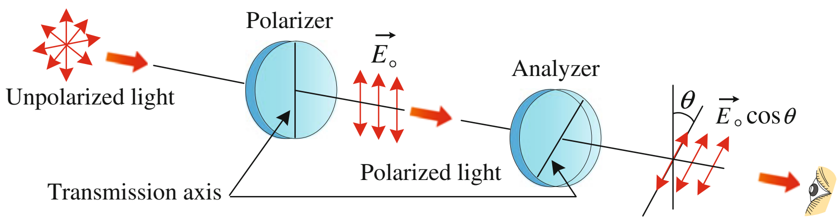


Fig. 18.16 Two polarizing sheets whose transmission axes make an angle θ with each other. Only a fraction of the polarized light incident on the analyzer is transmitted through it

Since the intensity of the transmitted beam varies as the square of the electric field, then the intensity of the polarized beam transmitted through the analyzer varies with θ, and is given by Malus's law as:

$$\boxed{I = I_\circ \cos^2 \theta} \tag{18.32}$$

where $I_\circ$ is the intensity of the polarized beam incident on the analyzer. This expression applies to any two polarizing materials where their transmission axes are placed at an angle θ to each other.

Example 18.11

A plane-polarized light wave $\vec{E}_\circ \sin \omega t$ of intensity $I_\circ$ makes an angle θ with the transmission axis of a Polaroid sheet. What fraction of the original light is transmitted through the Polaroid?

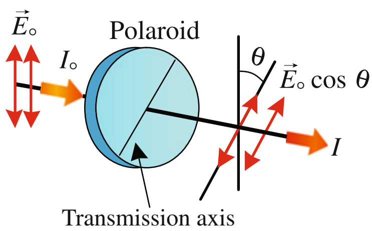
Solution: The incident polarized wave is equivalent to two mutually perpendicular components. One component is parallel to the transmission axis $\vec{E}_\parallel = (\vec{E}_\circ \cos\theta)$ $\sin\omega t$ and the other is perpendicular to it $\vec{E}_\perp = (\vec{E}_\circ \sin\theta) \sin \omega t$. Since $\vec{E}_\parallel$ is the only transmitted component through the Polaroid with an intensity I proportional to the square of its amplitude, see Fig. 18.17, then:

$$\frac{I}{I_\circ} = \frac{(\vec{E}_\circ \cos\theta) \bullet (\vec{E}_\circ \cos\theta)}{\vec{E}_\circ \bullet \vec{E}_\circ} = \cos^2\theta$$

Or: $\quad I = I_\circ \cos^2\theta$

Which is Malus's law.

Fig. 18.17



Example 18.12

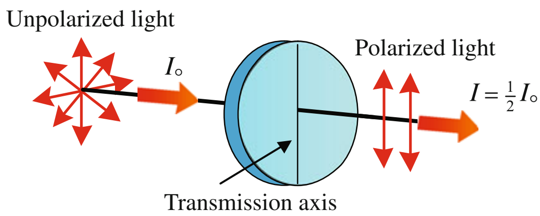
Unpolarized light of intensity $I_\circ$ is incident on a Polaroid sheet with a vertical transmission axis, see Fig. 18.18. What is the intensity I of the transmitted beam?

Solution: We recall that the incident wave consists of a multitude of randomly oriented electric fields. Then, Eq. 18.32 applies to each electric field, but with angle θ ranging from 0° to 360°. Because the orientation is random, all values of θ will occur equally. As a result, Eq. 18.32 will give us the transmitted intensity if we use the average value of $\cos^2\theta$.

Thus, $I = I_\circ\,\overline{\cos^2\theta}$, where $\overline{\cos^2\theta} = \frac{1}{2}$.

Then: $I = \frac{1}{2}I_\circ$ (as indicated in Fig. 18.18)

Fig. 18.18



18.7 Exercises

Sections 18.1 and 18.2 Interference of Light Waves and Young's Double-Slit Experiment

(1) Two identical narrow slits are separated by a distance $d = 0.3$ mm. The slits are illuminated by a monochromatic red light of wavelength $\lambda = 630$ nm. An interference pattern is observed on a screen at a distance $D = 1.2$ m from the plane of the slits. Find the separation between adjacent bright fringes.

(2) Two narrow slits are separated by a distance $d = 0.05$ mm and are 2 m away from a screen. When the slits are illuminated by a monochromatic light of unknown wavelength λ, we obtain a second-order bright fringe 4 cm from the central line. Find the wave length of the light.

(3) When white light is used instead of the monochromatic light in Exercise 2, the first-order fringe of the observed interference pattern resembles a rainbow of violet and red light at the fringe border. The approximate locations of the violet and red light on the screen are $y_{1V} = 16$ mm and $y_{1R} = 28$ mm from the central line. Estimate the wavelengths of the violet and red light.

(4) In a Young's double-slit experiment, monochromatic light is diffracted from two narrow slits 0.4 mm apart. Near the central line, successive bright and dark fringes that are 6 mm apart are both viewed on a screen 4 m away. Find the wavelength and frequency of the light.

(5) In a double-slit experiment, the fifth-order bright fringe produced by light of wavelength 450 nm is observed at an angle of 30° from the central line. How far apart are the two slits?

(6) What are the expected angles of all dark fringes preceding the fifth-order bright fringe of Exercise 5?

(7) A blue light of wavelength $\lambda_B = 475$ nm and yellow light of wavelength $\lambda_Y = 570$ nm pass through a Young's double-slit apparatus. Blue and yellow patterns of fringes are formed on the screen of the experiment. At the central bright fringe, where $m_V = m_B = 0$, both the blue and yellow light mix and form a green fringe. What is the next order of the blue and yellow fringes that overlap on the screen to form a green fringe?

(8) Two narrow slits that are 0.015 mm apart in a Young's double-slit experiment are illuminated by a green laser beam of wavelength $\lambda_G = 510$ nm. (a) What will be the total number of bright fringes that will be formed on both sides of a very large distant screen? (b) What angle does the most distant bright fringe from the central fringe make with respect to the original direction of the laser beam?

(9) Two very narrow slits are 1.5 μm apart and are 25 cm away from a screen. Light of wavelength 450 nm passes through the double slits, forming an interference pattern that does not satisfy the condition $\lambda \ll d$, i.e., $\tan\theta \neq \sin\theta$. What is the distance between the first and second dark fringes of the interference pattern on the screen?

(10) A double-slit experiment is designed for easy viewing in a classroom such that the distance between the central maximum and the first maximum is 30 cm. The wavelength of the He-Ne red laser light used in the experiment is 634 nm, and the viewing screen is 9 m away from the double slits. What slit separation is required for such an interference pattern?

(11) Light of wavelength 600 nm passes through two slits that are 0.5 mm apart. What is the phase difference between two parallel diffracted light rays making an angle 15° with the central line?

(12) The peak intensity of a two-slit interference pattern is denoted by $I_\circ$ and the variation of the intensity as a function of the phase difference is given by Eq. 18.12. Assume a point in the pattern where the phase difference between waves from the two slits is $\pi/3$ rad. (a) What is the intensity of the pattern at this point? (b) What is the path difference between waves at this point when light of wavelength $\lambda = 567$ nm is used?

(13) Two narrow slits 0.15 mm apart are illuminated by 634 nm light and their interference pattern is viewed on a screen 1.2 m meters away from the slits. Find the intensity (relative to the central maximum) of the interference pattern 3.5 cm above the central line.

(14) Show that the relation $2d \sin\theta = (m + \frac{1}{2})\lambda$, $m = 0, 1, 2, \ldots$ gives the angle θ at which the double-slit intensity is one-half of the peak value, i.e., when $I = \frac{1}{2} I_\circ$.

(15) Using the relation obtained in Exercise 14 for half the intensity at the peak, show that the angular displacement from the half-intensity position on one side of the central maximum to the half-intensity position on the other side is given by the relation $\Delta\theta = \lambda/2d$. (Hint: $\sin\theta \simeq \theta$ when θ is small)

Section 18.3 Thin Films—Change of Phase due to Reflection

(16) An oil film has an index of refraction $n = 1.5$. Monochromatic light of wavelength $\lambda = 600$ nm is incident normally on the film. (a) What is the smallest thickness of the film that will give a maximum interference in the reflected light view? (b) Would tripling the thickness calculated in part (a) produce maximum interference in the reflected light view?

(17) A soap bubble has an index of refraction $n = 1.32$ and is 130 nm thick. White light is incident normally on the outer surface of the bubble. What is the wavelength of the color that reflected from the bubble's outer surface?

(18) Two glass plates are in contact at one edge and separated by a very fine separator at the other end to form a wedge as shown in Fig. 18.19. The wedge is illuminated normally from above by light of wavelength $\lambda = 589$ nm. When the wedge is viewed vertically, six dark fringes are observed between the left edge and the last dark fringe located at the separator, see Fig. 18.19. Find the height of the separator.

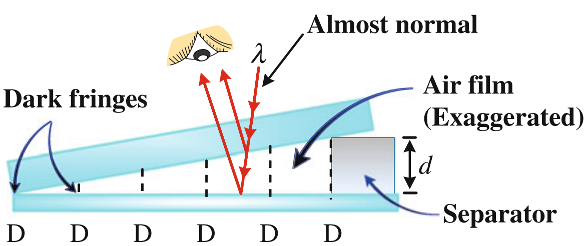


Fig. 18.19 See Exercise(18)

(19) Assume that the separator in Fig. 18.19 has a height $d = 6{,}000$ nm. (a) How many dark and bright bands will be seen in the wedge area? (b) Assume that the glass plates are 20.5 cm long and the dark and bright bands are equal in thickness. How far apart are the bright bands?

(20) A soap film has an index of refraction $n = 1.32$. The film is illuminated normally by light of wavelength $\lambda = 445$ nm. (a) What is the smallest thickness of the film that it will appear black? (b) Why would the film also appear black if the film thickness is much less than the wavelength?

(21) A lens of index of refraction $n_L = 1.52$ appears green ($\lambda = 510$ nm, i.e., reflects most of the green) when white light is shined on its surface. One solution to avoid this effect is to coat the surface of the lens with a film of material that has an index of refraction $n_C = 1.25$. What is the minimum thickness of coating required such that the coating interferes constructively with the green light?

(22) Figure 18.20 shows a very thin film of oil ($n_o = 1.5$) with variable thickness. The oil is floating on water ($n_w = 1.33$). The film is illuminated from above by white light, which causes a sequence of highly distinguished colors to appear as shown in the figure. Take blue to have $\lambda = 445$ nm, yellow to have $\lambda = 570$ nm, and red to have $\lambda = 650$ nm. (a) Find the minimum and maximum thickness of the variable thickness of the oil film. (b) Explain the existence of the dark region in the oil film.

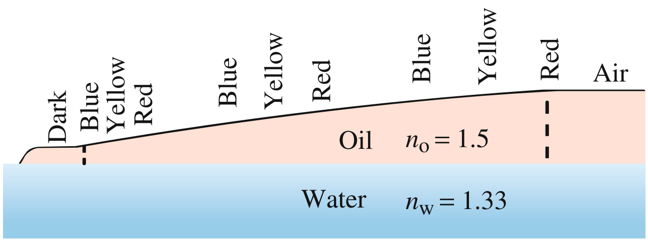


Fig. 18.20 See Exercise (22)

(23) The radius of curvature of the convex surface of a plano-convex thin lens is $R = 5$ m. The convex surface is placed down on a plane glass plate and illuminated from above by light of wavelength $\lambda = 450$ nm, see Fig. 18.21. (a) What is the change in thickness of the air film between the third d_3 and the sixth d_6 bright fringes in the reflected light view? (b) What is the radius r_4 of the fourth bright fringe?

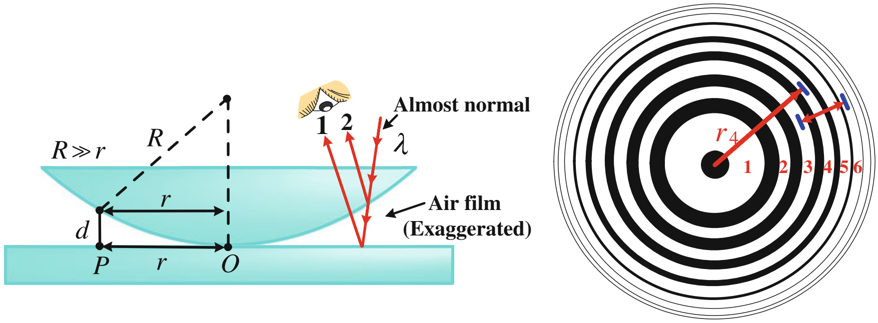


Fig. 18.21 See Exercise(23)

(24) When viewing the Newton's rings from above, use the geometry of Fig. 18.21 to show that the radius r_m of the m^{th} dark ring when $r_m \ll R$ is given by $r_m = \sqrt{m\lambda R/n_{\text{film}}}$, where n_{film} is refractive index of the film. [Hint: Use the binomial expansion $(1-x)^n \simeq 1 - nx$, when $x \ll 1$.]

(25) Use the result of Exercise 24 to show that the distance Δr between adjacent dark Newton's rings of order m and $m+1$ is given by $\Delta r = \sqrt{\lambda R/4mn_{\text{film}}}$, when $m \gg 1$.

(26) The maximum ring radius in Fig. 18.21 is $r = 1.5$ cm and corresponds to the 32^{th} dark ring. (a) What is the radius of curvature of the plano-convex lens if light of wavelength $\lambda = 570$ nm is used? (b) What is the focal length of the lens if its refractive index is $n = 1.52$?

Sections 18.4 and 18.5 Diffraction of Light Waves and Diffraction Gratings

(27) A beam of red light from a helium-neon laser is diffracted by a slit of width $a = 0.5$ mm. A diffraction pattern is formed on a screen at a distance $D = 1.9$ m from the slit. The distance between the zero intensities on either side of the central peak is 4.81 mm. (a) Find the wavelength of the laser light. (b) Calculate the ratio of the intensities of the third maximum to the central maximum.

(28) Monochromatic light of wavelength $\lambda = 480$ nm is diffracted by a single slit of width $a = 4.5 \times 10^{-3}$ mm. A diffraction pattern is formed on a screen at

a distance $D = 7$ m away from the slit. Assuming that the angle of the first maximum is equal to the average of the angles of the first and second minima, estimate how far the first maximum is from the central maximum.

(29) A single slit diffracts light of wavelength $\lambda = 650$ nm. When the screen is at a distance $D = 4$ m away from the slit, the central maximum is 2 cm wide. What is the width of the slit?

(30) (a) What will be the minimum value of a single slit of width a that will not produce diffraction minima for a given wavelength λ? (b) What will be the minimum value of the width a that will not produce diffraction minima for the whole range of visible light (with the approximate range from 400 nm to 700 nm)?

(31) A single slit 1.5 μm wide is illuminated by 634 nm light and its diffraction pattern is viewed on a screen 53.6 cm away from the slit. (a) What is the height of the first minimum above the central maximum? (b) As a fraction of the central maximum's intensity $I_\circ$, determine the light intensity 10 cm above the central maximum.

(32) The secondary maxima in Eq. 18.26 do not occur precisely at the maximum of the sine function. This is because the denominator of the intensity function causes the intensity to decrease more rapidly than the sine function causes it to increase. Consequently, the intensity reaches a maximum slightly before the sine function reaches its maximum. By differentiating Eq. 18.26 with respect to $\delta/2$, show that the secondary maxima occur when $\delta/2$ satisfies the condition $\tan \delta/2 = \delta/2$.

(33) A diffraction grating has 8,000 grooves per centimeter. The first order of the spectral line is observed to be diffracted at an angle of 30°. What is the wavelength of the light used?

(34) A diffraction grating has 5,000 lines per centimeter. The grating is illuminated normally with the green light of mercury, which has a wavelength $\lambda = 546.1$ nm. What is the angular separation between the first and second-order green lines?

(35) A diffraction grating has $N = 4{,}000$ equally spaced slits per centimeter. The diffraction grating is illuminated at normal incidence by the doublet colors of wavelengths $\lambda_1 = 732$ nm and $\lambda_2 = 733$ nm, emitted by a singly-ionized Oxygen atom. (a) What is the separation between the slits of the grating? (b) How many bright fringes are seen for both colors? (c) What should the

resolving power of the grating be if these two colors are to be distinguished? (d) How many slits of this grating must be illuminated in order to resolve these two colors in the second-order?

(36) A diffraction grating has 4,500 lines per centimeter. The grating is illuminated normally by white light (wavelengths ranging from 400 nm to 700 nm). How many spectral orders can be observed?

(37) Use the data of Exercise 36 to find the width of the first-order spectrum on a screen 0.5 m away from the grating.

(38) Find the range of wavelengths of the second-order spectra of a diffraction grating of white light that overlap with the range of wavelengths of the third-order spectra.

(39) A diffraction grating has 6,200 lines per centimeter and is illuminated by light with $\lambda = 525$ nm. The light falls normally on its surface. (a) What is the maximum possible order for this grating? (b) What order gives the best resolution? (c) How close can two wavelengths be if they are to be resolved in the maximum order?

Section 18.6 Polarization of Light Waves

(40) A beam of polarized light has one-fifth of its initial intensity after passing through an analyzer. What is the angle between the axis of the analyzer and the initial polarization direction of the beam?

(41) The transmission axes of two polarizers are oriented at 60° to one another. Unpolarized light of intensity $I_\circ$ falls on them. What fraction of light is transmitted through them?

(42) If the light in Exercise 41 was polarized and the transmitted intensity from the two polarizers is $0.125\, I_\circ$, what is the angle between the axis of the first polarizer and the initial polarization direction of the beam?

(43) Two polarizers are oriented 60° to one another. Light of intensity $I_\circ$ gets polarized as it passes through these polarizers at half the orientation angle between them. What fraction of light intensity is transmitted through both of them?

(44) Determine the angle in Exercise 43 that will make the two polarizers transmit only half of the incident light intensity.

(45) An ordinary light of intensity $I_\circ$ is incident on one Polaroid sheet and then falls on a second Polaroid sheet whose transmission axis makes an angle $\theta = 30°$

with the first, see Fig. 18.22. (a) Find the intensity fractions $I_1/I_\circ$, I_2/I_1, and $I_2/I_\circ$. (b) If the second Polaroid is rotated until the transmitted intensity is 10% of the incident intensity $I_\circ$, what is the new angle?

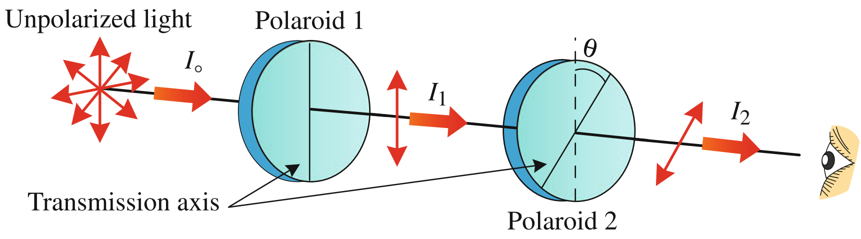


Fig. 18.22 See Exercise(45)

Part V
Electricity

Electric Force

19

In this chapter, we study one of the fundamental forces of nature, the electric force. Electrical forces play an important role in the structure of atoms, molecules, and nuclei. We will discuss the following:

(1) The existence of electric charges and electric forces.
(2) The basic properties of electrostatic forces.
(3) Coulomb's law, which is the fundamental law governing electric forces between charged particles.
(4) The application of Coulomb's law to simple charge distributions.

19.1 Electric Charge

Many simple experiments indicate the existence of electric forces and charges. It is possible to impart an electric charge to any solid material by rubbing it with another material. The rubbed solid material is said to be **electrified**, or **electrically charged**. For example, a comb becomes electrified when it is used to brush dry hair. This is justified by observing that the comb will attract bits of paper.

Many experiments conducted by Benjamin Franklin reveal that there are two types of electric charges: **positive** and **negative**. A glass rod that has been rubbed with silk is commonly used as an example for identifying positive and negative charges. Another common example is a hard rubber rod that has been rubbed with fur. Using Franklin's convention, positive charges are formed on a glass rod that has been rubbed with silk, and negative charges are formed on a rubber rod that has been rubbed with fur.

H. A. Radi and J. O. Rasmussen, *Principles of Physics*,
Undergraduate Lecture Notes in Physics, DOI: 10.1007/978-3-642-23026-4_19,

When a positively charged glass rod is brought close to a suspended negatively charged rubber rod, the two rods *attract* each other, see Fig. 19.1a. Conversely, if two positively charged glass rods (or two negatively charged rubber rods) are brought close to each other, the two rods *repel* each other, see Fig. 19.1b.

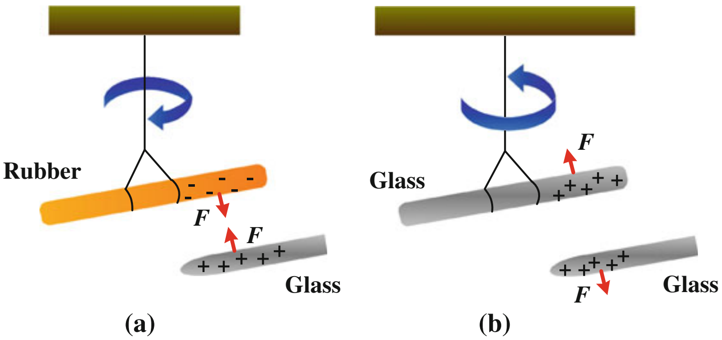


Fig. 19.1 (a) A negatively charged rubber rod attracting a positively charged glass rod. (b) A positively charged glass rod repelling another positively charged glass rod

Based on these observations, we conclude that there are two kinds of charges in nature; one is positive and the other is negative, and they obey the following properties:

Spotlight
Like charges repel each other and unlike charges attract each other.

Additionally, it was found that when one object is rubbed with another, charge is *transferred* between them, i.e. charge is not created in the rubbing process. That is:

Spotlight
The total charge in any isolated system is conserved.

In 1909, Robert Millikan discovered that an electric charge always occurs in integral multiples of a fundamental charge e. In a modern view, the electric charge q is said to be **quantized** and we can write $q = ne$, where n is an integer ($n = \pm 1, \pm 2, \ldots$). That is:

> Spotlight
> Charge is quantized.

In today's modern scientific views, an electric charge is considered to be a basic property of atoms. As we all know, an atom is the fundamental entity of which all matter is formed. Atoms themselves are composed of three types of particles—protons, electrons, and neutrons. A proton carries one unit of positive charge $+e$, an electron carries one unit of negative charge $-e$, and a neutron carries no charge; it is electrically neutral.

Based on the charge conservation and the atomic structure, we find that when a glass rod is rubbed with silk, electrons are transferred from the glass to the silk giving the silk a net negative charge and consequently leaving a net positive charge of the same magnitude on the glass, see Fig. 19.2. Similarly, when a rubber rod is rubbed with fur, electrons are transferred from the fur to the rod, giving the rod a net negative charge, leaving the fur with a net positive charge.

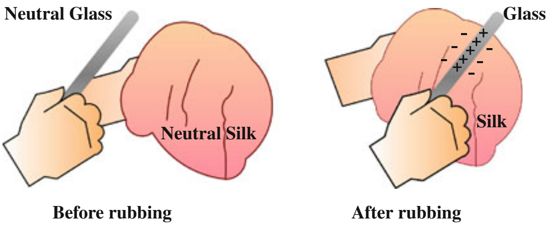


Fig. 19.2 When a neutral glass rod is rubbed with a neutral silk cloth, electrons are transferred from the glass to the silk, leaving the glass positively charged

19.2 Charging Conductors and Insulators

Materials can be classified according to their ability to *conduct* electrical charge. In some materials, such as metals (copper, aluminum, etc), tap water, and the human body, some of the negative charges (electrons) can move rather freely. We call such materials **conductors**.

> Spotlight
> Conductors are materials containing some electrons that can move freely.

In contrast, charges cannot move freely in some other materials such as glass, rubber, and plastic. We call such materials **nonconductors** or **insulators**.

Spotlight
Insulators are materials that contain electrons that are bound to their atoms and cannot move freely through the material.

Semiconductors are materials that lie somewhere between conductors and insulators, such as silicon and germanium. The electrical properties of semiconductors can be changed drastically by adding specific amounts of certain atoms (impurities). Generally, the conductivity of semiconductors increases with increasing temperature, in contrast to metallic conductors. The microelectronic revolution that has changed our lives is due to devices constructed of semiconductors.

Spotlight
Semiconductors are materials that have electrical properties that lie somewhere between conductors and insulators, such as silicon and germanium.

Charging a Conductor by Rubbing

When a person rubs a copper rod with wool while holding it in his hand, he will not be able to charge the rod. The reason is that both the rod and his body are conductors. The rubbing will cause a charge imbalance on the rod, but the excess charge will immediately flow from the rod through his body to the Earth, and the rod will be neutralized immediately. Conversely, if the experiment is repeated while the rod is held by an insulating handle, we would eliminate the conducting path to Earth, and the rod can then be charged.

Charging a Conductor by Induction

Another way of charging a conductor is shown in Fig. 19.3. Figure 19.3a shows a negatively charged plastic rod and an isolated neutral copper rod that is suspended by an insulated twistable wire. When the plastic rod is brought into the vicinity of the copper rod, many of the conduction electrons in the closer end of the copper rod are repelled by the negative charge on the plastic rod. They move to the far end of the copper rod, leaving the near end depleted in electrons. Such repulsion of negative charges from the near end leaves that end positively charged. This positive charge is attracted to the negative charge in the plastic rod as shown in Fig. 19.3b. Although as a whole, the copper rod is still neutral, it is said to have an *induced* charge. At this state, if we ground the copper rod, as shown in Fig. 19.3c, some of the negative charges move out of the rod through the wire into the Earth. *Earth can accept or provide electrons freely with negligible effect on its electrical characteristics*. When the wire to the ground is removed, the conducting copper rod remains in an induced positive charge state, see Fig. 19.3d. When the plastic rod is removed from the vicinity of the copper rod, this induced positive charge remains and is distributed on the rod, see Fig. 19.3e.

This process can be repeated with a positively charged glass rod to obtain a negatively charged copper rod.

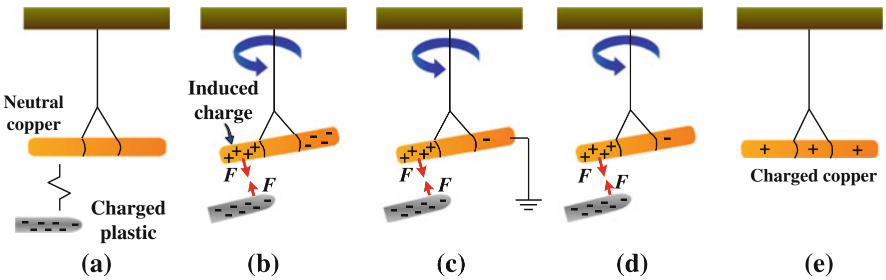


Fig. 19.3 (a) A negatively charged plastic rod is kept far away from a neutral copper rod. (b) The electrons on the copper rod are redistributed when the charged plastic rod is brought into the vicinity of the copper rod. (c) When the copper rod is grounded, some of the electrons move into the Earth. (d) Removing the ground connection. (e) Removing the plastic rod to obtain a positively charged conductor

Charging an Insulator by Induction

In most of the neutral molecules of insulators, the center of positive charge coincides with the center of negative charge. In the presence of a charged object, the charged centers of each molecule in the insulator may shift slightly. The molecule is then said to be electrically polarized. This produces a layer of induced charge on the insulator surface as shown in Fig. 19.4a. Consequently, the charged object and the insulator will attract each other, see Fig. 19.4b.

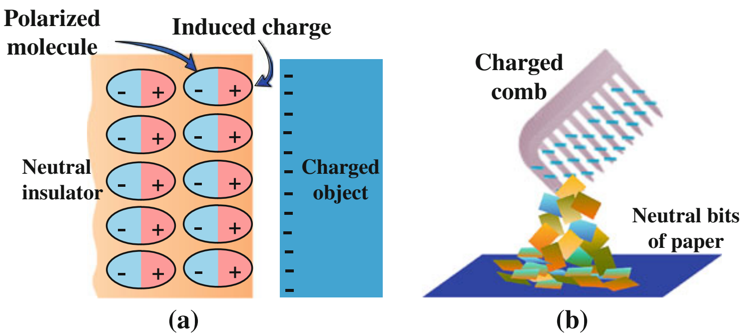


Fig. 19.4 (a) A negatively charged object produces an induced charge on the surface of an insulator because charges in the molecules of the insulator are electrically polarized. (b) A charged comb attracts small bits of dry paper due to the effect of molecular polarization

19.3 Coulomb's Law

In an experiment to measure the **magnitude of the electrical force** F between two charged particles separated by a distance r and having charges q_1 and q_2, Charles Coulomb was able to find that:

$$F = k\frac{|q_1|\,|q_2|}{r^2} \quad \text{(Coulomb's law)} \tag{19.1}$$

This formula is known as **Coulomb's Law**, and k is a constant called the **Coulomb constant**. Coulomb found that each charged particle (also called a *point charge*) exerts a force of this magnitude on the other particle, and the two forces form an action–reaction pair. It was found that charges of the same sign repel each other, while charges of opposite signs attract each other, see Fig. 19.5. The SI unit of a charge is the **coulomb** (abbreviated by C) and is derived from the SI unit of electric current, the **ampere** (abbreviated by A) which will be defined in Chap. 24.

The form given by Eq. 19.1 resembles Newton's force law that describes the universal gravitation between two objects of masses m_1 and m_2 that are separated by a distance r, see Fig. 19.6. That is:

$$F = G\frac{m_1\,m_2}{r^2} \quad \text{(Newton's gravitational law)} \tag{19.2}$$

where $G = 6.67 \times 10^{-11}$ N.m^2/kg^2 is the gravitational constant.

Both the inverse square laws describe a property of interacting objects where charges are involved in one case and masses in the other. The laws differ in that the electrostatic forces between two charged particles may be either attractive or repulsive, but gravitational forces are always attractive.

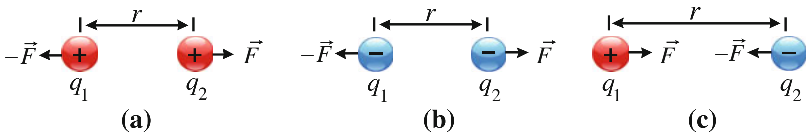


Fig. 19.5 In (a) and (b), two charged particles of the same sign repel. In (c), two charged particles of different signs attract each other. Notice that in all cases, the exerted forces are equal in magnitude but opposite in direction

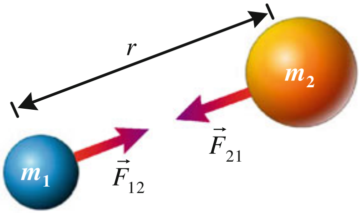


Fig. 19.6 Newton's law of universal gravitation states that the gravitational force between two objects of masses m_1 and m_2 is attractive. The magnitude of the force F_{12} exerted on object 1 by object 2 is equal to the magnitude of the force F_{21} exerted on object 2 by object 1. Note that $\vec{F}_{12} = -\vec{F}_{21}$.

The electrostatic constant k in Coulomb's law has the value:

$$\boxed{k = 8.9875 \times 10^9 \text{ N.m}^2/\text{C}^2 \approx 9 \times 10^9 \text{ N.m}^2/\text{C}^2} \qquad (19.3)$$

For historical reasons and for the aim of simplifying many other formulas, the constant k is usually written as:

$$\boxed{k = \frac{1}{4\pi\epsilon_\circ}} \qquad (19.4)$$

where the quantity $\epsilon_\circ$ (called the **permittivity constant** of free space) has the value:

$$\boxed{\epsilon_\circ = 8.8542 \times 10^{-12}\ \mathrm{C^2/N.m^2}} \tag{19.5}$$

Any positive or negative charge q that can be detected is written as:

$$q = ne, \quad n = \pm 1,\ \pm 2,\ \pm 3,\ \ldots \tag{19.6}$$

where e is the smallest unit charge in nature[1] and has the value:

$$\boxed{e = 1.60219 \times 10^{-19}\ \mathrm{C}} \tag{19.7}$$

As introduced earlier, the charge of an electron is $-e$ and of a proton is $+e$. Therefore, the number N of electrons or protons in 1 C is:

$$N = \frac{1\ \mathrm{C}}{1.60219 \times 10^{-19}\ \mathrm{C}} = 6.24 \times 10^{18}\ \text{electrons or protons} \tag{19.8}$$

Table 19.1 lists the charges and masses of the three elementary particles: the electron, the proton, and the neutron.

Table 19.1 Charge and mass of the electron, proton and neutron.

Particle	Charge (C)	Mass (kg)
Electron (e)	$-e$ $(= -1.60219 \times 10^{-19}$ C)	9.1095×10^{-31}
Proton (p)	$+e$ $(= +1.60219 \times 10^{-19}$ C)	1.67261×10^{-27}
Neutron (n)	0	1.67492×10^{-27}

Example 19.1

Consider the three point charges $q_1 = +2\ \mu$C, $q_2 = -5\ \mu$C, and $q_3 = +8\ \mu$C that are shown in Fig. 19.7. (a) Find the *resultant* force exerted on the charge q_2 by the two charges q_1 and q_3. (b) In a different layout (see Fig. 19.8), q_2 experiences a resultant force of zero. Find the position of q_2 and find the magnitude of each force exerted on q_2.

Solution: (a) Because q_2 is negative and both q_1 and q_3 are positive, the forces $\vec{F}_{21}$ and $\vec{F}_{23}$ are both attractive as displayed in Fig. 19.7. From Coulomb's law we can find F_{21} as follows:

[1] No charge smaller than e has yet been detected on a free particle. Recent theories propose the existence of particles called *quarks* having charges $-e/3$ and $+2e/3$ inside nuclear matter. Although a significant number of recent experiments indicate the existence of quarks inside nuclear matter, *free quarks* have not been detected yet.

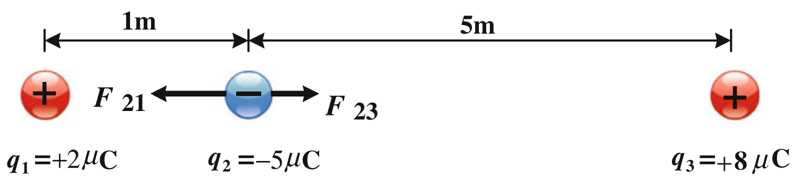


Fig. 19.7

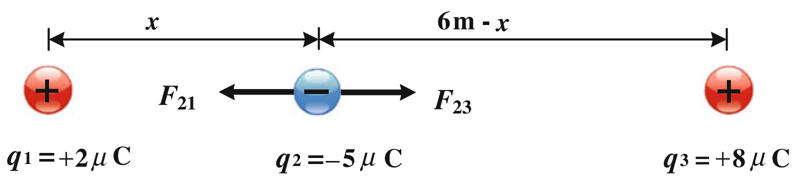


Fig. 19.8

$$F_{21} = k\frac{|q_2|\,|q_1|}{r^2} = 9\times10^9\ \text{N.m}^2/\text{C}^2\,\frac{(5\times10^{-6}\ \text{C})(2\times10^{-6}\ \text{C})}{(1\ \text{m})^2} = 0.09\ \text{N}$$

Also, we can find F_{23} as follows:

$$F_{23} = k\frac{|q_2|\,|q_3|}{r^2} = 9\times10^9\ \text{N.m}^2/\text{C}^2\,\frac{(5\times10^{-6}\ \text{C})(8\times10^{-6}\ \text{C})}{(5\ \text{m})^2} = 0.0144\ \text{N}$$

Since F_{21} is greater than F_{23}, the resultant force F exerted on q_2 will be toward the charge q_1, i.e. to the left. Therefore:

$$F = F_{21} - F_{23} = 0.09\ \text{N} - 0.0144\ \text{N} = 0.0756\ \text{N}$$

(b) When the resultant force on q_2 is zero, the magnitudes of F_{21} and F_{23} must be equal. Based on Fig. 19.8, the equality of the two forces F_{21} and F_{23} leads to the following steps:

$$F_{21} = F_{23} \Longrightarrow k\frac{|q_2|\,|q_1|}{x^2} = k\frac{|q_2|\,|q_3|}{(6\ \text{m}-x)^2} \Longrightarrow \frac{|q_1|}{x^2} = \frac{|q_3|}{(6\ \text{m}-x)^2}$$

We can now substitute the given values of q_1 and q_3 and this yields the following:

$$\frac{2\times10^{-6}\ \text{C}}{x^2} = \frac{8\times10^{-6}\ \text{C}}{(6\ \text{m}-x)^2} \Longrightarrow (6\ \text{m}-x)^2 = 4x^2 \Longrightarrow 6\ \text{m}-x = 2x \Longrightarrow x = 2\ \text{m}$$

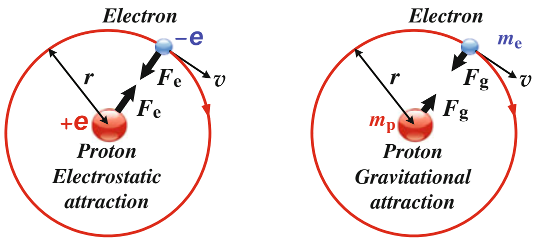
From Coulomb's law we can find either of the value of F_{21} or the value of F_{23} as follows:

$$F_{23} = F_{21} = k\frac{|q_2|\,|q_1|}{x^2} = 9\times10^9\ \text{N.m}^2/\text{C}^2\,\frac{(5\times10^{-6}\ \text{C})(2\times10^{-6}\ \text{C})}{(2\ \text{m})^2} = 0.0225\ \text{N}$$

Example 19.2

In the classical model of the hydrogen atom proposed by Niels Bohr, the electron rotates around a stationary proton in a circular orbit with an approximate radius $r = 0.053$ nm, see Fig. 19.9. (a) Find the magnitude of the electrostatic force of attraction, F_e, between the electron and the proton. (b) Find the magnitude of the gravitational force of attraction, F_g, between the electron and the proton, and then find the ratio F_e/F_g.

Fig. 19.9



A classical model of the hydrogen atom

Solution: (a) From Coulomb's law, the magnitude of the electrostatic force of attraction $\vec{F}_e$ between the electron and the proton is:

$$F_e = k\frac{|-e|\,|e|}{r^2} = 9\times10^9\ \text{N.m}^2/\text{C}^2\,\frac{(1.6\times10^{-19}\ \text{C})^2}{(0.053\times10^{-9}\ \text{m})^2} = 8.2\times10^{-8}\ \text{N}$$

(b) From Newton's law of gravitation, the magnitude of the gravitational force of attraction $\vec{F}_g$ between the two particles is:

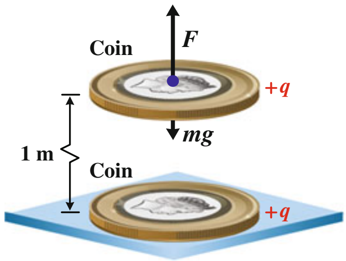
$$\begin{aligned}F_g &= G\frac{m_e\,m_p}{r^2}\\ &= 6.67\times10^{-11}(\text{N.m}^2/\text{kg}^2)\,\frac{(9.11\times10^{-31}\ \text{kg})(1.67261\times10^{-27}\ \text{kg})}{(0.053\times10^{-9}\ \text{m})^2}\\ &= 3.6\times10^{-47}\ \text{N}\end{aligned}$$

The ratio $F_e/F_g \approx 2.3\times10^{39}$. Thus, for elementary particles the gravitational force is negligible compared to the electrical forces.

Example 19.3

Two identical copper coins of mass $m = 2.5$ g contain about $N = 2 \times 10^{22}$ atoms each. A number of electrons n are removed from each coin to acquire a net positive charge q. Assume that when we place one of the coins on a table and the second above the first, the second coin stays at rest in air at a distance of 1 m, see Fig. 19.10. (a) Find the value of q that keeps the two coins in that configuration. (b) Find the number of removed electrons n from each coin. (c) Find the fraction of the copper atoms that lost those n electrons in each coin. Assume that each copper atom loses only one electron.

Fig. 19.10



Solution: (a) The upper coin is in equilibrium due to its weight and the electrostatic repulsion between the two charged coins. Therefore:

$$m\,g = k\frac{q \times q}{r^2}$$

$$q = \sqrt{\frac{m\,g\,r^2}{k}} = \sqrt{\frac{(2.5 \times 10^{-3}\ \text{kg})(9.8\ \text{N/kg})(1\ \text{m})^2}{9 \times 10^9\ \text{N}{\cdot}\text{m}^2/\text{C}^2}} = 1.65 \times 10^{-6}\ \text{C}$$

This small charge leads to a measurable force between large bodies.

(b) From the electronic charge ($-e$) and the total charge q on each coin, we can find the number of removed electrons n as follows:

$$n = \frac{q}{e} = \frac{1.65 \times 10^{-6}\ \text{C}}{1.6 \times 10^{-19}\ \text{C}} \approx 10^{13}\ \text{electrons} \quad \text{(Very big number)}$$

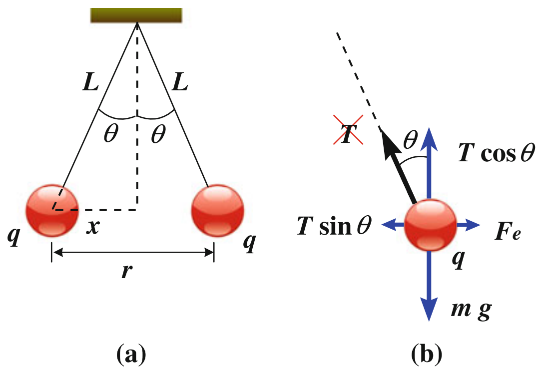
(c) The fraction of the copper atoms that loses the n electrons is:

$$f = \frac{n}{N} = \frac{10^{13}}{2 \times 10^{22}} = 5 \times 10^{-10} \quad \text{(Very small fraction)}$$

Example 19.4

Two identical tiny spheres of mass $m = 2$ g and charge q hang from non-conducting strings, each of length $L = 10$ cm. At equilibrium, each string makes an angle $\theta = 5°$ with the vertical, see Fig. 19.11a. Find the magnitude of the charge on each sphere.

Fig. 19.11



Solution: To analyze this problem, we draw the free-body diagram for the right sphere as shown in Fig. 19.11b. This sphere is in equilibrium under the tensional force $\vec{T}$ from the string, the electric force $\vec{F}_e$ from the left sphere, and the gravitational force $m\vec{g}$. After decomposing the tensional force $\vec{T}$ in the vertical and horizontal directions, we apply the condition of equilibrium as follows:

$$\sum F_x = F_e - T\sin\theta = 0 \quad \Rightarrow \quad F_e = T\sin\theta$$
$$\sum F_y = T\cos\theta - mg = 0 \quad \Rightarrow \quad T\cos\theta = mg$$

Eliminating T from the above two equations, we get the value of F_e:

$$F_e = mg\tan\theta = (2\times10^{-3}\ \text{kg})(9.8\ \text{N/kg})(\tan 5°) = 1.7\times10^{-3}\ \text{N}$$

From Fig. 19.11a, we find the distance r between the two charges:

$$r = 2\,x = 2\,L\sin\theta = 2(0.1\ \text{m})(\sin 5°) = 0.017\ \text{m}$$

Applying Coulomb's law, we find the magnitude of the charge to be:

$$F_e = k\frac{|q|\,|q|}{r^2} \quad \Rightarrow \quad |q| = \sqrt{\frac{F_e\,r^2}{k}} = \sqrt{\frac{(1.7\times10^{-3}\ \text{N})(0.017\ \text{m})^2}{9\times10^{9}\ \text{N.m}^2/\text{C}^2}} = 7.39\times10^{-9}\,\text{C}$$

Note that the charges of the two spheres could be positive or negative.

Example 19.5

Consider three charges $q_1 = +12\ \mu$C, $q_2 = +6\ \mu$C, and $q_3 = -4\ \mu$C are setup as shown in Fig. 19.12. Find the resultant force exerted on the charge q_2 by the two charges q_1 and q_3.

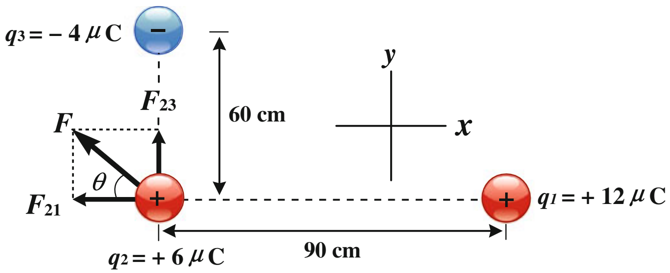


Fig. 19.12

Solution: Because q_1 and q_2 are positive, while q_3 is negative, the force $\vec{F}_{21}$ is repulsive and the force $\vec{F}_{23}$ is attractive as displayed in Fig. 19.12. From Coulomb's law we can find F_{21} as follows:

$$F_{21} = k\frac{|q_2|\ |q_1|}{r^2} = 9 \times 10^9\ \text{N.m}^2/\text{C}^2\,\frac{(6 \times 10^{-6}\ \text{C})(12 \times 10^{-6}\ \text{C})}{(0.9\ \text{m})^2}$$
$$= 0.8\ \text{N}$$

Similarly, we can find F_{23} as follows:

$$F_{23} = k\frac{|q_2|\ |q_3|}{r^2} = 9 \times 10^9\ \text{N.m}^2/\text{C}^2\,\frac{(6 \times 10^{-6}\ \text{C})(4 \times 10^{-6}\ \text{C})}{(0.6\ \text{m})^2}$$
$$= 0.6\ \text{N}$$

Since $\vec{F}_{21}$ is perpendicular to $\vec{F}_{23}$, we can use the Pythagorean theorem to find the magnitude of the resultant force $\vec{F}$, and we can use Fig. 19.12 to find its direction. Thus:

$$F = \sqrt{F_{21}^2 + F_{23}^2} = \sqrt{(0.8\ \text{N})^2 + (0.6\ \text{N})^2} = \sqrt{0.64\ \text{N}^2 + 0.36\ \text{N}^2} = 1\ \text{N}$$
$$\theta = \tan^{-1}\left(\frac{F_{23}}{F_{21}}\right) = \tan^{-1}(0.75) = 36.9^\circ$$

We can also write the resultant force $\vec{F}$ in vector form as follows:

$$\vec{F} = -F_{21}\,\vec{\mathrm{i}} + F_{23}\,\vec{\mathrm{j}} = \left(-0.8\vec{\mathrm{i}} + 0.6\vec{\mathrm{j}}\right)\ \text{N}$$

Example 19.6

Consider three charges $q_1 = +5\,\mu$C, $q_2 = +10\,\mu$C, and $q_3 = -2\,\mu$C are setup as shown in Fig. 19.13. Find the resultant force exerted on the charge q_2 by the two charges q_1 and q_3.

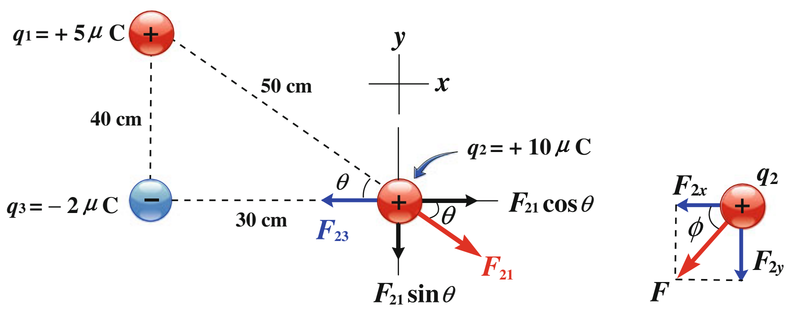


Fig. 19.13

Solution: Because q_1 and q_2 are positive, while q_3 is negative, the force $\vec{F}_{21}$ is repulsive and the force $\vec{F}_{23}$ is attractive as displayed in Fig. 19.13. From Coulomb's law we can find F_{21} and F_{23} as follows:

$$F_{21} = k\frac{|q_2|\,|q_1|}{r^2} = 9\times10^9\ \mathrm{N.m^2/C^2}\frac{(10\times10^{-6}\ \mathrm{C})(5\times10^{-6}\ \mathrm{C})}{(0.5\ \mathrm{m})^2} = 1.8\ \mathrm{N}$$

$$F_{23} = k\frac{|q_2|\,|q_3|}{r^2} = 9\times10^9\ \mathrm{N.m^2/C^2}\frac{(10\times10^{-6}\ \mathrm{C})(2\times10^{-6}\ \mathrm{C})}{(0.3\ \mathrm{m})^2} = 2\ \mathrm{N}$$

Using the coordinate system shown in Fig. 19.13, we have:

$$F_{2x} = F_{21}\cos\theta - F_{23} = (1.8\ \mathrm{N})\frac{30\ \mathrm{cm}}{50\ \mathrm{cm}} - 2\ \mathrm{N} = -0.92\ \mathrm{N}$$

$$F_{2y} = -F_{21}\sin\theta = -(1.8\ \mathrm{N})\frac{40\ \mathrm{cm}}{50\ \mathrm{cm}} = -1.44\ \mathrm{N}$$

Resultant: $F = \sqrt{F_{2x}^2 + F_{2y}^2} = \sqrt{(-0.92\ \mathrm{N})^2 + (-1.44\ \mathrm{N})^2} = 1.7\ \mathrm{N}$

Direction: $\phi = \tan^{-1}\left(\frac{|F_{2y}|}{|F_{2x}|}\right) = \tan^{-1}(1.565) = 57.4°$

We can also write the resultant force $\vec{F}$ in vector form as follows:

$$\vec{F} = -F_{2x}\vec{\mathrm{i}} + F_{2y}\vec{\mathrm{j}} = \left(-0.92\vec{\mathrm{i}} + 1.44\vec{\mathrm{j}}\right)N$$

19.4 Exercises

Section 19.1 Electric Charge

(1) Explain what is meant by the following: (a) a neutral atom, (b) a negatively charged atom, and (c) a positively charged atom.

(2) A neutral rubber rod is rubbed with fur as shown in Fig. 19.14. After rubbing, what would be the charge on each of these items? Is it possible to transfer positive charges from one of them to the other? Why so or why not?

Fig. 19.14 See Exercise (2)

Section 19.2 Charging Conductors and Insulators

(3) If we repeat the experiment illustrated in Fig. 19.3 but instead of using a charged plastic rod, we use a charged rubber rod, what will the final charge on the copper rod be?

(4) A charged plastic comb often attracts small bits of dry paper, as shown in the left part of Fig. 19.15. After a while, the bits of paper fall down, as shown in the right part of Fig. 19.15. Explain this observation.

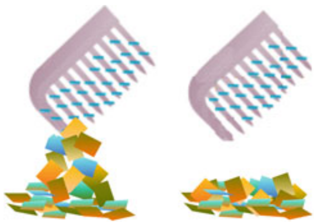


Fig. 19.15 See Exercise (4)

(5) In an oxygen-enriched atmosphere (as in hospital operation rooms), workers must wear special conducting shoes and avoid wearing rubber-soled shoes. Explain the reason behind this.

(6) A negatively charged balloon clings to a wall as shown in the right part of Fig. 19.16. Does this mean that the wall is positively charged? Why does the balloon fall afterwards?

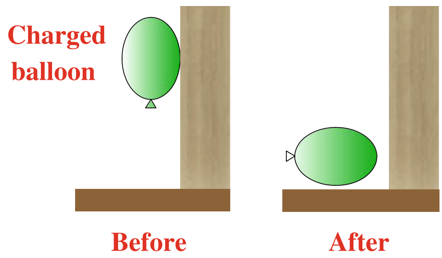


Fig. 19.16 See Exercise (6)

(7) Using a charged rubber rod, show the steps of how two uncharged metallic spheres mounted on insulating stands can be electrostatically charged with equal amount of charges, but opposite in sign.

Section 19.3 Coulomb's Law

(8) How many electrons exist in a -1 C charge? What is the total mass of these electrons?

(9) Find the magnitude of the electrostatic force between two 1 C charges separated by a distance (a) 1 cm, (b) 1 m, and (c) 1 km, if such a configuration could be set up. Are these forces substantial forces? Do they indicate that the coulomb is a very large unit of charge?

(10) Find the magnitude of the force between two electrons when they are separated by 0.1 nm (a typical atomic dimension).

(11) The uranium nucleus contains 92 protons. How large a repulsive force would a uranium proton experience when it is 0.01 nm from the nucleus center? (The nucleus can be treated as a point charge since the nuclear radius is of the order of 10^{-14} m).

(12) Two electrically neutral spheres are 0.1 m apart. When electrons are moved from one of the spheres to another, an attractive force of magnitude 10^{-3} N is established between them. How many electrons were transferred?

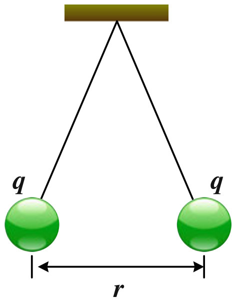
(13) Silver has 47 electrons per atom and a molar mass of 107.87 kg/kmol. An electrically neutral pin of silver has a mass of 10 g. (a) Calculate the number of electrons in the silver pin (Avogadro's number is 6.022×10^{26} atoms/kmol). (b) Electrons are added to the pin until the net charge is -1 mC. How many electrons are added for every billion (10^9) electrons in the neutral atoms?

(14) Two protons in an atomic nucleus are separated by 2×10^{-15} m (a typical inter-nuclear dimension). (a) Find the magnitude of the electrostatic repulsive force between the protons. (b) How does the magnitude of the electrostatic force compare to the magnitude of the gravitational force between the two protons?

(15) Two particles have an identical charge q and an identical mass m. What must the charge-mass ratio, q/m, of the two particles be if the magnitude of their electrostatic force equals the magnitude of the gravitational force.

(16) Two equally charged pith balls are at a distance $r = 3$ cm apart, as shown in Fig. 19.17. Find the magnitude of the charge on each ball if they repel each other with a force of magnitude 2×10^{-5} N. Does the answer give you any hint about the exact sign of each charge? Explain.

Fig. 19.17 See Exercise (16)



(17) Two point charges q_1 and q_2 are 3 m apart, and their combined charge is 40 μC. (a) If one repels the other with a force of 0.175 N, what are the two charges? (b) If one attracts the other with a force of 0.225 N, what are the two charges?

(18) Two point charges $q_1 = +4\ \mu$C and $q_2 = +6\ \mu$C are 10 cm apart. A point charge $q_3 = +2\ \mu$C is placed midway between q_1 and q_2. Find the magnitude and direction of the resultant force on q_3.

(19) Three 4 μC point charges are placed along a straight line as shown in Fig. 19.18. Calculate the net force on each charge.

(20) Three point charges $q_1 = q_2 = q_3 = -4\ \mu$C are located at the corners of an equilateral triangle as shown in Fig. 19.19. (a) Calculate the magnitude of

the net force on any one of the three charges. (b) If the charges are positive, i.e. $q_1 = q_2 = q_3 = +4\,\mu$C, would this change the magnitude calculated in part a?

Fig. 19.18 See Exercise (19)

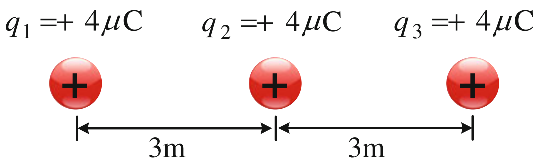
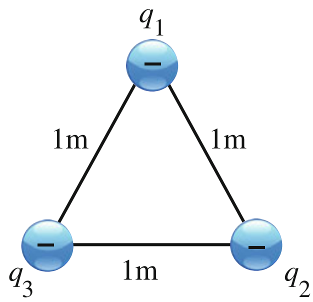
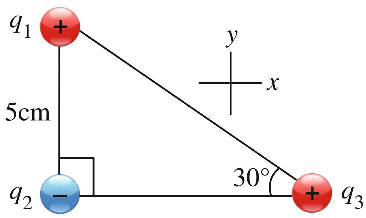


Fig. 19.19 See Exercise (20)



(21) Three point charges $q_1 = +2\,\mu$C, $q_2 = -3\,\mu$C, and $q_3 = +4\,\mu$C are located at the corners of a right angle triangle as shown in Fig. 19.20. Find the magnitude and direction of the resultant force on q_3.

Fig. 19.20 See Exercise (21)



(22) Three equal point charges of magnitude q lie on a semicircle of radius R as shown in Fig. 19.21. Show that the net force on q_2 has a magnitude $kq^2/\sqrt{2}R^2$ and points downward away from the center C of the semicircle.

(23) Four equal point charges, $q_1 = q_2 = q_3 = q_4 = +3\,\mu$C, are placed at the four corners of a square that has a side $a = 0.4$ m, see Fig. 19.22. (a) Find the force

on q_1. (b) Find the force exerted on a test charge of 1 C placed at the center P of the square.

Fig. 19.21 See Exercise (22)

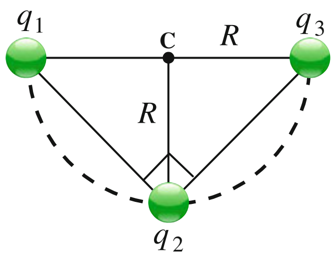
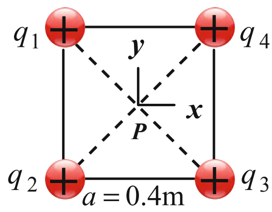
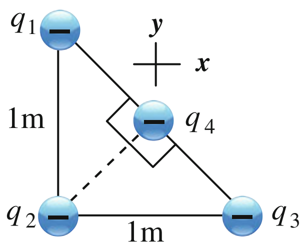


Fig. 19.22 See Exercise (23)



(24) Four equal point charges $q_1 = q_2 = q_3 = q_4 = -1\ \mu\text{C}$, are located as shown in Fig. 19.23. (a) Calculate the net force exerted on the charge q_4, which is located midway between q_1 and q_3. (b) Calculate the magnitude and direction of the net force on the charge q_2.

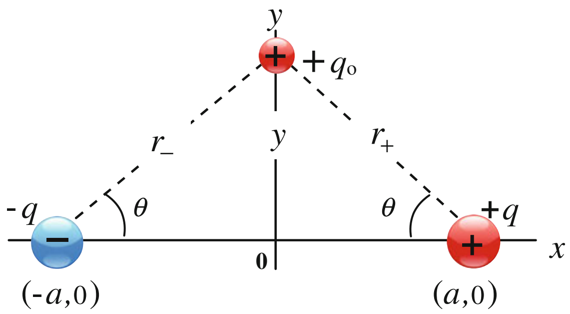
Fig. 19.23 See Exercise (24)



(25) A negative point charge of magnitude q is located on the x-axis at point $x = -a$, and a positive point charge of the same magnitude is located at $x = +a$, see Fig. 19.24. A third positive point charge $q_\circ$ is located on the y-axis with a coordinate

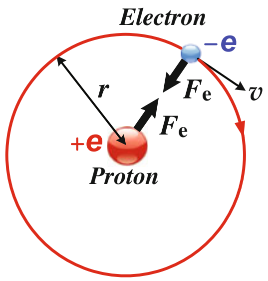
(0, y). (a) What is the magnitude and direction of the force exerted on $q_\circ$ when it is at the origin (0,0)? (b) What is the force on $q_\circ$ when its coordinate is (0, y)? (c) Sketch a graph of the force on $q_\circ$ as a function of y, for values of y between $-4a$ and $+4a$.

Fig. 19.24 See Exercise (25)



(26) In the Bohr model of the hydrogen atom, an electron of mass $m = 9.11 \times 10^{-31}$ kg revolves about a stationary proton in a circular orbit of radius $r = 5.29 \times 10^{-11}$ m, see Fig. 19.25. (a) What is the magnitude of the electrical force on the electron? (b) What is the magnitude of the centripetal acceleration of the electron? (c) What is the orbital speed of the electron?

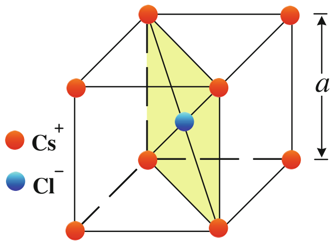
Fig. 19.25 See Exercise (26)



(27) In the cesium chloride crystal (CsCl), eight Cs^+ ions are located at the corners of a cube of side $a = 0.4$ nm and a Cl^- ion is at the center, see Fig. 19.26. What is the magnitude of the electrostatic force exerted on the Cl^- ion by: (a) the eight Cs^+ ions?, (b) only seven Cs^+ ions?

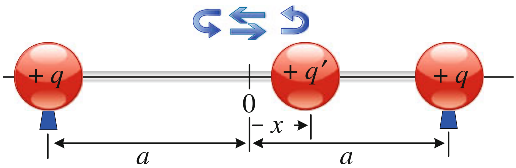
(28) Two positive point charges q_1 and q_2 are set apart by a fixed distance d and have a sum $Q = q_1 + q_2$. For what values of the two charges is the Coulomb force maximum between them?

Fig. 19.26 See Exercise (27)



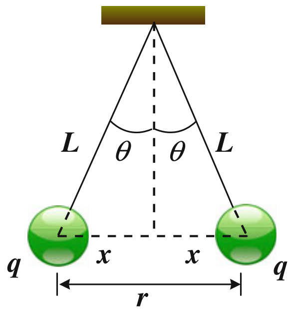
(29) Two equal positive charges q are held stationary on the x-axis, one at $x = -a$ and the other at $x = +a$. A third charge $+q'$ of mass m is in equilibrium at $x = 0$ and constrained to move only along the x-axis. The charge $+q'$ is then displaced from the origin to a small distance $x \ll a$ and released, see Fig. 19.27. (a) Show that $+q'$ will execute a simple harmonic motion and find an expression for its period T. (b) If all three charges are singly-ionized atoms ($q = q' = +e$) each of mass $m = 3.8 \times 10^{-25}$ kg and $a = 3 \times 10^{-10}$ m, find the oscillation period T.

Fig. 19.27 See Exercise (29)



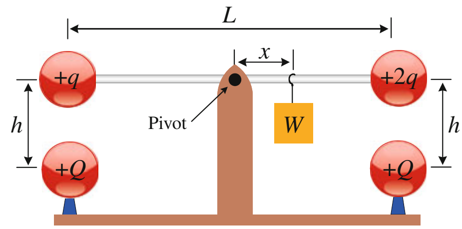
(30) Two identical small spheres of mass m and charge q hang from non-conducting strings, each of length L. At equilibrium, each string makes an angle θ with the vertical, see Fig. 19.28. (a) When θ is so small that $\tan\theta \simeq \sin\theta$, show that the separation distance r between the spheres is $r = (Lq^2/2\pi\epsilon_\circ mg)^{1/3}$. (b) If $L = 10$ cm, $m = 2$ g, and $r = 1.7$ cm, what is the value of q?

Fig. 19.28 See Exercise (30)



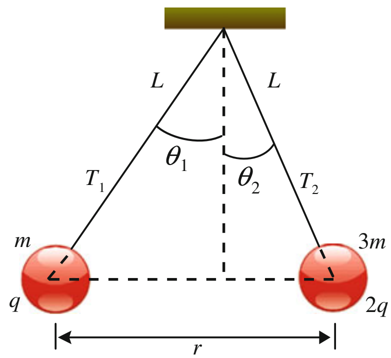
(31) For the charge distribution shown in Fig. 19.29, the long non-conducting massless rod of length L (which is pivoted at its center) is balanced horizontally when a weight W is placed at a distance x from the center. (a) Find the distance x and the force exerted by the rod on the pivot. (b) What is the value of h when the rod exerts no force on the pivot?

Fig. 19.29 See Exercise (31)



(32) Two small charged spheres hang from threads of equal length L. The first sphere has a positive charge q, mass m, and makes a small angle θ_1 with the vertical, while the second sphere has a positive charge $2\,q$, mass $3m$, and makes a smaller angle θ_2 with the vertical, see Fig. 19.30. For small angles, take $\tan\theta \simeq \theta$ and assume that the spheres only have horizontal displacements and hence the electric force of repulsion is always horizontal. (a) Find the ratio θ_1/θ_2. (b) Find the distance r between the spheres.

Fig. 19.30 See Exercise (32)



Electric Fields 20

In this chapter, we introduce the concept of an electric field associated with a variety of charge distributions. We follow that by introducing the concept of an electric field in terms of Faraday's electric field lines. In addition, we study the motion of a charged particle in a uniform electric field.

20.1 The Electric Field

Based on the electric force between charged objects, the concept of an electric field was developed by Michael Faraday in the 19th century, and has proven to have valuable uses as we shall see.

In this approach, an **electric field** is said to exist in the region of space around any charged object. To visualize this assume an electrical force of repulsion $\vec{F}$ between two positive charges q (called source charge) and $q_\circ$ (called test charge), see Fig. 20.1a.

Now, let the charge $q_\circ$ be removed from point P where it was formally located as shown in Fig. 20.1b. The charge q is said to set up an *electric field* $\vec{E}$ at P, and if $q_\circ$ is now placed at P, then a force $\vec{F}$ is exerted on $q_\circ$ by the field rather than by q, see Fig. 20.1c.

Since force is a vector quantity, the electric field is a vector whose properties are determined from both the magnitude and the direction of an electric force. We define the electric field vector $\vec{E}$ as follows:

H. A. Radi and J. O. Rasmussen, *Principles of Physics*,
Undergraduate Lecture Notes in Physics, DOI: 10.1007/978-3-642-23026-4_20,

> Spotlight
> The electric field vector $\vec{E}$ at a point in space is defined as the electric force $\vec{F}$ acting on a positive test charge $q_\circ$ located at that point divided by the magnitude of the test charge:

$$\boxed{\vec{E} = \frac{\vec{F}}{q_\circ}} \tag{20.1}$$

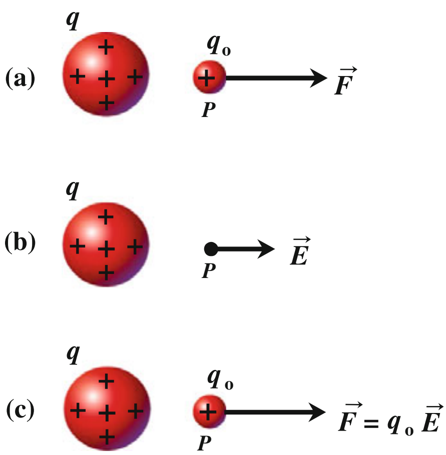


Fig. 20.1 (a) A charge q exerts a force $\vec{F}$ on a test charge $q_\circ$ at point P. (b) The electric field $\vec{E}$ established at P due to the presence of q. (c) The force $\vec{F} = q_\circ \vec{E}$ exerted by $\vec{E}$ on the test charge $q_\circ$

This equation can be rearranged as follows (see Fig. 20.1c):

$$\boxed{\vec{F} = q_\circ \vec{E}} \tag{20.2}$$

The SI unit of the electric field $\vec{E}$ is newton per coulomb (N/C).

The direction of $\vec{E}$ is the direction of the force on a positive test charge placed in the field, see Fig. 20.2.

20.2 The Electric Field of a Point Charge

To find the magnitude and direction of an electric field, we consider a positive point charge q as a source charge. A positive test charge $q_\circ$ is then placed at point P, a distance r away from q, see Fig. 20.3. From Coulomb's law, the force exerted on $q_\circ$ is:

$$\vec{F} = k\frac{q\,q_\circ}{r^2}\hat{\vec{r}} \tag{20.3}$$

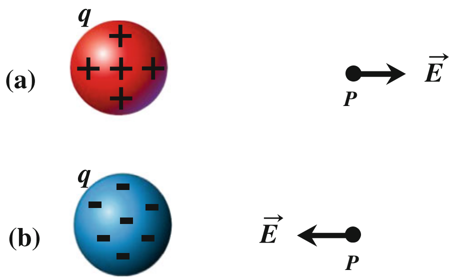


Fig. 20.2 (a) If the charge q is positive, then the force $\vec{F}$ on the test charge $q_\circ$ (not shown in the figure) at point P is directed away from q. Therefore, the electric field $\vec{E}$ at P is directed away from q. (b) If the charge q is negative, then the force $\vec{F}$ on $q_\circ$ at point P is directed toward q. Therefore, the electric field $\vec{E}$ at P is directed toward q

where $\hat{\vec{r}}$ is a unit vector directed from the source charge q to the test charge $q_\circ$. This force has the same direction as the unit vector $\hat{\vec{r}}$.

Since the electric field at point P is defined from Eq. 20.1 as $\vec{E} = \vec{F}/q_\circ$, then according to Fig. 20.3, the electric field created at P by q is an outward vector given by:

$$\boxed{\vec{E} = k\frac{q}{r^2}\hat{\vec{r}}} \tag{20.4}$$

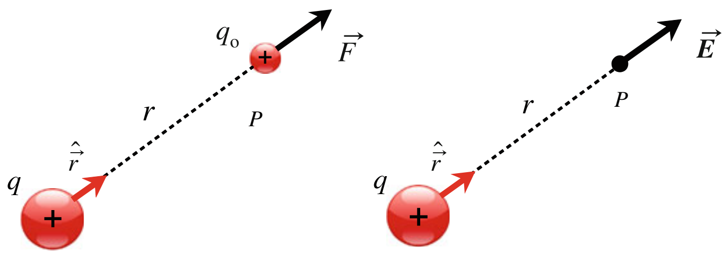


Fig. 20.3 If the point charge q is positive, then both the force $\vec{F}$ on the positive test charge $q_\circ$ and the electric field $\vec{E}$ at point P are directed away from q

When the source charge q is negative, the force $\vec{F}$ on $q_\circ$ and the electric field $\vec{E}$ at point P will be toward q, see Fig. 20.4.

Note that for both positive and negative charges, $\hat{\vec{r}}$ is a unit vector that is always directed from the source charge q to the point P, see Figs. 20.3 and 20.4.

In all previous and coming discussions, the positive test charge $q_\circ$ must be very small, so that it does not disturb the charge distribution of the source charge q. Mathematically, this can be done by taking the limit of the ratio $\vec{F}/q_\circ$ when $q_\circ$ approaches zero. Thus:

$$\boxed{\vec{E} = \lim_{q_\circ \to 0} \frac{\vec{F}}{q_\circ}} \tag{20.5}$$

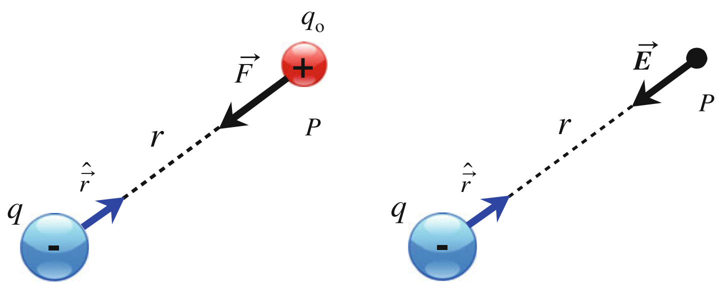


Fig. 20.4 If the point charge q is negative, then both the force $\vec{F}$ on the test positive charge $q_\circ$ and the electric field $\vec{E}$ at point P are directed toward q, but the unit vector $\hat{\vec{r}}$ remains pointed toward P

The electric field due to a group of point charges $q_1, q_2, q_3 \ldots$ at point P can be obtained by first using Eq. 20.4 to calculate the electric field of each individual charge, such that:

$$\boxed{\vec{E}_n = k\frac{q_n}{r_n^2}\hat{\vec{r}}_n} \quad (n = 1,\ 2,\ 3,\ \ldots) \tag{20.6}$$

Then we calculate the vector sum $\vec{E}$ of the electric fields of all the charges. This sum is expressed as follows:

$$\boxed{\vec{E} = \vec{E}_1 + \vec{E}_2 + \vec{E}_3 + \ldots = k\sum_n \frac{q_n}{r_n^2}\hat{\vec{r}}_n} \quad (n = 1,\ 2,\ 3,\ \ldots) \tag{20.7}$$

where r_n is the distance from the n^{th} source charge q_n to the point P and $\hat{\vec{r}}_n$ is a unit vector directed away from q_n to P.

It is clear that Eq. 20.7 exhibits the application of the superposition principle to electric fields.

Example 20.1

Four point charges $q_1 = q_2 = Q$ and $q_3 = q_4 = -Q$, where $Q = \sqrt{2}\,\mu\text{C}$, are placed at the four corners of a square of side $a = 0.4$ m, see Fig. 20.5a. Find the electric field at the center P of the square.

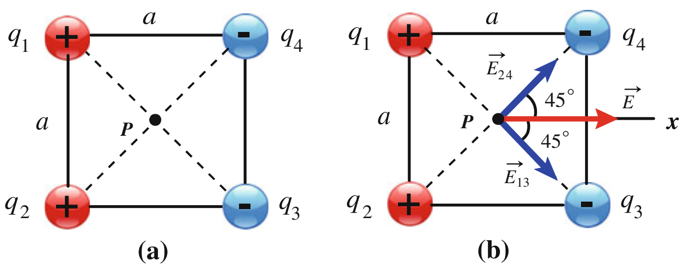


Fig. 20.5

Solution: The distance between each charge and the center P of the square is $a/\sqrt{2}$. At point P, the point charges q_1 and q_3 produce two diagonal electric field vectors $\vec{E}_1$ and $\vec{E}_3$, both directed toward q_3, see Fig. 20.5b. Hence, their vector sum $\vec{E}_{13} = \vec{E}_1 + \vec{E}_3$ points toward q_3 and has the magnitude:

$$E_{13} = E_1 + E_3 = k\frac{Q}{(a/\sqrt{2})^2} + k\frac{Q}{(a/\sqrt{2})^2} = 4k\frac{Q}{a^2}$$

At point P, the charges q_2 and q_4 produce two diagonal electric fields $\vec{E}_2$ and $\vec{E}_4$, both directed toward q_4, see Fig. 20.5b. Hence, their vector sum $\vec{E}_{24} = \vec{E}_2 + \vec{E}_4$ points toward q_4 and has the magnitude:

$$E_{24} = E_2 + E_4 = k\frac{Q}{(a/\sqrt{2})^2} + k\frac{Q}{(a/\sqrt{2})^2} = 4k\frac{Q}{a^2}$$

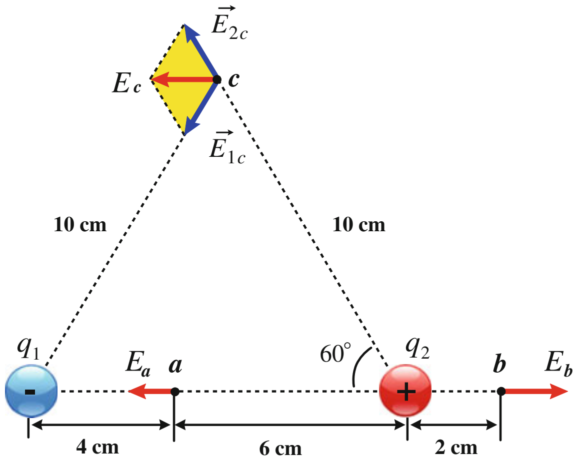
We now must combine the two electric field vectors $\vec{E}_{13}$ and $\vec{E}_{24}$ to form the resultant electric field vector $\vec{E} = \vec{E}_{13} + \vec{E}_{24}$ which is along the positive x-direction and has the magnitude:

$$E = E_{13}\cos 45^\circ + E_{24}\cos 45^\circ = 2\times\left(4k\frac{Q}{a^2}\times\frac{1}{\sqrt{2}}\right) = k\frac{8Q}{\sqrt{2}a^2}$$
$$= (9\times 10^9\ \text{N.m}^2/\text{C}^2)\frac{8(\sqrt{2}\times 10^{-6}\ \text{m})}{\sqrt{2}\ (0.4\ \text{m})^2} = 4.5\times 10^5\ \text{N/C}$$

Example 20.2 (Electric Dipole)

Consider two point charges $q_1 = -24$ nC and $q_2 = +24$ nC that are 10 cm apart, forming an *electric dipole*, see Fig. 20.6. Calculate the electric field due to the two charges at points a, b, and c.

Fig. 20.6



Solution: At point a, the electric field vector due to the negative charge q_1, is directed toward the left, and its magnitude is:

$$E_{1a} = k\frac{|q_1|}{r_{1a}^2} = (9\times 10^9\ \text{N.m}^2/\text{C}^2)\frac{(24\times 10^{-9}\ \text{C})}{(0.04\ \text{m})^2} = 135\times 10^3\ \text{N/C}$$

The electric field vector due to the positive charge q_2 is also directed toward the left, and its magnitude is:

$$E_{2a} = k\frac{|q_2|}{r_{2a}^2} = (9\times 10^9\ \text{N.m}^2/\text{C}^2)\frac{(24\times 10^{-9}\ \text{C})}{(0.06\ \text{m})^2} = 60\times 10^3\ \text{N/C}$$

Then, the resultant electric field at point a is toward the left and its magnitude is:

$$E_a = E_{1a} + E_{2a} = 135\times 10^3\ \text{N/C} + 60\times 10^3\ \text{N/C}$$
$$= 195\times 10^3\ \text{N/C} \qquad \text{(Toward the left)}$$

At point b, the electric field vector due to the negative charge q_1, is directed toward the left, and its magnitude is:

$$E_{1b} = k\frac{|q_1|}{r_{1b}^2} = (9 \times 10^9\,\text{N.m}^2/\text{C}^2)\frac{(24 \times 10^{-9}\,\text{C})}{(0.12\,\text{m})^2} = 15 \times 10^3\,\text{N/C}$$

In addition, the electric field vector due to the positive charge q_2 is directed toward the right, and its magnitude is:

$$E_{2b} = k\frac{|q_2|}{r_{2b}^2} = (9 \times 10^9\,\text{N.m}^2/\text{C}^2)\frac{(24 \times 10^{-9}\,\text{C})}{(0.02\,\text{m})^2} = 540 \times 10^3\,\text{N/C}$$

Since $E_{2b} > E_{1b}$, the resultant electric field at point b is toward the right and its magnitude is:

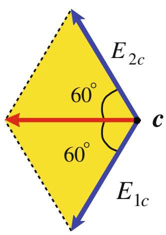
$$\begin{aligned} E_b &= E_{2b} - E_{1b} = 540 \times 10^3\,\text{N/C} - 15 \times 10^3\,\text{N/C} \\ &= 525 \times 10^3\,\text{N/C} \qquad \text{(Toward the right)} \end{aligned}$$

At point c, the magnitudes of the electric field vectors $\vec{E}_{1c}$ and $\vec{E}_{2c}$ established by q_1 and q_2 are the same because $|q_1| = |q_2| = 24\,\text{nC}$ and $r_{1c} = r_{2c} = 10\,\text{cm}$. Thus:

$$E_{2c} = E_{1c} = k\frac{|q_1|}{r_{1c}^2} = (9 \times 10^9\,\text{N.m}^2/\text{C}^2)\frac{(24 \times 10^{-9}\,\text{C})}{(0.1\,\text{m})^2} = 21.6 \times 10^3\,\text{N/C}$$

The triangle formed from q_1, q_2, and point c in Fig. 20.6 is an equilateral triangle of angle 60°. Hence, from geometry, the *vertical* components of the two vectors $\vec{E}_{1c}$ and $\vec{E}_{2c}$ cancel each other. The horizontal components are both directed toward the left and add up to give the resultant electric field E_c at point c, see the figure below.

$$E_c = E_{1c}\cos 60^\circ + E_{2c}\cos 60^\circ$$



$$\begin{aligned} \text{Thus:}\quad E_c &= E_{1c}\cos 60^\circ + E_{2c}\cos 60^\circ = 2E_{1c}\cos 60^\circ \\ &= 2(21.6 \times 10^3\,\text{N/C})(0.5) = 21.6 \times 10^3\,\text{N/C} \qquad \text{(Toward the left)} \end{aligned}$$

20.3 The Electric Field of an Electric Dipole

Generally, the electric dipole introduced in Example 20.3 consists of a positive charge $q_+ = +q$ and a negative charge $q_- = -q$ separated by a distance $2a$, see Fig. 20.7. In this figure, the dipole axis is taken to be along the x-axis and the origin of the xy plane is taken to be at the center of the dipole. Therefore, the coordinates of q_+ and q_- are $(+a, 0)$ and $(-a, 0)$, respectively.

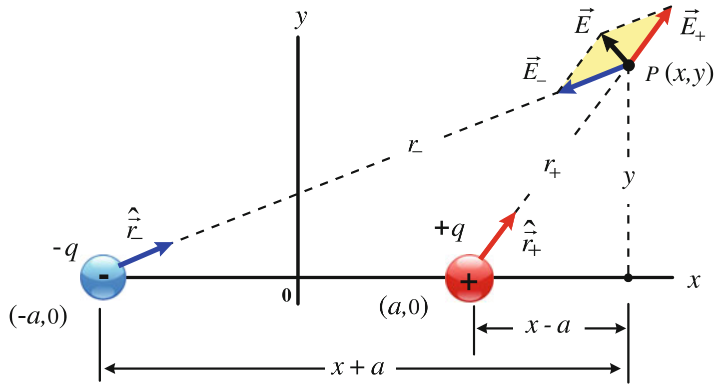


Fig. 20.7 The electric field $\vec{E} = \vec{E}_+ + \vec{E}_-$ at point $P(x, y)$ due to an electric dipole located along the x-axis. The dipole has a length $2a$

Let us assume that a point $P(x, y)$ exists in the xy-plane as shown in Fig. 20.7. We will call the electric field produced by the positive charge $\vec{E}_+$ and the electric field produced by the negative charge $\vec{E}_-$.

Using the superposition principle, the total electric field at P is:

$$\vec{E} = \vec{E}_+ + \vec{E}_- = k\frac{q_+}{r_+^2}\hat{\vec{r}}_+ + k\frac{q_-}{r_-^2}\hat{\vec{r}}_- \tag{20.8}$$

From the geometry of Fig. 20.7, we have $r_+^2 = (x-a)^2 + y^2$ and $r_-^2 = (x+a)^2 + y^2$. In addition, $\hat{\vec{r}}_+$ is a unit vector directed outwards and away from the positive charge q_+ at $(+a, 0)$. On the other hand, $\hat{\vec{r}}_-$ is a unit vector directed outwards and away from the negative charge q_- at $(-a, 0)$. Accordingly, Eq. 20.8 becomes:

$$\vec{E} = k\left[\frac{q}{(x-a)^2 + y^2}\hat{\vec{r}}_+ + \frac{-q}{(x+a)^2 + y^2}\hat{\vec{r}}_-\right] \tag{20.9}$$

Therefore, the general electric field will take the following form:

$$\vec{E} = kq\left[\frac{\hat{\vec{r}}_+}{(x-a)^2+y^2} - \frac{\hat{\vec{r}}_-}{(x+a)^2+y^2}\right] \tag{20.10}$$

The Electric Field Along the Dipole Axis

Let us first assume a point P exists on the dipole axis, i.e. $y = 0$, and satisfies the condition $x < -a$, as shown in Fig. 20.8a. In this case, $\hat{\vec{r}}_+ = \hat{\vec{r}}_- = -\vec{i}$, where $\vec{i}$ is a unit vector along the x-axis.

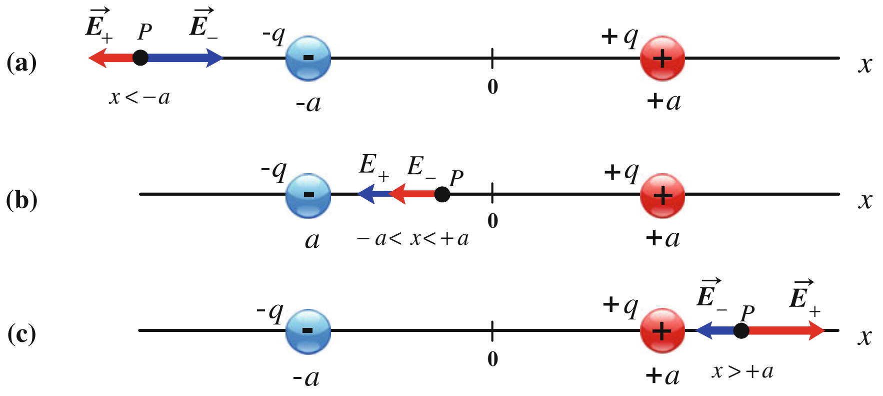


Fig. 20.8 The electric field $\vec{E} = \vec{E}_+ + \vec{E}_-$ at different points along the axis of a dipole that has a length $2a$

When P has an x-coordinate that satisfies $-a < x < +a$ as in Fig. 20.8b, then $\hat{\vec{r}}_+ = -\vec{i}$ and $\hat{\vec{r}}_- = +\vec{i}$. When P satisfies $x > +a$ as in Fig. 20.8c, then $\hat{\vec{r}}_+ = \hat{\vec{r}}_- = +\vec{i}$. Substituting in Eq. 20.10, we get:

$$\vec{E} = \begin{cases} -kq\left[\dfrac{1}{(x-a)^2} - \dfrac{1}{(x+a)^2}\right]\vec{i} & x < -a \quad \text{(Toward the right)} \\ -kq\left[\dfrac{1}{(x-a)^2} + \dfrac{1}{(x+a)^2}\right]\vec{i} & -a < x < +a \quad \text{(Toward the left)} \\ kq\left[\dfrac{1}{(x-a)^2} - \dfrac{1}{(x+a)^2}\right]\vec{i} & x > +a \quad \text{(Toward the right)} \end{cases} \tag{20.11}$$

When $x \gg a$ we can take out a factor of x^2 from each denominator of the brackets of the last formula for $x > +a$ and then expand each of these terms by binomial expansion. Therefore, we get:

$$\begin{aligned}
\vec{E} &= kq\left[\frac{1}{(x-a)^2} - \frac{1}{(x+a)^2}\right]\vec{\mathrm{i}} \\
&= \frac{kq}{x^2}\left[\left(1-\frac{a}{x}\right)^{-2} - \left(1+\frac{a}{x}\right)^{-2}\right]\vec{\mathrm{i}} \\
&= \frac{kq}{x^2}\left[\left(1+\frac{2a}{x} - \cdots\right) - \left(1-\frac{2a}{x}+\cdots\right)\right]\vec{\mathrm{i}} \\
&\simeq \frac{kq}{x^2}\frac{4a}{x}\vec{\mathrm{i}} = \frac{2\,k\,(2a\,q)}{x^3}\vec{\mathrm{i}} \qquad\qquad x \gg a
\end{aligned} \tag{20.12}$$

For $x \ll -a$, we can find an identical expression but with $|x|$ instead of x in the last formula. The product of the positive charge q and the length of the dipole $2a$ is called the magnitude of the *electric dipole moment*, $p = 2a\,q$. The direction of $\vec{p}$ is taken to be from the negative charge to the positive charge of the dipole, i.e. $\vec{p} = p\,\vec{\mathrm{i}}$. Using this definition, we have:

$$\boxed{\vec{E} = \begin{cases} 2\,k\dfrac{\vec{p}}{x^3} & x \gg a \\ 2\,k\dfrac{\vec{p}}{|x|^3} & x \ll -a \end{cases}} \qquad (\vec{p} = 2\,a\,q\,\vec{\mathrm{i}}) \tag{20.13}$$

Thus, at far distances, the electric field along the x-axis is proportional to the electric dipole moment $\vec{p}$ and varies as $1/|x^3|$.

Electric Field Along the Perpendicular Bisector of a Dipole Axis

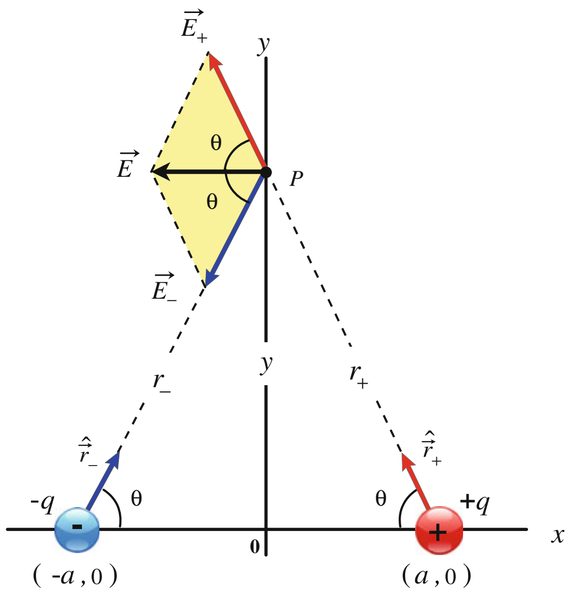
Let us assume that a point P lies on the y-axis, i.e. along the perpendicular bisector of the line joining the dipole charges, see Fig. 20.9. Substitute x with 0 in Eq. 20.10 to get:

$$\boxed{\vec{E} = \frac{k\,q}{a^2+y^2}\left[\hat{\vec{r}}_+ - \hat{\vec{r}}_-\right]} \tag{20.14}$$

From Fig. 20.9, we see that:

$$\hat{\vec{r}}_+ = -\cos\theta\,\vec{\mathrm{i}} + \sin\theta\,\vec{\mathrm{j}}, \quad \hat{\vec{r}}_- = \cos\theta\,\vec{\mathrm{i}} + \sin\theta\,\vec{\mathrm{j}}, \quad \cos\theta = a/\sqrt{a^2+y^2} \tag{20.15}$$

Fig. 20.9 The electric field $\vec{E} = \vec{E}_+ + \vec{E}_-$ at point $P(0, y)$ along the y-axis of an electric dipole lying along the x-axis with a length $2a$



Substituting these relations in Eq. 20.14 we get:

$$\vec{E} = -k\,\frac{2a\,q}{(a^2+y^2)^{3/2}}\,\vec{\mathrm{i}} = -k\,\frac{\vec{p}}{(a^2+y^2)^{3/2}} \qquad (\vec{p} = 2a\,q\,\vec{\mathrm{i}}) \tag{20.16}$$

When $|y| \gg a$, we can neglect a^2 when we compare it with y^2 in the denominator bracket and write:

$$\vec{E} = \begin{cases} -k\,\dfrac{\vec{p}}{y^3} & y \gg a \\ -k\,\dfrac{\vec{p}}{|y|^3} & y \ll -a \end{cases} \qquad (\vec{p} = 2\,a\,q\,\vec{\mathrm{i}}) \tag{20.17}$$

Thus, at far distances, the electric field along the perpendicular bisector of the line joining the dipole charges is proportional to the electric dipole moment $\vec{p}$ and varies as $1/|y|^3$. Generally, this inverse cube dependence at a far distance is a characteristic of a dipole.

Example 20.3 **(The Dipole Field Along the Dipole Axis)**

A proton and an electron separated by 2×10^{-10} m form an electric dipole, see Fig. 20.10. Use exact and approximate formulae to calculate the electric field on the x-axis at a distance 20×10^{-10} m to the right of the dipole's center.

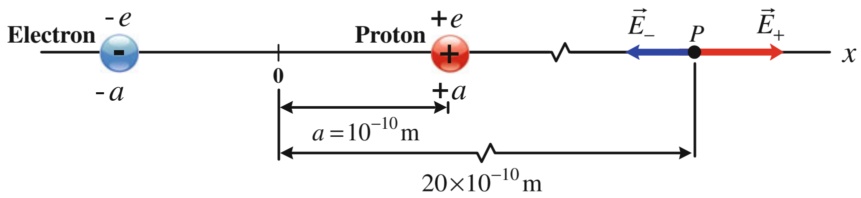


Fig. 20.10

Solution: In this problem we have $a = 10^{-10}$ m, $q = e = 1.6 \times 10^{-19}$ C, $x = 20 \times 10^{-10}$ m, $ke = (9 \times 10^9\ \text{N.m}^2/\text{C}^2)\ (1.6 \times 10^{-19}\ \text{C}) = 1.44 \times 10^{-9}\ \text{N.m}^2/\text{C}$, $x - a = 19 \times 10^{-10}$ m, and $x + a = 21 \times 10^{-10}$ m. Using the exact formula given by Eq. 20.11 in the case of $x > +a$, we have:

$$\begin{aligned}
E &= ke\left[\frac{1}{(x-a)^2} - \frac{1}{(x+a)^2}\right] \\
&= (1.44 \times 10^{-9}\ \text{N.m}^2/\text{C})\left[\frac{1}{(19 \times 10^{-10}\ \text{m})^2} - \frac{1}{(21 \times 10^{-10}\ \text{m})^2}\right] \\
&= (1.44 \times 10^{-9}\ \text{N.m}^2/\text{C})[2.770 \times 10^{17}\ \text{m}^{-2} - 2.268 \times 10^{17}\text{m}^{-2}] \\
&= 7.236 \times 10^7\ \text{N/C}
\end{aligned}$$

On the other hand, we have $x \gg a$ and we can use the approximate formula given by Eq. 20.13 as follows:

$$\begin{aligned}
E &= 2k\frac{p}{x^3} = 2k\frac{2a\,e}{x^3} = ke\frac{4a}{x^3} \\
&= (1.44 \times 10^{-9}\ \text{N.m}^2/\text{C})\frac{(4 \times 10^{-10}\ \text{m})}{(20 \times 10^{-10}\ \text{m})^3} \\
&= 7.200 \times 10^7\ \text{N/C}
\end{aligned}$$

Clearly this calculation is a good approximation when $x/a = 20$.

20.4 Electric Field of a Continuous Charge Distribution

The electric field at point P due to a continuous charge distribution shown in Fig. 20.11 can be evaluated by:

(1) Dividing the charge distribution into small elements, each of charge Δq_n that is located relative to point P by the position vector $\vec{r}_n = r_n\, \hat{\vec{r}}_n$.

(2) Using Eq. 20.4 to evaluate the electric field $\Delta\vec{E}_n$ due to the n^{th} element as follows:

$$\Delta\vec{E}_n = k\frac{\Delta q_n}{r_n^2}\,\hat{\vec{r}}_n \tag{20.18}$$

(3) Evaluating the order of the total electric field at P due to the charge distribution by the vector sum of all the charge elements as follows:

$$\vec{E} \approx k\sum_n \frac{\Delta q_n}{r_n^2}\,\hat{\vec{r}}_n \tag{20.19}$$

(4) Evaluating the total electric field at P due to the continuous charge distribution in the limit $\Delta q_n \to 0$ as follows:

$$\vec{E} = k\lim_{\Delta q_n\to 0}\sum_n \frac{\Delta q_n}{r_n^2}\,\hat{\vec{r}}_n = k\int\frac{dq}{r^2}\,\hat{\vec{r}} \tag{20.20}$$

where the integration is done over the entire charge distribution.

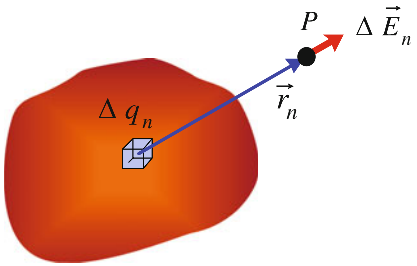


Fig. 20.11 The electric field $\vec{E}$ at point P due to a continuous charge distribution is the vector sum of all the fields $\Delta\vec{E}_n$ $(n = 1, 2, \cdots)$ due to the charge elements Δq_n $(n = 1, 2, \cdots)$ of the charge distribution

Now we consider cases were the total charge is uniformly distributed on a line, on a surface, or throughout a volume. It is convenient to introduce the *charge density* as follows:

(1) When the charge Q is uniformly distributed along a line of length L, the linear charge density λ is defined as:

$$\lambda = \frac{Q}{L} \tag{20.21}$$

where λ has the units of coulomb per meter (C/m).

(2) When the charge Q is uniformly distributed on a surface of area A, the surface charge density σ is defined as:

$$\sigma = \frac{Q}{A} \tag{20.22}$$

where σ has the units of coulomb per square meter ($\mathrm{C/m^2}$).

(3) When the charge Q is uniformly distributed throughout a volume V, the volume charge density ρ is defined as:

$$\rho = \frac{Q}{V} \tag{20.23}$$

where ρ has the units of coulomb per cubic meter ($\mathrm{C/m^3}$).

Accordingly, the charge dq of a small length dL, a small surface of area dA, or a small volume dV is respectively given by:

$$dq = \lambda\, dL, \quad dq = \sigma\, dA, \quad dq = \rho\, dV \tag{20.24}$$

20.4.1 The Electric Field Due to a Charged Rod

For a Point on the Extension of the Rod

Figure 20.12 shows a rod of length L with a uniform positive charge density λ and total charge Q. In this figure, the rod lies along the x-axis and point P is taken to be at the origin of this axis, located at a constant distance a from the left end. When we consider a segment dx on the rod, the charge on this segment will be $dq = \lambda\, dx$.

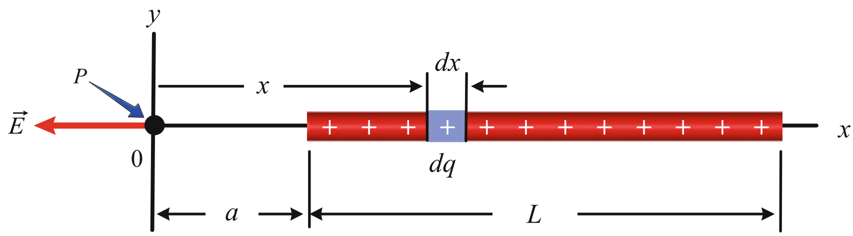


Fig. 20.12 The electric field $\vec{E}$ at point P due to a uniformly charged rod lying along the x-axis. The magnitude of the field due to a segment of charge dq at a distance x from P is $k\, dq/x^2$. The total field is the vector sum of all the segments of the rod

The electric field $d\vec{E}$ at P due to this segment is in the negative x direction and has a magnitude given by:

$$dE = k\frac{dq}{x^2} = k\frac{\lambda\, dx}{x^2} \tag{20.25}$$

The total electric field at P due to all the segments of the rod is given by Eq. 20.20 after integrating from one end of the rod ($x = a$) to the other ($x = a + L$) as follows:

$$\begin{aligned} E = \int dE &= \int_a^{a+L} k\frac{\lambda\, dx}{x^2} = k\lambda \int_a^{a+L} x^{-2}dx = k\lambda \left|-\tfrac{1}{x}\right|_a^{a+L} \\ &= k\lambda\left\{-\frac{1}{a+L} + \frac{1}{a}\right\} = \frac{k\lambda L}{a(a+L)} \end{aligned} \tag{20.26}$$

When we use the fact that the total charge is $Q = \lambda L$, we have:

$$\boxed{E = \frac{kQ}{a(a+L)}} \quad \text{(Toward the left)} \tag{20.27}$$

If P is a very far point from the rod, i.e. $a \gg L$, then L can be neglected in the denominator of Eq. 20.27. Accordingly, we have $E \approx kQ/a^2$, which resembles the magnitude of the electric field produced by a point charge.

For a Point on the Perpendicular Bisector of the Rod

A rod of length L has a uniform positive charge density λ and total charge Q. The rod is placed along the x-axis as shown in Fig. 20.13. Assume that point P is on the perpendicular bisector of the rod and is located at a constant distance a from the origin of the x-axis. The charge on a segment dx on the rod will be $dq = \lambda\, dx$.

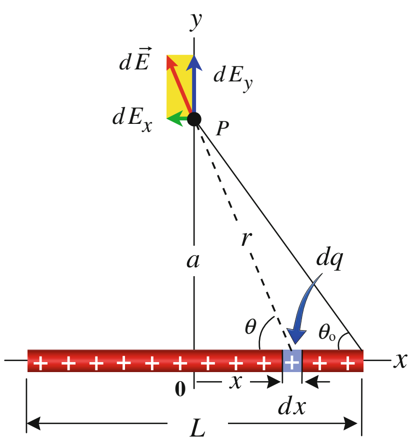


Fig. 20.13 A rod of length L has a uniform positive charge density λ and an electric field $d\vec{E}$ at point P due to a segment of charge dq, where P is located along the perpendicular bisector of the rod. From symmetry, the total field will be along the y-axis

The electric field $d\vec{E}$ at P due to this segment has a magnitude:

$$dE = k\frac{dq}{r^2} = k\frac{\lambda\, dx}{r^2} \tag{20.28}$$

This field has a vertical component $dE_y = dE \sin\theta$ along the y-axis and a horizontal component dE_x perpendicular to it, as shown in Fig. 20.13. An x-component at such a location is canceled out by a similar but symmetric charge segment on the opposite side of the rod. Thus:

$$E_x = \sum dE_x = 0 \tag{20.29}$$

The total electric field at P due to all segments of the rod is given by two times the integration of the y-component from the middle of the rod ($x = 0$) to one of the ends ($x = L/2$). Thus:

$$E = 2\int_{x=0}^{x=L/2} dE_y = 2\int_{x=0}^{x=L/2} dE \sin\theta = 2\, k\, \lambda \int_{x=0}^{x=L/2} \frac{\sin\theta\ dx}{r^2} \tag{20.30}$$

To perform the integration of this expression, we must relate the variables θ, x, and r. One approach is to express θ and r in terms of x. From the geometry of Fig. 20.13, we have:

$$r = \sqrt{x^2 + a^2} \quad \text{and} \quad \sin\theta = \frac{a}{r} = \frac{a}{\sqrt{x^2 + a^2}} \tag{20.31}$$

Therefore, Eq. 20.30 becomes:

$$E = 2\, k\lambda\, a \int_0^{L/2} \frac{dx}{(x^2 + a^2)^{3/2}} \tag{20.32}$$

From the table of integrals in Appendix B, we find that:

$$\int \frac{dx}{(x^2 + a^2)^{3/2}} = \frac{x}{a^2\sqrt{(x^2 + a^2)}} \tag{20.33}$$

Thus:

$$\begin{aligned} E &= 2k\lambda\, a \int_0^{L/2} \frac{dx}{(x^2 + a^2)^{3/2}} = 2\, k\lambda\, a \left| \frac{x}{a^2\sqrt{x^2 + a^2}} \right|_0^{L/2} \\ &= 2k\lambda\, a \left[\frac{L/2}{a^2\sqrt{(L/2)^2 + a^2}} - 0 \right] = \frac{k\lambda\, L}{a\sqrt{(L/2)^2 + a^2}} \end{aligned} \tag{20.34}$$

When we use the fact that the total charge is $Q = \lambda L$, we have:

$$\boxed{E = \frac{kQ}{a\sqrt{a^2 + (L/2)^2}}} \quad \text{or} \quad \boxed{E = \frac{k\lambda\, L}{a\sqrt{a^2 + (L/2)^2}}} \tag{20.35}$$

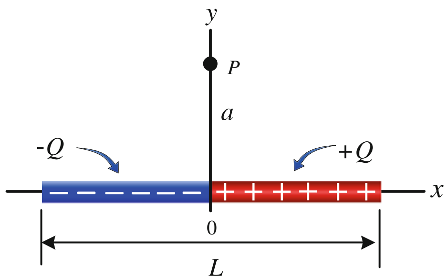
When P is a very far point from the rod, $a \gg L$, we can neglect $(L/2)^2$ in the denominator of Eq. 20.35. Thus, $E \approx kQ/a^2$. This is just the form of a point charge. For an infinitely long rod we get:

$$E = \lim_{L\to\infty} \frac{2k\lambda}{a\sqrt{(2a/L)^2 + 1}} \quad \Rightarrow \quad \boxed{E = 2k\frac{\lambda}{a}} \tag{20.36}$$

Example 20.4

Figure 20.14 shows a non-conducting rod that has a uniform positive charge density $+\lambda$ and a total charge Q along its right half, and a uniform negative charge density $-\lambda$ and a total charge $-Q$ along its left half. What is the direction and magnitude of the net electric field at point P that shown in Fig. 20.14?

Fig. 20.14



Solution: When we consider a segment dx on the right side of the rod, the charge on this segment will be $dq = \lambda\, dx$, see Fig. 20.15.

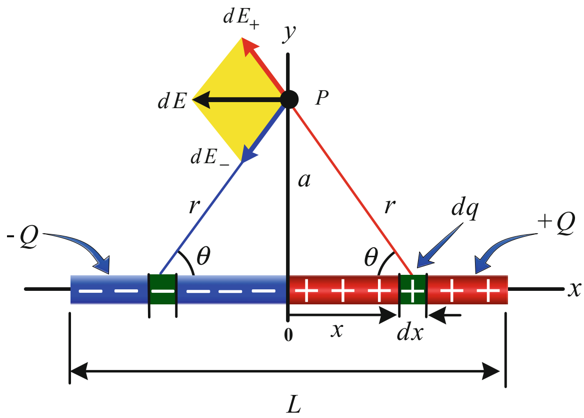
The electric field $d\vec{E}_+$ at P due to this segment is directed outwards and away from the positive charge dq and has a magnitude:

$$dE_+ = k\frac{dq}{r^2} = k\frac{\lambda\, dx}{r^2}$$

A symmetric segment on the opposite side of the rod, but with a negative charge, creates an electric field $d\vec{E}_-$ that is directed inwards and toward this segment and has the same magnitude as $d\vec{E}_+$, i.e. $dE_+ = dE_-$. The resultant electric field $d\vec{E}$ from both symmetric segments will be a vector to the left, see Fig. 20.15, and its magnitude will be given by:

$$dE = dE_+ \cos\theta + dE_+ \cos\theta = 2\, dE_+ \cos\theta$$
$$= 2\, k\frac{\lambda\, dx}{r^2}\,\frac{x}{r} = k\,\lambda(x^2 + a^2)^{-3/2}(2x)dx$$

Fig. 20.15



The total electric field at P due to all segments of the rod is found by integrating dE from $x = 0$ to only $x = L/2$, since the negative charge of the rod is considered in evaluating dE. Thus:

$$E = \int dE = k\,\lambda \int_{x=0}^{x=L/2} (x^2 + a^2)^{-3/2}(2x\, dx)$$

To evaluate the integral in this equation, we transform it to the form $\int u^n\, du = u^{n+1}/(n+1)$, as we shall do in solving Eq. 20.53. Thus:

$$E = k\,\lambda \left| \frac{(u^2 + a^2)^{-1/2}}{-1/2} \right|_{u=0}^{u=L/2} = k\,\lambda\left[\frac{-2}{\sqrt{(L/2)^2 + a^2}} - \frac{-2}{a}\right]$$
$$= 2k\,\lambda\left[\frac{1}{a} - \frac{1}{\sqrt{(L/2)^2 + a^2}}\right]$$

When we use the fact that the magnitude of the charge Q is given by $Q = \lambda L/2$, we get:

$$E = \frac{4k\,Q}{L}\left[\frac{1}{a} - \frac{1}{\sqrt{(L/2)^2 + a^2}}\right]$$

When P is very far away from the rod, i.e. $a \gg L$, we can neglect $(L/2)^2$ in the denominator of this equation and hence get $E \approx 0$. In this situation, the two oppositely charged halves of the rod would appear to point P as if they were two coinciding point charges and hence have a zero net charge.

Example 20.5

An infinite sheet of charge is lying on the xy-plane as shown in Fig. 20.16. A positive charge is distributed uniformly over the plane of the sheet with a charge per unit area σ. Calculate the electric field at a point P located a distance a from the plane.

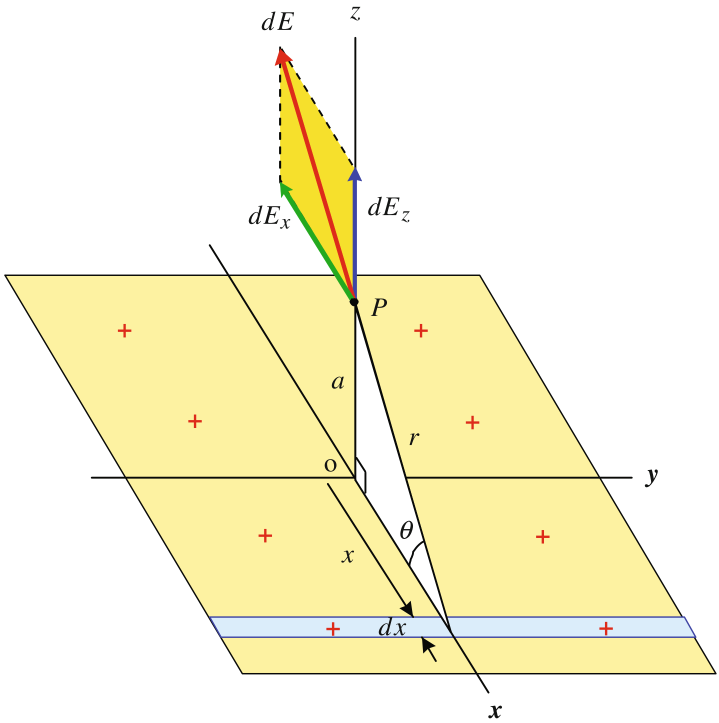


Fig. 20.16

Solution: Let us divide the plane into narrow strips parallel to the y-axis. A strip of width dx can be considered as an infinitely long wire of charge per unit length $\lambda = \sigma\, dx$. From Eq. 20.36, at point P, the strip sets up an electric field $d\vec{E}$ lying in the xz-plane of magnitude:

$$dE = 2k\frac{\lambda}{r} = 2k\frac{\sigma\, dx}{r}$$

This electric field vector can be resolved into two components $d\vec{E}_x$ and $d\vec{E}_z$. By symmetry the components $d\vec{E}_x$ will sum to zero when we consider the entire sheet of charge. Therefore, the resultant electric field at point P will be in the z-direction, perpendicular to the sheet. From Fig. 20.16, we find the following:

$$dE_z = dE \sin\theta$$

and hence:

$$E = \int dE_z = 2k\sigma \int_{-\infty}^{+\infty} \frac{\sin\theta\, dx}{r}$$

To perform the integration of this expression, we must first relate the variables θ, x, and r. One approach is to express θ and r in terms of x. From the geometry of Fig. 20.16, we have:

$$r = \sqrt{x^2 + a^2} \quad \text{and} \quad \sin\theta = \frac{a}{r} = \frac{a}{\sqrt{x^2 + a^2}}$$

Then, from the table of integrals in Appendix B, we find that:

$$\begin{aligned} E &= 2k\sigma a \int_{-\infty}^{+\infty} \frac{dx}{x^2 + a^2} = 2k\sigma a \left| \frac{1}{a} \tan^{-1} \frac{x}{a} \right|_{-\infty}^{+\infty} \\ &= 2k\sigma\left[\tan^{-1}(\infty) - \tan^{-1}(-\infty)\right] = 2k\sigma \left[\frac{\pi}{2} + \frac{\pi}{2}\right] \end{aligned}$$

Thus:

$$E = 2\pi k\sigma = \frac{\sigma}{2\epsilon_\circ}$$

This result is identical to the one we shall find in Sect. 20.4.4 for a charged disk of infinite radius. We note that the distance a from the plane to the point P does not appear in the final result of E. This means that the electric field set up at any point by an infinite plane sheet of charge is independent of how far the point is from the plane. In other words, the electric field is *uniform* and *normal* to the plane.

Also, the same result is obtained if the point P lies below the xy-plane. That is, the field below the plan has the same magnitude as that above the plane but as a vector it points in the opposite direction.

20.4.2 The Electric Field of a Uniformly Charged Arc

Assume that a rod has a uniformly distributed total positive charge Q. Also assume that the rod is bent into a circular section of radius R and central angle ϕ rad. To find the electric field at the center P of this arc, we place coordinate axes such that the axis of symmetry of the arc lies along the y-axis and the origin is at the arc's center, see Fig. 20.17a. If we let λ represent the linear charge density of this arc which has a length $R\phi$, then:

$$\lambda = \frac{Q}{R\phi} \tag{20.37}$$

For an arc element ds subtending an angle $d\theta$ at P, we have:

$$ds = R\,d\theta \tag{20.38}$$

Therefore, the charge dq on this arc element will be given by:

$$dq = \lambda\,ds = \frac{Q}{R\phi}R\,d\theta = \frac{Q}{\phi}d\theta \tag{20.39}$$

To find the electric field at point P, we first calculate the magnitude of the electric field dE at P due to this element of charge dq, see Fig. 20.17b, as follows:

$$dE = k\frac{dq}{R^2} = \frac{kQ}{R^2\phi}\,d\theta \tag{20.40}$$

This field has a vertical component $dE_y = dE\cos\theta$ along the y-axis and a horizontal component dE_x along the negative x-axis, as shown in Fig. 20.17b. The x-component created at P by any charge element dq is canceled by a symmetric charge element on the opposite side of the arc. Thus, the perpendicular components of all of the charge elements sum to zero. The vertical component will take the form:

$$dE_y = dE\cos\theta = \frac{kQ}{R^2\phi}\cos\theta\,d\theta \tag{20.41}$$

Consequently, the total electric field at P due to all elements of the arc is given by the integration of the y-component as follows:

$$E = \int dE_y = \frac{kQ}{R^2\phi} \int\limits_{-\phi/2}^{+\phi/2} \cos\theta \, d\theta = \frac{kQ}{R^2\phi} \left| \sin\theta \right|_{-\phi/2}^{+\phi/2} = \frac{kQ}{R^2\phi}\left[\sin\frac{\phi}{2} - \sin\left(-\frac{\phi}{2}\right)\right] \tag{20.42}$$

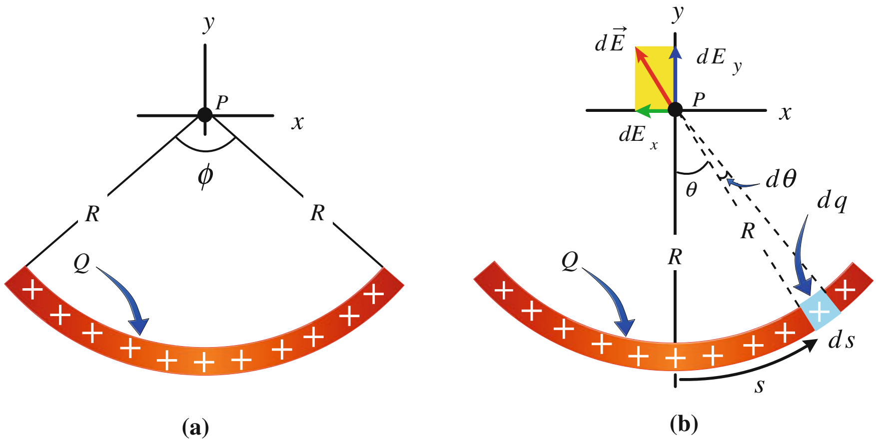


Fig. 20.17 (a) A circular arc of radius R, central angle ϕ, and center P has a uniformly distributed positive charge Q. (b) The figure shows the electric field $d\vec{E}$ at P due to an arc element ds having a charge dq. From symmetry, the horizontal components of all elements cancel out and the total field is along the y-axis

Finally, the total electric field at P will be along the y-axis and will have a magnitude given by:

$$\boxed{E = \frac{kQ}{R^2}\,\frac{\sin\phi/2}{\phi/2}} \tag{20.43}$$

There are three special cases to Eq. 20.43:

(1) $\phi = 0$ (Point charge)

When we apply the limiting case $\lim\limits_{\phi\to 0}[\sin(\phi/2)/(\phi/2)] = 1$, we get:

$$E = \frac{kQ}{R^2} \tag{20.44}$$

(2) $\phi = \pi$ (Half a circle of radius R)
When we substitute with $\sin(\pi/2)/(\pi/2) = 2/\pi$, we get:

$$\boxed{E = \frac{2kQ}{\pi R^2}} \tag{20.45}$$

(3) $\phi = 2\pi$ (A ring of radius R)
When we substitute with $\sin\pi = 0$, we get:

$$E = 0 \tag{20.46}$$

This is an expected result, since we shall see that Eq. 20.50 gives $E = 0$ when P is at the center of the ring, i.e. when $a = 0$.

20.4.3 The Electric Field of a Uniformly Charged Ring

Assume that a ring of radius R has a uniformly distributed total positive charge Q, see Fig. 20.18. Also, assume there is a point P that lies at a distance a from the center of the ring along its central perpendicular axis, as shown in the same figure.

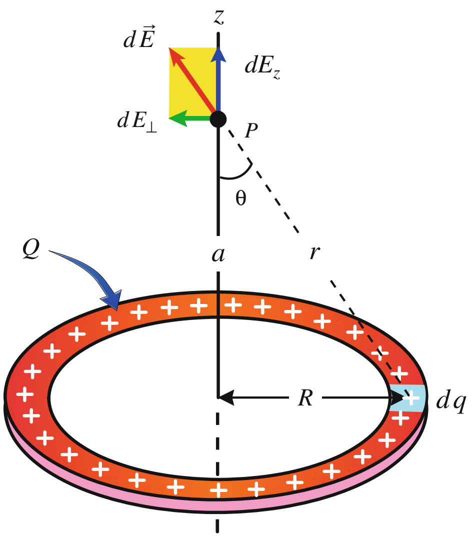


Fig. 20.18 A ring of radius R having a uniformly distributed positive charge Q. The figure shows the electric field $d\vec{E}$ at an axial point P due to a segment of charge dq. The horizontal components will cancel each other, and the total field will be along the z-axis

To find the electric field at P, we first calculate the magnitude of the electric field dE at P due to this segment of charge dq as follows:

$$dE = k\frac{dq}{r^2} \tag{20.47}$$

This field has a vertical component $dE_z = dE \sin\theta$ along the z-axis and a component $dE_\perp$ perpendicular to it, as shown in Fig. 20.18. The perpendicular component created at P by any charge segment is canceled by a symmetric charge segment on the opposite side of the ring. Thus, the perpendicular components of all of the charge segments sum to zero. Using $r = \sqrt{R^2 + a^2}$ and $\sin\theta = a/r$, the vertical component will take the form:

$$dE_z = dE \sin\theta = k\frac{dq}{r^2}\frac{a}{r} = \frac{ka\,dq}{(R^2 + a^2)^{3/2}} \tag{20.48}$$

The total electric field at P due to all segments of the ring is given by the integration of the z-component as follows:

$$\begin{aligned} E &= \int dE_z = \int \frac{ka\,dq}{(R^2 + a^2)^{3/2}} \\ &= \frac{ka}{(R^2 + a^2)^{3/2}} \int dq \end{aligned} \tag{20.49}$$

Since $\int dq$ represents the total charge Q over the entire ring, then the total electric field at P will be given by:

$$\boxed{E = \frac{kQa}{(R^2 + a^2)^{3/2}}} \tag{20.50}$$

This formula shows that the field is zero at the center of the ring, i.e., at $a = 0$.When point P is very far from the ring, i.e., $a \gg R$, then we can neglect R^2 in the denominator of Eq. 20.50 and get $E \approx kQ/a^2$. This form resembles the one we got for a point charge.

20.4.4 The Electric Field of a Uniformly Charged Disk

Assume that a disk of radius R has a uniform positive surface-charge density σ. Also, assume that a point P lies at a distance a from the disk along its central perpendicular axis, see Fig. 20.19.

To find the electric field at P, we divide the disk into concentric rings, then calculate the electric field at P for each ring by using Eq. 20.50, and finally we can sum up the contributions of all the rings.

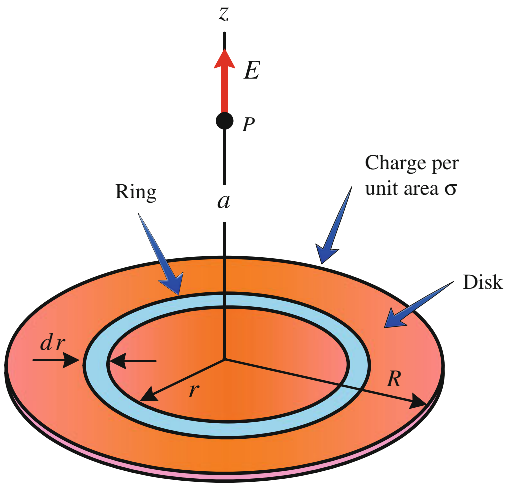


Fig. 20.19 A disk of radius R has a uniform positive surface charge density σ. The ring shown has a radius r and radial width dr. The total electric field at an axial point P is directed along this axis

Figure 20.19 shows one such ring, with radius r, radial width dr, and surface area $dA = 2\pi r\, dr$. Since σ is the charge per unit area, then the charge dq on this ring is:

$$dq = \sigma\, dA = 2\pi\, r\sigma\, dr \tag{20.51}$$

Using this relation in Eq. 20.50, and replacing E with dE, R with r, and Q with $dq = 2\pi\, r\sigma\, dr$, then we can calculate the field resulting from this ring as follows:

$$dE = \frac{ka}{(r^2 + a^2)^{3/2}}\,(2\pi\, r\sigma\, dr) = \pi k\sigma a \frac{2r\, dr}{(r^2 + a^2)^{3/2}} \tag{20.52}$$

To find the total electric field, we integrate this expression with respect to the variable r from $r = 0$ to $r = R$. This gives:

$$E = \int dE = \pi k\sigma a \int_0^R (r^2 + a^2)^{-3/2}(2r\, dr) \tag{20.53}$$

To solve this integral, we transform it to the form $\int u^n du = u^{n+1}/(n+1)$ by setting $u = r^2 + a^2$, and $du = 2r\, dr$. Thus, Eq. 20.53 becomes:

$$\begin{aligned} E &= \pi k\sigma a \int_0^R (r^2 + a^2)^{-3/2}(2r)dr = \pi k\sigma a \int_{u=a^2}^{u=R^2+a^2} u^{-3/2} du \\ &= \pi k\sigma a \left| \frac{u^{-1/2}}{-1/2} \right|_{u=\,a^2}^{u=R^2+\,a^2} = \pi k\sigma a \left[\frac{(R^2 + a^2)^{-1/2}}{-1/2} - \frac{a^{-1}}{-1/2} \right] \end{aligned} \tag{20.54}$$

Rearranging the terms, we find:

$$E = 2\pi k\sigma \left[1 - \frac{a}{\sqrt{R^2 + a^2}}\right] \tag{20.55}$$

Using $k = 1/4\pi\epsilon_\circ$, where $\epsilon_\circ$ is the permittivity of free space, it is sometimes preferable to write this relation as:

$$E = \frac{\sigma}{2\epsilon_\circ}\left[1 - \frac{a}{\sqrt{R^2 + a^2}}\right] \tag{20.56}$$

We can calculate the field when point P is very close to the disk (the near-field approximation) by assuming that $R \gg a$, or by assuming the disk to be an infinite sheet when $R \to \infty$ while keeping a finite. In both cases, the second term between the two brackets of Eq. 20.56 approaches zero, and the equation is reduced to:

$$\boxed{E = \frac{\sigma}{2\epsilon_\circ}} \qquad \left\{\begin{matrix} \text{(Points very close to the disk)} \\ \text{or} \\ \text{(Infinite sheet)} \end{matrix}\right\} \tag{20.57}$$

20.5 Electric Field Lines

The concept of electric field lines was introduced by Faraday as an approach to help us visualize electric fields.

> Spotlight
> An electric field line is an *imaginary* line drawn in such a way that the direction of its tangent at any point is the same as the direction of the electric field vector $\vec{E}$ at that point, see Fig. 20.20.

Since the direction of an electric field generally varies from one point to another, the electric field lines are usually drawn as curves, see Fig. 20.20.

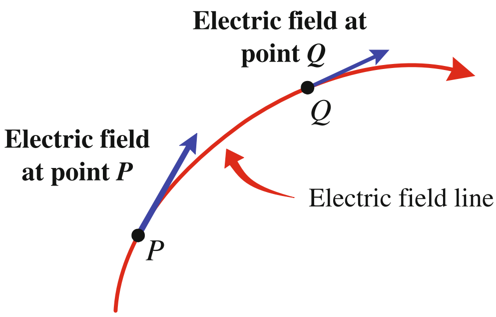


Fig. 20.20 The direction of the electric field at any point is the tangent to the electric field line at this point

The relation between electric field lines and electric field vectors is as follows:

Spotlight

- The electric field vector $\vec{E}$ is tangent to the electric field line at any point.
- The direction of the electric field line at any point is the same as the direction of the electric field.
- The number of electric field lines per unit area, measured in the plane of the lines, is proportional to the magnitude of $\vec{E}$. Thus, the electric field lines are closer together when the electric field is strong, and far apart when the field is weak.

The rules for drawing electric field lines are as follows:

Spotlight

- Electric field lines must emerge from a positive charge and end on a negative charge. For a system that has an excess of one type of charge, some lines will emerge or end infinitely far away.
- The number of lines emerging from a positive charge or ending at a negative charge is proportional to the magnitude of the charge.
- Electric field lines cannot cross each other.

The above rules are used in the six cases shown in Fig. 20.21.

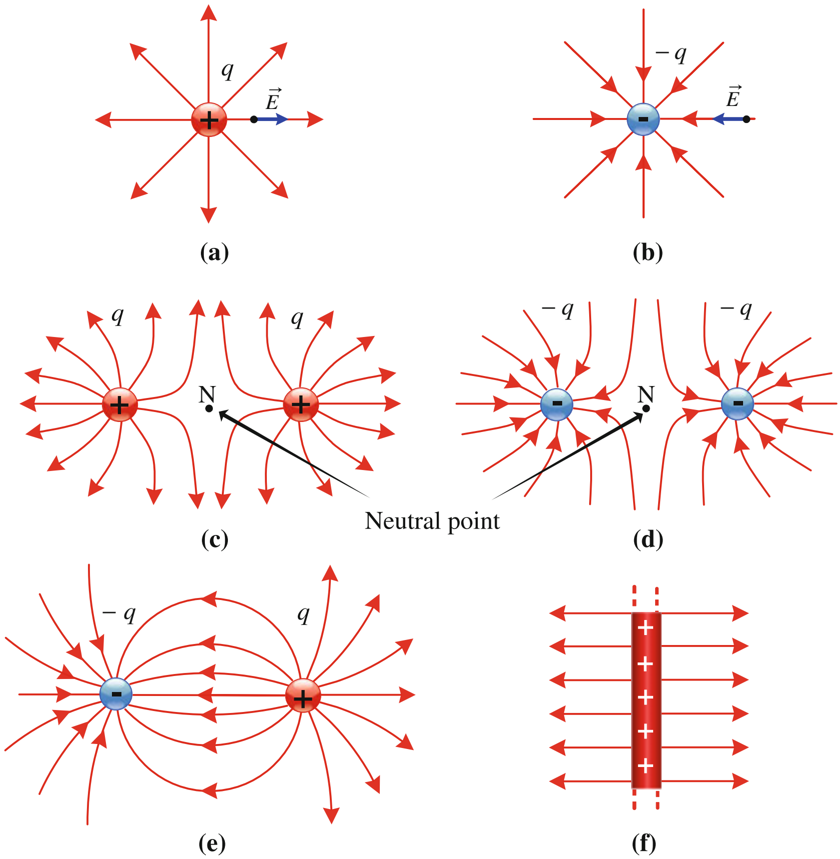


Fig. 20.21 The figure shows the electric field lines of: (a) a positive point charge, (b) a negative point charge, (c) two equal positive charges, (d) two equal negative charges, (e) an electric dipole, and (f) a side view of an infinite sheet of charge

20.6 Motion of Charged Particles in a Uniform Electric Field

When a particle of charge q and mass m is in an external electric field of strength $\vec{E}$, a force $q\vec{E}$ will be exerted on this particle. If $q\vec{E}$ is the *only acting force* on the particle, then according to Newton's second law, $\Sigma\vec{F} = m\vec{a}$, the acceleration of the particle will be given by:

$$\vec{a} = q\vec{E}/m \tag{20.58}$$

If $\vec{E}$ is uniform, then $\vec{a}$ will be constant vector.

Motion of a Charged Particle Along an Electric Field

Consider a particle of positive charge q and mass m in a uniform horizontal electric field $\vec{E}$ produced by two charged plates that are separated by a distance d as shown in Fig. 20.22.

If the particle is released from rest at the positive plate and $q\vec{E}$ is the *only force* that acts on the particle, then the particle will move horizontally along the x-axis with an acceleration $\vec{a} = q\vec{E}/m$. In such a case, we can apply the kinematics equations (see Chap. 3) on the initial and final motion as follows:

- The particle's time of flight t:

$$x = v_\circ\, t + \tfrac{1}{2} a\, t^2 \quad \Rightarrow \quad d = 0 + \tfrac{1}{2}\frac{qE}{m}\, t^2 \quad \Rightarrow \quad \boxed{t = \sqrt{\frac{2md}{qE}}} \tag{20.59}$$

- The speed of the particle v:

$$v = v_\circ + a\, t \quad \Rightarrow \quad v = 0 + \frac{qE}{m}\sqrt{\frac{2md}{qE}} \quad \Rightarrow \quad \boxed{v = \sqrt{\frac{2qEd}{m}}} \tag{20.60}$$

- The kinetic energy of the particle K:

$$K = \tfrac{1}{2} m v^2 \quad \Rightarrow \quad \boxed{K = qEd} \tag{20.61}$$

The last result can also be obtained from the application of the work-energy theorem $W = \Delta K$ because $W = (qE)d$ and $\Delta K = K_{\mathrm{f}} - K_{\mathrm{i}} = K$.

Example 20.6

In Fig. 20.22, assume that the charged particle is a proton of charge $q = +e$. The proton is released from rest at the positive plate. In this case, each of the two oppositely charged plates which are $d = 2$ cm apart has a charge per unit area of $\sigma = 5\,\mu\text{C/m}^2$. (a) What is the magnitude of the electric field between the two plates? (b) What is the speed of the proton as it strikes the second plate?

Solution: (a) The electric field arises from two infinite plates, Thus:

$$E = \frac{\sigma}{2\epsilon_\circ} + \frac{\sigma}{2\epsilon_\circ} = \frac{\sigma}{\epsilon_\circ} = \frac{5 \times 10^{-6}\ \text{C/m}^2}{8.85 \times 10^{-12}\ \text{C/N.m}^2} = 5.65 \times 10^5\ \text{N/C}$$

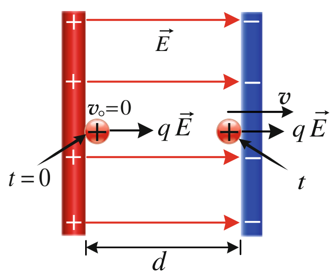


Fig. 20.22 A force $q\vec{E}$ exerted on a positive charge q by a uniform electric field $\vec{E}$ established between two oppositely charged plates

(b) We first find the proton's acceleration from Newton's second law:

$$a = \frac{F}{m} = \frac{eE}{m} = \frac{(1.6\times10^{-19}\,\text{C})(5.65\times10^{5}\,\text{N/C})}{1.67\times10^{-27}\,\text{kg}} = 5.41\times10^{13}\ \text{m/s}^2$$

Then, using $x = v_\circ\, t + \frac{1}{2}a\, t^2$, we find that $d = \frac{1}{2}a\, t^2$. Thus:

$$t = \sqrt{\frac{2d}{a}} = \sqrt{\frac{2(0.02\ \text{m})}{5.41\times10^{13}\ \text{m/s}^2}} = 2.72\times10^{-8}\ \text{s}$$

Finally, we use $v = v_\circ + a\, t$ to find the speed of the proton as follows:

$$v = a\, t = (5.41\times10^{13}\ \text{m/s}^2)(2.72\times10^{-8}\ \text{s}) = 1.47\times10^{6}\ \text{m/s}$$

Motion of a Charged Particle Perpendicular to an Electric Field

Consider an electron of charge $q=-e$ and mass m being projected in a uniform vertical electric field $\vec{E}$ that is established in a region of length L by two oppositely charged plates as shown in Fig. 20.23. If the initial speed $v_\circ$ of the electron at $t=0$ is along the nagative x-axis, and if $\vec{E}$ is along the y-axis, then the acceleration of the electron will be constant along the positive y-axis (ignoring the gravitational force and assuming vacuum conditions). That is:

$$a_x = 0 \quad a_y = \frac{eE}{m} \quad \text{(Upwards)} \tag{20.62}$$

When we apply the kinematics equations with $v_{x\circ} = v_\circ$ and $v_{y\circ} = 0$ while the electron is in the region of the electric field, we find that:

The components of the electron's velocity at time t will be:

$$\begin{aligned} &Along\ x \quad v_x = v_{x\circ} = v_\circ \\ &Along\ y \quad v_y = a_y\,t = \frac{eE}{m}t \end{aligned} \tag{20.63}$$

The components of the electron's position at time t will be:

$$\begin{aligned} &Along\ x \quad x = v_\circ t \\ &Along\ y \quad y = \tfrac{1}{2}a_y\,t^2 = \frac{eE}{2m}t^2 \end{aligned} \tag{20.64}$$

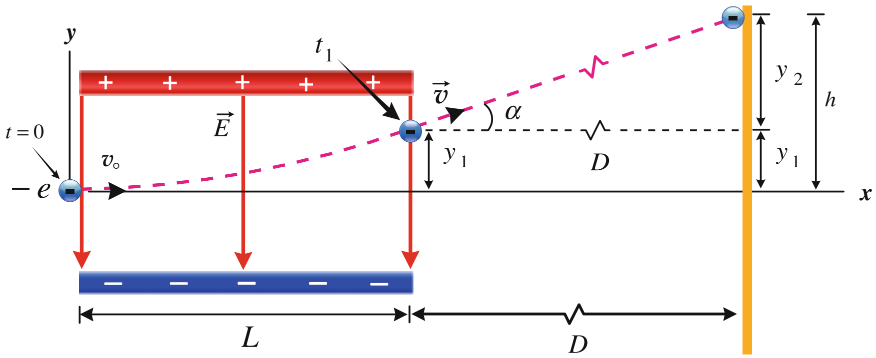


Fig. 20.23 The effect of an upward force $-e\vec{E}$ exerted on an electron projected horizontally with speed $v_\circ$ into a downward uniform electric field $\vec{E}$

The electron will move a distance L horizontally and a distance y_1 vertically before leaving the region of the electric field, see Fig. 20.23. According to Eq. 20.64, the time at this instant will be:

$$\boxed{t_1 = \frac{L}{v_\circ}} \tag{20.65}$$

The vertical position y_1 that corresponds to this time is:

$$\boxed{y_1 = \frac{eEL^2}{2mv_\circ^2}} \tag{20.66}$$

When the electron leaves the region of the electric field, with $v_x = v_\circ$ and $v_y = a_y t_1$, the electric force vanishes and the electron continues to move in a straight line with a constant velocity:

$$\vec{v} = v_\circ \vec{\mathrm{i}} + \frac{eEL}{mv_\circ}\vec{\mathrm{j}} \tag{20.67}$$

This velocity makes an angle α with the horizontal and so:

$$\tan\alpha = \frac{eEL/mv_\circ}{v_\circ} = \frac{eEL}{mv_\circ^2} \tag{20.68}$$

The extra vertical distance y_2 that the electron will move before hitting the screen, which is located at a horizontal distance D from the plates, is given by:

$$\boxed{y_2 = D\,\tan\alpha = D\,\frac{eEL}{mv_\circ^2}} \tag{20.69}$$

Finally, the total vertical distance h that the electron will move is:

$$\boxed{h = y_1 + y_2 = \frac{eEL}{mv_\circ^2}\left(\frac{L}{2} + D\right)} \tag{20.70}$$

Example 20.7

In Fig. 20.23, assume that the horizontal length L of the plates is 5 cm, and assume that the separation D between the plates and the screen is 50 cm. If the uniform electric field has $E = 250$ N/C, and the electron's initial speed $v_\circ$ is 2×10^6 m/s, then; (a) What is the acceleration of the electron between the two plates? (b) Find the time when the electron leaves the two plates. (c) Find the electron's vertical position before leaving the field region. (d) Find the electron's vertical distance before hitting the screen.

Solution: (a) Using the magnitude of the electronic charge $e = 1.6 \times 10^{-19}$ C and the electronic mass $m = 9.11 \times 10^{-31}$ kg in Eq. 20.62, we get:

$$a_x = 0 \quad \text{and} \quad a_y = \frac{eE}{m} = \frac{(1.6 \times 10^{-19}\,\mathrm{C})(250\,\mathrm{N/C})}{9.11 \times 10^{-31}\,\mathrm{kg}} = 4.391 \times 10^{13}\ \mathrm{m/s^2}$$

(b) Using Eq. 20.65 for the horizontal motion, we get:

$$t_1 = \frac{L}{v_\circ} = \frac{0.05\,\text{m}}{2 \times 10^6\,\text{m/s}} = 2.5 \times 10^{-8}\,\text{s}$$

(c) Using Eq. 20.66 for the vertical motion, we get:

$$y_1 = \frac{eEL^2}{2mv_\circ^2} = \frac{(1.6 \times 10^{-19}\,\text{C})(250\,\text{N/C})(0.05\,\text{m})^2}{2(9.11 \times 10^{-31}\,\text{kg})(2 \times 10^6\,\text{m/s})^2} = 0.0137\,\text{m} = 1.37\,\text{cm}$$

Alternatively, we can use Eq. 20.64 to find y_1 as follows:

$$y_1 = \tfrac{1}{2} a_y\, t_1^2 = \tfrac{1}{2}(4.391 \times 10^{13}\,\text{m/s}^2)(2.5 \times 10^{-8}\,\text{s})^2 = 0.0137\,\text{m} = 1.37\,\text{cm}$$

(d) We calculate y_2 from Eq. 20.69 as follows:

$$y_2 = D\,\frac{eEL}{mv_\circ^2} = \frac{(0.5\,\text{m})(1.6 \times 10^{-19}\,\text{C})(250\text{N/C})(0.05\,\text{m})}{(9.11 \times 10^{-31}\,\text{kg})(2 \times 10^6\,\text{m/s})^2} = 0.274\,\text{m} = 27.4\,\text{cm}$$

Therefore, the total vertical distance moved by the electron is:

$$h = y_1 + y_2 = 0.0137\ \text{m} + 0.274\ \text{m} = 0.2877\ \text{m} = 28.77\ \text{cm}$$

20.7 Exercises

Section 20.2 Electric Field of a Point Charge

(1) Find the electric field of a 1 μC point charge at a distance of: (a) 1 cm, (b) 1 m, and (c) 1 km.

(2) Find the value of a point charge if it has an electric field of 1 N/C at points: (a) 1 cm away, (b) 1 m away, and (c) 1 km away.

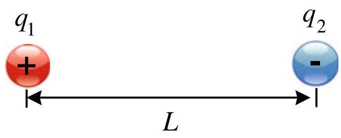
(3) A vertical electric field is set up in space to compensate for the gravitational force on a point charge. What is the required magnitude and direction of the field when the point charge is: (a) an electron? (b) a proton? Comment on the obtained values.

(4) An electron experiences a force of 8×10^{-14} N directed toward the front side of a TV tube (the positive x-direction). (a) What is the magnitude and direction of the electric field that produces this force? (b) What is the magnitude of the acceleration of the electron?

(5) A 4 μC point charge is placed at a point $P(x = 0.2\,\text{m}, y = 0.4\,\text{m})$. What is the electric field $\vec{E}$ due to this charge: (a) at the origin, (b) at $x = 1$ m and $y = 1$ m.

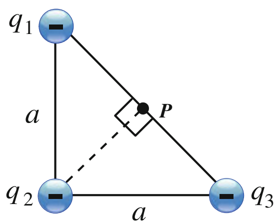
(6) Two point charges $q_1 = +9\,\mu$C and $q_2 = -4\,\mu$C are separated by a distance $L = 10$ cm, see Fig. 20.24. Find the point at which the resultant electric field is zero.

Fig. 20.24 See Exercise (6)



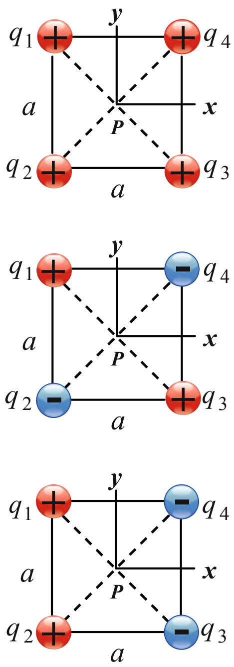
(7) Three negative point charges are placed at the vertices of an isosceles triangle as shown in Fig. 20.25. Given that $a = 10$ cm, $q_1 = q_3 = -2\,\mu$C, and $q_2 = -4\,\mu$C, find the magnitude and direction of the electric field at point P (which is midway between q_1 and q_3).

Fig. 20.25 See Exercise (7)



(8) Four charges of equal magnitude are located at the four corners of a square of side $a = 0.1$ m. Find the magnitude and direction of the electric field at the center P of the square if: (a) all the charges are positive, i.e. $q_i = 5\,\mu$C, where $i = 1, 2, 3, 4$, see top of Fig. 20.26. (b) the charges alternate in sign around the perimeter of the square, i.e. $q_1 = q_3 = 5\,\mu$C and $q_2 = q_4 = -5\,\mu$C, see middle of Fig. 20.26. (c) the anti-clockwise sequence of the charge signs around the perimeter are plus, plus, minus, and minus, i.e. $q_1 = q_2 = 5\,\mu$C and $q_3 = q_4 = -5\,\mu$C, see lower of Fig. 20.26.

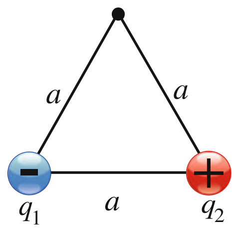
Fig. 20.26 See Exercise (8)



Section 20.3 Electric Field of an Electric Dipole

(9) Two point charges $q_1 = -6\,\mu$C and $q_2 = +6\,\mu$C are placed at two vertices of an equilateral triangle, see Fig. 20.27. If $a = 10$ cm, find the electric field at the third corner.

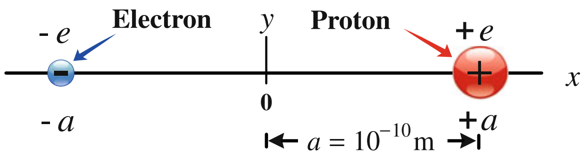
Fig. 20.27 See Exercise (9)



(10) A proton and an electron form an electric dipole and are separated by a distance of $2a = 2 \times 10^{-10}$ m, see Fig. 20.28. (a) Use exact formulas to calculate the electric field along the x-axis at $x = -10a$, $x = -2a$, $x = -a/2$,

$x = +a/2$, $x = +2a$, and $x = +10a$. (b) Show that at both points $x = \pm 10a$, the approximate formula given by Eq. 20.13 has a very close percentage difference from the exact value.

Fig. 20.28 See Exercise (10)



(11) Rework the calculations of Exercise 10 but on the y-axis at $y = -10\,a$, $y = -2\,a$, $y = -a/2$, $y = +a/2$, $y = +2a$, and $y = +10a$. In part (b), use Eq. 20.17.

Section 20.4 Electric Field of a Continuous Charge Distribution

(12) A non-conductive rod of length L has a total negative charge $-Q$ that is uniformly distributed along its length, see Fig. 20.29. (a) Find the linear charge density of the rod. (b) Use the coordinates depicted in the figure to prove that the electric field at point P, a distance a from the right end of the rod, has the same form as the one given by Eq. 20.27. (c) When P is very far from the rod, i.e. $a \gg L$, show that the electric field reduces to the electric field of a point charge (i.e. the rod would look like a point charge). (d) If $L = 15$ cm, $Q = 25\,\mu$C, and $a = 20$ cm, find the value of the electric field at P.

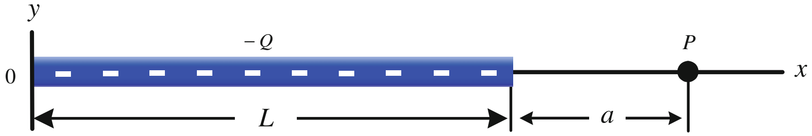


Fig. 20.29 See Exercise (12)

(13) A non-conductive rod lies along the x-axis with one of its ends located at $x = a$ and the other end located at ∞, see Fig. 20.30. Starting from the definition of an electric field of a differential element on the rod, find the electric field at the

origin if: (a) the rod carries a *uniform* positive linear charge density λ. (b) the rod carries a positive varying linear charge density $\lambda = \lambda_\circ a/x$.

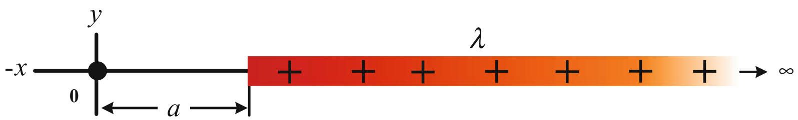
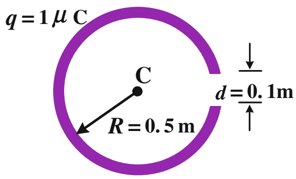


Fig. 20.30 See Exercise (13)

(14) A uniformly charged ring of radius 15 cm has a total charge of 50 μC. Find the electric field on the central perpendicular axis of the ring at: (a) 0 cm, (b) 1 cm, (c) 10 cm, and (d) 100 cm. (e) What do you observe about the values you just calculated?

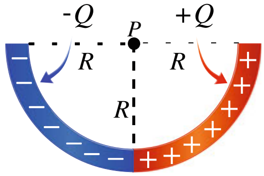
(15) A charged ring of radius $R = 0.5$ m has a gap $d = 0.1$ m, see Fig. 20.31. Calculate the electric field at its center C if it carries a uniform charge $q = 1\ \mu$C.

Fig. 20.31 See Exercise (15)



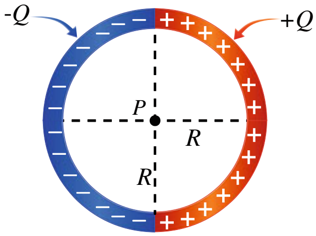
(16) Figure 20.32 shows a non-conductive semicircular arc of radius R that consists of two quarters. The semicircle has a uniform positive total charge Q along its right half, and a uniform negative total charge $-Q$ along its left half. Find the resultant electric field at the center of the semicircle.

Fig. 20.32 See Exercise (16)



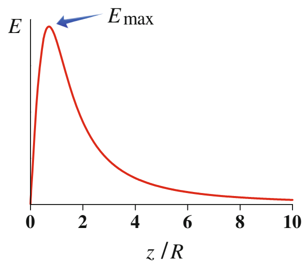
(17) Two non-conductive semicircular arcs, one of a uniform positive charge $+Q$ and the other of a uniform negative charge $-Q$, form a circle of radius R, see Fig. 20.33. Find the resultant electric field at the center of the circle, and compare it with the result of Exercise 16.

Fig. 20.33 See Exercise (17)



(18) If you consider a uniformly charged ring of total charge Q and a fixed radius R (as in Fig. 20.18), then the graph of Fig. 20.34 would map the electric field along the axis of such a ring as a function of z/R. Show that the maximum electric field is $E_{\max} = 2k\,Q/3\sqrt{3}R^2$ and occurs at $z = R/\sqrt{2}$.

Fig. 20.34 See Exercise (18)



(19) An electron is constrained to move along the central axis of a ring of radius R that has a uniform positive charge q, see Fig. 20.35. Show that when the position x of the electron is much less than the radius R ($x \ll R$), the electrostatic force exerted on the electron can cause it to oscillate through the center of the ring with an angular frequency given by $\omega = \sqrt{kqe/mR^3}$, where e and m are the electronic charge and mass, respectively.

(20) Two non-conductive rings having the same radius R are arranged with their central axes along a common horizontal line and separated by a distance of $4\,R$, see Fig. 20.36. Ring 1 has a uniform positive charge q_1, while ring 2 has a uniform positive charge q_2. Given that the net electric field is zero at point P, which is at a distance R from ring 1 and on the common central axis of the two rings, (a) find the ratio between the two charges. (b) If only the sign of q_1

is reversed, is it possible to have a point on the common axis where the net electric field is zero? If so, where would it be?

Fig. 20.35 See Exercise (19)

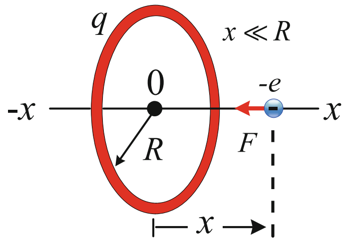
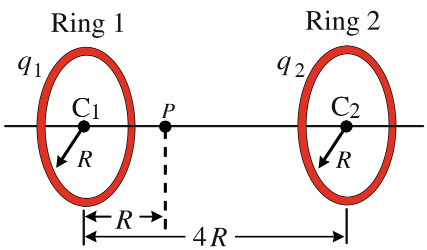


Fig. 20.36 See Exercise (20)



(21) A disk of radius $R = 5$ cm has a surface charge density $\sigma = 6\,\mu\text{C/m}^2$ on its surface. Calculate the magnitude of the electric field at points on the central axis of the disk located at: (a) 1 mm, (b) 1 cm, (c) 10 cm, and (d) 100 cm.

(22) A disk of radius R has a charge Q that is uniformly distributed over its surface area. Show that Eq. 20.55 transforms to:

$$E = \frac{2kQ}{R^2}\left[1 - \frac{a}{\sqrt{R^2 + a^2}}\right]$$

Show that when $a \gg R$, the electric field approaches that of a point charge formula:

$$E \approx k\frac{Q}{a^2} \quad (a \gg R)$$

You may use the binomial expansion $(1 + \delta)^p \approx 1 + p\delta$ when $\delta \ll 1$.

(23) Compare the obtained results of Exercise 21 to the near-field approximation $E = \sigma/2\epsilon_\circ$ as well as to the point charge approximation $E = k(\pi R^2\sigma)/a^2$, and find which result(s) of Exercise 21 match the two approximations.

(24) A disk of radius R has a surface charge density σ and an electric field of magnitude $E_\circ = \sigma/2\epsilon_\circ$ at the center of its surface, see Fig. 20.37. At what distance z along the central axis of the disk is the magnitude of the electric field E equal to one-half of $E_\circ$?

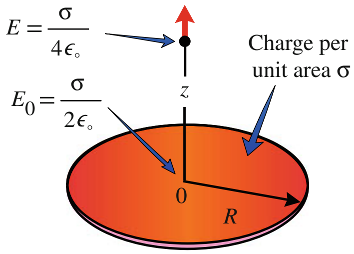


Fig. 20.37 See Exercise (24)

(25) Find the electric field between two oppositely-charged infinite sheets of charge, each having the same charge magnitude and surface charge density σ, but opposite signs, see Fig. 20.38.

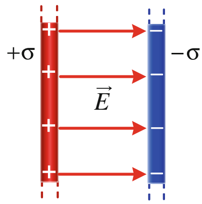


Fig. 20.38 See Exercise (25)

Section 20.5 Electric Field Lines

(26) (a) A negatively charged disk has a uniform charge per unit area. Sketch the electric field lines in the plane of the plane of the disk passing through its center. (b) Redo part (a) taking the disk to be positively charged. (c) A negatively charged rod has a uniform charge per unit length. Sketch the electric field lines in the plane of the rod. (d) Three equal positive charges are placed at the corners of an equilateral triangle. Sketch the electric field lines in the plane of the charges. (e) An infinite linear rod has a uniform charge per unit length. Sketch the electric field lines in the plane of the rod.

Section 20.6 Motion of Charged Particles in a Uniform Electric Field

(27) An electron and a proton are released simultaneously from rest in a uniform electric field of 10^5 N/C. Ignore the effect of the fields of the electron and proton on each other. (a) Find the speed and kinetic energy of the electron 50 ns after it has been released. (b) Repeat part (a) for the proton.

(28) Figure 20.39 shows two oppositely charged parallel plates that are separated by a distance $d = 1.5$ cm. Each plate has a charge per unit area of magnitude $\sigma = 4\,\mu\text{C/m}^2$. An electron is released from rest at $t = 0$ from the negative plate. (a) Calculate the electric field between the two plates. (b) Ignoring the effect of gravity, find the resultant force exerted on the electron? (c) Find the acceleration of the electron. (d) How long does it take the electron to strike the positive plate? (e) What is the speed and kinetic energy of the electron just before striking the positive plate?

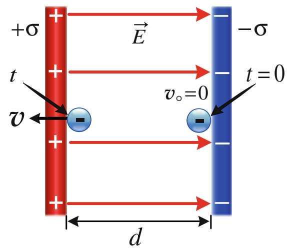


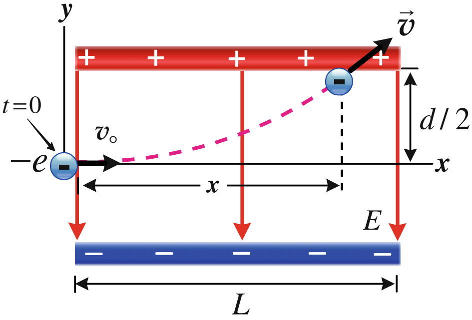
Fig. 20.39 See Exercise (28)

(29) In Exercise 28 assume that the electron is projected from the positive plate toward the negative plate with an initial speed $v_\circ$ at time $t = 0$. The electron travels the distance $d = 1.5$ cm between the two plates and stops temporarily before hitting the negatively charged plate. (a) Find the magnitude and direction of its acceleration. (b) Find the value of the electron's initial speed. (c) Find the time before the electron stops temporarily.

(30) Two oppositely charged horizontal plates are separated by a distance $d = 1$ cm and each has a length $L = 3$ cm, see Fig. 20.40. The electric field between the plates is uniform and has a magnitude $E = 10^2$ N/m. An electron is projected between the plates with a horizontal initial speed of $v_\circ = 10^6$ m/s as shown. Assuming this experiment is conducted in a vacuum, where will the electron strike the upper plate?

(31) Repeat Exercise 30 when a proton replaces the electron.

Fig. 20.40 See Exercise (30)



(32) To prevent the Electron in exercise 30 from striking the upper plate, its initial horizontal speed is increased to $v_\circ = 2 \times 10^6$ m/s, see Fig. 20.41, and it then strikes a screen at a distance $D = 30$ cm. (a) What is the acceleration of the electron in the region between the two plates? (b) Find the time when the electron leaves the two plates. (c) What is the vertical position of the electron just before leaving the region between the two plates? (d) Find the electron's total vertical distance just before hitting the screen.

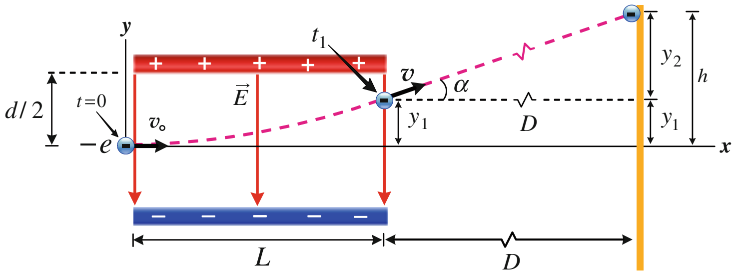


Fig. 20.41 See Exercise (32)

(33) Repeat Exercise 32 when a proton replaces the electron.

Gauss's Law

21

Although Coulomb's law is the governing law in electrostatics, its form does not always simplify calculations in situations involving symmetry. In this chapter, we introduce Gauss's law as an alternative method for calculating electric fields of certain highly symmetrical charge distribution systems. In addition to being simpler than Coulomb's law, Gauss's law permits us to use qualitative reasoning.

21.1 Electric Flux

Consider a uniform electric field $\vec{E}$ penetrating a small area A oriented perpendicularly to the field as shown in Fig. 21.1. Recall from Sect. 20.5 that the number of electric field lines per unit area (measured in a plane perpendicular to the lines) is proportional to the magnitude of $\vec{E}$. Therefore, the total number of lines penetrating the surface is proportional to EA. This product is called the **electric flux**[1] Φ_E. Thus:

$$\boxed{\Phi_E = EA} \tag{21.1}$$

The SI units for Φ_E is newton-meters square per coulomb (N.m^2/C).

> Spotlight
> Electric flux is proportional to the number of electric field lines penetrating a certain area.

If the area in Fig. 21.1 is tilted by an angle θ with respect to $\vec{E}$, the flux through it (the number of electric lines) will decrease. To visualize this, Fig. 21.2 shows an

[1] The word "flux" comes from the Latin word meaning "to flow".

H. A. Radi and J. O. Rasmussen, *Principles of Physics*,
Undergraduate Lecture Notes in Physics, DOI: 10.1007/978-3-642-23026-4_21,

area A that makes an angle θ with the field. The number of lines that cross the area A is equal to the number that cross the area A', which is *perpendicular* to $\vec{E}$ and hence $A' = A\cos\theta$. Thus, the flux through A, $\Phi_E(A)$, is equal to the flux through A', $\Phi_E(A')$. But according to Eq. 21.1, the flux through A' is defined as $\Phi_E(A') = EA'$. Therefore, the flux through A is:

$$\Phi_E(A) = \Phi_E(A') = EA' = EA\cos\theta \tag{21.2}$$

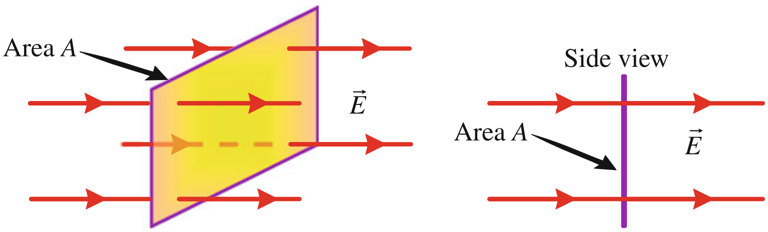


Fig. 21.1 Electric field lines representing a uniform electric field $\vec{E}$ that penetrates an area A perpendicularly (shown both in 3D and as a side view). The electric flux Φ_E through this area is EA

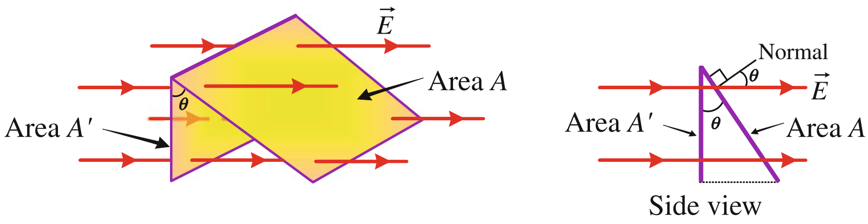


Fig. 21.2 Electric field lines representing a uniform electric field $\vec{E}$ penetrating an area A that is at an angle θ with the field (both three dimensional and side views are displayed). Since the flux through A is the same as through A', the flux through A is $\Phi_E = EA\cos\theta$

If we define a vector area $\vec{A}$ whose magnitude represents the surface area and whose direction is defined to be perpendicular to the surface area as in Fig. 21.3, then Eq. 21.2 can be written as:

$$\Phi_E = \vec{E} \bullet \vec{A} = EA\cos\theta \tag{21.3}$$

The flux through a surface of area A has a maximum value EA when the surface is perpendicular to the field (i.e. when $\theta = 0°$), and is zero when the surface is parallel to the field (i.e. when $\theta = 90°$).

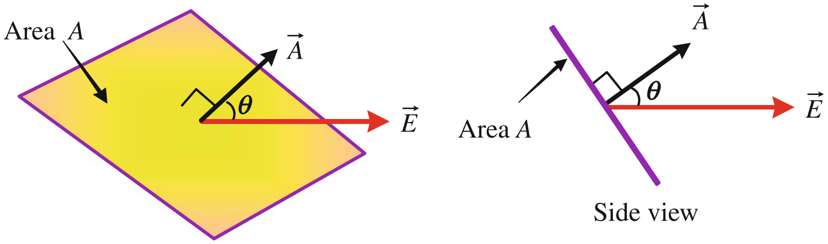


Fig. 21.3 The definition of a vector area $\vec{A}$ whose magnitude represents the surface area and whose direction is defined to be perpendicular to the surface area

Generally, the electric field may vary over the surface of any shape. Let us consider the general surface depicted by the shape in Fig. 21.4 and calculate the electric flux over the whole surface.

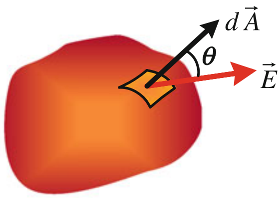


Fig. 21.4 The differential surface vector area $d\vec{A}$ of magnitude dA and direction perpendicular to the differential surface area and pointing outwards. When the electric field $\vec{E}$ makes an angle θ with that differential surface area, the differential flux $d\Phi_E$ is $\vec{E} \bullet d\vec{A}$

We start by considering a differential vector surface area $d\vec{A}$ to be normal to the surface and to point outwards at a specific location. If the electric field vector at this location is $\vec{E}$, then the differential electric flux $d\Phi_E$ through this differential area will be:

$$d\Phi_E = \vec{E} \bullet d\vec{A} \tag{21.4}$$

We integrate this relation over a surface S to get the electric flux as:

$$\Phi_E = \int_S \vec{E} \bullet d\vec{A} \tag{21.5}$$

Generally, Φ_E depends on the field pattern and the surface shape.

According to the definition of the vector area $d\vec{A}$ which always points outwards, the sign of the flux depends on the angle between $\vec{E}$ and $d\vec{A}$ as follows:

(1) If $\theta < 90°$, then $\vec{E}$ crosses the surface from the inside to the outside and hence $d\Phi_E = \vec{E} \bullet d\vec{A}$ is positive.

(2) If $\theta = 90°$, then $\vec{E}$ grazes the surface and hence $d\Phi_E = \vec{E} \bullet d\vec{A}$ is zero.

(3) If $90° < \theta < 180°$, then $\vec{E}$ crosses the surface from the outside to the inside and hence $d\Phi_E = \vec{E} \bullet d\vec{A}$ is negative.

The *net flux* through a surface is proportional to the *net number* of electric field lines leaving the surface. If more lines are entering than leaving, then the net flux is negative. If more lines are leaving than entering, then the net flux is positive.

We can write the net flux through a closed surface as:

$$\Phi_E = \oint \vec{E} \bullet d\vec{A} \tag{21.6}$$

where the symbol $\oint$ represents an integral over a closed surface.

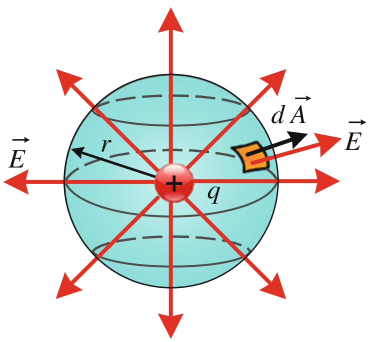
Example 21.1

Find the electric flux through a sphere of radius r enclosing at its center: (a) a positive charge q, and (b) a negative charge $-q$.

Solution: (a) When a positive charge q is at the center of such a sphere, the electric field would be directed outwards and be normal to the surface. It would also have a constant magnitude, $E = q/(4\pi\epsilon_\circ r^2)$, see Fig. 21.5. Therefore:

$$\vec{E} \bullet d\vec{A} = E\,dA \cos 0° = E\,dA = \frac{q}{4\pi\epsilon_\circ r^2} dA$$

Fig. 21.5



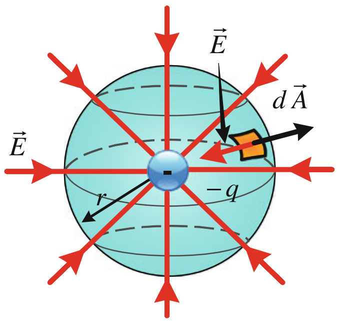
From Eq. 21.6 and the fact that $\oint dA$ over a spherical surface gives the area of the sphere, i.e. $\oint dA = A = 4\pi r^2$, we find the net flux through the sphere that encloses the positive charge q to be:

$$\Phi_E = \oint \vec{E} \bullet d\vec{A} = \frac{q}{4\pi \epsilon_\circ r^2} \oint dA = \frac{q}{4\pi \epsilon_\circ r^2} 4\pi r^2 = \frac{q}{\epsilon_\circ}$$

(b) When a negative charge $-q$ is at the center of such a sphere, the electric field would be directed inwards and be normal to the surface. It would also have a constant magnitude, $E = q/(4\pi \epsilon_\circ r^2)$, see Fig. 21.6. Therefore:

$$\vec{E} \bullet d\vec{A} = E\, dA \cos 180° = -E\, dA = -\frac{q}{4\pi \epsilon_\circ r^2} dA$$

Fig. 21.6



Performing similar steps as in part (a), we find the net flux to be:

$$\Phi_E = \oint \vec{E} \bullet d\vec{A} = -\frac{q}{4\pi \epsilon_\circ r^2} \oint dA = -\frac{q}{4\pi \epsilon_\circ r^2} 4\pi r^2 = -\frac{q}{\epsilon_\circ}$$

Negative electric flux means electric lines are entering the surface.

21.2 Gauss's Law

In this section, we introduce a new foundation of Coulomb's law, called **Gauss's law**. This law can be used to take advantage of symmetry in the problem under consideration. Central to Gauss's law is a *hypothetical* closed surface called a **Gaussian surface**.

In Example 21.1 we noticed that the flux over a sphere of radius r was equal to the charge q inside the sphere divided by the permittivity of free space $\epsilon_\circ$. Now consider several closed Gaussian surfaces surrounding the charge as shown in Fig. 21.7. The number of electric field lines passing through the spherical surface S_1 is the same as the number of lines passing through the non-spherical surfaces S_2 and S_3. Therefore, we conclude that the flux through any closed Gaussian surface surrounding the point charge q is $q/\epsilon_\circ$.

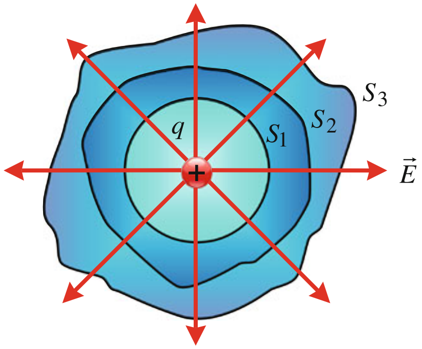


Fig. 21.7 Different closed Gaussian surfaces enclosing a point charge q. The net electric flux is the same through all surfaces

Gauss's law is a generalization to what we just described.

> Gauss's Law
> The net electric flux through any closed surface is equal to the net charge inside the surface divided by the permittivity of free space $\epsilon_\circ$.

That is:

$$\Phi_E = \oint \vec{E} \bullet d\vec{A} = \frac{q_{in}}{\epsilon_\circ} \tag{21.7}$$

where q_{in} represents the *net charge inside* the surface and $\vec{E}$ represents the *total electric field* at any point on the surface, which includes contributions from charges inside and/or outside (if any).

Note that Gauss's law is very useful in calculating electric fields in situations where the charge distributions have planar, cylindrical, or spherical symmetry. In these charge distribution systems, one must carefully construct the imaginary Gaussian surface such that it simplifies the integral in Eq. 21.7. This can be done by trying to satisfy one or more of the following conditions:

(1) The value of the field over the surface is constant, $E =$ constant.
(2) The dot product $\vec{E} \bullet d\vec{A}$ is $E\,dA$ because $\vec{E} \,//\, d\vec{A}$.
(3) The dot product $\vec{E} \bullet d\vec{A}$ is zero because $\vec{E} \perp d\vec{A}$.
(4) The value of the field over the surface is zero, $E = 0$.

21.3 Applications of Gauss's Law

Example 21.2

Using Gauss's law, find the electric field at a distance r from a positive point charge q, and compare it with Coulomb's law.

Solution: We apply Gauss's law to the spherical Gaussian surface of radius r in Fig. 21.5. From the symmetry of the problem, we know that at any point, the electric field $\vec{E}$ is perpendicular to the surface and directed outwards from the spherical center. Thus, $\vec{E} \mathbin{//} d\vec{A}$ and $\vec{E} \bullet d\vec{A} = E\,dA$. Then, with $q_{in} = q$, we can write Gauss's law as:

$$\Phi_E = \oint \vec{E} \bullet d\vec{A} = \frac{q_{in}}{\epsilon_\circ} \quad \Rightarrow \quad \oint \vec{E} \bullet d\vec{A} = \oint E\,dA = E \oint dA = 4\pi r^2 E = \frac{q}{\epsilon_\circ}$$

This leads to:
$$E = \frac{1}{4\pi\epsilon_\circ}\frac{q}{r^2} = k\frac{q}{r^2}$$

which is simply Coulomb's law. This proves that Gauss's law and Coulomb's law are equivalent.

Example 21.3

Prove that any excess positive charge q on the isolated conductor of Fig. 21.8 will lie entirely on its outer surface.

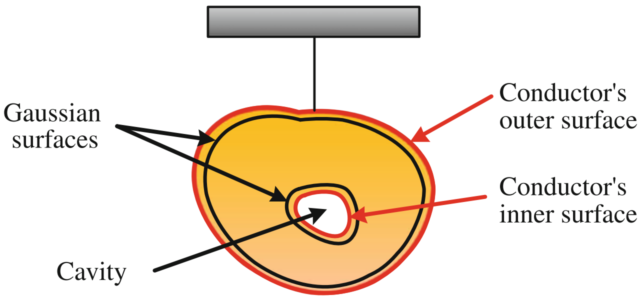


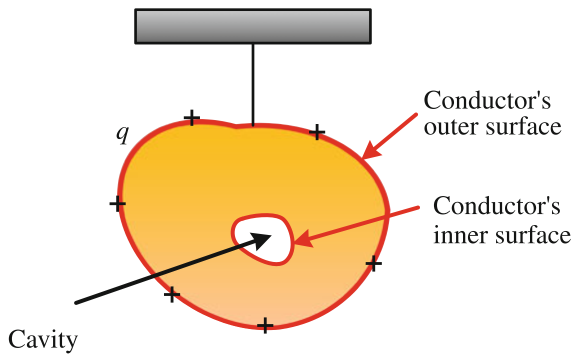
Fig. 21.8

Solution: The electric field inside the conductor must be zero. If this is not the case, the field would exert forces on the free electrons and a current would flow within the conductor. Of course, there are no such currents in an isolated conductor in electrostatic equilibrium.

First, we draw a Gaussian surface surrounding the conductor's cavity, close to its surface, as seen in Fig. 21.8. Since $\vec{E}=0$ inside the conductor, then $\Phi_E=0$ and hence according to Gauss's law, no net charge would exist on the inner walls of the cavity.

Then we draw a Gaussian surface just inside the outer surface of the conductor. Since $\vec{E}=0$ inside the conductor, then $\Phi_E=0$ and according to Gauss's law, no net charge will exist inside the Gaussian surface. If the excess charge is not inside the Gaussian surface, it must then be outside that surface, on the conductor's surface; see Fig. 21.9.

Fig. 21.9



Example 21.4

Using Gauss's law, find the electric field just outside the surface of a conductor carrying a positive surface charge density σ.

Solution: Consider a small section of the conductor's surface so as to neglect curvature. Then construct a cylindrical Gaussian surface normal to the conductor as shown in Fig. 21.10, where one end of the cylinder is inside the conductor while the other end is outside. Each end has an area A. The electric field $\vec{E}$ inside the conductor is zero, and the electric field $\vec{E}$ just outside the conductor's surface must be perpendicular to the surface. If this were not true, the component of the field along the surface of the conductor would force the free electrons to move, violating the conductor's electrostatic equilibrium.

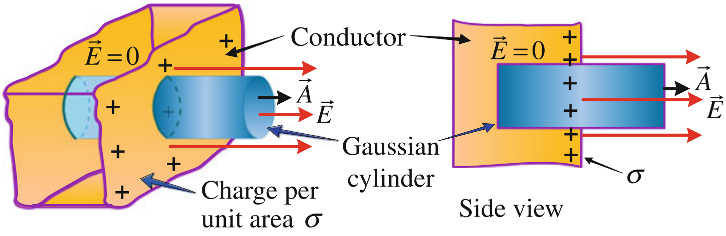


Fig. 21.10

To find the net flux through this cylindrical Gaussian surface, let us consider each of the four faces of the cylinder. (1) Because $\vec{E}=0$ inside the conductor, then the flux through the end of the cylinder inside the conductor is $\Phi_E(1)=0$. (2) For the same reason, the flux through the curved surface of the cylinder inside the conductor is $\Phi_E(2)=0$. (3) Since $\vec{E} \perp d\vec{A}$ along the curved surface of the cylinder outside the conductor, the flux there is also $\Phi_E(3)=0$. (4) Since $\vec{E} \,//\, d\vec{A}$ along the end of the cylinder that is outside the conductor, the flux there is $\Phi_E(4)=EA$. Thus, the net flux through the cylindrical Gaussian surface is $\Phi_E=EA$. Since $q_{in}=\sigma A$, we can then find electric field just outside the surface of the conductor as follows:

$$\Phi_E = \oint \vec{E} \bullet d\vec{A} = \frac{q_{in}}{\epsilon_\circ} \quad \Rightarrow \quad EA = \frac{\sigma A}{\epsilon_\circ} \quad \Rightarrow \quad E = \frac{\sigma}{\epsilon_\circ}$$

Example 21.5

Find the electric field due to an infinite plane sheet of charge with a uniform positive surface charge density σ.

Solution: By symmetry, the electric field $\vec{E}$ outside the infinite plane sheet must be: (1) uniform, (2) perpendicular to the sheet, (3) of the same magnitude at all points equidistant from the sheet, and (4) in opposite direction on the other side of the sheet, see Fig. 21.11. The choice that reflects that symmetry is a cylindrical

Gaussian surface normal to the sheet as shown in Fig. 21.11, where one end of the cylinder (of area A) is to the right of the sheet while the other end is to the left of it.

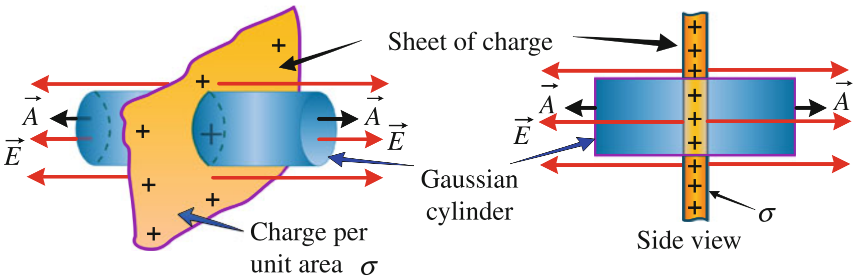


Fig. 21.11

As in Example 21.4, to find the net flux through this cylindrical Gaussian surface, let us consider each of the four faces of the cylinder. (1) Because $\vec{E}\ //\ d\vec{A}$ through the left end of the cylinder, then the flux there is $\Phi_E(1) = EA$. (2) Because $\vec{E} \perp d\vec{A}$ through the left curved surface of the cylinder then the flux there is $\Phi_E(2) = 0$. (3) For the same reason $(\vec{E} \perp d\vec{A})$, and the flux through the right curved surface of the cylinder is $\Phi_E(3) = 0$. (4) Because $\vec{E}\ //\ d\vec{A}$ through the right end of the cylinder, then the flux there is $\Phi_E(4) = EA$. Thus, the net flux through the Gaussian surface is $\Phi_E = 2EA$. Since $q_{in} = \sigma A$, we then can find electric field as follows:

$$\Phi_E = \oint \vec{E} \bullet d\vec{A} = \frac{q_{in}}{\epsilon_\circ} \quad \Rightarrow \quad 2\,EA = \frac{\sigma\,A}{\epsilon_\circ} \quad \Rightarrow \quad E = \frac{\sigma}{2\epsilon_\circ}$$

Example 21.6

Two infinite conducting plates have a uniform surface charge density of magnitude σ' on their faces. The excess charge is positive on one plate and negative on the other, see Fig. 7.12a, b. The two plates are brought together as shown in Fig. 21.12c. Find the electric field to the left and right of the plates in each part of the figure.

Solution: The charge on the positively charged conductor in Fig. 21.21a will spread over its two faces each with a surface density of magnitude σ'. From

Example 21.4, the electric field just outside each of these surfaces would be directed away from the two faces and would have a magnitude of:

$$E_+ = \frac{\sigma'}{\epsilon_\circ}$$

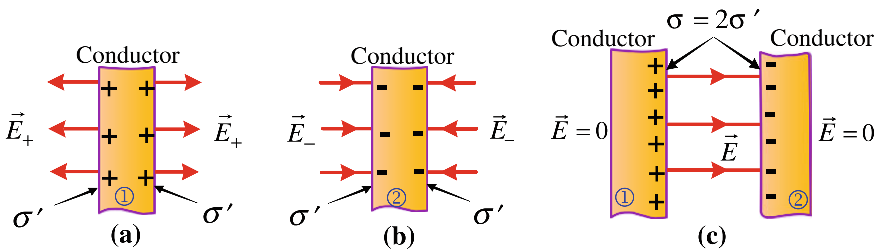


Fig. 21.12

We have a similar situation in Fig. 21.12b, except that the electric field is directed *toward* the two faces and has a magnitude given by:

$$E_- = \frac{\sigma'}{\epsilon_\circ}$$

When we bring the two plates together, the excess charge on one plate attracts the excess of charge on the other, and all the excess charge moves onto the inner surfaces of the plates, see Fig. 21.12c. The magnitude of the new surface charge density of the inner surfaces is $\sigma = 2\sigma'$. Thus, the electric field between the plates is to the right with a magnitude:

$$E = \frac{\sigma}{\epsilon_\circ} \qquad \text{(Between the plates)}$$

The electric field on the outer sides of the two plates of Fig. 21.12c is zero since no charge is left on those sides.

Example 21.7

Two infinitely long nonconductive sheets are aligned in parallel, see Fig. 21.13a. Each sheet has a fixed uniform charge. One sheet is positively charged and has a surface charge density of magnitude $\sigma_+ = 6.5\ \mu\text{C/m}^2$. The other sheet is negatively charged with $|\sigma_-| = 4.5\ \mu\text{C/m}^2$. Find the electric field: (a) to the left (L) of the sheets, (b) between (B) the sheets, and (c) to the right (R) of the sheets.

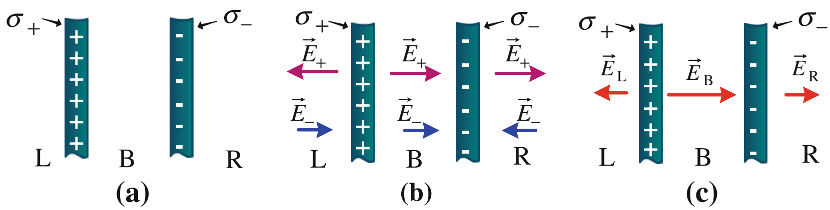


Fig. 21.13

Solution: For items of fixed charge, we can calculate the electric field of the items in the same manner as if each item were isolated, i.e. by adding the fields algebraically using the superposition principle. Thus, the magnitude of the electric field due to the positive sheet is:

$$E_+ = \frac{\sigma_+}{2\epsilon_\circ} = \frac{6.5 \times 10^{-6}\,\mathrm{C/m^2}}{2(8.85 \times 10^{-12}\,\mathrm{C^2/N.m^2})} = 3.67 \times 10^5\,\mathrm{N/C}$$

Also, the magnitude of the electric field due to the negative sheet is:

$$E_- = \frac{\sigma_-}{2\epsilon_\circ} = \frac{4.5 \times 10^{-6}\,\mathrm{C/m^2}}{2(8.85 \times 10^{-12}\,\mathrm{C^2/N.m^2})} = 2.54 \times 10^5\,\mathrm{N/C}$$

The directions of the fields to the left (L), between (B), and right (R) of the sheets are shown in Fig. 21.13b. The resultant field depends on the values of E_+ and E_-. Since $E_+ > E_-$, then E_L is:

$$E_L = E_+ - E_- = 3.67 \times 10^5\,\mathrm{N/C} - 2.54 \times 10^5\,\mathrm{N/C} = 1.13 \times 10^5\,\mathrm{N/C} \quad \text{(to the left)}$$

The field to the right of the sheets E_R has the same magnitude but is directed to the right. Between the sheets, we have:

$$E_B = E_+ + E_- = 3.67 \times 10^5\,\mathrm{N/C} + 2.54 \times 10^5\,\mathrm{N/C} = 6.21 \times 10^5\,\mathrm{N/C} \quad \text{(to the right)}$$

Example 21.8

Using Gauss's law, find the electric field at a distance r from a long thin rod that has a uniform charge per unit length λ.

Solution: By symmetry, the electric fields outside the rod are radial and lie in a plane perpendicular to the rod. Additionally, the field has the same magnitude at all points at the same radial distance from the rod. This suggests that we can construct a cylindrical Gaussian surface of an arbitrary radius r and height ℓ. Such a cylinder would have its ends perpendicular to the rod as shown in Fig. 21.14.

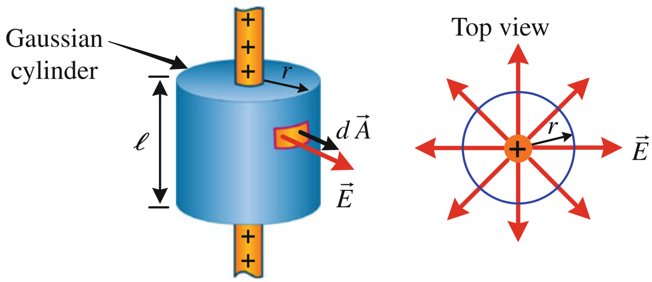


Fig. 21.14

We divide the flux into two cases: (1) The flux through the two ends of the Gaussian cylinder is zero because $\vec{E}$ is parallel to these surfaces, i.e. $\vec{E} \perp \vec{A}$. (2) The flux through the curved surface of the Gaussian cylinder can be obtained by taking into account that $E =$ constant and $\vec{E}$ is parallel to $d\vec{A}$, i.e. $\vec{E} \bullet d\vec{A} = E\,dA$. Therefore, $\oint \vec{E} \bullet d\vec{A} = \oint E\,dA = E \oint dA = EA$, where A is the area of the curved cylinder and is given by $A = 2\pi r\ell$. The net charge inside the Gaussian cylinder is $q_{\text{in}} = \lambda\ell$. We can now use Gauss's law to find the electric field as follows:

$$\Phi_E = \oint \vec{E} \bullet d\vec{A} = \frac{q_{\text{in}}}{\epsilon_\circ} \quad \Rightarrow \quad \oint \vec{E} \bullet d\vec{A} = \oint E\,dA = E \oint dA$$

$$= E(2\pi\, r\ell) = \frac{\lambda\ell}{\epsilon_\circ}$$

Then:
$$E = \frac{1}{2\pi\epsilon_\circ}\frac{\lambda}{r} = 2k\frac{\lambda}{r}$$

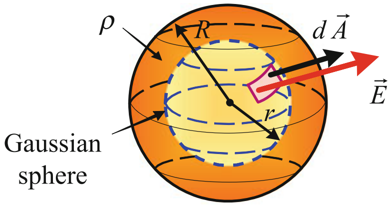
This relation was derived in Chap. 20 using Coulomb's law (Eq. 20.36).

Example 21.9

A solid sphere of radius R has a uniform volume charge density ρ and carries a total positive charge Q. Find and sketch the electric field at any distance r away from the sphere's center.

Solution: We divide the solution to $0 \le r \le R$ and $r \ge R$.
(1) For $0 \le r \le R$
When dealing with a spherically symmetric charge distribution, we chose a spherical Gaussian surface of radius $r < R$ concentric with the charged sphere as shown in Fig. 21.15.

Fig. 21.15



By symmetry, the magnitude of the electric field is constant everywhere on the spherical Gaussian surface and normal to the surface at any point, i.e. $\vec{E} \mathbin{//} d\vec{A}$. Thus:

$$\oint \vec{E} \bullet d\vec{A} = \oint E\, dA = E \oint dA = E(4\pi r^2)$$

It is important to notice that the volume, say V', of the Gaussian sphere encloses a net charge $q_{\text{in}} = \rho V'$; that is:

$$q_{\text{in}} = \rho V' = \rho(\tfrac{4}{3}\pi r^3)$$

We can now use Gauss's law to find electric field as follows:

$$\Phi_E = \oint \vec{E} \bullet d\vec{A} = \frac{q_{\text{in}}}{\epsilon_\circ} \quad \Rightarrow \quad E(4\pi r^2) = \frac{\rho(\frac{4}{3}\pi r^3)}{\epsilon_\circ}$$

Then: $$E = \frac{\rho}{3\epsilon_\circ} r \qquad (0 \le r \le R)$$

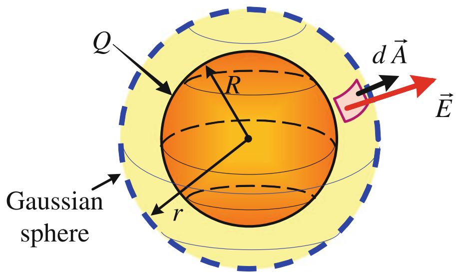
Using the definition $\rho = Q/(\frac{4}{3}\pi R^3)$ and $k = 1/(4\pi\epsilon_\circ)$, we get:

$$E = \frac{Q}{4\pi\epsilon_\circ R^3} r = k\frac{Q}{R^3} r \qquad (0 \le r \le R)$$

(2) For $r \geq R$

Again, because the charge distribution is spherically symmetric, we can construct a Gaussian sphere of radius $r > R$ concentric with the charged sphere, as shown in Fig. 21.16.

Fig. 21.16



Just as when $r < R$, $\oint \vec{E} \bullet d\vec{A} = E(4\pi r^2)$, but $q_{in} = Q$. Thus, we can use Gauss's law to find the electric field as follows:

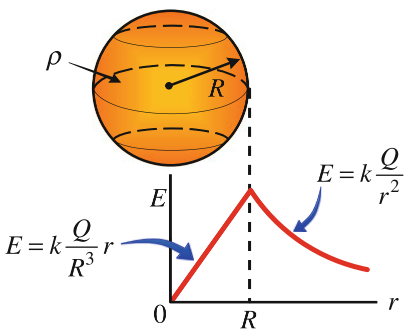
$$\Phi_E = \oint \vec{E} \bullet d\vec{A} = \frac{q_{\text{in}}}{\epsilon_\circ} \quad \Rightarrow \quad E(4\pi r^2) = \frac{Q}{\epsilon_\circ}$$

i.e.:

$$E = \frac{1}{4\pi\epsilon_\circ}\frac{Q}{r^2} = k\frac{Q}{r^2} \qquad (r \geq R)$$

Notice that this is identical to the result obtained for a point charge. Therefore, we conclude that the electric field outside any uniformly charged sphere is equivalent to that of a point charge located at the center of the sphere. At $r = R$, the two cases give identical results $E = kQ/R^2$. A plot of E versus r is shown in Fig. 21.17. This figure shows the continuation of E and its maximum at $r = R$.

Fig. 21.17

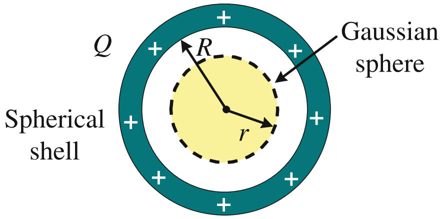


Example 21.10

A thin spherical shell of radius R has a total positive charge Q distributed uniformly over its surface. Find the electric field inside and outside the shell.

Solution: By symmetry, if any field exists inside the shell, it must be radial. Let us construct a spherical Gaussian surface of radius $r < R$ concentric with the shell, see the cross sectional view in Fig. 21.18.

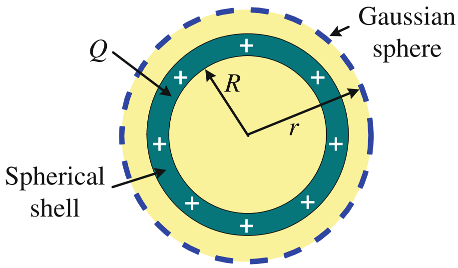
Fig. 21.18



Based on Gauss's law, the lack of charge inside the surface indicates that $\oint \vec{E} \bullet d\vec{A} = E(4\pi r^2) = 0$, or $E = 0$. Accordingly, we conclude that *there is no electric field inside a uniformly charged spherical shell*.

Outside the shell, we construct a spherical Gaussian surface of radius $r > R$ concentric with the charged shell as shown in Fig. 21.19.

Fig. 21.19



Symmetry suggests that $E =$ constant on that surface and $\vec{E}$ is parallel to $d\vec{A}$, i.e. $\oint \vec{E} \bullet d\vec{A} = E(4\pi r^2)$. Since the net charge q_{in} inside the Gaussian surface is equal to the total charge Q on the shell, the *shell is equivalent to a point charge located at the center*. That is:

$$E = \frac{1}{4\pi\epsilon_\circ}\frac{Q}{r^2} = k\frac{Q}{r^2} \qquad (r > R).$$

21.4 Conductors in Electrostatic Equilibrium

Conductors contain free electrons that can move freely. When there is no net motion of electrons within the conductor, the conductor is in electrostatic equilibrium and has the following four properties:

(1) The electric field inside the conductor is zero (see Example 21.3).

(2) The excess charge on an isolated conductor lies on its outer surface (see Example 21.3).

(3) The electric field just outside a charged conductor at any point is perpendicular to its surface and has a magnitude $E = \sigma/\epsilon_\circ$, where σ is the surface charge density at that point (see Example 21.4).

(4) The surface charge density is greatest at locations where the radius of curvature of the surface is smallest (see Chap. 22).

We can elaborate more about the first property by considering the conducting slab in electrostatic equilibrium on the left of Fig. 21.20, where the free electrons are uniformly distributed throughout the slab, i.e. $\vec{E}_{\text{int}} = 0$. When we place the slab in an external electric field $\vec{E}_{\text{ext}}$ as in the right part of Fig. 21.20, the free electrons move to the left. In time, more negative and positive charges accumulate on the left and right surfaces, respectively. These two planes of charge create an increasing internal electric field $\vec{E}_{\text{int}}$ inside the conductor. After awhile, $\vec{E}_{\text{int}}$ will compensate $\vec{E}_{\text{ext}}$, resulting in a zero net electric field inside the conductor, i.e. $\vec{E}_{\text{net}} = \vec{E}_{\text{ext}} - \vec{E}_{\text{int}} = 0$. The time to reach this new electrostatic equilibrium is of the order 10^{-6} s.

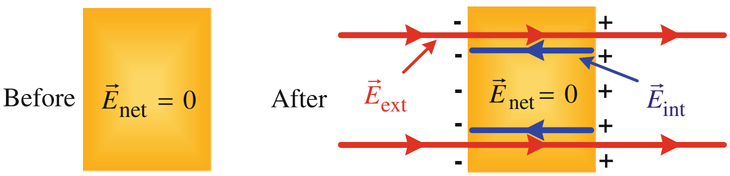


Fig. 21.20 An external electric field $\vec{E}_{\text{ext}}$ creates an internal electric field $\vec{E}_{\text{int}}$ in the conductor such that the net electric field $\vec{E}_{\text{net}}$ is zero

Example 21.11

A conducting sphere of radius R carries a net positive charge $2Q$. A conducting spherical shell of inner radius R_1 ($R_1 > R$) and outer radius R_2 carries a net negative charge $-Q$. This shell is concentric with the conducting sphere. Find the magnitude of the electric field at a distance r away from the common center when: (a) $r < R$, (b) $R < r < R_1$, (c) $R_1 < r < R_2$, and (d) $r > R_2$.

Solution: The charge distributions under consideration are characterized by being spherically symmetrical around the common center c. This suggests that a spherical Gaussian surface of radius r is to be constructed in each case such as S_1, S_2, S_3, and S_4 that are displayed in Fig. 21.21. In addition, we use the fact that the electric field inside a conductor is zero and all the excess charge will lie entirely on the outer surface of the isolated conductor.

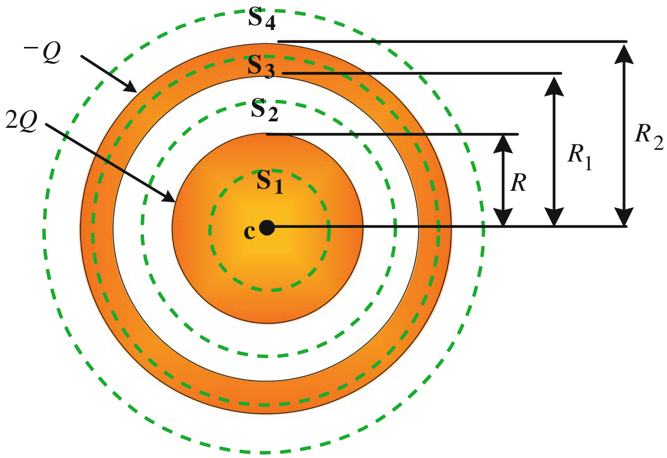


Fig. 21.21

(a) In this region the Gaussian sphere S_1 of Fig. 21.21 satisfies the condition $r < R$. Because there is no charge inside the conductor in this region, i.e. $q_{in} = 0$; then, $E_1 = 0$.

(b) In this region the Gaussian sphere S_2 of Fig. 21.21 satisfies the condition $R < r < R_1$. Because $q_{in} = 2Q$ inside this surface and because $\oint \vec{E}_2 \bullet d\vec{A} = E_2(4\pi r^2)$, we can use Gauss's law to find:

$$E_2 = \frac{1}{4\pi\epsilon_\circ}\frac{2Q}{r^2} = k\frac{2Q}{r^2} \qquad (R < r < R_1)$$

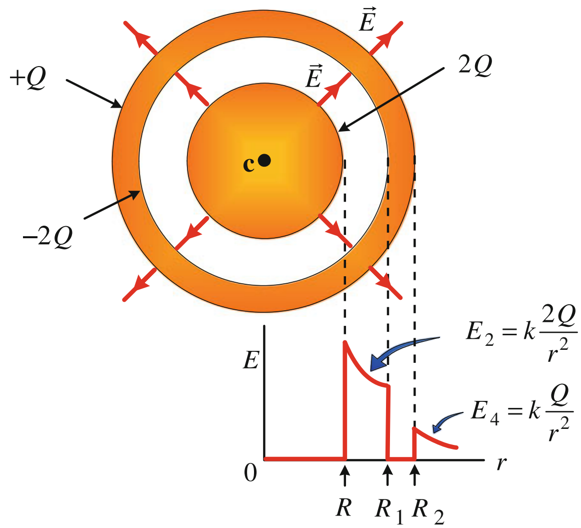
(c) In this region, the Gaussian sphere S_3 of Fig. 21.21 satisfies the condition $R_1 < r < R_2$. Because the electric field inside an equilibrium conductor is zero, i.e. $E_3 = 0$; then, based on Gauss's law, the net charge q_{in} must be zero. From this argument, we find that an induced charge $-2Q$ must be established on the inner surface of the shell to cancel the charge $+2Q$ on the solid sphere. In addition, because the net charge on the whole shell is $-Q$, we conclude that its outer surface must carry an induced charge $+Q$.

(d) In this region, the Gaussian sphere S_4 of Fig. 21.21 satisfies the condition $r > R_2$. Because $q_{in} = 2Q - Q = Q$ inside this surface and because $\oint \vec{E}_4 \bullet d\vec{A} = E_4(4\pi r^2)$, we can use Gauss's law to find:

$$E_4 = \frac{1}{4\pi\epsilon_\circ}\frac{Q}{r^2} = k\frac{Q}{r^2} \qquad (r > R_2)$$

Figure 21.22 shows a graphical representation of the variation of the electric field E with r. In addition, the figure shows the final distribution of the charge on the two conductors.

Fig. 21.22



The expressions that we have arrived at for the electric fields established by simple charge distributions are presented in Table 21.1.

Table 21.1 Electric fields due to simple charge distributions

Charge distribution	Electric field
Single point charge q	$E = k\dfrac{q}{r^2} \qquad r > 0$
Charge q uniformly distributed on the surface of a conducting sphere of radius R	$E = \begin{cases} k\dfrac{q}{r^2} & r \geq R \\ 0 & r < R \end{cases}$
Charge q uniformly distributed with uniform charge density ρ over an insulating sphere of radius R	$E = \begin{cases} k\dfrac{q}{r^2} & r \geq R \\ k\dfrac{q}{R^3}r & r \leq R \end{cases}$
Infinitely long thin rod of a uniform charge per unit length λ	$E = 2k\dfrac{\lambda}{r}$ r outside the line
Infinite plane sheet of charge of uniform surface charge density σ	$E = \dfrac{\sigma}{2\epsilon_\circ}$ Everywhere outside the plane
Conductor having surface charge density σ	$E = \begin{cases} \dfrac{\sigma}{\epsilon_\circ} & \text{Just outside the conductor} \\ 0 & \text{Inside the conductor} \end{cases}$
Two oppositely charged conducting plates with surface charge density of magnitude σ	$E = \begin{cases} \dfrac{\sigma}{\epsilon_\circ} & \text{Any point betwee the plates} \\ 0 & \text{Outside the plates} \end{cases}$

21.5 Exercises

Section 21.1 Electric Flux

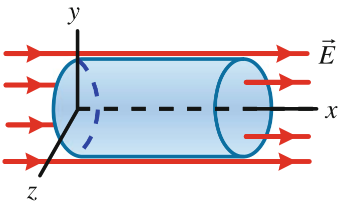
(1) A uniform electric field is directed from left to right along the positive x-axis. If the magnitude of the field is 2×10^5 N/C, what flux passes through a circular loop of area $0.5\,\text{m}^2$ if the normal to loop is: (a) in the positive x-direction, (b) in the negative x-direction, (c) in the positive y-direction, (d) in the negative y-direction, and (e) in a direction that makes an angle 60° from the x-axis.

(2) The maximum flux through a rectangle of area $0.2\,\text{m}^2$ is $5 \times 10^5\,\text{N.m}^2/\text{C}$. Find the magnitude of the electric field.

(3) A cylinder of length $L = 40$ cm and radius $R = 10$ cm has its axis along the x-axis, see Fig. 21.23. The electric field in this region is $\vec{E} = (10^5\,\vec{i}\,)$ N/C.

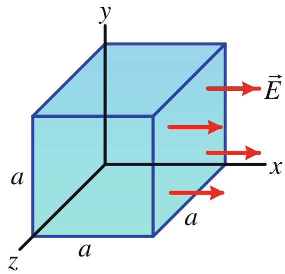
Find the flux through: (a) the cylindrical wall, (b) the cap at the left end of the cylinder, (c) the cap at the right end of the cylinder.

Fig. 21.23 See Exercise (3)



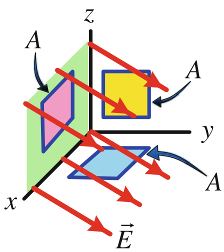
(4) An electric field $\vec{E} = (\alpha + \beta x)\ \vec{i}$ passes through a cube of side a, as shown in Fig. 21.24. (a) Find the net electric flux through the cube. (b) Calculate this flux given that $\alpha = 2\,\text{N/C}$, and $a = 0.2\,\text{m}$ for $\beta = 5\,\text{N.m/C}$ and $\beta = 0$.

Fig. 21.24 See Exercise (4)



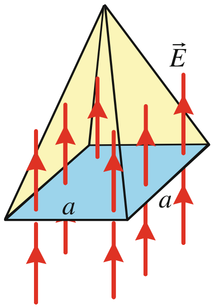
(5) A uniform electric field $\vec{E} = \alpha\vec{i} + \beta\vec{j}$ passes through a square surface of area A. What is the flux through this area if the surface lies: (a) in the xy-plane? (b) in the xz-plane? and (c) in the yz-plane? see Fig. 21.25.

Fig. 21.25 See Exercise (5)



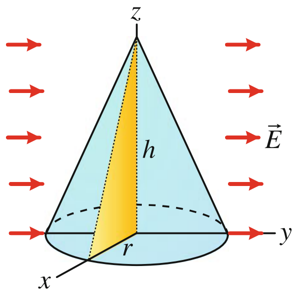
(6) A pyramid has a horizontal square base of side $a = 20\,\text{cm}$. The pyramid is placed in a uniform electric field E of $70\,\text{N/C}$ that is directed upwards, see Fig. 21.26. (a) Find the electric flux through the pyramid's base. (b) Find the electric flux through the pyramid's four slanted surfaces.

Fig. 21.26 See Exercise (6)



(7) A horizontal uniform electric field E penetrates a vertical cone of base radius r and height h, see Fig. 21.27. (a) Find the electric flux through the left-hand side of the cone. (b) Find the electric flux through the right-hand side of the cone. (c) Find the electric flux through the base of the cone.

Fig. 21.27 See Exercise (7)



Section 21.2 Gauss's Law

(8) A point charge q is located at the center of a charged ring of radius R. The ring has a linear charge density λ, see Fig. 21.28. (a) Find the total electric flux through the Gaussian sphere S_1 of radius $r < R$. (b) Find the total electric flux through the Gaussian sphere S_2 of radius $r > R$.

(9) Figure 21.29 shows four closed surfaces S_1, S_2, S_3, and S_4 and four point charges q, $-q$, $2q$, and $-2q$. (a) Find the electric flux through each surface. (b) Would the electric field lines produced by the point charge $-2q$ have an effect on the calculated fluxes? (c) Explain the reasoning behind your answer for (b).

Fig. 21.28 See Exercise (8)

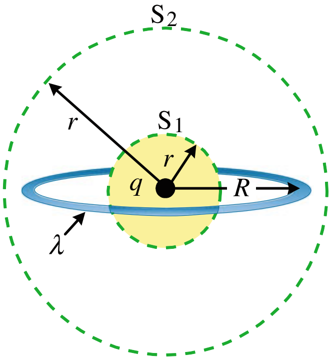
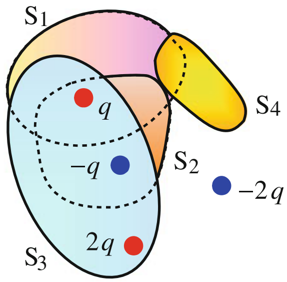
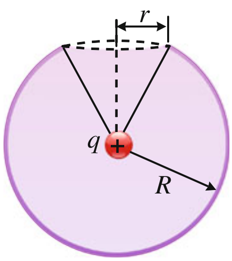


Fig. 21.29 See Exercise (9)



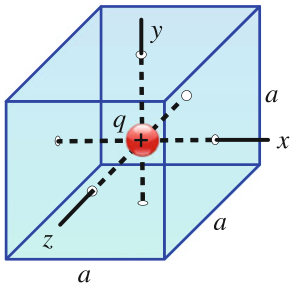
(10) A point charge $q = 25\,\mu\mathrm{C}$ is located at the center of a sphere of radius $R = 25$ cm. A circular cut of radius $r = 5$ cm is removed from the surface of the sphere, see Fig. 21.30. (a) Find the electric flux that passes through that cut. (b) Repeat when the cut has a radius $r = 25$ cm. Is the answer $q/2\epsilon_\circ$?

Fig. 21.30 See Exercise (10)



(11) A point charge $q = 53.1$ nC is located at the center of a cube of side $a = 5$ cm, see Fig. 21.31. (a) Find the electric flux through each face of the cube. (b) Find the flux through the four slanted surfaces of a pyramid formed from a vertex on the center of the cube and one of its six square faces.

Fig. 21.31 See Exercise (11)



(12) At an altitude $h_1 = 700$ m above the ground, the electric field in a particular region is $E_1 = 95$ N/C downwards. At an altitude $h_2 = 800$ m, the electric field is $E_2 = 80$ N/C downwards. Construct a Gaussian surface as a box of horizontal area A and height lying between h_1 and h_2, to find the average volume-charge density in the layer of air between these two elevations.

(13) A point P is at a distance $a = 10$ cm from an infinite rod, charged with a uniform charge per unit length $\lambda = 5$ nC/m. (a) Find the electric flux through a sphere of radius $r = 5$ cm centered at P, see left of Fig. 21.32. (b) Find the electric flux through a sphere of radius $r = 15$ cm centered at P, see right of Fig. 21.32.

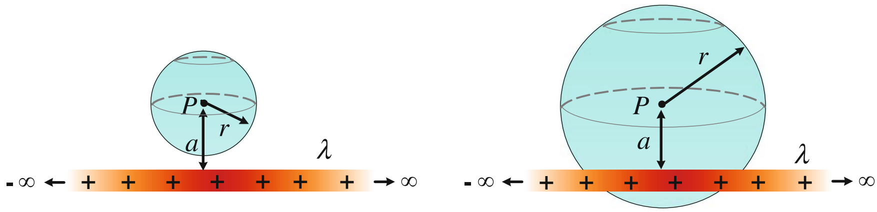
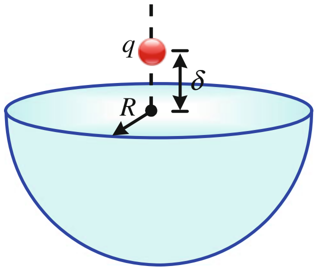


Fig. 21.32 See Exercise (13)

(14) A point charge q is located at a distance δ just above the center of the flat face of a hemisphere of radius R as shown in Fig. 21.33. (a) When δ is very small, use the argument of symmetry to find an approximate value for the electric flux Φ_{curved} through the curved surface of the hemisphere. (b) When δ is very small, use Gauss's law to find an approximate value of the electric flux Φ_{flat} through the flat surface of the hemisphere.

Fig. 21.33 See Exercise (14)



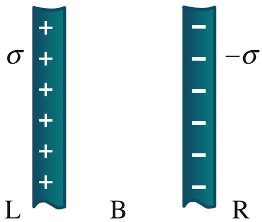
Section 21.3 Applications of Gauss's Law

(15) An infinite horizontal sheet of charge has a charge per unit area $\sigma = 8.85\ \mu\text{C/m}^2$. Find the electric field just above the sheet.

(16) A nonconductive wall carries a uniform charge density $\sigma = 8.85\ \mu\text{C/cm}^2$. Find the electric field 7 cm away from the wall. Does your result change as the distance from the wall increases such that it is much less than the wall's dimensions?

(17) Two infinitely long, nonconductive charged sheets are parallel to each other. Each sheet has a fixed uniform charge. The surface charge density on the left sheet is σ while on the right sheet is $-\sigma$, see Fig. 21.34. Use the superposition principle to find the electric field: (a) to the left of the sheets, (b) between the sheets, and (c) to the right of the sheets.

Fig. 21.34 See Exercise (17)



(18) Repeat the calculations for Exercise 17 when: (i) both the sheets have positive uniform surface charge densities σ, and (ii) both the sheets have negative uniform surface charge densities $-\sigma$.

(19) A thin neutral conducting square plate of side $a = 80$ cm lies in the xy-plane, see Fig. 21.35. A total charge $q = 5$ nC is placed on the plate. Assuming that

the charge density is uniform, find: (a) the surface charge density on the plate, (b) the electric field just above the plate, and (c) the electric field just below the plate.

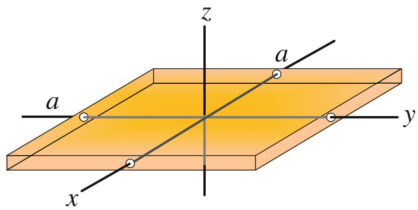


Fig. 21.35 See Exercise (19)

(20) A long filament has a charge per unit length $\lambda = -80\,\mu\text{C/m}$. Find the electric field at: (a) 10 cm, (b) 20 cm, and (c) 100 cm from the filament, where distances are measured perpendicular to the length of the filament.

(21) A uniformly charged straight wire of length $L = 1.5$ m has a total charge $Q = 5\,\mu\text{C}$. A thin uncharged nonconductive cylinder of height $\ell = 2$ cm and radius $r = 10$ cm surrounds the wire at its central axis, see Fig. 21.36. Using reasonable approximations, find: (a) the electric field at the surface of the cylinder and (b) the total electric flux through the cylinder.

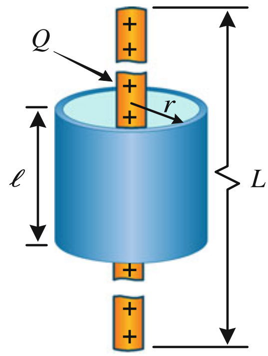


Fig. 21.36 See Exercise (21)

(22) A thin nonconductive cylindrical shell of radius $R = 10$ cm and length $L = 2.5$ m has a uniform charge Q distributed on its curved surface, see Fig. 21.37. The radial outward electric field has a magnitude 4×10^4 N/C at a distance $r = 20$ cm from its axis (measured from the midpoint of the shell).

Find: (a) the net charge on the shell, and (b) the electric field at a point $r = 5$ cm from its axis.

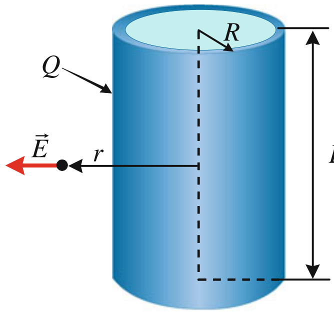


Fig. 21.37 See Exercise (22)

(23) A long non-conducting cylinder of radius R has a uniform charge distribution of density ρ throughout its volume. Find the electric field at a distance r from its axis where $r < R$?

(24) A thin spherical shell of radius $R = 15$ cm has a total positive charge $Q = 30\,\mu$C distributed uniformly over its surface, see Fig. 21.38. Find the electric field at: (a) 10 cm and (b) 20 cm from the center of the charge distribution.

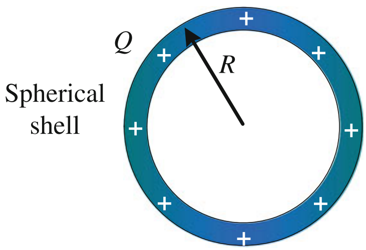


Fig. 21.38 See Exercise (24)

(25) Two concentric thin spherical shells have radii $R_1 = 5$ cm and $R_2 = 10$ cm. The two shells have charges of the same magnitude $Q = 3\,\mu$C, but different in sign, see Fig. 21.39. Use the shown three Gaussian surfaces S_1, S_2, and S_3 to find the electric field in the three regions: (a) $r < R_1$, (b) $R_1 < r < R_2$, and (c) $r > R_2$.

(26) A particle with a charge $q = -60$ nC is located at the center of a non-conducting spherical shell of volume $V = 3.19 \times 10^{-2}\ \mathrm{m}^3$, see Fig. 21.40. The spherical shell carries over its interior volume a uniform negative charge Q of volume density $\rho = -1.33\,\mu\mathrm{C/m}^3$. A proton moves outside the spherical shell in a circular orbit of radius $r = 25$ cm. Calculate the speed of the proton.

Fig. 21.39 See Exercise (25)

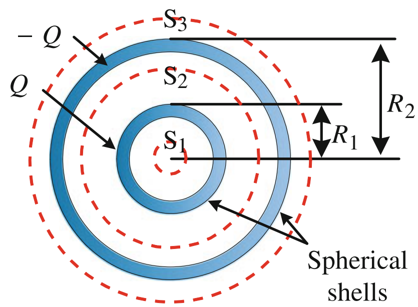
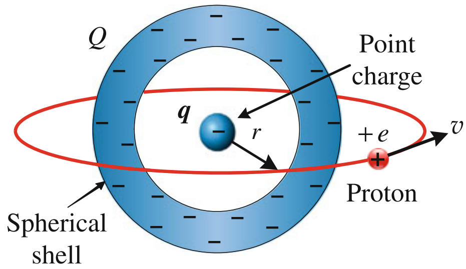


Fig. 21.40 See Exercise (26)



(27) A solid non-conducting sphere is 4 cm in radius and carries a 7.5 μC charge that is uniformly distributed throughout its interior volume. Calculate the charge enclosed by a spherical surface, concentric with the sphere, of radius (a) $r = 2$ cm and (b) $r = 8$ cm.

(28) A solid non-conducting sphere of radius $R = 20$ cm has a total positive charge $Q = 30\,\mu$C that is uniformly distributed throughout its volume. Calculate the magnitude of the electric field at: (a) 0 cm, (b) 10 cm, (c) 20 cm, (d) 30 cm, and (e) 60 cm from the center of the sphere.

(29) If the electric field in air exceeds the threshold value $E_{\text{thre}} = 3 \times 10^6$ N/C, sparks will occur. What is the largest charge Q can a metal sphere of radius 0.5 cm hold without sparks occurring?

(30) The charge density inside a non-conducting sphere of radius R varies as $\rho = \alpha\ r$ (C/m^3), where r is the radial distance from the center of the sphere. Use Gauss's law to find the electric field inside and outside the sphere.

(31) A solid sphere of radius R with a center at point C_1 has a uniform volume charge density ρ. A spherical cavity of radius $R/2$ with a center at point C_2 is then scooped out and left empty, see Fig. 21.41. Point A is at the surface of the big sphere and collinear with points C_1 and C_2. What is the magnitude and direction of the electric field at points C_1 and A?

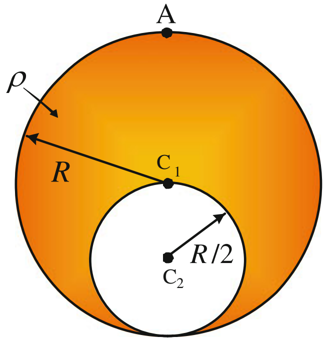


Fig. 21.41 See Exercise (31)

Section 21.4 Conductors in Electrostatic Equilibrium

(32) A non-conducting sphere of radius R and charge $+Q$ uniformly distributed throughout its volume is concentric with a spherical conducting shell of inner radius R_1 and outer radius R_2. The shell has a net charge $-Q$, see Fig. 21.42. Find an expression for the electric field as a function of the radius r when: (a) $r < R$ (within the sphere). (b) $R < r < R_1$ (between the sphere and the shell). (c) $R_1 < r < R_2$ (inside the shell). (d) $r > R_2$ (outside the shell). (e) What are the charges on inner and outer surfaces of the conducting shell?

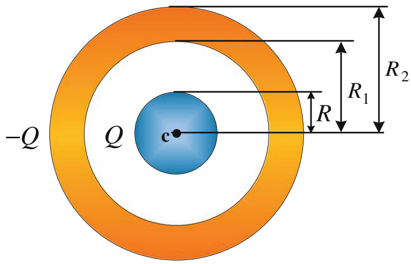


Fig. 21.42 See Exercise (32)

(33) A large, thin, copper plate of area A has a total charge Q uniformly distributed over its surfaces. The same charge Q is uniformly distributed over the upper surface of a glass plate, which is identical to the copper plate, see Fig. 21.43. (a) Find the surface charge density on each face of the two plates. (b) Compare the electric fields just above the center of the upper surface of each plate.

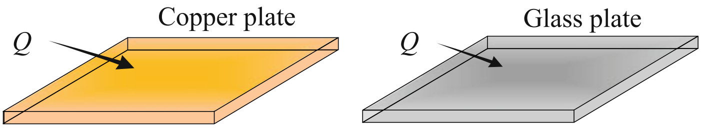
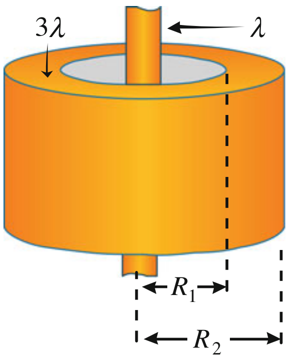


Fig. 21.43 See Exercise (33)

(34) A thin, long, straight wire carries a charge per unit length λ. The wire lies along the axis of a long conducting cylinder carrying a charge per unit length 3λ. The cylinder has an inner radius R_1 and an outer radius R_2, see Fig. 21.44. (a) Use a Gaussian surface inside the conducting cylinder to find the charge per unit length on its inner and outer surfaces. (b) Use Gauss's law to find the electric field E outside the wire. (c) Sketch the electric field E as a function of the distance r from the wire's axis.

Fig. 21.44 See Exercise (34)



(35) An uncharged solid conducting sphere of radius R contains two cavities. A point charge q_1 is placed within the first cavity, and a point charge q_2 is placed within the second one. Find the magnitude of the electric field for $r > R$, where r is measured from the center of the sphere.

Electric Potential

22

Newton's law of gravity and Coulomb's law of electrostatics are mathematically identical. Therefore, the general features of the gravitational force introduced in Chap. 6 apply to electrostatic forces. In particular, electrostatic forces are conservative. Consequently, it is more convenient to assign an *electric potential energy* U to describe any system of two or more charged particles. This idea allows us to define a *scalar quantity* known as the **electric potential**. It turns out that this concept is of great practical value when dealing with devices such as capacitors, resistors, inductors, batteries, etc, and when dealing with the flow of currents in electric circuits.

22.1 Electric Potential Energy

The electric force that acts on a test charge q placed in an electric field $\vec{E}$ (created by a source charge distribution) is defined by $\vec{F} = q\vec{E}$. For an infinitesimal displacement $d\vec{s}$ [1], the work done by the conservative electric field is:

$$dW = \vec{F} \bullet d\vec{s} = q\vec{E} \bullet d\vec{s} \tag{22.1}$$

According to Eq. 6.39, this amount of work corresponds to a change in the potential energy of the charge-field system given by:

$$dU = -dW = -\vec{F} \bullet d\vec{s} = -q\vec{E} \bullet d\vec{s} \tag{22.2}$$

For a finite displacement of the charge from an initial point A to a final point B, the change in electric potential energy $\Delta U = U_B - U_A$ of the charge-field system

[1] When dealing with electric and magnetic fields, it is common to use this notation to represent an infinitesimal displacement vector that is tangent to a path through space.

H. A. Radi and J. O. Rasmussen, *Principles of Physics*,
Undergraduate Lecture Notes in Physics, DOI: 10.1007/978-3-642-23026-4_22,

is given by integrating along any path that the charge can take between these two points. Thus:

$$\Delta U = U_B - U_A = -W_{AB} = -q \int_A^B \vec{E} \bullet d\vec{s} \tag{22.3}$$

This integral does not depend on the path taken from A to B because the electric force is conservative.

For convenience, we usually take the *reference configuration* of the charge-field system when the charges are infinitely separated. Moreover, we usually set the corresponding *reference potential energy* to be zero. Therefore, we assume that the charge-field system comes together from an initial infinite separation state at ∞ with $U_\infty = 0$ to a final state B with U_B. We also let $W_{\infty B}$ represent the work done by the electrostatic force during the movement of the charge. Thus:

$$U_B = -W_{\infty B} = -q \int_\infty^B \vec{E} \bullet d\vec{s} \tag{22.4}$$

Although the electric potential energy at a particular point U_B (or simply U) is associated with the *charge-field system*, one can say that the *charge* in the electric field has an electric potential energy at a particular point U_B. You should always keep in mind the fact that the electric potential energy is actually associated with the charge plus the source charge distribution that establishes the electric field $\vec{E}$.

Moreover, when defining the work done on a certain charged particle by the electric field, you are actually defining the work done on that certain particle by the electric force due to all the other charges that actually created the electric field $\vec{E}$ in which that certain particle moved.

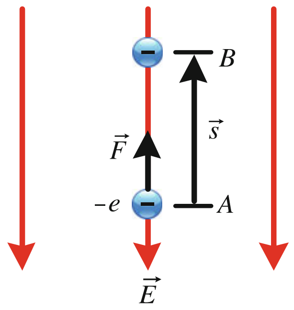
Example 22.1

In the air above a particular region near Earth's surface, the electric field is uniform, directed downwards, and has a value 100 N/C, see Fig. 22.1. Find the change in the electric potential energy of an electron released at point A such that the electrostatic force due to the electric field causes it to move up a distance $s = 50$ m.

Solution: The change in the electric potential energy of the electron is related to the work done on the electron by the electric field given by Eq. 22.3. Since the electron's displacement is upwards and the electric field is directed downwards, i.e. $\theta = 180°$, then we have:

$$\begin{aligned}\Delta U &= -q\int_A^B \vec{E} \bullet d\vec{s} = -(-e)\int_A^B E\,ds\cos 180° \\ &= -eE\int_A^B ds = -eE(s_B - s_A) = -eEs \\ &= -(1.6\times 10^{-19}\,\text{C})(100\,\text{N/C})(50\,\text{m}) \\ &= -8\times 10^{-16}\,\text{J}\end{aligned}$$

Fig. 22.1



Notice that the sign of the electron's charge is used in this calculation. Thus, during the 50 m ascent of the electron, the electric potential of the electron decreases by 8×10^{-16} J. Also, from Eq. 22.3, the work done by the electrostatic force on the electron is:

$$W_{AB} = -\Delta U = 8\times 10^{-16}\,\text{J}$$

22.2 Electric Potential

The electric potential energy depends on the charge q. However, the electric potential energy per unit charge U/q has a unique value at any point and depends only on the electric field (or alternatively on the source-charge distribution). This quantity is called the **electric potential** V (or simply the **potential**). Thus:

$$V = \frac{U}{q} \tag{22.5}$$

This equation implies that the electric potential is a scalar quantity.

Spotlight
Electric potential is a scalar quantity that characterizes an electric field and is independent of any charge that may be placed in the field.

The **electric potential difference** ΔV (or simply the **potential difference**) between two points A and B in an electric field is defined as the difference in the electric potential energy per unit charge between the two points. Thus, dividing Eq. 22.3 by q leads to:

$$\Delta V = V_B - V_A = \frac{\Delta U}{q} = -\frac{W_{AB}}{q} = -\int_A^B \vec{E} \bullet d\vec{s} \tag{22.6}$$

It is clear that the electric potential difference between A and B depends only on the source-charge distribution, and is equal to the negative of the work done by the electrostatic force per unit charge. The SI unit of both the electric potential and the electric potential difference is joule per coulomb, or **volt** (abbreviated by V). That is:

$$1\,\mathrm{V} = 1\,\mathrm{J/C} \tag{22.7}$$

From this unit, we see that the joule is one coulomb times 1 V (J = CV). Additionally, the volt unit allows us to adopt a more convenient unit for the electric field. From Eq. 22.6, the volt unit also has the units of electric field times distance. Then we have:

$$1\,\mathrm{N/C} = 1\,\mathrm{V/m} \tag{22.8}$$

Spotlight
Electric field can be expressed as the rate of change of electric potential with position.

From now on, we shall express values of electric fields in volts per meter (V/m) rather than newtons per coulomb (N/C).

Suppose we move a particle of an arbitrary charge q from point A to point B by applying a force on it and changing its kinetic energy. The applied force must perform work W_{AB}(app), while the electric field does work W_{AB}. From Eq. 6.54, the change in the kinetic energy of the particle is:

$$\Delta K = W_{AB}(\text{app}) + W_{AB} \tag{22.9}$$

Now suppose the kinetic energy of the particle does not change during the move. Then, Eq. 22.9 reduces to:

$$W_{AB}(\text{app}) = -W_{AB} \tag{22.10}$$

Since Eq. 22.6 gives $q\Delta V = -W_{AB}$, then we have:

$$\boxed{W_{AB}(\text{app}) = q\Delta V} \tag{22.11}$$

In atomic and nuclear physics, we use a convenient unit of energy called **electron-volt** (eV). One electron-volt (1 eV) is the energy equal to the work required to move a single elementary charge e, such as that of an electron or a proton, through an electric potential difference of exactly one volt (1 V). To find the value of this unit, one can use Eq. 22.11 where $q = 1.6 \times 10^{-19}$ C and $\Delta V = 1$ V, Thus:

$$\boxed{1\,\text{eV} = (1.6 \times 10^{-19}\,\text{C})(1\,\text{V}) = (1.6 \times 10^{-19}\,\text{C})(1\,\text{J/C}) = 1.6 \times 10^{-19}\,\text{J}} \tag{22.12}$$

An electron that strikes the screen of a typical TV set may have a speed of 4×10^7 m/s. This corresponds to a kinetic energy of 7.28×10^{-16} J, which is equivalent to 4.55 keV. In other words, such an electron has to be accelerated from rest through a potential difference of 4.55 kV to reach a speed of 4×10^7 m/s.

Equations 22.3 and 22.6 hold true for both discrete and continuous source distributions, and for both uniform and varying fields. In the following sections, we calculate the electric potential for various cases.

22.3 Electric Potential in a Uniform Electric Field

Displacement Parallel to a Field

Let us calculate the potential difference between two points A and B separated by a distance $|\vec{s}| = d$, where $\vec{s}$ is a displacement along a uniform field $\vec{E}$, see Fig. 22.2. Then, Eq. 22.6 gives:

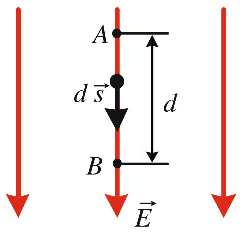


Fig. 22.2 A uniform electric field pointing downwards and $d\vec{s}$ is along $\vec{E}$. The electric potential at point B is lower than that at point A

$$\Delta V = V_B - V_A = -\int_A^B \vec{E} \bullet d\vec{s} = -\int_A^B (E\cos 0°)ds = -E\int_A^B ds \tag{22.13}$$

Integrating ds along a straight line gives $(s_B - s_A) = d$. Thus:

$$\boxed{\Delta V = V_B - V_A = -Ed} \qquad \text{(Along the field)} \tag{22.14}$$

The negative sign means that the electric potential at point B is lower than that at point A, i.e. $V_B < V_A$. In fact, *electric field lines always point in the direction of decreasing electric potential.*

If a charge q moves along the field from point A to point B, then according to Eq. 22.6, the change in the electric potential energy ΔU of the charge-field system is given by $\Delta U = q\Delta V$. Then, substituting with Eq. 22.14, we get the following:

$$\boxed{\Delta U = q\Delta V = -qEd} \qquad \text{(Along the field)} \tag{22.15}$$

(1) If q is *positive* (i.e. $q = +|q|$) and the charge moves in the direction of the field from A to B, then $\Delta U = -|q|Ed$ is *negative*. This means that the *positive* charge-field system *loses* electric potential energy. Also, since $W_{AB} = -\Delta U$, this means that the electric field does *positive* work W_{AB} on the *positive* charge during this motion. Furthermore, if this *positive* charge is released from rest at point A, it accelerates downwards, gaining kinetic energy $\Delta K = W_{AB}$ when it reaches point B. Thus:

$$\boxed{\Delta K = -\Delta U = +|q|Ed} \qquad \text{(Along the field for positive } q) \tag{22.16}$$

(2) If q is *negative* (i.e. $q = -|q|$), and the charge moves in the direction of the field from A to B, then $\Delta U = +|q|Ed$ is *positive*. This means that the *negative* charge-field system *gains* electric potential energy. In order for the *negative* charge to move along the field, an external agent must apply a force and do positive work W_{AB}(app) during that motion. For motion with zero acceleration from A to B,

this positive work compensates the *negative* work W_{AB} done on the *negative* charge by the field.

Similar steps can be performed when the displacement $\vec{s}$ is opposite to the field from point A to point B, see Fig. 22.3, which leads to:

$$\boxed{\Delta V = V_B - V_A = Ed} \qquad \text{(Opposite the field)} \qquad (22.17)$$

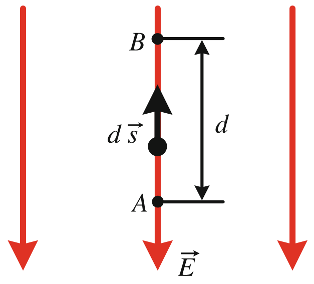


Fig. 22.3 Same as Fig. 22.2 except that $d\vec{s}$ is opposite to $\vec{E}$

If a charge q moves against the field from A to B, then according to Eq. 22.6, the change in the electric potential energy ΔU of the charge-field system is given by $\Delta U = q\Delta V$. By substitution with Eq. 22.17 we get the following:

$$\boxed{\Delta U = q\Delta V = qEd} \qquad \text{(Opposite the field)} \qquad (22.18)$$

(1) If q is *positive* (i.e. $q = +|q|$), and the charge moves against the field from A to B, then $\Delta U = +|q|Ed$ is *positive*. This means that the *positive* charge-field system *gains* electric potential energy. In order for the *positive* charge to move against the field, an external agent must apply a force and do positive work W_{AB}(app) during that motion. For motion with zero acceleration from A to B, this positive work compensates for the negative work W_{AB} done on the *positive* charge by the electric field.

(2) If q is *negative* (i.e. $q = -|q|$) and is released from rest at A, it will accelerate upwards to B. In this case $\Delta V = V_B - V_A = +Ed$, and $\Delta U = -|q|\Delta V = -|q|Ed$ (loss of electric potential energy). Since $W_{AB} = -\Delta U$, then $W_{AB} = +|q|Ed$, i.e. the electric field does *positive* work on the *negative* charge during its motion from A to B. The gain in kinetic energy $\Delta K = W_{AB}$ is thus:

$$\boxed{\Delta K = -\Delta U = +|q|Ed} \qquad \text{(Opposite the field for negative } q) \qquad (22.19)$$

This gain is the same as for a positive charge moving along the field.

Arbitrary Displacement in the Field

Now, let us calculate the potential difference between two points A and B when the displacement $\vec{s}$ has an arbitrary direction with respect to the uniform field $\vec{E}$, see Fig. 22.4a. Equation 22.6 gives:

$$\Delta V = V_B - V_A = -\int_A^B \vec{E}\bullet d\vec{s} = -\vec{E}\bullet \int_A^B d\vec{s} \tag{22.20}$$

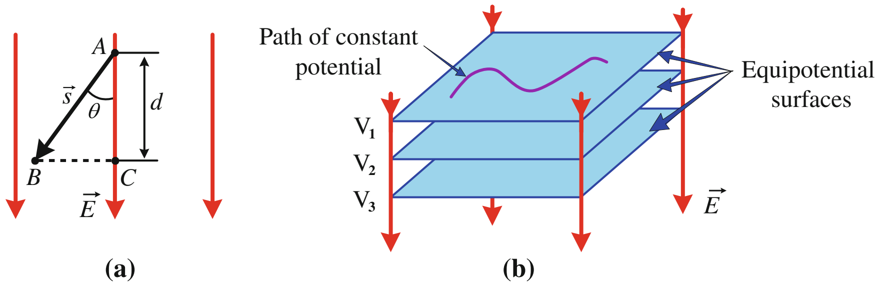


Fig. 22.4 (a) When the uniform electric field lines point down, the electric potential at point B is lower than that at point A. Points B and C are at the same electric potential. (b) Portions of equipotential surfaces that are perpendicular to the electric field $\vec{E}$

Integrating $d\vec{s}$ along the path gives $\vec{s}_B - \vec{s}_A = \vec{s}$. Thus:

$$\boxed{\Delta V = V_B - V_A = -\vec{E}\bullet\vec{s}} \tag{22.21}$$

When a charge q moves from A to B, the change in the electric potential energy ΔU of the charge-field system is given according to Eq. 22.6 by $\Delta U = q\Delta V$. Then, using Eq. 22.21 we get:

$$\boxed{\Delta U = q\Delta V = -q\vec{E}\bullet\vec{s}} \qquad \text{(From } A \text{ to } B\text{)} \tag{22.22}$$

From Fig. 22.4a, we notice that $\vec{E}\bullet\vec{s} = Es\cos\theta = Ed$. Thus, any point in a plane perpendicular to a uniform electric field has the same electric potential. We can see this in Fig. 22.4a, where the potential difference $V_B - V_A$ is equal to the potential difference $V_C - V_A$; that is $V_C = V_B$. All points that have the same electric potential form what is called an **equipotential surface**, see Fig. 22.4b.

Example 22.2

The electric potential difference between two opposing parallel plates is 12 V, while the separation between the plates is $d = 0.4$ cm. Find the magnitude of the electric field between the plates.

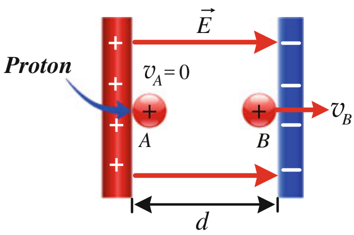
Solution: We use Eq. 22.14, namely $\Delta V = V_B - V_A = -Ed$, to find the magnitude of the uniform electric field as follows:

$$E = \frac{|V_B - V_A|}{d} = \frac{12\ \text{V}}{0.4 \times 10^{-2}\ \text{m}} = 3{,}000\ \text{V/m (or N/C)}$$

Example 22.3

Figure 22.5 shows two oppositely charged parallel plates that are separated by a distance $d = 5$ cm. The electric field between the plates is uniform and has a magnitude $E = 10$ kV/m. A proton is released from rest at the positive plate, see Fig. 22.5. (a) Find the potential difference between the two plates. (b) Find the change in potential energy of the proton just before striking the opposite plate. (c) Calculate the speed of the proton when it strikes the plate.

Fig. 22.5



Solution: (a) Since the potential must be lower in the direction of the field, then $V_A > V_B$ in Fig. 22.5. According to Eq. 22.14 the potential difference between the two plates along the field is:

$$\Delta V = V_B - V_A = -Ed = -(10 \times 10^3\ \text{V/m})(5 \times 10^{-2}\ \text{m}) = -500\ \text{V}$$

(b) Equation 22.15 gives the change in potential energy as:

$$\Delta U = q\Delta V = e\Delta V = (1.6 \times 10^{-19}\ \text{C})(-500\ \text{V}) = -8 \times 10^{-17}\ \text{J}$$

The negative sign indicates that the potential energy of the proton decreases as the proton moves in the direction of the field.

(c) Using Eq. 22.16, we can find for the proton that:

$$\Delta K = -\Delta U$$

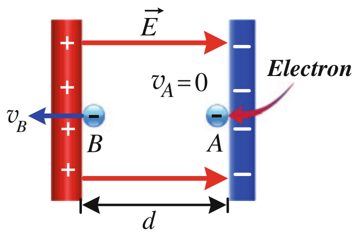
$$\tfrac{1}{2}m_p v_B^2 - 0 = -\Delta U$$

$$v_B = \sqrt{\frac{-2\Delta U}{m_p}} = \sqrt{\frac{-2(-8\times 10^{-17}\,\text{J})}{1.67\times 10^{-27}\,\text{kg}}} = 3.1\times 10^5\ \text{m/s}$$

Example 22.4

Redo Example 22.3, but this time with an *electron* that is released from rest at the *negative* plate, see Fig. 22.6.

Fig. 22.6



Solution: (a) Since the potential must be lower in the direction of the field, then $V_B > V_A$ in Fig. 22.6. Then, according to Eq. 22.17 the potential difference between the two plates is:

$$\Delta V = V_B - V_A = +Ed = +(10\times 10^3\ \text{V/m})(5\times 10^{-2}\ \text{m}) = +500\ \text{V}$$

This means that $V_B = V_A + 500\ \text{V}$ as expected, since point B lies on the positive plate.

(b) From Eq. 22.18, we find for the electron that:

$$\Delta U = q\Delta V = -e\Delta V = -(1.6\times 10^{-19}\ \text{C})(500\ \text{V}) = -8\times 10^{-17}\ \text{J}$$

The negative sign indicates that the potential energy of the electron decreases as the electron moves in the opposite direction of the field.

(c) From Eq. 22.19, we find for the electron that:

$$\Delta K = -\Delta U$$

$$\tfrac{1}{2} m_e v_B^2 - 0 = -\Delta U$$

$$v_B = \sqrt{\frac{-2\Delta U}{m_e}} = \sqrt{\frac{-2(-8 \times 10^{-17}\,\text{J})}{9.11 \times 10^{-31}\,\text{kg}}} = 1.33 \times 10^7\ \text{m/s}$$

Although we are using the same constants of Example 22.3, with the same change in potential energy of the electron as for the proton, the speed of the electron is much greater than the speed of the proton because the electron's mass is much smaller.

22.4 Electric Potential Due to a Point Charge

To find the electric potential at a point located at a distance r from an isolated positive point charge q, we begin with the general expression for potential difference:

$$V_B - V_A = -\int_A^B \vec{E} \bullet d\vec{s} \tag{22.23}$$

where A and B are two points having position vectors $\vec{r}_A$ and $\vec{r}_B$, respectively, see Fig. 22.7. In this representation, the origin is at q. According to Eq. 20.4, the electric field at a distance r from the point charge is $\vec{E} = kq\,\hat{\vec{r}}\,/r^2 = E_r\,\hat{\vec{r}}$, where $\hat{\vec{r}}$ is a unit vector directed from the charge to point P. The quantity $\vec{E} \bullet d\vec{s}$, can be written as:

$$\vec{E} \bullet d\vec{s} = E_r\,\hat{\vec{r}} \bullet d\vec{s} = k\frac{q}{r^2}\,\hat{\vec{r}} \bullet d\vec{s} \tag{22.24}$$

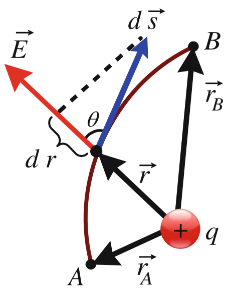


Fig. 22.7 The potential difference between points A and B due to a point charge q depends only on the radial coordinates r_A and r_B

From Fig. 22.7, we see that $\hat{\vec{r}} \bullet d\vec{s} = 1 \times ds \times \cos\theta = dr$. Then, $\vec{E} \bullet d\vec{s} = E_r\,dr$, and Eq. 22.23 can be written as:

$$V_B - V_A = -\int_{r_A}^{r_B} E_r\, dr = -kq \int_{r_A}^{r_B} r^{-2}\, dr = -kq\left[-r^{-1}\right]_{r_A}^{r_B}$$

Therefore:

$$\boxed{V_B - V_A = kq\left[\frac{1}{r_B} - \frac{1}{r_A}\right]} \qquad (22.25)$$

This result does not depend on the path between A and B, but depends only on the initial and final radial coordinates r_A and r_B.

It is common to choose a reference point where $V_A = 0$ at $r_A = \infty$. With this reference choice, the electric potential at any arbitrary distance r from a point charge q will be given by:

$$\boxed{V = k\frac{q}{r}} \qquad (22.26)$$

The sign of V depends on q. If $r \to 0$, then $V \to +\infty$ or $V \to -\infty$ depending on q. Figure 22.8 shows a plot for V in the xy-plane.

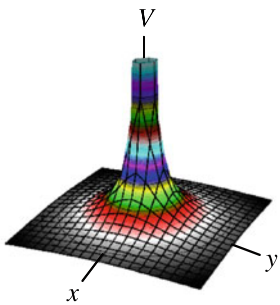


Fig. 22.8 A computer-generated plot for the electric potential $V(r)$ of a single point-charge in the xy-plane. The predicted infinite value of $V(r)$ is not plotted

When many point charges are involved, we can get the resulting electric potential at point P from the superposition principle. That is:

$$\boxed{V = k\sum_n \frac{q_n}{r_n}, \qquad (n = 1, 2, 3, \ldots)} \qquad (22.27)$$

where r_n is the distance from the point P to the charge q_n.

Electric Potential Energy of a System of Point Charges

The electric potential energy of a system of two point charges q_1 and q_2 can be obtained first by having both charges placed at rest and set infinitely apart. Then, by bringing q_1 by itself from infinity and putting it in place as shown in Fig. 22.9a, we do no work for such a move. The electric potential at point P which is at a distance r_{12} from q_1 is given by Eq. 22.26 as $V_P = kq_1/r_{12}$. Later, by bringing q_2 without acceleration from infinity to point P at a distance r_{12} from q_1, as shown in Fig. 22.9b, we must do work $W_{\infty P}(\text{app})$ for such a move, since q_1 exerts an electrostatic force on q_2 during the move.

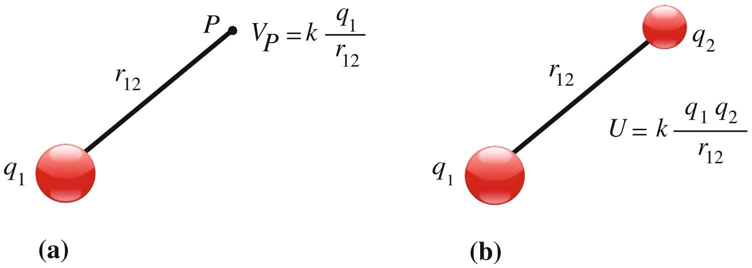


Fig. 22.9 (a) The potential V_P at a distance r_{12} from a point charge q_1. (b) The potential energy of two point charges is $U = kq_1q_2/r_{12}$

We can calculate $W_{\infty P}(\text{app})$ by using $\Delta K = W_{\infty P}(\text{app}) + W_{\infty P} = 0$, i.e. $W_{\infty P}(\text{app}) = -W_{\infty P}$. When we replace q_2 by the general charge q in Eq. 22.6, we find that:

$$V_P - V_\infty = \frac{U_P - U_\infty}{q_2} = -\frac{W_{\infty P}}{q_2} \tag{22.28}$$

Setting V_∞ as well as U_∞ to zero (our reference point at ∞), we get:

$$V_P = \frac{U_P}{q_2} = \frac{W_{\infty P}(\text{app})}{q_2} \quad \Rightarrow \quad U_P = W_{\infty P}(\text{app}) = q_2 V_P \tag{22.29}$$

Substituting with $V_P = kq_1/r_{12}$, we can generalize the electric potential energy of a system of two point charges q_1 and q_2 separated by a distance r_{12} as follows:

$$\boxed{U = k\frac{q_1\,q_2}{r_{12}}} \tag{22.30}$$

If the charges have the same sign, we have to do positive work to overcome their mutual repulsion, and then U is positive. If the charges have opposite signs, we have to do negative work against their mutual attraction to keep them stationary, and then U is negative.

When the system consists of more than two charges, we calculate the potential energy of each pair and add them algebraically. For instance, the total potential energy of three charges q_1, q_2, and q_3 is:

$$\boxed{U = k\left(\frac{q_1\, q_2}{r_{12}} + \frac{q_1\, q_3}{r_{13}} + \frac{q_2\, q_3}{r_{23}}\right)} \tag{22.31}$$

Example 22.5

Two charges $q_1 = 2\,\mu\text{C}$ and $q_2 = -4\,\mu\text{C}$ are fixed in their positions and separated by a distance $d = 10$ cm; see Fig. 22.10a. (a) Find the total electric potential at the point P in Fig. 22.10a. (b) Find the change in potential energy of the two charges when a third charge $q_3 = 6\,\mu\text{C}$ is brought from ∞ to P, see Fig. 22.10b. (c) What is the total electric potential energy of the three charges?

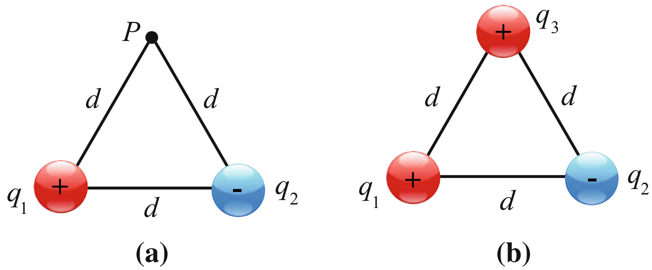


Fig. 22.10

Solution: (a) For two charges, Eq. 22.27 gives:

$$V_P = k\left(\frac{q_1}{d} + \frac{q_2}{d}\right) = (9 \times 10^9\,\text{N.m}^2/\text{C}^2)\left(\frac{2 \times 10^{-6}\,\text{C}}{0.1\,\text{m}} + \frac{-4 \times 10^{-6}\,\text{C}}{0.1\,\text{m}}\right)$$
$$= -1.8 \times 10^5\,\text{V}$$

(b) When we replace q_3 by the charge q of Eq. 22.6, and bring q_3 from infinity to point P, this equation gives:

$$\Delta U = q_3(V_P - V_\infty) = (6 \times 10^{-6}\,\text{C})(-1.8 \times 10^5\,\text{V} - 0) = -1.08\,\text{J}$$

(c) The total electric potential energy of the three charges is:

$$U = k\left(\frac{q_1\, q_2}{d} + \frac{q_1\, q_3}{d} + \frac{q_2\, q_3}{d}\right)$$
$$= (9 \times 10^9\,\mathrm{N.m^2/C^2})\frac{[(2)(-4) + (2)(6) + (-4)(6)] \times 10^{-12}\,\mathrm{C}^2}{0.1\,\mathrm{m}} = -1.8\,\mathrm{J}$$

22.5 Electric Potential Due to a Dipole

As introduced in Chap. 20, an electric dipole consists of a positive charge $q_+ = +q$ and an equal-but-opposite negative charge $q_- = -q$ separated by a distance $2a$. Let us find the electric potential at a point P in the xy-plane as shown in Fig. 22.11a. We assume that V_+ is the electric potential produced at P by the positive charge, and that V_- is the electric potential produced at P by the negative charge.

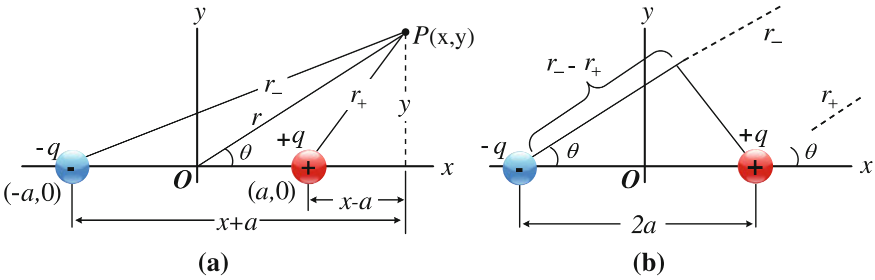


Fig. 22.11 (a) The electric potential V at point $P(x, y)$ due to an electric dipole located along the x-axis with a length $2a$. Point P is at a distance r from the midpoint O of the dipole, where OP makes an angle θ with the dipole's x-axis. (b) When P is very far, the lines of length r_+ and r_- are approximately parallel to the line OP

The total electric potential at P is thus:

$$V = V_+ + V_- = k\frac{q_+}{r_+} + k\frac{q_-}{r_-} = kq\left(\frac{1}{r_+} - \frac{1}{r_-}\right) = kq\left(\frac{r_- - r_+}{r_+\, r_-}\right) \qquad (22.32)$$

From the geometry of Fig. 4.11a, we find that $r_+^2 = (x-a)^2 + y^2$ and $r_-^2 = (x+a)^2 + y^2$. Accordingly, Eq. 22.32 becomes:

$$\boxed{V = kq\left(\frac{1}{\sqrt{(x-a)^2 + y^2}} - \frac{1}{\sqrt{(x+a)^2 + y^2}}\right)} \qquad (22.33)$$

Note that $V = \infty$ at $P(x = a,\ y = 0)$ and $V = -\infty$ at $P(x = -a,\ y = 0)$. Figure 22.12 shows a plot of the general shape of V in the xy-plane.

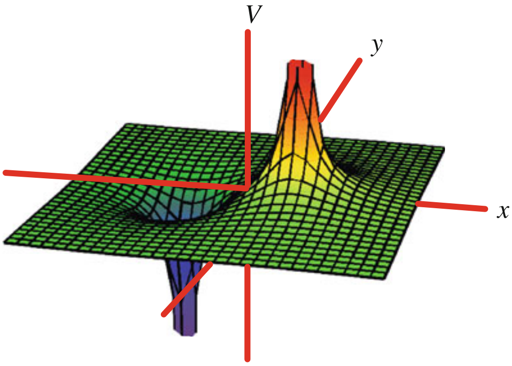


Fig. 22.12 A computer-generated plot of the electric potential V in the xy-plane for an electric dipole. The predicted infinite values of V are not plotted

Because naturally occurring dipoles have very small lengths, such as those possessed by many molecules, we are usually interested only in points far away from the dipole, i.e. $r \gg 2a$. Considering these conditions, we find from Fig. 22.11b that:

$$r_- - r_+ \simeq 2a\cos\theta, \qquad \text{and} \qquad r_-\, r_+ \simeq r^2 \tag{22.34}$$

When substituting these approximate quantities in Eq. 22.32, we find that:

$$\boxed{V = kq\frac{2a\cos\theta}{r^2}} \quad \boxed{(r \gg 2a)} \tag{22.35}$$

As introduced in Chap. 20, the product of the positive charge q and the length of the dipole $2a$ is called the magnitude of the *electric dipole moment*, $p = 2aq$. The direction of $\vec{p}$ is taken to be from the negative charge to the positive charge of the dipole, i.e. $\vec{p} = p\,\vec{\mathrm{i}}$. This indicates that the angle θ is measured from the direction of $\vec{p}$. Using this definition, we have:

$$\boxed{V = k\frac{p\cos\theta}{r^2}} \quad \boxed{(r \gg 2a)} \tag{22.36}$$

Example 22.6

Find the electric potential along the axis of the electric dipole at the four points A, B, C, and D in Fig. 22.13.

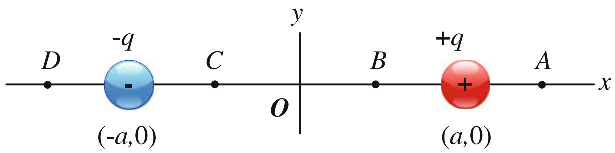


Fig. 22.13

Solution: We use Eq. 22.32 with r_+ and r_- as the distance from each point to the positive and negative charges, respectively:

(1) For point A in Fig. 22.13, we have $x > a$. Therefore, $r_+ = x - a$ and $r_- = a + x$. The electric potential V_A is:

$$V_A = kq\left(\frac{1}{r_+} - \frac{1}{r_-}\right) = kq\left(\frac{1}{x-a} - \frac{1}{a+x}\right) = \frac{2k\,qa}{x^2 - a^2} \qquad (V_A \text{ positive})$$
$$\simeq \frac{2k\,qa}{x^2} \qquad (x \gg a)$$

(2) For point B in Fig. 22.13, we have $0 < x < a$. Therefore $r_+ = a - x$ and $r_- = a + x$. The electric potential V_B is:

$$V_B = kq\left(\frac{1}{r_+} - \frac{1}{r_-}\right) = kq\left(\frac{1}{a-x} - \frac{1}{a+x}\right) = \frac{2kq}{a^2 - x^2}x \qquad (V_B \text{ positive})$$

(3) For point C in Fig. 22.13, we have $-a < x < 0$. Therefore $r_+ = a - x$ and $r_- = a + x$. The electric potential V_C is:

$$V_C = kq\left(\frac{1}{r_+} - \frac{1}{r_-}\right) = kq\left(\frac{1}{a-x} - \frac{1}{a+x}\right) = \frac{2kq}{a^2 - x^2}x \qquad (V_C \text{ negative})$$

(4) For point D in Fig. 22.13, we have $x < -a$. Therefore $r_+ = a - x$ and $r_- = -x - a$. The electric potential V_D is:

$$V_D = kq\left(\frac{1}{r_+} - \frac{1}{r_-}\right) = kq\left(\frac{1}{a-x} + \frac{1}{a+x}\right) = -\frac{2kqa}{x^2 - a^2} \qquad (V_D \text{ negative})$$
$$\simeq -\frac{2kqa}{x^2} \qquad (x \ll -a)$$

22.6 Electric Dipole in an External Electric Field

Consider an electric dipole of electric dipole moment $\vec{p}$ is placed in a uniform external electric field $\vec{E}$, as shown in Fig. 22.14. Do not get confused between the field produced by the dipole and this external field. In addition, we assume that the vector $\vec{p}$ makes an angle θ with the external field $\vec{E}$.

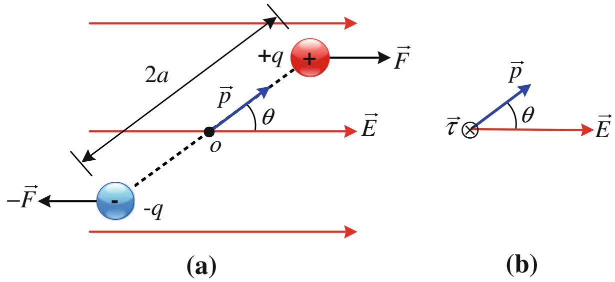


Fig. 22.14 (a) An electric dipole has an electric dipole moment $\vec{p}$ in an external uniform electric field $\vec{E}$. The angle between $\vec{p}$ and $\vec{E}$ is θ. The line connecting the two charges represents their rigid connection and their center of mass is assumed to be midway between them. (b) Representing the electric dipole by a vector $\vec{p}$ in the external electric field $\vec{E}$ and showing the direction of the torque $\vec{\tau}$ into the page by the symbol $\otimes$

Figure 22.14 shows a force $\vec{F}$, of magnitude qE in the direction of $\vec{E}$, is exerted on the positive charge, and a force $-\vec{F}$, of the same magnitude but in opposite direction, is exerted on the negative charge. The resultant force on the dipole is zero, but since the two forces do not have the same line of action, they establish a clockwise torque $\vec{\tau}$ about the center of mass of the two charges at o. The magnitude of this torque about o is:

$$\tau = (2a \sin\theta)F \tag{22.37}$$

Using $F = qE$ and $p = 2aq$, we can write this torque as:

$$\tau = (2aq \sin\theta)E = pE \sin\theta \tag{22.38}$$

The vector torque $\vec{\tau}$ on the dipole is therefore the cross product of the vectors $\vec{p}$ and $\vec{E}$. Thus:

$$\boxed{\vec{\tau} = \vec{p} \times \vec{E}} \tag{22.39}$$

The effect of this torque is to rotate the dipole until the dipole moment $\vec{p}$ is aligned with the electric field $\vec{E}$.

Potential energy can be associated with the orientation of an electric dipole in an electric field. The dipole has the least potential energy when it is in the equilibrium orientation, which occurs when $\vec{p}$ is along $\vec{E}$. On the other hand, the dipole has the greatest potential energy when $\vec{p}$ is antiparallel to $\vec{E}$. We chose the zero-potential-energy configuration when the angle between $\vec{p}$ and $\vec{E}$ is 90°.

According to Eq. 22.3, $\Delta U = U_B - U_A = -W_{AB}$, we can find the electric potential energy of the dipole by calculating the work done by the field from the initial orientation $\theta = 90°$, where $U_A \equiv U(90°) = 0$, to any orientation θ, where $U_B \equiv U(\theta)$. In addition, we use the relation $W = \int \tau \, d\theta$, to find $U(\theta)$ as follows:

$$U(\theta) - U(90°) = -W_{90° \to \theta} = -\int_{90°}^{\theta} \tau \, d\theta \tag{22.40}$$

Letting $U(\theta) \equiv U$, $U(90°) = 0$, $\tau = pE \sin\theta$, and integrating we get:

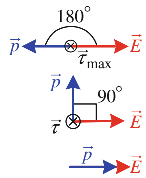
$$U = -pE \cos\theta \tag{22.41}$$

This relation can be written in vector form as follows:

$$U = -\vec{p} \bullet \vec{E} \tag{22.42}$$

Equation 22.42 shows the least and greatest value of U as follows:

$$U = -\vec{p} \bullet \vec{E} = -pE\cos\theta = \begin{cases} +pE & \text{if } \theta = 180° \\ 0 & \text{if } \theta = 90° \\ -pE & \text{if } \theta = 0° \end{cases}$$



22.7 Electric Potential Due to a Charged Rod

For a Point on the Extension of the Rod

Figure 22.15 shows a rod of length L with a uniform positive charge density λ and a total charge Q. In this figure, the rod lies along the x-axis and point P is taken to be at the origin of this axis, located at a distance a from the left end. When we consider a segment dx on the rod, the charge on this segment will be $dq = \lambda \, dx$.

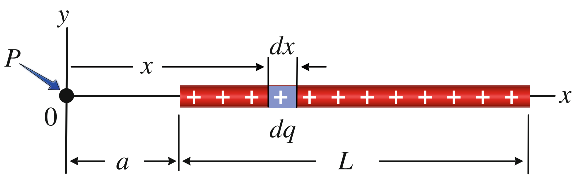


Fig. 22.15 The electric potential V at point P due to a uniformly charged rod lying along the x-axis. The electric potential due to a segment of charge dq at a distance x from P is $k\,dq/x$. The total electric potential is the algebraic sum of all the segments of the rod

The electric potential dV at P due to this segment is given by:

$$dV = k\frac{dq}{x} = k\frac{\lambda\,dx}{x} \tag{22.43}$$

We obtain the total electric potential at P due to all the segments of the rod by integrating from one end of the rod ($x = a$) to the other ($x = a + L$) as follows:

$$V = \int dV = \int_a^{a+L} k\frac{\lambda\,dx}{x} = k\lambda \int_a^{a+L} x^{-1}\,dx = k\lambda |\ln x|_a^{a+L} = k\lambda\,\{\ln(a+L) - \ln a\}$$

Therefore:

$$\boxed{V = k\lambda \ln\left(\frac{a+L}{a}\right) = \frac{kQ}{L}\ln\left(\frac{a+L}{a}\right)} \tag{22.44}$$

For a Point on the Perpendicular Bisector of the Rod

A rod of length L has a uniform positive charge density λ and a total charge Q. The rod is placed along the x-axis as shown in Fig. 22.16. Assuming that P is a point on the perpendicular bisector of the rod and is located a distance a from the origin of the x-axis, then the charge of a segment dx on the rod will be $dq = \lambda\,dx$.

The electric potential dV at P due to this segment is:

$$dV = k\frac{dq}{r} = k\frac{\lambda\,dx}{r} \tag{22.45}$$

The total electric potential at P due to all segments of the rod is given by two times the integral of dV from the middle of the rod ($x = 0$) to one of its ends ($x = L/2$). Thus:

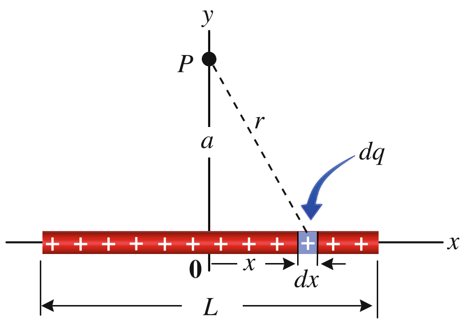


Fig. 22.16 A rod of length L with a uniform positive charge density λ and an electric potential dV at point P due to a charge segment

$$V = 2\int_{x=0}^{x=L/2} dV = 2k\lambda \int_{0}^{L/2} \frac{dx}{r} \tag{22.46}$$

To perform the integration of this expression, we relate the variables x and r. From the geometry of Fig. 22.16, we use the fact that $r = \sqrt{x^2 + a^2}$. Therefore, Eq. 22.46 becomes:

$$V = 2k\lambda \int_{0}^{L/2} \frac{dx}{(x^2 + a^2)^{1/2}} \tag{22.47}$$

From the table of integrals in Appendix B, we find that:

$$\int \frac{dx}{(x^2 + a^2)^{1/2}} = \ln(x + \sqrt{x^2 + a^2}) \tag{22.48}$$

Thus:

$$\begin{aligned} V &= 2k\lambda \left| \ln(x + \sqrt{x^2 + a^2}) \right|_{0}^{L/2} \\ &= 2k\lambda \left[\ln(L/2 + \sqrt{(L/2)^2 + a^2}) - \ln(a) \right] \end{aligned}$$

Therefore:

$$\boxed{V = 2k\lambda \ln\left(\frac{L/2 + \sqrt{(L/2)^2 + a^2}}{a} \right)} \tag{22.49}$$

When we use the fact that the total charge $Q = \lambda L$, we get:

$$\boxed{V = \frac{2kQ}{L} \ln\left(\frac{L/2 + \sqrt{(L/2)^2 + a^2}}{a} \right)} \tag{22.50}$$

For a Point Above One End of the Rod

When a point P is located at a distance a from one of the rod's ends, see Fig. 22.17, we can perform similar calculations to find that:

$$\boxed{V = \frac{kQ}{L} \ln\left(\frac{L + \sqrt{L^2 + a^2}}{a}\right)} \tag{22.51}$$

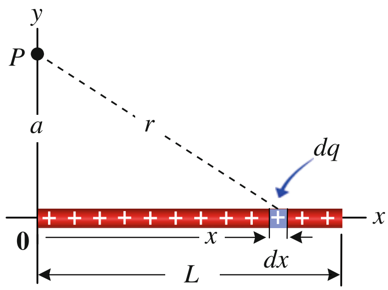


Fig. 22.17 A setup similar to Fig. 22.16 except P is above one end

22.8 Electric Potential Due to a Uniformly Charged Arc

Assume that a rod has a uniformly distributed total positive charge Q. Assume now that the rod is bent into an arc of radius R and central angle ϕ rad, see Fig. 22.18a. To find the electric potential at the center P of this arc, we first let λ represent the linear charge density of this arc, which has a length $R\phi$. Thus:

$$\lambda = \frac{Q}{R\phi} \tag{22.52}$$

For an arc element ds subtending an angle $d\theta$ at P, we have:

$$ds = R\, d\theta \tag{22.53}$$

Therefore, the charge dq on this arc element will be given by:

$$dq = \lambda\, ds = \lambda\, R\, d\theta \tag{22.54}$$

To find the electric field at P, we first calculate the differential electric potential dV at P due to the element ds of charge dq, see Fig. 22.18b, as follows:

$$dV = k\frac{dq}{R} = k\lambda\, d\theta \tag{22.55}$$

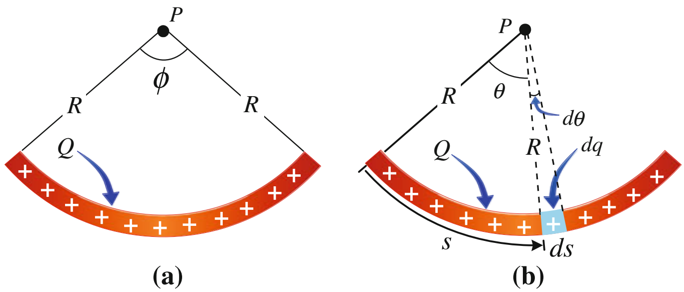


Fig. 22.18 (a) A circular arc of radius R, central angle ϕ, and center P has a uniformly distributed positive charge Q. (b) The figure shows how to calculate the electric potential dV at P due to an arc element ds having a charge dq

The total electric potential at P due to all elements of the arc is thus:

$$V = \int dV = k\lambda \int_0^{\phi} d\theta = k\lambda\phi$$

Using Eq. 22.52, we get:

$$\boxed{V = k\lambda\phi = k\frac{Q}{R}} \qquad (22.56)$$

This expression is identical to the formula of a point charge. The reason for this is that the distance between P and each charge element on the arc does not change and its orientation is irrelevant.

22.9 Electric Potential Due to a Uniformly Charged Ring

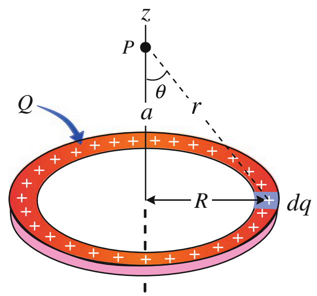
Assume that a ring of radius R has a uniformly distributed total positive charge Q, see Fig. 22.19. Additionally, assume that a point P lies at a distance a from the center of the ring along its central perpendicular axis, as shown in the same figure.

To find the electric potential at P, we first calculate the electric potential dV at P due to a segment of charge dq as follows:

$$dV = k\frac{dq}{r} \qquad (22.57)$$

where $r = \sqrt{R^2 + a^2}$ is a constant distance for all elements on the ring.

Fig. 22.19 A ring of radius R having a uniformly distributed positive charge Q. The figure shows how to calculate the electric potential dV at an axial point P due to a segment of charge dq on the ring



Thus, the total electric potential at P is:

$$V = \int dV = \int \frac{k\,dq}{\sqrt{R^2 + a^2}} = \frac{k}{\sqrt{R^2 + a^2}} \int dq \tag{22.58}$$

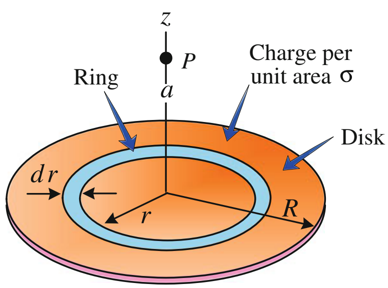
Since $\int dq$ represents the total charge Q over the entire ring, then the total electric potential at P will be given by:

$$\boxed{V = \frac{kQ}{\sqrt{R^2 + a^2}}} \tag{22.59}$$

22.10 Electric Potential Due to a Uniformly Charged Disk

Assume that a disk of radius R has a uniform positive surface charge density σ, and a point P lies at a distance a from the disk along its central perpendicular axis, see Fig. 22.20.

Fig. 22.20 A disk of radius R has a uniform positive surface charge density σ. The ring shown has a radius r and a radial width dr



To find the electric potential at P, we divide the disk into concentric rings, then calculate the electric potential at P for each ring by using Eq. 22.59, and summing up the contribution of all the rings.

Figure 22.20 shows one such ring, with radius r, radial width dr, and surface area $dA = 2\pi\, r dr$. Since σ is the charge per unit area, then the charge dq on this ring is:

$$dq = \sigma\, dA = 2\pi\, r\sigma\, dr \tag{22.60}$$

Using this relation in Eq. 22.59, and replacing V with dV, R with r, and Q with $dq = 2\pi\, r\sigma\, dr$, we can calculate the electric potential resulting from this ring as follows:

$$dV = \frac{k}{\sqrt{r^2 + a^2}}\,(2\pi\, r\sigma\, dr) = \pi\, k\sigma\, \frac{2rdr}{\sqrt{r^2 + a^2}} \tag{22.61}$$

To find the total electric potential, we integrate this expression with respect to the variable r from $r = 0$ to $r = R$. This gives:

$$V = \int dV = \pi\, k\sigma \int_0^R (r^2 + a^2)^{-1/2}(2r\, dr) \tag{22.62}$$

To solve this integral, we transform it to the form $\int u^n du = u^{n+1}/(n+1)$ by setting $u = r^2 + a^2$, and $du = 2r\, dr$. Thus, Eq. 22.62 becomes:

$$\begin{aligned} V &= \pi\, k\sigma \int_0^R (r^2 + a^2)^{-1/2}(2r\, dr) = \pi\, k\sigma \int_{u=a^2}^{u=R^2+a^2} u^{-1/2} du \\ &= \pi\, k\sigma \left| \frac{u^{1/2}}{1/2} \right|_{u=a^2}^{u=R^2+a^2} = \pi\, k\sigma \left[\frac{(R^2 + a^2)^{1/2}}{1/2} - \frac{a}{1/2} \right] \end{aligned} \tag{22.63}$$

Rearranging the terms, we find that:

$$\boxed{V = 2\pi\, k\sigma \left[\sqrt{R^2 + a^2} - a \right]} \tag{22.64}$$

Using $k = 1/4\pi\epsilon_\circ$, where $\epsilon_\circ$ is the permittivity of free space, it is sometimes preferable to write this relation as:

$$\boxed{V = \frac{\sigma}{2\epsilon_\circ} \left[\sqrt{R^2 + a^2} - a \right]} \tag{22.65}$$

22.11 Electric Potential Due to a Uniformly Charged Sphere

A solid sphere of radius R has a uniform volume charge density ρ and carries a total positive charge Q. First, we find the electric potential in the region $r \geq R$ by using the electric field obtained in Example 21.9. In this region, we found that E is radial and has a magnitude:

$$E_r = k\frac{Q}{r^2} \qquad (r \geq R) \tag{22.66}$$

This is the same as the electric field due to a point charge, and hence the electric potential at any point of radius r in this region is given by:

$$\boxed{V_r = k\frac{Q}{r} \qquad (r \geq R)} \tag{22.67}$$

The potential at a point on the surface of the sphere ($r = R$) is:

$$V_R = k\frac{Q}{R} \tag{22.68}$$

In the region $0 \leq r \leq R$ inside the sphere, we use the result of the electric field obtained in Example 21.9. In this region, we found that $\vec{E}$ is radial and has a magnitude:

$$E_r = k\frac{Q}{R^3}r \qquad (0 \leq r \leq R) \tag{22.69}$$

For a point that has a radius r in the region $0 \leq r \leq R$, we can find the potential difference between this point and any point on the surface with a radius R by using $\vec{E} \bullet d\vec{s} = E_r\, dr$ in Eq. 22.6. Thus:

$$\begin{aligned} V_r - V_R &= -\int_R^r \vec{E} \bullet d\vec{s} = -\int_R^r E_r\, dr = -k\frac{Q}{R^3}\int_R^r r\, dr \\ &= -k\frac{Q}{R^3}\left|\frac{r^2}{2}\right|_{r=R}^{r=r} = k\frac{Q}{2R^3}(R^2 - r^2) \end{aligned} \tag{22.70}$$

Using $V_R = kQ/R$ in the last result we reach to the following relation:

$$\boxed{V_r = k\frac{Q}{2R}\left(3 - \frac{r^2}{R^2}\right) \qquad (0 \leq r \leq R)} \tag{22.71}$$

At $r = 0$, we have $V_0 = 3kQ/2R$, and at $r = R$, we get $V_R = kQ/R$ as expected. Figure 22.21 sketches the electric potential in the two regions $0 \leq r \leq R$ and $r \geq R$.

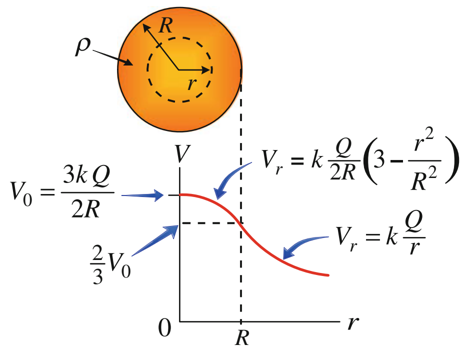


Fig. 22.21 A sketch of the electric potential $V(r)$ as a function of r in the two regions $0 \le r \le R$ and $r \ge R$. The *curve* for the region $0 \le r \le R$ is parabolic and joins smoothly with the *curve* for the region $r \ge R$, which is hyperbola

22.12 Electric Potential Due to a Charged Conductor

Assume that a solid conducting sphere of radius R carries a net positive charge Q as shown in Fig. 22.22a. We found in Chap. 21 that the charge on this equilibrium conductor must reside on its outer surface. Furthermore, the electric field inside the conductor is zero and the electric field just outside its surface is perpendicular to the surface.

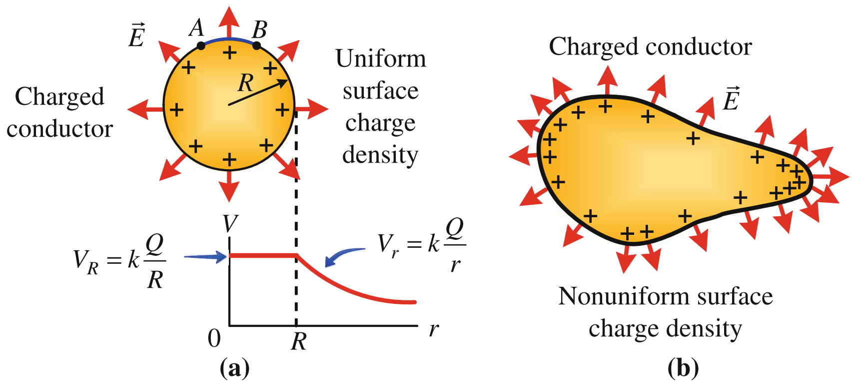


Fig. 22.22 (a) A sketch of the electric potential $V(r)$ as a function of r in the two regions $0 \le r \le R$ and $r \ge R$ for a charged spherical conductor. When $r = R$, the formulas in the regions match. (b) For a nonsymmetrical conductor, the surface charge density is greatest at the points where the radius of curvature of the surface is least

Consider two arbitrary points A and B on the surface of this spherical conductor, see Fig. 22.22a. Since $\vec{E} \perp d\vec{s}$ along the path AB on the surface of that conductor, then $\vec{E} \bullet d\vec{s} = 0$. Using this result along with Eq. 22.6, we find that:

$$V_B - V_A = -\int_A^B \vec{E} \bullet d\vec{s} = 0 \tag{22.72}$$

During equilibrium, the electric potential V is constant everywhere on the surface of this charged spherical conductor and equal to $V_R = kQ/R$, or $V_R = 4\pi kR\sigma$ in terms of the surface charge density σ.

Furthermore, because the electric field is zero inside the conductor, the electric potential would be constant everywhere inside the conductor and is equal to its value at the surface.

Outside this spherical conductor, the electric potential is $V_r = kQ/r$ for $r \geq R$. Figure 22.22a plots V against r and shows the dependence of $V(r)$ on r for the whole range of r.

If the conductor is not symmetric as in Fig. 22.22b, the electric potential is constant everywhere on its surface, but the surface charge density is not uniform. Since $V = \text{const.}$ and $V \propto R\sigma$, i.e. $R\sigma = \text{const.}$, the surface charge density increases as the radius of curvature decreases.

22.13 Potential Gradient

We defined the potential difference between two points A and B as the negative of the work done by the electric field $\vec{E}$ per unit charge in moving the charge from A to B, see Eq. 22.6. Thus:

$$V_B - V_A = -\int_A^B \vec{E} \bullet d\vec{s} \tag{22.73}$$

If we write $V_B - V_A = \int_A^B dV = -\int_A^B \vec{E} \bullet d\vec{s}$, then we must have:

$$\boxed{dV = -\vec{E} \bullet d\vec{s}} \tag{22.74}$$

If the electric field has only one component E_x along the x-axis, then $\vec{E} \bullet d\vec{s} = E_x\, dx$. The last equation becomes $dV = -E_x\, dx$, or:

$$\boxed{E_x = -\frac{dV}{dx}} \tag{22.75}$$

Thus, the x component of the electric field is equal to the negative of the derivative of the electric potential with respect to x.

If the field is radial, i.e. $V = V(r)$ as introduced in Sect. 22.4, then $\vec{E} \bullet d\vec{s} = E_r\,dr$ and we can express Eq. 22.74 as:

$$\boxed{E_r = -\frac{dV}{dr}} \tag{22.76}$$

Generally, $\vec{E} = E_x\,\vec{\mathrm{i}} + E_y\,\vec{\mathrm{j}} + E_z\,\vec{\mathrm{k}}$ and $d\vec{s} = dx\,\vec{\mathrm{i}} + dy\,\vec{\mathrm{j}} + dz\,\vec{\mathrm{k}}$. Then:

$$dV = -\vec{E} \bullet d\vec{s} = -E_x\,dx - E_y\,dy - E_z\,dz \tag{22.77}$$

When $V = V(x, y, z)$, the chain rule of differentiation gives:

$$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz \tag{22.78}$$

By comparing the last two equations, we get the potential gradients:

$$\boxed{E_x = -\frac{\partial V}{\partial x}} \quad \boxed{E_y = -\frac{\partial V}{\partial y}} \quad \boxed{E_z = -\frac{\partial V}{\partial z}} \tag{22.79}$$

Example 22.7

From the formulas of the electric potential given by Eqs. 22.26, 22.44, 22.59, and 22.65, find the formulas of the electric fields.

Solution: To get the electric field from the electric potential, we use Eqs. 22.76 or 22.79 depending on the system coordinates.

(1) From the point-charge formula given by Eq. 22.26, we have a radial electric field. Thus:

$$E_r = -\frac{dV}{dr} = -\frac{d}{dr}\left(k\frac{q}{r}\right) = -kq\frac{dr^{-1}}{dr} = k\frac{q}{r^2} \qquad \text{(Identical to Eq. 20.4)}$$

(2) From the charged-rod formula given by Eq. 22.44, our variable is the distance a from the end of the rod. Thus:

$$\begin{aligned} E_a &= -\frac{dV}{da} = -\frac{d}{da}\left\{k\lambda \ln\left(\frac{a+L}{a}\right)\right\} = -k\lambda\frac{d}{da}\{\ln(a+L) - \ln a\} \\ &= -k\lambda\left\{\frac{1}{a+L} - \frac{1}{a}\right\} = \frac{k\lambda L}{a(a+L)} \qquad \text{(Identical to Eq. 20.26)} \end{aligned}$$

(3) From the charged-ring formula given by Eq. 22.59, our variable is the distance a from the center of the ring. Thus:

$$E_a = -\frac{dV}{da} = -\frac{d}{da}\left\{\frac{kQ}{\sqrt{R^2+a^2}}\right\} = -kQ(-\tfrac{1}{2})(R^2+a^2)^{-3/2}(2a)$$
$$= \frac{kQa}{(R^2+a^2)^{3/2}} \qquad \text{(Identical to Eq. 20.50)}$$

(4) From the charged-disk formula given by Eq. 22.65, our variable is the distance a from the center of the disk. Thus:

$$E_a = -\frac{dV}{da} = -\frac{d}{da}\left\{\frac{\sigma}{2\epsilon_\circ}\left[\sqrt{R^2+a^2}-a\right]\right\} = -\frac{\sigma}{2\epsilon_\circ}\left[\left(\tfrac{1}{2}\right)(R^2+a^2)^{-1/2}(2a)-1\right]$$
$$= \frac{\sigma}{2\epsilon_\circ}\left[1-\frac{a}{\sqrt{R^2+a^2}}\right] \qquad \text{(Identical to Eq. 20.56)}$$

The expressions that we have arrived at for the electric potentials established by simple charge distributions are presented in Table 22.1.

Table 22.1 Electric potential due to simple charge distributions

Charge distribution	Electric potential
Two oppositely charged conducting plates separated by a distance d	$\Delta V = V_B - V_A = -Ed$ Along the field $\Delta V = V_B - V_A = Ed$ Opposite the field
Single point charge q	$V = k\dfrac{q}{r} \qquad r > 0$
Charged ring of radius R with a uniformly distributed total charge Q	$V = \dfrac{kQ}{\sqrt{R^2+a^2}} \qquad a \ge 0$
Disk of radius R having a uniform surface charge density σ	$V = \dfrac{\sigma}{2\epsilon_\circ}\left[\sqrt{R^2+a^2}-a\right] \qquad a > 0$
Charge q uniformly distributed on the surface of a conducting sphere of radius R	$V = \begin{cases} k\dfrac{q}{r} & r \ge R \\ k\dfrac{q}{R} & r \le R \end{cases}$

22.14 The Electrostatic Precipitator

Electrostatic precipitators are highly efficient filtration devices used to remove particles from a flowing gas (such as air). They do this using the force of an induced electrostatic charge. Such devices remove particulate matter from combustion gases, and as a result reduce air pollution. Most precipitators on the market today are capable of eliminating more than 99% of the ash from smoke.

A schematic diagram of an electrostatic precipitator is shown in Fig. 22.23. Applied between the central wire and the duct walls, where smoke is flowing up the duct, is a large voltage of several thousand volts (50–100 kV). To generate an electric field that is directed toward the wire, the wire is maintained at a negative electric potential with respect to the walls. Such a large electric field produces a discharge around the wire, which causes the air near the wire to contain electrons, positive ions, and negative ions such as O_2^-.

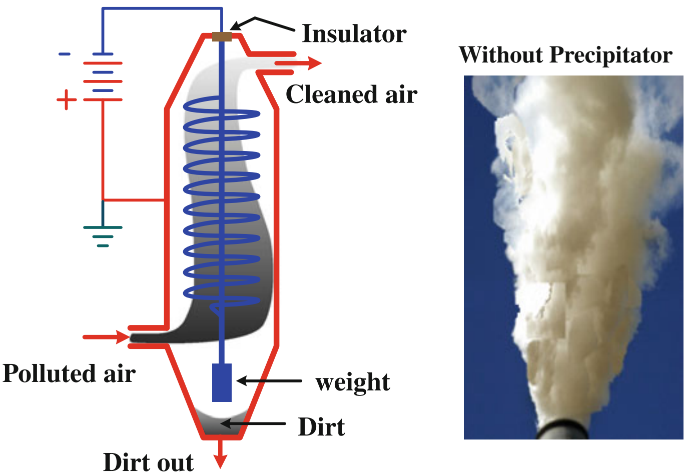


Fig. 22.23 (a) A schematic diagram of an electrostatic precipitator. The high negative electric potential on the central wire creates a discharge in the vicinity of the wire, causing dirt to fall down. (b) Pollution from a power-plant's chimney not equipped with an electrostatic precipitator

Polluted air enters the duct from the bottom and moves near the coiled wire. The discharge creates electrons and negative ions, which accelerate toward the outer wall due to the force of the electric field. Consequently, the dirt particles become charged by collisions and ion capture. Because most of the charged dirt particles are negative, they are drawn to the walls by the electric field. Periodic shaking of the duct loosens the particles, which are then collected at the bottom.

22.15 The Van de Graaff Generator

When a charged conductor is connected to the inside of a hollow conductor, all the charge is *transferred* to the outer surface of the hollow conductor regardless of any charge already retained by the conductor. The generator invented by Robert Van de Graaff makes use of this principle, where a “conveyor belt” carries out the charge continuously, see the schematic diagram of Fig. 22.24a.

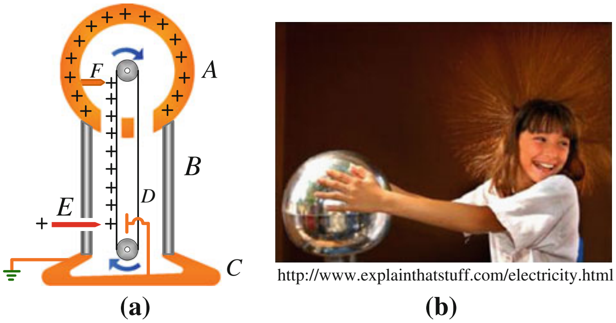


Fig. 22.24 (a) The charge in the Van de Graaff generator is deposited at E and transferred to the dome at F. (b) By touching the dome, each hair strand becomes charged and repels strands around it

This generator consists of a hollow metallic dome A supported by an insulating stand B mounted on a grounded metal base C and a non-conducting belt D running over two non-conducting pulleys. The belt is charged as a result of the discharge produced by the metallic needle at E, which is maintained at a positive electric potential of about 10^4 V. The positive charge on the moving belt is transferred to the dome by the needle at F, regardless of the dome’s electric potential. It is possible to increase the potential of the dome until electrical ionization occurs in the air. Since the ionization breakdown of air occurs at an electric field of about 3×10^6 V/m, a sphere of 1 m can be raised to maximum of $V_{max} = ER = (3 \times 10^6\ \text{V/m})(1\ \text{m}) = 3 \times 10^6$ V. The dome’s electric potential can be increased further by placing the dome in vacuum and by increasing the radius of the sphere.

22.16 Exercises

Section 22.1 Electric Potential Energy

(1) A charge $q = 2.5 \times 10^{-8}$ C is placed in an upwardly uniform electric field of magnitude $E = 5 \times 10^4$ N/C. What is the change in the electric potential energy of the charge-field system when the charge is moved (a) 50 cm to the right? (b) 80 cm downwards? (c) 250 cm upwards at an angle 30° from the horizontal?

(2) Redo question 1 to calculate the work done on the charge q by the electric field.

(3) Redo question 1 to calculate the work done on the charge q by an external agent such that the charge moves in each case without changing its kinetic energy.

Section 22.2 Electric Potential

(4) How much work is done by an external agent in moving a charge $q = -9.63 \times 10^4$ C from a point A where the electric potential is 10 V to a point B where the electric potential is -4 V? How many electrons are there in this charge? Is this number related to any of the known physical constants?

Section 22.3 Electric Potential in a uniform Electric Field

(5) The electric potential difference between the accelerating plates in the electron gun of a TV tube is 5,550 V, while the separation between the plates is $d = 1.5$ cm. Find the magnitude of the uniform electric field between the plates.

(6) An electron moves a distance $d = 2$ cm when released from rest in a uniform electric field of magnitude $E = 6 \times 10^4$ N/C. (a) What is the electric potential difference through which the electron has passed? (b) Find the electron's speed after it has moved that distance?

(7) Two large parallel metal plates are oppositely charged with a surface charge density of magnitude $\sigma = 1.2\ \text{nC/m}^2$, see Fig. 22.25. (a) Find the electric field between the plates. (b) If the electric potential difference between these two plates is 10 V, what is the distance between the plates?

(8) A uniform electric field of magnitude 3×10^2 N/C is directed in the positive x direction as shown in Fig. 22.26. In this figure, the coordinates of point A are (0.3, −0.2) m and the coordinates of point B are (−0.5, 0.4) m. Calculate the potential difference $V_B - V_A$ using: (a) the path $A \to C \to B$. (b) the direct path $A \to B$.

Fig. 22.25 See Exercise (7)

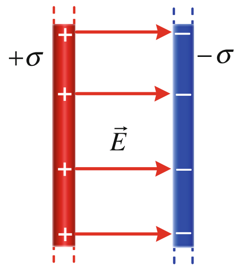
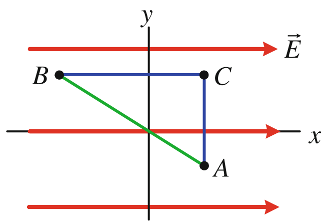
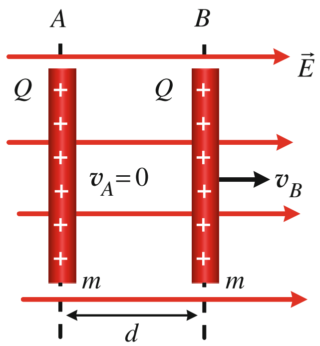


Fig. 22.26 See Exercise (8)



(9) An insulated rod has a charge $Q = 20\,\mu\text{C}$ and a mass $m = 0.05\,\text{kg}$. The rod is released from rest at a location A in a uniform electric field of magnitude 10^4 N/C directed perpendicular to the field, see Fig. 22.27 and neglect gravity. (a) Find the speed of the rod when it reaches location B after it has traveled a distance $d = 0.5\,\text{m}$. (b) Does the answer to part (a) change when the rod is released at an angle $\theta = 45°$ relative to the electric field?

Fig. 22.27 See Exercise (9)



Section 22.4 Electric Potential Due to a Point Charge

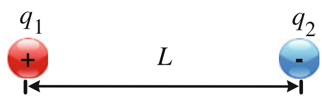
(10) (a) What is the electric potential at a distances 1 and 2 cm, from a proton? (b) What is the potential difference between these two points?

(11) Redo Exercise 10 for an electron.

(12) At a distance r from a particular point charge q, the electric field is 40 N/C and the electric potential is 36 V. Determine: (a) the distance r, (b) the magnitude of the point charge q.

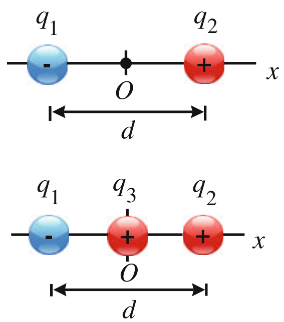
(13) Two point charges $q_1 = +2\,\mu\text{C}$ and $q_2 = -6\,\mu\text{C}$ are separated by a distance $L = 12$ cm, see Fig. 22.28. Find the point at which the resultant electric potential is zero.

Fig. 22.28 See Exercise (13)



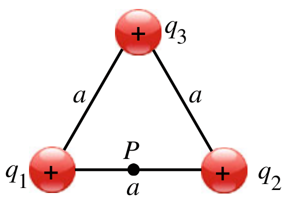
(14) Two charges $q_1 = -2\,\mu\text{C}$ and $q_2 = +2\,\mu\text{C}$ are fixed in their positions and separated by a distance $d = 10$ cm, see the top part of Fig. 22.29. (a) What is the electric field at the origin due these two charges? (b) What is the electric potential at the origin and the electric potential energy of the two charges? (c) Find the change in potential energy of the two charges when a third charge $q_3 = 2\,\mu\text{C}$ is brought from ∞ to O, see the bottom part of Fig. 22.29.

Fig. 22.29 See Exercise (14)



(15) Three equal charges $q_1 = q_2 = q_3 = 6$ nC are located at the vertices of an equilateral triangle of side $a = 6$ cm, see Fig. 22.30. Find the electric potential at point P, which is at the center of the base of the triangle.

Fig. 22.30 See Exercise (15)



(16) Three negative point charges are placed at the vertices of an isosceles triangle as shown in Fig. 22.31. Given that $a = \sqrt{2}$ cm, $q_1 = q_3 = -2$ nC, and $q_2 = -4$ nC, find the electric potential at point P (which is midway between q_1 and q_3).

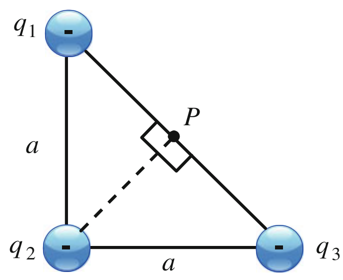


Fig. 22.31 See Exercise (16)

Section 22.5 Electric Potential Due to a Dipole

(17) An electric dipole is located along the x-axis with its center at the origin. The dipole has a negative charge $-q$ at $(-a, 0)$ and a positive charge $+q$ at $(+a, 0)$, see Fig. 22.11. Show that the electric potential on the y-axis is zero for any value of y.

(18) Assume that a third positive charge $+q$ is placed at the origin of the dipole of Fig. 22.11, so that the new configuration will be as show in Fig. 22.32. Show that the electric potential for far away points (such as P) on the dipole axis is given by:

$$V(x) = \frac{kq}{x}\left(1 + \frac{2a}{x}\right) \qquad (x \gg a)$$

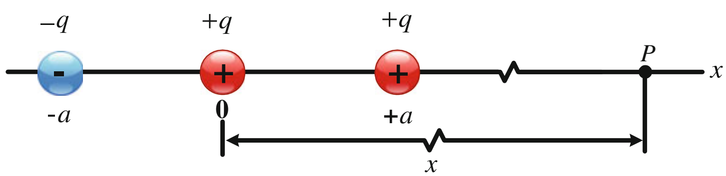


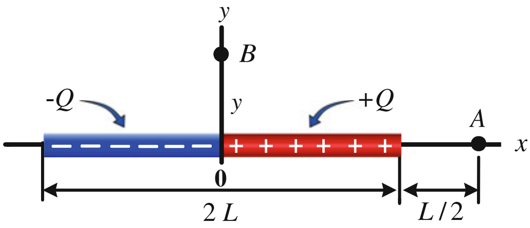
Fig. 22.32 See Exercise (18)

(19) The permanent electric dipole moment of the ammonia molecule NH_3 is $p = 4.9 \times 10^{-30}$ C.m. Find the electric potential due to an ammonia molecule at a distance 52 nm away along the axis of the dipole.

Section 22.6 Electric Potential Due to a Charged Rod

(20) A non-conductive rod has a uniform positive charge density $+\lambda$, a total charge Q along its right half, a uniform negative charge density $-\lambda$, and a total charge $-Q$ along its left half, see Fig. 22.33. (a) What is the electric potential at point A? (b) What is the electric potential at point B?

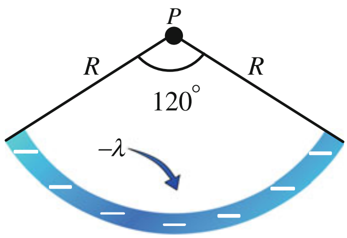
Fig. 22.33 See Exercise (20)



Section 22.7 Electric Potential Due to a Uniformly Charged Arc

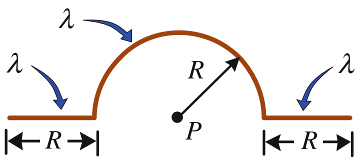
(21) A non-conductive rod has a uniformly distributed charge per unit length $-\lambda$. The rod is bent into a circular arc of radius R and central angle 120°, see Fig. 22.34. Find the electric potential at the center of the arc.

Fig. 22.34 See Exercise (21)



(22) A non-conductive rod has a uniformly distributed charge per unit length λ. Part of the rod is bent into a semicircular arc of radius R and the rest is left as two straight rod segments each of length R as shown in Fig. 22.35. Find the electric potential at point P.

Fig. 22.35 See Exercise (22)



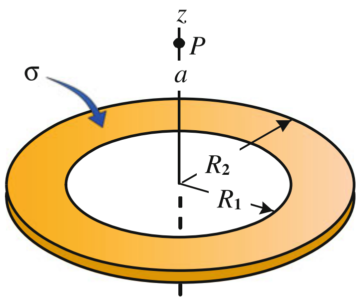
Section 22.8 Electric Potential Due to a Uniformly Charged Ring

(23) A uniformly charged insulated rod of charge $Q = -8\ \mu\text{C}$ and length $L = 15.0$ cm is bent into the shape of a circle. Find the electric potential at the center of the circle. If the rod is bent into the shape of a semicircle, find the electric field at its center.

(24) A ring of radius R has a uniformly distributed total positive charge Q, see Fig. 22.19. Find the point on the axis of the ring where the electric potential is half the value of the electric potential at the center.

(25) An annulus of inner radius R_1 and outer radius R_2 has a uniform surface charge per unit area σ. Calculate the electric potential at the point P which lies at a distance a from the center of the annulus along its central axis, see Fig. 22.36.

Fig. 22.36 See Exercise (25)



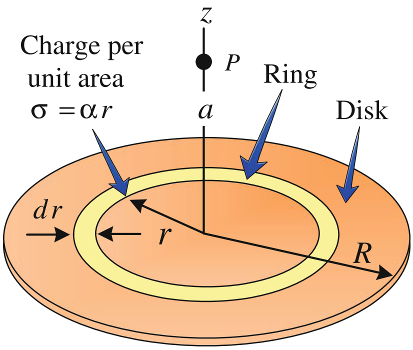
Section 22.9 Electric Potential Due to a Uniformly Charged Disk

(26) The disk of Fig. 22.20 has a radius $R = 4$ cm. If its surface charge density is $2\ \mu\text{C/m}^2$ from $r = 0$ to $R/2$ and $1.5\ \mu\text{C/m}^2$ from $r = R/2$ to R. Find the electric potential at point P on the central axis of the disk, at a distance $a = R/2$ from its center.

(27) Show that if the disk of Fig. 22.20 has a radius R and a fixed charge Q, the potential on the z-axis reduces to that of a point charge at the origin in the limit $R/a \to 0$, i.e. far away from the disk along the z-axis.

(28) Assume that a disk of radius R has a non-uniform surface charge density $\sigma = \alpha r$, where α is a constant and r is the distance from the center of the disk, see Fig. 22.37. Find the electric potential at point P on the central axis of the disk, at a distance a from its center. (Hint: use the electric potential produced by an element in the form of a ring of radius r and thickness dr.)

Fig. 22.37 See Exercise (28)



Section 22.10 Potential Due to a Uniformly Charged Sphere

(29) A charge Q is distributed uniformly throughout a spherical volume of radius R, see Fig. 22.21. (a) Find the point where the electric potential is half the value of the electric potential at the center. (b) What is the potential difference between a point on the surface and the sphere's center?

(30) The charge density inside a non-conductive sphere of radius R varies as $\rho = \alpha r$ (C/m^3), where α is a constant and r is the radial distance from the center of the sphere. The electric fields inside and outside the sphere are radial by symmetry and given by:

$$E = \alpha r^2/4\epsilon_\circ \qquad \text{for } r \le R$$
$$E = \alpha R^4/(4\epsilon_\circ r^2) \qquad \text{for } r \ge R$$

(a) Find the electric potential inside the sphere. (b) Find the electric potential outside the sphere.

Section 22.11 Electric Potential Due to a Charged Conductor

(31) A spherical conductor has a radius $R = 10\,\text{cm}$ and a positive charge of $20\,\mu\text{C}$. Find the electric potential at: (a) 0 cm, (b) 5 cm, (c) 10 cm, and (d) 15 cm from its center.

(32) An initially uncharged spherical conductor has a radius $R = 20\,\text{cm}$. (a) How many electrons should be removed from the sphere to produce an electrical potential of 3×10^6 V at its surface? (b) At that state, what is its surface charge density?

(33) A metal sphere of radius a has a charge q and is placed at the center of a hollow metal sphere of inner radius b and outer radius c, which carries a charge Q, see Fig. 22.38. (a) Use Gauss's law to show that:

$$\begin{aligned} E_r &= k(q+Q)/r^2 && r > c \\ E_r &= 0 && b < r < c \\ E_r &= kq/r^2 && a < r < b \\ E_r &= 0 && 0 < r < a \end{aligned}$$

(b) Show that the electric potential in all ranges is given by:

$$\begin{aligned} V_r &= k(q+Q)/r && r \geq c \\ V_r &= k(q+Q)/c && b \leq r \leq c \\ V_r &= k(q+Q)/c + kq(1/r - 1/b) && a \leq r \leq b \\ V_r &= k(q+Q)/c + kq(1/a - 1/b) && 0 \leq r \leq a \end{aligned}$$

(c) Consider the case where the sphere has a radius $a = 15\,\text{cm}$ and carries a charge $q = 10\,\text{nC}$. Additionally, consider the shell to be very thin when $c = b = 30\,\text{cm}$, see Fig. 22.39. Find the electric potential on the surface of the sphere if the shell carries a charge $Q = -15\,\text{nC}$.

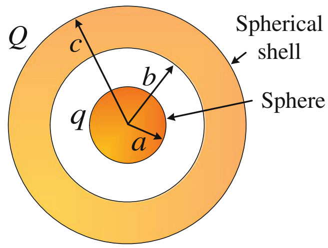
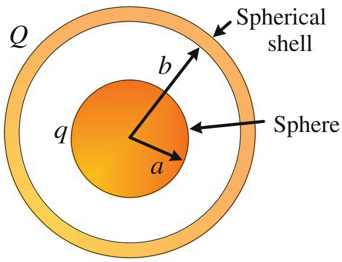


Fig. 22.38 See Exercise (33)

Fig. 22.39 See Exercise (33)



(34) A conducting sphere of radius a has a charge q and is placed at the center of a thin hollow conducting sphere of radius b that carries a charge $Q = -q$, see Fig. 22.39 of part c of the previous exercise. (a) Show that the electric potential difference between the sphere and the shell is:

$$\Delta V = V_a - V_b = kq\left(\frac{1}{a} - \frac{1}{b}\right)$$

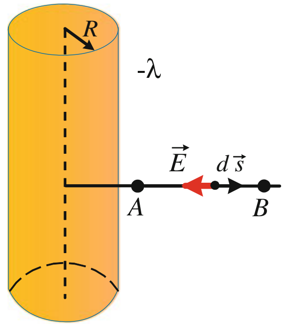
(b) Show that the electric field between the sphere and the shell is:

$$E_r = \frac{V_a - V_b}{(1/a - 1/b)}\,\frac{1}{r^2} \qquad a < r < b$$

(35) Two metal spheres of radii a and b are very far apart, but connected by a thin wire. Their combined charge is Q. Ignoring any charge on this thin wire, find (a) the charge on each sphere, and (b) the electric potential on each sphere.

(36) The electric field outside a long cylindrical conductor of radius R is directed radially toward the cylindrical axis and has a magnitude $E_r = 100/r$, see Fig. 22.40. Find the value of the electric potential difference $\Delta V = V_B - V_A$, if $r_A = 40$ cm and $r_B = 120$ cm.

Fig. 22.40 See Exercise (36)



Section 22.12 Potential Gradient

(37) The electric potential over a certain region of space is given by $V = (3x^2 + 6y^2 - 4z)$ V. (a) Find the expressions for the x, y, and z components of the electric fields over this region. (b) What are the values of the three components at a point P that has the coordinates $(-2, 2, 2)$ m?

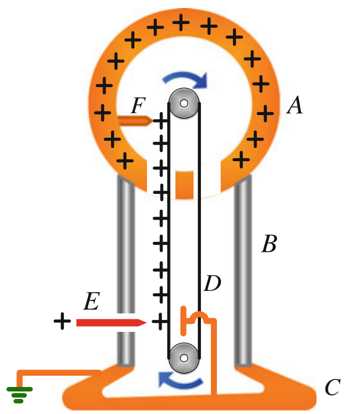
(38) The electric potential of a dipole at large distances is given in polar coordinates by $V(r) = kp\cos\theta/r^2$, where the angle θ is measured from the direction of electric dipole moment vector $\vec{p}$, see Eq. 22.36. Find the electric field due to the dipole at a point on the dipole axis.

Section 22.14 The Van de Graaff Generator

(39) Suppose the dome of a Van de Graaff generator has a radius $R = 0.3$ m and surrounded by dry air with ionization breakdown that occurs at an electric field of 3×10^6 V/m. (a) What is the maximum potential of the dome? (b) What is the maximum charge on the dome? (c) What is the magnitude of the electric potential and electric field 2 m away from the dome's center?

(40) Assume that the potential difference between the dome A and the point on the belt facing the needle F of the Van de Graaff generator shown in Fig. 22.41 is 2×10^6 V. If the belt D delivers a positive charge to the dome A at a rate of 4×10^{-3} C/s, what horsepower must be consumed to drive the belt against electrical forces?

Fig. 22.41 See Exercise (40)



Capacitors and Capacitance 23

In this chapter we introduce capacitors, which are one of the simplest *circuit elements*. Capacitors are charge-storing devices that can store energy in the form of an electric potential energy, and are commonly used in a variety of electric circuits.

Apart from being energy-storing devices, capacitors can be used to accumulate charges relatively slowly during the charging process, or to minimize voltage variations in electronic power supplies, or to detect electromagnetic waves, such as when tuning a radio receiver.

We shall first study the properties of capacitors and dielectrics, and follow that by studying capacitors in combination, and finally studying capacitors as electric charge-storing devices.

23.1 Capacitor and Capacitance

We can use a device called **capacitor** to store energy in the form of an electric potential. Beyond serving as storehouses for electric potential energy, capacitors have many uses in our electronic and microelectronic age.

Figure 23.1a shows the basic elements of an air-filled capacitor. It consists of two isolated conductors of any arbitrary shape, each of which carries an equal but opposite charge of magnitude Q.

Figure 23.1b shows a more convenient and practical arrangement of an air-filled capacitor, called a *parallel-plate capacitor*, consisting of two parallel conducting plates of area A separated by a distance d of air. We represent a capacitor of any geometry by the symbol ($\dashv\vdash$), which is based on the structure of a parallel plate capacitor.

H. A. Radi and J. O. Rasmussen, *Principles of Physics*,
Undergraduate Lecture Notes in Physics, DOI: 10.1007/978-3-642-23026-4_23,

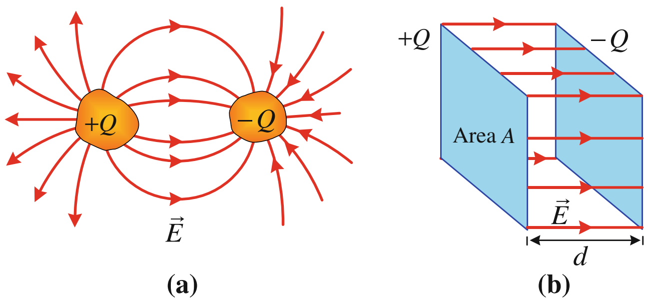


Fig. 23.1 (a) A capacitor made up of two conductors carrying an equal but opposite charge of magnitude Q. (b) A parallel-plate capacitor made up of two plates of area A separated by a distance d. Each plate carries an equal but opposite charge of magnitude Q

Experiments show that the magnitude of the charge on a capacitor is directly proportional to the potential difference between its conductors; i.e. $Q \propto \Delta V$; which can be written as $Q = C\,\Delta V$. Thus:

$$\boxed{C = \frac{Q}{\Delta V}} \tag{23.1}$$

The proportionality constant C is called the **capacitance** of the capacitor and depends on the shape and separation of the conductors. Furthermore, the charge Q and the potential difference ΔV are always expressed in Eq. 23.1 as positive quantities to produce a positive ratio $C = Q/\Delta V$. Hence:

> Spotlight
> The capacitance C of a capacitor is defined as the ratio of the magnitude of the charge on either conductor to the magnitude of the potential difference between the conductors.

The SI unit of the capacitance is coulomb per volt, or **farad** (abbreviated by F). That is:

$$1\,\mathrm{F} = 1\,\mathrm{C/V} \tag{23.2}$$

The farad is a very large unit of capacitance. In practice, typical devices have capacitances ranging from microfarads ($1\ \mu\mathrm{F} = 10^{-6}\,\mathrm{F}$), nanofarads ($1\ \mathrm{nF} = 10^{-9}\,\mathrm{F}$), to picofarads ($1\ \mathrm{pF} = 10^{-12}\,\mathrm{F}$).

23.2 Calculating Capacitance

For a capacitor with a charge of magnitude Q, we can calculate the potential difference ΔV using the technique described in the preceding chapter. Then we can use the expression $C = Q/\Delta V$ to calculate the capacitance for the capacitor under consideration.

A Parallel-Plate Capacitor

Figure 23.2a shows an uncharged parallel-plate capacitor of equal area A separated by a distance d. The capacitor is connected in a circuit containing a battery B that has a potential difference ΔV and an open switch S. When the switch is closed, the battery establishes an electric field in the wires and consequently charges flow in the circuit to charge the capacitor with a charge of magnitude Q, see Fig. 23.2b. Therefore, some of the stored chemical energy in the battery is transformed to the capacitor in the form of an electric field $\vec{E}$. Figure 23.2c shows the circuit schematic diagram, where we use the symbol $\overset{+}{\dashv}\!\vdash^{-}$ to represent the battery, the symbol $\dashv\,\vdash$ to represent the capacitor C, and the symbol —o-o— to represent the closed switch S. An open switch is represented by the symbol —o/ o—.

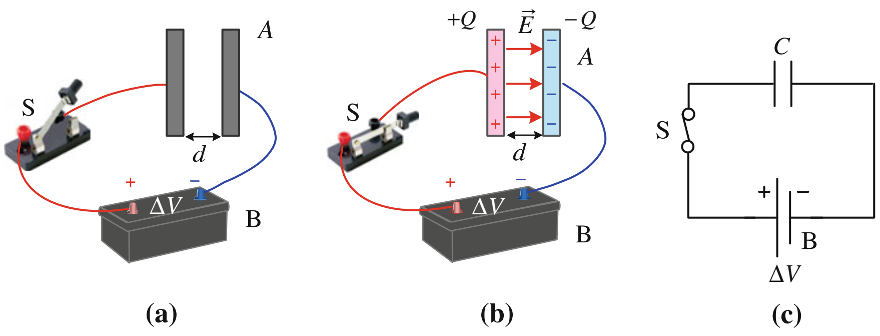


Fig. 23.2 (a) A parallel-plate capacitor is connected to a battery B and an open switch **S**. (b) When **S** is closed, each capacitor plate will carry equal but opposite charges of magnitude Q. (c) A schematic diagram of the circuit with symbols representing the elements used

To find the relation between the capacitance and the geometry of this parallel-plate capacitor, we first note that the magnitude of the surface charge density on either plate is $\sigma = Q/A$. Then according to Example 21.6, the magnitude of the electric field between the plates (assuming it uniform) is:

$$E = \frac{\sigma}{\epsilon_\circ} = \frac{Q}{\epsilon_\circ A} \tag{23.3}$$

Since the positive potential difference ΔV across the battery and the plates are identical, then according to Eq. 22.17 we have:

$$\Delta V = Ed = \frac{Qd}{\epsilon_\circ A} \tag{23.4}$$

Substituting this result into Eq. 23.1, we get:

$$C = \frac{Q}{\Delta V} = \frac{Q}{Qd/\epsilon_\circ A}$$

Thus, the capacitance of the parallel-plate capacitor is:

$$\boxed{C = \frac{\epsilon_\circ A}{d}} \quad \text{(Parallel-plate capacitor)} \tag{23.5}$$

A Cylindrical Capacitor

Figure 23.3a shows a cylindrical capacitor of length ℓ composed of a solid cylindrical conductor of radius a having a charge Q and a coaxial cylindrical conducting shell of radius b having a charge $-Q$. Thus, the magnitude of the linear charge density on either the cylinders is $\lambda = Q/\ell$. We assume that $\ell \gg b$ and hence neglect the fringing (non-uniformity) of the electric field at the cylinders' ends.

Figure 23.3b shows a cross-sectional view of the cylindrical capacitor. The electric field in the region between the cylinders is radial and perpendicular to the axis of the cylinders. In Chap. 21, we showed using Gauss's law that the electric field of a cylindrical charge distribution having a linear charge density λ is radial and is given by:

$$E_r = 2k\frac{\lambda}{r} \qquad (k = 1/4\pi\epsilon_\circ)$$

The same formula applies here since the charge on the outer shell does not contribute to any cylindrical Gaussian surface having $a < r < b$.

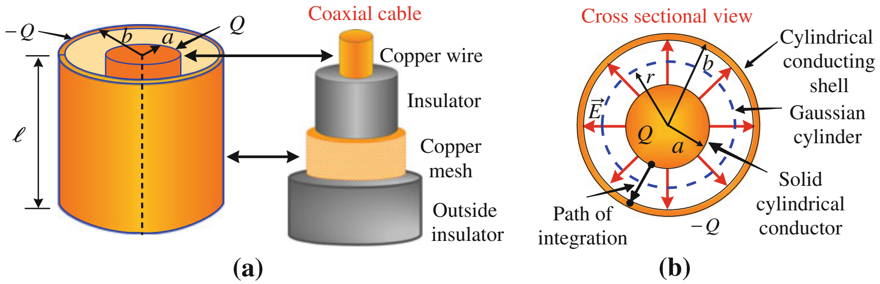


Fig. 23.3 (a) A cylindrical capacitor in the form of a cylindrical solid conductor surrounded by a coaxial shell. (b) A cross-sectional view of the capacitor showing a Gaussian cylinder of radius $a < r < b$

The potential difference $V_b - V_a$ between the cylinders is given by:

$$V_b - V_a = -\int_a^b \vec{E} \bullet d\vec{s} = -\int_a^b E_r\, dr = -2k\lambda \int_a^b \frac{dr}{r} = -2k\lambda \ln\left(\frac{b}{a}\right) \qquad (23.6)$$

Therefore, the magnitude of the potential difference between the cylinders is $\Delta V = |V_b - V_a| = 2k\lambda \ln(b/a)$. Substituting this result into Eq. 23.1 and using the fact that $\lambda = Q/\ell$, we get:

$$C = \frac{Q}{\Delta V} = \frac{Q}{2k(Q/\ell)\ln(b/a)}$$

Thus, the capacitance of a cylindrical capacitor of length ℓ is:

$$\boxed{C = \frac{\ell}{2k\ln(b/a)} = 2\pi\epsilon_\circ \frac{\ell}{\ln(b/a)}} \quad \text{(Cylindrical capacitor)} \qquad (23.7)$$

In addition, the capacitance per unit length of this configuration is:

$$\boxed{\frac{C}{\ell} = \frac{1}{2k\ln(b/a)} = 2\pi\epsilon_\circ \frac{1}{\ln(b/a)}} \quad \text{(Cylindrical capacitor)} \qquad (23.8)$$

A Spherical Capacitor

Figure 23.4a shows a three-dimensional spherical capacitor consisting of a solid spherical conductor of radius a having a charge Q and a concentric spherical shell of radius b having a charge $-Q$.

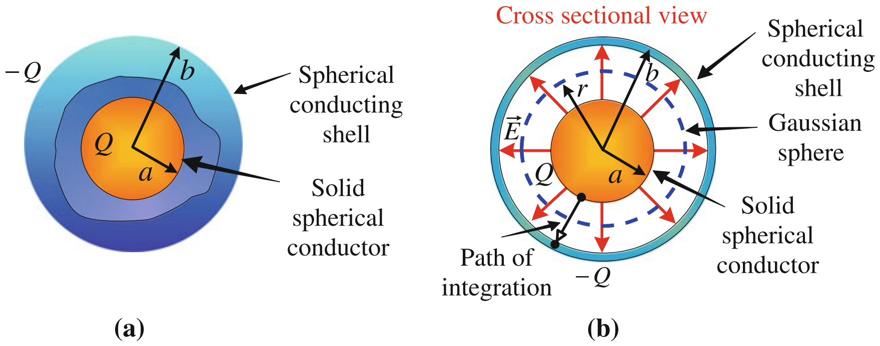


Fig. 23.4 (a) A spherical capacitor consists of a spherical solid conductor surrounded by a concentric spherical shell. (b) A cross-sectional view across the center of the spheres showing a Gaussian sphere of radius $a < r < b$

Figure 23.4b shows a cross-sectional view of the spherical capacitor. As shown in Chap. 21, the electric field outside a spherically symmetric charge distribution is radial and is given by:

$$E_r = k\frac{Q}{r^2}$$

This result applies only to the field between the spheres since the charge on the outer spherical shell does not contribute to any spherical Gaussian surface having $a < r < b$, see Fig. 23.4b.

The potential difference $V_b - V_a$ between the spheres is given by:

$$\begin{aligned} V_b - V_a &= -\int_a^b \vec{E} \bullet d\,\vec{s} = -\int_a^b E_r\,dr = -k\,Q\int_a^b \frac{dr}{r^2} \\ &= k\,Q\left|\frac{1}{r}\right|_a^b = k\,Q\left(\frac{1}{b} - \frac{1}{a}\right) \end{aligned} \tag{23.9}$$

Therefore, the magnitude of the potential difference between the spheres is $\Delta V = |V_b - V_a| = kQ\,(b-a)/ab$. Substituting this result into Eq. 23.1, we obtain:

$$C = \frac{Q}{\Delta V} = \frac{Q}{kQ\,(b-a)/ab}$$

Thus, the capacitance of the spherical capacitor is:

$$\boxed{C = \frac{ab}{k\,(b-a)} = 4\pi\epsilon_\circ\frac{ab}{(b-a)}} \qquad \text{(Spherical capacitor)} \tag{23.10}$$

An Isolated Sphere

The capacitance of a *single* isolated spherical conductor of radius R can be obtained by assuming that the missing second conducting sphere has an infinite radius. The electric field lines that leave or enter the isolated spherical conductor must therefore end at infinity. For practical purposes, the walls of the room in which the spherical conductor is housed can serve as our missing sphere of infinite radius. This proves that *any single conductor has a capacitance*.

To find the capacitance of the isolated spherical conductor, we rearrange Eq. 23.10 to be as follows:

$$C = \frac{a}{k\,(1 - a/b)}$$

Then we let $b \to \infty$ and replace a by R in this formula to find the following relation:

$$\boxed{C = \frac{R}{k} = 4\pi\epsilon_{\circ} R} \quad \text{(Isolated sphere)} \qquad (23.11)$$

Note that all the formulas derived so far for the capacitance [Eqs. 23.5, 23.7, 23.10, and 23.11] involve the constants $1/k$ or $\epsilon_{\circ}$ multiplied by a quantity that has the dimension of a length. Thus, the units of k and $\epsilon_{\circ}$ may be expressed as m/F and F/m, respectively.

Example 23.1

The plates of a parallel-plate capacitor are separated in air by a distance $d = 1$ mm. (a) Find the capacitance of this capacitor if its area is $A = 1\ \text{cm}^2$. (b) What must be the plate area if its capacitance is to be 1 F?

Solution: (a) From Eq. 23.5, we have:

$$C = \frac{\epsilon_{\circ} A}{d} = \frac{(8.85 \times 10^{-12}\ \text{F/m})(1 \times 10^{-4}\ \text{m}^2)}{(1 \times 10^{-3}\ \text{m})} = 8.85 \times 10^{-13}\ \text{F} = 0.885\ \text{pF}$$

(b) From Eq. 23.5, we have:

$$A = \frac{Cd}{\epsilon_{\circ}} = \frac{(1\ \text{F})(1 \times 10^{-3}\ \text{m})}{(8.85 \times 10^{-12}\ \text{F/m})} = 1.13 \times 10^{8}\ \text{m}^2$$

This is an area of a square that has a side of more than 10.6 km. Therefore, the farad is indeed a large unit. However, modern technology has permitted the construction of a 1 F capacitor of a very modest size. This capacitor is used as a backup power supply (up to many months) for computer memory chips in case of a power failure.

Example 23.2

Show that the capacitance of the cylindrical capacitor shown in Fig. 23.3a approaches the capacitance of a parallel-plate capacitor if the separation d between the two cylinders is very small.

Solution: When $d = b - a$ is very small, then d/a must also be very small. If we use the approximation $\ln(1 + x) \approx x$ for $x \ll 1$, in the natural logarithm of the denominator of Eq. 23.7, we find that:

$$\ln\left(\frac{b}{a}\right) = \ln\left(\frac{a+d}{a}\right) = \ln\left(1 + \frac{d}{a}\right) \approx \frac{d}{a} \quad \text{(When } d/a \ll 1\text{)}$$

Then, using the surface area of the inner cylinder $A = 2\pi a\ell$, we find that Eq. 23.7 approaches Eq. 23.5 as follows:

$$C = 2\pi\epsilon_\circ \frac{\ell}{\ln(b/a)} \approx 2\pi\epsilon_\circ \frac{\ell}{d/a} = \epsilon_\circ \frac{2\pi a\ell}{d} = \frac{\epsilon_\circ A}{d}$$

Example 23.3 **(Spherical Capacitor)**

(a) How much charge is stored in a spherical capacitor consisting of two concentric spheres of radii $a = 20$ cm and $b = 21$ cm if the potential difference between them is 200 V? (b) Show that if the separation d between the two spheres is small compared to their radii, then the capacitance is given by the parallel-plate capacitance formula $\epsilon_\circ A/d$. (c) Does the answer to part (b) apply to part (a)? (d) Find the capacitance of the inner sphere of part (a) if it is isolated.

Solution: (a) For concentric spheres, Eq. 23.10 is used to calculate the capacitance as follows:

$$C = \frac{ab}{k(b-a)} = \frac{(0.2\,\text{m})(0.21\,\text{m})}{(9 \times 10^9\,\text{m/F})(0.21\,\text{m} - 0.2\,\text{m})} = 4.67 \times 10^{-10}\,\text{F} = 0.467\,\text{nF}$$

Then, by using Eq. 23.1, the magnitude of the charge on each sphere will be:

$$Q = C\Delta V = (4.67 \times 10^{-10}\,\text{F})(200\,\text{V}) = 93.4\,\text{nC}$$

(b) When the separation $d = b - a$ is small, we can write the surface area of each sphere as $A \approx 4\pi a^2 \approx 4\pi b^2 \approx 4\pi ab$. Then, we have:

$$C = 4\pi\epsilon_\circ \frac{ab}{(b-a)} = \epsilon_\circ \frac{4\pi ab}{d} \approx \frac{\epsilon_\circ A}{d}$$

(c) Since the separation d in part (a) is very small compared to the radii of the spheres, then according to part (b) the capacitance is:

$$C \approx \frac{\epsilon_\circ A}{d} = \frac{4\pi a^2 \epsilon_\circ}{d} = \frac{4\pi(0.2\,\text{m})^2(8.85 \times 10^{-12}\,\text{F/m})}{(1 \times 10^{-2}\,\text{m})} = 4.45 \times 10^{-10}\,\text{F}$$

This is very close to the answer 4.67×10^{-10} F obtained in part (a).

(d) Substituting with $R = a = 20$ cm in Eq. 23.11, we find that:

$$C = 4\pi\epsilon_\circ R = 4\pi(8.85 \times 10^{-12}\,\text{F/m})(0.2\,\text{m}) = 2.22 \times 10^{-11}\,\text{F}$$

23.3 Capacitors with Dielectrics

An Electrical Description of Dielectrics

Capacitance was found to increase when a non-conducting material (such as oil, rubber, plastic, glass, or waxed paper) is inserted between the capacitor's plates. These non-conducting materials are called **dielectrics**. If the dielectric completely fills the space between the plates, the capacitance is found to increase by a dimensionless factor κ (the Greek alphabet Kappa), called the **dielectric constant**.

Fixed Charge

Consider a parallel-plate capacitor without a dielectric to have a capacitance $C_\circ$, a charge $Q_\circ$, and potential difference $\Delta V_\circ$, i.e. $C_\circ = Q_\circ/\Delta V_\circ$, see Fig. 23.5a. When a dielectric is inserted between the plates, see Fig. 23.5b, the potential difference between the plates is found to decrease to a value ΔV related to $\Delta V_\circ$ by the relation:

$$\boxed{\Delta V = \frac{\Delta V_\circ}{\kappa}} \tag{23.12}$$

Note that, $\kappa > 1$ because $\Delta V < \Delta V_\circ$.

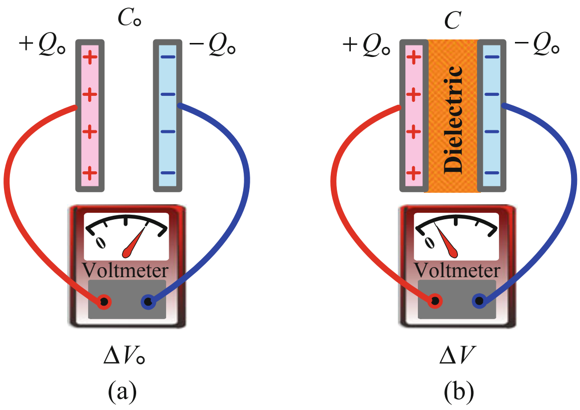


Fig. 23.5 (a) A capacitor with capacitance $C_\circ$ has a charge $Q_\circ$ when the potential difference between the plates is $\Delta V_\circ$. (b) When the capacitor's charge is maintained, inserting a dielectric reduces the potential difference to ΔV, where $\Delta V < \Delta V_\circ$

After inserting the dielectric, the capacitance C of the capacitor can be obtained from Eq. 23.1 as follows:

$$C = \frac{Q_\circ}{\Delta V} = \frac{Q_\circ}{\Delta V_\circ/\kappa} = \kappa \frac{Q_\circ}{\Delta V_\circ} \tag{23.13}$$

Using $C_\circ = Q_\circ/\Delta V_\circ$, we find that:

$$\boxed{C = \kappa\, C_\circ} \tag{23.14}$$

This indicates that the capacitance increases by a factor κ when the dielectric completely fills the space between the plates of the capacitor. Using Eq. 23.5, $C_\circ = \epsilon_\circ A/d$, the capacitance becomes:

$$\boxed{C = \frac{\kappa\, \epsilon_\circ A}{d} = \frac{\epsilon A}{d}} \tag{23.15}$$

where $\epsilon = \kappa\, \epsilon_\circ$ and is known as the *permittivity of the dielectric*.

On the other hand, if $\vec{E}_\circ$ is the electric field without the dielectric, then a reduction of the potential difference from $\Delta V_\circ$ to $\Delta V = \Delta V_\circ/\kappa$ means that the electric field decreases from $\vec{E}_\circ$ to $\vec{E} = \vec{E}_\circ/\kappa$. That is:

$$\boxed{\vec{E} = \frac{\vec{E}_\circ}{\kappa}} \tag{23.16}$$

Fixed Potential Difference

Now, consider a parallel-plate capacitor without a dielectric, having a capacitance $C_\circ$, a charge $Q_\circ$, and connected to a battery that has a potential difference $\Delta V_\circ$, i.e. $C_\circ = Q_\circ/\Delta V_\circ$, see Fig. 23.6a. If the dielectric is inserted between the plates while the potential difference is held constant by keeping the capacitor connected to the battery, see Fig. 23.6b, then the capacitance has to increase as before by the relation $C = \kappa C_\circ$. Consequently, the magnitude of the charge on the capacitor has to increase by a factor κ according to the relation:

$$\boxed{Q = \kappa Q_\circ} \tag{23.17}$$

The extra charge comes from the battery attached to the capacitor.

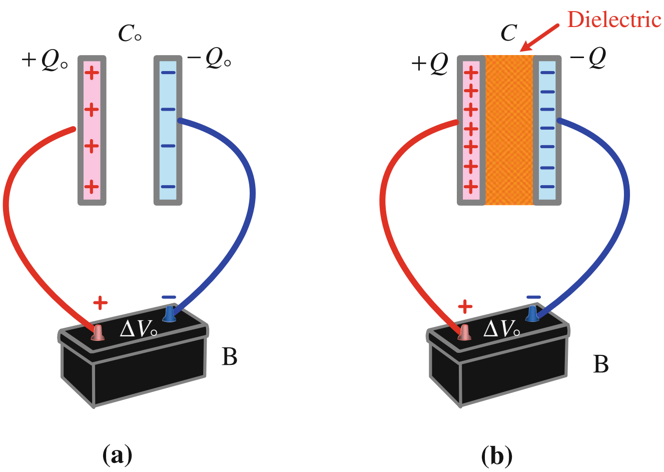


Fig. 23.6 (a) A capacitor with capacitance $C_\circ$ has a charge $Q_\circ$ when connected to a battery that has a potential difference $\Delta V_\circ$. (b) When the potential difference is maintained by the battery, inserting a dielectric increases the charge to Q, where $Q = \kappa Q_\circ$

An Atomic Description of Dielectrics

The molecules of some dielectrics have randomly oriented *permanent electric dipole moments* as shown in Fig. 23.7a. The presence of an external electric field $\vec{E}_\circ$ in such materials (called *polar dielectrics*), will exert a torque on the dipoles, causing them to partially align with the field, as shown in Fig. 23.7b. We can now describe the dielectric as being polarized, and the degree of alignment depends generally on the strength of $\vec{E}_\circ$.

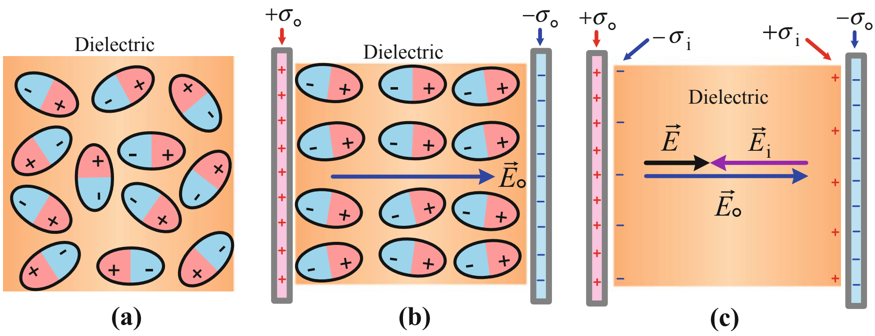


Fig. 23.7 (a) A dielectric that has randomly oriented molecules. (b) The partial alignment of molecules in the presence of an external electric field $\vec{E}_\circ$ due to a charged parallel plate capacitor with a surface charge density of magnitude $\sigma_\circ$. (c) The formation of an induced charge density $+\sigma_\text{i}$ and $-\sigma_\text{i}$ on either sides of the capacitor sets up an induced electric field $\vec{E}_\text{i}$. The resultant electric field $\vec{E}$ inside the dielectric has the same direction as $\vec{E}_\circ$ but is less in magnitude

Even when the dielectric material is *non-polar*, the applied external electric field $\vec{E}_\circ$ tends to separate the centers of the positive and negative charges of the molecules, producing *induced electric dipole moments*. Therefore, the induced electric dipole moments tend to align with the external electric field, and the dielectric is polarized.

The net effect on the dielectric is the formation of an *induced* positive and negative charge density $+\sigma_\text{i}$ and $-\sigma_\text{i}$ on the right and left faces of the dielectric, respectively, see Fig. 23.7c. Therefore, an *induced electric field* $\vec{E}_\text{i}$ will be established in a direction opposite to the external electric field $\vec{E}_\circ$. Accordingly, the net electric field $\vec{E}$ in the dielectric will have a magnitude given by:

$$\boxed{E = E_\circ - E_\text{i}} \tag{23.18}$$

In the case of the parallel-plate capacitor shown in Fig. 23.7c, we use the relations $E_\circ = \sigma_\circ/\epsilon_\circ$, $E_\text{i} = \sigma_\text{i}/\epsilon_\circ$, and $E = E_\circ/\kappa = \sigma_\circ/\epsilon$, to get:

$$\frac{\sigma_\circ}{\kappa\epsilon_\circ} = \frac{\sigma_\circ}{\epsilon_\circ} - \frac{\sigma_\text{i}}{\epsilon_\circ} \tag{23.19}$$

or

$$\boxed{\sigma_\text{i} = \frac{\kappa - 1}{\kappa}\sigma_\circ} \quad \text{(Parallel-plate capacitor)} \tag{23.20}$$

where $\sigma_i < \sigma_o$ because $\kappa > 1$. When the dielectric is replaced by a conductor, for which $E = 0$, then $E_i = E_o$ and hence $\sigma_i = \sigma_o$. This means that the induced charge on the conductor is equal in magnitude but opposite in sign to that on the plates of the parallel-plate capacitor.

Equation 23.15 indicates that the capacitance C increases drastically when d diminishes. However, d is limited by the electric discharge that could occur through the dielectric medium. Every dielectric material has a specific *dielectric strength* E_{max}, which is the maximum value of the electric field that the dielectric can withstand without *electrical breakdown*. Above this value the dielectric breaks down and forms a conducting path between the capacitor's plates. The largest potential difference ΔV_{max} that can be applied to a dielectric without exceeding the dielectric strength is called the *breakdown potential difference*. In fact, insulating materials have $\kappa > 1$ and their E_{max} is greater than that of air.

Table 23.1 displays approximate dielectric constants κ and dielectric strengths E_{max} of some materials at room temperature.

Table 23.1 Approximate values of the dielectric constants and dielectric strengths of some materials at room temperature

Material	κ	E_{max} (10^6 V/m $\equiv$ kV/mm)
Vacuum	1.00000	–
Air (1 atm)	1.00059	3
Teflon	2.1	60
Silicon oil	2.5	15
Mylar	3.2	7
Nylon	3.4	14
Paraffin-impregnated paper	3.5	11
Paper	3.7	16
Pyrex glass	5.6	14
Distilled Water	80	–

Types of Capacitors

Low-voltage capacitors are usually made of metallic foil interlaced with thin sheets of a dielectric material, made of either paraffin-impregnated paper or Mylar. The metallic foil and dielectric are rolled into a cylinder to form a small package, see Fig. 23.8a.

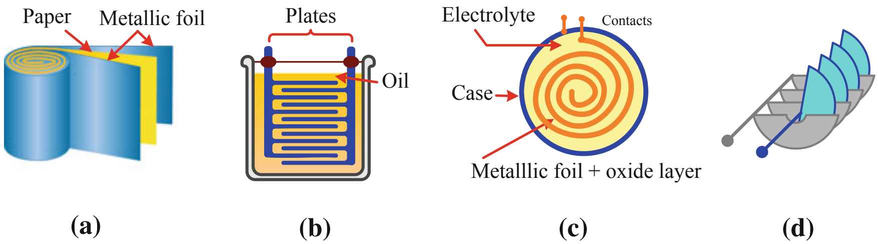


Fig. 23.8 (a) A low-voltage capacitor whose plates are separated by paper as a dielectric. (b) A high-voltage capacitor consisting of a number of plates separated by insulating oil as a dielectric. (c) An electrolytic capacitor used to store a large amount of charge. (d) A variable air capacitor

High-voltage capacitors are usually made of a number of interwoven metallic plates immersed in silicon oil, see Fig. 23.8b.

Large-charge storage capacitors consist of a metallic foil in contact with an electrolyte. When a voltage is applied between the foil and the electrolyte, a very thin layer of metal oxide is formed on the foil, and that layer serves as a dielectric, see Fig. 23.8c. Because the dielectric layer is very thin, the capacitance obtained with this type is very large. Such capacitors are assigned a polarity, which is indicated by positive and negative signs. If the polarity of the applied voltage is reversed, the oxide layer is removed, and the capacitor starts conducting electricity instead of storing charge.

Variable capacitors whose capacitance may vary are widely used in tuning circuits of radio receivers. They are constructed from a set of fixed parallel-plates connected together to form one plate of the capacitor, while the second set of movable plates are connected together to form the other plate. The plates are separated by air as a dielectric, see Fig. 23.8d.

Example 23.4

The parallel plates in Fig. 23.9a have an area $A = 0.2\ \text{m}^2$ and separation distance $d = 0.01$ m. The original potential difference between them is $\Delta V_\circ = 300$ V which decreases to $\Delta V = 100$ V when a dielectric sheet fills the space between the plates, see Fig. 23.9b. (a) Calculate the capacitance $C_\circ$, the magnitude of the charge $Q_\circ$, and the magnitude of the electric field $E_\circ$. (b) Calculate the final capacitance C and the dielectric constant κ. (c) Find the magnitudes of the induced charge density σ_i, the induced electric field E_i, and the final electric field E.

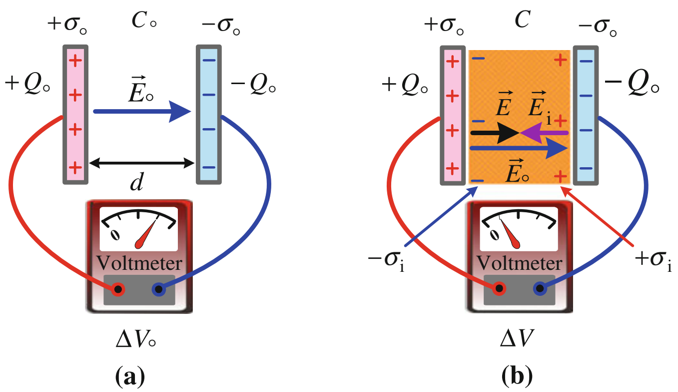


Fig. 23.9

Solution: (a) Using the parallel-plate capacitor Eq. 23.5, we get:

$$C_\circ = \frac{\epsilon_\circ A}{d} = \frac{(8.85\times 10^{-12}\ \mathrm{F/m})(0.2\ \mathrm{m}^2)}{0.01\ \mathrm{m}} = 1.77\times 10^{-10}\ \mathrm{F} = 177\ \mathrm{pF}$$

Then, when using Eq. 23.1, the magnitude of the charge on each plate will be the following:

$$Q_\circ = C_\circ\,\Delta V_\circ = (1.77\times 10^{-10}\ \mathrm{F})(300\ \mathrm{V}) = 5.31\times 10^{-8}\ \mathrm{C} = 53.1\ \mathrm{nC}$$

Finally, we use Eq. 22.17 to find the magnitude of the uniform electric field $E_\circ$ as follows:

$$E_\circ = \frac{\Delta V_\circ}{d} = \frac{300\ \mathrm{V}}{0.01\ \mathrm{m}} = 3\times 10^4\ \mathrm{V/m}$$

Alternatively, we can use the relation $E_\circ = \sigma_\circ/\epsilon_\circ$ to find $E_\circ$. First, we calculate $\sigma_\circ$ as follows:

$$\sigma_\circ = \frac{Q_\circ}{A} = \frac{5.31\times 10^{-8}\ \mathrm{C}}{0.2\ \mathrm{m}^2} = 2.655\times 10^{-7}\ \mathrm{C/m}^2$$

Then we find the value of $E_\circ$ as follows:

$$E_\circ = \frac{\sigma_\circ}{\epsilon_\circ} = \frac{2.655\times 10^{-7}\ \mathrm{C/m}^2}{8.85\times 10^{-12}\ \mathrm{F/m}} = 3\times 10^4\ \mathrm{C/F.m} = 3\times 10^4\ \mathrm{V/m}$$

(b) We first use Eq. 23.1 to find C as follows:

$$C = \frac{Q_\circ}{\Delta V} = \frac{5.31\times 10^{-8}\ \mathrm{C}}{100\ \mathrm{V}} = 5.31\times 10^{-10}\ \mathrm{F} = 531\ \mathrm{pF}$$

Then, by using equation $C = \kappa C_\circ$, we find that:

$$\kappa = \frac{C}{C_\circ} = \frac{5.31 \times 10^{-10}\ \text{F}}{1.77 \times 10^{-10}\ \text{F}} = 3$$

(c) The induced charge density σ_i can be obtained from Eq. 23.20 as follows:

$$\sigma_\text{i} = \frac{\kappa - 1}{\kappa}\sigma_\circ = \frac{(3-1)(2.655 \times 10^{-7}\ \text{C/m}^2)}{(3)} = 1.77 \times 10^{-7}\ \text{C/m}^2$$

The magnitude of the induced electric field is therefore:

$$E_\text{i} = \frac{\sigma_\text{i}}{\epsilon_\circ} = \frac{1.77 \times 10^{-7}\ \text{C}}{8.85 \times 10^{-12}\ \text{F/m}} = 2 \times 10^4\ \text{V/m}$$

The magnitude of the final electric field can be obtained from Eq. 23.16 as follows:

$$E = \frac{E_\circ}{\kappa} = \frac{3 \times 10^4\ \text{V/m}}{3} = 10^4\ \text{V/m}$$

Alternatively, we can find E from Eq. 23.18 as follows:

$$E = E_\circ - E_\text{i} = 3 \times 10^4\ \text{V/m} - 2 \times 10^4\ \text{V/m} = 10^4\ \text{V/m}$$

Example 23.5

Assume that the parallel-plate capacitor of Fig. 23.10a has a plate area $A = 0.2\ \text{m}^2$, separation distance $d = 10^{-2}$ m, and original potential difference $\Delta V_\circ = 300$ V. A dielectric slab of thickness $a = 5 \times 10^{-3}$ m and dielectric constant $\kappa = 2.5$ is inserted between the plates as shown in Fig. 23.10b. (a) Find the magnitudes of the final electric field E in the slab, the final potential difference ΔV between the plates, and the final capacitance C with the dielectric slab in place. (b) Find an expression for C in terms of $C_\circ$, a, d, and κ.

Solution: (a) From Example 23.4, we have $E_\circ = 3 \times 10^4$ V/m. Therefore, the magnitude of the final electric field in the slab can be obtained from Eq. 23.16 as follows:

$$E = \frac{E_\circ}{\kappa} = \frac{3 \times 10^4\ \text{V/m}}{2.5} = 1.2 \times 10^4\ \text{V/m}$$

By applying Eq. 22.6, we can find ΔV by integrating against the electric field along a straight line from the negative plate (−) to the positive plate (+). Within the dielectric, we must note that $\vec{E} \bullet d\vec{s} = -E\,ds$, the path length is a, and the magnitude of the field is E. But within the right and left gaps, the total path length is $d - a$ and the magnitude of the field is $E_\circ$. Thus, Eq. 22.6 yields:

$$\Delta V = V_+ - V_- = -\int_-^+ \vec{E} \bullet d\vec{s} = \int_-^+ E\,ds = E_\circ\,(d - a) + Ea$$
$$= (3 \times 10^4 \text{ V/m})(10^{-2} \text{ m} - 5 \times 10^{-3} \text{ m}) + (1.2 \times 10^4 \text{ V/m})(5 \times 10^{-3} \text{ m})$$
$$= 210 \text{ V}$$

From Example 23.4, we found that $Q_\circ = 5.31 \times 10^{-8}$ C and from Eq. 23.1 we can find the value of C as follows:

$$C = \frac{Q_\circ}{\Delta V} = \frac{5.31 \times 10^{-8}\,\text{C}}{210 \text{ V}} = 2.53 \times 10^{-10}\,\text{F} = 0.253\,\text{nF}$$

Note that we cannot use the relation $C = \kappa\, C_\circ$, because it is true only if the dielectric material *fills* the space between the capacitor's plates.

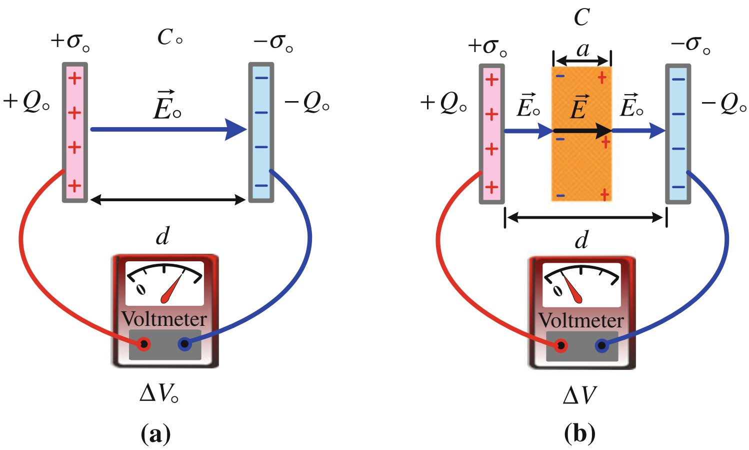


Fig. 23.10

(b) We start with the proven formula of part (a); that is:

$$\Delta V = E_\circ(d - a) + Ea$$

Then, using $\Delta V = Q_\circ/C$, $E_\circ = \sigma_\circ/\epsilon_\circ = Q_\circ/\epsilon_\circ A$, $C_\circ = \epsilon_\circ A/d$, and $E = E_\circ/\kappa = \sigma_\circ/\epsilon$, we can find an expression for C by performing the following steps:

$$\frac{Q_\circ}{C} = \frac{Q_\circ}{\epsilon_\circ A}(d-a) + \frac{Q_\circ}{\kappa\epsilon_\circ A}a$$

$$\frac{1}{C} = \frac{d-a}{\epsilon_\circ A} + \frac{a}{\kappa\epsilon_\circ A}$$

$$C = \frac{\epsilon_\circ A}{\left[(d-a) + \dfrac{a}{\kappa}\right]} \quad \Rightarrow \quad C = \frac{d}{\left[(d-a) + \dfrac{a}{\kappa}\right]}\frac{\epsilon_\circ A}{d}$$

$$C = \frac{d}{\left[(d-a) + \dfrac{a}{\kappa}\right]}C_\circ$$

In the second step, $(d-a)/\epsilon_\circ A$ is the inverse of the capacitance of an air capacitor of separation $d-a$, and $a/\kappa\epsilon_\circ A$ is the inverse of the capacitance of a capacitor of separation a but filled with a dielectric.

23.4 Capacitors in Parallel and Series

Capacitors in a circuit may be used in different combinations, and we can sometimes replace a combination of capacitors with one *equivalent capacitor*. In this section, we introduce two basic combinations of capacitors that allow such a replacement.

Capacitors in a Parallel Combination

Figure 23.11a shows two capacitors of capacitances C_1 and C_2, that are connected in **parallel** with a battery B. Figure 23.11b shows a circuit diagram for this combination of capacitors. The potential difference ΔV between the battery's terminals is the same as the potential difference across each capacitor. Figure 23.11c shows a single capacitance C_{eq} that is equivalent to this combination and has the same effect on the circuit. This means that when the potential difference ΔV is applied across the equivalent capacitor, it will store the same magnitude of the maximum total charge Q as stored in the combination being replaced.

When the circuit is first connected, electrons are transferred between the wires and the plates. This transfer leaves the top plates of the two capacitors positively charged, and the bottom plates negatively charged. If the magnitude of the maximum charges stored on the two capacitors are Q_1 and Q_2, then we must have:

$$Q = Q_1 + Q_2 \tag{23.21}$$

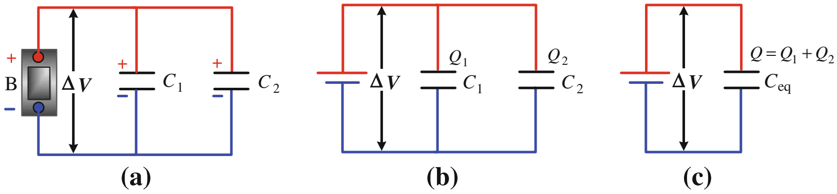


Fig. 23.11 (a) Two capacitors of capacitances C_1 and C_2 are connected in parallel to a battery B that has a potential difference ΔV. (b) The circuit diagram for this parallel combination. (c) The equivalent capacitance C_{eq} replaces the parallel combination

For the two capacitors in Fig. 23.11b, we have:

$$Q_1 = C_1 \Delta V \quad \text{and} \quad Q_2 = C_2 \Delta V \tag{23.22}$$

Substituting in Eq. 23.21, we get:

$$Q = (C_1 + C_2)\Delta V \tag{23.23}$$

The equivalent capacitor with the same total charge Q and applied potential difference ΔV has a capacitance C_{eq} given by:

$$C_{eq} = \frac{Q}{\Delta V} = C_1 + C_2 \quad \text{(Parallel combination)} \tag{23.24}$$

We can extend this treatment to n capacitors connected in parallel as:

$$\boxed{C_{eq} = C_1 + C_2 + C_3 + \cdots + C_n} \quad \text{(Parallel combination)} \tag{23.25}$$

Thus, the equivalent capacitance of a parallel combination of capacitors is simply the algebraic sum of the individual capacitances and is greater than any one of them.

Example 23.6

In Fig. 23.11, let $C_1 = 6\,\mu\text{F}$ and $C_2 = 3\,\mu\text{F}$, and $\Delta V = 18\,\text{V}$. Find the equivalent capacitance as well as the charges on C_1 and C_2.

Solution: The equivalent capacitance of the parallel combination is:

$$C_{eq} = C_1 + C_2 = 6\,\mu\text{F} + 3\,\mu\text{F} = 9\,\mu\text{F}$$

The magnitudes of the charges Q_1 and Q_2 on the two capacitors are:

$$Q_1 = C_1 \Delta V = (6\,\mu\,\mathrm{F})(18\ \mathrm{V}) \;=\; 108\ \mu\mathrm{C}$$
$$Q_2 = C_2 \Delta V = (3\,\mu\mathrm{F})(18\ \mathrm{V}) = 54\ \mu\mathrm{C}$$

Capacitors in a Series Combination

Figure 23.12a shows two capacitors of capacitances C_1 and C_2 that are connected in **series** with a battery B. Figure 23.12b shows a circuit diagram for this combination of capacitors.

When the circuit is first connected, the electrons are transferred out of the upper plate of C_1 (leaving it with an excess of positive charge) into the lower plate of C_2. As this negative charge accumulates on the lower plate of C_2, an exact amount of negative charge is forced off the upper plate of C_2 (leaving it with an excess positive charge) into the lower plate of C_1. As a result, all the upper plates acquire a positive charge $+Q$, and the lower plates acquire a negative charge $-Q$. Figure 23.11c shows a single capacitance C_{eq} that is equivalent to this combination and has the same effect on the circuit. This means that when the potential difference ΔV is applied across the equivalent capacitor, it must have a positive charge $+Q$ on its upper plate and a negative charge $-Q$ on its lower plate.

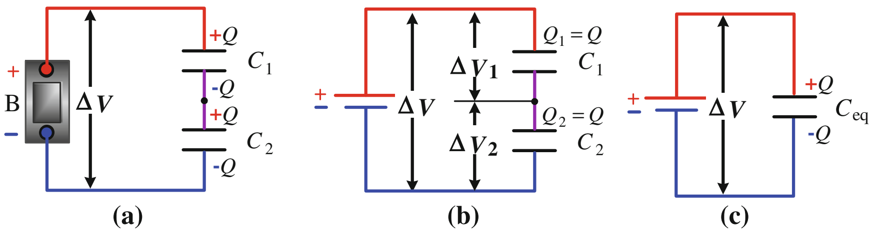


Fig. 23.12 (a) Two capacitors are connected in series to a battery B that has a potential difference ΔV. (b) The circuit diagram for this series combination. (c) An equivalent capacitance C_{eq} replacing the original capacitors set up in a series combination

The potential difference ΔV is divided to ΔV_1 and ΔV_2 across the capacitors C_1 and C_2, respectively. Thus:

$$\Delta V = \Delta V_1 + \Delta V_2 \tag{23.26}$$

For the two capacitors in Fig. 23.12b, we have:

$$\Delta V_1 = \frac{Q_1}{C_1} = \frac{Q}{C_1} \quad \text{and} \quad \Delta V_2 = \frac{Q_2}{C_2} = \frac{Q}{C_2} \tag{23.27}$$

Substituting in Eq. 23.26, we get:

$$\Delta V = \frac{Q}{C_1} + \frac{Q}{C_2} \tag{23.28}$$

The equivalent capacitor C_{eq} has the same charge Q and applied potential difference ΔV; thus:

$$\Delta V = \frac{Q}{C_{eq}} = \frac{Q}{C_1} + \frac{Q}{C_2} \tag{23.29}$$

Canceling Q, we arrive at the following relationship:

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} \quad \text{(Series combination)} \tag{23.30}$$

We can extend this treatment to n capacitors connected in series as:

$$\boxed{\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \cdots + \frac{1}{C_n}} \quad \text{(Series combination)} \tag{23.31}$$

Thus, the equivalent capacitance of a series combination of capacitors is simply the algebraic sum of the reciprocals of the individual capacitances and will always be less than any one of them.

Example 23.7

In Fig. 23.12, let $C_1 = 6\,\mu\text{F}$ and $C_2 = 3\,\mu\text{F}$, and $\Delta V = 18\text{ V}$. Find C_{eq}, Q, ΔV_1, and ΔV_2.

Solution: The equivalent capacitance of the series combination is:

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} = \frac{1}{6\,\mu\text{F}} + \frac{1}{3\,\mu\text{F}} = \frac{1}{2\,\mu\text{F}} \quad \Rightarrow \quad C_{eq} = 2\,\mu\text{F}$$

Consequently: $\quad Q = C_{eq}\,\Delta V = (2\,\mu\text{F})(18\text{ V}) = 36\,\mu\text{C}$

$$\Delta V_1 = \frac{Q}{C_1} = \frac{36\,\mu\text{V}}{6\,\mu\text{F}} = 6\text{ V} \quad \text{and} \quad \Delta V_2 = \frac{Q}{C_2} = \frac{36\,\mu\text{V}}{3\,\mu\text{F}} = 12\text{ V}$$

Example 23.8

For the combination of capacitors shown in Fig. 23.13a, assume that $C_1 = 2\ \mu\text{F}$, $C_2 = 4\ \mu\text{F}$, and $C_3 = 3\ \mu\text{F}$, and $\Delta V = 12\ \text{V}$. (a) Find the equivalent capacitance of the combination. (b) What is the charge on C_1?

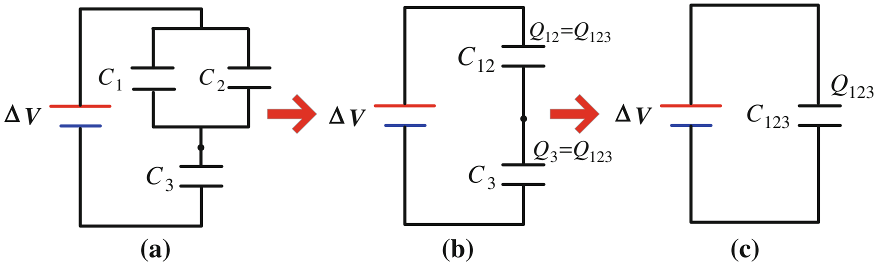


Fig. 23.13

Solution: (a) Capacitors C_1 and C_2 in Fig. 23.13a are in parallel and their equivalent capacitance C_{12} is:

$$C_{12} = C_1 + C_2 = 2\ \mu\text{F} + 4\ \mu\text{F} = 6\ \mu\text{F}$$

From Fig. 23.13b, we find that C_{12} and C_3 form a series combination and their equivalent capacitance C_{123} is given by:

$$\frac{1}{C_{123}} = \frac{1}{C_{12}} + \frac{1}{C_3} = \frac{1}{6\ \mu\text{F}} + \frac{1}{3\ \mu\text{F}} = \frac{1}{2\ \mu\text{F}} \quad \Rightarrow \quad C_{123} = 2\ \mu\text{F}$$

(b) We first find the charge Q_{123} on C_{123} in Fig. 23.13c as follows:

$$Q_{123} = C_{123}\Delta V = (2\ \mu\text{F})(12\ \text{V}) = 24\ \mu\text{C}$$

This same charge exists on each capacitor in the series combination of Fig. 23.13b. Therefore, if Q_{12} represents the charge on C_{12}, then $Q_{12} = Q_{123} = 24\ \mu\text{C}$. Accordingly, the potential difference across C_{12} is:

$$\Delta V_{12} = \frac{Q_{12}}{C_{12}} = \frac{24\ \mu\text{C}}{6\ \mu\text{F}} = 4\ \text{V}$$

This same potential difference exists across C_1, i.e. $\Delta V_1 = \Delta V_{12}$. Thus:

$$Q_1 = C_1 \Delta V_1 = (2\ \mu\text{F})(4\ \text{V}) = 8\ \mu\text{C}$$

23.5 Energy Stored in a Charged Capacitor

When the switch S of Fig. 23.14a is closed, the process of charging the capacitor starts by transferring electrons from the left plate (leaving it with an excess of positive charge) to the right plate. In the process of charging this capacitor, the battery must do work at the expense of its stored chemical energy.

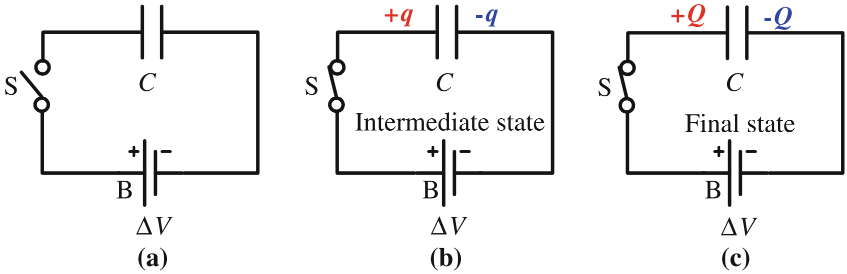


Fig. 23.14 (a) A circuit consisting of a battery B, a switch S, and a capacitor C. (b) An intermediate state when the magnitude of the charge on the capacitor is q. (c) A final state when $q = Q$.

In principle, the charging process occurs *as if* positive charges were pulled off from the right plate and transferred directly to the left plate. Suppose that, at a given instant during the charging process, as shown in Fig. 23.14b, the charge on the capacitor is q, i.e. $q = C\Delta V$. Moreover, according to Eq. 22.11, the differential applied work necessary to transfer a differential charge dq from the plate having charge $-q$ to the plate having a charge $+q$ is given by:

$$dW(\text{app}) = dq\ \Delta V = \frac{q}{C}dq \tag{23.32}$$

The total work required to charge the capacitor from a charge $q = 0$ to a final charge $q = Q$, see Fig. 23.14c, is thus:

$$W(\text{app}) = \int_0^Q \frac{q}{C}dq = \frac{1}{C}\int_0^Q q\ dq = \frac{Q^2}{2\,C} \tag{23.33}$$

According to Eqs. 22.6 and 22.10, this work done by the battery is stored as electrostatic potential energy U in the capacitor. Thus:

$$\boxed{U = \frac{Q^2}{2C}} \quad \text{(Electric potential energy)} \tag{23.34}$$

From Eq. 23.1, we can write this stored electric potential energy in the following forms:

$$\boxed{U = \tfrac{1}{2}C(\Delta V)^2} \quad \text{(Electric potential energy)} \tag{23.35}$$

or

$$\boxed{U = \tfrac{1}{2}Q\Delta V} \quad \text{(Electric potential energy)} \tag{23.36}$$

It is important to note that Eqs. 23.34 to 23.36 hold for any capacitor, regardless of its shape.

When we neglect the fringing effect (nonuniform $\vec{E}$) in a parallel-plate capacitor filled with a dielectric, we know that the electric field has the same value at any point between the plates. Thus, the potential energy per unit volume between the plates, known as the **energy density** u_E, should also be uniform. Then we can find u_E by dividing the electric potential energy U by the volume Ad between the plates:

$$u_E = \frac{U}{Ad} = \frac{C(\Delta V)^2}{2Ad} \tag{23.37}$$

Using $C = \kappa\,\epsilon_\circ A/d$ and $\Delta V = Ed$ for parallel-plate capacitors, we get:

$$\boxed{u_E = \tfrac{1}{2}\kappa\epsilon_\circ E^2} \quad \text{(Electric energy density)} \tag{23.38}$$

Although this equation is derived for a parallel-plate capacitor, it holds true for any source of electric field. When the electric field $\vec{E}$ exists at any point in a dielectric material of dielectric constant κ, the potential energy per unit volume at this point is given by Eq. 23.38. When $\kappa = 1$, this relation reduces to $u_E = \frac{1}{2}\epsilon_\circ E^2$.

Example 23.9

A capacitor $C_1 = 4\,\mu\text{F}$ is charged by an initial potential difference $\Delta V_i = 12\,\text{V}$, see Fig. 23.15a. The charging battery is then removed, as shown in Fig. 23.15b, and the capacitor is connected to the uncharged capacitor $C_2 = 2\,\mu\text{F}$, as shown in Fig. 23.15c. (a) Find the final potential difference ΔV_f as well as Q_{1f} and Q_{2f}. (b) Find the stored energy before and after the switch is closed.

Solution: (a) The original charge is now shared by C_1 and C_2, so:

$$Q_{1i} = Q_{1f} + Q_{2f}$$

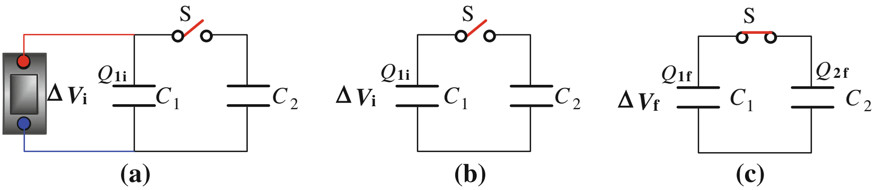


Fig. 23.15

Using of the relation $Q = C\Delta V$ in each term of this equation, we get:

$$C_1 \Delta V_i = C_1 \Delta V_f + C_2 \Delta V_f$$

Thus:

$$\Delta V_f = \frac{C_1}{C_1 + C_2}\Delta V_i = \frac{(4\,\mu\text{F})}{4\,\mu\text{F} + 2\,\mu\text{F}}(12\,\text{V}) = 8\,\text{V}$$

and:

$$Q_{1f} = C_1 \Delta V_f = (4\,\mu\text{F})(8\ \text{V}) = 32\ \mu\text{C}$$
$$Q_{2f} = C_2 \Delta V_f = (2\,\mu\text{F})(8\ \text{V}) = 16\ \mu\text{C}$$

(b) The initial potential energy is:

$$U_i = \tfrac{1}{2} C_1 (\Delta V_i)^2 = \tfrac{1}{2}(4\,\mu\text{F})(12\ \text{V})^2 = 288\ \mu\text{J}$$

The final potential energy is:

$$U_f = \tfrac{1}{2} C_1 (\Delta V_f)^2 + \tfrac{1}{2} C_2 (\Delta V_f)^2 = \tfrac{1}{2}(4\,\mu\text{F} + 2\,\mu\text{F})(8\ \text{V})^2 = 192\ \mu\text{J}$$

Although $U_i > U_f$, this is not a violation of the conservation of energy principle. The missing energy is transferred as thermal energy into the connecting wires and as radiated electromagnetic waves.

23.6 Exercises

Section 23.1 Capacitor and Capacitance

(1) A capacitor has a capacitance of 15 μF. How much charge must be removed to lower the potential difference between its conductors to 10 V?

(2) Two identical coins carry equal but opposite charges of magnitude 1.6 μC. The capacitance of this combination is 20 pF. What is the potential difference between the coins?

(3) A capacitor with a charge of magnitude 10^{-4} C has a potential difference of 50 V. What charge value is needed to produce a potential difference of 15 V?

Section 23.2 Calculating Capacitance

(4) A computer memory chip contains a large number of capacitors, each of which has a plate area $A = 20 \times 10^{-12}\ \text{m}^2$ and a capacitance of 50 f F (50 femtofarads). Assuming a parallel-plate configuration, find the order of magnitude of the separation distance d between the plates of such a capacitor.

(5) A parallel-plate capacitor has a plate area $A = 0.04\ \text{m}^2$ and a vacuum separation $d = 2 \times 10^{-3}$ m. A potential difference of 20 V is applied between the plates of the capacitor. (a) Find the capacitance of the capacitor. (b) Find the magnitude of the charge and charge density on the plates of the capacitor. (c) Find the magnitude of the electric field between the plates.

(6) An electric spark occurs if the electric field in air exceeds the value 3×10^6 V/m. Find the maximum magnitude of the charge on the plates of an air-filled parallel-plate capacitor of area $A = 30\ \text{cm}^2$ such that a spark is avoided.

(7) A parallel-plate capacitor has circular plates, each with a radius $r = 5$ cm. Assume a vacuum separation $d = 1$ mm exists between the plates, see Fig. 23.16. How much charge is stored on each plate of the capacitor when their potential difference has the value $\Delta V = 50$ V.

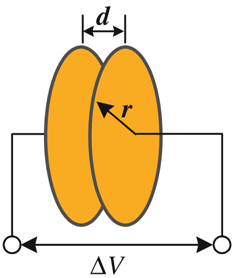
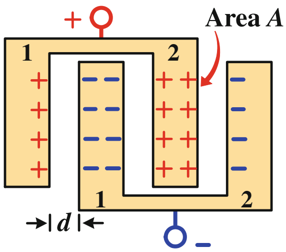


Fig. 23.16 See Exercise (7)

(8) Figure 23.17 shows a set of two parallel sheets of a conductor connected together to form one plate of a capacitor, while the second set is connected

together to form the other plate of the capacitor. Assume that the effective area of adjacent sheets is A and that the air separation is d. From the figure, confirm that the number of adjoining sheets of positive and negative charges is 3 and the capacitor has a capacitance $C = 3\epsilon_\circ A/d$.

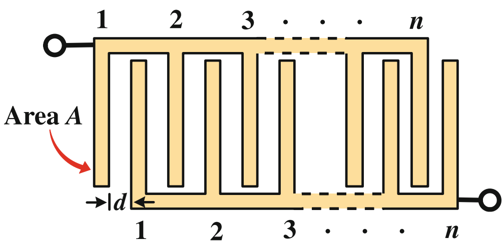
Fig. 23.17 See Exercise (8)



(9) If each set in Exercise 8 consists of n plates, see Fig. 23.18, then show that the capacitance of the capacitor will be given by:

$$C = \frac{(2n-1)\epsilon_\circ A}{d}$$

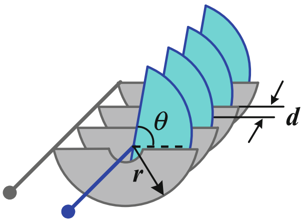
Fig. 23.18 See Exercise (9)



(10) A variable air capacitor used in radio tuning consists of a set of n fixed semi-circular plates, each of radius r, and located a distance d from a neighboring plate of an identical yet rotatable set, see Fig. 23.19. Show that when one set is rotated by an angle θ, the capacitance is:

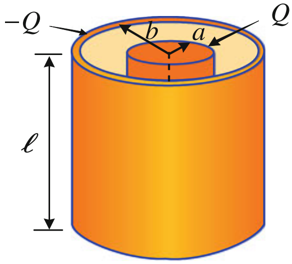
$$C = \frac{(2n-1)\epsilon_\circ(\pi - \theta)\, r^2}{2d}$$

Fig. 23.19 See Exercise (10)



(11) A coaxial cable of length $\ell = 5$ m consists of a solid cylindrical conductor surrounded by a cylindrical conducting shell. The inner conductor has a radius $a = 2.5$ mm and carries a charge Q, while the surrounding shell has a radius $b = 8.5$ mm and carries a charge $-Q$, see Fig. 23.20. Assume that $Q = +8 \times 10^{-8}$ C and that air fills the gap between the conductors. (a) What is the capacitance of this cable? (b) What is the magnitude of the potential difference between the two cylinders?

Fig. 23.20 See Exercise (11)



(12) An isolated spherical conductor carries a charge $Q = 4$ nC, see Fig. 23.21. The potential difference between the sphere and its surroundings is $\Delta V = 100$ V. What is the capacitance formed from the sphere and its surroundings?

Fig. 23.21 See Exercise (12)

(13) A capacitor consists of two concentric spheres of radii $a = 30$ cm and $b = 36$ cm, see Fig. 23.22. Assume the gap between the conductors is filled with air. (a) What is the capacitance of this capacitor? (b) How much charge is stored in the capacitor if the potential difference between the two spheres is $\Delta V = 50$ V?

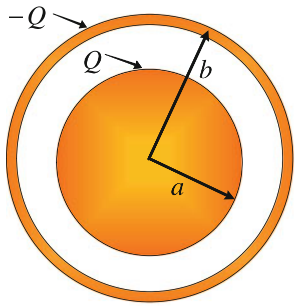


Fig. 23.22 See Exercise (13)

(14) Find the capacitance of Earth by assuming that the "missing second conducting sphere" has an infinite radius. The radius of Earth is $R = 6.37 \times 10^6$ m.

(15) A spherical drop of mercury has a capacitance of 2.78 fF. If two such drops combine into one, what would its capacitance be?

Section 23.3 Capacitors with Dielectrics

(16) Two parallel plates of area $A = 0.01\ \text{m}^2$ are separated by a distance $d = 5 \times 10^{-3}$ m. The region between these plates is filled with a dielectric material of $\kappa = 3$, and the plates are given equal but opposite charges of 2 μC. (a) What is the capacitance of this capacitor? (b) Find the potential difference between the plates.

(17) An air-filled parallel-plate capacitor of 15 μF is connected to a 50 V battery; then the battery is removed. (a) Find the charge on the capacitor. (b) If the air is replaced with oil having $\kappa = 2.2$, find the new values of the capacitance and the potential difference between the plates.

(18) A parallel-plate capacitor has an area $A = 4\ \text{cm}^2$. (a) Find the maximum stored charge on the capacitor if air fills the space between the plates. (b) Redo part (a) when paper is used instead of the air (use the dielectric strengths given in Table 23.1).

(19) The charged air capacitor shown in Fig. 23.23 is first placed at a pressure of 1 atm and found to have a potential difference $\Delta V = 10{,}376$ V. Then, the capacitor is placed in a vacuum chamber and the air is removed. The potential difference is found to rise to $\Delta V_\circ = 10{,}382$ V. Determine the dielectric constant of the air.

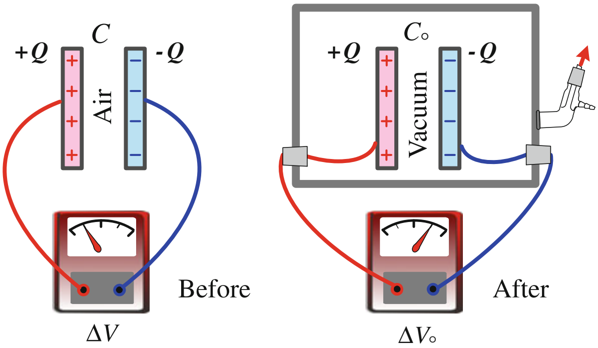


Fig. 23.23 See Exercise (19)

(20) A parallel-plate capacitor having an area $A = 0.2\,\text{m}^2$, and a plate separation $d = 1$ mm filled with air as an insulator, is connected to a battery that has a potential difference $\Delta V_\circ = 12$ V, see Fig. 23.24. While the battery is still connected to the capacitor, a sheet of glass ($\kappa = 4.5$) is inserted to fill the space between the plates, see the figure. (a) Determine both the initial capacitance ($C_\circ$) and the initial charge ($Q_\circ$), then find C and Q after inserting the glass. (b) If σ_i is the magnitude of the induced surface charge density on the glass and $\sigma_\circ$ is the magnitude of the charge density of the plates before the insertion of the glass, then show that:

$$\sigma_i = (\kappa - 1)\sigma_\circ$$

(c) Find the values of σ_i and the induced electric field E_i.

Section 23.4 Capacitors in Parallel and Series

(21) Two capacitors, $C_1 = 2\,\mu\text{F}$ and $C_2 = 3\,\mu\text{F}$, are connected in parallel to a battery that has a potential difference $\Delta V = 9$ V. (a) Find the equivalent capacitance of the combination. (b) Find the charge on each capacitor. (c) Find the potential difference across each capacitor.

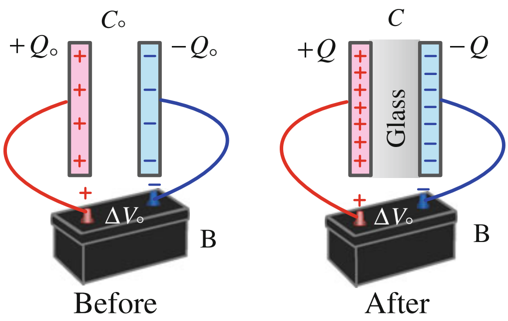


Fig. 23.24 See Exercise (20)

(22) The two capacitors of exercise 21 are now connected in series to the same battery (i.e. with a potential difference $\Delta V = 9$ V). (a) Find the equivalent capacitance of the combination. (b) Find the charge on each capacitor. (c) Find the potential difference across each capacitor.

(23) For the combination of capacitors shown in Fig. 23.25, assume that $C_1 = 1\ \mu$F, $C_2 = 2\ \mu$F, $C_3 = 3\ \mu$F, and $\Delta V = 6$ V. (a) Find the equivalent capacitance of the combination. (b) Find the charge on each capacitor. (c) Find the potential difference across each capacitor.

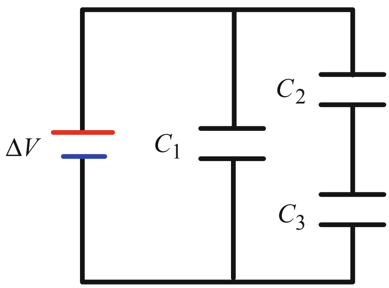


Fig. 23.25 See Exercise (23)

(24) Three capacitors, $C_1 = 6\ \mu$F, $C_2 = 4\ \mu$F, and $C_3 = 12\ \mu$F, are connected in four different ways, as shown in Fig. 23.26. In all configurations, the potential difference is 22 V. How many coulombs of charge pass from the battery to each combination?

(25) When the three capacitors $C_1 = 2\ \mu$F, $C_2 = 1\ \mu$F, and $C_3 = 4\ \mu$F are connected to a source of a potential difference ΔV, as shown Fig. 23.27, the charge Q_2 on C_2 is found to be 10 μC. (a) Find the values of the charges on the two capacitors C_1 and C_3. (b) Determine the value of ΔV.

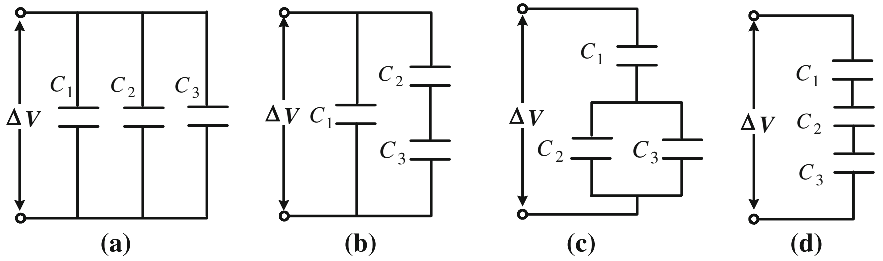
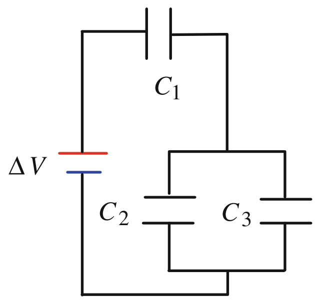


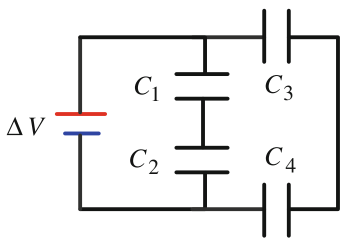
Fig. 23.26 See Exercise (24)

Fig. 23.27 See Exercise (25)



(26) For the circuit shown in Fig. 23.28, $C_1 = 3\,\mu\text{F}$, $C_2 = 6\,\mu\text{F}$, $C_3 = 6\,\mu\text{F}$, $C_4 = 12\,\mu\text{F}$, and $\Delta V = 12\,\text{V}$. (a) Find the equivalent capacitance of the combination. (b) Find the potential difference across each capacitor.

Fig. 23.28 See Exercise (26)



(27) For each of the combinations shown in Fig. 23.29, find a formula that represents the equivalent capacitance between the terminals A and B.

(28) Assume that in Exercise 27, $C = 12\,\mu\text{F}$ and $\Delta V_{BA} = 12\,\text{V}$. For each combination, find the magnitude of the total charge that the source between A and B will distribute on the capacitors.

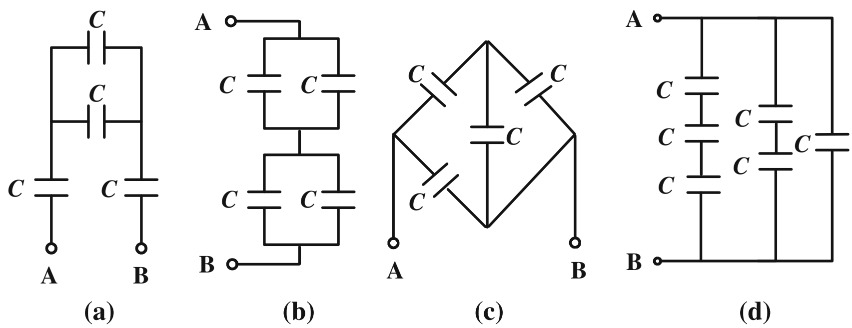


Fig. 23.29 See Exercise (27)

(29) Two capacitors, $C_1 = 25\,\mu\text{F}$ and $C_2 = 40\,\mu\text{F}$, are charged by being connected to batteries that have a potential difference $\Delta V = 50\,\text{V}$, see part (a) of Fig. 23.30. They are then disconnected from their batteries and connected to each other, with each positive plate connected to the other's negative plate; see part (b) of Fig. 23.30. (a) Find the equivalent capacitance between A and B. (b) What is the charge Q on the equivalent capacitor? (c) What is the potential difference ΔV_{BA} between A and B? (d) Find the final charge on each capacitor.

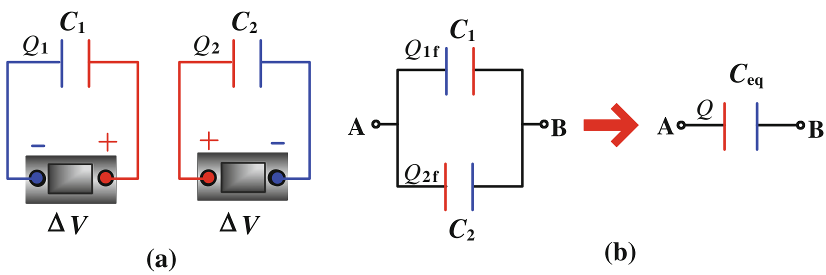
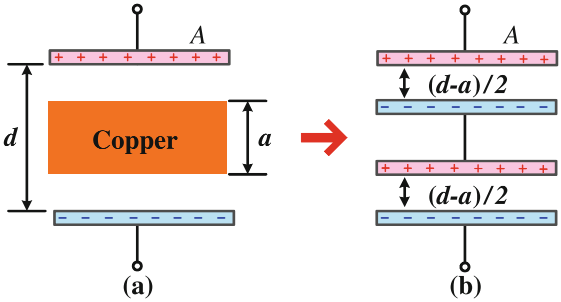


Fig. 23.30 See Exercise (29)

(30) A parallel-plate capacitor has an area A and separation d. A slab of copper of thickness a is inserted midway between the plates, see part (a) of Fig. 23.31. Show that the capacitor is equivalent to two capacitors in series, each having a plate separation $(d - a)/2$, as shown in part (b) of the figure, and show that the capacitance after inserting the slab is given by:

$$C = \frac{\epsilon_\circ A}{d - a}$$

Fig. 23.31 See Exercise (30)



(31) Show that the capacitance of the capacitor in Fig. 23.10b can be obtained by finding the equivalent capacitance of two capacitors in series, one capacitor with a dielectric of thickness a and the second an air-filled capacitor of thickness $d - a$.

(32) A parallel-plate capacitor of plate area A and separation d is filled in two different ways with two dielectrics κ_1 and κ_2 as shown in parts (a) and (b) of Fig. 23.32. Show that the capacitances of the two capacitors of parts (a) and (b) are:

$$C = \frac{\epsilon_\circ A}{d}\,\frac{\kappa_1 + \kappa_2}{2} \quad \text{and} \quad C = \frac{2\epsilon_\circ A}{d}\,\frac{\kappa_1\,\kappa_2}{\kappa_1 + \kappa_2} \quad \text{respectively,}$$

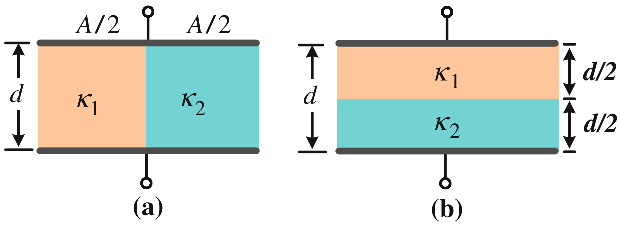


Fig. 23.32 See Exercise (32)

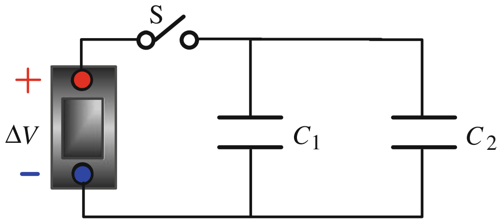
Section 23.6 Energy Stored in a charged Capacitor

(33) How much energy is stored in one cubic meter of air due to an electric field of magnitude 100 V/m?

(34) The two capacitors shown in Fig. 23.33 are uncharged when the switch S is open. Assume that $C_1 = 4\,\mu\text{F}$, $C_2 = 6\,\mu\text{F}$, and $\Delta V = 10\,\text{V}$. The two capacitors become fully charged when the switch S is closed. (a) Find the energy stored

in these two capacitors. (b) Does the stored potential energy in the equivalent capacitor equal the total stored energy in the two capacitors?

Fig. 23.33 See Exercise (34)



(35) Redo Example 23.9 when $C_1 = C_2 = 5\,\mu\text{F}$, and $\Delta V_i = 10\,\text{V}$. Does the initial and final stored potential energy remains the same?

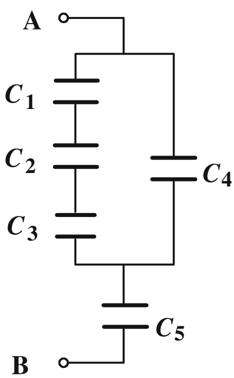
(36) A capacitor is charged to a potential difference ΔV. How much should you increase ΔV so that the stored potential energy is increased by 20%?

(37) Calculate the electric field, the energy density, and the stored potential energy in the parallel-plate capacitor of Exercise 7.

(38) A parallel-plate capacitor has a capacitance of $4\,\mu\text{F}$ when a mica sheet with dielectric constant $\kappa = 5$ fills the space between the plates. The capacitor is charged by a battery that has a potential difference 50 V, and is later disconnected. How much work must be done to slowly pull the dielectric from the capacitor?

(39) For the circuit shown in Fig. 23.34, $C_1 = 2\,\mu\text{F}$, $C_2 = 3\,\mu\text{F}$, $C_3 = 6\,\mu\text{F}$, $C_4 = 1\,\mu\text{F}$, and $C_5 = 2\,\mu\text{F}$. (a) Find the potential difference between A and B needed to give C_3 a charge of $20\,\mu\text{C}$. (b) Under these considerations, what is the electric potential energy stored in the combination?

Fig. 23.34 See Exercise (39)



(40) Confirm the relationships shown in Fig. 23.35, where $\Delta V_\circ$ is shortened by $V_\circ$ and ΔV is shortened by V.

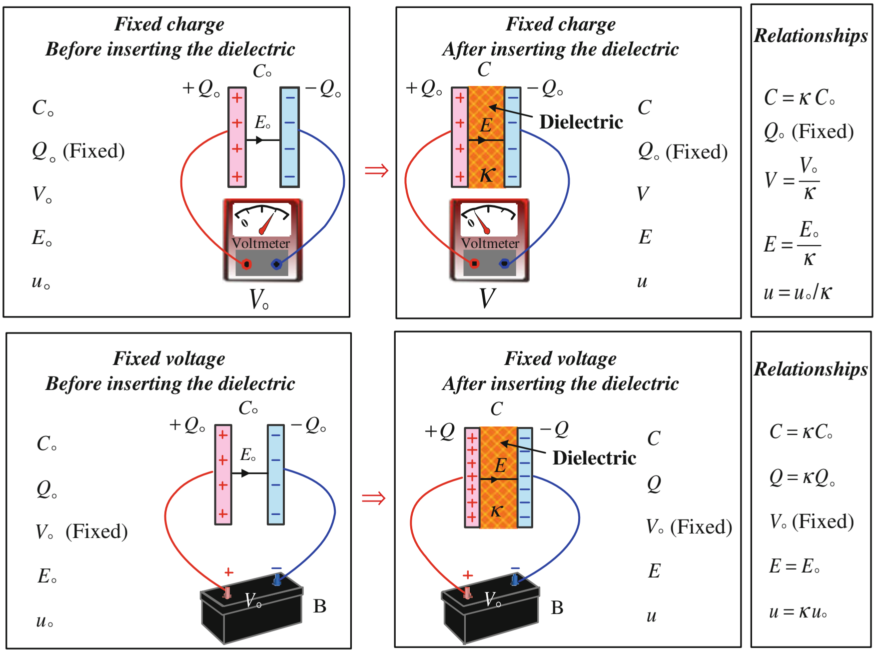


Fig. 23.35 See Exercise (40)

24 Electric Circuits

In this chapter we analyze simple electric circuits that contain devices such as batteries, resistors, and capacitors in various combinations. We begin by introducing **steady-state** electric circuits and the concept of a constant rate of flow of electric charges, known as **direct current (dc)**. We also introduce **Kirchhoff's two rules**, which are used to simplify and analyze more complicated circuits. Finally, we consider circuits containing resistors and capacitors, in which currents can *vary* with time.

24.1 Electric Current and Electric Current Density

Electric Current

When there is a net flow of charge across any area, we say there is an **electric current** (or simply **current**) across that area. To maintain a continuous current, we must maintain a net force on the mobile charge in some way. The net force may result, for example, from an electrostatic field. We assume that an electric field $\vec{E}$ is maintained within a conductor such that the charged particle q is acted on by a force $\vec{F} = q\vec{E}$. We refer to this force as the particle's *driving force*.

To define the current, we consider positive charges moving perpendicularly onto a surface area A as shown in Fig. 24.1.

Spotlight
The current I across an area A is defined as the *net charge* flowing perpendicularly to that area per unit time.

H. A. Radi and J. O. Rasmussen, *Principles of Physics*,
Undergraduate Lecture Notes in Physics, DOI: 10.1007/978-3-642-23026-4_24,

Thus, if a net charge ΔQ flows across an area A in a time Δt, the **average current** I_{av} across the area is:

$$I_{av} = \frac{\Delta Q}{\Delta t} \tag{24.1}$$

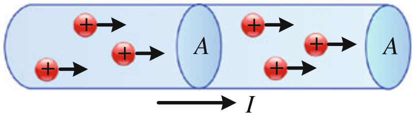


Fig. 24.1 Charged particles in motion perpendicular onto an area A. The current I represents the time rate of flow of charges and has by convention the direction of the motion of positive charges

When the rate of flow varies with time, we define the **instantaneous current** (or the **current**) I as:

$$I = \frac{dQ}{dt} \tag{24.2}$$

The SI unit of the current is **ampere** (abbreviated by A). That is:

$$1\,\text{A} = \frac{1\,\text{C}}{1\,\text{s}} \tag{24.3}$$

Thus, 1 A is equivalent to 1 C of charge passing through the surface area in 1 s. Small currents are more conveniently expressed in *milliamperes* (1 mA $= 10^{-3}$ A) or *microamperes* (1 μA $= 10^{-6}$ A).

Currents can be due to positive charges, or negative charges, or both. In conductors, the current is due to the motion of only negatively charged free electrons (called *conduction electrons*). **By convention**, *the direction of the current is the direction of the flow of positive charges*. Therefore, the direction of the current is opposite to the direction of the flow of electrons, see Fig. 24.2b. A moving charge, positive or negative, is usually referred to as a mobile **charge carrier**.

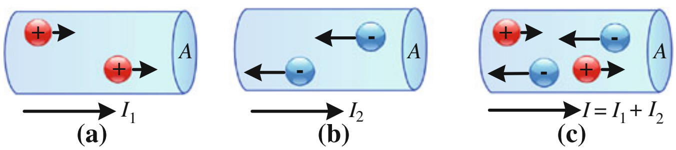


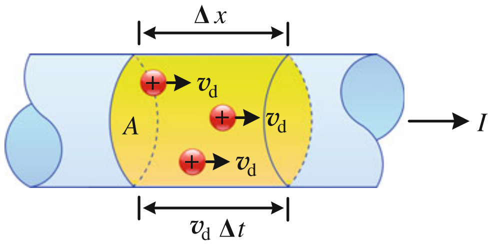
Fig. 24.2 Direction of current due to (a) positive charges, (b) negative charges, and (c) both positive and negative charges

Electric Current Density

The current across an area can be expressed in terms of the motion of the charge carriers. To achieve this we consider a portion of a cylindrical rod that has a cross-sectional area A, length Δx, and carries a *constant* current I, see Fig. 24.3. For convenience we consider positive charge carriers each having a charge q, and the number of carriers per unit volume in the rod is n. Therefore, in this portion, the number of carriers is $n\,A\Delta x$ and the total charge ΔQ is:

$$\Delta Q = (n\,A\Delta x)\,q \tag{24.4}$$

Fig. 24.3 A portion of a straight rod of uniform cross-sectional area A, carrying a constant current I. The mobile charge carriers are assumed to be positive and move with an average speed v_d



Suppose that all the carriers move with an average speed v_d (called the **drift speed**). Therefore, during a time interval Δt, all carriers must achieve a displacement $\Delta x = v_d\,\Delta t$ in the x direction. Now, let us choose Δt such that the carriers in the cylindrical portion move through a displacement whose magnitude is equal to the length of the cylinder, see Fig. 24.3. During such a time interval, all the charge carriers in this cylindrical portion must pass through the circular area A at the right end. Accordingly, we write the last relation as:

$$\Delta Q = (n\,A\,v_d\,\Delta t)\,q \tag{24.5}$$

Therefore, the current $I = \Delta Q/\Delta t$ in the rod will be given by:

$$\boxed{I = n\,q\,v_d\,A} \tag{24.6}$$

The charge carriers in a solid conductor are all free electrons. If the conductor is isolated, these electrons move with speeds of the order of 10^6 m/s, and because of their collisions with the scatterers (atoms or molecules in the conductor), they move randomly in all directions. This results in a *zero drift velocity* and hence no *net* charge transport, which means *zero current*. When an electric field $\vec{E}$ is established across the conductor, this field exerts an electric force $\vec{F} = -e\vec{E}$ on each electron, producing a current. Of course, the electrons do not move in a straight line along the conductor, but their resultant motion is complicated and zigzagged, see Fig. 24.4. Regardless of the collisions of these electrons, they move slowly along the conductor in a direction opposite to $\vec{E}$ with a drift velocity $\vec{v}_d$, see Fig. 24.4.

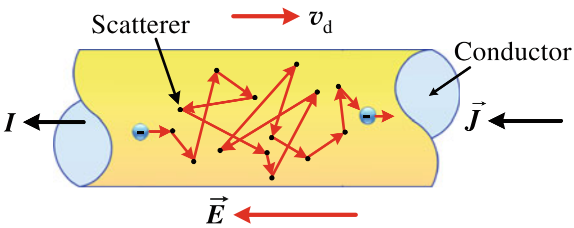


Fig. 24.4 A schematic representation of the random zigzag motion and the drift of a free electron with an average speed v_d in a conductor, due to the effect of an external electric field $\vec{E}$

The **current density** J is defined as the current per unit area, i.e.:

$$\boxed{J = \frac{I}{A}} \tag{24.7}$$

Using the relation $I = n\,q\,v_d\,A$, we get:

$$\boxed{J = n\,q\,v_d} \tag{24.8}$$

where the SI unit of the current density is A/m^2. Equation 24.8 is valid only if J is uniform and the direction of I is perpendicular to the cross-sectional area A. Generally, the current density is a **vector** quantity that has the direction of $q\,\vec{v}_d$, for both signs of q; that is:

$$\boxed{\vec{J} = n\,q\,\vec{v}_d} \tag{24.9}$$

The amount of current that passes through an element of area dA, can be written as $\vec{J} \bullet d\vec{A}$, where $d\vec{A}$ is the vector area of the element. The current that passes throughout the entire area A is thus:

$$\boxed{I = \int \vec{J} \bullet d\vec{A}} \tag{24.10}$$

If the current density is uniform across the area and parallel to $d\vec{A}$, then this equation leads to Eq. 24.7.

Example 24.1

Estimate the drift speed of the conduction electrons in a copper wire that is 2 mm in diameter and carries a current of 1 A. Comment on your result. The density of copper is 8.92×10^3 kg/m^3.
[Hint: Assume that each copper atom contributes one free conduction electron to the current.]

Solution: To get the drift speed v_d, we need to find the free-electron density n. To get n, we need to know the volume occupied by one kmol of copper. From the periodic table of elements, see Appendix C, the molar mass of copper is $M(\text{Cu}) = 63.546$ kg/kmol. Recall that the mass of one kmol of $^{63.5}$Cu contains Avogadro's number of atoms ($N_A = 6.022 \times 10^{26}$ atoms/kmol). Thus:

$$\text{Volume of 1 kmol} = \frac{\text{Mass of 1 kmol}}{\text{Density}} \quad \Rightarrow \quad V = \frac{M}{\rho}$$

$$\text{Number of copper atoms/m}^3 = \frac{\text{Avogadro's number}}{\text{Volume of 1 kmol}} \quad \Rightarrow \quad n = \frac{N_A}{V} = \frac{N_A\,\rho}{M}$$

Therefore:

$$\begin{aligned} n &= \frac{N_A\,\rho}{M} = \frac{(6.022 \times 10^{26}\,\text{atoms/kmol})(8.92 \times 10^3\,\text{kg/m}^3)}{63.546\ \text{kg/kmol}} \\ &= 8.45 \times 10^{28}\,\text{atoms/m}^3 \end{aligned}$$

Since the density of free-electrons is equal to the density of copper atoms, then we use Eq. 24.6 to find the drift speed as follows:

$$\begin{aligned} v_d &= \frac{I}{neA} = \frac{1\ \text{C/s}}{(8.45 \times 10^{28}\,\text{electrons/m}^3)(1.6 \times 10^{-19}\,\text{C})(\pi \times (10^{-3}\,\text{m})^2)} \\ &= 2.35 \times 10^{-5}\,\text{m/s} = 8.46\,\text{cm/h} \qquad \text{(Very small speed)} \end{aligned}$$

You might ask why, even though v_d is so small, that regular light bulbs light up very quickly when one turns on its circuit switch? The answer is that the electric field travels along the connecting wires of the circuit at almost the speed of light, so electrons everywhere in the wires all begin to *drift at once* with a small drift speed.

Example 24.2

One end of the copper wire in example 1 is welded to one end of an aluminum wire with a 4 mm diameter. The composite wire carries a steady current equal to that of Example 24.1 (i.e. $I = 1$ A). (a) What is the current density in each wire? (b) What is the value of the drift speed v_d in the aluminum? [Aluminum has one free electron per atom and density 2.7×10^3 kg/m^3]

Solution: (a) Except near the junction, the current density in a copper wire of radius $r_{Cu} = 1$ mm and aluminum wire of radius $r_{Al} = 2$ mm are:

$$J_{Cu} = \frac{I}{A_{Cu}} = \frac{I}{\pi r_{Cu}^2} = \frac{1\,\text{A}}{\pi \times (10^{-3}\,\text{m})^2} = 3.18 \times 10^5\ \text{A/m}^2$$

$$J_{Al} = \frac{I}{A_{Al}} = \frac{I}{\pi r_{Al}^2} = \frac{1\,\text{A}}{\pi \times (2 \times 10^{-3}\,\text{m})^2} = 7.96 \times 10^4\ \text{A/m}^2$$

(b) From the periodic table of elements, see Appendix C, the molar mass of aluminum is $M(\text{Al}) = 26.98$ kg/kmol. As in Example 24.1, we find:

$$n = \frac{N_A\,\rho}{M} = \frac{(6.022 \times 10^{26}\,\text{atoms/kmol})(2.7 \times 10^3\,\text{kg/m}^3)}{26.98\,\text{kg/kmol}}$$
$$= 6.03 \times 10^{28}\ \text{atoms/m}^3$$

$$v_d = \frac{I}{neA} = \frac{1\,\text{C/s}}{(6.03 \times 10^{28}\,\text{electrons/m}^3)(1.6 \times 10^{-19}\,\text{C})(\pi \times (2 \times 10^{-3}\,\text{m})^2)}$$
$$= 8.25 \times 10^{-6}\,\text{m/s} = 2.97\,\text{cm/h}$$

24.2 Ohm's Law and Electric Resistance

As a result of maintaining a potential difference ΔV across a conductor, an electric field $\vec{E}$ and a current density $\vec{J}$ are established in the conductor. For materials with electrical properties that are the same in all directions (isotropic materials), the electric field is found to be proportional to the current density. That is:

$$\boxed{\vec{E} = \rho\,\vec{J}} \qquad \text{(Ohm's law)} \tag{24.11}$$

where the constant ρ[1] is called the **resistivity** of the conductor. Materials that obey this relation are said to obey **Ohm's law**:

[1] Not to be confused with ρ referring to mass density or charge density.

Spotlight
For many materials and most metals, the ratio of the magnitude of the electric field to the magnitude of the current density is a constant and does not depend on the electric field producing the current.

Since it is difficult to measure $\vec{E}$ and $\vec{J}$ directly, we need to put Ohm's law into a more practical form. This can be obtained by considering a portion of a straight conductor that has a uniform cross-sectional area A and length L, as shown in Fig. 24.5. In addition, a potential difference $\Delta V = V_b - V_a$ between the ends of the conductor (denoted by a and b) will create a *straight* electric field and current, as also shown in Fig. 24.5. Since charge carriers in conductors are electrons, they will drift from face a to face b, against the field $\vec{E}$.

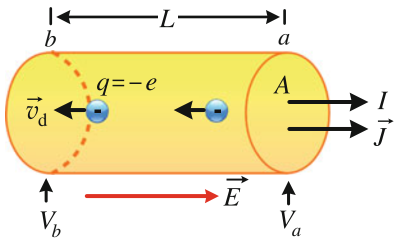


Fig. 24.5 A potential difference $\Delta V = V_b - V_a$ across a conductor of cross-sectional area A and length L sets up a field $\vec{E}$ and current I

Recall that for uniform electric fields we have:

$$\Delta V = E\,L \tag{24.12}$$

Using this relation to eliminate E from the scalar form of Eq. 24.11, we get:

$$\frac{\Delta V}{L} = \rho J \tag{24.13}$$

Also, using $J = I/A$, the potential difference ΔV can be written as:

$$\Delta V = \left(\rho \frac{L}{A}\right) I \tag{24.14}$$

The quantity in brackets is called the **electrical resistance** (or simply **resistance**) of the conductor and is denoted by the symbol R; that is:

$$\boxed{R = \rho \frac{L}{A}} \quad \Rightarrow \quad \boxed{R \propto \rho} \tag{24.15}$$

We can define the resistance R as a proportionality constant to the relation $\Delta V \propto I$ and write the equivalent Ohm's law as:

$$\boxed{\Delta V = I\,R \quad \left\{\begin{array}{l}\text{Equivalent form}\\ \text{of Ohm's law}\end{array}\right\}} \tag{24.16}$$

The SI unit of resistance is **ohm** (abbreviated by Ω). That is:

$$1\,\Omega = \frac{1\,\mathrm{V}}{1\,\mathrm{A}} \tag{24.17}$$

This means that if one applies a potential difference of 1 V across a conductor and this causes 1 A to flow, then the resistance of the conductor is 1 Ω. Note that according to Eq. 24.15, the SI unit of resistivity is ohm-meter (Ω.m). Also, since $\Delta V = V_b - V_a$, we note that the *direction of the current* is in the direction of *decreasing potential*.

The inverse of resistivity is called the conductivity σ, thus:

$$\boxed{\sigma = \frac{1}{\rho}} \tag{24.18}$$

where the SI unit of σ is $(\Omega.\mathrm{m})^{-1}$. The resistance of a conductor can also be written in terms of the conductivity as follows:

$$R = \frac{1}{\sigma}\frac{L}{A} \tag{24.19}$$

Equations 24.15 and 24.19 hold true only for isotropic conductors. Additionally, the resistance in Eq. 24.15 depends on the geometry of the resistor through the length L, area A, and resistivity ρ, which is a constant for a specific metallic conductor (assuming a constant temperature).

A material obeying Ohm's law is called an *ohmic* material or a *linear* material. If a material does not obeys Ohm's law, the material is called a *non-ohmic* or a *nonlinear material*.

Variation of Resistance with Temperature

The variation of resistivity with temperature is mostly linear over a broad range. Since $R \propto \rho$, then for most engineering purposes a good empirical linear approximation for ρ and R can be written as:

$$\boxed{\rho = \rho_\circ[1 + \alpha(T - T_\circ)]} \quad \text{or} \quad \boxed{R = R_\circ[1 + \alpha(T - T_\circ)]} \tag{24.20}$$

where ρ is the resistivity at temperature T (in degrees Celsius), $\rho_\circ$ is the resistivity at a reference temperature $T_\circ$ (usually selected to be 20 °C), and α is the **temperature coefficient of resistivity**. The same applies for the resistance. The coefficient α is selected such that Eq. 24.20 matches best with experimental measurements for the selected range of temperatures. From Eq. 24.20, we find that:

$$\boxed{\alpha = \frac{1}{\rho_\circ}\frac{\Delta\rho}{\Delta T} = \frac{1}{R_\circ}\frac{\Delta R}{\Delta T}} \quad \text{with} \quad \begin{cases} \Delta\rho = \rho - \rho_\circ \\ \Delta R = R - R_\circ \\ \Delta T = T - T_\circ \end{cases} \tag{24.21}$$

Table 24.1 lists the resistivity ρ and the temperature coefficient of resistivity α for some materials at 20 °C.

Table 24.1 The resistivity and temperature coefficient of resistivity for various materials at 20° Celsius

Material	Resistivity ρ (Ω.m)	Temperature coefficient of resistivity α [$(\mathrm{C}^\circ)^{-1}$]
Silver	1.59×10^{-8}	3.8×10^{-3}
Copper	1.7×10^{-8}	3.9×10^{-3}
Gold	2.44×10^{-8}	3.4×10^{-3}
Aluminum	2.82×10^{-8}	3.9×10^{-3}
Tungsten	5.6×10^{-8}	4.5×10^{-3}
Iron	10×10^{-8}	5.0×10^{-3}
Platinum	11×10^{-8}	3.98×10^{-3}
Lead	22×10^{-8}	3.9×10^{-3}
Nichrome[a]	1.50×10^{-6}	0.4×10^{-3}
Carbon	3.5×10^{-5}	-0.5×10^{-3}
Germanium	0.46	-48×10^{-3}
Silicon	640	-75×10^{-3}
Glass	10^{10}–10^{14}	
Hard rubber	$\sim 10^{13}$	
Sulfur	10^{15}	
Fused quartz	75×10^{16}	

[a] A nickel–chromium alloy commonly used in heating elements

Most electric circuits use elements called **resistors** to control the current flowing through the circuit. Values of the resistance are normally indicated by color-coding as shown in Tables 24.2 and 24.3.

Table 24.2 Color-coding for resistors

Color	Number	Multiplier
Black	0	1
Brown	1	10^1
Red	2	10^2
Orange	3	10^3
Yellow	4	10^4
Green	5	10^5
Blue	6	10^6
Violet	7	10^7
Gray	8	10^8
White	9	10^9
Gold	–	10^{-1}
Silver	–	10^{-2}

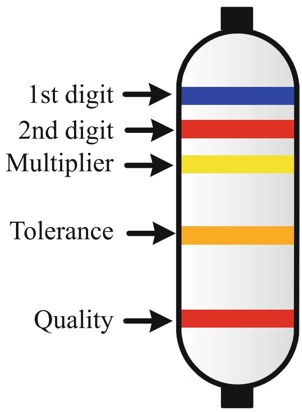
Table 24.3 Tolerance-coding

Color	Number	Multiplier	Tolerance
Gold	–	10^{-1}	5%
Silver	–	10^{-2}	10%
Colorless	–	–	20%

How to Read the Color-coding

- First find the tolerance band; it will typically be gold (5%) or silver (10%), and sometimes colorless (20%), see the example shown in Fig. 24.6. In this example, the color is Gold, so 5% tolerance.

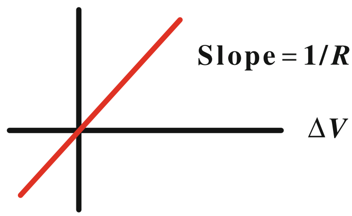
Fig. 24.6



- Starting from the other end, identify the first band, write down the number associated with that color; in this example Blue is '6'.
- Now read the next color, in this example it is Red, so write down a '2' next to the six (You should have '62' so far).
- Now read the third color, which indicates the multiplier exponent band, and write that down as the power of ten for the multiplier of the resistance value. In this example the multiplier is Yellow which represents 'four', so we get '$62 \times 10^4\ \Omega$'.
- If the resistor has one extra band past the tolerance band, it is a quality band. Read the number as the % Failure rate per 1,000 h. In this example it is Red, so that we can expect a 2% failure rate per 1,000 h.

All Ohmic resistors have a linear-potential-difference relationship over a broad band of applied potential differences. The slope of the I versus ΔV curve in the linear region yields a value for $1/R$, see Fig. 24.7.

Fig. 24.7



Example 24.3

At 20 °C, a copper wire has a diameter of 4 mm, a length of 10 m, a resistivity of $1.7 \times 10^{-8}\ \Omega$.m, a temperature coefficient of resistivity of $3.9 \times 10^{-3}\ (\mathrm{C}^\circ)^{-1}$, and carries a current of 1 A. (a) What is the current density in the wire? (b) What is the magnitude of the electric field applied to the wire? (c) What is the potential difference between the two ends of the wire? (d) What is the resistance of the wire? (e) When the wire is used in a thermometer for measuring the melting point of indium, the resistance calculated in part (d) increases to 0.0207 Ω. Find the melting point temperature of indium.

Solution: (a) The current density in a copper wire of radius 2 mm is:

$$J = \frac{I}{A} = \frac{I}{\pi r^2} = \frac{1\ \mathrm{A}}{\pi \times (2 \times 10^{-3}\ \mathrm{m})^2} = 7.96 \times 10^4\ \mathrm{A/m^2}$$

(b) From Eq. 24.11, the electric field is given by:

$$E = \rho J = (1.7 \times 10^{-8}\,\Omega.\text{m})(7.96 \times 10^{4}\,\text{A/m}^2) = 1.353 \times 10^{-3}\,\text{V/m}$$

(c) Using Eq. 24.12, the potential difference will be given by:

$$\Delta V = E L = (1.353 \times 10^{-3}\,\text{V/m})(10\,\text{m}) = 1.353 \times 10^{-2}\,\text{V}$$

(d) From Eq. 24.15, the resistance of the wire is:

$$R = \rho \frac{L}{A} = 1.7 \times 10^{-8}\,\Omega.\text{m} \frac{10\,\text{m}}{\pi \times (2 \times 10^{-3}\,\text{m})^2} = 0.0135\,\Omega$$

(e) Solving Eq. 24.21 for ΔT and then using: (1) the calculated resistance $R_\circ = 0.0135\,\Omega$ at the reference temperature $T_\circ = 20\,°\text{C}$, (2) the value of α, and (3) the final resistance $R = 0.0207\,\Omega$, we obtain:

$$\Delta T = \frac{\Delta R}{\alpha R_\circ} = \frac{R - R_\circ}{\alpha R_\circ} = \frac{0.0207 - 0.0135\,\Omega}{[3.9 \times 10^{-3}\,(\text{C}°)^{-1}](0.0135\,\Omega)} = 136.8\,\text{C}°$$

Since $T_\circ = 20\,°\text{C}$, we find that the melting point of indium is:

$$T = T_\circ + \Delta T = 20\,°\text{C} + 136.8\,\text{C}° = 156.8\,°\text{C}$$

Example 24.4

A cylindrical shell of length $L = 20\,\text{cm}$ is made of aluminum and has an inner radius $a = 2\,\text{mm}$ and an outer radius $b = 4\,\text{mm}$, see Fig. 24.8. Assume that the shell has a uniform current density $J = 2 \times 10^5\,\text{A/m}^2$ in the direction of the wire's length. (a) What is the current through the shell? (b) What is the resistance of the shell and the potential difference ΔV?

Solution: (a) Since the current density is uniform across any plane perpendicular to the length of the shell, we can use the relation $I = J A$ to find the current. First, we calculate the cross-sectional area of the shell as follows:

$$\begin{aligned} A &= \pi b^2 - \pi a^2 = \pi [b^2 - a^2] \\ &= \pi [(4 \times 10^{-3}\,\text{m})^2 - (2 \times 10^{-3}\,\text{m})^2] = 3.77 \times 10^{-5}\,\text{m}^2 \end{aligned}$$

Then, we use this result to find I as follows:

$$I = J A = (2 \times 10^5\,\text{A/m}^2)(3.77 \times 10^{-5}\,\text{m}^2) = 7.54\,\text{A}$$

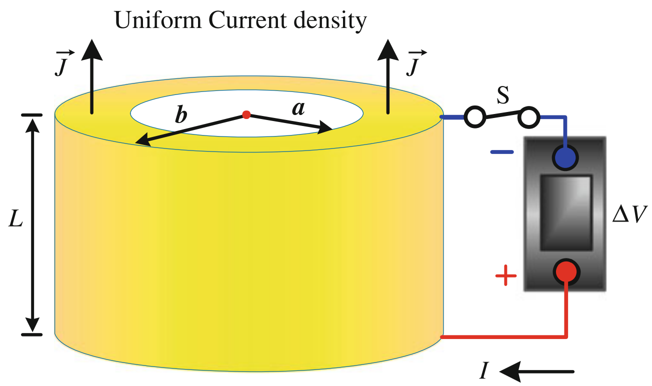


Fig. 24.8

(b) From Table 24.1, the resistivity of aluminum is $2.82 \times 10^{-8}\,\Omega$.m. We use Eq. 24.15 to find the resistance of the shell, and then use Eq. 24.16 to find the potential difference ΔV as follows:

$$R = \rho\frac{L}{A} = 2.82 \times 10^{-8}\,\Omega.\mathrm{m}\frac{0.2\,\mathrm{m}}{3.77 \times 10^{-5}\,\mathrm{m}^2} = 1.5 \times 10^{-4}\,\Omega$$

$$\Delta V = IR = (7.54\,\mathrm{A})(1.5 \times 10^{-4}\,\Omega) = 1.13 \times 10^{-3}\,\mathrm{V}$$

Example 24.5

A conducting rod of radius $a = 2$ mm is concentric with a conducting cylindrical shell that has a radius $b = 4$ mm and length $L = 2.94$ cm, see Fig. 24.9a. The space between the rod and the shell is tightly packed with silicon of resistivity $\rho = 640\,\Omega$.m. A battery of potential difference $\Delta V = 12$ V is connected in such a way that the current through the silicon flows in the radial direction. (a) Find the resistance of the silicon between the rod and the shell. (b) Find the radial current in the circuit. (c) Find the radial current density and electric field at the inner and outer surfaces of the silicon.

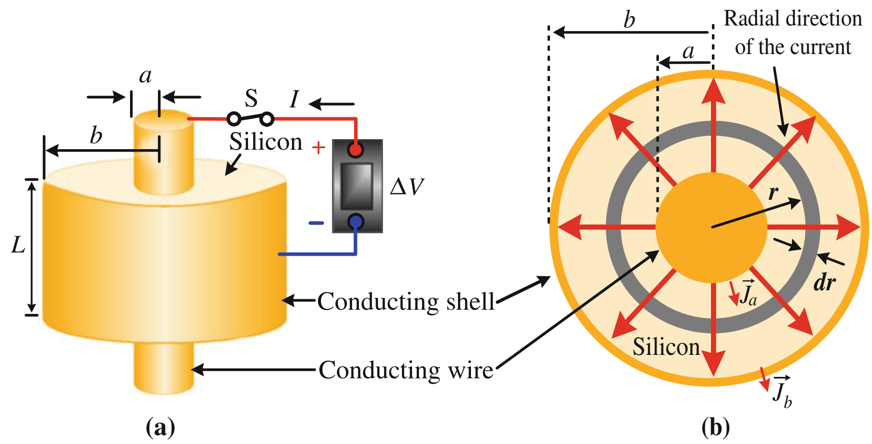


Fig. 24.9

Solution: (a) Cylindrical symmetry of the silicon suggests a radial flow of the current density $\vec{J}$. Equation 24.15 cannot be used directly because the cross section through which the charge travels varies from $2\pi\, a\, L$ (at the inner cylindrical face) to $2\pi\, b\, L$ (at the outer cylindrical face). Therefore, we consider a cylindrical silicon shell element of an inner radius r, height L, face area $A = 2\pi\, r\, L$, and thickness $d\,r$, see Fig. 24.9b. This shell element has a resistance $d\,R$. In this case, Eq. 24.15 will take the following form:

$$d\,R = \rho \frac{d\,r}{2\pi\, r\, L}$$

To find the total resistance across the entire silicon, we must integrate the previous expression from $r = a$ to $r = b$. Thus:

$$R = \int_a^b d\,R = \int_a^b \rho \frac{d\,r}{2\pi\, r\, L} = \frac{\rho}{2\pi\, L} \int_a^b \frac{d\,r}{r} = \frac{\rho}{2\pi\, L} \ln\left(\frac{b}{a}\right)$$

Now, substituting with the given values, we get:

$$R = \frac{\rho}{2\pi\, L} \ln\left(\frac{b}{a}\right) = \frac{640\ \Omega.\text{m}}{2\pi\,(2.94 \times 10^{-2}\ \text{m})} \ln\left(\frac{4\ \text{mm}}{2\ \text{mm}}\right) = 2.4 \times 10^3\ \Omega$$

(b) Knowing the resistance R and the potential difference ΔV, we use Ohm's law given by Eq. 24.16 to find the total current in the silicon (which is the current in the circuit) as follows:

$$I = \frac{\Delta V}{R} = \frac{12\,\text{V}}{2.4 \times 10^3\,\Omega} = 5 \times 10^{-3}\,\text{A} = 5\,\text{mA}$$

(c) At the inner and outer faces of the silicon, namely $2\pi\, a L$ and $2\pi\, b L$, respectively, we use Eq. 24.7 to find the corresponding current density as follows:

$$J_a = \frac{I}{A_a} = \frac{I}{2\pi\, a L} = \frac{5 \times 10^{-3}\,\text{A}}{2\pi (2 \times 10^{-3}\,\text{m})(2.94 \times 10^{-2}\,\text{m})} = 13.53\,\text{A/m}^2$$

$$J_b = \frac{I}{A_b} = \frac{I}{2\pi\, b L} = \frac{5 \times 10^{-3}\,\text{A}}{2\pi (4 \times 10^{-3}\,\text{m})(2.94 \times 10^{-2}\,\text{m})} = 6.77\,\text{A/m}^2$$

Finally, we use Ohm's law given by Eq. 24.11 to find the corresponding electric fields at the inner and outer faces of the silicon as follows:

$$E_a = \rho J_a = (640\,\Omega.\text{m})(13.53\,\text{A/m}^2) = 8.659 \times 10^3\,\text{V/m}$$
$$E_b = \rho J_b = (640\,\Omega.\text{m})(6.77\,\text{A/m}^2) = 4.333 \times 10^3\,\text{V/m}$$

24.3 Electric Power

When a battery is used to establish an electric current in a light bulb, the battery transforms its stored chemical energy to kinetic energy of the electrons. These electrons flow through the filament of the light bulb, and result in an increase in the temperature of the filament. It is important to calculate the rate of this energy transfer.

Figure 24.10 shows a battery of potential difference ΔV connected to a simple circuit (our system) containing a resistor of resistance R. The resistor is usually represented by the symbol . Unless noted otherwise, we assume that the connecting wires have zero resistance.

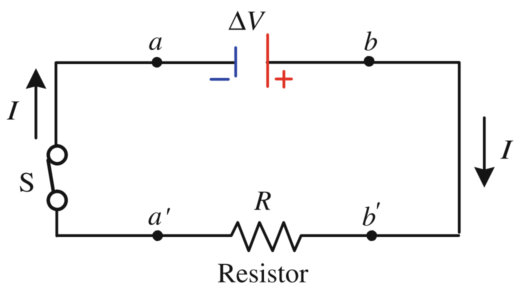


Fig. 24.10 A simple circuit containing one battery and one resistor

Now, imagine a positive charge dQ flowing clockwise from point a through the battery and the resistor, and back to the same point a. In a time interval dt a quantity of charge dQ enters point a, and an equal quantity leaves point b. Thus, the electric potential energy of the system *increases* by the amount $dU = dQ\, \Delta V$, see Eq. 22.18, while the stored chemical potential energy of the battery *decreases* by the same amount. On the other hand, as the charge enters the resistor at b' and an equal quantity leaves a' (which is identical to a) over the same time dt, the system *loses* this energy through collisions with the molecules of the resistor. The net result is that some of the chemical energy of the battery has been delivered to the resistor as internal energy associated with molecular vibration (rise in temperature). This rise in temperature will ultimately transfer to the surroundings through thermal radiation.

The rate at which the system loses energy as the charges pass through the resistor is:

$$\frac{dU}{dt} = \frac{dQ\, \Delta V}{dt} = \frac{dQ}{dt} \Delta V = I\, \Delta V \tag{24.22}$$

where I is the current. This rate is equal to the rate at which the resistor *gains* internal energy, and is defined as the power P:

$$P = I\, \Delta V \tag{24.23}$$

Using the relation $\Delta V = I\, R$ for a resistor of resistance R, the electric power P delivered in the resistor can be written in the following form:

$$\boxed{P = I\, \Delta V = I^2 R = \frac{(\Delta V)^2}{R}} \tag{24.24}$$

Because $P = I\, \Delta V$, the same amount of power P can be transported either at high I and low ΔV, or at low I and high ΔV.

Example 24.6

A 220 V potential difference is maintained across an electric heater that is made from a nichrome wire of resistance 20 Ω. (a) Find the current in the wire and the power rating of the heater. (b) At an estimated price of 0.35 LE (Egyptian pound) per kilowatt-hour of electricity, what is the cost of operating the heater for 2 h?

Solution: (a) Using $\Delta V = I\,R$, we get:

$$I = \frac{\Delta V}{R} = \frac{220\,\text{V}}{20\,\Omega} = 11\,\text{A}$$

Using the power expression $P = I^2\,R$, we find that:

$$P = I^2\,R = (11\,\text{A})^2(20\,\Omega) = 2{,}420\,\text{W}$$

(b) The amount of energy transferred in time Δt is $P\,\Delta t$. Thus:

$$P\,\Delta t = (2{,}420\,\text{W})(2\,\text{h}) = 4{,}840\,\text{Wh} = 4.84\,\text{kWh}$$

If energy is purchased at 35 piaster per kilowatt-hour then the cost is:

$$\text{Cost} = (4.84\,\text{kWh})(0.35\,\text{LE/kWh}) = 1.69\,\text{LE}$$

24.4 Electromotive Force

We previously introduced the battery as a device that produces a potential difference and causes charges to move. In fact, it is a device that works as an *energy converter*. A battery is often called a *source of electromotive force* or, a *source of emf* (this unfortunate historical name describes a potential difference in volts, but not a force).

> Spotlight
> The emf $\mathcal{E}$ of a battery is the maximum possible potential difference that the battery can provide between its terminals, usually the voltage at zero current.

Figure 24.11a shows a device (a battery) with an emf $\mathcal{E}$ that is used in a simple circuit containing a resistor of resistance R. The battery keeps one terminal (labeled with the sign $+$) at a higher electric potential than the other (labeled with the sign $-$). Therefore, within the battery, the *conventional positive charge carriers* move from a region of low electric potential (at the negative terminal) to a region of higher electric potential (at the positive terminal).

Because a real battery is made of matter, there is a resistance against the flow of charge within the battery. This resistance is called the battery's **internal resistance** and is usually denoted by r. For an *ideal battery* with *zero internal resistance*, the

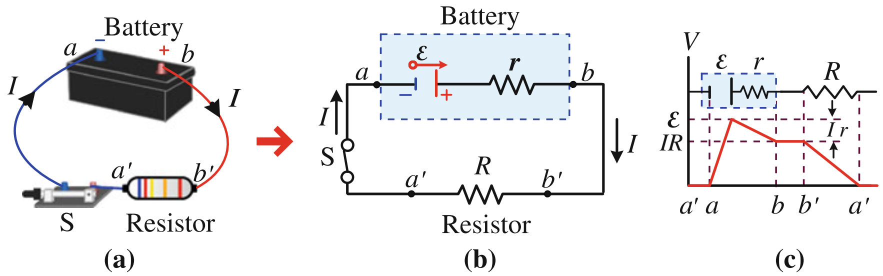


Fig. 24.11 (a) A simple circuit containing a resistor connected to a battery. (b) A circuit diagram of a source of emf $\mathcal{E}$ (the battery) of internal resistance r, connected to a resistor of resistance R. (c) Graphical representation of the electric potential at different points

potential difference between its terminals is equal to its emf $\mathcal{E}$ (*directed from the* $-$ *terminal to the* $+$ *terminal*). For *real batteries*, this is not the case.

We now consider the circuit diagram in Fig. 24.11b, which is the same as the *real* emf device of Fig. 24.11a, except we represent the battery with a dashed rectangular box containing an *ideal* emf $\mathcal{E}$ in series with an internal resistance r. Let us start at point a (where the potential is V_a), and move *clockwise* to point b (where the potential is V_b), and measure the electric potential at different locations. When we move from the negative terminal to the positive terminal, the potential *increases* by the amount of the emf $\mathcal{E}$. However, as we move through the internal resistance r in the *direction of the current* I, the potential *drops* by an amount Ir. Thus, the potential difference between the terminals of the battery $\Delta V = V_b - V_a$ is:

$$\boxed{\Delta V = \mathcal{E} - Ir} \quad (\Delta V = \mathcal{E} \text{ for an open-circuit}) \tag{24.25}$$

We always assume that the wires in the circuit have no resistance, unless otherwise indicated. This means that the potentials of points a and a' are the same. The same applies to points b and b'. Thus:

$$V_b - V_a = V_{b'} - V_{a'} = \Delta V \tag{24.26}$$

But according to Ohm's law, given by Eq. 24.16, $V_{b'} - V_{a'}$ must equal IR. Thus, $V_b - V_a = V_{b'} - V_{a'} = IR$. Combining this expression with Eq. 24.25, we find that:

$$\boxed{\mathcal{E} = IR + Ir} \tag{24.27}$$

Solving for the current, we get:

$$I = \frac{\mathcal{E}}{R + r} \tag{24.28}$$

Note that the current I depends on the resistance R of the external resistor (which is called the load) and the internal resistance r of the battery. Since $R \gg r$ in most circuits, we can usually neglect r.

Example 24.7

A device is connected to a battery that has an emf $\mathcal{E} = 9$ V and internal resistance $r = 0.02\ \Omega$. Find the current in the circuit and the terminal voltage of the battery when the device is a: (a) light bulb that has a resistance $R = 4\ \Omega$, see Fig. 24.12a. (b) conducting wire having zero resistance, i.e. the battery is short circuited by this conductor, see Fig. 24.12b.

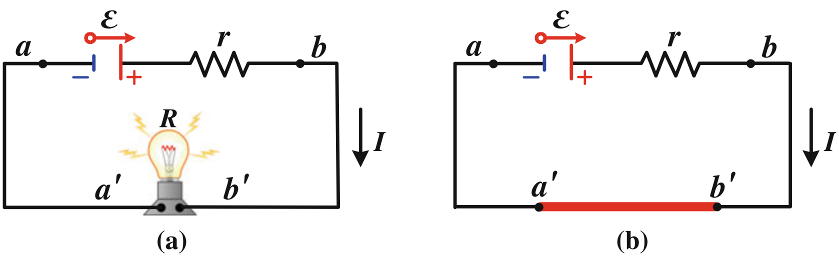


Fig. 24.12

Solution: (a) Equation 24.28 gives us the value of the current as:

$$I = \frac{\mathcal{E}}{R + r} = \frac{9\ \text{V}}{4\ \Omega + 0.02\ \Omega} = 2.24\ \text{A}$$

From Eq. 24.25, the terminal voltage of the battery will be given by:

$$\Delta V = \mathcal{E} - Ir = 9\ \text{V} - (2.24\ \text{A})(0.02\ \Omega) = 8.96\ \text{V}$$

(b) When we use a conducting wire, it is as if we have a device of $R = 0$. This results in a current and terminal voltage of the battery as follows:

$$I = \frac{\mathcal{E}}{r} = \frac{9\ \text{V}}{0.02\ \Omega} = 450\ \text{A}$$

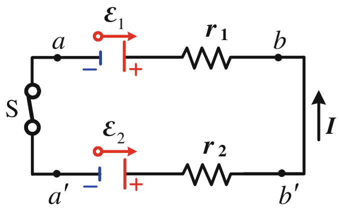
$$\Delta V = \mathcal{E} - Ir = 9\,\text{V} - (450\,\text{A})(0.02\,\Omega) = 0$$

Such large values for the current I would result in a very quick depletion of the battery as all of its stored energy would be quickly transferred to the conducting wire in the form of heat energy. The term "short circuit" is applied to such cases, and they can cause fire or burns.

Example 24.8

A battery that has an emf $\mathcal{E}_1 = 9$ V and internal resistance $r_1 = 0.02\ \Omega$ is connected to a second battery of $\mathcal{E}_2 = 12$ V and $r_2 = 0.04\ \Omega$, such that their like terminals are connected, see Fig. 24.13. Find the current in the circuit and the terminal voltage across each battery.

Fig. 24.13



Solution: The two batteries are oppositely directed around the circuit. Since $\mathcal{E}_2 > \mathcal{E}_1$, then the net emf $\mathcal{E}_{\text{net}}$ in this circuit will be in the counterclockwise direction, i.e.:

$$\mathcal{E}_{\text{net}} = \mathcal{E}_2 - \mathcal{E}_1 = 12 - 9\,\text{V} = 3\,\text{V} \quad \text{(Counterclockwise direction)}$$

Consequently, the current I in this circuit will also be in the counterclockwise direction as indicated in Fig. 24.13. This current is opposite to the *discharging* current that the $\mathcal{E}_1 = 9$ V battery should produce when connected to circuits containing only resistors. Actually, this current will *charge* the $\mathcal{E}_1 = 9$ V battery.

The total resistance of this circuit is only due to the presence of the internal resistances r_1 and r_2 of the two batteries. Therefore, Eq. 24.28 gives us the value of the current as follows:

$$I = \frac{\mathcal{E}_2 - \mathcal{E}_1}{r_1 + r_2} = \frac{12\,\text{V} - 9\,\text{V}}{0.02\,\Omega + 0.04\,\Omega} = \frac{3\,\text{V}}{0.06\,\Omega} = 50\,\text{A}$$

Depending on the direction of the current in each battery, the terminal voltages across the batteries are:

$$\begin{aligned}\Delta V &= V_b - V_a = \mathcal{E}_1 + Ir_1 = 9\text{ V} + (50\text{ A})(0.02\,\Omega)\\ &= 10\text{ V} \quad (\text{Gain from } a \text{ to } b)\\ \Delta V &= V_{b'} - V_{a'} = \mathcal{E}_2 - Ir_2 = 12\text{ V} - (50\text{ A})(0.04\,\Omega)\\ &= 10\text{ V} \quad (\text{Drop from } a' \text{ to } b')\end{aligned}$$

24.5 Resistors in Series and Parallel

Resistors in a circuit may be used in different combinations, and we can sometimes replace a combination of resistors with one *equivalent resistor*. In this section, we introduce two basic combinations of resistors that allow such a replacement.

Resistors in a Series Combination

Figure 24.14a shows two resistors R_1 and R_2 that are connected in **series** with a battery B. Figure 24.14b shows a circuit diagram for this combination of resistors.

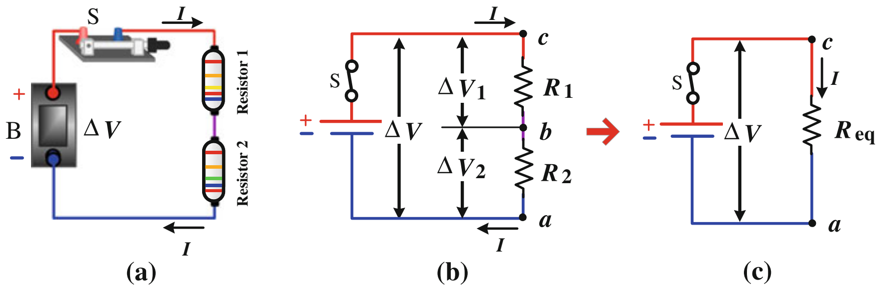


Fig. 24.14 (a) Two resistors are connected in series to a battery B that has a potential difference ΔV. (b) The circuit diagram for this series combination. (c) An equivalent resistance R_{eq} replacing the original resistors set up in a series combination

When the circuit is connected, the amount of charge that passes through R_1 must also pass through R_2 in the same time interval. Otherwise, charge will accumulate on the wire between resistors. Thus, for series combination of resistors, the current I is the same in both resistors. Figure 24.14c shows a single resistor R_{eq} that is equivalent to this combination and has the same effect on the circuit. This means that when the potential difference ΔV is applied across the equivalent resistor, it must produce the same current I as in the series combination.

The potential difference ΔV is divided to ΔV_1 and ΔV_2 across the resistors R_1 and R_2, respectively. Thus:

$$\Delta V = \Delta V_1 + \Delta V_2 \tag{24.29}$$

For the two resistors in Fig. 24.14b, we have:

$$\Delta V_1 = V_c - V_b = IR_1 \quad \text{and} \quad \Delta V_2 = V_b - V_a = IR_2 \tag{24.30}$$

Substituting in Eq. 24.29, we get:

$$\Delta V = IR_1 + IR_2 \tag{24.31}$$

The equivalent resistor R_{eq} has the same applied potential difference ΔV and the same circuit current I flowing through it; thus:

$$\Delta V = IR_{eq} = IR_1 + IR_2 \tag{24.32}$$

Canceling I, we arrive at the following relationship:

$$R_{eq} = R_1 + R_2 \quad \text{(Series combination)} \tag{24.33}$$

We can extend this treatment to n resistors connected in series as:

$$\boxed{R_{eq} = R_1 + R_2 + \cdots + R_n} \quad \text{(Series combination)} \tag{24.34}$$

Thus, the equivalent resistor of a series combination of resistors is simply the algebraic sum of the individual resistances and will always be greater than any one of them.

Example 24.9

In Fig. 24.14, let $R_1 = 6\,\Omega$, $R_2 = 3\,\Omega$, and $\Delta V = 18$ V. Find I in the circuit and the potential differences ΔV_1 and ΔV_2.

Solution: The equivalent resistance of the series combination is:

$$R_{eq} = R_1 + R_2 = 6\,\Omega + 3\,\Omega = 9\,\Omega$$

Using Ohm's law, given by Eq. 24.16, we find:

$$I = \frac{\Delta V}{R_{eq}} = \frac{18\,\text{V}}{9\,\Omega} = 2\,\text{A}$$

$$\Delta V_1 = IR_1 = (2\text{A})(6\,\Omega) = 12\,\text{V}$$

$$\Delta V_2 = IR_2 = (2\,\text{A})(3\,\Omega) = 6\,\text{V}$$

Resistors in a Parallel Combination

Figure 24.15a shows two resistors of resistances R_1 and R_2 that are connected in **parallel** with a battery B. Figure 24.15b shows a circuit diagram for this combination of resistors. The potential difference ΔV between the battery's terminals is the same as the potential difference across each resistor. Figure 24.15c shows a single resistance R_{eq} that is equivalent to this combination and has the same effect on the circuit.

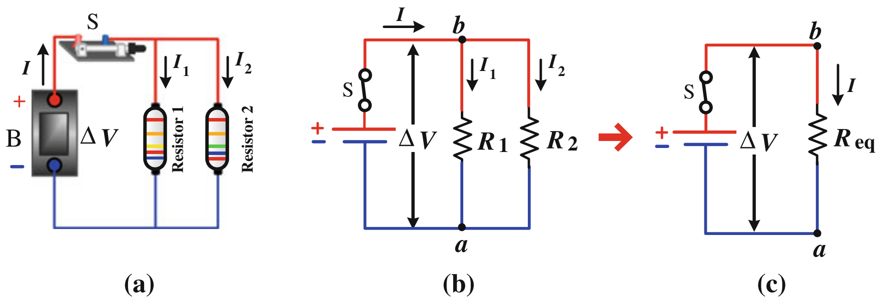


Fig. 24.15 (a) Two resistors of resistances R_1 and R_2 are connected in parallel to a battery B that has a potential difference ΔV. (b) The circuit diagram for this parallel combination. (c) The equivalent resistance R_{eq} replacing the parallel combination

When the current I reaches *junction b*, it will split into two parts, I_1 in R_1 and I_2 in R_2. Because electric charge is conserved, the current I that enters junction b must equal the total current leaving that junction; that is:

$$I = I_1 + I_2 \tag{24.35}$$

Because the potential difference ΔV across the resistors is the same, then from Fig. 24.15b, we have:

$$\Delta V = I_1 R_1 \quad \text{and} \quad \Delta V = I_2 R_2 \tag{24.36}$$

Substituting into Eq. 24.35, we get:

$$I = \frac{\Delta V}{R_1} + \frac{\Delta V}{R_2} = \left(\frac{1}{R_1} + \frac{1}{R_2}\right) \Delta V \tag{24.37}$$

An equivalent resistor with the same applied potential difference ΔV and total current I has a resistance R_{eq} given by $\Delta V = I\, R_{eq}$. Thus:

$$I = \frac{\Delta V}{R_{eq}} \tag{24.38}$$

Substituting in Eq. 24.37 and canceling ΔV, we arrive at the following relationship:

$$\boxed{\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2}} \quad \text{(Parallel combination)} \tag{24.39}$$

We can extend this treatment to n resistors connected in parallel as:

$$\boxed{\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} + \cdots + \frac{1}{R_n}} \quad \text{(Parallel combination)} \tag{24.40}$$

Thus, the equivalent resistance of a parallel combination of resistors is simply the algebraic sum of the reciprocal of the individual resistances and is less than any one of them.

Example 24.10

In Fig. 24.15, let $R_1 = 6\,\Omega$, $R_2 = 3\,\Omega$, and $\Delta V = 18$ V. Find the three currents I, I_1, and I_2 in the circuit.

Solution: The equivalent resistance of the parallel combination is:

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} = \frac{1}{6\,\Omega} + \frac{1}{3\,\Omega} = \frac{1}{2\,\Omega}$$

Then : $$R_{eq} = 2\,\Omega$$

Now we calculate the three currents in the circuit as follows:

$$I = \frac{\Delta V}{R_{eq}} = \frac{18\text{ V}}{2\,\Omega} = 9\text{ A} \quad I_1 = \frac{\Delta V}{R_1} = \frac{18\text{ V}}{6\,\Omega} = 3\text{ A} \quad I_2 = \frac{\Delta V}{R_2} = \frac{18\text{ V}}{3\,\Omega} = 6\text{ A}$$

Example 24.11

In Fig. 24.16, let $R_1 = 3\,\Omega$, $R_2 = 6\,\Omega$, $R_3 = 1\,\Omega$, $R_4 = 7\,\Omega$, and $\Delta V_{da} = V_a - V_d = 30\,\text{V}$. (a) What is the equivalent resistance between points a and d? (b) Evaluate the current passing through each resistor.

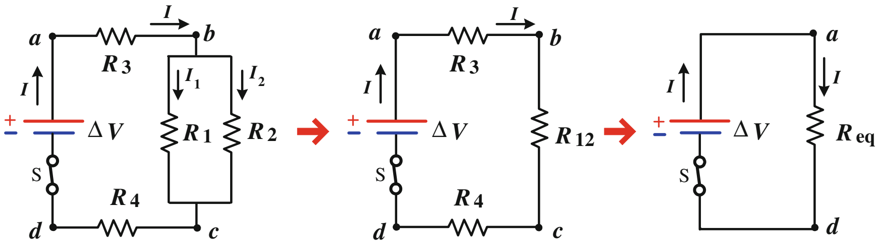


Fig. 24.16

Solution: (a) We can simplify the circuit by the rule of adding resistances in series and in parallel in steps. The resistors R_1 and R_2 are in parallel and their equivalent resistance R_{12} between b and c is:

$$\frac{1}{R_{12}} = \frac{1}{R_1} + \frac{1}{R_2} = \frac{1}{6\,\Omega} + \frac{1}{3\,\Omega} = \frac{1}{2\,\Omega}$$

Then : $$R_{12} = 2\,\Omega$$

Now R_3, R_{12}, and R_4 are in series between points a and d. Hence, their equivalent resistance R_{eq} is:

$$R_{\text{eq}} = R_3 + R_{12} + R_4 = 1\,\Omega + 2\,\Omega + 7\,\Omega = 10\,\Omega$$

(b) The current I that passes through the equivalent resistor also passes through R_3 and R_4. Thus, using Ohm's law, we find that:

$$I = \frac{\Delta V_{da}}{R_{\text{eq}}} = \frac{30\,\text{V}}{10\,\Omega} = 3\,\text{A} \quad \text{(Current through the battery, } R_3 \text{ and } R_4\text{)}$$

Since $\Delta V_{cb} = IR_{12} = I_1R_1 = I_2R_2$, then we find I_1 and I_2 as follows:

$$I_1 = \frac{IR_{12}}{R_1} = \frac{(3\,\text{A})(2\,\Omega)}{3\,\Omega} = 2\,\text{A} \quad \text{and} \quad I_2 = \frac{IR_{12}}{R_2} = \frac{(3\,\text{A})(2\,\Omega)}{6\,\Omega} = 1\,\text{A}$$

24.6 Kirchhoff's Rules

Not all circuits can be reduced to simple series and parallel combinations. A technique that is applied to loops in complicated circuits consists of two principles called Kirchhoff's Rules.

Kirchhoff's Rules:

1. Junction rule
At any junction in a circuit, the sum of the ingoing currents must equal the sum of the outgoing currents. That is:

$$\sum I_{\text{in}} = \sum I_{\text{out}} \tag{24.41}$$

2. Loop rule
For any closed loop in a circuit, the sum of the potential differences across all elements must be zero. That is:

$$\sum_{\text{closed loop}} \Delta V = 0 \tag{24.42}$$

The first rule merely states that no charge can accumulate at a junction. This rule is based on the principle of conservation of charge within any system. The second rule follows from the law of conservation of energy but is expressed in terms of potential energy.

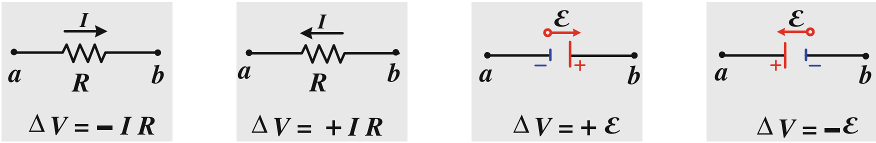
When we apply Kirchhoff's second rule to a loop, we should note the following sign conventions:

(1) When a resistor is traversed in the direction of the current, the potential difference ΔV is $-IR$ (Fig. 24.17a).

(2) When a resistor is traversed in the direction opposite the current, the potential difference ΔV is $+IR$ (Fig. 24.17b).

(3) When a source of emf is traversed in the direction of its emf (from $-$ to $+$), the potential difference ΔV is $+\mathcal{E}$ (Fig. 24.17c).

(4) When a source of emf is traversed in a direction opposite to its emf (from $+$ to $-$), the potential difference ΔV is $-\mathcal{E}$ (Fig. 24.17d).



(a) **(b)** **(c)** **(d)**

Fig. 24.17 The potential differences $\Delta V = V_b - V_a$ across a resistor of resistance R and a battery of emf $\mathcal{E}$ (assumed to have zero internal resistance), when each element is traversed from a to b

Example 24.12

Apply Kirchhoff's loop on the circuit of Example 24.8 to find the current in the circuit, see Fig. 24.13.

Solution: Applying Kirchhoff's loop rule to the loop $abb'a'a$ of Fig. 24.13, and traversing the loop clockwise, we obtain the following expression:

$$\text{Loop } abb'a'a: \qquad \mathcal{E}_1 + Ir_1 + Ir_2 - \mathcal{E}_2 = 0$$

$$9\,\text{V} + (0.02\,\Omega)I + (0.04\,\Omega)I - 12\,\text{V} = 0$$

$$\text{Then:} \qquad I = 50\text{A}$$

Of course, we do not need Kirchhoff's rules to solve this simple loop circuit. We are just using it to practice applying the loop rule.

Example 24.13

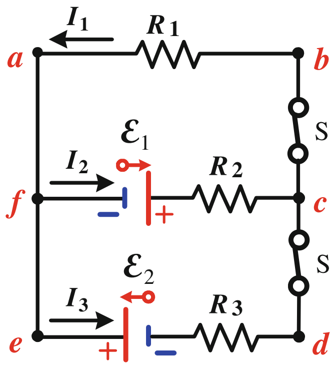
In Fig. 24.18, let $R_1 = 2\,\Omega$, $R_2 = 6\,\Omega$, $R_3 = 4\,\Omega$, $\mathcal{E}_1 = 10\,\text{V}$, and $\mathcal{E}_2 = 14\,\text{V}$. Find the currents I_1, I_2, and I_3 in the circuit.

Solution: We cannot simplify the circuit by the rule of adding resistances in series and in parallel. Thus, we must use Kirchhoff's rules. By applying Kirchhoff's junction rule to the junction f, we get:

$$\text{(1) Junction } f: \qquad I_1 = I_2 + I_3$$

We have three loops in this circuit, but we need only two loop equations to determine the three unknown currents. Applying Kirchhoff's loop rule to the loops *abcfa* and *fcdef* and traversing these loops clockwise, we obtain the following equations (after temporarily omitting the units, since they are all consistent SI units):

Fig. 24.18



(2) Loop *abcfa*: $\quad I_1R_1 + I_2R_2 - \mathcal{E}_1 = 0 \quad \Rightarrow \quad 2I_1 + 6I_2 - 10 = 0$

(3) Loop *fcdef*: $\quad \mathcal{E}_1 - I_2 R_2 + I_3 R_3 + \mathcal{E}_2 = 0 \quad \Rightarrow \quad 24 - 6I_2 + 4I_3 = 0$

Substituting Eqs. (1) into (3) gives:

$$24 - 10I_2 + 4I_1 = 0$$

Dividing this equation by 2 gives:

(4) $$12 - 5I_2 + 2I_1 = 0$$

Subtracting Eqs. (4) from (2) gives:

$$(2I_1 + 6I_2 - 10) - (12 - 5I_2 + 2I_1) = 0 \quad \Rightarrow \quad I_2 = 2\,\text{A}$$

Using this value of I_2 in Eq. (4) gives I_1 a value of:

$$12 - 5 \times 2 + 2I_1 = 0 \quad \Rightarrow \quad I_1 = -1\,\text{A}$$

Finally, from Eq. (1) we have:

$$I_3 = I_1 - I_2 = -1\,\text{A} - 2\,\text{A} = -3\,\text{A}$$

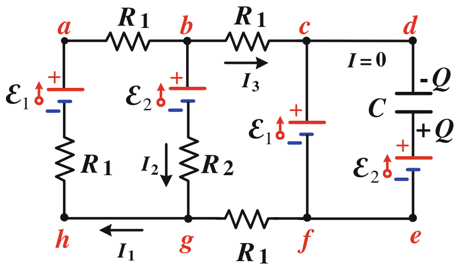
Thus $\qquad (I_1 = -1\text{A}, I_2 = 2\text{A}, I_3 = -3\text{A})$

We notice that I_1 and I_2 are both negative. This means that the currents are opposite to the direction we chose. However, the numerical values are correct.

Example 24.14

In Fig. 24.19, let $R_1 = 2\,\Omega$, $R_2 = 4\,\Omega$, $\mathcal{E}_1 = 6\,\text{V}$, $\mathcal{E}_2 = 3\,\text{V}$, and $C = 2\,\mu\text{F}$. Find the steady currents I_1, I_2, and I_3 and the charge Q.

Fig. 24.19



Solution: Applying Kirchhoff's junction rule at point b, we get:

(1) Junction b: $$I_1 = I_2 + I_3$$

The application of the loop rule to the loops *abgha* and *bcfgb* gives:

Loop *abgha*: $$-I_1 R_1 - \mathcal{E}_2 - I_2 R_2 - I_1 R_1 + \mathcal{E}_1 = 0$$

$$-2 I_1 - 3 - 4 I_2 - 2 I_1 + 6 = 0$$

(2) $$-4 I_1 - 4 I_2 + 3 = 0$$

Loop *bcfgb*: $$-I_3 R_1 - \mathcal{E}_1 - I_3 R_1 + I_2 R_2 + \mathcal{E}_2 = 0$$

$$-2 I_3 - 6 - 2 I_3 + 4 I_2 + 3 = 0$$

(3) $$-4 I_3 + 4 I_2 - 3 = 0$$

Substituting Eqs. (1) into (3) gives:

(4) $$-4 I_1 + 8 I_2 - 3 = 0$$

Subtracting Eqs. (4) from (2) gives:

$$(-4 I_1 - 4 I_2 + 3) - (-4 I_1 + 8 I_2 - 3) = 0 \quad \Rightarrow \quad I_2 = 0.5\ \text{A}$$

Using this value of I_2 in Eq. (4) gives I_1 a value of:

$$-4I_1 + 8(0.5) - 3 = 0 \quad \Rightarrow \quad I_1 = 0.25 \text{ A}$$

Finally, from Eq. (1) we have:

$$I_3 = I_1 - I_2 = 0.5 - 0.25 \text{ A} = 0.25 \text{ A}$$

Applying Kirchhoff's loop rule to the loop *cdefc* gives:

Loop *cdefc*: $\quad Q/C - \mathcal{E}_2 + \mathcal{E}_1 = 0 \quad \Rightarrow \quad Q = (\mathcal{E}_2 - \mathcal{E}_1)C = -6\,\mu\text{C}$

24.7 The RC Circuit

In all previous analyses, we considered steady-state situations where the current remains constant. However, when a circuit contains a resistor and a capacitor (called an RC circuit), the current in that circuit is found to vary with time until a steady current is reached.

Charging a Capacitor

Consider the simple RC series circuit shown in Fig. 24.20a. We assume that the capacitor is initially uncharged when the switch S is open, see Fig. 24.20a, b. Once the switch S is closed at time $t = 0$, charge begins to flow, setting up a current I in the circuit, until the capacitor is fully charged and the current becomes zero.

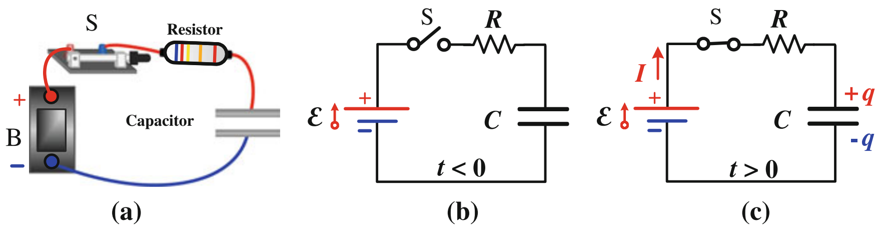


Fig. 24.20 (a) A capacitor in series with a resistor, switch, and battery. (b) The circuit diagram before the switch is closed ($t < 0$). (c) The circuit diagram at time $t > 0$ after the switch is closed at $t = 0$

Assume that the current in the circuit at time $t > 0$ is I and the magnitude of the charge on the capacitor is q (Fig. 24.20c). Applying Kirchhoff's loop rule and traversing the circuit clockwise, we get:

$$\mathcal{E} - IR - \frac{q}{C} = 0 \quad (\text{At time } t > 0) \tag{24.43}$$

Since the capacitor is uncharged at $t = 0$, then substituting $q = 0$, into this equation indicates that the current, denoted by $I_\circ$, is maximum:

$$I_\circ = \frac{\mathcal{E}}{R} \quad (\text{Current at time } t = 0) \tag{24.44}$$

At $t = 0$, the potential difference across the battery appears entirely on the resistor. At $t = \infty$, when the capacitor is fully charged to its maximum value Q, the current is zero and the potential difference across the battery appears entirely on the capacitor. Therefore, Eq. 24.43 gives:

$$Q = C\,\mathcal{E} \quad (\text{Maximum charge at } t = \infty) \tag{24.45}$$

To find the charge as a function of time, we substitute $I = dq/dt$ into Eq. 24.43 and rearrange the equation as follows:

$$\frac{dq}{dt} = \frac{\mathcal{E}}{R} - \frac{q}{R\,C} = -\frac{q - C\,\mathcal{E}}{R\,C}$$

$$\frac{dq}{q - C\,\mathcal{E}} = -\frac{dt}{R\,C}$$

Substituting with $C\,\mathcal{E} = Q$ in this expression and integrating from the initial charge $q = 0$ at $t = 0$ to an arbitrary charge q at time t, we get:

$$\int_0^q \frac{dq}{q - Q} = -\int_0^t \frac{dt}{R\,C} \quad \Rightarrow \quad \ln\frac{q - Q}{-Q} = -\frac{t}{R\,C}$$

By using the definition of the natural logarithm, we can rewrite the last expression as:

$$\boxed{q = Q(1 - e^{-t/RC}), \quad Q = C\,\mathcal{E}} \tag{24.46}$$

This relation conforms with the facts that we already know, i.e. that $q = 0$ at $t = 0$ and that $q = Q = C\,\mathcal{E}$ at $t = \infty$. Using $I = dq/dt$, we differentiate the charge q in Eq. 24.46 to find the current I as a function of time as follows:

$$\boxed{I = I_\circ\, e^{-t/RC}, \quad I_\circ = \frac{\mathcal{E}}{R}} \tag{24.47}$$

This relation shows that $I = I_\circ = \mathcal{E}/R$ at $t = 0$ as obtained in Eq. 24.44 and $I = 0$ at $t = \infty$ as expected.

The quantity RC in the exponents of Eqs. 24.46 and 24.47 is called the **time constant** τ of the circuit. Therefore, the quantity $\tau = RC$ represents the time interval during which the charge on the capacitor increases to $Q(1 - e^{-1}) = 0.632\,Q$, i.e., $\sim 63\%$ increase. Similarly, after a time interval τ, the current decreases to $1/e$ of its initial value; that is, $I = e^{-1}I_\circ = 0.368 I_\circ$ ($\sim$37% decrease).

Figure 24.21 shows the variation of the capacitor charge q and the circuit current I as a function of time.

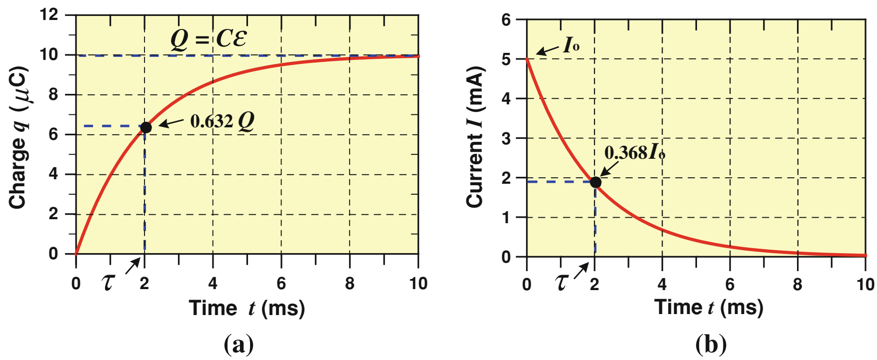


Fig. 24.21 (a) A plot of the charge q on the capacitor of Fig. 24.20 versus time t. (b) A plot of the current I in the same figure versus time t. The two curves are for $R = 2\,\text{k}\Omega$, $C = 1\,\mu\text{F}$, and $\mathcal{E} = 10\,\text{V}$

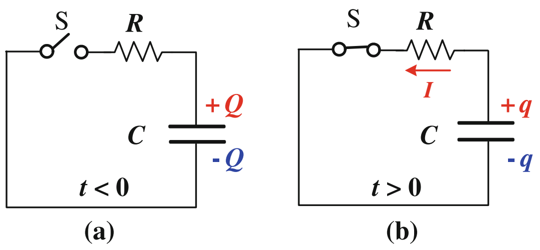
Discharging a Capacitor

Let us consider the circuit shown in Fig. 24.22a, in which we have a capacitor of capacitance C carrying an initial charge Q, a resistor of resistance R, and an open switch S. When the switch is closed at time $t = 0$, the capacitor begins to discharge through the resistor. If the current in the circuit at time $t > 0$ is I and the magnitude of the charge on the capacitor is q (Fig. 24.22b), then by applying Kirchhoff's loop rule and traversing the circuit clockwise, we get:

$$+IR - \frac{q}{C} = 0 \quad \text{(At time } t\text{)} \tag{24.48}$$

To find the charge as a function of time, we substitute with $I = -dq/dt$ (the rate of decrease of charge on the capacitor) into Eq. 24.48 and rearrange the equation as follows:

Fig. 24.22 (a) A capacitor with an initial charge Q is connected to a resistor and an open switch ($t < 0$). (b) A circuit diagram showing the charge and current at $t > 0$, after the switch is closed at time $t = 0$



$$\frac{dq}{dt} = -\frac{q}{R\,C}$$
$$\frac{dq}{q} = -\frac{dt}{R\,C}$$

Integrating this expression from the initial charge $q = Q$ at $t = 0$ to an arbitrary charge q at time t, we get:

$$\int_{Q}^{q} \frac{dq}{q} = -\int_{0}^{t} \frac{dt}{R\,C} \quad \Rightarrow \quad \ln\frac{q}{Q} = -\frac{t}{R\,C}$$

By using the definition of the natural logarithm, we can rewrite the last expression as:

$$\boxed{q = Qe^{-t/RC} = Qe^{-t/\tau}} \tag{24.49}$$

Using $I = -dq/dt$, we differentiate q in Eq. 24.49 to find the current I as a function of time as follows:

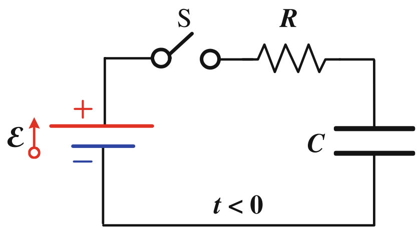
$$\boxed{I = I_{\circ}\, e^{-t/RC} = I_{\circ}\, e^{-t/\tau}, \quad I_{\circ} = \frac{Q}{RC}} \tag{24.50}$$

We must note that the *discharging current* in Fig. 24.22 is opposite to the direction of the *charging current* in Eq. 24.20.

Example 24.15

In the circuit of Fig. 24.23a, let $R = 2\,\mathrm{k\Omega}$, $\mathcal{E} = 10\,\mathrm{V}$, and $C = 1\,\mu\mathrm{F}$. The capacitor is uncharged before closing the switch S. (a) Find the time constant of the circuit. After closing S at $t = 0$, find the maximum current in the circuit and find the maximum charge on the capacitor at $t = \infty$. (b) Find the charge and current as a function of time.

Fig. 24.23



Solution: (a) The time constant of the circuit is:

$$\tau = RC = (2 \times 10^3\ \Omega)(1 \times 10^{-6}\ \text{F}) = 2 \times 10^{-3}\ \text{s} = 2\,\text{ms}$$

The maximum current in the circuit (see Eq. 24.44) is:

$$I_\circ = \frac{\mathcal{E}}{R} = \frac{10\ \text{V}}{2 \times 10^3\ \Omega} = 5 \times 10^{-3}\ \text{A} = 5\,\text{mA}$$

At $t = \infty$, the magnitude of the maximum charge on the capacitor is:

$$Q = C\mathcal{E} = (1 \times 10^{-6}\ \text{F})(10\ \text{V}) = 10^{-5}\ \text{C} = 10\,\mu\text{C}$$

(b) Substituting the obtained values of part (a) in Eqs. 24.46 and 24.47, we get for the charge the following relation:

$$\begin{aligned} q &= Q(1 - e^{-t/RC}) = Q(1 - e^{-t/\tau}) \\ &= (10\ \mu\text{C})(1 - e^{-t/(2\times10^{-3}\text{s})}) \end{aligned}$$

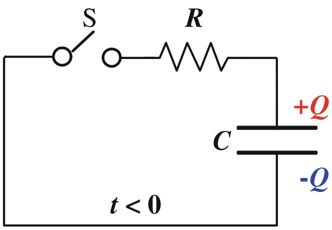
and for the current the following relation:

$$\begin{aligned} I &= I_\circ\, e^{-t/RC} = I_\circ\, e^{-t/\tau} \\ &= (5 \times 10^{-3}\text{A})\ (1 - e^{-t/(2\times10^{-3}\text{s})}) \end{aligned}$$

Example 24.16

In Fig. 24.24, let $R = 2\,\text{k}\Omega$, $C = 5\,\mu\text{F}$, and $Q = 50\,\mu\text{C}$. (a) After how many time constants, $\tau = RC$, will the charge on the capacitor be half of its initial value when the switch is closed? (b) When the stored energy in the capacitor becomes half of its initial value?

Fig. 24.24



Solution: (a) The time constant of the circuit is:

$$\tau = RC = (2 \times 10^3\,\Omega)(5 \times 10^{-6}\,\text{F}) = 10^{-2}\,\text{s} = 10\,\text{ms}$$

After closing the switch at $t = 0$, the charge on the capacitor is given by Eq. 24.49, $q = Qe^{-t/\tau}$. To find the time interval during which q drops to one-half its initial value, we substitute $q = Q/2$ into this equation and solve for the time t as follows:

$$\frac{Q}{2} = Qe^{-t/\tau} \quad \Rightarrow \quad \frac{1}{2} = e^{-t/\tau}$$

Taking the logarithm of both sides, we find:

$$-\ln 2 = -\frac{t}{\tau} \quad \Rightarrow \quad t = (\ln 2)\tau = 0.69\tau = 0.69 \times (10\,\text{ms}) = 6.9\,\text{ms}$$

(b) From Eq. 23.24, the initial stored energy in the capacitor is $U_\circ = Q^2/2C$. Using Eq. 24.49, the energy stored at time t is:

$$U = \frac{q^2}{2C} = \frac{Q^2}{2C}e^{-2t/\tau} = U_\circ e^{-2t/\tau}$$

As in part (a), we set $U = U_\circ/2$ and solve for t as follows:

$$\frac{U_\circ}{2} = U_\circ e^{-2t/\tau} \quad \Rightarrow \quad \frac{1}{2} = e^{-2t/\tau}$$

Again, taking the logarithm of both sides and solving for t, we find:

$$-\ln 2 = -2t/\tau \quad \Rightarrow \quad t = \tfrac{1}{2}(\ln 2)\tau = \tfrac{1}{2} \times 0.69\tau = \tfrac{1}{2} \times 0.69 \times (10\,\text{ms}) = 3.45\,\text{ms}$$

Note that the results of both parts are independent on the value of Q.

24.8 Exercises

Section 24.1 Electric Current and Electric Current Density

(1) How many electrons per second would pass through a given cross section of a conductor carrying a current $I = 1.6$ A?

(2) A current of 10 A is maintained in a wire for 1 min. (a) How much charge flows through the wire in this period? (b) How many electrons flow through the wire in this period?

(3) A 0.1 mol of electrons flows through a wire in 30 min. (a) What is the total charge that passes through the wire? (b) What is the value of the current in the wire?

(4) A copper wire contains 2×10^{21} free electrons in 1 cm of its length. The electrons move with a drift speed of 2.5×10^{-3} cm/s. (a) How many electrons pass through a given cross section of the wire each second? (b) How large is the current in the wire?

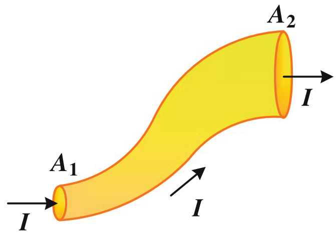
(5) The current through a cross-sectional area of a wire is given by the relation $I = 2 + 3t^3$; where I is in amperes and t is in seconds. (a) Find the total charge that passes through this area between $t = 2$ s and $t = 8$ s. (b) Find the average current needed to pass the same quantity of charge calculated in part (a) during the same time interval.

(6) The charge that passes a cross-sectional area $A = 10^{-4}\ \text{m}^2$ varies with time according to the relation $Q = 4 + 2t + t^2$, where Q is in coulombs and t is in seconds. (a) Find the relation that gives the instantaneous current at any time, and evaluate this current at time $t = 2$ s. (b) Find the relation that gives the current density at any time, and evaluate this current density at time $t = 2$ s.

(7) A wire carrying a current of 3 A has a circular cross section everywhere with a non-uniform radius, see Fig. 24.25. The radius of the cross section A_1 is 2 cm. (a) Find the current density across A_1. (b) Find the current density across A_2 if its radius is two times the radius at A_1.

(8) A copper wire with a 0.2 mm diameter and an iron wire with a 5 mm diameter are soldered together to form one wire in a circuit. A current of 8 A is found to pass through the copper wire. (a) What is the current and current density through the iron wire? (b) What is the current density through the copper wire?

Fig. 24.25 See Exercise (7)



(9) Given that the density of aluminum is 2.7×10^3 kg/m^3, find the drift speed of the conduction electrons in an aluminum wire that has a cross-sectional area of 10^{-6} m^2 and carries a current of 10 A. Assume that each aluminum atom contributes one free conduction electron to the current.

Section 24.2 Ohm's Law and Electric Resistance

(10) Use Table 24.1 to calculate the electric field that exists in a gold wire when the current density in the wire is 3×10^7 A/m^2.

(11) A metallic rod has a length $L = 1.5$ m and a diameter $D = 0.2$ cm. The rod carries a current of 5 A when a potential difference of 75 V is applied between its ends. (a) Find the current density in the rod. (b) Calculate the magnitude of the electric field applied to the rod. (c) Calculate the resistivity and conductivity of the material of the rod.

(12) Use Table 24.1 to calculate the resistance of a silver wire that has a length of 100 m and a cross section of 0.4 mm^2.

(13) At 20 °C, a silver wire has a diameter of 2 mm, a length of 0.5 m, a resistivity of 1.6×10^{-8} Ω.m, a temperature coefficient of resistivity of 4×10^{-3} (C°)$^{-1}$, and carries a current of 5 A. (a) What is the current density in the wire? (b) Find the magnitude of the electric field applied to the wire. (c) What is the potential difference between the ends of the wire? (d) What is the resistance of the wire? (e) Find the temperature of the wire when its resistance increases to 6.5×10^{-4} Ω.

(14) A cylindrical shell of length $L = 10$ cm is made of copper and has an inner radius $a = 2$ mm and an outer radius $b = 8$ mm, see Fig. 24.26. Assume that the shell has a uniform current density $J = 10^5$ A/m^2 directed upward as shown in the figure. (a) What is the current through the shell? (b) What are the values of the resistance of the shell and the potential difference ΔV?

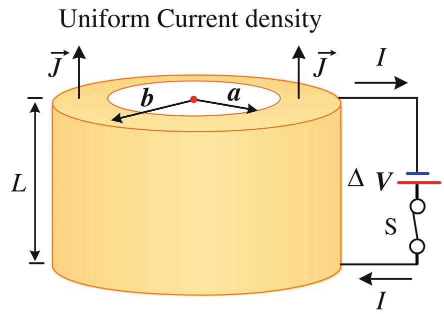


Fig. 24.26 See Exercise (14)

(15) A cube of copper has a mass $m = 50$ g, see Fig. 24.27. The copper has a density of 8.92×10^3 kg/m^3, a molar mass of 63.546 kg/kmol, a resistivity of 1.7×10^{-8} Ω.m, and contributes one conduction electron per atom. (a) What is the distance between opposite faces of the cube? (b) What is the resistance between opposite faces of the cube? (c) What is the current and the average drift speed of the conduction electrons when a potential difference ΔV of 10^{-4} V is applied between two opposite faces of the cube?

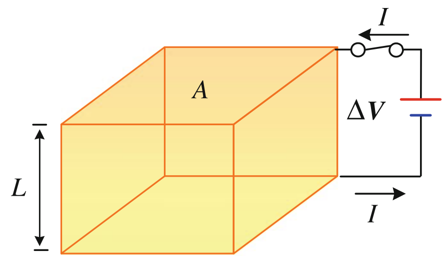


Fig. 24.27 See Exercise (15)

(16) The temperature coefficient of resistivity of copper at 20 °C is 3.9×10^{-3} (C°)$^{-1}$. Calculate the percentage increase in its resistivity when its temperature increases to 220 °C.

(17) At a temperature of 1,800 °C the tungsten filament of a light bulb has a resistance of 250 Ω. With the aid of Table 24.1, find its resistance at room temperature (assume it to be 20 °C).

(18) At 20 °C a copper wire has a resistance of 4×10^{-3} Ω and a temperature coefficient of resistivity of 3.9×10^{-3} (C°)$^{-1}$. What is its resistance at 100 °C?

(19) At 70 °C, an electric field $E = 0.2$ V/m is applied along a silver rod of length $L = 0.5$ m and radius $r = 0.05$ mm. Silver has a density of 10.5×10^3 kg/m^3,

a molar mass of 107.868 kg/kmol, a coefficient α at 20 °C of 3.8×10^{-3} $(\mathrm{C}^\circ)^{-1}$, and a resistivity of 1.59×10^{-8} Ω.m at 20 °C. Assuming 1 free electron per atom, find: (a) the resistivity of the silver wire (b) the current density in the silver wire. (c) the current in the silver wire. (d) the resistance of the silver wire. (e) the drift speed of the conduction electrons. (f) the potential difference between the ends of the silver wire.

(20) At 20 °C, a nichrome wire of resistance $R_{\circ n}$ and a carbon wire of resistance $R_{\circ c}$ are attached end-to-end to form one wire of resistance $R_\circ$, where $R_\circ = R_{\circ n} + R_{\circ c} = 9\,\Omega$. What values of $R_{\circ n}$ and $R_{\circ c}$ would give a combined resistance of R equal to $R_\circ$ regardless of the temperature T? [Hint: use Table 24.1.]

Section 25.3 Electric Power

(21) A light bulb rated 60 W at 240 V is operated from a 240 V source. (a) Find the current flowing through the bulb. (b) Find the resistance of the bulb. (c) Repeat (a) and (b) when the bulb is rated 100 W at 240 V.

(22) A 550 W electric heater is designed to operate from a 220 V source. (a) What is the resistance of the heater? (b) What current does the heater draw from the source? (c) If the source voltage drops to 120 V, what power does the heater consume from the source?

(23) A heating coil is made from a nichrome wire of radius 0.45 mm. The coil is designed to produce 240 W of thermal power when connected to a source that has a potential difference of 24 V. (a) What is the resistance of the coil? (b) What current does the heating coil draw from the source? (c) What is the length of the coil?

(24) A 1 kΩ carbon resistor used in an electric circuit is rated 0.4 W. (a) Find the maximum allowable current that can pass through the resistor. (b) Find the maximum allowable potential difference that can be applied across the resistor.

(25) Batteries are rated in terms of the quantity $I\,t$, i.e. rated in ampere-hours (A.h). For instance, a battery that can produce a current of 4 A for 5 h is rated as a 20 A.h battery. (a) Find the total energy stored in a 12 V battery rated at 75 A.h. (Express your answer in kW.h, where 1 kW.h $= 3.6 \times 10^3$ J). (b) At a price of 35 piaster per kilowatt-hour of electricity, what is the total cost of the electricity produced by this battery?

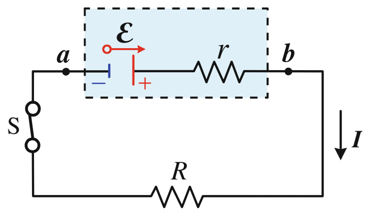
(26) A beam of electrons in a TV set has a radius of 0.1 cm. The electrons move from the cathode to the screen with an electron current of 0.1 mA and a kinetic energy of 5 ke V. (a) What is the current density in the beam? (b) How many electrons per second hit the screen? (c) How much power is dissipated at the screen? (d) What is the speed of each electron in the beam? (e) Find the number of electrons per unit volume in the beam.

(27) A heating coil operating from a 220 V source increases the temperature of 2 kg of water from 20 °C to 50 °C in 20 min. Find the coil's resistance if water's specific heat is 4,186 J/kg.C°.

Section 24.4 Electromotive Force

(28) In Fig. 24.28 the circuit contains a battery that has an emf $\mathcal{E} = 11$ V and internal resistance $r = 0.5\ \Omega$. The load in the circuit has a resistance $R = 5\ \Omega$. (a) Find the current in the circuit. (b) Find the potential difference between a and b.

Fig. 24.28 See Exercise (28)



(29) Assume a circuit similar to the one in Exercise 28 has an unknown emf $\mathcal{E}$ and internal resistance r. It is found that when the current is 0.5 A, the load resistance is 16 Ω. Similarly, it is found that when the current is 1.5 A, the load resistance is 5 Ω. (a) Find the internal resistance of the battery. (b) Find the emf of the battery.

(30) Two batteries, one old and the other new, each have an emf of 1.5 V. When each battery is short-circuited with a conducting wire of zero resistance, it is found that the new one establishes a 30 A current in the wire while the old one establishes a 10 A current. Find the internal resistance of the two batteries.

(31) A battery has an emf $\mathcal{E}_1 = 9$ V and an internal resistance $r_1 = 0.4\ \Omega$. This battery is connected to a second battery of $\mathcal{E}_2 = 12$ V and $r_2 = 0.6\ \Omega$, and a light bulb of resistance R. If the batteries are connected with their positive

terminals in the same direction as shown in Fig. 24.29, a current of 0.7 A is established in the circuit. (a) Find the resistance of the light bulb. (b) What fraction of the transferred chemical energy is dissipated in the two batteries? (c) If we reverse the polarity of the $\mathcal{E}_1 = 9$ V battery in the circuit, what is the value of the current in the circuit? Would the answer to part (b) change in this case?

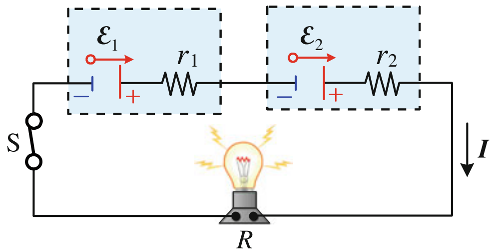


Fig. 24.29 See Exercise (31)

Section 24.5 Resistors in Series and Parallel

(32) When three resistors of resistances $R_1 = 2\ \Omega$, $R_2 = 1\ \Omega$, and $R_3 = 4\ \Omega$ are connected to a source of potential difference ΔV as shown in Fig. 24.30, the current in the circuit is found to be 5 A. (a) Find the equivalent resistance of the combination. (b) Determine the value of ΔV.

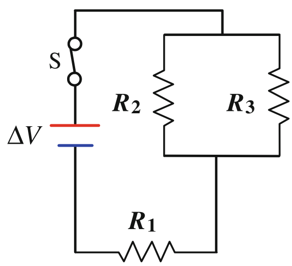


Fig. 24.30 See Exercise (32)

(33) For the circuit shown in Fig. 24.31, take $R_1 = 3\ \Omega$, $R_2 = 6\ \Omega$, $R_3 = 12\ \Omega$, $R_4 = 6\ \Omega$, and $\Delta V = 12$ V. (a) Find the equivalent resistance of the combination. (b) Find the current in the branch containing R_1 and R_2. (c) Repeat (b) for the branch containing R_3 and R_4. (d) Find the potential difference across each resistor.

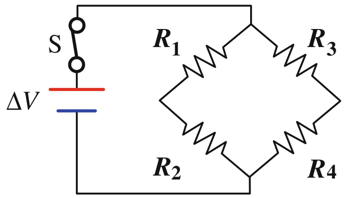


Fig. 24.31 See Exercise (33)

(34) For each of the combinations shown in Fig. 24.32, find a formula that represents the equivalent resistance between the terminals A and B.

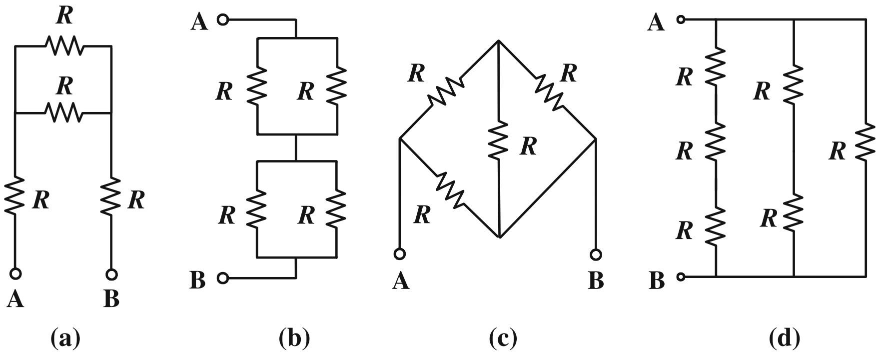


Fig. 24.32 See Exercise (34)

(35) Assume that in exercise 34, $R = 2\,\Omega$ and $\Delta V_{BA} = 12$ V. For each combination, find the current in each branch of the circuit. Always start from the branch closest to A and move toward B.

(36) It is recommended that the current through the human body not exceed 150 μA. Assume a person stands barefoot on the ground, holding a wire connected through a resistor of high resistance R to a power source of potential difference $\Delta V = 220$ V as shown in Fig. 24.33. Assume that the circuit's wire makes a low-resistance contact with the person's hand. Also, assume that the resistance through the person's body is negligible compared to the resistance R. (a) Find R_{min}, which is the safest resistance value of R. (b) While holding R_{min}, the person decided to wear shoes of resistance R_S to reduce the current to 100 μA. Find R_S.

(37) A light bulb is rated 60 W at 240 V. The bulb is connected to a source of 240 V with two equal length wires, each having a resistance $R/2 = 120\,\Omega$, see

Fig. 24.34. (a) What is the resistance R_b of the light bulb? (b) What is the value of the current I in the circuit? (c) What is the potential difference between the sockets of the light bulb? (d) What is the actual power delivered to the bulb in this circuit?

Fig. 24.33 See Exercise (36)

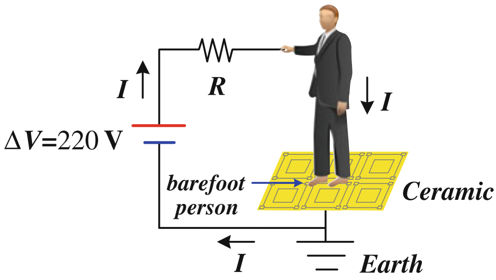
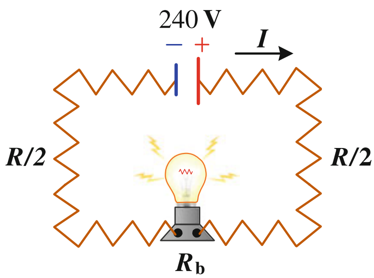
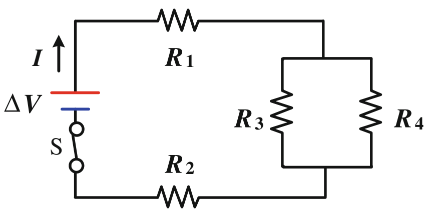


Fig. 24.34 See Exercise (37)



(38) The four resistances of Fig. 24.35 are $R_1 = 1\,\Omega$, $R_2 = 2\,\Omega$, $R_3 = 4\,\Omega$, and $R_4 = 12\,\Omega$. The power source has a potential difference $\Delta V = 12$ V. (a) Find the equivalent resistance of the combination. (b) What is the value of the current I in the circuit? (c) Find the currents in R_3 and R_4. (d) Calculate the power delivered to each resistor in the circuit.

Fig. 24.35 See Exercise (38)



Section 24.6 Kirchhoff's Rules

(39) For the circuit shown in Fig. 24.36, let $R_1 = 10\,\Omega$, $R_2 = 20\,\Omega$, $\mathcal{E}_1 = 10\,\text{V}$, and $\mathcal{E}_2 = 12\,\text{V}$. Find the values of the currents I_1, I_2, and I_3 in the circuit.

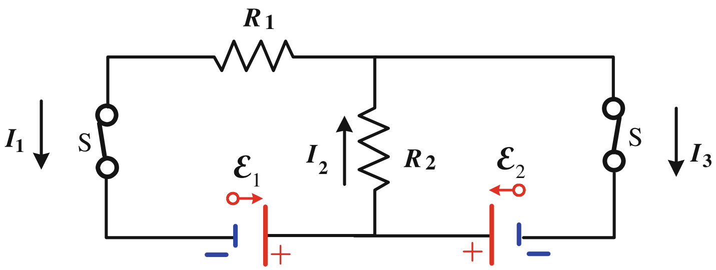
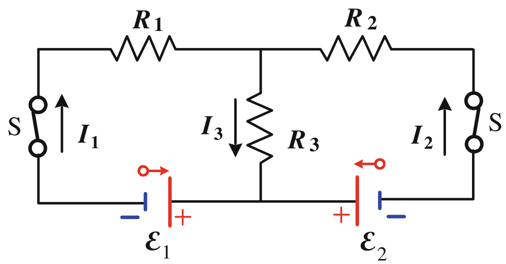


Fig. 24.36 See Exercise (39)

(40) For the circuit shown in Fig. 24.37, let $R_1 = 1\,\Omega$, $R_2 = 2\,\Omega$, $R_3 = 3\,\Omega$, $\mathcal{E}_1 = 10\,\text{V}$, and $\mathcal{E}_2 = 12\,\text{V}$. Find the values of the currents I_1, I_2, and I_3 in the circuit.

Fig. 24.37 See Exercise (40)



(41) For the circuit shown in Fig. 24.38, let $R_1 = 5\,\Omega$, $R_2 = R_3 = 15\,\Omega$, $\mathcal{E}_1 = 60\,\text{V}$, $\mathcal{E}_2 = 80\,\text{V}$, and $\mathcal{E}_3 = 10\,\text{V}$. Find the values of the currents I_1, I_2, and I_3 in the circuit.

(42) For the circuit shown in Fig. 24.39, let $R_1 = 2\,\Omega$, $R_2 = R_3 = 4\,\Omega$, $\mathcal{E}_2 = 20\,\text{V}$, and $\mathcal{E}_3 = 2\,\text{V}$. The ammeter, represented by the symbol —Ⓐ—, reads the current I_1 in the wire to be 0.5 A. Find the voltage of the unknown battery $\mathcal{E}_1$ and the values of the currents I_2, and I_3.

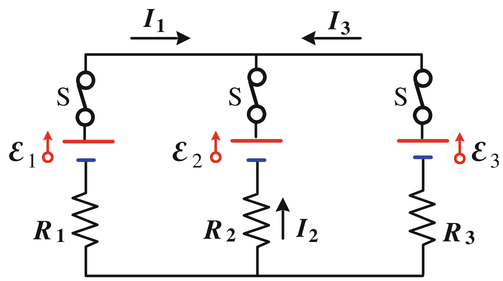


Fig. 24.38 See Exercise (41)

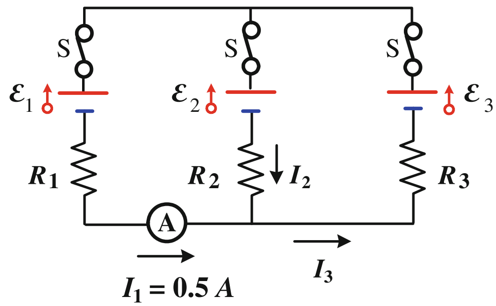


Fig. 24.39 See Exercise (42)

(43) For the circuit shown in Fig. 24.40, let $R_1 = 3\,\Omega$, $R_2 = 6\,\Omega$, $R_3 = 3\,\Omega$, $R_4 = 6\,\Omega$, and $\mathcal{E} = 7.5\,\text{V}$. Find the values of the currents I_1, I_2, I_3, and I_4 in the circuit.

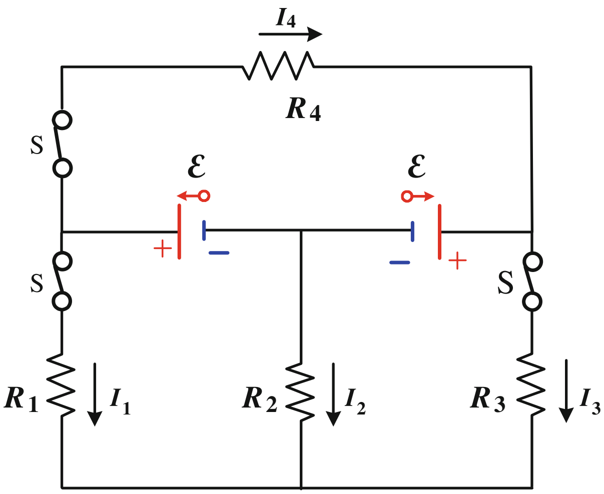


Fig. 24.40 See Exercise (43)

(44) Each resistor in the different configurations of Fig. 24.41 has the same resistance R. Show that the equivalent resistance of the four parts of the figure are: (a) $7R/5$, (b) $2R/3$, (c) R, and (d) $3R/4$, respectively.

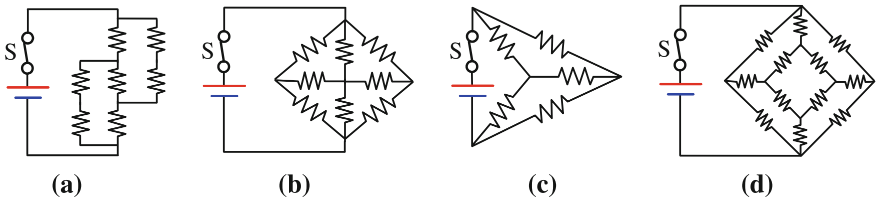
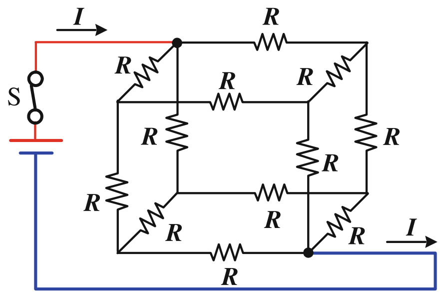


Fig. 24.41 See Exercise (44)

(45) Apply symmetry arguments to the equal-valued resistors of Fig. 24.42 to show that: (a) the current passing through any resistor in the figure is either $I/3$ or $I/6$. (b) the equivalent resistance of the circuit is $5\,R/6$.

Fig. 24.42 See Exercise (45)



Section 24.7 The RC Circuit

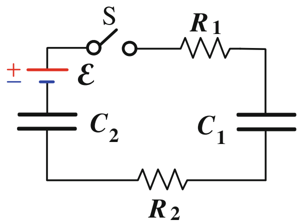
(46) In the process of charging a capacitor of capacitance C through a resistor of resistance R, about 63% of the maximum charge will accumulate on the capacitor in a time $t = R\,C$ (known as the time constant $\tau = R\,C$). In this time, what percentage of the maximum electrostatic energy is stored on the capacitor?

(47) An uncharged capacitor has a capacitance of $2\,\mu$F. A battery of 12 V charges this capacitor through a $1\,\mathrm{M}\,\Omega$ resistor. (a) Find the time constant of the circuit, the maximum charge on the capacitor, and the maximum current in the circuit.

(b) How much time is required for the potential difference across the capacitor to reach 6 V?

(48) Prove that when switch S in Fig. 24.43 is closed, the charge q at time t on any capacitor is $q = Q(1 - e^{-t/\tau})$, where $\tau = (R_1 + R_2)(C_1 + C_2)$ and $Q = (C_1 + C_2)\,\mathcal{E}$.

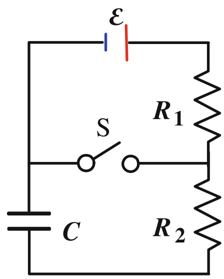
Fig. 24.43 See Exercise (47)



(49) A 5 μF capacitor is charged to 220 V. After disconnecting it from its source, a student holds its two lead wires with his bare hands. Assume that the resistance between the student's hands is 50 kΩ. (a) What is the initial charge on the capacitor and the maximum current that passes through the student's body? (b) Find the charge that remains on the capacitor, and calculate the current that passes through the student's body after 0.5 s.

(50) The switch in the circuit of Fig. 24.44 is left open for a long time, and then closed at $t = 0$. Let $R_1 = 50\,\mathrm{k\Omega}$, $R_2 = 150\,\mathrm{k\Omega}$, $C = 5\,\mu\mathrm{F}$, and $\mathcal{E} = 30\,\mathrm{V}$. Find the time constant before and after the switch is closed. Then find the current in the switch as a function of time.

Fig. 24.44 See Exercise (50)



Part VI
Magnetism

Magnetic Fields

25

It is of common knowledge that every magnet attracts pieces of iron and has two *poles: a north* pole (N) and a *south* pole (S). In addition, given two magnets, like poles (N–N or S–S) repel each other, and opposite poles (N–S) attract each other. Moreover, if we cut a magnet in half, we do not obtain isolated north and south poles. Instead, we get two magnets, each with its own north and south pole.

In 1819, Oersted observed the deflection of a pivoted magnet when it was in the vicinity of a current-carrying wire. Now, it is known that all magnetic phenomena result from forces arising from electric charges in motion. Based on these forces, the concept of a **magnetic field** was introduced as a mechanism for exerting a magnetic force on a moving charge. This is similar to the concept of an electric field surrounding an electric charge. That is, in the region of space around any moving charge, a magnetic field is established (as well as an electric field), and this magnetic field can exert a force on a second moving charge. Consequently, all atoms can exhibit magnetic effects, due to the motion of their electrons about their nuclei.

In this chapter, we discuss forces that act on moving charges as well as forces that act on current-carrying conductors in the presence of a magnetic field. We postpone discussing the sources of such fields.

25.1 Magnetic Force on a Moving Charge

A magnetic field exists at a particular point in space if a force is exerted on a moving charge at that point. The magnetic field, like the electric field, is a vector quantity and historically is denoted by the symbol $\vec{B}$. We can define the magnetic field $\vec{B}$ at any point in terms of the magnetic force $\vec{F}_B$ exerted by the field on a test charge q

H. A. Radi and J. O. Rasmussen, *Principles of Physics*,
Undergraduate Lecture Notes in Physics, DOI: 10.1007/978-3-642-23026-4_25,

moving with a velocity $\vec{v}$. If the smaller angle between the two vectors $\vec{B}$ and $\vec{v}$ is denoted by θ, then experiments show that:

- $F_B \propto |q| vB \sin\theta$
- $\vec{F}_B$ has the direction of $\vec{v} \times \vec{B}$ if q is positive
- $\vec{F}_B$ has the direction of $-\vec{v} \times \vec{B}$ if q is negative

In vector form, these results can be written as follows:

$$\vec{F}_B = q\,\vec{v} \times \vec{B} = q \begin{vmatrix} \vec{\mathrm{i}} & \vec{\mathrm{j}} & \vec{\mathrm{k}} \\ v_x & v_y & v_z \\ B_x & B_y & B_z \end{vmatrix} \tag{25.1}$$

Therefore, the magnitude of the magnetic force on q is:

$$F_B = |q|\, vB \sin\theta \tag{25.2}$$

To find the direction of $\vec{v} \times \vec{B}$ and the direction of $\vec{F}_B$ for both positive and negative q, we use the right-hand rule, as shown in Fig. 25.1.

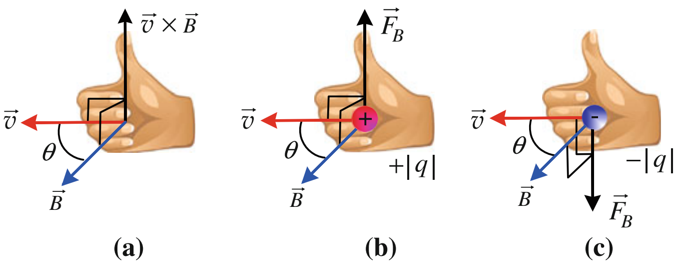


Fig. 25.1 (a) With the right-hand rule, the direction of the thumb points in the direction of $\vec{v} \times \vec{B}$ when the fingers curl $\vec{v}$ into $\vec{B}$. (b) When q is positive, the direction of $\vec{F}_B$ has the same sign as $\vec{v} \times \vec{B}$. (c) When q is negative, the directions of $\vec{F}_B$ is opposite to $\vec{v} \times \vec{B}$

Equation 25.1 indicates that:

- $F_B = 0$ (when $\vec{v}\,//\,\vec{B}$ and, of course, when $v = 0$)
- $F_B|_{\max} = q\,vB$ (when $\vec{v} \perp \vec{B}$)
- $\vec{F}_B \perp \vec{v}$ at all times, (hence $\vec{B}$ changes only the direction of $\vec{v}$)

From Eq. 25.1, we see that the SI unit for B is newton per coulomb-meter per second, which is called **tesla** (T). With the use of the SI unit: 1 ampere is 1 coulomb per second, so we have:

$$1\,\mathrm{T} = 1\frac{\mathrm{N}}{\mathrm{C.m/s}} = 1\frac{\mathrm{N}}{\mathrm{A.m}} \tag{25.3}$$

An earlier non-SI unit of B, still in common use, is *gauss* (G), and is related to tesla through the conversion formula:

$$1\,\mathrm{T} = 10^4\,\mathrm{G} \tag{25.4}$$

Table 25.1 lists some approximate values of B in a few situations.

Table 25.1 Some approximate values of the magnetic fields

Source of the field	Value of B(T)
New kind of neutron star called a "Magnetar"	10^{11}
Neutron star	10^{8}
Superconducting magnet	30
Strong magnet	2
Medical MRI unit	1.5
Small bar magnet	10^{-2}
Surface of the earth	10^{-4}
Inside human brain	10^{-13}
Smallest value in a magnetically shielded room	10^{-14}

For convenience, we label the magnetic field coming out of the page by black dots (or blue dots), as shown in Fig. 25.2a and the magnetic field going into the page by black crosses (or blue crosses), as shown in Fig. 25.2b. The same approach is used for both $\vec{v}$ and I but sometimes with different colors.

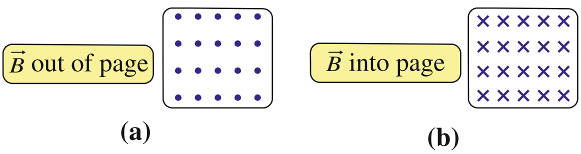
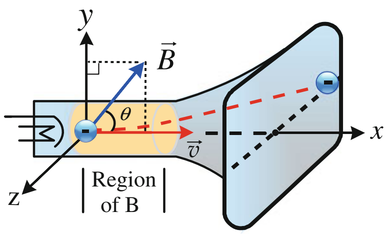


Fig. 25.2 Magnetic field lines: (a) coming out of the page are indicated by *dots*, (b) going into the page are indicated by *crosses*

Example 25.1

An electron in a television tube moves along the x-axis with a speed v of 10^7 m/s, see the sketch in Fig. 25.3. A uniform magnetic field in the xy plane has a magnitude 0.02 T and is directed at an angle of 30° from the x-axis. (a) Calculate the magnitude of the magnetic force on the electron. (b) Find the vector expression of the magnetic force on the electron in terms of the unit vectors $\vec{\mathrm{i}}$, $\vec{\mathrm{j}}$, and $\vec{\mathrm{k}}$ along x, y, and z axes.

Fig. 25.3



Solution: (a) using Eq. 25.2 we find that:

$$F_B = |q|\, vB \sin\theta = (1.6 \times 10^{-19}\,\mathrm{C})(10^7\,\mathrm{m/s})(0.02\,\mathrm{T})(\sin 30°) = 1.6 \times 10^{-14}\,\mathrm{N}$$

(b) We first express the velocity and the magnetic field in terms of the unit vectors $\vec{\mathrm{i}}$, $\vec{\mathrm{j}}$, and $\vec{\mathrm{k}}$ as follows:

$$\vec{v} = (10^7\,\vec{\mathrm{i}}\,)\,\mathrm{m/s}$$

$$\begin{aligned}\vec{B} &= B\cos\theta\,\vec{\mathrm{i}} + B\sin\theta\,\vec{\mathrm{j}}\\ &= [(0.02)(\cos 30°)\,\vec{\mathrm{i}} + (0.02)(\sin 30°)\,\vec{\mathrm{j}}\,]\,\mathrm{T}\\ &= (0.017\,\vec{\mathrm{i}} + 0.01\,\vec{\mathrm{j}}\,)\,\mathrm{T}\end{aligned}$$

We use Eq. 25.1 to find the force on the electron as follows:

$$\vec{F}_B = q\,\vec{v} \times \vec{B} = q\begin{vmatrix}\vec{\mathrm{i}} & \vec{\mathrm{j}} & \vec{\mathrm{k}}\\ v_x & v_y & v_z\\ B_x & B_y & B_z\end{vmatrix} = (-e)\begin{vmatrix}\vec{\mathrm{i}} & \vec{\mathrm{j}} & \vec{\mathrm{k}}\\ 10^7\,\mathrm{m/s} & 0 & 0\\ 0.017\,\mathrm{T} & 0.01\,\mathrm{T} & 0\end{vmatrix}$$

$$\begin{aligned}&= (-e)\left[(0)\,\vec{\mathrm{i}} - (0)\,\vec{\mathrm{j}} + (10^7\,\mathrm{m/s})(0.01\,\mathrm{T})\,\vec{\mathrm{k}}\right]\\ &= (-1.6 \times 10^{-19}\,\mathrm{C})(10^5\,\mathrm{T\,m/s})\,\vec{\mathrm{k}}\\ &= -(1.6 \times 10^{-14}\,\mathrm{N})\,\vec{\mathrm{k}}\end{aligned}$$

The magnetic force on the electron $\vec{F}_B$ has a magnitude that agrees with the result of part (a) and is directed along the negative z-axis.

25.2 Motion of a Charged Particle in a Uniform Magnetic Field

The fact that $\vec{F}_B \perp \vec{v}$ indicates that the magnetic field $\vec{B}$ does not work on the *charged* particle. Therefore, $\vec{F}_B$ never changes the magnitude of $\vec{v}$, but only changes its direction.

Let us consider a uniform magnetic field (coming out of the page). Now assume a positively charged particle q moving with an initial velocity vector $\vec{v}$ perpendicular to the field, as shown in Fig. 25.4. As the direction of the particle's velocity changes in response to the magnetic force, the new $\vec{F}_B$ at the new location remains perpendicular to the new direction of the particle. As a result, the path of the particle is a circle of radius r. The particle rotates in a clockwise sense if its charge is positive, as shown in Fig. 25.4, and in a counterclockwise sense if the charge is negative.

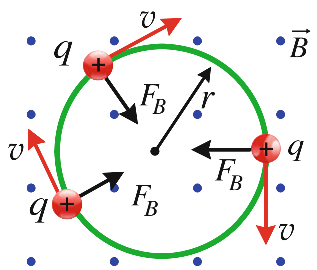


Fig. 25.4 When the initial velocity of a positively charged particle is perpendicular to the magnetic field, the particle's orbit is a *circle*

When we equate the magnitude of the magnetic force, $F_B = qvB$, to the product of the mass of the particle m and the magnitude of the centripetal acceleration, we get:

$$F_B = qvB = m \times \frac{v^2}{r} \tag{25.5}$$

Solving for r, we get:

$$\boxed{r = \frac{mv}{qB}} \tag{25.6}$$

That is, the radius of curvature is proportional to the magnitude of the momentum mv of the particle and inversely proportional to the *magnitude of the charge* and to the magnitude of the magnetic field.

The period of the motion $T = 2\pi r/v$, the frequency $f = 1/T$, and the angular frequency $\omega = 2\pi/T$, can be written as:

$$\boxed{T = \frac{2\pi m}{qB}} \tag{25.7}$$

$$\boxed{f = \frac{qB}{2\pi m}} \tag{25.8}$$

$$\boxed{\omega = \frac{qB}{m}} \tag{25.9}$$

These equations show that T, f, and ω are independent of the speed v of the particle and the radius r of the orbit.

If the velocity of the charged particle has two components, one perpendicular ($v_\perp$) to the uniform magnetic field and the other parallel ($v_\parallel$) to it, then the particle will move in a helical path about the direction of the magnetic field $\vec{B}$. For example, if $\vec{B}$ is along the x-axis, the perpendicular component $v_\perp$ (in the yz plane) determines the radius of the helix $r = mv_\perp/qB$, while the parallel component determines the distance between the turns of the helix (the pitch) $p = v_\parallel T$, see Fig. 25.5.

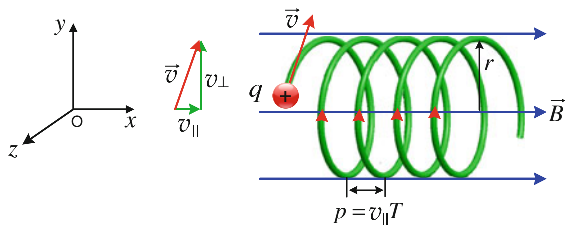


Fig. 25.5 When the initial velocity of a positively charged particle has a component parallel to the magnetic field $\vec{B}$, the particle will move in a helical path about the direction of the field

Example 25.2

A proton of mass $m = 1.67 \times 10^{-27}$ kg and charge $q = e = 1.6 \times 10^{-19}$ C is moving in a circular orbit of radius $r = 20$ cm perpendicular to a uniform magnetic field of magnitude $B = 0.25$ T. (a) Find the period of the proton. (b) Find the speed of the proton. (c) Find the magnitude of the magnetic force on the proton.

Solution: (a) From Eq. 25.7, we have:

$$T = \frac{2\pi m}{eB} = \frac{2\pi(1.67 \times 10^{-27}\,\text{kg})}{(1.6 \times 10^{-19}\,\text{C})(0.25\,\text{T})} = 2.6 \times 10^{-7}\,\text{s}$$

(b) Using the relation $T = 2\pi r/v$ [or Eq. 25.6], we have:

$$v = \frac{2\pi r}{T} = \frac{2\pi(0.2\,\text{m})}{2.6 \times 10^{-7}\,\text{s}} = 4.8 \times 10^{6}\,\text{m/s}$$

(c) From the relation $F_B = |q|vB\sin 90°$, we have:

$$F_B = evB = (1.6 \times 10^{-19}\,\text{C})(4.8 \times 10^{6}\,\text{m/s})(0.25\,\text{T}) = 1.9 \times 10^{-13}\,\text{N}$$

Example 25.3

An electron of mass $m = 9.11 \times 10^{-31}$ kg is moving with a speed $v = 2.8 \times 10^{6}$ m/s. The electron enters a uniform magnetic field of magnitude $B = 5 \times 10^{-4}$ T when the angle between $\vec{v}$ and $\vec{B}$ is 60°. Find the radius and pitch of the helical path taken by the electron.

Solution: The components $v_\perp$ and $v_\parallel$ with respect to $\vec{B}$ are:

$$v_\perp = v\sin\theta = (2.8 \times 10^{6}\,\text{m/s})\sin 60° = 2.42 \times 10^{6}\,\text{m/s}$$
$$v_\parallel = v\cos\theta = (2.8 \times 10^{6}\,\text{m/s})\cos 60° = 1.40 \times 10^{6}\,\text{m/s}$$

Using the relations $r = mv_\perp/qB$ and $p = v_\parallel T$, we have:

$$r = \frac{mv_\perp}{eB} = \frac{(9.11 \times 10^{-31}\,\text{kg})(2.42 \times 10^{6}\,\text{m/s})}{(1.6 \times 10^{-19}\,\text{C})(5 \times 10^{-4}\,\text{T})} = 0.0276\,\text{m} = 2.76\,\text{cm}$$

$$p = v_\parallel T = v_\parallel \frac{2\pi r}{v_\perp} = \frac{2\pi(0.0276\,\text{m})(1.4 \times 10^{6}\,\text{m/s})}{(2.42 \times 10^{6}\,\text{m/s})} = 0.1003\,\text{m} = 10.03\,\text{cm}$$

25.3 Charged Particles in an Electric and Magnetic Fields

In the presence of both an electric field $\vec{E}$ and a magnetic field $\vec{B}$, the total force $\vec{F}$ exerted on a charge q moving with velocity $\vec{v}$ is:

$$\vec{F} = q\vec{E} + q\vec{v} \times \vec{B} \tag{25.10}$$

which is often called the **Lorentz force**.

25.3.1 Velocity Selector

Sometimes it is required to select charged particles moving only with same constant velocity. This can be achieved by applying an upward electric field $\vec{E}$ perpendicular to a magnetic field $\vec{B}$ coming out of the page, as shown in Fig. 25.6. In this figure a positive charge q passes from the source through slits S_1 and S_2 and moves to the right in a straight line with velocity $\vec{v}$. Consequently, the electric force $q\vec{E}$ points upwards with a magnitude qE, while the magnetic force $q\vec{v} \times \vec{B}$ points downwards with a magnitude qvB.

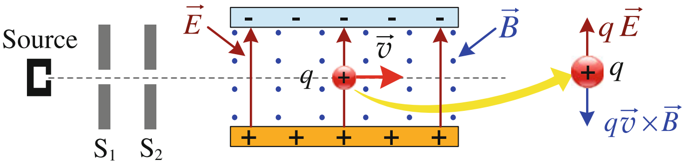


Fig. 25.6 In a velocity selector, the magnetic field $\vec{B}$, electric field $\vec{E}$, and the velocity $\vec{v}$ of the charged particle are perpendicular to each other. When the magnetic force $q\vec{v} \times \vec{B}$ cancels the electric force $q\vec{E}$, the charged particle will move in a *straight line*

If we choose the values of $\vec{E}$ and $\vec{B}$ such that $qE = q\,vB$, then:

$$\boxed{v = \frac{E}{B}} \tag{25.11}$$

and the particle will continue moving in a horizontal straight line through the region of the fields. For the chosen values of $\vec{E}$ and $\vec{B}$, all particles with speeds greater than $v = E/B$ will move downwards, while all particles with speeds less than $v = E/B$ will move upwards.

25.3.2 The Mass Spectrometer

A mass spectrometer is an instrument used to measure the mass or the mass-to-charge ratio for charged particles (or ions). The mass spectrometer of Fig. 25.7 has a source of charged particles behind S_1, and these particles pass through S_1 and S_2 into a velocity selector like the one shown in Fig. 25.6. Particles that have a speed of $v = E/B$ pass through slit S_3 and enter a *deflecting chamber* of uniform magnetic field $\vec{B}'$ that has the direction of $\vec{B}$ in the velocity selector. In this region the particles move in a circular path of radius r.

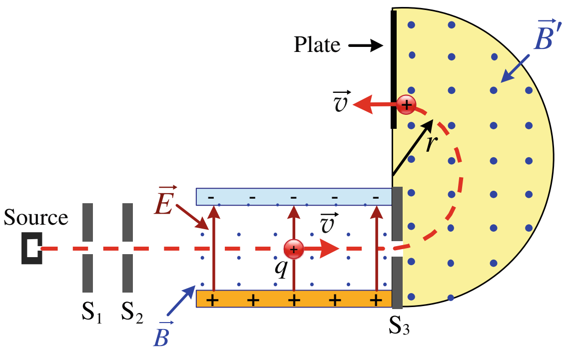


Fig. 25.7 The schematic drawing of a mass spectrometer. Positively charged particles from the source enter the velocity selector and then into a region where the magnetic field $\vec{B}'$ causes the particle to move in a *semicircle* of radius r before striking a plate

From Eq. 25.6, the mass m can be expressed as follows:

$$m = \frac{qB'r}{v} \tag{25.12}$$

Then we use $v = E/B$, to calculate the ratio m/q as follows:

$$\boxed{\frac{m}{q} = \frac{BB'r}{E}} \tag{25.13}$$

If the charge q is known, then the mass m of the charged particle can be calculated in terms of B, B', E, and r.

25.3.3 The Hall Effect

In 1879, Edwin Hall showed that when a current I passes through a strip of metal which is placed perpendicular to a magnetic field $\vec{B}$, a potential difference is established in a direction perpendicular to both I and $\vec{B}$. This phenomenon is known as *Hall effect*.

Figure 25.8a shows a thin flat strip of copper connected to a battery. Electrons flow with drift speed v_d opposite to the conventional current I. In Fig. 25.8b we show that when we apply to the strip a magnetic field $\vec{B}$ (into the page), electrons experience an upward transverse magnetic force $\vec{F}_M = q\vec{v}_d \times \vec{B} = -e\vec{v}_d \times \vec{B}$ and are deflected from their previous course. Because electrons cannot escape from the strip, negative charges accumulate on its upper side, leaving a net positive charge on its lower side. This separation of charges produces an upward transverse *Hall electric field* $\vec{E}_H$ that exerts a downward electric force on the electrons $\vec{F}_E = q\vec{E}_H = -e\vec{E}_H$. Charges accumulate, and $\vec{E}_H$ increases, until the electric force finally cancels the magnetic force and equilibrium is established.

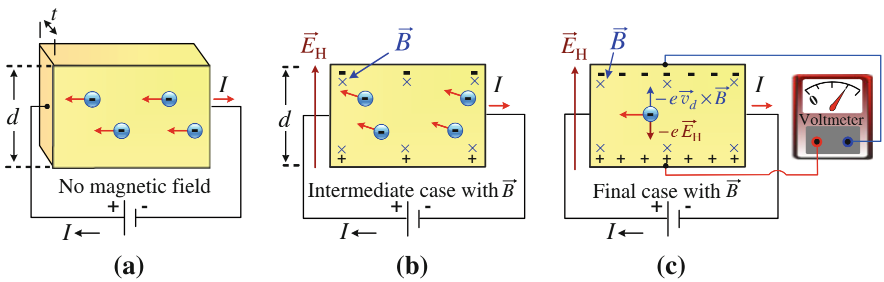


Fig. 25.8 (a) A conductor carrying a current I. (b) The situation immediately after applying the magnetic field into the page. Electrons experience an upward magnetic force $\vec{F}_M$, accumulate on the top surface, which creates an upward electric field that produces a downward electric force $\vec{F}_E$. (c) $\vec{F}_E$ cancels $\vec{F}_M$ at equilibrium

Equating the electric and magnetic forces on an electron gives:

$$eE_H = ev_d B \quad \Rightarrow \quad E_H = v_d B \tag{25.14}$$

When d is the width of the strip, the potential difference ΔV_H, called the *Hall voltage*, across the strip is related to electric field E_H by:

$$\Delta V_H = E_H d \tag{25.15}$$

From Eq. 24.6, the drift speed v_d is related to the current I by:

$$I = nev_d A \tag{25.16}$$

where $A = td$ is the cross-sectional area of the strip. Substituting with E_H from Eq. 25.15 and v_d from Eq. 25.16 into Eq. 25.14, we get $\Delta V_H = IB/net$. Usually this result is written as:

$$\boxed{\Delta V_H = R_H \frac{IB}{t}} \quad \text{where} \quad \boxed{R_H = \frac{1}{ne}} \tag{25.17}$$

where $R_H = 1/ne$ is the **Hall coefficient**. Equation 25.17 can be used to measure the magnitude of the magnetic fields and give information about the sign of the charge carriers and their density.

Example 25.4

The value of the Hall coefficient R_H for a copper strip is 5.4×10^{-11} m^3/C. The strip is 2 mm wide and 0.05 mm thick and carries a current $I = 100$ mA in a magnetic field $B = 1$ T, see Fig. 25.8. (a) How large is the Hall voltage across the strip? (b) Find the magnitude of the Hall electric field.

Solution: (a) From Eq. 25.17, we have:

$$\Delta V_H = R_H \frac{IB}{t} = 5.4 \times 10^{-11}\,\text{m}^3/\text{C}\frac{(100 \times 10^{-3}\,\text{A})(1\,\text{T})}{0.05 \times 10^{-3}\,\text{m}} = 1.08 \times 10^{-7}\,\text{V}$$

A Hall voltage of 0.108 μV needs a sensitive measuring instrument.

(b) From Eq. 25.15, we have:

$$E_H = \frac{\Delta V_H}{d} = \frac{1.08 \times 10^{-7}\,\text{V}}{2 \times 10^{-3}\,\text{m}} = 5.4 \times 10^{-5}\,\text{V/m}$$

25.4 Magnetic Force on a Current-Carrying Conductor

A net flow of charges through a wire is represented by a current. Since a magnetic field exerts a force on a moving charge, then one should expect that it should exert a force on a wire carrying a current.

- Figure 25.9a showns a horizontal flexible conducting wire carrying no current. In the presence of a uniform magnetic field $\vec{B}$ directed out of the page, the wire stays horizontal.
- However, when the wire carries a current in the left direction, as shown in Fig. 25.9b, the wire deflects upwards.
- Now, if the current direction is reversed, as shown in Fig. 25.9c, the wire deflects downwards.

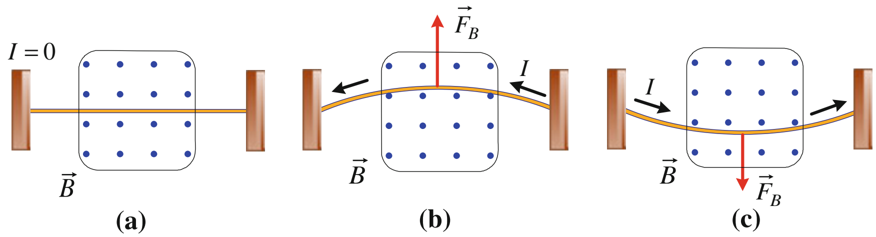


Fig. 25.9 A flexible wire is suspended horizontally and passes through a region of uniform magnetic field. (a) Without current in the wire, the wire stays horizontal. (b) With a left current, the deflection is *upwards*. (c) With a right current, the deflection is *downwards*

Figure 25.10 shows a segment of a horizontal straight wire of length L and cross-sectional area A, carrying a current I to the left in a uniform magnetic field $\vec{B}$ out of the page. First, we consider a conducting electron of charge $q=-e$ drifting to the right (opposite to the conventional left current I) with a drift speed v_d. According to Eq. 25.2, the magnetic force on this electron has a magnitude $ev_d B$ and is directed upwards.

To find the magnitude of the total *upward force* on this segment of wire, we multiply the force on one electron by the total number of conducting electrons in the segment, which is nAL, where n is the number of electrons per unit volume. Thus:

$$F_B = (ev_d B)nAL$$

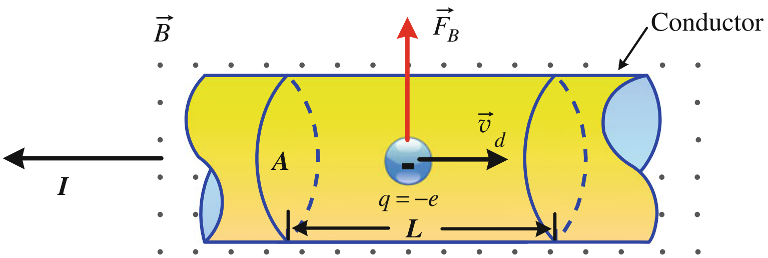


Fig. 25.10 Force on a moving charge in a current-carrying conductor. The current direction is to the left, which means that the electrons drift to the right. A magnetic field out of the page causes the electrons and the wire to be deflected *upwards*

From Eq. 25.16, the current in the wire is $I=nev_d A$. Then, the magnitude of the total upward force on this segment of wire will be:

$$F_B = ILB \tag{25.18}$$

When the uniform magnetic field $\vec{B}$ is not perpendicular to the straight wire, the magnetic force is given by a generalization of Eq. 25.18 as follows:

$$\boxed{\vec{F}_B = I\,\vec{L} \times \vec{B}} \tag{25.19}$$

where $\vec{L}$ is a *length vector* that points in the direction of the conventional current I.

If the wire is not straight, we consider a small straight segment of length ds and apply Eq. 25.19 to calculate the differential force:

$$\boxed{d\vec{F}_B = I\,d\vec{s} \times \vec{B}} \tag{25.20}$$

To calculate the total force on a wire of arbitrary shape, as shown in Fig. 25.11a, we integrate Eq. 25.20 over the length of the wire as follows:

$$\vec{F}_B = \int_a^b d\vec{F}_B = I\int_a^b d\vec{s} \times \vec{B} \tag{25.21}$$

where the current I runs from one endpoint a to another endpoint b.

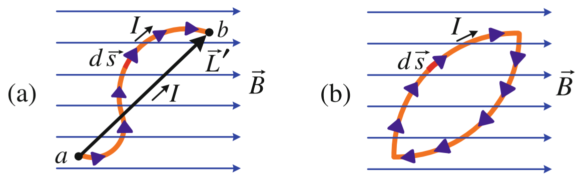


Fig. 25.11 (a) $\vec{F}_B$ on any curved wire carrying a current I in a uniform magnetic field is equal to the magnetic force on a straight wire of length L' from a to b. (b) $\vec{F}_B$ on a closed loop is zero

When the magnetic field is uniform, we take $\vec{B}$ outside the integrand of Eq. 25.21. Therefore, this equation reduces to:

$$\vec{F}_B = I\left(\int_a^b d\vec{s}\right) \times \vec{B} \tag{25.22}$$

When we integrate over $\vec{s}$, we get $\int_a^b d\vec{s} = \vec{L}'$, where $\vec{L}'$ is a length vector directed from a to b. Therefore, Eq. 25.21 becomes:

$$\boxed{\vec{F}_B = I\,\vec{L}' \times \vec{B}} \tag{25.23}$$

For a closed loop, see Fig. 25.11b, $\oint d\vec{s} = 0$ and hence $\vec{F}_B = 0$.

Therefore, in a uniform magnetic field, we conclude that:

- The net magnetic force on any curved wire carrying a current I flowing from one endpoint a to another endpoint b is the same as that for a straight wire carrying the same current from a to b.
- The net magnetic force on any closed loop of a wire carrying a current I is zero.

Example 25.5

A conducting wire has a *linear density* $\rho = 40 \times 10^{-3}$ kg/m and carries a current $I = 20$ A. Assume a magnetic field $\vec{B}$ perpendicular to the wire; find the minimum B and its direction in order to suspend the wire (that is to balance its weight) when the wire: (a) is in a horizontally straight configuration of a length L, (b) is bent into an upward vertical semicircular arc of radius R.

Solution: (a) Figure 25.12 shows the situations for both cases, with a selected direction of I. For a minimum magnetic field, the magnetic force must be upwards in both cases as shown in Fig. 25.12.

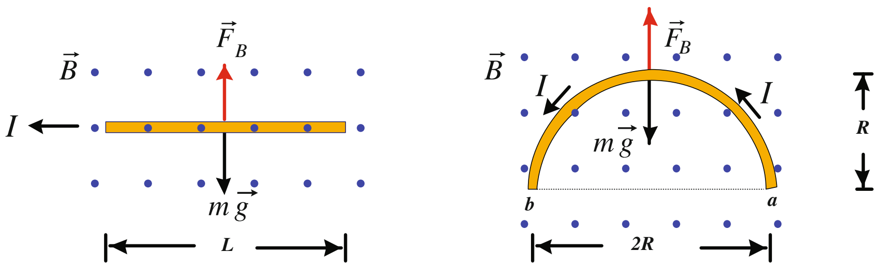


Fig. 25.12

In order to suspend the straight wire, the magnetic force F_B must equal to the wire's weight mg. Since $F_B = ILB$ and $m = \rho L$, we have:

$$F_B = mg \quad \Rightarrow \quad ILB = mg \quad \Rightarrow \quad ILB = \rho L g$$

Thus: $$B = \frac{\rho g}{I} = \frac{(40 \times 10^{-3}\,\text{kg/m})(10\,\text{m/s}^2)}{20\,\text{A}} = 0.02\,\text{T}$$

which is about 200 times the strength of the earth's magnetic field.

(b) The magnetic force F_B on a semicircular wire of radius R carrying a current I flowing from the one endpoint a to another endpoint b is the same as the magnetic force exerted on a straight wire having length $L' = 2R$ carrying the same current from a to b. That is $F_B = I(2R)B$. Since $m = \rho(\pi R)$ and F_B must equal mg, then:

$$2IRB = \pi \rho R g$$

Thus: $$B = \frac{\pi \rho g}{2I} = \frac{\pi \times (40 \times 10^{-3}\,\text{kg/m})(10\,\text{m/s}^2)}{2 \times 20\,\text{A}} = 0.0314\,\text{T}$$

Loudspeakers

The electrical output of a radio or TV set is connected to the leads of a device referred to as a loudspeaker, which converts electrical energy to sound energy. A loudspeaker has a permanent magnet that exerts a force on a current-carrying conductor. Those leads of the speaker are connected internally to a coil that is attached to the speaker cone, which is made of stiff cardboard that can move freely back and forth in front of the magnet, see Fig. 25.13.

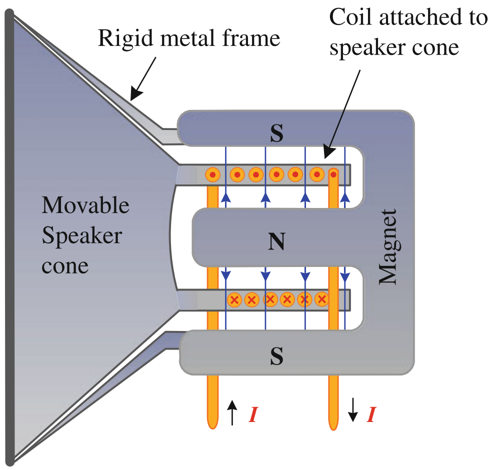


Fig. 25.13 A sketch showing a cross-sectional view of a typical loudspeaker, where both the coil and the speaker cone can move back and forth freely due to the magnetic force exerted by the permanent magnet on the current-carrying coil

When a current representing an audio signal flows through the coil, the magnetic field produced by the magnet will exert a force on the coil. As the current varies with the frequency of the audio signal, the coil and the speaker cone will move back and forth with the same frequency. This movement causes compressions and expansions of the air adjacent to the cone and consequently produces sound waves. As the electrical input to the speaker varies, the frequency and intensity of the generated sound waves also change to match.

25.5 Torque on a Current Loop

Most electric motors operate on the principle that a magnetic field exerts a torque on a loop of a current-carrying conductor. This torque has the ability to rotate the loop about a fixed rotational axis.

Consider a rectangular loop of two short sides ① and ② each of length a and two long sides ③ and ④ each of length b. The loop carries a current I in the presence of uniform magnetic field $\vec{B}$ which is always perpendicular to the long sides ③ and ④, and free to rotate about the axis OO', see Fig. 25.14.

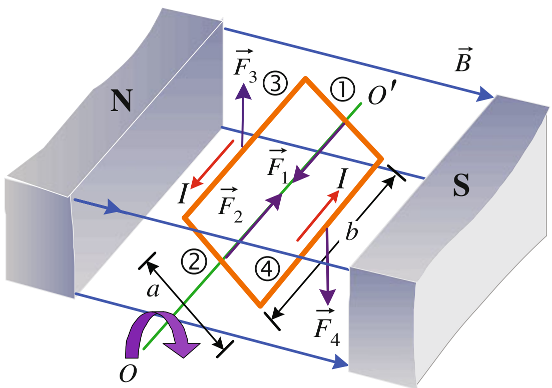


Fig. 25.14 A rectangular loop carrying a current I that can rotate freely about the axis OO' in the presence of a uniform magnetic field

In Fig. 25.14, we notice the following:

- The magnetic forces $\vec{F}_1$ and $\vec{F}_2$ on the short sides ① and ② cancel each other and *produce no torque*, since they pass through a common origin.
- The magnetic forces $\vec{F}_3$ and $\vec{F}_4$ on the long sides ③ and ④ cancel each other, but *produce a torque* about the rotational axis OO'.

We assume that $\vec{B}$ makes an angle $0 \le \theta \le 90°$ with the vector area $\vec{A}$, which is a vector perpendicular to the plane of the loop and has a magnitude equal to the area of the loop, see the side view of the loop shown in Fig. 25.15.

In Fig. 25.15, the side ③ is represented by a circle and the current passing through it is represented by a red dot, while the side ④ has the current represented by a red cross. From Eq. 25.19, the magnitudes of $\vec{F}_3$ and $\vec{F}_4$ are the same and given by:

$$F_3 = F_4 = IbB \tag{25.24}$$

The moment arm of F_3 and F_4 about O is $(a/2)\sin\theta$. Thus, the magnitude of the net torque about the rotational axis OO' is:

$$\begin{aligned}\tau &= F_3(a/2)\sin\theta + F_4(a/2)\sin\theta \\ &= [F_3 + F_4](a/2)\sin\theta = [2IbB](a/2)\sin\theta \\ &= IAB\sin\theta\end{aligned} \tag{25.25}$$

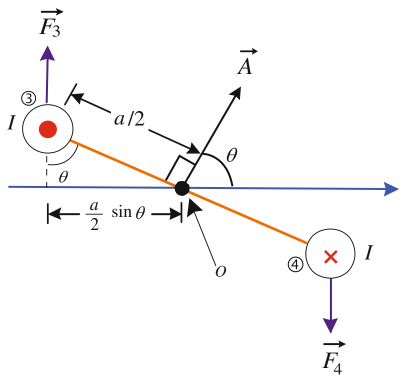


Fig. 25.15 A side view of the loop showing the two forces $\vec{F}_3$ and $\vec{F}_4$ that produce a torque on the current loop about point O

where $A = ab$ is the area of the loop. This equation shows that $\tau_{max} = IAB$ when $\vec{B}$ is perpendicular to the normal of the loop ($\theta = 90°$), and $\tau_{min} = 0$ when $\vec{B}$ is parallel to the normal to the plane of the loop ($\theta = 0$).

The direction of the torque exerted on the loop can be expressed in terms of the vector area as follows:

$$\vec{\tau} = I\vec{A} \times \vec{B} \qquad (\tau = IAB\sin\theta) \tag{25.26}$$

The product $I\vec{A}$ is defined as the magnetic dipole moment $\vec{\mu}$ (or simply the magnetic moment) of the loop and has the SI unit ampere-meter ^{2} (A.m^2). Thus:

$$\vec{\mu} = I\vec{A} \quad \text{(Single loop)} \tag{25.27}$$

If we replace the single loop of current with a *coil* of N loops, or turns, then the magnetic dipole moment $\vec{\mu}$ of the coil will be given by:

$$\vec{\mu} = NI\vec{A} \quad \text{(Coil of } N \text{ loops)} \tag{25.28}$$

Using this definition, Eq. 25.26 can be written as:

$$\vec{\tau} = \vec{\mu} \times \vec{B} \tag{25.29}$$

We can determine the direction of $\vec{A}$ and $\vec{\mu}$ by using the right-hand rule, which is described in Fig. 25.16.

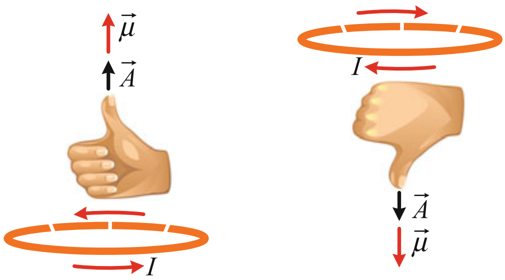


Fig. 25.16 Using the right-hand rule for determining the direction of $\vec{A}$ and $\vec{\mu}$ for a loop of wire carrying a current I

25.5.1 Electric Motors

A motor is an apparatus that converts electrical energy into rotational energy. A battery-powered motor uses the principle of torque exerted on a coil of wire wound onto a shaft that rotates 360°.

In order to allow the coil to continue rotating, the current through the coil must reverse the direction just as the coil reaches its vertical position. As shown in Fig. 25.17, several components are required to achieve this reversal. First, an electric connection is made using two brushes. These are contacts usually made of graphite. Second, a ring that is split into two halves, called a split-ring commutator. Brushes make contact with the commutator and allow current to flow into the coil. As the coil rotates, so does the commutator, which is arranged so that each of its halves changes brushes just as the coil reaches the vertical position. Changing brushes reverses the direction of the current in the coil. As a result, the direction of the force on each side of the coil is reversed and the coil continues to rotate. This process repeats at each half-turn, causing the coil to spin in the magnetic field.

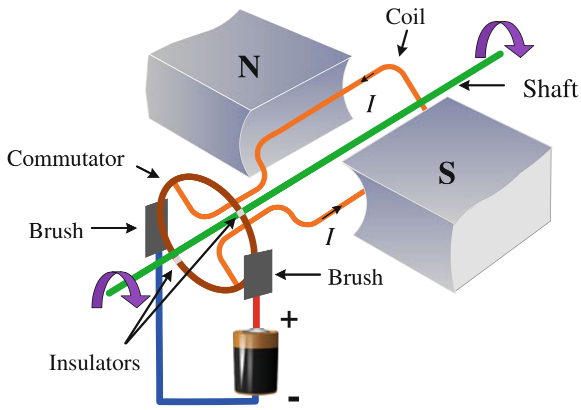
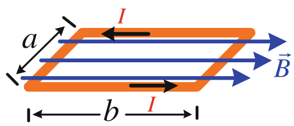


Fig. 25.17 The split-ring commutators in an electric motor allow the current in the wire coil to change direction and thus enable the coil in the motor to rotate continuously

Example 25.6

A rectangular coil of sides $a = 4\,\text{cm}$ and $b = 8\,\text{cm}$ consists of $N = 75$ turns of wire and carries a current $I = 10\,\text{mA}$. A magnetic field of magnitude $B = 0.2\,\text{T}$ is applied parallel to the plane of the coil, see Fig. 25.18. (a) Find the magnitude of the magnetic dipole moment of the coil. (b) What is the magnitude of the torque acting on the coil?

Fig. 25.18



Solution: (a) Using Eq. 25.28, we have:

$$\mu = NIA = (75)(10 \times 10^{-3}\,\text{A})[(4 \times 10^{-2}\,\text{m})(8 \times 10^{-2}\,\text{m})] = 2.4 \times 10^{-3}\,\text{A.m}^2$$

(b) Since $\vec{B}$ is perpendicular to $\vec{\mu}$, then Eq. 25.29 gives:

$$\tau = \mu B \sin 90^\circ = (2.4 \times 10^{-3}\,\text{A.m}^2)(0.2\,\text{T}) = 4.8 \times 10^{-4}\,\text{N.m}$$

25.5.2 Galvanometers

The basic component of analog ammeters, voltmeters, and ohmmeters is a galvanometer. Figure 25.19 displays the main features of a type of galvanometer called the D'Arsonval galvanometer. It consists of a coil of wire that has N loops, each of cross-sectional area A. That coil is attached to a pointer and a spring. The coil is also suspended so that it can rotate freely in a radial magnetic field produced by a circular cross-sectional permanent magnet.

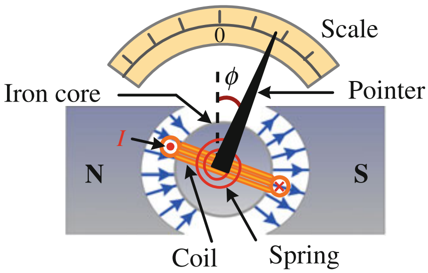


Fig. 25.19 Sketch of the structure of a moving-coil galvanometer

When a current I flows through the coil, the magnetic field exerts a torque on the coil given by Eq. 25.29, and this torque has a magnitude given by:

$$\tau = \mu B = NIAB \tag{25.30}$$

This torque is opposed by the torque τ_s exerted by the spring, which is approximately proportional to the coil deflecting angle ϕ. That is:

$$\tau_s = k\,\phi \tag{25.31}$$

where k is the stiffness constant of the spring. When the pointer is in equilibrium, we have $\tau_s = \tau$, and we get:

$$\boxed{\phi = \frac{NAB}{k} I} \quad \text{or} \quad \boxed{\phi \propto I} \tag{25.32}$$

Thus, the angular deflection ϕ of the pointer is directly proportional to the current I in the coil.

25.6 Non-Uniform Magnetic Fields

One of the useful types of non-uniform magnetic fields is the "magnetic bottle" shown in Fig. 25.20a. Such magnetic bottles can be used to trap charged particles, because the magnetic field is strong at the ends and weak in the middle. Charged particles spiral along the field lines back and forth almost indefinitely if they do not collide.

Therefore, this magnetic bottle can be used to confine a *plasma* (a gas consisting of electrons and ions). Such a confinement can help control nuclear fusion, a process that could supply us with energy indefinitely.

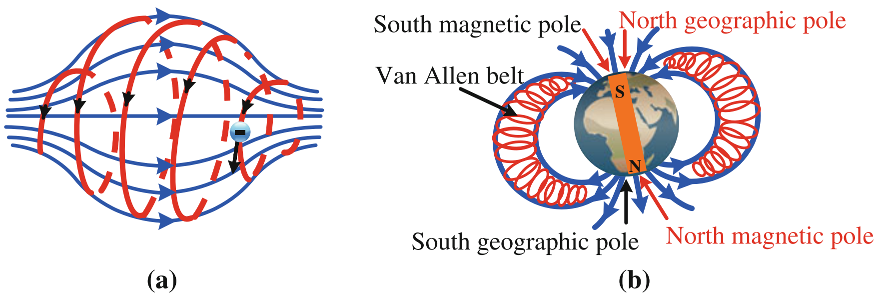


Fig. 25.20 (a) Trapping of charged particles in a non-uniform magnetic bottle. (b) A sketch of the Van Allen belt, which consists of charged particles trapped by Earth's non-uniform magnetic field

The Earth behaves like a gigantic magnet. Its north magnetic pole is actually near the geographic south pole, and its south magnetic pole is near the geographic north pole, see Fig. 25.20b. This non-uniform magnetic field traps charged particles (mostly electrons and protons) in a region of space known as *Van Allen belt*. In this belt, charged particles spiral around the field lines from pole to pole in a period of few seconds. The sun and stars are the sources of these particles (called *cosmic rays*). Most cosmic rays are deflected by the Earth's magnetic field and never reach the atmosphere. When some particles of the Van Allen belt are close to the poles, they collide with the atoms of the atmosphere causing them to emit light (Aurora Borealis or Aurora Australis).

25.7 Exercises

Section 25.1 Magnetic Force on a Moving Charge

(1) For each of the moving charges shown in Fig. 25.21, find the direction of the magnetic force, taking $\vec{v}$ to be the velocity of the particle and $\vec{B}$ to be the magnetic field.

(2) Consider a uniform magnetic field directed vertically up along the page of this paper. In which direction does an electron deflect if its velocity is directed: (a) into the paper, (b) up along the paper, (c) to the left, and (d) out of the paper.

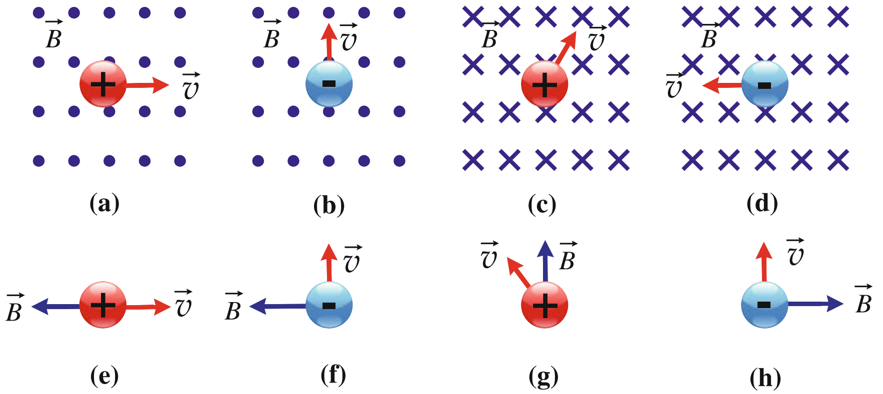


Fig. 25.21 See Exercise (1)

(3) When moving with a speed of 10^7 m/s in a magnetic field of magnitude 1.5 T, an electron experiences a magnetic force of magnitude 10^{-12} N. What is the angle between the electron's velocity and the field at this instant?

(4) A proton that has a velocity $\vec{v} = (3 \times 10^6\ \vec{i} + 4 \times 10^6\ \vec{j})$ (m/s) moves through a magnetic field $\vec{B} = (0.3\ \vec{i} + 0.02\ \mathrm{T}\ \vec{j})$ (T). Find the vector magnetic force exerted by the field on the proton, and then find the magnitude and direction of this force.

(5) Near the Earth's surface at the equator, the magnetic and electric fields are about 50 μT due North and 100 N/C downwards, respectively. Find the net force on an electron traveling with velocity 10^7 m/s due East.

Section 25.2 Motion of a Charged Particle in a Uniform Magnetic Field

(6) In a uniform magnetic field of magnitude of 10^{-4} T, an ion that has a charge $q = +2e$ completes two revolutions in 1.51 ms. Find the mass and the type of the ion.

(7) A proton travels with a speed of 8×10^7 m/s perpendicular to a uniform magnetic field of magnitude 5 T. (a) What is the radius of the proton's circular path? (b) What is the period of the motion? (c) Find the magnitude of the magnetic force on the proton.

(8) An alpha particle has a charge $q = 2\,e$ and mass $m \simeq 4\,m_p$, where m_p is the mass of a proton. The alpha particle has a kinetic energy of 5 MeV and enters a uniform magnetic field of 1.5 T directed perpendicular to its velocity. (a) Find the speed

of the alpha particle. (b) Find the magnetic force acting on the particle due to the field. (c) Find the radius of the particle's path. (d) Find the acceleration of the particle due the magnetic force.

(9) An electron of speed 5×10^6 m/s enters a uniform magnetic field of magnitude 0.01 T at an angle of 36.87°. (a) Determine the radius of the electron's helical path. (b) Determine the period of one helical path. (c) Determine the pitch of the electron's helical path.

(10) Figure 25.22 shows a region of uniform magnetic field $\vec{B}$ of magnitude 0.5 T which extends for a width $W = 0.4$ m. Consider a proton moving with a velocity $\vec{v}$ of magnitude 3×10^7 m/s, where $\vec{v}$ is perpendicular to $\vec{B}$. If the incident angle $\theta_\circ$ at the lower boundary is 60°, the proton emerges from the lower boundary as shown in the left part of the figure. However, if the incident angle $\theta_\circ$ at the lower boundary is 0°, the proton emerges from the upper boundary as shown in the right part of the figure. (a) At what angle θ and distance d does the proton exit from the lower boundary? (b) At what angle θ and distance d does the proton exit from the upper boundary? (c) At what critical incident angle $\theta_\circ$ does the proton barely touch the upper boundary?

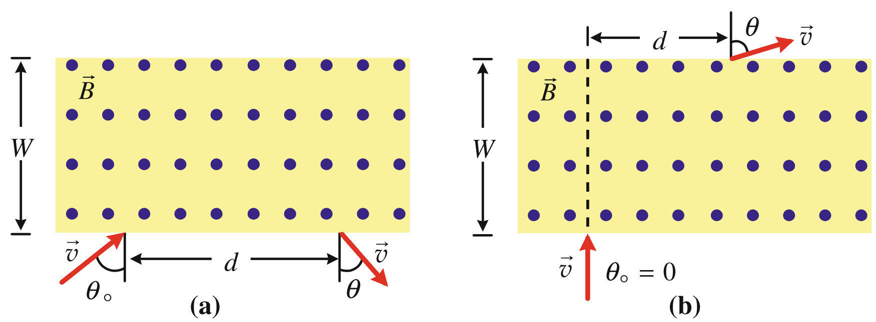


Fig. 25.22 See Exercise (10)

Section 25.3 Charged Particles in Electric and Magnetic Fields

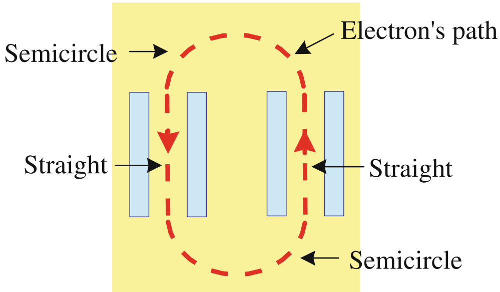
(11) A uniform magnetic field of magnitude 0.02 T is perpendicular to a uniform electric field of magnitude 750 V/m. What is the speed of an electron that goes undeflected when moving perpendicular to both fields?

(12) Assume that a 1 keV electron travels in a uniform electric field $\vec{E} = 385\,\vec{j}$ (kV/m) and a uniform magnetic field $\vec{B} = B_z\,\vec{k}$. Find the value of B_z such that

the electron would have a velocity $\vec{v} = v_x \,\vec{i}$ and would move undeflected in the presence of the two fields.

(13) Figure 25.23 shows the path of an electron in a region of uniform magnetic field. Each of the plates is uniformly charged. (a) Which plate is at the higher electric potential for each pair? (b) What is the direction of the magnetic field in this region? (c) For both pairs of plates, if the magnitude of the electric field between the plates is 6×10^4 V/m and the magnitude of the magnetic field is 2 mT, find the radius of the two semicircles.

Fig. 25.23 See Exercise (13)



(14) In the mass spectrometer shown schematically in Fig. 25.6, the magnitude of the electric and magnetic fields in the velocity-selector region are 3 kV/m and 40 mT, respectively. The magnitude of the magnetic field in the deflecting chamber is 75 mT. (a) What is the speed of ions in the velocity selector? (b) What is the radius of the path in the deflecting chamber for a singly-charged ion having a mass of 6.49×10^{-26} kg?

(15) Two single ions of the boron isotopes (of masses 10 u and 11 u) are studied in the mass spectrometer shown schematically in Fig. 25.6. Assume that the values $B = B' = 250$ mT and $E = 60$ kV/m are used in this experiment. (a) What is the speed of the ions in the velocity selector? (b) What is the spacing between the marks produced on the photographic plate by the ions of boron?

(16) A strip of copper of thickness $t = 0.4$ mm and width $d = 5$ mm is placed in a uniform magnetic field $\vec{B}$ of magnitude 1.5 T perpendicular to the strip, see Fig. 25.24. When a current $I = 20$ A passes though the strip, a Hall potential difference ΔV_H is generated across the width of the strip. The number of charge carriers per unit volume for copper is 8.47×10^{28} electrons/m^3. (a) Find the Hall coefficient R_H for the copper strip. (b) How large is the Hall voltage ΔV_H across the strip? (c) Find the magnitude of the Hall electric field E_H.

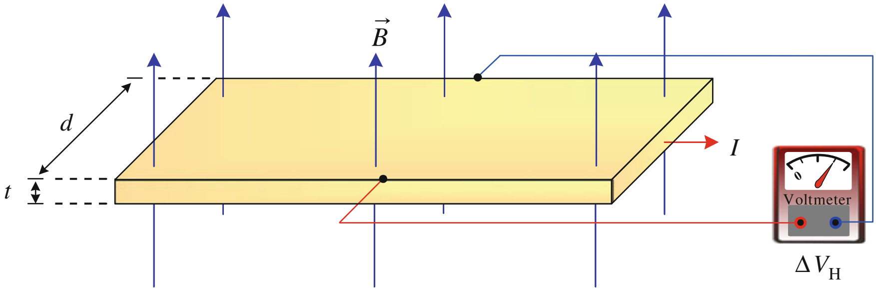
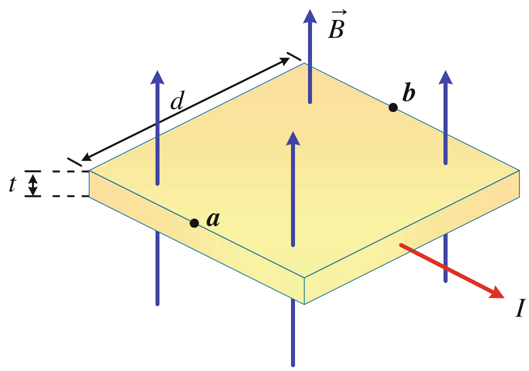


Fig. 25.24 See Exercise (16)

(17) A silver slab of thickness $t = 1.5$ mm and width $d = 2.5$ mm carries a current $I = 4$ A in a region in which there is a uniform magnetic field $\vec{B}$ of magnitude 1.25 T perpendicular to the slab. The Hall voltage ΔV_H across the slab is found to be 0.356 μV. (a) Calculate the density of the charge carriers in the slab. (b) Compare your answer in part (a) to the density of atoms in the silver slab, which has a density $\rho = 10.5 \times 10^3$ kg/m^3 and a molar mass $M = 107.9$ kg/kmol. What is the conclusion that you can find from this comparison? (c) Find the magnitude of the Hall electric field E_H.

(18) A metal strip of thickness $t = 1$ mm and width $d = 2$ cm carries a current $I = 12.5$ A in a region in which there is a uniform magnetic field $\vec{B}$ of magnitude 1.6 T perpendicular to the strip, as shown in Fig. 25.25. The Hall voltage ΔV_H across the strip is measured to be 2.135 μV. (a) Calculate the drift speed of the electrons in the strip. (b) Find the density of the charge carriers in the strip. (c) Which point is at the higher potential, a or b?

Fig. 25.25 See Exercise (18)

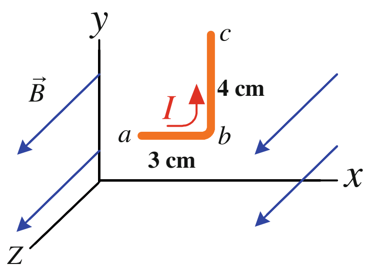


Section 25.4 Magnetic Force on a Current-Carrying Conductor

(19) A 1.5 m long straight stiff wire carries a current of 2 A and makes an angle 30° with a uniform magnetic field of 0.35 T. Find the magnitude of the force on the wire.

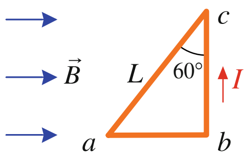
(20) The L-shaped wire shown in Fig. 25.26 lies in the xy plane. In the presence of a uniform magnetic field $\vec{B} = 1.5\ \vec{k}$ (T), the wire carries a current of 2.5 A from point a to point c. (a) Find the net force exerted on the wire. (b) Show that this net force is the same as if the wire were a straight segment from point a to point c.

Fig. 25.26 See Exercise (20)



(21) For the circuit shown in Fig. 25.27, find the magnitude and direction of the force on each side, and find the resultant force.

Fig. 25.27 See Exercise (21)



(22) A straight horizontal wire has a length $L = 20$ cm and mass $m = 0.02$ kg. The wire is hung by connecting it by massless flexible leads to an emf source. A uniform magnetic field of magnitude $B = 1.6$ T is perpendicular to the wire, as shown in Fig. 25.28. Find the necessary current needed to suspend the wire and hence remove the tension in the flexible wire.

(23) If $B = 0.2$ T and $I = 5$ A in Fig. 25.29, find the force exerted on each segment of the wire.

Fig. 25.28 See Exercise (22)

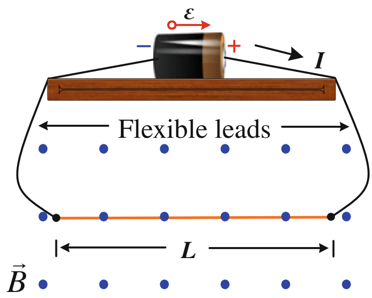
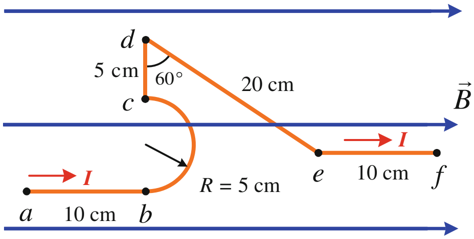
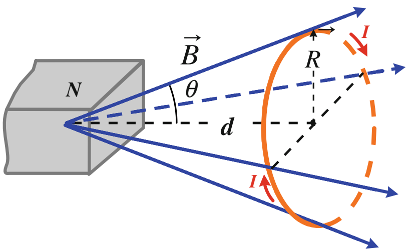


Fig. 25.29 See Exercise (23)



(24) A circular loop of wire has a radius R and carries a current I. The loop is placed in a magnetic field whose lines seem to diverge from a point on the perpendicular axis of the circular loop and at a distance d from its center, see Fig. 25.30. Find the total force on the loop.

Fig. 25.30 See Exercise (24)



Section 25.5 Torque on a Current Loop

(25) A circular coil of $N = 40$ turns has a radius $r = 5\,\text{cm}$ and carries a current $I = 2\,\text{A}$. The coil is placed in a uniform magnetic field of 0.5 T so that the normal to the coil makes an angle $\theta = 30°$ with the direction of $\vec{B}$. (a) What is the magnitude of the magnetic moment of the coil? (b) What is the magnitude of the torque exerted on the coil?

(26) For the current loop shown in the figure of exercise 21, find: (a) the magnitude and direction of the loop's magnetic moment. (b) the magnitude of the torque on the loop and the direction in which it will rotate.

(27) What is the maximum torque exerted on a 400-turn circular coil of radius 0.5 cm placed in a uniform magnetic field of magnitude 0.2 T if it carries a current of 1.5 A?

(28) A small, stiff, circular loop of radius R and mass m carries a current I. The loop lies horizontally on a rough flat table in the presence of a horizontal magnetic field of magnitude B. (a) What is the required minimum value of B so that one edge of the loop will lift off the table? (b) What is the required value of B so that one edge of the loop will lift off the table through an angle θ?

(29) The 240-turn rectangular coil shown in Fig. 25.31 carries a current of 1.5 A in a uniform magnetic field of $B = 0.25$ T. Find the magnitude of the torque on the loop and the direction in which it will rotate.

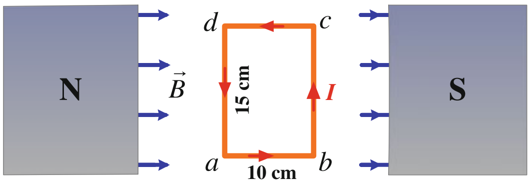


Fig. 25.31 See Exercise (29)

(30) A rectangular 100-turn coil carries a current $I = 1.75$ A and has sides $a = 40$ cm and $b = 30$ cm. The coil is hinged along the y-axis, so that its plane makes an angle $\theta = 73°$ with the x-axis as shown in Fig. 25.32. (a) What is the magnitude of the magnetic moment $\vec{\mu}$ of the coil? (b) What angle does the vector $\vec{\mu}$ make with the x-axis. (c) In the presence of a uniform magnetic field $\vec{B} = 0.8\,\vec{i}$ (T), what is the magnitude of the torque exerted on the coil and what is the expected direction of the coil's rotation?

(31) A current $I = 0.75$ A flows in a quarter of a single circular loop of wire that has a radius $R = 5$ cm. The loop lies in the xy plane and is hinged along the y-axis, so that it can rotate about this axis, see Fig. 25.33. (a) What is the magnitude of the magnetic moment $\vec{\mu}$ of the coil? (b) Express the vector $\vec{\mu}$ in terms of unit vectors. (c) When a uniform magnetic field $\vec{B} = [0.2\,\vec{i} + 0.3\,\vec{j} + 0.4\,\vec{k}]$ (T) is applied to the loop, express the torque acting on the coil in terms of unit vectors? In which direction will the loop rotate?

Fig. 25.32 See Exercise (30)

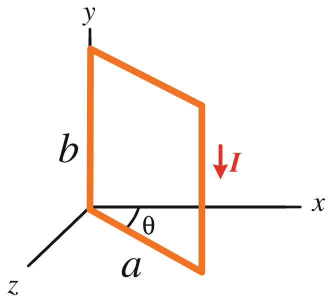
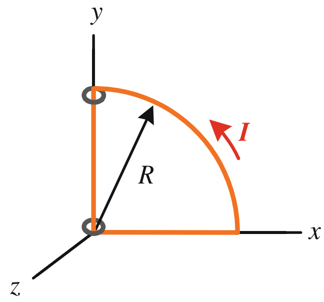
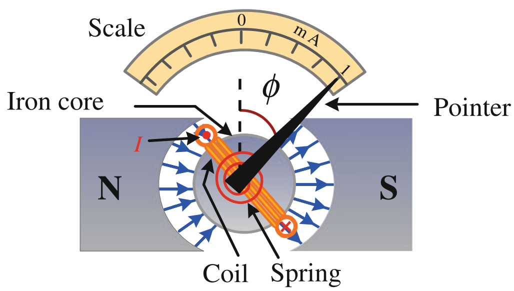


Fig. 25.33 See Exercise (31)



(32) The coil of the galvanometer shown in Fig. 25.34 has $N = 35$ turns where the dimensions of each rectangular turn are 2 cm by 2.5 cm. For any position of the coil, its plane is parallel to the magnetic field which has the value $B = 0.4$ T. The galvanometer has a spring with a stiffness constant $k = 5 \times 10^{-6}$ N.m/rad and gives a full-scale deflection if the current I going through it is 1 mA. What is the full-scale deflection angle ϕ in radians and degrees?

Fig. 25.34 See Exercise (32)



(33) Assume that the Earth's magnetic field at the equator is uniform and northerly directed at all points with a magnitude 5×10^{-5} T and that it extends out by

Earth's diameter (i.e. by 1.28×10^4 km). (a) Find the speed and time that a singly-ionized uranium atom ($m = 238$ u, $q = +e$) would take to circulate the Earth 20 km above the surface at the equator. (b) A cosmic-ray proton traveling with a speed of 2.5×10^7 m/s is heading directly towards the center of the Earth in the plane of the Earth's equator. Estimate the radius of the proton's path. Will the proton hit the Earth?

26 Sources of Magnetic Field

In this chapter we complete the description of magnetic interactions by briefly exploring the origins of magnetic fields.

26.1 The Biot-Savart Law

Based on quantitative experiments, Biot and Savart were able to arrive at a mathematical expression that describes the magnetic field at any point in terms of the current or the charge that produces the field.

Consider a point P at a distance r from: (a) an element $d\vec{s}$ chosen in the direction of a *steady* current I, (b) a point charge q moving with velocity $\vec{v}$, see Fig. 26.1. Biot and Savart proposed that the magnetic field produced by the element, or by the charge, would be:

$$\boxed{d\vec{B} = \frac{\mu_\circ}{4\pi}\frac{I\,d\vec{s} \times \hat{\vec{r}}}{r^2}} \quad \text{and} \quad \boxed{\vec{B} = \frac{\mu_\circ}{4\pi}\frac{q\,\vec{v} \times \hat{\vec{r}}}{r^2}} \quad \text{(Biot-Savart law)} \qquad (26.1)$$

where $\hat{\vec{r}}$ is a unit vector directed from $d\vec{s}$ or q toward point P. The product $I\,d\vec{s}$ is called the *differential current element*, and $\mu_\circ$ is a constant called the **permeability of free space**' which has the exact value:

$$\boxed{\mu_\circ = 4\pi \times 10^{-7}\,\text{T.m/A}} \qquad (26.2)$$

H. A. Radi and J. O. Rasmussen, *Principles of Physics*,
Undergraduate Lecture Notes in Physics, DOI: 10.1007/978-3-642-23026-4_26,

Fig. 26.1 (a) The differential magnetic field vector $d\vec{B}$ at point P, which is located by a position vector $\vec{r}$ drawn from a differential current element $I\,d\vec{s}$ to P. (b) In case of a point charge q moving with a velocity $\vec{v}$, the magnetic field $\vec{B}$ is related to the product $q\vec{v}$

To find the magnetic field $\vec{B}$ created at some point by a current of an extended circuit, we integrate Eq. 26.1 over all current elements as follows:

$$\vec{B} = \frac{\mu_\circ I}{4\pi}\int \frac{d\vec{s} \times \hat{\vec{r}}}{r^2} \tag{26.3}$$

It is useful to compare the Biot-Savart law with Coulomb's Law as follows:

Biot-Savart law ($d\vec{B}$)	Coulomb's law ($d\vec{E}$)
$d\vec{B}$ is due to differential current element $I\,d\vec{s}$, a vector	$d\vec{E}$ is due to differential charge dq, a scalar
$1/r^2$ distance dependence	$1/r^2$ distance dependence
Proportional to electric current I	Proportional to electric charge dq
Lateral, perpendicular to the $\vec{r}$ direction	Radial, in the $\vec{r}$ direction

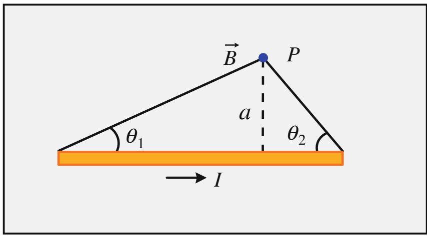
Some Applications of the Biot-Savart Law

In some situations, the integrand of Eq. 26.3 needs lengthy mathematical steps. For those interested, several mathematical and integration techniques are given at the end of this book. In this section we avoid the complexity arising from integrating Eq. 26.3 and only present the results for some cases.

1. Magnetic Field on the Extension of a Straight Wire

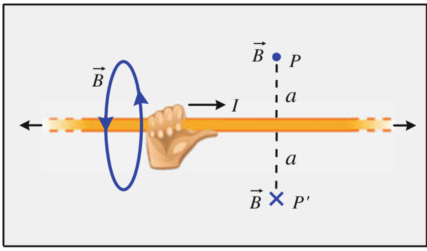
$$\vec{B} = 0 \tag{26.4}$$

2. Magnetic Field Surrounding a Thin Straight Wire



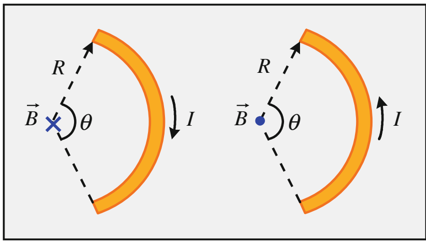
$$B = \frac{\mu_\circ I}{4\pi\, a}(\cos\theta_1 + \cos\theta_2) \tag{26.5}$$

3. Magnetic Field Surrounding a Very Long Straight Wire



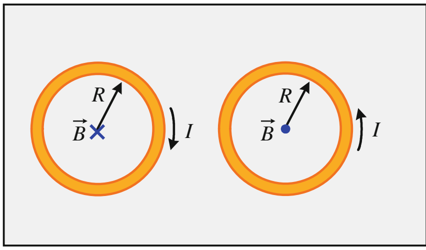
$$B = \frac{\mu_\circ I}{2\pi\, a} \tag{26.6}$$

4. Magnetic Field Due to a Curved Wire Segment



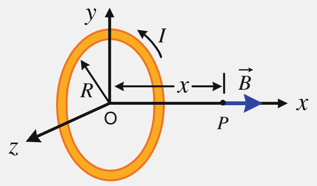
$$B = \frac{\mu_\circ I}{4\pi\, R}\theta \tag{26.7}$$

5. Magnetic Field at the Center of a Circular Wire Loop



$$B = \frac{\mu_\circ I}{2\, R} \tag{26.8}$$

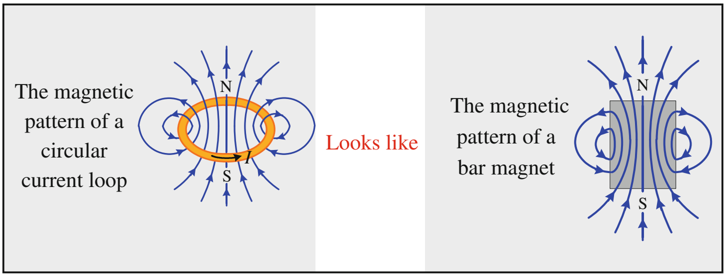
6. Magnetic Field on the Axis of a Circular Wire Loop



$$B = \frac{\mu_\circ I}{2} \frac{R^2}{(R^2 + x^2)^{3/2}} \qquad (26.9)$$

$$B = \frac{\mu_\circ I R^2}{2x^3} \quad (\text{for } x \gg R)$$

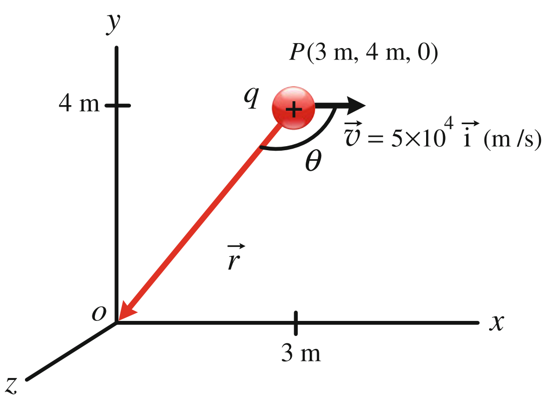
7. Sketch of $\vec{B}$ Along the Axis of a Loop and a bar Magnet



Example 26.1

A point charge $q = 6\,\mu\text{C}$ is moving in a straight line with a velocity $\vec{v} = 5 \times 10^4\,\vec{i}$ (m/s). When the charge is at the location $P(3\text{ m}, 4\text{ m}, 0)$, find the magnetic field produced by this point charge at the origin o, see Fig. 26.2.

Fig. 26.2



Solution: For a point charge q moving with a velocity $\vec{v}$, Eq. 26.1 leads to:

$$\vec{B} = \frac{\mu_\circ}{4\pi}\frac{q\vec{v}\times\hat{\vec{r}}}{r^2}$$

From the figure, we can find r and $\hat{\vec{r}}$ (from the point charge) as follows:

$$\vec{r} = [-3\,\vec{\mathrm{i}} - 4\,\vec{\mathrm{j}}](\mathrm{m})$$

$$r = \sqrt{(-3\,\mathrm{m})^2 + (-4\,\mathrm{m})^2} = 5\ \mathrm{m}$$

Thus: $$\hat{\vec{r}} = \frac{\vec{r}}{r} = \frac{[-3\,\vec{\mathrm{i}} - 4\,\vec{\mathrm{j}}](\mathrm{m})}{5\ \mathrm{m}} = -0.6\,\vec{\mathrm{i}} - 0.8\,\vec{\mathrm{j}}$$

Substituting the above results into the equation for $\vec{B}$ we obtain:

$$\vec{B} = \frac{\mu_\circ}{4\pi}\frac{q\vec{v}\times\hat{\vec{r}}}{r^2} = \frac{\mu_\circ}{4\pi}\frac{q\,(v\vec{\mathrm{i}})\times[-0.6\,\vec{\mathrm{i}} - 0.8\,\vec{\mathrm{j}}]}{r^2} = -\frac{\mu_\circ}{4\pi}\frac{q\,v\,(0.8\,\vec{\mathrm{k}})}{r^2}$$

$$= -(10^{-7}\,\mathrm{T.m/A})\frac{(6\times10^{-6}\,\mathrm{C})(5\times10^4\ \mathrm{m/s})(0.8)}{(5\ \mathrm{m})^2}\vec{\mathrm{k}}$$

$$= -9.6\times10^{-10}\,\vec{\mathrm{k}}\ (\mathrm{T})$$

Indeed this is a very small value for the magnetic field produced by this charge, which is equivalent to the charge of about 4×10^{13} protons.

Example 26.2

Two very long parallel straight wires carry currents that are perpendicular to the page. Wire 1 carries a current $I_1 = 3$ A out of the page and passes through the origin o of the x-axis, while wire 2 carries a current $I_2 = 2$ A into the page and passes through the x-axis at a distance $d = 0.6$ m from the origin. (a) On the x-axis, show the directions of the magnetic fields, to right of wire 2, between the two wires, and to the left of wire 1. (b) To the right of wire 2, find a distance a at which the resultant magnetic field is zero.

Solution: (a) Using the right hand rule presented in the figure of Eq. 26.6, we can draw the direction of $\vec{B}_1$ of wire 1 and $\vec{B}_2$ of wire 2 on the three regions of the x-axis as shown in Fig. 26.3:

(b) From Eq. 26.6 the magnitudes of the magnetic-field vectors $\vec{B}_1$ and $\vec{B}_2$ at point P are:

$$B_1 = \frac{\mu_\circ I_1}{2\pi(d+a)}, \quad \text{and} \quad B_2 = \frac{\mu_\circ I_2}{2\pi a}$$

For $I_1 > I_2$

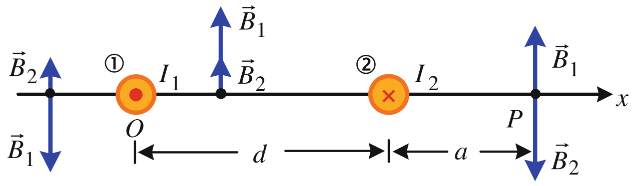


Fig. 26.3

When the magnitudes of the opposite two vectors $\vec{B}_1$ and $\vec{B}_2$ are equal, the resultant magnetic field becomes zero. Therefore, we have:

$$\frac{\mu_\circ I_1}{2\pi(d+a)} = \frac{\mu_\circ I_2}{2\pi a} \quad \Rightarrow \quad \frac{I_1}{d+a} = \frac{I_2}{a} \quad \Rightarrow \quad aI_1 = I_2(d+a)$$

$$\Rightarrow \quad a\left(\frac{I_1}{I_2} - 1\right) = d$$

Thus:
$$a = \frac{d}{\left(\dfrac{I_1}{I_2} - 1\right)} = \frac{0.6\ \text{m}}{\left(\dfrac{3\ \text{A}}{2\ \text{A}} - 1\right)} = 1.2\ \text{m}$$

Since $I_1 > I_2$, P is the only point at which $B_{net} = 0$ on the x-axis.

Example 26.3

Two straight wires ① and ③, each of length $L = 4$ cm, are connected by a quarter circular arc wire ② of radius $R = 3$ cm, as shown in Fig. 26.4. Determine the magnitude and direction of the magnetic field at the center P of the arc, when the current I is 2 A.

Solution: There is no contribution to the field at point P from the lower wire ①, since P is on the extension of the wire, i.e. $B_1 = 0$.

From Eq. 26.7, the quarter circular arc wire ② has a magnetic field:

$$B_2 = \frac{\mu_\circ I}{8R} \quad \text{(Directed out of the page)}$$

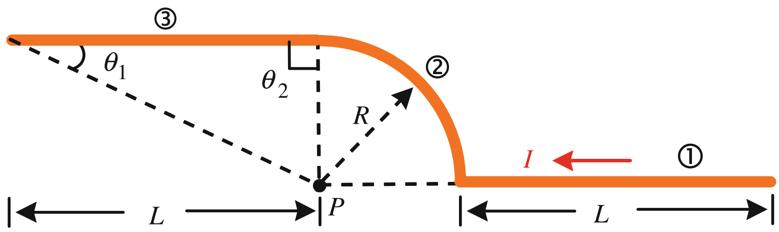


Fig. 26.4

According to Eq. 26.5, point P is at a distance $R = 3$ cm from the straight wire ③ and subtends two angles with the wire, θ_1 and θ_2. From the figure, we get:

$$\cos\theta_1 = L/\sqrt{L^2 + R^2} = 4/5 \quad \text{and} \quad \cos\theta_2 = \cos 90^\circ = 0$$

Thus: $$B_3 = \frac{\mu_\circ I}{4\pi R}(\cos\theta_1 + \cos\theta_2) = \frac{\mu_\circ I}{5\pi R} \quad \text{(Directed out of the page)}$$

The total magnetic field is the superposition of the fields from the three wires. Thus, the resultant magnetic field is:

$$\begin{aligned} B &= B_1 + B_2 + B_3 = 0 + \frac{\mu_\circ I}{8R} + \frac{\mu_\circ I}{5\pi R} \\ &= \frac{\mu_\circ I}{R}\left(\frac{1}{8} + \frac{1}{5\pi}\right) = \frac{(4\pi \times 10^{-7}\,\text{T.m/A})(2\text{A})}{3 \times 10^{-2}\,\text{m}}\left(\frac{1}{8} + \frac{1}{5\pi}\right) \\ &= 1.58 \times 10^{-5}\,\text{T} = 15.8\,\mu\text{T} \quad \text{(Directed out of the page)} \end{aligned}$$

26.2 The Magnetic Force Between Two Parallel Currents

Figure 26.5 shows a portion of length ℓ of two *long* straight parallel wires separated by a distance a and carrying currents I_1 and I_2 in the same direction. Since each wire lies in the magnetic field established by the other, each will experience a magnetic force.

Wire ② sets up a magnetic field $\vec{B}_2$ perpendicular to wire ①. According to Eq. 25.19, the magnetic force on a length ℓ of wire ① is $\vec{F}_1 = I_1\,\vec{\ell} \times \vec{B}_2$, where the direction of $\vec{F}_1$ is toward wire ②. Since $\vec{\ell} \perp \vec{B}_2$, the magnitude of $\vec{F}_1$ is $F_1 = I_1\,\ell\,B_2$. When we substitute with the magnitude of B_2 given by Eq. 26.6, we get:

$$F_1 = I_1\,\ell\,B_2 = I_1\,\ell\left(\frac{\mu_\circ I_2}{2\pi a}\right) = \frac{\mu_\circ I_1 I_2}{2\pi a}\ell \tag{26.10}$$

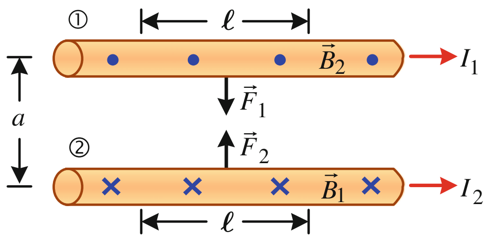


Fig. 26.5 Two parallel wires carrying currents in the same direction attract each other. Wire ② sets up a magnetic field $\vec{B}_2$ at wire ① and wire ① sets up a magnetic field $\vec{B}_1$ at wire ②

We can show that the magnetic force $\vec{F}_2$ on wire ② has the same magnitude as $\vec{F}_1$ but is opposite in direction, i.e. the two wires attract each other. We denote the magnitude of the force between the two wires by the symbol F_B and write this magnitude per unit length as:

$$\boxed{\frac{F_B}{\ell} = \frac{\mu_\circ}{2\pi}\,\frac{I_1 I_2}{a}} \tag{26.11}$$

If the two currents were antiparallel (i.e. the wires were parallel but the currents were opposite in direction), then the wires would repel.

Spotlight
Parallel currents attract and antiparallel currents repel.

Example 26.4

A battery of 12 V is connected to a resistor of resistance $R = 3\ \Omega$ by two parallel wires each of length $L = 50$ cm and separated by a distance $a = 2$ cm, see Fig. 26.6. All the connecting wires have negligible resistance. Find the magnitude of the magnetic force between the two wires. Will the wires repel or attract each other?

Solution: According to the figure, the battery sets a clockwise current I in the circuit, and the current in the parallel two wires have the same value but opposite direction. The value of this current is:

$$I = \frac{\Delta V}{R} = \frac{12\ \text{V}}{3\ \Omega} = 4\ \text{A}$$

From Eq. 26.11, the magnetic force between the two wires is:

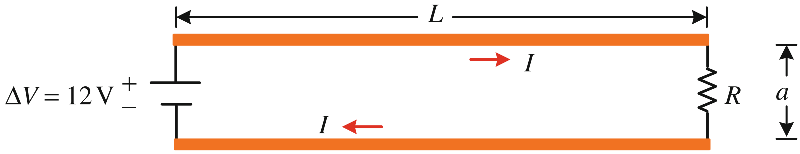


Fig. 26.6

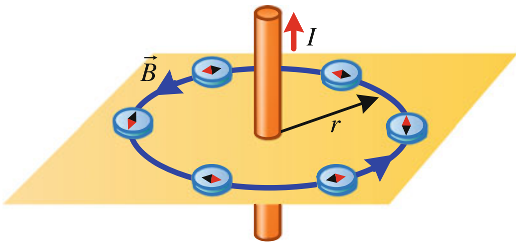
$$F_B = \frac{\mu_\circ}{2\pi}\frac{I^2}{a}L = \frac{4\pi \times 10^{-7}\,\mathrm{T.m/A}}{2\pi}\frac{(4\,\mathrm{A})^2}{2 \times 10^{-2}\,\mathrm{m}} \times 50 \times 10^{-2}\,\mathrm{m}$$
$$= 8 \times 10^{-5}\,\mathrm{N}$$

Since the currents in the two wires are antiparallel, the wires will repel each other, but with a very small force due the smallness of $\mu_\circ$.

26.3 Ampere's Law

When Oersted traced the magnetic field near a long vertical wire carrying a current I by a compass, he found that its needle deflects in a direction tangent to any circular path concentric with the wire, i.e. the needle points in the direction of $\vec{B}$, see Fig. 26.7.

Fig. 26.7 The compass needle deflects in a direction tangent to a circle of radius r, which is the direction of $\vec{B}$ created by I



The same results can be obtained when we use the Biot-Savart Law to calculate the magnetic field around a long straight wire carrying a current. The magnitude of $\vec{B}$ was given by Eq. 26.6.

The work of Oersted and Biot-Savart was continued by Ampere. Ampere's work lead to what is now known as Ampere's law, a law used in the cases of steady currents, which can be stated as follows:

Ampere's law
The *line integral* of the *tangential* magnetic field around a closed path is proportional to the net *conduction steady* current I enclosed by the path. That is:

$$\oint \vec{B} \bullet d\vec{s} = \mu_\circ I \quad \text{(Ampere's law)} \tag{26.12}$$

As a check for the long wire of Fig. 26.7, let us consider an element $d\vec{s}$ on the circular path and integrate the product $\vec{B} \bullet d\vec{s}$ over this closed path. Since $\vec{B}$ is parallel to $d\vec{s}$, then $\vec{B} \bullet d\vec{s} = B\,ds$. Thus:

$$\oint \vec{B} \bullet d\vec{s} = \oint B\,ds = B \oint ds = B(2\pi r) \tag{26.13}$$

By Ampere's law, this result should be equal to $\mu_\circ I$. Therefore:

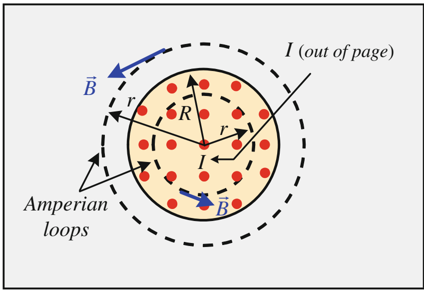
$$B = \frac{\mu_\circ I}{2\pi r} \tag{26.14}$$

This result is in complete agreement with Eq. 26.6 obtained by using the Biot-Savart law; however, Ampere's law saves considerable effort when we deal with problems that have some ***symmetry***.

Some Applications of Ampere's Law

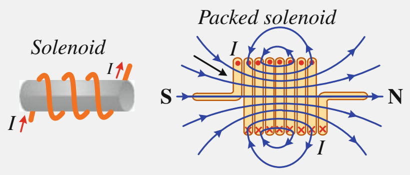
In these applications, we avoid solving the integrand of Eq. 26.12 and only present the results of some well-known cases.

1. Magnetic Field Inside and Outside a Long Straight Wire



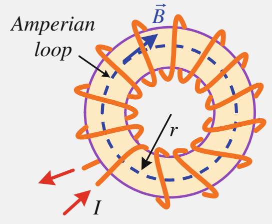
$$B = \begin{cases} \dfrac{\mu_\circ I}{2\pi R^2} r & (\text{for } r \le R) \\ \dfrac{\mu_\circ I}{2\pi r} & (\text{for } r \ge R) \end{cases} \tag{26.15}$$

2. Magnetic Field of a Solenoid of *n* Turns per Unit Length



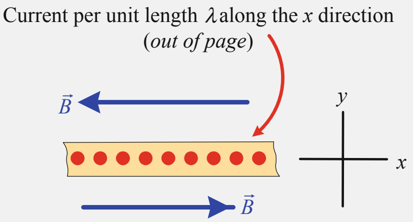
$$B = \mu_\circ n I \begin{Bmatrix} \text{inside the} \\ \text{solenoid and} \\ \text{if } n \text{ is large} \end{Bmatrix} \tag{26.16}$$

3. Magnetic Field of a Toroid of *N* Total Turns (or *n* turns/m)



$$B = \frac{\mu_\circ N I}{2\pi\, r} = \mu_\circ n I \tag{26.17}$$

4. Magnetic Field Produced by an Infinite Current Sheet



$$B = \frac{\mu_\circ \lambda}{2} \tag{26.18}$$

Example 26.5

A long wire of radius $R = 10\,\text{mm}$ carries a current $I = 3\,\text{A}$. What are the magnitudes of the magnetic field at a point 5 mm and a point 50 mm from the axis of the wire?

Solution: For a point inside the wire we use Eq. 26.15 for $r \leq R$:

$$B = \frac{\mu_\circ I}{2\pi R^2}\, r = \frac{(4\pi \times 10^{-7}\,\text{T.m/A})(3\,\text{A})}{(2\pi)(10 \times 10^{-3}\,\text{m})^2} \times (5 \times 10^{-3}\,\text{m}) = 3 \times 10^{-5}\,\text{T}$$

For a point outside the wire we use Eq. 26.15 for $r \geq R$:

$$B = \frac{\mu_\circ I}{2\pi r} = \frac{(4\pi \times 10^{-7}\,\text{T.m/A})(3\,\text{A})}{(2\pi)(50 \times 10^{-3}\,\text{m})} = 1.2 \times 10^{-5}\,\text{T}$$

Example 26.6

A solenoid of length $L = 0.5\,\text{m}$ carries a current $I = 2\,\text{A}$. The solenoid consists of six closely-packed layers, each of 800 turns. What is the magnitude of the magnetic field inside the solenoid?

Solution: The diameter of winding does not enter into the solenoid Eq. 26.16. The number of turns per unit length is:

$$n = \frac{(\text{No. of layers})(\text{No. of turns per layer})}{L} = \frac{6 \times 800\,\text{turns}}{0.5\,\text{m}} = 9{,}600\,\text{turns/m}$$

Since n is large, then from Eq. 26.16 we have:

$$B = \mu_\circ n I = (4\pi \times 10^{-7}\,\text{T.m/A})(9{,}600\,\text{turns/m})(2\,\text{A}) = 2.41 \times 10^{-2}\,\text{T}$$

Example 26.7

In a fusion reactor, a toroid has inner and outer radii $a = 0.5\,\text{m}$ and $b = 1.5\,\text{m}$, respectively. The toroid has 900 turns and carries a current of 12 kA. What is the magnitude of the magnetic field at a point located on a circle having the average radius of the toroid?

Solution: With $R = (a + b)/2 = (0.5 + 1.5)/2 = 1$ m, Eq. 26.17 gives:

$$B = \frac{\mu_\circ N\, I}{2\pi R} = \frac{(4\pi \times 10^{-7}\ \text{T.m/A})(900\ \text{turns})(12 \times 10^3\ \text{A})}{(2\pi)(1\ \text{m})} = 2.16\ \text{T}$$

26.4 Displacement Current and the Ampere-Maxwell Law

Ampere's law is incomplete when the conduction current is not steady. We can show this by considering the region near a parallel-plate capacitor while the capacitor is charging, see Fig. 26.8a. A variable conduction current $i = dq/dt$ reaches one plate and the same conduction current i leaves the other plate. There is no current flow across the space between the plates. Experiments show the establishment of a magnetic field between the two plates as well as on both sides of the plates. In addition, experiments show that the value of $\oint \vec{B} \bullet d\vec{s}$ is the same for the three circular loops labeled ①, ②, and ③ in Fig. 26.8a. But according to Ampere's law, $\oint \vec{B} \bullet d\vec{s}$ must be zero for loop ②, because the conduction current is zero.

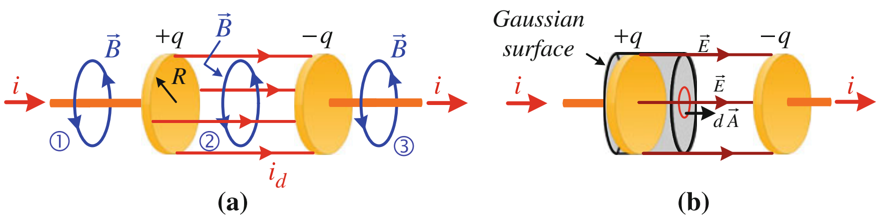


Fig. 26.8 (a) The displacement current i_d between the plates of a capacitor. (b) The Gaussian surface that encloses the varying charge q

Maxwell solved this problem by postulating an additional term to the right side of Ampere's law that is related to the changing electric field between the plates of the capacitor. This term is referred to as the *displacement current* i_d between the plates. This current is defined as:

$$i_d = \epsilon_\circ \frac{d\Phi_E}{dt} \tag{26.19}$$

The displacement current i_d between the plates is equivalent to the conduction current i in the wires, i.e. $i_d = i$, and hence produces the same magnetic effects observed experimentally, see Fig. 26.8a.

Maxwell added the displacement current i_d to the varying conduction current i and expressed Ampere's law as follows:

$$\oint \vec{B} \bullet d\vec{s} = \mu_\circ (i + i_d) = \mu_\circ \left(i + \epsilon_\circ \frac{d\Phi_E}{dt} \right) \quad \text{(Ampere–Maxwell law)} \tag{26.20}$$

When there is a conduction current but no change in electric flux (only like loops ① and ③), the second term is zero. When there is a change in electric flux but no conduction current (only like loop ②), the first term is zero.

Spotlight

Magnetic fields are produced both by conduction currents i and by displacement currents i_d, created by a time varying electric flux.

To establish the relation Eq. 26.20, we apply Gauss's law for the Gaussian surface shown in Fig. 26.8b. According to Gauss's law, see Eq. 21.7, this surface encloses a net charge q, and we have:

$$\Phi_E = \oint \vec{E} \bullet d\vec{A} = \frac{q}{\epsilon_\circ} \tag{26.21}$$

As q changes, $\vec{E}$ changes too, and the rate at which q changes gives the displacement current postulated by Maxwell. Thus:

$$i_d = \frac{dq}{dt} = \epsilon_\circ \frac{d\Phi_E}{dt} \tag{26.22}$$

Example 26.8

The circular capacitor of Fig. 26.8 a has a radius $R = 10\,\text{cm}$ and a charge $q = (4 \times 10^{-4}\,\text{C}) \sin(2 \times 10^4\, t)$ that varies with time t. In the region between the plates, find the displacement current and the maximum value of the magnetic field at radius $r = 15\,\text{cm}$.

Solution: From Eq. 26.22, we find the displacement current as:

$$i_d = \frac{dq}{dt} = \frac{d}{dt}[(4 \times 10^{-4}\,\text{C}) \sin(2 \times 10^4\, t)] = (8\,\text{A}) \cos(2 \times 10^4\, t)$$

For a maximum displacement current $(i_d)_{\text{max}}$ of 8 A at a point between the plates, we use Eq. 26.15 for $r \geq R$ to find B_{max}:

$$B_{\text{max}} = \frac{\mu_\circ (i_d)_{\text{max}}}{2\pi r} = \frac{(4\pi \times 10^{-7}\,\text{T.m/A})(8\,\text{A})}{(2\pi)(15 \times 10^{-2}\,\text{m})} = 1.07 \times 10^{-5}\,\text{T}$$

26.5 Gauss's Law for Magnetism

As in the case of an electric flux, we calculate the magnetic flux throughout a particular surface S, see Fig. 26.9, as follows:

$$\Phi_B = \int \vec{B} \bullet d\vec{A} \tag{26.23}$$

The SI unit for the magnetic flux is tesla-square meter, which is called weber (abbreviated Wb). Thus, 1 weber = 1 Wb = 1 Tm^2.

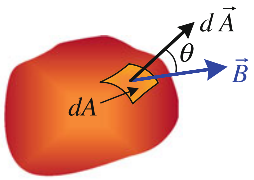


Fig. 26.9 The differential surface vector area $d\vec{A}$ is perpendicular to the differential area dA and pointing outwards. When the magnetic field $\vec{B}$ makes an angle θ with $d\vec{A}$, the differential flux $d\Phi_B$ is $\vec{B} \bullet d\vec{A}$

Since magnetic fields form closed loops, i.e. the magnetic field lines do not begin or end at any point, and for a closed surface the number of lines entering that surface equals the number of lines leaving it. Thus, the net magnetic flux over a closed surface is zero. This is known as Gauss's law for magnetism and can be stated as:

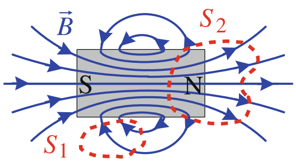
Gauss's Law for Magnetism
The *net magnetic flux* throughout any closed surface is always zero:

$$\oint \vec{B} \bullet d\vec{A} = 0 \quad \text{(Gauss's law for magnetism)} \tag{26.24}$$

Example 26.9

Find the net magnetic flux through the closed surfaces S_1 and S_2 of Fig. 26.10, which are represented by dashed lines intersecting the page.

Fig. 26.10



Solution: According to Gauss's law for magnetism, we must have:

$$\oint_{S_1} \vec{B} \bullet d\vec{A} = 0, \quad \text{and} \quad \oint_{S_2} \vec{B} \bullet d\vec{A} = 0$$

Notice that surface S_2 encloses only the north pole of the magnet, and that the south pole is associated with the left boundary of S_2.

26.6 The Origin of Magnetism

We have seen how to generate a magnetic field by allowing an electric current to pass through a wire. Moreover, we found that the magnetic pattern of a circular current loop has a North Pole and a South Pole with a magnetic dipole moment $\vec{\mu}$ producing a magnetic pattern that looks like the magnetic pattern produced by a bar magnet. (Searches for magnetic monopoles in cosmic rays or elsewhere have been negative.)

In addition, there are two subatomic ways that produce a magnetic field in space, each one involving a magnetic dipole moment. These require an understanding of *quantum physics*, which is beyond the scope of this study. Therefore, we shall only begin our study by presenting the results of the classical model of atoms and electrons.

Orbital Magnetic Dipole Moments of Atoms

In the *classical Bohr model of hydrogen atoms*, we assume that an electron of mass m_e and charge $-e$ moves around a fixed nucleus with a constant speed v in a circular orbit of radius r, see Fig. 26.11.

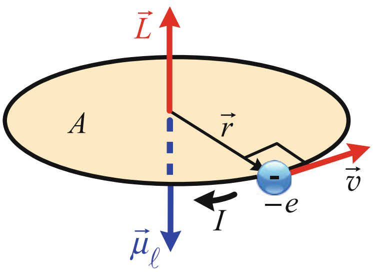


Fig. 26.11 The classical model of a hydrogen atom, where an electron moves with a constant speed in a circular orbit about a nucleus. The direction of the associated current is opposite to the direction of the electron's motion

Because the electron travels a circumference $2\pi r$ in an interval of time $T = 2\pi r/v$, the current I associated with this motion is:

$$I = \frac{e}{T} = \frac{ev}{2\pi r} \tag{26.25}$$

The magnitude of the orbital magnetic dipole moment associated with this orbiting current is $\mu_\ell = IA = \pi r^2 I$, where A is the circular area enclosed by the electron's orbit. Thus, using Eq. 26.25, we get:

$$\mu_\ell = \pi r^2 I = \tfrac{1}{2} e\, v\, r \tag{26.26}$$

From the definition of the orbital angular momentum $\vec{L} = \vec{r} \times \vec{p}$, where $\vec{p} = m_e \vec{v}$ is the momentum of the electron, we see that the angle between $\vec{r}$ and $\vec{p}$ is 90°. Then $L = m_e v r$ and μ_ℓ and L are given by:

$$\mu_\ell = \frac{e}{2m_e} L \tag{26.27}$$

Because the electron is a negatively charged particle, the vectors $\vec{\mu}_\ell$ and $\vec{L}$ are opposite to each other, see Fig. 26.11. Thus:

$$\boxed{\vec{\mu}_\ell = -\frac{e}{2m_e} \vec{L}} \tag{26.28}$$

The orbital angular momentum $\vec{L}$ cannot be measured. Instead, only its components along an axis can be measured. A fundamental outcome of quantum physics is that the orbital angular momentum and its components are quantized (which means having discrete restricted values). The quantization rules of $\vec{L}$ and its component along the z axis, L_z, have only the values given by:

$$\boxed{\begin{array}{ll} L = \sqrt{\ell(\ell+1)}\,\hbar, & (\ell = 0, 1, 2, \ldots) \\ L_z = m_\ell\,\hbar, & (m_\ell = -\ell, \ldots, -1, 0, +1, \ldots, +\ell) \end{array}} \tag{26.29}$$

where ℓ is the *orbital quantum number*, m_ℓ is the *orbital magnetic quantum number*, $\hbar = h/2\pi$, and h is an ever-present constant in quantum physics known as Planck's constant, which has the value:

$$h = 6.63 \times 10^{-34}\ \text{J.s} \quad \text{and} \quad \hbar = 1.05 \times 10^{-34}\ \text{J.s} \tag{26.30}$$

Figure 26.12 displays a vector model for the orbital angular momentum in case of $\ell = 1$.

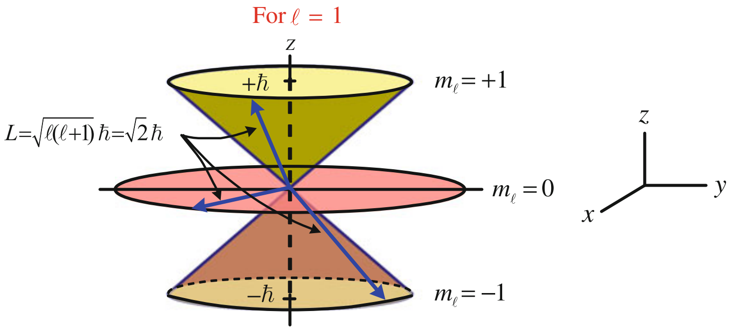


Fig. 26.12 For every value of $L_z = m_\ell\,\hbar$, there is an equal probability of finding $\vec{L}$ anywhere on the surface of a symmetrical cone about the z axis. The vector $\vec{L}$ rotates randomly about this axis, such that it has a constant value $\sqrt{\ell(\ell+1)}\,\hbar$ and a constant component $L_z = m_\ell\,\hbar$, but L_x and L_y are unknown and satisfy the average values $\overline{L}_x = \overline{L}_y = 0$

We can relate the component $\mu_{\ell,\mathrm{z}}$ to Lz by rewriting Eq. 26.28 in component form as follows:

$$\mu_{\ell,\mathrm{z}} = -m_\ell \frac{e\hbar}{2m_e} = -m_\ell\,\mu_\mathrm{B} \tag{26.31}$$

where the quantity μ_B is called the Bohr magneton and is given by:

$$\mu_\mathrm{B} = \frac{e\hbar}{2m_e} = 9.27 \times 10^{-24}\,\mathrm{J/T}(\equiv \mathrm{A.m^2}) = 5.79 \times 10^{-5}\,\mathrm{eV/T} \tag{26.32}$$

When an electron is placed in an external magnetic field $\vec{B}$, a torque $\vec{\tau} = \vec{\mu}_\ell \times \vec{B}$ is exerted on its orbital magnetic dipole moment. This reminds us of the corresponding equation for the torque exerted by an electric field $\vec{E}$ on an electric dipole moment $\vec{p}$, $\tau = \vec{p} \times \vec{E}$; see Eq. 22.39. In each case, the torque exerted by the field (either $\vec{B}$ or $\vec{E}$) is equal to the vector product of the dipole moment and the field. In strict analogy to the $U = -\vec{p} \bullet \vec{E}$, see Eq. 22.42, a potential energy U_ℓ can be associated with the orientation of the orbital magnetic dipole moment $\vec{\mu}_\ell$, and it is given by:

$$U_\ell = -\vec{\mu}_\ell \bullet \vec{B} \tag{26.33}$$

If the direction of the magnetic field is taken to be along the z-axis, then the orientation potential energy can be written as:

$$U_\ell = -\mu_{\ell,\mathrm{z}}\,B \tag{26.34}$$

Quantization of the component of the orbital magnetic moment gives:

$$\boxed{U_\ell = +m_\ell \frac{e\hbar}{2m_e} B} \quad \text{or} \quad \boxed{U_\ell = +m_\ell\, \mu_B B} \tag{26.35}$$

We used the word "orbital" in both classical and quantum studies, but in quantum physics we must make it clear that all electrons do not orbit the atomic nucleus like planets orbiting the sun.

Although all materials contain electrons, most of them do not exhibit magnetic properties. The main reason is due to the *cancelation of the randomly oriented orbital magnetic dipole moments of atoms*. Then, for most materials the magnetic effect produced by the electronic orbital motion is either zero or very small.

Spin Magnetic Dipole Moments of Electrons

In addition to the orbital angular momentum $\vec{L}$, an electron has an intrinsic angular momentum called the **spin angular momentum** (or just **spin**) $\vec{S}$. The vector $\vec{S}$ is a purely quantum-mechanical physical quantity that has no classical analog. Associated with this spin is an **intrinsic-spin magnetic dipole moment** $\vec{\mu}_s$. Experiments indicate that the $\vec{S}$ and S_z are quantized and related to $\vec{\mu}_s$ and $\mu_{s,z}$ as follows:

$$\begin{array}{llll} S = \sqrt{s(s+1)}\,\hbar \quad (s = \frac{1}{2}), & S_z = m_s\,\hbar & (m_s = -\frac{1}{2}, +\frac{1}{2}) \\ \vec{\mu}_s = -\frac{e}{m_e}\vec{S}\,, & \mu_{s,z} = -2m_s\mu_B & (m_s = -\frac{1}{2}, +\frac{1}{2}) \end{array} \tag{26.36}$$

where s is the *spin quantum number* and m_s is the *spin-projection magnetic quantum number*. There are two possibilities of finding the atomic electron, either in a state with $m_s = -\frac{1}{2}$ or in a state with $m_s = +\frac{1}{2}$.

When the electron is placed in an external magnetic field $\vec{B}$, the potential energy U_s associated with orientation of the spin magnetic dipole moment $\vec{\mu}_s$ is similarly given by:

$$\boxed{U_s = -\vec{\mu}_s \bullet \vec{B}} \tag{26.37}$$

When $\vec{B}$ is along the z-axis, $\mu_{s,z}$ can take only two possible values (up or down), and hence, the potential energy U_s takes the two values:

$$U_s = -\mu_{s,z}\,B = \pm\mu_B\,B = \begin{cases} +\mu_B\,B & \text{if}\, m_s = +\frac{1}{2} \quad (\text{then } \mu_{s,z} = -\mu_B) \\ \\ -\mu_B\,B & \text{if}\, m_s = -\frac{1}{2} \quad (\text{then } \mu_{s,z} = +\mu_B) \end{cases} \tag{26.38}$$

In both cases, $\vec{S}$ will rotate about $\vec{B}$ with angular frequency given by:

$$\vec{\omega} = \frac{\mu_B}{\hbar}\vec{B} \tag{26.39}$$

In addition, the lowest energy ($-\mu_B B$) occurs when $\mu_{s,z}$ is lined up with $\vec{B}$ and the highest energy ($+\mu_B B$) occurs when $\mu_{s,z}$ is in the opposite direction of $\vec{B}$, see Fig. 26.13. The difference in energy between these two orientation levels is $\Delta U_s = 2\,\mu_B\,B$.

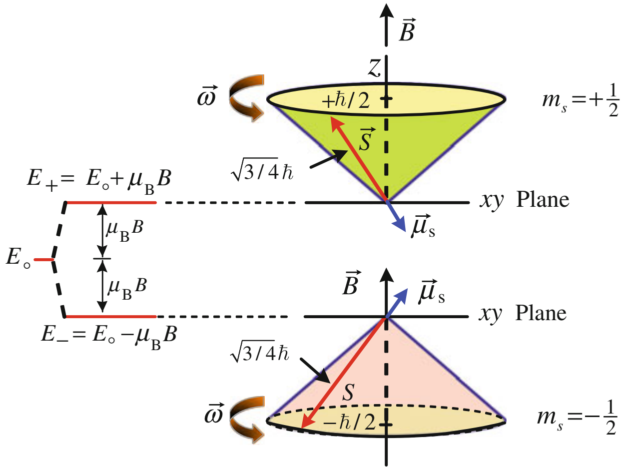


Fig. 26.13 In the presence of a magnetic field $\vec{B}$, the energy $E_\circ$ of the electron splits into two levels with a difference of $2\,\mu_B\,B$. In each level, $\vec{S}$ (or $\vec{\mu}_s$) will rotate about $\vec{B}$ with angular frequency $\vec{\omega} = \mu_B\,\vec{B}/\hbar$

Protons and neutrons have intrinsic magnetic dipole moments given by similar formulas, but are an order of 10^3 smaller than that of the electron. This is because the mass of proton m_p and the mass of neutron m_n are much greater than the mass of the electron m_e.

26.7 Magnetic Materials

Some materials exhibit weak magnetic properties, and others exhibit strong magnetic properties due to the alignment of the magnetic moments of their atoms. We consider a small volume V of one of these materials and assume that the magnetic moment of a typical atom/molecule is $\vec{\mu}_{\text{atomic}}$. Then the total magnetic moment within V is the vector sum $\sum \vec{\mu}_{\text{atomic}}$. The magnetic state of this material is described by a quantity called the **magnetization vector** $\vec{M}$ and is defined as:

$$\boxed{\vec{M} = \frac{\sum \vec{\mu}_{\text{atomic}}}{V}} \tag{26.40}$$

Spotlight

The magnetization of a material is defined as the magnetic moment per unit volume.

The unit of magnetization is A/m. If the atomic magnetic dipole moments of a magnetic material are randomly oriented, or there are none, then $\sum \vec{\mu}_{\text{atomic}} = 0$ and $\vec{M} = 0$.

Consider a region in which a current-carrying conductor produces a magnetic field $\vec{B}_\circ$. If this region is filled with a magnetic material that produces a magnetic field $\vec{B}_M$, then the total magnetic field in this region will be:

$$\boxed{\vec{B} = \vec{B}_\circ + \vec{B}_M} \tag{26.41}$$

To find the relation between $\vec{B}_M$ and $\vec{M}$, we consider a solenoid of length L having N turns and carrying a current I. In vacuum the magnetic field inside the solenoid is given by Eq. 26.16 as $B_\circ = \mu_\circ\, n\, I = \mu_\circ\, N\, I/L$. Multiplying and dividing the right hand side of this equation by the cross-sectional area A of the solenoid allows us to write this equation in terms of the total magnetic moment of all the solenoid loops $\sum \mu_{\text{coil}} = N\ I\ A$ and the solenoid volume $V = LA$ as:

$$B_\circ = \mu_\circ\, n\, I = \mu_\circ \frac{N\ I\ A}{LA} = \mu_\circ \frac{\sum \mu_{\text{coil}}}{V} \tag{26.42}$$

This relation can be written in vector form as:

$$\boxed{\vec{B}_\circ = \mu_\circ \frac{\sum \vec{\mu}_{\text{coil}}}{V}} \tag{26.43}$$

When a magnetic material fills the solenoid, the contribution resulting from the alignment of the atomic-induced magnetic dipole moments $\sum \vec{\mu}_{\text{atomic}}$ produces a magnetic field $\vec{B}_M$ that can be written in a form similar to Eq. 26.43 as:

$$\boxed{\vec{B}_M = \mu_\circ \frac{\sum \vec{\mu}_{\text{atomic}}}{V}} \tag{26.44}$$

The ratio $\sum \vec{\mu}_{\text{atomic}}/V$ was defined in Eq. 26.40 as the magnetization vector $\vec{M}$ of the magnetic material. Thus:

$$\boxed{\vec{B}_M = \mu_\circ \vec{M}} \tag{26.45}$$

Therefore, the total magnetic field inside the solenoid will be:

$$\vec{B} = \vec{B}_\circ + \mu_\circ \vec{M} \tag{26.46}$$

In Eq. 26.43, it is convenient to introduce the **magnetic field strength** $\vec{H} = \sum \vec{\mu}_{\text{coil}}/V$. This field is a quantity related to the magnetic field resulting from the conduction current. Therefore:

$$\boxed{\vec{B}_\circ = \mu_\circ \vec{H}} \tag{26.47}$$

Thus, Eq. 26.46 can be written as:

$$\boxed{\vec{B} = \mu_\circ (\vec{H} + \vec{M})} \tag{26.48}$$

Note that $\vec{B}$ is composed of $\mu_\circ \vec{H}$ (associated with the conduction current) and $\mu_\circ \vec{M}$ (resulting from the magnetization of the material that fills the solenoid). Since $B_\circ = \mu_\circ n I$ and $B_\circ = \mu_\circ H$, then:

$$H = n\, I \quad \text{(Solenoid or a toroid)} \tag{26.49}$$

Magnetic materials are classified into three categories:

Diamagnetic	*where atoms have* ***no*** *permanent magnetic moments*
Paramagnetic, Ferromagnetic	*where atoms have permanent magnetic moments*

26.8 Diamagnetism and Paramagnetism

When a diamagnetic or paramagnetic material is placed in an external magnetic field, the magnetization vector $\vec{M}$ is proportional to the magnetic field strength $\vec{H}$, and we can write:

$$\boxed{\vec{M} = \chi \vec{H}} \tag{26.50}$$

where χ is a dimensionless factor called the **magnetic susceptibility**, which measures the responsiveness of a material to being magnetized.

Substituting Eq. 26.50 for $\vec{M}$ into Eq. 26.48 gives:

$$\vec{B} = \mu_\circ(\vec{H} + \vec{M}) = \mu_\circ(\vec{H} + \chi\vec{H}) = \mu_\circ(1 + \chi)\vec{H} \tag{26.51}$$

or:

$$\boxed{\vec{B} = \mu_m \vec{H}} \tag{26.52}$$

where μ_m is called the **magnetic permeability** of the material and is related to its magnetic susceptibility χ by the relation:

$$\boxed{\mu_m = \mu_\circ(1 + \chi) \begin{cases} <\mu_\circ & \text{For diamagnetic materials} \\ >\mu_\circ & \text{For paramagnetic materials} \end{cases}} \tag{26.53}$$

The factor $K_m = \mu_m/\mu_\circ$ is called the relative permeability of the material.

Diamagnetic Materials

A material is considered diamagnetic if its atoms have zero net angular momentum and hence no permanent magnetic moment. Diamagnetic materials interact weakly with the applied magnetic field, in which case χ is very small negative value and $\vec{M}$ is opposite to $\vec{H}$. This causes diamagnetic materials to be weakly repelled by a magnet. Diamagnetism is present in all materials, but its effects are much smaller than those in paramagnetic or ferromagnetic materials.

To understand this interaction we consider the motion of two electrons orbiting a nucleus with the same speed but in opposite directions, see Fig. 26.14a. The magnetic moments of the two electrons in this figure are in opposite directions and therefore cancel.

In the presence of a uniform magnetic field $\vec{B}$ directed out of the page, as shown in Fig. 26.14b, both of the electrons experience an extra magnetic force $(-e)\,\vec{v} \times \vec{B}$. Thus:

- For the electron in the left of Fig. 26.14b, the extra magnetic force is radially inward, increasing the centripetal force. If this electron is to remain in the same circular path, it must speed up to $\vec{v}\,'$, so that mv'^2/r equals the total newly increased centripetal force. Therefore, its *inward magnetic moment thus increases*.

- For the electron in the right of Fig. 26.14b, the extra magnetic force is radially outward, decreasing the centripetal force. If this electron is to remain in the same circular path, it must slow down to $\vec{v}''$, so that mv''^2/r equals the total newly decreased centripetal force. Therefore, its *outward magnetic moment thus decreases*.

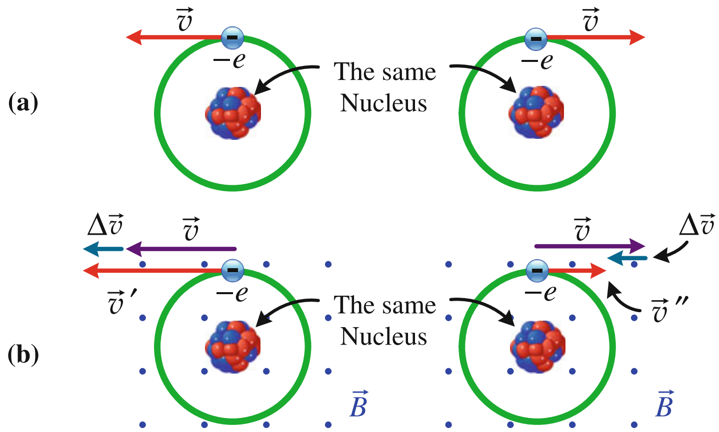


Fig. 26.14 (a) Two atomic electrons orbiting a fixed nucleus with the same speed but in opposite directions (separated for clarity). (b) When a magnetic field is applied out of the page, the magnetic force increases the speed of the left electron and decreases the speed of the right one

As a result, the change in the magnetic moment of the two electrons is into the page, opposite to the external applied magnetic field. Because the permanent magnetic moments of the two electrons cancel each other, only an induced magnetic moment opposite to the applied magnetic field will remain. The induced magnetic moments that cause diamagnetism are of the order of $10^{-5}\ \mu_B$. This value is much smaller than that of the permanent magnetic moments of the atoms of paramagnetic and ferromagnetic materials. However, the alignments produced in the diamagnetism decrease with temperature. Therefore, diamagnetism disappears in all materials at sufficiently high temperatures.

Certain types of superconductors (a substance of zero electric resistance) exhibit diamagnetism below some critical temperature. As a result, the superconductor can repel a permanent magnet.

Paramagnetic Materials

Atoms of paramagnetic materials have permanent magnetic moments that interact with each other very weakly, resulting in a very small positive magnetic susceptibility χ. Therefore, $\vec{M}$ is in the same direction as $\vec{H}$. However, the thermal motion of the molecules reduces the alignments, and this tends to randomize the magnetic dipole moments' orientations. The degree to which the magnetic moments line up with an external magnetic field depends on the strength of the field and on the temperature.

Even in a very strong magnetic field B of 1 T and a typical atomic magnetic moment μ of 1 μ_B, the difference in potential energy ΔU when the magnetic moment is parallel the field (lower energy) and when the moment antiparallel the field (higher energy) is:

$$\Delta U = 2\mu_B B = 2 \times (5.79 \times 10^{-5}\ \text{eV/T})(1\ \text{T}) = 1.2 \times 10^{-4}\ \text{eV}$$

At a normal temperature $T = 300$ K, the typical thermal energy $k_B T$ is:

$$k_B T = (8.62 \times 10^{-5}\ \text{eV/T})(300\ \text{K}) = 2.6 \times 10^{-2}\ \text{eV}$$

Therefore, $k_B T \gtrsim 200\ \ \Delta U$. Thus, at room temperature and even in a very strong magnetic field, most of the magnetic moments will be randomly oriented unless the temperature is very low.

In 1895, Pierre Curie discovered that M is directly proportional to the external magnetic field $B_\circ$ and inversely proportional to the kelvin temperature, when $B_\circ/T$ is very small; that is:

$$M = C\frac{B_\circ}{T} \quad \text{(Curie's law)} \tag{26.54}$$

where the constant C is a known as Curie's constant. This law shows that $M = 0$ when $B_\circ = 0$. Even if $B_\circ$ is very large ($\sim$2 T), deviation from Curie's law can be observed at extremely low temperatures (i.e. at a few kelvins). In addition, as $B_\circ$ increases (or T decreases), Eq. 26.54 will no longer be valid, and quantum physics indicates that the magnetization M approaches some maximum value M_{max}, which corresponds to a complete alignment of all permanent magnetic dipole moments.

Table 26.1 gives the magnetic susceptibility of some materials.

Table 26.1 Magnetic susceptibility of some diamagnetic and paramagnetic materials at 300 K

Diamagnetic material	χ	Paramagnetic material	χ
Bismuth	-1.7×10^{-5}	Aluminum	2.3×10^{-5}
Carbon (graphite)	-1.4×10^{-5}	Calcium	1.9×10^{-5}
Copper	-9.8×10^{-6}	Chromium	2.7×10^{-4}
Carbon (Diamond)	-2.2×10^{-5}	Lithium	2.1×10^{-5}
Gold	-3.6×10^{-5}	Magnesium	1.2×10^{-5}
Lead	-1.7×10^{-5}	Niobium	2.6×10^{-4}
Mercury	-2.9×10^{-5}	Oxygen	2.1×10^{-6}
Nitrogen	-5.0×10^{-9}	Platinum	2.9×10^{-4}
Silver	-2.6×10^{-5}	Potassium	5.8×10^{-6}
Silicon	-4.2×10^{-6}	Tungsten	6.8×10^{-5}

26.9 Ferromagnetism

Materials such as iron, cobalt, nickel, gadolinium, dysprosium, and alloys containing these materials usually exhibit strong magnetic properties and are called **ferromagnetic materials**. These materials contain permanent atomic magnetic moments that tend to align even in the presence of a weak external magnetic field and remain magnetized after the magnetic field is removed. These alignments can only be understood in quantum-mechanical terms.

Consider a specimen of ferromagnetic material, such as iron in its crystalline form. Such a crystal would be made of microscopic regions called **magnetic domains**. Each domain would be less than 1 mm wide and would have all its atomic magnetic moments aligned. The boundaries between domains that have different magnetic-moment orientations are called **domain walls**. Depending on the structure and type of the material, the volume of each magnetic domain would vary from about 10^{-12} to 10^{-8} m^3 and contain about 10^{18} to 10^{22} molecules.

If magnetic domains of a particular ferromagnetic material specimen are randomly oriented as shown in Fig. 26.15a, then the entire specimen would not display a net magnetic dipole moment.

As the unmagnetized ferromagnetic specimen is placed in an external magnetic field $\vec{B}_\circ$ that increases gradually, then the specimen would experience the following two types of domain interactions:

- Reversible magnetization by domain growth:
  When the applied magnetic field $\overrightarrow{B}_\circ$ is weak, a growth in volume of the domains that are oriented along $\overrightarrow{B}_\circ$ occurs at the expense of those that are not, see Fig. 26.15b. In this case, the specimen is magnetized, and this magnetization is reversible. That is, we have *reversible domains* when $\overrightarrow{B}_\circ$ is removed.
- Irreversible magnetization by domain alignments and rotations:
  As the applied magnetic field $\overrightarrow{B}_\circ$ strengthens, the domains align even more, and after a particular threshold the material manifests *irreversible domains* if $\overrightarrow{B}_\circ$ is removed. But if the magnetic field $\overrightarrow{B}_\circ$ becomes even stronger, the *irreversible domains rotate* and start to align more and more in the direction of $\overrightarrow{B}_\circ$, see Fig. 26.15c. In both cases, the specimen remains magnetized at ordinary temperatures even after $\overrightarrow{B}_\circ$ is removed.

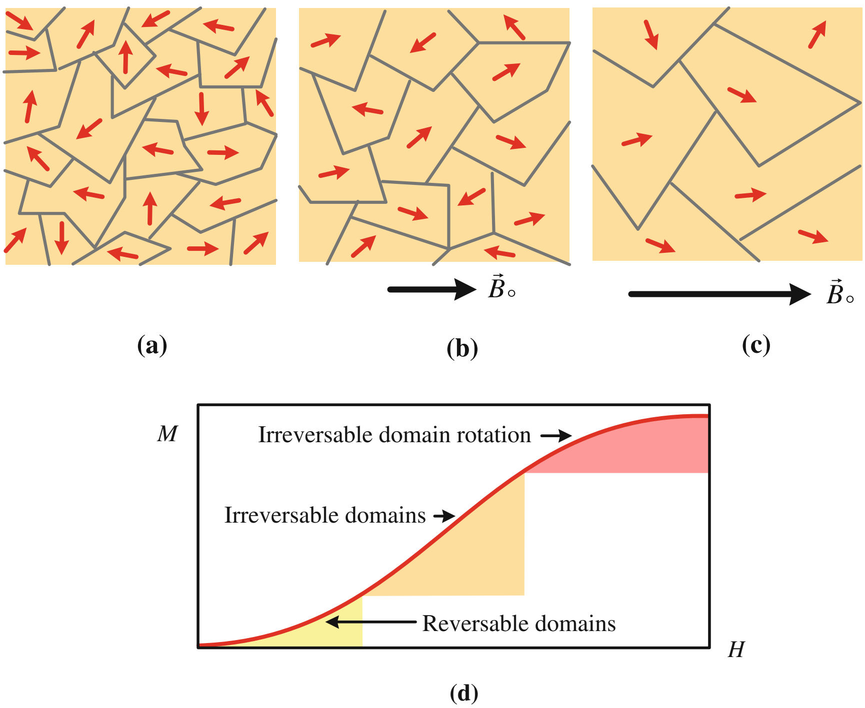


Fig. 26.15 (a) An unmagnetized specimen having magnetic domains with random magnetic dipole orientations. (b) A growth in volume of domains that are oriented along $\overrightarrow{B}_\circ$. (c) When the magnetic field becomes much stronger, the *domains rotate* and align more in the direction of $\overrightarrow{B}_\circ$. (d) Variation of the magnetization M as a function of H (or $B_\circ = \mu_\circ H$). As H increases, the domains become more and more aligned until saturation is reached

For a ferromagnetic material, χ and hence μ_M are very large, but the relation between $\vec{M}$ and $\vec{H}$ is not linear. This is because μ_M is not only a characteristic of ferromagnetic material, but also depends on $\vec{B}_\circ$ and on the previous state of the material, as we will see shortly.

Hysteresis

Measurements of the magnetic properties are usually done using a toroid (or a solenoid) of N turns with an initially unmagnetized ferromagnetic core, see Fig. 26.16.

Suppose that when the switch S in Fig. 26.16 is open (i.e. the current I in the windings is zero and $B_\circ = 0$), the ferromagnetic core is unmagnetized ($B = 0$). Then, we perform the following:

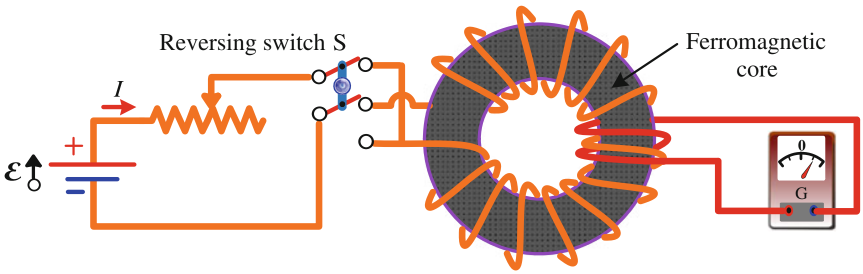


Fig. 26.16 A circuit used to study the properties of a ferromagnetic material that fills the core of a toroid, where the magnetic flux is measured by a galvanometer

1. When we close the switch and slowly increase the current in the circuit, the toroid magnetic field $B_\circ = \mu_\circ H$ increases linearly with I, but the total magnetic field $B = \mu_m H$ ($B \gg B_\circ$) follows the curve shown in the **magnetization curve** of Fig. 26.17. Initially, at point O, the domains of the core are randomly oriented. As $B_\circ$ increases gradually, the domains become more and more aligned until we reach the *saturation* point a where nearly all domains are aligned. Increasing $B_\circ$ further has a small effect on increasing B.
2. Next, we reduce the external magnetic field by decreasing the current in the coil until I becomes zero, We notice that the curve follows the path ab, where $B_\circ = 0$ at point b. This point indicates that $B \neq 0$ even though the external field $B_\circ$ is zero (that is $B = B_M$). In other words, some permanent magnetism remains, and the domains do not become completely random as they were initially.

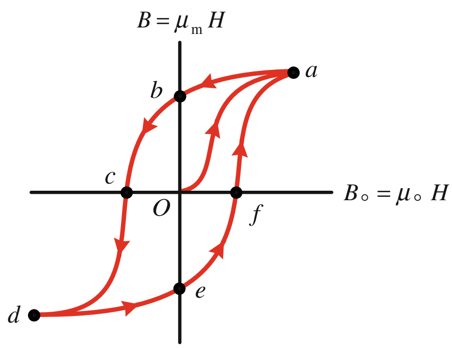


Fig. 26.17 Hysteresis curve for a ferromagnetic material

3. When the direction of the current is reversed and increased gradually (i.e. the direction of the external magnetic field $B_\circ$ is reversed), enough domains reorient their magnetic moments until the material is again unmagnetized at point c, where $B = 0$.
4. An increase in the reverse current causes the ferromagnetic material to be magnetized in the opposite direction, until we reach the *saturation* at point d.
5. Finally, if the current is again reduced to zero and then increased in the original positive direction, the total magnetic field follows the path *defa*.

We notice that the magnetic field did not pass through the origin (point O) in the loop *abcdefa*. This effect is called **magnetic Hysteresis**, while this loop is called the **Hysteresis loop**. Points b and e on the hysteresis loop indicate that the ferromagnetic material has a 'memory' because it remains magnetized even when the external field is removed. The area of this cycle is proportional to the thermal energy used to align the domains.

The area of the hysteresis loop depends on the properties of the ferromagnetic material under investigation. Two classifications arise as follows, depending on how big or small the loop area is:

1. **Hard ferromagnetic material** (Hard in a magnetic sense): If the hysteresis loop is wide as shown in Fig. 26.18a, the material can turn into a strong permanent magnet that cannot be easily demagnetized by an external magnetic field.
2. **Soft ferromagnetic material** (Soft in a magnetic sense): If the hysteresis loop is narrow, as shown in Fig. 26.18b, the material can be easily magnetized and demagnetized (such as iron, which is perfect for making electromagnets and transformers). An ideal soft ferromagnetic material would exhibit no hysteresis and would therefore have no residual magnetization at all.

A ferromagnetic material can be demagnetized by hitting it hard, heating it, or reversing the magnetizing current repeatedly while decreasing its magnitude, see Fig. 26.18c. As an example, the heads of a tape recorder can demagnetize tapes this way.

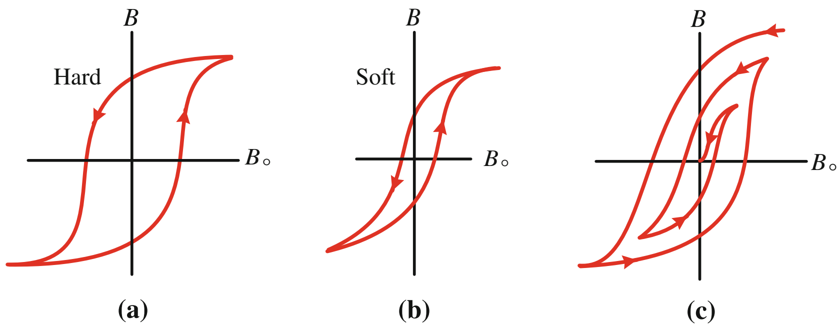


Fig. 26.18 Hysteresis curve for: (a) a hard ferromagnetic, (b) a soft ferromagnetic. (c) Demagnetizing a ferromagnetic material can be done by successive hysteresis loops

Ferromagnetic materials are no longer ferromagnetic above a critical temperature called the Curie temperature, T_{Curie}. Above this temperature, they are generally paramagnetic (for iron this temperature is about 1,040 K = 770°C).

Example 26.10

A toroid has 100 turns/m of wire carrying a current of 3 A. The core of the toroid is filled with powdered steel whose magnetic permeability μ_{m} is 100 $\mu_\circ$ (i.e. with relative permeability $K_{\text{m}} = \mu_{\text{m}}/\mu_\circ = 100$). Find the magnitude of the magnetic field strength H, the magnitude of the magnetic field $B_\circ$ produced by the toroid, and the magnitude of the magnetic field B inside the steel.

Solution: Using Eq. 26.49, we find H as follows:

$$H = n\,I = (100\ \text{turns/m})(3\ \text{A}) = 300\ \text{A/m}$$

Using Eq. 26.17, we find the $B_\circ$ as follows:

$$B_\circ = \mu_\circ H = (4\pi \times 10^{-7}\ \text{T.m/A})(300\ \text{A/m}) = 3.77 \times 10^{-4}\ \text{T}$$

Then using Eq. 26.52, we find B in the steel core as follows:

$$B = \mu_m H = 100 \times (4\pi \times 10^{-7}\,\text{T.m/A})(300\,\text{A/m}) = 0.038\,\text{T}$$

The value of B inside the steel is about 100 times the value $B_\circ$ in the absence of a steel core.

Example 26.11

(a) A substance has a magnetization of magnitude $M = 10^6$ A/m and a magnetic field of magnitude $B = 4$ T. Find the magnitude of the magnetic field strength H that produces this field. (b) A solenoid of $n = 590$ turns/m carries a current $I = 0.3$ A. If the solenoid's core is iron of magnetic permeability $\mu_m = 4{,}500\,\mu_\circ$, find the magnitude of the magnetic field in its core.

Solution: (a) Using Eq. 26.48, we find B as follows:

$$B = \mu_\circ(H + M) \quad \Rightarrow \quad H = \frac{B}{\mu_\circ} - M = \frac{4\,T}{4\pi \times 10^{-7}\,\text{T.m/A}} - 10^6\,\text{A/m}$$
$$= 2.2 \times 10^6\,\text{A/m}$$

(b) Using Eqs. 26.52 and 26.49, we find B as follows:

$$B = \mu_m H = 4{,}500\,\mu_\circ n\,I$$
$$= (4{,}500)(4\pi \times 10^{-7}\,\text{T.m/A})(590\,\text{turns/m})(0.3\,\text{A}) = 1\,\text{T}$$

26.10 Some Applications of Magnetism

Electromagnets

If a soft iron rod is placed inside a solenoid carrying a current, the magnetic field increases greatly due to the domain alignments. This setup is referred to as an **electromagnet**. The alloys of iron used in an electromagnet gain and lose magnetism quite quickly when the current in the solenoid is turned on or off. Electromagnets are used in many applications, such as in motors, generators, etc.

One simple use of electromagnets is in doorbells, where a rod of soft iron is attached to a spring and partially fitted inside a coil, see Fig. 26.19a. Pushing the doorbell button closes the circuit and the coil becomes a magnet and hence exerts a force on the rod. The rod is then pulled into the coil and strikes the bell, see Fig. 26.19b. If the circuit is then opened, the rod quickly loses its magnetization and the spring pulls the rod back to its initial position.

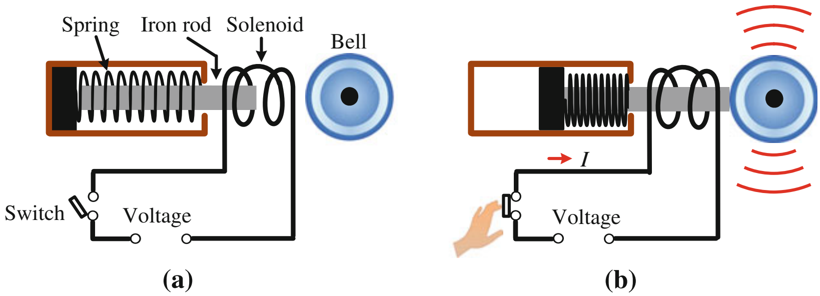


Fig. 26.19 Using the property of soft iron in doorbells. (a) The initial state when the circuit is open. (b) The circuit is closed

Magnetic Circuit Breakers

If the current in a circuit is larger than it should be, the circuit wires might become very hot and may burn. Circuit breakers are installed to prevent overloading by the current in a circuit. These ensure that the current never exceeds a particular value. Modern circuit breakers contain a magnetic sensing coil as shown in Fig. 26.20a. Inside the coil of this figure is a non-magnetic tube containing a spring-based moving iron rod.

When the contacts are closed by a switch and the operating current I is less than or equal to the maximum current I_{max} rated for this circuit breaker, the current flowing through the sensing coil establishes a magnetic field around it. In this case, the field is not strong enough to pull the armature, so the contacts are kept closed, as shown in Fig. 26.20a. However, when the current exceeds I_{max}, the strength of the magnetic field increases enough for the rod to compress the spring and move toward the pole piece. Once it reaches it, the pole piece gets magnetized and attracts the armature, pulling the contacts open. This unlatching of the trip mechanism happens very quickly ($<$10 ms) and thereby opens the contacts, see Fig. 26.20b.

In the case of a short circuit, the increase in magnetic field is so rapid that the armature is attracted to the pole instantaneously without any rod movement, allowing the circuit breaker to trip much faster.

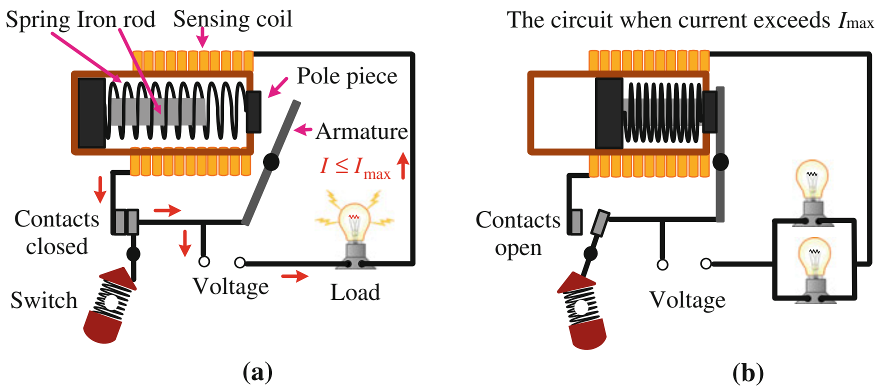


Fig. 26.20 (a) The circuit breaker is closed when the current I is maximum, $I = I_{max}$, or even when $I \le I_{max}$. (b) When I exceeds I_{max}, the armature unlatches a trip mechanism, and the circuit is opened

26.11 Exercises

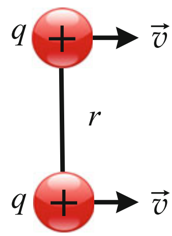
Section 26.1 The Biot-Savart Law

(1) A point charge $q = 25\,\mu\text{C}$ is moving in a straight line with a velocity $\vec{v} = 5 \times 10^4\,\vec{i}$ (m/s). If the charge is at location $P(0, 4\text{ m}, 0)$ at time t, find the magnetic field produced by this point charge at: (a) the origin $O(0, 0, 0)$, (b) the point $Q(3\text{ m}, 0, 0)$.

(2) An electron in the hydrogen atom orbits a fixed proton at a radius $r = 5.29 \times 10^{-11}$ m with a speed $v = 2.4 \times 10^6$ m/s. What is the magnitude of the magnetic field at the proton?

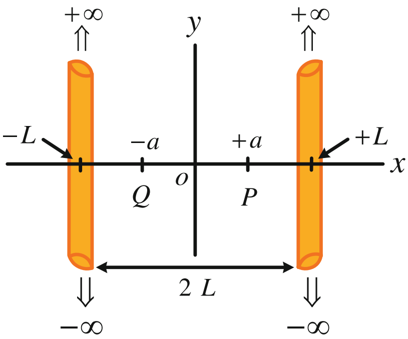
(3) Two point-particles of equal charge q are at a distance r apart. The particles are moving with the same velocity $\vec{v}$, see Fig. 26.21. What is the ratio of the magnitudes of the magnetic and electrostatic forces that each particle exerts on the other? [Hint: use $c^2 = 1/\mu_\circ \epsilon_\circ$, where c is the speed of light]

Fig. 26.21 See Exercise (3)



(4) Two very long straight conducting wires lie in the xy plane and are parallel to the y-axis. One wire is at $x = +L$ and the other is at $x = -L$, where $L = 8$ cm. Points P and Q are on the x-axis at $x = +a$ and $x = -a$, where $a = 4$ cm, see Fig. 26.22. The current in each wire is $I = 10$ A. If the currents in the wires are in the positive y direction, find the magnitude and direction of the magnetic field at point P and point Q.

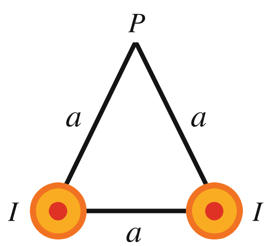
Fig. 26.22 See Exercise (4)



(5) Redo Exercise 4 when the current in the wire at $x = +L$ is in the positive y direction while the current in the wire at $x = -L$ is in the negative y direction. Then calculate the magnitude and direction of the magnetic field at point o.

(6) Two long parallel wires are at two corners of an equilateral triangle of side $a = 5$ cm, as shown in the cross-sectional view of Fig. 26.23. The current in each wire is 10 A. Find the magnitude and direction of the magnetic field at the unoccupied corner P.

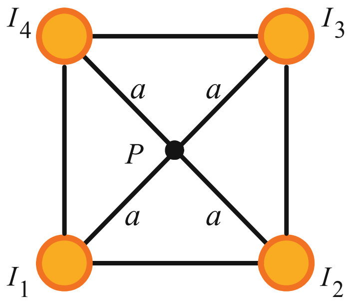
Fig. 26.23 See Exercise (6)



(7) Four long parallel wires are at the four corners of a square which has a diagonal of length $2a$, where $a = 10$ cm, see Fig. 26.24. The magnitudes of the currents in the four wires are the same, i.e. $I_1 = I_2 = I_3 = I_4 = 2$ A. Point P is at the

center of the square. Find $\vec{B}$ at P when: (a) all currents are out of the page, (b) I_1 and I_2 are out of the page while I_3 and I_4 are into th page, (c) I_1 and I_3 are out of the page while I_2 and I_4 are into the page.

Fig. 26.24 See Exercise (7)



(8) The wire shown in Fig. 26.25 carries a current $I = \sqrt{2}$ A. Find the magnetic field $\vec{B}$ at point P due to each wire segment and then find the resultant magnetic field.

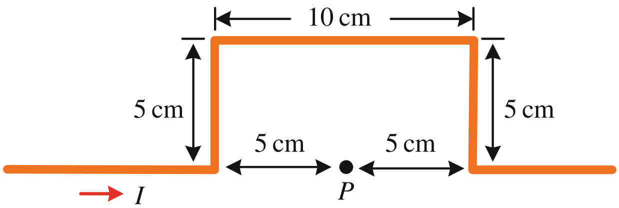


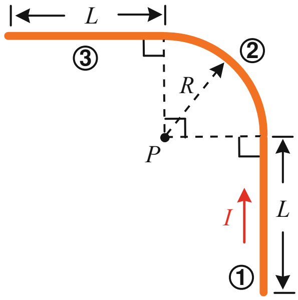
Fig. 26.25 See Exercise (8)

(9) A circular loop of radius $R = 4$ cm carries a current $I = 2$ A. What is the magnitude of the magnetic field on the axis of the loop at its center? What about 2 cm from its center, 4 cm from its center, and 10 cm from its center?

(10) For the circular loop of Exercise 9, how far from the center of the loop and along its axis are the points where the magnetic field is 10% of the field at the center? What about with 1%, and 0.1%?

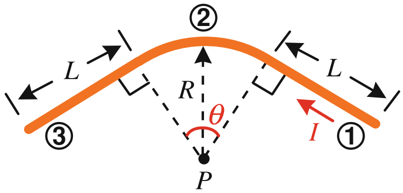
(11) Two straight wires ①and ③, each of length $L = 4$ cm, are connected by a quarter circular arc wire ② of radius $R = 3$ cm, as shown in Fig. 26.26. Determine the magnitude and direction of the magnetic field at the center P of the arc, when the current I is 2 A.

Fig. 26.26 See Exercise (11)



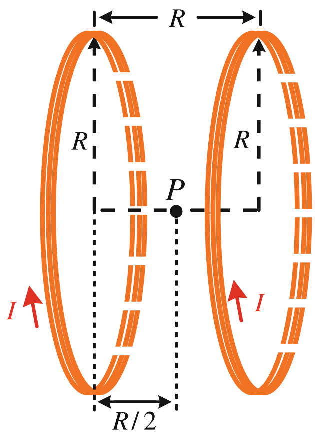
(12) Use the same values of Exercise 11 to calculate the magnitude and direction of the magnetic field at the center P of the arc, shown in Fig. 26.27, when the arc subtended an angle $\theta = \pi/3$.

Fig. 26.27 See Exercise (12)



(13) Figure 26.28 shows two identical coaxial coils, called *Helmholtz coils*, each having a radius R, N coil turns, and are separated by a distance R. The coils carry equal currents I such that their axial magnetic fields add. (a) When $R = 30$ cm, $N = 350$ turns, and $I = 20$ A, find the magnitude of the magnetic field B_P at point P, which is a point that exists midway between the coils centers. (b) Show that B_P can be written as $B_P = 8\,\mu_\circ\, N\, I/(5^{3/2}\, R)$.

Fig. 26.28 See Exercise (13)



Section 26.2 The Magnetic Force Between Two Parallel Currents

(14) Two long parallel wires are separated by a distance $a = 5$ cm and carry antiparallel currents of the same magnitude, $I_1 = I_2 = 4$ A. (a) What is the magnitude of the magnetic field created by each wire at the location of the other? (b) What is the magnitude of the force per unit length that each wire exerts on the other? Is this force attractive or repulsive?

(15) Two long parallel wires in the xy plane are separated by a distance $2a$ and carry equal currents I in opposite directions. The origin of the x-axis is taken to be midway between the wires and x is the position of an arbitrary point P from that origin, see the cross-sectional view of Fig. 26.29. (a) Derive an expression for the magnitude of the resultant magnetic field $B(x)$ as a function of the position x. (b) Plot $B(x)$ for $-60\,\text{mm} < x < 60\,\text{mm}$ for $I = 20$ A and $a = 30$ mm.

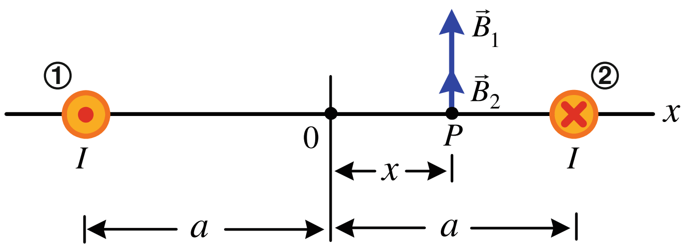


Fig. 26.29 See Exercise (15)

(16) Figure 26.30 shows a very long wire which carries a current $I_1 = 10$ A and a rectangular loop which carries a current $I_2 = 15$ A. Both the wire and the loop lie in one plane. Take $a = 0.1$ m, $b = 0.2$ m, and $c = 0.3$ m. (a) Find the magnitude and direction of the force exerted by the long wire on the wires ② and ④. (b) Find the direction of the force exerted by the long wire on the wires ① and ③. (c) Find the total force exerted by the long wire on the loop.

Section 26.3 Ampere's Law

(17) A long thin-walled conducting cylindrical shell of radius R carries a current I, see Fig. 26.31. Use Ampere's law to find the magnitude of the magnetic field inside and outside the shell.

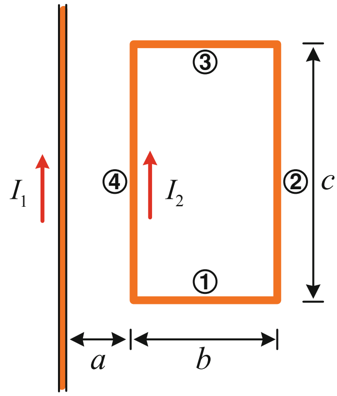


Fig. 26.30 See Exercise (16)

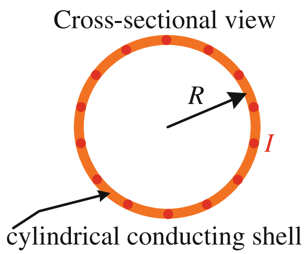


Fig. 26.31 See Exercise (17)

(18) Figure 26.32 shows two antiparallel currents of the same magnitude, $I = 10$ A. Evaluate the line integral $\oint \vec{B} \bullet d\vec{s}$ around the closed paths C_1, C_2, and C_3, where each line integral is taken with $d\vec{s}$ in a counterclockwise direction. Which path can be used to find the magnetic field at some point?

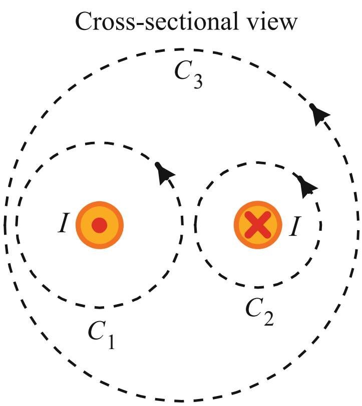


Fig. 26.32 See Exercise (18)

(19) A very long coaxial cable consists of a central wire, surrounded by a rubber layer, which is surrounded by a concentric conducting shell of radius $R = 3$ mm, which is surrounded by another rubber layer, see Fig. 26.33. The current I_1 in the inner wire is 1 A out of the page and the current I_2 in the outer conducting shell is 2 A into the page. Find the magnitude and direction of the magnetic field at $r_a = 2$ mm and $r_b = 4$ mm.

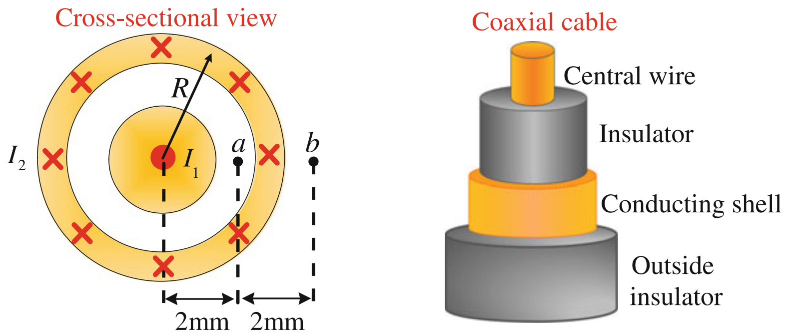


Fig. 26.33 See Exercise (19)

(20) A long wire of radius $R = 2$ cm carries a steady current $I = 50$ A. What are the magnitudes of the magnetic fields from the axis of the wire: (a) at a point 1 cm from the center, (b) on the surface, and (c) at a point 4 cm from the center?

(21) A long conducting cylindrical shell of inner radius a and outer radius b carries a current I uniformly distributed across the cross-sectional area of the shell. Find the magnitude of the magnetic field at points of radii: $r < a$, $a < r < b$, and $r > b$.

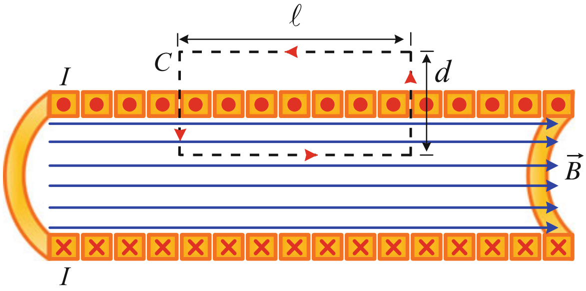
(22) A solenoid with n turns per unit length carries a current I, see Fig. 26.34. Apply Ampere's law to the rectangular path shown in the figure and derive an expression for the magnetic field B. For a packed solenoid such as this, assume that B is uniform inside and $B = 0$ outside.

(23) A solenoid of length $\ell = 0.25$ m carries a current $I = 10$ A. The solenoid consists of twenty closely packed layers, each of 500 turns. What is the magnitude of the magnetic field inside the solenoid?

(24) A solenoid has a length $\ell = 10$ cm. A superconducting fine wire (with almost zero resistance at low temperature) is wound in 10 layers such that $n = 4 \times 10^4$ turns per meter. (a) What is the number of turns per layer? (b) What is the

magnitude of the magnetic field B produced inside the solenoid when the current I in the wire is 60 A?

Fig. 26.34 See Exercise (22)



(25) Assume the wire of the solenoid of exercise 24 has a resistance of $10^4\ \Omega$ at room temperature. A 12 V battery is applied to the solenoid terminals. Find B under these conditions.

(26) An insulating cylindrical shell has a radius $R = 0.5$ cm and length $\ell = 10$ cm. A fine wire of diameter $d = 0.4$ mm is wound in many layers to establish a magnetic field of magnitude $\pi \times 10^{-2}$ T inside the cylindrical solenoid when the current is 2 A. (a) Determine the number of layers of wire needed. (b) Determine the length of the wire.

(27) Figure 26.35 shows a toroid with N turns that carries a current I. The toroid has an inner radius a and an outer radius b. Apply Ampere's law to the circular path C of radius r shown in the figure to derive an expression for the magnetic field B that is only confined to the space enclosed by the windings, and hence $B = 0$ anywhere else. Show that B is approximately uniform when $(b - a)/2 \ll R$, where $R = (a + b)/2$ is the mean radius of the toroid.

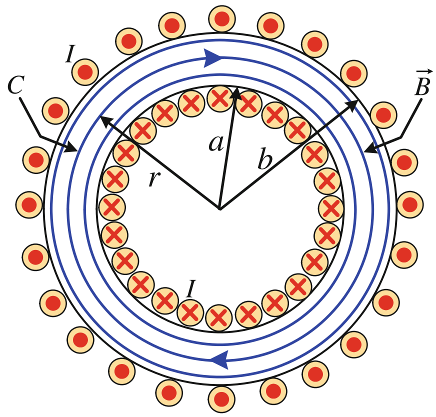
(28) A plastic ring of mean radius 6 cm is wound with $N = 1{,}000$ turns of wire. If the current in this toroid is $I = 0.6$ A, find the magnitude of the magnetic field on the mean circumference.

(29) A tightly wound toroid of inner radius $a = 5$ cm and outer radius $b = 7$ cm has $N = 3{,}300$ turns of wire and carries a current $I = 2$ A. Find the magnitude of the magnetic field: (a) at any point on the circumference of a circle of radius $r = 5.5$ cm. (b) on the mean circumference, which has a radius $r = 6$ cm.

(30) An infinite conducting sheet lying in the xz plane carries a current in the positive z direction, see Fig. 26.36. The current per unit length (or the linear current density) along the x-axis is λ. (a) Use the Biot-Savart law and the symmetry of

the problem to show that for every point P (such that $y > 0$) and every point P' (such that $y < 0$), the magnetic field $\vec{B}$ is parallel to the sheet and directed as shown in the figure. (b) Apply Ampere's law to the rectangular path shown in the figure to derive an expression for the magnitude of the magnetic field B.

Fig. 26.35 See Exercise (27)



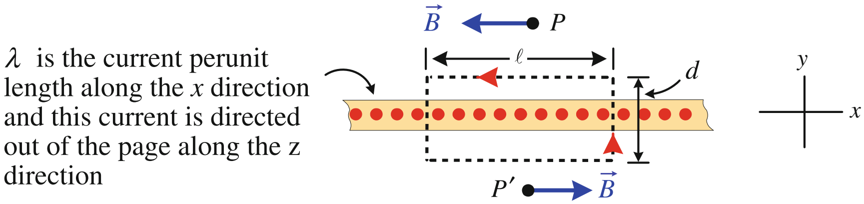
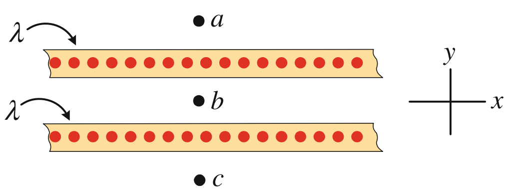


Fig. 26.36 See Exercise (30)

(31) Figure 26.37 shows two parallel infinite conducting sheets, each carries λ amperes of current per unit length out of the page. Find the magnetic field at points a, b, and c.

Fig. 26.37 See Exercise (31)

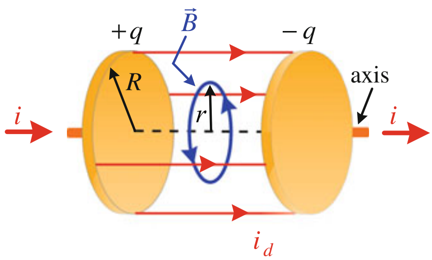


Section 26.4 Displacement Current and the Ampere–Maxwell Law

(32) A capacitor has circular plates, each of radius $R = 5$ cm. At a particular instant, the capacitor is charging by a current of 0.2 A. (a) What is the displacement current between the plates? (b) What is the rate of change of electric flux between the plates? (c) What is the magnitude of the magnetic field at $r = 8$ cm from the capacitor's axis in the region between the plates?

(33) A capacitor has circular plates, each of radius $R = 10$ cm. At a particular instant, the capacitor is charging by a current of 0.3 A. (a) What is the rate of change of electric field between the plates? (b) Apply Ampere-Maxwell Law to find the magnitude of the magnetic field at $r = 5$ cm from the capacitor's axis in the region between the plates, see Fig. 26.38.

Fig. 26.38 See Exercise (33)



Section 26.5 The Origin of Magnetism

(34) What is the value of the orbital angular momentum of an electron having orbital quantum number $\ell = 2$? For this electronic state, what is the measured component of the orbital magnetic dipole moment $\mu_{\ell,z}$ when its orbital magnetic quantum numbers are $m_\ell = 0$, $m_\ell = 1$, and $m_\ell = -2$?

(35) An atomic electron has an orbital angular momentum with $m_\ell = 0$. (a) What are the measured components L_z and $\mu_{\ell,z}$? (b) When an external magnetic field of magnitude $B = 40$ mT is applied along the z-axis, find the potential energy U_ℓ associated with the orientation of the orbital magnetic dipole moment $\vec{\mu}_\ell$. (c) Repeat parts (a) and (b) for orbital angular momentum with $m_\ell = -2$.

(36) What is the potential energy U_s associated with the orientation of the spin magnetic dipole moment $\vec{\mu}_s$ of an atomic electron when an external magnetic field of magnitude $B = 0.5$ T is applied along the z-axis. Then what is the energy difference between parallel and antiparallel alignment of $\mu_{s,z}$?

(37) An external magnetic field $\vec{B}$ of magnitude 35 T is produced along the z direction in a short period by a pulsed coil. An electron whose $\mu_{s,z}$ was parallel to $\vec{B}$ experiences a "spin flip" so that the final orientation of $\mu_{s,z}$ is antiparallel to $\vec{B}$. Find the change in the orientation potential energy of the electron.

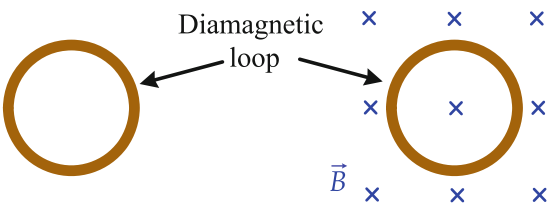
Section 26.7 Magnetic Materials

Section 26.8 Diamagnetism and Paramagnetism

(38) A small magnetic disk has a radius of 1.2 cm and thickness of 0.2 cm. The disk has a uniform magnetization throughout its volume and along its axis. The magnetic moment of the disk is 10^{-2} A.m^2. (a) What is the magnetization $\vec{M}$ of the disk? (b) If the magnetization is due to the alignment of N atoms along the disk axis, each with magnetic moment of 1 μ_B (1 Bohr Magneton), what is the value of N?

(39) Figure 26.39 shows a diamagnetic loop before and after applying an external magnetic field $\vec{B}$. (a) What is the direction of the loop's net magnetic dipole moment $\vec{\mu}$ before and after the application of $\vec{B}$? (b) Is the conventional current counterclockwise or clockwise? (c) What is the direction of the magnetic force on the loop?

Fig. 26.39 See Exercise (39)



(40) The magnetic field inside a solenoid carrying a current is decreased by 5×10^{-3}% when sample of liquid is inserted into its core. What is the magnetic susceptibility of the liquid?

(41) The number of turns for a toroid is $N = 1{,}200$ and the current it carries is $I = 1.5$ A. The core has an average circumference of 100 cm and a cross-sectional area of 2 cm^2. (a) If the core is air, find the magnetic field strength H, the magnetic field $B_\circ$, and the total flux Φ_B in the toroid. (b) If the core is filled with bismuth of magnetic susceptibility $\chi = -2 \times 10^{-6}$, find the mag-

netization M of bismuth, the magnetic field B in bismuth, and the total flux Φ_B.

(42) A solenoid 0.4 m long is tightly wound with 800 turns of copper wire. The current in the winding is $I = 2$ A. (a) If the core is air, find at the center of the core the magnetic field strength H and the magnetic field $B_\circ$. (b) If the solenoid has an aluminum core of magnetic susceptibility $\chi = 2.3 \times 10^{-5}$, find the magnetization M of aluminum and the magnetic field B.

(43) Repeat part (b) of Exercise 42 for a tungsten core of magnetic susceptibility $\chi = 6.8 \times 10^{-5}$.

(44) If all atoms in a material have their magnetic moments aligned, then the maximum magnetization is given by $M_{\text{max}} = n\mu_{\text{atomic}}$, where n is the number of atoms per unit volume. Aluminum has a density of 2.7×10^3 kg/m^3, molecular mass of 27 kg/kmol, and atomic magnetic moment $\mu_{\text{atomic}} = 9.27 \times 10^{-24}$ J/T $= 1\,\mu_{\text{B}}$, where the quantity μ_{B} is called the *Bohr magneton*. Find M_{max} and $\mu_\circ M_{\text{max}}$.

Section 26.9 Ferromagnetism

(45) Assume that all iron atoms in an iron rod are completely aligned and each atom has an approximate magnetic dipole moment $\mu_{\text{Iron}} = 1.9 \times 10^{-23}$ J/T. Iron has density of 7.8×10^3 kg/m^3 and molecular mass of 55.85 kg/kmol. (a) Find the maximum magnetization M_{max}. (b) Find the dipole moment of the rod if it is 10 cm long, 2 cm wide, and 0.5 cm thick. (c) When a magnetic field of 0.5 T is applied perpendicular to the rod, find the torque exerted by the field.

(46) A 4-A current flows through the wire of a toroid that has 250 turns per meter. The toroid's core is iron of magnetic permeability $\mu_{\text{m}} = 2{,}100\,\mu_\circ$, find the magnitude of the magnetic field in its core.

(47) A solenoid of 50 cm long, 1.5 cm in diameter, and 500 turns is filled with iron core. When a current of 10 A flows through the wire of the solenoid, the magnetic field inside it reaches 2 T. What is the permeability of the iron?

Faraday's Law, Alternating Current, and Maxwell's Equations

27

Experimentally, M. Faraday and J. Henry show that a changing magnetic field can establish a current in a circuit that has no battery.

27.1 Faraday's Law of Induction

When we *move a magnet* toward a *stationary loop* that is connected to a galvanometer, see Fig. 27.1a, the galvanometer's needle deflects in one direction. When the magnet stops, as shown in Fig. 27.1b, no deflection is observed. Now, when we move the magnet away from the loop, as shown in Fig. 27.1c, the needle deflects in the opposite direction.

The current produced in this loop is called an *induced current* and the work done per unit charge in producing that current is called an *induced emf*. This emf is due to the change in magnetic flux through the loop, and this process is known as **Faraday's law of induction** and stated as:

> Faraday's law of induction
> The magnitude of the induced emf $|\mathcal{E}|$ in a conducting loop is equal to the rate of change of the magnetic flux Φ_B through the loop.

Lenz's Law

Soon after Faraday proposed his law, Lenz devised a rule—now known as **Lenz's law**, for determining the direction of an induced emf and the direction of an induced current in a loop. This law states that:

H. A. Radi and J. O. Rasmussen, *Principles of Physics*,
Undergraduate Lecture Notes in Physics, DOI: 10.1007/978-3-642-23026-4_27,

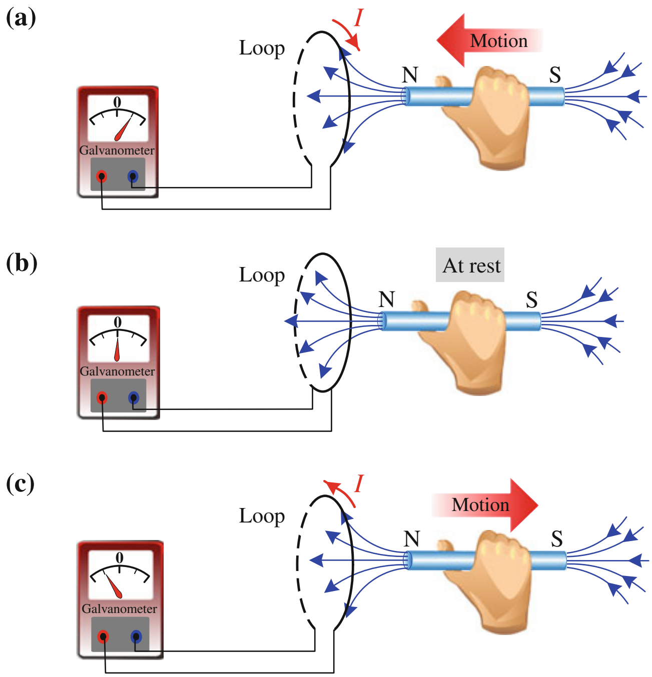


Fig. 27.1 A galvanometer registers an induced current in a loop when the magnet is moving with respect to the loop (parts a and c). In part b, the magnet is at rest, and no induced current is established

> Lenz's law
> An induced current in a loop is created such that the internal magnetic field of the loop opposes the changes in the external magnetic flux.

To get a sense of Lenz's law, let us consider the case of a bar magnet approaching the loop of Fig. 27.2a. During the motion toward the loop, the external magnetic field $\vec{B}_{\text{ext}}$ of the bar magnet increases the magnetic flux on the loop and thereby induces a current in the loop. The induced current produces its own internal magnetic field $\vec{B}_{\text{int}}$ that counteracts the increase in the external magnetic flux. In Fig. 27.2b, $\vec{B}_{\text{int}}$ opposes the decrease in the external flux.

Based on Faraday's law and Lenz's law, the induced emf in a coil of N loops of the same area is given by:

$$\mathcal{E} = -N\frac{d\Phi_B}{dt} \quad \text{(Faraday's law)} \tag{27.1}$$

If a coil lies in a uniform magnetic field $\vec{B}$, then $\Phi_B = \vec{B} \bullet \vec{A} = B\,A\cos\theta$, where any combination of the quantities A, B, and θ can change with time. The induced emf in this case will take the form:

$$\mathcal{E} = -N\frac{d}{dt}(B\,A\cos\theta) \tag{27.2}$$

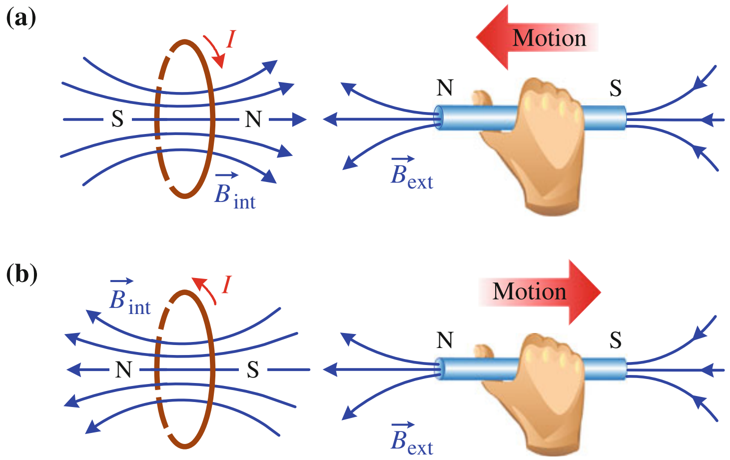


Fig. 27.2 The internal magnetic field $\vec{B}_{int}$: (a) opposes the *increase* in flux of $\vec{B}_{ext}$. (b) opposes the *decrease* in flux of $\vec{B}_{ext}$

Example 27.1

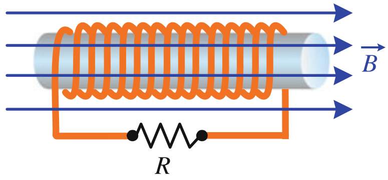
A coil of wire has $N = 15$ turns and each turn has an area $A = 0.04\,\text{m}^2$. The coil is placed in a uniform magnetic field directed perpendicular to the plane of the coil and connected to a resistor of resistance $R = 2\,\Omega$, see Fig. 27.3. The magnetic field changes linearly from 0.1 T at time $t = 0$ to 0.6 T at time $t = 0.5$ s. (a) What is the magnitude of the induced emf in the coil during this time interval? (b) What is the magnitude and direction of the induced current?

Solution: (a) The flux Φ_B through each turn at $t = 0$ and $t = 0.5$ s is:

$$\Phi_B|_{t=0} = B\,A = (0.1\,\text{T})\,(0.04\,\text{m}^2) = 0.004\,\text{Wb}$$

$$\Phi_B|_{t=0.5\,\text{s}} = B\,A = (0.6\,\text{T})(0.04\,\text{m}^2) = 0.024\,\text{Wb}$$

Fig. 27.3



Therefore, from Eq. 27.1, the magnitude of the induced emf is:

$$|\mathcal{E}| = N\frac{\Delta\Phi_B}{\Delta t} = 15\frac{0.024\,\text{Wb} - 0.004\,\text{Wb}}{0.5\,\text{s} - 0} = 0.6\,\text{V}$$

(b) According to Lenz's law, since the magnetic flux increases, then the induced current I established in the circuit must be in a clockwise direction. The value of the induced current is:

$$I = \frac{|\mathcal{E}|}{R} = \frac{0.6\,\text{V}}{2\,\Omega} = 0.3\,\text{A}$$

27.2 Motional emf

The **Motional emf** is an induced emf in a conductor moving through a constant magnetic field.

Figure 27.4 shows a conducting bar of length L moving to the right with a velocity $\vec{v}$ perpendicular to a uniform magnetic field $\vec{B}$ into the page. Each conduction electron is subjected to a downward magnetic force $\vec{F}_B = -e\vec{v} \times \vec{B}$. Consequently, an accumulation of negative charges on the lower end is established, leaving a net positive charge on the upper end. Because of this accumulation, a downwards electric field $\vec{E}$ is produced inside the conducting bar and hence an upwards electric force $\vec{F}_e = -e\vec{E}$ is exerted on each electron. The accumulation process will continue until the magnitude of the downwards magnetic force $F_B = evB$ is balanced by the upwards electric force $F_e = eE$. This condition of equilibrium requires that:

$$eE = evB \quad \text{or} \quad E = vB \tag{27.3}$$

The established electric field is related to the induced potential difference ΔV between the ends of the conducting bar according to the relation $\Delta V = EL$ or $|\mathcal{E}| = EL$, see Eq. 22.17. Thus, from Eq. 27.3, the equilibrium condition requires that:

$$\boxed{\Delta V = BLv} \quad \text{or} \quad \boxed{|\mathcal{E}| = BLv} \tag{27.4}$$

Of course, the potential of the upper end is higher than the lower end, and this polarity is reversed when the direction of motion is reversed.

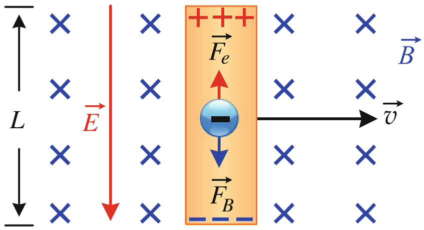


Fig. 27.4 A conductor of length L moving with velocity $\vec{v}$ across a uniform magnetic field $\vec{B}$. This establishes an electric field $\vec{E}$

Now, let us consider the sliding of this conducting bar on horizontal, frictionless, conducting rails connected to a resistor of a resistance R, as shown in Fig. 27.5 (top view). As the bar is pulled to the right with a velocity $\vec{v}$ under the influence of a force $\vec{F}_{\text{app}}$, a magnetic force acts on the free electrons, causing a counterclockwise induced conventional current I to pass in the circuit. At the same time, a magnetic force $\vec{F}_B = ILB$ will act on the bar opposite to $\vec{F}_{\text{app}}$. If both forces are equal, $F_B = F_{app}$, the bar will move with a constant speed v.

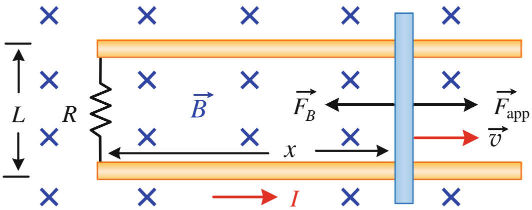


Fig. 27.5 A conducting bar of length L is connected to a resistor and moves on a horizontal conducting rails with velocity $\vec{v}$ across a uniform magnetic field $\vec{B}$. A counterclockwise current I is induced

The area of the circuit within the magnetic field is Lx, where x is the position of the bar from the resistor. Thus, the magnetic flux through this area is:

$$\Phi_B = BLx \tag{27.5}$$

Using Faraday's law, and noting that $v = dx/dt$, the induced emf is:

$$\mathcal{E} = -\frac{d\Phi_B}{dt} = -\frac{d}{dt}(BLx) = -BL\frac{dx}{dt}$$

Therefore,
$$\boxed{\mathcal{E} = -BLv} \tag{27.6}$$

and,
$$\boxed{I = \frac{|\mathcal{E}|}{R} = \frac{BLv}{R}} \tag{27.7}$$

The power delivered by the applied force is:

$$P = F_{app}\, v = (ILB)v = \frac{B^2L^2v^2}{R} = \frac{\mathcal{E}^2}{R} \tag{27.8}$$

This proves that the power input is equal to the rate at which energy is delivered to the resistor, which is confirmed by Eq. 24.24.

Example 27.2

The conducting bar in Fig. 27.5 is 0.1 m long and moves to the right with a speed of 10 m/s across a uniform magnetic field of 1.5 T.(a) Find the induced emf in the circuit. (b) Find the force per a unit charge in the conducting bar.

Solution: (a) From Eq. 27.6, the induced emf in the circuit is:

$$|\mathcal{E}| = BLv = (1.5\,\text{T})(0.1\,\text{m})(10\,\text{m/s}) = 1.5\,\text{V}$$

From Lenz's law, a counterclockwise emf is created in the circuit.

(b) Using Eq. 25.2, the force per unit charge is:

$$\frac{F_B}{|q|} = vB\sin 90^\circ = (10\,\text{m/s})(1.5\,\text{T}) = 15\,\text{N/C}$$

From $\vec{F}_B = q\,\vec{v} \times \vec{B}$, the force $\vec{F}_B$ must be along the bar and upwards.

Example 27.3

The conducting bar in Fig. 27.5 is 0.5 m long and moves to the right across a uniform magnetic field of 0.15 T. If the total resistance of the circuit is 3 Ω, calculate the force required to move the rod at a constant speed of 2 m/s. Find the power delivered.

Solution: From Eq. 27.6, the induced emf in the circuit is:

$$|\mathcal{E}| = BLv = (0.15\,\text{T})(0.5\,\text{m})(2\,\text{m/s}) = 0.15\,\text{V}$$

From Lenz's law, a counterclockwise emf is created in the circuit.

The current in the circuit is:

$$I = \frac{|\mathcal{E}|}{R} = \frac{0.15\,\mathrm{V}}{3\,\Omega} = 0.05\,\mathrm{A}$$

Equation 25.19 gives the magnitude of the magnetic force on the bar:

$$F_B = ILB\sin 90^\circ = (0.05\,\mathrm{A})(0.5\,\mathrm{m})(0.15\,\mathrm{T}) = 3.75\times 10^{-3}\,\mathrm{N}$$

This magnetic force is directed to the left and must be equal in magnitude to the applied force F_{app}, but opposite in direction. The power delivered is calculated from Eq. 27.8 as:

$$P = F_{\mathrm{app}}\,v = F_B\,v = (3.75\times 10^{-3}\,\mathrm{N})(2\,\mathrm{m/s}) = 7.5\times 10^{-3}\,\mathrm{W}$$

Example 27.4

A rectangular conducting loop has a resistance R, width L, and length a. The loop is pulled at a constant speed v to the right while approaching, entering, and leaving a uniform magnetic field directed out of the page, which extends over a distance b along the x-axis, see Fig. 27.6 for different times. Plot as a function of x: (a) the magnetic flux through the loop, (b) the induced motional emf.

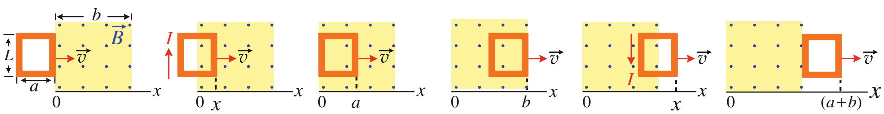


Fig. 27.6

Solution: (a) Figure 27.7 shows the flux Φ_B through the loop as a function of x for different times. The flux is zero when the loop is not in the field; it is BLx when the loop is entering the field; it is BLa when the loop is entirely in the field; it is $BL(a+b-x)$ when the loop is leaving the field; and finally zero when $x \geq a+b$.

(b) As the loop enters the field, the flux increases (with $\vec{B}$ out of the page). By Lenz's law, a clockwise current is created to produce a magnetic field into the page with emf $\mathcal{E} = BLv$. When the loop is entirely in the field, the change in flux is zero, and hence $\mathcal{E} = 0$. When the loop is leaving the field, the flux decreases and a counterclockwise current is created with $\mathcal{E} = -BLv$. When the loop leaves the field, the emf drops to zero, see Fig. 27.7. The current value is $I = BLv/R$.

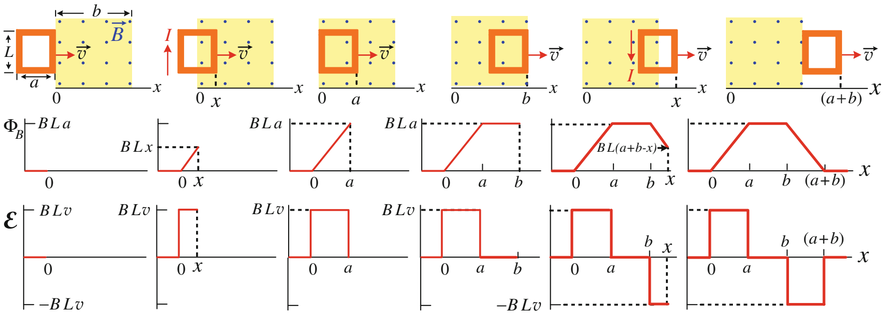


Fig. 27.7

27.3 Electric Generators

Electric generators are devices that convert rotational energy to electric energy.

A generator consists of a coil of wire wound on an *armature* that can rotate in a magnetic field between the poles of the magnet. The magnetic flux through the coil changes with time. Thus, according to Faraday's law, an induced emf and current will be created in the coil.

If θ is the angle between the magnetic field $\vec{B}$ and the normal to the plane of the coil, then the magnetic flux through each loop of the coil will be given by:

$$\Phi_B = \vec{B} \bullet \vec{A} = BA\cos\theta \tag{27.9}$$

If the shaft of the generator rotates with constant angular frequency ω (in rad/s), then the relation between the angular position θ (in rad) and the frequency ω is $\theta = \omega t$. Therefore, $\Phi_B = BA\cos\omega t$. Hence, according to Faraday's law, the induced emf of a coil of N loops will be:

$$\mathcal{E} = -N\frac{d\Phi_B}{dt} = -N\frac{d}{dt}(BA\cos\omega t) = N\,B\,A\,\omega\sin\omega t$$

which can be written in compact ($\mathcal{E} \equiv v$ and $\mathcal{E}_\circ \equiv V$) form as follows:

$$\boxed{v = V\sin\omega t} \quad \text{where} \quad \boxed{V = N\,B\,A\,\omega} \tag{27.10}$$

The output emf is sinusoidal with amplitude (or peak) V. From this relation we see that $v = 0$ when $\omega t = 0$ or $\omega t = \pi$, and this occurs when $\vec{B}$ is perpendicular to the plane of the coil. Furthermore, $v = V$ when $\omega t = \pi/2$ or $\omega t = 3\pi/2$, and this occurs when $\vec{B}$ is in the plane of the coil.

The Direct Current (dc) Generator

Figure 27.8 illustrates the simplest form of the **direct current** (**dc**) **generator**. The ends of the coil are connected to a split-ring commutator (as shown in Fig. 27.8a) that rotate with the coil. Those splits are in contact with two brushes that act as the output terminals of the generator. The output is always of the same polarity and varies with time as shown in Fig. 27.8b.

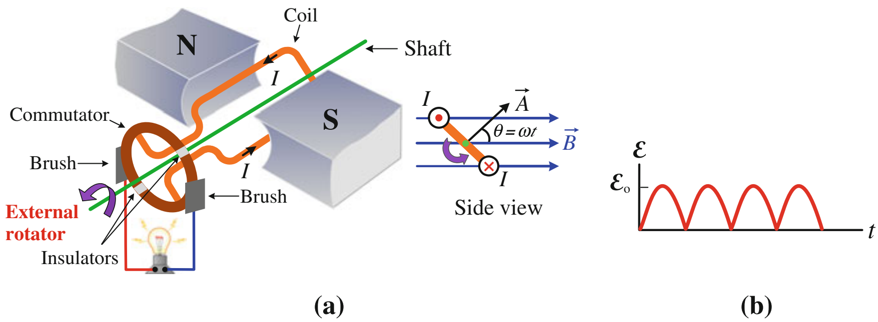


Fig. 27.8 (a) A sketch of a dc generator. An emf is induced in a coil when it rotates with constant angular frequency ω in a magnetic field $\vec{B}$. (b) The direct induced emf is plotted as a function of time

The Alternating Current (ac) Generator

Figure 27.9 illustrates the simplest form of an **alternating current** (**ac**) **generator**. The ends of the coil are connected to slip-rings (as shown in Fig. 27.9a) that rotate with the coil. Those slips are in contact with two brushes that act as the output terminals of the generator. The output varies sinusoidally with time. See Fig. 27.9b.

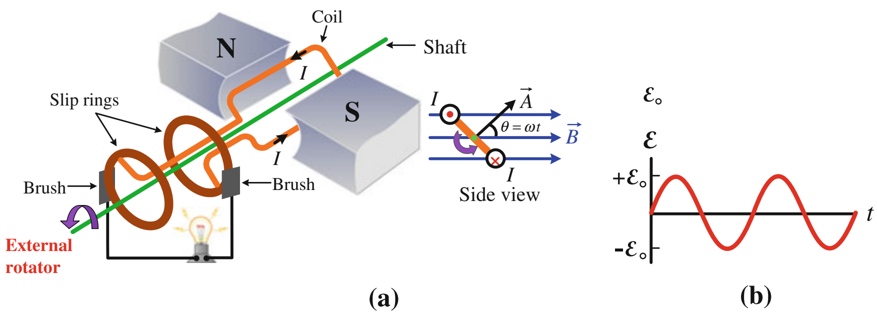


Fig. 27.9 (a) A sketch of an ac generator. An emf is induced in a coil when it rotates with constant angular frequency ω in a magnetic field. (b) The alternating, induced emf is plotted as a function of time

27.4 Alternating Current

When electric generators at electric power plants produce alternating emf we get **alternating current**, or **ac** (uppercase letters can be used) with the usual symbol . Alternating current reverses direction many times per second, which means that electrons in a wire will repeatedly move in one direction and then reverse their direction. Since the output emf of an ac generator is sinusoidal, as shown in Fig. 27.9b, then the current it produces is also sinusoidal.

Ohm's law, Eq. 24.8, is also valid for alternating voltage and current. Based on Eq. 27.10, when a sinusoidal voltage v exists across a resistance R, see Fig. 27.10a, then the alternating current i (we use the small letter i for ac) through the resistor is:

$$\boxed{i = \frac{v}{R} = \frac{V}{R}\sin\omega t = I\sin\omega\, t} \quad \text{where} \quad \boxed{I = \frac{V}{R}} \tag{27.11}$$

where I is the peak current, see Fig. 27.10b. From this figure we see that the average current is zero. This does not mean that no heat is produced in the resistor. On the contrary, electrons produce heat when they move back and forth in the resistor. The power delivered at time t to a resistor of resistance R, see Fig. 27.10c, is:

$$P(t) = i^2 R = I^2 R\sin^2\omega t \tag{27.12}$$

which indicates that the power is always positive because the current is squared. The quantity $\sin^2\omega t$ varies between 0 and 1 and we can prove that its mean (or average) value is $\frac{1}{2}$, i.e. $\overline{\sin^2\omega t} = 1/2$. Therefore, using $\overline{P} = \overline{i^2}\,R$ or $\overline{P} = \overline{v^2}/R^2$ as the average power delivered to the resistor, we get:

$$\overline{P} = I^2R/2 \quad \text{or} \quad \overline{P} = V^2/2R \tag{27.13}$$

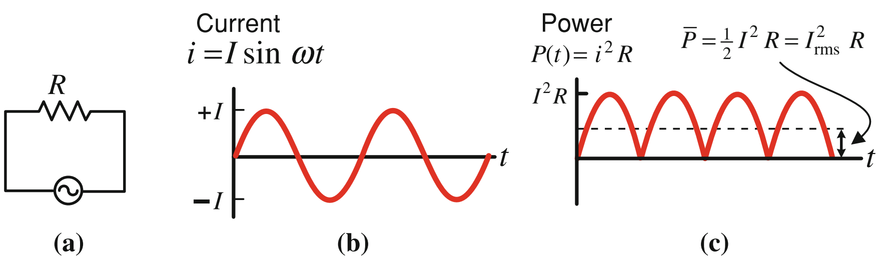


Fig. 27.10 (a) A resistor connected to an ac source. (b) Alternating current in a resistor as a function of time. (c) Power and average power delivered to a resistor as a function of time

As introduced in Sect. 13.1, the notation rms stands for *root-mean-square*, which here means *the square root of the mean value of the square of the current* $I_{rms} = \sqrt{\overline{i^2}}$ or the voltage $V_{rms} = \sqrt{\overline{\mathcal{E}^2}}$. Thus (remember $\mathcal{E} \equiv v$):

$$I_{rms} = \frac{I}{\sqrt{2}} = 0.707\, I \quad \text{and} \quad V_{rms} = \frac{V}{\sqrt{2}} = 0.707\, V \tag{27.14}$$

This means that an alternating current whose maximum value is 1 A delivers to a resistor the same power as a direct current of 0.707 A. Thus, Ohm's law and the average power delivered to a resistor give:

$$\boxed{V_{rms} = I_{rms}\, R} \qquad \text{and} \qquad \boxed{\overline{P} = I_{rms}\, V_{rms} = I_{rms}^2\, R = V_{rms}^2/R} \tag{27.15}$$

Alternating-current instruments are usually calibrated to read the rms values of the current defined by $I_{rms} = I/\sqrt{2}$ and voltage defined by $V_{rms} = V/\sqrt{2}$. More than 81% of countries around the globe use V_{rms} in the range from 220 to 240 V.

Example 27.5

Find the resistance and the peak current in a 1,000-W heater connected to a 220-V ac line.

Solution: Using Eq. 27.15, we can find the rms current:

$$I_{rms} = \frac{\overline{P}}{V_{rms}} = \frac{1{,}000\ \text{W}}{220\ \text{V}} = 4.55\ \text{A}$$

Then, the peak current and the resistance of the heater will be:

$$I = \sqrt{2}\, I_{rms} = 6.43\ \text{A}$$

$$R = \frac{V_{rms}}{I_{rms}} = \frac{220\ \text{V}}{4.55\ \text{A}} = 48.35\ \Omega$$

27.5 Transformers

A **transformer** is a device used to increase or decrease an ac voltage. Transformers are widely used in: reducing the high voltage from the electric power plant to a usable household electric ac outlet (120 or 220 V), in chargers for mobiles, laptops, and other electronic devices, in cars to increase the voltage to a high voltage needed to spark the plugs, in CRT monitors, and on many electrical applications.

An ideal transformer consists of two resistanceless coils known as the **primary** and **secondary** coils, wound around an iron core, see Fig. 27.11. We assume that the primary coil has N_P turns and the secondary coil has N_S turns. If all magnetic lines are confined to the iron core, then at any instant the magnetic flux per turn Φ_B produced by the current in the primary coil will pass through the secondary coil.

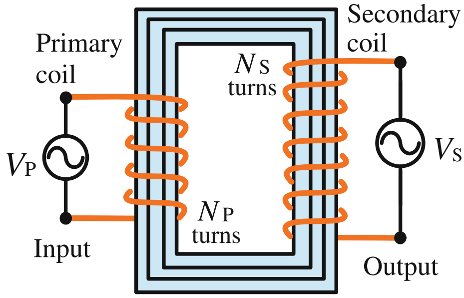


Fig. 27.11 An ac input of voltage v_P (with peak V_P) is connected to the primary coil of a transformer to get an ac output in the secondary coil. The figure shows a step-up transformer where $N_P = 5$ and $N_S = 7$

When an ac voltage v_P (with peak V_P) is applied to the primary coil, the magnetic flux change is the same in each turn of the primary and secondary coils. Thus, according to Faraday's law, the resulting induced emf in the primary and secondly coils will be:

$$\mathcal{E}_P = -N_P \frac{d\Phi_B}{dt} \quad \text{and} \quad \mathcal{E}_S = -N_S \frac{d\Phi_B}{dt} \tag{27.16}$$

where $\mathcal{E}_P$ and $\mathcal{E}_S$ have the same frequency as the ac input source v_P. Since the flux per turn Φ_B is the same, the ratio of the secondary emf to the primary emf at any instant is $\mathcal{E}_S/\mathcal{E}_P = N_S/N_P$. If the windings have zero resistance, the induced emf $\mathcal{E}_P$ and $\mathcal{E}_S$ are exactly balanced by the terminal voltage across the primary voltage v_P (with peak V_P) and the secondary voltage v_S (with peak V_S), respectively. Thus:

$$\boxed{\frac{V_S}{V_P} = \frac{N_S}{N_P}} \tag{27.17}$$

This **transformer equation** relates the output (the secondary) to the input (the primary) and can apply for the amplitude or the rms values.

If $N_S > N_P$, then $V_S > V_P$ and this kind of transformer is called a **step-up transformer**. Similarly, if $N_S < N_P$, then $V_S < V_P$ and this kind of the transformer is called a **step-down transformer**.

For step-up or step-down ideal transformers, the power output is equal to the power input. Using Eq. 24.24, we have:

$$\boxed{I_P\, V_P = I_S\, V_S} \quad \text{or} \quad \boxed{\frac{I_S}{I_P} = \frac{N_P}{N_S}} \tag{27.18}$$

Example 27.6

The transformer used to charge a laptop reduces 220-V ac to 19 V ac. Assume that the primary coil contains 400 turns and the charger supplies 5 A to the laptop. Find: (a) the number of turns in the secondary coil and the current in the primary coil, and (b) the power transformed.

Solution: (a) Using Eq. 27.17, we have:

$$\frac{V_S}{V_P} = \frac{N_S}{N_P} \quad \Rightarrow \quad N_S = N_P \frac{V_S}{V_P} = \frac{(400)(19\,\text{V})}{220\,\text{V}} = 35\,\text{turns}$$

Using Eq. 27.18, we have:

$$I_P\, V_P = I_S\, V_S \quad \Rightarrow \quad I_P = I_S\, \frac{V_S}{V_P} = \frac{(5\,\text{A})(19\,\text{V})}{220\,\text{V}} = 431.8\,\text{mA}$$

(b) The power transformed is:

$$P = I_S\, V_S = (5\,\text{A})(19\,\text{V}) = 95\,\text{W}$$

27.6 Induced Electric Fields

By Faraday's law a changing magnetic flux induces both an emf and a current in a conducting loop. But in Chap. 24, see Eq. 24.11, we related current to an electric field that applies electric forces on charged particles.

Similarly, we relate an induced current to an electric field such as:

Spotlight
A changing magnetic field in a conducting loop, or even in any hypothetical closed path, induces an electric field.

We can examine this statement by considering a circular copper loop of radius r placed in a uniform magnetic field $\vec{B}$ perpendicular to the loop, see Fig. 27.12. If the magnetic field *increases* with time, then according to Faraday's law, an induced emf and an induced current are created in the loop. But this induced current implies the existence of an induced electric field $\vec{E}$. The work done by $\vec{E}$ to move a test charge

$q_\circ$ around the loop is $W = q_\circ \mathcal{E}$. On the other hand, according to Eq. 22.3, this work is given for a closed loop by:

$$W = q_\circ \oint \vec{E} \bullet d\vec{s} \tag{27.19}$$

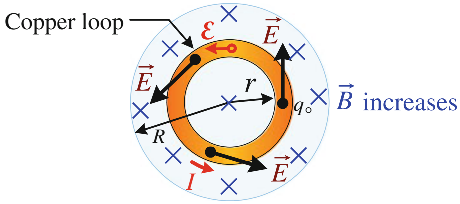


Fig. 27.12 A copper loop in a uniform magnetic field. If $\vec{B}$ changes with time, an induced electric field is produced tangent to the loop

Thus, equating the two expressions of work, we get:

$$q_\circ \oint \vec{E} \bullet d\vec{s} = q_\circ \mathcal{E} \tag{27.20}$$

Using $\mathcal{E} = -d\Phi_B/dt$, then Faraday's law of induction can be expressed in terms of the induced electric field as follows:

$$\boxed{\oint \vec{E} \bullet d\vec{s} = -\frac{d\Phi_B}{dt}} \qquad \text{(Faraday's law)} \tag{27.21}$$

The striking feature of Eq. 27.21 is that the electric field is induced even if there is no conducting loop of $r < R$. In addition, an induced electric field is established even if $r > R$, see Fig. 27.13.

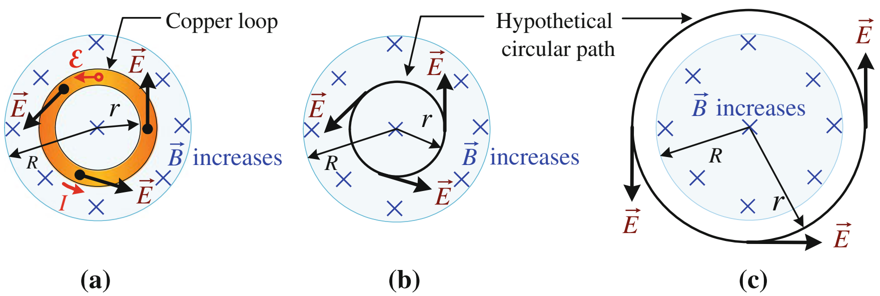


Fig. 27.13 (a) Same as Fig. 27.10 when the conducting loop is in place. (b) Induced electric field is established even for a hypothetical path of $r < R$. (c) Same as (b), but when $r > R$

Example 27.7

In Fig. 27.13, find an expression for the magnitude of the induced electric field E for $r < R$ and $r > R$. When $R = 8$ cm and the magnitude of the magnetic field increases at a rate given by $dB/dt = 0.2$ T/s, evaluate E for $r = 5$ cm and $r = 10$ cm

Solution: We evaluate the integral of Eq. 27.21 for any radius:

$$\oint \vec{E} \bullet d\vec{s} = \oint E\, ds = E \oint ds = E(2\pi r) = 2\pi r E$$

The flux Φ_B and its rate $d\Phi_B/dt$ through the circular path of $r < R$ are:

$$\Phi_B = BA = B(\pi r^2) \quad \text{and} \quad d\Phi_B/dt = \pi r^2 dB/dt$$

The flux Φ_B and its rate $d\Phi_B/dt$ through the circular path of $r > R$ are:

$$\Phi_B = BA = B(\pi R^2) \quad \text{and} \quad d\Phi_B/dt = \pi R^2 dB/dt$$

Thus, dropping the minus sign of Eq. 27.21 leads to:

$$E = \frac{r}{2}\frac{dB}{dt} \quad (\text{for } r < R) \quad \text{and} \quad E|_{r=5\,\text{cm}} = \frac{0.05\,\text{m}}{2} 0.2\,\text{T/s} = 5 \times 10^{-3}\,\text{V/m}$$

$$E = \frac{R^2}{2r}\frac{dB}{dt} \quad (\text{for } r > R) \quad \text{and} \quad E|_{r=10\,\text{cm}} = \frac{(0.08\,\text{m})^2}{2 \times (0.1\,\text{m})} 0.2\,\text{T/s} = 6.4 \times 10^{-3}\,\text{V/m}$$

27.7 Maxwell's Equations of Electromagnetism

From our previous studies, we can collect all the relationships between electric and magnetic fields and their sources. This collection consists of four equations, called **Maxwell's equations**. Maxwell used these equations to predict the existence of electromagnetic waves.

The first two equations involve a surface integral of $\vec{E}$ and $\vec{B}$ over a closed surface. The third and fourth two equations involve a line integral of $\vec{B}$ and $\vec{E}$ over a closed loop.

The first is simply Gauss's law for electric fields, Eq. 21.7, which states that "the net electric flux through any closed surface is equal to the net charge inside the surface divided by the permittivity of free space $\epsilon_\circ$". That is:

$$\boxed{\oint \vec{E} \bullet \vec{A} = \frac{q_{\text{in}}}{\epsilon_\circ}} \quad \text{(Gauss's law for } \vec{E}\,) \tag{27.22}$$

The second is the analogous relationship for magnetic field, Eq. 26.24, which states "The *net magnetic flux* throughout any closed surface is always zero". That is:

$$\boxed{\oint \vec{B} \bullet d\,\vec{A} = 0} \quad \text{(Gauss's law for } \vec{B}\,) \tag{27.23}$$

The third equation is Ampere–Maxwell law, Eq. 26.20, which states "Magnetic fields are produced both by varying conduction currents i and by displacement currents i_d, created by a time varying electric flux". That is:

$$\boxed{\oint \vec{B} \bullet d\vec{s} = \mu_\circ (i + i_d) = \mu_\circ \left(i + \epsilon_\circ \frac{d\Phi_E}{dt} \right)} \quad \text{(Ampere}-\text{Maxwell law)} \tag{27.24}$$

The fourth equation is Faraday's, Eq. 27.21, which states that "A changing magnetic field in a conducting loop, or even in any hypothetical closed path, induces an electric field". That is:

$$\boxed{\oint \vec{E} \bullet d\vec{s} = -\frac{d\Phi_B}{dt}} \quad \text{(Faraday's law)} \tag{27.25}$$

In all Maxwell's equations, $\vec{E}$ is the total electric field, which comes from an electrostatic field $\vec{E}_s$ caused by static charges and magnetically-induced, non-electrostatic field $\vec{E}_{\text{ind}}$.

It is worth noting that in empty space, where there is no charge and conduction current, the first two equations are identical in form. In addition to that, replacing Φ_E by $\int \vec{E} \bullet d\vec{A}$ and Φ_B by $\int \vec{B} \bullet d\vec{A}$, we can write the third and fourth equations in different but equivalent forms. Thus, in empty space Maxell's equations reduce to:

$$\begin{aligned} &\oint \vec{E} \bullet d\vec{A} = 0, && \oint \vec{B} \bullet d\vec{A} = 0 \\ &\oint \vec{B} \bullet d\vec{s} = \mu_\circ \epsilon_\circ \frac{d}{dt} \int \vec{E} \bullet d\vec{A}, && \oint \vec{E} \bullet d\vec{s} = -\frac{d}{dt} \int \vec{B} \bullet d\vec{A}. \end{aligned} \tag{27.26}$$

The most remarkable feature about these Maxwell's equations is that time-varying of $\vec{E}$ induces a field $\vec{B}$ and time-varying of $\vec{B}$ induces a field $\vec{E}$ in neighboring regions of space. Maxwell recognized that Eq. 27.26 predict the existence of electromagnetic disturbance of electric and magnetic fields that propagate from one point to another, even if no matter is present. This disturbance is called an electromagnetic wave (EMW), and this provides the physical basis for light and all the rest of the electromagnetic spectrum.

The properties of electromagnetic waves can be deduced from Maxwell's equations, but the mathematical treatment is beyond the scope of this book. Instead, one can focus attention on an electromagnetic wave traveling in the x-direction. By doing so, we can show that the line integral of the last two forms of Eq. 27.26 lead to the following two differential equations:

$$\frac{\partial^2 E}{\partial x^2} - \mu_\circ \epsilon_\circ \frac{\partial^2 E}{\partial t^2} = 0 \quad \text{and} \quad \frac{\partial^2 B}{\partial x^2} - \mu_\circ \epsilon_\circ \frac{\partial^2 B}{\partial t^2} = 0 \tag{27.27}$$

These two differential equations have the identical form as the general wave equation introduced in Chap. 14, see Eq. 14.58, for one-dimensional wave motion, but with speed c, given by:

$$\boxed{c = \frac{1}{\sqrt{\mu_\circ \epsilon_\circ}}} \tag{27.28}$$

Taking $\mu_\circ = 4\pi \times 10^{-7}$ T.m/A and $\epsilon_\circ = 8.8542 \times 10^{-12}$ C^2/N.m^2, we find that $c = 2.99792 \times 10^8$ m/s, which is precisely the speed of light in empty space. In addition $\vec{E}$ and $\vec{B}$ are perpendicular to one another, and both are perpendicular to the wave velocity $\vec{c}$, see Fig. 27.14a.

The simplest solution to Eq. 27.27 is a sinusoidal wave, where E and B vary with x and t according to the two expressions:

$$E = E_\circ \cos(kx - \omega t) \quad \text{and} \quad B = B_\circ \cos(kx - \omega t) \tag{27.29}$$

where $E_\circ$ and $B_\circ$ are the peak values of the electric and magnetic waves, respectively, while k and ω are defined in Chap. 14. Figure 27.14b displays the various types of electromagnetic waves.

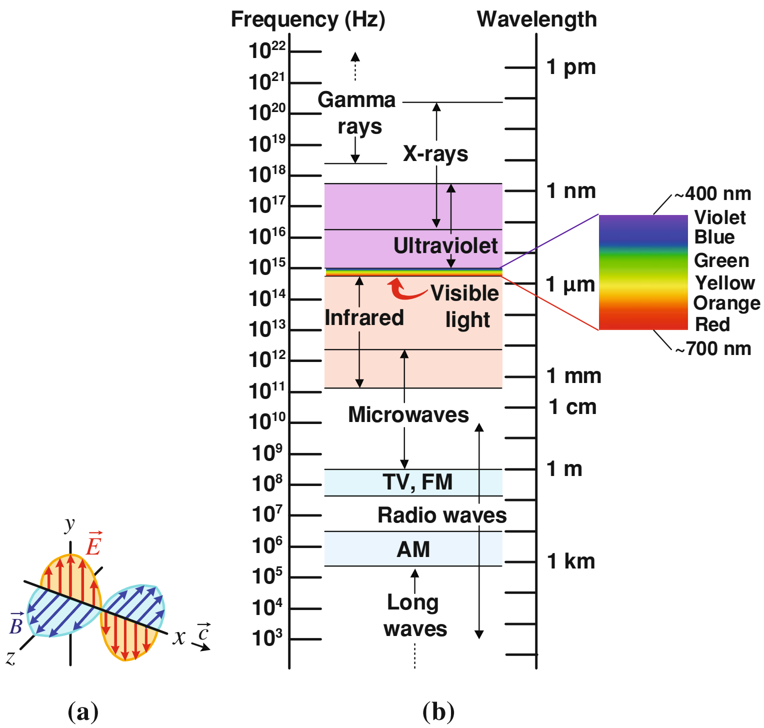


Fig. 27.14 (a) Propagation of EMW. (b) The EMW spectrum

27.8 Exercises

Section 27.1 Faraday's Law of Induction

(1) A loop of wire with a vector area $\vec{A} = (0.2\,\vec{i} + 0.3\,\vec{j} + 0.6\,\vec{k})\ \mathrm{m}^2$ is placed in a uniform magnetic field $\vec{B} = 0.2\,\vec{j}$ (T). (a) Find the magnetic flux through the loop. (b) What is the angle between $\vec{B}$ and $\vec{A}$?

(2) If the magnetic field of Exercise 1 changes to $\vec{B} = 0.2\,\vec{k}$ (T) in a period of 0.5 s, find the magnitude of the average induced emf and current in the loop, assuming the loop has a resistance of 1.5 Ω.

(3) In Fig. 27.15, the south pole of the magnet approaches the loop. What is the direction of the induced current in the resistor of the circuit?

(4) In Fig. 27.16, the north pole of the magnet recedes from the loop. What is the direction of the induced current in the resistor of the circuit?

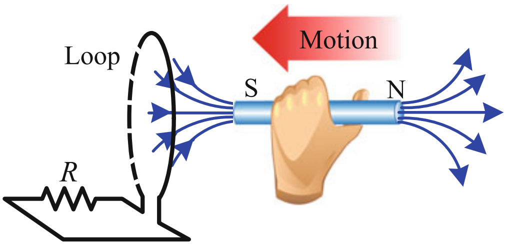


Fig. 27.15 See Exercise (3)

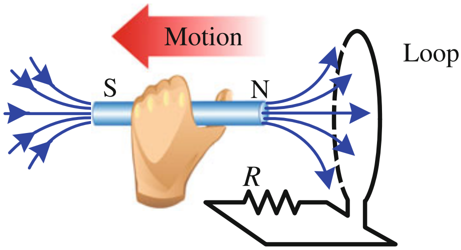


Fig. 27.16 See Exercise (4)

(5) The magnetic flux through a loop of wire changes from −20 to +20 Wb in 0.2 s. What is the average induced emf in the loop?

(6) Figure 27.17 shows a circuit containing a battery and a resistor whose resistance can vary. Two loops are located inside and outside the circuit. If the resistance is slowly decreased, what is the direction of the induced current in the two loops?

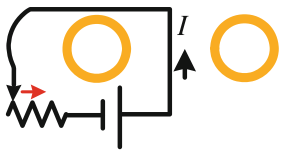


Fig. 27.17 See Exercise (6)

(7) A circular loop of radius $r = 5$ cm is perpendicular to a uniform magnetic field that is pointing out of the page and has an initial magnitude $B_i = 0.6$ T. During a time interval of 0.2 s the field is changed to a final magnitude $B_f = 0.2$ T and now points into the page. What is the average induced emf in the loop?

(8) A square loop of wire has a side $a = 5$ cm and is perpendicular to a uniform magnetic field of magnitude $B = 0.8$ T. The orientation of the loop changes in a period of 0.4 s until the surface of its plane is parallel to the field. What is the average induced emf in the loop?

(9) The plane of a circular loop of radius $r = 10\,\text{cm}$ is perpendicular to a uniform magnetic field of initial magnitude $B = 0.8\,\text{T}$. The field's magnitude then decreases at a constant rate of $dB/dt = -10^{-3}\,\text{T/s}$. (a) What is the magnitude of the field at any time? (b) What is the induced emf produced in the loop?

(10) For each situation in Fig. 27.18, show the direction of the induced current.

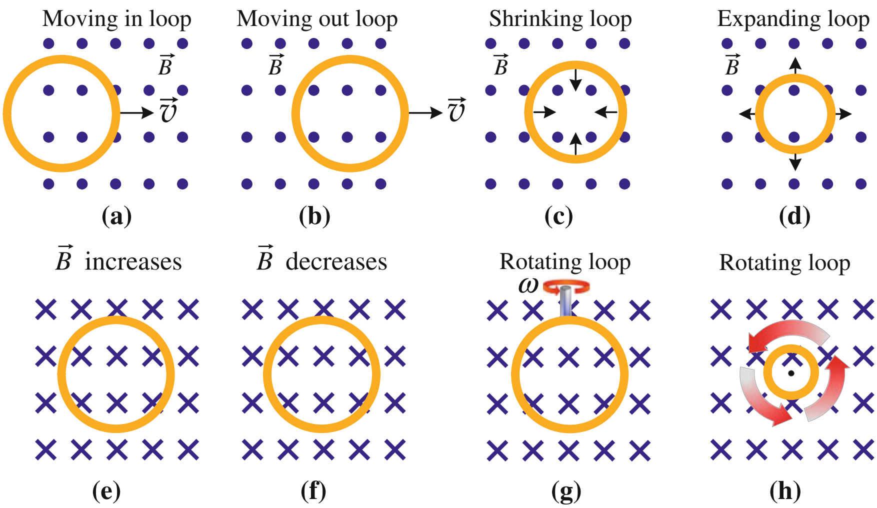


Fig. 27.18 See Exercise (10). (a) Moving in loop. (b) Moving out loop. (c) Shrinking loop. (d) Expanding loop. (e) $\vec{B}$ increases. (f) $\vec{B}$ decreases. (g) Rotating loop. (h) Rotating loop

(11) The diameter of the circular loop of Fig. 27.18c decreases from 20 to 5 cm in 0.6 s. The magnitude of the magnetic field in that figure has a value $B = 0.5\,\text{T}$. (a) What is the direction of the induced current? (b) What is the average induced emf in the loop? (c) What is the average magnitude of the induced current if the loop's resistance is $2\,\Omega$?

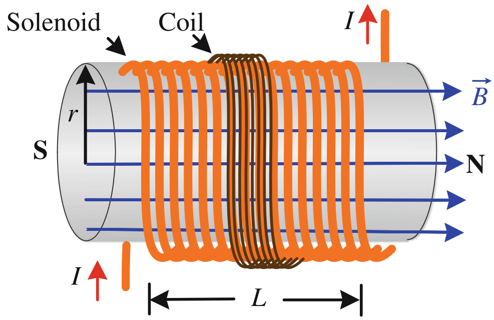
(12) The radius of the circular loop of Fig. 27.18d increases from 2 cm to 15 cm in 0.2 s. The magnitude of the magnetic field in that figure has a value $B = 0.25\,\text{T}$. (a) What is the direction of the induced current? (b) What is the average induced emf in the loop? (c) What is the average magnitude of the induced current if the loop's resistance is $1.5\,\Omega$?

(13) The plane of the circular loop of Fig. 27.18g has an area $A = 5\,\text{cm}^2$. In 0.2 s it rotates to make an angle $\theta = 60°$ with the field lines. The magnitude of the

magnetic field in that figure has a value $B = 0.75\,\text{T}$. (a) What is the direction of the induced current? (b) What is the average induced emf in the loop? (c) What is the average magnitude of the induced current if the loop's resistance is $2.5\,\Omega$?

(14) A solenoid of length $L = 0.25\,\text{m}$ and radius $r = 4\,\text{cm}$ has 100 turns. A coil of $N = 20$ turns and resistance $R = 4\,\Omega$ is wound tightly around the solenoid, see Fig. 27.19. The current in the direction shown in the figure increases uniformly from 0 to 2 A in 0.5 s. (a) What is the direction of the induced current in the coil during this period of time? (b) What is the magnitude of the induced emf in the coil? (c) What is the magnitude of the induced current in the coil? (d) Redo parts a, b, and c assuming this time that the solenoid's core is made entirely out of iron whose magnetic permeability μ_{m} is $1{,}000\mu_{\circ}$ and also assume that the direction of the solenoid's current is reversed.

Fig. 27.19 See Exercise (14)



(15) A thin circular gold ring has a diameter of 2 cm, a resistance of $60\,\mu\Omega$, a mass of 20 g, and a specific heat of 129 J/kg.C°. In a period of 40 ms, the ring moves from a zero-field location to a location where a magnetic field has magnitude $B = 0.75\,\text{T}$ and points perpendicular to the ring. (a) What is the average magnitude of the induced emf in the ring? (b) Find the thermal energy dissipated in the ring. (c) Find the temperature rise in the ring if all thermal energy converts to heat.

(16) A coil of radius $r_c = 10\,\text{cm}$ consists of $N = 30$ turns of copper wire. The wire of the coil has a radius of $r_w = 1.5\,\text{mm}$ and a resistivity $\rho_w = 1.68 \times 10^{-8}\,\Omega.\text{m}$. A uniform magnetic field perpendicular to the plane of the coil changes at a rate dB/dt of 8.5×10^{-3} T/s. (a) What is the induced emf produced in the coil? (b) What is the resistance of the wire of the coil? (c) What is the induced current

in the coil? (d) What is the rate at which the thermal energy is dissipated in the wire of the coil?

(17) A circular loop of wire has a radius $r = 15$ cm and resistance $R = 80\,\Omega$. The loop is initially in a uniform magnetic field perpendicular to the plane of the loop and has a magnitude $B = 0.5$ T. The loop is removed from the field in 150 ms. (a) What is the average induced emf produced in the loop? (b) Find the electric energy delivered by the process if it is equal to the energy dissipated in the loop.

(18) A vertical rectangular loop of width a and height b is at a distance x from a vertical long wire carrying a current I, see Fig. 27.20. (a) Find the magnetic flux through the loop. (b) If the rectangular loop is pulled away from the wire with a speed v, find the instantaneously-induced emf produced in the loop and the instantaneous force required.

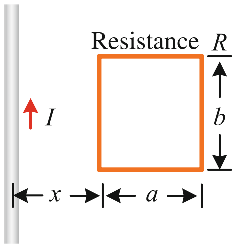


Fig. 27.20 See Exercise (18)

Section 27.2 Motional emf

(19) The rod in Fig. 27.4 has a length $L = 12$ cm and moves with a speed $v = 1$ m/s in a uniform magnetic field of magnitude $B = 1.5$ T. Find the induced motional emf developed in the rod.

(20) An induced emf of 2 V is established by moving a rod 0.8 m long at a speed of 5 m/s perpendicular to a uniform magnetic field. Find the magnitude of that field.

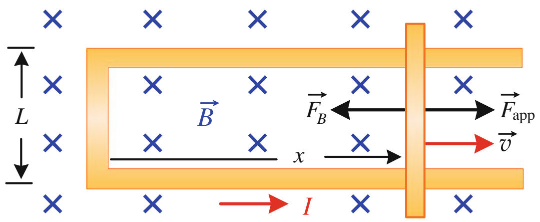
(21) A jet plane is flying horizontally at 250 m/s in a region where the vertical component of the Earth's magnetic field is 80 μT. The distance between the tips of the two wings of the plane is 30 m. What is the electric potential difference induced between the two wing tips?

(22) The rod in Fig. 27.5 has a length $L = 25$ cm and moves with a constant speed $v = 10$ m/s in a uniform magnetic field of magnitude $B = 1.5$ T. The resistor

has a resistance $R = 10\ \Omega$ and the rest of the circuit has a negligible resistance. (a) Find the induced motional emf developed in the circuit. (b) Find the force required to move the rod at this constant speed. (c) Find the power delivered to the resistor.

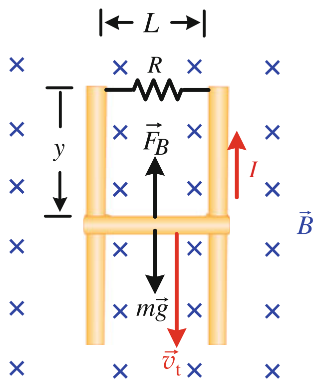
(23) Figure 27.21 shows a conducting rod that has a resistivity ρ and cross-sectional area A_{rod}. The rod makes contacts with horizontal conducting rails of the same type to complete the circuit. The circuit area is perpendicular to a uniform magnetic field of magnitude B. The rod starts from $x = 0$ at $t = 0$ and moves with constant speed v. (a) Find the induced current I as a function of time. (b) Find the power delivered by the applied force F_{app} as a function of time.

Fig. 27.21 See Exercise (23)



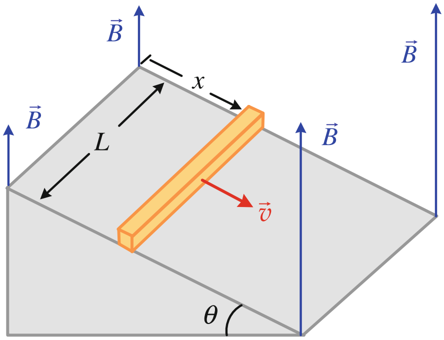
(24) A conducting rod of length $L = 25$ cm and mass $m = 3.5$ g slides along a pair of vertical metal guides connected to a resistor of resistance $R = 1.5 \times 10^{-3}\ \Omega$, see Fig. 27.22. The circuit is perpendicular to a magnetic field with $B = 0.05$ T. Friction and resistance of the rod and the guides are negligible. Find the terminal speed of the rod.

Fig. 27.22 See Exercise (24)



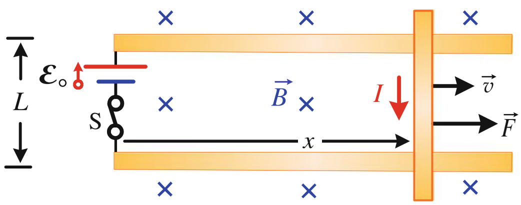
(25) A conducting rod of length L slides down from rest at the top of a frictionless incline of angle θ. Assume that a uniform vertical magnetic field $\vec{B}$ present throughout the motion of the rod, see Fig. 27.23. (a) Find the potential difference between the ends of the rod as a function of time. (b) Which side of the rod has a higher potential.

Fig. 27.23 See Exercise (25)



(26) Figure 27.24 shows a conducting rod of length L, mass m, and resistance R. The rod slides on two long horizontal frictionless and resistanceless parallel rails immersed in a uniform magnetic field $\vec{B}$. A battery of emf $\mathcal{E}_\circ$ and switch are connected to one end of the rails to complete the circuit. When the rod is at rest at $t = 0$, the switch S is closed. Find the speed v of the rod as a function of time, and find its terminal speed.

Fig. 27.24 See Exercise (26)



Section 27.3 Electric Generators

(27) When the rotating speed of the generator of a stationary car is 1,000 rpm its output is 12 V. What will the output of the generator be when its rotating speed is 2,500 rpm, assuming that the car is still stationary?

(28) If you plug an alternating-current voltmeter into a household electric outlet and the voltmeter reads 220 V, what is the peak value of the outlet voltage?

(29) The amplitude (or peak) of the sinusoidal output voltage of a generator is 311 V. The square coil of the generator has a side of $a = 10$ cm and rotates with an frequency $f = 50$ rev/s in a magnetic field with $B = 0.5$ T. How many loops of wire should the coil consist of?

(30) A generator has a coil of $N = 500$ loops, each of area $A = 5 \times 10^{-2}\ \text{m}^2$. The coil can rotate freely between the poles of a permanent magnet of uniform magnetic field $B = 0.45$ T. How fast must the coil rotate to produce a maximum output voltage of 311 V?

Section 27.4 Alternating Current

(31) Find the peak current when a 220-V rms source is connected to a resistor of resistance $R = 2\ \text{k}\Omega$.

(32) An ac power supply produces a peak voltage of 120 V and is connected to a 24-Ω resistor. What are the rms and peak currents in the resistor?

(33) Two light bulbs of 60-W and 100-W are connected in parallel to a 220-V rms source at your house. (a) What is the total resistance of the two bulbs as seen by the power company? (b) What is the resistance of each bulb?

(34) A heater rated as 1,000 W is connected to an ac source that allows a peak current of 12.86 A. What is the rms voltage of the source?

(35) A 1,100-W hair dryer is connected to 110-V ac source. Find the peak voltage applied to the dryer and the peak current passing through the dryer.

(36) A welding machine has a resistance $R = 22\ \Omega$ and is connected to a 220-V ac line. (a) What is the average electric power consumed in the welding machine? (b) What are the minimum and maximum values of the instantaneous consumed power?

Section 27.5 Transformers

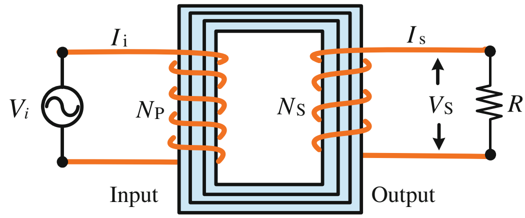
(37) A transformer has $N_P = 500$ turns in the primary coil and $N_S = 60$ turns in the secondary coil. (a) What kind of transformer is this? (b) By what factor does this transformer change the ac voltage and current?

(38) The transformer of a neon lamp operates from an alternating source of 220 V. The lamp requires 10 kV to operate. What is the ratio of the secondary to primary turns of the transformer coils?

(39) The input ac current of a 90-W transformer is 2 A and the output ac voltage is 12 V. (a) What kind of transformer is this? (b) By what factor does this transformer change the ac voltage?

(40) An ac source provides an output peak V_i and current peak I_i when connected to the primary coil of a transformer. The transformer has N_P turns in the primary coil and N_S turns in the secondary coil. A circuit of resistance R is connected to the transformer, see Fig. 27.25. What is the equivalent resistance of the circuit?

Fig. 27.25 See Exercise (40)



(41) Figure 27.26a shows the transmission of electric power from the generator of a power plant to a town, part b of the figure shows a simple equivalent circuit of part a, where the current and voltages are in rms values. The value of the ac voltage reaching the town is $V_{town} = 50$ kV with average power $\overline{P}_{town} = 80$ MW via a transmission line of resistance $R = 3.5\ \Omega$ from the generator. (a) Find the emf of the generator $\mathcal{E}_{gen}$. (b) What is the value of the average power generated by the power plant and the fraction of the lost generated power through the transmission line?

Section 27.6 Induced Electric Fields

(42) A positive charge $q = +20\ \mu$C is located on the right part of the circumference of the circle of Fig. 27.13b, where $R = 5$ cm. The magnetic field starts to decrease at a rate of -0.01 T/s. Find the initial force exerted on the charge.

(43) Repeat exercise 42 when the charge is $q = -20\ \mu$C and is located at $r = 25$ cm outside the region of the change of the magnetic field.

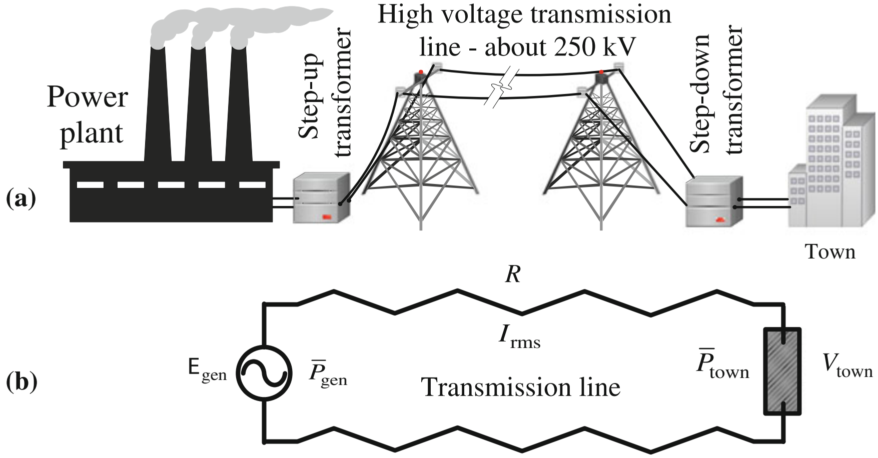


Fig. 27.26 See Exercise (41)

(44) A long solenoid has 500 turns per meter and a radius of 2 cm. The current in the solenoid is increasing at a rate of 2 A/s. What is the magnitude of the induced electric field at a point 1 cm from the axis of the solenoid?

(45) A long solenoid has a circular cross section of radius R. The magnetic field inside the solenoid is uniform and increasing at a rate dB/dt. The magnetic field outside the solenoid is essentially zero. (a) What is the rate of change of magnetic flux through a circle of radius $r < R$, normal to the axis of the solenoid and center coinciding with solenoid axis? (b) What is the magnitude of the induced electric field at a distance $r < R$ from the solenoid axis? (c) What is the magnitude of the induced electric field at a distance $r > R$ from the solenoid axis? (d) What is the magnitude of the induced emf in a circular loop of radius $r < R$, normal to the axis of the solenoid and center coinciding with the solenoid axis? (e) What is the magnitude of the induced emf if the radius in part d is R and $2R$?

Inductance, Oscillating Circuits, and AC Circuits

28

An emf produced by a physical source (like a battery) is quite different from that produced by changing magnetic flux. In this chapter, we study how an emf is induced as a result of a changing magnetic flux produced by the circuit itself or by a nearby circuit.

28.1 Self-Inductance

First, consider the loop shown in Fig. 28.1, which contains a battery of emf $\mathcal{E}$, a resistor of resistance R, and a switch S. When the switch is closed, the current does not jump immediately from 0 to its final value $\mathcal{E}/R$. Actually, Faraday's law and Lenz's law can be used as follows to describe what happens in this loop:

- *As the current I in the loop increases with time, the magnetic flux through the loop also increases*
- *The increasing magnetic flux creates an induced emf $\mathcal{E}_L$ in the loop*
- *The induced emf and induced current are opposing the battery's emf $\mathcal{E}$ and its current I*
- *This process causes a gradual increase in the battery's current to its final value $\mathcal{E}/R$*

This effect is called **self-induction** because it arises from the loop itself. The induced emf $\mathcal{E}_L$ is called a **self-induced emf,** or ***back emf***.

Consider a coil wound on a cylindrical core with a current passing through the coil, as shown in Fig. 28.2a. As a result, a magnetic field directed to the right is set up inside the coil. Faraday's law can be used to describe the effect of increasing or decreasing the current.

H. A. Radi and J. O. Rasmussen, *Principles of Physics*,
Undergraduate Lecture Notes in Physics, DOI: 10.1007/978-3-642-23026-4_28,

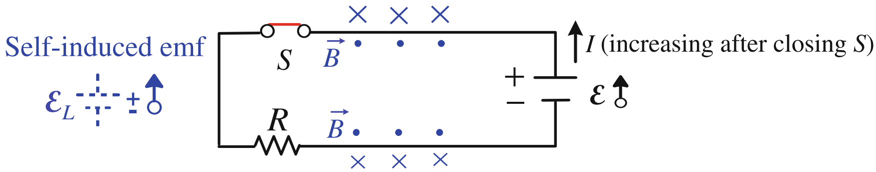


Fig. 28.1 After closing the switch S, the current increases and so does the magnetic flux through the loop. A self-induced emf $\mathcal{E}_L$ (the dashed battery) is created in the loop opposite to the battery's emf $\mathcal{E}$

When the current I in the coil *increases* with time as in Fig. 28.2b:

- *The magnetic flux through the coil also increases*
- *The increasing magnetic flux creates an induced emf $\mathcal{E}_L$ in the coil*
- *The induced emf and its induced current are opposing the emf and current generated by the source*

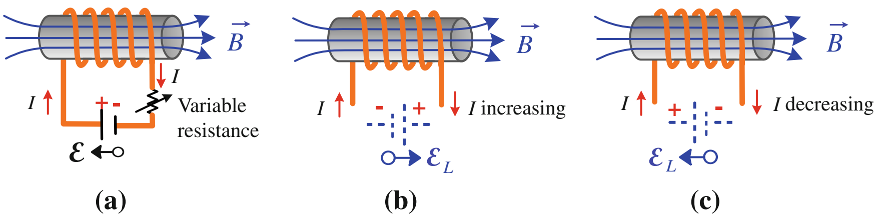


Fig. 28.2 (a) A current in the coil creates a magnetic field to the right. (b) When the current increases, the increasing magnetic flux creates a self-induced emf $\mathcal{E}_L$ (the dashed battery) opposite to the emf of the source. (c) The polarity of the self-induced emf $\mathcal{E}_L$ reverses if the current decreases

When the current I in the coil *decreases* with time as in Fig. 28.2c:

- *The magnetic flux through the coil also decreases*
- *The decreasing magnetic flux creates an induced emf $\mathcal{E}_L$ in the coil*
- *The induced emf and its induced current are in the same direction as the emf and current I generated by the source*

Spotlight

In conclusion, the self-induction of a coil prevents the current in the coil from increasing or decreasing instantaneously.

The magnetic flux Φ_B in a loop is proportional to the magnetic field $\vec{B}$, which in turn is proportional to the current I that produced this magnetic field in the loop. Therefore, $\Phi_B \propto I$. The proportionality constant between Φ_B and I is called the **self-inductance** (or inductance) of the loop, and is denoted by the symbol L. Thus:

$$\Phi_B = LI \tag{28.1}$$

According to both Faraday's law and Lenz's law, $\mathcal{E} = -d\Phi_B/dt$, the self-induced emf $\mathcal{E}_L$ is given by:

$$\boxed{\mathcal{E}_L = -L\frac{dI}{dt}} \tag{28.2}$$

The negative sign indicates that the self-induced emf $\mathcal{E}_L$ is a back emf that opposes the change in current. The SI unit of self-inductance is the henry (abbreviated by H). Thus:

$$1\,\text{H} = 1\frac{\text{V}\cdot\text{s}}{\text{A}} = 1\,\Omega\text{.s} \quad \text{or} \quad 1\,\text{H} = 1\frac{\text{Wb}}{\text{A}} = 1\,\text{T.m}^2/\text{A}$$

Comparing Eq. 28.2 with Faraday's law for N loops, we find:

$$\boxed{L = \frac{N\Phi_B}{I}} \tag{28.3}$$

When used in circuits, elements with large values of L are referred to as **inductors** and denoted by the circuit symbol .

Example 28.1

The current in a coil changes according to the formula: $I = 0.5 - 0.2\,t$ where t is in seconds and I is in amperes. Experimental measurements show that a self-induced emf of 0.5 mV is produced across the terminals of the coil. What is the self-inductance of the coil?

Solution: From the current-time relation, we have:

$$\frac{dI}{dt} = \frac{d}{dt}(0.5 - 0.2\,t) = -0.2\,\text{A/s}$$

which means that I decreases with time. Given that $\mathcal{E}_L = 0.5\,\text{mV}$, then:

$$L = -\frac{\mathcal{E}_L}{dI/dt} = -\frac{0.5 \times 10^{-3}\,\text{V}}{-0.2\,\text{A/s}} = 2.5 \times 10^{-3}\,\text{H} = 2.5\,\text{mH}$$

28.2 Mutual Inductance

We have seen that a changing current in a coil causes a changing magnetic flux, which in turn creates a self-induced emf $\mathcal{E}_L = -L dI/dt$ in the coil. If two coils exist in close proximity, then a changing current in one coil will result in a changing flux through the second coil; hence, there will be an induced emf in this second coil.

Figure 28.3 shows two coils with a common axis near each other. Coil 1 has N_1 turns and carries a current I_1 and coil 2 has N_2 turns. Part of the flux established by I_1 in coil 1 passes each turn of coil 2 and is represented by Φ_{21}. The total linkage flux through coil 2 is thus $N_2\Phi_{21}$. Since this total flux is directly proportional to the current I_1, then in analogy to Eq. 28.3, we define the **mutual inductance** M_{21} of coil 2 as:

$$M_{21} = \frac{N_2\Phi_{21}}{I_1} \tag{28.4}$$

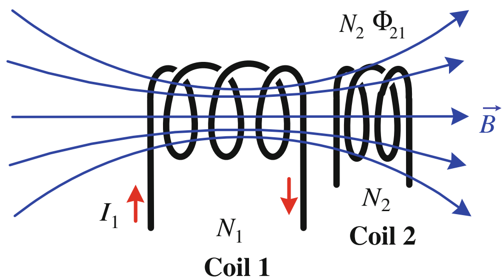


Fig. 28.3 If the current I_1 in coil 1 changes, a mutual induced emf $\mathcal{E}_2 = -M dI_1/dt$ will be established in a nearby coil 2

It is clear that the mutual inductance depends on the geometry of both coils. We can rearrange the last equation as:

$$M_{21} I_1 = N_2\Phi_{21} \tag{28.5}$$

If the current I_1 varies with time, then:

$$M_{21}\frac{dI_1}{dt} = N_2\frac{d\Phi_{21}}{dt} \tag{28.6}$$

According to Faraday's law, apart from a sign, the right side is just the emf $\mathcal{E}_{M2}$ established in coil 2. Thus, the mutual emf in coil 2 is:

$$\mathcal{E}_{M2} = -M_{21}\frac{dI_1}{dt} \tag{28.7}$$

The preceding steps can be repeated to show that if a current I_2 in coil 2 varies with time, then the mutual emf in coil 1 will be:

$$\mathcal{E}_{M1} = -M_{12}\frac{dI_2}{dt} \tag{28.8}$$

Thus, the emf produced in either coil is proportional to the rate of change of current in the other coil. It can be shown that $M_{12} = M_{21} = M$, i.e. the two coils have a single *mutual inductance M*. Then, we have:

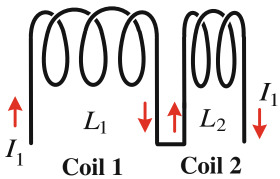
$$\boxed{\mathcal{E}_{M2} = -M\frac{dI_1}{dt}} \quad \text{and} \quad \boxed{\mathcal{E}_{M1} = -M\frac{dI_2}{dt}} \tag{28.9}$$

It is obvious that the unit of mutual inductance is the henry.

Example 28.2

Two nearby coils 1 and 2 have self-inductances $L_1 = 0.2\,\text{mH}$ and $L_2 = 0.1\,\text{mH}$, respectively. When the current in coil 1 changes at a rate of 4 A/s, it is found that a mutual emf of 10 mV is induced in coil 2. (a) What is the mutual inductance of the combination? (b) If the two coils are joined as shown in Fig. 28.4, find the total induced emf of the combination.

Fig. 28.4



Solution: (a) Using the magnitude values of Eq. 28.9, we get:

$$\mathcal{E}_{M2} = M\frac{dI_1}{dt} \quad \Rightarrow \quad M = \frac{\mathcal{E}_{M2}}{dI_1/dt} = \frac{10 \times 10^{-3}\,\text{V}}{4\,\text{A/s}} = 2.5 \times 10^{-3}\,\text{H}$$

(b) The mutually-induced and self-induced emfs in the two coils are in the same direction. Since the current is the same in both coils, then:

$$\begin{aligned}
\mathcal{E}_{\text{tot}} &= \mathcal{E}_{L1} + \mathcal{E}_{L2} + \mathcal{E}_{M1} + \mathcal{E}_{M2} = -(L_1 + L_2 + 2M)\frac{dI_1}{dt} \\
&= -(0.2 \times 10^{-3}\,\text{H} + 0.1 \times 10^{-3}\,\text{H} + 2 \times 2.5 \times 10^{-3}\,\text{H})(4\,\text{A/s}) \\
&= -21.2 \times 10^{-3}\,\text{V}
\end{aligned}$$

28.3 Energy Stored in an Inductor

It is necessary to do work in order to overcome the back emf in any conductor when the current is changing. Because energy is not dissipated by an ohm-less inductor, we may consider any work done as energy stored in the inductor in the form of a magnetic field.

Consider the circuit of Fig. 28.5a in which a battery of emf $\mathcal{E}$ is connected to an ohm-less inductor in series with an open switch S.

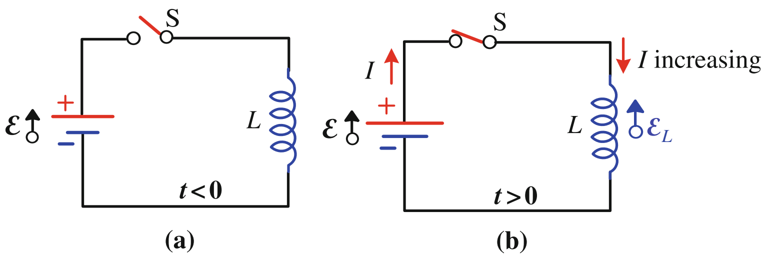


Fig. 28.5 (a) A battery, inductor, and open switch at $t < 0$. (b) The circuit at time $t > 0$ when I is increasing after S is closed at $t = 0$

When S is closed at time $t = 0$, the current I begins to increase, see Fig. 28.5b, and a back emf that opposes I is induced in the inductor; thus the induced emf is against $\mathcal{E}$. The loop theorem yields:

$$\mathcal{E} - L\frac{dI}{dt} = 0 \quad \Rightarrow \quad \mathcal{E} = L\frac{dI}{dt} \quad \text{(At time } t > 0\text{)} \tag{28.10}$$

Multiplying by the instantaneous value of the current, we get:

$$I\mathcal{E} = LI\frac{dI}{dt} \tag{28.11}$$

Since $I\mathcal{E}$ is the rate at which energy is being supplied by the battery, then $P = LI\,dI/dt$ must represent the rate at which energy is being stored in the inductor. The energy U_L stored is the conductor is thus:

$$U_L = \int_0^t P\,dt = \int_0^t LI\frac{dI}{dt}dt = L\int_0^I I\,dI = \frac{1}{2}LI^2$$

$$\boxed{U_L = \frac{1}{2}LI^2} \tag{28.12}$$

This is the energy stored in the magnetic field of an inductor when the current is I. We can prove that $u_B = B^2/2\mu_\circ$ is the energy density, see Eq. 23.38.

28.4 The *L* – *R* Circuit

We can place inductors in circuits to prevent the current in these circuits from increasing or decreasing instantaneously.

Figure 28.6 displays a circuit containing a battery of emf $\mathcal{E}$, a resistor of resistance R, an inductor of inductance L, and a switch S. Assume that the switch S is open for $t < 0$, as shown in Fig. 28.6a.

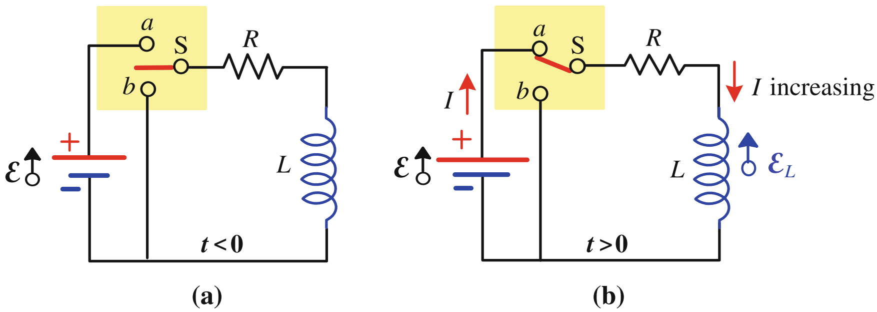


Fig. 28.6 (a) The circuit diagram of an inductor in series with a resistor, an open switch, and a battery. (b) The circuit diagram at time $t > 0$ when I is increasing after the switch S is connected to a at $t = 0$

Connecting Switch *S* to Position *a*

Once the switch S is connected to position a at time $t = 0$, the current begins to increase, and a back emf that opposes the increasing current is induced in the inductor; thus $\mathcal{E}_L$, is against the battery's emf.

Assume that the current in the circuit at time $t > 0$ is I, as shown in Fig. 28.6b. Applying Kirchhoff's loop rule and traversing the circuit clockwise, we get:

$$\mathcal{E} - IR - L\frac{dI}{dt} = 0 \quad \text{(At time } t > 0\text{)} \tag{28.13}$$

Using the condition $I = 0$ at $t = 0$ and changing the variables by letting $x = \mathcal{E}/R - I$, it is left as a problem to show that the solution of (28.13) is:

$$I = \frac{\mathcal{E}}{R}(1 - e^{-t/\tau}), \quad \tau = \frac{L}{R} \tag{28.14}$$

This relation shows that $I = 0$ at $t = 0$ and $I = \mathcal{E}/R$ at $t = \infty$, as expected.

We can generalize these results as follows:

Spotlight

Initially, an inductor acts to oppose the increase in the current, but after a long time it acts like an ordinary conductive connecting wire.

If we take the first time derivative of Eq. 28.14, we get:

$$\frac{dI}{dt} = \frac{\mathcal{E}}{L} e^{-t/\tau}, \quad \tau = \frac{L}{R} \tag{28.15}$$

Thus, dI/dt is a maximum and is equal to $\mathcal{E}/L$ at $t = 0$ and falls off exponentially to zero as t approaches infinity.

The quantity L/R in the exponents of Eqs. 28.14 and 28.15 is called the **time constant** τ of the circuit. Therefore, the quantity $\tau = L/R$ represents the time interval during which the current in the circuit increases to $(1 - e^{-1}) = 0.632 \equiv 63.2 \sim 63\%$ of its final value $\mathcal{E}/R$. Similarly, after a time interval τ, the current rate dI/dt decreases to $e^{-1} = 0.368 \equiv 36.8 \sim 37\%$ of its initial value $\mathcal{E}/L$.

Figure 28.7 shows the variation of the circuit current I and the current rate dI/dt as a function of time.

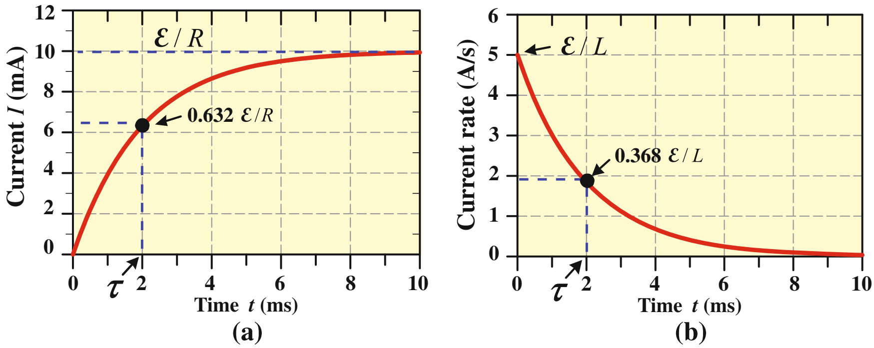


Fig. 28.7 (a) A *plot* of the current I in the circuit of Fig. 28.6 versus time t. (b) A *plot* of current rate dI/dt in the circuit of Fig. 28.6 versus time t. The two *curves* of parts (a) and (b) are based on the values $R = 200\,\Omega$, $L = 0.4\,\text{H}$, and $\mathcal{E} = 2\,\text{V}$

Connecting *S* is Changed to Position *b* After Being Connected to *a*

Suppose that the switch S has been connected to position a first for a long period to allow the current to reach to its equilibrium value $\mathcal{E}/R$, as shown in Fig. 28.8a.

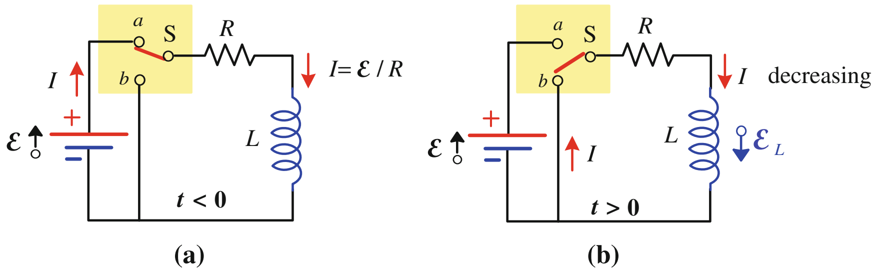


Fig. 28.8 (a) The circuit diagram with a saturated current of constant value $I = \mathcal{E}/R$. (b) The circuit diagram at time $t > 0$ when I is decreasing after the switch S is connected to position b at $t = 0$

At $t = 0$, the switch S is disconnected from position a and instantaneously connected to position b. At this moment, the current begins to decrease, and a self-induced emf that opposes the decreasing current is induced in the inductor; thus $\mathcal{E}_L$ is clockwise.

Assume that the current in the circuit at time $t > 0$ is I, as show in Fig. 28.8b. With the switch in position b, the battery's emf $\mathcal{E}$ is removed and Eq. 28.13 reduces to:

$$0 - IR - L\frac{dI}{dt} = 0 \quad (\text{At time } t > 0) \tag{28.16}$$

It is left as an exercise to show that the solution of Eq. 28.16 is:

$$\boxed{I = \frac{\mathcal{E}}{R}e^{-t/\tau}, \quad \tau = \frac{L}{R}} \tag{28.17}$$

This current falls exponentially from $\mathcal{E}/R$ to zero. In a time interval $\tau = L/R$, the current in the circuit declines to $e^{-1} = 0.368 \sim 37\%$ of its initial value $\mathcal{E}/R$. Note that the direction of the current is the same when the switch is connected to position a or position b.

Example 28.3

In Fig. 28.6, let $R = 12\,\Omega$, $\mathcal{E} = 24$ V, and $L = 60$ mH. (a) Find the time constant of the circuit. (b) After closing S at $t = 0$, find the current in the circuit at $t = 2$ ms. (c) Find the energy stored in the inductor when the current is 1.5 A.

Solution: (a) The time constant of the R–L circuit is given by:

$$\tau = \frac{L}{R} = \frac{60 \times 10^{-3}\,\mathrm{H}}{12\,\Omega} = 5 \times 10^{-3}\,\mathrm{s} = 5\,\mathrm{ms}$$

(b) Using Eq. 28.14, we find the current in the circuit at $t = 2$ ms:

$$I = \frac{\mathcal{E}}{R}(1 - e^{-t/\tau}) = \frac{24\,\mathrm{V}}{12\,\Omega}(1 - e^{-0.4}) = 0.659\,\mathrm{A}$$

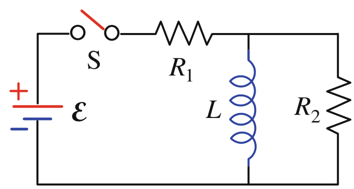
(c) Using Eq. 28.12, the energy stored when $I = 1.5$ A is:

$$U_B = \frac{1}{2}LI^2 = 0.5(60 \times 10^{-3}\,\mathrm{H})(1.5\,\mathrm{A})^2 = 67.5 \times 10^{-3}\,\mathrm{J} = 67.5\,\mathrm{mJ}$$

Example 28.4

In Fig. 28.9, determine the initial current at $t = 0$ (when the switch is closed) and the final current at $t \to \infty$ (when the switch is closed for a long time).

Fig. 28.9



Solution: When the switch is closed at $t = 0$, the current in the inductance coil cannot change instantaneously. Therefore, at $t = 0$ the current from the battery must flow through R_1 and R_2 only. Hence:

$$I(\text{at}\,t{=}0) = \frac{\mathcal{E}}{R_1 + R_2}$$

When the switch is closed for a long time, the current in the inductor is not changing and therefore the induced emf is zero. In this case the inductor (which has zero resistance) is short circuited with R_2. Thus, there is no current in R_2 and the same current will flow through R_1 and L. Hence:

$$I(\text{at}\, t \to \infty) = \frac{\mathcal{E}}{R_1}$$

28.5 The Oscillating $L-C$ Circuit

In Fig. 28.10a, assume that the switch S is open when the capacitor has an initial charge Q (the maximum charge), and hence the total energy stored in the capacitor is $U = Q^2/2C$. In addition, we assume a resistance-free, non-radiating LC circuit.

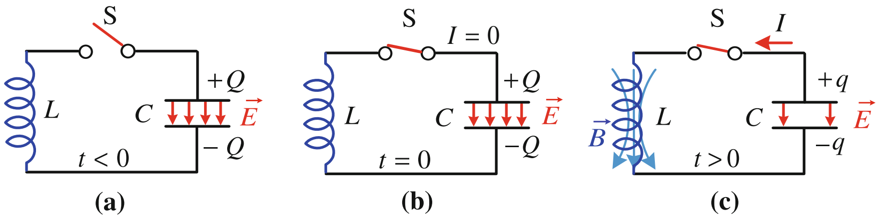


Fig. 28.10 (a) Before starting ($t < 0$), switch S is open and the capacitor has an initial maximum charge Q. (b) When the switch is closed at $t = 0$, the current in the circuit is zero and the charge begins decreasing. (c) For $t > 0$, the charge has decreased to $q(t)$ and the current in the circuit $I = -dq(t)/dt$ establishes a magnetic field $\vec{B}(t)$ in the inductor

When the switch is closed at $t = 0$, the current I in the circuit is zero, and the capacitor starts to discharge through the inductor, see Fig. 28.10b.

At $t > 0$, represented in Fig. 28.10c, the charge on the capacitor decreases to q (where $q < Q$) and the rate at which the charges leave (or enter) the capacitor is equal to the current I in the circuit. This current establishes a magnetic field $\vec{B}$ in the inductor.

When the capacitor is fully discharged, the current at this time reaches its maximum value $I_{\max}$, and all of the energy is now stored in the inductor. The current continues in the same direction, but it is now decreasing in magnitude and the capacitor is being charged with polarity opposite to the initial polarity. This is followed by another discharge until the circuit returns to its original state. In a system with zero resistance the energy continues to oscillate between the capacitor and inductor indefinitely. We refer to this as an "*oscillating circuit*".

At an arbitrary time t, the current in the circuit is related to the decreasing charge q by $I = -dq/dt$. In addition, at time t the sum of the stored energy in the capacitor U_C and the inductor U_L must equal the initial energy stored in the capacitor U at $t = 0$. Thus:

$$\begin{aligned} U_C + U_L &= U \\ \frac{q^2}{2C} + \frac{1}{2}LI^2 &= \frac{Q^2}{2C} \end{aligned} \tag{28.18}$$

Differentiating this equation with respect to the time t and noting that $dI/dt = -d^2q/dt^2$, we can reach the following differential equations:

$$\frac{d^2q}{dt^2} + \frac{1}{LC}q = 0$$

or:

$$\frac{d^2q}{dt^2} + \omega^2 q = 0 \tag{28.19}$$

where:

$$\boxed{\omega = \frac{1}{\sqrt{LC}}} \tag{28.20}$$

This equation is analogous to a block-spring system given by Eq. 14.8. By consideration of the initial conditions, $q = Q$ and $I = 0$ at $t = 0$, we find that Eq. 28.19 has a solution given by:

$$\boxed{q = Q\cos(\omega t)} \tag{28.21}$$

where ω is the angular frequency of the oscillations, which is a frequency solely depends on the capacitance C and inductance L of the circuit. The **undamped** frequency and period of the oscillations are given by:

$$\boxed{f = \frac{\omega}{2\pi} = \frac{1}{2\pi\sqrt{LC}}}\,, \quad \boxed{T = \frac{1}{f} = 2\pi\sqrt{LC}} \tag{28.22}$$

The current as a function of time is therefore given by:

$$I = -\frac{dq}{dt} = Q\omega\sin(\omega t) = I_{\max}\sin(\omega t) \tag{28.23}$$

where the maximum current and charge are related by $I_{\max} = Q\omega$. The general solution of (28.19) is $q = Q\cos(\omega t + \phi)$, with ϕ is a phase angle.

Figure 28.11 displays the electric and magnetic fields as well as current of a complete cycle of an $L - C$ circuit.

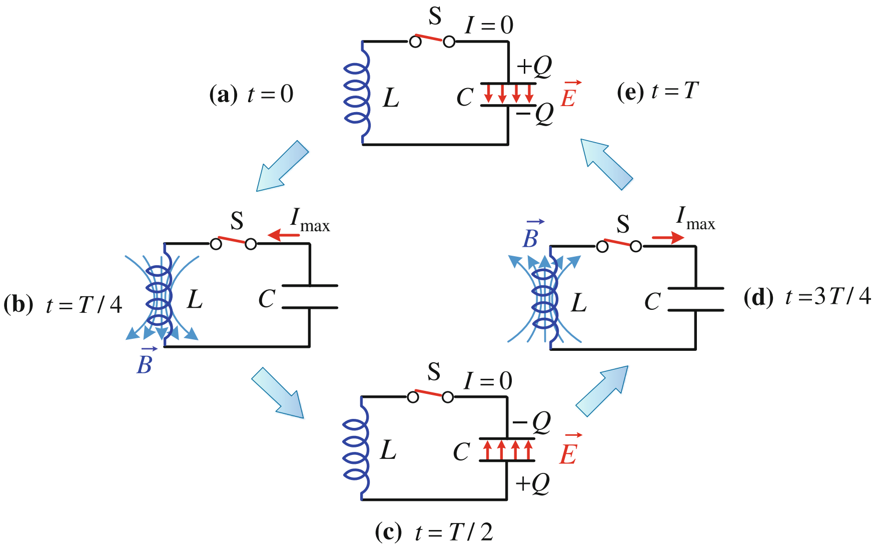
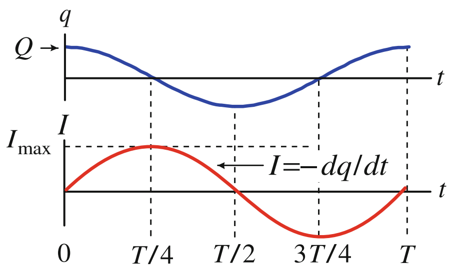


Fig. 28.11 (a) At $t=0$, all of the energy is stored as an electric energy $Q^2/2C$ in the capacitor. (b) At $t=T/4$, all of the energy is stored as a magnetic energy $\frac{1}{2}LI_{max}^2$ in the inductor. (c) At $t=T/2$, all of the energy is stored again in the capacitor, but with opposite polarity. (d) At $t=3T/4$, all of the energy is stored as a magnetic energy $\frac{1}{2}LI_{max}^2$ in the inductor. (e) At $t=T$, the circuit returns to its initial configuration at $t=0$

For a complete cycle, Fig. 28.12 displays both the charge and current versus time for a resistanceless nonradiating LC circuit.

Fig. 28.12 Variation of q and I as a function of time t



Example 28.5

When S_1 is closed and S_2 is opened, as shown in the left part of Fig. 28.13, a capacitor of capacitance $C=7.1$ pF is charged from a battery of emf $\mathcal{E}=12$ V. Switch S_1 is then opened, and the capacitor remains charged. Switch S_2 is then closed, so the capacitor is connected directly to an inductor of inductance $L = 3.56\,\text{mH}$, as shown in the right part of Fig. 28.13. (a) Find the frequency of oscillation of the circuit. (b) Find both the maximum charge on the capacitor and current in the circuit. (c) Find the charge and current as a function of time.

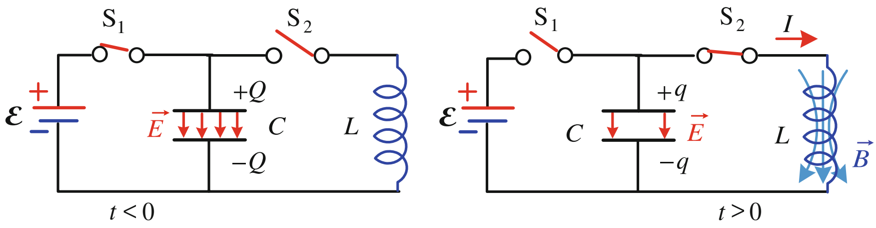


Fig. 28.13

Solution: (a) Eq. 28.22 gives for the frequency of the oscillating circuit as:

$$f = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{(3.56\times 10^{-3}\,\text{H})(7.1\times 10^{-12}\,\text{F})}} = 1\times 10^{6}\,\text{Hz} = 1\,\text{MHz}$$

(b) Using the relation $Q = C\Delta V = C\mathcal{E}$, we get the maximum charge as:

$$Q = C\mathcal{E} = (7.1\times 10^{-12}\,\text{F})(12\,\text{V}) = 8.52\times 10^{-11}\,\text{C} = 85.2\,\text{pC}$$

From Eq. 28.23 and the relation $\omega = 2\pi f$, the maximum current is given in terms of the maximum charge as:

$$I_{\text{max}} = Q\omega = (8.52\times 10^{-11}\,\text{C})(2\pi\times 10^{6}\,\text{s}^{-1}) = 5.35\times 10^{-4}\,\text{A}$$

(c) Using Eqs. 28.21 and 28.23, the charge and current as a function of time are given as follows:

$$q = Q\cos(\omega t) = (8.52\times 10^{-11}\,\text{C})\cos[(2\pi\times 10^{6}\,\text{s}^{-1})\,t]$$

$$I = I_{\text{max}}\sin\omega t = (5.35\times 10^{-4}\,\text{A})\sin[(2\pi\times 10^{6}\,\text{s}^{-1})\,t]$$

28.6 The *L – R – C* Circuit

Now we consider a realistic L–C circuit with some resistance R. In Fig. 28.14a, the switch S is open and the capacitor has an initial charge Q. This is the maximum charge that the capacitor can store. Consequently, the total energy stored in the capacitor is $U = Q^2/2C$.

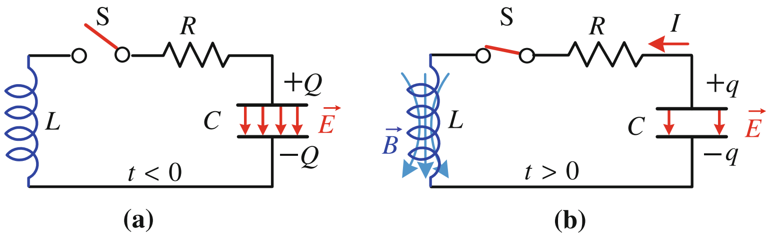


Fig. 28.14 (a) Before starting ($t<0$), the switch S is open and the capacitor has an initial maximum charge Q. (b) After the switch is closed ($t>0$), the charge has decreased to $q(t)$ and the current in the circuit $I=-dq(t)/dt$ establishes a magnetic field $\vec{B}$ in the inductor

The switch S is closed at $t=0$. Figure 28.14b represents the case at $t>0$. In this figure, the charge on the capacitor decreases to q (where $q<Q$) and the rate at which the charges leave (or enter) the capacitor is equal to the current I in the circuit. This current establishes a magnetic field $\vec{B}$ in the inductor. Applying Kirchhoff's loop rule and traversing the circuit counterclockwise (starting from the capacitor's negative plate), we get:

$$\frac{q}{C} - IR - L\frac{dI}{dt} = 0 \quad (\text{At time } t>0) \tag{28.24}$$

Since $I=-dq/dt$, this equation becomes:

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = 0 \tag{28.25}$$

This second-order differential equation in the variable q has the same form as the damped harmonic oscillator Eq. 14.25:

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + k_{\mathrm{H}}x = 0$$

Therefore, by comparison, Eq. 28.25 has the solution:

$$q(t) = Qe^{-(R/2L)t}\cos(\omega_d\, t + \phi) \tag{28.26}$$

where the angular frequency of the damped oscillation ω_d is given by:

$$\omega_d = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} \quad \left(\omega_d \xrightarrow[\text{or } R\to 0]{R\ll 2\sqrt{1/LC}} \sqrt{\frac{1}{LC}} = \omega\right) \tag{28.27}$$

If the resistance R is relatively small, the circuit oscillates, but with damped oscillations. We refer to this as an **underdamped** circuit, see Fig. 28.15. If we increase R, the oscillations die out more rapidly. When R reaches a certain critical value $R_c = \sqrt{4L/C}$, the circuit does not oscillate and it is said to be **critically damped**, see Fig. 28.15. When R is greater than R_c, the circuit is said to be **overdamped**.

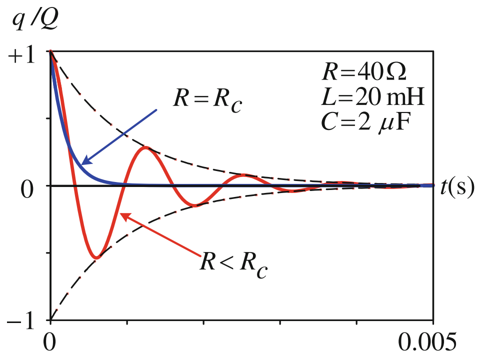


Fig. 28.15 When $\phi = 0$, the figure shows an underdamped circuit with $R < R_c$ (*red curve*) and critically damped circuit with $R = R_c$ (*blue curve*)

Example 28.6

In the circuit of Fig. 28.14, take $R = 40\,\Omega$, $L = 20\,\text{mH}$, and $C = 2\,\mu\text{F}$. (a) Show that this circuit oscillates. (b) Determine the frequency of the circuit. (c) When $\phi = 0$ and $t > 0$, find the first three times at which the cosine term of Eq. 28.26 becomes ∓ 1 and then find the ratio q/Q at these times. (d) What resistance R is required to make this circuit oscillate with one-half the undamped frequency?

Solution: (a) We first calculate the critical value R_c as follows:

$$R_c = \sqrt{\frac{4L}{C}} = \sqrt{\frac{4(20 \times 10^{-3}\,\text{H})}{2 \times 10^{-6}\,\text{F}}} = 200\,\Omega$$

Since $R < R_c$ is satisfied when $R = 40\,\Omega$, then this circuit oscillates.

(b) We use the relation $\omega_d = 2\pi f_d$ to find the frequency as follows:

$$f_d = \frac{\omega_d}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

$$= \frac{1}{2\pi}\sqrt{\frac{1}{(20 \times 10^{-3}\,\text{H})(2 \times 10^{-6}\,\text{F})} - \frac{(40\,\Omega)^2}{4(20 \times 10^{-3}\,\text{H})^2}} = 779.7\,\text{Hz}$$

(c) For $t>0$, the term $\cos(\omega_d\, t)$ equals ∓ 1 when $\omega_d\, t = \pi,\ 2\pi, \ldots,$ Thus, $t_n = n\pi/\omega_d = n/2f_d$, $(n = 1, 2, \ldots)$. Therefore, $\cos(\omega_d\, t) = \mp 1$ at:

$$t_1 = 0.641\,\text{ms}, \quad t_2 = 1.28\,\text{ms}, \quad t_3 = 1.924\,\text{ms}, \ \ldots$$

For these calculated times, the ratio q_n/Q is:

$$\frac{q_n}{Q} = e^{-\left(\frac{R}{2L}\right)t_n} = e^{-\left(\frac{40\,\Omega}{2(20\times 10^{-3}\,\text{H})}\right)t_n} = e^{-(1000\,\text{s}^{-1})\,t_n}$$

Thus:

$$\frac{q_1}{Q} = e^{-(1000\,\text{s}^{-1})\,t_1} = e^{-(1000\,\text{s}^{-1})(0.641\times 10^{-3}\,\text{s})} = e^{-0.641} = 0.53 \quad (53\%)$$
$$\frac{q_2}{Q} = e^{-(1000\,\text{s}^{-1})\,t_2} = e^{-(1000\,\text{s}^{-1})(1.28\times 10^{-3}\,\text{s})} = e^{-1.28} = 0.28 \quad (28\%)$$
$$\frac{q_3}{Q} = e^{-(1000\,\text{s}^{-1})\,t_3} = e^{-(1000\,\text{s}^{-1})(1.924\times 10^{-3}\,\text{s})} = e^{-1.924} = 0.15 \quad (15\%)$$

(d) Using Eqs. 28.22 and 28.27, the required resistance R that makes this circuit oscillate with one-half the undamped frequency is obtained by setting $\omega_d = \omega/2$. Thus:

$$\omega_d = \tfrac{1}{2}\omega \quad \Rightarrow \quad \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} = \tfrac{1}{2}\sqrt{\frac{1}{LC}}$$

When we square both sides, we get:

$$R = \sqrt{\frac{3L}{C}} = \sqrt{\frac{3(20\times 10^{-3}\,\text{H})}{2\times 10^{-6}\,\text{F}}} = 173.2\,\Omega$$

28.7 Circuits with an ac Source

In this section, we continue studying circuits containing elements such as resistors, inductors, and capacitors, but this time connecting them to a source of alternating voltage that produces an alternating current (ac). First, we examine these electronic components individually, considering a sinusoidal voltage (see Sect. 27.3) and current (see Sect. 27.4) that can be described by:

$$v = V\sin\omega t \tag{28.28}$$

$$i = I\sin\omega t \tag{28.29}$$

In these expressions, the lowercase v and i represent the instantaneous potential difference and current, respectively. The uppercase V and I represent the peak voltage and current, respectively. The angular frequency ω is equal to 2π times the frequency f of the oscillations.

Resistors in an ac Circuit

According to Eq. 28.28, we can write the alternating voltage across any resistor as:

$$v_R = V_R \sin \omega t \tag{28.30}$$

where V_R is the peak voltage across the resistor. From Ohm's law, the instantaneous current through such a resistor is:

$$i_R = \frac{v_R}{R} = \frac{V_R}{R} \sin \omega t = I_R \sin \omega t \tag{28.31}$$

Where the peak current I_R is given by $I_R = V_R/R$. According to this result, we have:

$$V_R = I_R\, R \tag{28.32}$$

In addition, the relations between the rms and peak values of the current and voltage; Ohm's law; and the average power delivered to a resistor as a heat, are all given Sect. 27.4 as follows:

$$I_{\text{rms}} = \frac{I}{\sqrt{2}} = 0.707\, I \quad \text{and} \quad V_{\text{rms}} = \frac{V}{\sqrt{2}} = 0.707\, V \tag{28.33}$$

$$V_{\text{rms}} = I_{\text{rms}}\, R \tag{28.34}$$

$$\overline{P} = I_{\text{rms}}\, V_{\text{rms}} = I_{\text{rms}}^2\, R = V_{\text{rms}}^2/R \tag{28.35}$$

Because the current i_R is zero when the voltage v_R is zero and the current reaches a peak when the voltage reaches a peak, they are both proportional to $\sin \omega t$ and we say that the current and voltage are **in phase**, see Fig. 28.16. That is:

Spotlight
The voltage across a resistor is in phase with current.

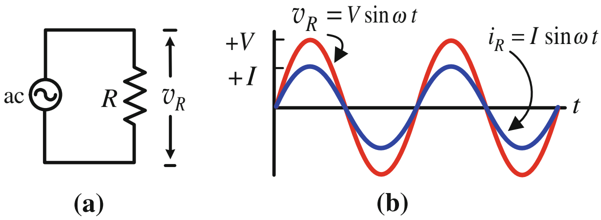


Fig. 28.16 (a) A resistor connected to an ac source. (b) Alternating voltage v_R (*red*) across R is in phase with alternating current i_R (*blue*)

Inductors in an ac Circuit

We replace the resistor in Fig. 28.16a with a pure inductor of inductance L and zero resistance as shown in Fig. 28.17a. The potential difference across the inductor can be written as:

$$v_L = V_L \sin \omega t \tag{28.36}$$

where V_L is the peak voltage. The voltage applied to the inductor will be equal to the back induced emf generated in the inductor by the changing alternating current. Thus:

$$v_L = L\frac{di_L}{dt} \tag{28.37}$$

If we combine the last two equations, we get:

$$\frac{di_L}{dt} = \frac{V_L}{L} \sin \omega t \tag{28.38}$$

To find the current, we integrate the last equation to get:

$$i_L = \frac{V_L}{L} \int \sin \omega t \, dt = -\frac{V_L}{\omega L} \cos \omega t \tag{28.39}$$

For reasons of symmetry of notation, we use trigonometric identities to replace $-\cos \omega t$ with a phase-shifted sine as follows:

$$-\cos \omega t = \sin(\omega t - \pi/2)$$

With this change, the current in the inductor becomes:

$$i_L = \frac{V_L}{\omega L} \sin(\omega t - \pi/2) = I_L \sin(\omega t - \pi/2) \tag{28.40}$$

where $I_L = V_L/\omega L$ is the peak current. Figure 28.17b shows the variation of v_L and i_L as a function of time. It is clear from the figure and Eqs. 28.36 and 28.40 that the voltage v_L and the current i_L are out of phase by a quarter cycle, which is equivalent to $\pi/2$ radians or 90°. That is:

> Soptlight
> The voltage across an inductor *leads* the current by 90°.

In other words, the current in an inductor reaches its peak quarter a cycle later than the voltage.

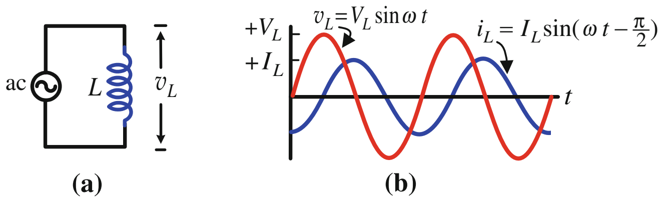


Fig. 28.17 (a) An inductor connected to an ac source. (b) Alternating voltage v_L (*red*) leads alternating current i_L (*blue*) by quarter a cycle or 90°

Because the current and voltage are out of phase by 90°, the average power dissipated is zero. Energy from the source is delivered to the inductor and stored as an increasing magnetic field between its turns. As the field decreases, the energy returns to the source. That is:

$$\overline{P}_L = 0 \tag{28.41}$$

Capacitors in an ac Circuit

Figure 28.18a shows a capacitor connected to a generator with an alternating emf. The applied potential difference of the ac source must equal the applied potential difference across the capacitor. Thus:

$$v_C = V_C \sin \omega t \tag{28.42}$$

where V_C is the peak voltage across the capacitor. According to the definition of capacitance, the instantaneous charge on the capacitor plates is:

$$q_C = C v_C = C V_C \sin \omega t \tag{28.43}$$

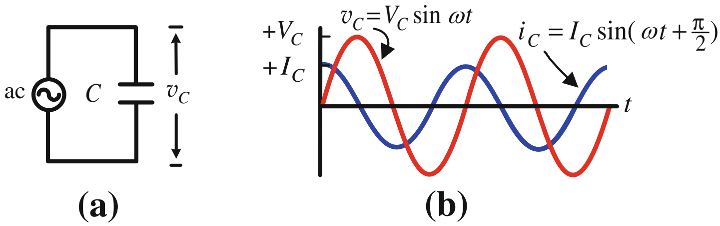


Fig. 28.18 (a) A capacitor connected to an ac source. (b) Alternating voltage v_C (*red*) lags alternating current i_C (*blue*) by quarter a cycle or 90°

The current in the circuit at any instant is thus:

$$i_C = \frac{dq_C}{dt} = \omega C V_C \cos \omega t \tag{28.44}$$

Again, for reasons of symmetry of notation, we use trigonometric identities to replace $\cos \omega t$ with a phase-shifted sine as follows:

$$\cos \omega t = \sin(\omega t + \pi/2)$$

With this change, the current in the capacitor becomes:

$$i_C = \omega\, C V_C \sin(\omega t + \pi/2) = I_C \sin(\omega t + \pi/2) \tag{28.45}$$

where $I_C = \omega\, C V_C$ is the peak current in the circuit. Figure 28.18b shows the variation of v_C and i_C as a function of time. It is clear from the figure and Eqs. 28.42 and 28.45 that the voltage v_C and the current i_C are out of phase by a quarter cycle, which is equivalent to $\pi/2$ radians or 90°. That is:

> Spotlight
> The voltage across a capacitor *lags* the current by 90°.

In other words, the current reaches its peak quarter a cycle ahead of the voltage.

Because the current and voltage are out of phase by 90°, the average power dissipated is zero. This is similar to an inductor. Energy from the source is delivered to the capacitor and stored as an increasing electric field between its plates. As the electric field decreases, the energy returns to the source. That is:

$$\overline{P}_C = 0 \tag{28.46}$$

Reactance and Phasors in an ac Circuit

We notice from Eqs. 28.40 and 28.45 that $V_L = I_L\,(\omega L)$ for inductors and $V_C = I_C(1/\omega C)$ for capacitors. As we search for additional symmetry in ac circuits, we introduce the two quantities X_L and X_C, called the **inductive reactance** of the inductor and the **capacitive reactance** of the capacitor, respectively, as follows:

$$X_L = \omega L \tag{28.47}$$

$$X_C = \frac{1}{\omega C} \tag{28.48}$$

where both quantities have the units of ohms. Just like the relation $V_R = I_R\,R$ for ohmic resistors, we can write similar relations for inductors and capacitors as follows:

$$V_L = I_L\,X_L \quad \text{(Peak or rms values)} \tag{28.49}$$

$$V_C = I_C\,X_C \quad \text{(Peak or rms values)} \tag{28.50}$$

Note that because the peak values of the current and voltage are not reached at the same time, these equations are valid only for peak or rms values and not for any other instant.

Note also that:

- The inductive reactance $X_L = \omega L$ is large for high frequencies f and/or larger inductances L. Consequently, the greater the value of X_L, the more it impedes the flow of charge and the smaller the current experienced in the inductor.
- The capacitive reactance $X_C = 1/\omega C$ is large for smaller frequencies f and/or smaller capacitances C. Consequently, the greater the value of X_C, the more it impedes the flow of charge and the smaller the current experienced in the capacitor. (For dc circuits $\omega = 0$ and $X_C = \infty$, and hence a capacitor does not pass dc current).

To simplify the analysis of complicated ac circuits, we use a graphical tool called the *phasor diagram*.

We define a phasor that represents a time-varying quantity to be a vector having the following properties:

- *Length*: Its length is proportional to the peak value of the variable
- *Angular frequency*: It rotates counterclockwise around the origin with the same angular frequency of the variable

- *Rotation angle*: Its rotation angle with respect to the horizontal axis is equal to the phase of the alternating quantity
- *Projection*: Its projection onto the vertical axis represents the instantaneous value of the variable

The time-varying quantities of v_R and i_R for a resistor, v_L and i_L for inductor, and v_C and i_C for capacitor are represented graphically in Fig. 28.19.

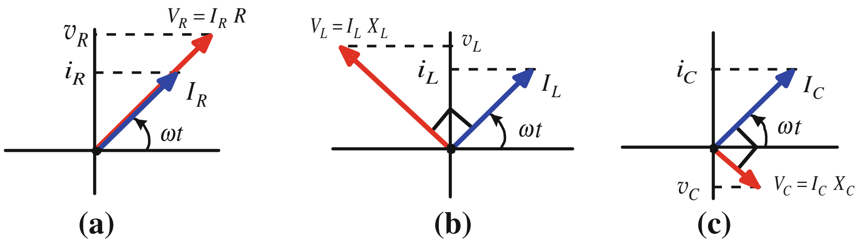


Fig. 28.19 Phasor diagrams. (a) For resistors, the voltage and current are in phase. (b) For inductors, the voltage leads the current by 90°. (c) For the capacitors, the voltage lags the current by 90°

Example 28.7

A coil has an inductance $L = 0.4$ H and a small resistance $R = 2\ \Omega$. Find the current in the coil when the applied voltage is: (a) 220-V dc, and (b) 220-V ac (rms) with a frequency $f = 50$ Hz.

Solution: (a) For a dc source, $\omega = 0$ and $X_L = 0$ and Ohm's law gives:

$$I_R = \frac{V_R}{R} = \frac{220\ \text{V}}{2\ \Omega} = 110\ \text{A}$$

(b) The value of the inductive reactance to be:

$$X_L = \omega L = 2\pi f L = 2\pi(50\ \text{cycle/s})(0.4\ \text{H}) = 126\ \Omega$$

Since X_L is much greater than R, we ignore its effect and use Eq. 28.49 to calculate the current as follows:

$$I_{\text{rms}} = \frac{V_{\text{rms}}}{X_L} = \frac{220\ \text{V}}{126\ \Omega} = 1.75\ \text{A}$$

Example 28.8

A capacitor has a capacitance $C = 2\,\mu\text{F}$. Find the current in the capacitor if you apply a 50 Hz and 220-V ac (rms) voltage.

Solution: The value of the capacitive reactance is:

$$X_C = \frac{1}{\omega C} = \frac{1}{2\pi f C} = \frac{1}{2\pi\,(50\,\text{cycle/s})\,(2\times 10^{-6}\,\text{F})} = 1{,}592\,\Omega$$

We use Eq. 28.50 to calculate the current as follows:

$$I_{\text{rms}} = \frac{V_{\text{rms}}}{X_C} = \frac{220\,\text{V}}{1{,}592\,\Omega} = 0.14\,\text{A}$$

28.8 $L-R-C$ Series in an ac Circuit

Figure 28.20 shows an ac source connected to a circuit containing three elements in series: a resistor of resistance R, an inductor of inductance L, and a capacitor of capacitance C. Let us find the effect of R, X_L, and X_C on the peak current and the relation of the phase between the voltage and the current.

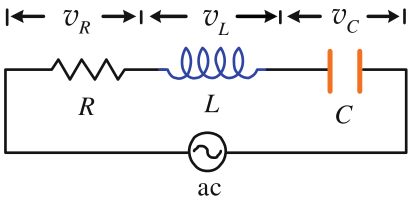


Fig. 28.20 An ac source connected in series with a resistor R, an inductor L, and a capacitor C

Since all elements in the circuit are in series, the current at any point in the circuit must be the same at any time. We choose the current i, at any time t to be:

$$i = I \sin \omega t$$

The peak currents in all elements are equal, i.e. $I_R = I_L = I_C = I$. Consequently, the peak voltages across the resistor, inductor, and capacitor are $V_R = IR$, $V_L = IX_L$, and $V_C = IX_C$, respectively. Based on the preceding section, the phases between the voltages across the elements and the current are summarized as follows:

1. The voltage across the resistor v_R is in phase with the current i
2. The voltage across the inductor v_L leads the current i by 90°
3. The voltage across the capacitors v_C lags the current i by 90°

We can express the relationships of these results as follows:

$$v_R = IR\sin\omega t = V_R\sin\omega t \tag{28.51}$$

$$v_L = IX_L\sin(\omega t + \pi/2) = V_L\sin(\omega t + \pi/2) \tag{28.52}$$

$$v_C = IX_C\sin(\omega t - \pi/2) = V_C\sin(\omega t - \pi/2) \tag{28.53}$$

The instantaneous voltage across the three elements equals the sum:

$$v = v_R + v_L + v_C \tag{28.54}$$

Although this analytical method is correct and leads to the final answer, it is actually simpler to use the phasor diagram. Figure 28.21a shows the phasor diagram for the three elements, based on the phasor diagram displayed in Fig. 28.19. To find the resultant phasor, we construct the difference phasor $V_L - V_C$ (assuming that the circuit is more inductive that capacitive), which is perpendicular to the phasor V_R, see Fig. 28.21b. From the Pythagorean theorem, the resultant voltage V is the hypotenuse of the right angle triangle shown in Fig. 28.21b. Thus:

$$V = \sqrt{V_R^2 + (V_L - V_C)^2} = I\sqrt{R^2 + (X_L - X_C)^2} \tag{28.55}$$

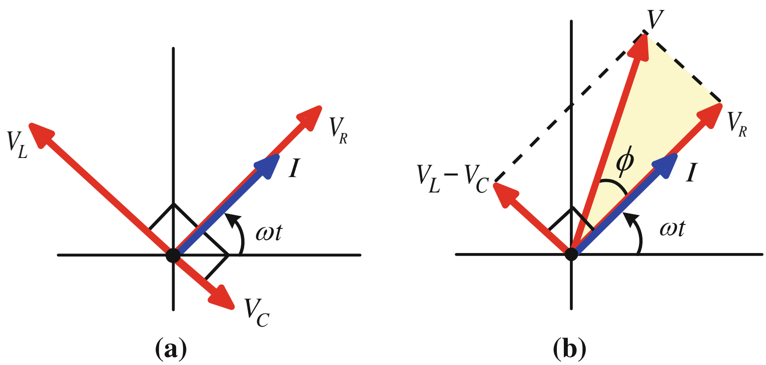


Fig. 28.21 (a) Phasor diagram for an ac source connected to a series L-R-C circuit. (b) The vector sum V of the three phasors V_R, V_L, and V_C

We define the impedance Z of an ac circuit as the ratio of the peak voltage across the circuit to the current peak in the circuit. Thus:

$$V = IZ \quad \text{or} \quad V_{\text{rms}} = I_{\text{rms}}Z \tag{28.56}$$

where

$$Z = \sqrt{R^2 + (X_L - X_C)^2} = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} \tag{28.57}$$

Eq. 28.56 is known as the impedance version of Ohm's law.

According to Fig. 28.21, the phase angle ϕ between the peak voltage V and peak current I is giving by the two relations:

$$\tan\phi = \frac{V_L - V_C}{V_R} = \frac{IX_L - IX_C}{IR} = \frac{X_L - X_C}{R} \tag{28.58}$$

or

$$\cos\phi = \frac{V_R}{V} = \frac{IR}{IZ} = \frac{R}{Z} \tag{28.59}$$

In a series L-R-C circuit, only the resistor dissipates the power. Then, the average power dissipated is given by:

$$\overline{P} = I_{\text{rms}}^2 R \tag{28.60}$$

We can use $R = Z\cos\phi$, from Eq. 28.59, and $V_{\text{rms}} = I_{\text{rms}} Z$ from Eq. 28.56 to write the average power as follows:

$$\overline{P} = I_{\text{rms}} V_{\text{rms}} \cos\phi \tag{28.61}$$

where the quantity $\cos\phi$ is called the **power factor** of the circuit. When the circuit contains only a resistor, then $\phi = 0$, $\cos\phi = 1$, and consequently $\overline{P} = I_{\text{rms}} V_{\text{rms}}$. However, when the circuit does not contain a resistor but contains either an inductor or capacitor, $\phi = +90°$ and $\phi = -90°$, respectively, then no power is dissipated since $\cos\phi = 0$.

Example 28.9

An ac source of 220-V (rms) and angular frequency $\omega = 314$ rad/s is connected to a series L-R-C circuit, where $R = 35\,\Omega$, $L = 100$ mH, and $C = 650\,\mu$F. Find: (a) the inductive reactance, the capacitive reactance, and the impedance of the circuit, (b) the peak and rms current, (c) the peak voltage, the instantaneous voltage, and the rms voltage across each element, (d) the phase angle ϕ and the average power dissipated in the circuit.

Solution: (a) The reactance of the inductor and capacitor are:

$$X_L = \omega L = (314\,\text{rad/s})(100 \times 10^{-3}\,\text{H}) = 31.4\,\Omega$$
$$X_C = \frac{1}{\omega C} = \frac{1}{(314\,\text{rad.s})(650 \times 10^{-6}\,\mu\text{F})} = 4.9\,\Omega$$

The impedance of the circuit is thus:

$$Z = \sqrt{R^2 + (X_L - X_C)^2} = \sqrt{(35\,\Omega)^2 + (31.4\,\Omega - 4.9\,\Omega)^2} = 43.9\,\Omega$$

(b) Using the impedance form of Ohm's law, Eq. 28.56, we get:

$$I_{\text{rms}} = \frac{V_{\text{rms}}}{Z} = \frac{220\,\text{V}}{43.9\,\Omega} = 5.01\,A \simeq 5\,\text{A} \quad \text{and} \quad I = \sqrt{2} I_{\text{rms}} = 7.09\,\text{A}$$

(c) The peak and instantaneous voltages across each element are:

$$V_{\text{R}} = IR = (7.09\,\text{A})(35\,\Omega) = 248.2\,\text{V}$$
$$V_{\text{L}} = IX_L = (7.09\,\text{A})(31.4\,\Omega) = 222.6\,\text{V}$$
$$V_{\text{C}} = IX_C = (7.09\,\text{A})(4.9\,\Omega) = 34.7\,\text{V}$$
$$v_{\text{R}} = (248.2\,\text{V})\sin(314\,t)$$
$$v_{\text{L}} = (222.6\,\text{V})\sin(314\,t + \pi/2)$$
$$v_{\text{C}} = (34.7\,\text{V})\sin(314\,t - \pi/2)$$

The rms voltage across each element is:

$$(V_{\text{R}})_{\text{rms}} = I_{\text{rms}}\,R = (5.01\,\text{A})(35\,\Omega) = 175\,\text{V}$$
$$(V_{\text{L}})_{\text{rms}} = I_{\text{rms}}\,X_L = (5.01\,\text{A})(31.4\,\Omega) = 157\,\text{V}$$
$$(V_{\text{C}})_{\text{rms}} = I_{\text{rms}}\,X_C = (5.01\,\text{A})(4.9\,\Omega) = 24.5\,\text{V}$$

Notice that the peak and rms voltages across the elements do not add to equal the source voltage, 311 V (peak value) or 220 V (rms). This is because the different voltages are not in phase with each other. At a particular instant, one voltage across a particular element may be negative in order to compensate for the large positive voltage on the other, but the instantaneous voltages must add up to the source voltage. On the other hand, the rms voltages are always positive by definition.

(d) The phase angle ϕ is given by Eq. 28.59 as:

$$\cos\phi = \frac{R}{Z} = \frac{35\,\Omega}{43.9\,\Omega} = 0.797 \quad \Rightarrow \quad \phi = \cos^{-1}(0.797) = 37.1^\circ$$

The average power dissipated is given by Eq. 28.61 as:

$$\overline{P} = I_{\text{rms}}\,V_{\text{rms}}\cos\phi = (5\,\text{A})(220\,\text{V})(0.797) = 876.7\,\text{W}$$

28.9 Resonance in *L* – *R* – *C* Series Circuit

As we saw in Eq. 28.56, the rms current in an $L\!-\!R\!-\!C$ series circuit depends on the source's frequency f. This can be rewritten as:

$$I_{\text{rms}} = \frac{V_{\text{rms}}}{Z} = \frac{V_{\text{rms}}}{\sqrt{R^2 + (X_L - X_C)^2}} \tag{28.62}$$

Such a circuit is said to be in **resonance** when the current is maximum at a certain frequency. The maximum current occurs when the impedance is minimum. This condition can happen when $X_L\!-\!X_C = 0$ at a certain frequency $\omega_\circ$, i.e. $X_L\!-\!X_C = \omega_\circ L - 1/\omega_\circ C = 0$. Therefore:

$$\omega_\circ = \frac{1}{\sqrt{LC}} \tag{28.63}$$

This frequency corresponds to the natural frequency of oscillation of an L – C circuit as introduced in Sect. 28.5.

Figure 28.22 shows the variation of I_{rms} as a function of the angular frequency ω for a particular $L\!-\!R\!-\!C$ series circuit. The current I_{rms} is maximum at $\omega = \omega_\circ$ and decreases when $\omega < \omega_\circ$ and also when $\omega > \omega_\circ$. At resonance, the energy transferred from the source to the circuit is maximum and increases for small values of R.

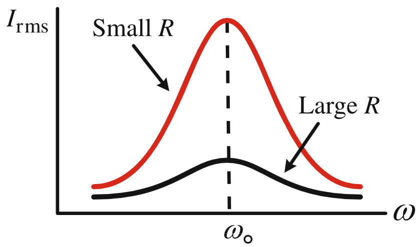


Fig. 28.22 Current in an $L\!-\!R\!-\!C$ circuit as a function of the angular frequency ω. At $\omega = \omega_\circ$ resonance occurs and the current is maximum

Resonance is used in Radio and TV sets for tuning to a station. By changing C of the $L\!-\!R\!-\!C$ circuit, the resonance frequency of the circuit matches a particular received EMW and the current flow is enhanced.

28.10 Exercises

Section 28.1 Self-Inductance

(1) The current in a coil of self-inductance $L = 75$ mH changes uniformly from zero to 1 A in 50 ms. What is the magnitude of the induced emf?

(2) The magnitude of the average induced emf in a coil is 5 V when its current changes from −30 to +120 mA during a period of 30 ms. What is the self-inductance (or inductance) of the coil?

(3) When a steady current $I = 10$ A passes through a solenoid of $N = 25$ turns, the magnetic flux through each turn is $\Phi_B = 10^{-2}$ Wb. What is the inductance of the coil?

(4) A solenoid has $N = 200$ turns, a length $\ell = 4$ cm, and a cross-sectional area $A = 10^{-2}\,\text{m}^2$. What is the inductance of the solenoid?

(5) An air-filled cylindrical inductor of cross-sectional area $A = 5 \times 10^{-3}\,\text{m}^2$ has a length $\ell = 4$ cm. How many turns of wire are required such that the inductor achieves an inductance $L = 125$ mH?

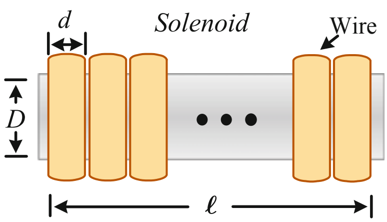
(6) If the core of the inductor of exercise 5 is filled with iron of relative permeability $K_m = \mu_m/\mu_\circ = 1{,}500$, how many turns are needed to obtain the same inductance?

(7) A solenoid of length $\ell = 20$ cm has $N = 500$ windings around an iron core of cross-sectional area $A = 2 \times 10^{-4}\,\text{m}^2$ and relative permeability 500. (a) What is the inductance of the solenoid? (b) What is the average emf induced in the solenoid when its current decreases from 1.8 to 0.5 A in a period of 20 ms?

(8) A steady current $I = 5$ A passes through a coil of $N = 400$ turns and creates a magnetic flux $\Phi_B = 10^{-3}$ Wb through each turn of the coil. (a) Find the average emf induced in the coil when the current drops to zero in 40 ms. (b) What is the inductance of the coil? (c) What was the initial magnetic energy stored in the coil?

(9) A student wants to build an air-filled solenoid of inductance $L = 0.1$ H and diameter $D = 20$ cm by tightly winding one layer of insulated copper wire of diameter $d = 0.5$ mm around a plastic hollow tube, see Fig. 28.23. (a) What is the length ℓ of the solenoid? (b) What is the length of the required copper wire? (c) What will be the resistance of this wire if the resistivity of copper is $\rho = 1.68 \times 10^{-8}\,\Omega$.m?

Fig. 28.23 See Exercise (9)



(10) Two inductors having inductances $L_1 = 0.1$ H and $L_2 = 0.2$ H are assumed to be well separated. What is the equivalent self-inductance L_{eq} between terminals a and b when the two inductors are placed in: (a) series, and (b) parallel, see Fig. 28.24?

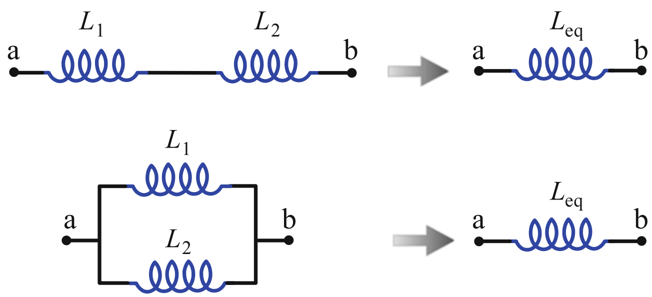


Fig. 28.24 See Exercise (10)

Section 28.2 Mutual Inductance

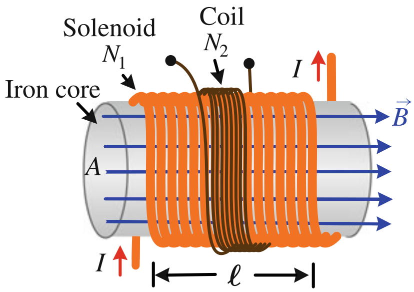
(11) When the current in a coil changes at a rate $dI/dt = 2$ A/s, an emf of 5 mV is induced in a nearby coil. What is the mutual inductance of the combination?

(12) The primary current in a transformer changes at a rate of 3 A/s. What is the induced emf in the secondary coil if the mutual inductance between the primary and secondary coils is 0.4 H?

(13) Primary and secondary coils have a common cylindrical iron core to allow for a common value of magnetic flux. A magnetic flux of 5×10^{-3} Wb is established in the primary coil when the current passing through it increases from zero to 5 A. What is the mutual inductance of the two coils if the secondary coil is an open circuit and has 20 loops?

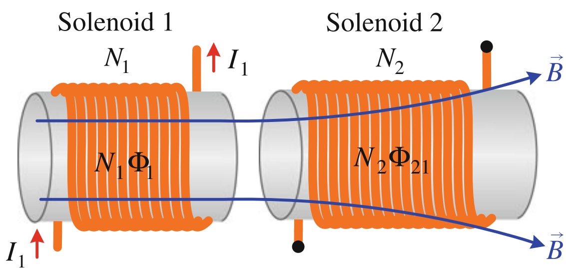
(14) A solenoid of length $\ell = 1.5$ m containing $N_1 = 500$ turns is wound around an iron core of cross-sectional area $A = 3 \times 10^{-3}$ m^2 and relative permeability $K_m = \mu_m/\mu_\circ = 2{,}100$. A second coil containing $N_2 = 40$ turns is wrapped around the solenoid such that the flux from the solenoid passes through the second coil, see Fig. 28.25. The current in the solenoid drops from 10 A to zero in 40 ms. (a) What is the mutual inductance of the combination? (b) What is the emf induced in the second coil?

Fig. 28.25 See Exercise (14)



(15) Two solenoids are close to each other and share the same cylindrical axle, see Fig. 28.26. The first solenoid has $N_1 = 250$ turns and the second solenoid has $N_2 = 500$ turns. A current $I_1 = 5$ A produces an internal magnetic flux per turn $\Phi_1 = 350\,\mu$Wb in the first solenoid and an external magnetic flux per turn $\Phi_{21} = 10\,\mu$Wb in the second solenoid. (a) What is the self-inductance of the first solenoid? (b) What is the mutual inductance of the two solenoids? (c) What is the emf induced in the second solenoid when the current in the first solenoid increases at a rate $dI_1/dt = 0.25$ A/s?

Fig. 28.26 See Exercise (15)



(16) Two inductors having self-inductances L_1 and L_2 and mutual inductance M_s when connected in series and M_p when connected in parallel, as shown in Fig. 28.27. Find the equivalent self-inductance L_{eq} of the system in both the series and parallel cases.

Section 28.3 Energy Stored in an Inductor

(17) An air-filled solenoid has length $\ell = 20$ cm and cross-sectional area $A = 10^{-4}$ m^2. The magnetic field inside the solenoid is uniform and has the value $B = 0.2$ T while the field outside the solenoid is very small (i.e. negligible).

(a) Find the magnetic energy density inside the solenoid. (b) How much magnetic energy is stored in this field?

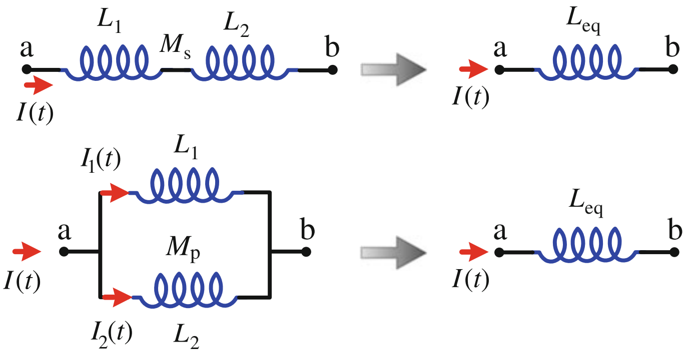


Fig. 28.27 See Exercise (16)

(18) An air-filled solenoid has $N = 500$ turns and carries a current $I = 1.5$ A in order to produce a magnetic flux per turn $\Phi_B = 3 \times 10^{-4}$ Wb. What is the energy stored in the magnetic field of the solenoid?

(19) An air-core solenoid has $N = 300$ turns, a length $\ell = 15$ cm, and a cross-sectional area $A = 10^{-4}$ m^2. How much magnetic energy is stored in its magnetic field when the current in the solenoid is $I = 0.5$ A?

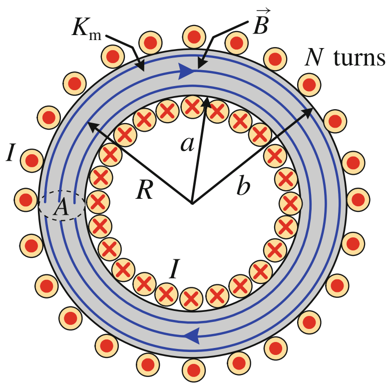
(20) Typical large experimental values of magnetic and electric fields that are used in laboratories are $B_{\text{large}} = 2$ T and $E_{\text{large}} = 10^4$ V/m. (a) Find and compare the energy density for each field. (b) Find the value of the electric field that produce the same energy as the magnetic field $B_{\text{large}} = 2$ T, and then compare this electric field with $E_{\text{breakdown}} = 3 \times 10^6$ V/m, the breakdown electric field in air.

(21) An electromagnet stores 800 J of magnetic energy when a current $I = 10$ A is used in its wires. What is the average emf induced if the current reduces to zero in 0.5 s?

(22) A loop of wire of radius $R = 30$ cm carries a current $I = 10$ A. What is the magnetic energy density at its center?

(23) A long narrow toroid has an average circumference $2\pi R$, cross-sectional area A, number of turns N, and permeability $K_m\,\mu_\circ$, see Fig. 28.28. (a) For circles of radii $a < r < b$ use the validity of $1/r \approx 1/R$ to show that the self-inductance of the toroid's coil is given by $L = K_m\,\mu_\circ N^2 A/(2\pi R)$. (b) Show that the energy stored per unit volume in the magnetic field of the toroid is $BH/2$.

Fig. 28.28 See Exercise (23)



(24) When the toroid of exercise 23 has $K_m = 250$, $A = 2.5 \times 10^{-4}\ \text{m}^2$, $R = 0.05$ m, $N = 3000$, and current $I = 0.4$ A, find the values of: (a) the self-inductance L, (b) the energy stored in the magnetic field U_B, (c) the magnetic field B, (d) the magnetization field H, and (e) the magnetic energy density u_B.

Section 28.4 The *L* – *R* Circuit

(25) After how many multiplications of the time constant $\tau = L/R$ does the current in Fig. 28.6 reach: (a) 10%, (b) 50%, and (c) 90% of its final value?

(26) An inductor of inductance $L = 2$ H and resistance $R = 1.5\ \Omega$ is connected to a 6-V battery. (a) Find the time required for the current to rise to 80% of its final value. (b) Find the final current through the inductor.

(27) An inductor of inductance $L = 80$ mH is connected in series with a resistor of resistance $R = 4\ \text{k}\Omega$, a switch S, and a battery of emf $\mathcal{E} = 24$ V. (a) What is the time constant of the circuit? (b) How long after the switch S is closed will the current take to reach 99% of its final value? (c) Find the final current through the resistor.

(28) A circuit contains two elements, an inductor of inductance $L = 50$ mH and a resistor. When a battery is connected in series with the two elements, the current increases from zero to 80% of its maximum value in 4 ms. (a) Find the time constant of the circuit. (b) Find the resistance of the resistor.

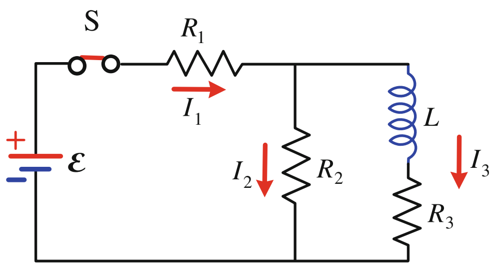
(29) When an air-core solenoid is connected to a 12-V battery, the current passing through it rises from zero to 63% of its maximum value in 5 ms. However, if the core of the solenoid is filled with iron, the current rises from zero to 63%

of its maximum value in 1.5 s. (a) What is the relative permeability K_m of this iron core? (b) What is the resistance R of the solenoid and the inductance L_{air} of the coil if the maximum current is 0.75 A?

(30) An inductor of inductance L is connected in series with a resistor of resistance R, a switch S, and a battery of emf $\mathcal{E}$. After the switch S is closed at time $t = 0$, find the following: (a) the induced emf in the inductor $\mathcal{E}_L(t)$, (b) the power output of the battery $P_{output}(t)$, (c) the power dissipated in the resistor $P_{diss}(t)$, (d) the rate at which energy is stored in the inductor $dU_B(t)/dt$, and (e) evaluate parts (a–d) when $\tau = L/R$, where τ is the time constant of the circuit.

(31) In Fig. 28.29, $\mathcal{E} = 12$ V, $R_1 = 4\,\Omega$, $R_2 = 6\,\Omega$, and $R_3 = 3\,\Omega$. Determine the currents I_1, I_2, and I_3 at: (a) $t = 0$, when S is closed, (b) $t = \infty$, when S is closed for a very long time, (c) $t = 0$, when S is reopened (after being closed for a long time in part b, and (d) after a long time following part c.

Fig. 28.29 See Exercise (31)



(32) In Fig. 28.8, take $\mathcal{E} = 9$ V, $R = 4\,\text{k}\Omega$, and $L = 40$ mH. The switch S in part a of the figure is connected to position a for a sufficient amount of time so that a steady current flows in the circuit. At $t = 0$, the switch S is disconnected from position a and connected instantaneously to position b to allow the current to decay exponentially through the resistor. (a) Find the induced emf $\mathcal{E}_L$ in the inductor as a function of time. (b) At what times does $\mathcal{E}_L(t)$ reach its maximum and minimum values?

Section 28.5 The Oscillating *L* – *C* Circuit

(33) Find the inductance of an L-C circuit that oscillates at 1 MHz when the capacitor's capacitance is 2 nF.

(34) An $L-C$ circuit has $L = 0.5$ H and $C = 8\ \mu$F. At $t = 0$, the initial charge on the capacitor is $Q = 400\ \mu$C. (a) What is the frequency of oscillation? (b) What is the maximum value of the current? (c) Represent the current as a function of time. (d) What is the maximum energy stored in the magnetic field of the inductor?

(35) When the capacitor of an $L-C$ circuit is originally charged to a potential difference of 10 V, the circuit oscillates at 1 kHz. A maximum current of 1 A is attained after quarter of a cycle and again after three quarters of a cycle. What are the values of the inductance L and capacitance C of the circuit?

(36) A radio tuner has an $L-C$ circuit of variable capacitance and a fixed inductance. The radio is tuned to a station of frequency 1.5 MHz when the tuner has a capacitance of 0.15 nF. (a) What must be the capacitance of the tuner in order to receive a station that broadcasts at a frequency of 0.8 MHz? (b) What is the inductance of the tuner?

(37) An $L-C$ circuit has an inductor of inductance $L = 20$ mH and a capacitor of capacitance $C = 2\ \mu$F. The capacitor is fully charged by a 50 V power supply and then discharged through the inductor. Use the concept of energy stored in the capacitor and inductor to find the maximum current in the oscillating circuit.

Section 28.6 The *L – R – C* Circuit

(38) In the circuit of Fig. 28.14, take $R = 8\ \Omega$, $L = 2.5$ mH, and $C = 2\ \mu$F. Does this circuit oscillate? If it does, then find the frequency of this oscillation.

(39) In the circuit of Fig. 28.14, take $R = 1.6\ \Omega$, $L = 1$ mH, and $C = 10^{-3}$ F. (a) Show that this circuit oscillates. (b) Determine the frequency of the circuit. (c) When $\phi = 0$ and $t > 0$, find the time when the cosine term of Eq. 28.26 first becomes -1 and then find the ratio q/Q at this time. (d) What resistance R is required to make this circuit oscillate with one-half the undamped frequency of the $L-C$ circuit?

(40) For the $L-R-C$ circuit of exercise 39, find the resistance R that will make the resistor dissipate only 5% of the circuit's energy in each cycle.

(41) An $L-R-C$ circuit executes a damped oscillation and its energy decreases by 2 % during each oscillation when it has a resistor of resistance $R = 10\ \Omega$. When the resistor is removed, the pure $L-C$ circuit oscillates at a frequency of 2 kHz. Find the inductance and capacitance of the circuit.

Sections 28.7 and 28.8 Circuits with ac Source—*L* – *R* – *C* Series in an ac Circuit

(42) A sinusoidal 50-cycle per second ac voltage is read to be 220 V by a voltmeter. (a) What is the peak (maximum) voltage of the source? (b) Find an equation that represents this voltage as a function of time.

(43) An ac voltage $v = (155.6\,\mathrm{V})\sin(100\,\pi\,t)$ is applied across a resistor of resistance $R = 20\,\Omega$. (a) What will be the reading of an ac voltmeter placed in parallel with the resistor? (b) What will be the reading of an ammeter placed in series with the resistor? (c) What is the frequency of the ac voltage?

(44) A coil has an inductance $L = 0.5$ mH and a small resistance $R = 1\,\Omega$. Find the current in the coil when the applied voltage is: (a) 110-V dc, and (b) 110-V (rms) with a frequency $f = 60$ Hz.

(45) Repeat exercise 44 when the coil is replaced by a capacitor of capacitance $C = 2\,\mu\mathrm{F}$.

(46) A coil has an inductance $L = 0.2$ mH and a resistance $R = 10\,\Omega$. When a voltage of 220-V (rms) with frequency $f = 50$ Hz is applied, find the impedance of the circuit and rms current in the coil.

(47) An ac source of frequency 50 Hz is connected in series with a resistor of resistance $R = 4\,\mathrm{k}\Omega$ and an inductor of inductance $L = 0.5$ H. At what frequency does the circuit's impendence double?

(48) An $R - C$ circuit of $R = 3\,\mathrm{k}\Omega$ and $C = 1\,\mu\mathrm{F}$ is connected to a 50 Hz ac source of 220 V(rms). (a) What is the impedance of the circuit? (b) What is the rms current in the circuit? (c) What is the phase angle between the current and the voltage? (d) What is the power dissipated in the circuit? (e) What are the voltmeter readings across the resistor and capacitor?

(49) In Fig. 28.30, $R = 1\,\mathrm{k}\Omega$ and $C = 1\,\mu\mathrm{F}$. Digital voltmeters are used to measure the voltages across the ac source, the resistor, and the capacitor. Their measurements are $(V_{\mathrm{rms}})_{\mathrm{ac}} = 104.80\,\mathrm{V}$, $(V_{\mathrm{rms}})_R = 31.42\,\mathrm{V}$, and $(V_{\mathrm{rms}})_C = 100.00\,\mathrm{V}$, respectively. (a) Find the frequency of the source. (b) Why is the voltage of the ac source not equal to the sum of the voltages across the resistor and the capacitor?

(50) An ac source of 110-V (rms) and frequency $f = 60$ Hz is connected to an $L - R - C$ series circuit which has a resistor of resistance $R = 8\,\Omega$, an inductor of inductive reactance $X_L = 9\,\Omega$, and capacitor of capacitive reactance $X_C = 3\,\Omega$.

(a) Find the impedance of the circuit. (b) Find the current in the circuit. (c) Find the voltage across the resistor, the inductor, and the capacitor.

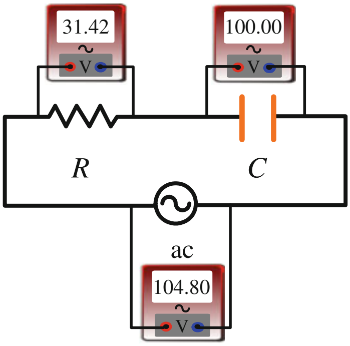


Fig. 28.30 See Exercise (49)

(51) For the circuit in exercise 50, find: (a) the inductance and capacitance of the circuit, (b) the power factor of the circuit, and (c) the power dissipated in the circuit.

(52) An ac source of 110-V (rms) and angular frequency $\omega = 377$ rad/s is connected to an L–R–C series circuit, where $R = 35\,\Omega$, $L = 100$ mH, and $C = 650\,\mu$F. Find: (a) the inductive reactance, the capacitive reactance, and the impedance of the circuit, (b) the peak and rms current, (c) the peak voltage, the instantaneous voltage, and the rms voltage across each element, (d) the phase angle ϕ and the average power dissipated in the circuit.

(53) Show that the charge q on the capacitor of the L–R–C series circuit of Fig. 28.20 has a peak value given by:

$$Q = \frac{V}{\sqrt{(\omega R)^2 + \left(\omega^2 L - \dfrac{1}{C}\right)^2}}$$

and show that $Q_{\max}$ occurs at an angular frequency ω' given by:

$$\omega' = \sqrt{\frac{1}{LC} - \frac{R^2}{2L^2}}$$

Section 28.9 Resonance in *L* – *R* – *C* Series Circuit

(54) An L–R–C series circuit has $R = 4\,\text{k}\Omega$ and $L = 6$ mH. (a) What must the value of the capacitance C be in order to produce a resonance at frequency of 40 kHz?

(b) What is the maximum rms current in the circuit when the rms voltage of the source is 150 V?

(55) In the $L–R–C$ series circuit of exercise 54, find: (a) the impedance of the inductor and capacitor, and (b) the power dissipated in the circuit.

(56) An $L–R–C$ series circuit has $R = 20\,\Omega$, $L = 0.16\,\mathrm{H}$, $C = 30\,\mu\mathrm{F}$, and an ac source of peak voltage 250 V. For a certain angular frequency, the power factor of the circuit becomes unity and the circuit consumes the maximum power. (a) Find this angular frequency. (b) Find the inductive reactance, the capacitive reactance, the impedance of the circuit. (c) Find the phase angle ϕ and the maximum current in the circuit. (d) Find the peak voltage across the resistor, the peak voltage across the inductor, and the peak voltage across the capacitor.

Conversion Factors

A

Table A.1 Length

	m	cm	km	in.	ft	mi
1 meter	1	10^2	10^{-3}	39.37	3.281	6.214×10^{-4}
1 centimeter	10^{-2}	1	10^{-5}	0.3937	3.281×10^{-2}	6.214×10^{-6}
1 kilometer	10^3	10^5	1	3.937×10^4	3.281×10^3	0.621 4
1 inch	2.540×10^{-2}	2.540	2.540×10^{-5}	1	8.333×10^{-2}	1.578×10^{-5}
1 foot	0.304 8	30.48	3.048×10^{-4}	12	1	1.894×10^{-4}
1 mile	1 609	1.609×10^5	1.609	6.336×10^4	5280	1

Table A.2 Time

	s	min	h	day	year
1 second	1	1.667×10^{-2}	2.778×10^{-4}	1.157×10^{-5}	3.169×10^{-8}
1 minute	60	1	1.667×10^{-2}	6.994×10^{-4}	1.901×10^{-6}
1 hour	3 600	60	1	4.167×10^{-2}	1.141×10^{-4}
1 day	8.640×10^4	1440	24	1	2.738×10^{-5}
1 year	3.156×10^7	5.259×10^5	8.766×10^3	365.2	1

Table A.3 Area

	m^2	cm^2	ft^2	$in.^2$
1 square meter	1	10^4	10.76	1550
1 square centimeter	10^{-4}	1	1.076×10^{-3}	0.1550
1 square foot	9.290×10^{-2}	929.0	1	144
1 square inch	6.452×10^{-4}	6.452	6.944×10^{-3}	1

Note 1 square kilometer = 247.108 acres

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Undergraduate Lecture Notes in Physics, DOI: 10.1007/978-3-642-23026-4,

Table A.4 Volume

	m^3	cm^3	L	ft^3	in.^{3}
1 cubic meter	1	10^6	1000	35.51	6.102×10^4
1 cubic centimeter	10^{-6}	1	1.000×10^{-3}	3.531×10^{-5}	6.102×10^{-2}
1 liter	1.000×10^{-3}	1000	1	3.531×10^{-2}	61.02
1 cubic foot	2.832×10^{-4}	1	28.32	1	1728
1 cubic inch	1.639×10^{-4}	16.39	1.639×10^{-2}	5.787×10^{-4}	1

Note 1 U.S. fluid gallon = 3.786 L

Table A.5 Speed

	m/s	cm/s	ft/s	mi/h	km/h
1 meter per second	1	10^2	3.281	2.237	3.6
1 centimeter per second	10^{-2}	1	3.281×10^{-2}	2.237×10^{-2}	3.6×10^{-2}
1 foot per second	0.304 8	30.48	1	0.681 8	1.097
1 mile per hour	0.447 0	44.70	1.467	1	1.609
1 kilometer per hour	0.277 8	27.78	0.9113	0.6214	1

Table A.6 Mass

	kg	g	slug	u
1 kilogram	1	10^3	6.852×10^{-2}	6.024×10^{26}
1 gram	10^{-3}	1	6.852×10^{-5}	6.024×10^{23}
1 slug	14.59	1.459×10^4	1	8.789×10^{27}
1 atomic mass unit	1.660×10^{-27}	1.660×10^{-24}	1.137×10^{-28}	1

Note 1 metric ton = 1000 kg

Table A.7 Force

	N	lb
1 newton	1	0.224 8
1 pound	4.448	1

Table A.8 Work, energy, and heat

	J	ft.lb	eV
1 joule	1	0.737 6	6.242×10^{18}
1 foot-pound	1.356	1	8.464×10^{18}
1 electron volt	1.602×10^{-19}	1.182×10^{-19}	1
1 calorie	4.186	3.087	2.613×10^{19}
1 British thermal unit	1.055×10^3	7.779×10^2	6.585×10^{21}
1 kilowatt hour	3.600×10^6	2.655×10^6	2.247×10^{25}

Table A.8 Continued

	cal	Btu	kWh
1 joule	0.2389	9.481×10^{-4}	2.778×10^{-7}
1 foot-pound	0.3239	1.285×10^{-3}	3.766×10^{-7}
1 electron volt	3.827×10^{-20}	1.519×10^{-22}	4.450×10^{-26}
1 calorie	1	3.968×10^{-3}	1.163×10^{-6}
1 British thermal unit	2.520×10^{2}	1	2.930×10^{-4}
1 kilowatt hour	8.601×10^{5}	3.413×10^{2}	1

Table A.9 Pressure

	Pa	atm	cm Hg	lb/in.^{2}	lb/ft^2
1 pascal	1	9.869×10^{-6}	7.501×10^{-4}	1.450×10^{-4}	2.089×10^{-2}
1 atmosphere	1.013×10^{5}	1	76	14.70	2.116×10^{3}
1 centimeter mercury[a]	1.333×10^{3}	1.316×10^{-2}	1	0.194 3	27.85
1 pound per square inch	6.895×10^{3}	6.805×10^{-2}	5.171	1	144
1 pound per square foot	47.88	4.725×10^{-4}	3.591×10^{-2}	6.944×10^{-3}	1

[a] At 0°C and at a location where the free-fall acceleration has its "standard" value, 9.806 65 m/s^2

Basic Rules and Formulas

B

Scientific Notation

When numbers in powers of 10 are expressed in scientific notation are being multiplied or divided, the following rules are very useful:

$$\boxed{\begin{aligned} &10^m \times 10^n = 10^{m+n} \\ &\frac{10^m}{10^n} = 10^{m-n} \end{aligned}} \tag{B.1}$$

When powers of a given quantity x are multiplied or divided, the following rules hold:

$$\boxed{\begin{aligned} &x^m \times x^n = x^{m+n} \\ &\frac{x^m}{x^n} = x^{m-n} \end{aligned}} \tag{B.2}$$

The Distance Between Two Points

In Fig. B.1, $P(x_1, y_1)$ and $Q(x_2, y_2)$ are two different points in the (x, y) plane. As we move from point P to point Q, the coordinates x and y change by amounts that we denote by Δx and Δy (read "delta x" and "delta y"). Thus:

$$\boxed{\begin{aligned} &\text{The change in } x = \Delta x = x_2 - x_1 \\ &\text{The change in } y = \Delta y = y_2 - y_1 \end{aligned}} \tag{B.3}$$

One can calculate the distance between the two points P and Q from the theorem of Pythagoras in geometry such that:

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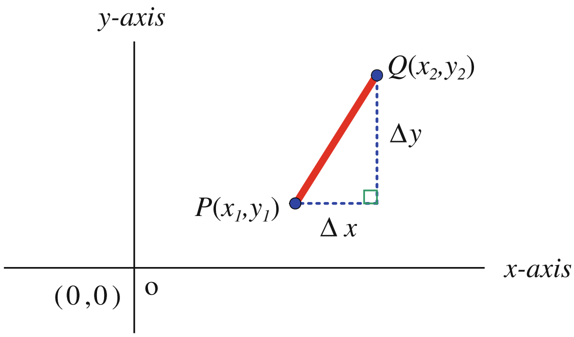
Undergraduate Lecture Notes in Physics, DOI: 10.1007/978-3-642-23026-4,

$$\boxed{\text{The distance PQ} = \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}} \tag{B.4}$$

Fig. B.1

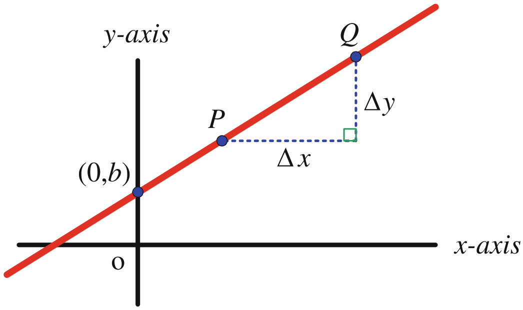


Slope and the Equation of a Straight Line

The slope of a line (usually given the symbol m) on which two points P and Q lie, is defined as the ratio $\Delta y/\Delta x$, see Fig. B.2. Thus:

$$\boxed{\text{slope} \equiv m = \frac{\Delta y}{\Delta x}} \tag{B.5}$$

Fig. B.2



Using this basic geometric property, we can find the equation of a straight line in terms of a general point (x, y), and the y intercept b of the line with the y-axis and the slope m of the line, as follows:

$$\boxed{y = mx + b} \tag{B.6}$$

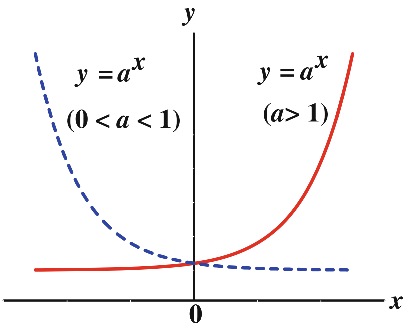
Exponential and Logarithmic Functions

An *exponential function with base a* has the following forms:

$$y = a^x \qquad (a > 0, a \neq 1) \tag{B.7}$$

where x is a variable and a is a constant, i.e., the exponential function is a constant raised to a variable power. Exponential functions are continuous on the interval $(-\infty, \infty)$ with a range $[0, \infty]$ and have one of the basic two shapes shown in Fig. B.3.

Fig. B.3



Moreover, some algebraic properties of exponential functions are:

$$\begin{aligned}
&1.\ a^x \times a^y = a^{x+y}\\
&2.\ (a\,b)^x = a^x \times b^x\\
&3.\ (a^x)^y = a^{x\,y}\\
&4.\ \frac{a^x}{a^y} = a^{x-y}\\
&5.\ a^{x/q} = \sqrt[q]{a^x} = (\sqrt[q]{a})^x,\ (q \text{ integer and } q > 0)\\
&6.\ a^0 = 1,\ (\text{for every positive real number } a)
\end{aligned} \tag{B.8}$$

The logarithmic function to the base a of x is introduced as the inverse of the exponential function $x = a^y$. That is, $y = \log_a x$ is the power (or exponent) to which a must be raised to produce x, so that:

$$y = \log_a x \quad (\text{is equivalent to}) \quad x = a^y \tag{B.9}$$

Additionally, some algebraic properties of logarithmic functions for any base a are as follows:

$$\begin{array}{lll} 1.\ \log_a(xy) = \log_a(x) + \log_a(y) & \text{Product property} \\ 2.\ \log_a(x/y) = \log_a(x) - \log_a(y) & \text{Quotient property} \\ 3.\ \log_a(x^r) = r\log_a(x) & \text{Power property} \\ 4.\ \log_a(1/x) = -\log_a(x) & \text{Reciprocal property} \end{array} \tag{B.10}$$

Historically, the first logarithmic base was 10, called the *common logarithm*. For such logarithms it is usual to suppress explicit reference to the base and write $\log x$ rather than $\log_{10} x$. However, the most widely used logarithms in applications are the *natural logarithms*, which have an irrational base denoted by the letter e in honor of L. Euler, who first suggested its application to logarithms. This constant's value to six decimal places is:

$$e \approx 2.718282 \tag{B.11}$$

This number arises as the horizontal asymptote of the graph of the equation $y = (1 + 1/x)^x$. Therefore, as $x \to \pm\infty$ this allows us to express e as a limit and e^x as an infinite sum such that:

$$e = \lim_{x \to \pm\infty} \left(1 + \frac{1}{x}\right)^x = \lim_{x \to 0} (1 + x)^{\frac{1}{x}} \tag{B.12}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \tag{B.13}$$

where the symbol $n!$ is read as "n factorial" and by definition $1! = 1$, $0! = 1$, and $n!$ are given by:

$$n! = n \times (n-1) \times (n-2) \ldots \times 3 \times 2 \times 1 \tag{B.14}$$

Both expressions (B.11) and (B.12) are sometimes taken to be the definition of the number e. Thus, $\log_e x$ is the natural logarithm to the base e of x, and it is usually denoted by $\ln x$, so that:

$$\ln x \equiv \log_e x \tag{B.15}$$

and thus:

$$y = e^x \quad \text{(is equivalent to)} \quad \ln y = x \tag{B.16}$$

The exponential function $f(x) = e^x$ is called the *natural exponential function*. To simplify the typography, this function is sometimes written as $\exp(x)$, that is $\exp(x) \equiv e^x$. As an example, Table B.1 displays some special cases of the last relation.

Table B.1 Some exponential and logarithmic functions

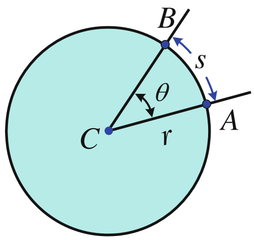
$y = e^x$	$1 = e^0$	$e = e^1$	$1/e = e^{-1}$	$e^x = e^x$
$\ln y = x$	$\ln\ 1 = 0$	$\ln\ e = 1$	$\ln(1/e) = -1$	$\ln\ e^x = x$

Radian Measures

The arc length s of a circular arc, see Fig. B.4, which is part of a circle of radius r is related to the radian measure θ of the angle ACB (measured in radians) by the relation:

$$\frac{s}{r} = \theta \quad \text{or} \quad s = r\theta \quad \text{(radian measure)} \tag{B.17}$$

Fig. B.4



Since the circumference of a unit circle is 2π and one complete revolution of a circle is $360°$, then the relation between revolutions, degrees, and radians is given by:

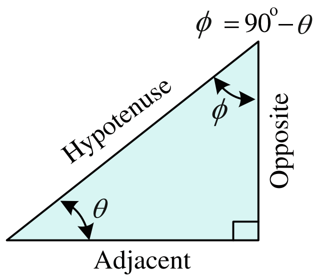
$$\begin{aligned} &1\ \text{rev} = 360° = 2\pi\ \text{rad} \quad \Rightarrow \quad \pi\ \text{rad} = 180° \\ &1° = \frac{\pi}{180}\text{rad} \approx 0.02\ \text{rad} \quad \text{and} \quad 1\ \text{rad} = \frac{180}{\pi}\ \text{deg} \approx 57.3° \end{aligned} \tag{B.18}$$

The Six Basic Trigonometric Functions

For an acute angle θ in a right-angled triangle, see Fig. B.5, we define the following six basic trigonometric functions:

Sine	$\sin\theta = \dfrac{\text{opp}}{\text{hyp}}$	Cosecant	$\csc\theta = \dfrac{\text{hyp}}{\text{opp}}$
Cosine	$\cos\theta = \dfrac{\text{adj}}{\text{hyp}}$	Secant	$\sec\theta = \dfrac{\text{hyp}}{\text{adj}}$
Tangent	$\tan\theta = \dfrac{\text{opp}}{\text{adj}}$	Cotangent	$\cot\theta = \dfrac{\text{adj}}{\text{opp}}$

Fig. B.5



To extend this definition to obtuse and negative angles, we place the angle in the *standard position* in a circle of radius r and define the trigonometric functions in terms of the point $P(x, y)$ where the angle's terminal ray intersects the circle, see Fig. B.6. Therefore, we get the following relations:

Sine	$\sin\theta = \dfrac{y}{r}$	Cosecant	$\csc\theta = \dfrac{r}{y} = \dfrac{1}{\sin\theta}$
Cosine	$\cos\theta = \dfrac{x}{r}$	Secant	$\sec\theta = \dfrac{r}{x} = \dfrac{1}{\cos\theta}$
Tangent	$\tan\theta = \dfrac{y}{x} = \dfrac{\sin\theta}{\cos\theta}$	Cotangent	$\cot\theta = \dfrac{x}{y} = \dfrac{\cos\theta}{\sin\theta}$

(B.19)

We see that $\tan\theta$ and $\sec\theta$ are not defined if $x = 0$. This means that they are not defined if θ is $\pm\pi/2, \pm 3\pi/2, \ldots$ Similarly, $\cot\theta$ and $\csc\theta$ are not defined if $y = 0$, namely $\theta = 0, \pm\pi, \pm 2\pi, \ldots$

Some properties of the trigonometric functions are:

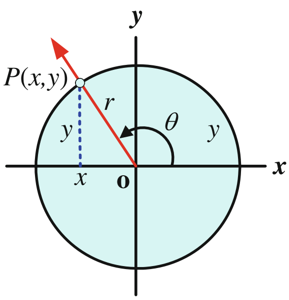


Fig. B.6

$$\begin{aligned} \sin(-\theta) &= -\sin(\theta) \\ \cos(-\theta) &= \cos(\theta) \\ \tan(-\theta) &= -\tan(\theta) \end{aligned} \tag{B.20}$$

From the right triangle of Fig. B.5, we can find the following:

$$\begin{aligned} \sin\theta &= \cos(90^\circ - \theta) \\ \cos\theta &= \sin(90^\circ - \theta) \\ \cot\theta &= \tan(90^\circ - \theta) \end{aligned} \tag{B.21}$$

Moreover, we list here the following trigonometric identities:

$$\begin{aligned}
&\cos^2\theta + \sin^2\theta = 1 \\
&1 + \cot^2\theta = \csc^2\theta && 1 + \tan^2\theta = \sec^2\theta \\
&\sin 2\theta = 2\sin\theta\,\cos\theta && \cos 2\theta = \cos^2\theta - \sin^2\theta \\
&\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} && \tan\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{1+\cos\theta}} \\
&\sin^2\frac{\theta}{2} = \frac{1}{2}(1-\cos\theta) && \cos^2\frac{\theta}{2} = \frac{1}{2}(1+\cos\theta) \\
&\sin(\alpha \pm \beta) = \sin\alpha\,\cos\beta \pm \cos\alpha\,\sin\beta \\
&\cos(\alpha \pm \beta) = \cos\alpha\,\cos\beta \mp \sin\alpha\,\sin\beta \\
&\sin\alpha \pm \sin\beta = 2\sin[(\alpha \pm \beta)/2]\cos[(\alpha \mp \beta)/2]
\end{aligned} \tag{B.22}$$

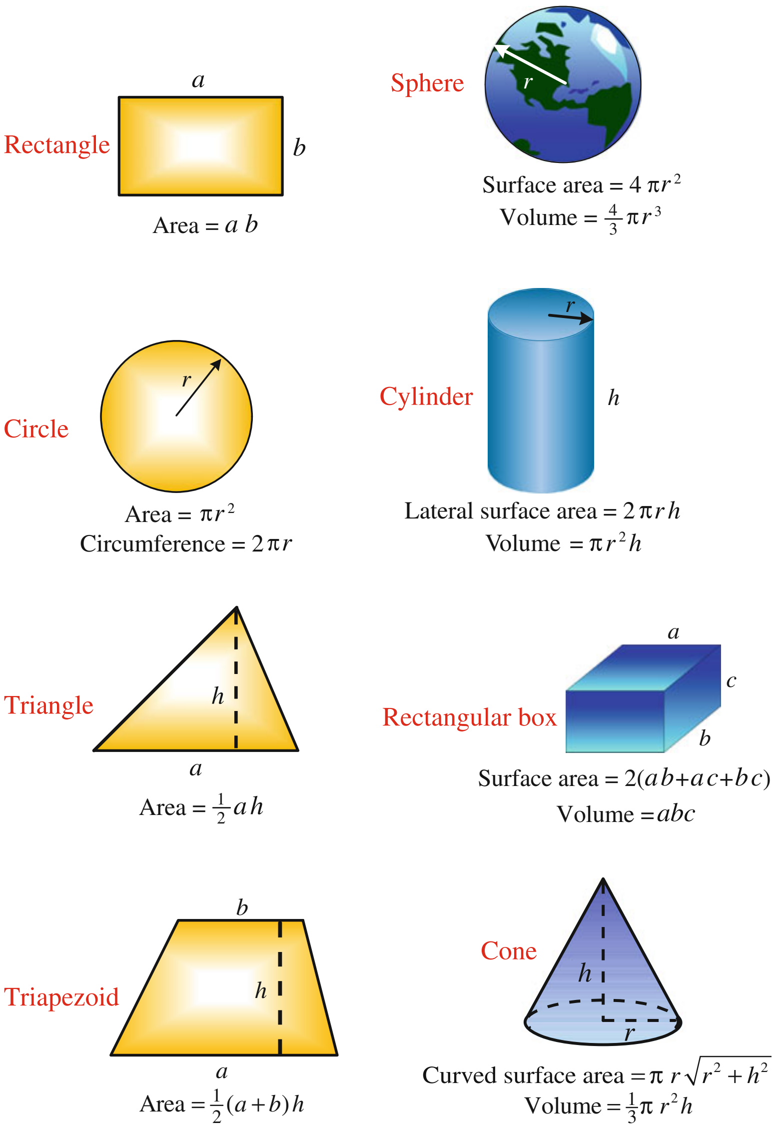
Table B.2 The results of differentiating several functions and their corresponding integrations

Differentiation formula	Integration formula
$\frac{d}{dx}[x] = 1$	$\int dx = x + C$
$\frac{d}{dx}\left[\frac{x^{n+1}}{n+1}\right] = x^n,\ (n \neq -1)$	$\int x^n dx = \frac{x^{n+1}}{n+1} + C, \quad (n \neq -1)$
$\frac{d}{dx}[\sin x] = \cos x$	$\int \cos x\ dx = \sin x + C$
$\frac{d}{dx}[\cos x] = -\sin x$	$\int \sin x\ dx = -\cos x + C$
$\frac{d}{dx}[\tan x] = \sec^2 x$	$\int \sec^2 x\ dx = \tan x + C$
$\frac{d}{dx}[\csc x] = -\csc x\ \cot x$	$\int \csc x\ \cot x\, dx = -\csc x + C$
$d\frac{d}{dx}[\sec x] = \sec x\ \tan x$	$\int \sec x\ \tan x\, dx = \sec x + C$
$\frac{d}{dx}[\cot x] = -\csc^2 x$	$\int \csc^2 x\ dx = -\cot x + C$
$\frac{d}{dx}[e^x] = e^x$	$\int e^x\ dx = e^x + C$
$\frac{d}{dx}[\ln x] = \frac{1}{x}$	$\int \frac{1}{x}\ dx = \ln x + C$

Table B.3 Some complicated indefinite integrals (an arbitrary constant should be added to each of these integrals)

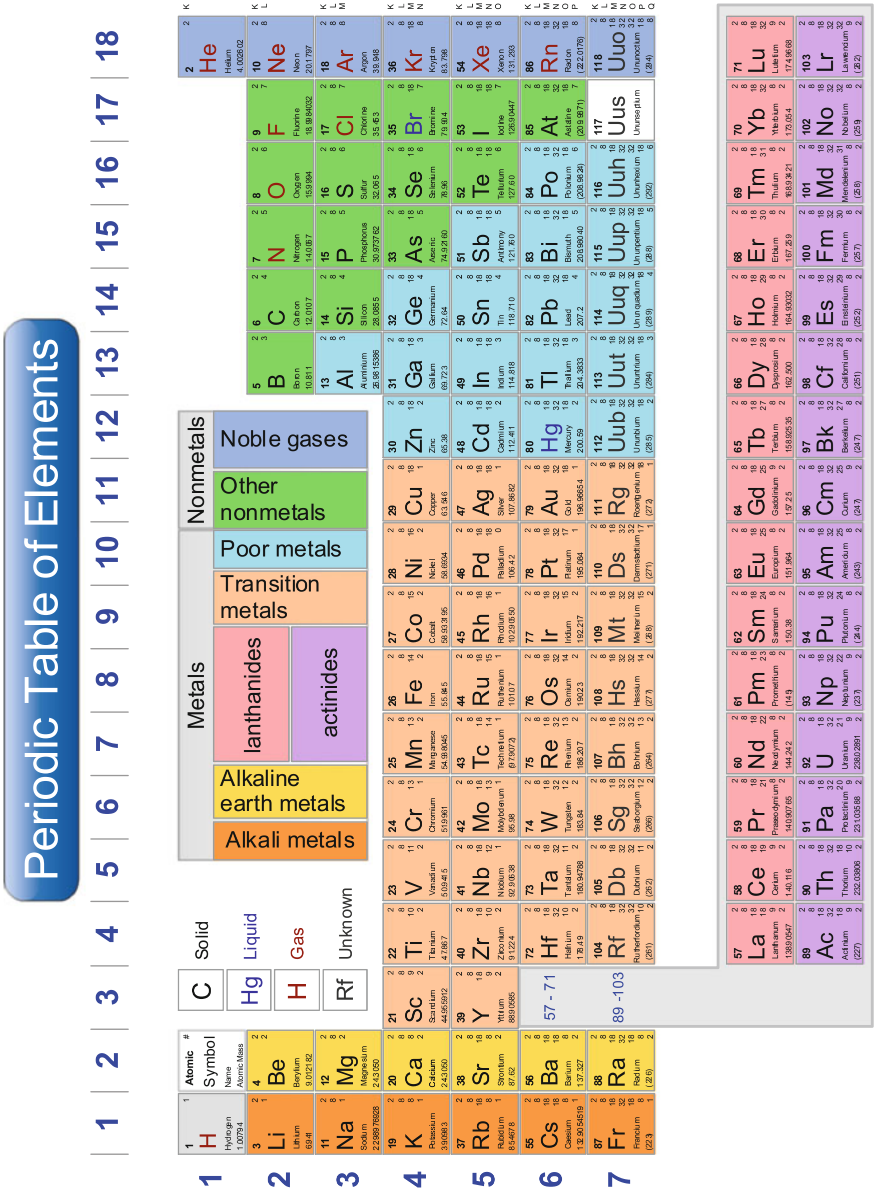
$\int \frac{dx}{a+bx} = \frac{1}{b} \ln(a+bx)$	$\int \frac{x\,dx}{\sqrt{a^2-x^2}} = -\sqrt{a^2-x^2}$
$\int \frac{x\,dx}{a+bx} = \frac{x}{b} - \frac{a}{b^2} \ln(a+bx)$	$\int \frac{x\,dx}{\sqrt{x^2 \pm a^2}} = \sqrt{x^2 \pm a^2}$
$\int \frac{dx}{x\,(x+a)} = -\frac{1}{a} \ln \frac{x+a}{x}$	$\int x\sqrt{a^2-x^2}\,dx = -\frac{1}{3}(a^2-x^2)^{3/2}$
$\int \frac{dx}{(a+bx)^2} = -\frac{1}{b(a+bx)}$	$\int x\sqrt{x^2 \pm a^2}\,dx = \frac{1}{3}(x^2 \pm a^2)^{3/2}$
$\int \frac{dx}{x^2+a^2} = \frac{1}{a} \tan^{-1} \frac{x}{a}$	$\int e^{a\,x}\,dx = \frac{1}{a} e^{a\,x}$
$\int \frac{dx}{a^2-x^2} = \frac{1}{2\,a} \ln \frac{a+x}{a-x},\ (a^2-x^2>0)$	$\int x\,e^{a\,x}\,dx = \frac{1}{a^2}(a\,x-1)\,e^{a\,x}$
$\int \frac{dx}{x^2-a^2} = \frac{1}{2\,a} \ln \frac{x-a}{x+a},\ (x^2-a^2>0)$	$\int \frac{dx}{a+b\,e^{c\,x}} = \frac{x}{a} - \frac{1}{a\,c} \ln(a+b\,e^{c\,x})$
$\int \frac{x\,dx}{a^2 \pm x^2} = \pm\,\frac{1}{2} \ln(a^2 \pm x^2)$	$\int \ln(a\,x)\,dx = x\,\ln(a\,x) - x$
$\int \frac{dx}{(x^2+a^2)^{3/2}} = \frac{x}{a^2\sqrt{x^2+a^2}}$	$\int \cos^2(ax)dx = \frac{x}{2} + \frac{\sin 2\,a\,x}{4\,a}$
$\int \frac{x\,dx}{(x^2+a^2)^{3/2}} = -\frac{x}{\sqrt{x^2+a^2}}$	$\int \sin^2(a\,x)\,dx = \frac{x}{2} - \frac{\sin 2\,a\,x}{4\,a}$
$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1} \frac{x}{a},\ (a^2-x^2>0)$	$\int \tan^2(a\,x)\,dx = \frac{1}{a} \tan(a\,x) - x$
$\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln(x+\sqrt{x^2 \pm a^2})$	$\int \cot^2(a\,x)\,dx = -\frac{1}{a} \cot(a\,x) - x$

Useful Information for Geometry



The Periodic Table of Elements

C



Notes: Elements with atomic weights between brackets have no stable isotopes.

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Undergraduate Lecture Notes in Physics, DOI: 10.1007/978-3-642-23026-4,

Answers to All Exercises

Chapter 1

(1) (a) kilo lambs, (b) mega bytes, (c) giga cars, (d) tera stars, (e) deci kelvin, (f) centi meter, (g) milli ampere, (h) micro newton, (i) nano kilogram, (j) femto second

(2) (a) 4×10^7 m, (b) 6.366×10^6 m, (c) 2.486×10^4 mi, 3.956×10^4 mi, (d) 4.02×10^6 mi which is very close to the answer of part a

(3) 2.362×10^5 mi, 3.8×10^8 m, 3.8×10^{10} cm, 3.8×10^{11} mm

(4) $0.02(\text{km})^3$

(5) (a) AU $= 1.5 \times 10^{11}$ m $= 1.5$ Gm, (b) ly $= 9.461 \times 10^{15}$ m $= 9.461$ Pm, (c) pc $= 3.084 \times 10^{16}$ m $= 30.84$ Pm, (d) Mpc $= 3.084 \times 10^{22}$ m $= 30.84$ Zm

(6) (a) 400, (b) $400^3 = 6.4 \times 10^7$, (c) 4.815×10^6 m

(7) (a) 6.3699×10^{11} m, (b) Estimated/Actual $= 1.7 \times 10^3$

(8) (a) 1.16×10^{34} days, (b) 5.78×10^{12} days, (c) 1.51×10^{12} days, (d) 1.83×10^4 days

(9) (a) 1 microyear $=$ 0.526 of a 1-minute TV commercial, (b) 1 microcentury $=$ 0.877 of a 60-minute TV commercial

(10) (a) 0.03 mi/h, (b) 1.243 mi/h, (c) 22.99 mi/h, (d) 136.73 mi/h, (e) 621.5 mi/h

(11) 48 months $=$ 1440 dy (if the clock doesn't show am/pm) or 96 months $=$ 2880 dy (if the clock shows am/pm)

(12) Atomic clock precession is about 1 part in 2×10^{15}, or about 5×10^{-16} s. So, the error for a 19-year interval is 2.9×10^{-7} s. Therefore, it is sufficiently

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precise to determine your age within 10^{-6} s, but certainly much more precise with 10^{-3} s.

(13) (a) After ten centuries, the day is longer by 0.01 s. The average day duration difference for these 10 centuries is 0.005 s, (b) The total cumulative effect is: (the average day duration difference for these 10 centuries) $\times$ (the number of days) $= 1826.25\,\mathrm{s} = 0.5073\,\mathrm{h}$

(14) $285714.3\,\mathrm{mg/day},\ 11904.8\,\mathrm{mg/h},\ 198.4\,\mathrm{mg/min},\ 3.3\,\mathrm{mg/s}$

(15) $5.95 \times 10^{24}\,\mathrm{kg}$

(16) (a) 5.01×10^{25} atoms/(1 kg), (b) 6.022×10^{26} atoms/(12 kg)

(17) (a) $(2.988\,897\,2 \pm 0.000\,001\,7) \times 10^{-26}\,\mathrm{kg}$, (b) 5.01×10^{46} molecules

(18) (a) $1.178 \times 10^{-26}\,\mathrm{m}^3$, (b) $2.28 \times 10^{-9}\,\mathrm{m}$

(19) $T = 2\pi\sqrt{L/g} \Rightarrow \mathrm{T} = \sqrt{\mathrm{L}/(\mathrm{L}/\mathrm{T}^2)} = \mathrm{T}$. Thus, the expression is dimensionally correct.

(20) $s = ka^m t^n \Rightarrow \mathrm{L} = \left(\mathrm{L}/\mathrm{T}^2\right)^m \times \mathrm{T}^n = \mathrm{L}^m \times \mathrm{T}^{n-2m} \Rightarrow m = 1,\ n - 2m = 0$. Therefore $m = 1$ and $n = 2$.

(21) (a) $v^2 = v_\circ^2 + 2as \Rightarrow (\mathrm{L}/\mathrm{T})^2 = (\mathrm{L}/\mathrm{T})^2 + (\mathrm{L}/\mathrm{T}^2) \times \mathrm{L} = (\mathrm{L}/\mathrm{T})^2$. Thus, the equation is dimensionally correct., (b) $s = s_\circ + v_\circ t + \frac{1}{2}at^2 \Rightarrow \mathrm{L} = \mathrm{L} + (\mathrm{L}/\mathrm{T}) \times \mathrm{T} + (\mathrm{L}/\mathrm{T}^2) \times \mathrm{T}^2 = \mathrm{L}$. Thus, the equation is dimensionally correct., (c) $s = s_\circ \cos kt \Rightarrow \mathrm{L} = \mathrm{L} \times \cos(\mathrm{T}^{-1} \times \mathrm{T}) = \mathrm{L} \times \cos(\text{number}) = \mathrm{L}$. Thus, the equation is dimensionally correct.

(22) $F \propto ma \Rightarrow F \propto \mathrm{kg} \times \mathrm{L}/\mathrm{T}^2 \Rightarrow F$ has the units $\mathrm{kg\,m/s^2}$in the SI units

(23) $G = Fr^2/m_1m_2 = (\mathrm{kg\,m/s^2})(\mathrm{m}^2)/(\mathrm{kg})^2 = \mathrm{m}^3/(\mathrm{kg\,s}^2)$

Chapter 2

(1) $11.18\,\mathrm{km},\ 26.6°$ W of N

(2) $5.29\,\mathrm{km},\ 40.9°$ N of E

(3) (a) 8.66 units at 90°, (b) 13.23 units at 40.9°

(4) (a) $20\,\mathrm{m},\ 10\pi\ \mathrm{m}$, (b) $0,\ 20\pi\ \mathrm{m}$

(5) -2 cm along the x-axis and 2 cm along the y-axis

(6) (a) 10.96 units along the x-axis and 5 units along the y-axis, (b) 12.1 units at $-24.5°$

(7) (a) 104 km/h, (b) No, because the radar unit measures only the component of the car's velocity along the radar beam. If the angle between the beam and the car's velocity is 90°, then the radar unit will measure zero velocity since the car is not moving perpendicularly to the highway.

(8) 15.62 km
(9) $\vec{R} = 3\vec{i} + 5\vec{j} + 5\vec{k}$ and $R = 7.68$
(12) 5 at 306.9°
(13) (a) 6 at 0°, (b) 6.3 at 108.4°
(14) (a)$\vec{A} + \vec{B} = 2\vec{i} - 3\vec{j} + 5\vec{k}$, (b)$\vec{A} - \vec{B} = -4\vec{i} + 5\vec{j} + 3\vec{k}$,(c)$\vec{C} = -2\vec{i} + 3\vec{j} - 5\vec{k}$
(15) (a) $\vec{A}\bullet\vec{B} = A\,B\cos\theta = -15.59$, (b) $\vec{A}\bullet\vec{B} = A_xB_x + A_yB_y = -15.59$
(18) (a) $\vec{A}\bullet\vec{B} = 0$, (b) $\vec{A}\bullet\vec{C} = -9$, (c) $\vec{B}\bullet\vec{C} = -16$, (d) $\vec{A} \times \vec{B} = 12\vec{k}$, (e) $\vec{A} \times \vec{C} = -12\vec{k}$, (f) $\vec{B} \times \vec{C} = 12\vec{k}$
(19) (b) $A^2B\sin\theta$
(22) (b) $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$, $\tan^{-1}[(y_2 - y_1)/(x_2 - x_1)]$
(23) $\vec{F} = 1.6 \times 10^{-14}(1.5\vec{i} - \vec{j})$ N
(24) $\vec{S} = -0.44\vec{i} - 0.4\vec{j} + 1.12\vec{k}$

Chapter 3

(1) (a) 0.25 km/min, (b) 4.17×10^{-3} km/s, (c) 15 km/h
(2) (a) 53.3 km/h, (b) 53.3 km/h
(3) (a) 24 m, (b) 12 m/s and 12 m/s
(4) 100 m
(5) (a) 6 m/s, (b) 8 m/s, (c) 9 m/s
(6) (a) $\bar{v} = \Delta x/\Delta t = 4$ m/s, $\bar{s} = d/\Delta t = 4$ m/s, (b) At $t_i = 0$, we find from the equation $x = 8t - 2t^2$ that $x_i = 0$, i.e., the body is at the origin. At $t = 2$ s, we find that x is maximum and equal to 8 m. At $t = 4$ s, we find that $x = 0$ again, which means that the body returns to the origin and moves a distance of 16 m. At $t_f = 5$ s, we find that $x_f = -10$ m, which means that the body moves a total distance of 26 m. Thus, $\bar{v} = \Delta x/\Delta t = -2$ m/s and $\bar{s} = d/\Delta t = 5.2$ m/s.
(7) (a) 19.2 m, 4.8 m/s for the interval $0 \le t \le 4$ and 100.8 m, 16.8 m/s for the interval $4 \le t \le 10$, (b) 9.6 m/s, 24 m/s
(8) (a) For $t = 1, 2, 3, 4,$ and 5 s we have $x = 1, -2, -3, -2,$ and 1 m, (b) For $t = 1, 2, 3, 4,$ and 5 s we have $v = -4, -2, 0, 2,$ and 4 m/s, (c) For $t = 1, 2, 3, 4,$ and 5 s we have: motion towards decreasing x, motion towards decreasing x, momentarily no motion, motion towards increasing x, and motion towards increasing x, (d) Yes, at $t = 3$ s, (e) No
(9) Negative, zero, positive, zero, zero, and negative

(10) (a) $0 < t < 1$ s, (b) $3\text{ s} < t < 5$ s, (c) $1\text{ s} < t < 3$ s and $5\text{ s} < t < 7$ s

(11) 28 m

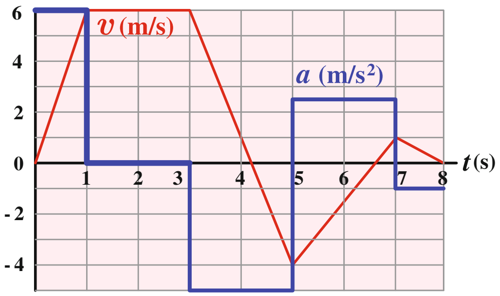
(12) -5 m/s^2

(13) (a) $v = 8 + 4t$, (b) $a = 4\text{ m/s}^2$, (c) 28 m/s, 4 m/s^2

(14) (a) 19.6 m/s^2, 15.6 m/s^2, (b) $a = 20 - 0.8t$

(15) $x = 10t^2 - 0.4t^3/3$, for $t = 0, 3,$ and 6 s we have: $x = 0, 86.4,$ and 331.3 m, $v = 0, 56.4,$ and 105.6 m/s, and $a = 20, 17.6,$ and 15.2 m/s^2

(16) (a)



(b) -0.8 m/s^2, (c) -5 m/s^2

(17) (a) 6 m/s, (b) 26 m/s, (c) $a = 4 + 6t$, 16 m/s^2, (d) $x - x_\circ = 6t + 2t^2 + t^3$

(18) 20 m/s, 50 m

(19) (a) 3 m/s^2, (b) 24 m

(20) (a) 20 m/s^2, (b) 200 m/s, (c) 4 km

(21) (a) -2 m/s^2, (b) 5 m/s, (c) 150 m

(22) (a) 31.9 m, (b) 2.55 s, (c) 26.9 m/s

(23) (a) 19.6 m, (b) 2 s, (c) 4 s, (d) -19.6 m/s

(24) (a) 49 m/s, (b) 122.5 m

(25) (a) 122.5 m, (b) 72.5 m, (c) 172.5 m

(26) (a) 36.72 m/s (downward), (b) 41.8 m

(27) (a) 9.28 m/s (upward), (b) 22.6 m

(28) (a) 5.1 s, (b) 127.6 m, (c) 10.2 s, (d) -50 m/s, (e) -53.8 m/s, (f) 10.6 s

(29) (a) 44.1 m, (b) 44.1 m for the fourth stone, 39.2 m for the third stone, 24.5 m for the second stone, 0 m for the first stone, (c) 3 s

(30) (a) $\sqrt{10}$ m/s, (d) $3\sqrt{10}/\left[\sqrt{2} + 1\right]^2$

(31) (a) 1.96 m/s, (b) 0.196 m, (c) -1.96 m/s

(33) (a) 1.5 s, (b) 11.25 m, (c) 25 m/s, 15 m/s

Chapter 4

(1) (a) $\Delta\vec{r} = (-5\vec{i} + 10\vec{j} - 5\vec{k})$m, (b) $\overline{\vec{v}} = (-\vec{i} + 2\vec{j} - \vec{k})$m/s

(2) (a) $\overline{\vec{v}} = (2\vec{i} + 4\vec{j})$m/s, (b) $\vec{v} = (2\vec{i} + 4t\vec{j})$m/s, $|\vec{v}|_{t=2s} = 8.25$ m/s at 76°, (c) $\vec{a} = (4\vec{j})$m/s^2

(3) (a) $\overline{\vec{v}} = (16\vec{i} + 6\vec{j})$m/s, (b) $\vec{v} = [(12t^2 - 12)\vec{i} + 6\vec{j}]$m/s, $|\vec{v}|_{t=1\,s} = 6$ m/s at 90°, (c) $\vec{v}|_{t=3\,s} = (96\vec{i} + 6\vec{j})$m/s, $|\vec{v}|_{t=3\,s} = 96.2$ m/s at 90°, (d) $\overline{\vec{a}} = (36\vec{i})$m/s^2, (e) $\vec{a} = (24\,t\,\vec{i})$m/s^2, $a_{t=2\,s} = 48$ m/s^2 at 0°, (f) At $t = 1$ s, x is minimum

(4) (a) $r = \sqrt{9t^2 + 4t^4 + 4}$(m), $r|_{t=2\,s} = 10.2$ m, (b) $\vec{v} = (3\vec{i} - 4t\vec{j})$m/s, $|\vec{v}|_{t=2\,s} \equiv v|_{t=2\,s} = 8.54$ m/s at 291°, (c) $\vec{a} = (-4\vec{j})$m/s^2, $a = 4$ m/s^2 at 270°

(5) $\vec{r} = (5\vec{i} - 1.25\vec{j})$(m), $\vec{v} = (10\vec{i} - 5\vec{j})$m/s

(6) (a) 13 m/s, (b) 32.7 m/s at 66.6° below the horizontal line

(7) (a) 6.708 m/s, (b) 1.265 s

(8) (a) 11.18 m/s, (b) 1.265 s

(9) (a) 18 m/s and 24 m/s, (b) $\vec{r} = 36\vec{i} + 28\vec{j}$, $\vec{v} = 18\vec{i} + 4\vec{j}$ ($v = 18.5$ m/s and $\theta = 13.7°$), (c) 28.8 m, 2.4 s, (d) 2.4 s, 86.4 m

(10) (a) 2 m/s, 6 m/s, (b) 1.2 s, 7.2 m, 7.2 m, (c) 2.4 s, 14.4 m

(11) (a) 48.2 m, (b) 60 m, (c) 25.24 m/s, (d) 100 m

(12) (a) 27.39 m/s = 98.59 km/h, (b) 17.83 m/s = 64.19 km/h

(13) (a) $v_\circ = 254.5$ m/s, (b) 50 s, (c) In the presence of air resistance, $v_\circ$ should increase so that the rock can reach the point $x = 9$ km

(14) (a) 7.45 s, (b) 438.2 m

(15) 63.44°

(17) $\theta_\circ = \frac{1}{2}\tan^{-1}(-1/\tan\phi)$

(18) $R = (v_0\cos\theta_0/g)\left[v_0\sin\theta_0 + \sqrt{v_0^2\sin^2\theta_0 - 2gh}\right]$

(19) $R = (v_0\cos\theta_0/g)\left[v_0\sin\theta_0 + \sqrt{v_0^2\sin^2\theta_0 + 2gh}\right]$

(20) 200 m/s

(21) 2.47 m/s^2

(22) (a) 1025 m/s, (b) 2.73×10^{-3} m/s^2

(23) (a) 30 m/s, downwards, (b) 60 m/s^2

(24) (a) 9×10^{22} m/s^2, (b) 1.52×10^{-16} s

(25) 0.029 m/s^2

(26) 1.64×10^6 m/s^2 = 167000 g

(27) 1.9 km

(28) (a) $2\,\text{m/s}^2$, (b) $5.66\,\text{m/s}^2$, (c) $5.05\,\text{m/s}$
(29) (a) $8.66\,\text{m/s}^2,\ 5\,\text{m/s}^2$, (b) $4.16\,\text{m/s}$
(30) (a) $7839\,\text{m/s},\ 9.38\,\text{m/s}^2$, (b) $26.7\,\text{m/s}^2$ at $\theta = 20.6°$

Chapter 5

(1) $1.25 \times 10^4\,\text{N}$
(2) (a) 5 s, (b) 25 m
(3) (a) $\vec{F} = 3\vec{\mathrm{i}} - 4\vec{\mathrm{j}}$, 5 N at 323.1°, (b) $2.5\,\text{m/s}^2$ at 323.1°
(4) (a) $5.25\vec{\mathrm{i}} + 1.5\vec{\mathrm{j}}$, (b) $5.46\,\text{m/s}^2$ at 15.9°
(5) (a) 30°, (b) Yes, this angle is independent of W
(6) $T_1 = 100\,\text{N},\ T_2 = 118.3\,\text{N}$
(7) $T_1 = 200\,\text{N},\ T_2 = 190.8\,\text{N},\ T_3 = 101.5\,\text{N}$
(8) (a) $-16\,\text{m/s}^2$, (b) $-16000\,\text{N}$, (c) $\mu_s = 1.6$ (In some cases μ_s can exceed 1 as in this case)
(9) (a) 8.7°, (b) 0°
(10) 14.3°
(11) (a) $0.75\,\text{m/s}^2$, (b) 9 N
(12) (a) $0.75\,\text{m/s}^2$, (b) 3 N
(13) (a) $0.25\,\text{m/s}^2$, (b) 0.03 N
(14) 0.25
(15) (a) $a_P/s_B = 1/2$, (b) $12\,\text{N},\ 2\,\text{m/s}^2$, (c) $12\,\text{N},\ 1\,\text{m/s}^2$
(16) (a) $T_1 = 60\,\text{N},\ T_2 = 100\,\text{N}$, (b) $T_1 = 72\,\text{N},\ T_2 = 120\,\text{N}$
(17) $a_2 = 0.5\,\text{m/s}^2,\ T_2 = 0.01\,\text{N}$
(18) (a) 500 N, (b) 560 N, (c) 300 N, 336 N
(19) 20 N
(20) 90 N
(21) (a) For m_1, $a = 4\,\text{m/s}^2$ up the plane and for m_2, $a = 4\,\text{m/s}^2$ downwards. (b) The magnitude of the tension in both cords is 36 N, (c) For $m_1,\ a = 1\,\text{m/s}^2$ up the plane and for $m_2,\ a = 1\,\text{m/s}^2$ downwards. The magnitude of the tension in both cords is also 36 N
(22) For m_1, $2\,\text{m/s}^2$ downwards and for m_2, $2\,\text{m/s}^2$ upwards, 48 N
(23) 3 m/s
(24) $a = (m_2 - m_1)g/(m_1 + m_2),\ T_1 = T_2 = T_3 = 2m_1 m_2 g/(m_1 + m_2)$
(25) (a) 0.5, 30 N

(26) $6\,\text{m/s}^2$, 12 N

(27) $3\,\text{m/s}^2$, 12 N

(28) $T_1 = \frac{3}{4}F,\ T_2 = \frac{2}{4}F,\ T_3 = \frac{1}{4}F$, when the number of the locomotive engine plus the cars is n, we get $T_i = \frac{n-1}{n}F,\ i = 1, 2, \ldots, (n-1)$

(29) (a) 0.58, (b) 0.36

(30) (a) 1154.7 N, (b) 2309.4 N

(31) 603.9 N

(32) $1.68\,\text{m/s}^2$, 4.62 N

(33) Block m_2 has $a_2 = 1.103\,\text{m/s}^2$, block m_1 has $a_1 = 2.835\,\text{m/s}^2$, and the tension is zero

(34) From 0 to $2.5\,\text{m/s}^2$

(35) (a) 491 N, 49.1 kg, (b) $2.04\,\text{m/s}^2$

(36) The same answers as exercise 35, but the maximum/minimum readings will be during the stopping/starting period of the elevator's descending motion

(37) 13.8 m/s

(38) (a) 7.97 m/s, when we take $g = 10\,\text{m/s}^2$ (b) 140 m/s ($\approx$50 km/h) (about 18 times the speed of the drop when the resistive drag force exists)

(40) $\theta = 0$

Chapter 6

(1) (a) 200 N, (b) 100 m, (c) $-20\,000$ J, (d) 400 N, 50 m, $-20\,000$ J

(2) (a) -39.2 J, (b) $+39.2$ J, (c) $+19.6$ J, -19.6 J

(3) (a) $W_g(\text{A} \to \text{B}) = -mgh$, (b) $W_g(\text{B} \to \text{A}) = +mgh$, (c) $W_g(\text{A} \to \text{B} \to C) = -mgh$, (d) $W_g(\text{A} \to \text{C}) = -mgh$, (e) $W_g(\text{A} \to \text{B} \to C \to A) = 0$

(4) -1.715 J

(5) (a) Fd, $-\mu_k mgd$, 0, 0, (b) 100 J, -49 J, 0, 0

(6) 60 J, 0, 69.3 J

(7) (a) -2 J, (b) -8 J, (c) -2 J, (d) 9 J, (e) -3 J

(8) (a) 32 J, (b) 32 J

(9) (a) 5.89 J, (b) -1.57×10^{-2} J

(10) (a) 0.54 J, (b) 0.3 J

(12) $-\frac{1}{5}kd^5$

(13) $2\pi RF\cos\theta$

(14) 3.86×10^5 J

(15) 1.5 J

(16) 4.06×10^5 J

(17) 420 J

(18) 40 J

(19) (a) 25 J, (b) -25 J, (c) 9 J, (d) -9 J, (e) 0

(20) (a) 2.5 J, (b) 7.5 J, (c) 11.875 J

(21) 0.21 m

(22) 187.5 J

(23) (a) 98 J, (b) 6.26 m/s

(24) (a) 7.35 J, (b) 5.78 m/s

(25) (a) 9.8 J, (b) 6.26 m/s, (c) 4.43 m/s

(26) (a) 4.43 m/s, (b) 0.25

(28) (a) $v = \pm\sqrt{3Fd/4\,m}$, (b) $v = \pm\sqrt{Fd/m}$

(29) 2.86 m/s

(30) 14 m

(31) $\theta = \cos^{-1}(2/3) = 48.2°$

(32) 20 m/s

(33) (a) 7.67 m/s, (b) -845 J (more energy loss than Ex. 6.8, but the percentage loss of energy by friction with respect to original potential energy of the boy is the same; about 58%)

(34) (a) -98 J, (b) The block will never reach point C if the track is more rough and might stop somewhere on the track once it goes past point A. The block will pass point C if the track is smoother

(35) -14.5 J, No, because its energy (stored in the spring) will be less than its potential energy at the edge of the rough surface

(36) (a) 29 N, (b) 8.57 cm

(37) 36750 J, 147 N

(38) (a) -25 J, (b) 25 J, (c) 125 N

(39) 0.327 kW $=$ 0.438 hp

(40) 223.8 piasters

(41) (a) -62400 W, (b) $(-18900\,t)$W

(42) 500 W

(43) 537.1 N opposite the velocity

(44) 7.234×10^4 W $\simeq$ 97 hp opposite the velocity

(45) (a) $ma + mg\sin\theta + \alpha + \beta\,v^2$, (c) 40 000 W, 50 728.5 W, 4 000 W, 4 000 W, 98 728.5 W

Chapter 7

(1) 2.71×10^{-22} kg.m/s
(2) (a) 160 000 kg.m/s, (b) 80 m/s, (c) 40 m/s
(3) -3.2 kg.m/s
(4) (a) -16 kg.m/s, -16 kg.m/s (b) -8000 N
(5) (a) -3 m/s (b) 15 N
(6) (a) 16 kg.m/s, -12 kg.m/s (b) 20 kg.m/s, 323.1°
(7) $(16\,\vec{i} - 8\,\vec{j})$ kg.m/s
(8) 300 N
(9) (a) 40 kg.m/s, (b) 20 N, (c) 30 N
(10) $m\sqrt{2\,g\,h}$
(11) (a) 2.4 kg.m/s upward, (b) 75%
(12) (a) 1.25×10^{-3} s, (b) 0.48 N.s (in the direction of penetration), (c) -384 N (opposite to the direction of penetration)
(13) (a) 5×10^{-5} meters every second, (b) 0.1 kg, (c) 0.6 N (downwards)
(14) (a) -86.6 kg.m/s (opposite to the x-axis), (b) -8660.3 N (opposite to the x-axis)
(15) The smallest value is for $\theta = 0$, where $\Delta p = 0$ and $\overline{F} = 0$. The largest value is for $\theta = 90°$, where $\Delta p = -100$ kg.m/s and $\overline{F} = -10000$ N
(16) (a) 0.8 m/s, (b) 4.8×10^4 J, (c) $-4/3$ m/s (in opposite direction)
(17) $m_1/m_2 = 0.5$
(18) -6.47×10^2 m/s (The negative sign indicates that the recoiling nucleus is moving in the opposite direction to the alpha particle)
(19) 0.125 m/s
(20) (a) $V = -(m/M)\ v$ (The negative sign indicates that the car is moving in the opposite direction to the man's motion), (b) $v_{\text{rel}} = v + |V| = [(m+M)/M]v$
(21) (a) 0.5 m/s, (b) 200 J, (c) 199.75 J
(22) 594 m/s
(23) 2.8 m/s, 3.8 m/s
(24) -1.67 m/s, 3.33 m/s
(25) $m_2 = 3\ m_1$
(26) (a) 1.5 kg, (b) $v_1' = -0.2\,v_1$ (The negative sign indicates that the first ball will move in the opposite direction to its original motion), (c) 0.96
(27) (a) 1, (b) 0.89, (c) 0.296, (d) 0.019

(28) (a) Yes, the collision is elastic because all involved forces are conservative forces, (b) 0.4 cm

(29) (a) Yes, as in Exercise 28, (b) 2 m/s, (c) 0.25 m, (d) $v_1' = -4$ m/s, $v_2' = 4$ m/s

(30) (a) Yes, as in Exercise 29, (b) 6 m/s, (c) 0.25 m (same compression as Exercise 29), (d) $v_1' = +4$ m/s, $v_2' = 12$ m/s

(31) $v_1' = v_1/\sqrt{3} = 10\sqrt{3/2}$ m/s, $v_2' = v_1/\sqrt{6} = 10/\sqrt{2}$ m/s, $\cos\theta = \sqrt{2/3}$, $K_{\text{target}}/K_{\text{projectile}} = 0.5$

(32) $v_1' = \sqrt{3}\, v_1/2 = 15\sqrt{3}$ m/s, $v_2' = v_1/2 = 15$ m/s

(33) $v_1' = v_2' = v_1/\sqrt{2} = 30/\sqrt{2}$ m/s

(35) Two times

(36) $-M/(m+M)$, -0.98. Thus, 98% of the energy is lost.

(37) (a) 10 m/s, (b) -0.923. Thus, 92.3% of the energy is lost.

(38) (a) 19 m/s, (b) -0.687. Thus, 68.7% of the energy is lost

(39) (a) The heavier nucleus will move with half the speed of the lighter nucleus, but in an opposite direction, (b) 4×10^{-17} J for the lighter nucleus and 2×10^{-17} J for the heavier one

(40) (a) $\vec{v}\,'_2 = (2\vec{\text{i}} + 3\vec{\text{j}})$ (m/s), (b) 50 J are lost

(41) $|\vec{p}\,'_3| = 1.3 \times 10^{-22}$ kg.m/s, $\vec{p}\,'_3$ is 157.4° from the vector $\vec{p}\,'_2$ and 112.6° from the vector $\vec{p}\,'_1$

(42) (a) $v_2' = 2.506$ (m/s), $\theta = 60.8°$ (b) 22.14 J

(43) (b) 18.47 m/s, $\phi = 22.5°$, -0.146. Thus 14.6% of the energy is lost

(44) 0.048 nm

(45) $x_{\text{CM}} = 0.286$ m, $y_{\text{CM}} = 0.571$ m (this answer does not depend on the value of m because it appears as a common factor in both the numerator and denominator)

(47) $z_{\text{CM}} = 0.03$ nm

(48) $\vec{r}_{\text{CM}} = 2.8\vec{\text{i}} + 3.8\vec{\text{j}}$

(49) $x_{\text{CM}} = L/2$, $y_{\text{CM}} = L/2$ (from the center of the left rod)

(50) $x_{\text{CM}} = (3/4)h$

(51) $z_{\text{CM}} = H/4$, 34.7 m

(52) $\vec{v}_{\text{CM}} = (2.8\vec{\text{i}} + 0.2\vec{\text{j}})$ (m/s)

(53) (a) -7.8 m/s, 11.2 m/s, (b) 3.6 m/s

(54) (a) 0, (b) 1.2 m, (c) 1.2 m/s, 0.8 m/s

(55) (a) 7.5 m from the man, 5 m/s, (b) 4.5 m/s, (c) 10 s, 45 m/s

(56) 120 m

(57) (a) 49 N (b) 171.5 W, (c) 85.75 W

(58) 50 m/s^2

(59) -1000 m/s, 1.5×10^5 N
(60) (a) 3.75×10^6 N, (b) 6056.5 m/s

Chapter 8

(1) $\pi/6 = 0.52$ rad, $\pi/4 = 0.79$ rad, $\pi/3 = 1.05$ rad, $\pi/2 = 1.57$ rad, $\pi = 3.14$ rad, $3\pi/2 = 4.71$ rad, $\pi = 6.28$ rad
(2) 1.327×10^3 km
(3) (a) 0.75 rad = 42.97°, (b) 2.4 m
(4) 636.6 rev
(5) 930.8 rad/s^2
(6) -188.5 rad/s^2
(7) 1.2 rev
(8) (a) $\omega = 8t - 14$, $\alpha = 8$ rad/s^2, (b) $\theta = 0$ at $t = 0.5$ s and $t = 3$ s, $\omega = 0$ at $t = 1.75$ s
(9) $\omega = \omega_\circ - 2\,bt + 3\,at^2$, $\theta = \theta_\circ + \omega_\circ\, t - bt^2 + at^3$
(10) (a) 2.4 m/s, (b) No, but may be the best location is somewhere close to the rim of the wheel if the spokes and the dart are not very thin
(11) 9.425×10^3 rad = 1.5×10^3rev = $3 \times 10^3\pi$ rad = 5.4×10^5 degrees, 2.356×10^3 rad = 3.75×10^2rev = $7.5 \times 10^2\pi$ rad = 1.35×10^5 degrees
(12) (a) $-10\,\pi$ rad/s^2 = -31.42 rad/s^2, (b) 4 s
(13) (a) $-50\,\pi$ rad/s^2 = -1.571×10^2 rad/s^2, (b) 116.7 rev
(14) (a) 23.56 s, (b) 167.7 rev
(15) $-30\,\pi$ rad/s^2 = -188.5 rad/s^2
(16) (a) $3\,\vec{\mathrm{i}}$ rad/s, $4\,\vec{\mathrm{k}}$ rad/s, (b) 5 rad/s and at angle 53.13° above the x-axis, (c) $\vec{\alpha}_1(t) = 12[-\sin 4t\,\vec{\mathrm{i}} + \cos 4t\,\vec{\mathrm{j}}\,]$ (rad/s^2), $\vec{\alpha}_1(0) = 12\vec{\mathrm{j}}$ (rad/s^2)
(17) (a) $12\,\pi$ rad/s = 37.699 rad/s, (b) $2.4\,\pi$ m/s = 7.539 m/s, $a_\mathrm{t} = 0$, $a_\mathrm{r} = 28.8\,\pi^2$ m/s^2 = 284.24 m/s^2
(18) (a) $\pi/30$ rad/s = 1.05×10^{-1} rad/s, (b) $\pi/1800$ rad/s = 1.75×10^{-3} rad/s, (c) $\pi/21600$ rad/s = 1.45×10^{-4} rad/s, (d) zero
(19) 20.9 mm/s, 0.26 mm/s, 0.015 mm/s
(20) (a) 12.57 m/s, (b) $a_\mathrm{t} = 0$, $a_\mathrm{r} = 16\,\pi^2/3$ m/s^2 = 52.64 m/s^2 towards the center
(21) (a) 7.272×10^{-5} rad/s, (b) 403 m/s, $a_\mathrm{t} = 0$, $a_\mathrm{r} = 0.029$ m/s^2 (perpendicular to the Earth's axis), (c) 465.4 m/s
(22) 5.373 rad/s if we take $g = 10$ m/s^2
(23) (a) 2.5 rad/s^2, (b) 500 rad = 79.58 rev

(24) 102.9 m.N
(25) 2.05 m.N, clockwise
(26) (a) 2, (b) 7.84×10^{-2} m·N, counterclockwise
(27) $(m_A + m_B)L^2/3$
(28) (a) $I = \frac{2}{5}MR^2 + M(L+R)^2$, (b) $I_{\text{app}} = M(L+R)^2$, (c) 1.1%
(29) 0.12π m.N $= 0.377$ m·N
(30) 4 m/s^2, 20 rad/s^2, 18 N
(31) (a) $\alpha = 3g\cos\theta/2L - 3\tau_f/ML^2$, $\alpha_{\max} = 60$ rad/s^2, (b) 29.9°
(32) (a) 0.095 kg·m^2, (b) It is greater than the value 0.05 kg·m^2 obtained from $I = MR^2/2$. This is because the pulley with the wrapped cord has more mass concentrated around its edge
(33) $\tau = mgL\sin\theta$, $\alpha = g\sin\theta/L$
(34) $\alpha = g\sin\theta[mL + 2M(L+R)]/[\frac{2}{3}mL^2 + M(3R^2 + 4RL + 2L^2)]$
(35) 1.67 m/s^2, $T_2 = 50$ N, $T_1 = 46.67$ N
(36) 616.9 J
(37) (a) 432 J, $F_M = 384$ N, $F_m = 192$ N, (b) 384 J, $F_M = F_m = 256$ N
(38) $v = \sqrt{54ga/7} = 8.695\sqrt{a}$
(39) 0.792 kg.m^2
(40) (a) 398.4 W, (b) -6.56 m·N
(41) (a) 16 rad/s, (b) 1.6 m/s, $a_t = 0$, $a_r = 12.8$ m/s^2, 1.6 m/s, (c) 3.84 J
(42) (a) To the right, (b) 10 m/s^2, (c) 10 N

Chapter 9

(1) $34\,\vec{k}$ (kg.m^2/s or J.s)
(2) $-1.5 \times 10^5\,\vec{k}$ (kg.m^2/s) for the clockwise motion, $1.5 \times 10^5\,\vec{k}$ (kg.m^2/s) for the counterclockwise motion
(3) $-24\,t^2\,\vec{k}$ (kg.m^2/s)
(4) $\vec{L}_i = mvd$ (into the page for $i = 1, 2, 3$), $\vec{L}_i = mvd$, (out the page for $i = 5, 6, 7$), $\vec{L}_i = 0$ (for $i = 4, 8$)
(5) (a) 15 kg.m^2/s (into the page), (b) 25.46 m.N (out of the page)
(9) 11.27 kg.m^2/s (out of the page)
(10) (a) 0.1047 rad/s, (b) 3.421×10^{-6} kg.m^2/s (into of the page)
(11) (a) 7.1 kg.m^2, (b) 14.22 kg.m^2/s (out of the page)
(12) (a) $(0.24\,\vec{i} + 0.16\vec{j})$kg·m^2/s, (b) $3.2 \times 10^{-2}\,\vec{j}$ (kg·m^2/s), (c) 0°

(13) (a) $I = \frac{17}{6} m R^2$, $L = \frac{17}{3} \pi m R^2/T$ along z-axis, (b) 1.417×10^{-2} kg.m^2, 4.451×10^{-2} kg.m^2/s

(14) (a) $m_2\, g\, R$ clockwise, (b) $[m_2 + m_1 + \frac{1}{2}M]\, R\, v$ clockwise, (c) $m_2\, g/[m_2 + m_1 + \frac{1}{2}M]$, 3 m/s^2

(15) (a) $\alpha = 24\, t$ (rad/s), $L = 24\, t^2$ (J·s), $\alpha = 48$ rad/s, $L = 96$ (J·s), (b) $\sum \tau_{\text{ext}} = I\alpha = 48\, t$ (m·N), $\sum \tau_{\text{ext}} = dL/dt = 48\, t$ (m·N), $\sum \tau_{\text{ext}} = 96$ m·N

(16) 0.7 kg.m^2/s along z-axis

(17) 3.848×10^3 kg.m^2/s upwards

(18) (a) 4, 1/4, (b) 256, 16

(20) $I_a\, R_a\, R_b \omega_a/(I_a\, R_b^2 + I_b\, R_a^2)$

(21) (b) 1.5 m/s^2, $T_1 = 10$ N, $T_2 = 17.5$ N, (c) $0.75\, t$ (kg.m^2/s)

(22) (a) 7.149×10^{33} kg.m^2/s, (b) 2.69×10^{40} kg.m^2/s

(23) (a) $\sum \tau_{\text{ext},1} = m_1\, g\, R$ clockwise, $\sum \tau_{\text{ext},2} = -m_2\, g\, R$ counterclockwise, $\sum \tau_{\text{ext,sys}} = (m_1 - m_2)\, g\, R$ clockwise, (b) $L_1 = R\, m_1\, v + M\, R\, v/2$ clockwise, $L_2 = R\, m_2\, v + M\, R\, v/2$ clockwise, $L_{\text{sys}} = (m_1 + m_2 + M) R\, v$ clockwise, (c) $a = (m_2 - m_1)\, g\,/(m_1 + m_2 + M)$, $T_1 = (2\, m_2 + M)\, m_1\, g\,/(m_1 + m_2 + M)$, $T_2 = (2\, m_1 + M)\, m_2\, g\,/(m_1 + m_2 + M)$

(24) 100 %

(25) 1.8 kg.m^2, by pulling her arms to the center of her body

(26) 0.41 rev/s

(27) $\omega_i/(1 + 6\, m/M)$

(28) 5.45 rev/min

(29) 0.316 rev/s

(30) 0.2 rev/s (same as before)

(31) -0.8 rad/s

(32) (a) 0.643 rad/s, (b) 1080 J, 463 J

(33) 1.2 rev/s

(34) (a) 2 rev/s $= 4\,\pi$ rad/s, (b) 66.67% decrease

(35) -3.7×10^{-15}%

(36) -2.6×10^{-15}%

(37) (a) $\omega_f = 2m\, v/[(4\,M/3 + m)d]$, (b) $H = m^2\, v^2/[(M + m)(4\,M/3 + m)g]$

(38) (a) $\omega_f = 2m\, v/[(M/3 + m)d]$, (b) $-(1 + 3m/M)^{-1}$

(39) (a) $\omega_f = 5$ rad/s, (b) -74.8%

(40) $v_{\text{CM}} = mv/(M + m)$, ω(about CM) $= [12m/(7m + 4M)](v/d)$

(41) (a) 3 rev/s, (b) $K_i = 3$ J, $K_f = 18$ J, the increase in the rotational kinetic energy came from the work that the student did in pulling his arms with the dumbbells

(42) 1.974 J
(43) 2.34 rad/s = 22.34 rev/min
(44) 3.466 rad/s = 33.1 rev/min
(45) (a) 5.516×10^{-4} kg.m^2, (b) 3.313×10^{-2} m.N

Chapter 10

(1) 4×10^4 kg/m^3, 40
(2) 9549.3 kg/m^3, 9.55
(3) 11.36 kg, 111.328 N
(4) 6.24×10^7 N/m^2
(5) 1.96×10^{11} N/m^2
(6) 2.352×10^{11} N/m^2
(7) 8.04×10^{-3} m, -5.03×10^{-6} m
(8) 6.57×10^{-4} m
(9) (a) 2.5 N/m^2, (b) 0.025, (c) 100 N/m^2
(10) 2×10^{-7} m, $(4.6 \times 10^{-5})°$
(11) 3.82×10^{-4} rad $= 2.19 \times 10^{-2}$ deg
(12) 6.67×10^{-7}
(13) -1.024×10^{-5} m^3
(14) 84 000 N/m^2
(15) 3.92×10^7 N/m^2
(16) 345000 N/m^2 = 3.45 P_a, 245000 N/m^2 = 2.45 P_a [$P_a = 10^5$ N/m$^2 \equiv 10^5$ Pa]
(17) 117 268 N/m$^2 \equiv$ 117 268 Pa
(18) 28.57 m
(19) 0.8×10^3 kg/m^3
(20) 113 328 Pa
(21) (a) 10.31 m, (b) 13.05 m, (Both values are not practical)
(22) 498 N
(23) (a) 3.27 N, (b) 0.817 N
(24) 3800 kg/m^3
(25) (a) 2.205×10^{-3} N, (b) 533.3 kg/m^3
(26) (a) $v_2 = 25\, v_1$, (b) No effect, because the continuity equation does not depend on altitude
(27) (a) 400 Pa, (b) 5400 Pa
(28) (a) 8.854 m/s, (b) 5.657 m

(30) (a) $v_C = \sqrt{2\,g\,h}$, (b) $P_B = P_a - \rho\,g\,(h+H)$, (c) $H_{max} = P_a/\rho\,g - h$, (d) 7.67 m/s, 52 kPa, 7.3 m

(31) 1.5×10^{-3} N

(32) 5×10^{-3} m/s

(33) 4.36×10^{-4} m/s

(35) 2.64×10^{3} Pa $\equiv$ 19.9 mm Hg

Chapter 11

(1) $-30\,°C \equiv 243.15\,K \equiv -22\,°F$, $10\,°C \equiv 283.15\,K \equiv 50\,°F$, $50\,°C \equiv 323.15\,K \equiv 122\,°F$

(2) $37\,°C \equiv 98.6\,°F \equiv 310.15$ K, $6\,000\,°C \equiv 10\,832\,°F \equiv 6\,273.15$ K

(3) $-40\,°C \equiv -40\,°F \equiv 233.15$ K, $\Delta T = 10\,°C - (-40\,°C) = 50\,C° \equiv 90\,F°$

(4) (a) $1064.5\,°C \equiv 1948.1\,°F \equiv 1337.65$ K, $2660\,°C \equiv 4820\,°F \equiv 2933.15$ K, (b) $\Delta T = 1595.5$ C°, (c) $\Delta T = 1595.5$ K

(5) $T = 5\,°C$

(6) 0.12 m

(7) 2.72×10^{-3} m

(8) 3.6 mm

(9) 100.1 m

(10) 2.88×10^{-4} m^2

(11) 0.048 %

(12) 8.95 cm

(13) (b) 0.5 m, 0.3 m

(14) $(b-a)_T \to (b-a)_{T+\Delta T} = \alpha\,(b-a)\Delta T$, i.e., $a_T \to a_{T+\Delta T} = a(1+\alpha\,\Delta T)$ and $b_T \to b_{T+\Delta T} = b(1+\alpha\,\Delta T)$. Thus, $b_T/a_T = b_{T+\Delta T}/a_{T+\Delta T}$

(15) $-113.16\,°C$

(16) 8.395×10^{7} N/m^2

(17) $r = [2 + (\alpha_2 + \alpha_1)\,\Delta T]\,d/[2(\alpha_2 - \alpha_1)\,\Delta T] \approx d/[(\alpha_2 - \alpha_1)\,\Delta T]$

(18) 1.13×10^{-5} m^3

(19) 50.0135 cm^3

(20) 3.64×10^{7} N/m^2

(21) 2.688 cm^3

(22) 1.25 kg/m^3, 1.43 kg/m^3

(23) (a) 3×10^{4} N/m^2, (b) 3×10^{5} N/m^2

(24) 1.59 atm

(25) 4.15%

(26) 31.18 atm

(27) 3.214

(28) 2.42×10^{22} molecules

(29) 1.14 atm

(30) 1.155×10^{-3} kg

(31) 500 K

(32) (a) 1.270 kg, (b) 0.726 kg, (c) $0.566\ m^3 = 566$ L

(33) 1.43 times the original volume

(34) 0.588 kg/m^3. The difference in density between 0.588 kg/m^3 and the value 0.598 kg/m^3 arises from the fact that water vapor is very "near" to the state phase change. Therefore, we would not expect the steam to act like an ideal gas, because water vapor molecules will have other interactions besides purely elastic collisions. This is evident from the fact that steam can form droplets, indicating an attractive force between the molecules.

Chapter 12

(1) 2.592×10^6 J

(2) 1526 m

(3) 16.5 °C

(4) 3×10^5 J, 71.66 kcal

(5) 8561.9 cal

(6) 6.279×10^6 J

(7) (a) 4500 J/C°, (b) 45000 J

(8) 450 J/kg.C°

(9) 4 °C

(10) 91.8 °C

(11) 4867 J/kg.C°

(12) 754.9 J/kg.C°

(13) 1.45×10^7 J

(14) 0.285 kg

(15) 152695 J

(16) 0.0329 kg = 32.9 g

(17) 8.09 °C

(18) 26.45 g

(19) 6.1 g

(20) (a) 1.2×10^6 J, (b) -3×10^5 J

(21) (a) 300 J, 225 J, 150 J, (b) −300 J, −225 J, −150 J

(22) (a) 3.174×10^6 J, (b) -3.174×10^6 J

(23) 2.5×10^5 J

(24) −100 J, −418.6 J, −318.6 J

(25) (a) 1.65×10^{-3} J, (b) 17 550 J, (c) 17 549.995 J

(26) (a) 538.85 K, 44.8 m^3, 269.43 K, (b) 8.65×10^5 J

(27) (a) 0.289 K, (b) 2.27×10^{-3} m^3

(28) (a) 3.2×10^3 kJ, (b) 0, (c) -1.6×10^3 kJ, (d) 1.6×10^3 kJ

(29) (a) 164.6 kJ, (b) 2200 kJ, (c) 2035.4 kJ

(30) (a) 6 000 J, (b) 3 500 J, (c) 627 °C

(31) (a) 28 J, (b) 62 J, (c) −68 J, (d) −96 J

(32) (a) 1000 J, (b) 1000 Pa, (c) 6907.7 J, (d) 6907.7 J

(33) (a) 3×10^{-3} cal/cm.C°.s = 1.256 W/m.C°, 3×10^{-3} cal/cm.C°.s = 0.075 Btu/ft.F°.h., (b) 7.963×10^{-3} m^2.C°/W

(34) 4.32×10^6 J

(35) 1656 W

(36) 45°C, $H = 45\ k_F\, A/L$

(37) 350.4 W

(38) (a) 4825 W, (b) 0.24 cm

(40) 0.018 W/m.C°

Chapter 13

(1) 26.5×10^4 N/m^2

(2) (a) 5.65×10^{-21} J, (b) 6813 J

(3) 1200 K = 927 °C

(4) (a) 1.368×10^4 m/s, (b) 10 times faster

(5) (a) 240.6 K, (b) 4.98×10^{-21} J

(6) (a) 6.642×10^{-27} kg, (b) 2.415×10^{21} atoms, (c) 1368 m/s

(7) (a) 6.21×10^{-21} J, (b) 7480 J

(8) (a) 7.721×10^{-21} J, (b) 1525 m/s, 483 m/s

(9) (a) 498.8 J, (b) Yes, because the monatomic gas model does not include the energy associated with the internal motions of the gas, such as vibrational and rotational motions of molecules.

(10) (a) 3.73×10^{-26} m^3/molecule, (b) 3.34×10^{-9} m

(11) $v_{\mathrm{rms}} = \sqrt{3P/\rho}$

(12) (a) 493.1 m/s, (b) 5.269×10^{-3} s, (c) 94.9 round/s

(13) (a) 3.7413×10^6 J, (b) 6.2355×10^6 J, 2.4942×10^6 J, 3.7413×10^6 J

(14) (a) 2×10^5 J, (b) 43 °C

(15) 131.293 kg/kmol, Xenon gas

(16) 22.7 C°

(17) (a) 1247.1 J, (b) 831.4 J, (c) 2078.5 J

(18) 28284.2 J

(20) 4.65×10^{-21} J

(21) (a) $C_P = 29.09$ J/mol.K, $C_V = 20.79$ J/mol.K (b) $\Delta T = 85.94$ K, (c) $\Delta E_{\mathrm{int}} = 3.5726 \times 10^3$ J, (b) $V_f = 7.72 \times 10^{-3}$ m^3

(23) (a) 483 m/s, (b) 445 m/s, (c) 395 m/s

(24) 1.5

(25) 1900, 60

Chapter 14

(1) (a) 2 s, (b) 0.5 Hz, (c) π rad/s

(2) (a) 0.25 s, 4 Hz, 8π rad/s, (b) $x(t) = A\cos(8\pi t)$

(3) (a) 1.5 m, 1 Hz, 1 s, (b) $v = -(3\pi \text{ m/s}) \sin(2\pi t - \pi/4)$, $a = -(6\pi^2 \text{ m/s}^2) \times \cos(2\pi t - \pi/4)$, (c) 3π m/s, $6\pi^2$ m/s^2, (d) zero

(4) The new amplitude is $\sqrt{2}$ times the old one

(5) (a) 39 N/m, (b) 1.42 kg

(6) (a) 0.5 s, 2 Hz, 4π rad/s, (b) $8\pi^2$N/m, 1.4π m/s, $2.8\pi^2$N

(7) (a) $k_{\mathrm{eff}} = k_1 + k_2$, (b) $k_{\mathrm{eff}} = k_1 + k_2$, (c) $1/k_{\mathrm{eff}} = 1/k_1 + 1/k_2$

(8) (a) $f = \sqrt{2k/m}/2\pi$, (b) $f = \sqrt{2k/m}/2\pi$, (c) $f = \sqrt{k/2m}/2\pi$

(9) (a) 6×10^4 N/m, (b) 2.52 Hz

(11) 4 J

(12) (a) 6.25×10^{-3} J, (b) 0.25 m/s, (c) $v = \pm 2.291 \times 10^{-1}$ m/s, $K = 5.25 \times 10^{-3}$ J, $U = 10^{-3}$ J

(13) (a) $T = \pi/2$ s, $f = 2/\pi$ Hz, $\omega = 4$ rad/s, (b) $E = \frac{1}{2} m v_i^2 + \frac{1}{2} k x_i^2 = 0.004$ J, $A = \sqrt{2}/10$ m, $\phi = -\pi/4$ rad, $v_{\max} = 0.4\sqrt{2}$ m/s, $a_{\max} = 1.6\sqrt{2}$ m/s^2, (c) $x = (0.1\sqrt{2}\text{ m}) \cos(4t - \pi/4)$, $v = -(0.4\sqrt{2}\text{ m}) \sin(4t - \pi/4)$, $a = -(1.6\sqrt{2}\text{ m}) \cos(4t - \pi/4)$, $x = +0.1$ m, $v = -0.4$ m/s, $a = -1.6$ m/s^2

(14) (a) 196.2 m/s, (b) 1.03 s, 0.97 Hz

(15) (a) 0.3408 s, (b) 5%, (c) 5 s

(16) (c) 2.3×10^{-3} kg/s, 6.67×10^{-6} (about 7 parts per million)

(17) 0.2 m, 0.5π m, $4/\pi$ Hz, 2 m/s

(18) (a) 0.25 m, 3 rad/m, 40 rad/s, 13.3 m/s, (b) 20.9 m, 0.157 s, 6.37 Hz

(19) 519.6 m/s, No

(20) $y = (0.05\text{ m})\ \sin(5\pi x - 100\pi t)$, 0.08 N

(21) 55.1 Hz

(22) (a) 16 m/s, 628.3 rad/s, (b) 157.9 W, (c) 1.6 cm

(23) (a) 20 m/s, π m, 6.4 Hz, (b) 75 W

(26) 60 m/s

(27) (a) 0.02 m, (b) 36 m/s, (c) 64.8 N

(28) (a) 40 Hz, (b) 80 Hz, 120 Hz, 160 Hz

(29) 1 m, π m, $10/\pi$ Hz, 10 m/s

(30) (a) 2.3 cm, (b) $n(\frac{\pi}{2.3})$ cm, $(n = 0, 1, 2, \ldots)$, $(n + \frac{1}{2})(\frac{\pi}{2.3})$ cm, $(n = 0,1,2,\ldots)$, (c) 4 cm

(31) 3.7 cm

(32) 437 Hz

(33) (a) 25 Hz, (b) $25/\sqrt{2}$ Hz, (c) $25\sqrt{2}$ Hz

(36) (a) 40 Hz, (b) 400 kg

(37) (a) 6 loops, (b) 1.67 Hz

(38) (a) $\mu_1/\mu_2 = 4$, (b) $\mu_1/\mu_2 = 2.25$

(39) 2 loops in string 1 and 5 loops in string 2, 395.2 Hz

(40) 8 nodes positioned at 0.32 m, 0.64 m, 0.8 m, 0.96 m, 01.12 m, 1.28 m, and 1.44 m from the left end of string 1

Chapter 15

(1) 351.6 m/s

(2) 422.3 m/s

(3) 5064 m/s

(4) 0.272 s

(6) 1321 m/s

(7) (a) 8.746×10^{-3} s = 8.8 ms, (b) 2.915×10^{-2} s = 29.2 ms

(8) 1170 m

(9) 1400 m

(10) (a) 2 Pa, (b) 1 m, 343 Hz, (c) 343 m/s

(11) (a) 4 μm, 0.314 m, 1091.8 Hz, 343 m/s, (b) 1.766 μm, (c) 2.74 cm/s

(13) 5.81 m

(14) 22.9 W

(15) (a) 2 W/m^2, (b) 1.125 W/m^2

(16) 1.77 μW

(17) (a) $\lambda_w = 4.51\,\lambda_a$, (b) $(s_{max})_a = 59.13\,(s_{max})_w$, (c) $(\Delta P_{max})_w = 59.13\,(\Delta P_{max})_a$, (d) $\lambda_a = 0.331$ m, $\lambda_w = 1.49$ m, $(s_{max})_a = 1.09 \times 10^{-8}$ m, $(s_{max})_w = 1.84 \times 10^{-10}$ m, $(\Delta P_{max})_a = 0.0292$ Pa, $(\Delta P_{max})_w = 1.73$ Pa

(18) 5×10^{-17} W, 5×10^{-5} W

(19) 120.8 dB

(20) 1000

(21) (a) 10^{-4} W/m^2, (b) 82.1 dB

(23) 1.76 dB (This would barely be perceptible)

(24) (a) 133.8 dB, (b) 132 dB, (c) 129 dB

(25) (a) increased by a factor of 5, (b) increased by 7 dB

(26) (a) 4.0×10^{-5} W/m^2, (b) 10 dB

(27) (a) about 10^9, (b) about 10^{12}

(28) from about 100 Hz to about 20000 Hz

(29) 9 Hz difference

(30) 40 kHz

(31) 36 kHz

(32) (a) 5.92 Hz, (b) 4.34 m/s

(33) 20.58 m/s

(34) (a) 313 Hz, (b) 524 Hz, (c) 480 Hz

(35) (a) 471 Hz, (b) 480 Hz, (c) 9 beats/s

(36) (a) 0.364 m, (b) 0.398 m, (c) 982 Hz, (d) 900 Hz

(37) (a) The plane has a speed which is 1.5 times the speed of sound (or Mach 1.5), (b) 41.8°

(38) (a) 42.4 km, (b) 41.5 s

(39) (a) 23.6°, (b) 17.2 s

(40) (a) 73.4°, (b) 29.4 s, (c) 33.5 km

Chapter 16

(1) 2.83 cm, zero

(2) (a) 8 rad, (b) 0.073 m

(3) (a) 5(2n+1) cm, n = 0,1,2,..., (b) 10 n cm, n = 0,1,2,...

(4) 40.4 Hz, 80.9 Hz, 121.3 Hz

(6) The listener hears three minima.

(8) (a) 2.18 cm, (b) $0.4\,n\pi$ cm $(n=0,1,2,\ldots)$, $0.4\,(n+\frac{1}{2})\,\pi$ cm $(n=0,1,2,\ldots)$, (c) 4 cm
(11) 1429 Hz, 1143 Hz, 1715 Hz
(12) 286 Hz, 1429 Hz, 858 Hz
(13) (a) 0.75 m, (b) 1.5 m
(14) (a) 120 Hz, (b) 1.43 m
(15) 850 Hz, 1133 Hz
(16) 0.85 cm (for the upper limit) to 850 cm for the lower limit
(17) 67 cm, 111.7 cm
(18) (a) 15.5 cm, (b) 119 cm, (c) 440 Hz, 78 cm
(19) (a) 66 cm, (b) 262 Hz, 132 cm, (c) 262 Hz, 132 cm (the frequency and wavelength are the same in the air, because it is the air that is resonating in the organ pipe)
(20) -1.72%
(21) 476 m/s
(22) (a) The difference between successive harmonics is 140 Hz. The difference between successive overtones for an open pipe is the fundamental frequency, and each overtone is an integer multiple of it. Since 210 Hz is not a multiple of 140 Hz, then 140 Hz cannot be the fundamental frequency, and so the pipe cannot be open at both ends. Thus, it must be a closed pipe. (b) For a closed pipe, the successive harmonics differ by twice the fundamental frequency. Thus 140 Hz must be twice the fundamental frequency, which is 70 Hz.
(23) (a) 85 Hz, (b) 340 m/s
(24) (a) 291 harmonics with $n=1, 2, 3, \ldots, 291$, (b) 291 harmonics with $n=1, 3, 5, \ldots, 583$
(25) (a) 348 m/s, (b) 125 cm
(26) 6 Hz
(27) 2%
(28) 516 Hz
(29) (a) 259 Hz or 265 Hz, (b) The frequency must have started at 265 Hz to become 266 Hz, (c) The tension should be reduced by 2.99%
(30) (a) 10.13 Hz, (b) 34.45 m

Chapter 17

(1) 1.25×10^8 m/s, 208.3 nm

(2) (a) 3×10^8 m/s, (b) 2.256×10^8 m/s, (c) 5×10^{14} Hz, (d) 451.113×10^{-9} m, (e) 5×10^{14} Hz

(3) (a) 0.1 ns, (b) 50 000

(4) 43.6°

(5) (a) 2.143×10^8 m/s, (b) 38.2°

(6) (a) 1.43, (b) 2.098×10^8 m/s

(7) 0.9 cm

(8) (a) 32.1°, (b) 25.7°, (d) 0.387 cm

(9) 24.4°

(10) (a) 61.3°, (b) 53.7°

(11) (a) 1.3, (b) 50.3°, (c) 66.8°

(12) (a) 48.8°, (b) 41.2°

(13) (a) 50.3°, (b) 33°, 387 μm, Yes $\theta = 57° > \theta_c$ fulfill the condition of total internal reflection (c) 3902 reflections

(14) 56.2°

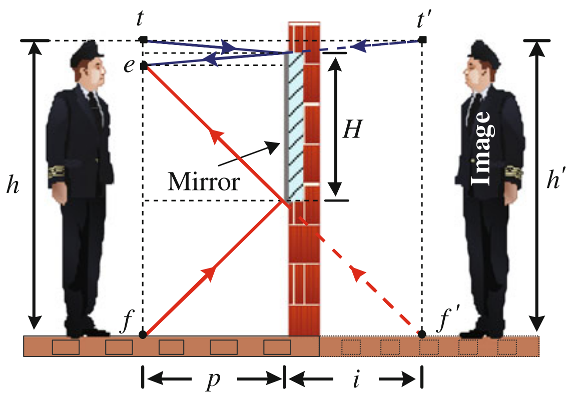
(15) 1.2 μs

(16) 58.47°

(17) 0.34°

(18) (a) 15.68°, (b) 22.84°

(19) (a) $H = 100$ cm, (b) $h' = h = 200$ cm, $i = -p$ (virtual), see the figure



(20) 0.75 m from its center

(21) $i = -0.2$ m and $M = +2$. The image is *virtual* because i is negative, *upright* because M is positive, and *twice as large as* the object ($h' = 6$ cm) because $M = 2$

(22) $i = -0.2/3$ m and $M = +2/3$. The image is *virtual* because i is negative, *upright* because M is positive, and *reduced* ($h' = 2$ cm) because M is less than unity

(23) We found that choosing $i = p$ from the condition $M = |-i/p| = 1$ satisfies the mirror equation $1/p + 1/i = 1/|f|$ and gives $p = 2|f|$, i.e. the object must be placed at a distance $2f$ from the concave mirror. Note that, choosing $i = -p$ from the condition $M = |-i/p| = 1$ cannot satisfy the mirror equation.

(24) Choosing either $i = p$ or $i = -p$ from the condition $M = |-i/p| = 1$ does not satisfy the mirror equation $1/p + 1/i = -1/|f|$. Note that the mirror equation for convex mirrors leads always to a *virtual*, *upright*, and *reduced* image for all values of p.

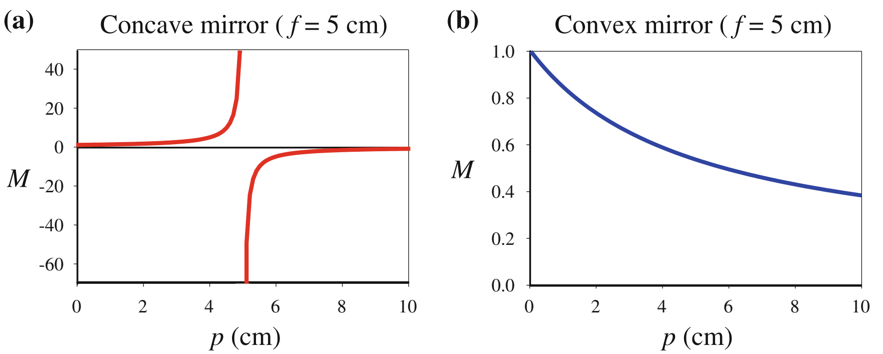
(25) (a) $f = +5$ cm for the concave mirror:

(i)	$p = \infty$,	$i = 5$ cm,	$M = 0$	(real, focus, reduced)
(ii)	$p = 15$ cm,	$i = 7.5$ cm,	$M = -0.5$	(real, inverted, reduced)
(iii)	$p = 10$ cm,	$i = 10$ cm,	$M = -1$	(real, inverted, equal)
(iv)	$p = 7.5$ cm,	$i = 15$ cm,	$M = -2$	(real, inverted, enlarged)
(v)	$p = 5$ cm,	$i = \infty$ cm,	$M = -\infty$	(real, inverted, enlarged)
(vi)	$p = 2.5$ cm,	$i = -5$ cm,	$M = +2$	(virt., upright, enlarged)

(b) $f = -5$ cm for the convex mirror:

(i)	$p = \infty$,	$i = -5$ cm,	$M = 0$	(virt., focus, reduced)
(ii)	$p = 15$ cm,	$i = -3.75$ cm,	$M = +0.25$	(virt., upright, reduced)
(iii)	$p = 10$ cm,	$i = -3.3$ cm,	$M = +0.33$	(virt., upright, reduced)
(iv)	$p = 7.5$ cm,	$i = -3$ cm,	$M = +0.4$	(virt., upright, reduced)
(v)	$p = 5$ cm,	$i = -2.5$ cm,	$M = +0.5$	(virt., upright, reduced)
(vi)	$p = 2.5$ cm,	$i = -1.67$ cm,	$M = +0.67$	(virt., upright, reduced)

(26)



(27) (a) $i = 18$ cm and $M = -2$. The image is *real* because i is positive, *inverted* because M is negative, and *enlarged* ($h' = -0.4$ cm) because $|M|$ is greater than unity. (b) $i = -6$ cm and $M = 2$. The image is *virtual* because i is negative, *upright* because M is positive, and *enlarged* ($h' = 0.4$ cm) because M is greater than unity

(28) $i = -22.5$ cm (both the object and image are in *front* of the spherical surface in water) and $M = 1.5$. The image is *virtual* because i is negative, *upright* because M is positive, and *enlarged* because M is greater than unity

(29) (a) $i = 15$ cm and $M = -0.5$. The image is *real* because i is positive, *inverted* because M is negative, and *reduced* because M is less than unity, (b) $i = -10$ cm and $M = 2$. The image is *virtual* because i is negative, *upright* because M is positive, and *enlarged* because M is greater than unity

(30) (a) $i = -7.5$ cm and $M = 0.25$. The image is *virtual* because i is negative, *upright* because M is positive, and *reduced* because M is less than unity, (b) $i = -3.3$ cm and $M = 0.67$. The image is *virtual* because i is negative, *upright* because M is positive, and *reduced* because M is less than unity

(32) (a) $i = -5$ cm and $f = -20/3$ cm. The image is *virtual* and the lens is a diverging lens. (b) $R = 18$ cm

(33) $f_2 = -37.5$ cm

(34) $i = -1.75$ cm. The image is *virtual* and 1.75 cm in front the diverging lens

(35) $i = +9.6$ cm. The image is *real* and 9.6 cm behind the diverging lens

(36) $i = +40$ cm. The image is *real* and 40 cm behind the second lens, *reduced* because $M = 0.5$, and *upright* because M is positive

(37) $i = +4$ cm. The image is *real* and 4 cm behind the second lens, *reduced* because $M = -0.4$, and *inverted* because M is negative

(38) $f_1 = -5$ cm

(39) $f_1 = -5$ cm as in Exercise 38

(40) $di/dt = f^2/(p-f)^2\ v, p = 2f$

Chapter 18

(1) $\Delta y = 2.52$ mm

(2) 500 nm (in the range of green light)

(3) $\lambda_V = 400$ nm, $\lambda_R = 700$ nm

(4) 600 nm, 5×10^{14} Hz

(5) 4.5×10^{-6} m

(6) 2.9°, 8.6°, 14.5°, 20.5°, and 26.7°, (corresponding to the order $m = 1, 2, \ldots, 5$ for dark fringes)

(7) After the central fringe, the 12th blue fringe will overlap with the 10th yellow fringe to produce a green fringe

(8) (a) 58, (b) 80.4°

(9) 8.8 cm

(10) 0.019 mm

(11) 1355 rad

(12) (a) 0.75 $I_\circ$, (b) 94.5 nm

(13) 0.864

(16) (a) $d = 100$ nm, (b) Yes, with $m = 2$

(17) 686.4 nm

(18) $d = 1473$ nm

(19) (a) 21 dark bands and 20 bright bands between them, (b) 0.5 cm

(20) (a) 168.6 nm, (b) If the thickness were much less than one wavelength, then there would be a very little phase change introduced by an additional path length, and so the two reflected waves would have about π rad phase difference. This would produce destructive interference.

(21) 102 nm

(22) (a) 74.2 nm, 541.7 nm (b) A light ray reflected from the air-oil interface undergoes a phase shift $\phi_1 = \pi$. A ray reflected at the oil-water interface undergoes no phase shift. When the oil thickness is negligible compared to the wavelength of the light, then there is no significant shift in phase due to a path distance traveled by a ray in the oil, i.e., $\phi_2 \approx 0$. Thus, the light reflected from the two surfaces will destructively interfere for all visible wavelengths and the oil will appear black.

(23) (a) 675 nm, (b) 2.8 mm

(26) 12.33 m, 23.71 m

(27) (a) $\lambda = 632.9$ nm, (b) $I_3/I_{\max} = 8.3 \times 10^{-3} \equiv 0.83\%$

(28) 114 cm

(29) 0.26 mm

(30) (a) $a = \lambda$, (a) $a = 400$ nm

(31) (a) 25 cm, (a) 51.5%

(33) 625 nm

(34) 17.3°

(35) (a) $d = 2.5 \times 10^3$ nm, (b) $m = 0, 1, 2, 3$, (c) $R = 732.5$, (d) $N' = 366$ slits

(36) For $\lambda = 700$ nm, $m_{\text{max}} = 3.2$. Three full spectral orders can be observed on each side of the central maximum as well as a portion of the fourth order. For $\lambda = 400$ nm, $m_{\text{max}} = 5.6$. Five full spectral orders can be observed on each side of the central maximum as well as a portion of the six order.

(37) 16.6 cm for $\lambda = 700$ nm and 9.1 cm for $\lambda = 400$ nm

(38) The wavelengths 600–700 nm of the second order overlap with the wavelengths 400–467 nm of the third order.

(39) (a) 3, (b) The resolution is best for the third order, since it is more spread out than the second and first order, (c) 0.028 nm

(40) 63.4°

(41) 0.125 $I_\circ$

(42) 45°

(43) 0.5625 $I_\circ$ (56.25%)

(44) 65.53°

(45) (a) $I_1/I_\circ = \frac{1}{2}$, $I_2/I_1 = \frac{3}{4}$, and $I_2/I_\circ = \frac{3}{8}$, (b) 63.4°

Chapter 19

(1) A neutral atom has the same number of electrons orbiting a nucleus having the same number of protons. A negatively charged atom has an excess of one or more electrons, while a positively charged atom has one or more missing electrons.

(2) The rubber rod will be negatively charged while the fur will be positively charged. It is not possible to transfer positive charges from rubber to fur or vice versa, because positively charged nuclei (or protons) are massive and immobile, unlike electrons.

(3) Negative charged copper rod.

(4) When the comb is near the bits of paper, molecules in the paper are polarized with an opposite charge facing the comb, and the paper is attracted. During contact, charge from the comb is transferred to the paper by conduction. Then the paper may be neutralized and fall off. It may even become equally charged as the comb, and then get repelled.

(5) Wearing rubber-soled shoes allows for an accumulation of charge by friction with the floor. Upon discharging, a spark may result, and if the area is enriched with oxygen, then it would result in an explosion.

(6) No. Molecules in the wall are polarized with an opposite charge facing the balloon, and the balloon is attracted to the wall. During contact, ionization of the air between the balloon and the wall provide ions so the excess electrons in the balloon can be transferred to the ions, reducing the charge on the balloon and eventually causing the attractive force to be insufficient to support the weight of the balloon.

(7) We first allow the two uncharged metallic spheres to touch. The charged rubber rod is then brought near one of the spheres. The positive charge on the rubber rod will repel the electrons in the nearby sphere and cause them to move to the far end of the second sphere (this is known as charging by induction). If the spheres are now separated, one of them will retain a negative charge while the other will retain an equal amount of positive charge. Finally, we take away the charged rubber rod.

(8) 6.24×10^{18} electrons, 5.68×10^{-12} kg

(9) (a) 9×10^{13} N, (b) 9×10^{9} N, (c) 9000 N, Yes, Yes

(10) 2.3×10^{-8} N

(11) 2.1×10^{-4} N

(12) 2.1×10^{11} electrons

(13) (a) 2.62×10^{24} electrons, (b) 2.39 electrons per billion (10^9)

(14) (a) 57.6 N, (b) Larger by 1.24×10^{36} times

(15) $q/m = 8.61 \times 10^{-11}$ C/kg

(16) $q = \pm 1.4 \times 10^{-9}$ C, No, both positive and negative charges repel each other.

(17) (a) 35 μC and 5 μC, (b) 45 μC and -5 μC or -45 μC and 5 μC

(18) 14.4 N away from q_2

(19) 0.02 N on q_1 and directed to the left, zero force on q_2, and 0.02 N on q_3 and directed to the left.

(20) 0.25 N, No, only the direction will be reversed

(21) 8.9 N at 204° or $\vec{F} = (-8.1\vec{i} - 3.6\vec{j})$ N

(23) 0.97 N at 135° or $\vec{F} = (-0.69\vec{i} + 0.69\vec{j})$ N

(24) (a) 0.018 N at 45° or $\vec{F} = (0.013\vec{i} + 0.013\vec{j})$ N, (b) 3.1×10^{-2} N at 225° or $\vec{F} = (-2.2 \times 10^{-2}\vec{i} - 2.2 \times 10^{-2}\vec{j})$ N

(25) (a) $2kqq_\circ/a^2$, negative x-direction, (b) $2kqq_\circ/(a^2 + y^2)^{3/2}$, negative x-direction

(26) (a) 82.3×10^{-9} N, (b) 9.04×10^{22} m/s^2, (c) 2.19×10^{6} m/s

(27) (a) zero, (b) 1.9×10^{9} N

(28) $q_1 = q_2 = Q/2$

(29) (a) $T = 2\pi\sqrt{\pi\epsilon_\circ m a^3/q q'}$, (b) 6.63×10^{-13} s $\simeq 0.7$ ps
(30) (b) 7.318×10^{-9} C
(31) (a) $x = k q Q L/2 W h^2$, $P = W - 3 k q Q/h^2$, (b) $h = \sqrt{3 k q Q/W}$
(32) (a) $\theta_1/\theta_2 = 3$, (b) $r = (8 k L q^2/3 m g)^{1/3}$

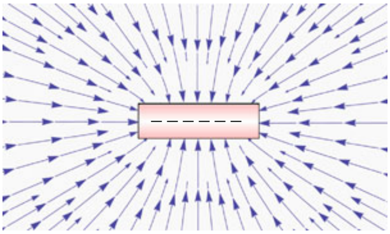
Chapter 20

(1) (a) 9×10^7 N/C, (b) 9×10^3 N/C, (c) 9×10^{-3} N/C
(2) (a) 1.1×10^{-14} C, (b) 1.1×10^{-10} C, (c) 1.1×10^{-4} C
(3) (a) 5.6×10^{-11} N/C, down, (b) 1.0×10^{-7} N/C, up, very small values
(4) (a) 5×10^5 N/C, negative x-direction, (b) 8.8×10^{16} m/s^2
(5) (a) 1.8×10^3 N/C, 243.4°, (b) 3.6×10^4 N/C, 36.9°
(6) At 20 cm to the right of the $-4\ \mu$C charge
(7) 7.2×10^7 N/C directed toward q_2
(8) (a) zero, (b) zero, (c) $1.3 \times 10^7 \vec{i}$ (N/C)
(9) 5.4×10^5 N/C to the left
(10) (a) $+5.88 \times 10^8$N/C, $+1.28 \times 10^{11}$N/C, -6.41×10^{11}N/C, -6.41×10^{11}N/C, $+1.28 \times 10^{11}$N/C, $+5.88 \times 10^8$N/C, (b) about 98%
(11) (a) $-2.84 \times 10^8 \vec{i}$ (N/C), $-2.58 \times 10^{10} \vec{i}$ (N/C), $-2.06 \times 10^{11} \vec{i}$ (N/C), $-2.06 \times 10^{11} \vec{i}$ (N/C), $-2.58 \times 10^{10} \vec{i}$ (N/C), $-2.84 \times 10^8 \vec{i}$ (N/C), (b) about 102%
(12) (a) $\lambda = -Q/L$, (d) 3.2×10^6 N/C directed toward the rod
(13) (a) $E = k\lambda/a$ to the left, (a) $E = k\lambda_\circ/2a$ to the left
(14) (a) zero, (b) 1.32×10^6 N/C, (c) 7.68×10^6 N/C, (d) 4.35×10^5 N/C, (e) The electric field is zero at the center of the ring, then increases as a increases, and finally starts to decrease as a increases
(15) 1182 N/C to the right
(16) $E = 4kQ/\pi R^2$ to the left, where Q is the magnitude of the charge on each quarter circle, i.e. with $|\lambda| = 2|Q|/(\pi R)$
(17) $E = 4kQ/\pi R^2$ to the left, the same formula as in Exercise 16, but Q here is the magnitude of the charge on each half circle, i.e. with $|\lambda| = |Q|/(\pi R)$
(20) (a) $q_1/q_2 = 3/5^{3/2} \simeq 0.3$, (b) Yes, to the left of C_1
(21) (a) 3.32×10^5 N/C, (b) 2.72×10^5 N/C, (c) 3.58×10^4 N/C, (d) 4.23×10^2 N/C
(23) The near-field approximation matches the 1 mm location and the point charge approximation matches the 100 cm location.
(24) $z = R/\sqrt{3}$

(25) $E = \sigma/\epsilon_\circ$

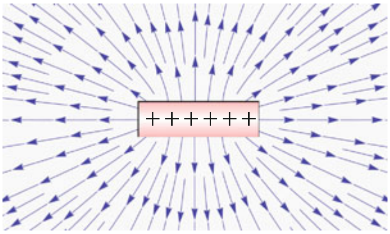
(27) (a) 8.78×10^8 m/s, 3.51×10^{-13} J, (b) 4.79×10^5 m/s, 1.92×10^{-16} J

$-\sigma$ on the disk and $-\lambda$ on the rod

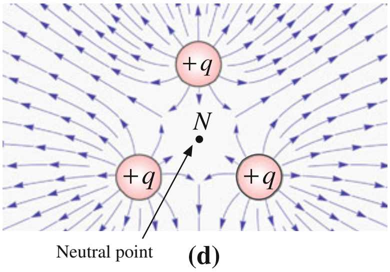


(a) and (c)

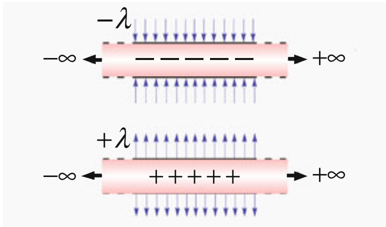
$+\sigma$ on the disk



(b)



(d)



(e)

(28) (a) 4.52×10^5 N/C, (b) 7.23×10^{-14} N to the left, (c) 7.95×10^{16} m/s^2 to the left, (d) 6.14×10^{-10} s, (e) 4.88×10^7 m/s and 1.08×10^{-15} J

(29) (a) 7.95×10^{16} m/s^2 to the left, (b) 4.88×10^7 m/s, (c) 4.61×10^{-10} s

(30) The electron will hit the upper plate at $x = 2.386 \times 10^{-2}$ m $\simeq 2.4$ cm

(31) The proton will never hit the lower plate and at $y = -d/2$, the x-coordinate of the proton will be $x \simeq 102$ cm

(32) (a) 1.76×10^{13} m/s^2, (b) 1.5×10^{-8} s, (c) 1.98×10^{-3} m $= 0.198$ cm, (d) 4.15×10^{-2} m $= 4.15$ cm

(33) (a) 9.581×10^9 m/s^2, (b) 1.5×10^{-8} s, (c) 1.078×10^{-6} m (almost no deflection), (d) 2.263×10^{-5} m (little deflection)

Chapter 21

(1) (a) 10^5 N.m^2/C, (b) -10^5 N.m^2/C, (c) zero, (d) zero, (e) 5×10^4 N.m^2/C

(2) 2.5×10^6 N/C

(3) (a) zero, (b) $-\pi \times 10^3$ N.m^2/C, (c) $\pi \times 10^3$ N.m^2/C

(4) (a) $a^3\beta$ N.m^2/C, (b) 0.04 N.m^2/C, zero

(5) (a) zero, (b) βA, (c) αA
(6) (a) 2.8 N.m^2/C, (b) -2.8 N.m^2/C
(7) (a) $-E\ r\ h$, (b) $+E\ r\ h$, (c) zero
(8) (a) $q/\epsilon_\circ$, (b) $(q + 2\,\pi\, R\,\lambda)/\epsilon_\circ$
(9) (a) $q/\epsilon_\circ$, zero, $2\,q/\epsilon_\circ$, and zero, (b) No, (c) Because the number of electric field lines that enter any surface will emerge from it and hence do not contribution to the electric flux.
(10) (a) 2.856×10^4 N.m^2/C, (b) 1.414×10^6 N.m^2/C, Yes
(11) (a) 10^3 N.m^2/C, (b) zero
(12) 4.425×10^{-3} C/m^3
(13) (a) zero, (b) 126.3 N.m^2/C
(14) (a) $q/2\epsilon_\circ$, (b) $-q/2\epsilon_\circ$
(15) 5×10^5 N/C upwards
(16) 5×10^9 N/C away from the wall. The field is uniform as long as the distance from the wall is much less than the wall's dimensions.
(17) (a) zero, (b) $\sigma/\epsilon_\circ$ to the right, (c) zero
(18) (i) (a) $\sigma/\epsilon_\circ$ to the left, (b) zero, (c) $\sigma/\epsilon_\circ$ to the right (ii) (a) $\sigma/\epsilon_\circ$ to the right, (b) zero, (c) $\sigma/\epsilon_\circ$ to the left
(19) (a) 3.9×10^{-9} C/m^2, (b) (441.4 N/C) $\vec{\mathrm{k}}$, (c) $-$(441.4 N/C) $\vec{\mathrm{k}}$
(20) (a) 14.4 M N/C inwards, directed to the filament, (b) 7.2 M N/C inwards, directed to the filament, (c) 1.44 M N/C inwards, directed to the filament
(21) (a) 6×10^5 N/C, (b) 7.54×10^3 N.m^2/C
(22) (a) 1.1×10^{-6} C, (b) zero
(23) $\rho\, r/2\epsilon_\circ$ radially outward if ρ is positive
(24) (a) zero, (b) 6.75×10^6 N/C
(25) (a) zero, (b) $E = k\,Q/r^2$, (c) zero
(26) 5.94×10^5 m/s
(27) (a) 9×10^{-7} C, (b) 7.5×10^{-6} C
(28) (a) zero, (b) 3.38×10^6 N/C, (c) 6.75×10^6 N/C, (d) 3.0×10^6 N/C, (e) 7.5×10^5 N/C
(29) 8.34×10^{-9} C
(30) $E = \alpha\, r^2/4\epsilon_\circ$ for $r \le R$ radially outward, and $E = \alpha\, R^4/4\epsilon_\circ\, r^2$ for $r \ge R$ radially outward
(31) $E_{C_1} = -\rho R/6\epsilon_\circ$ downwards, $E_{C_1} = 17\rho R/54\epsilon_\circ$ upwards

(32) (a) $E = (kQ/R^3)\, r$, (b) $E = kQ/r^2$, (c) zero, (d) zero, (e) inner charge is $-Q$, outer charge is 0

(33) (a) $\sigma_{\text{Copper}} = Q/2A$, $\sigma_{\text{Glass}} = Q/A$, (b) $E_{\text{Copper}} = \sigma_{\text{Copper}}/\epsilon_\circ = Q/2A\,\epsilon_\circ$, $E_{\text{Glass}} = \sigma_{\text{Glass}}/\epsilon_\circ = Q/2A\,\epsilon_\circ$, the magnitude of the two fields are the same, and both are perpendicular to the plates

(34) (a) $\lambda_{\text{inner}} = -\lambda$, $\lambda_{\text{outer}} = 4\,\lambda$, (b) $E = 2\,k\,\lambda/r$ (radius of the wire $< r < R_1$), $E = 0$, $(R_1 < r < R_2)$, $E = 8\,k\,\lambda/r (r > R_2)$

(35) $E = k(q_1 + q_2)/r^2$, directed outward if $(q_1 + q_2) > 0$ and inwards if $(q_1 + q_2) < 0$

Chapter 22

(1) (a) Zero, (b) 10^{-3} J, (c) -1.56×10^{-3} J

(2) (a) Zero, (b) -10^{-3} J, (c) $+1.56 \times 10^{-3}$ J

(3) (a) Zero, (b) 10^{-3} J, (c) -1.56×10^{-3} J

(4) 1.35×10^6 J, 6.02×10^{23} electrons, Avogadro's number

(5) 3.7×10^5 N/C

(6) (a) 1200 V, (b) 2.05×10^7 m/s

(7) (a) 135.6 N/C, (b) 7.38 cm

(8) (a) 240 V, (b) 240 V

(9) (a) 2 m/s, (b) The same

(10) (a) 1.44×10^{-7} V, 7.2×10^{-8} V, (b) -7.2×10^{-8} V

(11) (a) -1.44×10^{-7} V, -7.2×10^{-8} V, (b) 7.2×10^{-8} V

(12) (a) 0.9 m, (b) 3.6×10^{-9} C

(13) 3 cm

(14) (a) $-1.44 \times 10^7 \vec{i}$ (V), (b) Zero, -0.36 J, (c) Zero

(15) 4639 V

(16) -7.2×10^3 V

(19) $16.3\,\mu$ V

(20) (a) $k\,Q \ln(1.8)/L$, (b) Zero

(21) $-2\,\pi\,k\,\lambda/3$

(22) $k\,\lambda(\pi + 2\ln 2)$

(23) (a) -3.02×10^6 V, (b) -1.51×10^6 V

(24) $z = \pm\sqrt{3}\,R$

(25) $V = 2\,\pi\,\sigma\,k \left(\sqrt{R_2^2 + a^2} - \sqrt{R_1^2 + a^2}\right)$

(26) 2331 V

(28) $V = \pi\,\alpha\,k\left(R\sqrt{R^2+a^2} + a^2 \ln\left[a/\left\{R+\sqrt{R^2+a^2}\right\}\right]\right)$

(29) (a) $r = \sqrt{3/2}\,R$, (b) $V_R - V_0 = -kQ/2R$

(30) (a) $V_r = (\alpha R^3/12\,\epsilon_\circ)(4 - r^3/R^3)$ for $0 \le r \le R$, (b) $V_r = \alpha R^4/(4\,\epsilon_\circ r)$ for $r \ge R$

(31) (a) 1.8×10^6 V, (b) 1.8×10^6 V, (c) 1.8×10^6 V, (d) 1.2×10^6 V

(32) (a) 4.2×10^{14} electrons, (b) 1.33×10^{-4} C/m^2

(33) 150 V

(35) (a) $q_a = Q\,a/(a+b)$, $q_b = Q\,b/(a+b)$, (b) $V = k\,Q/(a+b)$

(36) 109.86 V

(37) (a) $E_x = (-6\,x - 6\,y^2 + 4\,z)$ V/m, $E_y = (-3\,x^2 - 12\,y^2 + 4\,z)$ V/m, $E_z = (-3\,x^2 - 6\,y^2 + 4)$ V/m, (b) $E_x = -4$ V/m, $E_y = -28$ V/m, $E_z = -32$ V/m

(38) $E_r = \pm\,2\,k\,p/r^3$ (+ when $\theta = 0$ and − when $\theta = \pi$)

(39) (a) 900 kV, (b) 3×10^{-5} C = 30 μC, (c) 135 kV, 67.5 kV/m

(40) 10.7 hp

Chapter 23

(1) 150 μ C

(2) 80 000 V

(3) 3×10^{-5} C

(4) 3.54 n m

(5) (a) 177 pF, (b) 3.54 n C, 8.85×10^{-8} C/m^2 (c) 10 kV

(6) 79.65 n C

(7) 3.475 n C

(11) (a) 227 pF, (b) 353 V

(12) 40 pF

(13) (a) 2×10^{-10} F, (b) 10 nC

(14) 708 μ F

(15) 4.41 f F

(16) (a) 53.1 pF, (b) 376.6 V

(17) (a) 750 μ C, (b) 33 μ C, 22.7 V

(18) (a) 10.6 n C, (b) 210 n C

(19) 1.000 578

(20) (a) 1.77 nF, 21.24 n C, 7.97 nF, 95.58 n C, (b) 30.98 n C/m^2, 3500 N/C

(21) (a) 5 μ F, (b) $Q_1 = 18$ μ C, $Q_2 = 27$ μ C, (c) $\Delta V_1 = \Delta V_2 = 9$ V

(22) (a) 1.2 μ F, (b) $Q_1 = Q_2 = 10.8$ μ C, (c) $\Delta V_1 = 5.4$ V, $\Delta V_2 = 3.6$ V

(23) (a) 2.2 μ F, (b) $Q_1 = 6\ \mu$ C, $Q_2 = Q_3 = 7.2\ \mu$ C, (c) $\Delta V_1 = 6$ V, $\Delta V_2 = 3.6$ V, $\Delta V_3 = 2.4$ V
(24) (a) 484 μ C, (b) 198 μ C, (c) 96 μ C, (d) 44 μ C
(25) (a) $Q_1 = 50\ \mu$ C, $Q_3 = 40\ \mu$ C, (b) $\Delta V = 35$ V
(26) (a) 6 μ F, (b) $\Delta V = 35$ V
(27) (a) 2 C/5, (b) C, (c) 5 C/3, (d) 11 C/6
(28) (a) 9.6 μ C, (b) 24 μ C, (c) 40 μ C, (d) 44 μ C
(29) (a) 65 μ F, (b) 750 μ C, (c) 11.54 V, (d) $Q_{1\mathrm{f}} = 28.85\ \mu$ C, $Q_{2\mathrm{f}} = 46.13\ \mu$ C
(33) 44.25 n J
(34) (a) $U_1 = 200\ \mu$ J, $U_2 = 300\ \mu$ J, (b) Yes, $U_{\mathrm{eq}} = U_1 + U_2$
(35) (a) $\Delta V_f = 5$ V, $Q_{1\mathrm{f}} = Q_{2\mathrm{f}} = 25\ \mu$ C, (b) $U_{\mathrm{i}} = 250\ \mu$ J, $U_{\mathrm{f}} = 125\ \mu$ J, $U_{\mathrm{i}} > U_{\mathrm{f}}$
(36) 9.5%
(37) 50 kV/m, 0.011 J/m^3, 8.69×10^{-8} J
(38) 0.02 J
(39) (a) 40 V, (b) 800 J

Chapter 24

(1) 10^{19} elecrons/s
(2) (a) 600 C, (b) 3.75×10^{21} electrons
(3) (a) 9632 C, (b) 5.35 A
(4) (a) 5×10^{18} electrons per second, (b) 0.8 A
(5) (a) 60 C, (b) 30 A
(6) (a) $I = 2(1 + t)$, 6 A, (b) $J = 2 \times 10^4(1 + t)$, 60 kA/m^2
(7) (a) 2387.3 A/m^2, (b) 596.8 A/m^2
(8) (a) $I_{\mathrm{Iron}} = 8$ A, $J_{\mathrm{Iron}} = 4.07 \times 10^5$ A/m^2, (b) $J_{\mathrm{Copper}} = 2.55 \times 10^8$ A/m^2
(9) 1.04×10^{-3} m/s $\simeq 1$ mm/s
(10) 7.32×10^{-1} V/m
(11) (a) 1.59×10^6 A/m^2, (b) 50 V/m, (c) $3.14 \times 10^{-5}\ \Omega$.m, $3.18 \times 10^4\ (\Omega.\mathrm{m})^{-1}$
(12) 3.975 Ω
(13) (a) 3.9×10^5 A/m^2, (b) 6.357×10^{-3} V/m, (c) 3.184×10^{-3} V, (d) $6.366 \times 10^{-4}\ \Omega$, (e) 23.91°C
(14) (a) 18.85 A, (b) $5.3 \times 10^{-6}\ \Omega$, 10^{-4} V
(15) (a) 1.776×10^{-2} m, (b) $9.57 \times 10^{-7}\ \Omega$, (c) 10.45 A, 2.45 μm/s
(16) 78%

(17) $27.8\ \Omega$

(18) $5.25 \times 10^{-3}\ \Omega$

(19) (a) $1.892 \times 10^{-8}\ \Omega$.m, (b) 1.06×10^{7} A/m^2, (c) 8.33 A, (d) $0.012\ \Omega$, (e) 1.13 mm/s, (f) 1 V

(20) $R_{\circ n} = 4\ \Omega, R_{\circ c} = 5\ \Omega$

(21) (a) 0.25 A, (b) 960 Ω, (c) 0.42 A, 576 Ω

(22) (a) 88 Ω, (b) 2.5 A, (c) 163.6 W

(23) (a) 2.4 Ω, (b) 10 A, (c) 102 mm

(24) (a) 0.02 A, (b) 20 V

(25) (a) 0.9 kW. h, (b) 31.5 piaster

(26) (a) 31.83 A/m^2, (b) 6.25×10^{14} electrons/s, (c) 0.5 W, (d) 4.19×10^{7} m/s, (e) 4.75×10^{12} electrons/m^3

(27) 231.25 Ω

(28) (a) 2 A, (b) 10 V

(29) (a) 0.5 Ω, (b) 8.25 V

(30) (a) 0.05 Ω, (b) 0.15 Ω

(31) (a) 29 Ω, (b) 3.3%, (c) 0.1 A, No

(32) (a) 2.8 Ω, (b) 14 V

(33) (a) 6 Ω, (b) 14 V

(34) (a) $5\,R/2$, (b) R, (c) $3\,R/5$, (d) $6\,R/11$

(35) (a) 2.4 A, 1.2 A, 1.2 A, 2.4 A, (b) 3 A, 3 A, 3 A, 3 A, (c) 4 A, 6 A, 2 A, 2 A, (d) 2 A, 3 A, 6 A

(36) (a) 1.5 MΩ, (b) 0.7 MΩ

(37) (a) 960 Ω, (b) 0.2 A, (c) 192 V, (d) 38.4 W

(38) (a) 6 Ω, (b) 2 A, (c) $I_3 = 1.5$ A, $I_4 = 0.5$ A, (d) $P_1 = 4$ W, $P_2 = 9$ W, $P_3 = 9$ W, $P_4 = 3$ W

(39) $I_1 = -1$ A, $I_2 = 1$ A, $I_3 = 2$ A

(40) $I_1 = -14/11$ A, $I_2 = -18/11$ A, $I_3 = -32/11$ A

(41) $I_1 = 2$ A, $I_2 = 2$ A, $I_3 = -4$ A

(42) $\varepsilon_1 = 9$ V, $I_2 = -2.5$ A, $I_3 = -2$ A

(43) $I_1 = 0.5$ A, $I_2 = -1$ A, $I_3 = -0.5$ A, $I_4 = 0$

(46) 39. 7%

(47) (a) 2 s, 24 μC, 12 μA, (b) 1.39 s

(49) (a) 1.1 mC, 4.4 mA, (b) 0.15 mC, 0.6 μA

(50) $\tau_{\text{Before}} = 2$ s, $\tau_{\text{After}} = 0.75$ s, $I_{\text{Switch}} = 0.6\ \text{mA} + (0.2\ \text{mA})e^{-t/0.75}$

Chapter 25

(1) (a) down, (b) to the left, (c) in the plane of the page and perpendicular to $\vec{v}$ and $\vec{B}$, (d) up, (e) no force, (f) into the page, (g) into the page, (h) out of the page

(2) (a) to the left, (b) no deflection, (c) out of the page, (d) to the right

(3) 24.6° or 155.4°

(4) $-1.82 \times 10^{-13}\,\vec{\mathrm{k}}$ (N), along the negative z-axis

(5) 6.4×10^{-17} N, downwards

(6) 3.845×10^{-26} kg, Sodium ion

(7) (a) 0.167 m, (b) 131 μs, (c) 6.4×10^{-11}N

(8) (a) 1.548×10^{7} m/s, (b) 7.43×10^{-12} N, (c) 0.215 m, (d) 1.1×10^{15} m/s^2

(9) (a) 1.708×10^{-3} m, (b) 3.577 ns, (c) 0.014 m

(10) (a) 60°, 62.625 cm, (b) 39.7°, 14.44 cm, (c) 21.2°

(11) 3.75×10^{4} m/s

(12) 20.5 mT

(13) (a) and (b) The magnetic field is out of page, the left plate is at a higher electric potential for the left pair, and the right plate is at a higher electric potential for the right pair. Note that, these polarities are reversed when the magnetic field is into the page in the case of a clockwise path, (c) 8.54 cm

(14) (a) 7.5×10^{4} m/s, (b) 40.56 cm

(15) (a) 2.4×10^{5} m/s, (b) 9.96 mm

(16) (a) 7.38×10^{-11} m^3/C, (b) 5.53 μV, (c) 1.11×10^{-3} V/m

(17) (a) 5.85×10^{28} electrons/m^3, (b) 5.86×10^{28} atoms/m^3, the number of charge carriers in silver is almost one electron per atom, (c) 1.424×10^{-4} V/m

(18) (a) 6.67×10^{-5} m/s, (b) 5.854×10^{28} electrons/m^3, (c) point b is at higher potential

(19) 0.525 N

(20) (a) 1.875×10^{-1} N, 323.1° from the x-axis in the xy plane

(21) $F_{ab} = 0$, $F_{bc} = 0.5\,I\,LB$, into the page, $F_{cd} = 0.5\,I\,LB$ out of the page, $\sum F = 0$ as must be for a closed loop

(22) 0.625 A (when $g = 10$ m/s^2)

(23) $F_{ab} = 0$, $F_{bc} = 0.1$ N, $F_{cd} = 0.05$ N, $F_{de} = 0.1$ N, $F_{ef} = 0$

(24) $F = 2\pi R\,I\,B \sin\theta$, to the right

(25) (a) 0.628 A·m^2, (b) 6.283×10^{-2} A·m

(26) (a) $\mu = 0.2171\,I\,L^2$ out of the page, (b) $\tau = 0.2171\,I\,L^2 B$ up

(27) 9.425×10^{-3} N·m

(28) (a) $B = mg/\pi\, I\, R$, (b) The same $B = mg/\pi\, IR$ for $0° \leq \theta \leq 90°$

(29) 1.35 N·m, counterclockwise

(30) (a) 21 A·m^2, (b) 53°, (c) 13.42 N·m, the coil will rotate so that $\vec{\mu}$ aligns with $\vec{B}$. Looking down along the y-axis, the loop will rotate in a clockwise direction

(31) (a) $\mu = 1.473 \times 10^{-3}$ A·m^2, (b) $\vec{\mu} = 1.473 \times 10^{-3} \vec{\mathrm{k}}$ (A·m^2), (c) $\vec{\tau} = [2.946 \times 10^{-4}\, \vec{\mathrm{j}} - 4.419 \times 10^{-4} \vec{\mathrm{i}}\,]$ (N·m), only the component of torque along y-axis cause a torque about this axis, while the one along the negative x-axis has no effect on the loop. Looking down along the y-axis, the loop will rotate in a counterclockwise direction

(32) 1.4 rad = 80.2°

(33) (a) 1.3×10^8 m/s, 0.31 s, (b) 5.2 km, no

Chapter 26

(1) (a) $-7.8 \times 10^{-9}\, \vec{\mathrm{k}}\, (T)$, (b) $-4 \times 10^{-9}\, \vec{\mathrm{k}}\, (T)$

(2) 13.7 T

(3) υ^2/c^2

(4) At P, 33.3 μT out of the page and at Q, 33.3 μT out the page

(5) At P, 66.7 μT out of the page and at Q, 66.7 μT into the page

(6) 69.3 μT to the left

(7) (a) Zero, (b) 11.3 μT to the left, (c) Zero

(8) Zero for the two wires that point extends along their length, 2 μT (into the page) for the two vertical wires that they have a 5 cm length, 4 μT (into the page) for the horizontal wire that has a 10 cm length, $B_{\text{tot}} = 8\,\mu$T

(9) 31.42 μT, 22.48 μT, 1.11 μT, 1.16 μT

(10) 7.63 cm, 18.13 cm, 39.79 cm

(11) 21.14 μT out of the page

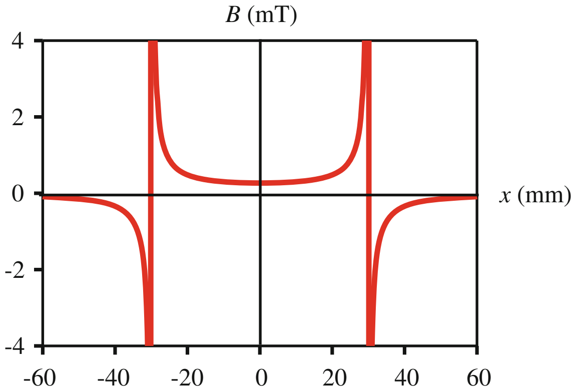
(12) 17.65 μT out of the page

(13) (a) 0.021 T

(14) (a) 16 μC, (b) 64 μN repulsive force

(15) (a) $B(x) = \mu_\circ Ia/\pi(a^2 + x^2)$

(b)



(16) (a) $F_{(2)} = 30\,\mu$T to the right, $F_{(4)} = 90\,\mu$T to the left, (b) $F_{(3)}$ is up, $F_{(1)}$ is down, (c) 60 μT to the left

(17) (a) $B = 0$ (for $r < R$) and $B = \mu_\circ I/2\pi r$ (for $r \geq R$)

(18) $\oint_{C_1} \vec{B} \bullet d\vec{s} = 10\mu_\circ$, $\oint_{C_3} \vec{B} \bullet d\vec{s} = -10\mu_\circ$, $\oint_{C_3} \vec{B} \bullet d\vec{s} = 0$, No one

(19) $B_a = 100\,\mu$T toward top of page, $B_b = 50\,\mu$T toward bottom of page

(20) 250 μT, 500 μT, 250 μT

(21) $B_{r<a} = 0$, $B_{a<r<b} = [\mu_\circ I/2\pi r][(r^2 - a^2)/(b^2 - a^2)]$, $B_{r>b} = \mu_\circ I/2\pi r$

(22) $B = \mu_\circ nI$

(23) $B = 0.503$ T

(24) (a) 400 turns per layer, (b) 3.0 T

(25) 60.3 μT

(26) (a) 5 layers, (b) 47.43 m

(27) $B(r) = \mu_\circ NI/2\pi r$

(28) 2×10^{-3} T

(29) 0.024 T, 0.022 T

(30) (b) $B = \frac{1}{2}\mu_\circ\lambda$

(31) $\vec{B}_a = -\mu_\circ\lambda\vec{\mathrm{i}}$, $\vec{B}_b = 0$, $\vec{B}_c = +\mu_\circ\lambda\vec{\mathrm{i}}$

(32) (a) 0.2 A, (b) 22.6×10^9 V·m/s, (c) 0.5 μT

(33) (a) 1.079×10^{12} V/m.s, (b) 3×10^{-7} T

(34) 2.57×10^{-34} J·s, 0, -9.27×10^{-24} J/T, 1.85×10^{-23} J/T

(35) (a) For $m_\ell = 0$ we get $L_z = 0$, $\mu_{\ell,z} = 0$, (b) For $m_\ell = 0$ we get $U_\ell = 0$, (c) For $m_\ell = -2$ we get $L_z = 2.1 \times 10^{-34}$ J·s, $\mu_{\ell,z} = 1.85 \times 10^{-23}$ J/T, $U_\ell = -7.42 \times 10^{-24}$ J

(36) $U_s = \pm 4.635 \times 10^{-24}$ J, $\Delta U_s = 9.27 \times 10^{-24}$ J

(37) 6.489×10^{-22} J = 4.056×10^{-3} eV
(38) 1.105×10^{2} A/m along the disk axis, 1.192×10^{27} atoms
(39) (a) $\vec{\mu}_{\text{Before}} = 0$, $\vec{\mu}_{\text{After}}$ is out of page, (b) Counterclockwise, (c) Into of the page
(40) -5×10^{-5}
(41) (a) 1.8×10^{3} A/m, 2.2619467 mT, 0.452 389 34 μWb, (b) 3.6×10^{-3} A/m, 2.2619422 mT, 0.452 388 44 μWb
(42) (a) 4×10^{3} A/m, 5.026548 m T (b) 9.2×10^{-2} A/m, 5.026663 m T,
(43) 2.72×10^{-1} A/m, 5.026890 mT
(44) 5.58×10^{5} A/m, 0.7 T
(45) (a) 1.6×10^{6} A/m, (b) 15.98 A.m^2, (c) 8 m.N
(46) 2.64 T
(47) 2×10^{-4} T.m/A = 159.2 $\mu_\circ$

Chapter 27

(1) (a) 0.06 Wb, (b) 64.6°
(2) 0.12 V, 0.08 A
(3) As the south pole of the magnet is pushed into the loop, the magnetic flux *increases out* of the right face of the loop. To oppose this increase, the flux produced by the induced current must be *into* the right face of the loop, so the induced current must be from *right* to *left* in the resistor
(4) As the north pole of the magnet recedes from the loop, the magnetic flux *decreases into* the left face of the loop. To oppose this decrease, the flux produced by the induced current must be *into* the left face of the loop, so the induced current must be from *left* to *right* in the resistor
(5) -200 V
(6) Clockwise for the inside loop and Counterclockwise for the outside loop
(7) 1.57×10^{-2} V
(8) 0.005 V
(9) (a) $(0.8 - 10^{-3}t)$ (T), (b) $\pi \times 10^{-5}$ V
(10) (a) Clockwise, (b) counterclockwise, (c) counterclockwise, (d) clockwise, (e) counterclockwise, (f) clockwise, (g) clockwise when Φ_B decreases and counterclockwise when Φ_B increases, (h) no induced current
(11) (a) Clockwise, (b) 4.91×10^{-2} V, (c) 2.5×10^{-2} A
(12) (a) Clockwise, (b) 3.47×10^{-1} V, (c) 2.31×10^{-1} A

(13) (a) Clockwise, (b) 0.94 mV, (c) 0.38 mA

(14) (a) Opposite to the solenoid's current, (b) 0.2 mV, (c) 51 μA, (d) Opposite to the solenoid's new current, 0.2 V, 51 mA

(15) (a) 5.89 mV, (b) 23.1 mJ, (c) 8.97×10^{-3} °C

(16) (a) 8.011×10^{-3} V, (b) $4.48 \times 10^{-2}\ \Omega$, (c) 178.8 mA, (d) 1.43×10^{-3} W

(17) (a) 235.6 mV, (b) 1.04×10^{-4} J,

(18) (a) $\Phi_B = \mu_\circ I\, b\, \mathrm{In}(1 + x/a)/2\pi$, (b) $\varepsilon = \mu_\circ I\, b\, v/[2\pi(x+a)]$, $F = \{\mu_\circ I\, b\, v/[2\pi(x+a)]\}^2/(Rv)$

(19) 0.18 V

(20) 0.5 T

(21) 0.6 V

(22) (a) 3.75 V, (b) 140.6 mN, (c) 1.406 W

(23) (a) $I = BLvA_{\mathrm{rod}}/[2(vt+L)\rho]$, (b) $P = B^2L^2v^2A_{\mathrm{rod}}/[2(vt+L)\rho]$

(24) $v_t = m\, g\, R/B^2L^2 = 0.33$ m/s

(25) (a) $B\, L\, g\ \sin\theta \cos\theta\ t$, (b) The near side has a higher potential

(26) $v = (\varepsilon_\circ/BL)[1 - e^{-(B^2L^2/mR)t}]$, $v_t = \varepsilon_\circ/BL$

(27) 30 V

(28) 311 V

(29) 198 turns

(30) 4.4 rev/s

(31) 0.156 A

(32) 3.536 A, 5 A

(33) (a) 302.5 Ω, (b) 806.7 Ω for the 60-W bulb and 484 Ω for the 100-W bulb

(34) 110 V

(35) 155.6 V, 14.14 A

(36) (a) 2200 W, (b) 0 and 4400 W

(37) (a) Step-down, (b) 0.12, 8.3

(38) 46

(39) (a) Step-down, (b) 2.7

(40) $R_{\mathrm{eq}} = (N_P/N_S)^2\ R$

(41) (a) 55.6 kV, (b) 88.96 MW, 10.1%

(42) 5×10^{-9} N down

(43) 10^{-9} N up

(44) 6.283×10^{-6} N/C

(45) (a) $\pi r^2 dB/dt$, $r < R$, (b) $\frac{1}{2} r dB/dt$, $r < R$, (c) $\frac{1}{2}(R^2/r) dB/dt$, $r < R$, (d) $\pi r^2 dB/dt$, $r < R$, (e) $\pi R^2 dB/dt$, $r = R$, $\pi R^2 dB/dt$, $r > R$

Chapter 28

(1) 1.5 V
(2) 1 H
(3) 25 mH
(4) 12.57 mH
(5) 892 turns
(6) 23 turns
(7) (a) 157.1 mH, (b) 10.2 V
(8) (a) 10 V, (b) 8 mH, (C) 0.1 J
(9) (a) 63.33 cm, (b) 795.8 m, (c) 68.1 Ω
(10) (a) (3/10) H (when in series), (b) (2/30) H (when in parallel)
(11) 2.5 mH
(12) 1.2 V
(13) 20 mH
(14) (a) 105.6 mH, (b) 26.39 V
(15) (a) 17.5 mH, (b) 1 mH, (c) −0.25 V
(16) $L_{\text{eq}} = L_1 + L_2 + M_{\text{s}}$, $L_{\text{eq}} = (L_1 L_2 - M_{\text{p}})/(L_1 + L_2 - 2M_{\text{p}})$
(17) (a) 1.592×10^4 J/m^3, (b) 318 mJ
(18) 112.5 mJ
(19) 9.43 μ J
(20) (a) $u_B \simeq 1.6 \times 10^6$ J/m^3, $u_E \simeq 4.4 \times 10^{-4}$ J/m^3, (b) $E = 6 \times 10^8$ N$/C$ =200 $E_{\text{breakdown}}$
(21) 320 V
(22) 174.5 μ J
(24) (a) 2.25 H, (b) 180 mJ, (c) 1.2 T, (d) 3819 A/m, (e) 573 kJ/m^3
(25) (a) 0.11 τ, (b) 0.69 τ, (c) 2.3 τ
(26) (a) 2.15 s, (b) 4 A
(27) 20 μs, (b) 92.1 μs, (c) 6 mA
(28) (a) 2.49 s, (b) 20.1 Ω
(29) (a) 300, (b) 16 Ω, 80 mH
(30) (a) $\varepsilon_L(t) = -\varepsilon \exp(-t/\tau)$, (b) $P_{\text{output}}(t) = (\varepsilon^2/R)[1 - \exp(-t/\tau)]$, (c) $P_{\text{diss}}(t) = (\varepsilon^2/R)[1 - \exp(-t/\tau)]^2$, (d) $dU_B(t)/dt = (\varepsilon^2/R)[1 - \exp(-t/\tau)]\exp(-t/\tau)$, (e) $-0.368\,\varepsilon$, 0.632 (ε^2/R), 0.3996 (ε^2/R), 0.2326 (ε^2/R)
(31) (a) $I_1 = I_2 = 1.2$A, $I_3 = 0$, (b) $I_1 = 2$ A, $I_2 = 2/3$ A, $I_3 = 4/3$ A, (c) $I_1 = 0$, $I_2 = -2.25$A, $I_3 = -2.25$A, (d) $I_1 = I_2 = I_3 = 0$

(32) (a) $\varepsilon_L(t) = +(9V)\exp(-[10^{-5}\,\text{s}^{-1}]t)$, (b) 9 V, 0

(33) 12.7 μH

(34) (a) 79.6 Hz, (b) 0.2A, (c) (0.2 A) $\sin[(500\,\text{s}^{-1})t]$ (d) 10^{-2} J

(35) 1.59 mH, 15.92 μF

(36) (a) 0.35 nF, (b) 75.1 μH

(37) 0.5 A

(38) Yes, the circuit oscillates with frequency 2236 Hz

(39) (a) $R_c = 2\,\Omega$, and the circuit will oscillate since $R < R_c$, (b) 95.5 Hz (c) 5.236 ms, 1.5 %, (d) 1.73 Ω

(40) $8.163 \times 10^{-3}\,\Omega$

(41) 0.248 H, 25.6 nF

(42) (a) 311 V, (b) $v = (311\text{ V})\sin(100\,\pi t)$

(43) (a) 110 V, (b) 5.5 A, (c) 50 cycle/s

(44) (a) 110 A, (b) 0.58 A

(45) (a) zero, (b) 82.9 mA

(46) 10 Ω, 22 A

(47) 2.21 kHz

(48) (a) 4.375 kΩ, (b) 5.029×10^{-2} A, (c) $-46.7°$ (The current leads the source voltage by 46.7°), (d) 7.587 W, (e) 150.9 V, 160.08 V

(49) (a) 50 Hz, (b) The voltages across the resistor and across the capacitor are not in phase, the rms voltage across the source will not be the sum of their rms voltages

(50) (a) 10 Ω, (b) 11 A, (c) 88 V, 99 V, 33 V

(51) (a) 23.9 mH, 884 μF, (b) 0.8, (c) 968 W

(52) (a) 37.7 Ω, 4.1 Ω, 48.5 Ω, (b) 2.267 A, 3.206 A, (c) 112.2 V, 120.9 V, 13.1 V, $v_R = (112.2\text{ V})\sin(377\,t)$, $v_L = (120.9\text{ V})\sin(377\,t + \pi/2)$, $v_C = (13.1\text{ V})\sin(377\,t - \pi/2)$, 79.3 V, 85.5 V, 9.3V, (d) 43.8°, 179.9 W

(54) (a) 2.639 nF, (b) 37.5 mA

(55) (a) 1508 Ω, 1508 Ω, (b) 5.63 W

(56) (a) 456.4 rad/s, (b) 73.03 Ω, 73.03 Ω, 20 Ω (c) 0, 12.5 A, (d) 250 V, 913 V, 913 V

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