Pietro Giuseppe Frè

Gravity, a Geometrical Course

Volume 2: Black Holes, Cosmology and Introduction to Supergravity

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Volume 2: Black Holes, Cosmology and Introduction to Supergravity

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ISBN 978-94-007-5442-3 ISBN 978-94-007-5443-0 (eBook)
DOI 10.1007/978-94-007-5443-0
Springer Dordrecht Heidelberg New York London

Library of Congress Control Number: 2012950601

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

This book is dedicated to my beloved daughter Laura and to my darling wife Olga.

Preface

This book grew out from the Lecture Notes of the course in General Relativity which I gave for more than 15 years at the University of Torino. That course has a long tradition since it was attached to the Chair of Relativity created at the beginning of the 1960s for prof. Tullio Regge. In the years 1990–1996, while prof. Regge was Member of the European Parliament the course was given by my long time excellent friend and collaborator prof. Riccardo D'Auria. In 1996 I had the honor to be appointed on Regge's chair[1] and I left SISSA of Trieste to take this momentous and challenging legacy. Feeling the burden of the task laid on my shoulders I humbly tried to do my best and create a new course which might keep alive the tradition established by my so much distinguished predecessors. In my efforts to cope with the expected standards, I obviously introduced my own choices, view-points and opinions that are widely reflected in the present book. The length of the original course was of about 120 hours (without exercises). In the new $3 + 2$ system introduced by the Bologna agreements it was split in two courses but, apart from minor readjustments, I continued to consider them just as part one and part two of a unique entity. This was not a random choice but it sprang from the views that inspired my teaching and the present book. I always held the opinion that University courses should be long, complex and articulated in many aspects. They should not aim at a quick transmission of calculating abilities and of ready to use information, rather they should be as much formative as informative. They should offer a general overview of a subject as seen by the professor, in this way giving the students the opportunity of developing their own opinions through the critical absorption of those of the teacher.

One aspect that I always considered essential is the historical one, concerning on one side the facts, the life and the personalities of the scientists who shaped our present understanding, on the other hand concerning the usually intricate development of fundamental ideas.

The second aspect to which I paid a lot of attention is the use of an updated and as much as possible rigorous mathematical formalism. Moreover I always tried to

[1] At that time Regge had shifted from the University to the Politecnico of Torino.

convey the view that Mathematics should not be regarded as a technical tool for the solution of Physical Problems or simply as a language for the formulation of Physical Laws, rather as an essential integral part of the whole game.

The third aspect taken not only into account but also into prominence, is the emphasis on important physical applications of the theory: not just exercises, from which I completely refrained, but the full-fledged *ab initio* development of relevant applications in Astrophysics, Cosmology or Particle Theory. The aim was that of showing, from A to Z, as one goes from the first principles to the actual prediction of experimentally verifiable numbers. For the reader's or student's convenience I included the listing of some computer codes, written in MATHEMATICA, that solve some of the posed problems or parts thereof. The aim was, once again formative. In the course of their theoretical studies the students should develop the ability to implement formal calculations on a machine, freeing themselves from the slavery to accidental errors and focusing instead all their mental energies on conceptual points. Furthermore implementation of formulae in a computer code is the real test of their comprehension by the learners, more efficient in its task than any *ad-hoc prepared* exercise.

As for the actual choice of the included and developed material, I was inspired by the following view on the role of the course I used to gave, which I extended as a mission to the present book. General Relativity, Quantum Mechanics, Gauge Theories and Statistical Mechanics are the four pillars of the Physical Thought developed in the XXth century. That century laid also the foundations for new theories, whose actual relations with the experimental truth and with observations will be clarified only in the present millennium, but whose profound influence on the current thought is so profound that no-one approaching theoretical studies can ignore them: I refer to supersymmetry, supergravity, strings and branes. The role of the course in *General Relativity* which I assumed as given, was not only that of presenting Einstein Theory, in its formulation, historical development and applications, but also that of comparing the special structure of Gravity in relation with the structure of the Gauge-Theories describing the other fundamental interactions. This was specially aimed at the development of critical thinking in the student and as a tool of formative education, preparatory to the study of unified theories.

The present one is a Graduate Text Book but it is also meant to be a self-contained account of Gravitational Theory attractive for the person with a basic scientific education and a curiosity for the topic who would like to learn it from scratch, being his/her own instructor.

Just as the original course given in Torino after the implementation of the Bologna agreements, this book is divided in two volumes:

1. **Volume 1**: *Development of the Theory and Basic Physical Applications*.
2. **Volume 2**: *Black Holes, Cosmology and Introduction to Supergravity*.

Volume 1, starting from a summary of Special Relativity and a sketchy historical introduction of its birth, given in Chap. 1, develops the general current description of the physical world in terms of gauge connections and sections of the bundles on which such connections are constructed. The special role of Gravity as the gauge theory of the tangent bundle to the base manifold of all other bundles is emphasized. The mathematical foundations of the theory are contained in Chaps. 2 and 3. Chapter 2 introduces the basic notions of differential geometry, the definition of manifolds and fibre-bundles, differential forms, vector fields, homology and cohomology. Chapter 3 introduces the theory of connections and metrics. It includes an extensive historical account of the development of mathematical and physical ideas which eventually lead to both general relativity and modern gauge theories of the non-gravitational interactions. The notion of geodesics is introduced and exemplified with the detailed presentation of a pair of examples in two dimensions, one with Lorentzian signature, the other with Euclidian signature. Chapter 4 is devoted to the Schwarzschild metric. It is shown how geodesics of the Schwarzschild metric retrieve the whole building of Newtonian Physics plus corrections that can be very tiny in weak gravitational fields, like that of the Solar System, or gigantic in strong fields, where they lead to qualitatively new physics. The classical tests of General Relativity are hereby discussed, perihelion advance and the bending of light rays, in particular. Chapter 5 introduces the Cartan approach to differential geometry, the vielbein and the spin connection, discusses Bianchi identities and their relation with gauge invariances and eventually introduces Einstein field equations. The dynamical equations of gravity and their derivation from an action principle are developed in a parallel way to their analogues for electrodynamics and non-Abelian gauge theories whose structure and features are constantly compared to those of gravity. The linearization of Einstein field equations and the spin of the graviton are then discussed. After that the bottom-up approach to gravity is discussed, namely, following Feynman's ideas, it is shown how a special relativistic linear theory of the graviton field could be uniquely inferred from the conservation of the stress-energy tensor and its non-linear upgrading follows, once the stress-energy tensor of the gravitational field itself is taken into account. The last section of Chap. 5 contains the derivation of the Schwarzschild metric from Einstein equations. Chapter 6 addresses the issue of stellar equilibrium in General Relativity, derives the Tolman Oppenheimer Volkhoff equation and the corresponding mass limits. Next, considering the role of quantum mechanics the Chandrasekhar mass limits for white dwarfs and neutron stars are derived. Chapter 7 is devoted to the emission of gravitational waves and to the tests of General Relativity based on the slowing down of the period of double star systems.

Volume 2, after a short introductory chapter, the following two chapters are devoted to Black Holes. In Chap. 2 we begin with a historical account of the notion of black holes from Laplace to the present identification of stellar mass black holes in the galaxy and elsewhere. Next the Kruskal extension of the Schwarzschild solution is considered in full detail preceded by the pedagogical toy example of Rindler space-time. Basic concepts about Future, Past and Causality are introduced next. Conformal Mappings, the Causal Structure of infinity and Penrose diagrams are discussed and exemplified.

Chapter 3 deals with rotating black-holes and the Kerr-Newman metric. The usually skipped form of the spin connection and of the Riemann tensor of this metric is calculated and presented in full detail, together with the electric and magnetic field strengths associated with it in the case of a charged hole. This is followed by a careful discussion of the static limit, of locally non-rotating observers, of the horizon and of the ergosphere. In a subsequent section the geodesics of the Kerr metric are studied by using the Hamilton Jacobi method and the system is shown to be Liouville integrable with the derivation of the fourth Hamiltonian (the Carter constant) completing the needed shell of four, together with the energy, the angular momentum and the mass. The last section contains a discussion of the analogy between the Laws of Thermodynamics and those of Black Hole dynamics including the Bekenstein-Hawking entropy interpretation of the horizon area.

Chapters 4 and 5 are devoted to cosmology. Chapter 4 contains a historical outline of modern Cosmology starting from Kant's proposal that nebulae might be different island-universes (galaxies in modern parlance) to the current space missions that have measured the Cosmic Microwave Background anisotropies. The crucial historical steps in building up the modern vision of a huge expanding Universe, which is even accelerating at the present moment, are traced back in some detail. From the Olbers paradox to the discovery of the stellar parallax by Bessel, to the Great Debate of 1920 between Curtis and Shapley, how the human estimation of the Universe's size enlarged, is historically reported. The discovery of the Cepheides law by Henrietta Leavitt, the first determination of the distance to nearby galaxies by Hubble and finally the first measuring of the universal cosmic recession are the next episodes of this tale. The discovery of the CMB radiation, predicted by Gamow, the hunt for its anisotropies and the recent advent of the Inflationary Universe paradigm are the subsequent landmarks, which are reported together with biographical touches upon the life and personalities of the principal actors in this exciting adventure of the human thought.

Chapter 5, entitled *Cosmology and General Relativity: Mathematical Description of the Universe*, provides a full-fledged introduction to Relativistic Cosmology. The chapter begins with a long mathematical interlude on the geometry of coset manifolds. These notions are necessary for the mathematical formulation of the Cosmological Principle, stating homogeneity and isotropy, but have a much wider spectrum of applications. In particular they will be very important in the subsequent chapters about Supergravity. Having prepared the stage with this mathematical preliminaries, the next sections deal with homogeneous but not isotropic cosmologies. Bianchi classification of three dimensional Lie groups is recalled, Bianchi metrics are defined and, within Bianchi type I, the Kasner metrics are discussed with some glimpses about the cosmic billiards, realized in Supergravity. Next, as a pedagogical example of a homogeneous but not isotropic cosmology, an exact solution, with and without matter, of Bianchi type II space-time is treated in full detail. After this, we proceed to the Standard Cosmological Model, including both homogeneity and isotropy. Freedman equations, all their implications and known solutions are discussed in detail and a special attention is given to the embedding of the three type of standard cosmologies (open, flat and closed) into de Sitter space. The concept of particle and event horizons is next discussed together with the derivation of exact formulae for read-shift distances. The conceptual problems (horizon and flatness) of the Standard Cosmological Model are next discussed as an introduction to the new inflationary paradigm. The basic inflationary model based on one scalar field and the slow rolling regime are addressed in the following sections with fully detailed calculations. Perturbations, the spectrum of fluctuations up to the evaluation of the spectral index and the principles of comparison with the CMB data form the last part of this very long chapter.

The last four chapters of the book provide a conceptual, mathematical and descriptive introduction to Supergravity, namely to the Beyond GR World.

Chapter 6 starts with a historical outline that describes the birth of supersymmetry both in String Theory and in Field Theory, touching also on the biographies and personalities of the theorists who contributed to create this entire new field through a complicated and, as usual, far from straight, path. The chapter proceeds than with the conceptual foundations of Supergravity, in particular with the notion of Free Differential Algebras and with the principle of rheonomy. Sullivan's structural theorems are discussed and it is emphasized how the existence of p-forms, that close the supermultiplets of fundamental fields appearing in higher dimensional supergravities, is at the end of the day a consequence of the superPoincaré Lie algebras through their cohomologies. The structure of M-theory, the constructive principles to build supergravity Lagrangians and the fundamental role of Bianchi identities is emphasized. The last two sections of the chapter contain a thorough account of type IIA and type IIB supergravities in $D = 10$, the structure of their FDAs, the rheonomic parameterization of their curvatures and the full-fledged form of their field equations.

Chapter 7 deals with the brane/bulk dualism. The first section contains a conceptual outline where the three sided view of branes as 1) classical solitonic solutions of the bulk theory, 2) world volume gauge-theories described by suitable world-volume actions endowed with κ-supersymmetry and 3) boundary states in the superconformal field theory description of superstring vacua is spelled out. Next a New First Order Formalism, invented by the author of this book at the beginning of the XXIst century and allowing for an elegant and compact construction of κ-supersymmetric Born-Infeld type world-volume actions on arbitrary supergravity backgrounds is described. It is subsequently applied to the case of the $D3$-brane, both as an illustration and for the its intrinsic relevance in the gauge/gravity correspondence. The last sections of the chapter are devoted to the presentations of branes as classical solitonic solutions of the bulk theory. General features of the solutions in terms of harmonic functions are presented including also a short review of domain walls and some sketchy description of the Randall-Sundrun mechanism.

Chapter 8 is a bestiary of Supergravity Special Geometries associated with its scalar sector. The chapter clarifies the codifying role of the scalar geometry in constructing the bosonic part of a supergravity Lagrangian. The dominant role among the scalar manifolds of homogeneous symmetric spaces is emphasized illustrating the principles that allow the determination of such U/H cosets for any supergravity theory. The mechanism of symplectic embedding that allows to extend the action of U-isometries from the scalar to the vector field sector are explained in detail within the general theory of electric/magnetic duality rotations. Next the chapter provides a self-contained summary of the most important special geometries appearing in

$D = 4$ and $D = 5$ supergravity, namely Special Kähler Geometry, Very Special Real Geometry and Quaternionic Geometry.

Chapter 9 presents a limited anthology of supergravity solutions aimed at emphasizing a few relevant new concepts. Relying on the special geometries described in Chap. 8 a first section contains an introduction to supergravity spherical Black Holes, to the attraction mechanism and to the interpretation of the horizon area in terms of a quartic symplectic invariant of the U duality group. The second and third sections deal instead with flux compactifications of both M-theory and type IIA supergravity. The main issue is that of the relation between supersymmetry preservation and the geometry of manifolds of restricted holonomy. The problem of supergauge completion and the role of orthosymplectic superalgebras is also emphasized.

Appendices contain the development of gamma matrix algebra necessary for the inclusion of spinors, details on superalgebras and the user guide to Mathematica codes for the computer aided calculation of Einstein equations.

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Acknowledgements

With great pleasure I would like to thank my collaborators and colleagues Pietro Antonio Grassi, Igor Pesando and Mario Trigiante for the many suggestions and discussions we had during the writing of the present book and also for their critical reading of several chapters. Similarly I express my gratitude to the Editors of Springer-Verlag, in particular to Dr. Maria Bellantone, for their continuous assistance, constructive criticism and suggestions.

My thoughts, while finishing the writing of these volumes, that occurred during solitary winter week-ends in Moscow, were frequently directed to my late parents, whom I miss very much and I will never forget. To them I also express my gratitude for all what they taught me in their life, in particular to my father who, with his own example, introduced me, since my childhood, to the great satisfaction and deep suffering of writing books.

Furthermore it is my pleasure to thank my very close friend and collaborator Aleksander Sorin for his continuous encouragement and for many precious consultations.

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Chapter 1
Introduction

The Two Most Powerful Warriors Are Patience and Time
Leo Tolstoy

The goal of this second volume is two-fold.

On one hand we want to complete the presentation of General Relativity by analyzing two of its main fields of application:

1. Black Holes,
2. Cosmology.

On the other hand we want to introduce the reader to Theory of Gravitation Beyond General Relativity which is Supergravity. The latter invokes, in a way which we hope to be able to explain, Superstrings and also other Branes.

Sticking to the method followed in Volume 1 we will trace the conceptual development of fundamental ideas through history. At the same time we will recast all equations in a mathematical formalism adapted to the embedding of General Relativity into its modern extensions like Supergravity. This is done in order to retrieve the logical development of ideas, which differs from the historical one and constantly requires revisiting Old Theories from the stand-point of New Ones. This was the motivation for the particular and sometimes unconventional way of presenting General Relativity we adopted in the first volume. The reader will fully appreciate the relevance of this strategy when coming to Chap. 6 and to the constructive principles underlying supergravity. The prominence given to the Cartan formulation in terms of vielbein and spin connection and to the role of Bianchi identities will reveal its profound rationale in that chapter. There the reader will find the end-point of a long argument that, starting from Lorentz symmetry leads first to the distinctive features of a gauge theory of the Poincaré connection and then, if one admits the supersymmetry charges, to a new algebraic category, that of Free Differential Algebras encompassing p-forms and a totally new viewpoint on gauging. The p-forms open the window on the world of *branes* and on their dualism with the gravitational theories living in the bulk. In the rich and complex new panorama provided by the Bestiary of Supergravities and of their solutions also Black Holes and Cosmology acquire new perspectives and possibilities.

P.G. Frè, *Gravity, a Geometrical Course*, DOI 10.1007/978-94-007-5443-0_1,

Introducing step by step the necessary mathematical structures and framing historically the development of ideas we promise our patient reader to conduct him smoothly and, hopefully without logical jumps, to the current frontier of Gravitational Theory.

Chapter 2
Extended Space-Times, Causal Structure and Penrose Diagrams

O radiant Dark! O darkly fostered ray
Thou hast a joy too deep for shallow Day!
George Eliot (The Spanish Gypsy)

2.1 Introduction and a Short History of Black Holes

It seems that the first to conceive the idea of what we call nowadays a black-hole was the English Natural Philosopher and Geologist John Michell (1724–1793). Member of the Royal Society, Michell already before 1783 invented a device to measure Newton's gravitational constant, namely the torsion balance that he built independently from its co-inventor Charles Augustin de Coulomb. He did not live long enough to put into use his apparatus which was inherited by Cavendish. In 1784 in a letter addressed precisely to Cavendish, John Michell advanced the hypothesis that there could exist heavenly bodies so massive that even light could not escape from their gravitational attraction. This letter surfaced back to the attention of contemporary scientists only in the later seventies of the XXth century [1]. Before that finding, credited to be the first inventor of black-holes was Pierre Simon Laplace (see Fig. 2.1). In the 1796 edition of his monumental book *Exposition du Système du Monde* [2] he presented exactly the same argument put forward in Michell's letter, developing it with his usual mathematical rigor. All historical data support the evidence that Michell and Laplace came to the same hypothesis independently. Indeed the idea was quite mature for the physics of that time, once the concept of escape velocity v_e had been fully understood.

Consider a spherical celestial body of mass M and radius R and let us pose the question what is the minimum initial vertical velocity that a point-like object located on its surface, for instance a rocket, should have in order to be able to escape to infinite distance from the center of gravitational attraction. Energy conservation provides the immediate answer to such a problem. At the initial moment $t = t_0$ the energy of the missile is:

$$E = \frac{1}{2} m_m v_e^2 - \frac{G M m_m}{R} \tag{2.1.1}$$

where G is Newton's constant. At a very late time, when the missile has reached $R = \infty$ with a final vanishing velocity its energy is just $0 + 0 = 0$. Hence E vanished

P.G. Frè, *Gravity, a Geometrical Course*, DOI 10.1007/978-94-007-5443-0_2,

Fig. 2.1 Pierre Simon Laplace (1749–1827) was born in Beaumont en Auge in Normandy in the family of a poor farmer. He could study thanks to the generous help of some neighbors. Later with a recommendation letter of d'Alembert he entered the military school of Paris where he became a teacher of mathematics. There he started his monumental and original research activity in Mathematics and Astronomy that made him one of the most prominent scientists of his time and qualified him to the rank of founder of modern differential calculus, his work being a pillar of XIXth century Mathematical Physics. A large part of his work on Astronomy was still done under the Ancien Regime and dates back to the period 1771–1787. He proved the stability of the Solar System and developed all the mathematical tools for the systematic calculus of orbits in Newtonian Physics. His results were summarized in the two fundamental books *Mecanique Cèleste* and *Exposition du Système du Monde*. Besides introducing the first idea of what we call nowadays a black-hole, Laplace was also the first to advance the hypothesis that the Solar System had formed through the cooling of a globular-shaped, rotating, cluster of very hot gas (a nebula). In later years of his career Laplace gave fundamental and framing contributions to the mathematical theory of probability. His name is attached to numberless corners of differential analysis and function theory. He received many honors both in France and abroad. He was member of all most distinguished Academies of Europe. He also attempted the political career serving as Minister of Interiors in one of the first Napoleonic Cabinets, yet he was soon dismissed by the First Consul as a person not qualified for that administrative job notwithstanding Napoleon's recognition that he was a great scientist. Politically Laplace was rather cynic and ready to change his opinions and allegiance in order to follow the blowing wind. Count of the First French Empire, after the fall of Napoleon he came on good terms with the Bourbon Restoration and was compensated by the King with the title of marquis

also at the beginning, which yields:

$$v_e = \sqrt{2\frac{GM}{R}} \tag{2.1.2}$$

If we assume that light travels at a finite velocity c, then there could exist heavenly bodies so dense that:

$$\sqrt{2\frac{GM}{R}} > c \tag{2.1.3}$$

In that case not even light could escape from the gravitational field of that body and no-one on the surface of the latter could send any luminous signal that distant observers could perceive. In other words by no means distant observers could see the surface of that super-massive object and even less what might be in its interior.

Obviously neither Michell nor Laplace had a clear perception that the speed of light c is always the same in every reference frame, since Special Relativity had to wait its own discovery for another century. Yet Laplace's argument was the following: let us assume that the velocity of light is some constant number a on the surface of the considered celestial body. Then he proceeded to an estimate of the speed of light on the surface of the Sun, which he could do using the annual light aberration in the Earth-Sun system. The implicit, although unjustified, assumption was that light velocity is unaffected, or weakly affected, by gravity. Analyzing such an assumption in full-depth it becomes clear that it was an anticipation of Relativity in disguise.

Actually condition (2.1.3) has an exact intrinsic meaning in General Relativity. Squaring this equation we can rewrite it as follows:

$$R > r_S \equiv 2\frac{GM}{c^2} \equiv 2m \tag{2.1.4}$$

where r_S is the Schwarzschild radius of a body of mass M, namely the unique parameter which appears in the Schwarzschild solution of Einstein Equations.

So massive bodies are visible and behave qualitatively according to human common sense as long as their dimensions are much larger then their Schwarzschild radius. Due to the smallness of Newton's constant and to the hugeness of the speed of light, this latter is typically extremely small. Just of the order of a kilometer for a star, and about 10^{-23} cm for a human body. Nevertheless, as we extensively discussed in Chap. 6 of Volume 1, sooner or later all stars collapse and regions of space-time with outrageously large energy-densities do indeed form, whose typical linear size becomes comparable to r_S. The question of what happens if it is smaller than r_S is not empty, on the contrary it is a fundamental one, related with the appropriate interpretation of what lies behind the apparent singularity of the Schwarzschild metric at $r = r_S$.

As all physicists know, any singularity is just the signal of some kind of criticality. At the singular point a certain description of physical reality breaks down and it must be replaced by a different one: for instance there is a phase-transition and the degrees of freedom that capture most of the energy in an ordered phase become negligible with respect to other degrees of freedom that are dominating in a disordered phase. What is the criticality signaled by the singularity $r = r_S$ of the Schwarzschild metric? Is it a special feature of this particular solution of Einstein Equations or it is just an instance of a more general phenomenon intrinsic to the laws of gravity as stated by General Relativity? The answer to the first question is encoded in the wording *event horizon*. The answer to the second question is that event horizons are a generic feature of static solutions of Einstein equations.

An event-horizon $\mathfrak{H}$ is a hypersurface in a pseudo-Riemannian manifold $(\mathscr{M}, g)$ which separates two sub-manifolds, one $\mathfrak{E} \subset \mathscr{M}$, named the exterior, can communicate with infinity by sending signals to distant observers, the other $\mathrm{BH} \subset \mathscr{M}$, named the black-hole, is causally disconnected from infinity, since no signal produced in

BH can reach the outside region $\mathfrak{E}$. The black-hole is the region deemed by Michell and Laplace where the escape velocity is larger than the speed of light.

In order to give a precise mathematical sense to the above explanation of event-horizons a lot of things have to be defined and interpreted. First of all what is infinity and is it unique? Secondly which kind of hypersurface is an event-horizon? Thirdly can we eliminate the horizon singularity by means of a suitable analytic extension of the apparently singular manifold? Finally, how do we define causal relations in a curved Lorentzian space-time?

The present chapter addresses the above questions. The answers were found in the course of the XXth century and constitute the principal milestones in the history of black-holes.

Although Schwarzschild metric was discovered in 1916, less than six months after the publication of General Relativity, its analytic extension, that opened the way to a robust mathematical theory of black-holes, was found only forty-five years later, six after Einstein's death. In 1960, the American theorist Martin Kruskal (see Fig. 2.2) found a one-to-many coordinate transformation that allowed him to represent Schwarzschild space-time as a portion of a larger space-time where the locus $r = r_S$ is non-singular, rather it is a well-defined light-like hypersurface constituting precisely the event-horizon [6]. A similar coordinate change was independently proposed the same year also by the Australian-Hungarian mathematician Georges Szekeres [7].

These mathematical results provided a solid framework for the description of the final state in the gravitational collapse of those stars that are too massive to stop at the stage of white-dwarfs or neutron-stars. In Chap. 6 of Volume 1 we already mentioned the intuition of Robert Openheimer and H. Snyder who, in their 1939 paper, wrote: *When all thermonuclear sources of energy are exhausted, a sufficiently heavy star will collapse. Unless something can somehow reduce the star's mass to the order of that of the sun, this contraction will continue indefinitely...past white dwarfs, past neutron stars, to an object cut off from communication with the rest of the universe*. Such an object, could be identified with the interior of the event horizon in the newly found Kruskal space-time. Yet, since the Kruskal-Schwarzschild metric is spherical symmetric such identification made sense only in the case the parent star had vanishing angular momentum, namely was not rotating at all. This is quite rare since most stars rotate.

In 1963 the New Zealand physicist Roy Kerr, working at the University of Texas, found the long sought for generalization of the Schwarzschild metric that could describe the end-point equilibrium state in the gravitational collapse of a rotating star. Kerr metric, that constitutes the main topic of Chap. 3, introduced the third missing parameter characterizing a black-hole, namely the angular momentum J. The first is the mass M, known since Schwarzschild's pioneering work, the second, the charge Q (electric, magnetic or both) had been introduced already in the first two years of life of General Relativity. Indeed the Reissner-Nordström metric,[1] which

[1]Hans Jacob Reissner (1874–1967) was a German aeronautical engineer with a passion for mathematical physics. He was the first to solve Einstein's field equations with a charged electric source

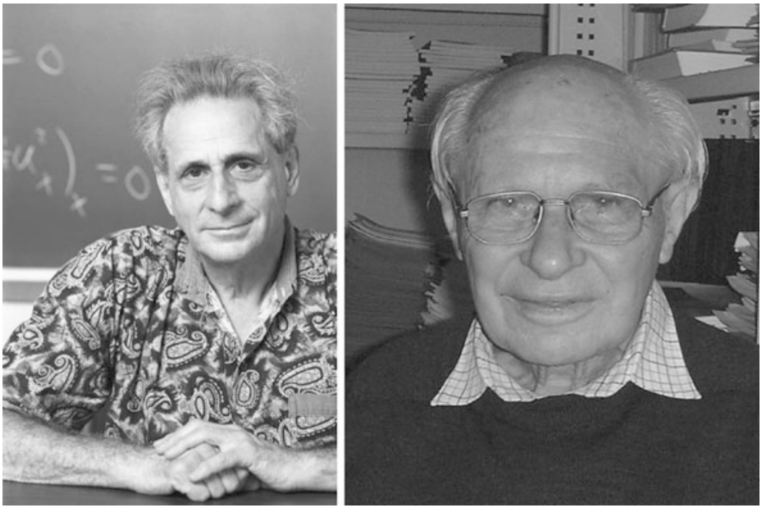

Fig. 2.2 Martin David Kruskal (1925–2006) on the *left* and George Szekeres (1911–2005) on the *right*. Student of the University of Chicago, Kruskal obtained his Ph.D from New York University and was for many years professor at Princeton University. In 1989 he joined Rutgers University were he remained the rest of his life. Mathematician and Physicist, Martin Kruskal gave very relevant contributions in theoretical plasma physics and in several areas of non-linear science. He discovered exact integrability of some non-linear differential equations and is reported to be the inventor of the concept of *solitons*. Kruskal 1960 discovery of the maximal analytic extension of Schwarzschild space-time came independently and in parallel with similar conclusions obtained by Georges Szekeres. Born in Budapest, Szekeres graduated from Budapest University in Chemistry. As a Jewish he had to escape from Nazi persecution and he fled with his family to China where he remained under Japanese occupation till the beginning of the Communist Revolution. In 1948 he was offered a position at the University of Adelaide in Australia. In this country he remained the rest of his life. Notwithstanding his degree in chemistry Szekeres was a Mathematician and he gave relevant contributions in various of its branches. He is among the founders of combinatorial geometry

solves coupled Einstein-Maxwell equations for a charged spherical body, dates back to 1916–1918.

The long time delay separating the early finding of the spherical symmetric solutions and the construction of the axial symmetric Kerr metric is explained by the high degree of algebraic complexity one immediately encounters when spherical

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and he did that already in 1916 [3]. Emigrated to the United States in 1938 he taught at the Illinois Institute of Technology and later at the Polytechnic Institute of Brooklyn. Reissner's solution was retrieved and refined in 1918 by Gunnar Nordström (1881–1923) a Finnish theoretical physicist who was the first to propose an extension of space-time to higher dimensions. Independently from Kaluza and Klein and as early as 1914 he introduced a fifth dimension in order to construct a unified theory of gravitation and electromagnetism. His theory was, at the time, a competitor of Einstein's theory. Working at the University of Leiden in the Netherlands with Paul Ehrenfest, in 1918 he solved Einstein field equations for a spherically symmetric charged body [4] thus extending the Hans Reissner's results for a point charge.

symmetry is abandoned. Kerr's achievement would have been impossible without the previous monumental work of the young Russian theoretician A.Z. Petrov [5]. Educated in the same University of Kazan where, at the beginning of the XIXth century Lobachevskij had first invented non-Euclidian geometry, in his 1954 doctoral dissertation, Petrov conceived a classification of Lorentzian metrics based on the properties of the corresponding Weyl tensor. This leads to the concept of principal null-directions. According to Petrov there are exactly six types of Lorentzian metrics and, in current nomenclature, Schwarzschild and Reissner Nordström metrics are of Petrov type D. This means that they have two double principal null directions. Kerr made the hypothesis that the metric of a rotating black-hole should also be of Petrov type D and searching in that class he found it.

The decade from 1964 to 1974 witnessed a vigorous development of the mathematical theory of black-holes. Brandon Carter solved the geodesic equations for the Kerr-metric, discovering a fourth hidden first integral which reduces these differential equations to quadratures. In the same time through the work of Stephen Hawking, George Ellis, Roger Penrose and several others, general analytic methods were established to discuss, represent and classify the causal structure of space-times. Slowly a new picture emerged. Similarly to soliton solutions of other non-linear differential equations, black-holes have the characteristic features of a new kind of particles, mass, charge and angular momentum being their unique and defining attributes. Indeed it was proved that, irrespectively from all the details of its initial structure, a gravitational collapsing body sets down to a final equilibrium state parameterized only by (M, J, Q) and described by the so called Kerr-Newman metric, the generalization of the Kerr solution which includes also the Reissner Nordström charges (see Chap. 3, Sect. 3.2).

This introduced the theoretical puzzle of *information loss*. Through gravitational evolution, a supposedly coherent quantum state, containing a detailed fine structure, can evolve to a new state where all such information is unaccessible, being hidden behind the event horizon. The information loss paradox became even more severe when Hawking on one side demonstrated that black-holes can evaporate through a quantum generated thermic radiation and on the other side, in collaboration with Bekenstein, he established, that the horizon has the same properties of an entropy and obeys a theorem similar to the second principle of thermodynamics.

Hence from the theoretical view-point black-holes appear to be much more profound structures than just a particular type of classical solutions of Einstein's field equations. Indeed they provide a challenging clue into the mysterious realm of quantum gravity where causality is put to severe tests and needs to be profoundly revised. For this reason the study of black-holes and of their higher dimensional analogues within the framework of such candidates to a Unified Quantum Theory of all Interactions as Superstring Theory is currently a very active stream of research.

Ironically such a Revolution in Human Thought about the Laws of Causality, whose settlement is not yet firmly acquired, was initiated two century ago by the observations of Laplace, whose unshakable faith in determinism is well described by the following quotation from the *Essai philosophique sur les probabilités*. In

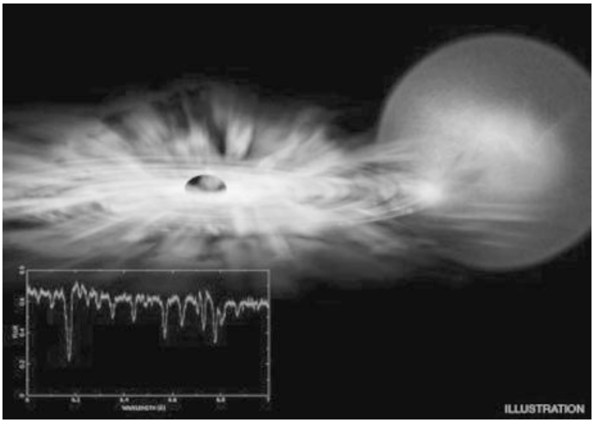

Fig. 2.3 J1655 is a binary system that harbors a black hole with a mass seven times that of the sun, which is pulling matter from a normal star about twice as massive as the sun. The Chandra observation revealed a bright X-ray source whose spectrum showed dips produced by absorption from a wide variety of atoms ranging from oxygen to nickel. A detailed study of these absorption features shows that the atoms are highly ionized and are moving away from the black hole in a high-speed wind. The system J1655 is a galactic object located at about 11,000 light years from the Sun

that book he wrote: *We may regard the present state of the universe as the effect of its past and the cause of its future. An intellect which at a certain moment would know all forces that set nature in motion, and all positions of all items of which nature is composed, if this intellect were also vast enough to submit these data to analysis, it would embrace in a single formula the movements of the greatest bodies of the universe and those of the tiniest atom; for such an intellect nothing would be uncertain and the future just like the past would be present before its eyes.* The vast intellect advocated by Pierre Simon and sometimes named the Laplace demon might find some problems in reconstructing the past structure of a star that had collapsed into a black hole even if that intellect had knowledge of all the conditions of the Universe at that very instant of time.

From the astronomical view-point the existence of black-holes of stellar mass has been established through many overwhelming evidences, the best being provided by binary systems where a visible normal star orbits around an invisible companion which drags matter from its mate. An example very close to us is the system J1655 shown in Fig. 2.3. Giant black-holes of millions of stellar masses have also been indirectly revealed in the core of active galactic nuclei and also at the center of our Milky Way a black hole is accredited.

In the present chapter, starting from the Kruskal extension of the Schwarzschild metric we establish the main framework for the analysis of the causal structure of space-times and we formulate the general definition of black-holes. In the next chapter we study the Kerr metric and the challenging connection between the laws of black-hole mechanics and those of thermodynamics.

2.2 The Kruskal Extension of Schwarzschild Space-Time

According to the outlined programme in this section we come back to the Schwarzschild metric (2.2.1) that we rewrite here for convenience

$$ds^2 = -\left(1 - 2\frac{m}{r}\right) dt^2 + \left(1 - 2\frac{m}{r}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right) \qquad (2.2.1)$$

and we study its causal properties. In particular we investigate the nature and the significance of the coordinate singularity at the Schwarzschild radius $r = r_S \equiv 2m$ which, as anticipated in the previous section, turns out to correspond to an *event horizon*. This explains the nomenclature *Schwarzschild emiradius* that in Chap. 4 of Volume 1 we used for the surface $r = m$.

2.2.1 Analysis of the Rindler Space-Time

Before analyzing the Kruskal extension of the Schwarzschild space-time, as a preparatory exercise we begin by considering the properties of a two-dimensional toy-model, the so called Rindler space-time. This is $\mathbb{R}^2$ equipped with the following Lorentzian metric:

$$ds^2_{Rindler} = -x^2\, dt^2 + dx^2 \qquad (2.2.2)$$

which, apparently, has a singularity on the line $H \subset \mathbb{R}^2$ singled out by the equation $x = 0$. A careful analysis reveals that such a singularity is just a coordinate artefact since the metric (2.2.2) is actually flat and can be brought to the standard form of the Minkowski metric via a suitable coordinate transformation:

$$\xi : \mathbb{R}^2 \to \mathbb{R}^2 \qquad (2.2.3)$$

The relevant point is that the diffeomorphism ξ is not surjective since it maps the whole of Rindler space-time, namely the entire $\mathbb{R}^2$ manifold into an open subset $I = \xi(\mathbb{R}^2) \subset \mathbb{R}^2 = \text{Mink}_2$ of Minkowski space. This means that Rindler space-time is incomplete and can be extended to the entire 2-dimensional Minkowski space Mink_2. The other key point is that the image $\xi(H) \subset \text{Mink}_2$ of the singularity in the extended space-time is a perfectly regular null-like hypersurface. These features are completely analogous to corresponding features of the Kruskal extension of Schwarzschild space-time. Also there we can find a suitable coordinate transformation $\xi_K : \mathbb{R}^4 \to \mathbb{R}^4$ which removes the singularity displayed by the Schwarzschild metric at the Schwarzschild radius $r = 2m$ and such a map is not surjective, rather it maps the entire Schwarzschild space-time into an open sub-manifold $\xi_K(\text{Schwarzschild}) \subset \text{Krusk}$ of a larger manifold named the Kruskal space-time. Also in full analogy with the case of the Rindler toy-model the image $\xi_K(H)$ of the coordinate singularity H defined by the equation $r = 2m$ is a regular null-like hypersurface of Kruskal space-time. In this case it has the interpretation of event-horizon delimiting a black-hole region.

The basic question therefore is: how do we find the appropriate diffeomorphism ξ or ξ_K? The answer is provided by a systematic algorithm which consists of the following steps:

1. derivation of the equations for geodesics,
2. construction of a complete system of incoming and outgoing null geodesics,
3. transition to a coordinate system where the new coordinates are the affine parameters along the incoming and outgoing null geodesics,
4. analytic continuation of the new coordinate patch beyond its original domain of definition.

We begin by showing how this procedure works in the case of the metric (2.2.2) and later we apply it to the physically significant case of the Schwarzschild metric.

The metric (2.2.2) has a coordinate singularity at $x=0$ where the determinant $\det g_{\mu\nu} = -x^2$ has a zero. In order to understand the real meaning of such a singularity we follow the programme outlined above and we write the equation for null geodesics:

$$g_{\mu\nu}(x)\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = 0; \qquad -x^2\left(\dot{t}^2\right) + \left(\dot{x}^2\right) = 0 \tag{2.2.4}$$

from which we immediately obtain:

$$\left(\frac{dx}{dt}\right)^2 = \frac{1}{x^2} \quad \Rightarrow \quad t = \pm\int\frac{dx}{x} = \pm\ln x + \text{const} \tag{2.2.5}$$

Hence we can introduce the null coordinates by writing:

$$\begin{aligned} t + \ln x = v; \quad v = \text{const} \quad &\Leftrightarrow \quad \text{(incoming null geodesics)} \\ t - \ln x = u; \quad u = \text{const} \quad &\Leftrightarrow \quad \text{(outgoing null geodesics)} \end{aligned} \tag{2.2.6}$$

The shape of the corresponding null geodesics is displayed in Fig. 2.4. The first change of coordinates is performed by replacing x, t by u, v. Using:

$$x^2 = \exp[v-u]; \qquad \frac{dx}{x} = \frac{dv-du}{2}; \qquad dt = \frac{dv+du}{2} \tag{2.2.7}$$

the metric (2.2.2) becomes:

$$ds^2_{Rindler} = -\exp[v-u]\,du\,dv \tag{2.2.8}$$

Next step is the calculation of the affine parameter along the null geodesics. Here we use a general property encoded in the following lemma:

Lemma 2.2.1 *Let* $\mathbf{k}$ *be a Killing vector for a given metric* $g_{\mu\nu}(x)$ *and let* $\mathbf{t} = \frac{dx^\mu}{d\lambda}$ *be the tangent vector to a geodesic. Then the scalar product*:

$$E \equiv -(\mathbf{t},\mathbf{k}) = -g_{\mu\nu}\frac{dx^\mu}{d\lambda}k^\nu \tag{2.2.9}$$

is constant along the geodesic.

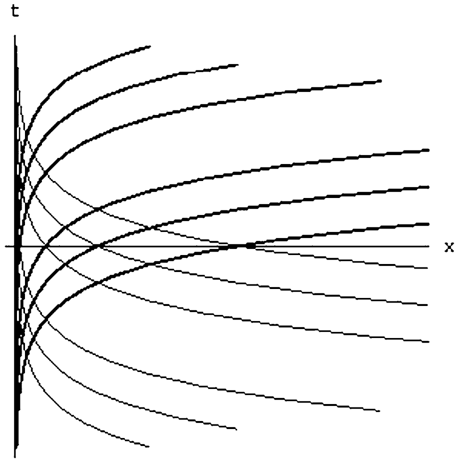

Fig. 2.4 Null geodesics of the Rindler metric. The *thin curves* are incoming ($v = \text{const}$), while the *thick* ones are outgoing ($u = \text{const}$)

Proof The proof is immediate by direct calculation. If we take the $d/d\lambda$ derivative of E we get:

$$\frac{dE}{d\lambda} = -\underbrace{\nabla_\rho g_{\mu\nu}\frac{dx^\rho}{d\lambda}\frac{dx^\mu}{d\lambda}k^\nu}_{\substack{=0 \text{ since metric} \\ \text{is cov. const.}}} - \underbrace{g_{\mu\nu}\left(\nabla_\rho\frac{dx^\mu}{d\lambda}\right)\frac{dx^\rho}{d\lambda}k^\nu}_{=0 \text{ for the geodesic eq.}}$$
$$- \underbrace{g_{\mu\nu}\nabla_\rho k^\nu, \frac{dx^\rho}{d\lambda}\frac{dx^\mu}{d\lambda}}_{=0 \text{ for the Killing vec. eq.}} \tag{2.2.10}$$

So we obtain the sum of three terms that are separately zero for three different reasons. □

Relying on Lemma 2.2.1 in Rindler space time we can conclude that $E = x^2 \frac{dt}{d\lambda}$ is constant along geodesics. Indeed the vector field $\mathbf{k} \equiv \frac{d}{dt}$ is immediately seen to be a Killing vector for the metric (2.2.2). Then by means of straightforward manipulations we obtain:

$$d\lambda = \frac{1}{E}\exp[v-u]\frac{du+dv}{2} \quad \Rightarrow$$
$$\lambda = \begin{cases} \frac{\exp[-u]}{2E}\exp[v] & \text{on } u = \text{const outgoing null geodesics} \\ -\frac{\exp[v]}{2E}\exp[-u] & \text{on } v = \text{const incoming null geodesics} \end{cases} \tag{2.2.11}$$

The third step in the algorithm that leads to the extension map corresponds to a coordinate transformation where the new coordinates are proportional to the affine parameters along incoming and outgoing null geodesics. Hence in view of (2.2.11) we introduce the coordinate change:

$$U = -e^{-u} \quad \Rightarrow \quad dU = e^{-u}\,du; \qquad V = e^v \quad \Rightarrow \quad dV = e^v\,dv \tag{2.2.12}$$

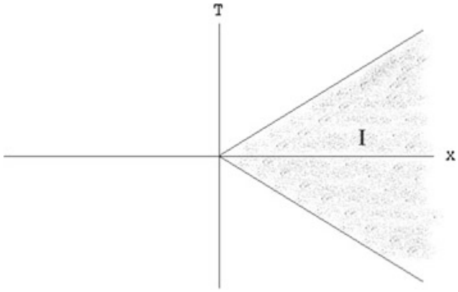

Fig. 2.5 The image of Rindler space-time in two-dimensional Minkowski space-time is the shaded region I bounded by the two null surfaces $X = T$ $(X > 0)$ and $X = -T$ $(X > 0)$. These latter are the image of the coordinate singularity $x = 0$ of the original metric

by means of which the Rindler metric (2.2.8) becomes:

$$ds^2_{Rindler} = -dU \otimes dV \tag{2.2.13}$$

Finally, with a further obvious transformation:

$$T = \frac{V+U}{2}; \qquad X = \frac{V-U}{2} \tag{2.2.14}$$

the Rindler metric (2.2.13) is reduced to the standard two-dimensional Minkowski metric in the plane $\{X, T\}$:

$$ds^2_{Rindler} = -dT^2 + dX^2 \tag{2.2.15}$$

Putting together all the steps, the coordinate transformation that reduces the Rindler metric to the standard form (2.2.15) is the following:

$$x = \sqrt{X^2 - T^2}; \qquad t = \text{arctanh}\left[\frac{T}{X}\right] \tag{2.2.16}$$

In this way we have succeeded in eliminating the apparent singularity $x = 0$ since the metric (2.2.15) is perfectly regular in the whole $\{X, T\}$ plane. The subtle point of this procedure is that by means of the transformation (2.2.12) we have not only eliminated the singularity, but also *extended* the space-time. Indeed the definition (2.2.12) of the U and V coordinates is such that V is always positive and U always negative. This means that in the $\{U, V\}$ plane the image of Rindler space-time is the quadrant $U < 0$; $V > 0$. In terms of the final X, T variables the image of the original Rindler space-time is the angular sector I depicted in Fig. 2.5. Considering the coordinate transformation (2.2.16) we see that the image in the extended space-time of the apparent singularity $x = 0$ is the locus $X^2 = T^2$ which is perfectly regular but has the distinctive feature of being *a null-like surface*. This surface is also the boundary of the image I of Rindler space-time in its maximal extension. Furthermore setting $X = \pm T$ we obtain $t = \pm\infty$. This means that in the original Rindler space any test particle takes an infinite amount of *coordinate time* to reach the boundary locus $x = 0$: this is also evident from the plot of null geodesics in Fig. 2.4. On the other hand the *proper time* taken by a test particle to reach such a locus from any other point is just finite.

All these features of our toy model apply also to the case of Schwarzschild space-time once it is extended with the same procedure. The image of the coordinate singularity $r = 2m$ will be a null-like surface, interpreted as *event horizon*, which can be reached in a *finite proper-time* but only after an *infinite interval of coordinate time*. What will be new and of utmost physical interest is precisely the interpretation of the locus $r = 2m$ as an *event horizon* $\mathfrak{H}$ which leads to the concept of Black-Hole. Yet this interpretation can be discovered only through the Kruskal extension of Schwarzschild space-time and this latter can be systematically derived via the same algorithm we have applied to the Rindler toy model.

2.2.2 Applying the Same Procedure to the Schwarzschild Metric

We are now ready to analyze the Schwarzschild metric (2.2.1) by means of the tokens illustrated above. The first step consists of reducing it to two-dimensions by fixing the angular coordinates to constant values $\theta = \theta_0$, $\phi = \phi_0$. In this way the metric (2.2.1) reduces to:

$$ds^2_{Schwarz.} = -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 \tag{2.2.17}$$

Next, in the reduced space spanned by the coordinates r and t we look for the null-geodesics. From the equation:

$$-\left(1 - \frac{2m}{r}\right)\dot{t}^2 + \left(1 - \frac{2m}{r}\right)^{-1} \dot{r}^2 = 0 \tag{2.2.18}$$

we obtain:

$$\frac{dt}{dr} = \pm \frac{r}{r - 2m} \quad \Rightarrow \quad t = \pm r^*(r) \tag{2.2.19}$$

where we have introduced the so called Regge-Wheeler *tortoise coordinate* defined by the following indefinite integral:

$$r^*(r) \equiv \int \frac{r}{r - 2m} dr = r + 2m \log\left(\frac{r}{2m} - 1\right) \tag{2.2.20}$$

Hence, in full analogy with (2.2.6), we can introduce the null coordinates

$$\begin{aligned} t + r^*(r) = v; \quad & v = \text{const} \quad \Leftrightarrow \quad \text{(incoming null geodesics)} \\ t - r^*(r) = u; \quad & u = \text{const} \quad \Leftrightarrow \quad \text{(outgoing null geodesics)} \end{aligned} \tag{2.2.21}$$

and the analogue of Fig. 2.4 is now given by Fig. 2.6. Inspection of this picture reveals the same properties we had already observed in the case of the Rindler toy model. What is important to stress in the present model is that each point of the

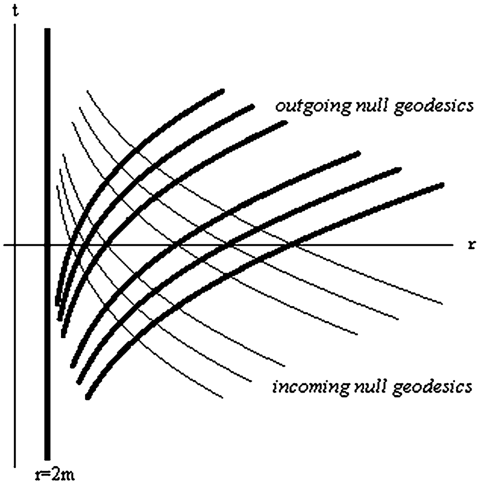

Fig. 2.6 Null geodesics of the Schwarzschild metric in the r, t plane. The *thin curves* are incoming ($v = \text{const}$), while the *thick* ones are outgoing ($u = \text{const}$). Each point in this picture represents a 2-sphere, parameterized by the angles θ_0 and ϕ_0. The *thick vertical line* is the surface $r = r_S = 2m$ corresponding to the coordinate singularity. As in the case of the Rindler toy model the null–geodesics incoming from infinity reach the coordinate singularity only at asymptotically late times $t \rightarrow> +\infty$. Similarly outgoing null-geodesics were on this surface only at asymptotically early times $t \rightarrow -\infty$

diagram actually represents a 2-sphere parameterized by the two angles θ and ϕ that we have freezed at the constant values θ_0 and ϕ_0. Since we cannot make four-dimensional drawings some pictorial idea of what is going on can be obtained by replacing the 2-sphere with a circle $\mathbb{S}^1$ parameterized by the azimuthal angle ϕ. In this way we obtain a three-dimensional space-time spanned by coordinates t, $x = r\cos\phi$, $y = r\sin\phi$. In this space the null-geodesics of Fig. 2.6 become two-dimensional surfaces. Indeed these null-surfaces are nothing else but the projections $\theta = \theta_0 = \pi/2$ of the true null surfaces of the Schwarzschild metric. In Fig. 2.7 we present two examples of such projected null surfaces, one incoming and one outgoing.

Having found the system of incoming and outgoing null-geodesics we go over to point (iii) of our programme and we make a coordinate change from t, r to u, v. By straightforward differentiation of (2.2.20), (2.2.21) we obtain:

$$dr = -\left(1 - \frac{r_S}{r}\right)\frac{du - dv}{2}; \qquad dt = \frac{du + dv}{2} \tag{2.2.22}$$

so that the reduced Schwarzschild metric (2.2.17) becomes:

$$ds^2_{Schwarz.} = -\left(1 - \frac{r_S}{r}\right) du \otimes dv \tag{2.2.23}$$

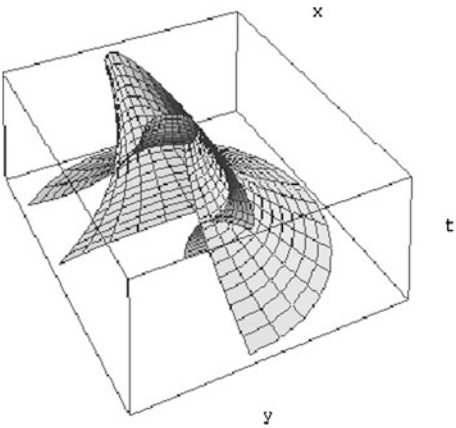

Fig. 2.7 An example of two null surfaces generated by null geodesics of the Schwarzschild metric in the r, t plane

Using the definition (2.2.20) of the tortoise coordinate we can also write:

$$\left(1 - \frac{r_S}{r}\right) = -\exp\left[\frac{v-u}{2r_S}\right]\exp\left[-\frac{r}{r_S}\right] \tag{2.2.24}$$

which combined with (2.2.22) yields:

$$ds^2_{Schwarz.} = \exp\left[-\frac{r}{r_S}\right]\exp\left[\frac{v-u}{2r_S}\right]\frac{r_S}{r}\, du \otimes dv \tag{2.2.25}$$

In complete analogy with (2.2.12) we can now introduce the new coordinates:

$$U = -\exp\left[-\frac{u}{2r_S}\right]; \qquad V = \exp\left[-\frac{u}{2r_S}\right] \tag{2.2.26}$$

that play the role of affine parameters along the incoming and outgoing null geodesics.

Then by straightforward differentiation of (2.2.26) the reduced Schwarzschild metric (2.2.25) becomes:

$$ds^2_{Schwarz.} = -4\frac{r_S^3}{r}\exp\left[-\frac{r}{r_S}\right] dU \otimes dV \tag{2.2.27}$$

where the variable $r = r(U, V)$ is the function of the independent coordinates U, V implicitly determined by the transcendental equation:

$$r + r_S \log\left(\frac{r}{r_S} - 1\right) = r_S \log(-UV) \tag{2.2.28}$$

In analogy with our treatment of the Rindler toy model we can make a final coordinate change to new variables X, T related to U, V as in (2.2.14). These, together with the angular variables θ, ϕ make up the Kruskal coordinate patch which, putting together all the intermediate steps, is related to the original coordinate patch t, r, θ, ϕ by the following transition function:

$$\begin{array}{l}\text{polar}\\ \text{versus}\\ \text{Kruskal}\\ \text{coord.}\end{array}\begin{cases}\theta = \theta \\ \phi = \phi \\ (\frac{r}{r_S} - 1)\exp[\frac{r}{r_S}] = T^2 - X^2 \\ \frac{t}{r_S} = \log(\frac{T+X}{T-X}) \equiv 2\,\mathrm{arctanh}\,\frac{X}{T}\end{cases} \tag{2.2.29}$$

In Kruskal coordinates the Schwarzschild metric (2.2.1) takes the final form:

$$ds^2_{\text{Krusk}} = 4\frac{r_S^3}{r}\exp\left[\frac{r}{r_S}\right]\left(-dT^2 + dX^2\right) + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right) \tag{2.2.30}$$

where the $r = r(X, T)$ is implicitly determined in terms of X, T by the transcendental equations (2.2.29).

2.2.3 A First Analysis of Kruskal Space-Time

Let us now consider the general properties of the space-time $(\mathscr{M}_{\text{Krusk}}, g_{\text{Krusk}})$ identified by the metric (2.2.30) and by the implicit definition of the variable r contained in (2.2.29). This analysis is best done by inspection of the two-dimensional diagram displayed in Fig. 2.8. This diagram lies in the plane $\{X, T\}$, each of whose points represents a two sphere spanned by the angle-coordinates θ and ϕ. The first thing to remark concerns the physical range of the coordinates X, T. The Kruskal manifold $\mathscr{M}_{\text{Krusk}}$ does not coincide with the entire plane, rather it is the infinite portion of the latter comprised between the two branches of the hyperbolic locus:

$$T^2 - X^2 = -1 \tag{2.2.31}$$

This is the image in the X, T-plane of the $r = 0$ locus which is a genuine singularity of both the original Schwarzschild metric and of its Kruskal extension. Indeed from (5.9.6)–(5.9.11) of Volume 1 we know that the intrinsic components of the curvature tensor depend only on r and are singular at $r = 0$, while they are perfectly regular at $r = 2m$. Therefore no geodesic can be extended in the X, T plane beyond (2.2.31) which constitutes a boundary of the manifold.

Let us now consider the image of the constant r surfaces. Here we have to distinguish two cases: $r > r_S$ or $r < r_S$. We obtain:

$$\begin{aligned}&\{X, T\} = \{h\cosh p, h\sinh p\}; \qquad h = e^{\frac{r}{r_S}}\sqrt{\frac{r}{r_S} - 1} \quad \text{for } r > r_S \\ &\{X, T\} = \{h\sinh p, h\cosh p\}; \qquad h = e^{\frac{r}{r_S}}\sqrt{1 - \frac{r}{r_S}} \quad \text{for } r < r_S\end{aligned} \tag{2.2.32}$$

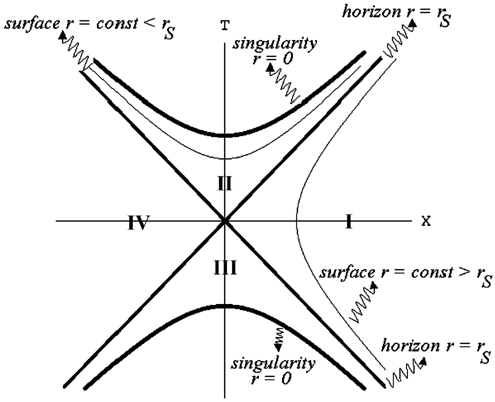

Fig. 2.8 A two-dimensional diagram of Kruskal space-time

These are the hyperbolae drawn in Fig. 2.8. Calculating the normal vector $N^\mu = \{\partial_p T, \partial_p X, 0, 0\}$ to these surfaces, we find that it is time-like $N^\mu N^\nu g_{\mu\nu} < 0$ for $r > r_S$ and space-like $N^\mu N^\nu g_{\mu\nu} > 0$ for $r < r_S$. Correspondingly, according to a discussion developed in the next section, the constant r surfaces are space-like outside the sphere of radius r_S and time-like inside it. The dividing locus is the pair of straight lines $X = \pm T$ which correspond to $r = r_S$ and constitute a null-surface, namely one whose normal vector is light-like. This null-surface is the *event horizon*, a concept whose precise definition needs, in order to be formulated, a careful reconsideration of the notions of Future, Past and Causality in the context of General Relativity. The next two sections pursue such a goal and by their end we will be able to define Black-Holes and their Horizons. Here we note the following. If we solve the geodesic equation for time-like or null-like geodesics with arbitrary initial data inside region II of Fig. 2.8 then the end point of that geodesic is always located on the singular locus $T^2 - X^2 = -1$ and the whole development of the curve occurs inside region II. The formal proof of this statement is involved and it will be overcome by the methods of Sects. 2.3 and 2.4. Yet there is an intuitive argument which provides the correct answer and suffices to clarify the situation. Disregarding the angular variables θ and ϕ the Kruskal metric (2.2.30) reduces to:

$$ds^2_{\text{Krusk}} = F(X,T)\big(-dT^2 + dX^2\big); \quad F(X,T) = 4\frac{r_S^3}{r}\exp\left[\frac{r}{r_S}\right] \tag{2.2.33}$$

so that it is proportional to two-dimensional Minkowski metric $ds^2_{\text{Mink}} = -dT^2 + dX^2$ through the positive definite function $F(X,T)$. In the language of Sect. 2.4 this fact means that, reduced to two-dimensions, Kruskal and Minkowski metrics are conformally equivalent. According to Lemma 2.4.1 proved later on, conformally equivalent metrics share the same light-like geodesics, although the time-like and space-like ones may be different. This means that in two-dimensional Kruskal space-time light travels along straight lines of the form $X = \pm T + k$ where k is some constant. This is the same statement as saying that at any point p of the $\{X, T\}$ plane the tangent vector to any curve is time-like or light-like and oriented to the future if

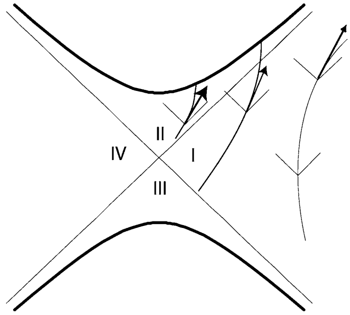

Fig. 2.9 The light-cone orientations in Kruskal space-time and the difference between physical geodesics in regions I and II

its inclination α with respect to the X axis is in the following range $3\pi/4 \geq \alpha \geq \pi/4$. This applies to the whole plane, yet it implies a fundamental difference in the destiny of physical particles that start their journey in region I (or IV) of the Kruskal plane, with respect to the destiny of those ones that happen to be in region II at some point of their life. As it is visually evident from Fig. 2.9, in region I we can have curves (and in particular geodesics) whose tangent vector is time-like and future oriented at any of their points which nonetheless avoid the singular locus and escape to infinity. In the same region there are also future oriented time-like curves which cross the horizon $X = \pm T$ and end up on the singular locus, yet these are not the only ones, as already remarked. On the contrary all curves that at some point happen to be inside region II can no longer escape to infinity since, in order to be able to do so, their tangent vector should be space-like, at least at some of their points. Hence the horizon can be crossed from region I to region II, never in the opposite direction. This leads to the existence of a Black-Hole, namely a space-time region, (II in our case) where gravity is so strong that not even light can escape from it. No signal from region II can reach a distant observer located in region I who therefore perceives only the presence of the gravitational field of the black hole swapping infalling matter.

To encode the ideas intuitively described in this section into a rigorous mathematical framework we proceed next to implement our already announced programme. This is the critical review of the concepts of Future, Past and Causality within General Relativity, namely when we assume that all physical events are points p in a pseudo-Riemannian manifold $(\mathscr{M}, g)$ with a Lorentzian signature.

2.3 Basic Concepts about Future, Past and Causality

Our discussion starts by reviewing the basic properties of the light-cone (see Fig. 2.10). In Special Relativity, where space-time is Minkowski-space, namely a pseudo-Riemannian manifold which is also affine, the light cone has a global meaning, while in General Relativity light-cones can be defined only locally, namely at each point $p \in \mathscr{M}$. In any case the Lorentzian signature of the metric implies that $\forall p \in \mathscr{M}$, the tangent space $T_p\mathscr{M}$ is isomorphic to Minkowski space and it admits the same decomposition in time-like, null-like and space-like sub-manifolds. Hence

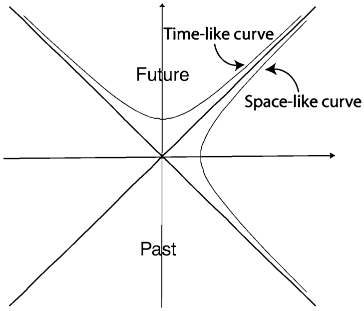

Fig. 2.10 The structure of the light-cone

the analysis of the light-cone properties has a general meaning also in General Relativity, although such analysis needs to be repeated at each point. All the complexities inherent with the notion of global causality arise from the need of gluing together the locally defined light-cones. We will develop appropriate conceptual tools to manage such a gluing after our review of the local light-cone properties.

2.3.1 The Light-Cone

When a metric has a Lorentzian signature, vectors t can be of three-types:

1. Time-like, if $(t, t) < 0$ in mostly plus convention for $g_{\mu\nu}$.
2. Space-like, if $(t, t) > 0$ in mostly plus convention for $g_{\mu\nu}$.
3. Null-like, if $(t, t) = 0$ both in mostly plus and mostly minus convention for $g_{\mu\nu}$.

At any point $p \in \mathscr{M}$ the light-cone $\mathscr{C}_p$ is composed by the set of vectors $t \in T_p\mathscr{M}$ which are either time-like or null-like. In order to study the properties of the light-cones it is convenient to review a few elementary but basic properties of vectors in Minkowski space.

Theorem 2.3.1 *All vectors orthogonal to a time-like vector are space-like.*

Proof Using a mostly plus signature, we can go to a diagonal basis such that:

$$g(X, Y) = g_{00}X^0Y^0 + (\mathbf{X}, \mathbf{Y}) \tag{2.3.1}$$

where $g_{00} < 0$ and $(\, , \,)$ denotes a non-degenerate, positive-definite, Euclidian bilinear form on $\mathbb{R}^{n-1}$. In this basis, if $X \perp T$ and T is time-like we have:

$$\begin{aligned} -g_{00}T^0T^0 &> (\mathbf{T}, \mathbf{T}) \\ -g_{00}T^0X^0 &= (\mathbf{T}, \mathbf{X}) \leq \sqrt{(\mathbf{T}, \mathbf{T})(\mathbf{X}, \mathbf{X})} \end{aligned} \tag{2.3.2}$$

Then we get:

$$\frac{-g_{00}T^0X^0}{\sqrt{-g_{00}T^0T^0}} < \frac{(\mathbf{T}, \mathbf{X})}{\sqrt{(\mathbf{T}, \mathbf{T})}} \leq \sqrt{(\mathbf{X}, \mathbf{X})} \tag{2.3.3}$$

Squaring all terms in (2.3.3) we obtain

$$-g_{00}X^0X^0 < (\mathbf{X},\mathbf{X}) \quad \Rightarrow \quad g(X,X) > 0 \tag{2.3.4}$$

namely the four-vector X is space-like as asserted by the theorem. □

Another useful property is given by the following

Lemma 2.3.1 *The sum of two future-directed time-like vectors is a future-directed time-like vector.*

Proof Let t and T be the two vectors under considerations. By hypothesis we have

$$\begin{aligned} g(t,t) < 0; &\qquad t^0 > 0 \\ g(T,T) < 0; &\qquad T^0 > 0 \end{aligned} \tag{2.3.5}$$

Since:

$$\begin{aligned} \sqrt{-g_{00}}\,t^0 &> (\mathbf{t},\mathbf{t}) \\ \sqrt{-g_{00}}\,T^0 &> (\mathbf{T},\mathbf{T}) \\ \sqrt{-g_{00}}\,t^0T^0 &> \sqrt{(\mathbf{t},\mathbf{t})(\mathbf{T},\mathbf{T})} > (\mathbf{t},\mathbf{T}) \end{aligned} \tag{2.3.6}$$

we have:

$$\begin{aligned} g(t+T,t+T) &= g(t,t)+g(T,T)+2g(t,T) \\ &\Downarrow \\ -g_{00}\big((t^0)^2+(T^0)^2+2t^0T^0\big) &> (\mathbf{t},\mathbf{t})+(\mathbf{T},\mathbf{T})+2(\mathbf{t},\mathbf{T}) \end{aligned} \tag{2.3.7}$$

which proves that $t+T$ is time-like. Moreover $t^0+T^0>0$ and so the sum vector is also future-directed as advocated by the lemma. □

On the other hand with obvious changes in the proof of Theorem 2.3.1 the following lemma is established

Lemma 2.3.2 *All vectors X, orthogonal to a light-like vector L are either light-like or space-like.*

Let us now consider in the manifold $(\mathcal{M},g)$ surfaces Σ defined by the vanishing of some smooth function of the local coordinates:

$$p\in\Sigma \quad \Leftrightarrow \quad f(p)=0 \quad \text{where } f\in\mathbb{C}^\infty(\mathcal{M}) \tag{2.3.8}$$

By definition the normal vector to the surface Σ is the gradient of the function f:

$$n^{(\Sigma)}_\mu = \nabla_\mu f = \partial_\mu f \tag{2.3.9}$$

Indeed any tangent vector to the surface is by construction orthogonal to $n^{(\Sigma)}$:

$$g\big(t^{(\Sigma)}, n^{(\Sigma)}\big) = 0 \tag{2.3.10}$$

Definition 2.3.1 A surface Σ is said to be *space-like* if its normal vector $n^{(\Sigma)}$ is everywhere *time-like* on the surface. Conversely Σ is *time-like* if $n^{(\Sigma)}$ is **space-like**. We name *null surfaces* those Σ whose normal vector $n^{(\Sigma)}$ is *null-like*.

Null surfaces have very intriguing properties. First of all their normal vector is also tangent to the surface. This follows from the fact that the normal vector is orthogonal to itself. Furthermore we can prove that any null-surface is generated by null-geodesics. Indeed we can easily prove that the normal vector $n^{(\Sigma)}$ to a null surface is the tangent vector to a null-geodesics. Indeed we have:

$$\begin{aligned} 0 &= \nabla_\mu\big(\nabla_\nu f \nabla^\nu f\big) = 2\nabla^\nu f \nabla_\nu \nabla_\mu f \\ &= n^\nu \nabla_\nu n_\mu \end{aligned} \tag{2.3.11}$$

and the last equality is precisely the geodesic equation satisfied by the integral curve to the normal vector $n^{(\Sigma)}$.

A typical null-surface is the event-horizon of a black-hole.

2.3.2 Future and Past of Events and Regions

Let us now consider the pseudo-Riemannian space-time manifold $(\mathscr{M}, g)$ and at each point $p \in \mathscr{M}$ introduce the local light-cone $\mathscr{C}_p \subset T_p\mathscr{M}$. In this section we find it convenient to change convention and use a mostly minus signature where $g_{00} > 0$.

Definition 2.3.2 The local light-cone $\mathscr{C}_p$ (see Fig. 2.11) is the set of all tangent vectors $t \in T_p\mathscr{M}$, such that:

$$g_{\mu\nu} t^\mu t^\nu \geq 0 \tag{2.3.12}$$

and it is the union of the future light-cone with the past light-cone:

$$\mathscr{C}_p = \mathscr{C}_p^+ \bigcup \mathscr{C}_p^- \tag{2.3.13}$$

where

$$\begin{aligned} t \in \mathscr{C}_p^+ \quad &\Leftrightarrow \quad g(t,t) \geq 0 \quad \text{and} \quad t^0 > 0 \\ t \in \mathscr{C}_p^- \quad &\Leftrightarrow \quad g(t,t) \geq 0 \quad \text{and} \quad t^0 < 0 \end{aligned} \tag{2.3.14}$$

The vectors in $\mathscr{C}_p^+$ are named *future-directed*, while those in $\mathscr{C}_p^-$ are named *past-directed*.

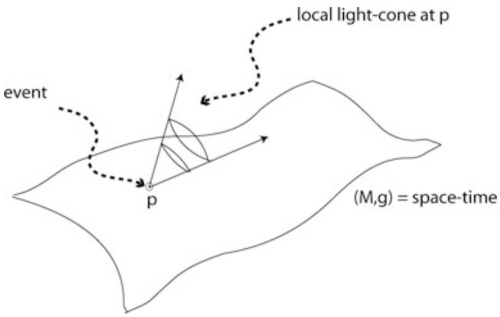

Fig. 2.11 At each point of the space-time manifold, the tangent space $T_p\mathscr{M}$ contains the sub-manifold $\mathscr{C}_p$ of time-like and null-vectors which constitutes the local light-cone

We can now transfer the notions of time orientation from vectors to curves by means of the following definitions:

Definition 2.3.3 A differentiable curve $\lambda(s)$ on the space-time manifold $\mathscr{M}$ is named a *future-directed time-like curve* if at each point $p \in \lambda$, the tangent vector to the curve t^μ is future-directed and time-like. Conversely $\lambda(s)$ is *past-directed time-like* if such is t^μ.

Similarly we have:

Definition 2.3.4 A differentiable curve $\lambda(s)$ on the space-time manifold $\mathscr{M}$ is named a *future-directed causal curve* if at each point $p \in \lambda$, the tangent vector to the curve t^μ is either a future-directed *time-like* or a future-directed *null-like* vector. Conversely $\lambda(s)$ is a *past-directed causal* curve when the tangent t^μ, time-like or null-like, is past directed.

Relying on these concepts we can introduce the notions of Chronological Future and Past of a point $p \in \mathscr{M}$.

Definition 2.3.5 The Chronological Future (Past) of a point p, denoted $I^\pm(p)$ is the subset of points of $\mathscr{M}$, defined by the following condition:

$$I^\pm(p) = \left\{ q \in \mathscr{M} \;\middle|\; \begin{array}{l} \exists \text{ future- (past-)directed time-like} \\ \text{curve } \lambda(s) \text{ such that} \\ \lambda(0) = p; \quad \lambda(1) = q \end{array} \right\} \tag{2.3.15}$$

In other words the Chronological Future or Past of an event are all those events that can be connected to it by a future-directed or past-directed time-like curve.

Let now $S \subset \mathscr{M}$ be a region of space-time, namely a continuous sub-manifold of the space-time manifold.

Definition 2.3.6 The Chronological Future (Past) of the region S, denoted $I^\pm(S)$ is defined as follows:

$$I^\pm(S) = \bigcup_{p \in S} I^\pm(p) \tag{2.3.16}$$

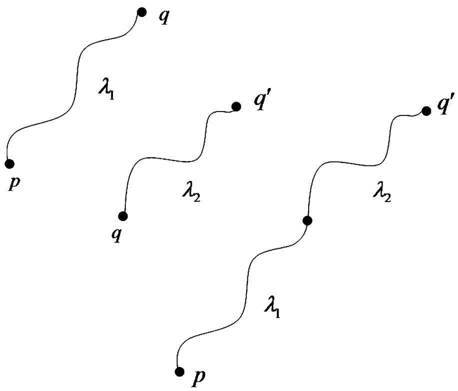

Fig. 2.12 The union of two time-like future-directed curves is still a time-like future directed curve

An elementary property of the Chronological Future is the following:

$$I^{\pm}\big(I^{\pm}(S)\big) = I^{\pm}(S) \tag{2.3.17}$$

The proof is illustrated in Fig. 2.12.

If $q' \in I^{\pm}(I^{\pm}(S))$ then, by definition, there exists at least one point $q \in I^{\pm}(S)$ to which q' is connected by a time-like future directed curve $\lambda_2(s)$. On the other hand, once again by definition, q is connected by a future-directed time-like curve $\lambda_1(s)$ to at least one point $p \in S$. Joining λ_1 with λ_2 we obtain a future-directed time-like curve that connects q' to p, which implies that $q \in I^{+}(S)$.

In a similar way, if $\overline{S}$ denotes the closure, in the topological sense, of the region S, we prove that:

$$I^{+}(\overline{S}) = I^{+}(S) \tag{2.3.18}$$

In perfect analogy with Definition 2.3.5 we have:

Definition 2.3.7 The Causal Future (Past) of a point p, denoted $J^{\pm}(p)$ is the subset of points of $\mathscr{M}$, defined by the following condition:

$$J^{\pm}(p) = \left\{ q \in \mathscr{M} \;\middle|\; \begin{array}{l} \exists \text{ future- (past-)directed causal} \\ \text{curve } \lambda(s) \text{ such that} \\ \lambda(0) = p; \quad \lambda(1) = q \end{array} \right\} \tag{2.3.19}$$

and the Causal Future(Past) of a region S, denoted $J^{\pm}(S)$ is:

$$J^{\pm}(S) = \bigcup_{p \in S} J^{\pm}(p) \tag{2.3.20}$$

An important point which we mention without proof is the following. In flat Minkowski space $J^{\pm}(p)$ is always a closed set in the topological sense, namely it contains its own boundary. In general curved space-times $J^{\pm}(p)$ can fail to be closed.

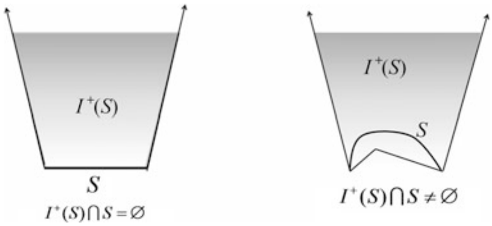

Fig. 2.13 In two-dimensional Minkowski space we show an example of achronal set. In the picture on *left* the segment S parallel to the space axis is achronal because it does not intersect its chronological future. On the other hand, in the picture on the *right*, the line S, although one dimensional is not achronal because it intersects its own chronological future

Achronal Sets

Definition 2.3.8 Let $S \subset \mathscr{M}$ be a region of space-time. S is said to be *achronal* if and only if

$$I^+(S) \bigcap S = \emptyset \tag{2.3.21}$$

The relevance of achronal sets resides in the following. When considering classical or quantum fields $\phi(x)$, conditions on these latter specified on an achronal set S are consistent, since all the events in S do not bear causal relations to each other. On the other hand one cannot freely specify initial conditions for fields on regions that are not achronal because their points are causally related to each other. In Fig. 2.13 we illustrate an example and a counterexample of achronal sets in two-dimensional Minkowski space.

Time-Orientability We mentioned above the splitting of the local light-cones in the future $\mathscr{C}_p^+$ and past $\mathscr{C}_p^-$ cones. Clearly, just as all the tangent spaces are glued together to make a fibre-bundle, the same is true of the local light-cones. The subtle point concerns the nature of the transition functions. Those of the tangent bundle $T\mathscr{M} \to \mathscr{M}$ to an n-dimensional manifold take values in $\mathrm{GL}(n, \mathbb{R})$. The light-cone, on the other hand, is left-invariant only by the subgroup $\mathrm{O}(1, n-1) \subset \mathrm{GL}(n, \mathbb{R})$. Furthermore the past and future cones are left invariant only by the subgroup of the former connected with the identity, namely $\mathrm{SO}(1, n-1) \subset \mathrm{O}(1, n-1)$. Hence the tipping of the light-cones from one point to the other of the space-time manifold are described by those transition functions of the tangent bundle that take values in the cosets $\mathrm{GL}(n, \mathbb{R})/\mathrm{O}(1, n-1)$ or $\mathrm{GL}(n, \mathbb{R})/\mathrm{SO}(1, n-1)$. The difference is subtle. Let $\mathrm{H_p} \subset \mathrm{GL}(n, \mathbb{R})$ be the subgroup isomorphic to $\mathrm{SO}(1, n-1)$, which leaves invariant the future and past light-cones at $p \in \mathscr{M}$ and let $\mathrm{H_q} \subset \mathrm{GL}(n, \mathbb{R})$ be the subgroup, also isomorphic to $\mathrm{SO}(1, n-1)$, which leaves invariant the future and past light cones at the point $q \in \mathscr{M}$. The question is the following. Are $\mathrm{H_p}$ and $\mathrm{H_q}$ conjugate to each other under the transition function $g(p, q) \in \mathrm{GL}(n, \mathbb{R})$ of the tangent bundle, that connects the tangent plane at p with the tangent plane at q, namely is it true that $\mathrm{H_q} = g(p, q)\mathrm{H_p} g^{-1}(p, q)$? If the answer is yes for all pair of points

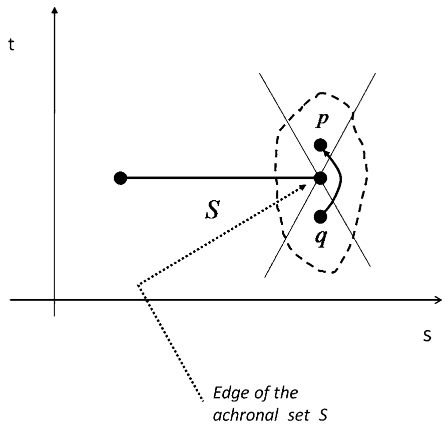

Fig. 2.14 The edge of an achronal set in two-dimensional Minkowski space. Notwithstanding how small can be the neighborhood $\mathscr{O}$ of the end point of the segment S, which we singled out with the dashed line, it contains a pair of points q and p, the former in the past of the end-point, the latter in its future, which can be connected by a time-like curve getting around the segment S and not intersecting it. Clearly this property does not hold for any of the interior points of the segment

p, q in $\mathscr{M}$, then the manifold $(\mathscr{M}, g)$ is said to be **time-orientable**. In this case the definition of future and past orientations varies continuously from one point to the other of the manifold without singular jumps. Yet there exist cases where the answer is no. When this happens the corresponding manifold is not time-orientable and all global notions of causality loose their meaning. In all the sequel we assume time-orientability.

For time orientable space-times we have the following theorem that we mention without proof

Theorem 2.3.2 *Let $(\mathscr{M}, g)$ be time-orientable and let $S \subset \mathscr{M}$ be a continuous connected region. The boundary of the chronological future of S, denoted $\partial I^+(S)$ is an achronal $(n-1)$-dimensional sub-manifold.*

Domains of Dependence The future domains of dependence are those sub-manifolds of space-time which are completely causally determined by what happens on a certain achronal set S. Alternatively the past domains of dependence are those that completely causally determine what happens on S. To discuss them we begin by introducing one more concept, that of edge.

Definition 2.3.9 Let S be an achronal and closed set. We define edge of S the set of points $a \in S$ such that for all open neighborhoods $\mathscr{O}_a$ of a, there exists two points $q \in I^-(a)$ and $p \in I^+(a)$ both contained in $\mathscr{O}_a$ which are connected by at least one time-like curve that does not intersect S.

The definition of edge is illustrated in Fig. 2.14. A very important theorem that once again we mention without proof is the following:

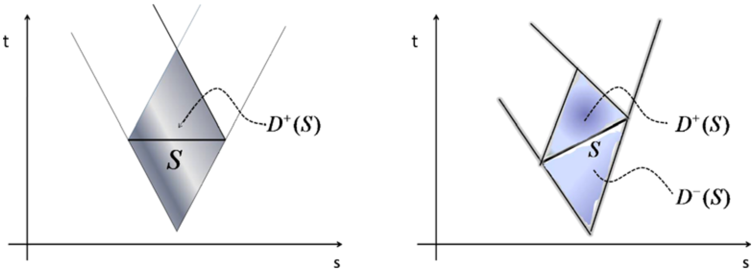

Fig. 2.15 Two examples of Future and Past domains of dependence for an achronal region S of two-dimensional Minkowski space

Theorem 2.3.3 *Let $S \subset \mathscr{M}$ be an achronal closed region of a time-orientable n-dimensional space-time $(\mathscr{M}, g)$ with Lorentz signature. Let us assume that* $\mathrm{edge}(S) = \emptyset$. *Then S is an $(n-1)$-dimensional sub-manifold of $\mathscr{M}$.*

The relevance of this theorem resides in that it establishes the appropriate notion of places in space-time, where one can formulate initial conditions for the time development. These are achronal sets without an edge and, as intuitively expected, they correspond to the notion of space ($(n-1)$-dimensional sub-manifolds) as opposed to time.

These ideas are made more precise introducing the appropriate mathematical definitions of domains of dependence.

Definition 2.3.10 Let S be a closed achronal set. We define the Future (Past) Domain of Dependence of S, denoted $D^{\pm}(S)$ as follows:

$$D^{\pm}(S) = \left\{ p \in \mathscr{M} \;\middle|\; \begin{array}{l} \text{every past- (future-)directed time-like} \\ \text{curve through } p \text{ intersects } S \end{array} \right\} \qquad (2.3.22)$$

The above definition is illustrated in Fig. 2.15. The meaning of $D^{\pm}(S)$ was already outlined above. What happens in the points $p \in D^{+}(S)$ is completely determined by the knowledge of what happened in S. Conversely what happened in S is completely determined by the knowledge of what happened in all points of $p \in D^{-}(S)$.

The *Complete Domain of Dependence* of the achronal set S is defined below:

$$D(S) \equiv D^{+}(S) \bigcup D^{-}(S) \qquad (2.3.23)$$

All the introduced definitions were preparatory for the appropriate formulation of the main concept, that of Cauchy surface.

Cauchy surfaces

Definition 2.3.11 A closed achronal set $\Sigma \subset \mathscr{M}$ of a Lorentzian space-time manifold $(\mathscr{M}, g)$ is named a Cauchy surface if and only if its domain of dependence coincides with the entire space-time, as follows:

$$D(\Sigma) = \mathscr{M} \tag{2.3.24}$$

A Cauchy surface is without edge by definition. Hence it is an $(n-1)$-dimensional hypersurface. If a Cauchy surface Σ exists, data on Σ completely determine their future development in time. This is true for all fields lying on $\mathscr{M}$ but also for the metric. Knowing for instance the perturbations of the metric on a Cauchy surface we can calculate (analytically or numerically) their future evolution without ambiguity.

Definition 2.3.12 A Lorentzian space-time $(\mathscr{M}, g)$ is named *Globally Hyperbolic* if and only if it admits at least one Cauchy surface.

Globally Hyperbolic space-times are the good, non-patological solutions of Einstein equations which allow a consistent and global formulation of causality. A major problem of General Relativity is to pose appropriate conditions on matter fields such that Global Hyperbolicity of the metric is selected. Unified theories should possess such a property.

2.4 Conformal Mappings and the Causal Boundary of Space-Time

Given the appropriate definitions of Future and Past discussed in the previous section, in order to study the causal structure of a given space-time $(\mathscr{M}, g)$, one has to cope with a classical problem met in the theory of analytic functions, namely that of bringing the point at infinity to a finite distance. Only in this way the behavior at infinity can be mastered and understood. Behavior of what? This is the obvious question. In complex function theory the behavior under investigation is that of functions, in our case is that of geodesics or, more generally, of causal curves. These latter are those that can be traveled by physical particles and the issue of causality is precisely the question of who can be reached by what. Infinity plays a distinguished role in this game because of an intuitively simple feature that characterizes those systems which the space-times $(\mathscr{M}, g)$ under consideration here are supposed to describe. The feature alluded above corresponds to the concept of an isolated dynamical system. A massive star, planetary system or galaxy is, in any case, a finite amount of energy concentrated in a finite region which is separated from other similar regions by extremely large spatial distances. The basic idea of General Relativity foresees that space-time is curved by the presence of energy or matter so that, far away from concentrations of the latter, the metric should become the flat one of empty Minkowski space. This was the boundary condition utilized in the solution of Einstein equations which lead to the Schwarzschild metric and it is the generic one assumed whenever we use Einstein equations to describe any type of star or of other localized energy lumps. Mathematically, the property of $(\mathscr{M}, g)$

which encodes such a physical idea is named *asymptotic flatness*. The point at infinity corresponds to the regions of the considered space-time $(\mathscr{M}, g)$ where the metric g becomes indistinguishable from the Minkowski metric g_{Mink} and, by hypothesis, these are at very large distances from the center of gravitation. We would like to study the structure of such an asymptotic boundary and its causal relations with the finite distance space-time regions. Before proceeding in this direction it is mandatory to stress that asymptotic flatness is neither present nor required in other physical contexts, notably that of cosmology. When we apply General Relativity to the description of the Universe and of its Evolution, energy is not localized rather it is overall distributed. There is no asymptotically far empty region and most of what we discuss here has to be revised.

This being clarified let us come back to the posed problem. Assuming that a flat boundary at infinity exists how can we bring it to a finite distance and study its structure? The answer is suggested by the analogy with the theory of analytic functions we already anticipated and it is provided by the notion of *conformal transformations*. In the complex plane, conformal transformations change distances but preserve angles. In the same way the conformal transformations we want to consider here are allowed to change the metric, that is the instrument to calculate distances, yet they should preserve the causal structure. In plain words this means that time-like, space-like and null-like vector fields should be mapped into vector fields with the same properties. Under these conditions causal curves are mapped into causal curves, although geodesics are not necessarily mapped into geodesics. Shortening the distances, infinity can come close enough to be inspected.

We begin by presenting an explicit instance of such conformal transformations corresponding to a specifically relevant case, namely that of Minkowski space. From the analysis of this example we will extract the general rules of the game to be applied also to the other cases.

2.4.1 Conformal Mapping of Minkowski Space into the Einstein Static Universe

Let us consider flat Minkowski metric in polar coordinates:

$$ds^2_{\mathrm{Mink}} = -dt^2 + dr^2 + r^2\big(d\theta^2 + \sin^2\theta\, d\phi^2\big) \tag{2.4.1}$$

and let us perform the following change of coordinates:

$$t + r = \tan\left[\frac{T+R}{2}\right] \tag{2.4.2}$$

$$t - r = \tan\left[\frac{T-R}{2}\right] \tag{2.4.3}$$

$$\theta = \theta \tag{2.4.4}$$

$$\phi = \phi \tag{2.4.5}$$

where T, R are the new coordinates replacing t, r. By means of straightforward calculations we find that in the new variables the flat metric becomes:

$$ds^2_{\text{Mink}} = \Omega^{-2}(T, R)\, ds^2_{\text{ESU} } \tag{2.4.6}$$

$$ds^2_{\text{ESU}} = -dT^2 + dR^2 + \sin^2 R\big(d\theta^2 + \sin^2\theta\, d\phi^2\big) \tag{2.4.7}$$

$$\Omega(T, R) = \frac{1}{2}\cos\left[\frac{T+R}{2}\right]\cos\left[\frac{T+R}{2}\right] \tag{2.4.8}$$

This apparently trivial calculation leads to many important conclusions.

First of all let us observe that, considered in its own right, the metric ds^2_{ESU}, named the *Einstein Static Universe*, is the natural metric on a manifold $\mathbb{R} \times \mathbb{S}^3$. To see this it suffices to note that because of its appearance as argument of a sine, the variable R is an angle, furthermore, parameterizing the points of a three-sphere:

$$1 = X_1^2 + X_2^2 + X_3^2 + X_4^2 \tag{2.4.9}$$

as follows:

$$\begin{aligned} X_1 &= \cos R \\ X_2 &= \sin R \cos\theta \\ X_3 &= \sin R \sin\theta \cos\phi \\ X_4 &= \sin R \sin\theta \sin\phi \end{aligned} \tag{2.4.10}$$

another straightforward calculation reveals that:

$$\sum_{i=1}^{4} dX_i^2 = dR^2 + \sin^2 R\big(d\theta^2 + \sin^2\theta\, d\phi^2\big) \tag{2.4.11}$$

This demonstrates that $ds^2_{\text{ESU}} = -dT^2 + ds^2_{\mathbb{S}^3}$. The metric ds^2_{ESU} receives the name of Einstein Static Universe for the following reason. It is just an instance of a family of metrics, which we will consider in later chapters while studying cosmology, that are of the following type:

$$ds^2 = -dt^2 + a^2(t)\, ds^2_{3\text{D}} \tag{2.4.12}$$

where $ds^2_{3\text{D}}$ is the Euclidian metric of a homogeneous isotropic three-manifold, in the present case the three-sphere, and $a(t)$ is a function of the cosmic time, named the scale-factor. In the case of ds^2_{ESU} the scale factor is just one and for this reason the corresponding universe is static. Einstein, who was opposed to the idea of an evolving world discovered that by the addition of the cosmological constant his own equations admitted static cosmological solutions, in particular ds^2_{ESU}. Yet it was soon proved that Einstein's static universe is unstable and the great man later considered the cosmological constant the biggest mistake of his life. He was, in this respect, twice wrong, since the cosmological constant does indeed exist, yet the universe evolves nonetheless. All these questions we shall address in later chapters; at present what is important for us is the following. By means of the coordinate transformation (2.4.2)–(2.4.5), we have realized a mapping:

$$\psi : \mathscr{M}_{\text{Mink}} \to \mathscr{M}_{\text{ESU}} \simeq \mathbb{R} \otimes \mathbb{S}^3 \tag{2.4.13}$$

that injects the whole of Minkowski space into a finite volume region of the Einstein Static Universe, whose corresponding differentiable manifold is isomorphic to $\mathbb{R} \otimes \mathbb{S}^3$. In order to verify the statement we just made it suffices to compare the ranges of the coordinates T, R, θ, ϕ respectively corresponding to the whole $\mathbb{R} \otimes \mathbb{S}^3$ and to the image of Minkowski-space through the ψ-mapping:

$$\psi(\mathscr{M}_{\text{Mink}}) \subset \mathbb{R} \otimes \mathbb{S}^3 \tag{2.4.14}$$

This comparison is presented below:

$\mathbb{R} \otimes \mathbb{S}^3$	Minkowski
$-\infty < T < +\infty$	$-\pi < T + R < \pi$
$0 \le R \le \pi$	$-\pi < T - R < \pi$
$0 \le \theta \le \pi$	$0 \le \theta \le \pi$
$0 \le \phi \le 2\pi$	$0 \le \phi \le 2\pi$

(2.4.15)

The specified ranges of the $T \pm R$ variables in Minkowski case are elementary properties of the function $\arctan(x)$ which maps the infinite interval $\{-\infty, \infty\}$ into the finite one $\{-\pi, \pi\}$. To each point T, R is attached a two-sphere $\mathbb{S}^2$ parameterized by the angles θ, ϕ. It is difficult to visualize four-dimensional spaces, yet, if we replace the three-sphere by a circle, we can visualize $\mathbb{R} \otimes \mathbb{S}^3$ as an infinite cylinder and the sub-manifold $\psi(\mathscr{M}_{\text{Mink}})$ corresponds to the finite shaded region of the cylinder displayed in Fig. 2.16. The reader will notice that we have decomposed the boundary of $\psi(\mathscr{M}_{\text{Mink}})$ into various components $i^0, i^\pm, J^\pm$. To understand the meaning of such a decomposition we need to stress another fundamental property of the mapping ψ defined by (2.4.2)–(2.4.5). As it is evident from (2.4.6) Minkowski metric and the metric of ESU are not identical, yet they differ only by the square of an overall function of the coordinates. This property is precisely what defines the concept of a *conformal mapping*.

Definition 2.4.1 Let $(\mathscr{M}, g)$ be a (pseudo-)Riemannian manifold of dimension m and $(\widetilde{\mathscr{M}}, \tilde{g})$ another (pseudo-)Riemannian manifold with the same dimension. A differentiable map:

$$\psi : \mathscr{M} \to \widetilde{\mathscr{M}} \tag{2.4.16}$$

is named *conformal* if and only if on the image $\operatorname{Im}\psi \equiv \psi(\mathscr{M})$ the following condition holds true:

$$\exists \Omega \in \mathbb{C}^\infty(\operatorname{Im}\psi) \setminus \tilde{g}|_{\operatorname{Im}\psi} = \Omega^2 \psi^* g \tag{2.4.17}$$

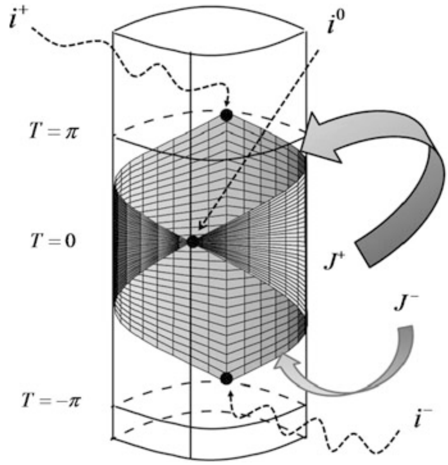

Fig. 2.16 The shaded region corresponds to the image, inside the Static Einstein Universe, of Minkowski space by means of the conformal mapping ψ. This picture visualizes the causal boundary of Minkowski space composed of a spatial infinity i^0 a future and a past time-like infinity $i^\pm$ and a future and past light-like infinity $J^\pm$

where ψ^*g denotes the pull-back of the metric g. The function Ω is named the *conformal factor*.

As anticipated above, the basic property of conformal mappings is that they preserve the causal structure. On $\psi(\mathcal{M}) \subset \widetilde{\mathcal{M}}$ we have two metrics, namely $\tilde{g}|_{\mathrm{Im}\,\psi}$ and ψ^*g. Generically curves that are geodesics with respect to the former are not geodesics with respect to the latter; yet curves that are causal in one metric are causal also in the other and vice-versa. Furthermore light-like geodesics are common to $\tilde{g}|_{\mathrm{Im}\,\psi}$ and ψ^*g. Indeed we have the following:

Lemma 2.4.1 *Consider two metrics G and g on the same manifold $\mathcal{M}$ related by a positive definite conformal factor $\Omega^2(x)$, namely*:

$$G_{\mu\nu}\,dx^\mu \otimes dx^\nu = \Omega^2(x) g_{\mu\nu}\,dx^\mu \otimes dx^\nu \tag{2.4.18}$$

The light-like geodesics with respect to the metric G are light-like geodesics also with respect to the metric g and vice-versa.

Proof The proof is performed in two steps. First of all let us note that the differential equation for geodesics takes the form

$$\frac{d^2x^\rho}{d\lambda^2} + \Gamma^\rho_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = 0 \tag{2.4.19}$$

when we use an affine parameter λ. It can be rewritten with respect to an arbitrary parameter $\sigma = \sigma(\lambda)$. By means of direct substitution equation (2.4.19) transforms into:

$$\frac{d^2x^\rho}{d\sigma^2} + \Gamma^\rho_{\mu\nu}\frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\sigma} = -\left(\frac{d\sigma}{d\lambda}\right)^{-2}\frac{d^2\sigma}{d\lambda^2}\frac{dx^\rho}{d\sigma} \tag{2.4.20}$$

Secondly let us compare the Christoffel symbols of the metric G, named $\Gamma^{\rho}_{\mu\nu}$ with those of the metric g, named $\gamma^{\rho}_{\mu\nu}$. Once again by direct evaluation we find:

$$\Gamma^{\rho}_{\mu\nu} = \gamma^{\rho}_{\mu\nu} + 2\partial_{\{\mu} \ln \Omega\, \delta^{\rho}_{\nu\}} - g_{\mu\nu}\partial^{\rho} \ln \Omega \tag{2.4.21}$$

Hence we obtain:

$$\begin{aligned}\frac{d^2x^{\rho}}{d\sigma^2} + \Gamma^{\rho}_{\mu\nu}\frac{dx^{\mu}}{d\sigma}\frac{dx^{\nu}}{d\sigma} &= \frac{d^2x^{\rho}}{d\sigma^2} + \gamma^{\rho}_{\mu\nu}\frac{dx^{\mu}}{d\sigma}\frac{dx^{\nu}}{d\sigma} - \left(g_{\mu\nu}\frac{dx^{\mu}}{d\sigma}\frac{dx^{\nu}}{d\sigma}\right)\partial^{\rho}\ln\Omega \\ &\quad + \left(\frac{d}{d\sigma}\ln\Omega\right)\frac{dx^{\rho}}{d\sigma}\end{aligned} \tag{2.4.22}$$

Let us now apply the identity (2.4.22) to the case where the curve $x^{\mu}(\sigma)$ is a light-like geodesics for the metric $g_{\mu\nu}$ and σ is an affine parameter for it. Then all terms on the right hand side of equation (2.4.22) written in the first line vanish. Indeed:

$$\frac{d^2x^{\rho}}{d\sigma^2} + \gamma^{\rho}_{\mu\nu}\frac{dx^{\mu}}{d\sigma}\frac{dx^{\nu}}{d\sigma} = 0 \tag{2.4.23}$$

is the geodesic equation in the affine parameterization and

$$g_{\mu\nu}\frac{dx^{\mu}}{d\sigma}\frac{dx^{\nu}}{d\sigma} = 0 \tag{2.4.24}$$

is the light-like condition on the tangent vector to the considered curve. It follows that the same curve $x^{\mu}(\sigma)$ satisfies the geodesic equation also with respect to the metric $G_{\mu\nu}$ provided we are able to solve the following differential equation:

$$-\left(\frac{d\sigma}{d\lambda}\right)^{-2}\frac{d^2\sigma}{d\lambda^2} = \frac{d}{d\sigma}\ln\Omega \tag{2.4.25}$$

for a function $\lambda(\sigma)$ which will play the role of affine parameter with respect to the new metric. Such an integration is easily performed. Indeed by means of straight-forward steps we first reduce (2.4.25) to:

$$\ln\left(\frac{d\sigma}{d\lambda}\right) = -\ln\Omega + \text{const} \tag{2.4.26}$$

and then with a further integration we obtain:

$$\lambda = k_1 \int \Omega(\sigma)\, d\sigma + k_2 \tag{2.4.27}$$

where $k_{1,2}$ are the two integration constants. So a light-like geodesics with respect to the metric $g_{\mu\nu}$ satisfies the geodesic equation also with respect to any metric G conformal to g with an affine parameter λ given by (2.4.27). Moreover the tangent vector to the curve is obviously light-like in the metric G if it is light-like in the metric g. This concludes the proof of the lemma. □

Let us summarize. We have found a conformal mapping of Minkowski space into a finite region of another pseudo-Riemannian manifold so that the boundary at infinity has been brought to finite distance and can be inspected. This boundary is decomposed into the following pieces:

$$\partial\psi(\mathscr{M}_{\text{Mink}}) = i^0 \bigcup i^+ \bigcup i^- \bigcup J^+ \bigcup J^- \tag{2.4.28}$$

that have been appropriately marked in Fig. 2.16. What is their meaning? It is listed below:

(1) i^0, named *Spatial Infinity* is the endpoint of the ψ image of all space-like curves in $(\mathscr{M}, g)$.
(2) i^+, named *Future Time Infinity* is the endpoint of the ψ image of all future-directed time-like curves in $(\mathscr{M}, g)$.
(3) i^-, named *Past Time Infinity* is the endpoint of the ψ image of all past-directed time-like curves in $(\mathscr{M}, g)$.
(4) J^+, named *Future Null Infinity* is the endpoint of the ψ image of all future-directed light-like curves in $(\mathscr{M}, g)$.
(5) J^-, named *Past Null Infinity* is the endpoint of the ψ image of all past-directed light-like curves in $(\mathscr{M}, g)$.

In the above listing we have denoted by $(\mathscr{M}, g)$ Minkowski space with its flat metric. The reason to use such a notation is that the same structure of the boundary applies to all *asymptotically flat space-times* in the definition we shall shortly provide.

In order to verify the above interpretation of the boundary it is convenient to disregard the two-sphere coordinates θ, ϕ restricting one's attention to radial geodesics or curves only. In this way Minkowski space becomes effectively two-dimensional with the metric $ds^2 = -dt^2 + dr^2$. The conformal transformation (2.4.2), (2.4.3) maps the half plane ($\infty \geq t \geq -\infty$, $\infty \geq r \geq 0$) into a finite region of the half-plane ($\infty \geq T \geq -\infty$, $\infty \geq R \geq 0$). This finite region is the triangle displayed in Fig. 2.17. Radial geodesics in Minkowski space are straight lines in the (t, r) half-plane. They are time-like if the angular coefficient is bigger than 45 degrees, space-like if it is less than 45 degrees and they are light-like when it is exactly $\pi/2$. In Fig. 2.18 we display the conformal transformation of these geodesics from which it is evident that the time-like ones end up in the time-infinities while the space-like ones end up in spatial infinity. The image of the light-like geodesics are still segments of straight-lines at 45 degrees which end on the null-infinities defined above. Analytically the above statements can be verified by calculating some elementary limits. The image of a straight line $t = \alpha r$ is given by:

$$\begin{aligned} T(\alpha, r) &= \arctan\big[(\alpha+1)r\big] + \arctan\big[(\alpha-1)r\big] \\ R(\alpha, r) &= \arctan\big[(\alpha+1)r\big] - \arctan\big[(\alpha-1)r\big] \end{aligned} \tag{2.4.29}$$

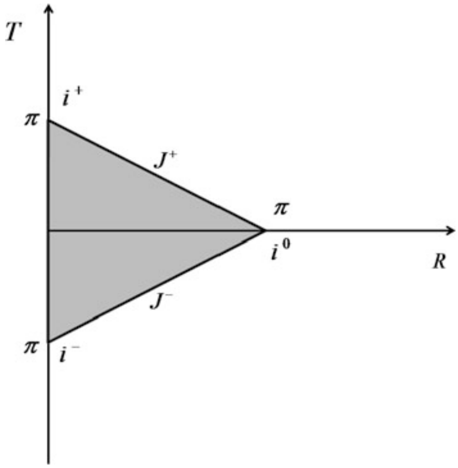

Fig. 2.17 The Penrose diagram of Minkowski space

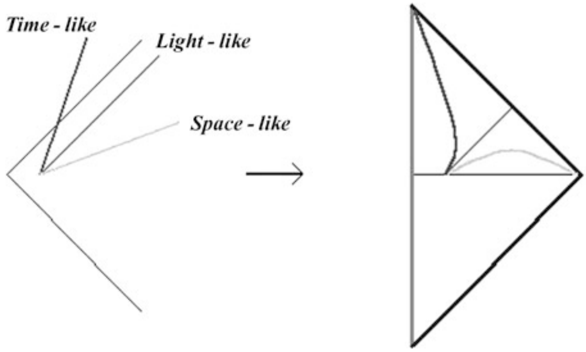

Fig. 2.18 The conformal mapping of Minkowski geodesics into the Penrose triangle

and we find:

$$\lim_{r\to\infty} T(\alpha, r) = \begin{cases} \pi & \text{if } \alpha > 1 \\ 0 & \text{if } 1 > \alpha > -1 \\ -\pi & \text{if } \alpha < -1 \end{cases} \tag{2.4.30}$$

$$\lim_{r\to\infty} R(\alpha, r) = \begin{cases} 0 & \text{if } \alpha > 1 \\ \pi & \text{if } 1 > \alpha > -1 \\ 0 & \text{if } \alpha < -1 \end{cases} \tag{2.4.31}$$

More generally we can consider curves $t = f(r)$. The same limits as above hold true if we replace α with $f'(r)$.

This concludes our discussion of the causal boundary of Minkowski space which was possible thanks to the conformal mapping of the latter into a finite region of the Einstein Static Universe. From this discussion we learnt a lesson that enables us to extract some general definition of conformal flatness allowing the inspection of the causal boundary of more complicated space-times such as, for instance, the Kruskal extension of the Schwarzschild solution.

2.4.2 Asymptotic Flatness

In this section we describe the definition of asymptotic flatness according to Ashtekar [8].

Definition 2.4.2 A space-time $(\mathscr{M}, g)$ is asymptotically flat if there exists another larger space-time $(\widetilde{\mathscr{M}}, \tilde{g})$ and a *conformal mapping*:

$$\psi : \mathscr{M} \to \psi(\mathscr{M}) \subset \widetilde{\mathscr{M}} \tag{2.4.32}$$

with conformal factor Ω:

$$\tilde{g} = \Omega^2 \psi^* g \quad \text{on } \psi(\mathscr{M}) \tag{2.4.33}$$

such that the following conditions are verified:

(1) Naming i^0 *spatial infinity*, namely the locus in $\psi(\mathscr{M})$ where terminate all space-like curves, which is required to be a single point, we have:

$$\widetilde{\mathscr{M}} - \psi(\mathscr{M}) = \overline{J^+(i^0)} \bigcup \overline{J^-(i^0)}$$

(2) The boundary of $\mathscr{M}$, named $\partial\mathscr{M}$ is decomposed as follows:

$$\partial\mathscr{M} = i^0 \bigcup \mathscr{I}^+ \bigcup \mathscr{I}^-$$

where by definition we have set:

$$\mathscr{I}^{\pm} = \partial J^{\pm}(i^0) - i^0$$

(3) There exists a neighborhood $V \subset \partial\psi(\mathscr{M})$ such that for every $p \in V$ and every neighborhood $\mathscr{O}_p$ of that point we can find a sub-neighborhood $\mathscr{U}_p \subset \mathscr{O}_p$ with the property that no causal curve intersects $\mathscr{U}_p$ more than once.
(4) The conformal factor Ω can be extended to an overall function on the whole $\widetilde{\mathscr{M}}$
(5) The conformal factor Ω vanishes on $\mathscr{I}^+$ and $\mathscr{I}^-$ but its derivative $\nabla_\mu \Omega$ does not on the same locus.

In order to appreciate all the points of the above definition it is convenient to look at Fig. 2.19 and compare with the case of Minkowski space. The starting point of the analysis is the obvious observation that any causal curve which departs from spatial infinity $i^0 \equiv (\pi, 0)$ cannot penetrate in the triangle representing Minkowski space and therefore lies in $\widetilde{\mathscr{M}} - \psi(\mathscr{M})$. If the causal curve is future-directed it goes up, while if it is past directed it goes down so that point (1) of Definition 2.4.2 is indeed verified. Let us next consider the boundary of the causal future and causal past of spatial infinity. They are given by the upper and lower side, respectively, of the triangle in Fig. 2.19, which intersect in i^0. Hence point (2) of Definition 2.4.2 is also verified. Let us note that according to this definition $\mathscr{I}^{\pm}$ are just the Causal

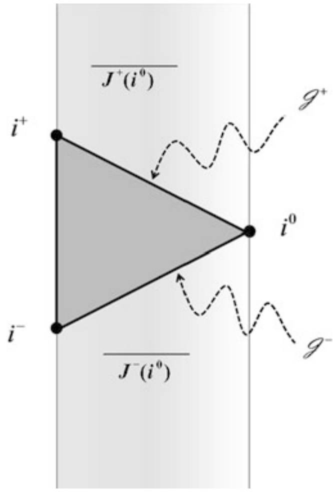

Fig. 2.19 The causal boundary of Minkowski space following Ashtekar definition

Future and Causal Past of the considered space-time, namely the locus where terminate future-directed and past-directed causal curves, respectively. In the case of Minkowski space we were able to make a finer distinction by decomposing:

$$\mathscr{I}^{\pm} = i^{\pm} \bigcup J^{\pm} \tag{2.4.34}$$

where $i^{\pm}$ correspond to Future and Past Time-Infinity, while $J^{\pm}$ are Future and Past Null-Infinities.

Point (3) of the definition is also visually evident in the case of Minkowski space and aims at excluding pathological space-times where causal curves might have chaotic behavior.

Points (4) and (5) are also extracted from the example of Minkowski space mapped into the Einstein Static Universe. There the conformal factor is

$$\Omega = \frac{1}{2} \cos\left[\frac{T+R}{2}\right] \cos\left[\frac{T-R}{2}\right] \equiv \cos T + \cos R$$

which vanishes on the two straight-lines:

$$\{\xi, \xi + \pi\}; \qquad \{\xi, -\xi + \pi\} \tag{2.4.35}$$

so, in particular on the two loci $\mathscr{I}^{\pm}$.

2.5 The Causal Boundary of Kruskal Space-Time

Let us now consider the Kruskal extension of the Schwarzschild metric given in (2.2.30) where the variable r is implicitly defined by its relation with T and X,

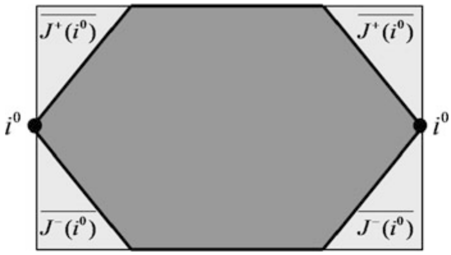

Fig. 2.20 The Spatial Infinity of Kruskal space-time and its Future and Past

namely:

$$T^2 - X^2 = \left(\frac{r}{r_S} - 1\right) \exp\left[\frac{r}{r_S}\right] \tag{2.5.1}$$

Let us introduce the further change of variables defined below:

$$\begin{aligned} T &= \frac{1}{2}\tan\left(\frac{\tau+\rho}{2}\right) + \frac{1}{2}\tan\left(\frac{\tau-\rho}{2}\right) \\ X &= \frac{1}{2}\tan\left(\frac{\tau+\rho}{2}\right) - \frac{1}{2}\tan\left(\frac{\tau-\rho}{2}\right) \end{aligned} \tag{2.5.2}$$

By means of straightforward substitutions we find that:

$$ds^2_{\text{Krusk}} = \Omega^{-2}\, \widetilde{ds}^2_{\text{Krusk}} \tag{2.5.3}$$

$$\begin{aligned} \widetilde{ds}^2_{\text{Krusk}} &= 4\frac{r^3}{r_S} \exp\left[-\frac{r}{r_S}\right]\left(-d\tau^2 + d\rho^2\right) \\ &\quad + r^2(\cos\tau + \cos\rho)^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right) \end{aligned} \tag{2.5.4}$$

$$0 = \tan\left(\frac{\tau+\rho}{2}\right)\tan\left(\frac{\tau-\rho}{2}\right) + \left(\frac{r}{r_S} - 1\right)\exp\left[\frac{r}{r_S}\right] \tag{2.5.5}$$

$$\Omega = (\cos\tau + \cos\rho) \tag{2.5.6}$$

This calculation shows that the map ψ defined by the coordinate substitution (2.5.2) is indeed a conformal map, the new metric being $\widetilde{ds}^2_{\text{Krusk}}$ defined by (2.5.4) and the conformal factor being Ω defined in (2.5.6). Let us then verify that Kruskal space-time is asymptotically flat and study the causal structure of its boundary. To this effect let us consider Fig. 2.20. Just as in the case of Minkowski space we represent the four-dimensional space-time by means of a two-dimensional picture where each point actually stands for a two-sphere spanned by the coordinates $\{\theta, \phi\}$. The points are located in the $\{\tau, \rho\}$ plane and such kind of visualization receives the name of *Penrose diagram* (Fig. 2.21).

As in the case of Minkowski space we first look for *Spatial Infinity* and we find that in the plane $\{\tau, \rho\}$ it is given by the following pair of points:

$$i^0 \equiv \{\pi, 0\} \bigcup \{-\pi, 0\} \tag{2.5.7}$$

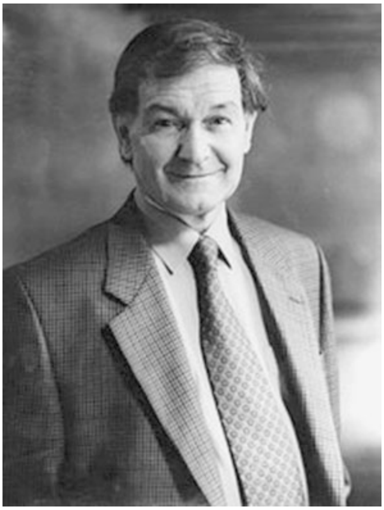

Fig. 2.21 Sir Roger Penrose, was born in 1931 in Colchester (England) and he is Emeritus Rouse Ball Professor of Mathematics at the University of Oxford. His main contributions have been to Mathematical Physics in the fields of Relativity and Quantum Field Theory. He invented the *twistor* approach to Lorentzian field theories which maps geometrical metric data of a real manifold into holomorphic data in a complex manifold with signature $(2, 2)$. He was the first to propose the cosmic censorship hypothesis according to which space-time singularities are always hidden behind event-horizons and he conceived the idealized *Penrose mechanism* which shows how energy can be extracted from rotating black-holes. Probably the most famous of his results is the quasi-periodic *Penrose tiling* of the plane with five-fold rotational symmetry. Roger Penrose is also an amateur philosopher whose views on consciousness and its relation with quantum physics are quite original and source of intense debate

Indeed this is the locus where terminate the images of all space-like curves. The duplication of i^0 is due to the periodicity of the trigonometric functions and it occurs also in Minkowski case. There it was disregarded because all copies of i^0, namely $\{(2n+1)\pi, 0\}$, $(n \in \mathbb{Z})$ could be identified without ambiguity. In the Kruskal case, instead, as we are going to see, $i^0_{\mathrm{I}} = \{\pi, 0\}$ and $i^0_{\mathrm{IV}} = \{-\pi, 0\}$ must be considered as distinct physical points since they are separated by the black-hole region which we are now going to discuss.

Following the scheme outlined in previous section, we search for the causal future and causal past of i^0 inside the extended manifold $(\widetilde{\mathscr{M}}_{\mathrm{Krusk}}, \tilde{g}_{\mathrm{Krusk}})$. At this level a fundamental new feature appears with respect to Minkowski case where, reduced to the plane $\{T, R\}$, the manifold $\widetilde{\mathscr{M}}_{\mathrm{Mink}}$ was identified with the infinite vertical strip depicted in Fig. 2.19. In the Kruskal case, on the contrary, also the embedding manifold $\widetilde{\mathscr{M}}_{\mathrm{Krusk}}$ corresponds to a finite region of the $\{\tau, \rho\}$ plane, namely the following rectangular region:

$$\{\tau, \rho\} \in \widetilde{\mathscr{M}}_{\mathrm{Krusk}} \quad \Leftrightarrow \quad -\frac{\pi}{2} \leq \tau \leq \frac{\pi}{2} \quad \text{and} \quad -\pi \leq \rho \leq \pi \tag{2.5.8}$$

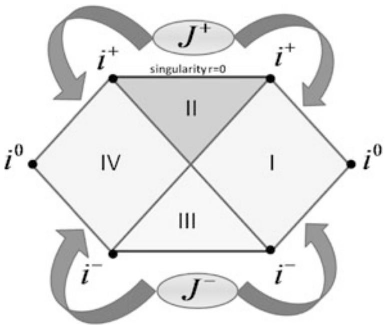

Fig. 2.22 The Penrose diagram of Kruskal space-time

The upper and lower limits on the variable τ are consequences of the form of the metric $\tilde{g}_{\text{Krusk}}$ defined in (2.5.4). This latter becomes singular when $r = 0$ and from (2.5.5) we realize that this singularity is mapped into $\tau = \pm\frac{\pi}{2}$. Hence no causal curve can trespass such limits which become a boundary for the manifold $(\widetilde{\mathscr{M}}_{\text{Krusk}}, \tilde{g}_{\text{Krusk}})$. The range of the variable ρ is fixed instead by modding out the periodicity $\rho \to \rho + 2n\pi$.

Once (2.5.8) is established, it is fairly easy to conclude that the Causal Future and Causal Past of Spatial Infinity are indeed the lighter regions of the rectangle depicted in Fig. 2.20. The corresponding boundaries are:

$$\begin{aligned}
\partial\overline{J^+(i^0)} &= \left\{\frac{\pi}{2}\xi, -\frac{\pi}{2}\xi + \pi\right\} \bigcup \left\{\frac{\pi}{2}\xi, \frac{\pi}{2}\xi - \pi\right\}; \quad \xi \in [0, 1] \\
\partial\overline{J^-(i^0)} &= \left\{-\frac{\pi}{2}\xi, -\frac{\pi}{2}\xi + \pi\right\} \bigcup \left\{-\frac{\pi}{2}\xi, \frac{\pi}{2}\xi - \pi\right\}; \quad \xi \in [0, 1]
\end{aligned} \tag{2.5.9}$$

and on these boundaries the conformal factor (2.5.6) vanishes. Hence all necessary conditions are satisfied and the Kruskal extension of Schwarzschild space-time is indeed asymptotically flat.

We can now inspect the causal structure of conformal infinity and we are led to consider the more detailed diagram of Fig. 2.22, which is the conformal image in the $\{\tau, \rho\}$ plane of the diagram 2.8 drawn in the $\{T, X\}$ plane. We easily identify in Fig. 2.22 the points $i^{\mp}$ that correspond to time-like Past and Future Infinity, respectively. Just as it was the case for Spatial Infinity also these Infinities have a double representation in the diagram. Similarly Past and Future Null Infinities are twice represented and correspond to the segments with ± 45 degrees orientation shown in Fig. 2.22. The conformal image of the singularity $r = 0$ is also double and it is provided by the two segments, upper and lower, parallel to the ordinate axis depicted in Fig. 2.22. The conformal image of the event horizon $X^2 - T^2 = 0$ is provided by the two internal lines splitting the hexagon of Fig. 2.22 into four separate regions.

Let us know consider, using the language developed in previous sections, the Causal Past of Future-Null Infinity namely $J^-(\mathscr{J}^+)$. By definition this is the set of all space-time events p such that there exists at least one causal curve starting at p

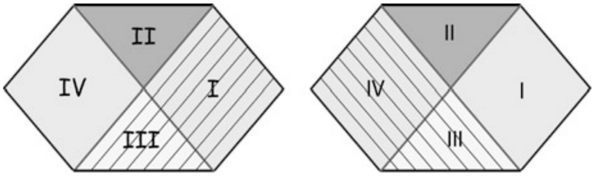

Fig. 2.23 The Causal Past of Future Null Infinity is composed of two-sheets. The Causal Past of $\mathscr{I}^+_{\mathrm{I}}$ and the Causal Past of $\mathscr{I}^+_{\mathrm{IV}}$. The first corresponds to the region shaded by lines in the picture on the *left*, the second to the region shaded by lines in the picture on the *right*

and ending on $\mathscr{I}^+$. Since $\mathscr{I}^+$ is the union of two disconnected loci:

$$\mathscr{I}^+ = \mathscr{I}^+_{\mathrm{I}} \bigcup \mathscr{I}^+_{\mathrm{IV}} \tag{2.5.10}$$

we actually have:

$$J^-\left(\mathscr{I}^+\right) = J^-\left(\mathscr{I}^+_{\mathrm{I}}\right) \bigcup J^-\left(\mathscr{I}^+_{\mathrm{IV}}\right) \tag{2.5.11}$$

A simple inspection of the Penrose diagram shows that the Causal Past of Future Null Infinity is given by the regions shown in Fig. 2.23, namely we have:

$$\begin{aligned} J^-\left(\mathscr{I}^+\right) &= \mathrm{I} \bigcup \mathrm{III} \bigcup \mathrm{IV} \\ J^-\left(\mathscr{I}^+_{\mathrm{I}}\right) &= \mathrm{I} \bigcup \mathrm{III} \\ J^-\left(\mathscr{I}^+_{\mathrm{IV}}\right) &= \mathrm{III} \bigcup \mathrm{IV} \end{aligned} \tag{2.5.12}$$

This conclusion is simply reached with the following argument. The image of light-like geodesics in the Penrose diagram is given by the straight lines with ±45 degrees orientation; hence it suffices to trace all lines that have such an inclination and which intersect Future Null Infinity. The result is precisely that of (2.5.12), depicted in Fig. 2.23.

In this way we discover a very important feature of region II, namely we find that it has empty intersection with the Causal Past of Future Null Infinity: $\mathrm{II} \bigcap J^-(\mathscr{I}^+) = \emptyset$. This property provides a rigorous mathematical formulation of that *object cut off from communication with the rest of the universe* which was firstly conceived by Openheimer and Snyder as end-point result of the gravitational collapse of super massive stars.

Inspired by the case of Kruskal space-time we can now present the general definition of black-holes:

Definition 2.5.1 Let $(\mathscr{M}, g)$ be an asymptotically flat space time and let $\mathscr{I}^+$ denote the Future Null Infinity component of its causal boundary. A black-hole region $\mathrm{BH} \subset \mathscr{M}$ is a sub-manifold of this space-time with the following defining property:

$$\mathrm{BH} \bigcap J^-\left(\mathscr{I}^+\right) = \emptyset \tag{2.5.13}$$

The event horizon is the boundary of the black-hole region separating it from the Causal Past of Future Null Infinity, namely:

$$\mathfrak{H} = \partial \mathrm{BH} \bigcap \partial J^{-}\left(\mathscr{I}^{+}\right) \tag{2.5.14}$$

Let us now comment on the properties of region III of Kruskal space-time. Differently from the black-hole region II, where all future-directed causal curves end up on the singularity, in region III this is the inevitable property of past-directed causal curves. Namely every one who happens to be in region III at some instant of time had origin in the singularity and came out from there. Furthermore all future-directed causal curves starting in III necessarily cross the horizon and end up either in the flat asymptotic region I or in its copy IV. Hence III is just the time reversal of a black-hole, named a *white hole*. A white hole emits matter rather than swallowing it and therefore evaporates as soon as it is formed.

Although white holes are a part of the classical Kruskal vacuum solution, it is doubtful that they might exist in Nature. When one considers the gravitational collapse of realistic stars, the presence of matter fields in the Einstein equations removes the presence of the white hole sector from their solutions. Furthermore, as it will become clearer in next chapter where we study the fascinating relation between the Laws of Thermodynamics and those of Black-Hole Mechanics, white holes violate the second principle of thermodynamics, reducing rather than increasing the entropy and this is one more reason for their absence from the physical universe.

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Chapter 3
Rotating Black Holes and Thermodynamics

Tu vedresti 'l Zodiaco rubecchio
Ancora all'Orse piú stretto rotare
Se non uscisse fuor dal cammin vecchio.
Sí ch'ambo e due hann'un solo orizzon,
E diversi emisperi: ...
Dante Alighieri (Purgatorio Canto IV, 64)

3.1 Introduction

In this chapter we study in considerable detail the quite intriguing and challenging properties of rotating black-holes encoded in the Kerr-Newman metric which contains only three parameters (m, J, q) corresponding, respectively, to the mass, to the angular momentum and to the charge of the hole. As anticipated in the previous chapter, irrespectively from all the details of its initial structure, a gravitational collapsing body sets down to a final equilibrium state described by the Kerr-Newman metric, which is the unique one, in $D = 4$, to be static, stationary, axial symmetric and asymptotically flat. The geodesic problem for this metric is still a completely integrable one, since there are enough first integrals, yet the explicit integration is very much laborious because it involves higher transcendental functions and the classification of trajectories turns out very complicated. We will derive the final integration formulae but we will present only a simple example of their application in view of such a complexity. We will instead dwell on the general new properties displayed by rotating black-holes that allow for a mechanism of energy extraction whose features have a surprising analogy with the laws of thermodynamics. Such an analogy is actually only the tip of an iceberg. The horizon area of the black holes behaves as an entropy and this makes it clear that, in a fundamental quantum theory of gravity, black holes must necessarily be endowed with a statistical interpretation in terms of some kind of microstates.

3.2 The Kerr-Newman Metric

Let us consider the standard set up of polar coordinates r, θ, ϕ for $\mathbb{R}^3$ plus the parameter t for time. For the angular variables θ, ϕ labeling the points of each 2-

P.G. Frè, *Gravity, a Geometrical Course*, DOI 10.1007/978-94-007-5443-0_3,

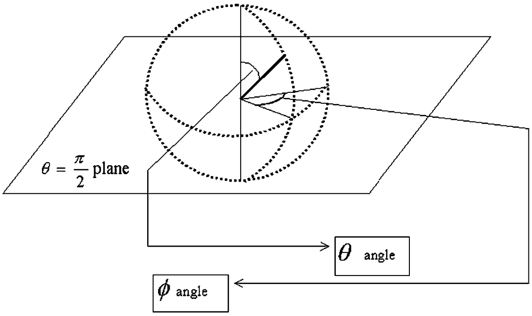

Fig. 3.1 Our conventions for the angular coordinates on the S^2 sphere are as follows: the azimuthal angle ϕ takes the values in the range $[0, 2\pi]$, while the ascension angle θ runs from 0 (the North Pole) to π (the South Pole). The metric $ds^2 = d\theta^2 + \sin^2\theta\, d\phi^2$ is singular at $\theta = 0$ and $\theta = \pi$. These are coordinate singularities that can be removed by redefining θ

sphere of radius r we adopt the same conventions already utilized in establishing the Schwarzschild solution. For reader's convenience we recall them here in Fig. 3.1.

Using this coordinate patch let us introduce a metric depending on three parameters m, α, q whose physical interpretation will be that of *mass, angular momentum* and *electric/magnetic charge* of the black hole, respectively.

It is convenient to introduce the following functions which will play the role of building blocks for the metric:

$$\rho(r,\theta) = \sqrt{r^2 + \alpha^2\cos^2\theta} \tag{3.2.1}$$

$$\Delta = r^2 + \alpha^2 - 2mr + q^2 \tag{3.2.2}$$

In terms of these notations the Kerr-Newman metric is given by the following expression for the line-element:

$$\begin{aligned} ds^2_{KN} &= -\frac{\Delta}{\rho^2}\big(dt - \alpha\sin^2\theta\, dt\big)^2 + \frac{\rho^2}{\Delta}\, dr^2 \\ &\quad + \rho^2\, d\theta^2 + \frac{1}{\rho^2}\sin^2\theta\big[\big(r^2 + \alpha^2\big) - \alpha\, dt\big]^2 \\ &\equiv -g_{\mu\nu}\, dx^\mu \otimes dx^\nu = -d\tau^2_{KN} \end{aligned} \tag{3.2.3}$$

Before studying the properties of such a metric it is useful to emphasize its notable limits in parameter space. They are listed below.

Minkowski If we set all parameters to zero $m = \alpha = q = 0$ the Kerr-Newman metric (3.2.3) degenerates into the flat Minkowski metric:

$$ds^2_{KN} \mapsto ds^2_{Mink} = -dt^2 + dr^2 + r^2\big(d\theta^2 + \sin^2 d\phi^2\big) \tag{3.2.4}$$

Schwarzschild If we set $\alpha = q = 0$, but we keep different from zero the mass parameter $m \neq 0$ the Kerr-Newman metric (3.2.3) degenerates into the spherical symmetric Schwarzschild metric. Indeed, under these assumptions we have:

$$\rho = r; \qquad \frac{\Delta}{\rho^2} = \left(1 - \frac{2m}{r}\right) \tag{3.2.5}$$

so that:

$$ds^2_{KN} \mapsto ds^2_{Schw} = -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2 d\phi^2\right) \tag{3.2.6}$$

Reissner-Nordström If we put $\alpha = 0$ keeping both m and q non-vanishing we obtain the so-called Reissner-Nordström metric which is spherical symmetric but not Ricci-flat. As we shall discuss later on, this metric corresponds to an *electrovac* solution namely to a solution of the coupled system of Maxwell-Einstein equations. This solution describes the gravitational field generated by an electrically or magnetically charged monopole of mass m and charge q. If $\alpha = 0$, we have

$$\rho = r; \qquad \frac{\Delta}{\rho^2} = \left(1 - \frac{2m}{r} + \frac{q^2}{r^2}\right) \tag{3.2.7}$$

and

$$ds^2_{KN} \mapsto ds^2_{RN} = -\left(1 - \frac{2m}{r} + \frac{q^2}{r^2}\right) dt^2 + \left(1 - \frac{2m}{r} + \frac{q^2}{r^2}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2 d\phi^2\right) \tag{3.2.8}$$

Kerr If we put the charge parameter to zero, namely $q = 0$, the Kerr-Newman metric degenerates into the Kerr metric which is Ricci-flat but not spherically symmetric. It is only axial-symmetric and it describes a rotating black-hole of mass m and angular momentum $J = m\alpha$. In this case we have

$$\rho \neq r; \qquad \Delta = \Delta_0 \equiv r^2 + \alpha^2 - 2mr \tag{3.2.9}$$

and we find:

$$ds^2_{KN} \mapsto ds^2_{Kerr} = -\frac{\Delta_0}{\rho^2}\left(dt - \alpha \sin^2\theta\, dt\right)^2 + \frac{\rho^2}{\Delta_0} dr^2 + \rho^2 d\theta^2 + \frac{1}{\rho^2} \sin^2\theta \left[\left(r^2 + \alpha^2\right) - \alpha\, dt\right]^2 \tag{3.2.10}$$

3.2.1 Riemann and Ricci Curvatures of the Kerr-Newman Metric

The next step in the analysis of the proposed metric (3.2.3) is the construction of the corresponding curvature forms. As usual we adopt the vielbein formalism and we aim at the construction first of the spin connection ω^{ab}, secondly of the curvature 2-form R^{ab}, from which we will extract the Riemann and Ricci tensors.

As written in (3.2.3) the Kerr-Newman metric is already presented as a sum of four squares so that singling out the vielbein 1-forms is an immediate task. Indeed if we define:

$$\begin{aligned} V^0 &= \frac{\sqrt{\Delta}}{\rho}\big(dt - \alpha \sin^2\theta \, d\phi\big); \qquad & V^1 &= \frac{\rho}{\sqrt{\Delta}} dr \\ V^2 &= \rho \, d\theta; & V^3 &= \frac{\sin\theta}{\rho}\big((r^2+\alpha^2)\, d\phi - \alpha \, dt\big) \end{aligned} \tag{3.2.11}$$

we obtain:

$$d\tau^2_{KN} \equiv -ds^2_{KN} = V^a \otimes V^b \eta_{ab}; \qquad \eta_{ab} = \mathrm{diag}(+,-,-,-) \tag{3.2.12}$$

Next we consider the construction of the torsionless spin-connection defined by:

$$dV^a + \omega^{ab} \wedge V^c \eta_{bc} = 0 \tag{3.2.13}$$

The solution of (3.2.13) is the following one:

$$\begin{aligned} \omega^{01} &= \frac{(2rq^2 - 2mr^2 + m\alpha^2 + r\alpha^2 + (m-r)\alpha^2 \cos 2\theta)}{2\sqrt{\Delta}\rho^3} V^0 + \frac{r\alpha \sin\theta}{\rho^3} V^3 \\ \omega^{02} &= \frac{\alpha^2 \cos\theta \sin\theta}{\rho^3} V^0 + \frac{\alpha\sqrt{\Delta}\cos\theta}{\rho^3} V^3 \\ \omega^{03} &= \frac{\alpha r \sin\theta}{\rho^3} V^1 - \frac{\alpha\sqrt{\Delta}\cos\theta}{\rho^3} V^2 \\ \omega^{12} &= \frac{\alpha^2 \cos\theta \sin\theta}{\rho^3} V^1 + \frac{r\sqrt{\Delta}}{\rho^3} V^2 \\ \omega^{13} &= \frac{\alpha r \sin\theta}{\rho^3} V^0 + \frac{r\sqrt{\Delta}}{\rho^3} V^3 \\ \omega^{23} &= \frac{\alpha\sqrt{\Delta}\sin\theta \cot\theta}{\rho^3} V^0 + \frac{(r^2+\alpha^2)\cot\theta}{\rho^3} V^3 \end{aligned} \tag{3.2.14}$$

Relying on the above result we can proceed to the calculation of the curvature 2-form, defined by:

$$R^{ab} = d\omega^{ab} + \omega^{ac} \wedge \omega^{db} \eta_{cd} \tag{3.2.15}$$

and we obtain the following result:

$$
\begin{aligned}
R^{01} &= \frac{1}{2\rho^6}\big[-\big(4mr^3 - 6m\alpha^2 r + q^2\big(\alpha^2 - 6r^2\big) + \big(q^2 - 6mr\big)\alpha^2 \cos 2\theta\big) V^0 \wedge V^1 \\
&\quad - 2\alpha \cos\theta \big(4rq^2 + m\big(\alpha^2 - 6r^2\big) + m\alpha^2 \cos 2\theta\big) V^2 \wedge V^3\big] \\
R^{02} &= \frac{1}{2\rho^6}\big[\big(\big(\alpha^2 - 2r^2\big)q^2 + mr\big(2r^2 - 3\alpha^2\big) + \big(q^2 - 3mr\big)\alpha^2 \cos 2\theta\big) V^0 \wedge V^2 \\
&\quad - \alpha \cos\theta \big(4rq^2 + m\big(\alpha^2 - 6r^2\big) + m\alpha^2 \cos 2\theta\big) V^1 \wedge V^3\big] \\
R^{03} &= \frac{1}{2\rho^6}\big[\big(\big(\alpha^2 - 2r^2\big)q^2 + mr\big(2r^2 - 3\alpha^2\big) + \big(q^2 - 3mr\big)\alpha^2 \cos 2\theta\big) V^0 \wedge V^3 \\
&\quad + \alpha \cos\theta \big(4rq^2 + m\big(\alpha^2 - 6r^2\big) + m\alpha^2 \cos 2\theta\big) V^1 \wedge V^2\big] \\
R^{12} &= \frac{1}{2\rho^6}\big[\big(\big(\alpha^2 - 2r^2\big)q^2 + mr\big(2r^2 - 3\alpha^2\big) + \big(q^2 - 3mr\big)\alpha^2 \cos 2\theta\big) V^1 \wedge V^2 \\
&\quad - \alpha \cos\theta \big(4rq^2 + m\big(\alpha^2 - 6r^2\big) + m\alpha^2 \cos 2\theta\big) V^0 \wedge V^3\big] \\
R^{13} &= \frac{1}{2\rho^6}\big[\alpha \cos\theta \big(4rq^2 + m\big(\alpha^2 - 6r^2\big) + m\alpha^2 \cos 2\theta\big) V^0 \wedge V^2 \\
&\quad + \big(\big(\alpha^2 - 2r^2\big)q^2 + mr\big(2r^2 - 3\alpha^2\big) + \big(q^2 - 3mr\big)\alpha^2 \cos 2\theta\big) V^1 \wedge V^3\big] \\
R^{23} &= \frac{1}{2\rho^6}\big[2\alpha \cos\theta \big(4rq^2 + m\big(\alpha^2 - 6r^2\big) + m\alpha^2 \cos 2\theta\big) V^0 \wedge V^1 \\
&\quad + \big(q^2 - 2mr\big)\big(2r^2 - 3\alpha^2 - 3\alpha^2 \cos 2\theta\big) V^2 \wedge V^3\big]
\end{aligned}
\tag{3.2.16}
$$

Inspecting (3.2.16) we see that the intrinsic components of the curvature 2-form, namely the flat index components of the Riemann tensor, are functions only of the coordinates θ and r, while they do not depend on the time t and on the azimuthal angle ϕ. This is so because the Kerr-Newman metric is static and axial symmetric namely it admits the following two Killing vector fields:

$$k \equiv \frac{\partial}{\partial t}; \qquad \tilde{k} \equiv \frac{\partial}{\partial \phi} \tag{3.2.17}$$

Furthermore we also note that the non-vanishing components of the Riemann tensor are of the form:

$$R^{ab}{}_{cd} = (\cdots)_{ab} \times \delta^{ab}_{cd} + (\cdots)_{ab} \times \varepsilon_{abcd} \tag{3.2.18}$$

Extracting from (3.2.16) the Riemann tensor $R^{ab}{}_{cd}$, we can calculate the Ricci tensor defined by:

$$\mathrm{Ric}_{ab} \equiv \eta_{am} R^{mn}{}_{bn} \tag{3.2.19}$$

and we obtain the following result:

$$\mathrm{Ric}_{ab} = \frac{q^2}{2\rho^4}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.2.20}$$

We wonder which kind of matter can produce a stress-energy tensor of the form (3.2.20) so that the constructed metric might be an exact solution of Einstein field equations. The answer is very simple: an electromagnetic field!

Let us consider the general form of the stress energy tensor for a Maxwell field. From the Maxwell action:

$$\mathscr{A}_{Maxwell} = -\frac{1}{4}\int d^4x\,\sqrt{-\det g}\,F_{\mu\rho}F_{\nu\sigma}g^{\mu\nu}g^{\rho\sigma} \tag{3.2.21}$$

varying with respect to the metric we obtain:

$$T^{(Maxw)}_{\mu\nu} = -\frac{1}{2}F_{\mu\rho}F_{\nu\sigma}g^{\rho\sigma} + \frac{1}{8}g_{\mu\nu}|F|^2 \tag{3.2.22}$$

where we defined:

$$|F|^2 \equiv F_{\mu\rho}F_{\nu\sigma}g^{\mu\nu}g^{\rho\sigma} = F_{ac}F_{bd}\eta^{ab}\eta^{cd} \tag{3.2.23}$$

The stress-energy tensor $T^{(Maxw)}_{\mu\nu}$ is traceless ($g^{\mu\nu}T^{(Maxw)}_{\mu\nu} = 0$) and in flat indices takes the same form as in curved indices:

$$T^{(Maxw)}_{ab} = -\frac{1}{2}F_{ac}F_{bd}\eta^{cd} + \frac{1}{8}\eta_{ab}|F|^2 \tag{3.2.24}$$

For the particular case of an electromagnetic field of the form:

$$F_{ab} = \begin{pmatrix} 0 & F_{01} & 0 & 0 \\ -F_{01} & 0 & 0 & 0 \\ 0 & 0 & 0 & F_{23} \\ 0 & 0 & -F_{23} & 0 \end{pmatrix} \tag{3.2.25}$$

Equation (3.2.24) yields the result:

$$T^{(Maxw)}_{ab} = \frac{1}{4}\left(F_{01}^2 + F_{23}^2\right)\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.2.26}$$

Due to tracelessness of the stress-energy tensor, Einstein field equations reduce to:

$$\mathrm{Ric}_{ab} = \kappa T^{(Maxw)}_{ab} \tag{3.2.27}$$

which is compatible with result (3.2.20) for the Ricci tensor of the Kerr-Newman metric if

$$\frac{1}{4}\left(F_{01}^2 + F_{23}^2\right) = \frac{q^2}{2\kappa\rho^4} \tag{3.2.28}$$

We conclude that the Kerr-Newman metric provides a consistent solution of the coupled Maxwell Einstein field equations if there exist two functions $F_{01}(r, \theta)$ and $F_{23}(r, \theta)$ of the coordinates r, θ such that:

1. (3.2.28) is verified,
2. the 2-form:

$$F \equiv 2F_{01} V^0 \wedge V^1 + 2F_{23} V^2 \wedge V^3 \tag{3.2.29}$$

   is closed:

$$dF = 0 \tag{3.2.30}$$

3. and also coclosed, namely:

$$d \star F = 0 \tag{3.2.31}$$

   where

$$\star F \equiv 2F_{23} V^0 \wedge V^1 + 2F_{01} V^2 \wedge V^3 \tag{3.2.32}$$

   is the Hodge dual of the 2-form F.

Indeed (3.2.30) and (3.2.31) are the two Maxwell equations.

By direct evaluation one can verify that all the above conditions are met by the following two functions:

$$\begin{aligned} F_{01} &= -\frac{\sqrt{2}q}{\kappa}\frac{r^2 - \alpha^2\cos^2\theta}{\rho^4} \\ F_{23} &= \frac{2\sqrt{2}q}{\kappa}\frac{\alpha r\cos\theta}{\rho^4} \end{aligned} \tag{3.2.33}$$

It follows from (3.2.33) that the Kerr-Newmann solution has both an electric and a magnetic field, while the non-rotating spherical symmetric limit ($\alpha \to 0$) which is the Reissner Nodström solution has only an electric field. The electric and magnetic charges of the rotating black-hole are rigidly related to each other in order to obtain a consistent solution of Maxwell Einstein equations.

3.3 The Static Limit in Kerr-Newman Space-Time

The key feature of the Kerr-Newman space-time is that it describes the gravitational field of a *rotating black hole*. This will become evident by studying the properties of the world-lines of test particles or observers around the hole.

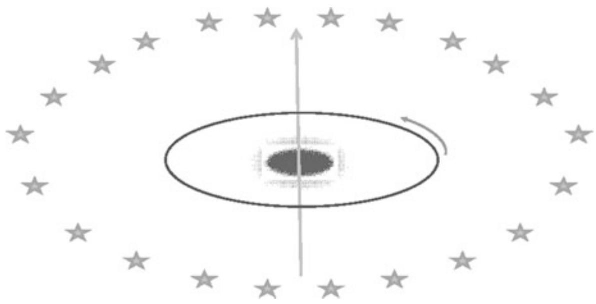

Fig. 3.2 An observer rotating in the equatorial plane around the hole has an angular velocity $\Omega = \frac{d\phi}{dt}$ with respect to the fixed stars

Differently from the Schwarzschild metric, which is both static and spherically symmetric, the KN-metric is static and only axial symmetric. Indeed, instead of four, the KN-space-time admits only two Killing vector fields corresponding to translations in the time variable t and in the axial angle ϕ respectively. In the coordinate patch we have utilized these Killing vectors have the following simple form:

$$k = \frac{\partial}{\partial t}; \qquad \tilde{k} = \frac{\partial}{\partial \phi} \tag{3.3.1}$$

and their norms and scalar products are directly related to the metric coefficients in the following way:

$$(k, k) = g_{tt} = 1 - \frac{2mr - q^2}{\rho} \tag{3.3.2}$$

$$(\tilde{k}, \tilde{k}) = g_{\phi\phi} = \frac{\sin^2\theta}{\rho^2}\left[\left(r^2 + \alpha^2\right)^2 - \Delta\alpha^2 \sin^2\theta\right] \tag{3.3.3}$$

$$(k, \tilde{k}) = g_{t\phi} = \frac{2mr - q^2}{\rho^2}\alpha \sin^2\theta \tag{3.3.4}$$

Consider next an observer which rotates around the black hole along a circular orbit lying in its equatorial plane. The trajectory of such a test-particle is characterized by the following simple equation:

$$r = \text{const}; \qquad \theta = \frac{\pi}{2}; \qquad t = s; \qquad \phi = \Omega s \tag{3.3.5}$$

where s is an affine parameter and Ω is the angular velocity of the particle perceived by an observer that is at rest with respect to the distant fixed stars (see Fig. 3.2). The 4-velocity of such a test-particle is given by:

$$u = (1, \Omega, 0, 0) = k + \Omega\tilde{k} \tag{3.3.6}$$

For a physical particle the norm of the 4-velocity is necessarily non-negative and this yields the following interesting quadratic condition on the angular velocity Ω:

$$(u, u) \geq 0 \quad \Rightarrow \quad (k, k) + 2(k, \tilde{k})\Omega + (\tilde{k}, \tilde{k})\Omega^2 \geq 0 \tag{3.3.7}$$

The roots of the above quadratic form are given by:

$$\Omega_{\pm} = \frac{-g_{t\phi} \pm \sqrt{g_{t\phi}^2 - g_{tt} g_{\phi\phi}}}{g_{\phi\phi}} \tag{3.3.8}$$

and we have physical non-tachyonic observers as long as their angular velocity lyes in the range between the two roots:

$$\Omega_- \leq \Omega \leq \Omega_+ \tag{3.3.9}$$

Naming:

$$\omega \equiv -\frac{g_{t\phi}}{g_{\phi\phi}} = \alpha \frac{2mr - q^2}{(r^2 + \alpha^2)^2 - \Delta \alpha \sin^2 \theta} \tag{3.3.10}$$

the two roots of the quadratic form (3.3.7) can also be rewritten as:

$$\Omega_{\min} \equiv \Omega_- = \omega - \sqrt{\omega^2 - \frac{g_{tt}}{g_{\phi\phi}}} \tag{3.3.11}$$

$$\Omega_{\max} \equiv \Omega_+ = \omega + \sqrt{\omega^2 - \frac{g_{tt}}{g_{\phi\phi}}} \tag{3.3.12}$$

The interest of this rewriting comes from the fact that the quantity ω has a distinctive physical interpretation, namely it is the angular velocity of a locally non-rotating observer.

Locally Non-rotating Observers Which observers deserve the name of locally non-rotating? Clearly those whose angular momentum vanish! We now prove that this happens for those test-bodies whose 4-velocity is orthogonal to the constant time hypersurfaces $t = \text{const}$, so that they are at rest on them. For a similar observer the 4-velocity is just the gradient of time, namely:

$$u^\mu = \nabla^\mu t = g^{\mu\nu} \partial_\nu t = g^{\mu t} \tag{3.3.13}$$

Taking into account the specific form of the Kerr-Newman metric we obtain that the 4-velocity of a locally non-rotating observer is:

$$u = \left(g^{tt}, g^{\phi t}, 0, 0\right) \tag{3.3.14}$$

Consider now the general problem of computing time-like geodesics for the KN-metric. Just as in the case of the Schwarzschild metric we can address such a problem starting from the effective Lagrangian[1] and writing the corresponding Euler-Lagrange equations:

$$0 = \frac{d}{d\tau} \frac{\partial L}{\partial \dot{x}^\mu} - \frac{\partial L}{\partial x^\mu} \tag{3.3.15}$$

[1]Compare with (3.8.5)–(3.8.9) of Volume One.

Also here we immediately derive two-conserved quantities associated with the cyclic Lagrangian coordinates t and ϕ. The first integral of motion ℓ associated with the azimuthal angle ϕ is the angular momentum and (3.3.15) provides its definition. For a test-body moving on a world-line of type (3.3.5) we find

$$\ell = g_{\phi\phi}\dot{\phi} + g_{\phi t}\dot{t} \tag{3.3.16}$$

According to (3.3.14) the angular momentum of a locally non-rotating observer (LNRO) vanishes since we obtain:

$$\ell_{LNRO} = g_{\phi\phi}g^{\phi t} + g_{\phi t}g^{tt} \equiv \delta^t_\phi = 0 \tag{3.3.17}$$

and this concludes the proof of our statement.

The crucial point, however, is that a locally non-rotating observer has a non-vanishing angular velocity with respect to the reference frame of the fixed stars. In other words a test-body with a null angular momentum is perceived to rotate around the hole by a distant observer who is at rest in the asymptotic flat geometry. What is the actual angular velocity of such a locally non-rotating test body? It is given by the quantity ω which we introduced in (3.3.10). Indeed the equation:

$$0 = g_{\phi\phi}\dot{\phi} + g_{\phi t}\dot{t} = g_{\phi\phi}\Omega + g_{\phi t} \tag{3.3.18}$$

is solved by $\Omega = \omega$. Hence

$$\omega = \frac{(q^2 - 2mr)\alpha \sin^2\theta}{q^2 + r^2 + \alpha^2 - \alpha^2 \sin^2\theta - 2mr} \tag{3.3.19}$$

is the angular velocity with which rotates with respect to the fixed stars an observer which is at rest with respect to its local geometry. The behavior of this angular velocity with respect to the radius r and to the declination angle θ is displayed in Fig. 3.3.

Static Observers We have seen that those observers who have zero angular momentum and are not rotating with respect to the local geometry have a non-vanishing angular velocity $\Omega = \omega$ with respect to the fixed stars. We can now consider the case of the *static observers* defined as those whose angular velocity in the fixed star frame vanishes, namely $\Omega = 0$. The angular momentum of the static observers does not vanish. It is equal to:

$$\ell = g_{t\phi} = -\frac{(q^2 - 2mr)\alpha \sin^2\theta}{r^2 + \alpha^2 \cos^2\theta} \tag{3.3.20}$$

The physical interpretation of this fact is clear. The black hole rotates and drags all reference frames along its rotation. In order to stand still, a test-body needs to have an angular momentum which counterbalance the dragging of the inertial frames. The question is: how far can the dragging be opposed? The answer is simple: as long as the 4-velocity of a static observer is time-like. For a static observer (3.3.6)

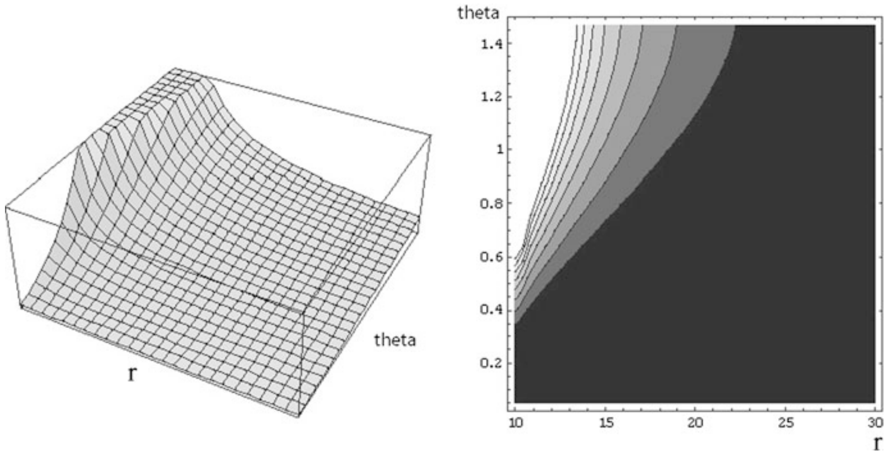

Fig. 3.3 Value of the angular velocity ω of a locally non-rotating observer plotted against the radius r and the declination angle θ. In the planes at $\theta = 0, \pi$ namely at the North and South poles of the hole, we have $\omega = 0$, *i.e.* there is no rotational dragging of the inertial frames. On the other hand ω is maximal on the equatorial plane $\theta = \frac{\pi}{2}$. On the other hand ω decreases with the distance r from the hole and vanishes at $r = \infty$ where it is uniformly zero for all values of θ

implies that the 4-velocity coincides with the time-translation Killing vector k. In asymptotic geometry this Killing vector is time-like, but as we get closer to the hole its norm (k, k) shrinks and there is a surface where it vanishes, namely $(k, k) = 0$. This equation defines the *static limit*:

$$\begin{aligned} \Sigma_{SL} \;:\; 0 = (k,k) &\equiv g_{tt} \\ &\Downarrow \\ 0 = q^2 + r^2 + \alpha^2 &- \alpha^2 \sin^2(\theta) - 2mr \\ &\Downarrow \\ r = r_\pm(\theta) \equiv m \pm &\sqrt{m^2 - q^2 - \alpha^2 + \alpha^2 \sin^2(\theta)} \end{aligned} \tag{3.3.21}$$

Corresponding to the two roots of the quadratic equation there are two vanishing surfaces for the norm of Killing vector k: one outer $r = r_+(\theta)$ and one inner $r = r_-(\theta)$. The *static limit* corresponds to the outer surface $r = r_+(\theta)$. An image of the static limit surface is displayed in Fig. 3.4.

3.4 The Horizon and the Ergosphere

As we have seen in the previous section, a physical observer has an angular velocity Ω falling in the range (3.3.9) comprised between the two roots $\Omega_\pm$ of the quadratic

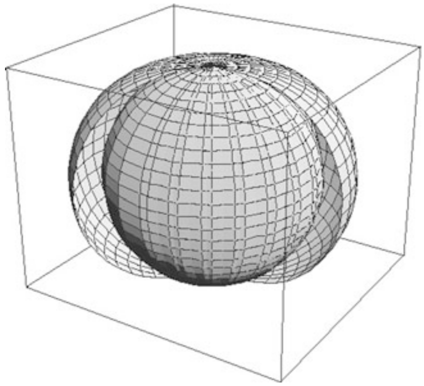

Fig. 3.4 The static limit surface Σ_{SL} defined by $r = r_+(\theta)$ is an ellipsoid and contains inside itself the spherical surface $r = r_H = r_+(\frac{\pi}{2})$ which, as we discuss in the main text, is the event horizon Σ_H. The static limit surface is tangential to the horizon at the North and South poles of the hole. The region contained between Σ_{SL} and Σ_H is named the *ergosphere*

form (3.3.7). When the discriminant of that quadratic form vanishes, the two-roots coincide and we have:

$$\Omega_{\max} = \Omega_{\min} = \Omega_H \tag{3.4.1}$$

Inspecting (3.3.7) we see that its discriminant is given by the expression:

$$\Delta = g_{t\phi}^2 - g_{tt} g_{\phi\phi} = r^2 + \alpha^2 - 2mr + q^2 \tag{3.4.2}$$

which is indeed the building block function $\Delta(r)$ introduced in (3.2.2). The reason for the choice of its name becomes now apparent.

We claim that the bigger root of the quadratic equation $\Delta(r) = 0$ is the event-horizon of the black-hole. Let us first spell out the two roots and then motivate our statement. We have

$$\Delta = 0 \to r = r_\pm = m \pm \sqrt{m^2 - \left(q^2 + \alpha^2\right)} \tag{3.4.3}$$

Let us now argue in the following way. Given the two Killing vectors (3.3.1) let us define the family of Killing fields:

$$\chi(\Omega) = k + \Omega \tilde{k} \tag{3.4.4}$$

which, as we know from (3.3.6), correspond to the 4-velocities of test-bodies having angular velocities Ω with respect to the fixed stars. For each $\chi(\Omega)$ let us consider the light-like radial curves that admit $\chi(\Omega)$ as the tangent vector field. Explicitly we set:

$$dt = dp; \qquad d\phi = \Omega\, dp; \qquad d\theta \tag{3.4.5}$$

and we obtain the equation:

$$0 = g_{tt}\, dp^2 + 2\Omega g_{t\phi}\, dp^2 + \Omega^2 g_{\phi\phi}\, dp^2 + g_{rr}\, dr^2 \tag{3.4.6}$$

so that for each Ω we have an effective 2-dimensional metric:

$$ds^2 = g_{pp}(r, \Omega)\, dp^2 + g_{rr}\, dr^2 \tag{3.4.7}$$

where

$$g_{pp}(r,\Omega) = g_{tt} + 2\Omega g_{t\phi} + \Omega^2 g_{\phi\phi} \tag{3.4.8}$$

If $g_{pp}(r,\Omega)$ never changes sign (namely it is positive definite) then the effective 2-dimensional metric displays no horizon. If $g_{pp}(r,\Omega)$ goes through zero, then in the p, r plane there is a horizon. However if there is a horizon for a certain time $p(\Omega)$ light can still escape to infinity along some other time $p(\Omega')$ for which $g_{pp}(r,\Omega')$ is positive-definite. In other words we look for the norm of the Killing vectors $\chi(\Omega)$:

$$\big(\chi(\Omega), \chi(\Omega)\big) = g_{tt} + 2\Omega g_{t\phi} + \Omega^2 g_{\phi\phi} \tag{3.4.9}$$

If all the possible vectors $\chi(\Omega)$ have negative norm then we are below the horizon. This implies that we are below the horizon when the discriminant of the quadratic form (3.4.8) is negative, so that the horizon is indeed given by the condition (3.4.3) as we claimed. On the horizon $r = r_+$ the equation:

$$\big(\chi(\Omega), \chi(\Omega)\big) = 0 \tag{3.4.10}$$

admits only one solution:

$$\Omega = \Omega_H \equiv \frac{\alpha}{r_+^2 + \alpha^2} = \frac{\alpha}{\alpha^2 + (m + \sqrt{m^2 - q^2 - \alpha^2})^2} \tag{3.4.11}$$

The above quantity Ω_H can be interpreted as the angular velocity of the event-horizon in the sense that any physical test-body sitting on the horizon necessarily rotates with such a velocity with respect to the fixed stars.

The Horizon Area We can now easily calculate the area of the horizon. By definition we have:

$$\begin{aligned} \text{Area}_H &= \int_{r=r_+} \sqrt{g_{\theta\theta} g_{\phi\phi}}\, d\theta\, d\phi = \left(r_+^2 + \alpha^2\right) \int \sin\theta\, d\theta\, d\phi \\ &= 4\pi\left(r_+^2 + \alpha^2\right) \end{aligned} \tag{3.4.12}$$

and by comparison with (3.4.11) we obtain the following very interesting relation of the horizon area with the mass m and the angular momentum $J \equiv m\alpha$ of the black-hole:

$$\text{Area}_H = 4\pi \frac{J}{\Omega_H m} \tag{3.4.13}$$

3.5 Geodesics of the Kerr Metric

The Kerr metric was discovered at the beginning of the sixties of the XXth century but it took several years before the problem of integrating its geodesics equations was solved. For the Schwarzschild field the geodesics equations are almost immediately reduced to quadratures by regarding them as Euler Lagrange equations of a mechanical problem with 4 Lagrangian coordinates $q^\mu = (t, r, \theta, \phi)$ and exploiting two facts:

1. There are three *first integrals of the motion* respectively given by the *energy* $\mathscr{E}$, *the angular momentum* L and *the mass* μ of the particle
2. One Lagrangian coordinate can be eliminated from start, since all orbits are planar and the declination angle θ can be conventionally fixed to the value $\theta = \frac{\pi}{2}$ without loss of generality.

In this way, after elimination of θ we have a number of conserved charges equal to the number of effective Lagrangian coordinates and the mechanical system is necessarily reduced to the quadratures. The really crucial point, therefore, is the elimination of θ which, in the Schwarzschild case might be seen as a consequence of the full-spherical symmetry, absent in the Kerr case. At $\alpha \neq 0$ there is dynamics also in the declination angle θ, while at first glance, the integrals of motion seem to be just three as at $\alpha = 0$. Hence integrability seem to be lost for the Kerr metric.

As Carter[2] discovered, the truth is more subtle and the Kerr geodesic system is still fully integrable. The reason for that is the existence of a fourth hidden integral of motion, the Carter constant K, which exists at all values of α and is, in the limit $\alpha \mapsto 0$, the real source for the trivialization of the θ motion.

In order to discover the Carter constant one has to reformulate the geodesic problem within the framework of the Hamilton Jacobi approach to classical mechanics and this is what we shall do in the present section. As a preparation to this task let us first review the construction of the three integral of motion associated with manifest symmetries.

3.5.1 The Three Manifest Integrals, $\mathscr{E}$, L and μ

The two first integrals $\mathscr{E}$ and L are associated with symmetries of the metric via Noether theorem (see Sect. 1.7 in Chap. 1 of Volume 1). They exist just because the two Lagrangian coordinates t and ϕ are cyclic. On its turn this cyclicity follows from the existence of the two Killing vectors $k = \partial/\partial t$ and $\tilde{k} = \partial/\partial\phi$. These properties are true for the Kerr metric as much as for the Schwarzschild one. Hence also the Kerr metric admits the first integrals $\mathscr{E}$ and L.

Defining the Lagrangian according to the conventions of used in Chaps. 3 and 4 of Volume 1 and using the form (3.2.10) of the Kerr metric in Boyes-Lindquist coordinates, namely:

$$\mathscr{L} \equiv -\frac{1}{2} g_{\mu\nu}(x) \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}$$

[2] Brandon Carter is an Australian born theoretical physicist working at Meudon (CNRS), France.

$$= -\frac{1}{2}\left(\frac{\rho^2 \dot{r}^2}{\Delta} + \rho^2 \dot{\theta}^2 + \frac{((r^2+\alpha^2)\dot{\phi} - \alpha \dot{t})^2 \sin^2\theta}{\rho^2} - \frac{\Delta(\dot{t} - \alpha\dot{\phi}\sin^2\theta)^2}{\rho^2}\right) \tag{3.5.1}$$

we find the Kerr definition of the first integrals of motion $\mathscr{E}$ and L. Explicitly:

$$\mathscr{E} = p_t \equiv \frac{\partial \mathscr{L}}{\partial \dot{t}} = \left(1 - \frac{2m}{\rho^2}\right)\dot{t} + \alpha\left(\frac{2mr}{\rho^2}\sin^2\theta\right)\dot{\phi} \tag{3.5.2}$$

$$-L = p_\phi \equiv \frac{\partial \mathscr{L}}{\partial \dot{\phi}} = \alpha\left(\frac{2mr}{\rho^2}\sin^2\theta\right)\dot{t} - \frac{\Sigma^2}{\rho^2}\sin^2\theta\dot{\phi} \tag{3.5.3}$$

where, adding the new shorthand Σ^2 to the already introduced ones ρ^2 and Δ, we have:

$$\begin{aligned} \rho^2 &= r^2 + \alpha^2\cos^2\theta \\ \Delta &= r^2 - 2mr + \alpha^2 \\ \Sigma^2 &= \left(r^2+\alpha^2\right)^2 - \alpha^2\left(r^2 - 2mr + \alpha^2\right)\sin^2\theta \end{aligned} \tag{3.5.4}$$

Equation (3.5.3) replace the homologous ones of the Schwarzschild case (see Chap. 3 of Volume One). In the limit $\alpha \mapsto 0$ the Kerr metric degenerates into the Schwarzschild metric and the definitions (3.5.3) of the energy and angular momentum of a test particle flow to the Schwarzschild ones. This is easily checked, noting that at $\alpha = 0$ we have $\rho^2 = \Delta = r^2$ and $\Sigma^2 = r^4$.

Equation (3.5.3) can be effectively interpreted in the following matrix form:

$$\begin{pmatrix} \mathscr{E} \\ -L \end{pmatrix} = \mathfrak{M}(r,\theta)\begin{pmatrix} \dot{t} \\ \dot{\phi} \end{pmatrix} \tag{3.5.5}$$

where the key point is that the 2×2 matrix:

$$\mathfrak{M}(r,\theta) = \begin{pmatrix} 1 - \frac{2m}{\rho^2} & \frac{2mr\alpha\sin^2\theta}{\rho^2} \\ \frac{2mr\alpha\sin^2\theta}{\rho^2} & -\frac{\Sigma^2}{\rho^2} \end{pmatrix} \tag{3.5.6}$$

is function only of the coordinates r and θ. The same, obviously is true also of the inverse matrix.

$$\mathfrak{M}^{-1}(r,\theta) = \begin{pmatrix} \frac{\rho^2\Sigma^2}{4m^2r^2\alpha^2\sin^4\theta + (\rho^2-2m)\Sigma^2} & \frac{2mr\alpha\rho^2\sin^2\theta}{4m^2r^2\alpha^2\sin^4(\theta) + (\rho^2-2m)\Sigma^2} \\ \frac{2mr\alpha\rho^2\sin^2\theta}{4m^2r^2\alpha^2\sin^4\theta + (\rho^2-2m)\Sigma^2} & \frac{2m\rho^2-\rho^4}{4m^2r^2\alpha^2\sin^4\theta + (\rho^2-2m)\Sigma^2} \end{pmatrix} \tag{3.5.7}$$

Hence if the geodesic flow of the coordinates r, θ has already been determined in terms of the first integral of motion, namely if we have the two proper-time functions:

$$r = r(\tau, E, L); \qquad \theta = \theta(\tau, E, \lambda) \tag{3.5.8}$$

then the matrix $\mathfrak{M}^{-1}$ is reduced to a known function of τ and the inverse relation:

$$\begin{pmatrix} \dot{t} \\ \dot{\phi} \end{pmatrix} = \mathfrak{M}^{-1}(\tau) \begin{pmatrix} \mathscr{E} \\ -L \end{pmatrix} \tag{3.5.9}$$

reduces also the integration of the cyclic variables t and ϕ to quadratures.

The constant of motion μ^2 is associated with fixing the reparameterization invariance of the geodesics equation. Indeed, in order for the Euler-Lagrange equations obtained from the Lagrangian (3.5.1) to be equivalent to the original geodesics equations it is necessary that the *Lagrangian time* τ should coincide with the proper time defined by the metric. This implies that we have to enforce the constraint:

$$\mathscr{L} = \frac{1}{2}\mu^2 \quad \text{where} \begin{cases} \mu^2 = 1; & \text{time-like geodesics} \\ \mu^2 = 0; & \text{light-like geodesics} \end{cases} \tag{3.5.10}$$

This condition yields the third manifest integral of motion:

$$\mu^2 = \frac{\rho^2 \dot{r}^2}{\Delta} + \rho^2 \dot{\theta}^2 + \frac{((r^2+\alpha^2)\dot{\phi} - \alpha \dot{t})^2 \sin^2\theta}{\rho^2} - \frac{\Delta(\dot{t} - \alpha\dot{\phi}\sin^2\theta)^2}{\rho^2} \tag{3.5.11}$$

3.5.2 The Hamilton-Jacobi Equation and the Carter Constant

Let us recall the essential points of the Hamilton Jacobi method of integration of a Hamiltonian system.

Given the Hamiltonian:

$$H(p,q) = p_i \dot{q}^i - \mathscr{L}(q,\dot{q}) \tag{3.5.12}$$

where the canonical momenta $p_i \equiv \frac{\partial \mathscr{L}}{\partial \dot{q}^i}$ are defined as usual, the Hamilton Jacobi method consists of constructing the generating function $S(\tau, p, q)$ of a canonical transformation which reduces the new Hamiltonian $\widetilde{H}$ to an identically vanishing function of the new canonical variables (P, Q). In this way we will be guaranteed that both the new canonical coordinates Q^i and the new canonical momenta P_i are constant, since:

$$0 = \dot{Q}^i = \frac{\partial \widetilde{H}}{\partial P_i}; \qquad 0 = \dot{P}_i = -\frac{\partial \widetilde{H}}{\partial Q^i} \tag{3.5.13}$$

Calling $S(q, P, \tau)$ the generating function of such a canonical transformation, where q^i are the old coordinates and P_i the new momenta, by definition we have the relations:

$$\begin{aligned} p_i &= \frac{\partial S(\tau, q, P)}{\partial q^i} \\ Q^i &= \frac{\partial S(\tau, q, P)}{\partial P^i} \end{aligned} \tag{3.5.14}$$

which, provided that $S(\tau, q, P)$ is known, yields the explicit solution of the mechanical problem under consideration. Such solution is the complete integral since it involves exactly $2n$ integration constants that are nothing else but the new canonical momenta and coordinates (P, Q). The function S is named the Jacobi principal function and, as a consequence of its definition, it satisfies the Hamilton Jacobi equation:

$$\frac{\partial S}{\partial \tau} + H\left(q^i, \frac{\partial S}{\partial q^i}\right) = 0 \tag{3.5.15}$$

The question is whether the Hamilton-Jacobi equation can be integrated more easily than the original Hamiltonian equations. This happens when (3.5.15) is such that it allows for a separation of the variables. By this we mean that it is consistent to write the following ansatz for the function $S(\tau, q, P)$:

$$S(\tau, q, P) = \mathscr{E}\tau + \sum_{i=1}^{n} W_i\left(q^i, P_i\right) \tag{3.5.16}$$

where each function W_i depends only on the corresponding *old* canonical variable q^i. When this is the case the integration of the Hamilton Jacobi equation can be reduced to the quadratures.

Applying the Hamilton-Jacobi method to the problem of geodesics, the first thing that we note is one of a general character, common to any metric. Since the Lagrangian is a quadratic form in the velocities, with coordinate dependent coefficients, the Hamiltonian will be a quadratic form in the canonical momenta with coordinate dependent coefficients. Indeed in full generality we obtain:

$$p_\mu \equiv -g_{\mu\nu}(q)\dot{q}^\nu \quad \Rightarrow \quad H(q, p) = -\frac{1}{2} g^{\mu\nu}(q) p_\mu p_\nu \tag{3.5.17}$$

where $g^{\mu\nu}$ is the controvariant metric with upper indices.

In the case of the Kerr-metric we have:

$$g^{\mu\nu} = \begin{pmatrix} \frac{\Sigma^2}{\Delta\rho^2} & 0 & 0 & \frac{2mr\alpha}{\Delta\rho^2} \\ 0 & -\frac{\Delta}{\rho^2} & 0 & 0 \\ 0 & 0 & -\frac{1}{\rho^2} & 0 \\ \frac{2mr\alpha}{\Delta\rho^2} & 0 & 0 & \frac{\csc^2\theta(\alpha^2\sin^2\theta - \Delta)}{\Delta\rho^2} \end{pmatrix} \tag{3.5.18}$$

where the chosen order of the coordinates is (t, r, θ, ϕ). Correspondingly the Hamiltonian takes the explicit form:

$$H(p, q) = \frac{-\Delta^2 p_r^2 + \Sigma^2 p_t^2 - \Delta p_\theta^2 + \alpha^2 p_\phi^2 - \Delta\csc^2(\theta) p_\phi^2 + 4mr\alpha p_t p_\phi}{2\Delta\rho^2} \tag{3.5.19}$$

which provides the explicit form of the Hamilton-Jacobi equation. Recalling the first integrals (3.5.3) we try the following factorized ansatz for the principal Jacobi function:

$$S(\tau, q, P) = \frac{1}{2}\mu^2\tau + \mathscr{E}t - L\phi + \sigma(\theta) + \varpi(r) \tag{3.5.20}$$

where $\sigma(\theta)$ is some function of the variable θ and $\varpi(r)$ some function of the variable r. The ansatz (3.5.20) is consistent with the definitions:

$$\mathscr{E} = \frac{\partial S}{\partial t}; \qquad -L = \frac{\partial S}{\partial \phi} \tag{3.5.21}$$

and yields:

$$p_\theta = \frac{\partial S}{\partial \theta} = \partial_\theta \sigma(\theta); \qquad p_r = \frac{\partial S}{\partial r} = \partial_r \varpi(r) \tag{3.5.22}$$

Furthermore, upon insertion into the Hamilton Jacobi equation (3.5.15), the chosen ansatz reproduces the constraint:

$$\frac{1}{2}\mu^2 = H\left(q, \frac{\partial S}{\partial q}\right) \tag{3.5.23}$$

provided the following equation holds:

$$\mathfrak{H}_\theta(\theta) + \mathfrak{H}_r(r) = 0 \tag{3.5.24}$$

where we have introduced the following two functions of the declination angle θ and of the radius r, respectively:

$$\begin{aligned} \mathfrak{H}_\theta(\theta) &= \alpha^2\mu^2\cos^2\theta + (\alpha\mathscr{E}\sin\theta - L\csc\theta)^2 + \sigma'(\theta)^2 \\ \mathfrak{H}_r(r) &= -\frac{((r^2+\alpha^2)\mathscr{E} - L\alpha)^2}{r^2 - 2mr + \alpha^2} + r^2\mu^2 + \left(r^2 - 2mr + \alpha^2\right)\varpi'(r)^2 \end{aligned} \tag{3.5.25}$$

Since $\mathfrak{H}_\theta$ depends only on the θ variable and $\mathfrak{H}_r(r)$ depends only on the r variable, (3.5.24) can be true if and only if both functions are constant throughout the geodesic motion and their constant values are opposite. Namely we must have:

$$K = \mathfrak{H}_\theta(\theta) = -\mathfrak{H}_r(r) \tag{3.5.26}$$

The constant K, named the Carter constant is the fourth missing integral of motion which ensures full-integrability of the mechanical system.

3.5.3 Reduction to First Order Equations

Thanks to the above introduced Carter constant the geodesic equations can be completely reduced to a first order system. The procedure is straightforward. Varying the Lagrangian (3.5.1) with respect to $\dot{r}$ and $\dot{\theta}$ we find the explicit form of p_r and p_θ, respectively:

$$p_r = \frac{\rho^2}{\Delta}\dot{r} \tag{3.5.27}$$

$$p_\theta = \rho^2\dot{\theta} \tag{3.5.28}$$

Solving (3.5.26) for $\varpi'(r) = p_r$ and $\sigma'(\theta) = p_\theta$ and equating the results to those of (3.5.28), we obtain:

$$\rho^2\dot{r} = \pm\sqrt{\mathscr{E}^2 - \mu^2}\sqrt{\mathfrak{p}(r)} \tag{3.5.29}$$

$$\rho^2\dot{\theta} = \pm\left\{-\alpha^2\mu^2\cos^2\theta - (L\csc\theta - \alpha\varepsilon\sin\theta)^2 + K\right\}^{1/2} \tag{3.5.30}$$

where

$$\begin{aligned}\mathfrak{p}(r) \equiv \frac{1}{\mathscr{E}^2 - \mu^2}\Big\{&\big(\mathscr{E}^2 - \mu^2\big)r^4 + 2m\mu^2 r^3 - \big(K + \alpha\big(-2\alpha\mathscr{E}^2 + 2L\mathscr{E} + \alpha\mu^2\big)\big)r^2 \\ &+ 2Kmr + \alpha^2\big((L - \alpha\mathscr{E})^2 - K\big)\Big\}\end{aligned} \tag{3.5.31}$$

is a quartic polynomial in the radial variable r whose coefficients depend algebraically on the first integrals of motion $\mathscr{E}$, L, K, μ^2.

Changing variable in the second of equations (3.5.30) by setting $u = \cos\theta$ we can rewrite it as follows:

$$\rho^2\dot{u} = \pm\alpha\sqrt{\mu^2 - \mathscr{E}^2}\sqrt{\mathfrak{q}(u)} \tag{3.5.32}$$

where also $\mathfrak{q}(u)$ is a quartic polynomial, but it as the special property that it contains only the even powers of u:

$$\mathfrak{q}(u) = u^4 + \frac{(K + \alpha(2L\varepsilon + \alpha(\mu^2 - 2\varepsilon^2)))u^2}{\alpha^2(\varepsilon^2 - \mu^2)} + \frac{(L - \alpha\varepsilon)^2 - K}{\alpha^2(\varepsilon^2 - \mu^2)} \tag{3.5.33}$$

Hence we have the differential system:

$$\rho^2\dot{r} = \pm\sqrt{\mathscr{E}^2 - \mu^2}\sqrt{\mathfrak{p}(r)} \tag{3.5.34}$$

$$\rho^2\dot{u} = \pm\alpha\sqrt{\mu^2 - \mathscr{E}^2}\sqrt{\mathfrak{q}(u)} \tag{3.5.35}$$

Let us name e_i, $(i = 1, \ldots, 4)$ the roots of the polynomial, $\mathfrak{p}(r)$, namely let us set:

$$\mathfrak{p}(r) = \prod_{i=1}^{4}(r - e_i) \tag{3.5.36}$$

and let us name g_1, g_2 the two independent roots of the polynomial $\mathfrak{q}(u)$ which is necessarily of the form:

$$\mathfrak{q}(u) = \prod_{i=1}^{2} \left(u^2 - g_i^2\right) \tag{3.5.37}$$

Eliminating τ from (3.5.35) we conclude that the relation between the variables r and u is reduced to quadratures, namely:

$$\int \frac{dr}{\sqrt{\mathfrak{p}(r)}} = \mathrm{i}\alpha \int \frac{du}{\sqrt{\mathfrak{q}(u)}} + \text{cost} \tag{3.5.38}$$

One finds that the relevant integrals appearing in the above relation can be analytically evaluated and expressed in terms of the elliptic integral function:

$$F(\xi|m) \equiv \int_0^{\xi} \frac{d\phi}{\sqrt{1 - m \sin^2 \phi}} \tag{3.5.39}$$

Indeed we find:

$$\mathfrak{P}(r, e_i) \equiv \int \frac{dr}{\sqrt{\mathfrak{p}(r)}} = -2 \frac{F\left(\arcsin\left(\sqrt{\frac{(r-e_2)(e_1-e_4)}{(r-e_1)(e_2-e_4)}}\right) | \frac{(e_1-e_3)(e_2-e_4)}{(e_2-e_3)(e_1-e_4)}\right)}{\sqrt{(e_2 - e_3)(e_1 - e_4)}} \tag{3.5.40}$$

$$\mathfrak{W}(u, g_i) \equiv \int \frac{du}{\sqrt{\mathfrak{q}(u)}} = \frac{F\left(\arcsin\left(\frac{u}{g_1}\right) | \frac{g_1^2}{g_2^2}\right)}{g_2} \tag{3.5.41}$$

and the final relation between u and r along the geodesics is implicitly given by:

$$\mathfrak{P}(r, e_i) - \mathrm{i}\alpha \mathfrak{W}(u, g_i) = c_1 \tag{3.5.42}$$

where c_1 is the first found of the remaining four integration constants.

3.5.4 *The Exact Solution of the Schwarzschild Orbit Equation as an Application*

The Schwarzschild metric is a particular limit of the Kerr metric for $\alpha \mapsto 0$. Hence the above formal integration of the geodesic equations in the Kerr case should provide, as a by-product, also the exact analytic equation of the Schwarzschild orbit equation, which in Chap. 4 of Volume 1 we treated only perturbatively. As an illustration of the method, in this section we derive the complete analytic form of the orbit for a massive test-particle moving around a spherical symmetric Schwarzschild black-hole.

In the Schwarzschild case the equation for the derivatives of the time and azimuthal coordinates (3.5.9) reduce to:

$$\dot{\phi} = \frac{L}{r^2} \tag{3.5.43}$$

$$\dot{t} = \frac{r^2 \mathscr{E}}{r^2 - 2m} \tag{3.5.44}$$

while the equation for the derivative of the declination angle θ is:

$$r^2 \dot{\theta} = \sqrt{K - L^2 \csc^2(\theta)} \tag{3.5.45}$$

which follows from (3.5.30) by setting $\alpha = 0$. From the above relation we conclude that we can always impose the vanishing of the θ-derivative for any value of θ by choosing the Carter constant K appropriately. Since in a spherical symmetric field the actual value of θ is purely conventional, we can just choose to confine all motions to the equatorial plane by setting:

$$\theta = \frac{\pi}{2}; \qquad K = L^2 \tag{3.5.46}$$

Fixing $\alpha = 0$ and $K = L^2$ the quartic polynomial (3.5.31) becomes:

$$\mathfrak{p}(r) = r^4 + \frac{2mr^3}{\mathscr{E}^2 - 1} - \frac{L^2 r^2}{\mathscr{E}^2 - 1} + \frac{2L^2 mr}{\mathscr{E}^2 - 1} \tag{3.5.47}$$

which is still quartic but has the property that one of its roots is $r = 0$. Hence we can write:

$$\mathfrak{p}(r) = r \prod_{i=1}^{3} (r - e_i) \tag{3.5.48}$$

and the relation between the three non-trivial roots e_i and the physical first integrals is the following:

$$L^2 = \frac{e_1 e_2 e_3}{e_1 + e_2 + e_3} \tag{3.5.49}$$

$$m = \frac{e_1 e_2 e_3}{2(e_2 e_3 + e_1(e_2 + e_3))} \tag{3.5.50}$$

$$\mathscr{E}^2 = \frac{(e_1 + e_2)(e_1 + e_3)(e_2 + e_3)}{(e_2 + e_3)e_1^2 + (e_2^2 + 3e_3 e_2 + e_3^2)e_1 + e_2 e_3(e_2 + e_3)} \tag{3.5.51}$$

At this point we can directly obtain the analytic form of the orbit eliminating $d\tau$ from the two equations:

$$r^2 \frac{dr}{d\tau} = \left(\mathscr{E}^2 - 1\right)\sqrt{\mathfrak{p}(r)} \tag{3.5.52}$$

$$r^2 \frac{d\phi}{d\tau} = L \tag{3.5.53}$$

In this way we get:

$$\frac{\mathscr{E}^2 - 1}{L} \int d\phi = \int \frac{dr}{\sqrt{\mathfrak{p}(r)}} \tag{3.5.54}$$

From which we immediately get:

$$\frac{\mathscr{E}^2 - 1}{L} \phi = -\frac{2F\left(\arcsin\left(\sqrt{\frac{(\frac{e_1}{r}-1)e_3}{e_1-e_3}}\right) | \frac{e_2(e_1-e_3)}{(e_1-e_2)e_3}\right)}{\sqrt{(e_2-e_1)e_3}} \tag{3.5.55}$$

which can be rewritten as:

$$F\left(\arcsin\sqrt{X}\,|z\right) = Y \tag{3.5.56}$$

$$X \equiv \frac{(\frac{e_1}{r} - 1)e_3}{e_1 - e_3} \tag{3.5.57}$$

$$z \equiv \frac{e_2(e_1 - e_3)}{(e_1 - e_2)e_3} \tag{3.5.58}$$

$$Y \equiv -\frac{\varphi}{2} \frac{e_3\sqrt{\frac{e_1e_2(e_2-e_1)}{e_1+e_2+e_3}}}{(e_2e_3 + e_1(e_2+e_3))} \tag{3.5.59}$$

The first of (3.5.59) can be analytically inverted in terms of special functions since, by very definition, we have:

$$F\left(\arcsin\sqrt{X}\,|z\right) = Y \quad \Leftrightarrow \quad \sqrt{X} = \mathfrak{sn}(Y|z) \tag{3.5.60}$$

where $\mathfrak{sn}(Y|z)$ is the Jacobi special elliptic function $\mathfrak{sn}$ while $F(t|z)$ denotes the elliptic integral of the first kind, whose definition we have already recalled in (3.5.39).

In this way we obtain the final explicit analytic form of the Schwarzschild orbit for a massive particle depending on the three integration constants e_1, e_2, e_3 which parameterize the angular momentum L, the energy $\mathscr{E}$ and the Schwarzschild emiradius m. We find:

$$r(\phi) = \frac{e_1e_3}{(e_1 - e_3)\left(\mathfrak{sn}\left[-\frac{\varphi}{2}\frac{e_3\sqrt{\frac{e_1e_2(e_2-e_1)}{e_1+e_2+e_3}}}{(e_2e_3+e_1(e_2+e_3))} | \frac{e_2(e_1-e_3)}{(e_1-e_2)e_3}\right]\right)^2 + e_3} \tag{3.5.61}$$

Equation (3.5.61) contains both closed and open orbit depending on whether the energy $\mathscr{E}^2$ is less or larger than one. Two examples of orbits described by formula (3.5.61) are displayed in Figs. 3.5 and 3.6.

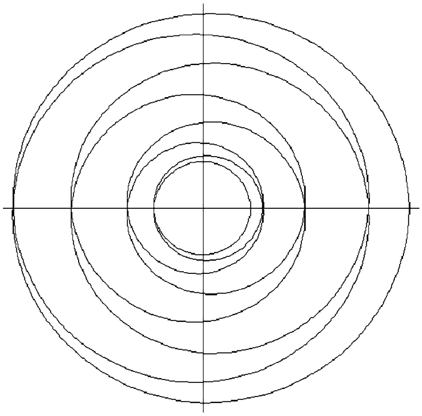

Fig. 3.5 An example of a closed orbit described by the exact analytic solution (3.5.61) of the geodesic equations for the Schwarzschild metric. The values of the roots chosen in this example are: $\{e_1, e_2, e_3\} = \{2.7, 11.7, 25.6\}$ corresponding to $\{L^2, m, \mathscr{E}^2\} = \{19.98, 1, 0.95\}$. As we see, in this case $\mathscr{E}^2 < 1$ and for this reason the orbit is closed

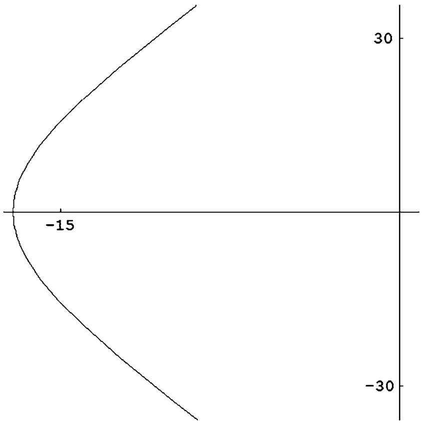

Fig. 3.6 An example of an open orbit described by the exact analytic solution (3.5.61) of the geodesic equations for the Schwarzschild metric. The values of the roots chosen in this example are: $\{e_1, e_2, e_3\} = \{-17.1, 2.1, 10.9\}$ corresponding to $\{L^2, m, \mathscr{E}^2\} = \{100, 1, 1.5\}$. As we see, in this case $\mathscr{E}^2 > 1$ and for this reason the orbit is open

3.5.5 About Explicit Kerr Geodesics

In the Schwarzschild case we demonstrated the use of the complete integration formulae. The classification of all time-like and null-like geodesics encoded in the final integration formulae is still very laborious for the general Kerr case because of the implicit form of the solution. Indeed there are very many different type of geodesics spherical, and non-spherical, open and closed, retrograding and advancing and so on. We stop our discussion at this level and we turn to the most intriguing analogy with thermodynamics.

3.6 The Kerr Black Hole and the Laws of Thermodynamics

Let us now focus on the case of a neutral rotating black-hole by setting $q = 0$ and let us reconsider the results we obtained for the horizon area A_H of a pure Kerr solution and for its angular velocity Ω_H. In terms of the black-hole mass m and of its angular momentum J, (3.4.13) and (3.4.11) can be rewritten as follows:

$$A_H(m, J) = 8\pi m\left(m + \sqrt{m^2 - \frac{J^2}{m^2}}\right) \tag{3.6.1}$$

$$\Omega_H(m, J) = \frac{J}{2m^2(m + \sqrt{m^2 - \frac{J^2}{m^2}})} \tag{3.6.2}$$

Let us now introduce an additional function, whose interpretation we will later retrieve:

$$\kappa(m, J) = \frac{\sqrt{m^2 - \frac{J^2}{m^2}}}{2m(m + \sqrt{m^2 - \frac{J^2}{m^2}})} \tag{3.6.3}$$

Calculating the variation of $A_H(m, J)$ in the standard way:

$$\delta A_H = \partial_m A_H \delta m + \partial_J A_H \delta J \tag{3.6.4}$$

we can verify the following variational identity:

$$\delta m = \kappa \frac{1}{8\pi} \delta A_H + \Omega_H \delta J \tag{3.6.5}$$

What is it special about this identity? The answer is striking: it is formally identical to the first law of thermodynamics if we introduce the following interpretations:

$$m = U \qquad \text{internal energy} \tag{3.6.6}$$

$$\frac{1}{8\pi} A_H = S \qquad \text{entropy} \tag{3.6.7}$$

$$\kappa = \frac{1}{T} \qquad \text{inverse temperature} \tag{3.6.8}$$

$$\Omega_H = -p \qquad \text{pressione} \tag{3.6.9}$$

$$J = V \qquad \text{volume} \tag{3.6.10}$$

At first sight this might seem just an arbitrary, meaningless, formal exercise yet a little bit of further consideration starts revealing the profound significance of the analogy. First of all if (3.6.5) is the first law of thermodynamics then the second law should also apply in the form:

$$\delta A_H \geq 0 \quad \text{in all physical processes} \tag{3.6.11}$$

thirdly if κ is the inverse temperature, it should be an intensive quantity, namely constant over the body which in our analogy is the event horizon. Clearly the function $\kappa(m, J)$ introduced in (3.6.3) as such a property yet the interesting point is that we can identify this expression with a quantity defined in terms of the black-hole geometry that is constant over the horizon and has a well defined physical interpretation. Let us postpone this identification for a moment and consider the last implication of the thermodynamical interpretation of (3.6.5). Indeed if all the rest is as we claimed the term

$$\delta W = \Omega_H \delta J \tag{3.6.12}$$

should be interpreted as some *work* extracted from a thermodynamical process involving the black-hole. The whole point is precisely this. Do such processes exist by means of which we can extract energy from a rotating black-hole and do they satisfy the second law of thermodynamics (3.6.11)? The answer is yes and involves in a crucial way the near horizon region that we named ergosphere in previous pages. The *gedanken experiment* showing the mechanism of energy extraction was found by Penrose in 1969.

3.6.1 The Penrose Mechanism

The Killing vector field k defined in (3.3.1) which becomes the standard time translation in the asymptotic flat space-time far from the hole is instead space-like inside the ergosphere as we already noted. Thus for a massive test particle of four momentum $p^\mu = \mu u^\mu$ the energy:

$$E \equiv p^\mu k_\mu \tag{3.6.13}$$

is not necessarily positive inside the ergosphere. Therefore, by making a black hole absorb a particle with negative total energy we can actually extract energy from the black hole! Let us see how we can do this. Suppose that from our laboratory, located far from the hole and at rest with respect to the reference frame of the fixed stars, we throw a rocket towards the black-hole. Let us denote p_0^μ the momentum of our missile that will navigate along a time-like geodesic. Its energy:

$$E_0 \equiv (p_0, k) \tag{3.6.14}$$

stays constant along the trajectory since it is the scalar product of a Killing vector with the tangent vector to a geodesic. Suppose that when it enters the ergosphere the rocket splits into two fragments as illustrated in Fig. 3.7. Conceptually this can be arranged for instance by means of an explosive connected to a suitable clock. By local conservation of the energy-momentum we have:

$$p_0^\mu = p_1^\mu + p_2^\mu \tag{3.6.15}$$

where $p_{1,2}^\mu$ are the four-momenta of the two fragments. Contracting equation (3.6.15) with the Killing vector k_μ we obtain:

$$E_0 = E_1 + E_2 \tag{3.6.16}$$

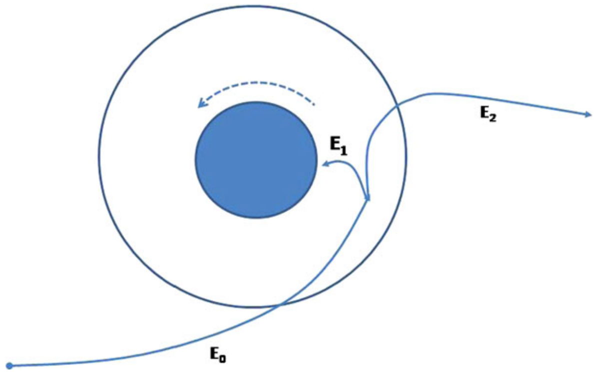

Fig. 3.7 Schematic view of the Penrose gedanken experiment

However, inside the ergosphere, we can arrange the breakup of the rocket in such a way that one of his fragments has negative total energy:

$$E_1 < 0 \tag{3.6.17}$$

Therefore, if the other fragment will make return to asymptotically flat infinity following its own geodesic it will have an energy E_2 which is greater than the initial energy of our projectile. In other words we have extracted energy from the black hole which has made some *work* for us! What has it happened? It is easily understood. The fragment with negative energy from the ergosphere has crossed the event horizon and it has fallen inside the whole. The latter having absorbed a negative energy particle has now a slightly smaller mass: $m' = m - |E_1|$. Let us now consider the angular momentum of the infalling negative energy particle. By definition we have:

$$\ell_1 = -(\tilde{k}, p_1) \tag{3.6.18}$$

where $\tilde{k}$ is the rotational Killing vector defined in (3.3.1). On the other hand since the Killing vector $\chi(\Omega_H)$ is null-like and future-directed on the horizon it follows that for any physical particle of momentum p^μ crossing the horizon we must have:

$$\big(p, \chi(\Omega_H)\big) \equiv E - \Omega_H \ell > 0 \tag{3.6.19}$$

This applies to all particles also to our negative energy rocket-fragment. It follows that, not only the energy, but also the angular momentum of this latter is negative and we have:

$$\ell_1 < \frac{E_1}{\Omega_H} \tag{3.6.20}$$

At the end of the process our black hole has swallowed an object of energy $E_1 < 0$ and of angular momentum $\ell_1 < 0$. As a result both its mass and its angular momentum have been decreased since:

$$\begin{aligned} m' &= m - |E_1| \\ J' &= J - |\ell_1| \end{aligned} \tag{3.6.21}$$

The hole is lighter and rotates slower. The important thing is that as a consequence of (3.6.20) we have:

$$\delta J < \frac{\delta m}{\Omega_H} \tag{3.6.22}$$

3.6.2 The Bekenstein Hawking Entropy and Hawking Radiation

Inserted into the identity (3.6.5) the inequality (3.6.22) implies that, for the Penrose process, we have:

$$\kappa \frac{1}{8\pi} \delta A_H \geq 0 \quad \Rightarrow \quad \delta A_H \geq 0 \tag{3.6.23}$$

and the thermodynamical interpretation is consistent since both the first and the second law are respected. It is obviously important to establish that such conditions hold true for any physically conceivable process. This was advocated with many arguments and in 1971 a very important result in classical differential geometry was rigorously proved by Hawking [1], stating that in any time development, governed by Einstein field equations and involving black-holes, the total sum of all the horizon areas can never decrease.

Hence the interpretation of the horizon area as an entropy got momentum and in 1974 it was proposed by Bekenstein [2] that the formula:

$$S_{BH} = \frac{1}{8\pi} A_H \tag{3.6.24}$$

should be interpreted as stating that in all thermodynamical processes of the universe the black-hole entropy takes part as an addendum to the total statistical entropy.

This stimulated the hunt for the statistical interpretation of the horizon area. Indeed, if this latter behaves as a true entropy, it means that classical black holes actually correspond to a very large number N_{micro} of quantum microstates and we have

$$A_H \propto \log N_{micro} \tag{3.6.25}$$

Which microstate and in which quantum theory was not clear for a long time and it is not completely clarified to the present time. Yet the statistical interpretation of black-holes obtained further evidence from the parallel discovery of the phenomenon of *Hawking radiation* [3], which gave an independent argument to identify the above introduced function κ with a temperature. The actual intrinsic definition of κ is the following. Let us simply name χ the Killing vector $\chi(\Omega_H)$ which is null-like and future directed on the horizon. Since the horizon is a null surface, χ is both tangential and orthogonal to it. The norm of χ vanishes on the horizon and as such it is constant on it. The gradient of this norm is therefore normal to the horizon and as such it is proportional to χ. In other words we necessarily have:

$$\nabla^\mu (\chi, \chi) = -2\kappa \chi^\mu \tag{3.6.26}$$

The proportionality factor $\kappa(x)$ is a space-time function which can be proved to be constant on the horizon and there precisely equal to the expression introduced in (3.6.3). It is named *surface gravity* since it can be shown to be the limiting force that must be exerted at infinity to hold a unit test mass in place when approaching the horizon. This interpretation becomes obvious in the Schwarzschild limit ($J \to 0$). In this case we have $\kappa = \frac{1}{2m}$ and, by reinstalling the physical constants, we obtain

$$\kappa \propto \frac{GM}{R_s^2} \tag{3.6.27}$$

which is indeed the Newton force on the horizon ($r = R_s = 2\frac{GM}{c^2}$).

By using quantum field theory in the background metric of a black-hole and carefully dealing with the creation of particle-antiparticle pairs near the horizon, Hawking found that all black-holes (including the Schwarzschild one) actually emit a faint thermal radiation whose temperature is the inverse of κ, evaluated at the horizon.

This intriguing semiclassical phenomenon gave the final evidence that the thermodynamical interpretation of the laws of black hole dynamics is quite sound and that a statistical interpretation of the Bekenstein Hawking entropy is compelling. The first example of such an interpretation was obtained by Strominger in 1996 in the context of string theory. We will not address such a topic in this book but we shall come back, in Chap. 9, to the structure of the black hole entropy, while discussing the classical black hole solutions of supergravity. There the entropy acquires a group theoretical interpretation that is also the main clue to its statistical interpretation in terms of string microstates.

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Chapter 4
Cosmology: A Historical Outline from Kant to WMAP and PLANCK

Vos calculs sont corrects, mais votre physique est abominable
Albert Einstein

4.1 Historical Introduction to Modern Cosmology

From most remote antiquity all humans have looked at the sky, have observed the stars and tried to obtain from them some answers to the most profound and intriguing questions that challenged their minds. Immanuel Kant (see Fig. 4.1), who was the first to understand the existence of other galaxies beyond the Milky Way, disposed that the inscription on his grave-stone should specify the most relevant targets of his life-long philosophical meditation, namely the star globe above our head and the moral law within it.

Essentially no human civilization, at any time of history and in whatsoever corner of the Earth was deprived of some kind of cosmology, namely of some general overview of the sky, of its contents, of its order and structure. Yet, until very recent times, all theories of the world have been very far from providing some realistic description of the existing cosmos, since all of them underestimated by several orders of magnitude the actual dimensions of the Universe and our distance from the closest stars. General Relativity came just at the time when the true ladder of cosmic distances started to be unveiled. Modern cosmology developed from these two seeds: a new geometrical theory of gravity, which is the leading force at very large scales, and the discovery that the Universe is indeed very large, its constituents being separated from each other by distances of a previously never suspected magnitude.

4.2 The Universe Is a Dynamical System

Einstein once said: *The most incomprehensible thing about the Universe is that it is comprehensible*.

Indeed his theory, General Relativity, proved to be the conceptual framework where, for the first time in human history, questions about the large scale structure of the visible universe could be formulated in an algorithmic way obtaining answers and predictions. To a certain extent such answers surprised and disappointed Einstein, whose fundamental philosophical attitude is revealed by his frequent ex-

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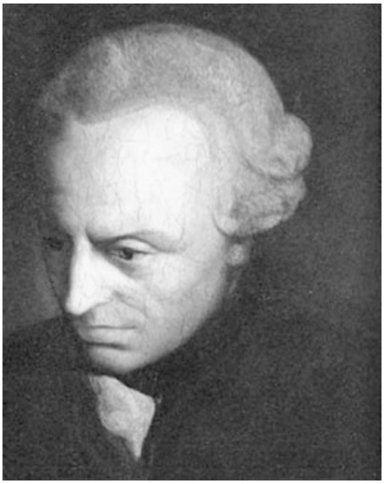


Fig. 4.1 Portrait of Immanuel Kant (1724–1804)

pressions of faith in *Spinoza's God*, namely in a God which manifests his own image through the harmony of the world. Spinoza and Einstein's God is not a person and quite different from the personalized divinities of all human religions: he is not characterized by a human psychology, he is completely orthogonal to the concept of Divine Providence, whose model comes from the egoistic desires of the human species. Yet, as Einstein used to say, *he does not cast dices*.

This well known sentence explains Einstein's stubborn opposition to the probabilistic interpretation of quantum mechanics and his life long efforts to avoid it. Moreover Einstein-Spinoza God is infinite and eternal and just for this reason incompatible with the idea of *creatio ex nihilo*, namely of creation from nothing.

The Universe that Einstein cheered is an eternal and static one. Quite ironically the field equations of General Relativity lead instead to solutions where the Universe expands or, in any case, evolves. This is precisely what observational data, starting from 1929, showed in a more and more definite and persuasive way.

Today we know that the Universe is a dynamical system undergoing a continuous, complicated and, somehow, chaotic evolution. The world at large scales is far from being a place of order and harmony, it is on the other hand a place characterized by violent and catastrophic phenomena, encompassing explosions, gravitational collapses, collisions of galaxies and continuous, gigantic displacements of energy that remodel its structure, while, in the background, persists an accelerated expansion, possessing all the features of a *creation from nothing*, disordered and probably stochastic.

4.3 Expansion of the Universe

Let us review the historical path through which the contemporary vision of a constantly expanding dynamical Universe replaced Aristotle's view of a static eternal world.

4.3.1 Why the Night is Dark and Olbers Paradox

Heinrich Wilhelm Olbers was born in 1758 at Arbergen in the vicinity of Bremen. As he reached the appropriate age for that, he went to Saxony where he enrolled as a student of the Göttingen Medical Faculty. In 1780 he graduated from Göttingen University and made return to Bremen where, during the day, he practiced the medical profession, while at nights he looked at the sky. At home, doctor Olbers had transformed his own loft into an astronomical observatory and he was the first to invent a method to compute the orbits of comets. In 1815 he discovered a new comet that to the present day bears his name. Earlier he was among the firsts who observed asteroids. In 1801 the Italian monk Giuseppe Piazzi discovered the first asteroid, in 1802 Olbers spotted the second and named it *Pallas*. Five years later, in march 1807, he discovered a third one that was christened *Vesta* by Gauss, specially solicited by its discoverer to name the new celestial body.

Olbers' obsession was the darkness of the night sky and in 1823 he formulated the thesis that such fact is in conflict with the hypothesis of an infinite eternal and static universe. The thesis was published in 1826 and since then it is named the *Olbers paradox*, although it seems that the same idea had been put forward by other people years or centuries before and among them also by Kepler. The paradox consists of the rather obvious observation that at any angle from the Earth the sight line should end at the surface of a star, so that the night sky should be completely white. This would certainly happen if the Universe were infinite and infinitely old: in that case in any direction we should find at some distance a star and, no matter how large that distance were, an infinite age of the Universe would grant a sufficient time for the light emerging from such a star to travel to the Earth. Quite simply, according to Olbers, darkness of the night points to the fact that the Universe is finite both in space and time and probably expanding.

4.3.2 Hubble, the Galaxies and the Great Debate

Edwin Hubble (see Fig. 4.2) was born in 1889 at Marshfield in Missouri: his father was a lawyer and Edwin was the fifth of seven brothers. Very tall and athletic, in the course of his life he practiced many sports almost professionally. Astronomy was a passion for him since early youth but became his stable profession only after some time. At the beginning of his career he was a student of Chicago University where he actually devoted most of his time to sports. In 1910, being the recipient of a prestigious fellowship, he went to England where he studied Law at Oxford University. However, coming back to the States after graduation, he did not become a lawyer, rather a high school teacher in Kentucky. After one year he quitted also that profession and went to Indiana where he worked as the coach of a basket team. In 1914 he left that job and returned to Chicago University where he studied Astronomy. In 1917 he graduated defending a doctoral thesis entitled: *Photographic Investigations of Faint Nebulae*. In his work the candidate came to the conclusion

Fig. 4.2 Edwin Hubble 1889–1953

that spiral nebulae are not galactic objects. In this way Hubble was addressing the most fundamental question of Astronomy that had raised the passionate interest of scientists for more than a century and that would culminate three years later in the famous *Great Debate* that opposed Harlow Shapley to Herbert Curtis in front of the National Academy of Sciences on April 26th 1920. The question at the root of the Great Debate is a very relevant and fundamental one and can be summarized as follows: *how large is the cosmos and what is its large scale structure?* Indeed it is correct to state that the real vastity of the Universe was immensely underestimated in the whole course of human history and that this situation still persisted in the first decades of the XXth century although some hints of the truth had already been collected.

For the Ancients the World was essentially constituted by the Sun and the Planets, namely by the solar system, which they considered surrounded by the fixed star sphere on whose nature ideas were always very vague (see for instance [1] for more details).

When the heliocentric theory of Copernicus replaced the Ptolemaic geocentric system, a new conceptual problem arose whose only resolution resided in the following conclusion: the distance between the Sun and the other stars is actually gigantic to an extent never suspected before. Indeed if the Earth orbits around the Sun and the fixed stars are fixed with respect to this latter, why we do not see them moving in the sky progressively changing their angular position from the winter to the summer solstice? This necessarily implied angular displacement was named *parallax* and it was immediately noted that measuring it and knowing the distance from the Earth to the Sun one might compute the distance of the latter from the observed star. Yet no star displayed such an angular motion and until 1838 no star-parallax was ever detected. There were just two options: either the heliocentric system was wrong or all the stars were so enormously far away from the Sun that parallax angles resulted immensely tiny and smaller than the resolution of all the available instruments.

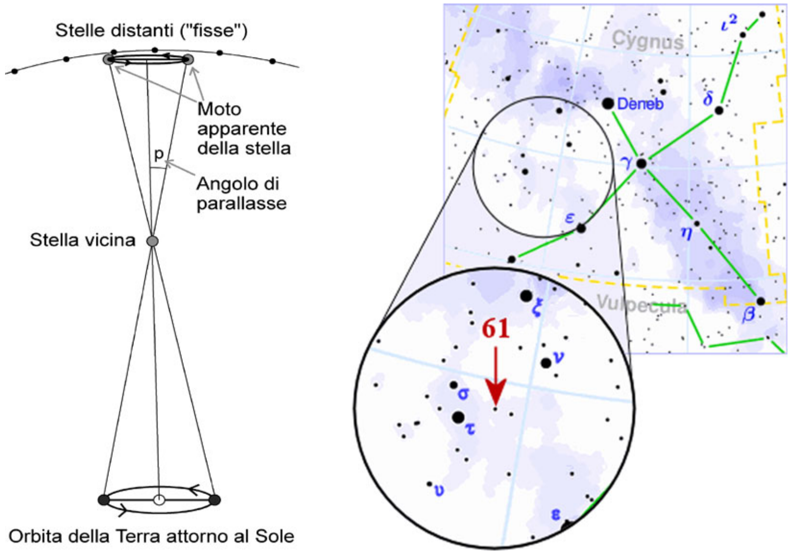


Fig. 4.3 The parallax angle is the angular amplitude of the apparent motion of a star due to the real motion of the Earth around the Sun. The first parallax angle was measured for the star 61 Cygni in the Cygnus constellation. In 1838, Bessel evaluated it as 0.314 seconds of arc. Explanation of the Italian wording in the table: *Orbita della Terra attorno al Sole* = Orbit of the Earth around the Sun. *Stella vicina* = Close Star. *Angolo di parallasse* = parallax angle. *Moto apparente della stella* = Apparent motion of the star. *Stelle distanti "fisse"* = Distant stars, "fixed"

At the beginning of the XIXth century no one could doubt about the correctness of the Copernican system, explained by Newton's gravitational law and tested by more than a century long extensive use of perturbation theory in the calculation of planetary orbits. Therefore stars were enormously distant from us, yet how far, enormously meant, could be established only if some parallax angle, no matter how tiny, could be finally measured.

Such a measure was provided by *Friedrich Wilhelm Bessel*, the German astronomer and mathematician whose family name is associated with one of the most important class of special functions. In 1838 he succeeded in determining the parallax angle of *61 Cygni*. His measure provided the figure of 0.314 arc-seconds which corresponds to a distance of 11.4 light-years from the Sun (see Fig. 4.3). In this way we finally knew that one of the stars closest to us, so close that we can measure its parallax, is at a distance not smaller than 100 billions kilometers.

This was the first step of the ladder on which Human Mind started to step up in order to estimate the real spacial extension of the visible Universe.

The same year Friedrich Georg Wilhelm von Struve e Thomas Henderson measured the parallax of *Vega* and *Alpha Centaury*. In the meantime astronomers began to detect *nebulae* and the discussion started about their physical nature and about the nature of the Milky Way, known to Mankind since the very beginning of civilization.

The Kantian conception of Euclidian Geometry, considered a *necessary a priori representation that is the foundation of all external perceptions*, constituted an ob-

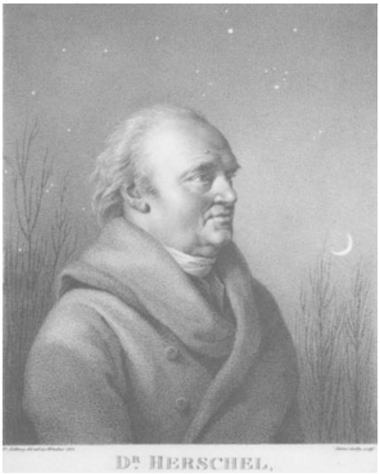


Fig. 4.4 Portrait of Sir William Herschel (1738–1822) who was probably the most famous astronomer of the 18th century. Born in Hannover in 1738, he was the son of a musician, officer of the Military Guard. When his father's regiment visited England in 1755, young Herschel began to study English. As the Princes of his native German country became the Royal Family of the United Kingdom, the future Sir William followed them to Britain, where he continued his musical career as a teacher, concert player and organist. He was also a composer and his music is worth consideration. His interest for astronomy led him to be the first builder of large telescopes and to make few sensational discoveries that granted him world-wide fame. Helped in his astronomical ventures by his sister Carolina, he discovered the planet Uranus. Then he extensively studied nebulae and listed and classified more than 1500 of them. His giant telescope with a 48 inch aperture was constructed in 1788 with special funds from the English Crown and until 1840 it remained the largest of the world. In 1816 Herschel was made "Sir" by the King and in 1821 he was elected first president of the Royal Astronomical Society

stacle to the development of non-Euclidian Geometry and forced Gauss, who had the first intuition of its existence, not to publish his own results, in order to avoid, as he confessed later, the *Loud Cries of the Beotes*, namely of the post-Kantian philosophers dominating the German culture of his times. So, from this point of view, Immanuel Kant was responsible for delaying the comprehension of our Universe with a heavy conceptual prejudice. From another viewpoint, we have to acknowledge Kant as an enlightened pioneer of a realistic representation of the Cosmos, representation that was not yet established on stable grounds in 1920 and became such only a few years later thanks to the work of Edwin Hubble.

In 1755, in his juvenile work entitled *Allgemeine Naturgeschichte und Theorie des Himmels*, namely General History of Nature and Theory of the Sky, Kant was the first to present the theory of Island-Universes and interpret the Milky Way as a vast cluster of stars that forms a plane disk of almost perfect circular shape. The nebulae that we see in the sky and which, few years later, Sir William Herschel (see Fig. 4.4) started to observe attentively and depict in his famous tables, are nothing else, said Kant, than similar clusters of stars organized in a way fully analogous to that of the Milky Way.

What impressed the young philosopher was the elliptical form of many nebulae. This led him to think that they might be gigantic circular arrays of stars which appear elliptical if seen at an angle different from zero. In this way Immanuel Kant was the first to hint at the existence of galaxies and correctly identified the Milky Way as the Galaxy which encompasses the Solar System.

In 1920, notwithstanding Kant's first intuition and notwithstanding the vast amount of data accumulated in one and a half century of observations that Herschel had just initiated, the theory of Island-Universes was just only a hypothesis accepted by some scientists and opposed by many others.

On April 26th of that year, in front of the National Academy, Harlow Shapley, who was born in 1885 in the State of Missouri and held, at that time a permanent job at the Mount Wilson Observatory, claimed that the Galactic System, *i.d.* the Milky way, has a radial extension of about 300.000 light-years and that *there is evidence against the vision suggesting that the spirals are galaxies made of stars just as ours... Actually the spirals are not made out of stars, rather they are clouds of gas.* Shapley, however, concluded that, *although the spirals are galactic gas clouds and not new galaxies, yet somewhere else there might exist other stellar systems even bigger than ours which so far have not yet been identified and lie at such enormous distance from us that they are unaccessible to present day observation instruments*.

In other words the evaluation of the Universe's spatial extension implicit in Shapley's arguing was that this latter might be of the order of magnitude of the Milky Way's size just as it might be immensely larger. In any case Shapley thought that at his time there was no experimental chance of answering such a question.

The same day and in front of the same scientific gathering, Herber Curtis, who had been recently appointed Director of Pittsburg Observatory in Pennsylvania and who was thirteen year older than Shapley, presented a completely opposite thesis. In his conclusions he said: *Henceforth I subscribe to the following opinion, namely that the Galaxy is not larger than 30.000 light-years, that spirals are not galactic objects rather Island-Universes analogous to our own Galaxy and that these facts point to a much bigger Universe where we may push our sight to distances ranging from ten to hundred millions of light-years* (see [2] for more details on the Great Debate).

By means of his arguing, Curtis had raised the spatial extension of the Cosmos by a factor one thousand with respect to the cautious estimate of Shapley. In the same decade the visible Universe was to be extended of another factor 100 reaching the order of tens of billions of light-years.

Both Shapley and Curtis who respectively died in 1972 and 1942, were to witness, in the years following 1920, Hubble's results that, on one side confirmed Curtis' thesis of island-universes and on the other gave, for the first time, not only an evaluation of the Universe's spatial extension but also of its age.

In 1923 Hubble succeeded in enlarging the images of the M31 and M33 galaxies, respectively known as *Andromeda* and *Triangle Galaxy* (see Fig. 4.5). On October 3rd of 1923 he was able to spot a variable star within Andromeda and in a few month time he had singled out a conspicuous number of Cepheides inside both Andromeda and the Triangle. As a newcomer to Astronomy, young Hubble had already piled up such results as to win for himself an imperishable fame and become also the true

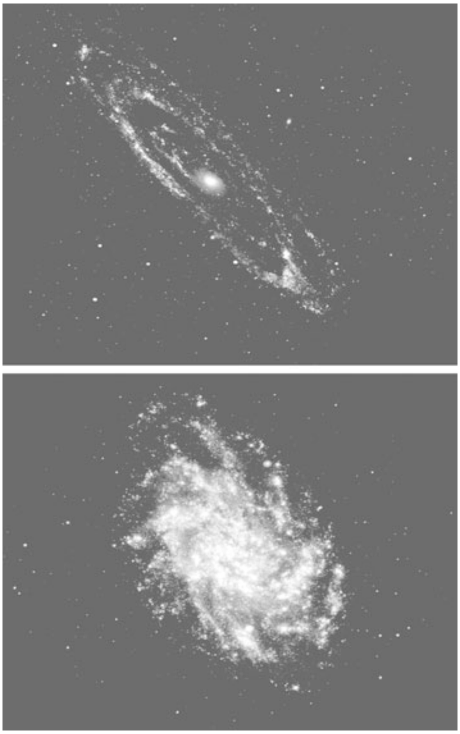


Fig. 4.5 The Galaxies M31 and M33, also known as Andromeda and the Triangle Galaxy are, together with the Milky Way the principal members of the local group, composed by these giant spirals plus a pair of dozens of smaller galaxies

father of Modern Cosmology. Indeed his discovery clarified once for all the issue of island-universes. The spirals are not galactic objects, as Shapley affirmed not earlier than three years before, rather they are other galaxies formed by billions of typical stars. Furthermore in two of these galaxies Hubble had found a few *standard candles* that allowed him to measure their distance. As we further explain in Fig. 4.6 a standard candle in Astronomy is a luminous source whose absolute luminosity is known a priori from some other identifier. In this way by using the inverse square law, obeyed by the apparent luminosity with respect to the absolute one, we can evaluate the distance of the emitting source.

In 1912, Henrietta Leavitt (see Fig. 4.7), from the Harvard Astronomical Observatory, discovered 25 variable Cepheides within the Small Magellanic Cloud. Of each of these pulsating stars, Henrietta determined the period P. She noticed that longer was the period, more luminous was the star, following a precise analytical relation that she was able to fit to the data (see Fig. 4.8). No one had before noted such relation because of lack of information on the absolute luminosity of the observed stars. On the contrary the Cepheides of the Small Magellanic Cloud could be considered all at the same distance from us, since their relative distances are negligible with respect to distance of the Cloud from us. Hence although the absolute magnitudes of the 25 Cepheides were unknown, the ratios of their apparent magnitudes were essentially equal to the ratio of their absolute ones and Miss Leavitt was able to draw the period-luminosity curve. Later on by determining the precise distance

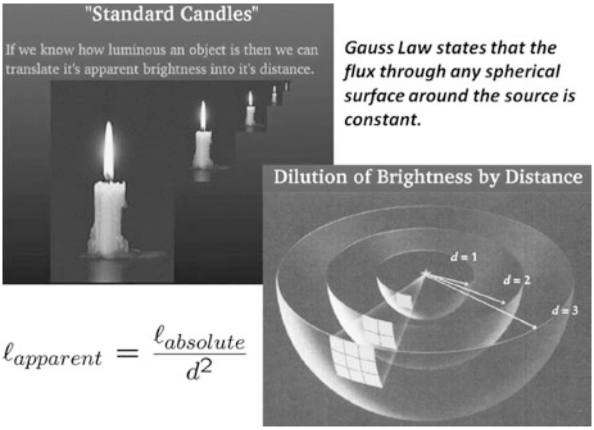


Fig. 4.6 The concept of standard candle is based on Gauss Law of fluxes which is satisfied by any type of massless radiation like the electromagnetic one. Since the flux through any spherical surface that surrounds the source is constant, it follows that the apparent luminosity decreases, with respect to the absolute one, with the square of the distance from the surface. If we happen to know the absolute luminosity then, by measuring the apparent one, we can easily evaluate the distance which separates our terrestrial environment from the observed object

Fig. 4.7 Henrietta Leavitt (1868–1921) was completely deaf. This notwithstanding she was one of the first women who made a scientific career in Astronomy. After graduation from the Radcliffe College, an allied institution of Harvard University, she obtained a job from the Harvard Observatory as a human-computer, namely as a human resource for those calculations that at that time, deprived of electronic computers, had to be done by hand. In 1912 she made her great discovery about variable Cepheides. She died from cancer in 1921. Proposed for the Nobel Prize by Mittag-Leffler she could not get it since she had already died at the time of assignment

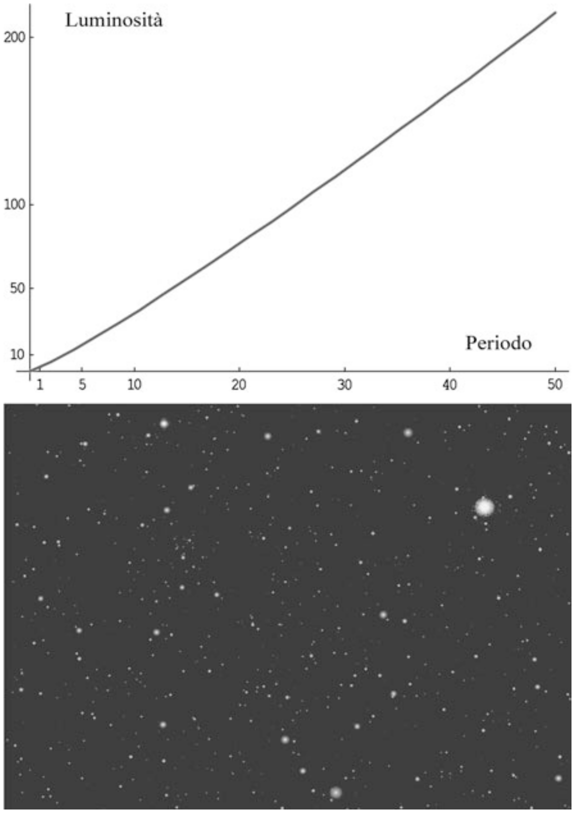


Fig. 4.8 The empirical Law of Cepheides determined by Henrietta Leavitt. The absolute luminosity L of the star grows with the period P of variability expressed in days according to a curve of type $L = P^{1.124}$. Under the plot, the reader can see a picture of the Small Magellanic Cloud within which, in 1912, Henrietta Leavitt discovered 25 Cepheides. The Large and Small Magellanic Clouds actually are two Dwarf Galaxies, satellites of our own Galaxy, the Milky Way. They are visible with naked eye in the Austral Emisphere and were observed by Magellan in his famous trip around the world in 1519

of some close-by Cepheides, this type of variable stars was turned into a precious system of standard candles. In 1913 Hertzsprung established that a Cepheid with a period of 6.6 days had an absolute magnitude equal to $+2.3$ and on the basis of this result, using Leavitt's curve he determined the absolute magnitude of all the Cepheides. Few years later Hertzsprung 6.6 value was corrected by Shapley into 5.96 which was not yet the completely correct result but almost it.

By a long tradition that dates back to the Ancient Astronomers Star luminosities are measured on a logarithmic scale organized as follows. Larger the magnitude dimmer the star, and stars of magnitude n are always a factor 2.512 brighter than the stars of magnitude $n + 1$.

Magnitude of the star	Times dimmer than 1st magnitude stars
1	0
2	2.5
3	6.25
4	15.63
5	39.06
6	97.66
7	244.14

(4.3.1)

The Greeks had organized all visible stars in six classes of magnitudes where the magnitude 1 stars were the most luminous. With the invention of the telescope and modern equipment to measure star magnitudes the scale has been extended in both directions. Dimmer stars are assigned magnitudes larger than 6 while stars brighter than first magnitude stars are assigned negative magnitudes. The absolute scale is fixed by the magnitude of the Moon and of the sun that are respectively given by -12.7 and -26.75. This means that a magnitude one star has a luminosity which is $\frac{1}{(2.512)^{26.75}} \simeq \frac{1}{5.02} \times 10^{-10}$ dimmer than the luminosity of the Sun. The new method of measuring distances based on Leavitt's discovery produced a new significant leap forward of Mankind on the cosmic ladder. New unprecedented possibilities opened up to evaluate the actual dimensions and structure of immensely far away objects. Hubble took advantage of these possibilities.

Having discovered standard candles in the two spirals Andromeda and Triangle, he was able to determine their distance from us. As we know today with higher precision, these are 2.5 and 2.81 millions of light years, respectively. Hubble's calculation produced two numbers of the same order of magnitude but underestimated by a factor 2. Responsible for this was the unprecise absolute normalization of the luminosity-period curve provided by Shapley in 1916.

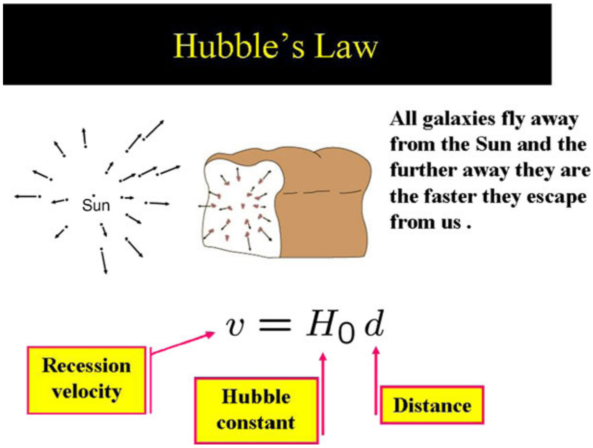
Apart from this, the relevant point is that Curtis' viewpoint was entirely confirmed. Spirals are not galactic objects, rather they are other galaxies similar to our own and those closest to us are millions of light-years away from the Sun. Hubble's paper containing the determination of the distance of M31 and M33 by means of Cepheides was presented to the American Astronomical Society in Washington on January 1st 1925. It was awarded a 1000 dollar prize ex aequo with some other contribution already forgotten in history. After that communication the human perception of the visible Universe had completely changed. It was now established that Kant's theory of island-universes was true: the world is made of a number at that time unknown, but probably enormous, of galaxies, each of which contains a number of stars of the order of the billion and the distances separating galaxies one from the other is of the order of the million of light-years.

4.3.3 The Discovery of Hubble's Law

The most important astronomical discovery of the XXth century that started Modern Cosmology and transformed it from metaphysical arguing into an observative and experimental science is Hubble's law on the universal recession of galaxies [3].

Already in 1914 Slipher had measured the radial velocity of 13 spiral nebulae and with great surprise he had found that they were all recessional, namely corresponded to an outgoing motion from the Sun, and were quite large. When in 1925 appeared Hubble's results on M31 and M33, Slipher had already measured the radial velocities of more than 39 galaxies. Then Hubble began to work with the large 100 inch. reflector of Mount Wilson and in 1929 he already mastered data for 46 galaxies and for 24 of them he also determined their distance using Cepheides, Blue

Fig. 4.9 Pictorial description of Hubble's Law which implies the expansion of the Universe



Stars and Novae. Assembling all this information he found a very simple and much intriguing relation. The radial velocities of all galaxies, once the Sun motion within the Milky Way was subtracted, appeared to be recessional and proportional to their distance. In other words all galaxies escape from us and they escape with a velocity which is faster as they are further away from us.

So in 1929, in the same year of the Wall Street disaster that started the Great Depression period, Hubble presented the linear law of universal recession:

$$v = H_0\, d \tag{4.3.2}$$

where v is the recession velocity of a galaxy, d is its distance from the center of the Milky Way and H_0 is a universal constant with the dimension of the inverse of a time which, since then, was named the *Hubble constant*.

Hubble's law is the fundamental observational datum which reveals the expansion of the Universe (see Fig. 4.9). With this 1929 result the dark night paradox of Olbers found its profound resolution and Aristotle's conception of an ethernal and static Universe was definitely dismissed. Modern Cosmology was born, whose object of study is a Universe in constant evolution: a turbulent physical system of high complexity whose large scale dynamics is dominated by gravitational interactions described by Einstein's General Relativity.

Yet it is interesting to note that nowhere in his 1929 article did Hubble mention the expansion of the Universe and also in the following decades he was always quite critical and substantially opposed to the interpretation of his own law as a manifestation of cosmic expansion, interpretation which was instead adopted by the whole scientific community. For instance seven years after his discovery in a paper published on the Astrophysical Journal in 1936, Hubble wrote: *If the redshifts are actually velocity-shifts that measure the expansion speed of the Universe, than the theoretical models are all inconsistent with observations and the expansion is an unjustified interpretation of experimental results*.

In this reluctance to accept the consequences of his own discovery, Hubble was influenced by the doubts of Einstein with whom he met many times in California. All

Fig. 4.10 Georges Edouard Lemaitre (1894–1966), Belgian by nationality, was at the same time a catholic priest and a mathematical physicist. He studied first in a Jesuit College at Charleroi and then he studied Mathematics and Physics at the Catholic University of Leuven. He entered a seminar in 1920 and became a priest in 1923. In the following years he was particularly attracted and got involved in the General Theory of Relativity, meeting Einstein several times. He worked at the Astronomical Observatory of Cambridge in England under the supervision of Eddington and after that at the Massachusetts Institute of Technology in the USA where he wrote his doctoral thesis. In 1925 he went back to Belgium where he was appointed professor of Leuven University. There he thought until 1964. Essentially Lemaitre is the father of the Big Bang Theory which was named such for the first time by Fred Hoyle in a radio broadcast of 1949. For the same concept Lemaitre used the different name of *primeval atom*

of its life Einstein could not accept the idea of an expanding Universe and constantly looked for an alternative way out from the predictions of his own theory. Indeed Einstein's field equations, as we will extensively discuss in this chapter, are not only in agreement with Hubble's results, rather they codify them into an effective and consistent theoretical frame.

To Georges Lemaitre (see Fig. 4.10) who, in 1927, independently from Friedman, Robertson and Walker derived that solution of General Relativity [4], which nowadays constitutes the basis of the *Standard Cosmological Model* and which we extensively discuss later on, Einstein wrote in French: *Vos calculs sont corrects, mais votre physique est abominable*, namely, your calculations are right but your physical interpretation is abominable.

In order to avoid similar conclusions Einstein introduced the *cosmological constant* Λ that allowed him to obtain static solutions for the cosmic metric. At the end of his life Einstein agreed to withdraw such a constant that all experimental data of that time suggested should be zero. Ironically, suitable reinterpreted, the cosmological constant is a manifestation of the *Dark Energy*, which is responsible not only for the cosmic expansion but also for its acceleration.

4.3.4 The Big Bang

How was Hubble's Law derived and how can it be verified? The answer is by means of the *redshift* of atomic spectral lines.

In order to clarify this point it is convenient to consider its analogy with the familiar Doppler effect in acoustical waves. We all made experience of what happens when an ambulance goes by, horning his siren. When the vehicle approaches the tune of its siren is high pitched while, when it runs away from us, the siren tune falls off and off. Moreover the faster the vehicle runs away, the lower falls the tune. The same happens for light waves, namely for photons.

The faster a luminous source recedes from an observer the redder it appears to him. Hence by performing the spectral analysis of the light which comes to us from distant galaxies we can recognize the structure of spectral lines for all atomic transitions but we also find that they are all shifted towards low frequencies and that they are the more shifted the larger is the distance of the observed galaxy. Defining redshift the percentual change of spectral lines and plotting it against the distance one obtains a line whose slope is Hubble's constant H_0.

Hence the redshift factor is defined:

$$z = \frac{\lambda - \lambda_0}{\lambda_0} \tag{4.3.3}$$

where λ is the wave-length of a spectral line observed in a distant galaxy while λ_0 is the wave-length of the corresponding spectral line observed in laboratory experiments on the Earth (see Fig. 4.11).

What is the interpretation of Hubble's Law?

At first sight one might think that it denotes our privileged position in the Universe. If all cosmic objects radially recede from us, it follows that we are at the center of the Universe which, once upon a time, was all concentrated in the place where we are now. Furthermore a linear relation between the recession velocity and the distance suggests the scenario of a gigantic primeval explosion. At the time when a bomb explodes all of his fragments are expelled in all directions with different velocities. After some time the faster fragments have run the further and for this reason they are more distant.

This interpretation which corresponds to an anthropic principle is what suggested the naming BIG BANG, yet it is somehow naive and conflicts with the homogeneity and isotropy of the Universe. As a consequence of this homogeneity and isotropy we should rather suppose that what we see is exactly the same picture seen by any other observer in any other galaxy. How can we then interpret Hubble's Law?

The intuitive model is the following one.

The galaxies are like balls arranged on a elastic sheet (the three-dimensional space) and with respect to that sheet they do not move. Yet it is that sheet that is uniformly stretched in all directions and as a consequence of this stretching every ball recedes from every other one. This way of thinking leads us to the concept of time dependent scale-factor (see Fig. 4.12). Imagine that our three-dimensional

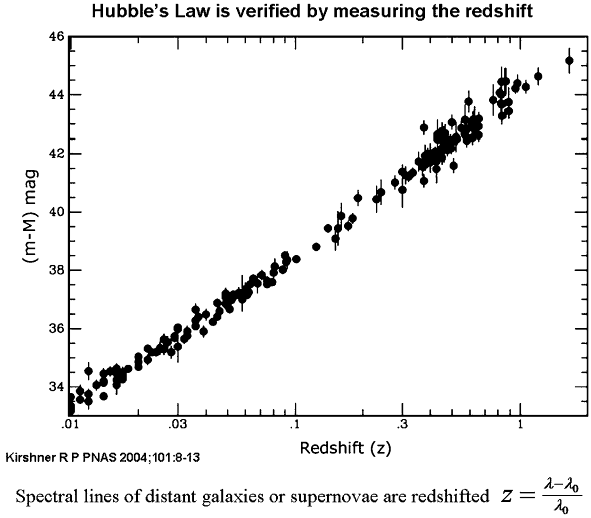


Fig. 4.11 The redshift of distant galaxies

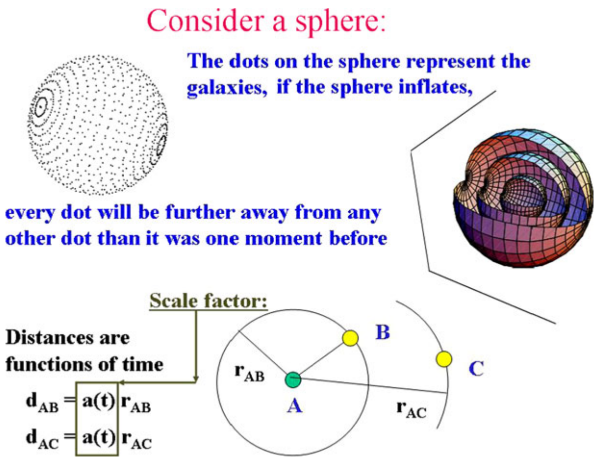


Fig. 4.12 The expansion of three-dimensional space

space is something like the surface of a two-sphere and that the galaxies are arranged and soldered at fixed locations on that spherical surface. Let us now imagine that some demon inflates the sphere, namely that he enlarges its radius while times goes on. All distances between each of the galaxies with all the others have fixed ratios but they are all proportional to the radius of the sphere which grows in time and so they also uniformly grow. It is like if the unit of measure increased constantly and were a function of time. We denote this time dependent unit of measure *the scale factor* and we denote it as $a(t)$.

Velocity is the derivative of the distance with respect to time. A simple calculation shows that we can deduce Hubble's Law from the above reasoning and identify Hubble's constant with the logarithmic derivative of the scale factor at the present cosmic time t_0:

$$
\begin{aligned}
v(t) &= \frac{d}{dt} d(t) = \dot{a}(t) r \\
&\Downarrow \\
v(t) &= \frac{\dot{a}(t)}{a(t)} d(t) = H(t)\, d(t) \\
&\Downarrow \\
H_0 &= H(t_0)
\end{aligned}
\tag{4.3.4}
$$

The Hubble constant actually is not a constant, rather it is a function of the cosmic time and it encodes information about the first derivative of the scale factor, namely about the *velocity of the expansion* of the Universe. The parameter H_0 originally measured by Hubble and determined with increasing precision in subsequent observations is the value at the present time of the Hubble function $H(t)$.

The first who introduced the notion of Big Bang, namely the theory according to which the present Universe evolved by expansion in the course of time starting from an initial state of enormous density, very tiny and extremely hot was *Monsegneur Georges Lemaitre*, on the basis of the solution of Einstein equations that bears his name together with those of Friedman, Robertson and Walker. He never used the wording Big Bang, rather referred to his own hypothesis as to that of the *primeval atom*. As far as we know the name Big Bang was invented as a despising joke by Fred Hoyle during a radio interview in 1949. Hoyle, like Einstein did not like expanding universes.

It is almost a historical nemesis that this ironical nickname of a serious but audacious theory became the official scientific name of the *standard cosmological model*. The idea of the primeval explosion has so much penetrated the common language and has become so popular that when my daughter, now twenty-four of age, started studying history at the elementary school, her textbook started no longer with the Great Flood rather with the Big Bang.

4.4 The Cosmological Principle

The mathematical basis needed to postulate the type of metric which now goes under the name of Friedman, Lemaitre, Robertson and Walker (FLRW), arriving at those conclusions that Einstein so much hated and from which emerges the Big Bang scenario, are provided by the *Cosmological Principle*.

This latter assumes two properties that are supposed to characterize the structure of space-time on very large scales, namely:

1. Isotropy.
2. Homogeneity.

Isotropy means invariance against rotations, namely in whatever direction it is pointed, our telescope should reveal approximately the same panorama. Homogeneity, on the other hand means invariance against translations. In other words what we

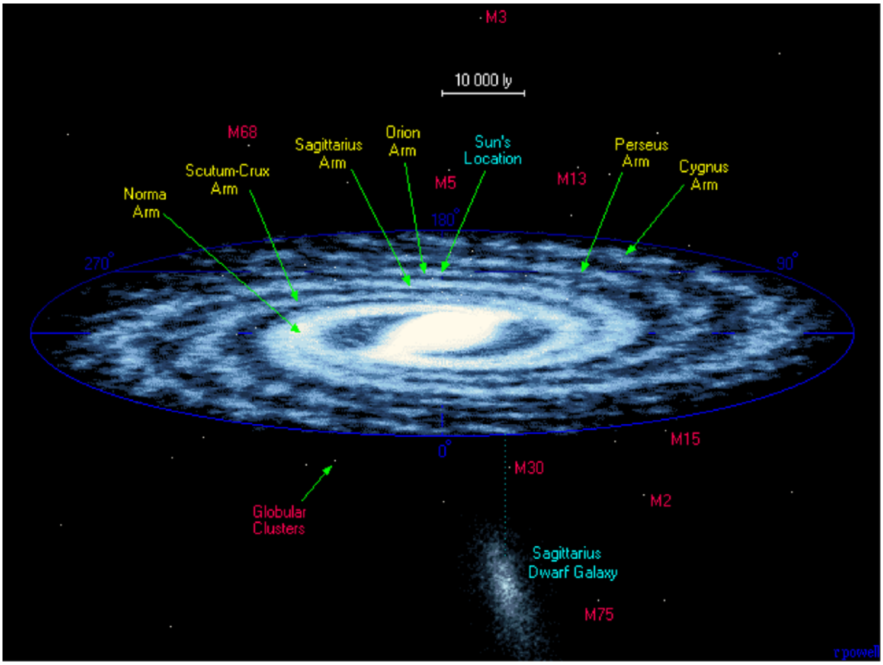


Fig. 4.13 The hierarchy of cosmic distances. First step in the ladder: 100.000 light years, the Galaxy

see from our own galaxy should be the same landscape observed by any other observer placed in any other galaxy no matter how far from us.

There is no a priori reason to assume the Cosmological Principle and at first sight no empirical basis for it appears to exist, given the granular structure of our universe made of stars grouped into galaxies that are, in turn, grouped into galaxy clusters. Cosmology, however, aims at studying the history of the Universe, analyzing its evolution at so large scales that we can consider galaxies as the point particles of a cosmic dust.

Let us then consider the scale-hierarchy.

Indeed the Universe appears granular only at short distant scales.

- 100.000 light-years is the typical dimension of medium size galaxies like our own, the Milky Way. See Fig. 4.13.
- 10 millions of light years is the scale of galactic clusters. See Fig. 4.14.
- 100 millions of light years is the scale of galactic super clusters. See Fig. 4.15.
- At the scale of one billion of light years, our Universe appears as a homogeneous soup of galaxies and it may be modeled as a perfect fluid. See Fig. 4.16.

The first basis for the Cosmological Principle is this matter of fact evidence on the homogeneous distribution of galactic clusters at very large scales.

Mathematically the Cosmological Principle is enforced by assuming that the space-time metric possesses a set of *isometries*, namely a set of continuous trans-

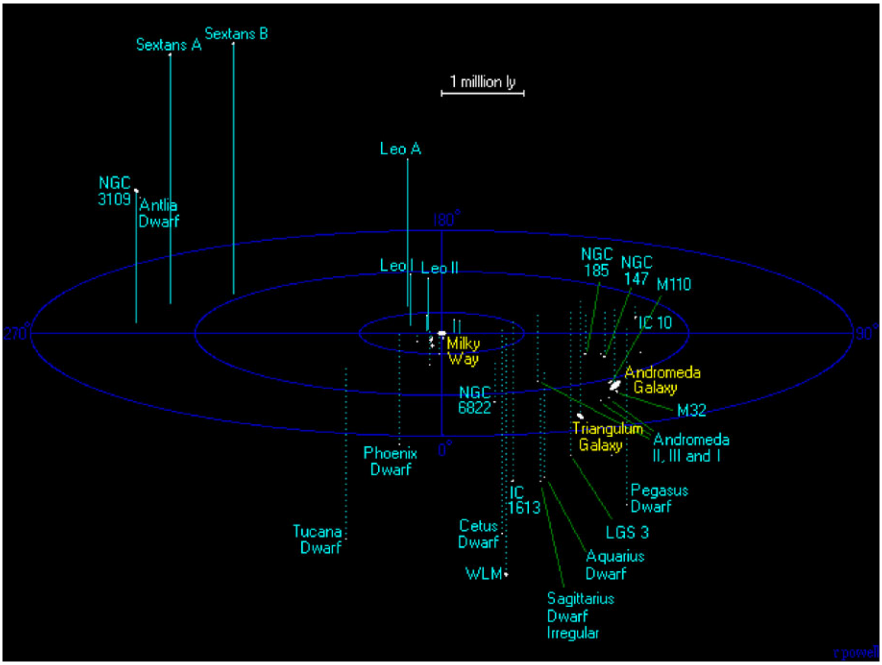


Fig. 4.14 The hierarchy of cosmic distances. Second step in the ladder: 10 millions of light years, the Local Group and its neighbors

formations that leave it invariant and form among themselves a Lie group. Isotropy requires that all rotations contained in SO(3) should act as isometries on the cosmological metric. Similarly homogeneity requires that there should be three translational isometries namely as many as the spatial dimensions of the Universe. Imposing such conditions is equivalent to selecting the geometry of constant time sections of space-time. The proper mathematical treatment of isometries is encoded in the theory of coset manifolds and symmetric spaces which is a chapter of Lie algebra and Lie group theory. The geometry of coset manifolds will be summarized in an appropriate mathematical section in next chapter.

At each instant of time the Universe is a three-dimensional space. Assuming the Cosmological Principle means that such a space should admit the maximal possible number of isometries. The mathematical theory of coset manifolds shows that in dimension $d = n$ such maximal number is $\frac{1}{2}n(n+1)$, namely precisely 6 for $d = 3$. Furthermore the same mathematical theory shows that there are just only three *maximally symmetric manifolds* in $d = 3$. In these spaces the curvature is constant over the manifold and we just have to decide its sign, namely the sign of the constant curvature scalar, positive, negative or null. This choice is encoded in a parameter κ whose possible values are $\kappa = 1, -1, 0$, corresponding to the three possibilities we just mentioned. The three maximally symmetric spaces in $d = 3$ are $\mathbb{S}^3$, namely the three-dimensional generalization of the sphere, corresponding to $\kappa = 1$, the three-

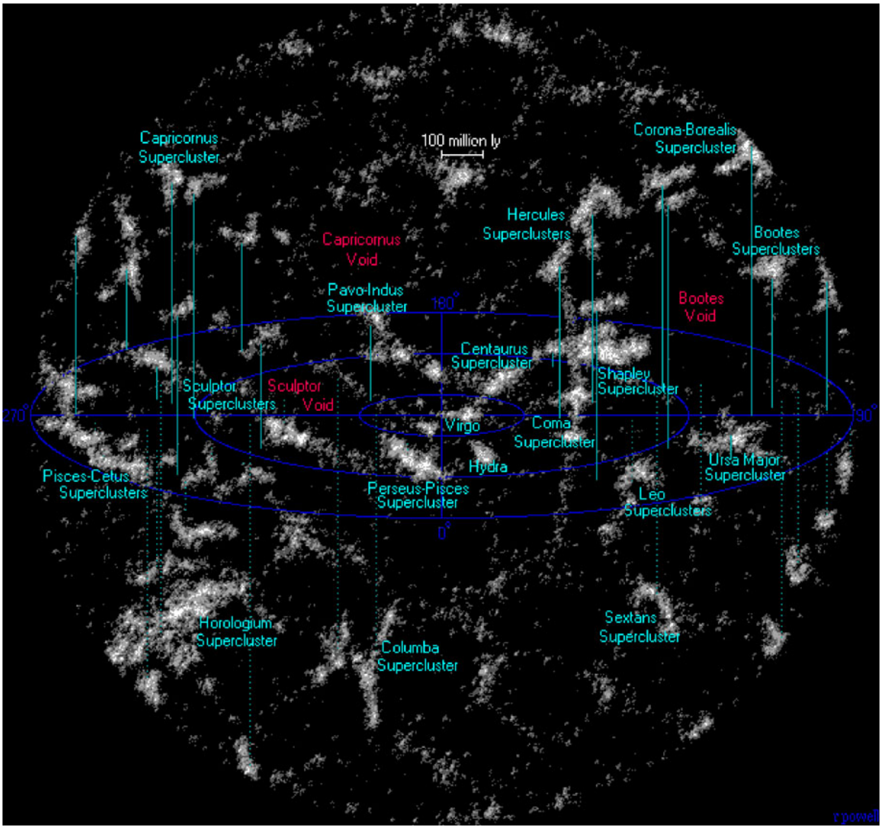


Fig. 4.15 The hierarchy of cosmic distances. Third step in the ladder: 100 millions of light years, the galactic superclusters

dimensional analogue $\mathbb{H}^3$ of the pseudo-sphere, corresponding to $\kappa = -1$, and $\mathbb{R}^3$, corresponding to $\kappa = 0$, which is the standard three-dimensional Euclidian space.

As we will discuss in great detail later on, having imposed such conditions the four-dimensional line-element which encodes the cosmic gravitational field, takes an extremely simple form which is indeed that of the FLRW metric. Naming t the time coordinate and collectively x the spatial coordinates that label the points of the chosen three-dimensional manifold, we can write:

$$ds^2 = -dt^2 + a^2(t)\, d\Omega_\kappa^2(x) \tag{4.4.1}$$

where $d\Omega_\kappa^2(x)$ denotes the line-element of the three-dimensional maximally symmetric space selected by the value of κ. Substituting the ansatz (4.4.1) in Einstein equations and modeling the matter content of the universe as a perfect fluid one obtains certain differential equations for the scale factor that are named *Friedman equations* and have radically different solutions for different signs of the scale fac-

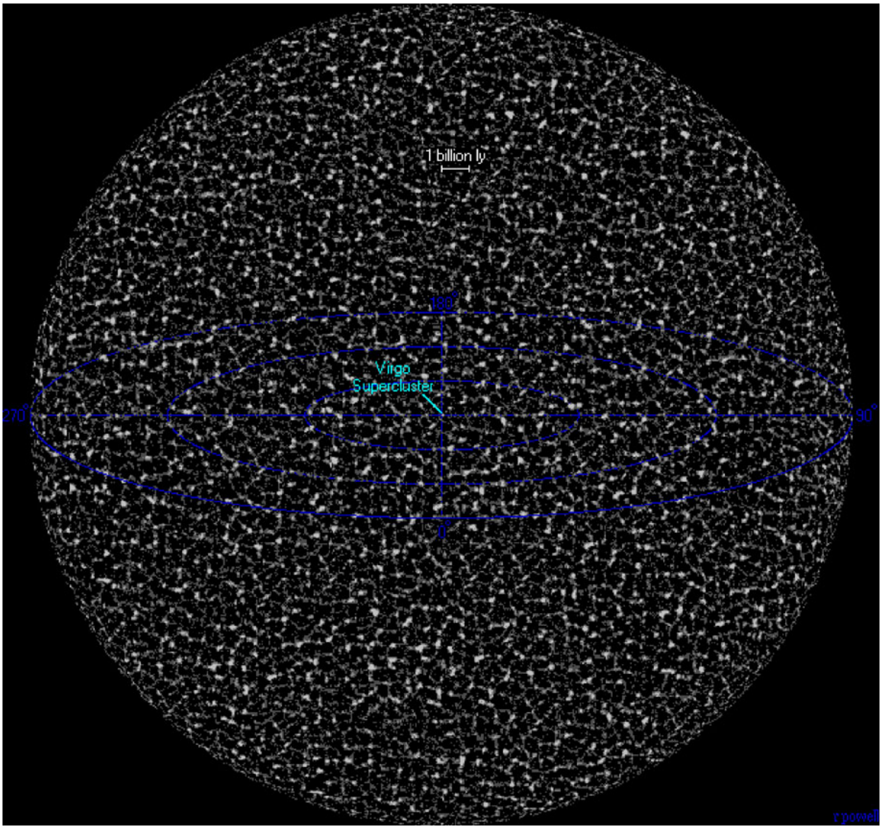


Fig. 4.16 The hierarchy of cosmic distances. Fourth step in the ladder: 1 billion of light years: homogeneous distribution of super-clusters

tor. In all cases there is a fast initial expansion, later, however, the expansion velocity decreases and the deceleration is stronger and stronger as the curvature increases.

The case of negative curvature $\kappa = 1$ is named the *open universe*. As we will derive in the sequel from Friedman equations the expansion of the open universe continues indefinitely, yet it slows down until it reaches a linear behavior. When the open universe is very old, the scale factor grows like $a(t) \approx t$.

The case $\kappa = 0$ is named the *flat universe*. Also here the expansion is endless, yet, as we will see, it tends asymptotically to a weaker growth than linear. When the flat universe is old, its scale factor grows as: $a(t) \approx t^{2/3}$.

The case $\kappa = 1$ is named the *closed universe*. For positive curvature the scale factor growth slows down up to zero velocity in a finite time. After that the expansion reverts into a contraction. The galaxies no longer recede from each other rather they begin to come together and the further apart they are the faster they approach each other. The redshift is turned into a *blueshift*. The universe becomes progressively smaller and smaller, hotter and hotter. In a finite time the closed universe collapses

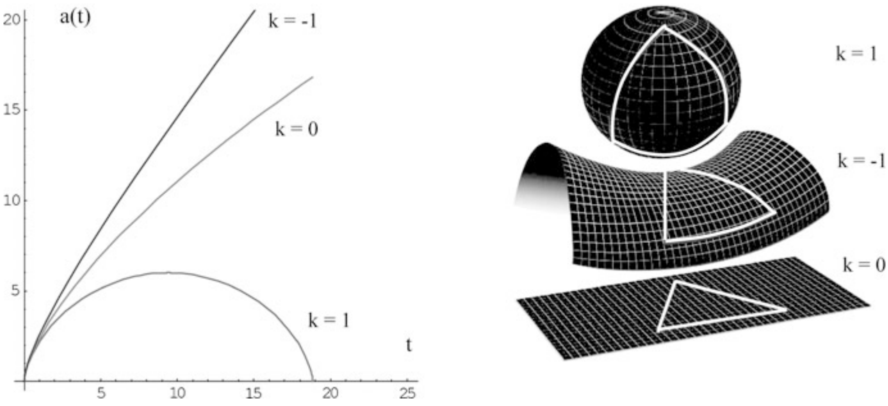


Fig. 4.17 The behavior of the scale factor in a matter filled universe for the three cases of positive, negative and null spatial curvature

into a state of infinite energy density. The Big Bang is followed, in the closed universe, by a *Big Crunch*.

These are the implications, visually summarized in Fig. 4.17 and accurately derived in the next chapter, of the Cosmological Principle, namely of the assumption that the Universe is isotropic and homogeneous. Is it really such?

4.5 The Cosmic Background Radiation

The final answer to the question posed at the end of the previous subsection came in 1965 thanks to Arno Allan Penzias and Robert Woodrow Wilson who, for their discovery of that year were awarded the Nobel Prize in 1978.

Penzias was born in a Jewish family in Munich in 1933, the very same year when Hitler got to power. Wilson was born three years later in that Texas which at the end of World-War Two boasted the victory on Germany, as a contemporary newspaper ironically wrote one of those days: *Texas has defeated the Third Reich*.

In 1939 Arno Penzias was among the 10.000 Jewish children who were evacuated from Germany and transported to England with the naval operation later known as *Kindertransport*. He was luckier than the majority of his fellow travelers who lost their parents and relatives in the Nazi lagers of the Holocaust. Arno's father and mother succeeded to flee to the United States just six months later than the evacuation of their son who could reach them in New York. There he lived, studied and in 1962 he got his Ph.D. at Columbia University.

Penzias and Wilson, who had obtained his Ph.D. from the California Institute of Technology at Pasadena, met a short time later at the Laboratories of the Bell Telephone Company in New Jersey. Both young researchers had been hired by Bell and in the little village of Holmdel, near the Company Headquarters at Crawford Hill, they worked at the construction of a new large radio antenna. The *Horn Antenna*

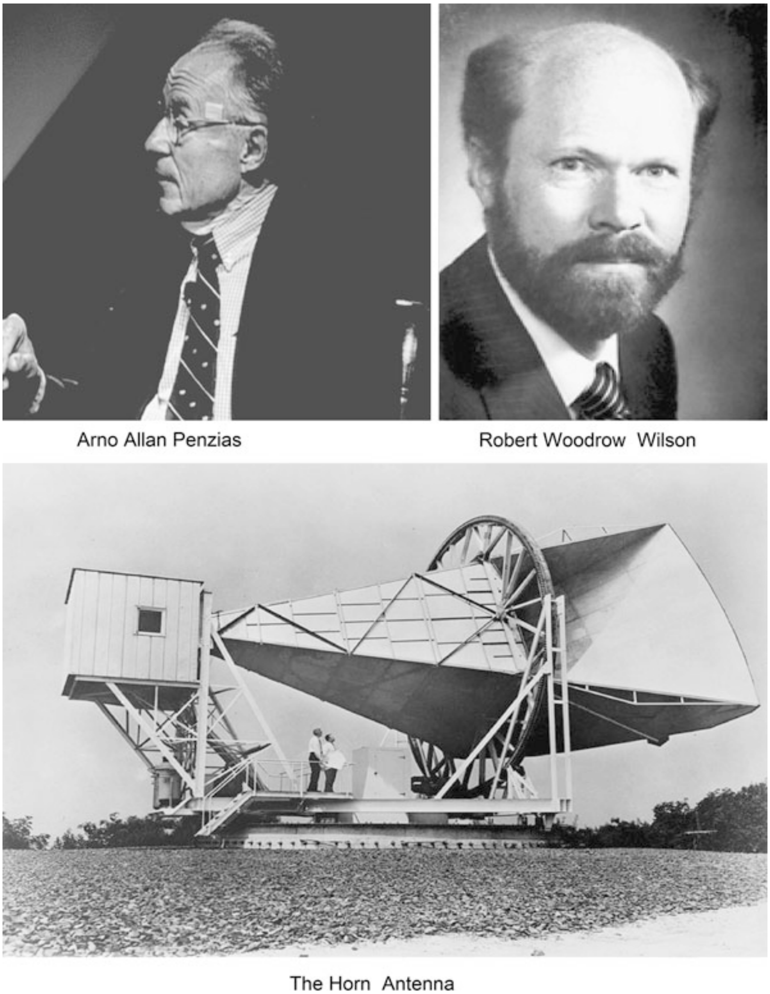


Fig. 4.18 The two discoverers of the Cosmic Background Radiation, Penzias and Wilson with their instrument, the Horn Antenna

they had designed was conceived for experiments in radioastronomy and telecommunications between the Earth and the artificial satellites (see Fig. 4.18). However there was a problem.

The sophisticated apparatus displayed an excess of antenna temperature of 3.5 Kelvin degrees that the two brilliant engineers could not explain. From Crawford Hill they phoned Princeton University and discussed their problem with Dicke, Wilkinson and Roll who were constructing another similar radio antenna. Immediately after that conversation Dicke exclaimed to his collaborators: *Guys, they got it! They made the scoop!* Which was the scoop alluded to by that distinguished professor who, during World War Two, at the Radiation Laboratory of the MIT, had

Fig. 4.19 Alexander Friedman (1888–1925) and his student Georgij Gamow (1904–1968)

created the *Dicke radiometer*, a sophisticated detector of electromagnetic waves? The scoop was the discovery of the Cosmic Background Radiation. With great intuition Dicke had immediately guessed the origin of that 3.5 K excess. No terrestrial phenomenon and no instrumental error could explain it. Behind that ultra-cold remnant lurked a cosmic phenomenon that had been predicted, few decades before by another brilliant fugitive.

Georgij Antonovich Gamow (see Fig. 4.19) was born from Russian parents in 1904 in the imperial town of Odessa, refunded in 1794 by Catherine the Great on the ruins of the Turkish Town of Khadjibey, just captured by the Russians from the Ottomans. At the beginning Gamow studied in his natal town at the University of Novorossya, but in 1922 he went to Saint Petersburg, transformed into Leningrad after the October Revolution. Here he became student of Alexander Friedman.

This latter, a brilliant Russian Mathematician and Physicist, spent the whole of its life in Leningrad and prematurely died at the age of thirty-seven in 1925. His name corresponds to the F in the denomination of the standard cosmological metric. In a 1924 article, published in German on the Zeitshrift für Physik and bearing the title *Uber die Möglichkeit einer Welt mit konstanter negativer Krümmung*,[1] Friedman, independently from Lemaitre, presented the cosmological solutions, isotropic and homogeneous, of the Einstein equations for the three cases of positive, negative and null curvature ($\kappa = \pm 1, 0$) [5]. Robertson and Walker reobtained the same solutions ten years later. It is also interesting to stress that the mathematical solution of 1924

[1] On the possibility of a world with negative spatial curvature.

had no motivation in experimental data since Hubble's law was discovered only five years later when Friedman was already dead.

In Leningrad, Gamow, who lost his thesis advisor before the end of his Ph.D., obtained in 1929, established a close friendship relation with another student who was to become one of the greatest physicists of the XXth century: Lev Davidovich Landau. Nobel laureate in 1962 for its theory of superfluidity, Landau was the greatest master of Soviet Physics and the endless series of volumes of his course on theoretical physics, written with his younger collaborator Lifshitz has been the educational basis of thousand of physicists around the world.

As for intellectual phantasy and scientific successes, Gamow was not that much inferior to his friend Landau. Differently from the latter who, except for two short trips to Copenhagen and Zürich, never left the Soviet Union, and suffered also one year in jail at the time of Stalin Purges, Gamow, who had worked both in Copenhagen and Göttingen tried to emigrate as early as 1932. With his wife he tried an escape on a small boat once on the Black Sea towards Turkey, a second time from Murmansk to Norway. Both times they failed because of very bad weather conditions, but in 1933 the Gamows succeeded in their intent having obtained from Soviet authorities the permission to participate to one of the famous Solvay Conferences in Brussels. There they deserted the Soviet Union and became political refugees. In 1934 from Belgium they went to the United States where, becoming Americans, they spent the rest of their life.

Gamow contributed fundamental results in nuclear physics explaining the β-decay of heavy nuclei and was the inventor of the *drop model* of the nucleus.

In a paper [6], based on previous results of Alpher [7] and published in 1948 on the Physical Review, Gamow advocated that the Universe should be filled with an electromagnetic Black Body radiation produced by all the atomic and subatomic transitions occurred after the Big Bang. For a certain period during the cosmic expansion this primeval radiation was in thermal equilibrium with ionized matter and the rest of the energy content of the Universe. However, as the expansion went on, the primeval radiation fell out of equilibrium with matter that had recombined into atoms and had become too rarefied in order to interact with radiation. Since that moment, known as the *decoupling time*, the cosmic radiation became, according to Gamow, a fossil which pervades the entire space-time but essentially does not interact with anything. Furthermore because of the cosmological red-shift, due to the universe expansion, which stretches all the wave-lengths,[2] the effective black-body temperature of the fossil radiation has cooled down to incredibly low values, close to the absolute zero. Indeed knowing the age of the universe through Hubble's law, Alpher and Gamow evaluated the red-shift factor and predicted that the Cosmic Microwave Background Radiation should have a black-body temperature of the order of few Kelvins. They advanced the prediction of 5 K.

In an almost accidental way, Penzias and Wilson had discovered the primeval radiation predicted by Alpher and Gamow. Its temperature was not exactly 5 K but very close to such number. In the first estimate of the discoverers 3.5 K, in

[2]The cosmological redshift will be explained in mathematical terms in later sections, at the end of this historical introduction.

subsequent more precise measurements 2.75 K. Dicke, Wilkinson and Roll were constructing a Dicke radiometer in order to detect this cosmic background but they were not fast enough. Penzias and Wilson had made the scoop and for that scoop they were awarded the 1978 Nobel Prize in Physics.

The 1965 detection of the CMB (Cosmic Microwave Background) not only confirmed Gamow hypothesis and gave the first direct evidence of the Big Bang but also provided an image of the primeval Universe.

Thirteen billion of years old, the CMB is the light emitted by the Last Scattering Surface (LSS). By this name we denote the space occupied by the entire energy content of the Universe at the time of decoupling. Hence the spatial distribution of the CMB at the present time is a quite faithful image of the spatial distribution of light sources at that very remote age. As it was immediately evident at that time and as we could later verify with incredibly high precision, the CMB presents an absolutely perfect black-body spectrum. The distribution of the energy amplitude emitted at frequency ν, for unit of surface, unit of solid angle and unit of time, follows the Planck curve (see Fig. 4.20). The experimental data for the CMB are reproduced with incredibly high accuracy by the Planck curve corresponding to temperature $T = 2.725$ K and this happens in the same way independently from the direction in which our spectrometer points.

Recalling that what we see through the CMB is a uniformly redshifted image of the LSS, namely of the primeval Universe we can conclude that this latter was absolutely homogeneous and isotropic to a very high accuracy.

Therefore the detection of the CMB has been the experimental confirmation of the Cosmological Principle. The Universe where we live has evolved from an isotropic and homogeneous state and therefore is accurately described by a FLRW metric. Assuming this latter, the Einstein equations imply without any possible escape the expansion of the Universe, presently visible through the Hubble law. That this expansion occurred through the last thirteen billion of years is also confirmed by the gigantic redshift of the CMB. Starting from a temperature of about 3000 K that was that of the Universe at the decoupling time, corresponding to about 400.000 years after the Big Bang, in the following 13 billions of years the radiation cooled down to the present $T = 2.725$ K.

A trivial calculation provides a shaking evidence of this truth. As we shall prove later on, the cosmological redshift goes as follows $\nu_0 = \frac{a_e}{a_0}\nu_e$ where ν_e is the frequency of a photon at the time of emission in the remote past while ν_0 is the frequency of the same photon detected at the present time. Similarly a_0 and a_e denote the values of the scale factor at the two considered instants of time. From this consideration it follows that the temperature of the cosmic black-body radiation follows the same law and we have:

$$\frac{T_e}{T_0} = \frac{a_0}{a_e} \tag{4.5.1}$$

From this we obtain the estimate:

$$\frac{a_0}{a_e} \sim \frac{3000}{2.725} \sim 10^3 \tag{4.5.2}$$

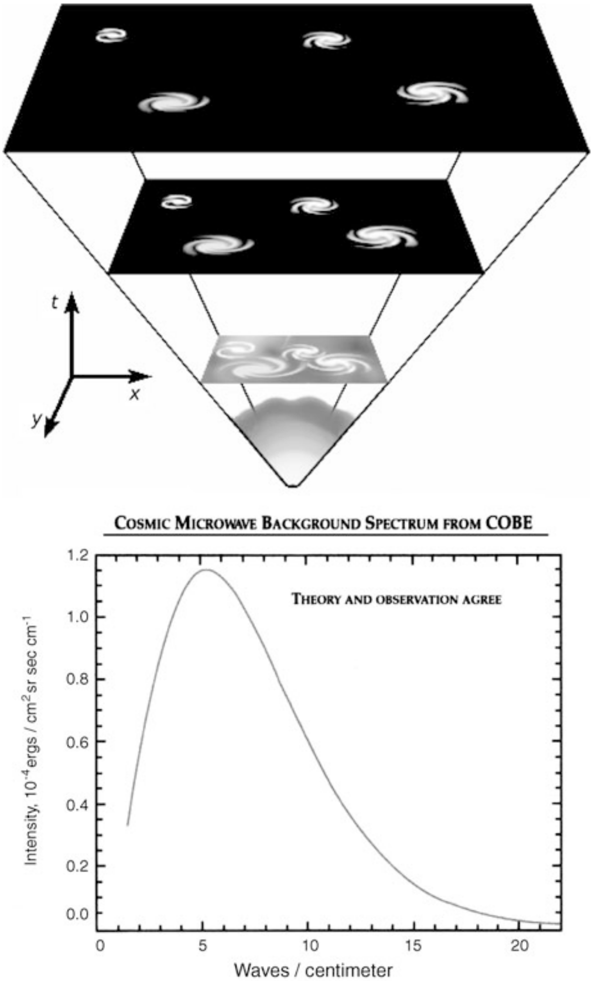


Fig. 4.20 The perfect Black-Body spectrum of the Cosmic Microwave Background Radiation

On the other hand the ratio between the two considered times is:

$$\frac{t_0}{t_e} \sim \frac{13 \times 10^9}{4 \times 10^5} \sim 3 \times 10^4 \sim 10^{4.5} \tag{4.5.3}$$

According to Friedman equations, the asymptotic behavior of the scale factor in a flat matter dominate universe is $a \sim t^{2/3}$. Now it is remarkable that:

$$\left(10^{4.5}\right)^{\frac{2}{3}} \sim 10^3 \tag{4.5.4}$$

In other words the observed redshift of the CMB over the last 12.5 billions of years from the decoupling is consistent with the expansion of the universe predicted by Friedman equations in the case of a flat, matter dominated Universe.

4.6 The New Scenario of the Inflationary Universe

The great success of the *Standard Cosmological Model*, based on the hypothesis of the hot Big Bang and the principles of homogeneity and isotropy, which are mathematically rephrased by assuming the FLRW metric (4.4.1), should not induce the reader to think that everything has been understood and solved. In Physics the absolute agreement of a theoretical model with experimental data is often the source of a conceptual problem, rather than being its solution.

At first sight, such a statement might seem paradoxical, yet a short discussion can clarify its profound meaning. If reality agrees perfectly and not only approximately with some modeling of its behavior, that means that the hypothesis underlying our model do not correspond to some accidental circumstances, rather to some fundamental law, and the problem is that of explaining such a law in terms of more profound reasons and principles. A historical example is that of the identity between the inertial and the gravitational mass. These latter are not approximately equal rather they are equal with extraordinary precision. This means that a good theory of gravitation should include such an identity as a necessary and founding condition, not as an accidental fact. Starting from this consideration, as we know, Einstein discovered General Relativity.

Similarly the *Standard Model of Strong, Weak and Electromagnetic Interactions* has proved to be a very precise and accurate description of elementary particle physics. Yet, this model includes a large number of parameters and the problem is that of creating a more fundamental theory, within whose frame the values of the standard model parameters can be predicted equal to those experimentally measured.

In the case of Cosmology, the high accuracy of the predictions of the *Big Bang Model* implies the necessity of explaining in more profound terms the two hypotheses that constitute its foundation, namely homogeneity and isotropy, alias the Cosmological Principle.

Considering this issue with a clear and not biased mind, we easily convince ourselves that there is no a priori reason for the Universe to be so homogeneous and symmetric, as it proves to be in observations. On the contrary, it would be natural for it to be highly disordered and inhomogeneous. Indeed one can show that, starting from a situation that includes anisotropies and inhomogeneities, Einstein equations tend to enlarge them during time evolution. Therefore, if we confine ourselves to consider Einstein theory with a Universe content made only of conventional matter and radiation, then the extraordinary isotropy and homogeneity of the Universe at present time requires that its initial state was prepared homogeneous and isotropic with almost infinite precision, which is quite unnatural in any stochastic process.

Different is the perspective if we discover a physical mechanism that can prepare such isotropic and homogeneous state starting from a generic one.

In 2002 the Dirac Medal of Trieste ICTP,[3] which, after the Nobel Prize, is probably the most prestigious honor available to theoretical physicists, was awarded to Alan Guth, Andrei Linde and Paul Steinhardt, for their fundamental contributions

[3]International Centre of Theoretical Physics.

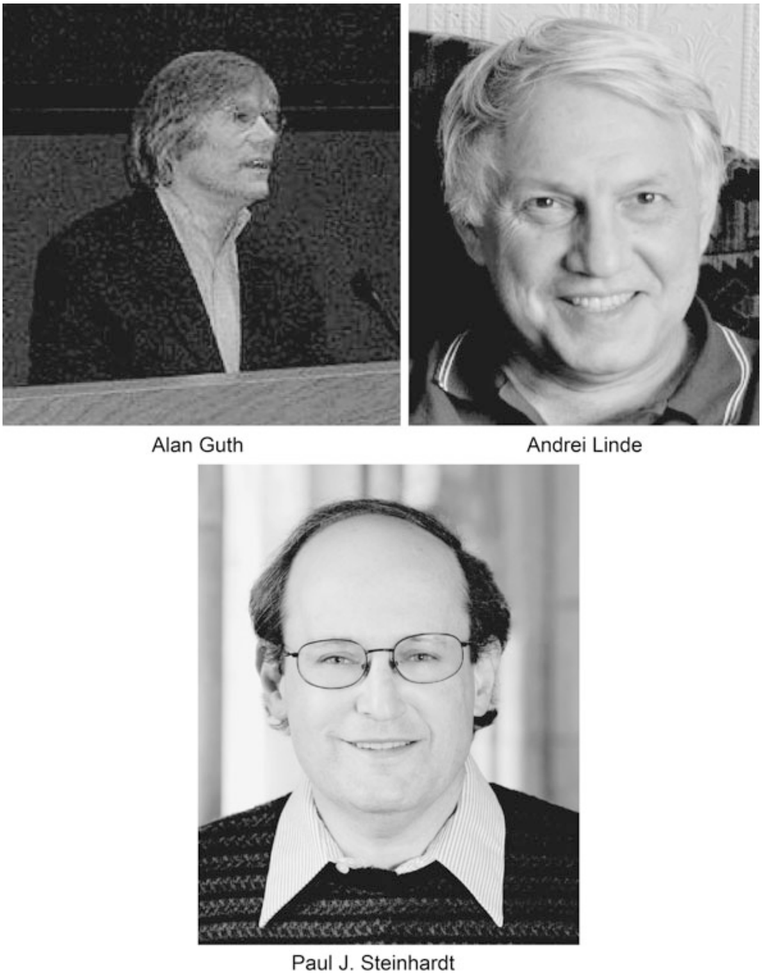


Fig. 4.21 Alan Guth, Andrei Linde and Paul J. Steinhardt, the fathers of the Inflationary Universe scenario

to the creation of the Inflationary Universe paradigm (see Fig. 4.21). Alan Guth, born in the USA in 1947 is professor at the Massachussetts Institute of Technology, Paul J. Steinhardt, also born in the US, is Einstein professor of Physics at Princeton University, while Andrei Linde, born in Moscow, studied and worked there, becoming one of the most famous and distinguished cosmologists of the world. In the mid nineties of the XXth century he accepted the invitation of Stanford University to join the faculty of its Physics Department.

There are many formulations of the inflationary theory and its details crucially depend on the structure of the unified theory of all interactions that will prove to be the one chosen by Nature. For instance, within the framework of supergravity, regarded as the low-energy limit of superstring theory, there are several interesting possibilities to implement the inflationary scenario and determine its parameters in agreement with the experimental data that are piling up. However, beyond its detailed structure, the great value of the Inflationary Universe is that it provides a very simple conceptual paradigm, up to now without any rivals, capable of explaining the isotropy, homogeneity and spatial flatness of the Universe.

Here we do not dwell too much on explanations of Inflation, which will be discussed in a detailed mathematical way in later sections. We just mention that the generic mechanism, capable of preparing homogeneous, isotropic and spatially flat boundary conditions, consists of a primeval phase of exponential expansion that should have taken place before the age of decoupling and should have also gracefully ended. On its turn, an exponential expansion takes place when gravity becomes repulsive and this happens when the energy content of the Universe is mainly provided by vacuum energy, for instance the potential energy $V(\varphi)$ of one or more scalar fields φ. Hence, the inflationary universe scenario is just a generic property of any fundamental theory of particle interactions that contains scalar fields. Fundamental spin zero fields, namely scalars, have not yet been detected, but their presence is ubiquitous in all approaches to unification, they are essential in all versions of supergravity theory and they are necessary because of symmetry breaking. From this point of view we can say that Cosmology provides another indirect evidence for the existence of this type of particles whose detection is by now overdue.[4]

4.7 The End of the Second Millennium and the Dawn of the Third Bring Great News in Cosmology

The end of the XXth century and the beginning of the XXIst brought new developments into Cosmology, almost of the same relevance as the discovery of the Hubble law in 1929. A new series of data which have become available starting from 1998 caused a substantial revolution in the subject that, by now, has entered an entirely new phase. Before 1998, theoretical Cosmology was mostly a matter of conjectures and speculations with a remote chance of verification or disproval. At the end of the next decade, in mid 2009, when the European Satellite Planck was launched from the French basis in Guyana towards the Lagrangian point L2 (see Fig. 4.25), theoretical cosmology had already evolved into a science that deals with the explanation of a series of facts established in a substantially firm way. Let us list these facts:

1. Our Universe is spatially flat.
2. Our Universe is presently in a phase of accelerating expansion.[5]

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[4] July 4th 2012 it was officially announced by CERN that both ATLAS and CMS detectors had discovered a new bosonic particle that seems to be the long sought for spin zero particle implementing the Higgs symmetry breaking mechanism.

[5] The 2011 Nobel Prize in Physics was awarded to Saul Perlmutter of the Lawrence Berkeley National Laboratory, Brian Schmidt of the Australian National Laboratory and to Adam Reiss of the Johns Hopkins University, for their 1998 discovery of the present accelerated expansion of the Universe (see Fig. 4.22).

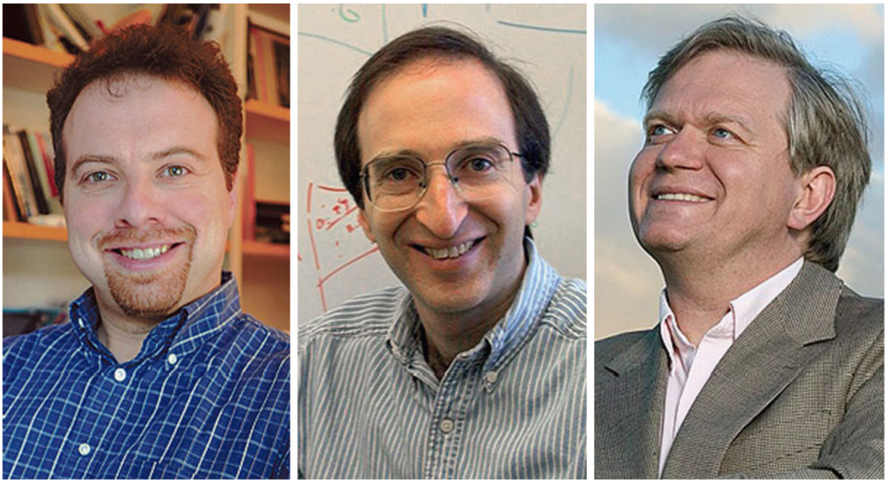

Fig. 4.22 The three recipients of the 2011 Nobel Prize in Physics that was awarded for the discovery of the present accelerated phase in the expansion of the Universe. From the right, Adam Reiss, Saul Perlmutter and Brian Schmidt

3. The energy content of our Universe is so distributed. The *baryonic matter* forming galaxies and providing the luminous content of the world is roughly 6 *percent of the total*. *Dark matter*, whatever it might be, amounts to about 24 *percent*. The remaining 70 *percent*, or even more, is just *vacuum energy* or, if you prefer, *dark energy*.
4. The structure of anisotropies of the CMB is in substantial agreement with the spectrum of primeval quantum fluctuations as predicted by the inflationary scenario.[6]

How were these facts established?

The first very important news came around 1998–1999 with the results of two ambitious surveys of the sky, independently performed by two large international collaborations of astronomers. The two research groups, involving many observatories around the world and also the orbiting Hubble Telescope, are respectively named the *Supernova Cosmology Project*, which developed from an original team of Berkeley University and the *High-Z Supernova Search*, led by the Australia's Mount Stromlo Observatory. Common task of the two projects was the observation of *supernovae of type IA* in very distant galaxies, characterized by a high redshift factor z.

Why were astronomers particularly interested in this type of exploding stars? The reason is simple and analogous to the reason that motivated Hubble to study the Cepheides in not too far galaxies. By the end of the eighties, after two decades of study of the supernova spectra, a new powerful class of *standard candles* had been

[6]The 2006 Nobel Prize in Physics was awarded to John C. Mather of the NASA Goddard Space Flight Center and to George F. Smoot, of the University of California at Berkeley for the first experimental detection of anisotropies in the Cosmic Microwave Background Radiation.

found. Indeed the spectra and the intrinsic luminosity of all known, nearby, type IA supernovae had been revealed to be equal. A fascinating theoretical explanation of these standard candles was also guessed. It was conjectured that type IA supernovae explosions originate from the following phenomenon. In a binary stellar system one of the two companions reaches the end of its life transforming first into a red-giant and then into a *white dwarf*, sustained against gravitational collapse by the degeneracy pressure of the electron gas, as we explained in Chap. 6 of Volume 1. The other star is still alive and active. If conditions of proximity and relative mass are right, there will be a steady stream of material from the active star slowly accreting onto the white dwarf. Over the millions of years, the dwarf's mass increases steadily until it reaches the Chandrasekhar limit explained in Chap. 6 of Volume 1. At that point a runaway thermonuclear explosion is triggered which destroys the dwarf and manifests itself in observations as a type IA supernova. The crucial point is that the Chandrasekhar mass, whose value $1.4M_\odot$ is determined in terms of fundamental constants of Nature, is the same for any supernova IA. This fixes the intrinsic luminosity of the event in an absolute way giving rise to an ideal standard candle which is luminous enough to be seen also in very distant galaxies. Indeed at the time of explosion and typically for a week after that, a supernova is as luminous as an entire galaxy.

Using systematically these standard candles and surveying the sky at very high redshifts z, namely at very large distances from our observation point in the Milky Way, by the end of 1998, the two collaborations groups were ready to present their Hubble plots of the redshift versus distance which, for the first time in history, showed their deviation from linearity (see Fig. 4.23). In this way, we got the first estimate of the deceleration parameter which is defined as follows:

$$q_0 \equiv -\frac{\ddot{a}(t_0)}{a(t_0)H_0^2}; \qquad H_0 \equiv \frac{\dot{a}(t_0)}{a(t_0)} \tag{4.7.1}$$

and parameterizes the aforementioned deviations. To see that, it suffices to consider the Taylor expansion of the scale factor around the present time:

$$a(t) = a(t_0)\left(1 + H_0(t - t_0) - \frac{1}{2}q_0 H_0^2 (t - t_0)^2 + \cdots\right) \tag{4.7.2}$$

and recall the definition (4.3.3) of the redshift factor which can be rewritten as follows:

$$z = \frac{\lambda(t_0)}{\lambda(t)} - 1 = \frac{a(t_0)}{a(t)} - 1 \tag{4.7.3}$$

since, as we already mentioned and as we will prove later on, the ratio between the wave-length of a photon at the present time t_0 and at the time of emission t is the same as the ratio of the scale factors at the same instants of time: $\lambda(t_0)/\lambda(t) = a(t_0)/a(t)$. Inserting the Taylor expansion (4.7.2) into (4.7.3) and inverting the relation we find:

$$cz = H_0\, d + \left(1 + \frac{q_0}{2}\right) H_0^2\, d^2 + \cdots \tag{4.7.4}$$

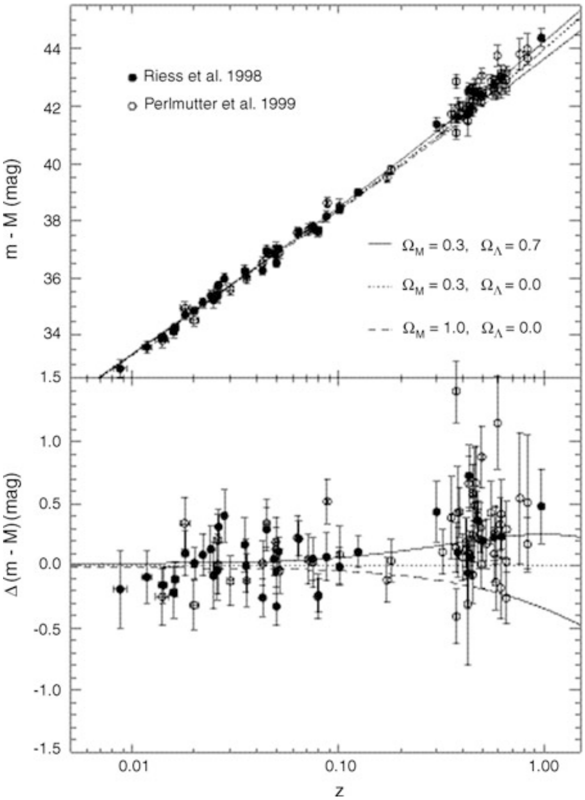

Fig. 4.23 Using type IA supernovae as standard candles and systematically detecting them in very remote galaxies has provided the means to determine the deviation of the Hubble law from linearity at high redshifts, in other words to estimate the acceleration of the Universe expansion which has proved to be positive. This is consistent with the existence of *dark energy*, alias of a positive cosmological constant

where c is the speed of light and the distance between us and the source as been approximated as $d \simeq c(t - t_0)$. Equation (4.7.4) presents the form of the first quadratic correction to the linear Hubble law which was experimentally evaluated for the first time in history in 1998. The surprise was immense since the deceleration parameter q_0 turned out to be negative, in other words it was revealed that our Universe is actually accelerating its expansion at the present time, since $\ddot{a}(t_0) > 0$.

In later sections, studying Friedman equations, we will show that the acceleration parameter can be positive only if the energy content of the Universe is dominated by vacuum energy rather than by ordinary matter and radiation. The evaluation of q_0 was therefore a direct evaluation of the percentage of vacuum energy filling our Universe: approximately the 70 percent, as it turned out by taking into account the other important results about the CMB anisotropies which became available in the following years.

The satellite WMAP (Wilkinson Microwave Anisotropy Probe) was launched in June 2001 from Cape Kennedy and reached the Lagrangian point L2 wherefrom, during seven years it collected and streamed to Earth very important data on the space distribution of the Cosmic Microwave Background Radiation, in particular measuring its temperature in each direction of the sky. The oscillations of the temperature with respect to its average value $T = 2.725$ K are of the order of

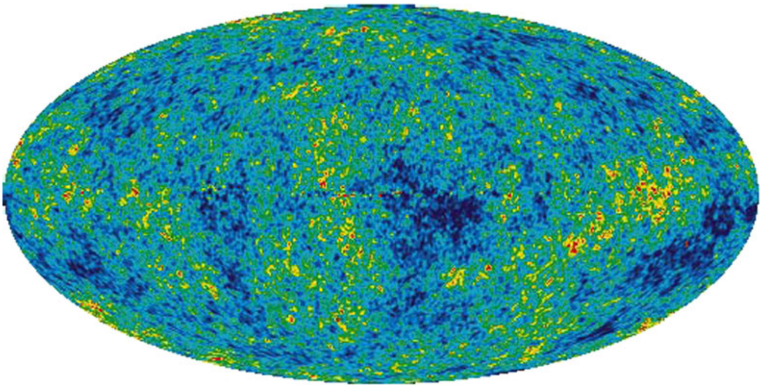

Fig. 4.24 The microwave image of the primeval sky obtained by the seven year mission WMAP, that has measured the temperature anisotropies of the Cosmic Background Radiation. With variations of the order of few milliKelvin the microwave sky displays hotter and colder spots. As shown in later sections, the temperature variations are a direct measure of the variations in the gravitational potential at the time of decoupling, 400.000 years after the Big Bang and approximately 13 billions of years ago

the milliKelvin; when such hotter and colder spots are reported on a two-sphere representing the sky one obtains an image of the same type as shown in Fig. 4.24.

Such a plot can be regarded as an image of the Last Scattering Surface at the time of decoupling of radiation. Furthermore, because of an effect named Sachs-Wolffe-effect, which we will mathematically explain in later sections, measuring the temperature variation function $\frac{\delta T(x)}{T}$ is nothing else but measuring the primeval gravitational potential $\Phi(x)$ that encodes the perturbation of the metric around its homogeneous and isotropic form.

In this way the WMAP mission, which was extremely successful, provided us with a direct measure of the cosmic primeval perturbations just before the time of radiation decoupling and established a new vision of the early Universe.

From the analysis of the CMB spectrum we learnt that our Universe is spatially flat $\kappa = 0$ and we could confirm its acceleration, obtaining a more precise evaluation of the amount of vacuum energy (around 72 %).

Furthermore the multipole analysis of the correlation function:

$$C(\mathbf{x} - \mathbf{y}) = \left\langle \frac{\delta T(\mathbf{x})}{T} \frac{\delta T(\mathbf{y})}{T} \right\rangle \tag{4.7.5}$$

confirmed the generic predictions of the Inflationary Universe scenario showing that, with high probability, the physical mechanism which explains the mysterious homogeneity and isotropy of our Universe, alias the Cosmological Principle, is indeed the one for which the 2003 Dirac Medal was awarded. In 2009 a new radio-telescope was launched by the European Space Agency, named Planck (see Fig. 4.25). It also was placed in the Lagrangian point L2 from which it started collecting data on the CMB anisotropies with still higher precision than its predecessor

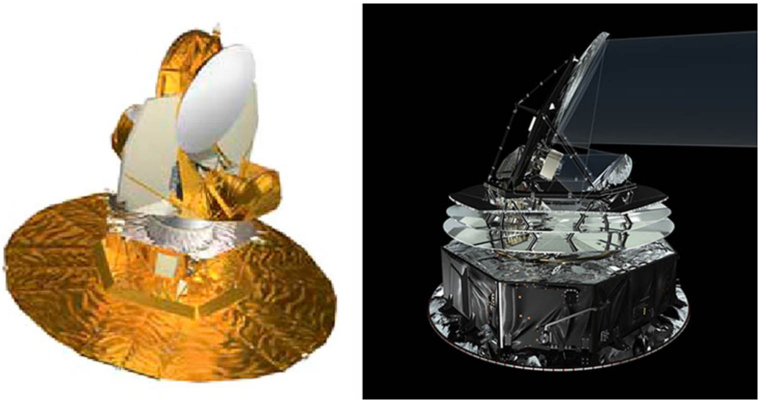

Fig. 4.25 On the *left* a view of the WMAP spacecraft which, for seven years, has inspected the Cosmic Microwave Background Radiation from the second Lagrangian point L2. WMAP was launched by NASA from Cape Kennedy in 2001. On the *right* a view of the Planck Telescope, constructed and launched by ESA in may 2009. Planck is also positioned in the Lagrangian point L2 and it is monitoring the CMB temperature anisotropies, as WMAP already did, yet with a much higher precision. In particular the sensitivity of the instrument for low frequencies on board of Planck reaches the one millionth of a Kelvin

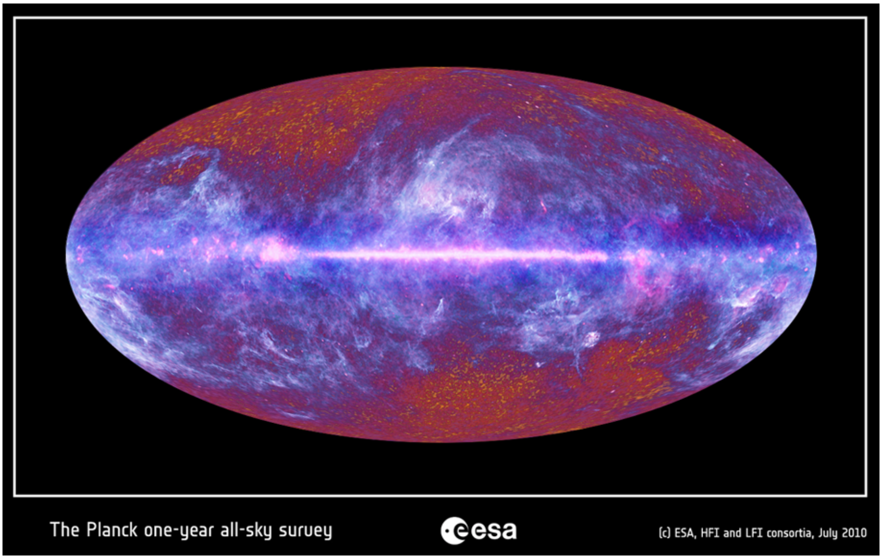

Fig. 4.26 The microwave image of the primeval sky obtained by the Planck satellite, after one year of data taking

WMP (see Fig. 4.26). With the Planck mission a truly new era of Cosmology has begun. We expect to obtain detailed information about the spectrum of primeval perturbations which might shed light on the detailed structure of the inflationary mechanism and even discriminate among different candidates for its realization within a fundamental unified theory of all interactions as superstring theory and its supergravity descendants.

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Chapter 5
Cosmology and General Relativity: Mathematical Description of the Universe

Ma sedendo e mirando, interminati
Spazi di là da quella, e sovrumani
Silenzi, e profondissima quiete
Io nel pensier mi fingo; ove per poco
Il cor non si spaura...
Giacomo Leopardi

5.1 Introduction

Having completed in the previous chapter our historical review of Physical Cosmology, from its very beginning at the end of the XVIIIth century to the challenging discoveries that reshaped it at the beginning of the XXIst century, it is time to enter its rigorous mathematical formulation in terms of General Relativity, which constitutes the main goal of the present chapter. As we already extensively pointed out throughout Chap. 4, the two crucial issues in cosmology are those of homogeneity and isotropy, whose physical explanation is the goal of the inflationary theory. In order to understand and correctly utilize these two geometrical concepts within the context of our geometrical theory of gravitation we have to address in full some mathematical questions that were only touched upon in the first volume and in previous chapters of the second. These questions relate to the concept of isometries for Riemannian and pseudo-Riemannian manifolds. The proper treatment of isometries leads us to develop the Differential Geometry of *Coset Manifolds* and *Symmetric Spaces* which, besides being ubiquitous in Mathematical Physics, is also very relevant to our subsequent chapters devoted to an introduction to Supergravity, Branes and Supersymmetric Black-Holes.

After this preparatory step we address the mathematical description of the Universe by means of metrics that possess the two properties required by the Cosmological Principle: homogeneity and isotropy.

In order to clarify the independent role of the two symmetry requirements we begin by discussing homogeneous but not isotropic metrics and we present some examples. We discuss *Kasner solutions* and some more intriguing ones based on non-Abelian three dimensional groups. We outline the celebrated Bianchi classification of such homogeneous but not isotropic universes. We emphasize that the curious mechanisms associated with anisotropic homogeneous universes might play a

P.G. Frè, *Gravity, a Geometrical Course*, DOI 10.1007/978-94-007-5443-0_5,

relevant role in higher dimensional gravitational theories like those that emerge in supergravity and superstring theory.

Next, introducing also isotropy, we go over to the standard cosmological model and to its back-bone that are Friedman equations. The latter are analyzed in all respects and consequences, discussing the role of the spatial curvature, the available types of hydrodynamical equations of state and the exact solutions that are known for them, corresponding to various energy fillings of the Universe. Particular attention is paid to the embedding of cosmological metrics within de Sitter space.

After discussing horizons and the conceptual problem of homogeneous initial boundary conditions we go over to discuss the mathematical modeling of the inflationary scenario by means of the coupling of gravity to a scalar field, endowed with a potential. The general framework of the *slow rolling phase* is presented together with examples of numerical solutions of the coupled Einstein-Klein-Gordon equations.

The next addressed topic is *perturbations*. We discuss in detail the general form of the scalar perturbations in the coupled Einstein-Klein-Gordon system and we derive the form of the independent scalar degree of freedom which we canonically quantize. In this way we are able to outline the derivation of the *power spectrum* of the primeval quantum fluctuations that is currently experimentally observed in the anisotropies of the Cosmic Microwave Background. The relation between the fluctuations of the radiation temperature T and those of the gravitational quantized potential Φ is due to the so called *Sachs Wolfe effect* whose derivation we also present.

5.2 Mathematical Interlude: Isometries and the Geometry of Coset Manifolds

The existence of continuous isometries is related with the existence of Killing vector fields which we already utilized in various occasions. Now we have to explain the underlying mathematical theory in full and this leads us to introduce a relevant chapter of differential geometry which is the study of coset manifolds and symmetric spaces. The present section is devoted to these topics.

5.2.1 *Isometries and Killing Vector Fields*

Finite isometries of a (pseudo-)Riemannian manifold $\mathcal{M}_g$ are diffeomorphisms:

$$\phi : \mathcal{M} \to \mathcal{M} \tag{5.2.1}$$

such that their pull-back[1] on the metric form leaves it invariant:

$$\phi^\star\big[g_{\mu\nu}(x)\,dx^\mu\,dx^\nu\big] = g_{\mu\nu}(x)\,dx^\mu\,dx^\nu \tag{5.2.2}$$

[1] See Sect. 3.3 of Volume 1 for the definition of the pull-back and of the push-forward.

Suppose now that the considered diffeomorphism is infinitesimally close to the identity:

$$x^\mu \to \phi^\mu(x) \simeq x^\mu + k^\mu(x) \tag{5.2.3}$$

The condition for this diffeomorphism to be an isometry, is a differential equation for the components of the vector field $\mathbf{k} = k^\mu \partial_\mu$ which immediately follows from (5.2.2):

$$\nabla_\mu k_\nu + \nabla_\nu k_\mu = 0 \tag{5.2.4}$$

Hence given a metric one can investigate the nature of its isometries by trying to solve the linear homogeneous equations (5.2.4) determining its general integral. The important point is that, if we have two Killing vectors $\mathbf{k}$ and $\mathbf{w}$ also their commutator $[\mathbf{k}, \mathbf{w}]$ will be a Killing vector. This follows from the fact that the product of two finite isometries is also an isometry. Hence Killing vector fields form a finite dimensional Lie algebra $\mathbb{G}_{iso}$ and one can turn the question around. Rather then calculating the isometries of a given metric one can address the problem of constructing (pseudo-)Riemannian manifolds that have a prescribed isometry algebra. Due to the well established classification of semi-simple Lie algebras this becomes a very fruitful point of view.

In particular, also in view of the Cosmological Principle, one is interested in homogeneous spaces, namely in (pseudo-)Riemannian manifolds where each point of the manifold can be reached from a reference one by the action of an isometry.

Homogeneous spaces are identified with coset manifolds, whose differential geometry can be thoroughly described and calculated in pure Lie algebra terms.

5.2.2 Coset Manifolds

Coset manifolds are a natural generalization of group manifolds and play a very important, ubiquitous, role both in Mathematics and in Physics.

In group-theory (irrespectively whether the group G is finite or infinite, continuous or discrete) we have the concept of *coset space* G/H which is just the set of equivalence classes of elements $g \in \mathrm{G}$, where the equivalence is defined by right multiplication with elements $h \in \mathrm{H} \subset \mathrm{G}$ of a subgroup:

$$\forall g, g' \in \mathrm{G} : g \sim g' \quad \text{iff} \quad \exists h \in \mathrm{H} \setminus gh = g' \tag{5.2.5}$$

Namely two group elements are equivalent if and only if they can be mapped into each other by means of some element of the subgroup. The equivalence classes, which constitute the elements of G/H are usually denoted $g\mathrm{H}$, where g is any representative of the class, namely any one of the equivalent G-group elements the class is composed of. The definition we have just provided by means of right multiplication can be obviously replaced by an analogous one based on left-multiplication. In this case we construct the coset H\G composed of *right lateral classes* Hg while gH are named the *left lateral classes*. For non-Abelian groups G and generic subgroups H the left G/H and right H\G coset spaces have different not coinciding elements. Working with one or with the other definition is just a matter of conventions. We choose to work with *left classes*.

Coset manifolds arise in the context of Lie group theory when G is a Lie group and H is a Lie subgroup thereof. In that case the set of lateral classes gH can be endowed with a manifold structure inherited from the manifold structure of the parent group G. Furthermore on G/H we can construct *invariant metrics* such that all elements of the original group G are isometries of the constructed metric. As we show below, the curvature tensor of invariant metrics on coset manifolds can be constructed in purely algebraic terms starting from the structure constants of the G Lie algebra, by-passing all analytic differential calculations.

The reason why coset manifolds are relevant to Cosmology is encoded in the concept of *homogeneity*, that is one of the two pillars of the Cosmological Principle. Indeed coset manifolds are easily identified with *homogeneous spaces* which we presently define.

Definition 5.2.1 A Riemannian or pseudo-Riemannian manifold $\mathscr{M}_g$ is said to be homogeneous if it admits as an isometry the transitive action of a group G. A group acts transitively if any point of the manifold can be reached from any other by means of the group action.

A notable and very common example of such homogeneous manifolds is provided by the spheres $\mathbb{S}^n$ and by their non-compact generalizations, the pseudo-spheres $\mathbb{H}_{\pm}^{(n+1-m,m)}$. Let x^I denote the Cartesian coordinates in $\mathbb{R}^{n+1}$ and let:

$$\eta_{IJ} = \mathrm{diag}(\underbrace{+, +\ldots, +}_{n+1-m}, \underbrace{-, -, \ldots, -}_{m}) \tag{5.2.6}$$

be the coefficient of a non-degenerate quadratic form with signature $(n+1-m, m)$:

$$\langle \mathbf{x}, \mathbf{x} \rangle_\eta \equiv x^I x^J \eta_{IJ} \tag{5.2.7}$$

We obtain a pseudo-sphere $\mathbb{H}_{\pm}^{(n+1-m,m)}$ by defining the algebraic locus:

$$\mathbf{x} \in \mathbb{H}_{\pm}^{(n+1-m,m)} \quad \Leftrightarrow \quad \langle \mathbf{x}, \mathbf{x} \rangle_\eta \equiv \pm 1 \tag{5.2.8}$$

which is a manifold of dimension n. The spheres $\mathbb{S}^n$ correspond to the particular case $\mathbb{H}_{+}^{n+1,0}$ where the quadratic form is positive definite and the sign in the right hand side of (5.2.8) is positive. Obviously with a positive definite quadratic form this is the only possibility.

All these algebraic loci are invariant under the transitive action of the group $\mathrm{SO}(n+1, n+1-m)$ realized by matrix multiplication on the vector $\mathbf{x}$ since:

$$\forall g \in \mathrm{G}: \quad \langle \mathbf{x}, \mathbf{x} \rangle_\eta = \pm 1 \quad \Leftrightarrow \quad \langle g\mathbf{x}, g\mathbf{x} \rangle_\eta = \pm 1 \tag{5.2.9}$$

namely the group maps solutions of the constraint (5.2.8) into solutions of the same and, furthermore, all solutions can be generated starting from a standard reference vector:

$$\langle \mathbf{x}, \mathbf{x} \rangle_\eta = \pm 1 \quad \Rightarrow \quad \exists g \in \mathrm{G} \backslash \mathbf{x} = g \mathbf{x}_0^\pm \tag{5.2.10}$$

where:

$$x_0^+ = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ \hline 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}; \qquad x_0^- = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \hline 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{5.2.11}$$

the line separating the first $n + 1 - m$ entries from the last m. Equation (5.2.10) guarantees that the locus is invariant under the action G, while (5.2.11) states that G is transitive.

Definition 5.2.2 In a homogeneous space $\mathcal{M}_g$, the subgroup $\mathrm{H}_p \subset \mathrm{G}$ which leaves a point $p \in \mathcal{M}_g$ fixed ($\forall h \in \mathrm{H}_p$, $hp = p$) is named the *isotropy subgroup* of the point. Because of the transitive action of G, any other point $p' = gp$ has an isotropy subgroup $\mathrm{H}_{p'} = g\mathrm{H}_p g^{-1}$ which is conjugate to H_p and therefore isomorphic to it.

It follows that, up to conjugation, the isotropy group of a homogeneous manifold $\mathcal{M}_g$ is unique and corresponds to an intrinsic property of such a space. It suffices to calculate the isotropy group H_0 of a conventional properly chosen reference point p_0: all other isotropy groups will immediately follow. For brevity H_0 will be just renamed H.

In our example of the spaces $\mathbb{H}_\pm^{(n+1-m,m)}$ the isotropy group is immediately derived by looking at the form of the vectors $\mathbf{x}_0^\pm$: all elements of G which rotate the vanishing entries of these vectors among themselves are clearly elements of the isotropy group. Hence we find:

$$\begin{aligned} \mathrm{H} &= \mathrm{SO}(n, m) && \text{for } \mathbb{H}_+^{(n+1-m,m)} \\ \mathrm{H} &= \mathrm{SO}(n+1, m-1) && \text{for } \mathbb{H}_-^{(n+1-m,m)} \end{aligned} \tag{5.2.12}$$

It is natural to label any point p of a homogeneous space by the parameters describing the G-group element which carries a conventional point p_0 into p. These parameters, however, are redundant: because of the H-isotropy there are infinitely many ways to reach p from p_0. Indeed, if g does that job, any other element of the lateral class $g\mathrm{H}$ does the same. It follows by this simple discussion that the homogeneous manifold $\mathcal{M}_g$ can be identified with the coset manifold G/H defined by the transitive group G divided by the isotropy group H.

Focusing once again on our example we find:

$$\mathbb{H}_{+}^{(n+1-m,m)} = \frac{\mathrm{SO}(n+1-m,m)}{\mathrm{SO}(n-m,m)}; \qquad \mathbb{H}_{-}^{(n+1-m,m)} = \frac{\mathrm{SO}(n+1-m,m)}{\mathrm{SO}(n+1-m,m-1)} \tag{5.2.13}$$

In particular the spheres correspond to:

$$\mathbb{S}^n = \mathbb{H}_{+}^{(n+1,0)} = \frac{\mathrm{SO}(n+1)}{\mathrm{SO}(n)} \tag{5.2.14}$$

Other important examples, relevant for cosmology are:

$$\mathbb{H}_{+}^{(n+1,1)} = \frac{\mathrm{SO}(n+1,1)}{\mathrm{SO}(n,1)}; \qquad \mathbb{H}_{-}^{(n+1,1)} = \frac{\mathrm{SO}(n+1,1)}{\mathrm{SO}(n+1)} \tag{5.2.15}$$

The general classification of homogeneous (pseudo-)Riemannian spaces corresponds therefore to the classification of the coset manifolds G/H for all Lie groups G and for their closed Lie subgroups $\mathrm{H} \subset \mathrm{G}$.

The equivalence classes constituting the points of the coset manifold can be labeled by a set of d coordinates $y \equiv \{y^1, \dots, y^d\}$ where:

$$d = \dim \frac{\mathrm{G}}{\mathrm{H}} \equiv \dim \mathrm{G} - \dim \mathrm{H} \tag{5.2.16}$$

There are of course many different ways of choosing the y-parameters since, just as in any other manifold, there are many possible coordinate systems. What is specific of coset manifolds is that, given any coordinate system y by means of which we label the equivalence classes, within each equivalence class we can choose a representative group element $\mathbb{L}(y) \in \mathrm{G}$. The choice must be done in such a way that $\mathbb{L}(y)$ should be a smooth function of the parameters y. Furthermore for different values y and y', the group elements $\mathbb{L}(y)$ and $\mathbb{L}(y')$ should never be equivalent, in other words no $h \in \mathrm{H}$ should exist such that $\mathbb{L}(y) = \mathbb{L}(y')h$. Under left multiplication by $g \in \mathrm{G}$, $\mathbb{L}(y)$ is in general carried into another equivalence class with coset representative $\mathbb{L}(y')$. Yet the g image of $\mathbb{L}(y)$ is not necessarily $\mathbb{L}(y')$: it is typically some other element of the same class, so that we can write:

$$\forall g \in G: \quad g\mathbb{L}(y) = \mathbb{L}\big(y'\big)h(g,y); \quad h(g,y) \in \mathrm{H} \tag{5.2.17}$$

where we emphasized that the H-element necessary to map $\mathbb{L}(y')$ into the g-image of $\mathbb{L}(y)$, depends, in general both from the point y and from the chosen transformation g. Equation (5.2.17) is pictorially described in Fig. 5.1. For the spheres a possible set of coordinates y can be obtained by means of the stereographic projection described, for the case of the two-sphere, in chapter two of Volume 1. Its conception is recalled here in Fig. 5.2.

As an other explicit example, which will be useful in the sequel, we consider the case of the Euclidian hyperbolic spaces $\mathbb{H}_{-}^{(n,1)}$ identified as coset manifolds in (5.2.15). In this case, to introduce a coset parameterization means to write a family of $\mathrm{SO}(n,1)$ matrices $\mathbb{L}(\mathbf{y})$ depending smoothly on an n-component vector $\mathbf{y}$

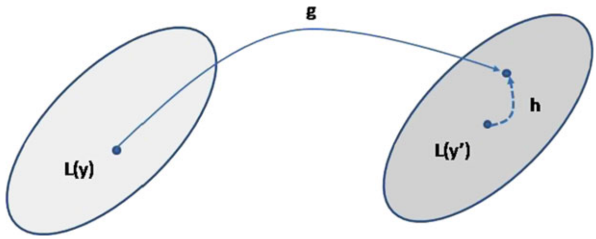

Fig. 5.1 Pictorial description of the action of the group G on the coset representatives

in such a way that for different values of $\mathbf{y}$ such matrices cannot be mapped one in the other by means of right multiplication with any element h of the subgroup $\mathrm{SO}(n) \subset \mathrm{SO}(n,1)$:

$$\mathrm{SO}(n,1) \supset \mathrm{SO}(n) \ni h = \left(\begin{array}{c|c} \mathscr{O} & 0 \\ \hline 0 & 1 \end{array}\right); \qquad \mathscr{O}^T \mathscr{O} = \mathbf{1}_{n\times n} \tag{5.2.18}$$

An explicit parameterization of this type can be written as follows:

$$\mathbb{L}(\mathbf{y}) = \left(\begin{array}{c|c} \mathbf{1}_{n\times n} + 2\frac{\mathbf{y}\mathbf{y}^T}{1-\mathbf{y}^2} & -2\frac{\mathbf{y}}{1-\mathbf{y}^2} \\ \hline -2\frac{\mathbf{y}^T}{1-\mathbf{y}^2} & \frac{1+\mathbf{y}^2}{1-\mathbf{y}^2} \end{array}\right) \tag{5.2.19}$$

where $\mathbf{y}^2 \equiv \mathbf{y}\cdot\mathbf{y}$ denotes the standard $\mathrm{SO}(n)$ invariant scalar product in $\mathbb{R}^n$. Why the matrices $\mathbb{L}(\mathbf{y})$ form a good parameterization of the coset? The reason is simple, first of all observe that:

$$\mathbb{L}(\mathbf{y})^T \eta \mathbb{L}(\mathbf{y}) = \eta \tag{5.2.20}$$

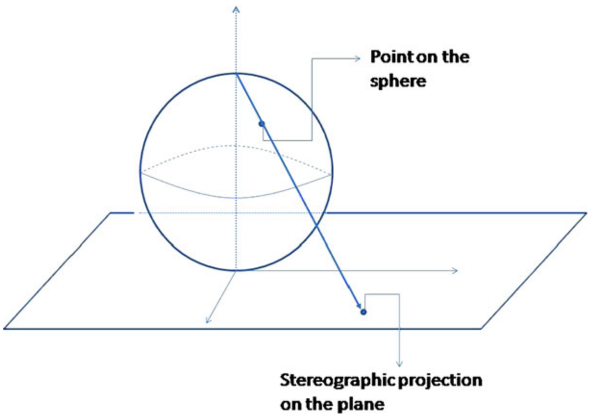

Fig. 5.2 The idea of the stereographic projection. Considering the $\mathbb{S}^n$ sphere immersed in $\mathbb{R}^{n+1}$, from the North-Pole $\{1,0,0,\ldots,0\}$ one draws the line that goes through the point $p \in \mathbb{S}^n$ and considers the point $\pi(p) \in \mathbb{R}^n$ where such a line intersects the $\mathbb{R}^n$ plane tangent to sphere in the South Pole and orthogonal to the line that joins the North and the South Pole. The n-coordinates $\{y^1,\ldots,y^n\}$ of $\pi(p)$ can be taken as labels of an open chart in $\mathbb{S}^n$

where

$$\eta = \mathrm{diag}(+, +, \dots, +, -) \tag{5.2.21}$$

This guarantees that $\mathbb{L}(\mathbf{y})$ are elements of $\mathrm{SO}(n, 1)$, secondly observe that the image $\mathbf{x}(\mathbf{y})$ of the standard vector $\mathbf{x}_0$ through $\mathbb{L}(\mathbf{y})$,

$$\mathbf{x}(\mathbf{y}) \equiv \mathbb{L}(\mathbf{y})\mathbf{x}_0 = \mathbb{L}(\mathbf{y}) \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \hline 1 \end{pmatrix} = \frac{1}{1 - \mathbf{y}^2} \begin{pmatrix} 2y^1 \\ \vdots \\ 2y^n \\ \hline \frac{1+\mathbf{y}^2}{1-\mathbf{y}^2} \end{pmatrix} \tag{5.2.22}$$

lies, as it should, in the algebraic locus $\mathbb{H}_-^{(n,1)}$,

$$\mathbf{x}(\mathbf{y})^T \eta \mathbf{x}(\mathbf{y}) = -1 \tag{5.2.23}$$

and has n linearly independent entries (the first n) parameterized by $\mathbf{y}$. Hence the lateral classes can be labeled by y and this concludes our argument to show that (5.2.19) is a good coset parameterization. $\mathbb{L}(0) = \mathbf{1}_{(n+1)\times(n+1)}$ corresponds to the identity class which is usually named the *origin* of the coset.

5.2.3 The Geometry of Coset Manifolds

In order to study the geometry of a coset manifold G/H, the first important step is provided by the orthogonal decomposition of the corresponding Lie algebra, namely by

$$\mathbb{G} = \mathbb{H} \oplus \mathbb{K} \tag{5.2.24}$$

where $\mathbb{G}$ is the Lie algebra of G and the subalgebra $\mathbb{H} \subset \mathbb{G}$ is the Lie algebra of the subgroup H and where $\mathbb{K}$ denotes a vector space orthogonal to $\mathbb{H}$ with respect to the Cartan Killing metric of $\mathbb{G}$. By definition of subalgebra we always have:

$$[\mathbb{H}, \mathbb{H}] \subset \mathbb{H} \tag{5.2.25}$$

while in general one has:

$$[\mathbb{H}, \mathbb{K}] \subset \mathbb{H} \oplus \mathbb{K} \tag{5.2.26}$$

Definition 5.2.3 Let G/H be a Lie coset manifold and let the orthogonal decomposition of the corresponding Lie algebra be as in (5.2.24). If the condition:

$$[\mathbb{H}, \mathbb{K}] \subset \mathbb{K} \tag{5.2.27}$$

applies, the coset G/H is named *reductive*.

Equation (5.2.27) has an obvious and immediate interpretation. The complementary space $\mathbb{K}$ forms a linear representation of the subalgebra $\mathbb{H}$ under its adjoint action within the ambient algebra $\mathbb{G}$.

Almost all of the "reasonable" coset manifolds which occur in various provinces of Mathematical Physics are reductive. Violation of *reductivity* is a sort of pathology whose study we can disregard in the scope of this book. We will consider only reductive coset manifolds.

Definition 5.2.4 Let G/H be a reductive coset manifold. If in addition to (5.2.27) also the following condition:

$$[\mathbb{K}, \mathbb{K}] \subset \mathbb{H} \tag{5.2.28}$$

applies, then the coset manifold G/H is named a *symmetric space*.

Let T_A $(A = 1, \ldots, n)$ denote a complete basis of generators for the Lie algebra $\mathbb{G}$:

$$[T_A, T_B] = C^C{}_{AB}\, T_C \tag{5.2.29}$$

and T_i $(i = 1, \ldots, m)$ denote a complete basis for the subalgebra $\mathbb{H} \subset \mathbb{G}$. We also introduce the notation T_a $(a = 1, \ldots, n - m)$ for a set of generators that provide a basis of the complementary subspace $\mathbb{K}$ in the orthogonal decomposition (5.2.24). We nickname T_a the *coset generators*. Using such notations, (5.2.29) splits into the following three ones:

$$[T_j, T_k] = C^i{}_{jk}\, T_i \tag{5.2.30}$$

$$[T_i, T_b] = C^a{}_{ib}\, T_a \tag{5.2.31}$$

$$[T_b, T_c] = C^i{}_{bc}\, T_i + C^a{}_{bc}\, T_a \tag{5.2.32}$$

Equation (5.2.30) encodes the property of $\mathbb{H}$ of being a subalgebra. Equation (5.2.31) encodes the property of the considered coset of being reductive. Finally if in (5.2.32) we have $C^a{}_{bc} = 0$, the coset is not only reductive but also symmetric.

We will be able to provide explicit formulae for the Riemann tensor of reductive coset manifolds equipped with G-invariant metrics in terms of such structure constants. Prior to that we consider the infinitesimal transformation and the very definition of the Killing vectors with respect to which the metric has to be invariant.

5.2.3.1 Infinitesimal Transformations and Killing Vectors

Let us consider the transformation law (5.2.17) of the coset representative. For a group element g infinitesimally close to the identity, we have:

$$g \simeq 1 + \varepsilon^A T_A \tag{5.2.33}$$

$$h(y, g) \simeq 1 - \varepsilon_A W^i_A(y) T_i \tag{5.2.34}$$

$$y'^\alpha \simeq y^\alpha + \varepsilon^A k^\alpha_A \tag{5.2.35}$$

The induced h transformation in (5.2.17) depends in general on the infinitesimal G-parameters ε^A and on the point in the coset manifold y, as shown in (5.2.34). The y-dependent rectangular matrix $W^i_A(y)$ is usually named the $\mathbb{H}$-compensator. The shift in the coordinates y^α is also proportional to ε^A and the vector fields:

$$\mathbf{k}_A = k^\alpha_A(y)\frac{\partial}{\partial y^\alpha} \tag{5.2.36}$$

are named the *Killing vectors of the coset*. The reason for such a name will be justified when we will show that on G/H we can construct a (pseudo-)Riemannian metric which admits the vector fields (5.2.36) as generators of infinitesimal isometries. For the time being those in (5.2.36) are just a set of vector fields that, as we prove few lines below, close the Lie algebra of the group G.

Inserting (5.2.33)–(5.2.35) into the transformation law (5.2.17) we obtain:

$$T_A\mathbb{L}(y) = \mathbf{k}_A\mathbb{L}(y) - W^i_A(y)\mathbb{L}(y)T_i \tag{5.2.37}$$

Consider now the commutator $g_2^{-1}g_1^{-1}g_2g_1$ acting on $\mathbb{L}(y)$. If both group elements $g_{1,2}$ are infinitesimally close to the identity in the sense of (5.2.33), then we obtain:

$$g_2^{-1}g_1^{-1}g_2g_1\mathbb{L}(y) \simeq \big(1 - \varepsilon_1^A\varepsilon_2^B[T_A, T_B]\big)\mathbb{L}(y) \tag{5.2.38}$$

By explicit calculation we find:

$$\begin{aligned}[T_A, T_B]\mathbb{L}(y) &= T_AT_B\mathbb{L}(y) - T_BT_A\mathbb{L}(y)\\ &= [\mathbf{k}_A, \mathbf{k}_B]\mathbb{L}(y) - \big(\mathbf{k}_A W^i_B - \mathbf{k}_B W^i_A + 2C^i{}_{jk}\, W^j_A W^k_B\big)\mathbb{L}(y)T_i\end{aligned} \tag{5.2.39}$$

On the other hand, using the Lie algebra commutation relations we obtain:

$$[T_A, T_B]\mathbb{L}(y) = C^C{}_{AB}\, T_C\mathbb{L}(y) = C^C{}_{AB}\big(\mathbf{k}_C\mathbb{L}(y) - W^i_C\mathbb{L}(y)T_i\big) \tag{5.2.40}$$

By equating the right hand sides of (5.2.39) and (5.2.40) we conclude that:

$$[\mathbf{k}_A, \mathbf{k}_B] = C^C{}_{AB}\mathbf{k}_C \tag{5.2.41}$$

$$\mathbf{k}_A W^i_B - \mathbf{k}_B W^i_A + 2C^i{}_{jk}\, W^j_A\, W^k_B = C^C{}_{AB}\, W^i_C \tag{5.2.42}$$

where we separately compared the terms with and without W's, since the decomposition of a group element into $\mathbb{L}(y)h$ is unique.

Equation (5.2.41) shows that the Killing vector fields defined above close the commutation relations of the $\mathbb{G}$-algebra.

Equation (5.2.42) will be used to construct a consistent $\mathbb{H}$-covariant Lie derivative.

In the case of the spaces $\mathbb{H}^{(n,1)}_-$, which we choose as illustrative example, the Killing vectors can be easily calculated by following the above described procedure step by step. For later purposes we find it convenient to present such a calculation in a slightly more general set up by introducing the following coset representative that depends on a discrete parameter $\kappa = \pm 1$:

$$\mathbb{L}_\kappa(\mathbf{y}) = \left(\begin{array}{c|c} \mathbf{1}_{n\times n} + 2\mathbf{y}\mathbf{y}^T \frac{\kappa}{1+\kappa\mathbf{y}^2} & -2\frac{\mathbf{y}}{1+\kappa\mathbf{y}^2} \\ \hline 2\kappa\frac{\mathbf{y}^T}{1+\kappa\mathbf{y}^2} & \frac{1-\kappa\mathbf{y}^2}{1+\kappa\mathbf{y}^2} \end{array}\right) \tag{5.2.43}$$

An explicit calculation shows that:

$$\mathbb{L}_\kappa(\mathbf{y})^T \underbrace{\left(\begin{array}{cccc|c} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \\ \hline 0 & \cdots & 0 & 0 & \kappa \end{array}\right)}_{\eta_\kappa} \mathbb{L}_\kappa(\mathbf{y}) = \underbrace{\left(\begin{array}{cccc|c} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \\ \hline 0 & \cdots & 0 & 0 & \kappa \end{array}\right)}_{\eta_\kappa} \tag{5.2.44}$$

Namely $\mathbb{L}_{-1}(\mathbf{y})$ is an $\mathrm{SO}(n, 1)$ matrix, while $\mathbb{L}_1(\mathbf{y})$ is an $\mathrm{SO}(n+1)$ group element. Furthermore defining, as in (5.2.22):

$$\mathbf{x}_\kappa(\mathbf{y}) \equiv \mathbb{L}_\kappa(\mathbf{y}) \left(\begin{array}{c} 0 \\ \vdots \\ 0 \\ \hline 1 \end{array}\right) \tag{5.2.45}$$

we find that:

$$\mathbf{x}_\kappa(\mathbf{y})^T \eta_\kappa \mathbf{x}_\kappa(\mathbf{y}) = \kappa \tag{5.2.46}$$

Hence by means of $\mathbb{L}_1(\mathbf{y})$ we parameterize the points of the n-sphere $\mathbb{S}^n$, while by means of $\mathbb{L}_{-1}(\mathbf{y})$ we parameterize the points of $\mathbb{H}_-^{(n,1)}$ named also the n-pseudo-sphere or the n-hyperboloid. In both cases the stability subalgebra is $\mathfrak{so}(n)$ for which a basis of generators is provided by the following matrices:

$$J_{ij} = \mathscr{I}_{ij} - \mathscr{I}_{ji}; \quad i, j = 1, \ldots, n \tag{5.2.47}$$

having named:

$$\mathscr{I}_{ij} = \left(\begin{array}{cccc|c} 0 & \cdots & \cdots & 0 & 0 \\ 0 & \cdots & 1 & 0 & 0 \,\}\ i\text{th row} \\ 0 & \cdots & \cdots & 0 & 0 \\ \hline 0 & \cdots & \underbrace{\cdots}_{j\text{th column}} & 0 & 0 \end{array}\right). \tag{5.2.48}$$

the $(n+1) \times (n+1)$ matrices whose only non-vanishing entry is the ijth one, equal to 1.

The commutation relations of the $\mathfrak{so}(n)$ generators are very simple and were already considered several times in Volume 1. We have:

$$[J_{ij}, J_{k\ell}] = -\delta_{ik} J_{j\ell} + \delta_{jk} J_{i\ell} - \delta_{j\ell} J_{ik} + \delta_{i\ell} J_{jk} \tag{5.2.49}$$

The coset generators can instead be chosen as the following matrices:

$$P_i = \begin{array}{cl} \left(\begin{array}{cccc|c} 0\cdots & \cdots & 0 & 0 \\ 0\cdots & 0 & 0 & 1 \\ 0\cdots & \cdots & 0 & 0 \\ \hline 0\cdots & \underbrace{-\kappa}_{i\text{th column}} & 0 & 0 \end{array}\right) & \begin{array}{l} \\ \}\ i\text{th row} \\ \\ \\ \\ \end{array} \end{array} \tag{5.2.50}$$

and satisfy the following commutation relations:

$$[J_{ij}, P_k] = -\delta_{ik} P_j + \delta_{jk} P_i \tag{5.2.51}$$

$$[P_i, P_j] = -\kappa J_{ij} \tag{5.2.52}$$

Equation (5.2.51) states that the generators P_i transform as an n-vector under $\mathfrak{so}(n)$ rotations (reductivity) while (5.2.52) shows that for both signs $\kappa = \pm 1$ the considered coset manifold is a symmetric space. Correspondingly we name $\mathbf{k}_{ij} = k^{\ell}_{ij}(y)\frac{\partial}{\partial y^{\ell}}$ the Killing vector fields associated with the action of the generators J_{ij}:

$$J_{ij}\mathbb{L}_{\kappa}(\mathbf{y}) = \mathbf{k}_{ij}\mathbb{L}_{\kappa}(\mathbf{y}) + \mathbb{L}_{\kappa}(\mathbf{y}) J_{pq} W^{pq}_{ij}(y) \tag{5.2.53}$$

while we name $\mathbf{k}_i = k^{\ell}_i(y)\frac{\partial}{\partial y^{\ell}}$ the Killing vector fields associated with the action of the generators P_i:

$$P_i\mathbb{L}_{\kappa}(\mathbf{y}) = \mathbf{k}_i\mathbb{L}_{\kappa}(\mathbf{y}) + \mathbb{L}_{\kappa}(\mathbf{y}) J_{pq} W^{pq}_{i}(y) \tag{5.2.54}$$

Resolving conditions (5.2.53) and (5.2.54) we obtain:

$$\mathbf{k}_{ij} = y_i \partial_j - y_j \partial_i \tag{5.2.55}$$

$$\mathbf{k}_i = \frac{1}{2}\big(1 - \kappa \mathbf{y}^2\big)\partial_i + \kappa y_i \mathbf{y}\cdot\partial \tag{5.2.56}$$

The $\mathbb{H}$-compensators W^{pq}_i and W^{pq}_{ij} can also be extracted from the same calculation but since their explicit form is not essential for our future discussion we skip them.

5.2.3.2 Vielbeins, Connections and Metrics on G/H

Consider next the following 1-form defined over the reductive coset manifold G/H:

$$\Sigma(y) = \mathbb{L}^{-1}(y)\, d\mathbb{L}(y) \tag{5.2.57}$$

which generalizes the Maurer Cartan form defined over the group manifold G, discussed in Sect. 3.3.1.2 of Volume 1. As a consequence of its own definition the 1-form Σ satisfies the equation:

$$0 = d\Sigma + \Sigma \wedge \Sigma \tag{5.2.58}$$

which provides the clue to the entire (pseudo-)Riemannian geometry of the coset manifold. To work out this latter we start by decomposing Σ along a complete set of generators of the Lie algebra $\mathbb{G}$. According with the notations introduced in the previous subsection we put:

$$\Sigma = V^a T_a + \omega^i T_i \tag{5.2.59}$$

The set of $(n-m)$ 1-forms $V^a = V^a_\alpha(y)\,dy^\alpha$ provides a covariant frame for the cotangent bundle CT(G/H), namely a complete basis of sections of this vector bundle that transform in a proper way under the action of the group G. On the other hand $\omega = \omega^i T_i = \omega^i_\alpha(y)\,dy^\alpha T_i$ is called the $\mathbb{H}$-connection. Indeed, according to the theory exposed in Chap. 3 of Volume 1, ω turns out to be the 1-form of a bona-fide principal connection on the principal fibre bundle:

$$\mathscr{P}\left(\frac{\mathrm{G}}{\mathrm{H}}, \mathrm{H}\right) : \mathrm{G} \stackrel{\pi}{\to} \frac{\mathrm{G}}{\mathrm{H}} \tag{5.2.60}$$

which has the Lie group G as total space, the coset manifold G/H as base space and the closed Lie subgroup $\mathrm{H} \subset \mathrm{G}$ as structural group. The bundle $\mathscr{P}(\frac{\mathrm{G}}{\mathrm{H}}, \mathrm{H})$ is uniquely defined by the projection that associates to each group element $g \in \mathrm{G}$ the equivalence class $g\mathrm{H}$ it belongs to.

Introducing the decomposition (5.2.59) into the Maurer Cartan equation (5.2.58), this latter can be rewritten as the following pair of equations:

$$dV^a + C^a{}_{ib}\,\omega^i \wedge V^b = -\frac{1}{2} C^a{}_{bc}\, V^b \wedge V^c \tag{5.2.61}$$

$$d\omega^i + \frac{1}{2} C^i{}_{jk} \omega^j \wedge \omega^k = -\frac{1}{2} C^i{}_{bc}\, V^b \wedge V^c \tag{5.2.62}$$

where we have used the Lie algebra structure constants organized as in (5.2.30)–(5.2.32).

Let us now consider the transformations of the 1-forms we have introduced.

Under left multiplication by a constant group element $g \in \mathrm{G}$ the 1-form $\Sigma(y)$ transforms as follows:

$$\begin{aligned}\Sigma\big(y'\big) &= h(y,g)\mathbb{L}^{-1}(y) g^{-1}\, d\big(gL(y)h^{-1}\big)\\ &= h(y,g)^{-1}\Sigma(y) h(y,g) + h(y,g)^{-1}\, dh(y,g)\end{aligned} \tag{5.2.63}$$

where $y' = g.y$ is the new point in the manifold G/H whereto y is moved by the action of g. Projecting the above equation on the coset generators T_a we obtain:

$$V^a\big(y'\big) = V^b(y)\mathscr{D}_b{}^a\big(h(y,g)\big) \tag{5.2.64}$$

where $\mathscr{D} = \exp[\mathfrak{D}_{\mathbb{H}}]$, having denoted by $\mathfrak{D}_{\mathbb{H}}$ the $(n-m)$ dimensional representation of the subalgebra $\mathbb{H}$ which occurs in the decomposition of the adjoint representation of $\mathbb{G}$:

$$\operatorname{adj} \mathbb{G} = \underbrace{\operatorname{adj} \mathbb{H}}_{=\mathfrak{A}_{\mathbb{H}}} \oplus \mathfrak{D}_{\mathbb{H}} \tag{5.2.65}$$

Projecting on the other hand on the $\mathbb{H}$-subalgebra generators T_i we get:

$$\omega\big(y'\big) = \mathscr{A}\big[h(y,g)\big]\omega(y)\mathscr{A}^{-1}\big[h(y,g)\big] + \mathscr{A}\big[h(y,g)\big]d\mathscr{A}^{-1}\big[h(y,g)\big] \tag{5.2.66}$$

where we have set:

$$\mathscr{A} = \exp[\mathfrak{A}_{\mathbb{H}}] \tag{5.2.67}$$

Considering a complete basis T_A of generators for the full Lie algebra $\mathbb{G}$, the adjoint representation is defined as follows:

$$\forall g \in \mathrm{G}: \quad g^{-1} T_A g \equiv \operatorname{adj}(g)_A{}^B\, T_B \tag{5.2.68}$$

In the explicit basis of T_A generators the decomposition (5.2.65) means that, once restricted to the elements of the subgroup $\mathrm{H} \subset \mathrm{G}$, the adjoint representation becomes block-diagonal:

$$\forall h \in \mathrm{H}: \quad \operatorname{adj}(h) = \left(\begin{array}{c|c} \mathscr{D}(h) & 0 \\ \hline 0 & \mathscr{A}(h) \end{array}\right) \tag{5.2.69}$$

Note that for such decomposition to hold true the coset manifold has to be reductive according to definition (5.2.27).

The infinitesimal form of (5.2.64) is the following one:

$$V^a(y+\delta y) - V^a(y) = -\varepsilon^A W^i_A(y) C^a{}_{ib}\, V^b(y) \tag{5.2.70}$$

$$\delta y^\alpha = \varepsilon^A k^\alpha_A(y) \tag{5.2.71}$$

for a group element $g \in \mathrm{G}$ very close to the identity as in (5.2.33).

Similarly the infinitesimal form of (5.2.66) is:

$$\omega^i(y+\delta y) - \omega^i(y) = -\varepsilon^A \big(C^i{}_{kj}\, W^k_A \omega^j + dW^i_A\big) \tag{5.2.72}$$

5.2.3.3 Lie Derivatives

The Lie derivative of a tensor $T_{\alpha_1 \dots \alpha_p}$ along a vector field v^μ provides the change in shape of that tensor under an infinitesimal diffeomorphism:

$$y^\mu \mapsto y^\mu + v^\mu(y) \tag{5.2.73}$$

Explicitly one sets:

$$\ell_{\mathbf{v}} T_{\alpha_1 \ldots \alpha_p}(y) = v^{\mu} \partial_{\mu} T_{\alpha_1 \ldots \alpha_p} + \left(\partial_{\alpha_1} v^{\gamma}\right) T_{\gamma \alpha_2 \ldots \alpha_p} + \cdots + \left(\partial_{\alpha_p} v^{\gamma}\right) T_{\alpha_1 \alpha_2 \ldots \gamma} \quad (5.2.74)$$

In the case of p-forms, namely of antisymmetric tensors the definition (5.2.74) of Lie derivative can be recast into a more intrinsic form using both the exterior differential d and the *contraction operator*.

Definition 5.2.5 Let $\mathscr{M}$ be a differentiable manifold and let $\Lambda_k(\mathscr{M})$ be the vector bundles of differential k-forms on $\mathscr{M}$, let $\mathbf{v} \in \Gamma(T\mathscr{M}, \mathscr{M})$ be a vector field. The contraction $\mathrm{i}_{\mathbf{k}}$ is a linear map:

$$\mathrm{i}_{\mathbf{v}} : \Lambda_k(\mathscr{M}) \to \Lambda_{k-1}(\mathscr{M}) \quad (5.2.75)$$

such that for any $\omega^{(k)} \in \Lambda_k(\mathscr{M})$ and for any set of $k-1$ vector fields $\mathbf{w}_1, \ldots, \mathbf{w}_{k-1}$, we have:

$$\mathrm{i}_{\mathbf{v}} \omega^{(k)}(\mathbf{w}_1, \ldots, \mathbf{w}_{k-1}) \equiv k \omega^{(k)}(\mathbf{v}, \mathbf{w}_1, \ldots, \mathbf{w}_{k-1}) \quad (5.2.76)$$

Then by going to components we can verify that the tensor definition (5.2.74) is equivalent to the following one:

Definition 5.2.6 Let $\mathscr{M}$ be a differentiable manifold and let $\Lambda_k(\mathscr{M})$ be the vector bundles of differential k-forms on $\mathscr{M}$, let $\mathbf{v} \in \Gamma(T\mathscr{M}, \mathscr{M})$ be a vector field. The Lie derivative $\ell_{\mathbf{v}}$ is a linear map:

$$\ell_{\mathbf{v}} : \Lambda_k(\mathscr{M}) \to \Lambda_k(\mathscr{M}) \quad (5.2.77)$$

such that for any $\omega^{(k)} \in \Lambda_k(\mathscr{M})$ we have:

$$\ell_{\mathbf{v}} \omega^{(k)} \equiv \mathrm{i}_{\mathbf{v}} d\omega^{(k)} + d\mathrm{i}_{\mathbf{v}} \omega^{(k)} \quad (5.2.78)$$

On the other hand for vector fields the tensor definition (5.2.74) is equivalent to the following one.

Definition 5.2.7 Let $\mathscr{M}$ be a differentiable manifold and let $T\mathscr{M} \to \mathscr{M}$ be the tangent bundle, whose sections are the vector fields. Let $\mathbf{v} \in \Gamma(T\mathscr{M}, \mathscr{M})$ be a vector field. The Lie derivative $\ell_{\mathbf{v}}$ is a linear map:

$$\ell_{\mathbf{v}} : \Gamma(T\mathscr{M}, \mathscr{M}) \to \Gamma(T\mathscr{M}, \mathscr{M}) \quad (5.2.79)$$

such that for any $\mathbf{w} \in \Gamma(T\mathscr{M}, \mathscr{M})$ we have:

$$\ell_{\mathbf{v}} \mathbf{w} \equiv [\mathbf{v}, \mathbf{w}] \quad (5.2.80)$$

The most important properties of the Lie derivative, which immediately follow from its definition are the following ones:

$$\begin{aligned} [\ell_{\mathbf{v}}, d] &= 0 \\ [\ell_{\mathbf{v}}, \ell_{\mathbf{w}}] &= \ell_{[\mathbf{v},\mathbf{w}]} \end{aligned} \tag{5.2.81}$$

The first of the above equations states that the Lie derivative commutes with the exterior derivative. This is just a consequence of the invariance of the exterior algebra of k-forms with respect to diffeomorphisms. The second equation states on the other hand that the Lie derivative provides an explicit representation of the Lie algebra of vector fields on tensors.

The Lie derivatives along the Killing vectors of the frames V^a and of the $\mathbb{H}$-connection ω^i introduced in the previous subsection are:

$$\ell_{\mathbf{v}_A} V^a = W^i_A C^a{}_{ib}\, V^b \tag{5.2.82}$$

$$\ell_{\mathbf{v}_A} \omega^i = -\left(dW^i_A + C^i{}_{kj}\, W^k_A \omega^j\right) \tag{5.2.83}$$

This result can be interpreted by saying that, associated with every Killing vector $\mathbf{k}_A$ there is a an infinitesimal $\mathbb{H}$-gauge transformation:

$$\mathbf{W}_A = W^i_A(y) T_i \tag{5.2.84}$$

and that the Lie derivative of both V^a and ω^i along the Killing vectors is just such local gauge transformation pertaining to their respective geometrical type. The frame V^a is a section of an H-vector bundle and transforms as such, while ω^i is a connection and it transforms as a connection should do.

5.2.3.4 Invariant Metrics on Coset Manifolds

The result (5.2.82), (5.2.83) has a very important consequence which constitutes the fundamental motivation to consider coset manifolds. Indeed this result instructs us to construct G-invariant metrics on G/H, namely metrics that admit all the above discussed Killing vectors as generators of true isometries.

The argument is quite simple. We saw that the one-forms V^a transform as a linear representation $\mathfrak{D}_{\mathbb{H}}$ of the isotropy subalgebra $\mathbb{H}$ (and group H). Hence if τ_{ab} is a symmetric H-invariant constant two-tensor, by setting:

$$ds^2 = \tau_{ab} V^a \otimes V^b = \underbrace{\tau_{ab} V^a_\alpha(y) V^b_\beta(y)}_{g_{\alpha\beta}(y)}\, dy^\alpha \otimes dy^\beta \tag{5.2.85}$$

we obtain a metric for which all the above constructed Killing vectors are indeed Killing vectors, namely:

$$\begin{aligned}\ell_{\mathbf{k}_A}\, ds^2 &= \tau_{ab}\big(\ell_{\mathbf{k}_A} V^a \otimes V^b + V^a \otimes \ell_{\mathbf{k}_A} V^b\big) \\ &= \underbrace{\tau_{ab}\big(\big[\mathfrak{D}_{\mathbb{H}}(W_A)\big]^a{}_c\, \delta^b_d + \big[\mathfrak{D}_{\mathbb{H}}(W_A)\big]^b{}_c\, \delta^a_d\big)}_{=\,0 \text{ by invariance}} V^c \otimes V^d \\ &= 0\end{aligned} \tag{5.2.86, 5.2.87}$$

The key point, in order to utilize the above construction, is the decomposition of the representation $\mathfrak{D}_{\mathbb{H}}$ into irreducible representations. Typically, for most common cosets, $\mathfrak{D}_{\mathbb{H}}$ is already irreducible. In this case there is just one invariant H-tensor τ and the only free parameter in the definition of the metric (5.2.85) is an overall scale constant. Indeed if τ_{ab} is an invariant tensor, any multiple thereof $\tau'_{ab} = \lambda\tau_{ab}$ is also invariant. In the case $\mathfrak{D}_{\mathbb{H}}$ splits into $\mathfrak{r}$ irreducible representations:

$$\mathfrak{D}_{\mathbb{H}} = \left(\begin{array}{c|c|c|c|c} \mathfrak{D}_1 & 0 & \cdots & 0 & 0 \\ \hline 0 & \mathfrak{D}_2 & 0 & \cdots & 0 \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots \\ \hline 0 & \cdots & 0 & \mathfrak{D}_{\mathfrak{r}-1} & 0 \\ \hline 0 & 0 & \cdots & 0 & \mathfrak{D}_{\mathfrak{r}} \end{array}\right) \tag{5.2.88}$$

we have $\mathfrak{r}$ irreducible invariant tensors $\tau^{(i)}_{a_i b_i}$ in correspondence of such irreducible blocks and we can introduce $\mathfrak{r}$ independent scale factors:

$$\tau = \left(\begin{array}{c|c|c|c|c} \lambda_1\tau^{(1)} & 0 & \cdots & 0 & 0 \\ \hline 0 & \lambda_2\tau^{(2)} & 0 & \cdots & 0 \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots \\ \hline 0 & \cdots & 0 & \lambda_{p-1}\tau^{(p-1)} & 0 \\ \hline 0 & 0 & \cdots & 0 & \lambda_p\tau^{(p)} \end{array}\right) \tag{5.2.89}$$

Correspondingly we arrive at a continuous family of G-invariant metrics on G/H depending on $\mathfrak{r}$-parameters or, as it is customary to say in this context, of $\mathfrak{r}$ *moduli*. The number $\mathfrak{r}$ defined by (5.2.88) is named the *rank of the coset manifold* G/H.

In this section we confine ourself to the most common case of rank one cosets ($\mathfrak{r} = 1$), assuming, furthermore, that the algebras $\mathbb{G}$ and $\mathbb{H}$ are both semi-simple. By an appropriate choice of basis for the coset generators T^a, the invariant tensor τ_{ab} can always be reduced to the form:

$$\tau_{ab} = \eta_{ab} = \operatorname{diag}(\underbrace{+,+,\ldots,+}_{n_+}, \underbrace{-,-,\ldots,-}_{n_-}) \tag{5.2.90}$$

where the two numbers n_+ and n_- sum up to the dimension of the coset:

$$n_+ + n_- = \dim\frac{\mathrm{G}}{\mathrm{H}} = \dim\mathbb{K} \tag{5.2.91}$$

and provide the dimensions of the two eigenspaces, $\mathbb{K}_\pm \subset \mathbb{K}$, respectively corresponding to real and pure imaginary eigenvalues of the matrix $\mathfrak{D}_{\mathbb{H}}(W)$ which represents a generic element W of the isotropy subalgebra $\mathbb{H}$.

Focusing on our example (5.2.43), that encompasses both the spheres and the pseudo-spheres, depending on the sign of κ, we find that:

$$n_+ = 0; \qquad n_- = n \tag{5.2.92}$$

so that in both cases ($\kappa = \pm 1$) the invariant tensor is proportional to a Kronecker delta:

$$\eta_{ab} = \delta_{ab} \tag{5.2.93}$$

The reason is that the subalgebra $\mathbb{H}$ is the compact $\mathfrak{so}(n)$, hence the matrix $\mathfrak{D}_{\mathbb{H}}(W)$ is antisymmetric and all of its eigenvalues are purely imaginary.

If we consider cosets with non-compact isotropy groups, then the invariant tensor τ_{ab} develops a non-trivial Lorentzian signature η_{ab}. In any case, if we restrict ourselves to rank one cosets, the general form of the metric is:

$$ds^2 = \lambda^2 \eta_{ab} V^a \otimes V^b \tag{5.2.94}$$

where λ is a scale factor. This allows us to introduce the *vielbein*

$$E^a = \lambda V^a \tag{5.2.95}$$

and calculate the *spin connection* from the vanishing torsion equation:

$$0 = dE^a - \omega^{ab} \wedge E^c \eta_{bc} \tag{5.2.96}$$

Using the Maurer Cartan equations (5.2.61)–(5.2.62), (5.2.96) can be immediately solved by:

$$\omega^{ab}\eta_{bc} \equiv \omega^a{}_c = \frac{1}{2\lambda} C^a{}_{cd}\, E^d + C^a{}_{ci}\, \omega^i \tag{5.2.97}$$

Inserting this in the definition of the curvature two-form

$$\mathfrak{R}^a{}_b = d\omega^a{}_b - \omega^a{}_c \wedge \omega^c{}_b \tag{5.2.98}$$

allows to calculate the Riemann tensor defined by:

$$\mathfrak{R}^a{}_b = R^a{}_{bcd}\, E^c \wedge E^d \tag{5.2.99}$$

Using once again the Maurer Cartan equations (5.2.61)–(5.2.62), we obtain:

$$R^a{}_{bcd} = \frac{1}{\lambda^2}\left(-\frac{1}{4}\frac{1}{2\lambda} C^a{}_{be}\, C^e{}_{cd} - \frac{1}{8} C^a{}_{ec}\, C^e{}_{bd} + \frac{1}{8} C^a{}_{ed}\, C^e{}_{bc} - \frac{1}{2} C^a{}_{bi}\, C^i{}_{cd}\right) \tag{5.2.100}$$

which, as previously announced provides the expression of the Riemann tensor in terms of structure constants.

In the case of symmetric spaces $C^a{}_{be} = 0$ formula (5.2.100) simplifies to:

$$R^a{}_{bcd} = -\frac{1}{2\lambda^2} C^a{}_{bi}\, C^i{}_{cd} \tag{5.2.101}$$

5.2.3.5 For Spheres and Pseudo-Spheres

In order to illustrate the structures presented in the previous section we consider the explicit example of the spheres and pseudo-spheres. Applying the outlined procedure to this case we immediately get:

$$\begin{aligned} E^a &= -\frac{2}{\lambda} \frac{dy^a}{1+\kappa \mathbf{y}^2} \\ \omega^{ab} &= 2\frac{\kappa}{\lambda^2} E^a \wedge E^b \end{aligned} \tag{5.2.102}$$

This means that for spheres and pseudo-spheres the Riemann tensor is proportional to an antisymmetrized Kronecker delta:

$$R^{ab}{}_{cd} = \frac{\kappa}{\lambda^2} \delta^{[a}_{[c} \delta^{b]}_{d]} \tag{5.2.103}$$

5.3 Homogeneity Without Isotropy: What Might Happen

Having prepared the stage with our discussion of coset manifolds, among which group manifolds are a particular case, we enter the main issue of mathematical cosmology by utilizing the above developed techniques in order to construct space-time metrics that display homogeneity with the possible addition of full or partial isotropy. The goal is that of understanding how Einstein equations turn into differential equations for the free functions $f_i(t)$ of the time variable t that parameterize such homogeneous metrics and solve them if possible. The behavior of the solutions encodes the possible scenarios of cosmic evolutions.

It is quite important to understand that the two features advocated by the Cosmological Principle, namely homogeneity and isotropy are completely independent. For this reason, in this section, we present solutions of the Einstein equations based on homogeneous but not isotropic metrics. The corresponding cosmic evolution is very different from the overall expansion motivated by Hubble Law. Without isotropy the space-like sections of the Universe not only expand or contract, but also continuously deform during time-evolution.

5.3.1 Bianchi Spaces and Kasner Metrics

A very simple way to realize a four-dimensional cosmological metric which is homogeneous without any a priori enforcement of isotropy relies on the use of the

Maurer Cartan forms Ω^i of a three-dimensional Lie group G_3 satisfying the equation:

$$d\Omega^i = t^i{}_{jk}\,\Omega^j \wedge \Omega^k \tag{5.3.1}$$

where $t^i{}_{jk}$ are the structure constants of the corresponding Lie algebra $\mathbb{G}_3$. Assuming that the Ω^i are, for instance, left-invariant we have that:

$$\ell_{\mathbf{k}_I}\Omega^i = 0; \quad I = 1, 2, 3 \tag{5.3.2}$$

where the vector fields $\mathbf{k}_I$ are the infinitesimal generators of the left translations. Introducing the ansatz:

$$ds_G^2 = -dt^2 + A_{ij}(t)\Omega^i \otimes \Omega^j \tag{5.3.3}$$

where $A_{ij}(t)$ is a *time-dependent symmetric positive definite matrix* we obtain a metric which is Lorentzian and admits the three vector fields $\mathbf{k}_I$ as space-like translational Killing vectors. The group G_3 has a transitive action on the constant time sections of such a space-time, which therefore realizes a homogeneous but a priori not isotropic cosmology.

One can insert the ansatz (5.3.3) into the Einstein equations and look for solutions with various types of matters.

The very interesting point is that in 1898 the Italian geometer Luigi Bianchi, the same who is responsible for Bianchi identities, succeeded in obtaining a complete classification of all possible three-dimensional real Lie algebras [1]. The key argument utilized by Bianchi which exploits the peculiar features of three dimensions is the following. Given the structure constants $t^i{}_{jk}$ one can define the following new tensors:

$$A_k = t^i{}_{ik} \tag{5.3.4}$$

$$M^{\ell i} = \frac{1}{2}\varepsilon^{\ell jk}\left(t^i{}_{jk} - \delta^i_j A_k\right) \tag{5.3.5}$$

As a consequence of its definition the matrix $M^{\ell i} = M^{i\ell}$ is symmetric. In terms of these new objects, the vector A_k and the matrix $M^{\ell i}$, the Jacobi identities reduce to the very simple condition:

$$M^{\ell k}A_k = 0 \tag{5.3.6}$$

Hence the classification of all three-dimensional Lie algebras was reduced to the classification of solutions of (5.3.6). By means of rotations in the basis of generators the vector A_k can be oriented in a conventional direction, say the first, and the matrix $M^{\ell k}$ can be diagonalized. Using this liberty the form taken by the Maurer Cartan equations of the Bianchi algebras is the following one:

$$\begin{aligned} d\Omega^1 &= \lambda_1\Omega^2 \wedge \Omega^3 \\ d\Omega^2 &= \lambda_2\Omega^3 \wedge \Omega^1 - a\Omega^2 \wedge \Omega^1 \\ d\Omega^3 &= \lambda_3\Omega^1 \wedge \Omega^2 - a\Omega^3 \wedge \Omega^1 \end{aligned} \tag{5.3.7}$$

Table 5.1 The classification by Bianchi of three-dimensional Lie algebras

Bianchi type	a	λ_1	λ_2	λ_3	Identification
I	0	0	0	0	$\mathbb{R}^3$
II	0	1	0	0	Heisenberg algebra
III	1	0	1	-1	
IV	1	0	0	0	
V	1	0	0	0	$Solv(\mathfrak{so}(1,3)/\mathfrak{so}(3))$
VI_0	0	1	-1	0	$\mathfrak{iso}(1,1)$
VI_a	a	0	1	-1	
VII_0	0	1	1	0	$\mathfrak{iso}(2)$
VII_a	a	0	1	1	
VIII	0	1	1	-1	$\mathfrak{so}(1,2) \sim \mathfrak{sl}(2,\mathbb{R})$
IX	0	1	1	1	$\mathfrak{so}(3)$

The algebras VI_a and VII_a are actually two continuous families of solvable non-isomorphic Lie algebras, distinguished by the value of the parameter a. Some of the Bianchi algebras can be identified with other well known, simple or solvable Lie algebras. Bianchi IX and Bianchi VIII are simple and correspond to the two possible real sections, respectively compact and non-compact, of the unique complex Lie algebra A_1. Bianchi VII_0 is the Lie algebra of the Euclidian group of the plane $\mathbb{E}^2 \sim \mathfrak{iso}(2)$, while Bianchi VI_0 is the Poincaré Lie algebra in two dimensions $\mathfrak{iso}(1,1)$. Bianchi I is just the translation algebra $\mathbb{R}^3$, while Bianchi II is the Heisenberg algebra in two dimension. Finally according to a general theorem every non-compact simple coset manifold G/H, where H is a maximally compact subgroup, is metrically equivalent to a solvable group manifold $\exp Solv$ where the corresponding solvable Lie algebra $Solv$ can be constructed from the Lie algebra $\mathbb{G}$, according to a well defined procedure. Bianchi V is the solvable Lie algebra associated with the pseudo-sphere $\mathrm{SO}(1,3)/\mathrm{SO}(3)$.

and the various solutions are classified by specifying the four numbers $\lambda_{1,2,3}$ and a. In this way Bianchi obtained the list of eleven algebras displayed in Table 5.1. An interesting class of solutions of Einstein equations is obtained by using the Abelian algebras of Bianchi type I.

5.3.1.1 Bianchi Type I and Kasner Metrics

Within the general class of Bianchi I metrics that can be written as follows:

$$ds^2_{\text{Bianchi I}} = -dt^2 + A_{ij}(t)\, dx^i \otimes dx^j \tag{5.3.8}$$

we can consider the subclass of Kasner metrics [2], defined below:

$$ds^2_{\text{Kasner}} = -dt^2 + \sum_{i=1}^{3} t^{2p_i} \left(dx^i\right)^2 \tag{5.3.9}$$

where p_i are real exponents. The Vielbein description of the metric (5.3.9) is the following one:

$$E^0 = dt; \qquad E^i = t^{p_i}\, dx^i \tag{5.3.10}$$

which leads to the following spin connection

$$\omega^{0i} = \frac{1}{t} p_i E^i; \qquad \omega^{ij} = 0 \tag{5.3.11}$$

and curvature 2-form:

$$\mathfrak{R}^{0i} = \frac{1}{t^2} p_i (p_i - 1) E^0 \wedge E^i; \qquad \mathfrak{R}^{ij} = \frac{1}{t^2} p_i p_j E^i \wedge E^j \tag{5.3.12}$$

yielding the following Ricci tensor (in flat indices):

$$\begin{aligned}
\mathrm{Ric}_{00} &= \frac{1}{2t^2}\left(\sum_{j=1}^{3} p_j^2 - \sum_{j=1}^{3} p_j\right) \\
\mathrm{Ric}_{ii} &= \frac{1}{2t^2} p_i \left(1 - \sum_{j=1}^{3} p_j\right) \\
\mathrm{Ric}_{ab} &= 0 \quad \text{if } a \neq b
\end{aligned} \tag{5.3.13}$$

It follows that the Kasner metric is a solution of vacuum Einstein equations, namely it is Ricci flat whenever the exponent p_i satisfies the following two algebraic equations:

$$\sum_{j=1}^{3} p_j^2 = 1; \qquad \sum_{j=1}^{3} p_j = 1 \tag{5.3.14}$$

Geometrically the locus singled out by (5.3.14) is the intersection of a two-sphere with a plane and corresponds to a curve in three-dimensions that is displayed in Fig. 5.3. A parametric solution of equations is given below:

$$\left\{\begin{array}{c} p_1 \\ p_2 \\ p_3 \end{array}\right\} = \left\{\begin{array}{c} \frac{1}{2}(-\omega + \sqrt{-3\omega^2 + 2\omega + 1} + 1) \\ \frac{1}{2}(-\omega - \sqrt{-3\omega^2 + 2\omega + 1} + 1) \\ \omega \end{array}\right\}; \quad \omega \in \left[-\frac{1}{3}, 1\right] \tag{5.3.15}$$

Equation (5.3.15) provides just one branch of the overall solution. The other branches are obtained by applying the permutation group S_3 to it and altogether they fill the curve presented in Fig. 5.3.

Any point on this curve $\{p_1, p_2, p_3\}$ yields a possible vacuum solution of Einstein equations that is named a *Kasner epoch*. In such Kasner epochs the destiny of the various space-dimensions is very different: some contracts, other expands, since

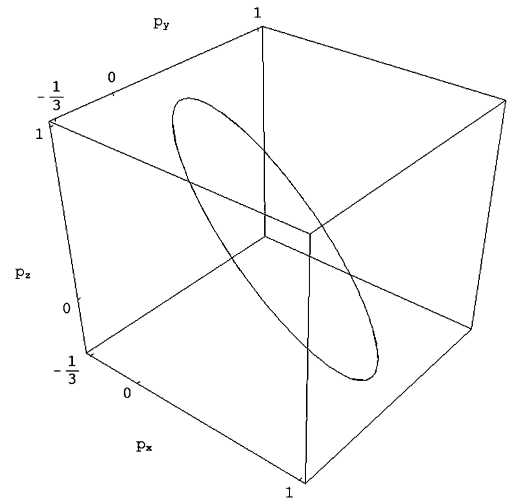

Fig. 5.3 The curve of Kasner exponents $\{p_x, p_y, p_z\}$ corresponding to Ricci flat metrics

the exponents p_i have typically different signs. For instance a nice rational solution of the Kasner constraints is provided by:

$$\{p_1, p_2, p_3\} = \left\{\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}\right\} \tag{5.3.16}$$

Let us now consider metrics of the following type:

$$ds_K^2 = -dt^2 + \sum_{i=1}^{3} a_i^2(t)\Omega_i^2 \tag{5.3.17}$$

where Ω_i are the Maurer Cartan forms of a Bianchi Lie algebra not necessarily of type I. One can draw a mechanical analogy by identifying:

$$h_i(t) = \log a_i(t) \tag{5.3.18}$$

with the coordinates of a fictitious ball that is moving in a three-dimensional space with velocity:[2]

$$v_i(t) = \frac{d}{dt}\log a_i(t) \tag{5.3.19}$$

Kasner epochs correspond to constant velocity trajectories.

A very interesting feature arising while discussing homogeneous non isotropic solutions of Einstein equations is that of *cosmic billiards*. These latter are exact solutions of matter coupled higher dimensional gravity where a succession of different Kasner epochs are glued together, one after the other, in a smooth but sharp way (see

[2]In higher dimensional gravity theories the ball moves in n-dimensions.

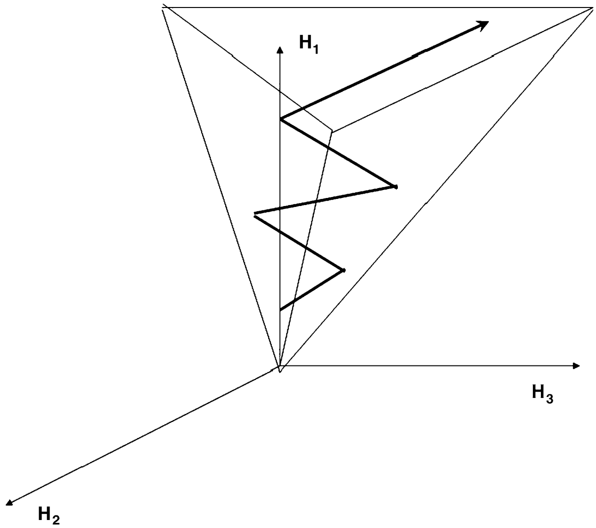

Fig. 5.4 The cosmic billiard mechanism envisages solutions of Einstein equations that include a series of Kasner epochs following each other as a result of a bounce on the walls of a billiard table. In higher dimensional supergravities the billiard table turns out to be the Weyl chamber of the duality Lie algebra pertaining to the considered model

Fig. 5.4). This provides a possible *new paradigm* for the interpretation of the extra-dimensions that occur in superstring based supergravity models. Not necessarily compact, such extra dimensions might be effectively small because depressed by decreasing (even exponentially decreasing) scale factors.

In this perspective the billiard mechanism implies that the effective dimensions of space-time might change with time. While entering a new Kasner phase, the Universe might acquire new dimensions which were previously contracting and now they might start expanding, while old expanding dimensions might contract and progressively disappear. The first idea of such a scenario was put forward by the Russian physicists Belinsky, Khalatnikov and Lifshitz in [3–6].

5.3.2 *A Toy Example of Cosmic Billiard with a Bianchi II Space-Time*

Here we do not present the very rich systematics of supergravity and gravity billiards, for which we refer the interested reader to some comprehensive research papers and lecture notes [7]. Our goal is just that of illustrating the main conception of the billiard mechanism by means of a simple toy model that we can realize in $d = 4$ space-time dimensions. The chosen toy model corresponds also to a Bianchi type of homogeneous but not isotropic universe which helps us to emphasize the role of isotropy in our subsequent discussion of the Standard Cosmological Model.

5.3.2.1 A Ricci Flat Bianchi II Metric

To begin with we study a particular cosmological metric which is Ricci flat and therefore corresponds to an empty Universe that, nonetheless, expands in some of its space-like directions. The Bianchi type II exact solutions presented in this and in the following subsection were derived a few years ago by this author with his collaborators Trigiante and Rulik in [8]. The metric is of the following form:

$$ds^2_{(d4)} = -A(t)\,dt^2 + \Lambda(t)\big(\Omega_2^2 + \Omega_3^2\big) + \Delta(t)\Omega_1^2 \tag{5.3.20}$$

where the three 1-forms Ω_i are explicitly given by:

$$\begin{aligned}\Omega_1 &= -dz + \frac{\omega r^2}{4} d\theta \\ \Omega_2 &= r\cos\theta\, d\theta + \sin\theta\, dr \\ \Omega_3 &= -\cos\theta\, dr + r\sin\theta\, d\theta\end{aligned} \tag{5.3.21}$$

and satisfy the following Cartan Maurer equations:

$$d\Omega_1 = \frac{\omega}{2}\Omega_2 \wedge \Omega_3$$

$$d\Omega_2 = 0 \tag{5.3.22}$$

$$d\Omega_3 = 0 \tag{5.3.23}$$

This means that the constant time sections of the space-time (5.3.20) are 3-dimensional homogeneous spaces, namely copies of a three dimensional group manifold $\mathscr{G}_\omega$ whose corresponding Lie algebra is the following *non-semisimple one*

$$\begin{aligned}&[T_i, T_j] = t_{ij}^k T_k \\ &\quad t_{23}^1 = \frac{\omega}{4}; \quad \text{all other components of } t_{ij}^k \text{ vanish}\end{aligned} \tag{5.3.24}$$

As it is the case for any group manifold, there exist on $\mathscr{G}_\omega$ two mutually commuting sets of vector fields that separately satisfy the Lie algebra of the group, namely the generators of left translations and the generators of right translations. Let us agree that the 1-forms (5.3.21) are left invariant. Then the triplet of vector fields that generate the left translations $\overrightarrow{k}_i$ will be such that they satisfy the Lie algebra (5.3.24) and the Lie derivative of the Ω_i along them vanishes.

$$[\overrightarrow{k}_i, \overrightarrow{k}_j] = t_{ij}^\ell\, \overrightarrow{k}_\ell \tag{5.3.25}$$

$$\ell_{\overrightarrow{k}_i} \Omega_j = 0 \tag{5.3.26}$$

The explicit form of such vector fields is the following one:

$$\begin{aligned}\overrightarrow{k}_1 &= \frac{\partial}{\partial z} \\ \overrightarrow{k}_2 &= \frac{2}{\sqrt{\omega}}\sin\theta\frac{\partial}{\partial r} + \frac{2}{\sqrt{\omega} r}\cos\theta\frac{\partial}{\partial\theta} + \frac{\sqrt{\omega}}{2} r\cos\theta\frac{\partial}{\partial z}\end{aligned} \tag{5.3.27}$$

$$\vec{k}_3 = -\frac{2}{\sqrt{\omega}} \cos\theta \frac{\partial}{\partial r} + \frac{2}{\sqrt{\omega} r} \sin\theta \frac{\partial}{\partial \theta} + \frac{\sqrt{\omega}}{2} r \sin\theta \frac{\partial}{\partial z}$$

and because of (5.3.26) they are Killing vectors of the 4-dimensional metric (5.3.20). These three are not the only Killing vectors. There is a fourth one generating O(2) rotations, namely:

$$\vec{k}_O = \frac{\partial}{\partial \theta} \tag{5.3.28}$$

The Lie derivatives of the 1-forms Ω_i are not all zero along $\vec{k}_O$, since we have:

$$\ell_{\vec{k}_O} \Omega_1 = 0; \qquad \ell_{\vec{k}_O} \Omega_2 = -\Omega_3; \qquad \ell_{\vec{k}_O} \Omega_3 = \Omega_2 \tag{5.3.29}$$

which means that under O(2) the three Ω_i arrange into a singlet and into a doublet. Yet $\vec{k}_O$ is a Killing vector for (5.3.20), since this metric is written in terms of O(2) invariants.

An alternative way of writing the metric (5.3.20) uses Cartesian coordinates, through the standard change of variables:

$$x = r\cos\theta; \qquad y = r\sin\theta \tag{5.3.30}$$

In these coordinates (5.3.20) reads:

$$ds^2_{(d4)} = -A(t)\,dt^2 + \Lambda(t)\big(dx^2 + dy^2\big) + \Delta(t)\left(dz + \frac{\omega}{4}(x\,dy - y\,dx)\right)^2 \tag{5.3.31}$$

and the four killing vectors take the very simple form:

$$\begin{aligned} \vec{k}_1 &= \frac{\partial}{\partial z} \\ \vec{k}_2 &= \frac{\partial}{\partial x} - \frac{\omega}{4} y \frac{\partial}{\partial z} \\ \vec{k}_3 &= \frac{\partial}{\partial y} + \frac{\omega}{4} x \frac{\partial}{\partial z} \\ \vec{k}_O &= -x\frac{\partial}{\partial y} + y\frac{\partial}{\partial x} \end{aligned} \tag{5.3.32}$$

which will be very useful in our subsequent discussion of geodesics.

5.3.3 Einstein Equation and Matter for This Billiard

Let us now study under which conditions the metric (5.3.20) is a solution of Einstein field equations. To this effect we use the vielbein formalism and we write the vierbein as follows:

$$E^0 = \sqrt{A(t)}\,dt; \qquad E^1 = \sqrt{\Delta(t)}\Omega_1; \qquad E^{2,3} = \sqrt{\Lambda(t)}\Omega^{2,3} \tag{5.3.33}$$

We can immediately calculate the spin connection from the vanishing torsion equation:

$$dE^A + \omega^{AB} \wedge E^C \eta_{BC} = 0 \tag{5.3.34}$$

where for the flat metric we have used the mostly plus convention:

$$\eta_{ab} = \mathrm{diag}\{-,+,+,+\} \tag{5.3.35}$$

We obtain the following result for the spin connection

$$\begin{aligned}
\omega^{01} &= \frac{\dot{\Delta}}{2\sqrt{A}\Delta}E^1; & \omega^{02} &= \frac{\dot{\Lambda}}{2\sqrt{A}\Lambda}E^2 \\
\omega^{03} &= \frac{\dot{\Lambda}}{2\sqrt{A}\Lambda}E^3; & \omega^{12} &= -\omega\frac{\dot{\Delta}}{4\Lambda}E^3 \\
\omega^{13} &= \omega\frac{\dot{\Delta}}{4\Lambda}E^2; & \omega^{23} &= \omega\frac{\dot{\Delta}}{4\Lambda}E^1
\end{aligned} \tag{5.3.36}$$

which can be used to calculate the curvature 2-form and the Ricci tensor from the standard formulae:

$$\begin{aligned}
R^{AB} &\equiv d\omega^{AB} + \omega^{AC} \wedge \omega^{DB}\eta_{CD} = R^{AB}{}_{CD}\, e^C \wedge e^D \\
\mathrm{Ric}_{FG} &= \eta_{FA} R^{AB}{}_{GB}
\end{aligned} \tag{5.3.37}$$

The Ricci tensor turns out to be diagonal and has the following eigenvalues:

$$\begin{aligned}
\mathrm{Ric}_{00} &= \frac{A'(t)\Delta'(t)}{8A(t)^2\Delta(t)} + \frac{\Delta'(t)^2}{8A(t)\Delta(t)^2} + \frac{A'(t)\Lambda'(t)}{4A(t)^2\Lambda(t)} + \frac{\Lambda'(t)^2}{4A(t)\Lambda(t)^2} \\
&\quad - \frac{\Delta''(t)}{4A(t)\Delta(t)} - \frac{\Lambda''(t)}{2A(t)\Lambda(t)} \\
\mathrm{Ric}_{11} &= \frac{\omega^2\Delta(t)}{16\Lambda(t)^2} - \frac{A'(t)\Delta'(t)}{8A(t)^2\Delta(t)} - \frac{\Delta'(t)^2}{8A(t)\Delta(t)^2} + \frac{\Delta'(t)\Lambda'(t)}{4A(t)\Delta(t)\Lambda(t)} + \frac{\Delta''(t)}{4A(t)\Delta(t)} \\
\mathrm{Ric}_{22} &= \mathrm{Ric}_{33} \\
\mathrm{Ric}_{33} &= \frac{-(\omega^2\Delta(t))}{16\Lambda(t)^2} - \frac{A'(t)\Lambda'(t)}{8A(t)^2\Lambda(t)} + \frac{\Delta'(t)\Lambda'(t)}{8A(t)\Delta(t)\Lambda(t)} + \frac{\Lambda''(t)}{4A(t)\Lambda(t)}
\end{aligned} \tag{5.3.38}$$

With little more effort we can calculate the Einstein tensor defined by:

$$\begin{aligned}
G_{AB} &= \mathrm{Ric}_{AB} - \tfrac{1}{2}\eta_{AB}R \\
R &= \eta^{FG}\mathrm{Ric}_{FG}
\end{aligned} \tag{5.3.39}$$

and we obtain a diagonal tensor with the following eigenvalues:

$$\begin{aligned}
G_{00} &= \frac{-(\omega^2 \Delta(t))}{32\Lambda(t)^2} + \frac{\Delta'(t)\Lambda'(t)}{4A(t)\Delta(t)\Lambda(t)} + \frac{\Lambda'(t)^2}{8A(t)\Lambda(t)^2} \\
G_{11} &= \frac{3\omega^2 \Delta(t)}{32\Lambda(t)^2} + \frac{A'(t)\Lambda'(t)}{4A(t)^2\Lambda(t)} + \frac{\Lambda'(t)^2}{8A(t)\Lambda(t)^2} - \frac{\Lambda''(t)}{2A(t)\Lambda(t)} \\
G_{22} &= G_{33} \\
G_{33} &= \frac{-(\omega^2 \Delta(t))}{32\Lambda(t)^2} + \frac{A'(t)\Delta'(t)}{8A(t)^2\Delta(t)} + \frac{\Delta'(t)^2}{8A(t)\Delta(t)^2} + \frac{A'(t)\Lambda'(t)}{8A(t)^2\Lambda(t)} \\
&\quad - \frac{\Delta'(t)\Lambda'(t)}{8A(t)\Delta(t)\Lambda(t)} + \frac{\Lambda'(t)^2}{8A(t)\Lambda(t)^2} - \frac{\Delta''(t)}{4A(t)\Delta(t)} - \frac{\Lambda''(t)}{4A(t)\Lambda(t)}
\end{aligned} \tag{5.3.40}$$

It is a remarkable fact that we can obtain an exact solution of the evolution equations in the absence of any matter content. What we get is an empty Ricci flat Universe with rather peculiar properties. Imposing that the Ricci tensor (5.3.38) vanishes (and hence also the Einstein tensor (5.3.40) does) we get differential equations for $\Lambda(t)$, $\Delta(t)$ and $A(t)$ that are exactly solved by the following choice of transcendental functions:

$$\begin{aligned}
A(t) &= \exp[t\omega]\cosh\left[\frac{t\omega}{2}\right] \\
\Lambda(t) &= \exp\left[\frac{t\omega}{2}\right]\cosh\left[\frac{t\omega}{2}\right] \\
\Delta(t) &= \frac{1}{\cosh[\frac{t\omega}{2}]}
\end{aligned} \tag{5.3.41}$$

In order to write the metric in a standard cosmological form we need to redefine the time variable by setting:

$$d\tau = \sqrt{A(t)}\,dt \tag{5.3.42}$$

so that in the new cosmic time (5.3.20) becomes:

$$ds^2_{(d4)} = d\tau^2 + \Lambda(\tau)\big(\Omega_2^2 + \Omega_3^2\big) + \Delta(\tau)\Omega_1^2 \tag{5.3.43}$$

Equation (5.3.42) can be exactly integrated in terms of hypergeometric functions. We obtain:

$$\tau(t) = \frac{2\sqrt{2}}{3\omega}\exp\left[\frac{t\omega}{4}\right]\left(\sqrt{1+\exp[\tau\omega]} + 2\,{}_2F_1\left[\frac{1}{4}, \frac{1}{2}, \frac{5}{4}, -\exp[t\omega]\right]\right) \tag{5.3.44}$$

Although inverting (5.3.44) is not analytically possible, yet it suffices to plot the behavior of the scale factors Λ and Δ as functions of the cosmic time τ. This behavior

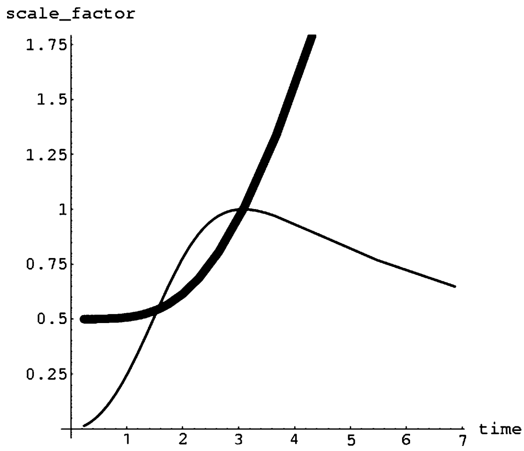

Fig. 5.5 Evolution of the cosmological scale factors $\Lambda(t)$ (*thick line*) and $\Delta(t)$ (*thin line*) for very early times, when the Universe is very young. Λ starts at a finite value 0.5 and always grows, while Δ starts at zero, grows for some time up to the maximum value 1 and then starts decreasing

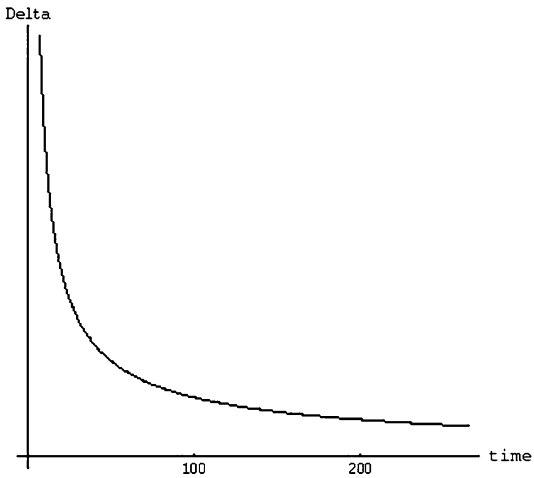

Fig. 5.6 Evolution of the cosmological scale factor $\Delta(t)$ for late times, when the Universe grows old. Δ tends exponentially to zero

is shown in several graphics. In Fig. 5.5 we see the behavior of the scale factors for very early times.

The early finite behavior of the scale factors has a very important consequence. This space-time has no initial singularity. Indeed for $\tau \mapsto 0$ the curvature 2-form is perfectly well behaved and tends to the following finite limit:

$$\begin{array}{lll} R^{01} = -\frac{1}{2}E^2 \wedge E^3; & R^{02} = -\frac{1}{4}E^1 \wedge E^3; & R^{03} = \frac{1}{4}E^1 \wedge E^2 \\ R^{12} = 0; & R^{13} = 0; & R^{23} = \frac{1}{2}E^0 \wedge E^1 \end{array} \tag{5.3.45}$$

In Fig. 5.6 we see the evolution of the Δ scale factor for late times just after reaching its maximum. As we see it rapidly and exponentially tends to zero.

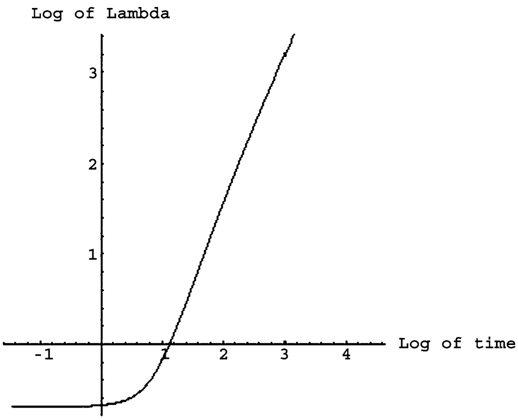

Fig. 5.7 Evolution of the cosmological scale factor $\Lambda(t)$ for very late times. By now Δ is essentially zero but Λ continues to grow and indefinitely in time with a power law. The graphic plots the logarithm of the scale factor against the logarithm of time and we see an almost perfect *straight line*

In Fig. 5.7 we see instead the very late time behavior of the scale factor Λ. At asymptotically late times this scale factor grows as a power of time which is slightly smaller than one.

We can summarize by saying that this funny homogeneous but not isotropic Universe, which is empty of matter, has a curious history. It has no initial singularity but it is born finite, small and essentially two-dimensional. It begins to expand and the third dimension starts to develop. It reaches a state when it is effectively three-dimensional, although still very small, the two scale factors being of equal size. Then the third dimension rapidly squeezes and the Universe becomes again effectively two dimensional growing monotonously large in the two dimensions in which it was born.

This is an example of the billiard mechanism. The effective dimensions of space-time change more than once in the course of the cosmic evolution. This is further illustrated in Fig. 5.8 where we show the motion of the fictitious cosmic ball whose coordinates are:

$$h_1(t) = h_2(t) = \frac{1}{2}\log\Lambda(t), \qquad h_3(t) = \frac{1}{2}\log\Delta(t) \tag{5.3.46}$$

It is evident from the figure that we have two Kasner epochs joined by a smooth bounce. For very early times $t \to -\infty$ and for $\omega > 0$ we have

$$h_i(t) \approx p_i t: \quad \{p_1, p_2, p_3\} = \left\{0, 0, \frac{\omega}{2}\right\} \tag{5.3.47}$$

while for very late times $t \to \infty$ and for $\omega > 0$ we find

$$h_i(t) \approx p_i t: \quad \{p_1, p_2, p_3\} = \left\{\frac{\omega}{2}, \frac{\omega}{2}, -\frac{\omega}{2}\right\} \tag{5.3.48}$$

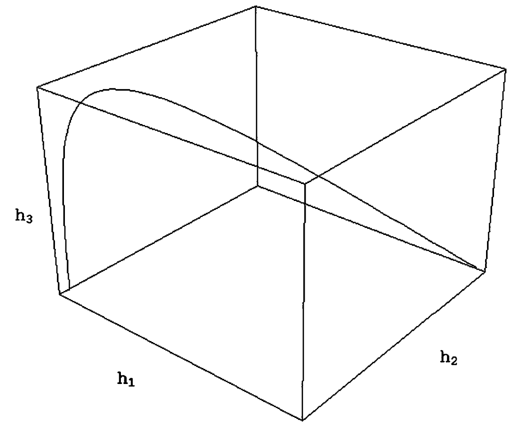

Fig. 5.8 Motion of the fictitious cosmic ball corresponding to the exact Ricci flat metric of Bianchi type II

5.3.4 The Same Billiard with Some Matter Content

We can find an exact solution of the Einstein equations for the above homogeneous but anisotropic Universe if we add some matter content. In order to write the Einstein differential equations in this case, we still need to consider the structure of the stress energy tensor. As usual, in curved indices this is given by:[3]

$$T^{\mu\nu} = \rho U^\mu U^\nu + p\big(U^\mu U^\nu - g^{\mu\nu}\big) \tag{5.3.49}$$

where ρ is the energy density, p the pressure and U^μ the four-velocity field of the fluid out of which we assume the Universe to be made of. In an isotropic and homogeneous Universe this fluid is assumed to be comoving. Namely, just as we did in the case of stellar equilibrium we assume that the velocity field is orthogonal to the *constant time slices of space-time* or equivalently that it has vanishing scalar product with all the six space-like Killing vectors:

$$(\overrightarrow{U}, \overrightarrow{k}) = 0 \tag{5.3.50}$$

In our chosen coordinate system this means $U = (1, 0, 0, 0)$. More intrinsically we can just state that in flat coordinates the stress energy tensor has the following diagonal form:

$$T_{AB} = \begin{pmatrix} \rho(t) & 0 & 0 & 0 \\ 0 & p(t) & 0 & 0 \\ 0 & 0 & p(t) & 0 \\ 0 & 0 & 0 & p(t) \end{pmatrix} \tag{5.3.51}$$

It is very instructive and of the outmost relevance to calculate the exterior covariant derivative of the above tensor using the spin connection as determined in (5.3.36).

[3]In the mostly minus conventions we have $ds^2 = g_{\mu\nu}\, dx^\mu \otimes dx^\nu$ and $g_{\mu\nu} U^\mu U^\nu = 1$.

We get:

$$\nabla T^{AB} = dT^{AB} + \omega^{AF} T^{GB} \eta_{FG} + \omega^{BF} T^{AF} \eta_{FG}$$
$$= \begin{pmatrix} E^0 \rho'(t) & \frac{E^1(p(t)+\rho(t))\Delta'(\mu)}{2\sqrt{A(\mu)}\Delta(\mu)} & \frac{E^2(p(t)+\rho(t))\Lambda'(\mu)}{2\sqrt{A(\mu)}\Lambda(\mu)} & \frac{E^3(p(t)+\rho(t))\Lambda'(\mu)}{2\sqrt{A(\mu)}\Lambda(\mu)} \\ \frac{E^1(p(t)+\rho(t))\Delta'(\mu)}{2\sqrt{A(\mu)}\Delta(\mu)} & E^1 p'(t) & 0 & 0 \\ \frac{E^2(p(t)+\rho(t))\Lambda'(\mu)}{2\sqrt{A(\mu)}\Lambda(\mu)} & 0 & E^2 p'(t) & 0 \\ \frac{E^3(p(t)+\rho(t))\Lambda'(\mu)}{2\sqrt{A(\mu)}\Lambda(\mu)} & 0 & 0 & E^3 p'(t) \end{pmatrix} \tag{5.3.52}$$

Then we can easily calculate the divergence of the stress-energy tensor, obtaining:

$$D_A T^{A0} = \frac{(p(t)+\rho(t))(\Lambda(\mu)\Delta'(\mu) + 2\Delta(\mu)\Lambda'(\mu))}{2\sqrt{A(\mu)}\Delta(\mu)\Lambda(\mu)} + \rho'(t) = 0 \tag{5.3.53}$$

$$D_A T^{Ai} = 0; \quad (i = 1, \ldots, 3) \tag{5.3.54}$$

Equation (5.3.53) is a conservation equation that can be easily integrated once one knows the equation of state, namely the relation between pressure and energy density:

$$p = f(\varrho) \tag{5.3.55}$$

The equation of state characterizes the type of fluid which is filling up the universe. In the present anisotropic case we are able to find an exact solution of Einstein field equations by using the equation of state of a free scalar field. This is the simple relation:

$$p = \rho \tag{5.3.56}$$

To see that this is the equation of state of a free scalar field, it suffices to calculate the stress energy tensor of such a field, assuming that it depends only on time. Anticipating the formula:

$$T^{(scal)}_{\mu\nu} = \frac{1}{2}\partial_\mu\phi\partial_\nu\phi - \frac{1}{4}g_{\mu\nu}\partial_\rho\phi\partial_\sigma\phi g^{\rho\sigma} \tag{5.3.57}$$

which we derive later in (5.8.4), with a cosmological metric of type $ds^2 = g_{00}\,dt^2 + g_{ij}\,dx^i\,dx^j$, we get:

$$T_{00} = \frac{1}{4}\dot{\phi}^2; \qquad T_{ij} = -\frac{1}{4}g_{ij}g^{00}\dot{\phi}^2 \tag{5.3.58}$$

and comparing this with (5.3.49) we identify:

$$\rho = \frac{1}{4}\dot{\phi}^2 g^{00}; \qquad p = \frac{1}{4}\dot{\phi}^2 g^{00} \tag{5.3.59}$$

This implies the equation of state (5.3.56). Substituting such a relation into the conservation equation (5.3.53) we obtain the following differential relation:

$$\frac{\rho(t)\Delta'(\mu)}{\sqrt{A(\mu)}\Delta(\mu)} + \frac{2\rho(t)\Lambda'(\mu)}{\sqrt{A(\mu)}\Lambda(\mu)} + \rho'(t) = 0 \tag{5.3.60}$$

which is immediately integrated to:

$$\rho(t) = \frac{\text{cost}}{\Lambda(t)^2\Delta(t)} \tag{5.3.61}$$

If we choose the following linear behavior of the scalar field:

$$\phi = \frac{1}{4}\kappa t \tag{5.3.62}$$

where κ is some constant and we choose the following scale factors,

$$\begin{aligned} A(t) &= e^{t\sqrt{\frac{\kappa^2}{3}+\omega}} \cosh\frac{t\omega}{2} \\ \Lambda(t) &= e^{\frac{1}{2}t\sqrt{\frac{\kappa^2}{3}+\omega}} \cosh\frac{t\omega}{2} \\ \Delta(t) &= \frac{1}{\cosh\frac{t\omega}{2}} \end{aligned} \tag{5.3.63}$$

by inserting into (5.3.59) we obtain:

$$\rho = \frac{\kappa^2}{64}\frac{1}{A(t)} = \frac{\kappa^2}{64} e^{-t\sqrt{\frac{\kappa^2}{3}+\omega}} \operatorname{sech}\frac{t\omega}{2}; \qquad p(t) = \rho(t) \tag{5.3.64}$$

Comparison with (5.3.61) shows that indeed the energy density in (5.3.64) is of the required form and obeys the conservation law, *i.e.* the field equation of the scalar field. On the other hand calculating the Einstein tensor, namely substituting (5.3.63) into (5.3.40) we get:

$$G_{00} = G_{11} = G_{22} = G_{33} = \frac{\kappa^2}{64} e^{-t\sqrt{\frac{\kappa^2}{3}+\omega}} \operatorname{sech}\frac{t\omega}{2} \tag{5.3.65}$$

and in this way we verify that Einstein equations are indeed satisfied.

We can now investigate the properties of this solution. First of all we reduce it to the standard form (5.3.43) as we did in the previous case. The procedure is the same, but now the cosmic time τ has a different analytic expression in terms of the original parametric time t. Indeed, substituting the new form of the scale function $A(t)$ as given in (5.3.63) into (5.3.42) we obtain the following definition of the cosmic time:

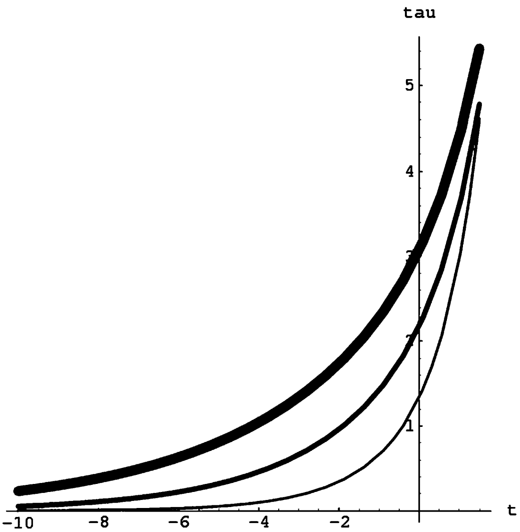

Fig. 5.9 The cosmic time τ versus the parameter t for various values of the parameter κ. The bigger κ the thinner the corresponding line. Here $\kappa = 0$ is the *thickest line*. The other two correspond to $\kappa = 1, 2$ respectively

$$\tau(t) = \frac{2(1+e^{t\omega})\,{}_2F_1\left(-(\frac{1}{4})+\frac{\sqrt{\frac{\kappa^2}{2}+\omega^2}}{2\omega}, -(\frac{1}{2}), \frac{3}{4}+\frac{\sqrt{\frac{\kappa^2}{2}+\omega^2}}{2\omega}, -e^{t\omega}\right)\sqrt{\frac{e^{t(-\omega+\sqrt{\frac{\kappa^2}{2}+\omega^2})}\operatorname{sech}(\frac{t\omega}{2})}{1+e^{t\omega}}}}{-\omega+2\sqrt{\frac{\kappa^2}{2}+\omega^2}} \tag{5.3.66}$$

A plot of the function $\tau(t)$ for various values of κ (see Fig. 5.9) shows that τ has always the same qualitative behavior. It tends to zero for $t \mapsto -\infty$ and it grows exponentially for $t \mapsto \infty$.

Hence we conclude that there is an initial time of this Universe at $\tau = 0$ and we can explore the initial conditions. In a completely different way from the previous vacuum solution, this Universe displays an initial singularity and has a Standard Big Bang behavior. The singularity can be seen in two ways. We can plot the energy density as given in (5.3.64) and realize that for all values of $\kappa \neq 0$ it diverges at the origin of time (see Fig. 5.10).

Alternatively, substituting the scale functions in the expression for the curvature 2-form, we can calculate its limit for $t \mapsto -\infty$ and we find that the intrinsic components diverge for all non-vanishing values of κ, while they are finite at $\kappa = 0$ which corresponds to the empty universe previously discussed.

Let us now analyze the behavior of the two scale factors $\Lambda(\tau)$ and $\Delta(\tau)$. This is displayed in Fig. 5.11. For late and intermediate times the behavior is just the same as in the vacuum solution with $\kappa = 0$, but the novelty is the behavior of Λ at the initial time. Rather than starting from a finite value as in the vacuum solution Λ starts at zero just as Δ. This is the cause of the initial singularity and the Standard Big Bang behavior. Further insight in the behavior of this solution is obtained by considering the evolution plots of the scale factor $\Lambda(\tau)$ for various values of κ, see Fig. 5.12. We can also look at the behavior of Δ which is plotted in Fig. 5.13.

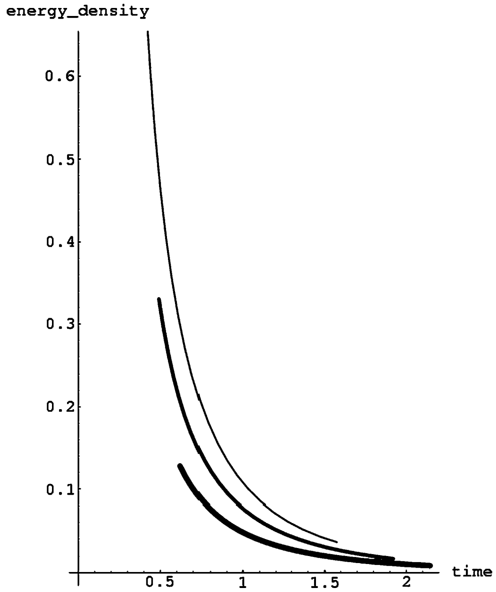

Fig. 5.10 The evolution of the energy density of the scalar field as function of the cosmic time, for various values of κ. The bigger κ, the thinner the corresponding line. Here $\kappa = 0.5$ is the *thickest line*. The other two correspond to $\kappa = 0.7$ and $\kappa = 1$, respectively

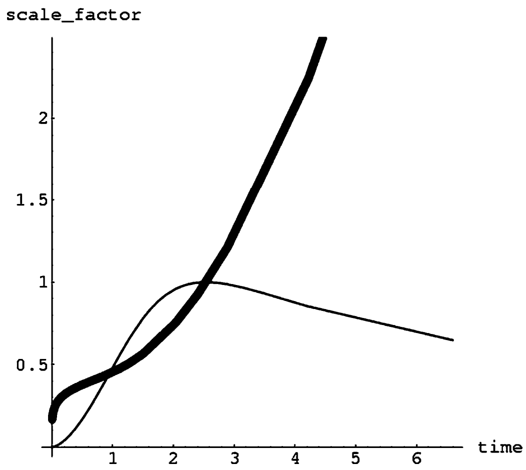

Fig. 5.11 The evolution of the two scale factors as function of the cosmic time τ in the dilaton gravity solution. The *thicker line* is Λ while the *thinner* one is Δ. The chosen value of the parameter kappa is $\kappa = 0.7$

5.3.5 Three-Space Geometry of This Toy Model

In order to better appreciate the structure of the cosmological solutions we have been considering in the previous subsection it is convenient to study the geometry of the *constant time sections* and the shape of its geodesics. At every instant of time we have the $3D$-metric:

$$ds^2_{3D} = \Lambda\left(dx^2 + dy^2\right) + \Delta\left[dz + \frac{\omega}{4}(x\,dy - y\,dx)\right]^2 \tag{5.3.67}$$

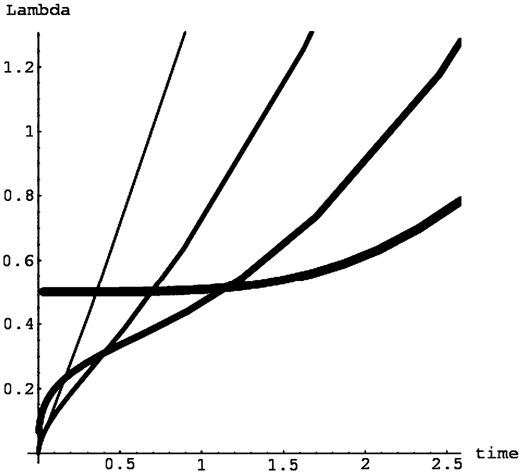

Fig. 5.12 The evolution of the Λ scale factor as function of the cosmic time τ in the dilaton gravity solution and for different values of kappa. The *thickest line* corresponds to $\kappa = 0$. The bigger κ, the thinner the line as in the other plots. Here we have $\kappa = 0, 1, 2, 4$. For all $\kappa \neq 0$, Λ begins at zero

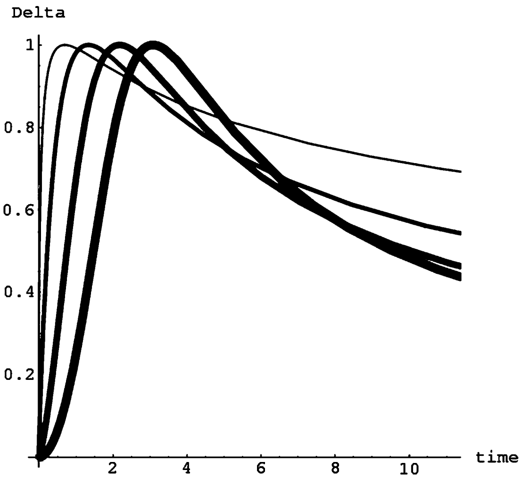

Fig. 5.13 The evolution of the Δ scale factor as function of the cosmic time τ in the dilaton gravity solution and for different values of kappa. The bigger κ, the thinner the line. Here we have $\kappa = 0, 1, 2, 4$. Δ has always the same behavior and increasing κ corresponds only to an anticipation of the peak

which admits the Killing vectors (5.3.27) as generators of isometries. As we explained several times, the scalar product of Killing vectors with the tangent vector to a geodesic is constant along the geodesic. Hence if λ is the affine parameter along a geodesic and $\overrightarrow{t} = \{x'[\lambda], y'[\lambda], z'[\lambda]\}$ is the tangent vector to the same, then we have the following four constants of motion:

$$
\begin{aligned}
A_1 &\equiv (\overrightarrow{k}_1, \overrightarrow{t}) = \frac{\Delta(-(\omega y(\lambda)x'(\lambda)) + \omega x(\lambda)y'(\lambda) + 4z'(\lambda))}{4} \\
A_2 &\equiv (\overrightarrow{k}_O, \overrightarrow{t}) = \frac{1}{16}\Big[\big(16\Lambda + \Delta\omega^2 x(\lambda)^2\big)y(\lambda)x'(\lambda) + \Delta\omega^2 y(\lambda)^3 x'(\lambda) \\
&\quad - \Delta\omega y(\lambda)^2\big(\omega x(\lambda)y'(\lambda) + 4z'(\lambda)\big) \\
&\quad - x(\lambda)\big(\big(16\Lambda + \Delta\omega^2 x(\lambda)^2\big)y'(\lambda) + 4\Delta\omega x(\lambda)z'(\lambda)\big)\Big] \qquad (5.3.68)
\end{aligned}
$$

$$A_3 \equiv (\overrightarrow{k}_2, \overrightarrow{t}) = \frac{(8\Lambda + \Delta\omega^2 y(\lambda)^2)x'(\lambda) - \Delta\omega y(\lambda)(\omega x(\lambda)y'(\lambda) + 4z'(\lambda))}{8}$$

$$A_4 \equiv (\overrightarrow{k}_3, \overrightarrow{t}) = \frac{8\Lambda y'(\lambda) + \Delta\omega^2 x(\lambda)^2 y'(\lambda) + \Delta\omega x(\lambda)(-(\omega y(\lambda)x'(\lambda)) + 4z'(\lambda))}{8}$$

Then the geodesics are characterized by the equations:

$$A_2 = \frac{-4A_4 x(\lambda) + 4A_3 y(\lambda) + \omega A_1(x(\lambda)^2 + y(\lambda)^2)}{4} \tag{5.3.69}$$

and

$$z'(\lambda) = \frac{8\Delta\omega A_2 + A_1(8\Lambda - \Delta\omega^2 x(\lambda)^2 - \Delta\omega^2 y(\lambda)^2)}{8\Delta\Lambda} \tag{5.3.70}$$

We also have:

$$\begin{aligned} x'(\lambda) &= \frac{2A_3 + \omega A_1 y(\lambda)}{2\Lambda} \\ y'(\lambda) &= \frac{2A_4 - \omega A_1 x(\lambda)}{2\Lambda} \end{aligned} \tag{5.3.71}$$

We conclude that the projection of all geodesics on the xy plane are circles with centers at:

$$(x_0, y_0) = \left(\frac{2A_4}{\omega A_1}, \frac{-2A_3}{\omega A_1}\right) \tag{5.3.72}$$

and radii:

$$R = 2\sqrt{\frac{\omega A_1 A_2 + {A_3}^2 + {A_4}^2}{\omega^2 {A_1}^2}} \tag{5.3.73}$$

and in terms of the new geometrically identified constants (5.3.70) becomes:

$$z'(\lambda) = \frac{A_1(8\Lambda + 2\Delta\omega^2(R^2 - {x_0}^2 - {y_0}^2) - \Delta\omega^2 x(\lambda)^2 - \Delta\omega^2 y(\lambda)^2)}{8\Delta\Lambda} \tag{5.3.74}$$

If we use a polar coordinate system in the xy-plane, namely if we write:

$$\begin{aligned} &x_0 = \rho\cos[\theta]; && y_0 = \rho\sin[\theta] \\ &x = \rho\cos[\theta] + R\cos\big[\phi(\lambda)\big]; && x = \rho\sin[\theta] + R\sin\big[\phi(\lambda)\big] \end{aligned} \tag{5.3.75}$$

where ρ and θ are constant parameters, we obtain that the derivative of the angle ϕ with respect to the affine parameter λ is just:

$$\frac{d\phi}{d\lambda} = -\frac{\omega A_1}{2\Lambda} \tag{5.3.76}$$

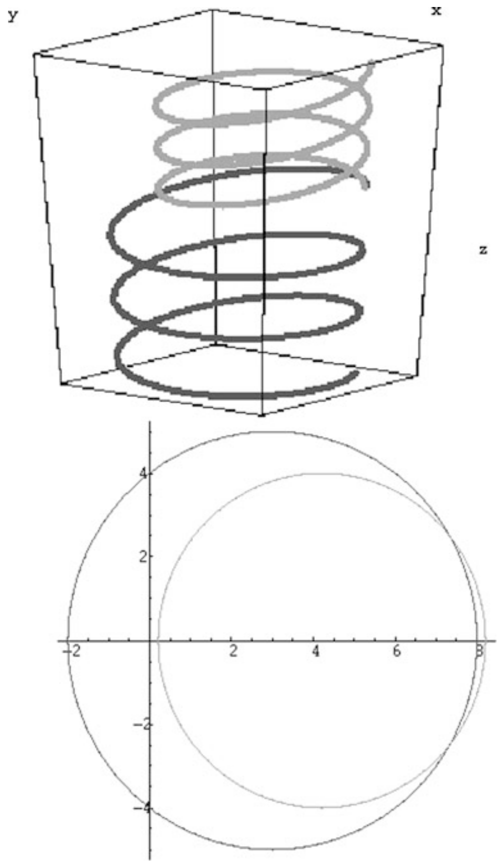

Fig. 5.14 In the *first picture* we see two geodesics in three space, while in the *second* we see their projection onto the plane xy

This means that ϕ itself, being linearly related to λ, is an affine parameter. On the other hand, the equation for the coordinate z, (5.3.70), becomes:

$$\frac{dz}{d\phi} = \frac{-(8\Lambda + \Delta(R^2 - 3\rho^2)\omega^2 - 2R\Delta\rho\omega^2 \cos(\theta - \varphi(\lambda)))}{4\Delta\omega} \tag{5.3.77}$$

which is immediately integrated and yields:

$$z[\phi] = \frac{(\theta - \varphi)(8\Lambda + \Delta(R^2 - 3\rho^2)\omega^2) - 2R\Delta\rho\omega^2 \sin(\theta - \varphi)}{4\Delta\omega} \tag{5.3.78}$$

Hence the possible geodesic curves in the three-dimensional sections of the cosmological solutions we have been discussing are described by (5.3.78) plus the second of (5.3.75). The family of such geodesics is parameterized by $\{R, \theta, \rho\}$, namely by the position of the center in the xy plane and by the radius. The shape of such geodesics is that of spirals (see Fig. 5.14).

A more illuminating visualization of this three-dimensional geometry is provided by the picture of a congruence of geodesics. Given a point in this $3D$ space, we can

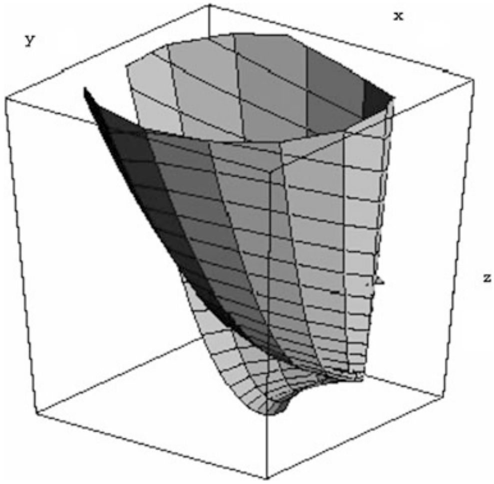

Fig. 5.15 In this picture we present a congruence of geodesics for the space with $\Lambda = \Delta = \omega = 1$. All the curves start from the same point and are distinguished by the value of the radius R in their circular projection onto the xy plane

consider all the geodesics that begin at that point and that have a radius R falling in some interval:

$$R_A < R < R_B \tag{5.3.79}$$

Following each of them for some amount of *parametric time* λ we generate a two dimensional surface. An example is given in Fig. 5.15.

The evolution of the Universe can now be illustrated by its effect on a congruence of geodesics. Chosen a congruence like in Fig. 5.15, the shape of the surface generated by such a congruence depends on the value of the scale parameters Λ and Δ. We can follow the evolution of the congruence while the Universe expands obtaining a movie.

Having illustrated the shape and the properties of the geodesics for the three dimensional sections of space-time we can now address the question of geodesics for the full space-time. To this effect we calculate first the three dimensional line element along the geodesics and we obtain the following result

$$\begin{aligned}
\frac{d\ell^2(t,\lambda)}{d\lambda^2} &\equiv \Lambda(t)\big[\dot{x}^2(\lambda)+\dot{y}^2(\lambda)\big]+\Delta(t)\left[\dot{z}(\lambda)+\frac{\omega}{4}\big(x(\lambda)\dot{y}(\lambda)-y(\lambda)\dot{x}(\lambda)\big)\right]^2 \\
&= \frac{(16R^2\Lambda(t)\omega^2+\frac{(-8\Lambda(t)+3\Delta(t)\rho^2\omega^2+3R\Delta(t)\rho\omega^2\cos(\theta-\varphi(\lambda)))^2}{\Delta(t)}){A_1}^2}{64\Lambda(t)^2} \\
&\equiv F^2(t,\phi)\left(\frac{d\phi}{d\lambda}\right)^2
\end{aligned} \tag{5.3.80}$$

In the last step of (5.3.80) we have introduced the notation:

$$F^2(t,\phi) = \frac{(16R^2\Lambda(t)\omega^2+\frac{(-8\Lambda(t)+3\Delta(t)\rho^2\omega^2+3R\Delta(t)\rho\omega^2\cos(\theta-\varphi(\lambda)))^2}{\Delta(t)})}{16\omega^2} \tag{5.3.81}$$

and we have used the relation (5.3.76).

Hence we obtain the complete space-time geodesics from those of three space by solving the following equation that relates the time coordinate t to the angular coordinate ϕ:

$$-A(t)\left(\frac{dt}{d\phi}\right)^2 + F^2(t,\phi) = k\frac{4}{\omega^2}\frac{A_1}{\Lambda^2(t)}; \quad \begin{cases} k=-1 & \text{time-like} \\ k=0 & \text{null-like} \\ k=1 & \text{space-like} \end{cases} \tag{5.3.82}$$

Furthermore, the constant A_1 is inessential and can always be fixed to 1 since it can be traded for the constant A_2 which does not appear in the equation. The differential (5.3.82) appears rather involved since $F^2(t,\phi)$ depends both on time and the angle ϕ. Yet we can take advantage of the homogeneous character of our space-time and simplify the problem very much. Indeed due to homogeneity it suffices to consider the geodesics whose projection in the xy plane is a circle centered at the origin and of radius R. All other geodesics with center in some point $\{x_0, y_0\}$ can be obtained from these ones by a suitable isometry that takes $\{0,0\}$ into $\{x_0, y_0\}$. So let us consider geodesics centered at the origin of the xy plane. This corresponds to setting $\rho = 0$. In this case we obtain:

$$F^2(t,\phi)|_{\rho=0} \equiv F_0^2(t) = \frac{\Lambda(t)(4\Lambda(t) + R^2\Delta(t)\omega^2)}{\Delta(t)\omega^2} \tag{5.3.83}$$

which depends only on time and the geodesic equations are reduced to quadratures since we get:

$$\int_0^{\phi_0} d\phi = \int_{-\infty}^{t_0} \frac{\sqrt{A(t)}}{\sqrt{F^2(t) - \frac{4k}{\omega^2\Lambda^2(t)}}}\, dt \tag{5.3.84}$$

The convergence or divergence of the second integral in (5.3.84) determines whether or not there are particle horizons in the considered cosmology. We will discuss the general concept of particle horizons for isotropic cosmologies later on. There the particle horizon appears as a spherical surface and is characterized by a radius. In non-isotropic cosmology as the present one, particle horizon may have a completely different much less intuitive shape. Curiously, in the above geometry horizons appear as an angular deficit. For each chosen radius R one can explore the geodesic (which is a spiral) only up to some maximal angle $\phi_{\max}$ at each chosen instant of time.

5.4 The Standard Cosmological Model: Isotropic and Homogeneous Metrics

Having analyzed the implications of homogeneity without the enforcement of complete isotropy, we turn to the Standard Cosmological Model, by assuming that the candidate cosmological metric is not only homogeneous but also isotropic, namely it admits a 6-parameter group $\mathscr{G}_6$ of isometries, generated by space-like Killing vectors. Furthermore isotropy means that $\mathscr{G}_6$ includes necessarily an SO(3) rotation subgroup. The classification of three-dimensional homogeneous spaces with such a $\mathscr{G}_6$ group of isometries reduces to the classification of three-dimensional Euclidian coset manifolds and we just have three possibilities:

$$\mathscr{M}_3 = \begin{cases} \frac{\mathrm{SO}(4)}{\mathrm{SO}(3)} \simeq S^3 \simeq \mathrm{SO}(3) & (\kappa = 1) \\ \frac{\mathrm{SO}(1,3)}{\mathrm{SO}(3)} \simeq Solv(\frac{\mathrm{SO}(1,3)}{\mathrm{SO}(3)}) & (\kappa = -1) \\ \frac{\mathrm{ISO}(3)}{\mathrm{SO}(3)} \simeq \mathbb{R}^3 & (\kappa = 0) \end{cases} \tag{5.4.1}$$

In the above formula we have emphasized the fact that the three selected coset manifolds, possessing the required isometry type, are metrically equivalent to three group-manifolds of dimension three and therefore fall into the Bianchi classification. In particular we have the following identifications at the level of Lie algebras:

$$\begin{aligned} \kappa = 1 \quad &\Leftrightarrow \quad \text{Bianchi Type IX} \\ \kappa = 0 \quad &\Leftrightarrow \quad \text{Bianchi Type I} \\ \kappa = -1 \quad &\Leftrightarrow \quad \text{Bianchi Type V} \end{aligned} \tag{5.4.2}$$

We can describe the geometries of these three spaces simultaneously with a single formula by writing the following $3D$-metric:

$$ds^2_{3D} = \frac{dr^2}{1-\kappa r^2} + r^2\big(d\theta^2 + \sin^2\theta\, d\phi^2\big) \tag{5.4.3}$$

We show below that when κ takes the two values ± 1 the metric (5.4.3) can be identified with the pull-back of the flat $\mathbb{R}^4$ metric on either the 3-sphere or the three-dimensional hyperboloid and hence just describes the SO(4) or SO(1, 3) invariant metrics on such coset manifolds, respectively. In the first case the variable r is actually compact and takes values in the range $[0, 2\pi]$. In the second case it is non compact and takes values in the infinite interval $[-\infty, +\infty]$. In the case $\kappa = 0$, the metric (5.4.3) is manifestly identical with the flat Euclidian metric in three dimension, written in polar coordinates and the variable r takes values in the semiinfinite interval $[0, +\infty]$. As such it admits the Euclidian group of isometries ISO(3).

The only parameter in (5.4.3) which is not fixed by isometries is the global scale of the three-dimensional space and this we can take to be time dependent: $a(t)$. Hence we can write the following ansatz for the isotropic and homogeneous cosmological metric

$$ds^2 = -dt^2 + a(t)^2\left[\frac{dr^2}{1-\kappa r^2} + r^2\big(d\theta^2 + \sin\theta^2\, d\phi^2\big)\right] \tag{5.4.4}$$

and correspondingly we introduce the following vierbein:

$$\begin{aligned} E^0 &= dt; & E^1 &= a(t)\tfrac{dr}{\sqrt{1-\kappa r^2}} \\ E^2 &= a(t) r\, d\theta; & E^3 &= a(t) r \sin\theta\, d\phi \end{aligned} \tag{5.4.5}$$

The calculation of the spin connection is immediate and we obtain:

$$\begin{aligned}
\omega^{01} &= \frac{\dot{a}(t)}{a(t)}E^1; & \omega^{02} &= \frac{\dot{a}(t)}{a(t)}E^2 \\
\omega^{03} &= \frac{\dot{a}(t)}{a(t)}E^3; & \omega^{12} &= -\frac{\sqrt{1-\kappa\rho^2}}{\rho a(t)}E^2 \\
\omega^{13} &= -\frac{\sqrt{1-\kappa\rho^2}}{\rho a(t)}E^3; & \omega^{23} &= -\frac{\cot(\theta)}{\rho a(t)}E^3
\end{aligned} \tag{5.4.6}$$

Next we evaluate the curvature 2-form:

$$\begin{aligned}
R^{01} &= \frac{\ddot{a}(t)}{a(t)}E^0 \wedge E^1; & R^{02} &= \frac{\ddot{a}(t)}{a(t)}E^0 \wedge E^2 \\
R^{03} &= \frac{\ddot{a}(t)}{a(t)}E^0 \wedge E^3; & R^{12} &= \frac{\kappa + \dot{a}(t)^2}{a(t)^2}E^1 \wedge E^2 \\
R^{13} &= \frac{\kappa + \dot{a}(t)^2}{a(t)^2}E^1 \wedge E^3; & R^{23} &= \frac{\kappa + \dot{a}(t)^2}{a(t)^2}E^2 \wedge E^3
\end{aligned} \tag{5.4.7}$$

which turns out to be diagonal, in the sense that $R^{AB} \propto E^A \wedge E^B$ but with different time-dependent eigenvalues.

Given these results we can evaluate the Ricci tensor and the Einstein tensor with flat indices defined by (5.3.37), (5.3.39). We get:

$$\begin{aligned}
G_{00} &= \frac{3}{2}\left[\left(\frac{\dot{a}(t)}{a(t)}\right)^2 + \frac{\kappa}{a(t)^2}\right] \\
G_{11} &= G_{22} = G_{33} \\
&= -\left[\frac{\ddot{a}(t)}{a(t)} + \frac{1}{2}\left(\frac{\dot{a}(t)}{a(t)}\right)^2 + \frac{1}{2}\frac{\kappa}{a(t)^2}\right]
\end{aligned} \tag{5.4.8}$$

In order to write the Einstein differential equations, we still need to consider the structure of the stress energy tensor. As usual, in curved indices this is given by (5.3.49) and in flat indices by (5.3.51). Analogously to (5.3.52) we can calculate the exterior derivative of (5.3.51) in the background metric (5.4.4) and we obtain:

$$\begin{aligned}
&\nabla T^{AB} = dT^{AB} + \omega^{AF}T^{GB}\eta_{FG} + \omega^{BF}T^{AF}\eta_{FG} \\
&= \begin{pmatrix}
E^0\rho'(t) & \frac{(p(t)+\rho(t))\dot{a}(t)}{a(t)}E^1 & \frac{(p(t)+\rho(t))\dot{a}(t)}{a(t)}E^2 & \frac{(p(t)+\rho(t))\dot{a}(t)}{a(t)}E^3 \\
\frac{(p(t)+\rho(t))\dot{a}(t)}{a(t)}E^1 & E^0\dot{p}(t) & 0 & 0 \\
\frac{(p(t)+\rho(t))\dot{a}(t)}{a(t)}E^2 & 0 & E^0\dot{p}(t) & 0 \\
\frac{(p(t)+\rho(t))\dot{a}(t)}{a(t)}E^3 & 0 & 0 & E^0\dot{p}(t)
\end{pmatrix}
\end{aligned} \tag{5.4.9}$$

from which we easily calculate the divergence whose vanishing provides a differential equation for the energy density:

$$\nabla_A T^{AB} = \left\{ \dot{\rho}(t) + 3\frac{\dot{a}(t)}{a(t)}\big(\rho(t) + p(t)\big), 0, 0, 0 \right\} = 0 \tag{5.4.10}$$

Having computed all the ingredients we can finally analyze the Einstein equations, that take the following form:

$$G_{AB} = 4\pi T_{AB} + \frac{1}{2}\eta_{AB}\Lambda \tag{5.4.11}$$

where Λ is a new constant originally introduced by Einstein and named by him the *cosmological constant*. It corresponds to the presence in the gravitational action of an additional term of the form $\int \Lambda\sqrt{-\det g}\, d^4x$ which is allowed by the principle of general covariance.

Inserting the results (5.4.8) and (5.3.51) into (5.4.11) we finally obtain the following two equations:

$$\begin{aligned} \left(\frac{\dot{a}}{a}\right)^2 &= \frac{8\pi G}{3}\rho - \frac{\kappa}{a^2} + \Lambda \\ \left(\frac{\ddot{a}}{a}\right) &= -\frac{4\pi G}{3}(\rho + 3p) \end{aligned} \tag{5.4.12}$$

that are currently known in the literature as *Friedman equations*.

Obviously these latter have to be supplemented with (5.4.10) which expresses the conservation of the stress energy tensor. It turns out that these three equations are not independent. This is just what should happen, since the Einstein tensor is conserved as a consequence of Bianchi identities. Indeed multiplying the first of (5.4.12) by a^2, taking a further derivative and combining it with the second we obtain the following result:

$$\dot{\varrho} + 3(\varrho + p)\frac{\dot{a}}{a} = 0 \tag{5.4.13}$$

which is nothing else but the already obtained conservation equation (5.4.10). So we can just focus on this latter equation and on the first of Friedman equations (5.4.12).

5.4.1 Viewing the Coset Manifolds as Group Manifolds

Before studying Friedman equations, and in order to better appreciate the role of isotropy versus homogeneity, we reconsider the statement made at the beginning of the present section, namely that each of the three coset manifolds mentioned in (5.4.1) can be also viewed as a group manifold and therefore that each of the three isotropic, cosmological metrics (5.4.4) admits an alternative description of the Bianchi type, namely:

$$ds^2_{[\kappa]} = -dt^2 + a(t)^2 \left[\sum_{i=1}^{3} \left(\Omega^i_{[\kappa]}\right)^2 \right] \tag{5.4.14}$$

where the 1-forms $\Omega^i_{[\kappa]}$ are left-invariant 1-forms satisfying the Maurer Cartan equations of three different appropriate Lie algebras:

$$d\Omega^i_{[\kappa]} = t^i_{[\kappa]|jk} \Omega^j_{[\kappa]} \wedge \Omega^k_{[\kappa]} \tag{5.4.15}$$

identified by their structure constants. Explicitly the appropriate algebras are:

$$\begin{aligned}
&\left.\begin{array}{c}\text{Bianchi}\\ \text{type IX}\end{array}\right\} \kappa = 1 \quad \Rightarrow \quad \begin{cases} d\Omega^1 = \Omega^2 \wedge \Omega^3 \\ d\Omega^2 = \Omega^3 \wedge \Omega^1 \\ d\Omega^3 = \Omega^1 \wedge \Omega^2 \end{cases} \\
&\left.\begin{array}{c}\text{Bianchi}\\ \text{type V}\end{array}\right\} \kappa = -1 \quad \Rightarrow \quad \begin{cases} d\Omega^1 = \Omega^1 \wedge \Omega^3 \\ d\Omega^2 = \Omega^2 \wedge \Omega^3 \\ d\Omega^3 = 0 \end{cases} \\
&\left.\begin{array}{c}\text{Bianchi}\\ \text{type I}\end{array}\right\} \kappa = 0 \quad \Rightarrow \quad \begin{cases} d\Omega^1 = 0 \\ d\Omega^2 = 0 \\ d\Omega^3 = 0 \end{cases}
\end{aligned} \tag{5.4.16}$$

From this point of view the candidate cosmological metric might have been much more general, *i.e.*

$$ds^2_{[\kappa]} = -dt^2 + \sum_{i,j} a_{ij}(t) \Omega^i_{[\kappa]} \otimes \Omega^j_{[\kappa]} \tag{5.4.17}$$

However such a metric as the above one has only three translational Killing vectors and describes a homogeneous but not isotropic universe. Isotropy follows only from the more restrictive $\mathfrak{so}(3)$ invariant choice:

$$a_{ij}(t) = a^2(t)\delta_{ij} \tag{5.4.18}$$

5.5 Friedman Equations for the Scale Factor and the Equation of State

In order to study the evolution of the cosmic scale factor we need to supplement the conservation equation (5.4.13) with an equation of state for the fluid filling up the universe:

$$p = f(\varrho) \tag{5.5.1}$$

Indeed, upon use of (5.5.1), (5.4.13) reduces to a first order differential equation for the energy density in terms of the scale factor. We shall consider two extreme cases of equations of state:

$$p = \begin{cases} 0 & \text{dust universe} \\ \frac{1}{3}\varrho & \text{radiation universe} \end{cases} \tag{5.5.2}$$

Combining (5.5.2) with (5.4.13) we immediately find:

$$\begin{cases} \varrho a^3 = \widehat{C}_d = \text{const}; & \text{dust universe} \\ \varrho a^4 = \widehat{C}_r = \text{const}; & \text{radiation universe} \end{cases} \tag{5.5.3}$$

Equations (5.5.3) are conservation laws and their physical interpretation will become clear through our discussion. For the dust case, its meaning should be apparent already at this stage. In a universe uniquely filled with matter, the energy density is, by definition:

$$\varrho_{matter} = \frac{\text{Total mass of the Universe}}{\text{Volume of the Universe}} \tag{5.5.4}$$

while, the volume of the Universe at cosmological time t can be identified with:

$$\text{Volume} = a(t)^3 \tag{5.5.5}$$

so that (5.5.3) states that the total mass of the universe is constant in time.

On the other hand for a universe filled with radiation, things are more subtle. The energy of a photon is:

$$E_{photon} = \hbar\nu \tag{5.5.6}$$

where ν denotes its frequency. Now assume that the frequency of a photon is *red-shifted* by the expansion according to the law:

$$\frac{\nu_{emission}}{\nu_{absorption}} = \frac{a(t_{absorption})}{a(t_{emission})} \tag{5.5.7}$$

it follows that the energy density of radiation at any cosmological time is:

$$\varrho_{radiation}(t) = \frac{\text{Number of photons}}{\text{Volume of the Universe}} \times \hbar\nu_{emission} \times \frac{1}{a(t)} \tag{5.5.8}$$

and the second of (5.5.3) is the statement that the total number of photons in the universe is approximately conserved. As we are going to see, the redshift law (5.5.7) is indeed true and a fundamental consequence of general relativity.

A realistic universe is neither pure dust nor pure radiation: it contains both components since there is both granular matter in the form of galaxies and radiation in the form of photons or other ultrarelativistic particles. Their relative contribution to Einstein equations, however, is different at different cosmological times since in an expanding or contracting universe the ratio of the energy densities is:

$$\frac{\varrho_{radiation}(t)}{\varrho_{matter}} = \text{const} \times \frac{1}{a(t)} \tag{5.5.9}$$

Consequently it makes sense to analyze the solution of the Einstein equations in the two idealized cases where either the radiation or the dust is present. The second solution applies to the present cosmological time when the Universe has already expanded so much that the radiation contribution has become irrelevant, while the first solution applies to early times when radiation was, because of (5.5.9), dominating. Indeed as we are presently going to see from our equations in both cases the behavior of the scale factor $a(t)$ is that of an increasing function of time, at least in a certain initial interval. Later, depending on the value of the curvature κ, the universe can also contract.

5.5.1 *Proof of the Cosmological Red-Shift*

The overall cosmological red-shift is a consequence of the homogeneity and isotropy of the universe. Let us proof this statement.

Consider the vierbein of a cosmological homogeneous and isotropic space time. It can be written in the form:

$$E^0 = dt; \qquad E^i = a(t)e^i \tag{5.5.10}$$

where e^i denote the vielbein of a three dimensional manifold admitting the transitive action of the symmetry group whose infinitesimal generators are represented by the Killing vectors $\mathbf{k}_I$. By definition we have:

$$\ell_{\mathbf{k}} e^i = W^{ij}_{\mathbf{k}} e^j \tag{5.5.11}$$

where the antisymmetric 3×3 matrix $W^{ij}_{\mathbf{k}}$ is the $\mathfrak{so}(3)$-compensator. Equation (5.5.11) implies that we also have:

$$\ell_{\mathbf{k}} E^0 = 0; \qquad \ell_{\mathbf{k}} E^i = W^{ij}_{\mathbf{k}} E^j \tag{5.5.12}$$

The Killing vectors $\mathbf{k}_I$ have purely space-like components. Correspondingly their squared norm is of the following form:

$$(\mathbf{k}, \mathbf{k}) = a^2(t) \underbrace{h_{ij} k^i k^j}_{= \langle \mathbf{k}, \mathbf{k} \rangle} \tag{5.5.13}$$

where $\langle \mathbf{k}, \mathbf{k} \rangle$ denotes the norm of the same Killing vector in the metric of the constant time sections and it is time independent.

It follows that the ratio of the Killing vector norms at different instant of time equals the corresponding ratio of the scale factors:

$$\frac{\sqrt{(\mathbf{k}, \mathbf{k})_{t_1}}}{\sqrt{(\mathbf{k}, \mathbf{k})_{t_2}}} = \frac{a(t_1)}{a(t_2)} \tag{5.5.14}$$

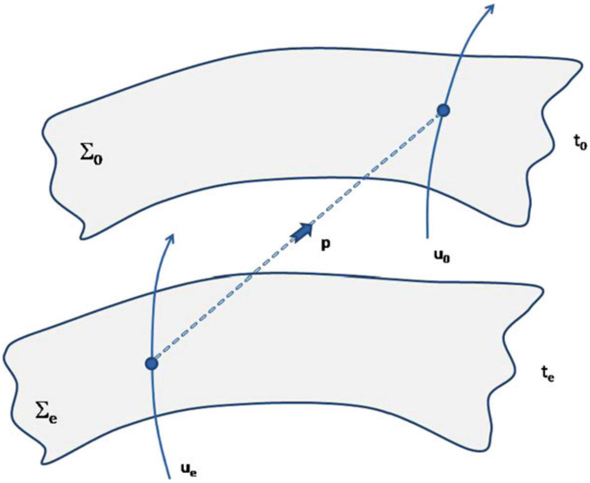

Fig. 5.16 At time t_e a source having quadri-velocity u_e^μ emits a photon of quadri-momentum k^μ that a the later time t_0 is absorbed by an observer having quadri-velocity u_0. Σ_e and Σ_0 are the constant time slices at the time of emission and of absorption. Both of them are Euclidian three-manifolds admitting the transitive action of the same translation group of isometries

Consider now the situation described in Fig. 5.16. At an early time t_e a source having quadri-velocity u_e^μ emits a photon of momentum p^μ which is later absorbed at time t_0 by an observer having four-velocity u_0^μ.

By definition the frequency of photon at emission and at absorption are:

$$\omega_e = p^\mu u_e^\nu g_{\mu\nu}; \qquad \omega_0 = p^\mu u_0^\nu g_{\mu\nu} \tag{5.5.15}$$

Since the photon is massless we always have that the time and space components of its four-momentum must be equal. On the other hand since the constant time slices of space-time admit the transitive action of a group of isometries, every direction in three space can always be viewed as aligned to a suitable translation space-like Killing vector k^ν. It follows from this argument that the frequency of the photon at the time of emission and of absorption can also be represented as follows:

$$\begin{aligned} \omega_e &= \frac{p^\mu k_\nu}{\sqrt{(\mathbf{k},\mathbf{k})_{t_e}}} \\ \omega_0 &= \frac{p^\mu k_\nu}{\sqrt{(\mathbf{k},\mathbf{k})_{t_0}}} \end{aligned} \tag{5.5.16}$$

Next we recall that the scalar product $p^\mu k_\nu$ where p^μ is tangent to a geodesic and k_ν is a Killing vector is constant along the geodesic. This implies that $p^\mu k_\nu$ will be the same at the emission and at the absorption time; consequently, in view of (5.5.14) we obtain:

$$\frac{\omega_e}{\omega_0} = \frac{a(t_0)}{a(t_e)} \tag{5.5.17}$$

which is the proof of the cosmological red-shift, already anticipated in previous pages. As we see the key point in the proof is that any direction taken by the three-momentum can be considered aligned to a Killing vector and this is true if our space-time is homogeneous. In addition we need the same scale factor $a(t)$ in all directions and this is the outcome of isotropy.

5.5.2 Solution of the Cosmological Differential Equations for Dust and Radiation Without a Cosmological Constant

If we substitute the first integral given by the conservation law (5.5.3) into the first of the differential equations (5.4.12) we get:

$$\begin{cases} 3(\frac{\dot{a}}{a})^2 = \frac{C_d}{a^3} - 3\frac{\kappa}{a^2}; & \text{dust} \\ 3(\frac{\dot{a}}{a})^2 = \frac{C_r}{a^4} - 3\frac{\kappa}{a^2}; & \text{radiation} \end{cases} \tag{5.5.18}$$

where we have defined:

$$C_{\substack{d\\r}} = \frac{8\pi G}{3} \widehat{C}_{\substack{d\\r}} \tag{5.5.19}$$

Equations (5.5.18) are easily reduced to quadratures obtaining:

$$\begin{cases} \frac{da}{dt} = \sqrt{\frac{C_d}{a} - \kappa}; & \text{dust} \\ \frac{da}{dt} = \sqrt{\frac{C_r}{a^2} - \kappa}; & \text{radiation} \end{cases} \tag{5.5.20}$$

The differential equation for the scale factor in the case of a radiation filled universe is immediately integrated and yields the following simple result:

$$a(t) = \begin{cases} \sqrt{2\sqrt{C_r}t - t^2} & \text{for } \kappa = 1 \\ \sqrt{t^2 - 2\sqrt{C_r}t} & \text{for } \kappa = -1 \\ \sqrt{2}\sqrt[4]{C_r}\sqrt{t} & \text{for } \kappa = 0 \end{cases} \tag{5.5.21}$$

where the integration constant has been fixed by means of the boundary condition $a(0) = 0$ (see Fig. 5.17).

As it is evident from the above analytic form the solution for a positively curved universe ($\kappa = 1$) makes sense only in the interval, $0 \leq t \leq 2\sqrt{C_r}$ where the function under square root is positive. Hence while for the open and flat universe ($\kappa \leq 0$), the scale factor grows indefinitely and the expansion never ceases, the closed universe undergoes an expansion phase followed by a contraction one which finally concentrates again all the radiation into a single point with a diverging energy density.

Although the analytic form of the solution is slightly different, the qualitative behavior of the scale factor follows exactly the same pattern also in the case of a dust filled universe, as we demonstrate in the following subsections, separately analyzing the three cases.

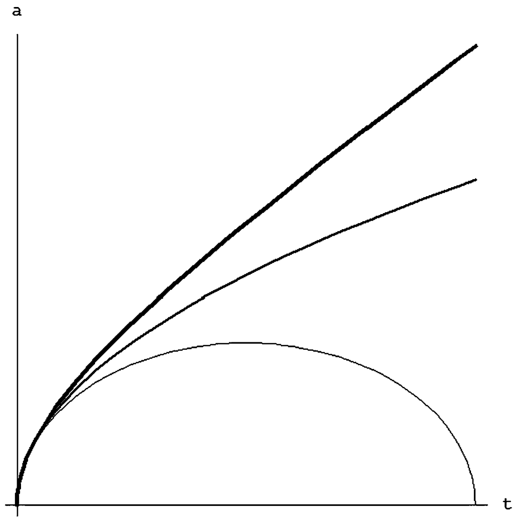

Fig. 5.17 Evolution of the cosmological scale factor $a(t)$ in the case of a radiation filled universe, for the three cases of positive ($\kappa = 1$), negative ($\kappa = -1$) and vanishing spatial curvature ($\kappa = 0$). The *thickest line* corresponds to the hyperbolic case ($\kappa = -1$) where, for late times, the scale factor grows asymptotically as $a \sim t$. The *medium thick line* corresponds to the flat case where the late time asymptotic behavior of the scale factor is $a \sim \sqrt{t}$. Finally the *thinnest line* correspond to the elliptic case ($\kappa = 1$), where the scale factor reaches a maximum and then contracts again to zero

5.5.2.1 Parametric Solution in the Dust Case of a Positively Curved Universe $\kappa = 1$

We solve the differential equation pertaining to this case by means of a suitable change of variables. We introduce the new variable η and we set:

$$a = \frac{1}{2} C_d (1 - \cos\eta) \tag{5.5.22}$$

Then we immediately get:

$$dt = \frac{da}{\sqrt{\frac{C_d}{a} - 1}} = \frac{1}{2} C_d (1 - \cos\eta)\, d\eta \tag{5.5.23}$$

and hence, by straightforward integration, we find the parametric solution for the curve describing the evolution of the scale factor in the plane t, a:

$$\begin{aligned} a(\eta) &= \frac{1}{2} C_d (1 - \cos\eta) \\ t(\eta) &= \frac{1}{2} C_d (\eta - \sin\eta) \end{aligned} \tag{5.5.24}$$

In Fig. 5.18 we show two instances of these evolutions. As one sees, in a positively curved universe, an initial expansion is always followed by a contraction phase.

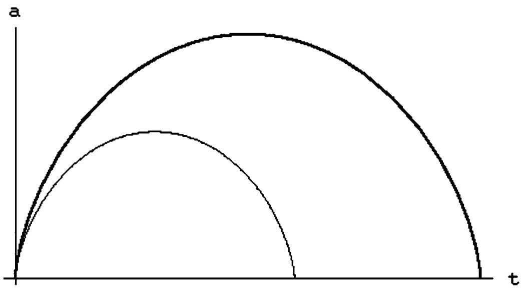

Fig. 5.18 Evolution of the cosmological scale factor $a(t)$ in the case of a closed ($\kappa = 1$) dust universe. The amplitude of the expansion, before the contraction sets on, depends on the total matter content of the universe codified by the constant C_d. In the figure we show two cases $C_d = 1$ (*thicker line*) and $C_d = 0.6$ (*thinner line*)

The amplitude of the expansion before the contraction depends on the total matter content of the universe.

5.5.2.2 Parametric Solution in the Dust Case of a Negatively Curved Universe $\kappa = -1$

The solution for the hyperbolic universe is obtained in a similar way. Rather than (5.5.22) we pose:

$$a = -\frac{1}{2} C_d (1 - \cosh \eta) \tag{5.5.25}$$

and, in complete analogy to the previous case, we obtain:

$$\begin{aligned} da &= \frac{1}{2} C_d \sinh \eta \, d\eta \\ \int dt &= \frac{1}{2} C_d (\sinh \eta - \eta) \end{aligned} \tag{5.5.26}$$

so that the parametric description of the scale factor evolution is the following one:

$$\begin{aligned} a(\eta) &= \frac{1}{2} C_d (\cosh \eta - 1) \\ t(\eta) &= \frac{1}{2} C_d (\sinh \eta - \eta) \end{aligned} \tag{5.5.27}$$

In this case the universe expands indefinitely and there is no contraction phase. Also here, the rate of the expansion depends on the total matter content of the universe: the bigger it is the faster the universe expands. Examples of this evolution are shown in Fig. 5.19.

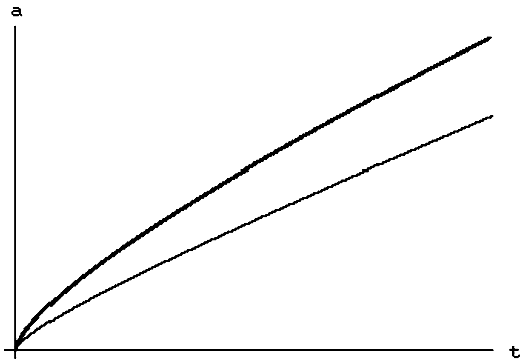

Fig. 5.19 Evolution of the cosmological scale factor $a(t)$ in the case of an open ($\kappa = -1$) dust universe. Also here the amplitude of the expansion depends on the total matter content of the universe codified by the constant C_d. The *thicker line* corresponds to the case $C_d = 1$ while the *thinner line* corresponds to the case $C_d = 0.2$

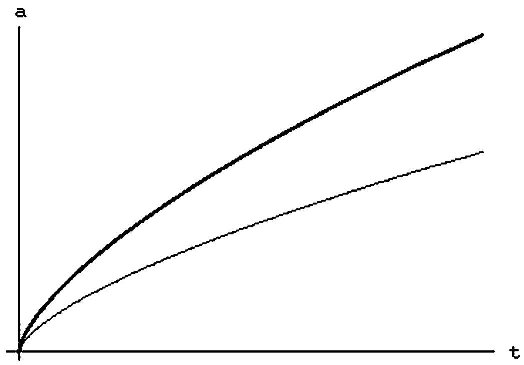

Fig. 5.20 Evolution of the cosmological scale factor $a(t)$ in the case of a flat ($\kappa = 0$) dust universe. Also here the amplitude of the expansion depends on the total matter content of the universe codified by the constant C_d. The *thicker line* corresponds to the case $C_d = 1$ while the *thinner line* corresponds to the case $C_d = 0.5$

5.5.2.3 Parametric Solution in the Dust Case of a Spatially Flat Universe $\kappa = 0$

In the case of zero spatial curvature, (5.5.20) reduce, for a dust universe, to:

$$\int dt = \frac{1}{\sqrt{C_d}} \int \sqrt{a}\, da \tag{5.5.28}$$

and hence we find:

$$a = \left(\frac{9C_d}{4}\right)^{2/3} t^{2/3} \tag{5.5.29}$$

We conclude that also the flat, dust filled, universe expands indefinitely and the scale factor raises as $a \sim t^{2/3}$ (see Fig. 5.20), to be compared with the weaker growth $a \sim t^{1/2}$ of the same flat universe when it is radiation dominated.

In Fig. 5.21 we have compared the three kinds of behavior of the cosmological scale factor for the positively, negatively curved and flat, dust filled universe.

As one sees the qualitative behavior is exactly the same as in the case of radiation.

Such behavior changes dramatically when we consider the case of universes with a positive space-time curvature, in particular the maximally symmetric de Sitter space.

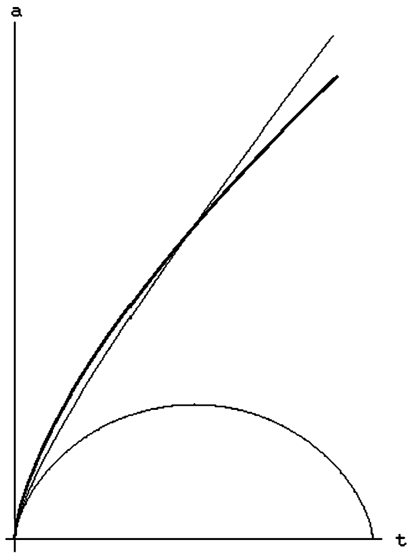

Fig. 5.21 Comparison between the three types of dust filled universes. With the same total matter content $C_d = 1$ we have plotted the behavior of the scale factor in the three cases of a closed, open and flat universe. The *line that goes back* to $a = 0$ is the closed universe $\kappa = 1$. Of the *two indefinitely growing lines* the *thinner* is the open universe $\kappa = -1$, the *thicker* is the flat universe $\kappa = 0$. The flat universe, initially expands faster than the open one, but at later times it is overcome by the open universe whose scale factor grows faster than $t^{2/3}$ for $t \to \infty$

Let us explain.

By assuming the cosmological principle, namely homogeneity and isotropy, we have imposed that the metric of space-time, at the scales of interest for cosmology, has a large symmetry, admitting six Killing vectors, three rotational ones closing the $\mathfrak{so}(3)$ Lie algebra and three translational ones. In the case of positive spatial curvature $\kappa = 1$ the six Killing vectors close the $\mathfrak{so}(4)$ Lie algebra, for negative curvature $\kappa = -1$ they close the Lorentz algebra $\mathfrak{so}(1,3)$, while for the flat universe they close the Lie algebra of the three dimensional Euclidian group $\mathbb{E}^3$.

Yet six is not the maximal number of Killing vectors that we can have in a four-dimensional manifold. The actual value of such maximal number is 10, namely the dimension of the Poincaré Lie algebra, $\mathfrak{iso}(1,3)$, but also of the Lie algebra $\mathfrak{so}(1,4)$ and $\mathfrak{so}(2,3)$. Indeed there are three maximally symmetric pseudo- Riemannian manifolds with Lorentzian signature that, respectively, admit the corresponding group of isometries, namely Minkowski space $Mink_4$, de Sitter space dS_4 and anti de Sitter space AdS_4. It follows that among the various isotropic and homogeneous universes, classified by the behavior of the scale factor $a(t)$, there should be special ones where the six-dimensional isometry algebra is promoted to a ten dimensional one. Clearly imposing the existence of extra Killing vectors puts differential constraints on the scale factor $a(t)$ which eventually will determine it uniquely.

In the next subsection we analyze in detail de Sitter space and we show that, in the framework of its geometry we cam embed all the three types of cosmological metrics ($\kappa = \pm 1, 0$), clearly with different forms of the scale factor $a(t)$. This might seem paradoxical, but it is not. The key point is that the choice of the time t in the three embeddings is different.

5.5.3 Embedding Cosmologies into de Sitter Space

The Lorentzian manifold dS, named de Sitter space is identified with the following coset manifold:

$$\mathrm{dS} = \frac{\mathrm{SO}(1,4)}{\mathrm{SO}(1,3)} \tag{5.5.30}$$

and therefore admits the 10-parameter group of isometries:

$$\mathrm{SO}(1,4) \tag{5.5.31}$$

The entire coset manifold can be identified as an algebraic locus in $\mathbb{R}^5$, namely as the set of points satisfying the following quadratic equation:

$$Y_0^2 - \sum_{i=1}^{4} Y_i^2 = -H_0^{-2} \tag{5.5.32}$$

where H_0 is a real number, whose physical interpretation will be that of Hubble constant.

Using rescaled variables $\overline{Y}^I = Y^I/H_0$, we see that de Sitter space corresponds to the manifold $\mathbb{H}_+^{(4,1)}$ in the language of Sect. 5.2.2. Here we introduce other coordinates for the coset manifold by solving explicitly the constraint (5.5.32), in various ways.

One parametric solution of the above algebraic equation is as follows:

$$\begin{aligned}
Y_0 &= H_0^{-1} \sinh H_0 t \\
Y_1 &= H_0^{-1} \cosh H_0 t \cos R \\
Y_2 &= H_0^{-1} \cosh H_0 t \sin R \cos\theta \\
Y_3 &= H_0^{-1} \cosh H_0 t \sin R \sin\theta \cos\phi \\
Y_4 &= H_0^{-1} \cosh H_0 t \sin R \sin\theta \sin\phi
\end{aligned} \tag{5.5.33}$$

and the pull-back of the Lorentzian metric in $\mathbb{R}^5$:

$$ds_5^2 = dY_0^2 - \sum_{i=1}^{4} dY_i^2 \tag{5.5.34}$$

on the locus (5.5.32) by means of the parameterization (5.5.33) leads to the following metric:

$$\begin{aligned}
ds^2_{\mathrm{dS}_+} &= -dt^2 + \frac{\cosh^2 H_0 t}{H_0^2}\left[dR^2 + \sin^2 R\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right] \\
&= -dt^2 + \left(\frac{\cosh H_0 t}{H_0}\right)^2 \left[\frac{dr^2}{1-r^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right]
\end{aligned} \tag{5.5.35}$$

where in the last step we have further posed:

$$\sin R = r \tag{5.5.36}$$

In this way we have shown that de-Sitter space can be identified with a cosmological model characterized by:

$$\kappa = 1; \qquad a(t) = \frac{\cosh H_0 t}{H_0} \tag{5.5.37}$$

The corresponding Hubble function and acceleration parameters are:

$$H(t) = H_0 \tanh(H_0 t) \tag{5.5.38}$$

$$\frac{\ddot{a}(t)}{a(t)} = H_0^2 \tag{5.5.39}$$

An alternatively equally good parametric solution of the quadric (5.5.32) is given by:

$$\begin{aligned} Y_0 &= H_0^{-1} \sinh H_0 t \cosh R \\ Y_1 &= H_0^{-1} \sinh H_0 t \sinh R \cos\theta \\ Y_2 &= H_0^{-1} \sinh H_0 t \sinh R \sin\theta \cos\phi \\ Y_3 &= H_0^{-1} \sinh H_0 t \sinh R \sin\theta \sin\phi \\ Y_4 &= H_0^{-1} \cosh t \end{aligned} \tag{5.5.40}$$

The pull back of the flat metric (5.5.34) on the locus (5.5.32) by means of this second parameterization (5.5.40) is:

$$\begin{aligned} ds^2_{\mathrm{dS}_-} &= -dt^2 + H_0^{-2} \sinh^2 H_0 t \left[dR^2 + \sinh^2 R \left(d\theta^2 + \sin^2\theta \, d\phi^2\right)\right] \\ &= -dt^2 + \left(\frac{\sinh H_0 t}{H_0}\right)^2 \left[\frac{dr^2}{1+r^2} + r^2\left(d\theta^2 + \sin^2\theta \, d\phi^2\right)\right] \end{aligned} \tag{5.5.41}$$

where in the last step we have further posed:

$$\sinh R = r \tag{5.5.42}$$

In this way we have shown that de-Sitter space can be also identified with a cosmological model characterized by:

$$\kappa = -1; \qquad a(t) = \frac{\sinh H_0 t}{H_0} \tag{5.5.43}$$

The corresponding Hubble function and acceleration parameter are:

$$H(t) = H_0 \frac{1}{\tanh(H_0 t)} \tag{5.5.44}$$

$$\frac{\ddot{a}(t)}{a(t)} = H_0^2 \tag{5.5.45}$$

A third possibility to obtain a parametric solution of the quadric (5.5.32) is the following one. First redefine

$$U = Y_0 - Y_4; \qquad V = Y_0 + Y_4 \tag{5.5.46}$$

and rewrite the quadric and the associated Lorentzian metric as follows:

$$UV - \sum_{i=1}^{3} Y_i^2 = -H_0^{-2} \tag{5.5.47}$$

$$ds_{(5)}^2 = -dU\,dV + \sum_{i=1}^{3} dY_i^2 \tag{5.5.48}$$

Then solve parametrically (5.5.47) as shown below:

$$\begin{aligned} U &= H_0^{-1}\rho \\ Y_i &= H_0^{-1}\rho x_i \\ V &= H_0^{-1}\left(-\frac{1}{\rho} + \rho\,\overrightarrow{x}^2\right) \end{aligned} \tag{5.5.49}$$

By means of this parameterization the pull-back of the Lorentz metric (5.5.48) on the locus (5.5.47) becomes:

$$\begin{aligned} ds_{\mathrm{dS}_0}^2 &= H_0^{-2}\left(-\frac{d\rho^2}{\rho^2} + \rho^2\,d\overrightarrow{x}^2\right) \\ &= -dt^2 + \left(\frac{\exp H_0 t}{H_0}\right)^2 \left[dr^2 + r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right] \end{aligned} \tag{5.5.50}$$

where in the last step we have set:

$$\begin{aligned} \rho &= \exp H_0 t \\ x_1 &= r\cos\theta \\ x_2 &= r\sin\theta\cos\phi \\ x_3 &= r\sin\theta\sin\phi \end{aligned} \tag{5.5.51}$$

In this way we have shown that de Sitter space can also be seen as a cosmological metric characterized by:

$$\kappa = 0; \qquad a(t) = \frac{\exp H_0 t}{H_0} \tag{5.5.52}$$

namely a flat Universe with an exponentially growing scale factor. The corresponding Hubble function and acceleration parameter are:

$$H(t) = H_0 \tag{5.5.53}$$

$$\frac{\ddot{a}(t)}{a(t)} = H_0^2 \tag{5.5.54}$$

The important lesson told by this analysis is that there are cases of cosmological homogeneous and isotropic metrics where, irrespectively from the sign of the spatial curvature, the universe expands indefinitely, even exponentially. The question is which kind of energy filling of the universe can yield such solutions, in particular de Sitter space. The answer will be *vacuum-energy*. To address such a question and similar ones we ought to consider the general properties and consequences of Friedman equations.

5.6 General Consequences of Friedman Equations

Let us reconsider the differential equation (5.4.13) and inspect its solution for a class of equations of state of the form

$$p = w\rho \tag{5.6.1}$$

where w is a constant coefficient. We already saw that $w = 0$ corresponds to baryonic matter (dust universe), while $w = \frac{1}{3}$ provides the equation of state of relativistic radiation. Other notable cases will be met soon.

Inserting (5.6.1) into (5.4.13) we get:

$$\frac{\dot{\rho}}{\rho} = -3(1+w)\frac{\dot{a}}{a} \tag{5.6.2}$$

which is immediately integrated to:

$$\frac{\rho}{\rho_0} = \left(\frac{a_0}{a}\right)^{3(1+w)} \tag{5.6.3}$$

where ρ_0 and a_0 are, respectively, the energy density and the scale factor at a reference instant of time, which we choose to be our own.

Evaluating the first of Friedman equations at current time, we obtain:

$$\kappa = a_0^2\left(\frac{8\pi G}{3}\rho_0 - H_0^2\right) \tag{5.6.4}$$

where it is proper to recall that H_0, the Hubble constant, is an experimentally evaluated parameter. It follows that the sign of the space curvature of the Universe depends on whether the present energy density is bigger, equal or less than the critical density defined by:

$$\rho_{crit} = \frac{3}{8\pi G} H_0^2 \tag{5.6.5}$$

In view of this and of (5.6.3), assuming that the energy filling of the Universe consists of various components:

$$\rho = \sum_{i=1}^{n} \rho_i \tag{5.6.6}$$

obeying the equation of state (5.6.1) with various values w_i of the proportionality parameter, the first of Friedman equations can be rewritten in the following inspiring form:

$$\left(\frac{H}{H_0}\right)^2 = \sum_{i=1}^{n} \Omega_0^i \left(\frac{a_0}{a}\right)^{3(1+w_i)} + \Omega_\kappa \left(\frac{a_0}{a}\right)^2 \tag{5.6.7}$$

where the so named dimensionless cosmological parameters have been defined as follows:

$$\Omega_0^i = \frac{\rho_0^i}{\rho_{crit}} \tag{5.6.8}$$

$$\Omega_\kappa = -\frac{\kappa}{H_0^2 a_0^2} \tag{5.6.9}$$

and as a consequence of (5.6.7) obey the consistency condition:

$$1 = \sum_{i=1}^{n} \Omega_0^i + \Omega_\kappa \tag{5.6.10}$$

The numbers Ω_0^i express the percentage contributed at the present time by the various components to the energy-filling of the Universe. It is interesting to note that the contribution of spatial curvature to the equation can be assimilated to that of a type of matter obeying the following equation of state:

$$p = -\rho \tag{5.6.11}$$

displaying a negative pressure.

By the same token, the second Friedman equation can be rewritten as follows:

$$q = \frac{1}{2} \sum_{i=1}^{n+1} (1 + 3w_i) \Omega_0^i \left(\frac{a_0}{a}\right)^{3(1+w_i)} \tag{5.6.12}$$

where the *deceleration function* is defined below:

$$q(t) = -\frac{\ddot{a}(t)}{a(t) H_0^2} \tag{5.6.13}$$

Evaluating (5.6.12) at the present time we obtain:

$$q_0 = \frac{1}{2}\sum_{i=1}^{n+1}(1+3w_i)\Omega_0^i \tag{5.6.14}$$

which is to be paired with (5.6.10).

Let us now consider the possible energy filling of the Universe at the present time. Radiation density decays very fast because of the $1/a^4$ law. Hence its contribution is certainly negligible and we can forget it. As for matter we can divide it into two parts:

(a) the visible *baryonic matter* composed of galaxies and their clusters, whose contribution we name Ω_0^B (the corresponding coefficient is $w_B = 0$),
(b) the invisible *dark matter* composed of possibly existing stable massive particles predicted in unified theories of particle interactions and/or by other non-radiating conventional matter filling galactic interstellar space, whose contribution we name Ω_0^D (the corresponding coefficient is $w_D = 0$).

In addition to that we envisage the possible presence of:

(c) vacuum energy, whose contribution we name Ω_0^Λ and whose defining equation of state is characterized by $w_\Lambda = -1$.

As we are going to see in next sections, the equation of state $p = -\rho$ describes the contribution to the overall stress-energy tensor of a cosmological constant or better of the potential energy of scalar fields. On the other hand, it is evident from (5.6.13) and (5.6.14) that an accelerating expansion of the universe ($\ddot{a} > 0$) is possible if and only if there are components of its energy filling that have $w < -\frac{1}{3}$ and if they are dominant.

With this assumption we obtain the following two equations for the four cosmological parameters Ω_0^B, Ω_0^D, Ω_0^Λ and Ω_0^κ:

$$q_0 = \frac{1}{2}\left(\Omega_0^B + \Omega_0^D\right) - \Omega_0^\Lambda \tag{5.6.15}$$

$$\Omega^\kappa = 1 - \underbrace{\left(\Omega_0^B + \Omega_0^D + \Omega_0^\Lambda\right)}_{\Omega_0} \tag{5.6.16}$$

Up to the end of the second millennium the only known parameter was Ω_0^B estimated to be $\Omega_0^B \simeq 0.06$ by the observation and counting of galaxies. After 1999, the measure of the deceleration parameter q_0 and the discovery that it is negative (the universe actually accelerates) revealed that $\Omega_0^\Lambda > 0$ and provided a constraint on the remaining parameters that could be completely solved when, both from the observation of the anisotropies in the Cosmic Background Radiation and from the supernova project, it was established that $\Omega_0 \sim 1$, namely that our universe is spatially flat.

With such an information we obtain:

$$\begin{aligned} \Omega_0^\Lambda &= \frac{1-2q_0}{3} \cong 0.72 \\ \Omega_0^D &= \frac{2}{3}(1+q_0) - \Omega_0^B \cong 0.22 \\ \Omega_0^B &\cong 0.06 \end{aligned} \tag{5.6.17}$$

where the numerical evaluation depends on the experimental result for q_0, whose determination was the motivation for the award of the 2011 Nobel Prize in Physics.

Let us now reconsider the general form of the Friedman Lemaitre Robertson Walker metric (5.4.4). Introducing the following functions:

$$R_\kappa^2(\chi) = \begin{cases} R_1^2(\chi) = \sin^2\chi \\ R_0^2(\chi) = \chi^2 \\ R_{-1}^2(\chi) = \sinh^2\chi \end{cases} \tag{5.6.18}$$

and denoting the volume element of the two-sphere by

$$d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2 \tag{5.6.19}$$

the metric (5.4.4) can be rewritten as follows:

$$ds^2 = -dt^2 + a^2(t)\big[d\chi^2 + R_\kappa^2(\chi)\, d\Omega^2\big] \tag{5.6.20}$$

where we have performed the coordinate change $r = R_\kappa(\chi)$. It is also convenient to introduce a further coordinate change to the so named conformal time:

$$dt = a(\eta)\, d\eta \tag{5.6.21}$$

upon which (5.6.20) transforms into:

$$ds^2 = a^2(\eta)\big(-d\eta^2 + d\chi^2 + R_\kappa^2(\chi)\, d\Omega^2\big) \tag{5.6.22}$$

Using such coordinates the radial light-like geodesics are very easily characterized by the following equation:

$$0 = -d\eta^2 + d\chi^2 \tag{5.6.23}$$

which is immediately integrated to:

$$\chi(\eta) = \pm\eta + \chi_0 \tag{5.6.24}$$

Relying on this result we can now introduce the concepts of *particle and event horizons*.

5.6.1 Particle Horizon

The concept of particle horizon arises from the finite age of the universe and it is the correct mathematical formulation, in terms of General Relativity, of the brilliant intuitions of Olbers (see Sect. 4.3.1). In a finite time, light can travel only finite distances. Hence the volume of space from which we can receive information at any given time is limited by a maximum radial distance. Naming η_i the date of birth of the Universe in the conformal coordinate system, the maximal observable distance at time $\eta \leftrightarrow t$ corresponds to the coordinate given below

$$\chi_p(t) = \eta - \eta_i = \int_{t_i}^{t} \frac{dt}{a(t)} \tag{5.6.25}$$

and in physical units is the following one:

$$d_p(t) = a(t) R_\kappa \big(\chi_p(t)\big) \tag{5.6.26}$$

We name it the particle horizon.

We are interested in the particle horizon at the current time and therefore we set:

$$\begin{aligned} d_p &\equiv d_p(t_0) = a_0 R_\kappa(\chi_p) \\ \chi_p &= \int_{t_i}^{t_0} \frac{1}{a(t)}\, dt \end{aligned} \tag{5.6.27}$$

By means of a change of variable we can now rewrite the limiting coordinate χ_p as follows:

$$\chi_p = \int_0^{a_0} \frac{1}{a^2 H(a)}\, da \tag{5.6.28}$$

where we identified the initial time t_i as the moment when the scale factor vanished $a(t_i) = 0$. Furthermore in (5.6.28) the dependence of the Hubble function from the scale factor a is given by the first Friedman equation as written in (5.6.7). For a matter dominated Universe ($w = 0$), we get:

$$\chi_p = \int_0^{a_0} \frac{1}{a_0 H_0 \sqrt{a(a + (a_0 - a)\Omega_0)}}\, da = \begin{cases} \dfrac{2 \operatorname{arcsin h}(\sqrt{\frac{1}{\Omega_0} - 1})}{a_0 H_0 \sqrt{1-\Omega_0}} & \text{for } \Omega_0 < 1 \\ \dfrac{2}{a_0 H_0} & \text{for } \Omega_0 = 1 \\ \dfrac{2 \arcsin(\sqrt{1 - \frac{1}{\Omega_0}})}{a_0 H_0 \sqrt{\Omega_0 - 1}} & \text{for } \Omega_0 > 1 \end{cases} \tag{5.6.29}$$

and we reach the following conclusion:

$$d_p = \frac{2}{H_0 \Omega_0} \mathfrak{f}_\kappa(d_0, \Omega_0); \quad d_0 = a_0 H_0 \tag{5.6.30}$$

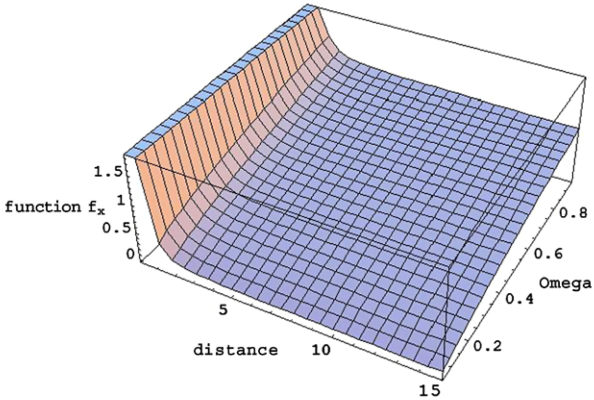

Fig. 5.22 Plot of the function $\mathfrak{f}_{-1}(d_0, \Omega_0)$ showing that it is bounded and of order unity

where the functions $\mathfrak{f}_\kappa(d_0, \Omega_0)$ depend on the sign of the spatial curvature and are listed below

$$\mathfrak{f}_\kappa(d_0, \Omega_0) = \begin{cases} \mathfrak{f}_{-1}(d_0, \Omega_0) = \frac{1}{2}\sinh\left(\frac{2\sinh^{-1}(\sqrt{\frac{1-\Omega_0}{\Omega_0}})}{d_0\sqrt{1-\Omega_0}}\right) d_0\Omega_0 \\ \mathfrak{f}_0(d_0, \Omega_0) = 1 \\ \mathfrak{f}_1(d_0, \Omega_0) = \frac{1}{2}\sin\left(\frac{2\sin^{-1}(\sqrt{\frac{\Omega_0-1}{\Omega_0}})}{d_0\sqrt{\Omega_0-1}}\right) d_0\Omega_0 \end{cases} \tag{5.6.31}$$

The relevant point of the above calculation is that the three functions $\mathfrak{f}_\kappa(d_0, \Omega_0)$ are all bounded and of order unity in the range where they are physically significant. For the case of the spatially flat Universe ($\kappa = 0$) this is obvious. In the case of an open Universe $\kappa = -1$ the expansion is indefinite and there is no limit to the distance d_0 that we can consider. On the other hand the cosmological parameter is defined only in the range $0 < \Omega_0 < 1$. Therefore we have to restrict our attention to the open strip $(]0, \infty[) \times (]0, 1[) \subset \mathbb{R}^2$. There the function $\mathfrak{f}_{-1}(d_0, \Omega_0)$ is bounded and takes values in the interval $[0, 2]$. Its very smooth plot is shown in Fig. 5.22. In the case of a closed Universe ($\kappa = 1$) the cosmological parameter is bounded only from below ($\Omega_0 > 1$), but there is always a maximal distance that we can explore $\frac{a}{a_0} < \frac{\Omega_0}{\Omega_0 - 1}$ corresponding to the absolute maximum reached by the scale factor during the whole history of such a Universe. The plot of the function $\mathfrak{f}_1(d_0, \Omega_0)$ in the physically available range is slightly more structured yet it is also bounded and of order unity as it is shown in Fig. 5.23. The outshot of such a discussion is that in a matter dominated Universe the *particle horizon* is *finite* and its *scale* is fixed by the *inverse of the Hubble constant*:

$$d_p \simeq \frac{1}{H_0} \tag{5.6.32}$$

Completely different is the conclusion one reaches in an exponentially expanding Universe dominated by vacuum energy. Suppose we sit at time t_0 and suppose that

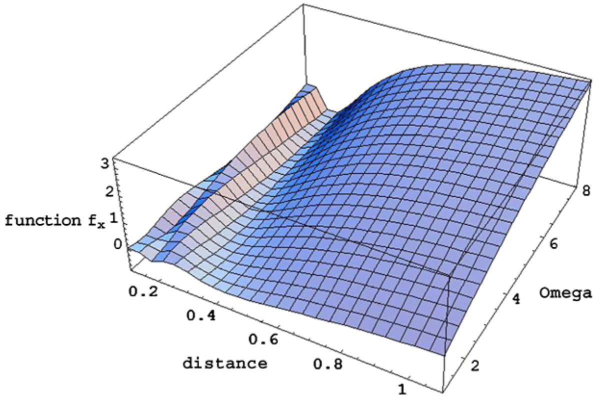

Fig. 5.23 Plot of the function $\mathfrak{f}_1(d_0, \Omega_0)$ showing that it is bounded and of order unity in the physically available range

in our past the Universe was expanding according to the law dictated by $w = -1$. To simplify matters let us also assume that we deal with a spatially flat Universe $\kappa = 0$. In that case the Hubble function is actually a Hubble constant H_0 and from the integral in (5.6.28) we obtain:

$$d_p = \frac{a_0}{H_0} \int_{a_i}^{a_0} \frac{da}{a^2} = \frac{1}{H_0} \frac{a_0 - a_i}{a_i} \tag{5.6.33}$$

In the limit $a_i \to 0$ we see that $d_p \to \infty$, in other words if the exponential expansion phase started at the very beginning of the Universe, there is no particle horizon and the distance that can be seen at any later moment of the exponential expansion extends to the whole physical space. This has a very important consequence which is the basic motivation to consider *inflation*. What we can see from the past is what can influence our present, namely what is in causal contact with us. Therefore we can conclude that at the end of an exponential expansion, if that expansion started early enough ($a_i \simeq 0$), the entire resulting Universe originated from a single causally connected region. If the subsequent expansion of the Universe proceeds through a matter dominated regime, from the perspective of an observer living in that age, the same Universe appears instead to be made of a plethora of causally disconnected regions. This observation will play a fundamental role in understanding why the inflationary scenario solves the puzzles of the Standard Cosmological Model in a robust way.

5.6.2 Event Horizon

The event-horizon is the complement of the particle horizon. By definition the event-horizon is the boundary of the space-time region from which no signal will ever be received by an observer in its future. According to this definition the events inside the event-horizon are characterized by radial coordinates χ larger than the following limiting one:

$$\chi_e(\eta) = \int_{\eta}^{\eta_{end}} d\eta = \eta_{end} - \eta \tag{5.6.34}$$

where η is the conformal time when the considered observer lives while η_{end} is the conformal date of death of the Universe. Therefore, in full analogy to our treatment of the particle horizon, the size of the event horizon at current cosmic time t_0 is determined by:

$$\chi_e(t_0) = \int_{t_0}^{t_{end}} \frac{1}{a(t)}\, dt = \int_{a_0}^{a_{end}} \frac{1}{a^2 H(a)}\, da \tag{5.6.35}$$

Hence for a matter dominated Universe we obtain:

$$\chi_e = \int_{a_0}^{a_{end}} \frac{1}{a_0 H_0 \sqrt{a(a + (a_0 - a)\Omega_0)}}\, da \tag{5.6.36}$$

In both cases of an open and a flat universe the above integral diverges for $a_{end} \to \infty$ and therefore there is no event horizon. Different is the case of a matter dominated, closed Universe. There the scale factor reaches a maximum and then decreases to zero at the *Big Crunch*. Recalling (5.5.22)–(5.5.23) we see that in the case of this universe the conformal time is bounded by $\eta_{end} = 2\pi$, so that we obtain:

$$\begin{aligned} \chi_e &= 2\pi - \eta \\ &\Downarrow \\ d_e(\eta) &= \frac{1}{2} a_{\max}(1 - \cos\eta)|\sin\eta| = d_p(\eta) \end{aligned} \tag{5.6.37}$$

where $a_{\max}$ is the maximal value of the scale factor attained in the history of the closed universe, which after that decreases until it vanishes again at the Big Crunch. The last identity in (5.6.37) is a consequence of the periodicity of trigonometric functions, $\sin^2(2\pi - \eta) = \sin^2(\eta)$, which implies a remarkable consequence: in closed universes the event and the particle horizons exactly coincide. In the first expansion phase the event horizon grows along with the scale factor but while the expansion slows down, it reaches a maximum corresponding to $\frac{2}{3}a_{\max}$ and then starts decreasing again until it shrinks to zero at the same time when the Universe attains its maximal extension. After that, while the Universe begins to contract, the event horizon grows once more and attains again the same maximum $\frac{2}{3}a_{\max}$. Then it shrinks along with the Universe and becomes zero at the Big Crunch. The plot of the event/particle horizon and of the scale factor are compared in Fig. 5.24.

Hence the visible portion of a closed Universe enlarges and shrinks with a different periodicity with respect to the expansion scale-time. This leads to the surprising result that the visible portion of the Universe shrinks almost to zero when it attains its maximal extension.

Something quite different happens in an exponentially expanding universe. For simplicity focusing once again on the case of a spatially flat de Sitter space, we

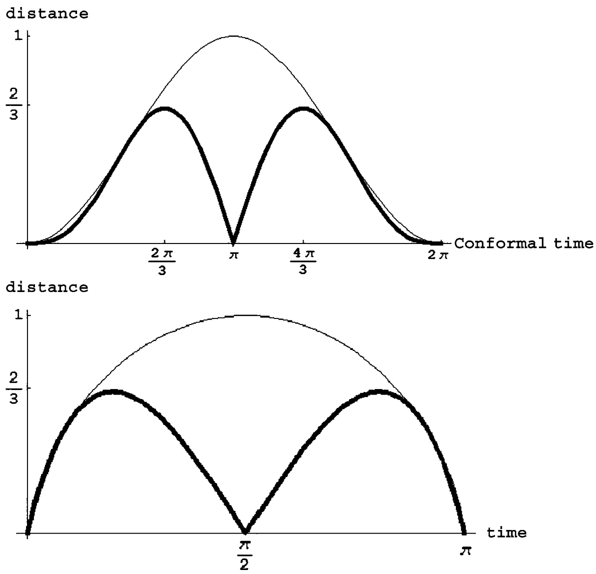

Fig. 5.24 Plot of the scale factor and of the event/particle horizon in a spatially positively curved, matter dominated universe. In both pictures the *thin line* corresponds to the scale factor, while the *thicker* one corresponds to the event/particle-horizon. In the *first picture* both a and $d_{e/p}$ are plotted against the conformal time η, while in the *second* they are plotted against the physical time t

calculate there the event horizon:

$$d_e(t) = a(t)\int_t^{\infty}\frac{d\tau}{a(\tau)} = \exp[H_0 t]\int_t^{\infty}\exp[-H_0\tau]\,d\tau = \frac{1}{H_0} \qquad (5.6.38)$$

where we have made use of (5.5.54). Hence while the Universe undergoes an *exponential expansion* the event horizon remains constant and its size is fixed by the inverse Hubble constant. The set of events causally disconnected from the observer fills a region which becomes bigger and bigger as time goes on. This elementary fact has striking implications within the inflationary scenario.

Consider a quantum particle emitted in some physical process at some instant of time during an exponential expansion phase of the Universe. Suppose that its wave-length at the emission time is $\lambda_e < H_0^{-1}$, which we describe by saying that it is inside the *Hubble radius*. Stretched by the cosmological red-shift, at a time t defined by $\frac{a(t)}{a(t_e)}\lambda_e = H_0^{-1}$ that wave-length exits the Hubble horizon and becomes larger than the current event horizon. After that time all physical quantities associated with such a wave-length freeze out, since no physical process can any more alter them. If the exponential expansion phase is followed by another one where the Universe continues to expand according to a power-like law, then the Hubble scale $H^{-1}(t)$ will grow once again and the considered primordial wave-length might reenter, at a later time, the Hubble radius. At that moment the physical quantities associated with it will be transformed by interactions with the other components of the post-inflationary Universe and their state at the present time will depend also on the post-inflationary evolution. At distances larger than the Hubble radius, we see instead faithful images of the remote age associated with the conjectured exponential expansion.

5.6.3 Red-Shift Distances

In our observation of the sky at extremely large distances, which means looking at very ancient objects, the only available method to measure both the space and the time separation of these sources from the vantage point of our observatory consists of determining their red-shift factor:

$$z \equiv \frac{\lambda_0 - \lambda_e}{\lambda_e} = \frac{a(t_0)}{a(t_e)} \tag{5.6.39}$$

where t_0 is the current cosmic time when the observed photons are absorbed and t_e is the remote time when they were emitted from their source.

It is therefore quite useful to use the red-shift factor as a label both for time and space distances. Indeed in the conformal coordinate system centered in our laboratory, the radial coordinate $\chi(z)$ of a distant source at red-shift z is unambiguously defined as:

$$\chi(z) = \eta_0 - \eta_e = \int_{t_e}^{t_0} \frac{dt}{a(t)} = \frac{1}{a_0} \int_0^z \frac{dz'}{H(z')} \tag{5.6.40}$$

and the Hubble function is expressed in terms of z through an immediate manipulation of the Friedman equation (5.6.7):

$$H(z) = H_0 \sqrt{\Omega_0 (z+1)^{3(w+1)} + (1-\Omega_0)(z+1)^2} \tag{5.6.41}$$

Let us also observe that for non-spatially flat universes the current value of the scale factor can be determined from Friedman equation as:

$$a_0 = \left(H_0 \sqrt{|1-\Omega_0|}\right)^{-1} \tag{5.6.42}$$

Hence for the case of a closed or open universe the physical distance of a source at red-shift z can be written as follows:

$$D_\pm(z) = \left(H_0 \sqrt{|1-\Omega_0|}\right)^{-1} R_\pm\big(\chi_\pm(z)\big) \tag{5.6.43}$$

whose explicit form we shall presently evaluate for matter dominated universes. For the case of a matter dominated flat universe we can make a separate very simple calculation. Setting $\Omega_0 = 1$ and $w = 0$ in (5.6.41) we get:

$$\begin{aligned}
\chi_0(z) &= \frac{1}{a_0} \int_0^z \frac{dz'}{H(z')} = \frac{1}{a_0} \int_0^z \frac{dz'}{\sqrt{(z'+1)^3} H_0} \\
&= \frac{2}{a_0 H_0} \left(1 - \frac{1}{\sqrt{z+1}}\right) \\
&\Downarrow \\
D_0(z) &= \frac{2}{H_0} \left(1 - \frac{1}{\sqrt{z+1}}\right)
\end{aligned} \tag{5.6.44}$$

Let us now evaluate $\chi(z)$ for a matter dominated universe ($w = 0$) with negative spatial curvature ($\Omega_0 < 1$). In this case, the defining integral yields:

$$\begin{aligned}\chi_-(z) &= H_0\sqrt{|1-\Omega_0|}\int_0^z \frac{dz}{H(z')}\\ &= 2\left(\coth^{-1}(\sqrt{1-\Omega_0}) - \tanh^{-1}\left(\sqrt{\frac{z\Omega_0+1}{1-\Omega_0}}\right)\right)\end{aligned} \tag{5.6.45}$$

If we perform the same calculation for a matter dominated ($w = 0$) closed universe ($\Omega_0 > 1$) we obtain instead the result:

$$\begin{aligned}\chi_+(z) &= H_0\sqrt{|1-\Omega_0|}\int_0^z \frac{dz}{H(z')}\\ &= 2\left(\tan^{-1}\left(\sqrt{z+\frac{z+1}{\Omega_0-1}}\right) - \cot^{-1}(\sqrt{\Omega_0-1})\right)\end{aligned} \tag{5.6.46}$$

Let us now calculate $R_\pm(\chi_\pm(z))$. With some algebraic effort we can verify that in both case the result is the same namely:

$$R_\pm\big(\chi_\pm(z)\big) = \frac{2\sqrt{|\Omega_0-1|}}{(1+z)\Omega_0^2}\big(z\Omega_0 + (\Omega_0-2)(\sqrt{1+z\Omega_0}-1)\big) \tag{5.6.47}$$

Using once again (5.6.42) and inserting the above results into (5.6.43) we conclude that:

$$D_\pm(z) = D(z,\Omega_0) \equiv \frac{2}{H_0\Omega_0^2(1+z)}\big(z\Omega_0 + (\Omega_0-2)(\sqrt{1+z\Omega_0}-1)\big) \tag{5.6.48}$$

The interesting point is that in the limit $\Omega_0 \to 1$ we exactly retrieve $D_0(z)$:

$$\lim_{\Omega_0\to 1} D(z,\Omega_0) = D_0(z) \tag{5.6.49}$$

This allows us to consider $D(z,\Omega_0)$ as the spatial distance of any source at red-shift z for any possible value of the cosmological parameter Ω_0.

5.7 Conceptual Problems of the Standard Cosmological Model

As we anticipated in Sect. 4.6, the great success of the Standard Cosmological Model based on the principles of homogeneity and isotropy does not remove a fundamental conceptual problem which can be summarized in the following question: *why is our Universe so much homogeneous and isotropic?*

The main source of the problem is the quality of Einstein equations that, in the course of time evolution enlarge perturbations and anisotropies rather than damping them. Therefore if our Universe is so much homogeneous at the present time it must have been even more so in the past and this is quite unnatural. Who has prepared such fine tuned homogeneous initial conditions? The paradox becomes evident if we compare the extension of the Universe with the causal horizon at various times. The argument is masterly presented in chapter five of [9] and we just follow the reasoning of that author.

Let us consider as initial time that fixed by the Planck scale which corresponds to:

$$t_{Planck} \sim 10^{-43} \text{ s} \tag{5.7.1}$$

The present time is instead fixed by the Hubble scale H_0 and we have:

$$t_0 \sim 14 \text{ billion years} \sim 10^{17} \text{ s} \tag{5.7.2}$$

The present size of the homogeneous region covers all the visible Universe and therefore is of the order of the present horizon scale namely:

$$\ell_{hom}(t_0) = \ell_{hor}(t_0) \sim ct_0 \sim 10^{28} \text{ cm} \tag{5.7.3}$$

Since, as we said, anisotropies and inhomogeneities cannot be washed away by the expansion of the Universe when it proceeds according to power laws, then assuming that this was the case, it follows that the size of the homogeneous region at the Planckian time must have been the following:

$$\ell_{hom}(t_{Planck}) = \ell_{hom}(t_0)\frac{a(t_{Planck})}{a(t_0)} \tag{5.7.4}$$

and we can compare it to the size of a causally connected region at the same time, which is the horizon scale at Planckian time:

$$\ell_{hor}(t_{Planck}) \sim ct_{Planck} \sim 10^{-32} \text{ cm} \equiv \ell_{Planck} \tag{5.7.5}$$

In this way we obtain:

$$\frac{\ell_{hom}(t_{Planck})}{\ell_{hor}(t_{Planck})} = 10^{60} \times \frac{a(t_{Planck})}{a(t_0)} \tag{5.7.6}$$

How can we estimate the ratio $\frac{a(t_{Planck})}{a(t_0)}$? The answer is simple: from the temperatures of the black-body radiation. Because of the cosmological red-shift the ratio between scale factors is proportional to the inverse ratio of radiation temperatures:

$$\frac{a(t_{Planck})}{a(t_0)} \sim \frac{T_0}{T_{Planck}} \tag{5.7.7}$$

At the present time we have $T_0 \sim 1$ K while at the Planckian time the radiation temperature must have been the temperature equivalent of the Planck length, namely $k_B T_{Planck} \sim \hbar \ell_{Planck}^{-1}$. This yields:

$$T_{Planck} \sim 10^{32} \text{ K} \tag{5.7.8}$$

so that:

$$\frac{\ell_{hom}(t_{Planck})}{\ell_{hor}(t_{Planck})} = 10^{28} \tag{5.7.9}$$

This means that according to the Standard Cosmological Model our Universe has evolved from a completely homogeneous region that was $10^{3\times 28}$ bigger than a causally connected region. How could it be homogeneous if there was no possibility of communication among the various causally disconnected cells and of establishing thermal equilibrium?

This paradox is named the *horizon problem*. Another conceptual problem is named the *flatness problem*. It appears from all our present cosmological data that our Universe is spatially flat, namely that the cosmological parameter Ω is nearly equal to 1. Why is that so? Who prepared once again the initial conditions in such a precise way as to make Ω exactly equal to one?

The answer to both problems can be provided by the scenario of a primeval cosmic inflation.

As we observed in Sect. 5.6.1, during a phase of exponential expansion of the scale factor the horizon scale remains constant so that, assuming the existence of such a phase in our remote past, explains how a single causally connected region could split into many apparently disconnected ones. Similarly as we will see in the sequel we can argue that the Universe always exits flat from an exponential expansion phase irrespectively of the spatial curvature it had when it entered such a phase.

Hence an exponential inflation provides a generic mechanism able to solve the conceptual problems of cosmology. The question is: which kind of matter can provide the means of realizing such an inflationary phase. The answer is simple enough. It suffices to have a microscopic dynamical theory that besides other fields includes also scalar ones, self-interacting through the presence of some potential. In the next section we will see that the requirements on the structure of the potential in order to realize a reasonable inflationary phase are rather mild and generic. This implies that the inflationary universe scenario is robust.

5.8 Cosmic Evolution with a Scalar Field: The Basis for Inflation

The dynamical basis of inflation is fairly simple. The paradigm is provided by the simple model of a scalar field $\varphi(x)$ interacting with gravity and with itself by means of some potential $V(\varphi)$.

Let us write the following action:

$$\begin{aligned}
\mathscr{A} &= \int d^4x \, (\mathscr{L}_{grav} + \mathscr{L}_{scalar}) \\
\mathscr{L}_{grav} &= \sqrt{-\operatorname{Det} g}\,\mathscr{R}[g] \\
\mathscr{L}_{scalar} &= \sqrt{-\operatorname{Det} g}\left[\frac{1}{2}\partial_\mu\varphi\partial_\nu\varphi g^{\mu\nu} - V(\varphi)\right]
\end{aligned} \tag{5.8.1}$$

Varying it respectively in the metric g and in the scalar field φ we obtain the following coupled equations:

$$G_{\mu\nu} \equiv \mathscr{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \mathscr{R}[g] = 4\pi G T_{\mu\nu}[\varphi] \tag{5.8.2}$$

$$\frac{1}{\sqrt{-\operatorname{Det} g}} \partial_\mu \left(\sqrt{-\operatorname{Det} g}\, g^{\mu\nu} \partial_\nu \varphi \right) + \frac{d}{d\varphi} V(\varphi) = 0 \tag{5.8.3}$$

where the stress energy tensor of the scalar field is given by:

$$T_{\mu\nu}[\varphi] = \frac{1}{2} \partial_\mu \varphi \partial_\nu \varphi - \frac{1}{2} g_{\mu\nu} \left[\frac{1}{2} g^{\rho\sigma} \partial_\rho \varphi \partial_\sigma \varphi - V(\varphi) \right] \tag{5.8.4}$$

If we introduce the homogeneous, isotropic ansatz (5.4.4) for the metric g and if we assume that the scalar field $\varphi = \varphi(t)$ depends only on the cosmic time t, then the two equations (5.8.2) and (5.8.3) are easily worked out and reduce to a very simple form. It suffices to observe that, under the above conditions, the stress energy tensor of the scalar field has the canonical form (5.3.49) of a fluid, with the following identification of the energy density and of the pressure:

$$\rho = \frac{1}{4} \dot{\varphi}^2 + \frac{1}{2} V(\varphi) \tag{5.8.5}$$

$$p = \frac{1}{4} \dot{\varphi}^2 - \frac{1}{2} V(\varphi) \tag{5.8.6}$$

Inserting this result into the Friedman equations (5.4.12) and choosing a vanishing cosmological constant ($\Lambda = 0$)[4] we obtain:

$$H^2 + \frac{\kappa}{a^2} = \frac{8\pi G}{3} \left(\frac{1}{4} \dot{\varphi}^2 + \frac{1}{2} V(\varphi) \right) \tag{5.8.7}$$

$$\frac{\ddot{a}}{a} = -\frac{8\pi G}{3} \frac{1}{2} \left(\dot{\varphi}^2 - V(\varphi) \right) \tag{5.8.8}$$

where:

$$H(t) \equiv \frac{\dot{a}(t)}{a(t)} \tag{5.8.9}$$

is the Hubble function. The differential system is completed by the explicit form of the propagation equation (5.8.3) of the scalar field which, in the chosen metric, is the following one:

$$\ddot{\varphi} + 3H\dot{\varphi} + V' = 0; \quad V' \equiv \frac{dV}{d\varphi} \tag{5.8.10}$$

[4]This choice has the following motivation. In presence of a generic potential for the scalar field, the cosmological constant is redundant. Indeed any constant contribution to $V(\varphi)$ just plays the role of a cosmological constant.

Equations (5.8.7), (5.8.8), (5.8.10) encode the inflationary Universe paradigm. To explain this let us explore some possible solutions of this differential system.

5.8.1 de Sitter Solution

Consider a value φ_0 of the scalar field corresponding to an extremum of the scalar potential where this latter attains a finite positive value:

$$V_0' \equiv \left.\frac{dV(\varphi)}{d\varphi}\right|_{\varphi=\varphi_0} = 0; \qquad V_0 = V(\phi_0) > 0 \tag{5.8.11}$$

In this case we can solve (5.8.7), (5.8.8), (5.8.10) by setting:

$$\varphi(t) = \varphi_0 \tag{5.8.12}$$

and

$$a(t) = \begin{cases} a_+(t) \equiv \frac{\cosh H_0 t}{H_0} & \text{for } \kappa = 1 \\ a_-(t) \equiv \frac{\sinh H_0 t}{H_0} & \text{for } \kappa = -1 \\ a_0(t) \equiv \frac{\exp H_0 t}{H_0} & \text{for } \kappa = 0 \end{cases} \tag{5.8.13}$$

where in all cases:

$$H_0 = \sqrt{\frac{4\pi G}{3} V_0} \tag{5.8.14}$$

Recalling the results of Sect. 5.5.3 it follows that, for a constant scalar field sitting at an extremal point of the potential, Einstein equations are solved by de Sitter space, which can be represented in the three versions of a closed, open or spatially flat Universe, in any case exponentially expanding. Indeed, differently from the case of all other fluids, characterized by a pressure $p > -\frac{1}{3}\rho$, the equation of state $p = -\rho$, satisfied when $\dot{\varphi} = 0$, implies, irrespectively of the sign κ of the spatial curvature, a constant positive acceleration of the Universe expansion, which proceeds indefinitely with an exponential asymptotic behavior. For large t the three scale factors $a_\pm(t)$ and $a_0(t)$ have the same form and tend to merge. This is quite clear from Friedman equations. The curvature term $\kappa/a^2(t)$ becomes negligible for large values of $a(t)$, which are always attained in an expanding Universe.

Considering the three metrics $ds^2_{\mathrm{dS}_+}$, $ds^2_{\mathrm{dS}_-}$ and $ds^2_{\mathrm{dS}_0}$, we see that, not only they describe the same intrinsic geometry, but they also approximately coincide in an open region $\mathscr{O}_{late}$ of de Sitter space, corresponding to late times and relatively small distances:

$$M_{\mathrm{dS}} \supset \mathscr{O}_{late} = \{t \gg 1, r \ll 1, \underbrace{\theta, \phi}_{\text{all range}}\} \tag{5.8.15}$$

This observation is the corner-stones of the inflationary scenario.

Imagine that, at some very ancient instant of time t_i, a small region of the space-time manifold, containing matter and energy of various types, begins to inflate since, over that region, the scalar field φ is approximately constant and equal to the critical value φ_0. The approximate behavior of this expanding *space-time bubble* will be the same as that of a region of de Sitter space. At the beginning t_i, the spatial curvature of the bubble might have been positive, negative or null, yet at a sufficiently later time t_{end}, the effective metric of the bubble will be that of a *spatially flat* de-Sitter manifold $ds^2_{\mathrm{dS}_0}$. Suppose now that at $t = t_{end}$ the scalar field, which was so far approximately constant, begins to evolve more rapidly and the kinetic energy $\dot{\varphi}^2$ starts dominating over the potential energy $V(\varphi)$ which has in the meantime decreased. According to Friedman equations the acceleration parameter changes sign, the expansion rate decreases and, in a finite time, the behavior of the scale factor approaches that of a Universe filled with ordinary baryonic matter and radiation. Yet, the considered space-time bubble is spatially-flat as a result of the inflation which has occurred in the previous phase. This fundamental property of an inflationary phase in the history of the Universe is the most relevant feature of the inflationary scenario. It explains in a natural way and without ad hoc fine tuning why our Universe is spatially flat as it happens to reveal itself through experimental observations.

As we plan to analyze more explicitly in the sequel, the inflationary phase explains also why the observable Universe is homogeneous to very high accuracy over regions that are causally disconnected at the present time.

5.8.2 Slow-Rolling Approximate Solutions

In view of the qualitative considerations discussed above, let us now consider approximate solutions of (5.8.7), (5.8.8), (5.8.10) corresponding to a *slow-roll phase*. This latter is defined by the following two conditions:

$$\frac{1}{2}\dot{\varphi}^2 \ll V(\varphi) \tag{5.8.16}$$

$$\ddot{\varphi} \ll 3H\dot{\varphi} + V'(\varphi) \sim 0 \tag{5.8.17}$$

whose physical interpretation is quite transparent, the former requiring that the kinetic energy of the scalar field should be negligible with respect to the potential energy, the latter requiring that the acceleration in the evolution of φ should also be small with respect to the velocity.

In order to discuss the existence and the properties of such a regime, it is convenient to manipulate a little bit the evolution equations (5.8.7), (5.8.8), (5.8.10) casting them in a more manageable form. First of all it is convenient to introduce rescaled variables so as to get rid of all the constants. Introducing the Planck mass:

$$m_P = \frac{1}{\sqrt{2\pi G}} \tag{5.8.18}$$

we set:

$$\varphi = m_P \phi; \qquad V(\varphi) = m_P^2 W\left(\frac{\varphi}{m_P}\right) = m_P W(\phi) \tag{5.8.19}$$

Substituting in (5.8.7), (5.8.8), (5.8.10) and using the identity $\frac{\ddot{a}}{a} = \dot{H} + H^2$ we obtain:

$$H^2 = \frac{1}{3}\dot{\phi}^2 + \frac{2}{3}W \tag{5.8.20}$$

$$\dot{H} + H^2 = -\frac{2}{3}\dot{\phi}^2 + \frac{2}{3}W \tag{5.8.21}$$

$$0 = \ddot{\phi} + 3H\dot{\phi} + W' \tag{5.8.22}$$

where we have already set $\kappa = 0$ in view of the discussion of the previous section. This shows that our system of equations is actually a (redundant[5]) differential system for the scalar field $\phi(t)$ and the Hubble function $H(t)$. The de Sitter solution is nothing else but the constant solution $\phi = \phi_0$, $H = H_0$, which requires that ϕ_0 should sit at an extremum of the potential $V'(\phi_0) = 0$. The main idea of the slow rolling consists of considering the evolution of the two functions $H(t)$ and $\phi(t)$ starting from an initial configuration which is very close to de Sitter one. At $t = t_i$ the scalar field does not sit at an extremum of the potential, yet not too far from it ($V'(\phi_i) \approx$ *small*), its initial velocity is small $\dot{\phi}_i \approx$ *small* and small is the initial derivative of the Hubble function $\dot{H}(t_i) \approx$ *small*. The latter condition is not independent rather it is just a consequence of the smallness of $\dot{\phi}_i$. Indeed from (5.8.22), it follows:

$$\dot{H} = -\dot{\phi}^2 \tag{5.8.23}$$

Obviously such initial conditions can always be postulated, yet the question is whether they can be maintained throughout evolution at least for a certain non-negligible period of time during which the Universe will experience an almost exponential expansion like if it were a true de Sitter space. The answer to such a question depends on the properties of the scalar potential. Several types of potentials have been used in the explicit modeling of Inflation, trying also to relate them with the fundamental theories of Particle Interactions, in particular with the scenario of symmetry breaking and with the Higgs mechanism. Indeed it has been conjectured that the *inflaton*, *i.e.* the scalar field responsible for the primeval exponential growth of the Universe should be a condensate of the scalar degrees of freedom available in the fundamental unified theory of all interactions which describes Physics at the Planck scale and hence at the Big Bang. Inspired by this natural point of view, in more recent time attentive consideration has been given to the many field potentials $V(\phi_1 \dots \phi_n)$ which arise in supergravity theory, induced by the brane-superstring

[5]Any one of the three equations is actually a consequence of the other two as a legacy of the Bianchi identities which constrain Einstein equations.

scenarios. We shall not enter the fine structure of inflation modeling, which is still in lively evolution, since the most relevant and appealing feature of the inflationary scenario is precisely its robustness. The main physical mechanism and the essence of its predictions are largely independent from the detailed structure of the utilized potential, provided it is sufficiently smooth and reasonable. What this concretely means we shall presently show. Therefore, for the sake of simplicity we focus on the most widely utilized type of potentials that are the polynomial ones, in particular the following one

$$W(\phi) = -\mu\phi^{\alpha} + \lambda\phi^{2\alpha} + \frac{\nu}{4\lambda} \tag{5.8.24}$$

which for various values of the parameters λ, μ, ν captures the main features of three distinct well-known modelings of inflation.

(a) For $\mu = \nu = 0$ we have the so named *large field* modeling of inflation. The potential has just a minimum at $\phi = 0$ and slow roll, as we shall prove below is obtained by starting at some sufficiently high value of ϕ_i.
(b) For $\mu = 0$ we have the so named *small field* modeling. The potential has just a maximum at $\phi = 0$ and a minimum at $\phi = (\frac{\nu}{4\lambda^2})^{1/2\alpha}$. Slow rolling is obtained by starting at small values ϕ_i of the field near the unstable maximum.
(c) For $\nu = \mu^2$ the potential becomes:

$$W(\phi) = \left(\sqrt{\lambda}\phi - \frac{\mu}{2\sqrt{\lambda}}\right)^2 \tag{5.8.25}$$

which again has a maximum and a minimum and corresponds to the sort of potentials appearing in the symmetry breaking scenarios. In this case, as in the previous ones, slow rolling is obtained by starting at small values ϕ_i of the field, near the maximum.

Independently from the explicit form of the potential let us consider the consistency conditions imposed on it by the assuming the existence of a slow rolling phase. From the exact result (5.8.23) we know that if the scalar field has a slow roll, even slower will be the change of the Hubble function. Consider next the slow-roll condition (5.8.17) from which we work out

$$\dot{\phi} \sim -\frac{W'}{3H} \tag{5.8.26}$$

and insert this result into the first condition (5.8.16). We obtain:

$$\frac{1}{6}\frac{(W')^2}{W} \ll W \quad \Rightarrow \quad \varepsilon_W(\phi) \equiv \frac{1}{6}\left(\frac{W'}{W}\right)^2 \ll 1 \tag{5.8.27}$$

We see that a necessary condition for slow roll corresponds to the smallness of a certain function $\varepsilon_W(\phi)$ of the scalar field, defined in (5.8.27) and completely determined by the potential. This condition is not the only one. Taking a further derivative of condition (5.8.16) we get:

$$3\ddot{\phi}H + 3\dot{\phi}\dot{H} + W''\dot{\phi} \sim 0 \quad \Rightarrow \quad \ddot{\phi} \sim -\frac{W''}{9H^2}W' + \frac{W'}{3H^2}\dot{H} \tag{5.8.28}$$

where $\dot{\phi}$ has been eliminated by means of the slow-roll condition (5.8.26). From the condition $\ddot{\phi} \ll W'$, using (5.8.28) and eliminating H^2 by means of (5.8.16) and H first by means of the exact result (5.8.23) then by means of (5.8.16) we get:

$$\frac{1}{6}\frac{W''}{W} + \frac{1}{3H}\varepsilon_W \ll 1 \tag{5.8.29}$$

Since we already know that $\varepsilon_W \ll 1$ and $3H > 1$, it follows that:

$$\eta_W(\phi) \equiv \frac{1}{6}\frac{W''}{W} \ll 1 \tag{5.8.30}$$

Hence we conclude that the smallness of the two functions $\varepsilon_W(\phi)$ and $\eta_W(\phi)$ is a necessary condition for slow-rolling.

In the case of the potential (5.8.25), the explicit form of the two slow roll index functions is:

$$\varepsilon_W(\phi) = \frac{8\alpha^2\lambda^2\phi^{2\alpha-2}(\mu - 2\lambda\phi^\alpha)^2}{3(4\lambda(\lambda\phi^\alpha - \mu)\phi^\alpha + \nu)^2} \tag{5.8.31}$$

$$\eta_W(\phi) = \frac{2\alpha\lambda\phi^{\alpha-2}(2(2\alpha-1)\lambda\phi^\alpha - \alpha\mu + \mu)}{3(4\lambda(\lambda\phi^\alpha - \mu)\phi^\alpha + \nu)} \tag{5.8.32}$$

which, in the two particular subcases that we respectively named *large field* and *symmetry breaking*, reduces to:

$$\begin{array}{ll} \text{large field} & \begin{cases} \varepsilon_W = \frac{2\alpha^2}{3\phi^2} \\ \eta_W = \frac{\alpha(2\alpha-1)}{3\phi^2} \end{cases} \\ \text{symm. break.} & \begin{cases} \varepsilon_W = \frac{8\alpha^2\lambda^2\phi^{2\alpha-2}}{3(\mu-2\lambda\phi^\alpha)^2} \\ \eta_W = \frac{2\alpha\lambda\phi^{\alpha-2}(2(2\alpha-1)\lambda\phi^\alpha-\alpha\mu+\mu)}{3(\mu-2\lambda\phi^\alpha)^2} \end{cases} \end{array} \tag{5.8.33}$$

The rationale for the given names becomes now clear. In the large field case the conditions $\varepsilon_V \ll 1$ and $\eta_V \ll 1$ translate into:

$$\sqrt{\frac{1}{3}\alpha(2\alpha-1)} < \sqrt{\frac{1}{3}2\alpha^2} \ll \phi \tag{5.8.34}$$

while in the symmetry breaking case the same conditions have two solutions, either

$$\phi \gg \left(\frac{\mu}{2\lambda}\right)^{\frac{1}{\alpha}} \tag{5.8.35}$$

Fig. 5.25 Picture of the potential (5.8.24) for an explicit choice of the parameters that corresponds to the symmetry breaking case: $\lambda = 1$, $\mu = 16$, $\nu = 256$. In this case we obtain: $W(\phi) = (-8 + \phi^2)^2$. Such a function has a maximum at $\phi = 0$ and a minimum at $\phi = 2\sqrt{2}$. On both sides of the maximum we have values named ϕ_{end}, whose precise definition is explained in the text, at which inflation ends for a slow rolling starting on the left or on the right of the minimum

or

$$\phi \ll \left(\frac{\mu}{2\lambda}\right)^{\frac{1}{\alpha}} \tag{5.8.36}$$

Hence in the large field case the slow-roll conditions are realized when the field is sufficiently large, while in the symmetry breaking case they are realized when it is either sufficiently large or sufficiently small as to be reasonably distant from the minimum of the potential where both indices $\varepsilon_W(\phi)$ and $\eta_W(\phi)$ develop a pole (see Fig. 5.26). This means that in the cosmic evolution which starts from initial conditions corresponding to slow-roll we will have slow-rolling and inflation while the field ϕ remains in the region where the two indices are small. Inflation will end as soon as these indices become of finite size and conventionally we can fix the end of inflation at:

$$\phi_{end} : \quad \varepsilon_W(\phi_{end}) = 1 \tag{5.8.37}$$

For the large field case we can give the general formula:

$$\text{large field scenario} : \quad \phi_{end} = \sqrt{\frac{2}{3}}\alpha \tag{5.8.38}$$

while in the symmetry breaking case a general solution of the implicit equation (5.8.37) cannot be written for all values of the power α and they have to be worked out case by case. Relying on these general observations we can now illustrate the mechanism of inflation by considering a specific case of the symmetry breaking scenario whose corresponding differential equations we will solve numerically. The qualitative picture is presented in Fig. 5.27.

We have an attractive mechanical analogy with a ball that is rolling down a hill towards the bottom of a valley that corresponds to the minimal of the potential. At the beginning the ball rolls slowly and its kinetic energy is negligible with respect

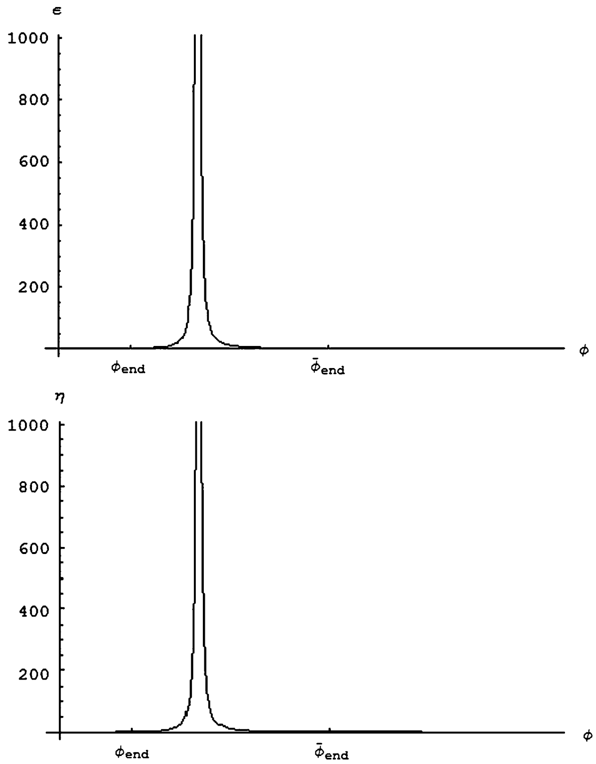

Fig. 5.26 In this picture we present the plot of the slow roll indices $\varepsilon_W(\phi)$ and $\eta_W(\phi)$ for the example of symmetry breaking potential presented in Fig. 5.25 corresponding to (5.8.24) with parameters: $\lambda = 1$, $\mu = 16$, $\nu = 256$. In this case we obtain: $\varepsilon_W(\phi) = \frac{8\phi^2}{3(\phi^2-8)^2}$ and $\eta_W(\phi) = \frac{2(3\phi^2-8)}{3(\phi^2-8)^2}$. Correspondingly we have two solutions for the end of inflation field value, namely $\phi_{end} = 2\sqrt{\frac{1}{3}(7-\sqrt{13})}$ and $\overline{\phi}_{end} = 2\sqrt{\frac{1}{3}(7+\sqrt{13})}$ which respectively sit before and after the minimum of the potential where the indices develop a pole

to the potential energy that drives an exponential growth of the scale factor. This is the inflation phase during which, dominated by vacuum energy, the Universe is very cold. Notwithstanding the slow rolling the ball goes down and it reaches the point where the slow rolling conditions cease to be valid. This is the end of inflation. After that the ball continues its descent, accelerating its motion while the growth of the scale factor experiences a rapid change of gear from an exponential to a power law one. Reaching the minimum the scalar field oscillates around it rapidly but with damped amplitude until it definitely sits down in the stable position. In order to verify the above qualitative description numerically it is convenient to rewrite the evolution equations (5.8.22) as a pair of second order coupled differential equations which we achieve as follows:

$$\frac{1}{2}\frac{\ddot{a}(t)}{a(t)} + \left(\frac{\dot{a}(t)}{a(t)}\right) - W\big(\phi(t)\big) = 0 \tag{5.8.39}$$

$$\ddot{\phi}(t) + 3\left(\frac{\dot{a}(t)}{a(t)}\right)\dot{\phi}(t) + W'\big(\phi(t)\big) = 0 \tag{5.8.40}$$

Starting from (5.8.25), if we choose the power $\alpha = 2$ and the parameters $\lambda = 1$, $\mu = 16$, $\nu = 256$, which is the choice graphically displayed in Fig. 5.25 and already

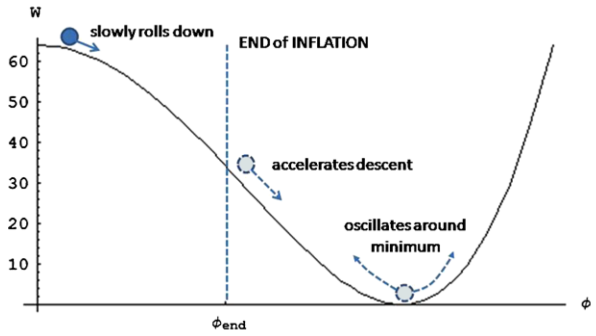

Fig. 5.27 The initial condition for cosmic evolution corresponds to a situation where the scalar field is at some value not too far from a maximum of the potential. Then the scalar starts rolling down from the hill towards the minimum. At the beginning the condition of slow rolling are satisfied. The scalar changes slowly while the scale factor increases almost exponentially. Going downhill the scalar reaches such a value where the slow-roll parameter ε_W becomes of order unity. There inflation ends since the kinetic energy of ϕ becomes comparable to its potential energy. The exponential growth of the scalar field ceases while the field ϕ accelerates towards the minimum. Around the minimum the scalar oscillates rapidly and its energy is dispersed by reheating the Universe

used also in the other plots, then the potential becomes:

$$W(\phi) = \left(\phi^2 - 8\right)^2; \qquad W'(\phi) = \quad 4\phi\left(\phi^2 - 8\right) \tag{5.8.41}$$

Replacing such explicit functions in (5.8.40), we can solve them numerically if we provide initial conditions at the initial time t_i which we can conventionally fix at zero $t_i = 0$. This means that we have to give the following data:

$$a(0) = a_0; \qquad \dot{a}(0) = \dot{a}_0; \qquad \phi(0) = \phi_0; \qquad \dot{\phi}(0) = \dot{\phi}_0 \tag{5.8.42}$$

It is important to stress that inflation will take place if and only if such initial conditions are given in such a way as to be consistent with the slow roll. This means first of all that ϕ_0 must be chosen in the region where $\varepsilon_W(\phi_0) \ll 1$ and $\eta_W(\phi_0) \ll 1$. Furthermore, once ϕ_0 has been chosen, $\dot{a}_0$ and $\dot{\phi}_0$ must be chosen in such a way that:

$$\begin{aligned} \left\| 3\frac{\dot{a}_0}{a_0}\phi_0 + W'(\phi_0) \right\| &\ll 1 \\ \left\| \left(\frac{\dot{a}_0}{a_0}\right)^2 - \frac{2}{3}W(\phi_0) \right\| &\ll 1 \end{aligned} \tag{5.8.43}$$

In this way we begin with slow roll and the slow roll phase hopefully will last for a sufficiently long interval of time as to allow the expansion of the scale factor by several order of magnitudes. In Fig. 5.28 we present the numerical solution of

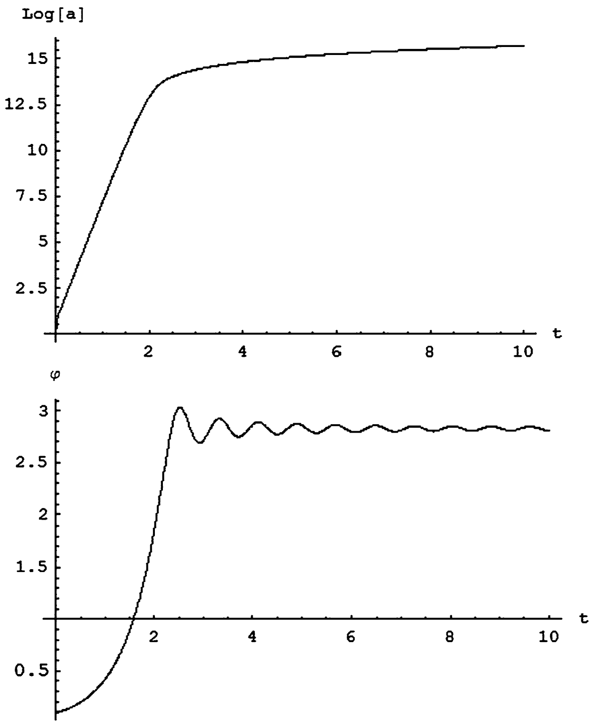

Fig. 5.28 Overall plot of the solution of the coupled differential equations (5.8.40) for the scale factor $a(t)$ and the scalar field $\phi(t)$ clearly displaying an early phase of slow rolling. The potential is that given in (5.8.41) and the initial conditions are those displayed in (5.8.44). In the inflation phase, lasting approximately in the interval of time from $t = 0$ to $t = 2.5$, the scale factor increases of almost 14 orders of magnitudes. As discussed in the text, this is not yet the sufficient amount of inflation to solve the horizon problem of cosmology. For that we need about 60 order of magnitudes (e-foldings)

(5.8.40) with the potential (5.8.41) and the following initial conditions:

$$\begin{aligned} \phi_0 &= 0.1; \qquad \dot{\phi}_0 = \dot{\phi}_0^\star \equiv -\frac{1}{a_0\sqrt{\frac{2}{3}W(\phi_0)}}W'(\phi_0) = 0.133333 \\ a_0 &= 1; \qquad \dot{a}_0 = \dot{a}_0^\star \equiv a_0\sqrt{\frac{2}{3}W(\phi_0)} = 7.99 \end{aligned} \tag{5.8.44}$$

which saturate the bounds (5.8.43). Clearly we have an inflation phase which lasts up to ϕ_{end} and inflates the Universe by about 14 orders of magnitudes. In Fig. 5.29 we present an enlargement of the same plot for later times, namely for $t > 3$. It is evident that the exponential expansion of the scale factor is now replaced by a power law growth which is weaker than linear as in a matter dominated Universe where $a \sim t^{2/3}$. At the same time the scalar field displays a damped rapid oscillation around the value corresponding to a minimum of the potential.

The existence of a good bona-fide slow roll phase in the numerical solution corresponding to the chosen initial conditions is best certified by the plot of the function

$$f(t) \equiv -\frac{3H(t)\dot{\phi}(t)}{W'(\phi(t))} \tag{5.8.45}$$

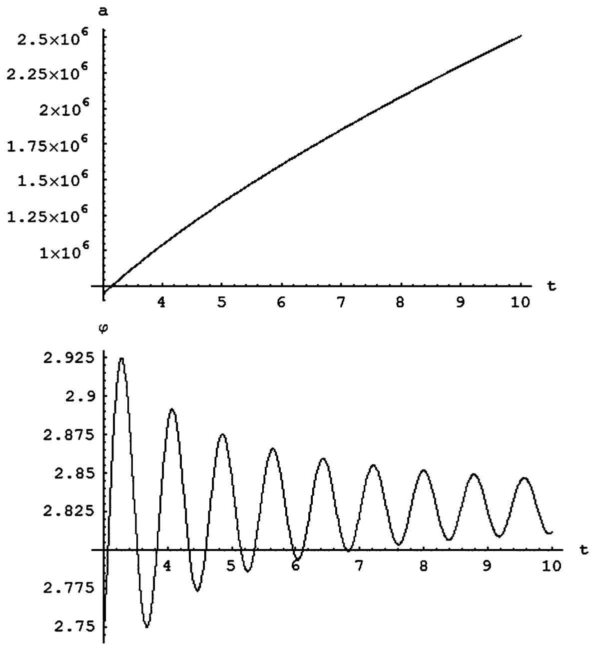

Fig. 5.29 After the end of inflation the growth of the scalar factor becomes of power type $a \sim t^p$ with $p < 1$. At the same time the scalar field undergoes rapid damped oscillations around the value $\phi_{\min}$ corresponding to a minimum of the potential

evaluated on the solution. This plot is presented in the first picture of Fig. 5.30. As we see, $f(t)$ is essentially constant and equal to one in the whole interval of time from $t = 0$ to $t = 2.3$. This is the epoch of inflation. After that $f(t)$ rapidly diverges signaling the end of the slow roll.

The important question which so far has no analytic answer is the following one. How large is the domain of initial conditions around the critical values $\dot{a}_0^\star$ and $\dot{\phi}_0^\star$ for which there is a slow roll phase? The relevance of this question can be appreciated a posteriori considering numerical solutions with initial conditions:

$$\dot{\phi}(0) = \dot{\phi}_0^\star + \Delta\dot{\phi}; \qquad \dot{a}(0) = \dot{a}_0^\star + \Delta\dot{a} \tag{5.8.46}$$

Experimentally for $\Delta\dot{\phi}$ and $\Delta\dot{a}$ not too large we find solutions that are very similar to the solution in Fig. 5.28. Yet for somewhat larger deviations in the initial conditions the qualitative picture of slow roll and inflation is completely destroyed.

For instance if we choose $\Delta\dot{a} = 100$ and $\Delta\dot{\phi} = 1500$ we obtain a solution where there is essentially no period of exponential growth of the scale factor (the Hubble function is never approximately constant) and the scalar field almost immediately oscillates. The essential absence of a slow rolling phase is certified by the plot of the function $f(t)$ which for such a solution is displayed in the second picture of Fig. 5.30.

It is evident that a decisive progress in the mathematical foundations of the inflationary scenario and on its robustness will come only from a proper definition of the domain of attraction of the slow rolling regime. Indeed it is important to establish how generic are the initial conditions that lead to inflation and therefore produce a flat, homogeneous universe like ours.

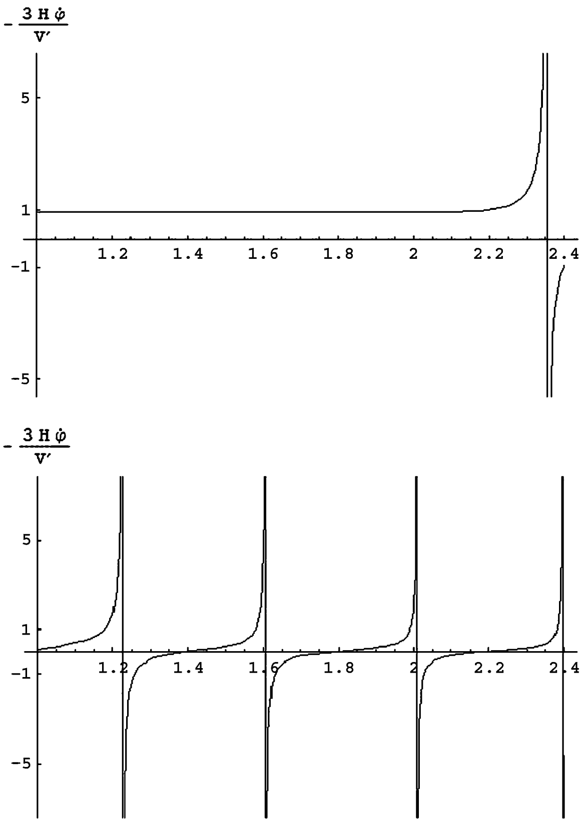

Fig. 5.30 In this picture the plot of the function $f(t)$ defined in (5.8.45) is presented for two different solutions of the same equations (5.8.40) corresponding to different boundary conditions. In the *first plot* where the initial conditions saturate the bound (5.8.43) we clearly see the slow role phase characterized by $f(t) \sim 1$. The *second plot* where the initial conditions violate in a severe way the bound (5.8.43), $f(t)$ is never approximately constant and equal to one. No slow roll phase is therefore present

5.8.2.1 Number of *e*-Folds

The primary motivation to consider the inflationary scenario is the solution of the paradox provided by (5.7.6). If in that equation we regard t_0 not as our present time rather as the time at which inflation ended, then the paradox can be solved in case the exponential increase of the scale factor is of order 10^{60}. The success of an inflationary model can be measured by estimating the so called number of e-folds:

$$N_e = \log \frac{a_{end}}{a_{init}}. \tag{5.8.47}$$

where a_{end} is the scale factor at the end of inflation and a_{init} is the scale factor at its beginning. The number of e-folds necessary to solve the horizon problem is $N_e \sim 60$–70.

Assuming the slow-roll we can estimate the number of e-folds purely in terms of the potential. Indeed we can write:

$$\begin{aligned} dN &= -d\log = H\,dt - \frac{H}{\dot{\phi}}d\phi \\ &\simeq 2\frac{W(\phi)}{W'(\phi)} \\ &\Downarrow \\ N_e &= 2\int_{\phi_{init}}^{\phi_{end}} \frac{W(\phi)}{W'(\phi)}\,d\phi \end{aligned} \tag{5.8.48}$$

The above integral formula is very useful in order to assess the validity of proposed potentials and as an a priori constraint on their parameters.

5.9 Primordial Perturbations of the Cosmological Metric and of the Inflaton

The most significant success of the inflationary paradigm is its ability to interpret the observed anisotropies of the Cosmic Background Radiation in terms of primordial perturbations of the cosmological metric and of the scalar fields.

In this and in the next section we present an introduction to this very challenging field which is presently under very rapid development. For simplicity we focus on the simplest model of *one scalar inflation*, aiming at illustrating the basic ideas. The reader, however, should be conscious that the most promising scenarios, developed within the framework of supergravity, correspond to *many scalar inflation* with a lot of extra complicacies.

In order to study the small perturbations of the cosmological metric it turns out to be more convenient to utilize the so called conformal frame described in the following subsection.

5.9.1 The Conformal Frame

Starting from the general isotropic and homogeneous ansatz (5.4.4), for the reasons discussed above, we choose the flat universe case $\kappa = 0$ and we introduce the conformal time η by setting:

$$dt = a(t)\,d\eta \equiv a(\eta)\,d\eta \tag{5.9.1}$$

At the same time we set

$$dr^2 + r^2\big(d\theta^2 + \sin^2(\theta)\,d\phi^2\big) = \sum_{i=1}^{3} dx_i^2 \tag{5.9.2}$$

which is the standard conversion formula from polar coordinates to a set of orthogonal Cartesian coordinates in the standard Euclidian space $\mathbb{R}^3$.

In terms of the coordinates $\{\eta, x_i\}$ the background cosmological metric takes the form:

$$ds_0^2 = a(\eta)^2 \left(d\eta^2 - \sum_{i=1}^{3} dx_i^2 \right) \tag{5.9.3}$$

Using the same coordinates also for the scalar field, the coupled system of Einstein and Klein-Gordon equations, displayed in (5.8.7), (5.8.8) and (5.8.10), is turned into the following one:

$$\mathscr{H}^2 = \frac{2}{3}\pi G\big((\varphi')^2 + 2a^2 V\big) \tag{5.9.4}$$

$$\frac{a''}{a} = \frac{2}{3}\pi G\big(-(\varphi')^2 + 4a^2 V\big) \tag{5.9.5}$$

$$0 = \varphi'' + 2\mathscr{H}\varphi' + a^2 \frac{dV}{d\varphi} \tag{5.9.6}$$

where the prime denotes the derivative with respect to η, and where we have introduced the conformal Hubble function:

$$\mathscr{H} = \frac{a'}{a} \tag{5.9.7}$$

From these background equations we can also derive the following identity:

$$\mathscr{H}' - \mathscr{H}^2 = -2\pi G\big(\varphi'\big)^2 \tag{5.9.8}$$

which is quite useful in further manipulations.

It is also convenient to inspect the form taken by the de Sitter solution in the conformal frame. This latter is characterized by a constant scalar field $\varphi = \varphi_0$ sitting at an extremum of the potential, as described in (5.8.11). Under these conditions the solution of the differential system is:

$$\begin{aligned} a(\eta) &= -\frac{2}{H_0 \eta} \\ \mathscr{H} &= -\frac{1}{\eta} \end{aligned} \tag{5.9.9}$$

where H_0 was defined in (5.8.14).

5.9.2 *Deriving the Equations for the Perturbation*

As a next step we parameterize the perturbations of the metric and of the scalar field around a homogeneous isotropic solution, namely around a solution of the differential system (5.9.4), (5.9.5), (5.9.6). Since the scalar field has spin zero it follows that the relevant perturbations of the metric which are driven by scalar perturbations have also spin zero. Modulo redefinitions induced by linearized diffeomorphisms, the scalar perturbations of the metric can be encoded in two scalar functions $\Phi(\eta, x_i)$ and $\Psi(\eta, x_i)$ that deform the line-element (5.9.3) in the following way:

$$ds^2 = a(\eta)^2 \left[\big(1 + 2\Phi(\eta, x_i)\big)\, d\eta^2 - \big(1 - 2\Psi(\eta, x_i)\big) \sum_{i=1}^{3} dx_i^2 \right] \tag{5.9.10}$$

where:

$$1 \gg \Phi(\eta, x_i); \qquad 1 \gg \Psi(\eta, x_i) \tag{5.9.11}$$

Similarly the perturbation of the scalar field is parameterized as follows:

$$\varphi(\eta, x_i) = \varphi(\eta) + \delta\varphi(\eta, x_i) \tag{5.9.12}$$

where we have:

$$\varphi(\eta) \gg \delta\varphi(\eta, x_i) \tag{5.9.13}$$

We are interested in expanding the Klein-Gordon and Einstein equations to first order in the perturbations in order to derive the linearized equations for these latter in the background of the considered isotropic homogeneous solution. It turns out to be convenient to write Einstein equations in the following way:

$$G^{\mu}{}_{\nu} = 4\pi G T^{\mu}{}_{\nu} \tag{5.9.14}$$

where the first index of both the Einstein tensor and of the stress-energy tensor have been raised by means of the inverse metric $g^{\mu\nu}$. Relying on this form of the equation we write:

$$G^{\mu}{}_{\nu} = G^{\mu}_{(0)\nu} + \delta G^{\mu}{}_{\nu} \tag{5.9.15}$$

$$T^{\mu}{}_{\nu} = T^{\mu}_{(0)\nu} + \delta T^{\mu}{}_{\nu} \tag{5.9.16}$$

and we obtain the linearized Einstein equations in the form:

$$\delta G^{\mu}{}_{\nu} = 4\pi G \delta T^{\mu}{}_{\nu} \tag{5.9.17}$$

In a similar way we have to expand to first order in the perturbation fields the Klein-Gordon equation. The results are obtained by means of a straightforward, although lengthy calculation. Let us spell it. For the perturbation of the Einstein tensor we obtain

$$\delta G^{0}{}_{0} = \frac{1}{a^2}\left[\nabla^2 \Psi - 3\mathscr{H}\Psi' - 3\mathscr{H}^2 \Phi\right] \tag{5.9.18}$$

$$\delta G^{0}{}_{i} = \frac{1}{a^2}\left[\partial_i\big(\Psi' + \mathscr{H}\Phi\big)\right] \tag{5.9.19}$$

$$\delta G^i{}_j = \frac{1}{a^2}\Bigg[\frac{1}{2}\partial_i\partial_j(\Phi-\Psi) \\ \times \delta^i{}_j\left(-\frac{1}{2}\nabla^2(\Phi-\Psi)-\Psi''+\mathscr{H}\big(2\Psi'+\Phi'\big)+\Phi\big(\mathscr{H}^2+2\mathscr{H}'\big)\right)\Bigg] \tag{5.9.20}$$

where:

$$\nabla^2 \equiv \sum_{i=1}^{3}\frac{\partial^2}{\partial x_i^2} \tag{5.9.21}$$

denotes the standard Laplacian operator in flat three-space.

On the other hand, the first order perturbation of the scalar field stress-energy tensor is the following one:

$$\delta T^0{}_0 = \frac{1}{2a^2}\left[\varphi'\delta\varphi'-\big(\varphi'\big)^2+a^2\frac{dV}{d\varphi}\right] \tag{5.9.22}$$

$$\delta T^0{}_i = \frac{1}{2a^2}\varphi'\partial_i\delta\varphi \tag{5.9.23}$$

$$\delta T^i{}_j = \delta^i{}_j\frac{1}{2a^2}\left[\frac{dV}{d\varphi}\delta\varphi'+\big(\varphi'\big)^2\Phi-\varphi'\delta\varphi'\right] \tag{5.9.24}$$

Finally the first order perturbation of the Klein-Gordon equation takes the following form:

$$\delta\varphi''-\nabla^2\delta\varphi-\varphi'(3\Psi+\Phi)'+2\mathscr{H}\varphi'+2a^2\frac{d^2V}{d\varphi^2}\varphi\delta\varphi=0 \tag{5.9.25}$$

The unknown functions are three Φ, Ψ, $\delta\varphi$, but the field equations are much more, although not all independent, since Einstein equations are constrained by the Bianchi identities, as we extensively discussed in Volume 1. The net balance of these arguments is that in the perturbed Einstein-Klein-Gordon system there is just one independent scalar degree of freedom $u(\eta,x_i)$, which obeys the following propagation equation:

$$u''-\nabla^2u-\frac{\theta''}{\theta}u=0 \tag{5.9.26}$$

where:

$$\theta(\eta)\equiv\frac{\mathscr{H}(\eta)}{a(\eta)\varphi'(\eta)}\equiv\frac{1}{z(\eta)} \tag{5.9.27}$$

All the other functions Φ, Ψ, $\delta\varphi$ can be expressed in terms of $u(\eta,x_i)$.

5.9.2.1 Meaning of the Propagation Equation

Before deriving it, let us briefly discuss the interpretation of (5.9.26). Consider a free scalar field ξ living in a conformally flat Minkowskian manifold with a metric of the following form:

$$ds_\lambda^2 = \lambda^2(\eta)\left(d\eta^2 - \sum_{i=1}^{2} dx_i^2\right) \tag{5.9.28}$$

in such a background, the action of the free field ξ, takes the following explicit form:

$$\begin{aligned}\mathscr{A}_\xi &= \int \sqrt{-\operatorname{Det} g}\,\frac{1}{2}\partial_\mu\xi\,\partial_\nu\xi\,g^{\mu\nu}\,d^4x \\ &= \int d\eta\,d^3x\,\frac{1}{\lambda^2}\left[\left(\xi'\right)^2 - \nabla\xi\cdot\nabla\xi\right]\end{aligned} \tag{5.9.29}$$

By means of the field redefinition:

$$u(\eta, x_i) = \lambda(\eta)\xi(\eta, x_i) \tag{5.9.30}$$

after an integration by parts, the action (5.9.29) becomes:

$$\mathscr{A}[u,\lambda] = \frac{1}{2}\int d\eta\,d^3x\left[\left(u'\right)^2 - \nabla u\cdot\nabla u + \frac{\lambda''}{\lambda}u^2\right] \tag{5.9.31}$$

and its variation yields the general field equation:

$$\Box_\lambda u \equiv u'' - \nabla^2 u - \frac{\lambda''}{\lambda}u = 0 \tag{5.9.32}$$

which is formally the same as the field equation of a free scalar field in a Minkowski space with an effective time-dependent mass:

$$m^2(\eta) \equiv \frac{\lambda''}{\lambda}(\eta) \tag{5.9.33}$$

In particular a free scalar field propagating in the background cosmological metric corresponds to this class of actions with $\lambda(\eta) = a(\eta)$, which, for de Sitter space is given by (5.9.9). Hence the effective degree of freedom encoding the combined gravitational and scalar field perturbation is just a free field propagating in an effective background manifold which has just the same structure as the physical universe but with an effective scale factor $\theta(\eta)$, as defined in (5.9.27), which replaces the actual scale factor $a(\eta)$.

For the pure de Sitter case the time evolving effective mass is:

$$m_{\mathrm{dS}}^2 = \frac{a''}{a}(\eta) = \frac{2}{\eta^2} \tag{5.9.34}$$

5.9.2.2 Evaluation of the Effective Mass Term in the Slow Roll Approximation

Let us now evaluate the effective η-dependent mass term

$$m_{eff}^2(\eta) = \frac{\theta''}{\theta}(\eta) \tag{5.9.35}$$

in the slow roll approximation.

Starting from its definition in (5.9.27), the function $\theta(\eta)$ can be rewritten as:

$$\theta = \frac{\mathscr{H}}{a\varphi'} = \frac{\dot{a}\frac{dt}{da}\frac{1}{a}}{a\frac{dt}{da}\dot{\varphi}} = \frac{H}{a\dot{\varphi}} \tag{5.9.36}$$

Calculating the first derivative of θ we obtain:

$$\begin{aligned}\theta' &= a\frac{d}{dt}\left(\frac{H}{a\dot{\varphi}}\right) = a\left(\frac{\dot{H}}{a\dot{\varphi}} - \frac{H}{a^2\dot{\varphi}^2}\dot{a}\dot{\varphi} - \frac{Ha}{a^2\dot{\varphi}^2}\ddot{\varphi}\right) \\ &= \dot{\varphi} - \frac{H^2}{\dot{\varphi}} - \frac{H}{\dot{\varphi}^2}\ddot{\varphi} \approx \dot{\varphi} - \frac{H^2}{\dot{\varphi}}\end{aligned} \tag{5.9.37}$$

where we used the exact result (5.8.23) and where the last approximate equality follows from the slow-rolling condition $\ddot{\varphi} \approx 0$. Using this approximate result in the calculation of the second derivative we obtain:

$$\begin{aligned}\theta'' &= a\frac{d}{dt}\left(\dot{\varphi} - \frac{H^2}{\dot{\varphi}}\right) \simeq -a\frac{d}{dt}\frac{H^2}{\dot{\varphi}} = -a\left(\frac{2H\dot{H}}{\dot{\varphi}} - \frac{H^2}{\dot{\varphi}^2}\ddot{\varphi}\right) \\ &\approx -2\frac{H\dot{H}}{\dot{\varphi}}\end{aligned} \tag{5.9.38}$$

Consequently we obtain:

$$m_{eff}^2(\eta) \equiv \frac{\theta''}{\theta}(\eta) \overset{\text{slow roll}}{\simeq} -2a^2\dot{H} \tag{5.9.39}$$

During an almost exponential expansion the scale factor behaves as:

$$a \sim -\frac{1}{H\eta} \tag{5.9.40}$$

Hence we conclude that in the slow roll approximation we have

$$m_{\text{slow-roll}}^2 \approx \frac{2}{\eta^2} \times \mu; \quad \mu \equiv \left(-\frac{\dot{H}}{H^2}\right) \tag{5.9.41}$$

In other words the effective scalar degree of freedom propagates as if it were in de Sitter space yet with an effective η-dependent mass term depressed by the almost constant factor μ which is very small since $\dot{H} \ll H^2$ in the slow roll phase. Note also that μ is positive since $\dot{H}$ is negative. Indeed in the slow rolling phase the expansion is slowly decelerating.

5.9.2.3 Derivation of the Propagation Equation

Having discussed its meaning, let us derive the propagation equation (5.9.26) of the scalar perturbation.

Combining the result (5.9.18)–(5.9.20) for the Einstein tensor with the result (5.9.22)–(5.9.24) for the stress-energy tensor we obtain a set of equations which immediately yield the following constraints. Since the perturbation δT_{ij} of the stress-energy tensor is diagonal and proportional to the Kronecker δ_{ij}, the same must be true of the corresponding Einstein tensor. This occurs if and only if:

$$\Phi = \Psi \tag{5.9.42}$$

which therefore has to be imposed. Next let us consider the implication of the equation $\delta G^0{}_i = 4\pi G \delta T^0{}_i$. This latter can be rewritten as follows:

$$\frac{1}{a^2}\partial_i\big[\Psi' + \mathscr{H}\Phi - 2\pi G\varphi'\delta\varphi\big] = 0 \tag{5.9.43}$$

Using (5.9.42) and fixing the boundary condition at some reference time, (5.9.43) implies:

$$\Psi' + \mathscr{H}\Psi - 2\pi G\varphi'\delta\varphi = 0 \tag{5.9.44}$$

By means of identities following from the equations satisfied by the background fields, the constraint (5.9.44) can be rewritten in the following way:

$$\frac{d}{\partial\eta}\left(\frac{a^2\Psi}{\mathscr{H}}\right) = 2\pi G\left(\frac{a\varphi'}{\mathscr{H}}\right)^2\left(\mathscr{H}\frac{\delta\varphi}{\varphi'} + \Psi\right) \tag{5.9.45}$$

To prove such a result we just compare the following elaborations of the l.h.s. and r.h.s. of (5.9.45).

$$\begin{aligned}
\text{l.h.s.} &= \frac{d}{\partial\eta}\left(\frac{a^2\Psi}{\mathscr{H}}\right) \\
&= \frac{2aa'}{\mathscr{H}}\Psi + \frac{a^2}{\mathscr{H}}\Psi' - \frac{a^2}{\mathscr{H}}\mathscr{H}'\Psi \\
&= 2a^2\Psi + \frac{a^2}{\mathscr{H}}\Psi' - \frac{a^2}{\mathscr{H}}\mathscr{H}'\Psi
\end{aligned} \tag{5.9.46}$$

$$\begin{aligned}
\text{r.h.s.} &= 2\pi G\left(\frac{a\varphi'}{\mathscr{H}}\right)^2\left(\mathscr{H}\frac{\delta\varphi}{\varphi'} + \Psi\right) \\
&= a^2\left[\frac{2\pi G(\varphi')^2}{\mathscr{H}^2}\left(\mathscr{H}\frac{\delta\varphi}{\varphi'}\right) + \frac{2\pi G(\varphi')^2}{\mathscr{H}^2}\Psi\right] \\
&= a^2\left[\frac{2\pi G\varphi'\delta\varphi}{\mathscr{H}} + \frac{-\mathscr{H}' + \mathscr{H}^2}{\mathscr{H}^2}\Psi\right] \\
&= a^2\left[\frac{\Psi'}{\mathscr{H}} - \frac{\mathscr{H}'}{\mathscr{H}^2} + 2\Psi\right]
\end{aligned} \tag{5.9.47}$$

showing that they are indeed equal. In going from the second to the third line of (5.9.47) we used the background field equation (5.9.8), while in going from the third to the fourth we used the constraint equation (5.9.44).

Let us next consider the equation $\delta G^0{}_0 = 4\pi G \delta T^0{}_0$. In view of (5.9.18) and (5.9.22) we can write:

$$\begin{aligned}\nabla^2 \Psi - 3\mathscr{H}\big(\Psi' + \mathscr{H}\Phi\big) &= 2\pi G\left[\varphi'\delta\varphi' - \big(\varphi'\big)^2\Phi + a^2\frac{dV}{d\varphi}\delta\varphi\right] \\ &= 2\pi G\big[\varphi'\delta\varphi' - \big(\varphi'\big)^2\Phi - \big(\varphi'' + 2\mathscr{H}\varphi'\big)\delta\varphi\big] \\ &= 2\pi G\big(\varphi'\big)^2\left[\left(\frac{\delta\varphi}{\varphi'}\right)' - 2\mathscr{H}\frac{\delta\varphi}{\varphi'} - \Phi\right] \end{aligned} \tag{5.9.48}$$

where, in stepping from the first to the second line we have used the back-ground equation (5.9.6). Using the constraint equations (5.9.42) and (5.9.44) we can further rewrite (5.9.48) as follows:

$$\begin{aligned}\nabla^2\Psi &= 2\pi G\big(\varphi'\big)^2\left[\left(\frac{\delta\varphi}{\varphi'}\right)' + \mathscr{H}\frac{\delta\varphi}{\varphi'} - \Psi\right] \\ &= \frac{2\pi G(\varphi')^2}{\mathscr{H}}\frac{d}{d\eta}\left[\mathscr{H}\frac{\delta\varphi}{\varphi'} + \Psi\right]\end{aligned} \tag{5.9.49}$$

In stepping from the first to the second line of the above equation one makes once again use of (5.9.8) and (5.9.44).

Consider now the following redefined fields:

$$u = \frac{a}{\varphi'}\Psi \tag{5.9.50}$$

$$v = 2\pi G a\left(\delta\varphi + \frac{\varphi'}{\mathscr{H}}\Psi\right) \tag{5.9.51}$$

in terms of them and of the function $z(\eta)$ of the conformal time η defined in (5.9.27), (5.9.49) and (5.9.45) can be rewritten as:

$$\nabla^2 u = z\frac{d}{d\eta}\left(\frac{v}{z}\right) \tag{5.9.52}$$

$$v = \frac{1}{z}\frac{d}{d\eta}(zu) \tag{5.9.53}$$

By taking the Laplacian ∇^2 of the latter equation and the derivative $\frac{d}{d\eta}$ of the former one obtains a system from which we can eliminate v obtaining the second order equation (5.9.26) satisfied by the field $u(\eta, x_i)$. Alternatively we can eliminate u obtaining the following field equation for the $v(\eta, x_i)$ field:

$$v'' - \nabla^2 v - \frac{z''}{z}v = 0 \tag{5.9.54}$$

This concludes the proof of what we stated above, namely that there is only one independent scalar degrees of freedom, corresponding to the free field v which propagates in the effective conformally flat space-time (5.9.28) with scale factor $\lambda(\eta) = \theta(\eta)$. Indeed the relevant point is that the fields u and v are not independent being related by a first order differential equation in η that can always be integrated yielding u in terms of v. Hence the effective field u can be quantized and the modes of both $\delta\varphi$ and Ψ can be uniquely expressed in terms of the modes of u.

5.9.3 Quantization of the Scalar Degree of Freedom

As a next step we can proceed to the canonical quantization of the scalar degree of freedom we have singled out in the previous sections. Following standard procedures we turn the classical field $u(\eta, \mathbf{x})$ into an operator-valued distribution $\hat{u}(\eta, \mathbf{x})$ and we introduce the expansion of the latter into Fourier modes:

$$u(\eta, \mathbf{x}) = \frac{1}{(2\pi)^{3/2}} \int d^3\mathbf{k} \left(\hat{a}_\mathbf{k} u_\mathbf{k}(\eta) \exp[\mathrm{i}\mathbf{k} \cdot \mathbf{x}] + \hat{a}^\dagger_\mathbf{k} \overline{u}_\mathbf{k}(\eta) \exp[-\mathrm{i}\mathbf{k} \cdot \mathbf{x}]\right) \tag{5.9.55}$$

Inserting (5.9.55) into (5.9.54) we find that for each *momentum vector* $\mathbf{k}$ the corresponding wave function $u_\mathbf{k}(\eta)$ satisfies the following equation:

$$\begin{aligned} u''_\mathbf{k} + \left(\kappa^2 - \frac{\theta''}{\theta}\right) u_\mathbf{k} &= 0 \\ \kappa^2 &= \mathbf{k} \cdot \mathbf{k} \end{aligned} \tag{5.9.56}$$

We consider the case where the mass term $\frac{\theta''}{\theta}$ takes the slow-rolling approximate form derived in (5.9.41) and we obtain the following equation:

$$u''_\mathbf{k} + \left(\kappa^2 - \frac{2}{\eta^2}\mu\right) u_\mathbf{k} = 0 \tag{5.9.57}$$

where the parameter μ allows to interpolate between various notable cases. If $\mu = 0$ we are actually discussing propagation in Minkowski space. For $\mu = 1$ we retrieve the propagation equation in de Sitter space (see (5.9.34)). For all the intermediate values $0 < \mu < 1$ we describe propagation in the background of a slow-rolling universe and the almost constant small parameter μ is given in (5.9.41).

Equation (5.9.57) is actually the Bessel equation in slightly modified variables and its solutions can be constructed by means of Bessel functions for all values of μ. Introducing the index

$$\nu = \frac{1}{2}\sqrt{1 + 8\mu} \tag{5.9.58}$$

we can easily verify that the following two functions

$$\psi_\pm(\eta, \kappa, \nu) = -\frac{\sqrt{\pi}}{2\kappa}\sqrt{\eta}\left(J_\nu(\eta\kappa) \pm \mathrm{i} Y_\nu(\eta\kappa)\right) \tag{5.9.59}$$

form a complete basis of solutions of the second order equation (5.9.57), namely we have:

$$u_{\mathbf{k}}(\eta) = c^{+}\psi_{+}(\eta,\kappa,\nu) + c^{-}\psi_{-}(\eta,\kappa,\nu) \tag{5.9.60}$$

where $c_{\pm}$ are the two integration constants to be fixed by means of boundary conditions.

In view of the remarks put forward few lines above, we expect that the lower and upper extremes of the μ interval, namely $\mu = 0 \Leftrightarrow \nu = \frac{1}{2}$ and $\mu = 1 \Leftrightarrow \nu = \frac{3}{2}$ should present distinguished features corresponding to Minkowski and de Sitter space, respectively. Indeed we find:

$$\psi_{\pm}\left(\eta,\kappa,\frac{1}{2}\right) = \pm \mathrm{i}\frac{e^{\pm \mathrm{i}\eta\kappa}}{\kappa\sqrt{2\kappa}} \tag{5.9.61}$$

$$\psi_{\pm}\left(\eta,\kappa,\frac{3}{2}\right) = \frac{e^{\pm \mathrm{i}\eta\kappa}(1 \pm \frac{\mathrm{i}}{\eta\kappa})}{\kappa\sqrt{2\kappa}} \tag{5.9.62}$$

In Minkowski space the wave function is a pure phase, its modulus being constant and equal to $\frac{1}{\sqrt{2\kappa}}$. In de Sitter space there are two regimes. For $\kappa\eta \gg 1$ the wave function behaves as in Minkowski space with an oscillating phase and a constant modulus. For small values of $\kappa\eta$, instead, the modulus of the wave function diverges as $\frac{1}{\kappa\eta}$.

As we are going to see shortly below, these two regimes have a profound cosmological significance, being related with the distinction between *frozen modes* that have exited the event-horizon and *active modes* which, being within the horizon, are subject to modification by means of interactions with the other modes. This two-regime structure of the de Sitter solution is actually generic for all values of μ and follows from the asymptotic expansions of Bessel functions at low and large values of their arguments. Indeed we have:

$$\psi_{\pm}(\eta,\kappa,\nu) \overset{\kappa\eta\to 0}{\approx} \frac{2^{-\nu-1}\sqrt{\eta}(\pm \mathrm{i}4^{\nu}(\frac{1}{\eta\kappa})^{\nu}\Gamma(\nu)\Gamma(\nu+1) - \pi(\eta\kappa)^{\nu})}{\kappa\sqrt{\pi}\,\Gamma(\nu+1)} \tag{5.9.63}$$

$$\psi_{\pm}(\eta,\kappa,\nu) \overset{\kappa\eta\to \infty}{\approx} \frac{(-1)^{3/4}e^{\pm \mathrm{i}(\eta\kappa - \frac{\pi\nu}{2})}}{\kappa\sqrt{\pi\kappa}} \tag{5.9.64}$$

Let us now implement the canonical quantization of the free-field system by imposing the standard canonical commutation relations on the creation-annihilation operators:

$$\left[\hat{a}_{\mathbf{k}}, \hat{a}^{\dagger}_{\mathbf{k}'}\right] = \delta^{3}\left(\mathbf{k} - \mathbf{k}'\right) \tag{5.9.65}$$

This corresponds to the standard canonical equal time commutation relations:

$$\left[\hat{u}(\eta,\mathbf{x}), \hat{\pi}_{u}(\eta,\mathbf{y})\right] = \mathrm{i}\hbar\delta^{3}\left(\mathbf{x} - \mathbf{y}'\right) \tag{5.9.66}$$

where $\hat{\pi}_u(\eta, \mathbf{y}) = \partial_\eta \hat{u}(\eta, \mathbf{x})$, if the wave-function (5.9.60) is properly normalized in such a way that:

$$u_{\mathbf{k}}(\eta)\partial_\eta \overline{u}_{\mathbf{k}}(\eta) - \overline{u}_{\mathbf{k}}(\eta)\partial_\eta u_{\mathbf{k}}(\eta) = \mathrm{i}\hbar \tag{5.9.67}$$

There are many choices of the wave function (coefficients $c_\pm$) consistent with the normalization condition (5.9.67). Every such choice is associated with a different decomposition into creation and annihilation modes and therefore with a different vacuum $|0\rangle$ which, as usual, is defined by the condition:

$$\hat{a}_{\mathbf{k}}|0\rangle = 0 \tag{5.9.68}$$

A proper normalization of the wave function is provided by the observation that for late times $\eta \to \infty$, or, equivalently, for very short wave-lengths $\kappa \to \infty$, we approach an effective Minkowski scenario, where the effects of space-time curvature are negligible. Hence we can just choose the normalization of the wave-function which corresponds to the association with the creation operator of a standard outgoing wave in the late time regime, namely:

$$c^+ = 0; \qquad c^- = 1 \tag{5.9.69}$$

Having so done we are in a position to calculate the two-point correlation function of the field $\hat{u}(\eta, \mathbf{x})$ or better of the gravitational potential $\widehat{\Phi}(\eta, \mathbf{x})$, which is related to $\hat{u}(\eta, \mathbf{x})$ by (5.9.50).

Setting:

$$\sigma_{\mathbf{k}} = \frac{\varphi'}{a} u_{\mathbf{k}} \tag{5.9.70}$$

by means of a standard calculation we find:

$$\begin{aligned}
\langle 0|\widehat{\Phi}(\eta, \mathbf{x})\widehat{\Phi}(\eta, \mathbf{y})|0\rangle &= \frac{1}{(2\pi)^3}\int d^3\mathbf{k}\int d^3\mathbf{k}'\,\delta^{(3)}\big(\mathbf{k}+\mathbf{k}'\big) \times |\sigma_{\mathbf{k}}|^2 \exp\big[\mathrm{i}\big(\mathbf{k}\cdot\mathbf{x}+\mathbf{k}'\cdot\mathbf{y}\big)\big] \\
&= \frac{1}{(2\pi)^3}\int d^3\mathbf{k}\exp\big[\mathrm{i}\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})\big]|\sigma_{\mathbf{k}}|^2 \\
&= \frac{1}{2\pi^2}\int \kappa^2\,d\kappa\,d\cos\theta\,d\phi\exp\big[\mathrm{i}\kappa|\mathbf{x}-\mathbf{y}|\cos\theta\big]|\sigma_\kappa|^2 \\
&= \frac{1}{2\pi^2}\int \mathscr{P}_\Phi(\kappa)\frac{\sin(\kappa r)}{\kappa r}\frac{d\kappa}{\kappa}
\end{aligned} \tag{5.9.71}$$

where, in the last line, we have used the definitions:

$$r \equiv \sqrt{(\mathbf{x}-\mathbf{y})\cdot(\mathbf{x}-\mathbf{y})} \tag{5.9.72}$$

$$\mathscr{P}_\Phi(\kappa) \equiv \big|\sigma_{\mathbf{k}}(\eta)\big|^2\kappa^3 = \left(\frac{\varphi'}{a}\right)^2\big|u_\kappa(\eta)\big|^2\kappa^3 \tag{5.9.73}$$

The function $\mathscr{P}_\Phi(\kappa)$ is named the power spectrum and it is the main target of all calculations since, supposedly, it is an experimentally accessible datum through the observation of anisotropies in the cosmic background radiation.

5.9.4 Calculation of the Power Spectrum in the Two Regimes

Let us now consider the power spectrum for short and long wave-lengths respectively.

5.9.4.1 Short Wave-Lengths

According to our previous discussion in the short wave-length regime, which can be defined as

$$\kappa\eta \gg 1 \tag{5.9.74}$$

we just have $|u_\kappa(\eta)|^2 \simeq \frac{1}{\pi\kappa^3}$ so that we find:

$$\begin{aligned}\mathscr{P}_\Phi(\kappa) &\overset{\kappa\eta\gg 1}{\simeq} \left(\varphi'\right)^2 \left(\frac{1}{a}\right)^2 \\ &= -\frac{m_P^2}{\pi}\dot{H}\end{aligned} \tag{5.9.75}$$

The last line of the above equation follows from use of the exact result (5.9.8) and further transformation of the η-derivatives into t-derivatives.

5.9.4.2 Long Wave-Lengths

The method to obtain information on the wave-function and hence on the power spectrum for long wave-lengths:

$$\kappa\eta \ll 1 \tag{5.9.76}$$

relies on solving once again the propagation equation in the approximation $\kappa^2 \to 0$. This means that in (5.9.56) we forget the term in κ^2 and we are left with the equation:

$$u''_{\mathbf{k}} - \frac{\theta''}{\theta} u_{\mathbf{k}} = 0 \tag{5.9.77}$$

A basis of two independent solutions of the above ordinary differential equation of the second order is immediately found as follows:

$$u_1 = \theta \tag{5.9.78}$$

$$u_2 = \theta \int_{\eta_0}^{\eta} \frac{d\eta}{\theta^2} \tag{5.9.79}$$

Indeed one can easily verify that the Wronskian of these two solutions is:

$$u_1 u'_2 - u_2 u'_1 = 1 \tag{5.9.80}$$

Correspondingly we can write the generic solution of (5.9.77) as follows:

$$\begin{aligned} u_{\mathbf{k}}(\eta) &= c_1\theta + c_2\theta \int_{\eta_0}^{\eta} \frac{d\eta}{\theta^2} \\ &= \mathscr{A}_{\mathbf{k}}\theta \int_{\overline{\eta}_0}^{\eta} \frac{d\eta}{\theta^2} \end{aligned} \tag{5.9.81}$$

Indeed the integral $\int_{\overline{\eta}_0}^{\eta_0} \frac{d\eta}{\theta^2}$ is just some number so that the contribution from the first solution can always be reabsorbed into a redefinition of the initial point of integration. The integration constant $\mathscr{A}_{\mathbf{k}}$ has instead to be fixed by means of boundary conditions. Using the exact result (5.9.8) in the definition (5.9.27) of the function θ we can rewrite it as it follows:

$$\theta = \frac{1}{\sqrt{2\pi G}} \frac{1}{a} \left(1 - \frac{\mathscr{H}'}{\mathscr{H}^2}\right)^{-1/2} \tag{5.9.82}$$

Using this expression and the definition of the conformal Hubble function $\mathscr{H}$ we get:

$$\begin{aligned} \int \frac{d\eta}{\theta^2} &= 2\pi G \int d\eta\, a^2 \left(1 - \frac{\mathscr{H}'}{\mathscr{H}^2}\right) \\ &= 2\pi G \left[\frac{a^2}{\mathscr{H}} - \int a^2\, d\eta\right] \end{aligned} \tag{5.9.83}$$

Using this result and multiplying by $\theta = \frac{\mathscr{H}}{a\varphi'}$ and by the factor $\frac{\varphi'}{a}$ necessary to convert a u-mode into a mode of the gravitational potential we obtain the following long wave-length result:

$$\begin{aligned} \sigma_{\mathbf{k}} &\simeq \overline{\mathscr{A}}_{\mathbf{k}} \left[1 - \frac{\mathscr{H}}{a^2} \int a^2\, d\eta\right] \\ &= \overline{\mathscr{A}}_{\mathbf{k}} \left[1 - \frac{H}{a} \int a\, dt\right] = \frac{d}{dt}\left[\frac{1}{a}\int a\, dt\right] \end{aligned} \tag{5.9.84}$$

The last line follows from conversion of the η-derivatives into t-ones; furthermore we have reabsorbed the factor $2\pi G$ into the integration constant $\overline{\mathscr{A}}_{\mathbf{k}}$

Apart from the initial approximation consisting in neglecting the κ^2 term for large wave-lengths the above result is exact. No approximation about cosmic evolution has been introduced so far. When the propagation of perturbations takes place in a slow-rolling universe we are in an approximately exponential phase where:

$$a(t) \simeq \exp[Ht] \quad \Rightarrow \quad \int a\, dt \simeq \frac{1}{H} a(t) \tag{5.9.85}$$

In this regime from (5.9.84) we obtain:

$$\sigma_{\mathbf{k}} \overset{\text{slow-roll}}{\simeq} \overline{\mathscr{A}}_{\mathbf{k}} \frac{d}{dt}\frac{1}{H} = -\overline{\mathscr{A}}_{\mathbf{k}} \frac{\dot{H}}{H^2} \tag{5.9.86}$$

On the other hand if we consider the same Fourier component $\sigma_{\mathbf{k}}$ during the post-inflationary *radiation-dominated* era we are in an approximately square root phase where:

$$\begin{aligned} a(t) \simeq m\sqrt{t} \quad &\Rightarrow \quad \int a\, dt \simeq \frac{2}{3} m t^{\frac{3}{2}} \\ \frac{d}{dt}\left[\frac{1}{a}\int a\, dt\right] = \frac{2}{3} & \end{aligned} \tag{5.9.87}$$

and we get:

$$\sigma_{\mathbf{k}} \overset{\text{radiation era}}{\simeq} \frac{2}{3}\overline{\mathscr{A}}_{\mathbf{k}} \tag{5.9.88}$$

We conclude that, as observed, in the post-inflationary age, the power spectrum of the gravitational potential has the following form:

$$\mathscr{P}_{\Phi}(\kappa) = \begin{cases} \frac{m_P^2}{\pi}\dot{H} & \text{for } \kappa|\eta| \gg 1 \quad \Rightarrow \quad \kappa > aH \\ \frac{4}{9}|\overline{\mathscr{A}}_{\mathbf{k}}|^2\kappa^3 & \text{for } \kappa|\eta| \ll 1 \quad \Rightarrow \quad \kappa < aH \end{cases} \tag{5.9.89}$$

The last column yielding the separating condition between the short and long wave-length regimes, follows from the approximate behavior of the scale factor in the almost exponential phase of inflation. There, in conformal time, we have: $a \sim -1/(H\eta)$ and therefore $\eta \sim (aH)^{-1}$. The physical interpretation of (5.9.89) is quite clear. At every cosmic time t,

$$\lambda_{\kappa}(t) \equiv \frac{a(t)}{\kappa} \tag{5.9.90}$$

is the effective wave-length of the Fourier mode κ, which is constantly stretched by the expansion of the Universe. Short wave-lengths are those that are shorter than the Hubble radius at the same time:

$$\lambda_{\kappa}(t) < H(t)^{-1} \tag{5.9.91}$$

Long wave lengths are those larger than the Hubble radius. In an exponential phase of expansion the Hubble radius is also the event horizon (see (5.6.38)) which remains approximately constant while the scale factor and hence all the wave-lengths rapidly grow. Hence if the exponential phase lasts long enough the wave-lengths of almost all modes κ exit the Hubble radius and becomes frozen. Indeed no physical process can influence a mode whose characteristic scale is larger than the event horizon.

Quite different is the evolution of wave-lengths in radiation and matter dominated universes. In both these cases the Hubble radius grows linearly in time:

$$H(t)^{-1} \sim t \tag{5.9.92}$$

while the wave-lengths grow either as $t^{\frac{1}{2}}$ or as $t^{\frac{2}{3}}$. Hence no mode which is inside the Hubble radius (particle horizon in these cases) at some time t will exit it in the future. On the contrary modes which were out of the Hubble horizon at the end of inflation can reenter it in the subsequent radiation dominated or matter dominated era.

5.9.4.3 Gluing the Long and Short Wave-Length Solutions Together

In view of these considerations the formula (5.9.89) for the power spectrum is quite exhaustive provided we can fix the integration constant $\overline{\mathscr{A}}_{\mathbf{k}}$ which encodes all the information. This step can be achieved by equating the long and short wave-length form of the mode $\sigma_{\mathbf{k}}$ at the transition time $\kappa|\eta| = 1$, namely by setting:

$$\frac{\varphi'}{a}\frac{(-1)^{3/4}e^{\pm \mathrm{i}(\eta\kappa - \frac{\mathrm{i}\pi\nu}{2})}}{\kappa\sqrt{\pi\kappa}} \simeq -\overline{\mathscr{A}}_{\mathbf{k}}\frac{\dot{H}}{H^2} \qquad \text{at } \kappa|\eta| = 1 \tag{5.9.93}$$

From (5.9.93) we obtain:

$$\overline{\mathscr{A}}_{\mathbf{k}} = e^{\mathrm{i}\psi}\frac{m_P^2}{\kappa^3\sqrt{\pi}}\frac{H^2}{\dot{\varphi}} \tag{5.9.94}$$

where we used the exact result (5.8.23) and where $e^{\mathrm{i}\psi}$ is an η dependent phase factor whose explicit form is irrelevant since we are interested in the square modulus of $\overline{\mathscr{A}}_{\mathbf{k}}$.

In this way the long wave-length form of the power spectrum becomes:

$$\mathscr{P}_{\Phi}(\kappa) = -\frac{4}{9}\frac{m_P^2}{\pi}\left(\frac{H^4}{\dot{H}}\right)_{\kappa = aH} \tag{5.9.95}$$

The value of the above expression resides in the following. In the post inflationary age we can use (5.9.95) for all those modes whose wave-length was inside the Hubble radius at the beginning of inflation but exited it before the end of inflation. This condition gives the range:

$$(Ha)_f > \kappa > (Ha)_i \tag{5.9.96}$$

where the suffix i/f means that we have to evaluate the specified quantity at the beginning and at the end of inflation, respectively. If inflation lasts long enough and produces 60 or 70 e-foldings the range described in (5.9.96) goes over as many order of magnitudes and encompasses the whole observable universe.

The power spectrum is observed today but the Hubble function and its derivative appearing in it refer to the inflation-age, when they were almost constant.

5.9.4.4 The Spectral Index

It is customary to characterize the behavior of the power spectrum by means of a so called spectral index, defined as follows:

$$n_S = 1 + \frac{d\ln\mathscr{P}_{\Phi}(\kappa)}{d\ln\kappa} = 1 + \frac{1}{\mathscr{P}_{\Phi}}\kappa\frac{d}{d\kappa}\mathscr{P}_{\Phi} \tag{5.9.97}$$

Imagine that the power spectrum has a power-like behavior:

$$\mathscr{P}_\Phi(\kappa) \sim \kappa^\alpha \tag{5.9.98}$$

then the spectral index would just be:

$$n_S = 1 + \alpha \tag{5.9.99}$$

In case of scale invariant spectra, namely $\mathscr{P}_\Phi(\lambda\kappa) = \mathscr{P}_\Phi(\kappa)$, the spectral index is exactly $n_S = 1$. It is very interesting that, by implementing the slow-roll approximation, the spectral index can be calculated and related to the slow-roll parameters of the potential. To this effect let us observe that, by definition, we have $d\ln\kappa = d\ln(aH)$. On the other hand in the slow roll approximation $d\ln(aH) \simeq d\ln a$ and we have:

$$\frac{d\ln a}{d\phi} = \frac{d\ln a}{dt}\frac{dt}{d\phi} = \frac{H}{\dot{\phi}} \simeq -\frac{3H^2}{W'} \simeq -\frac{2W}{W'} \tag{5.9.100}$$

In the above equations ϕ and W are the dimensionless scalar field and the dimensionless potential, introduced in (5.8.19) and we have used the two slow-roll equations:

$$H^2 \simeq \frac{2}{3}W(\phi); \qquad \dot{\phi} \simeq -\frac{W'}{3H} \tag{5.9.101}$$

Using these tools we can rewrite:

$$\frac{d\ln\mathscr{P}_\Phi(\kappa)}{d\ln\kappa} = -\frac{W'(\phi)}{2W}\frac{d}{d\phi}\mathscr{P}_\Phi(\kappa) \tag{5.9.102}$$

Substituting in (5.9.102) the expression (5.9.95) of the power-spectrum and using once again the slow-roll approximation (5.9.101) we finally obtain:

$$\frac{d\ln\mathscr{P}_\Phi(\kappa)}{d\ln\kappa} = -\frac{1}{2}\left(3\left(\frac{W'}{W}\right)^2 - 2\frac{W''}{W}\right) \tag{5.9.103}$$

This result immediately yields:

$$n_S = 1 + 6\eta_W - 9\varepsilon_W \tag{5.9.104}$$

where we used the slow-roll parameters of the potential defined in (5.8.27) and (5.8.30).

Thus the shape of the primeval inflationary potential defines the behavior of the power spectrum occurring in the two-point function of the quantized scalar field. The power spectrum, on its turn, is an experimentally accessible datum since it is directly related to the anisotropies of the Cosmic Microwave Background. How this can happen is outlined in the next section.

5.10 The Anisotropies of the Cosmic Microwave Background

Let us consider a spatially flat Universe described, in the conformal frame, by the following metric:

$$ds^2 = a^2(\eta)\big(d\eta^2 - d\mathbf{x}\cdot d\mathbf{x}\big) \tag{5.10.1}$$

Small perturbations of the above metric can be organized according to their spin ($s = 0, 1, 2$). Dominant contributions to physically observable phenomena are the lowest spin ones, namely the scalar fluctuations. As we have seen in Sect. 5.9, modulo gauge-transformations corresponding to diffeomorphisms, such scalar perturbations can be encoded into a potential function $\Phi(\eta, \mathbf{x})$ that deforms the homogeneous isotropic metric (5.10.1) in the following way:

$$ds^2_{pert} = a^2(\eta)\big[(1+2\Phi)\,d\eta^2 - (1-2\Phi)\,d\mathbf{x}\cdot d\mathbf{x}\big] \tag{5.10.2}$$

Perturbation means that $\Phi(\eta, \mathbf{x}) \ll 1$. The relativistic potential Φ plays a role very similar to that of the Newtonian potential and it describes local variations of the average gravitational field that can be somewhat stronger here and somewhat weaker there. In Sect. 5.9 we discussed the relation between Φ and the single *quantized* scalar degree of freedom of the Einstein-Klein-Gordon system. It is an incredibly interesting fact that such inhomogeneities of the average gravitational field can be directly observed as fluctuations in the temperature of the cosmic background radiation. Not only that: the relation between temperature fluctuations and fluctuations of the gravitational field is preserved during cosmic evolution so that, by observing present day inhomogeneities of the CMB, we directly measure the inhomogeneities of the gravitational field at the time of recombination or *last scattering*, namely when electromagnetic radiation fell off thermal equilibrium with respect to baryon matter. According to the thermal history of the Universe, this happened about 400 thousand years after the Big Bang, namely about 14 billion years ago. This crucial link between Φ and the temperature fluctuations is named the *Sachs-Wolfe effect* whose derivation is the issue addressed in the following subsection.

5.10.1 The Sachs-Wolfe Effect

Let us define the distribution function $f(x^i, p_i, \eta)$ which informs us about the number of photons dN at time η, which have three-momentum p_i at place x^i:

$$dN = f\big(x^i, p_i, \eta\big)\, d^3\mathbf{x}\, d^3\mathbf{p} \tag{5.10.3}$$

When we deal with a black-body radiation, like the cosmic background one, the distribution function is Planckian and we have:

$$f = f\big(\omega, T(\mathbf{x}, \mathbf{n})\big) \equiv \frac{2}{\exp[\frac{\omega}{T(\mathbf{x},\mathbf{n})}] - 1} \tag{5.10.4}$$

where $T(\mathbf{x}, \mathbf{n})$ is the temperature, which can depend both on the place $\mathbf{x}$ and on the direction $\mathbf{n}$ in which we observe the thermal spectrum, and ω is the energy of the considered photon. Calling u^α the four-velocity of the observer and p_α the four-momentum of the photon the energy measured by the observer is given by:

$$\omega = p_\alpha u^\alpha \tag{5.10.5}$$

Naming $\mathbf{p} = p_i$ the spatial part of the momentum and calling:

$$p \equiv \sqrt{\sum_{i=1}^{3} p_i^2} = \sqrt{\mathbf{p} \cdot \mathbf{p}} \tag{5.10.6}$$

we can easily calculate ω in the reference frame where the observer is at rest, namely $u^\alpha = \{\sqrt{g_{00}}, 0, 0, 0\}$. Since the photon is a massless particle we have $p_\alpha p_\beta g^{\alpha\beta} = 0$ which in the metric (5.10.2) implies

$$p_0 = \sqrt{\frac{1+2\Phi}{1-2\Phi}}\, p \tag{5.10.7}$$

so we get:

$$\omega = \frac{p_0}{\sqrt{g_{00}}} = \frac{p}{a\sqrt{1-2\Phi}} \underbrace{\simeq \frac{1+\Phi}{a} p}_{\Phi \ll 1} \tag{5.10.8}$$

the last identity corresponding to the first order contribution in the perturbation Φ.

For any metric, the distribution function must obey the Boltzmann transport equation:

$$0 = \frac{\partial f}{\partial \eta} + \frac{dx^i}{d\eta}\frac{\partial f}{\partial x^i} + \frac{dp_i}{d\eta}\frac{\partial f}{\partial p_i} = \frac{d}{d\eta} f\big(\mathbf{x}(\eta), \mathbf{p}(\eta), \eta\big) \tag{5.10.9}$$

which, as specified by the last equality in (5.10.9), is the statement that the total time derivative of f should vanish so that the total number of photons in the Universe is conserved.

Equation (5.10.9) can be simplified using the explicit form of the metric and the equation for null geodesics that are those traveled by the photons. Naming λ an affine parameter along the geodesics, the four-momentum vector of the photon can be identified as:

$$p^\alpha = \frac{dx^\alpha}{d\lambda}; \qquad p_\alpha = g_{\alpha\beta}\frac{dx^\beta}{d\lambda} \tag{5.10.10}$$

and, relying on the explicit form of the Christoffel symbols, the geodesic equation takes the following form:

$$\frac{d}{d\lambda} p_\alpha = \frac{1}{2}\partial_\alpha g_{\beta\gamma} p^\beta p^\gamma \tag{5.10.11}$$

Then (5.10.9) is rewritten as:

$$0 = \frac{\partial f}{\partial \eta} + \frac{d\lambda}{d\eta}\left[p^i \frac{\partial f}{\partial x^i} + \frac{1}{2}\partial_i g_{\beta\gamma} p^\beta p^\gamma \frac{\partial f}{\partial p_i}\right] \tag{5.10.12}$$

In the metric (5.10.2), from the null-like condition $p_\alpha p_\beta g^{\alpha\beta} = 0$ we derive the result:

$$\frac{d\lambda}{d\eta} = \frac{a^2\sqrt{1-4\Phi^2}}{p} \tag{5.10.13}$$

and:

$$\frac{1}{2}\partial_i g_{\beta\gamma} p^\beta p^\gamma = \frac{2p^2}{a^2(1+2\Phi)(1-2\Phi)^2}\partial_i \Phi \tag{5.10.14}$$

so that (5.10.12) becomes:

$$0 = \frac{\partial f}{\partial \eta} + \sqrt{\frac{1+2\Phi}{1-2\Phi}} n^i \frac{\partial f}{\partial x^i} + \frac{2p}{(1+2\Phi)^{\frac{1}{2}}(1-2\Phi)^{\frac{3}{2}}}\partial_i \Phi \frac{\partial f}{\partial p_i} \tag{5.10.15}$$

where we have introduced the directional unit vector:

$$n^i = -\frac{p_i}{p} \tag{5.10.16}$$

Developing (5.10.15) to first order in the small perturbation Φ we obtain the approximate transport equation:

$$0 = \frac{\partial f}{\partial \eta} + (1+2\Phi) n^i \frac{\partial f}{\partial x^i} + 2p\partial_i \Phi \frac{\partial f}{\partial p_i} \tag{5.10.17}$$

Let us apply the above transport equation to the Planckian distribution function (5.10.4). It reads as follows:

$$0 = \frac{\partial Q}{\partial \eta} + (1+2\Phi) n^i \frac{\partial Q}{\partial x^i} + 2p\partial_i \Phi \frac{\partial Q}{\partial p_i} \tag{5.10.18}$$

where we have defined:

$$Q = \frac{\omega}{T} \tag{5.10.19}$$

The quantity Q can be developed in power series of the perturbations. As for the photon energy ω we have already derived such a development in (5.10.8) that can be restated as follows:

$$\omega \simeq \omega_0(\eta) + \delta\omega(\eta, \mathbf{x}) = \frac{p}{a(\eta)} + \frac{p}{a(\eta)}\Phi(\eta, \mathbf{x}) \tag{5.10.20}$$

where the 0th order term ω_0 depends only on time being homogeneous and isotropic, while the perturbation $\delta\omega(\eta, \mathbf{x})$ varies from place to place. We can introduce a similar development for the CMB temperature:

$$T \simeq T_0(\eta) + \delta T(\eta, \mathbf{x}) \tag{5.10.21}$$

Combining (5.10.20) and (5.10.21) we obtain:

$$Q \simeq Q_0 + \delta Q = \frac{p}{aT_0} + \frac{p}{aT_0}\left(\Phi - \frac{\delta T}{T_0}\right) \tag{5.10.22}$$

Inserting the above development of Q into the transport equation (5.10.18), at 0th order we obtain:

$$\frac{\partial}{\partial \eta}\left[a(\eta)T_0(\eta)\right]^{-1} = 0 \tag{5.10.23}$$

while at first order in the perturbations we get:

$$0 = \left(\frac{\partial}{\partial \eta} + n^i \frac{\partial}{\partial x^i}\right)\left(\Phi + \frac{\delta T}{T_0}\right) = 2\frac{\partial \Phi}{\partial \eta} \tag{5.10.24}$$

Equation (5.10.23) implies that the average temperature of the cosmic background radiation is a monotonic function of the cosmic time and decreases inversely to the scale factor while the Universe expands. In other words the temperature T_0 is a very precise cosmological clock. On the other hand (5.10.24) for the perturbation has a very simple and most profound interpretation. It suffices to note that the operator

$$\left(\frac{\partial}{\partial \eta} + n^i \frac{\partial}{\partial x^i}\right) = \frac{d}{d\eta} \tag{5.10.25}$$

is just a total time derivative. Hence if the relativistic potential $\Phi(\mathbf{x})$ depends only on space and not on time, then the combination

$$\left(\Phi + \frac{\delta T}{T_0}\right) = \text{const} \tag{5.10.26}$$

is constant not only in time but also in space, as a consequence of (5.10.24). Therefore, measuring the inhomogeneities of the CMB temperature $\frac{\delta T}{T_0}$ at our time is the same thing as measuring the inhomogeneities of the gravitational potential Φ at the last scattering time 14 billions of years ago. The spectrum of such perturbations is predicted by the theory of inflation which therefore becomes, to a certain extent, experimentally verifiable.

5.10.2 The Two-Point Temperature Correlation Function

What is actually observed by CMB experiments is the spatial distribution of the temperature fluctuations, namely $\frac{\delta T}{T_0}(\mathbf{n})$, having denoted by $\mathbf{n}$ a unit vector on the three-sphere, just as we did in the previous section. Using these data, that are visualized in sky-maps like that of Fig. 4.24, one can construct the correlation function:

$$C(\theta) \equiv \left\langle \frac{\delta T}{T_0}(\mathbf{n}_1)\frac{\delta T}{T_0}(\mathbf{n}_2)\right\rangle \tag{5.10.27}$$

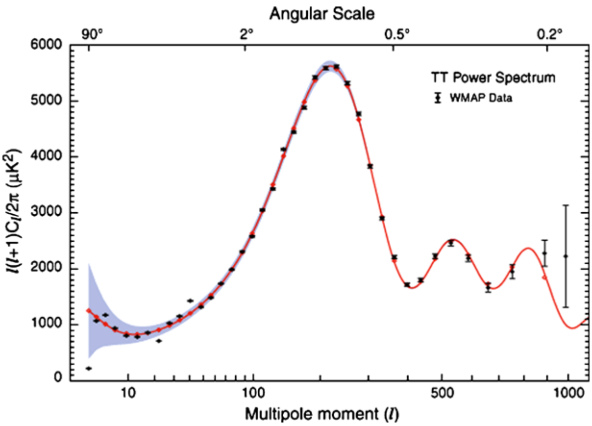

Fig. 5.31 Dependence of the CMB anisotropy multipole moments on ℓ, as measured by the WMAP satellite

where the bracket $\langle\rangle$ denotes averaging over all directions of the sky $\mathbf{n}_1$ and $\mathbf{n}_2$ satisfying:

$$\mathbf{n}_1 \cdot \mathbf{n}_2 = \cos\theta \tag{5.10.28}$$

Next the correlation function is expanded in multipoles by setting:

$$C(\theta) = \frac{1}{4\pi} \sum_{\ell=2}^{\infty} (2\ell + 1) C_\ell P_\ell(\cos\theta) \tag{5.10.29}$$

and the experimental data are encoded in the angular momentum dependence ℓ of the multipole moment C_ℓ, producing a graph such as that displayed in Fig. 5.31 (see [10–14]). Note that the multipole expansion excludes the first two moments, the monopole $\ell = 0$ and the dipole $\ell = 1$, which are sensitive to the position of the Sun in the Galaxy and to its motion around the Galaxy center. All the other moments automatically exclude these effects and provide therefore a clean information on primeval perturbations. The existence of the Sachs Wolfe effect allows to write down an analytic formula which predicts the multipole coefficients in terms of power spectrum we discussed in the previous section. Explicitly one finds:

$$C_\ell = \frac{2}{\pi} \int \left| \left(\sigma_\kappa(\eta_r) + \frac{\delta_\kappa(\eta_r)}{4} \right) j_\ell(\kappa\eta_0) - \frac{3\delta'_\kappa(\eta_r)}{4\kappa} \frac{dj_\ell(\kappa\eta_0)}{d(\kappa\eta_0)} \right|^2 \kappa^2 \, d\kappa \tag{5.10.30}$$

The ingredients entering the above formula are:

1. By η_r we denote the conformal time of recombination, after which the background radiation fell off equilibrium with matter.
2. By $\eta_0 r$ we denote the present conformal time at which we observe CMB.
3. By $\sigma_\kappa(\eta)$ we denote the Fourier component of the scalar potential Φ defined in (5.9.70), from which the power spectrum is calculated according to (5.9.73).
4. By $j_\ell(r)$ we denote the spherical Bessel functions.
5. The two functions $\delta_\kappa(\eta_r)$ and $\delta'_\kappa(\eta_r)$ encode a description of the CMB temperature fluctuations at the time of recombination.

The interested reader can find a detailed derivation of the above formula and a full-fledged discussion of its consequences and applications in Chap. 8 of the recent book [9] on the Physical Foundations of Cosmology by Mukhanov, who is one of the main actors in the new season of Theoretical Cosmology opened by the observation of the CMB anisotropies. Here we confine ourselves to sketch the logical connection between the power spectrum of primordial fluctuations and the correlation function of observed anisotropies.

As we already stressed, the main point is the Sachs Wolfe effect (5.10.26) which implies:

$$\underbrace{\frac{\delta T}{T_0}(\eta_0, \mathbf{x}_0, \mathbf{n})}_{\text{fluc. today}} = \underbrace{\frac{\delta T}{T_0}\big(\eta_r, \mathbf{x}(\eta_r), \mathbf{n}\big)}_{\text{fluc. at ricom}} + \underbrace{\Phi\big(\eta_r, \mathbf{x}(\eta_r)\big)}_{\text{grav. pot. at recom}} - \Phi\big(\eta_r, \mathbf{x}(\eta_r)\big) \qquad (5.10.31)$$

Here $\mathbf{x}(\eta)$ denotes the geodesic followed by a photon that arrives today into our measuring instrument from a direction $\mathbf{n}$ and was emitted at conformal time η_r from the Last scattering Surface. Such a geodesic is the straight line:

$$\mathbf{x}(\eta) = \mathbf{x}_0 + \mathbf{n}(\eta - \eta_0) \qquad (5.10.32)$$

An elaboration of formula (5.10.31), which we skip, allows to encode into two functions δ and δ_r the contribution of the primordial temperature fluctuations

$$\frac{\delta T}{T_0}\big(\eta_r, \mathbf{x}(\eta_r), \mathbf{n}\big)$$

leading to the following Fourier decomposition of the temperature fluctuations at the present time:

$$\frac{\delta T}{T_0}(\eta_0, \mathbf{x}_0, \mathbf{n}) = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \left[\left(\Phi + \frac{\delta}{4}\right)_{\mathbf{k}} - \frac{3\delta'_{\mathbf{k}}}{4\kappa^2}\frac{\partial}{\partial \eta_0} \right]_{\eta_r} \exp\big[i\mathbf{k}\cdot\big(\mathbf{x}_0 + \mathbf{n}(\eta_r - \eta_0)\big)\big] \qquad (5.10.33)$$

where the suffix $\mathbf{k}$ means that of the corresponding expression one has taken the $\mathbf{k}$th Fourier component. Inserted into the correlation function (5.10.27), the above decomposition of the temperature fluctuation field yields the result (5.10.30) upon use of the standard multipole expansion formula:

$$\frac{\sin(\kappa|\mathbf{n}_1\eta_1 - \mathbf{n}_2\eta_2|)}{\kappa|\mathbf{n}_1\eta_1 - \mathbf{n}_2\eta_2|} = \sum_{\ell=0}^{\infty}(2\ell+1)\, j_\ell(\kappa\eta_1)\, j_\ell(\kappa\eta_2)\, P_\ell(\cos\theta) \qquad (5.10.34)$$

5.10.3 Conclusive Remarks on CMB Anisotropies

Without the ambition of presenting too much detailed derivations, which would be out of the scope of the present *Course in General Relativity*, we have tried to outline the logical path which connects the spectrum of primordial quantum fluctuations to the observed data on CMB anisotropies.

One very important feature of the so far obtained experimental data is that they are consistent with a nearly flat power spectrum, namely the best fit of the spectral index on WMAP data is the following:

$$n_S = 0.963 \pm 0.012 \tag{5.10.35}$$

Another important general result of the CMB data concerns the possibilities of determining the cosmological parameters, quite unambiguously confirming the main information $\Omega_0 = 1$, namely that our Universe is spatially flat.

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Chapter 6
Supergravity: The Principles

Tiger, tiger, burning bright
In the forests of the night,
What immortal hand or eye
Could frame thy fearful (super) symmetry?
William Blake

6.1 Historical Outline and Introduction

The year 1968 was a crucial one for the political history of the world: the Prague Spring, which had started in January, was ruthlessly suppressed by soviet tanks in August, once for all destroying the dream of communism with a human face. The student revolt, that had started in American Universities as a protest against Vietnam war, migrated to Europe and from West Berlin diffused to all European capitals, culminating in the Paris upraising of May. The events of 1968 heavily marked the history of Western Countries since nothing after that year was the same as before. Also in Physics 1968 constitutes a hallmark since in those months a seed was planted from which a robust tree developed presently going under the name of Superstring Theory.

In 1968 Gabriele Veneziano (see Fig. 6.2) was 26 of age and was temporarily at the Theoretical Division of CERN, on leave of absence from the Weizman Institute, where he had obtained his Ph.D. just the year before. At that time the theory of strong interactions was still very vague: Quantum Chromodynamics had still to be invented and the minds of physicists were fascinated by the richness of the hadronic spectrum revealed by high energy experiments. The interpretation of all those particles as stable or unstable states created by the dynamics of quarks and gluons was not yet available. On the other hand, many scientists pursued the idea of describing the *scattering amplitude* of all hadrons by means of a universal formula such that in each reaction channel the dominant contribution should come from the sum over the intermediate states, provided by a unique infinite spectrum of particles of increasing mass.

The idea, as it usually happens with the fundamentals ones, is quite simple, at least *in nuce*. Two particles A and C collide and from the collision two new particles emerge B and D. We have to calculate the probability amplitude of such an event $\mathscr{A}_{ABCD}$ as a function of the momenta of the incoming and outgoing particles.

P.G. Frè, *Gravity, a Geometrical Course*, DOI 10.1007/978-94-007-5443-0_6,

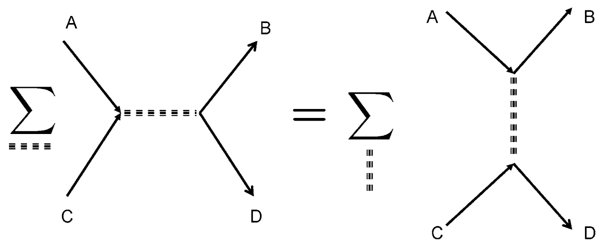

Fig. 6.1 A schematic view of Veneziano duality in hadron scattering

The process can be thought first as the fusion of A with C into an intermediate state, that successively decays into B and D. The probability of the process is essentially provided by a weighted sum over all possible intermediate states. Alternatively one could interpret the same process as the fusion of A with the antiparticle of B into an intermediate state that decays into D and the antiparticle of C. Also in this interpretation the probability is given by a weighted sum over all possible intermediate states. These two interpretations of the same process are respectively named the s and the t channel of the considered reaction (see Fig. 6.1).

The idea that fascinated the physicists of that time was the following one. Might it exist a scattering amplitude $\mathscr{A}_{ABCD}$ such the first and the second interpretation are simultaneously valid and the sum over the intermediate states in the s channel is exactly equal to the same sum in the t channel? Such a property was christened *duality* and preserves such a name to the present day.

The posed question was of a complete mathematical nature. If such a function $\mathscr{A}_{ABCD}$ existed, the next necessary step was to invent a theory capable of yielding it as scattering amplitude for the considered process.

In a paper sent to the *Rivista del Nuovo Cimento* in that dense 1968 year, Gabriele Veneziano singled out a function that realizes the desired duality in a mathematical exact way: it is the Euler beta-function introduced two hundred years before by the great swiss mathematician. The same Veneziano contributed a couple of years later, together with Sergio Fubini from Torino University and the MIT, to open the way for the identification of the physical system capable of producing *dual scattering amplitudes*. Just in a couple of years, by means of the contributions of many scientists throughout the world, Veneziano's formula for the dual scattering amplitude of four particles was generalized to processes with an arbitrary number N of external legs: in 1970, in another fundamental paper published on Nuovo Cimento, Fubini and Veneziano organized the construction of such amplitudes within a new algorithm defined *operatorial formalism* which involved the use of an infinite number of harmonic oscillators with frequencies that are integer multiples of a fundamental one.

This infinite spectrum of harmonic oscillators induced an intuition in the brilliant mind of Yoichiro Nambu (see Fig. 6.2), the same Nippon-American physicist who in 1965 had proposed the color charge for the quarks. Nambu observed that anyone who is familiar with string musical instruments perfectly knows a very simple physical system endowed with the spectrum used by Fubini and Veneziano: the vibrating string. A very short and tiny string that besides traveling through space-time can also

Fig. 6.2 The fathers of string theory. From the *left* Gabriele Veneziano, in the *middle* Sergio Fubini, on the *right* Yoichiro Nambu. Gabriele Veneziano was born in Florence, where he studied before transferring to the Weizman Institute in Israel, where he got his Ph.D. Later on, for many years he was permanent staff member of the Theoretical Division of CERN, which he left at retirement age to fill a highly prestigious position at the French Academy in Paris. Sergio Fubini was born in Torino, where he studied and became quite early full professor of Theoretical Physics. Appointed professor of Physics at the Massachusetts Institute of Technology, he lived several years in Boston, until he left it to become permanent staff member of the Theoretical Division of CERN. After retirement he continued to live in Geneva where he died in 2005. Yoichiro Nambu, born in Japan, studied in the United States of America and up to the present day has been full professor of Physics at the University of Chicago. In 2008, professor Nambu was awarded the Nobel prize in Physics for his early contributions to the theory of symmetry breaking

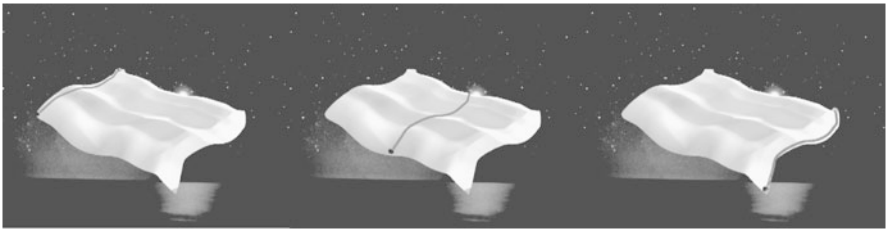

Fig. 6.3 An idealized view of an open string that propagates through space-time, tracing a world-sheet with the topology of a strip

vibrate! This had to be the typical hadron! The infinite spectrum of hadronic states and resonances was thus explained with the infinite number of vibrational modes of the microscopic string. Once started, the string concert rapidly grew and developed. In a series of papers produced by several authors from all countries of the world, the physical system of the quantum-relativistic string was analyzed from all viewpoints. The string can be *closed* or *open*, namely its end points can coincide, or not. In the first case the string has the topology of a circle, in the second that of a segment. In both cases propagating through an ambient space-time the strings sweeps a world-sheet that in the closed case has the topology of a cylinder, in the second case that of a strip with boundary (see Figs. 6.3, 6.4). The interpretation of Veneziano ampli-

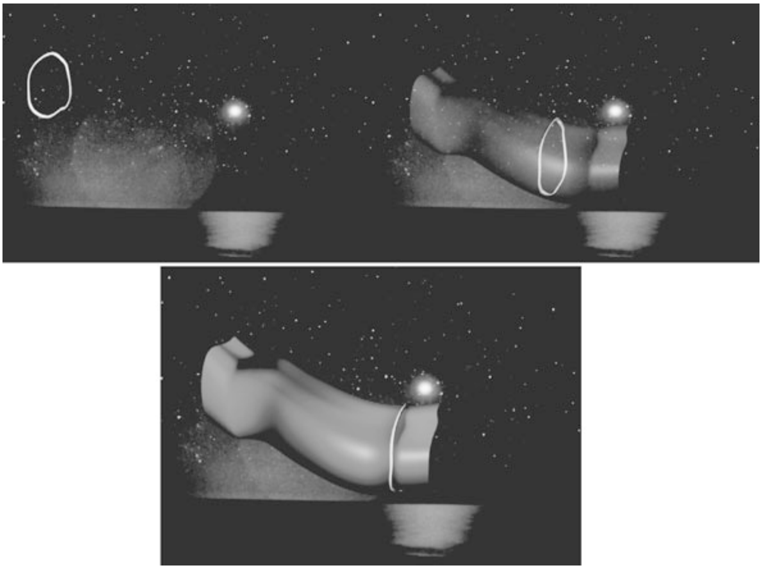

Fig. 6.4 An idealized view of a closed string that propagates through space-time, tracing a world-sheet with the topology of a tube

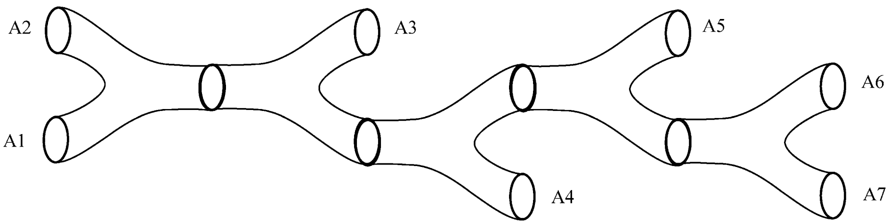

Fig. 6.5 The closed string interpretation of a scattering amplitude of seven particles

tudes as the result of the propagation of tiny strings that can join and split became standard and it is schematically illustrated in Fig. 6.5. In the quantization of the system two problems were met, whose solution led the theory very far in the direction of unexpected scenarios of incredible mathematical depth and unparalleled wealth of physical implications. The first problem related with the number of space-time dimensions. The usual four-dimensional space-time was too narrow for the strings to propagate freely without developing deadly anomalies capable of destroying the quantum consistency of the two-dimensional world-sheet theory. In quantum field theory anomalies are obstructions that forbid the extension to the quantum level of global or local symmetries present at the classical level. In the case of local symmetries, anomalies are deadly blows since the quantum theory acquires spurious degrees of freedom and becomes both meaningless and inconsistent. In the case of the strings the anomalous symmetry is the conformal one, namely the invariance against transformations that rescale lengths and shapes preserving only angles. It was discovered that the quantum string is free from conformal anomalies only if it propagates in a space-time with 26 dimensions, precisely one time and twenty-five space directions.

The second problem met in the stringy interpretation of Veneziano amplitudes related with the absence of fermions. So far the harmonic oscillators associated with the string vibrational modes were just bosonic and no state corresponding to particles with half-integer spin could be constructed. Yet the hadronic spectrum contains both bosons and fermions. Where were the latter hidden? The answer to both questions came soon and it opened new broad horizons.

6.1.1 Fermionic Strings and the Birth of Supersymmetry

To the question where in the theory of tiny strings the fermions were hidden, Neveu and Schwarz on one side and Pierre Ramond on the other (see Fig. 6.6) gave two answers which, although different, are not alternative, rather complementary. The followed approach was algebraic in both cases.

In 1971 while traveling back from Europe to the US on board of a big ship, John Schwarz met André Neveu[1] and during the Atlantic crossing they found a generalization of the infinite dimensional symmetry algebra of Nambu string that goes under the name of Virasoro algebra.[2] In its original approach Virasoro found that the physical string states constructed with the harmonic oscillators of the Fubini Veneziano approach should be annihilated by the action of an infinite number of operators L_n, that are in one-to-one correspondence with the integer numbers: $n \in \mathbb{Z}$:

$$\forall n \in \mathbb{Z}: \quad L_n|\text{phys.}\rangle = 0 \tag{6.1.1}$$

and satisfy the following infinite dimensional Lie algebra:

$$\begin{aligned} [L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}\big(m^3 - m\big)\delta_{m+n,0} \\ [c, L_n] &= 0 \end{aligned} \tag{6.1.2}$$

The operator c commutes with all other generators of the algebra and for this reason is called the central charge. Therefore in every irreducible representation, c takes a fixed numerical value which, as Virasoro showed [3], equals the number of space-time dimensions D in which the string is propagating. The cancellation of anomalies

[1]This episode is known to the author by private conversations with Prof. Schwarz who told him this story while visiting him at his place Torino in 1981.

[2]The name of the algebra refers to his discoverer, the brilliant Italo-Argentinian physicist Miguel Virasoro born in Buenos Aires in 1940, who is presently full professor of Theoretical Physics at La Sapienza of Rome and from 1995 to 2002 was Director of the International Centre of Theoretical Physics of Trieste, founded by Abdus Salam.

Andre Neveu
(nato a Parigi 1946)

John Schwarz
(nato nel Massachusetts1941)

Pierre Ramond
(nato a Neully/Seine
nel 1943)

Fig. 6.6 The three founders of fermionic strings: on the *left* André Neveu, in the *middle* John Schwarz, on the *right* Pierre Ramond. Born in Paris in 1946, Neveu studied at the Ecole normale superériure (ENS). In the early seventies he was for some time at Princeton where he collaborated with John Schwarz and David Gross. These collaborations resulted in two very important results: the Neveu-Schwarz algebra on one side and the Gross-Neveu toy model of quantum chromodynamics on the other. Later Neveu worked at the Laboratory of Theoretical Physics of ENS in Paris, at the CERN Theory Division in Geneve and from 1989 he has been director of the Laboratory of Theoretical Physics of the University of Montpellier II. Born in Massachusetts in 1941, John Schwarz studied as an undergraduate at Harvard and as a graduate at Berkeley University. Assistant professor in Princeton from 1966 to 1972 he moved next to the California Institute of Technology where he is currently the Harold Brown Professor of Theoretical Physics. For several years one of the very few believers in superstring theory, John Schwarz was responsible for the first string revolution in 1984 when, together with Michael Green he found the mechanism of anomaly cancellation establishing the set of five consistent perturbative string theories. John Schwarz was awarded the Dirac Medal in 1989 and the Dannie Heineman Prize for Mathematical Physics in 2002. Born in France in 1943, Pierre Ramond studied in the United States where he graduated in 1969 from Syracuse University. Assistant Professor at Yale University and at the California Institute of Technology, Ramond joined the University of Florida at Gainesville in 1980. There he is currently Distinguished Professor of Physics. His contribution to the development of superstring theory has been a fundamental one, the Ramond sector being an essential part of the superstring spectrum where all the fermionic particles are located

and the quantum consistency of the theory required a value $c = 26$ which explains the unexpected result quoted above.

During their boat trip André and John derived an extension of Virasoro algebra by means of another infinite set of operators $G_{m+\frac{1}{2}}$, in one to one correspondence with the half integer numbers ($n + \frac{1}{2} \in \mathbb{Z} + \frac{1}{2}$), that satisfy the following commutation, anti-commutation relations:

$$
\begin{aligned}
[L_m, G_{n+\frac{1}{2}}] &= \left(\frac{1}{2}m - n - \frac{1}{2}\right) G_{m+n+\frac{1}{2}} \\
\{G_{m+\frac{1}{2}}, G_{n+\frac{1}{2}}\} &= 2L_{n+m+1} + \frac{c}{3}\left(m^2 + m\right)\delta_{m+n+1,0} \\
[c, G_{n+\frac{1}{2}}] &= 0
\end{aligned}
\tag{6.1.3}
$$

The union of (6.1.2) with (6.1.3) constitutes the Neveu Schwarz algebra [7] which is a first example of a *super Lie Algebra*. The novelty that entitles it to the new qualifier "*super*" is the presence of a grading that splits the set of all generators in two classes, the *even* ones (in our case the L_n) and the *odd* ones (in our case the $G_{n+\frac{1}{2}}$). The algebra is specified by providing the *commutators* of the *even* operators with the *even* ones that necessarily produces another *even* operator, of the *even*s with the *odd*s that produces an *odd* and finally the *anti-commutator* of two *odd*s that necessarily produces an *even*.

Chronologically the Neveu Schwarz superalgebra was not really the very first, since a few months before, also in 1971, Pierre Ramond had found another very similar extension of the Virasoro algebra adding to it a set of *odd generators* G_n that are in correspondence with the integer numbers. Ramond superalgebra [4] is obtained by adjoining to (6.1.2) the following commutation, anti-commutation relations:

$$\begin{aligned} [L_m, G_n] &= \left(\frac{1}{2}m - n\right) G_{m+n} \\ \{G_m, G_n\} &= 2L_{n+m} + \frac{c}{3}\left(m^2 - \frac{1}{4}\right)\delta_{m+n,0} \\ [c, G_n] &= 0 \end{aligned} \tag{6.1.4}$$

Which algebra was the right one for superstrings? Both were right since they have a common origin in an extension of Nambu string theory by means of a new fermionic field that lives on the world-sheet traced by the string while propagating through space-time. The evolution history of the string, for instance that drawn in Fig. 6.3 or in Fig. 6.4 is described by giving the coordinate X^μ of the ambient space as functions $X^\mu(\sigma, \tau)$ of the two Gaussian coordinates $\{\sigma, \tau\}$ that label the world-sheet points. So doing we can regard the world-sheet itself as a two-dimensional space-time and the functions $X^\mu(\sigma, \tau)$ as a set of scalar fields living on it. Adopting this point of view why not consider, besides spin 0 fields also new fermionic fields of spin $\frac{1}{2}$ living in the same space-time? Let us do it and let us introduce as many new fields of this type as there were scalar fields: let us name such newcomers $\Psi^\mu(\sigma, \tau)$. What has it happened at the end of such a procedure? A surprising miracle! The little field theory that we have constructed on the world-sheet has the following marvelous properties:

- It is *supersymmetric* since it is invariant under a set of appropriate transformations that exchange the bosonic fields $X^\mu(\sigma, \tau)$ with the fermionic ones $\Psi^\mu(\sigma, \tau)$.
- It is anomaly free and quantum consistent no longer in a $D = 26$ space-time, rather in a significantly smaller one $D = 10$.
- The states encompassed in the spectrum of this quantum theory divide in two sectors, one named **NS** realizes the Neveu Schwarz algebra (6.1.2)+(6.1.3), the second named **R** realizes the Ramond algebra (6.1.2)+(6.1.4). Both sectors are necessary to construct Veneziano amplitudes for bosonic and fermionic particles.

6.1.2 Supersymmetry

While trying to insert fermions into the structure of Veneziano amplitudes, a new algebraic structure had been discovered, *supersymmetry*, which was destined to mark heavily the development of Theoretical Physics in the subsequent years. Actually in the same 1971 year, the supersymmetry algebra had been constructed in a completely different context by two Russian scientists, Golfand and Likhtman [1, 2], whose almost tragic personal story is a sort of emblem of the incredible contradictions of Soviet times, also in relation with pure science.[3] Results similar to those of Golfand and Likhtman were obtained also in Kharkov, by other two Soviet scientists, Volkov and Akulov [5] who constructed a non-linear field theoretical realization of the same super Poincaré Lie algebra discovered in Moscow.

If we analyse the meaning of the operators in the Virasoro algebra and in its extensions we understand the following: the operators L_n correspond to a mode-expansion of the stress-energy tensor, namely of the Noether current of translations P_μ, while the fermionic operators G_n correspond to the mode-expansion of a spinor-vector current J^α_μ. Which symmetry is such a Noether current associated with? The answer is unique: to some new symmetry generator Q^α which transforms as a spinor under the Lorentz group and whose anti-commutator with itself must be proportional to the translation generator P_μ.

In $D = 4$ the super Lie Algebra corresponding to such a symmetry, contains $\mathscr{N}$ of such spinor generators, named supercharges, and has the following general structure:

$$
\begin{aligned}
&[J_{\mu\nu}, J_{\rho\sigma}] = -\eta_{\mu\rho} J_{\nu\sigma} + \eta_{\nu\rho} J_{\mu\sigma} - \eta_{\nu\sigma} J_{\mu\rho} + \eta_{\mu\sigma} J_{\nu\rho} \\
&[J_{\mu\nu}, P_\rho] = -\eta_{\mu\rho} P_\nu + \eta_{\nu\rho} P_\mu \\
&[J_{\mu\nu}, \overline{Q}_{A\beta}] = -\frac{1}{4}\overline{Q}_{A\alpha}(\gamma_{\mu\nu})^\alpha{}_\beta \\
&\{\overline{Q}_{A\alpha}, \overline{Q}_{B\beta}\} = \mathrm{i}\big(\mathscr{C}\gamma^\mu\big)_{\alpha\beta} P_\mu \delta_{AB} - \mathscr{C}_{\alpha\beta} Z_{AB}; \quad (A, B = 1, \ldots, \mathscr{N}) \\
&[Z_{AB}, \text{anything}] = 0
\end{aligned}
\tag{6.1.5}
$$

[3] Born in Kharkov in 1922, Yuri Abramovich Golfand got his mathematical-physical education in that Ukrainian city. Later, since 1951, he joined the Tamm group at the Lebedev Physical Institute of the Soviet Academy of Sciences in Moscow (FIAN), an institution that collected seven Nobel Prizes in Physics in the course of sixty years. Golfand and his student Likhtman conducted there, at the end of the 1960s, the studies that led them to discover the super Poincaré Lie algebra and to construct its first field theoretical realizations, published in 1971 after a long procedure of checks and inspections by the Soviet censorship authorities (see [10] for a detailed account of these facts). The next year, in the course of a routine campaign of personnel cuts, Yuri Golfand was fired from FIAN and decided to apply for an exit visa to Israel. This put him in a very bad light in front of Soviet authorities who refused the visa and treated him as a renegate. For 7 years he lived unemployed and was readmitted to FIAN only in 1980. Golfand obtained permission to emigrate to Israel only in 1990 and there he lived his last four years in Haifa, where he died in 1994. Because of his association with the renegate Golfand, also Likhtman had very difficult times with Soviet authorities and could never get a proper academic position.

Fig. 6.7 The two western founders of supersymmetric field theories. On the *left* Bruno Zumino, born in 1923 in Rome, graduated from the University La Sapienza in 1945. He is currently emeritus professor of Berkeley University in California. For many years he was permanent member of the Theoretical Division at CERN. Julius Wess born in 1934 in Oberwölz Stadt in Austria died in 2007 in Hamburg. Austrian by nationality, Wess graduated from Vienna University and was professor first in Karlsruhe University and then in the Ludwig Maximilians University of Munich. Zumino and Wess have given many important contributions to Theoretical Physics in several directions. Jointly they introduced the first example of a supersymmetric field theory that bears their name

Equations (6.1.5) define the $\mathscr{N}$-extended super Poincaré Lie algebra; the antisymmetric generators $Z_{AB} = -Z_{BA}$ which are present only for $\mathscr{N} \geq 2$ are named the central charges. The case $\mathscr{N} = 1$ is the algebra introduced by Golfand and Likhtman who also tried to construct examples of field theories invariant against transformations closing such an algebra.

In the western world the date of birth supersymmetry is 1974. In that year Bruno Zumino and Julius Wess (see Fig. 6.7) published a paper [6] where they constructed the following very simple example of a field theory with supersymmetry invariance. Let $A(x)$, $F(x)$ be two scalar fields, $B(x)$, $G(x)$ two pseudo-scalar fields and let $\lambda(x)$ be a Majorana spinor field.[4] Consider the following very simple Lagrangian:

$$\begin{aligned} \mathscr{L}_{tot} &= \mathscr{L}_{kin} + \mathscr{L}_{mass} \\ \mathscr{L}_{kin} &= -\frac{1}{2}\left(\partial_\mu A \partial^\mu A + \partial_\mu B \partial^\mu B\right) + \frac{\mathrm{i}}{2}\overline{\lambda}\gamma^\mu \partial_\mu \lambda + \frac{1}{2}\left(F^2 + G^2\right) \\ \mathscr{L}_{mass} &= -m(FA - GB) + \frac{m}{2}\overline{\lambda}\lambda \end{aligned} \tag{6.1.6}$$

The action:

$$A = \int \mathscr{L}_{tot}\, d^4x \tag{6.1.7}$$

is easily verified to be invariant under the following infinitesimal transformations:

$$\delta_\varepsilon A = \frac{1}{2}\overline{\varepsilon}\lambda$$

[4]For the definition of Majorana spinors see (A.4.3) in Appendix A.4.

$$\begin{aligned}
\delta_\varepsilon B &= -\frac{1}{2}\mathrm{i}\,\overline{\varepsilon}\,\gamma_5\lambda \\
\delta_\varepsilon\lambda &= \frac{1}{2}\mathrm{i}\partial_\mu A\gamma^\mu\varepsilon + \frac{1}{2}\partial_\mu B\gamma^\mu\gamma_5\varepsilon + \frac{1}{2}(F + \mathrm{i}\gamma_5 G)\varepsilon \\
\delta_\varepsilon F &= \overline{\varepsilon}\,\lambda \\
\delta_\varepsilon G &= \mathrm{i}\,\overline{\varepsilon}\,\gamma_5\lambda
\end{aligned} \tag{6.1.8}$$

where ε^α is a constant anti-commuting spinor parameter ($\varepsilon^\alpha\varepsilon^\beta = -\varepsilon^\beta\varepsilon^\alpha$). The above equations provide the explicit form of the supersymmetry transformations that for each field Φ of the theory can be thought as the result of acting on it with $\overline{Q}\varepsilon$, namely:

$$\delta_\varepsilon\Phi = [\overline{Q}\varepsilon, \Phi] \tag{6.1.9}$$

the operator $\overline{Q}_\alpha$ being the spinorial supercharge. The $\mathscr{N} = 1$ case of the super Lie algebra (6.1.5) is realized, since it is immediately verified that for any field Φ we have:

$$\begin{gathered}
\delta_{\varepsilon_1}\delta_{\varepsilon_2}\Phi - \delta_{\varepsilon_1}\delta_{\varepsilon_2}\Phi = \overline{\varepsilon}_1\gamma^\mu\varepsilon_2\partial_\mu\Phi \\
\Updownarrow \\
\big[[\overline{Q}\varepsilon_1, \overline{Q}\varepsilon_2], \Phi\big] = \big[\overline{\varepsilon}_1\gamma^\mu\varepsilon_2 P_\mu, \Phi\big]
\end{gathered} \tag{6.1.10}$$

and the anticommutativity of the spinor parameters implies:

$$[\overline{Q}\varepsilon_1, \overline{Q}\varepsilon_2] = \varepsilon_1^\alpha\varepsilon_2^\beta\{\overline{Q}_\alpha, \overline{Q}_\beta\} \tag{6.1.11}$$

The Wess-Zumino model encoded in (6.1.6) has a very simple physical content. It contains two spin zero degrees of freedom and one spin one-half degree of freedom, which constitute the simplest supersymmetric multiplet: $\{0^+, 0^-, \frac{1}{2}\}$. Indeed the fields F and G, named auxiliary, can be eliminated through their own equations of motion which yield:

$$F = -mA; \qquad G = mB \tag{6.1.12}$$

After substitution of these equations into the original Lagrangian we obtain the standard action for a system composed by a free scalar A of mass m, a free pseudo-scalar B with the same mass and finally by a free spinor λ also with mass m. The interesting point is that the partial actions $\int \mathscr{L}_{kin}\, d^4x$ and $\int \mathscr{L}_{mass}\, d^4x$ are separately invariant under the transformations (6.1.8). These means that the mass term $\mathscr{L}_{mass}$ can be substituted by other more complicated but invariant functions of the four fields $\{F, G, A, B, \lambda\}$ leading, after substitution of the new field equations for F and G, to more complicated dynamics.

In the years after 1974 a lot of work was devoted to constructing supersymmetric field theories with several multiplets that extend up to spin one and to exploring the general form of the interactions allowed by this new powerful symmetry. In parallel, representation theory of the supersymmetry algebra was worked out for all numbers

Table 6.1 Structure of the massless multiplets in $D = 4$ space-time. In each column we write the multiplicity of fields of spin J contained in the considered multiplet

$\mathcal{N}$	Mult.	$J = 2$	$J = \frac{3}{2}$	$J = 1$	$J = \frac{1}{2}$	$J = 0$
1	WZ mult.				1	2
	vect mult.			1	1	
	gravitino mult.		1	1		
	graviton mult.	1	1			
2	hyper.				2	4
	vect mult.			1	2	2
	gravitino mult.		1	2	1	
	graviton mult.	1	2	1		
3	vect mult.			1	4	6
	gravitino mult.		1	3	3	2
	graviton mult.	1	3	3	1	
4	vect mult.			1	4	6
	gravitino mult.		1	4	7	8
	graviton mult.	1	4	6	4	2
5	gravitino mult.		1	6	15	10
	graviton mult.	1	5	10	11	10
6	gravitino mult.		1	6	15	20
	graviton mult.	1	6	16	26	30
7, 8	graviton mult.	1	8	28	56	70

$1 \leq \mathcal{N} \leq 8$ of the supercharges and the available irreducible field representations, named *supermultiplets* were established. For instance all the massless multiplets in space-time dimensions $d = 4$ are displayed in Table 6.1.

6.1.3 Supergravity

One point was immediately clear to every one after 1974. Supersymmetry may be global, as in the proposed Wess-Zumino model, but it might also be a candidate local symmetry. In that case all generators of the algebra should generate local symmetries, in particular the translations P_μ. Yet local translations is another word for general coordinate transformations and that means General Relativity. Hence it appeared that a local supersymmetric field theory is necessarily an extension of gravity including also a gauge field for each supercharge Q^α. Such a gauge field ψ_μ^α appeared to be a spin $\frac{3}{2}$ field. Thus the hunt was open for *supergravity*, an interacting

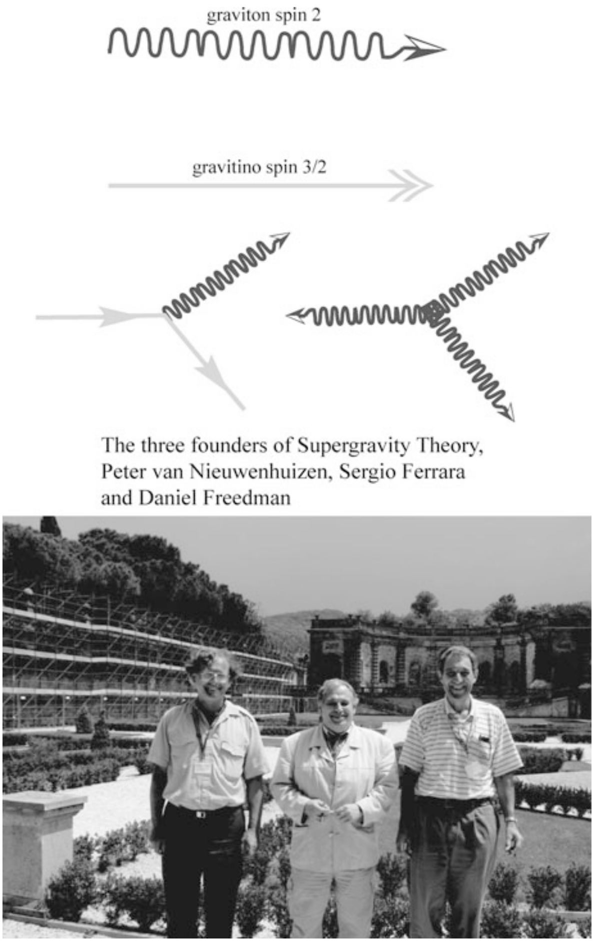

Fig. 6.8 The three founders of supergravity in a picture taken in Rome in 2007 at Villa Mondragone on the occasion of the Laurea Honoris Causa to Sergio Ferrara

theory of spin 2 *gravitons* and spin $\frac{3}{2}$ *gravitinos* that should be invariant under appropriate local supersymmetry transformations and should reduce to pure Einstein gravity when the gravitinos ψ_μ are frozen.

For $\mathscr{N} = 1$ the algebraic analysis showed that $\{2, \frac{3}{2}\}$ corresponds indeed to a massless multiplet in $D = 4$: hence the conjectured interacting theory was likely to exist and be consistent. In the case of extended supersymmetry, the supergravity Lagrangian had to include all the fields contained in the appropriate graviton multiplet as displayed in Table 6.1. Several researchers addressed the question in different approaches. In 1976 the race was won at the *Ecole Normale Superieure* of Paris by Daniel Freedman, Sergio Ferrara and Peter van Nieuwenhuizen (see Fig. 6.8), who constructed the Lagrangian of $\mathscr{N} = 1$ supergravity, using a second order formalism [8] and who were later awarded the Dirac Medal for such an achievement. A few week later appeared also a paper [9] by Stanley Deser and Bruno Zumino who obtained the same result in a more compact way using a first order formalism for the spin connection ω_μ^{ab}.

In this way 1976 opened a new important season in the theory of gravitation. General Relativity was found to admit a class of natural extensions dictated by a new powerful local symmetry that mixed fermions and bosons. Such supergravity theories, that can be constructed in various dimensions, are nothing else but Einstein gravity coupled to matter fields, both fermionic and bosonic, with very special choices of the spectrum, of the interactions and of the couplings.

In a few year time the complete park of all possible supergravities was constructed showing that they are all codified by a rich set of special geometric structures that can appear in the scalar sector (see Chap. 8).

At the beginning supergravity was developed independently from string theory, but in 1978, on the basis of a fundamental paper [11, 12] also written in Paris at the *Ecole Normale* by Gliozzi, Olive and Scherck, it became clear that the finite number of supergravities one can construct in $D = 10$ space-time dimensions, are in association with the corresponding consistent superstring models in that they just describe the low energy interaction of the massless modes of the superstring spectrum.

In 1978 in another fundamental paper [13] written by Cremmer and Julia in the same Paris location, there appeared the Lagrangian of the unique supergravity in $D = 11$ space-time dimensions which is the highest possible for such theories. Indeed, starting from $D = 12$, any closed supersymmetry multiplet includes spins higher than two and General Relativity is overcome. To the present moment, no one has been able to construct interacting theories with a finite number of spins higher than two and the unanswered question is whether they exist.

By dimensional reduction, compactification or direct construction a gigantic *bestiary* of pure and matter coupled supergravity theories has been derived in diverse dimensions and several type of classical solutions thereof have been found that have enormously enriched the landscape of General Relativity and Gravity providing new insights in the relation of gravity with strings, branes and gauge theories. A quick bird-eye survey of these topics will follow in Chaps. 7, 8, 9. In the present chapter we are interested in analyzing in depth the mathematical structure of supergravity theory developing further, in presence of supersymmetry, the principles that yield Einstein Theory as we presented it in Volume 1. This algebro-geometric approach leads us to single out in *free differential algebras* the appropriate environment for the construction of supergravities by means of the general principle of rheonomy which works as a sort of generalized analiticity.

6.2 Algebro-Geometric Structure of Supergravity

Let us now consider the generic structure of a candidate locally supersymmetric theory. This means that, in some appropriate way to be established, its field equations and eventually its action are invariant against infinitesimal transformations that are elements of the super-Poincaré Lie algebra. Hence the structure of this latter must be our primary concern. Looking at

$$\begin{aligned}\{\overline{Q}_\alpha, \overline{Q}_\beta\} = \mathrm{i}\big(\mathscr{C}\Gamma^a\big)_{\alpha\beta} P_a + \big(\mathscr{C}\Gamma^{a_1 a_2}\big)_{\alpha\beta} Z_{a_1 a_2} \\ + \mathrm{i}\big(\mathscr{C}\Gamma^{a_1\ldots a_5}\big)_{\alpha\beta} Z_{a_1\ldots a_5}\end{aligned} \tag{6.2.1}$$

that describes the supersymmetry algebra in $D = 11$, or at (6.1.5) which displays extended supersymmetry in $D = 4$, we see that the essential new ingredient is provided by the *supercharges* $\overline{Q}_\alpha$. These generators transform as Lorentz spinors:

$$[J_{ab}, \overline{Q}_\beta] = -\frac{1}{4}\overline{Q}_\alpha (\Gamma_{ab})^\alpha{}_\beta \tag{6.2.2}$$

and are fermionic, in the sense that the associated transformation parameter ε^α is not a real commuting number, rather an anticommuting Grassmann number:

$$\varepsilon^\alpha \varepsilon^\beta = -\varepsilon^\beta \varepsilon^\alpha \tag{6.2.3}$$

A would-be connection on a would-be supersymmetric principal bundle must be supersymmetry Lie algebra valued and therefore must contain a fermionic one-form ψ which couples to the supercharges $\overline{Q}_\alpha$. In full analogy with (5.2.27) of the first volume we can introduce a one-form:

$$\widehat{\Omega} = \widehat{T}_I \widehat{\Omega}^I \equiv P_a E^a + J_{ab}\omega^{ab} + \overline{Q}_\alpha \psi^\alpha \tag{6.2.4}$$

which is Poincaré super Lie algebra valued and whose supercurvature takes a form analogous to (5.2.28) of the first volume:

$$\begin{aligned}\widehat{\Theta} &= d\widehat{\Omega} + \widehat{\Omega}\wedge\widehat{\Omega} \\ &= \Big(d\widehat{\Omega}^K + \frac{1}{2} f_{IJ}{}^K\, d\widehat{\Omega}^I \wedge d\widehat{\Omega}^J\Big)\widehat{T}_K \\ &= P_a \mathfrak{T}^A + J_{ab}\mathfrak{R}^{ab} + \overline{Q}_\alpha \rho^\alpha\end{aligned} \tag{6.2.5}$$

By explicit calculations we find:

$$\begin{aligned}\mathfrak{T}^a &= \mathscr{D}V^a - \mathrm{i}\frac{1}{2}\overline{\psi}\wedge\Gamma^a\psi \\ \mathfrak{R}^{ab} &= d\omega^{ab} - \omega^{ac}\wedge\omega^{cb} \\ \rho &= \mathscr{D}\psi \equiv d\psi - \frac{1}{4}\omega^{ab}\wedge\Gamma_{ab}\psi\end{aligned} \tag{6.2.6}$$

In relation with the above equations we must remember that p-forms have now a double grading with respect to their degree and with respect to their bosonic / fermionic character. Let us convene that the fermion number f is 0 for bosons and 1 for fermions. Then, for a pair of forms, respectively of degrees $p_{1,2}$ and fermion numbers $f_{1,2}$, we have the following commutation relations under exterior product:

$$\omega^{\{p_1,f_1\}} \wedge \omega^{\{p_2,f_2\}} = (-1)^{(p_1 p_2 + f_1 f_2)} \omega^{\{p_2,f_2\}} \wedge \omega^{\{p_1,f_1\}} \tag{6.2.7}$$

In particular this implies that the *gravitino one-forms* ψ^α commute among themselves:

$$\psi^\alpha \wedge \psi^\beta = \psi^\beta \wedge \psi^\alpha \tag{6.2.8}$$

while they anti-commute with the vielbein and the spin connection:

$$\psi^\alpha \wedge E^b = -E^b \wedge \psi^\alpha; \qquad \psi^\alpha \wedge \omega^{ab} = -\omega^{ab} \wedge \psi^\alpha \tag{6.2.9}$$

Why did we name ψ^α gravitino one-forms? This will become clear in the sequel. Just as the vielbein one-forms E^a encode the spin-two particle named the *graviton*, in the same way the ψ^α one-forms encode its supersymmetric partner of spin $s = 3/2$, whose name has been agreed to be the *gravitino*, as we already stressed.

Independently from the number D of space-time dimensions, by setting $\mathfrak{T}^a = \mathfrak{R}^{ab} = \rho = 0$ we obtain the dual description of the super Poincaré Lie algebra in terms of Maurer Cartan equations.

In the case of General Relativity the dynamical theory was constructed first by considering a principal Poincaré bundle admitting D-dimensional space-time $\mathscr{M}_D$ as its base manifold and the D-dimensional Poincaré group as structural group:

$$\mathscr{P}(D\text{-Poincaré}, \mathscr{M}_D) \stackrel{\pi}{\Longrightarrow} \mathscr{M}_D \tag{6.2.10}$$

secondly by imposing the soldering condition:

$$\mathfrak{T}^a = 0 \tag{6.2.11}$$

which identifies the Lorentz sub-bundle of $\mathscr{P}(D\text{-Poincaré}, \mathscr{M}_D)$ with the tangent bundle $T\mathscr{M}_D$. In this way the spin connection could be solved in terms of the vielbein and its derivatives and local translations could be identified with diffeomorphisms of $\mathscr{M}_D$, the physical degrees of freedom being represented by the symmetric part of the square matrix $E^a_\mu(x)$. This could work because the Lorentz algebra is a closed subalgebra of the Poincaré Lie algebra, so that, at the end of the day, the spin connection behaves as a true principal connection on a Lorentz bundle. Rather than being an independent dynamical field, such a connection is a composite one in terms of the graviton degrees of freedom, yet mathematically it is a *bona-fide* principal connection.

In the case of supersymmetry, the generators $\{\overline{Q}_\alpha, J_{ab}\}$ do not close a subalgebra, since the anti-commutator of two $\overline{Q}$s produces a translation. Hence we cannot interpret local supersymmetry transformations as gauge-transformations in a principal bundle having the space-time manifold $\mathscr{M}_D$ as its base-manifold and a group generated by $\{\overline{Q}_\alpha, J_{ab}\}$ as structural group.

The alternative is that of enlarging the base-manifold by means of as many fermionic coordinates θ^α as there are supercharges. This imitates the structure of General Relativity where the space-time manifold $\mathscr{M}_D$ is a curved deformation of Minkowski space which, on its turn, can be viewed as the coset manifold:

$$\mathscr{M}_D^{(\text{Mink})} \equiv \frac{\text{Poincaré}_D}{\text{Lorentz}_D} = \frac{\text{ISO}(1, D-1)}{\text{SO}(1, D-1)} \tag{6.2.12}$$

In a similar way one can introduce *flat superspace*

$$\widehat{\mathscr{M}}^{(\text{flat})}_{(D|q)} \equiv \frac{\text{super-Poincaré}_D}{\text{Lorentz}_D} = \frac{\text{super-Poincaré}_D}{\mathrm{SO}(1, D-1)} \tag{6.2.13}$$

that is a supermanifold with D bosonic coordinates, named x^μ, and q fermionic ones named θ^α.[5] One might conclude that the degrees of freedom of supergravity are those encoded in the supervielbein of superspace:

$$\widehat{E}^A = \left\{E^a, \psi^\alpha\right\}; \quad A = \begin{cases} a \\ \alpha \end{cases} \tag{6.2.14}$$

namely those described by the supermatrix $\widehat{E}^A{}_M(x,\theta)$ defined by the expansion of $\widehat{E}^A$ in differentials of the supercoordinates:

$$\begin{aligned} \widehat{E}^A &= \widehat{E}^A{}_M(x,\theta)\, dz^M \\ dz^M &\equiv \left\{dx^\mu, d\theta^\alpha\right\} \end{aligned} \tag{6.2.15}$$

yet this turns out to be naive and leads to a wrong track. Differently from the pure bosonic case, the number of components contained in the graded symmetric part[6] of $\widehat{E}^A{}_M(x,\theta)$ is too big. It does not correspond to the physical off-shell degrees of freedom of a spin 2 and a spin 3/2 particle, as it should. This means that supergravity is not the theory of supermetrics in superspace. A new constructive principle should be added which should be simple, economic, universal and should introduce those appropriate constraints, that reduce the number of components parameterizing superspace geometry to that of the physical degrees of freedom of the relevant fermionic and bosonic particles.

Such a principle was found in the early years of supergravity theory and it is named the *rheonomy principle*. We explain it in Sect. 6.5. Before addressing this issue we have to dwell on another point of equal fundamental relevance. Not only supergravity theories are characterized by local supersymmetry transformations that are midway between gauge-transformations in a principle bundle and diffeomorphisms requiring the principle of rheonomy to obtain an adequate geometrical interpretation; they also involve, in all higher space-time dimensions, a new type of gauge fields, namely $(p+1)$-forms. From the physical view-point this fact is related to the existence of the so named *p-branes*, since such $(p+1)$-forms naturally couple to the world-volumes of p-extended objects, just as standard gauge fields couple to the world-lines of charged particles; from the mathematical side the presence of higher degree gauge forms is a clear indication that the algebraic structure

[5] In this discussion the index α incorporates both the spinor index running on the dimension of the relevant spinor representation of $\mathrm{SO}(1, D-1)$ and the replica index related with extended supersymmetry.

[6] By graded symmetric we mean $\widehat{E}^A{}_M(x,\theta) = (-)^{f_A f_B}\widehat{E}^M{}_A(x,\theta)$.

underlying supergravities goes beyond that of Lie (super-)algebras. What is it that substitutes the notion of Lie algebra of the structural group in a principle bundle? The answer is Free Differential Algebras, whose notion is discussed in the following section.

6.3 Free Differential Algebras

All higher dimensional supergravities and in particular the maximal one in $D = 11$ are based on the gauging of a new type of algebraic structure named *Free Differential Algebras*. What goes under this name was independently discovered at the beginning of the eighties in Mathematics by Sullivan [14] and in Physics by the author of this book in collaboration with R. D'Auria [15]. Indeed, Free Differential Algebras (FDA) are a categorical extension of the notion of Lie algebra and constitute the natural mathematical environment for the description of the algebraic structure of higher dimensional supergravity theory, hence also of string theory. The reason was anticipated few lines above: it is the ubiquitous presence in the spectrum of string/supergravity theories of antisymmetric gauge fields (p-forms) of rank greater than one. The very existence of FDAs is a consequence of the Chevalley cohomology of ordinary Lie algebras and Sullivan has provided us with a very elegant classification scheme of these algebras based on two structural theorems rooted in the set up of such an elliptic complex.

Another question which is of utmost relevance in physical applications is that of *gauging* of the FDAs. Just in the same way as physics gauges standard Lie algebras by means of Yang Mills theory, through the notion of gauge connections and curvatures, one expects to gauge FDAs by introducing their curvatures. A surprising feature of the FDA setup, which was noticed and explained by the author of this book in a paper of 1985 [16], is that, differently from Lie algebras, the algebraic structure of FDA already encompasses both the notion of connection and the notion of curvature and there is a well defined mathematical way of separating the two, which relies on the two structural theorems by Sullivan. Indeed the first of Sullivan's theorems, which is in some sense analogous to Levi's theorem for Lie algebras, states that the most general FDA is a semidirect sum of a so called minimal algebra $\mathbb{M}$ with a contractible one $\mathbb{C}$. The generators of the minimal algebra are physically interpreted as the connections or *potentials*, while the contractible generators are physically interpreted as the *curvatures*. The real hard-core of the FDA is the minimal algebra and it is obtained by setting the contractible generators (the curvatures) to zero. The structure of the minimal algebra $\mathbb{M}$, on its turn, is beautifully determined by the Chevalley cohomology of a standard Lie subalgebra $\mathbb{G} \subset \mathbb{M}$. This happens to be the content of Sullivan's second structural theorem.

6.3.1 Chevalley Cohomology

As a necessary preparatory step for our discussion of FDAs let us shortly recall the setup of the Chevalley elliptic complex leading to Lie algebra cohomology. This will also fix our notations and conventions.

Let us consider a (super) Lie algebra $\mathbb{G}$ identified through its structure constants $\tau^I{}_{JK}$ which are alternatively introduced through the commutation relation of the generators[7]

$$[T_I, T_K] = \tau^I{}_{JK} T_I \tag{6.3.1}$$

or the Cartan Maurer equations:

$$\partial e^I = \frac{1}{2} \tau^I{}_{JK} e^J \wedge e^K \tag{6.3.2}$$

where e^I is an abstract set of left-invariant 1-forms. The isomorphism between the two descriptions (6.3.1) and (6.3.2) of the Lie algebra is provided by the duality relations:

$$e^I(T_J) = \delta^I_J \tag{6.3.3}$$

A p-cochain $\Omega^{[p]}$ of the Chevalley complex is just an exterior p-form on the Lie algebra with constant coefficients, namely:

$$\Omega^{[p]} = \Omega_{I_1 \dots I_p} e^{I_1} \wedge \cdots \wedge e^{I_p} \tag{6.3.4}$$

where the antisymmetric tensor $\Omega_{I_1 \dots I_p} \in \bigwedge^p \operatorname{adj} \mathbb{G}$, which belongs to the pth antisymmetric power of the adjoint representation of $\mathbb{G}$, has constant components. Using the Maurer Cartan equations (6.3.2) the coboundary operator ∂ has a pure algebraic action on the Chevalley cochains:

$$\begin{aligned} \partial \Omega^{[p]} &= \partial \Omega_{I_1 \dots I_{p+1}} e^{I_1} \wedge \cdots \wedge e^{I_{p+1}} \\ \partial \Omega_{I_1 \dots I_{p+1}} &= (-)^{p-1} \frac{p}{2} \tau^R{}_{[I_1 I_2} \Omega_{I_1 \dots I_{p+1}]R} \end{aligned} \tag{6.3.5}$$

and Jacobi identities guarantee the nilpotency of this operation $\partial^2 = 0$. The cohomology groups $H^{[p]}(\mathbb{G})$ are constructed in standard way. The p-cocycles $\Omega^{[p]}$ are the closed forms $\partial \Omega^{[p]} = 0$ while the exact p-forms, or p-coboundaries, are those $\Lambda^{[p]}$ such that they can be written as $\Lambda^{[p]} = \partial \Phi^{[p-1]}$ for some suitable $(p-1)$-forms $\Phi^{[p-1]}$. The pth cohomology groups is spanned by the p-cocycles modulo

[7]For simplicity in this section we adopt a pure Lie algebra notation. Yet every definition presented here has a straightforward extension to superalgebras and indeed when we recall the discussion of how the FDA of M-theory or type II supergravity emerges from the application of Sullivan structural theorems it is within the scope of super Lie algebra cohomology.

the p-coboundaries. Calling $C^p(\mathbb{G})$ the linear space of p-chains the operator ∂ defined in (6.3.5) induces a sequence of linear maps ∂_p:

$$C^0(\mathbb{G}) \xrightarrow{\partial_0} C^1(\mathbb{G}) \xrightarrow{\partial_1} C^2(\mathbb{G}) \xrightarrow{\partial_2} C^3(\mathbb{G}) \xrightarrow{\partial_3} C^4(\mathbb{G}) \xrightarrow{\partial_4} \cdots \tag{6.3.6}$$

and we can summarize the definition of the Chevalley cohomology groups in the standard form used for all elliptic complexes:

$$H^{(p)}(\mathbb{G}) \equiv \frac{\ker \partial_p}{\operatorname{Im} \partial_{p-1}} \tag{6.3.7}$$

Contraction and Lie Derivative On the Chevalley complex it is also convenient to introduce the operation of contraction with a tangent vector and then of Lie derivative that are the algebraic counterparts of the corresponding operations introduced in (5.2.75)–(5.2.80) for generic differentiable manifolds.

The contraction operator i_X associates to every tangent vector, namely to every element of the Lie algebra $X \in \mathbb{G}$ a linear map from the space $C^p(\mathbb{G})$ of the p-cochains to the space $C^{p-1}(\mathbb{G})$ of the $(p-1)$-cochains:

$$\forall X \in \mathbb{G}; \quad i_X : C^p(\mathbb{G}) \mapsto C^{p-1}(\mathbb{G}) \tag{6.3.8}$$

Explicitly we set:

$$\forall X = X^M T_M \in \mathbb{G}; \quad i_X \Omega^{[p]} = p X^M \Omega_{M I_1 \ldots I_{p-1}} e^{I_1} \wedge \cdots \wedge e^{I_{p-1}} \tag{6.3.9}$$

Next we introduce the Lie derivative ℓ which to every element of the Lie algebra $X \in \mathbb{G}$ associates a map from the space of p-cochains into itself:

$$\forall X = X^M T_M \in \mathbb{G}; \quad \ell_X : C^p(\mathbb{G}) \mapsto C^p(\mathbb{G}) \tag{6.3.10}$$

In full analogy with (5.2.80) the map ℓ_X is defined as follows:

$$\ell_X \equiv i_X \circ \partial + \partial \circ i_X \tag{6.3.11}$$

and satisfies the necessary property in order to be a representation of the Lie algebra:

$$[\ell_X, \ell_Y] = \ell_{[X,Y]} \tag{6.3.12}$$

By explicit calculation we find that in components the Lie derivative is realized as follows:

$$\ell_X \Omega^{[p]} = (-)^{p-1} p X^M \tau^R{}_{M[I_1} \Omega_{I_2 I_3 \ldots I_p]R} e^{I_1} \wedge \cdots \wedge e^{I_p} \tag{6.3.13}$$

Furthermore if X and Y are any two $\mathbb{G}$ Lie algebra-valued space-time forms, respectively of degree x and y, by direct use of the above definitions, one can easily verify the following identity which holds true on any p-cochain $\mathscr{C}^{[p]}$:

$$\left(i_X \circ \ell_Y + (-)^{xy+1} \ell_Y \cdot i_X\right) \mathscr{C}^{[p]} = -i_{[X,Y]} \mathscr{C}^{[p]} \tag{6.3.14}$$

which is of great help in many calculations.

6.3.2 General Structure of FDAs and Sullivan's Theorems

As we just recalled, Free Differential Algebras are a natural categorical extension of the notion of Lie algebra and constitute the natural mathematical environment for the description of the algebraic structure of higher dimensional supergravity theories, hence also of string theory.

Definition of FDA The starting point for FDAs is the generalization of Maurer Cartan equations. As we already emphasized in Sect. 6.3.1 a standard Lie algebra is defined by its structure constants which can be alternatively introduced, either through the commutators of the generators, as in (6.3.1), or through the Maurer Cartan equations obeyed by the dual 1-forms, as in (6.3.2). The relation between the two descriptions is provided by the duality relation in (6.3.3). Adopting the Maurer Cartan viewpoint, FDAs can now be defined as follows. Consider a formal set of exterior forms $\{\theta^{A(p)}\}$ labeled by the index A and by the degree p, which may be different for different values of A. Given this set of p-forms we can write the corresponding set of generalized *Maurer Cartan equations* as follows:

$$d\theta^{A(p)} + \sum_{n=1}^{N} C^{A(p)}{}_{B_1(p_1)\ldots B_n(p_n)}\theta^{B_1(p_1)} \wedge \cdots \wedge \theta^{B_n(p_n)} = 0 \qquad (6.3.15)$$

where $C^{A(p)}{}_{B_1(p_1)\ldots B_n(p_n)}$ are generalized structure constants with the same symmetry as induced by permuting the θs in the wedge product. They can be non-zero only if:

$$p + 1 = \sum_{i=1}^{n} p_i \qquad (6.3.16)$$

Equations (6.3.15) are self-consistent and define an FDA if and only if $d\,d\theta^{A(p)} = 0$, upon substitution of (6.3.15) into its own derivative. This procedure yields the generalized Jacobi identities of FDAs.

Classification of FDA and the Analogue of Levi Theorem: Minimal Versus Contractible Algebras A basic theorem of Lie algebra theory states that the most general Lie algebra $\mathbb{A}$ is the semidirect product of a semisimple Lie algebra $\mathbb{L}$, called the Levi subalgebra, with Rad($\mathbb{A}$), namely with the radical of $\mathbb{A}$. By definition this latter is the maximal solvable ideal of $\mathbb{A}$. Sullivan [14] has provided an analogous structural theorem for FDAs. To this effect one needs the notions of *minimal FDA* and *contractible FDA*. A minimal FDA is one for which:

$$C^{A(p)}{}_{B(p+1)} = 0 \qquad (6.3.17)$$

This excludes the case where a $(p + 1)$-form appears in the generalized Maurer Cartan equations as a contribution to the derivative of a p-form. In a minimal algebra all non-differential terms are products of at least two elements of the algebra, so that all forms appearing in the expansion of $d\theta^{A(p)}$ have at most degree p, the degree $p+1$ being ruled out.

On the other hand a *contractible FDA* is one where the only form appearing in the expansion of $d\theta^{A(p)}$ has degree $p+1$, namely:

$$d\theta^{A(p)} = \theta^{A(p+1)} \quad \Rightarrow \quad d\theta^{A(p+1)} = 0 \tag{6.3.18}$$

A contractible algebra has a trivial structure. The basis $\{\theta^{A(p)}\}$ can be subdivided in two subsets $\{\Lambda^{A(p)}\}$ and $\{\Omega^{B(p+1)}\}$ where A spans a subset of the values taken by B, so that:

$$d\Omega^{B(p+1)} = 0 \tag{6.3.19}$$

for all values of B and

$$d\Lambda^{A(p)} = \Omega^{A(p+1)} \tag{6.3.20}$$

Denoting by $\mathbb{M}^k$ the vector space generated by all forms of degree $p \leq k$ and C^k the vector space of forms of degree k, a minimal algebra is shortly defined by the property:

$$d\mathbb{M}^k \subset \mathbb{M}^k \wedge \mathbb{M}^k \tag{6.3.21}$$

while a contractible algebra is defined by the property

$$dC^k \subset C^{k+1} \tag{6.3.22}$$

In analogy to Levi's theorem, the first theorem by Sullivan states that: *The most general FDA is the semidirect sum of a contractible algebra with a minimal algebra.*

Sullivan's First Theorem and the Gauging of FDAs Twenty five years ago in [16] the present author observed that the above mathematical theorem has a deep physical meaning relative to the gauging of algebras. Indeed he proposed the following identifications:

1. The *contractible generators* $\Omega^{A(p+1)} + \cdots$ of any given FDA $\mathbb{A}$ are to be physically identified with the *curvatures*.
2. The Maurer Cartan equations that begin with $d\Omega^{A(p+1)}$ are *the Bianchi identities*.
3. The algebra which is gauged is the *minimal subalgebra* $\mathbb{M} \subset \mathbb{A}$.
4. The Maurer Cartan equations of the minimal subalgebra $\mathbb{M}$ are consistently obtained by those of $\mathbb{A}$ by setting all contractible generators to zero.

Sullivan's Second Structural Theorem and Chevalley Cohomology The second structural theorem proved by Sullivan[8] deals with the structure of minimal algebras and it is constructive. Indeed it states that the most general minimal FDA $\mathbb{M}$

[8] The reader interested in very much detailed explanations on this point can find them both in the original article [14] and in the book [17]. Yet the present book is logically self-contained and the presented view-point is upgraded to a contemporary perspective.

necessarily contains an ordinary Lie subalgebra $\mathbb{G} \subset \mathbb{M}$ whose associated 1-form generators we can call e^I, as in (6.3.2). Additional p-form generators $A^{[p]}$ of $\mathbb{M}$ are necessarily, according to Sullivan's theorem, in one-to-one correspondence with Chevalley $p+1$ cohomology classes $\Gamma^{[p+1]}(e)$ of $\mathbb{G} \subset \mathbb{M}$. Indeed, given such a class, which is a polynomial in the e^I generators, we can consistently write the new higher degree Maurer Cartan equation:

$$\partial A^{[p]} + \Gamma^{[p+1]}(e) = 0 \tag{6.3.23}$$

where $A^{[p]}$ is a new object that cannot be written as a polynomial in the old objects e^I. Considering now the FDA generated by the inclusion of the available $A^{[p]}$, one can inspect its Chevalley cohomology: the cochains are the polynomials in the extended set of forms $\{A, e^I\}$ and the boundary operator is defined by the enlarged set of Maurer Cartan equations. If there are new cohomology classes $\Gamma^{[p+1]}(e, A)$, then one can further extend the FDA by including new p-generators $B^{[p]}$ obeying the Maurer Cartan equation:

$$\partial B^{[p]} + \Gamma^{[p+1]}(e, A) = 0 \tag{6.3.24}$$

The iterative procedure can now be continued by inspecting the cohomology classes of type $\Gamma^{[p+1]}(e, A, B)$ which lead to new generators $C^{[p]}$ and so on. Sullivan's theorem states that those constructed in this way are, up to isomorphisms, the most general minimal FDAs.

To be precise, this is not the whole story. There is actually one generalization that should be taken into account. Instead of *absolute Chevalley cohomology* one can rather consider *relative Chevalley cohomology*. This means that rather then being $\mathbb{G}$- singlets, the Chevalley p-cochains can be assigned to some linear representation of the Lie algebra $\mathbb{G}$. In this case (6.3.4) is replaced by:

$$\Omega^{\alpha[p]} = \Omega^{\alpha}_{I_1 \dots I_p} e^{I_1} \wedge \cdots \wedge e^{I_p} \tag{6.3.25}$$

where the index α runs in some representation D:

$$D : T_I \to \left[D(T_I)\right]^{\alpha}{}_{\beta} \tag{6.3.26}$$

and the boundary operator is now the covariant ∇:

$$\nabla \Omega^{\alpha[p]} \equiv \partial \Omega^{\alpha[p]} + e^I \wedge \left[D(T_I)\right]^{\alpha}{}_{\beta} \Omega^{\beta[p]} \tag{6.3.27}$$

Since $\nabla^2 = 0$, we can repeat all previously explained steps and compute cohomology groups. Each non-trivial cohomology class $\Gamma^{\alpha[p+1]}(e)$ leads to new p-form generators $A^{\alpha[p]}$ which are assigned to the same $\mathbb{G}$-representation as $\Gamma^{\alpha[p+1]}(e)$. All successive steps go through in the same way as before and Sullivan's theorem actually states that all minimal FDAs are obtained in this way for suitable choices of the representation D, in particular the singlet.

6.4 The Super FDA of M Theory and Its Cohomological Structure

We discussed Sullivan's theorems for Lie algebras and their corresponding FDA extensions but, as we stressed, they hold true, with obvious modifications, also for superalgebras $\mathbb{G}_s$ and for their FDA extensions. Actually, in view of superstring and supergravity, it is precisely in the supersymmetric context that FDAs have found their most relevant applications. As an illustration of the general set up and also for its intrinsic interest, by recalling the results of [15] and [16], we present here the structure of the M-theory FDA, namely the algebraic basis of maximal supergravity in eleven space-time dimensions. Indeed M-theory is the name frequently given in contemporary literature to $D = 11$ supergravity. This results from the so called second string revolution that showed that all consistent ten-dimensional string theories can be related to each other by non-perturbative dualities and can be regarded as special limits of a unified non-perturbative mother-theory (this is the meaning of M) whose exact microscopic definition has not yet been given, yet is somehow defined by its own low energy limit that is precisely $D = 11$ supergravity.

Within this context we are also able to illustrate the bearing of a quite relevant question: *is an FDA* $\mathbb{M}$ *always equivalent to a normal Lie algebra* $\widehat{\mathbb{G}} \supset \mathbb{G}$ *larger than the Lie algebra of which* $\mathbb{M}$ *is a cohomological extension?* How we can mathematically formulate and answer such a question we will show below by recalling results of [15] and also more recent literature [18–20].

Let us begin by writing the complete set of curvatures, plus their Bianchi identities. This will define the complete FDA:

$$\mathbb{A} = \mathbb{M} \biguplus \mathbb{C} \tag{6.4.1}$$

the curvatures being the contractible generators $\mathbb{C}$. By setting them to zero we retrieve, according to Sullivan's first theorem, the minimal algebra $\mathbb{M}$. This latter, according to Sullivan's second theorem, has to be explained in terms of cohomology of the normal subalgebra $\mathbb{G} \subset \mathbb{M}$, spanned by the 1-forms. In this case $\mathbb{G}$ is just the $D = 11$ superalgebra spanned by the following 1-forms:

1. the vielbein V^a,
2. the spin connection ω^{ab},
3. the gravitino ψ.

The higher degree generators of the minimal FDA $\mathbb{M}$ are:

1. the bosonic 3-form $\mathbf{A}^{[3]}$,
2. the bosonic 6-form $\mathbf{A}^{[6]}$.

The complete set of curvatures is given below [15, 16]:

$$\begin{aligned}
\mathfrak{T}^a &= \mathscr{D}V^a - \mathrm{i}\frac{1}{2}\overline{\psi} \wedge \Gamma^a \psi \\
\mathfrak{R}^{ab} &= d\omega^{ab} - \omega^{ac} \wedge \omega^{cb}
\end{aligned}$$

$$\rho = \mathscr{D}\psi \equiv d\psi - \frac{1}{4}\omega^{ab} \wedge \Gamma_{ab}\psi \tag{6.4.2}$$

$$\mathbf{F}^{[4]} = d\mathbf{A}^{[3]} - \frac{1}{2}\overline{\psi} \wedge \Gamma_{ab}\psi \wedge V^a \wedge V^b$$

$$\mathbf{F}^{[7]} = d\mathbf{A}^{[6]} - 15\mathbf{F}^{[4]} \wedge \mathbf{A}^{[3]} - \frac{15}{2}V^a \wedge V^b \wedge \overline{\psi} \wedge \Gamma_{ab}\psi \wedge \mathbf{A}^{[3]}$$
$$- \mathrm{i}\frac{1}{2}\overline{\psi} \wedge \Gamma_{a_1\dots a_5}\psi \wedge V^{a_1} \wedge \cdots \wedge V^{a_5}$$

From their very definition, by taking a further exterior derivative one obtains the Bianchi identities which play an even more fundamental role in constructing supergravity theories then they played in constructing General Relativity:

$$\mathscr{D}\mathfrak{R}^{ab} = 0 \tag{6.4.3}$$

$$\mathscr{D}\mathfrak{T}^a + \mathfrak{R}^{ab} \wedge V_b - \mathrm{i}\overline{\psi} \wedge \Gamma^a \rho = 0 \tag{6.4.4}$$

$$\mathscr{D}\rho + \frac{1}{4}\Gamma^{ab}\psi \wedge \mathfrak{R}^{ab} = 0 \tag{6.4.5}$$

$$d\mathbf{F}^{[4]} - \overline{\psi} \wedge \Gamma_{ab}\rho \wedge V^a \wedge V^b + \overline{\psi}\Gamma_{ab}\psi \wedge \mathfrak{T}^a \wedge V^b = 0 \tag{6.4.6}$$

$$\begin{aligned} & d\mathbf{F}^{[7]} - \mathrm{i}\overline{\psi} \wedge \Gamma_{a_1\cdots a_5}\rho \wedge V^{a_1} \wedge \cdots \wedge V^{a_5} \\ & - \frac{5}{3}\mathrm{i}\overline{\psi} \wedge \Gamma_{a_1\cdots a_5}\psi \wedge \mathfrak{T}^{a_1} \wedge V^{a_2} \wedge \cdots \wedge V^{a_5} \\ & - 15\overline{\psi} \wedge \Gamma_{ab}\rho \wedge V^a \wedge V^b \wedge \mathbf{F}^{[4]} - 15\mathbf{F}^{[4]} \wedge \mathbf{F}^{[4]} = 0 \end{aligned} \tag{6.4.7}$$

The dynamical theory is defined, according to a general constructive scheme of supersymmetric theories which we explain in Sect. 6.5, by the *principle of rheonomy*, implemented into Bianchi identities. Indeed, as we discuss in that section, there is a unique rheonomic parameterization of the curvatures (6.4.2) which solves the Bianchi identities and it is the following one:

$$\begin{aligned} \mathfrak{T}^a &= 0 \\ \mathbf{F}^{[4]} &= F_{a_1\dots a_4} V^{a_1} \wedge \cdots \wedge V^{a_4} \\ \mathbf{F}^{[7]} &= \frac{1}{84} F^{a_1\dots a_4} V^{b_1} \wedge \cdots \wedge V^{b_7} \varepsilon_{a_1\dots a_4 b_1\dots b_7} \\ \rho &= \rho_{a_1a_2} V^{a_1} \wedge V^{a_2} - \mathrm{i}\frac{1}{2}\left(\Gamma^{a_1a_2a_3}\psi \wedge V^{a_4} + \frac{1}{8}\Gamma^{a_1\dots a_4 m}\psi \wedge V^m\right) F^{a_1\dots a_4} \\ \mathfrak{R}^{ab} &= R^{ab}{}_{cd} V^c \wedge V^d + \mathrm{i}\rho_{mn}\left(\frac{1}{2}\Gamma^{abmn} - \frac{2}{9}\Gamma^{mn[a}\delta^{b]c} + 2\Gamma^{ab[m}\delta^{n]c}\right)\psi \wedge V^c \\ & \quad + \overline{\psi} \wedge \Gamma^{mn}\psi F^{mnab} + \frac{1}{24}\overline{\psi} \wedge \Gamma^{abc_1\dots c_4}\psi F^{c_1\dots c_4} \end{aligned} \tag{6.4.8}$$

The expressions (6.4.8) satisfy the Bianchi provided the space-time components of the curvatures satisfy the following constraints

$$\begin{aligned} 0 &= \mathscr{D}_m F^{mc_1c_2c_3} + \frac{1}{96}\varepsilon^{c_1c_2c_3a_1a_8} F_{a_1\dots a_4} F_{a_5\dots a_8} \\ 0 &= \Gamma^{abc}\rho_{bc} \\ R^{am}{}_{cm} &= 6F^{ac_1c_2c_3}F^{bc_1c_2c_3} - \frac{1}{2}\delta^a_b F^{c_1\dots c_4}F^{c_1\dots c_4} \end{aligned} \tag{6.4.9}$$

which are the space-time field equations. We will come back to these equations and study some of their solutions. We postpone this to the sequel. Here we concentrate on the structure of the FDA.

6.4.1 The Minimal FDA of M-Theory and Cohomology

Setting $\mathfrak{T}^a = \mathfrak{R}^{ab} = \rho = \mathbf{F}^{[4]} = \mathbf{F}^{[7]} = 0$ in (6.4.2) we obtain the Maurer Cartan equations of the minimal algebra $\mathbb{M}$. In particular we have:

$$\begin{aligned} d\mathbf{A}^{[3]} &= \Gamma^{[4]}(V,\psi) \equiv \frac{1}{2}\overline{\psi}\wedge\Gamma_{ab}\psi\wedge V^a\wedge V^b \\ d\mathbf{A}^{[6]} &= \Gamma^{[7]}\big(V,\psi,\mathbf{A}^{[3]}\big) \\ &\equiv \frac{15}{2}V^a\wedge V^b\wedge\overline{\psi}\wedge\Gamma_{ab}\psi\wedge\mathbf{A}^{[3]} \\ &\quad + \mathrm{i}\frac{1}{2}\overline{\psi}\wedge\Gamma_{a_1\dots a_5}\psi\wedge V^{a_1}\wedge\cdots\wedge V^{a_5} \end{aligned} \tag{6.4.10}$$

The reason why the three-form generator $\mathbf{A}^{[3]}$ does exist and also why the six-form generator $\mathbf{A}^{[6]}$ can be included is, in this set up, a direct consequence of the cohomology of the super Poncaré algebra in $D = 11$, via Sullivan's second theorem. Indeed the 4-form $\Gamma^{[4]}(V,\psi)$ defined in the first line of (6.4.10) is a cohomology class of the super Poincaré Lie algebra whose Maurer Cartan equations are the first three of (6.4.2) upon setting $\mathfrak{T}^a = \mathfrak{R}^{ab} = \rho = 0$. We have:

$$d\Gamma^{[4]}(V,\psi) = 0 \tag{6.4.11}$$

and there is no $\Phi^{[3]}(V,\psi)$ such that $\Gamma^{[4]}(V,\psi) = d\Phi^{[3]}(V,\psi)$.

An important issue to be stressed at this point is the following one. As we will emphasize in Chap. 7, the main new idea underlying superstring/supergravity theories is the interplay between gravitational-like bulk-theories and world-volume gauge-like brane-theories. In bulk-theories there are p-forms. These latter couple to the degrees of freedom corresponding to $(p-1)$-extended objects spanning a p-dimensional world-volume in the same way as the electromagnetic field (a 1-form) couples to charged particles (0-branes) spanning a world-line. The quantum theory living on the world-volume provides a spectrum of states whose light-ones fill-up the multiplet of fields contained in the gravitational theory. Strings appear because there is a 2-form in $D = 10$ gravitational super-theories. In $D = 11$ we have M2-branes and M5-branes since $D = 11$ supergravity contains a 3-form and a 6-form. On the other hand the existence of such forms is an yield of the cohomology of the super-Poincaré Lie algebra in $D = 11$. Hence we can conclude that the entire setup of the brane-world (including superstrings) is a logical algebraic consequence of the supersymmetric extension of Lorentz symmetry.

The black holes of General Relativity that can be seen as new particles provided by exact localized solutions of the bulk theory are the first instance of branes, actually 0-branes.

The algebraic reason why $\Gamma^{[4]}(V, \psi)$ is a closed form is also rooted in Lie algebra theory and can be expressed in intrinsic group-theoretical terms. It follows from the following Fierz identity:

$$\overline{\psi} \wedge \Gamma^{ab}\psi \wedge \overline{\psi} \wedge \Gamma_a \psi = 0 \tag{6.4.12}$$

Let us analyze the meaning of the above equation. The object constructed in the left hand side of (6.4.12) transforms as an 11-dimensional vector under the Lorentz group SO(1, 10). The building blocks of the construction are the gravitino one-forms ψ^α that transform under SO(1, 10) according to the 32-dimensional spinor representation. Since ψ is a fermionic one-form its components are commutative in wedge products. This implies that we can interpret (6.4.12) in the following way. The left hand side is a projection operator on the **11** *irrep*[9] out of the symmetric product of four *irreps* **32**. The reason why the result is zero is that in the Clebsch Gordan expansion of such a four product the *irrep* **11** is not contained. Indeed we have:

$$\begin{aligned}(\mathbf{32} \otimes \mathbf{32} \otimes \mathbf{32} \otimes \mathbf{32})_{\mathrm{symm}} = {} & \mathbf{1} \oplus \mathbf{165} \oplus \mathbf{330} \oplus \mathbf{462} \oplus \mathbf{65} \oplus \mathbf{429} \\ & \oplus \mathbf{4290} \oplus \mathbf{1144} \oplus \mathbf{17160} \oplus \mathbf{32604} \end{aligned} \tag{6.4.13}$$

6.4.2 FDA Equivalence with Larger (Super) Lie Algebras

Before proceeding with supergravity constructions we consider a natural question that arises, which is of fundamental interest in view of the central role played by the FDA algebraic structure. The question was alluded to above and it is the following one: *are FDAs eventually equivalent to normal (super) Lie algebras?* For minimal algebras the question can be nicely rephrased in the following way: *can a non-trivial cohomology class of a Lie algebra $\mathbb{G}$ be trivialized by immersing $\mathbb{G}$ into a larger*

[9] According to standard nomenclature *irrep* means irreducible representation.

algebra $\widehat{\mathbb{G}}$*?* Indeed by adding new 1-form generators ϕ^p which, together with the generators e^I of $\mathbb{G}$ satisfy the Maurer Cartan equations of the larger algebra $\widehat{\mathbb{G}} \supset \mathbb{G}$, it may happen that we are able to construct a polynomial $\Phi^{[p-1]}(e,\phi)$ such that:

$$d\Phi^{[p-1]}(e,\phi) = \Gamma^{[p]}(e) \tag{6.4.14}$$

In this case the generator $\mathbf{A}^{[p-1]}$ of the FDA associated with the cohomology class $\Gamma^{[p]}(e)$ can be simply deleted by the list of independent generators and simply identified with the polynomial $\Phi^{[p-1]}(e,\phi)$.

In these terms the question was already posed twenty three years ago by D'Auria and the author of this book in [15] obtaining a positive answer [15] which was revisited in [18, 19].

The enlarged algebra $\widehat{\mathbb{G}}$ contains, besides the generators of $\mathbb{G}$ a bosonic 1-form $B^{a_1a_2}$ which is in the rank two antisymmetric representation of the Lorentz group, a bosonic 1-form $B^{a_1a_2,\dots a_5}$ which is in the rank five antisymmetric representation and finally a fermionic 1-form η which is in the spinor representation just as the generator ψ.

The Maurer Cartan equations of $\widehat{\mathbb{G}}$ are:

$$0 = \mathfrak{R}^{ab} \equiv d\omega^{ab} - \omega^{ac} \wedge \omega^{cb} \tag{6.4.15}$$

$$0 = \mathfrak{T}^{a} \equiv \mathscr{D}V^{a} - \mathrm{i}\frac{1}{2}\overline{\psi} \wedge \Gamma^{a}\psi \tag{6.4.16}$$

$$0 = \mathfrak{T}^{a_1a_2} \equiv \mathscr{D}B^{a_1a_2} - \frac{1}{2}\overline{\psi} \wedge \Gamma^{a_1a_2}\psi \tag{6.4.17}$$

$$0 = \mathfrak{T}^{a_1\dots a_5} \equiv \mathscr{D}B^{a_1\dots a_5} - \mathrm{i}\frac{1}{2}\overline{\psi} \wedge \Gamma^{a_1\dots a_5}\psi \tag{6.4.18}$$

$$0 = \rho \equiv \mathscr{D}\psi \equiv d\psi - \frac{1}{4}\omega^{ab} \wedge \Gamma^{ab}\psi \tag{6.4.19}$$

$$\begin{aligned} 0 = \sigma \equiv \mathscr{D}\eta - \mathrm{i}\delta\Gamma^{a}\psi \wedge V^{a} - \gamma_1\Gamma^{ab}\psi \wedge B^{ab} \\ - \gamma_2\Gamma^{a_1\dots a_5}\psi \wedge B^{a_1\dots a_5} \end{aligned} \tag{6.4.20}$$

These Maurer Cartan equations are consistent, namely closed, provided the following equation is satisfied by the coefficients:

$$\delta + 10\gamma_1 - 720\gamma_2 = 0 \tag{6.4.21}$$

Using all the generators of $\widehat{\mathbb{G}}$ one can construct a cubic polynomial

$$\begin{aligned} &\Phi^{[3]}\big(V,\psi,B^{(2)},B^{(5)}\big) \\ &\quad = \lambda B^{a_1a_2} \wedge V^{a_1} \wedge V^{a_2} + \alpha_1 B^{a_1a_2} \wedge B^{a2a3} \wedge B^{a_3a_1} \\ &\qquad + \alpha_2 B^{b_1a_1\dots a_4} \wedge B^{b_1b_2} \wedge B^{b_2a_1\dots a_4} + \alpha_3\varepsilon^{a_1\dots a_5b_1\dots b_4m} B^{a_1\dots a_5} \wedge B^{b_1\dots b_5} \wedge V^{m} \\ &\qquad + \alpha_4\varepsilon^{m_1\dots m_6n_1\dots n_5} B^{m_1m_2m_3p_1p_2} \wedge B^{m_4m_5m_6p_1p_2} \wedge B^{n_1\dots n_5}\mathrm{i}\beta_1\overline{\psi} \wedge \Gamma^{a}\eta \wedge V^{a} \end{aligned}$$

$$+\beta_2\overline{\psi}\wedge\Gamma^{a_1a_2}\eta\wedge B^{a_1a_2}+\mathrm{i}\beta_3\overline{\psi}\wedge\Gamma^{a_1\ldots a_5}\eta\wedge B^{a_1\ldots a_5} \tag{6.4.22}$$

such that:

$$d\Phi^{[3]}\big(V,\psi,B^{(2)},B^{(5)}\big)=\Gamma^{[4]}(V,\psi)\equiv\frac{1}{2}\overline{\psi}\wedge\Gamma_{ab}\psi\wedge V^a\wedge V^b \tag{6.4.23}$$

The coefficients appearing in (6.4.22) are completely fixed by (6.4.23) for any of the 1-parameter family of algebras described by (6.4.15)–(6.4.20). Indeed the closure condition (6.4.21) is one equation on three parameters which are therefore reduced to two. One of them, say γ_1 can be reabsorbed into the normalization of the extra fermionic generator η, but the other remains essential and its value selects one algebra within a family of non-isomorphic ones. Following [18] it is convenient to set:

$$\delta=2\gamma_1(s+1);\qquad\gamma_2=2\gamma_1\left(\frac{s}{6!}+\frac{1}{5!}\right) \tag{6.4.24}$$

and s is the parameter which parameterizes the inequivalent algebras $\widehat{\mathbb{G}}_s$. For each of them we have a solution of (6.4.23) realized by

$$\begin{aligned}\alpha_1&=\frac{2(3+s)}{15s^2}; &\qquad \alpha_4&=\frac{-(6+s)^2}{259200s^2}\\ \alpha_2&=\frac{-(6+s)^2}{720s^2}; & \alpha_3&=\frac{(6+s)^2}{432000s^2}\\ \lambda&=\frac{6+2s+s^2}{5s^2}; & \beta_1&=\frac{3-2s}{10s^2\gamma_1}\\ \beta_2&=\frac{3+s}{20s^2\gamma_1}; & \beta_3&=\frac{6+s}{2400s^2\gamma_1}\end{aligned} \tag{6.4.25}$$

In the original paper [15] only two of this infinite class of solutions were found, namely those corresponding to the values:

$$s=-1;\qquad s=\frac{3}{2} \tag{6.4.26}$$

which are the roots of the equation $\lambda=1$. Indeed in [15] the additional condition $\lambda=1$ was imposed, which is unnecessary as it has been shown in [18, 19] where the more general solution (6.4.24), (6.4.25) was found.

It remains to be seen whether the equivalence between the minimal FDA $\mathbb{M}$ and the Lie algebra $\widehat{\mathbb{G}}$ can be promoted to a dynamical equivalence between their gaugings. In other words whether one can consistently parameterize the curvatures of $\widehat{\mathbb{G}}$ in such a way that identifying the three form $\mathbf{A}^{[3]}$ with the polynomial $\Phi^{[3]}$, the rheonomic parameterizations (6.4.8) are automatically reproduced? This is a rather formidable algebraic problem and to the present time no one has been able to answer it in the positive way. Actually although this has not been established like a theorem it appears from all undertaken attempts that the correct answer is the negative one. Hence the dynamical theory of supergravity is nicely and necessarily based on the categorical extension of (super) Lie Algebras provided by Free Differential Algebras.[10]

6.5 The Principle of Rheonomy

The principle of rheonomy was introduced by D'Auria and the present author in a paper of 1979 [22], formalizing a previous idea of Ne'eman and Regge [23] (see Fig. 6.9). The basic motivation to introduce such a concept was the geometrical interpretation of *local supersymmetry transformations* at the basis of the newly found theory of supergravity, which, at that time, was less than two year old. In this respect the key problem is that supersymmetry transformations, as they were case by case found in the early construction of supersymmetric theories, look similar to *gauge-transformations*, yet their gauge-field ψ_μ, which ultimately encodes the spin $\frac{3}{2}$ particles, has not a horizontal field-strength and therefore is *not* a proper *connection* on a *principal fibre-bundle*. To explain this point let us remind the reader of the basic structure of the supersymmetry algebras. Consider for instance the supersymmetry algebra in the maximal $D = 11$ dimensions (6.2.1). The supercharges $\overline{Q}_\alpha$ anticommute to the translation generators P_a and this is the key feature in all cases. It follows that horizontality of the curvatures in the directions associated with the supercharges would imply horizontality also in the directions associated with the translations. This is absurd. Indeed, as we explained at length in the first volume, translations are eventually identified, through the soldering condition, with the diffeomorphisms on the base manifold; hence they are horizontal by definition and cannot become vertical. By means of the above argument neither the supersymmetry directions can be vertical. Yet, as already observed, considering the fermionic directions of the manifold just on the same footing as the bosonic ones is equally misleading. Indeed a metric, or vielbein, theory in superspace leads to no good physics: one has too many degrees of freedom deprived of physical meaning that have to be got rid of. What is the outcome from this dilemma? It is a revision of the concept of horizontality and, hence, of principal connections on fibre-bundles. In its *strong formulation*, used so far, horizontality requires that the components of the curvatures should be zero in the vertical directions. A *weaker formulation* of the same idea is easily deemed of: one could just require that the vertical components should just be dependent on the horizontal ones, in particular *linear combinations* of the latter. This very simple idea is the *principle of rheonomy*.

Recalling that curvatures are essentially "derivatives" of the connections, the principle of rheonomy, which "equates" vertical derivatives to horizontal ones, is reminiscent of a very classical set of equations of mathematical analysis, namely Cauchy-Riemann equations satisfied by the real and imaginary parts of an analytic

[10] It must also be noted that the algebras defined by (6.4.15)–(6.4.20) and by some authors named D'Auria-Frè algebras have been discussed as a possible basis for a Chern-Simons formulation of fundamental M-theory [20]. They have also been retrieved as part of a wider set of gauge algebras by Castellani [21], using his method of extended Lie derivatives.

Fig. 6.9 Born in 1931 in Torino, Tullio Regge is probably the most famous Italian physicist of the second half of the XXth century. His first achievement, that gave him world-wide fame, dates 1957 when he was only twenty-six of age. It consists of the discovery of a subtle mathematical property of potential scattering in non-relativistic quantum mechanics, namely that the scattering amplitude can be thought of as an analytic function of the angular momentum which admits an extension to the complex plane, and that the positions of the poles determine power-law growth rates for the amplitude. Easily extended to the relativistic case, *Regge poles* opened a new era in scattering theory and provided the framework in which, ten years later, Veneziano introduced dual amplitudes and gave birth to String Theory. In the early 1960s, Regge introduced *Regge Calculus*, a simplicial formulation of General Relativity where space-time is approximated by gluing together polyhedra. Regge calculus was the first instance of discretization of a gauge theory suitable for numerical simulation, and an early relative of *lattice gauge theory*. Very important contributions were given by him, in collaboration with Wheeler, also to the early *theory of Black-Hole perturbations*. Tullio Regge received the Dannie Heineman Prize for Mathematical Physics in 1964, the Città di Como prize in 1968, the Albert Einstein Award in 1979, and the Cecil Powell Medal in 1987. In 1996 he was awarded the Dirac Medal. Full Professor of Relativity of Torino University since 1961, he was member of the Institute of Advanced Studies in Princeton from the early sixties to 1979, when he resumed his chair in Torino. Elected to the European Parliament in 1989, when he finished his term in 1995, he was called on a special chair by the Politecnico di Torino, where he taught until his retirement. Tullio Regge is also full member of the Accademia dei Lincei and a public figure in Italy for his frequent participation to TV debates on a variety of problems ranging from Energetics to Bioethics. He is also an appreciated writer of quite original popularizing books and articles

function on the complex plane:

$$
\begin{aligned}
& f(x+\mathrm{i}y) = u(x,y) + \mathrm{i}v(x,y) \\
& \frac{\partial}{\partial y} u(x,y) = \frac{\partial}{\partial x} v(x,y) \\
& \frac{\partial}{\partial x} v(x,y) = -\frac{\partial}{\partial x} u(x,y)
\end{aligned}
\tag{6.5.1}
$$

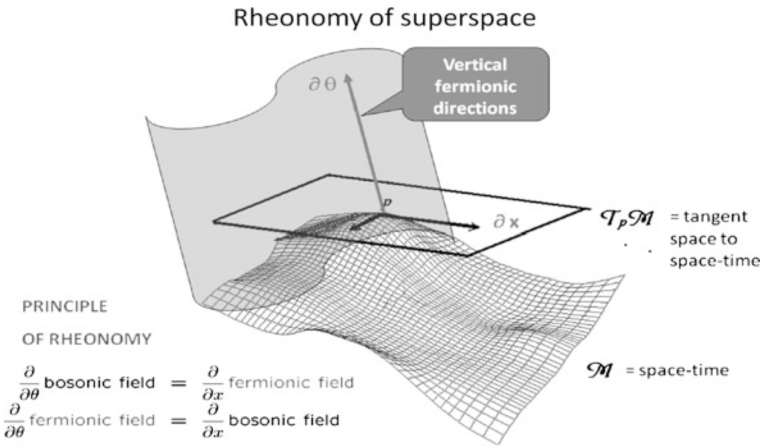

Fig. 6.10 The principle of rheonomy is reminiscent of the Cauchy-Riemann equations satisfied by the real and imaginary parts of analytic functions. Hence it encodes a sort of analyticity condition for the superconnections that constitute the field content of supergravity theories

In the suggested analogy, horizontal directions correspond to the real axis x, while the role of vertical ones is played by the imaginary axis y. Furthermore the real part $u(x, y)$ corresponds to the bosonic fields, while the imaginary part $v(x, y)$ corresponds to the fermionic ones. Indeed in order to respect the Bose/Fermi grading, vertical components of the curvatures can be restricted to be linear functions of the horizontal ones only by relating the vertical legs of bosonic curvatures to the horizontal ones of fermionic curvatures and vice-versa. The idea of rheonomy is graphically summarized in Fig. 6.10. The analogy with Cauchy-Riemann equations and analyticity immediately suggests one important consequence of rheonomy. As it is well known, the functions u and v are not arbitrary, rather, as a consequence of the integrability of Cauchy-Riemann equations, they are harmonic functions, namely each of them satisfies Laplace equation $\Delta u = \Delta v = 0$. In the same way we expect that the bosonic and fermionic connections, whose curvatures are rheonomic, should obey some differential equations of the second order in the horizontal variables as a consequence of integrability of the rheonomy conditions. This is indeed the case. What are the appropriate integrability conditions in this context? The answer is simple: they are the Bianchi identities of the considered Free Differential algebra with which the rheonomic conditions must be consistent. Writing the most general rheonomic parameterization of the curvatures with arbitrary coefficients and inserting it into the Bianchi identities, one finds that all such coefficients are uniquely determined: furthermore some algebraic constraints have to be satisfied by the horizontal curvature components. These constraints are differential equations in the space-time coordinates imposed on the connection components and, in our analogy, correspond to the Laplace equation satisfied by u and v. The physical interpretation of these constraints is fascinating: they are nothing else but the appropriate field equations of supergravity theory!

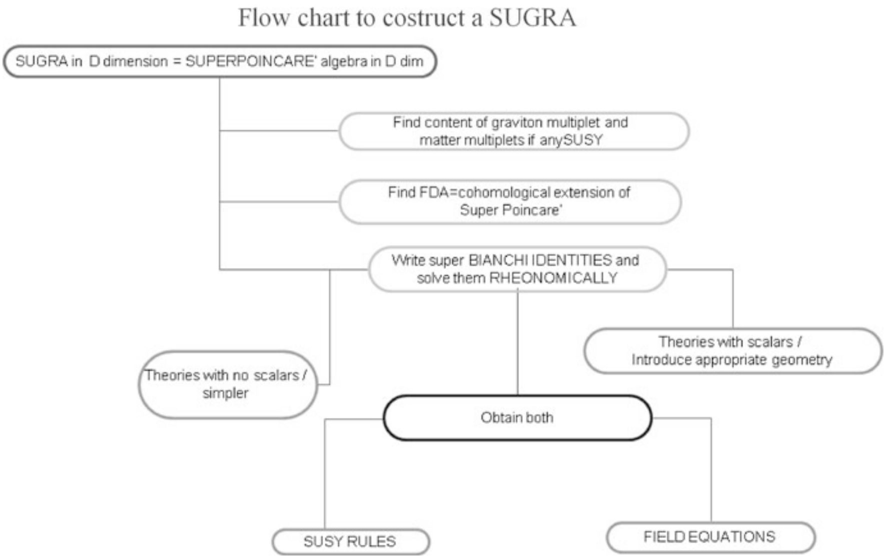

Fig. 6.11 A schematic graphical description of the steps involved in the construction of a supergravity theory

6.5.1 The Flow Chart for the Construction of a Supergravity Theory

From the considerations reviewed in the previous section a well-defined schema underlying the construction of a supergravity theory emerges quite clearly. Its logic is summarized in Fig. 6.11.

There are two preliminary steps.

The first is the construction of the relevant supermultiplets, namely of the irreducible unitary representations (UIR) of the supersymmetry algebra that will be included in the theory under consideration. By definition a supermultiplet is a finite collection of unitary irreducible representations of the Poincaré Lie algebra, in other words a stack of particles labeled by their *mass* and their *spin*, which, in higher dimensions D, means the representation of the little group, $\mathrm{SO}(D-1)$ for massive particles and $\mathrm{SO}(D-2)$ for massless ones, to which all the available states can be assigned. This step is purely algebraic and is based on a straightforward extension to the supersymmetry algebra of the method of induced representations utilized in constructing UIR of the Poincaré Lie algebra. Knowing the supermultiplets one obtains the field content of the considered supergravity theory.

The second step consists of determining the Free Differential Algebra in which the previously fixed field content will be accommodated. According to Sullivan's second theorem we have to consider the cohomology classes of the relevant D-dimensional super-Poincaré algebra and from that study determine the appropriate p-forms that have to be included in the list of minimal FDA generators.

Once the minimal FDA has been constructed the third step consists in its gauging. This is done by relaxing the condition that all curvatures should be zero. In this way we are able to write all the Bianchi identities and the fourth step, which is the most laborious, yet it is straightforward, consists in working out the rheonomic solution of the Bianchi identities together with its consistency conditions, coinciding with the classical field equations satisfied by the supergravity space-time fields. In the next subsection we illustrate such procedure by considering in some detail the master example of $D = 11$ supergravity. This is the largest possible supergravity and is thought to be the low energy effective field theory of M-theory, the so far mysterious non-perturbative theory that unifies in one more space-time dimensions all perturbative ten-dimensional superstrings.

6.5.2 Construction of $D = 11$ Supergravity, Alias M-Theory

In (6.4.2) we already introduced the Free Differential Algebra of M-theory and we justified its structure on the basis of Sullivan's second theorem and of the cohomology groups of the $D = 11$ super-Poincaré algebra. We found that in addition to the vielbein V^a, encoding the graviton degrees of freedom, the fermionic one-form ψ^α, encoding the degrees of freedom of a spin $\frac{3}{2}$ particle, and the spin-connection ω^{ab} providing, through soldering, the propagation mechanism of the graviton, we have a three-form $\mathbf{A}^{[3]}$ and a six-form $\mathbf{A}^{[6]}$. Going one step back we show here that this structure of the FDA perfectly matches with the field content of the massless multiplet of the $D = 11$ supersymmetry algebra which contains the spin two graviton. The structure of such a multiplet is summarized in Table 6.2 which anticipates the result. To derive such a result we argue as follows. First we construct a basis of gamma matrices well adapted to the case of massless particles propagating in a given direction, say along the 10th axis. In this case the transverse little group is SO(9) and a look at Table A.1 shows that we can represent the SO(9) Clifford

Table 6.2 Structure of the graviton multiplet in $D = 11$ supergravity

SO(1, 10) rep.	# of states	Name
$(2, 0, 0, 0, 0)$	44	graviton
$(\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$	128	gravitino
$(1, 1, 1, 0, 0)$	84	3-form

The five numbers given in brackets in column one are the Young labels of the corresponding irreducible representation of SO(1, 10). According to a well-established rule, for bosonic representations these labels denote the number of boxes in each row of a Young tableau which gives the symmetry of an irreducible tensor, $t_{a_1,\dots,a_n}$, having named n the sum of the five Young labels. For fermionic representations the Young labels are $n_i + \frac{1}{2}$ where once again n_i give, as in the bosonic case, the description of a Young tableau. The irreducible SO(1, 10) representation is provided by an irreducible spinor tensor $T^{\alpha}_{a_1,\dots,a_n}$ whose bosonic indices have the symmetry specified by the Youn tableau. All traces and gamma-traces of the irreducible spinor tensor vanish.

algebra by means of 16×16 symmetric matrices γ^i, fulfilling the relations:

$$\left\{\gamma^i, \gamma^j\right\} = -\delta^{ij} \tag{6.5.2}$$

Indeed from Table A.1 we obtain the information that in $d = 9$ there exists only a $\mathscr{C}_+$ charge conjugation matrix that is symmetric and squares to the identity. When $\mathscr{C}_+$ is chosen to be the unit matrix, which is always possible by means of a change of basis, the gamma matrices become symmetric. Relying on this we can construct the following basis of 32×32 gamma matrices fulfilling the $D = 11$ Minkowskian Clifford algebra with the standard mostly minus metric:

$$\eta^{ab} = \mathrm{diag}\{+, \underbrace{-, \ldots, -}_{10\ \text{times}}\} \tag{6.5.3}$$

We set:

$$\begin{aligned} \Gamma^0 &= \sigma_1 \otimes \mathbf{1} \\ \Gamma^i &= \sigma_3 \otimes \gamma^i \quad (i = 1, \ldots, 9) \\ \Gamma^{10} &= \mathrm{i}\sigma_2 \otimes \mathbf{1} \end{aligned} \tag{6.5.4}$$

With this choice the $D = 11$ antisymmetric charge conjugation matrix can be chosen as follows

$$\mathscr{C}_- = \mathrm{i}\sigma_2 \otimes \mathbf{1} = \Gamma^{10} \tag{6.5.5}$$

Consider next the supersymmetry algebra as given in (6.2.1) and specialize it to the case where the momentum vector P^μ is null-like and oriented along the 10th-axis: $P^\mu = p^0(1, 0, 0, \ldots, 0, 1)$. We obtain:

$$\{\overline{Q}_\alpha, \overline{Q}_\beta\} = \mathrm{i}\sigma_2(\sigma_1 - \mathrm{i}\sigma_2) p^0 = \mathrm{i} p^0 \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \tag{6.5.6}$$

Imposing the Majorana condition $Q = \mathscr{C}\overline{Q}^T$ on the supercharges, in the present gamma matrix basis we find the following result:

$$Q = \begin{pmatrix} q^\alpha \\ \mathrm{i} w^\beta \end{pmatrix} \tag{6.5.7}$$

where both q and w are real operators. Hence the anticommutation relations (6.5.6) representing the supersymmetry algebra reduce to the following form:

$$\begin{aligned} \left\{w^\alpha, w^\beta\right\} &= 0 \\ \left\{w^\alpha, q^\beta\right\} &= 0 \\ \left\{q^\alpha, q^\beta\right\} &= -\mathrm{i} p^0 \delta^{\alpha\beta} \end{aligned} \tag{6.5.8}$$

We conclude that in a UIR massless representation of the algebra we can consistently put to zero all the operators w^α and we are left with the sixteen q^α which close the standard algebra of eight fermionic harmonic oscillators. Indeed we can organize the q^α in two subsets of eight elements each, the former containing eight independent destruction operators, the latter containing their conjugate creation operators.

Since the sixteen operators q^α transform in the spinor representation of the transverse group SO(9), we can associate them with the sixteen weights of that representation which are 4-component vectors of the following form:

$$\mathfrak{W}_{spin\ 9} = \left(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}\right) \tag{6.5.9}$$

all possible choices of the signs being allowed. Furthermore we can arrange matters in such a way that the creation operators are associated with the positive weights, while the destruction operators are associated with the negative ones. The positive weights can be identified with:

$$\mathfrak{W}^{>}_{spin\ 9} = \left(+\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}\right) \tag{6.5.10}$$

while the negative ones are:

$$\mathfrak{W}^{<}_{spin\ 9} = \left(-\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}\right) \tag{6.5.11}$$

Let us now consider the three SO(9) representations that admit the following vectors as highest weights:

$$\mathfrak{W}^{graviton}_{\max} = (+2, 0, 0, 0) \tag{6.5.12}$$

$$\mathfrak{W}^{gravitino}_{\max} = \left(+\frac{3}{2}, +\frac{1}{2}, +\frac{1}{2}, +\frac{1}{2}\right) \tag{6.5.13}$$

$$\mathfrak{W}^{3\ form}_{\max} = (+1, +1, +1, 0) \tag{6.5.14}$$

As anticipated by their names, the UIR representations of SO(1, 10) induced by the above SO(9) irreducible representations correspond to the massless graviton, to the massless gravitino and to a massless gauge particle with three antisymmetric indices, respectively. These are the particles forming the $D = 11$ supermultiplet as described in Table 6.2 and the number of degrees of freedom of each them is just the dimension of the corresponding SO(9) representation, in other words the number of its weights. These weights can be regarded as the possible polarizations of the corresponding massless particle propagating at the speed of light in the 10th space direction. Remains the question why precisely these representations are the content of the supermultiplet, namely why they build up an irreducible representation of the supersymmetry algebra. The answer is easily obtained arguing in terms of highest vectors. Consider the highest helicity state of the graviton:

$$|\Omega\rangle = |2,0,0,0\rangle \tag{6.5.15}$$

and let us assume that it is the highest state of the entire supermultiplet. This means that it is annihilated by all supercharges that are creation operators, namely:

$$q^{(\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2})}|2,0,0,0\rangle = 0 \tag{6.5.16}$$

A non-vanishing result is obtained applying to $|2,0,0,0\rangle$ products of the operators $q^{(-\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2})}$, where all factors in the product are different. So we find:

$$\begin{aligned}
&|2,0,0,0\rangle\\
&\left|\frac{3}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right\rangle = q^{(-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})}|2,0,0,0\rangle\\
&|1,1,1,0\rangle = q^{(-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})}q^{(-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2})}|2,0,0,0\rangle\\
&\cdots = \cdots\\
&\cdots = \cdots\\
&|-2,0,0,0\rangle = \left(\prod_{8\text{ sign choices}} q^{-\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2}}\right)|2,0,0,0\rangle
\end{aligned} \tag{6.5.17}$$

Applying the destruction operators to all other positive weights of the spin two representation we find that the set of weights we can construct is just the union of the weights belonging to the three mentioned irreducible representations. Hence the $D = 11$ supermultiplet is indeed constituted by the fields mentioned in Table 6.2.

6.6 Summary of Supergravities

Having clarified the construction principles of supergravities and their fundamental algebraic basis, rooted in the Free Differential Algebra structure we pass to a scan of the available theories.

In $D = 11$ there is just a unique supergravity, M-theory, whose structure we have thoroughly discussed.

In $D = 10$ we have just five different supergravities, displayed in Fig. 6.12 which are in one-to-one correspondence with the available consistent superstring theories. Actually, from the supergravity viewpoint there are only three possible theories in $D = 10$. The type II theories that have two Majorana-Weyl supercharges (A and B, according to the choice of their chirality, as we explain below) and the type I theory that has only one Majorana-Weyl supercharge and can be coupled to a vector multiplet. The different choice of the gauge group is the only distinction among type

Theories	BOSE STATES		FERMI STATES	
		R-R	Left handed	Right handed
Type II A	$\phi, g_{\mu\nu}, B_{\mu\nu}$	$A^{[1]}, A^{[3]}$	$\psi_\mu^{[L]}, \chi^{[L]}$	$\psi_\mu^{[R]}, \chi^{[R]}$
Type II B	$\phi, g_{\mu\nu}, B_{\mu\nu}$	$A^{[2]}, A^{[4]}$	$\psi_\mu^{[L]}, \chi^{[L]}$ $\psi_\mu^{[L]*}, \chi^{[L]*}$	
Heterotic SO(32)	$\phi, g_{\mu\nu}, B_{\mu\nu}, A_\mu^{\alpha\in\mathrm{adj}[SO(32)]}$		$\psi_\mu^{[L]}$, $\lambda^{\alpha\in\mathrm{adj}[SO(32)]}$	$\chi^{[R]}$
Heterotic $E_8 \times E_8$	$\phi, g_{\mu\nu}, B_{\mu\nu}, A_\mu^{\alpha\in\mathrm{adj}[E_8\times E_8]}$		$\psi_\mu^{[L]}$, $\lambda^{\alpha\in\mathrm{adj}[E_8\times E_8]}$	$\chi^{[R]}$
Type I SO(32)	$\phi, g_{\mu\nu}, B_{\mu\nu}, A_\mu^{\alpha\in\mathrm{adj}[SO(32)]}$		$\psi_\mu^{[L]}$, $\lambda^{\alpha\in\mathrm{adj}[SO(32)]}$	$\chi^{[R]}$

Fig. 6.12 The field content of the five $D = 10$ supergravities. The fields of each of these theories are the massless modes of the corresponding superstring theory. The bosonic fields are organized according to their string origin in the NS-NS or R-R sector, while the fermionic fields are organized according to their chirality

I theories. However a very complex mechanism displayed by these latter resides in the possibility of introducing their coupling to gauge and Lorentz Chern-Simons three-forms: this provides the cancellation of anomalies and selects the three models that complete the list of consistent superstring theories. In this book we do not dwell on $D = 10$, $\mathcal{N} = 1$ supergravities and on their Lorentz Chern-Simons coupling for which we refer the reader to the book [28]. We rather focus on type II theories of which we give a complete account.

The key point in $D = 10$ is the existence of Majorana-Weyl spinors satisfying the double condition:

$$\psi_{L/R} = \mathscr{C}\overline{\psi}^T_{L/R}; \qquad \Gamma_{11}\psi_{L/R} = \pm\psi_{L/R} \tag{6.6.1}$$

where $\mathscr{C}$ is the charge conjugation matrix and Γ_{11} is the chirality matrix (see Appendix A.4 for details). The type IIA theory is based on the super-Poincaré Lie algebra containing two Majorana-Weyl supercharges, one Q_L which is left-handed, the other Q_R which is right-handed. The type IIB theory is instead based on the super-Poincaré Lie algebra that contains two Majorana-Weyl supercharges, Q_L^A $(A = 1, 2)$ of the same chirality, say left-handed. The presence of this doublet of chiral supercharges introduces an SL$(2, \mathbb{R})$ symmetry which is an essential item in the construction of the whole theory.

In dimensions $D < 10$ the number of available supergravity theories starts growing because we can reduce the number of supercharges and introduce an increasing available choice of matter supermultiplets. As soon as scalar fields appear in the spectrum they introduce a new quality: the scalars can be regarded as the coordinates of a differentiable target manifold for whose geometry supersymmetry selects a variety of *special structures*. Chapter 8 presents a bird-eye review of supergravity geometries and couplings.

Chapter 7 is instead devoted to enlighten the vital dualism between the bulk and the brane world-volume perspectives. For that we shall need the explicit structure of the type II theories which we present in the next two sections.

6.7 Type IIA Supergravity in $D = 10$

The full-fledged rheonomic construction of type IIA supergravity was obtained only recently in [29] which we follow in the present section. The field content of the theory is given in Table 6.3.

This field content corresponds to the basic forms of a specific Free Differential Algebra including the 0-form items entering the rheonomic parameterizations of its curvatures.

The starting point is, as usual, the super-Poincaré algebra. In $D = 10$ we have two super-Poincaré algebras with 32 supercharges, the type IIA and the type IIB. If we also include the dilaton, as we will do, there are various equivalent definitions of curvatures that are named *frames* and differ by dilaton pre-factors. The *Einstein frame* is that which leads to an action where the Einstein kinetic term is canonical without any dilaton pre-factors. The *string frame*, which has distinguished advantages when writing the string action in its background, corresponds instead to non-canonical Einstein terms in the action. The two frames are just related by a suitable Weyl transformation depending on the dilaton. For type IIA supergravity we use the string frames for two reasons. The first is that in this frame the FDA has a simpler and more elegant form. The second is pedagogical. We want to emphasize the freedom of using different but equivalent frames. For type IIB supergravity we will rather use the Einstein frame in which the $\mathrm{SL}(2,\mathbb{R})$ symmetry of that theory is manifest.

The Maurer Cartan description of the type IIA superalgebra is obtained by setting to zero the following curvatures:

Table 6.3 Structure of the graviton multiplet in Type IIA supergravity

$\mathrm{SO}(1,9)$ rep.	# of states	Name
$(2,0,0,0,0)$	35	graviton
$(\frac{3}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})_L$	56	left gravitino
$(\frac{3}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})_R$	56	right gravitino
$(1,1,0,0,0)$	28	NS B-field
$(1,0,0,0,0)$	8	RR 1-form
$(1,1,1,0,0)$	56	RR 3-form
$(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})_L$	8	left dilatino
$(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})_R$	8	right dilatino
$(0,0,0,0,0)$	1	dilaton

Type IIA Super-Poicaré Algebra in the String Frame

$$\mathfrak{R}^{ab} \equiv d\omega^{ab} - \omega^{ac} \wedge \omega^{cb} \tag{6.7.1}$$

$$\mathfrak{T}^{a} \equiv \mathscr{D}V^{a} - \mathrm{i}\frac{1}{2}\left(\overline{\psi}_L \wedge \Gamma^a \psi_L + \overline{\psi}_R \wedge \Gamma^a \psi_R\right) \tag{6.7.2}$$

$$\rho_{L,R} \equiv \mathscr{D}\psi_{L,R} \equiv d\psi_{L,R} - \frac{1}{4}\omega^{ab} \wedge \Gamma_{ab}\psi_{L,R} \tag{6.7.3}$$

$$\mathbf{G}^{[2]} \equiv d\mathbf{C}^{[1]} + \exp[-\varphi]\overline{\psi}_R \wedge \psi_L \tag{6.7.4}$$

$$\mathbf{f}^{[1]} \equiv d\varphi \tag{6.7.5}$$

$$\nabla\chi_{L/R} \equiv d\chi_{L,R} - \frac{1}{4}\omega^{ab} \wedge \Gamma_{ab}\chi_{L,R} \tag{6.7.6}$$

where the 0-form dilaton φ appearing in (6.7.4) introduces a mobile coupling constant. Furthermore, V^a, ω^{ab} denote the vielbein and the spin connection 1-forms, respectively, while the two fermionic 1-forms $\psi_{L/R}$ are Majorana-Weyl spinors of opposite chirality:

$$\Gamma_{11}\psi_{L/R} = \pm\psi_{L/R} \tag{6.7.7}$$

The flat metric $\eta_{ab} = \mathrm{diag}(+, -, \dots, -)$ is the mostly minus one and Γ_{11} is Hermitian and squares to the identity $\Gamma_{11}^2 = \mathbf{1}$.

Setting $R^{ab} = T^a = \mathbf{G}^{[2]} = \mathbf{f}^{[1]} = 0$ one obtains the Maurer Cartan equations of a superalgebra where the spinor charges, $Q_{L,R}$ dual to the spinor 1-forms $\psi_{L,R}$ not only anticommute to the translations P_a but also to a central charge Z dual to the (Ramond Ramond) 1-form $\mathbf{C}^{[1]}$.

According to Sullivan's second theorem the FDA extension of the above superalgebra is dictated by its cohomology. In a first step one finds that there exists a cohomology class of degree three which motivates the introduction of a new 2-form generator $\mathbf{B}^{[2]}$ which in the superstring interpretation is just the Kalb-Ramond field. Considering then the cohomology of the FDA-extended algebra one finds a degree four cohomology class which motivates the introduction of a 3-form generator $\mathbf{C}^{[3]}$. In the superstring interpretation, this is just the second R-R field, the first being the gauge field $\mathbf{C}^{[1]}$. Altogether the complete type IIA FDA is obtained by adjoining the following curvatures to those already introduced:

The FDA Extension of the Type IIA Superalgebra in the String Frame

$$\mathbf{H}^{[3]} = d\mathbf{B}^{[2]} + \mathrm{i}(\overline{\psi}_L \wedge \Gamma_a\psi_L - \overline{\psi}_R \wedge \Gamma_a\psi_R) \wedge V^a \tag{6.7.8}$$

$$\begin{aligned}\mathbf{G}^{[4]} &= d\mathbf{C}^{[3]} + \mathbf{B}^{[2]} \wedge d\mathbf{C}^{[1]} \\ &\quad - \frac{1}{2}\exp[-\varphi](\overline{\psi}_L \wedge \Gamma_{ab}\psi_R + \overline{\psi}_R \wedge \Gamma_{ab}\psi_L) \wedge V^a \wedge V^b\end{aligned} \tag{6.7.9}$$

Equations (6.7.1)–(6.7.5) together with (6.7.8)–(6.7.9) provide the complete definition of the type IIA Free Differential Algebra.

The next task is that of writing the Bianchi identities and construct their rheonomic solution.

The Bianchi Identities The curvature definitions listed above lead immediately to the following Bianchi identities which we write, already under the assumption that the torsion is zero $\mathfrak{T}^a = 0$:

$$0 = \mathscr{D}\mathfrak{R}^{ab} \tag{6.7.10}$$

$$0 = \mathfrak{R}^{ab} \wedge V_b - \mathrm{i}\big(\overline{\psi}_L \wedge \Gamma^a \rho_L + \overline{\psi}_R \wedge \Gamma^a \rho_R\big) \tag{6.7.11}$$

$$0 = \mathscr{D}\rho_{L/R} + \frac{1}{4}\mathfrak{R}^{ab} \wedge \Gamma_{ab}\psi_{L/R} \tag{6.7.12}$$

$$0 = d\mathbf{G}^{[2]} + \mathbf{f}^{[1]} \wedge \exp[-\varphi]\overline{\psi}_R \wedge \psi_L + \exp[-\varphi](\overline{\psi}_R \wedge \rho_L - \overline{\psi}_L \wedge \rho_R) \tag{6.7.13}$$

$$0 = d\mathbf{f}^{[1]} \tag{6.7.14}$$

$$0 = d\mathbf{H}^{[3]} + 2\mathrm{i}(\overline{\psi}_L \wedge \Gamma_a \rho_L - \overline{\psi}_R \wedge \Gamma_a \rho_R) \wedge V^a \tag{6.7.15}$$

$$\begin{aligned} 0 = {} & d\mathbf{G}^{[4]} - \mathbf{H}^{[3]} \wedge \mathbf{G}^{[2]} + \mathrm{i}(\overline{\psi}_L \wedge \Gamma_a \psi_L - \overline{\psi}_R \wedge \Gamma_a \psi_R) \wedge V^a \wedge \mathbf{G}^{[2]} \\ & + \mathbf{H}^{[3]} \wedge \exp[-\varphi]\overline{\psi}_R \wedge \psi_L \\ & - \frac{1}{2}\mathbf{f}^{[1]} \wedge \exp[-\varphi](\overline{\psi}_L \wedge \Gamma_{ab}\psi_R + \overline{\psi}_R \wedge \Gamma_{ab}\psi_L) \wedge V^a \wedge V^b \\ & - \exp[-\varphi](\overline{\psi}_L \wedge \Gamma_{ab}\rho_R + \overline{\psi}_R \wedge \Gamma_{ab}\rho_L) \wedge V^a \wedge V^b \end{aligned} \tag{6.7.16}$$

$$0 = \mathscr{D}^2\chi_{L/R} + \frac{1}{4}R^{ab} \wedge \Gamma_{ab}\chi_{L/R} \tag{6.7.17}$$

As it is the case for all supergravities and for all FDAs the above Bianchi identities admit a unique rheonomic solution up to field redefinitions. The rheonomic solution of the Bianchis implies also the field equations of the theory given as a set of constraints to be satisfied by the space-time curvature components. The choice of a frame is performed by imposing an additional condition which fixes the field redefinitions. In particular we define the string frame by requiring both the vanishing of the torsion

$$\mathfrak{T}^a = 0 \tag{6.7.18}$$

and the vanishing of all of the fermionic sectors of the 3-form curvature $\mathbf{H}^{[3]}$. This amounts to setting:

$$\mathbf{H}^{[3]} = \mathscr{H}_{abc} V^a \wedge V^b \wedge V^c \tag{6.7.19}$$

One can verify that the fulfillment of the above conditions requires a Weyl rescaling of the fields which yields the pre-factor $e^{-2\varphi}$ in front of the NS-NS part and the fermionic sector of the action.

6.7.1 *Rheonomic Parameterizations of the Type IIA Curvatures in the String Frame*

In order to present the result in its most compact form it is convenient to introduce a set of tensors, which involve both the field strengths $\mathscr{G}_{ab}$, $\mathscr{G}_{ABCD}$ of the Ramond-Ramond p-forms and also bilinear currents in the dilatino field $\chi_{L/R}$. The needed tensors are those listed below:

$$\begin{aligned}
\mathscr{M}_{ab} &= \left(\frac{1}{8}\exp[\varphi]\mathscr{G}_{ab} + \frac{9}{64}\overline{\chi}_R\Gamma_{ab}\chi_L\right) \\
\mathscr{M}_{abcd} &= -\frac{1}{16}\exp[\varphi]\mathscr{G}_{abcd} - \frac{3\mathrm{i}}{256}\overline{\chi}_L\Gamma_{abcd}\chi_R \\
\mathscr{N}_0 &= \frac{3}{4}\overline{\chi}_L\chi_R \\
\mathscr{N}_{ab} &= \frac{1}{4}\exp[\varphi]\mathscr{G}_{ab} + \frac{9}{32}\overline{\chi}_R\Gamma_{ab}\chi_L = 2\mathscr{M}_{ab} \\
\mathscr{N}_{abcd} &= \frac{1}{24}\exp[\varphi]\mathscr{G}_{abcd} + \frac{1}{128}\overline{\chi}_R\Gamma_{abcd}\chi_L = -\frac{2}{3}\mathscr{M}_{abcd}
\end{aligned} \tag{6.7.20}$$

The above tensors are conveniently assembled into the following spinor matrices

$$\mathscr{Z} = \mathscr{N}_{ab}\Gamma^{ab} + 3\mathscr{N}_{abcd}\Gamma^{abcd} \tag{6.7.21}$$

$$\mathscr{M}_{\pm} = \mathrm{i}\left(\mp\mathscr{M}_{ab}\Gamma^{ab} + \mathscr{M}_{abcd}\Gamma^{abcd}\right) \tag{6.7.22}$$

$$\mathscr{N}_{\pm}^{(even)} = \mp\mathscr{N}_0\mathbf{1} + \mathscr{N}_{ab}\Gamma^{ab} \mp \mathscr{N}_{abcd}\Gamma^{abcd} \tag{6.7.23}$$

$$\mathscr{N}_{\pm}^{(odd)} = \pm\frac{\mathrm{i}}{3}f_a\Gamma^a \pm \frac{1}{64}\overline{\chi}_{R/L}\Gamma_{abc}\chi_{R/L}\Gamma^{abc} - \frac{\mathrm{i}}{12}\mathscr{H}_{abc}\Gamma^{abc} \tag{6.7.24}$$

$$\mathscr{L}_{a\pm}^{(odd)} = \mathscr{M}_{\mp}\Gamma_a; \qquad \mathscr{L}_{a\pm}^{(even)} = \mp\frac{3}{8}\mathscr{H}_{abc}\Gamma^{bc} \tag{6.7.25}$$

In terms of these objects the rheonomic parameterizations of the curvatures, solving the Bianchi identities can be written as follows:

Bosonic Curvatures

$$\mathfrak{T}^a = 0 \tag{6.7.26}$$

$$\begin{aligned}
\mathfrak{R}^{ab} &= \mathscr{R}^{ab}{}_{mn}V^m \wedge V^n + \overline{\psi}_R\Theta^{ab}_{m|L} \wedge V^m + \overline{\psi}_L\Theta^{ab}_{m|R} \wedge V^m \\
&\quad + \mathrm{i}\frac{3}{4}(\overline{\psi}_L \wedge \Gamma_c\psi_L - \overline{\psi}_R \wedge \Gamma_c\psi_R)\mathscr{H}^{abc} \\
&\quad + \overline{\psi}_L \wedge \Gamma^{[a}\mathscr{Z}\Gamma^{b]}\psi_R
\end{aligned} \tag{6.7.27}$$

$$\mathbf{H}^{[3]} = \mathscr{H}_{abc}V^a \wedge V^b \wedge V^c \tag{6.7.28}$$

$$\mathbf{G}^{[2]} = \mathscr{G}_{ab} V^a \wedge V^b + \mathrm{i}\frac{3}{2} \exp[-\varphi](\overline{\chi}_L \Gamma_a \psi_L + \overline{\chi}_R \Gamma_a \psi_R) \wedge V^a \tag{6.7.29}$$

$$\mathbf{f}^{[1]} = f_a V^a + \frac{3}{2}(\overline{\chi}_R \psi_L - \overline{\chi}_L \psi_R) \tag{6.7.30}$$

$$\begin{aligned}\mathbf{G}^{[4]} &= \mathscr{G}_{abcd} V^a \wedge V^b \wedge V^c \wedge V^d \\ &\quad - \mathrm{i}\frac{1}{2} \exp[-\varphi](\overline{\chi}_L \Gamma_{abc} \psi_L - \overline{\chi}_R \Gamma_{abc} \psi_R) \wedge V^a \wedge V^b \wedge V^c\end{aligned} \tag{6.7.31}$$

Fermionic Curvatures

$$\rho_{L/R} = \rho^{L/R}_{ab} V^a \wedge V^b + \mathscr{L}^{(even)}_{a\pm} \psi_{L/R} + \mathscr{L}^{(odd)}_{a\mp} \psi_{R/L} + \rho^{(0,2)}_{L/R} \tag{6.7.32}$$

$$\nabla \chi_{L/R} = \mathscr{D}_a \chi_{L/R} V^a + \mathscr{N}^{(even)}_{\pm} \psi_{L/R} + \mathscr{N}^{(odd)}_{\mp} \psi_{R/L} \tag{6.7.33}$$

Note that the components of the generalized curvatures along the bosonic vielbeins do not coincide with their spacetime components, but rather with their supercovariant extension. Indeed expanding for example the four-form along the spacetime differentials one finds that

$$\begin{aligned}\widetilde{G}_{\mu\nu\rho\sigma} &\equiv \mathscr{G}_{abcd} V^a_\mu \wedge V^b_\nu \wedge V^c_\rho \wedge V^d_\sigma \\ &= \partial_{[\mu} C^{[4]}_{\nu\rho\sigma]} + B^{[2]}_{[\mu\nu} \partial_\rho C^{[1]}_{\sigma]} - \frac{1}{2} e^{-\varphi} (\overline{\psi}_{L[\mu} \Gamma_{\nu\rho} \psi_{R\sigma]} + \overline{\psi}_{R[\mu} \Gamma_{\nu\rho} \psi_{L\sigma]}) \\ &\quad + \mathrm{i}\frac{1}{2} \exp[-\varphi](\overline{\chi}_L \Gamma_{[\mu\nu\rho} \psi_{L\sigma]} - \overline{\chi}_R \Gamma_{[\mu\nu\rho} \psi_{R\sigma]})\end{aligned} \tag{6.7.34}$$

where $\widetilde{G}$ is the supercovariant field strength. In the parameterization (6.7.27) of the Riemann tensor we have used the following definition:

$$\Theta_{ab|cL/R} = -i(\Gamma_a \rho_{bcR/L} + \Gamma_b \rho_{caR/L} - \Gamma_c \rho_{abR/L}) \tag{6.7.35}$$

Finally by $\rho^{(0,2)}_{L/R}$ we have denoted the fermion-fermion part of the gravitino curvature whose explicit expression can be written in two different forms, equivalent by Fierz rearrangement:

$$\begin{aligned}\rho^{(0,2)}_{L/R} &= \pm \frac{21}{32} \Gamma_a \chi_{R/L} \overline{\psi}_{L/R} \wedge \Gamma^a \psi_{L/R} \\ &\qquad \mp \frac{1}{2560} \Gamma_{a_1 a_2 a_3 a_4 a_5} \chi_{R/L} \left(\overline{\psi}_{L/R} \Gamma^{a_1 a_2 a_3 a_4 a_5} \psi_{L/R}\right)\end{aligned} \tag{6.7.36}$$

or

$$\rho^{(0,2)}_{L/R} = \pm \frac{3}{8} \mathrm{i} \psi_{L/R} \wedge \overline{\chi}_{R/L} \psi_{L/R} \pm \frac{3}{16} \mathrm{i} \Gamma_{ab} \psi_{L/R} \wedge \overline{\chi}_{R/L} \Gamma^{ab} \psi_{L/R} \tag{6.7.37}$$

6.7.2 *Field Equations of Type IIA Supergravity in the String Frame*

As usual the rheonomic parameterizations of the supercurvatures imply, via Bianchi identities, a certain number of constraints on the inner components of the same curvatures which can be recognized as the field equations of type IIA supergravity. In [29] the authors derived the bosonic part of these field equations in two steps: first they performed the Einstein frame dimensional reduction on a circle of the field equations of $D = 11$ supergravity. Then they applied the Weyl transformation which relates the Einstein frame to the string frame:

$$V^a_{(E)} = V^a_{(S)} e^{-\varphi/4} \tag{6.7.38}$$

Obviously they could have obtained the same result directly from the Bianchi identities in the string frame, yet this would have been much more laborious.

In any case the result is the following one. There is an Einstein equation of the following form:

$$\mathscr{R}_{ab} = \widehat{T}_{ab}(f) + \widehat{T}_{ab}(\mathscr{G}_2) + \widehat{T}_{ab}(\mathscr{H}) + \widehat{T}_{ab}(\mathscr{G}_4) \tag{6.7.39}$$

where the stress-energy tensor on the right hand side are defined as

$$\widehat{T}_{ab}(f) = -\mathscr{D}_a\mathscr{D}_b\varphi + \frac{8}{9}\mathscr{D}_a\varphi\mathscr{D}_b\varphi - \eta_{ab}\left(\frac{1}{6}\Box\varphi + \frac{5}{9}\mathscr{D}^m\varphi\mathscr{D}_m\varphi\right) \tag{6.7.40}$$

$$\widehat{T}_{ab}(\mathscr{G}_2) = \exp[2\varphi]\mathscr{G}_{ax}\mathscr{G}_{by}\eta^{ab} \tag{6.7.41}$$

$$\widehat{T}_{ab}(\mathscr{H}) = -\exp\left[\frac{1}{3}\varphi\right]\left(\frac{9}{8}\mathscr{H}_{axy}\mathscr{H}_{bwt}\eta^{xw}\eta^{yt} - \frac{1}{8}\eta_{ab}\mathscr{H}_{xyz}\mathscr{H}^{xyz}\right) \tag{6.7.42}$$

$$\widehat{T}_{ab}(\mathscr{G}_4) = \exp[2\varphi]\left(6\mathscr{G}_{ax_1x_2x_3}\mathscr{G}_{by_1y_2y_3}\eta^{x_1y_1}\eta^{x_2y_2}\eta^{x_3y_3} - \frac{1}{2}\eta_{ab}\mathscr{G}_{x_1\dots x_4}\mathscr{G}^{x_1\dots x_4}\right) \tag{6.7.43}$$

Next we have the equations for the dilaton and the Ramond 1-form:

$$\begin{aligned} 0 &= \Box\varphi - 2f_a f^a + \frac{3}{2}\exp[2\varphi]\mathscr{G}^{x_1x_2}\mathscr{G}_{x_1x_2} \\ &\quad + \frac{3}{2}\exp[2\varphi]\mathscr{G}^{x_1x_2x_3x_4}\mathscr{G}_{x_1x_2x_3x_4} + \frac{3}{4}\exp\left[\frac{4}{3}\varphi\right]\mathscr{H}^{x_1x_2x_3}\mathscr{H}_{x_1x_2x_3} \end{aligned} \tag{6.7.44}$$

$$0 = \mathscr{D}_m\mathscr{G}^{ma} - \frac{5}{3}f^m\mathscr{G}_{ma} + 3\mathscr{G}^{ax_1x_2x_3}\mathscr{H}_{x_1x_2x_3} \tag{6.7.45}$$

and the equations for the NS 2-form and for the RR 3-form:

$$0 = \mathscr{D}_m\mathscr{H}^{mab} - \frac{2}{3}f^m\mathscr{H}_{mab}$$

$$- \exp\left[\frac{4}{3}\varphi\right]\left(4\mathscr{G}^{x_1x_2ab}\mathscr{G}_{x_1x_2} - \frac{1}{24}\varepsilon^{abx_1\ldots x_8}\mathscr{G}_{x_1x_2x_3x_4}\mathscr{G}_{x_5x_6x_7x_8}\right) \qquad (6.7.46)$$

$$\begin{aligned} 0 &= \mathscr{D}_m\mathscr{G}^{ma_1a_2a_3} + \frac{1}{3}f_m\mathscr{G}^{ma_1a_2a_3} \\ &\quad + \exp\left[\frac{2}{3}\varphi\right]\left(\frac{3}{2}\mathscr{G}^{m[a_1}H^{a_2a_3]n}\eta_{mn} + \frac{1}{48}\varepsilon^{a_1a_2a_3x_1\ldots x_7}\mathscr{G}_{x_1x_2x_3x_4}H_{x_5x_6x_7}\right) \end{aligned} \qquad (6.7.47)$$

Any solution of these bosonic set of equations can be uniquely extended to a full superspace solution involving 32 theta variables by means of the rheonomic conditions. The implementation of such a fermionic integration is the *supergauge completion*.

6.8 Type IIB Supergravity

The formulation of type IIB supergravity as it appears in string theory textbooks [30–33] is tailored for the comparison with superstring amplitudes and is quite appropriate to this goal. Yet, from the viewpoint of the general geometrical set up of supergravity theories this formulation is somewhat unwieldy. Specifically it neither makes the $\mathrm{SU}(1,1)/\mathrm{U}(1)$ coset structure of the theory manifest, nor it relates the supersymmetry transformation rules to the underlying algebraic structure which, as in all other instances of supergravities, is a simple and well defined *Free Differential algebra*.

The Free Differential Algebra of type IIB supergravity was singled out many years ago by Castellani in [35] and the geometric manifestly $\mathrm{SU}(1,1)$ covariant formulation of the theory was constructed by Castellani and Pesando in [34]. In this section we summarize their formulae giving also their transcription from a complex $\mathrm{SU}(1,1)$ basis to a real $\mathrm{SL}(2,\mathbb{R})$ basis. Furthermore we provide the translation vocabulary between these intrinsic notations and those of Polchinski's textbook [32, 33] frequently used in current superstring literature.

6.8.1 *The* $\mathrm{SU}(1,1)/\mathrm{U}(1) \sim \mathrm{SL}(2,\mathbb{R})/\mathrm{O}(2)$ *Coset*

As it is later emphasized in Chap. 8, a basic ingredient in all supergravity constructions is the parameterization of the scalar manifold geometry that, with few exceptions, corresponds to a homogeneous scalar manifold. In all these cases the essential building block appearing in the Lagrangian and supersymmetry transformation rules is the coset representative $\mathbb{L}(\phi_i)$ that provides a parameterization of the coset manifold G/H in terms of some chosen patch of coordinates. A very useful choice is given by the so called solvable Lie algebra parameterization.[11] This is true also in the present case where the solvable parameterization of the coset $\mathrm{SU}(1,1)/\mathrm{U}(1) \sim \mathrm{SL}(2,\mathbb{R})/\mathrm{O}(2)$ is precisely that which allows for the identification of the massless superstring fields inside the covariant formulation of supergravity.

Our notations are as follows.

SL(2, ℝ) Lie Algebra

$$[L_0, L_\pm] = \pm L_\pm; \qquad [L_+, L_-] = 2L_0 \tag{6.8.1}$$

with the following explicit 2-dimensional representation:

$$L_0 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \qquad L_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}; \qquad L_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{6.8.2}$$

Coset Representative of SL(2, ℝ)/O(2) in the Solvable Parameterization

$$\mathbb{L}(\varphi, C_{[0]}) = \exp[\varphi L_0]\exp\big[C_{[0]}e^{\varphi}L_-\big] = \begin{pmatrix} \exp[\varphi/2] & 0 \\ C_{[0]}e^{\varphi/2} & \exp[-\varphi/2] \end{pmatrix} \tag{6.8.3}$$

where $\varphi(x)$ and $C_{[0]}$ are respectively identified with the dilaton and with the Ramond-Ramond 0-form of the superstring massless spectrum. The isomorphism of $\mathrm{SL}(2,\mathbb{R})$ with $\mathrm{SU}(1,1)$ is realized by conjugation with the Cayley matrix:

$$\mathscr{C} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -\mathrm{i} \\ 1 & \mathrm{i} \end{pmatrix} \tag{6.8.4}$$

Introducing the $\mathrm{SU}(1,1)$ coset representative

$$\mathrm{SU}(1,1) \ni \Lambda = \mathscr{C}\mathbb{L}\mathscr{C}^{-1} \tag{6.8.5}$$

from the left invariant 1-form $\Lambda^{-1}\,d\Lambda$ we can extract the 1-forms corresponding to the scalar vielbein P and the $\mathrm{U}(1)$ connection Q

The SU(1, 1)/U(1) Vielbein and Connection

$$\Lambda^{-1}\,d\Lambda = \begin{pmatrix} -\mathrm{i}Q & P \\ P^\star & \mathrm{i}Q \end{pmatrix} \tag{6.8.6}$$

Explicitly

$$\begin{aligned} P &= \frac{1}{2}\big(d\varphi - \mathrm{i}e^{\varphi}\,dC_{[0]}\big) && \text{scalar vielbein} \\ Q &= \frac{1}{2}\exp[\varphi]\,dC_{[0]} && \text{U(1)-connection} \end{aligned} \tag{6.8.7}$$

[11] For a review see either [25] or [24] and all references therein.

Table 6.4 Field content of type IIB supergravity

Field in SU(1, 1) basis	SU(1, 1) repres.	U(1) charge	Superstring zero modes
V_μ^a	$J=0$	0	graviton $h_{\mu\nu}$
ψ_μ	$J=0$	$\frac{1}{2}$	gravitinos $\psi_{A\mu}$
$A_{\mu\nu}^\alpha$	$J=\frac{1}{2}$	0	$B_{[2]}$, $C_{[2]}$
$C_{\mu\nu\rho\sigma}$	$J=0$	0	$C_{[4]}$
λ	$J=0$	$\frac{3}{2}$	dilatinos λ_A
$\mathbb{L}^\alpha{}_\beta$	$J=\frac{1}{2}$	± 1	φ, $C_{[0]}$

The early Greek indices $\alpha, \beta, \ldots = 1, 2$ run in the fundamental representation of SU(1, 1), while the early capital Latin indices $A, B, \ldots = 1, 2$ run in the fundamental representation of $\mathrm{SL}(2, \mathbb{R})$. The p-gauge forms of the Ramond Ramond sector are denoted by $C_{[p]}$.

6.8.2 The Free Differential Algebra, the Supergravity Fields and the Curvatures

Following Castellani and Pesando the field content of type IIB supergravity is organized into representations of SU(1, 1) as displayed in Table 6.4. In order to write down the free differential algebra the critical issue is the correct identification of the fermionic terms contributing to the curvature of the complex 2-form doublet $A_{\mu\nu}^\alpha$. These latter transform in the 2-dimensional representation of SU(1, 1) and are related by the Cayley matrix of (6.8.4) to a doublet of real 2-forms $\mathbf{A}_{\mu\nu}^\Lambda$ that transform in the 2-dimensional representation of $\mathrm{SL}(2, \mathbb{R})$:

$$\begin{pmatrix} A_{\mu\nu}^1 \\ A_{\mu\nu}^2 \end{pmatrix} = \mathscr{C} \begin{pmatrix} \mathbf{A}_{\mu\nu}^1 \\ \mathbf{A}_{\mu\nu}^2 \end{pmatrix} \tag{6.8.8}$$

We introduce a doublet of Majorana-Weyl spinor 1-forms (the gravitinos) having the same chirality:

$$\Gamma_{11}\psi_A = -\psi_A; \qquad \mathbb{C}\overline{\psi}_A = \psi_A, \quad A = 1, 2 \tag{6.8.9}$$

In terms of these we define the complex doublet of gravitinos:

$$\begin{pmatrix} \psi^\star \\ \psi \end{pmatrix} = \mathscr{C} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \tag{6.8.10}$$

and we introduce the following doublet made by a complex 3-form current and its complex conjugate:

$$\mathbb{J}_{\mathrm{SU}}^x = \begin{pmatrix} \mathrm{i}\overline{\psi}^\star \wedge \Gamma_a \psi \wedge V^a \\ \mathrm{i}\overline{\psi} \wedge \Gamma_a \psi^\star \wedge V^a \end{pmatrix}; \quad (x = \pm) \tag{6.8.11}$$

By means of an inverse Cayley transformation we get a doublet of real currents:

$$\begin{aligned}\mathbb{J}^A_{\mathrm{SL}} = \left[\mathscr{C}^{-1}\right]^A{}_x \mathbb{J}^x_{\mathrm{SU}} = \begin{pmatrix} \mathrm{i}(\overline{\psi}_1 \wedge \Gamma_a \psi_1 - \overline{\psi}_2 \wedge \Gamma_a \psi_2) \wedge V^a \\ -2\mathrm{i}\overline{\psi}_1 \wedge \Gamma_a \psi_2 \wedge V^a \end{pmatrix} \\ \equiv d^{A|BC} \mathrm{i}\overline{\psi}_B \Gamma_a \psi_C \wedge V^a \end{aligned} \tag{6.8.12}$$

The formula (6.8.12) is understood as follows. Recall that the fermions transform only with respect to the isotropy subgroup $\mathrm{H} = \mathrm{U}(1) \sim \mathrm{O}(2)$ of the scalar coset (are neutral under G) and that all irreducible representations of O(2) are 2-dimensional. The coefficients $d^{A|BC}$ defined by (6.8.12) are the Clebsch Gordon coefficients that extract the doublet of helicity $s = 2$ from the tensor product of two representations of helicity $s = 1$. Relying on these notations we can write the type IIB curvature definitions in two equivalent bases related by a Cayley transformation:

1. the complex SU(1, 1) basis originally used by Castellani and Pesando [34],
2. the real $\mathrm{SL}(2,\mathbb{R})$, introduced here and best suited for comparison with string theory massless modes.

The Curvatures of the Free Differential Algebra in the Complex Basis Using the complex basis the curvatures are as follows[12]

$$\mathfrak{T}^a = \mathscr{D}V^a - i\overline{\psi} \wedge \Gamma^a \psi \tag{6.8.13}$$

$$\mathfrak{R}^{ab} = d\omega^{ab} - \omega^{ac} \wedge \omega^{db}\eta_{cd}$$

$$\rho = \mathscr{D}\psi \equiv d\psi - \frac{1}{4}\omega^{ab} \wedge \Gamma_{ab}\psi - \frac{1}{2}\mathrm{i}Q \wedge \psi \tag{6.8.14}$$

$$\mathscr{H}^\alpha_{[3]} = \sqrt{2}dA^\alpha_{[2]} + 2i\Lambda^\alpha_+ \overline{\psi} \wedge \Gamma_a \psi^* \wedge V^a + 2\mathrm{i}\Lambda^\alpha_- \overline{\psi}^* \wedge \Gamma_a \psi \wedge V^a \tag{6.8.15}$$

$$\begin{aligned}\mathscr{F}_{[5]} = dC_{[4]} + \frac{1}{16}\mathrm{i}\varepsilon_{\alpha\beta}\sqrt{2}A^\alpha_{[2]} \wedge \mathscr{H}^\beta_{[3]} + \frac{1}{6}\overline{\psi} \wedge \Gamma_{abc}\psi \wedge V^a \wedge V^b \wedge V^c \\ + \frac{1}{8}\varepsilon_{\alpha\beta}\sqrt{2}A^\alpha_{[2]} \wedge \left(\Lambda^\beta_+ \overline{\psi}\Gamma_a\psi^\star + \Lambda^\beta_- \overline{\psi}^\star \Gamma_a \psi\right) \wedge V^a\end{aligned} \tag{6.8.16}$$

$$\mathscr{D}\lambda = d\lambda - \frac{1}{4}\omega^{ab}\Gamma_{ab}\lambda - \mathrm{i}\frac{3}{2}Q\lambda \tag{6.8.17}$$

$$\mathscr{D}\Lambda^\alpha_\pm = d\Lambda^\alpha_\pm \mp \mathrm{i}Q\Lambda^\alpha_\pm. \tag{6.8.18}$$

alternatively using the real $\mathrm{SL}(2,\mathbb{R})$ basis we can write:

The Curvatures of the Free Differential Algebra in the Real Basis

$$\mathfrak{T}^a = \mathscr{D}V^a - i\overline{\psi}_A \wedge \Gamma^a \psi_A \tag{6.8.19}$$

[12] Comparing with the original paper by Castellani and Pesando, note that we have changed the normalization: $A_\alpha \to \sqrt{2}A_\alpha$ and $B_{\lambda\mu\nu\rho} = 6C_{\lambda\mu\nu\rho}$ so that eventually the 4-form $C_{[4]}$ will be identified with that used in Polchinski's book [32, 33].

$$\mathfrak{R}^{ab} = d\omega^{ab} - \omega^{ac} \wedge \omega^{db} \eta_{cd}$$

$$\rho_A = \mathscr{D}\psi_A \equiv d\psi_A - \frac{1}{4}\omega^{ab} \wedge \Gamma_{ab}\psi_A + \frac{1}{2}Q \wedge \varepsilon_{AB}\psi_B \tag{6.8.20}$$

$$\mathbf{H}^{\Lambda}_{[3]} = d\mathbf{A}^{\Lambda}_{[3]} + \mathrm{i}\mathbb{L}^{\Lambda}_{A} d^{A|BC}\overline{\psi}_B \wedge \Gamma_a \psi_C \wedge V^a \tag{6.8.21}$$

$$\mathscr{F}_{[5]} = dC_{[4]} - \frac{1}{16}\varepsilon_{\Lambda\Sigma}\mathbf{A}^{\Lambda}_{[3]} \wedge \mathbf{H}^{\Sigma}_{[3]} + \mathrm{i}\frac{1}{6}\overline{\psi}_A \wedge \Gamma_{abc}\psi_B \varepsilon^{AB} V^a \wedge V^b \wedge V^c$$

$$+ \mathrm{i}\frac{1}{4}\varepsilon_{\Lambda\Sigma}\mathbf{A}^{\Lambda}_{[2]}\mathbb{L}^{\Sigma}_{A} d^{A|BC}\overline{\psi}_B \wedge \Gamma_a \psi_C \wedge V^a \tag{6.8.22}$$

$$\mathscr{D}\lambda = d\lambda - \frac{1}{4}\omega^{ab}\Gamma_{ab}\lambda - \frac{3}{2}\mathrm{i}Q\lambda \tag{6.8.23}$$

$$\mathscr{D}\mathbb{L}^{\Lambda}_{\pm} = d\mathbb{L}^{\Lambda}_{A} + \varepsilon_{AB} Q \mathbb{L}^{\Lambda}_{B}. \tag{6.8.24}$$

In the above formulae, (6.8.18) and (6.8.24) define the covariant derivative of the coset representative of the scalar coset in the $\mathrm{SU}(1,1)$ and $\mathrm{SL}(2,\mathbb{R})$ basis respectively. They follow from the Maurer Cartan equations of G/H.

Next, using the results of Castellani and Pesando [34], we can write the rheonomic parameterizations of the curvatures (6.8.13)–(6.8.18) (alternatively (6.8.19)–(6.8.24)) that determine the supersymmetry transformation rules of all the fields. Prior to that, in order to make contact with superstring massless modes as normalized in Polchinski's book, it is convenient to introduce the following identifications:

$$\mathbf{A}^{1}_{[2]} = 2\sqrt{2}B_{[2]}; \qquad \mathbf{A}^{2}_{[2]} = 2\sqrt{2}C_{[2]} \tag{6.8.25}$$

where $B_{[2]}$ is the 2-form gauge field of the Neveu-Schwarz sector that couples to ordinary fundamental strings, while $C_{[2]}$ is the 2-form of the Ramond-Ramond sector that couples to $D1$-branes. For simplicity we write the rheonomic parameterizations only in the complex basis and we disregard the bilinear fermionic terms calculated by Castellani and Pesando. We have:

$$\mathfrak{T}^a = 0 \tag{6.8.26}$$

$$\rho = \rho_{ab}V^a \wedge V^b + \frac{5}{16}\mathrm{i}\Gamma^{a_1-a_4}\psi V^{a_5}\left(F_{a_1-a_5} + \frac{1}{5!}\varepsilon_{a_1-a_{10}}F_{a_6-a_{10}}\right)$$

$$+ \frac{1}{32}\left(-\Gamma^{a_1-a_4}\psi^* V_{a_1} + 9\Gamma^{a_2a_3}\psi^* V^{a_4}\right)\Lambda^{\alpha}_{+}\mathscr{H}^{\beta}_{a_2-a_4}\varepsilon_{\alpha\beta}$$

$$+ \text{fermion bilinears} \tag{6.8.27}$$

$$\mathscr{H}^{\alpha}_{[3]} = \mathscr{H}^{\alpha}_{abc}V^a \wedge V^b \wedge V^c + \Lambda^{\alpha}_{+}\overline{\psi}^*\Gamma_{ab}\lambda^* V^a \wedge V^b$$

$$+ \Lambda^{\alpha}_{-}\overline{\psi}\Gamma_{ab}\lambda V^a \wedge V^b \tag{6.8.28}$$

$$\mathscr{F}_{[5]} = F_{a_1-a_5}V^{a_1} \wedge \cdots \wedge V^{a_5} \tag{6.8.29}$$

$$\mathscr{D}\lambda = \mathscr{D}_a\lambda V^a + \mathrm{i}P_a\Gamma^a\psi^* - \frac{1}{8}\mathrm{i}\Gamma^{a_1-a_3}\psi\varepsilon_{\alpha\beta}\Lambda^{\alpha}_{+}\mathscr{H}^{\beta}_{a_1-a_3} \tag{6.8.30}$$

$$\mathscr{D}\Lambda^\alpha_+ = \Lambda^\alpha_- P_a V^a + \Lambda^\alpha_- \overline{\psi}^* \lambda \tag{6.8.31}$$

$$\mathscr{D}\Lambda^\alpha_- = \Lambda^\alpha_+ P^*_a V^a + \Lambda^\alpha_+ \overline{\psi} \lambda^* \tag{6.8.32}$$

$$\mathfrak{R}^{ab} = \mathscr{R}^{ab}{}_{cd} V^c \wedge V^d + \text{fermionic terms} \tag{6.8.33}$$

6.8.3 The Bosonic Field Equations and the Standard Form of the Bosonic Action

Following Castellani and Pesando we write next the general form of the bosonic field equations and using the identifications of (6.8.25), (6.8.3), (6.8.7) we reduce them to those following from a *standard supergravity action for p-branes*. As discussed in the literature [24, 26, 27], the standard form of a supergravity action truncated to the graviton, the dilaton and the $n_i = p_i + 2$ field strengths that can couple to the world-volume actions of p_i-branes is as follows:

$$\begin{aligned} \mathscr{A}_{standard} = & \int d^D x \det V \left[-2\mathscr{R}\big[\omega(V)\big] - \frac{1}{2}\partial^\mu \varphi \partial_\mu \varphi \right] \\ & - \int \sum_i \frac{1}{2} \exp[-a_i \varphi] F_{[n_i]} \wedge \star F_{[n_i]} \\ & + \text{Chern Simons couplings} \end{aligned} \tag{6.8.34}$$

where $\mathscr{R} = \mathscr{R}^{ab}{}_{ab}$ is the scalar curvature in the geometric normalizations always adopted in the rheonomic framework [17],[13] and a_i are characteristic exponents dictated by the structure of supergravity and playing an essential role in dictating the properties of p-brane solutions.[14] Furthermore in (6.8.34) we have defined:

$$|F_{[n]}|^2 \equiv g^{\mu_1 \nu_1} \cdots g^{\mu_n \nu_n} F_{\mu_1 \ldots \mu_n} F_{\nu_1 \ldots \nu_n} \tag{6.8.35}$$

$$F_{[n]} = F_{\mu_1 \ldots \mu_n} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_n} \tag{6.8.36}$$

and we have not made explicit the Chern Simons couplings between field strengths that are on the other hand essential in the derivation of the exact field equations.

Introducing the definition of the dressed 3-form field strengths:

$$\widehat{\mathscr{H}_\pm}_{|a_1 a_2 a_3} = \varepsilon_{\alpha\beta} \Lambda^\alpha_\pm \mathscr{H}_{a_1 a_2 a_3}; \qquad \widehat{\mathbf{H}}_{\Lambda|a_1 a_2 a_3} = \varepsilon_{\Lambda\Sigma} \mathbb{L}^\Lambda{}_A \mathbf{H}^\Sigma_{a_1 a_2 a_3} \tag{6.8.37}$$

---

[13] Note that our $\mathscr{R}$ is equal to $-\frac{1}{2}\mathscr{R}^{old}$, $\mathscr{R}^{old}$ being the normalization of the scalar curvature usually adopted in General Relativity textbooks. The difference arises because in the traditional literature the Riemann tensor is not defined as the components of the curvature 2-form $\mathfrak{R}^{ab}$ rather as -2 times such components.

[14] In the next Chap. 7 we will emphasize the role of the dilaton factors $\exp[-a\varphi]$ in front of the p-form kinetic terms.

it was shown by Castellani and Pesando [34] that the exact bosonic field equations implied by the closure of the supersymmetry algebra have the following form:

$$
\begin{aligned}
\mathscr{R}^{pr}{}_{qr} - \frac{1}{2}\delta^p_q \mathscr{R}^{ab}{}_{ab} = {} & -75\left(F_{qa_1-a_4} F^{pa_1-a_4} - \frac{1}{10}\delta^p_q F_{a_1-a_5} F^{a_1-a_5}\right) \\
& - \frac{9}{16}\left(\widehat{\mathscr{H}}_+^{pa_1a_2} \widehat{\mathscr{H}}_{-|qa_1a_2} + \widehat{\mathscr{H}}_-^{pa_1a_2} \widehat{\mathscr{H}}_{+|qa_1a_2} \right. \\
& \left. - \frac{1}{3}\delta^p_q \widehat{\mathscr{H}}_+^{a_1a_2a_3} \widehat{\mathscr{H}}_{-|a_1a_2a_3} \right) \\
& - \frac{1}{2}\left(P^p P^*_q + P_q P^{*p} - \delta^p_q P^a P^*_a \right)
\end{aligned} \tag{6.8.38}
$$

$$
\mathscr{D}^a P_a = -\frac{3}{8} \widehat{\mathscr{H}}_+^{a_1a_2a_3} \widehat{\mathscr{H}}_{+|a_1a_2a_3} \tag{6.8.39}
$$

$$
\mathscr{D}^b \widehat{\mathscr{H}}_{+|a_1a_2b} = -\mathrm{i}20 F_{a_1a_2b_1b_2b_3} \widehat{\mathscr{H}}_+^{b_1b_2b_3} - P^b \widehat{\mathscr{H}}_{-|a_1a_2b} \tag{6.8.40}
$$

$$
\mathscr{D}^b F_{a_1a_2a_3a_4b} = \mathrm{i}\frac{1}{960} \varepsilon_{a_1a_2a_3a_4b_1\ldots b_6} \widehat{\mathscr{H}}_+^{b_1b_2b_3} \widehat{\mathscr{H}}_-^{b_4b_5b_6} \tag{6.8.41}
$$

At the purely bosonic level (*i.e.* disregarding all fermionic contributions), using the solvable parameterization (6.8.3) of the $\mathrm{SL}(2,\mathbb{R})/\mathrm{O}(2)$ coset and inserting the identifications (6.8.25) we obtain the following expression for the dressed 3-forms in terms of string massless fields denoted *NS* or *RR* according to their origin in the Neveu Schwarz or Ramond Ramond sector:

$$
\begin{aligned}
\widehat{\mathscr{H}}_\pm &= \pm 2 e^{-\varphi/2} F^{NS}_{[3]} + \mathrm{i}2 e^{\varphi/2} F^{RR}_{[3]} \\
P &= \frac{1}{2} d\varphi - \mathrm{i}\frac{1}{2} e^{\varphi} F^{RR}_{[1]} \\
F^{NS}_{[3]} &= dB_{[2]} \\
F^{RR}_{[1]} &= dC_{[0]} \\
F^{RR}_{[3]} &= (dC_{[2]} - C_{[0]} dB_{[2]}) \\
F^{RR}_{[5]} &= \mathscr{F}_{[5]} = dC_{[4]} - \frac{1}{2}(B_{[2]} \wedge dC_{[2]} - C_{[2]} \wedge dB_{[2]})
\end{aligned} \tag{6.8.42}
$$

Using the Hodge dual of ℓ-forms in space-time dimensions D, the field equations (6.8.39)–(6.8.41) can be written in a more compact form. Let us begin with the scalar equation (6.8.39), it becomes:

$$
d(\star P) - 2\mathrm{i} Q \wedge \star P + \frac{1}{16} \widehat{\mathscr{H}}_+ \wedge \star \widehat{\mathscr{H}}_+ = 0 \tag{6.8.43}
$$

and separating its real from imaginary part we obtain the two equations:

$$d\star d\varphi - e^{2\varphi} F^{RR}_{[1]} \wedge F^{RR}_{[1]} = -\frac{1}{2}\big(e^{-\varphi} F^{NS}_{[3]} \wedge \star F^{NS}_{[3]} - e^{\varphi} F^{RR}_{[3]} \wedge \star F^{RR}_{[3]}\big) \quad (6.8.44)$$

$$d\big(e^{2\varphi} * F^{RR}_{[1]}\big) = -e^{\varphi} F^{NS}_{[3]} \wedge \star F^{RR}_{[3]} \quad (6.8.45)$$

Considering next the 3-form (6.8.40) it can be rewritten as:

$$d\star\widehat{\mathscr{H}_+} - \mathrm{i}Q \wedge *\widehat{\mathscr{H}_+} = \mathrm{i}\mathscr{F}_{[5]} \wedge \widehat{\mathscr{H}_+} - P \wedge \star\widehat{\mathscr{H}_-} \quad (6.8.46)$$

Separating the real and imaginary parts of (6.8.46) we obtain:

$$\begin{aligned} d\big(e^{-\varphi} \star F^{NS}_{[3]}\big) + e^{\varphi} F^{RR}_{[1]} \wedge \star F^{RR}_{[3]} &= -F^{RR}_{[3]} \wedge F^{RR}_{[5]} \\ d\big(e^{\varphi} \star F^{RR}_{[3]}\big) &= -F^{RR}_{[5]} \wedge F^{NS}_{[3]} \end{aligned} \quad (6.8.47)$$

Finally the equation for the Ramond-Ramond 5-form, namely (6.8.41) is rewritten as follows:

$$d\star F^{RR}_{[5]} = \mathrm{i}\frac{1}{8}\widehat{\mathscr{H}_+} \wedge \widehat{\mathscr{H}_-} = -F^{NS}_{[3]} \wedge F^{RR}_{[3]} \quad (6.8.48)$$

6.9 About Solutions

The main interest in the perspective of the present book focus on the wealth of new gravitational backgrounds that higher dimensional supergravities do introduce. Some type of solutions of both type IIA, type IIB and M-theory are presented in Chap. 9.

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Chapter 7
The Branes: Three Viewpoints

Tu se' certo il cantor del trino regno,
Tu lo spirto magnanimo e sovrano
Cui, quasi cervo a puro fonte, io vegno.
Giovanni Marchetti

7.1 Introduction and Conceptual Outline

Supergravity developed originally as the supersymmetric generalization of Einstein Gravity and, for several years, the construction of its various formulations in diverse dimensions, with diverse number of supercharges, went on independently from the theory of superstrings, whose origin was instead within the framework of the *dual models* of hadronic scattering amplitudes, namely within tentative theories of strong interactions. In the course of time, however, and as a result of the two *string revolutions*,[1] the subjects of supergravity and of superstring theory merged completely, as soon as it became clear that the $D = 10$ theories described in the previous chapter are just effective low energy Lagrangians that encode the interactions of the massless modes of the corresponding perturbative string models.

Since the mid nineties the relation between supergravity and superstrings underwent a further substantial upgrading which is the essence of the second string revolution.

On one hand it became clear that each superstring theory, besides the elementary string states, includes also additional non-perturbative excitations, similar to

[1] By first string revolution it is meant the discovery by Green and Schwarz of the mechanism of anomaly cancellation which singled out five perturbatively consistent superstring models, namely:

1. Type II A
2. Type II B
3. Type I with SO(32) gauge group
4. Heterotic $E_8 \times E_8$
5. Heterotic SO(32).

By second string revolution it is meant the series of discoveries around 1995–1996 that demonstrated that all the perturbatively consistent string models are related to each other by non-perturbative dualities pointing out to the fact that there is just one non-perturbative superstring theory.

P.G. Frè, *Gravity, a Geometrical Course*, DOI 10.1007/978-94-007-5443-0_7,

the solitons of non-linear field theories, that can be associated with the propagation of extended objects of higher dimensions, the p-branes. Among them, particularly relevant are the Dp-branes that can be alternatively regarded as the loci where open strings have their end-points, or as space-time boundaries that can absorb or emit closed strings. On the other hand the p-branes could be identified with classical solutions of the relevant low energy supergravity and it was discovered that the symmetries of supergravity realize those non-perturbative duality transformations that can map string states into solitonic ones and vice-versa, demonstrating that all string theories are just different perturbative limits of a single theory, usually named M-theory.

From these considerations a new more profound understanding of (super-)gravity and (super-)branes emerged that is the goal of the present chapter to outline, emphasizing that superstrings are just a particularly relevant instance in a broader landscape.

Our starting point is the action of a charged particle in the background of an electromagnetic field. Naming $x^\mu(\tau)$ the coordinates of the charged particle at proper time τ, we can write the following action:

$$\mathscr{A}_{part} = \underbrace{\int \sqrt{g_{\mu\nu}(x)\dot{x}^\mu \dot{x}^\nu}\, d\tau}_{\int ds} + q \underbrace{\int A_\mu(x)\dot{x}^\mu\, d\tau}_{\int \mathbf{A}} \tag{7.1.1}$$

where A_μ is the electromagnetic field, $g_{\mu\nu}$ the metric of the ambient space-time manifold, q the electric charge of the particle and $\dot{x}^\mu = \frac{d}{d\tau}x^\mu$. Varying the action (7.1.1) with respect to the trajectory function $\delta x^\mu(\tau)$ we obtain the equation of motion of the charged particle subject to the Lorentz force and to the gravitational field encoded in the metric. On the other hand we can add the action $\mathscr{A}_{part}$, which is a one-dimensional integral, to the action

$$\mathscr{A}_{Max} = -\frac{1}{4}\int F^{\mu\nu} F_{\mu\nu}\, d^4x \tag{7.1.2}$$

which is a 4-dimensional integral and we can vary $\mathscr{A}_{Max} + \mathscr{A}_{part}$ in the electromagnetic field δA_μ. What we obtain are Maxwell equations with a source term provided by the electric current localized on the world-line swept by the charged particle:

$$J^\mu(x) = q\int \delta^{(4)}\big(x - x(\tau)\big)\, d\tau \tag{7.1.3}$$

Similarly we can vary the action $\mathscr{A}_{part}$ with respect to the metric $\delta g_{\mu\nu}$ and this yields a stress-energy tensor, also localized on the particle world-line, that provides a source for the gravitational field in Einstein equation.

This short discussion puts into evidence the following two facts:

(A) If a field theory contains a gauge field that is a d-form $\mathbf{A}^{[d]}$ then, setting $p = d-1$, we can introduce a p-dimensional object which, by evolving through the

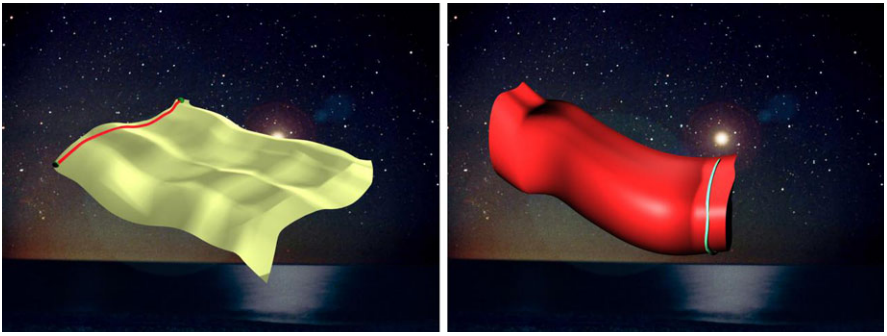

Fig. 7.1 In the above two pictures we present an intuitive image of the world-volume traced by p-branes in the ambient space-time. Since we cannot draw in higher dimensions we show the world volumes traced by 1-branes, namely strings. In the string case these world-volumes are actually world-sheets, namely 2-dimensional surfaces. Furthermore the string can be open, namely can admit end-points, or close. In the first case the world-sheet is of the type depicted on the *left*. In the second case the world-sheet is a sort of tube like that depicted on the *right*

ambient D-dimensional space-time $\mathscr{M}_D$, traces in this latter a d-dimensional world-volume (see Fig. 7.1):

$$\mathscr{W}_d \subset \mathscr{M}_D \tag{7.1.4}$$

The dynamics of such an extended object, which we name a p-brane, is described by an action given by a d-dimensional integral localized on the world-volume $\mathscr{W}_d$. Such a p-brane action is typically made of two terms

$$\mathscr{A}_{brane} = \mathscr{A}_{\text{Area}} + q \int_{\mathscr{W}_d} \mathbf{A}^{[d]} \tag{7.1.5}$$

the first term being the area of the world-volume or generalization thereof, the second, often named the Wess-Zumino term, being the integral of the form $\mathbf{A}^{[d]}$ on the world-volume.

(B) The extended objects described above provide *sources* for the bulk field-theory. These sources are localized on singular boundaries of space-time.

When $p > 0$, namely when the object tracing the world-volume is really extended and not simply a point-particle, on top of moving it can deform its own shape, vibrate, split and join with other similar entities. In other words there are dynamical processes occurring on the world-volume and besides the *bulk field theory*, that we name *macroscopic*, we have also a *world-volume field theory* that we name *microscopic*. The microscopic field theory can be quantized for its own sake and its elementary excitations are paired in a precise way to the classical fields of the macroscopic theory. This is what we do in the case of superstrings. In this case the microscopic field theory lives in two-dimensions and has distinctive miraculous properties: it is typically conformal, which means invariant with respect to a very specific infinite Lie group, its spectrum can be derived by means of algebraic techniques and its Green functions can be calculated exactly in a large variety of cases.

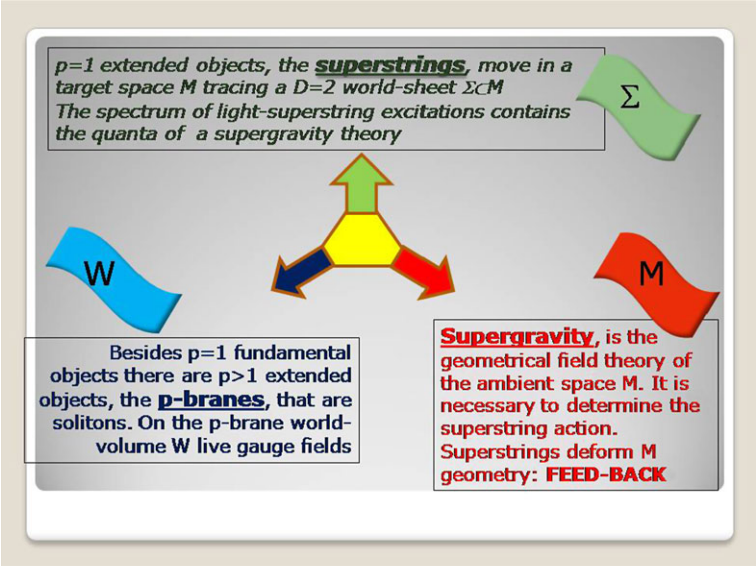

Fig. 7.2 The three intertwined aspects of brane theory

We do not dwell on the microscopic aspects of string theory that form the topics of large specialized text books. We just emphasize that the relation of supergravity to strings is the same as the relation of the former with other p-branes allowed by the existence of suitable $(p+1)$-forms. The difference is that the quantum microscopic theory of $p>1$ branes usually cannot be solved exactly: the spectrum is mostly unknown and the Green functions are out of reach of exact calculations.

It must be stressed that introducing p-brane boundary actions gives rise to solutions of the bulk field theory that are determined by such sources and have singularities on the world-volume of the source. This is just a generalization of the electric and magnetic fields generated by point-like charged particles. It follows that p-branes can also be identified with appropriate classical solutions of supergravity. Hence the new theory of strings and branes that emerged from the second string revolution has a challenging triadic structure which we have tried to summarize in Fig. 7.2. On one hand the superstring massless modes perfectly match the field content of that supergravity which is necessary to write the considered superstring microscopic action. On the other hand the field spectrum of supergravity determines which additional $p>1$ branes can be coupled to it. The microscopic action of such p-branes can be constructed according to a procedure which we outline in the following section issuing a generalized gauge theory living on the world-volume. Finally for each allowed p-brane we have a corresponding classical solution of supergravity. Therefore there are three complementary aspects and as many complementary approaches to the study of p-branes that can be alternatively viewed as:

(a) classical solutions of the low energy supergravity field equations in the bulk,
(b) world-volume gauge theories described by suitable world-volume actions characterized by κ-supersymmetry,
(c) boundary states in the superconformal field theory description (SCFT) of superstring vacua.

As we explained the three descriptions are intertwined. The relation between (a) and (b) was already illustrated. To the constructive principle of κ-supersymmetry we dedicate the next sections. The viewpoint (c) is explained as follows. When we adopt the abstract language of superconformal field theories (SCFT), classical string backgrounds are identified with a specific SCFT and the brane is identified with a suitable composite state constructed in the framework of the same. The world-volume action encodes the interactions within the chosen boundary conformal field theory.

The above discussions can be summarized in the following list of statements:

1. There is just one non-perturbative ten dimensional *string theory* that can also be identified as the mysterious *M-theory* having $D = 11$ supergravity as its low energy limit.
2. All *p-branes*, whether electric or magnetic, whether coupled to Neveu Schwarz or to Ramond $(p+1)$-forms encode noteworthy aspects of the unique M-theory.
3. *Microscopically* the p-brane degrees of freedom are described by a suitable *gauge theory* $\mathscr{G}\mathscr{T}_{p+1}$ living on the $p+1$ dimensional *world volume* $\mathscr{W}\mathscr{V}_{p+1}$ that can be either conformal or not.
4. *Macroscopically* each p-brane is a generalized soliton in the following sense. It is a classical solution of $D = 10$ or $D = 11$ supergravity interpolating between two asymptotic geometries that, with some abuse of language, we respectively name the *the geometry at infinity* geo^{∞} and the *the near horizon geometry* geo^{H}. The latter which only occasionally corresponds to a true event horizon is instead universally characterized by the following property. It can be interpreted as a solution of some suitable $p+2$ dimensional supergravity $\mathscr{S}\mathscr{G}_{p+2}$ times an appropriate *internal space* Ω_{D-p-2}.
5. Because of the statement above, all space-time dimensions $11 \geq D \geq 3$ are relevant and supergravities in these *diverse dimensions* describe various perturbative and non-perturbative aspects of superstring theory. In particular we have a most intriguing *gauge/gravity* correspondence implying that classical supergravity $\mathscr{S}\mathscr{G}_{p+2}$ expanded around the vacuum solution geo^{H} is dual to the quantum gauge theory $\mathscr{G}\mathscr{T}_{p+1}$ in one lower dimension.

In line with our previous choices, we do not address the superconformal aspects of p-branes which relate with the microscopic theory of superstrings. We just focus on the following two aspects that complete the landscape of far reaching consequences of General Relativity when enlarged by supersymmetry:

(A) Construction of the world-volume actions with κ-supersymmetry,
(B) p-brane solutions of bulk supergravity as classical solitons.

7.2 *p*-Branes as World Volume Gauge-Theories

From this discussion it is evident that a full command on the world-volume actions of p-branes is a most essential weapon in the arsenal of the modern string theorist. The distinctive feature of these actions, which guides their construction, is the so called κ-supersymmetry [1, 2]. This corresponds to the fermionic local symmetry that allows to halve the number of Fermi fields, originally equal to the number of θ-coordinates for the relevant superspace, and obtain, on-shell, an equal number of bosonic and fermionic degrees of freedom as required by the general brane-scan [3] where it was investigated which p-form actions can be supersymmetrized with the help of which gauge fields. As it is well understood in the literature since many years [4–8] the κ-supersymmetries are nothing else but suitable *chiral projections* of the original supersymmetry transformation rules defined by supergravity. This was made particularly evident and handy by the construction of world-volume actions within the framework of the rheonomy approach to supergravity [7–11]. In this geometric approach all Fermi fields are implicitly hidden in the definition of the geometric p-form potentials of supergravity and *formally* the action is the same as it would be in a purely bosonic theory. Yet it is supersymmetric and this supersymmetry, which fixes the relative coefficients of the kinetic terms with respect to the Wess-Zumino terms,[2] can be shown through a simple calculation starting from the rheonomic parameterizations of the supergravity curvatures. In order to apply such a powerful method, the world-volume action must be presented in first-order rather than in second order formalism, namely á la Polyakov [12] rather than á la Nambu-Goto [13, 14]. As a consequence, the rheonomic method was successfully applied to those instances of p-brane actions where the Polyakov formulation (further generalized with the introduction of an additional auxiliary field representing the derivative of scalar fields) did exist: in particular the string or 1-brane [7], the M2-brane [8] and the particle or 0-brane [9]. More general Dp-branes were out of reach because of the following reason: their second order action is of the Born-Infeld type and a suitable first order formalism for the Born-Infeld Lagrangian was not known.

In a paper [27] of the present author with one of his students this gap was filled by constructing a *new first order formalism* that is able of generating second order actions of the Born-Infeld type. This formulation which turns out to be particularly compact and elegant is based on the introduction of an additional auxiliary field, besides the world volume vielbein, and on the enlargement of the local symmetry from the Lorentz group to the general linear group:

$$\mathrm{SO}(1, d-1) \quad \overset{\text{enlarged}}{\Longrightarrow} \quad \mathrm{GL}(d, \mathbb{R}) \tag{7.2.1}$$

Within this formalism one can easily apply the rheonomic method. As an example we present here the κ-supersymmetric action of a $D3$-brane, that has many applica-

[2] As already mentioned in the main text, by Wess-Zumino terms it is generally understood terms of the form $\int_{\mathscr{W}_{p+1}} \mathbb{A}^{[p+1]}$ where $\mathscr{W}_{p+1}$ denotes the world volume spanned by a p-brane and $\mathbb{A}^{[p+1]}$ denotes a suitable $(p+1)$-form present in the considered background supergravity.

tions in the context of the gauge gravity correspondence (see, for instance, [15] and the more complicated examples in [16–19]).

In Sect. 7.3 we review the rheonomic formulation of κ-supersymmetry based on an essential use of the old 1st-order formalism. In Sect. 7.4 we present the new first order formalism and we show how, within this framework, we can recover the Born-Infeld action by eliminating the auxiliary fields through their own equation of motion. In Sect. 7.5 we apply this machinery to the case of the $D3$-brane and we explicitly show its κ-supersymmetry.

7.3 From 2nd to 1st Order and the Rheonomy Setup for to κ Supersymmetry

In this section we summarize the 1st order formulation of world-volume actions and we recall their essential role in setting up a simple, compact, rheonomic approach to κ-supersymmetry. Then we point out the problem arising with Dp-branes, related to the presence of the gauge-field A_μ. In this way we establish the need for the new first order formalism which is explained in the next section.

7.3.1 Nambu-Goto, Born-Infeld and Polyakov Kinetic Actions for p-Branes

The prototype of p-branes is furnished by the bosonic string (1-brane), whose 2nd order action was proposed by Nambu and Goto [13, 14] many years ago at the very beginning of the history of dual models. The string is a one-dimensional object, that moving through a D-dimensional space-time endowed with a metric $g_{\underline{\mu\nu}}$, sweeps a two-dimensional world sheet. The action functional governing the dynamics of the string is simply given by the area of such a world-sheet. Namely we have:

$$\mathscr{A}^{Nambu\text{-}Goto}_{string} = \int d^2\xi \sqrt{-\det G_{\mu\nu}} \tag{7.3.1}$$

where:

$$G_{\mu\nu} \equiv \partial_\mu X^{\underline{\mu}} \partial_\nu X^{\underline{\nu}} g_{\underline{\mu\nu}} \tag{7.3.2}$$

denotes the pull-back of the bulk metric $g_{\underline{\mu\nu}}(X)$ onto the world-sheet.[3] Such an action admits a straightforward generalization to the case of a p-brane, the area of the world-sheet being replaced by the value of the $d = p + 1$-dimensional world-volume:

$$\mathscr{A}^{Nambu\text{-}Goto}_{p\text{-}brane} = \int d^d\xi \sqrt{-\det G_{\mu\nu}} \tag{7.3.3}$$

[3] In this section we use the notations and conventions described in Appendix B.1.

As it is well known from the literature and thoroughly discussed in many string theory textbooks [24, 25], the kinetic part of Dp-brane actions is provided by a further generalization of the Nambu-Goto action (7.3.3) where the symmetric matrix $G_{\mu\nu}$ is modified by the addition of an antisymmetric part $F_{\mu\nu}$ that represents the field strength of a world volume gauge field A_μ:

$$\begin{aligned} \text{for } D\text{-branes} \quad & G_{\mu\nu} \mapsto G_{\mu\nu} + F_{\mu\nu} \\ \text{where } \; & F_{\mu\nu} = \frac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu) \end{aligned} \tag{7.3.4}$$

Seen from a different perspective the resulting second order action:

$$\mathscr{A}^{kinetic}_{D\text{-}brane} = \int d^d\xi \sqrt{-\det(G_{\mu\nu} + F_{\mu\nu})} \tag{7.3.5}$$

is a generalization of the Born-Infeld [26] action of non-linear electromagnetism. Indeed the latter was early shown to be the effective action for the zero mode gauge field of an open string theory [28, 29].

In the context of superstrings and in the analysis of Dp-brane systems the important issue is to write world-volume actions that possess both reparameterization invariance and κ-supersymmetry [1, 2]. The former is needed to remove the unphysical degrees of freedom of the bosonic sector, while the latter removes the unphysical fermions. In this way we end up with an equal number of physical bosons and physical fermions as it is required by supersymmetry [4, 7, 8, 10, 11]. The appropriate κ-supersymmetry transformation rules are nothing else but the supersymmetry transformation rules of the bulk supergravity background fields with a special supersymmetry parameter ε that is projected onto the brane. For those κ-supersymmetric branes where the gauge field strength $F_{\mu\nu}$ is not required (for example the string itself or the M2-brane) such a projection is realized by imposing that the spinor ε satisfies the following condition:

$$\varepsilon = \frac{1}{2}\left(1 + (\mathrm{i})^{d+1}\frac{1}{d!}\Gamma_{\underline{a_1\ldots a_d}} V^{\underline{a_1}}_{i_1}\ldots V^{\underline{a_d}}_{i_d}\varepsilon^{i_1\ldots i_d}\right)\varepsilon \tag{7.3.6}$$

where $\Gamma_{\underline{a}}$ are the gamma matrices in D-dimensions and $V^{\underline{a}}_m$ are the components of the bulk vielbein $V^{\underline{a}}$ onto a basis of world-volume vielbein e^m. Explicitly we write

$$V^{\underline{b}}_m e^m = \varphi^*\left[V^{\underline{b}}\right] \tag{7.3.7}$$

where $\varphi^*[V^{\underline{b}}]$ denotes the pull-back of the bulk vielbein on the world volume,

$$\varphi : \mathscr{W}_d \hookrightarrow \mathscr{M}_D \tag{7.3.8}$$

being the injection map of the latter into the former. For all other branes with a full-fledged Born-Infeld type of action the projection (7.3.6) becomes more complicated and involves also $F_{\mu\nu}$.

Certainly one can address the problem of κ-supersymmetrizing the 2nd-order action (7.3.5) and this programme was carried through in the literature to some extent [30, 32, 35, 36]. Yet due to the highly non-linear structure of such a bosonic action its supersymmetrization turns out to be quite involved. Furthermore the geometric structure is not transparent and any modification is very difficult and obscure in such an approach.

On the contrary it was shown in [7] and illustrated with the case of the M2-brane in [8] and with the case of the 0-brane in four-dimensions in [9] that, by using a first order formalism on the world volume, the implementation of κ-supersymmetry is reduced to an almost trivial matter once the rheonomic parameterizations, consistent with superspace Bianchi identities, are given for all the curvatures of the bulk background fields. It follows that an appropriate first order formulation of the Born-Infeld action (7.3.5) is an essential step for an easy and elegant approach to κ-supersymmetric Dp-brane world volume actions that are also sufficiently versatile to adapt to all type of bulk backgrounds.

The first order formulation of the Nambu-Goto action (7.3.3) is the Polyakov action for p-branes:

$$\mathcal{L}^{Polyakov}_{p\text{-}brane} = \frac{1}{2(d-1)} \int d^d\xi \sqrt{-\det h_{\mu\nu}} \left\{ h^{\rho\sigma} \partial_\rho X^{\underline{\mu}} \partial_\sigma X^{\underline{\nu}} g_{\underline{\mu\nu}} + (d-2) \right\} \quad (7.3.9)$$

where the auxiliary field $h_{\rho\sigma}$ denotes the world-volume metric. Varying the action (7.3.9) with respect to $\delta h_{\rho\sigma}$ we obtain the equation:

$$h_{\rho\sigma} = G_{\rho\sigma} \quad (7.3.10)$$

and substituting (7.3.10) back into (7.3.9) we retrieve the second order action (7.3.3).

The Polyakov action (7.3.9) is not yet in a suitable form for a simple rheonomic implementation of κ-supersymmetry but can be easily converted to such a form. The required steps are:

1. replacing the world-volume metric $h_{\mu\nu}(\xi)$ with a world-volume vielbein $e^i = e^i_\rho\, d\xi^\rho$,
2. using a first order formalism also for the derivatives of target space coordinates $X^{\underline{\mu}}$ with respect to the world volume coordinates ξ^ρ,
3. write everything only in terms of flat components both on the world volume and in the target space.

This programme is achieved by introducing an auxiliary 0-form field $\Pi^{\underline{a}}_i(\xi)$ with an index $\underline{a}$ running in the vector representation of $\mathrm{SO}(1, D-1)$ and a second index i running in the vector representation of $\mathrm{SO}(1, d-1)$ and writing the action:[4]

[4]The need of a cosmological term for p-brane actions with $p \neq 1$ was first noted by Tucker and Howe in [20]. We also would like to attract the attention of the reader on the Sect. 5.3 of Volume 1 where the auxiliary fields needed to realize a systematic first order formalism in geometrical gravity were first discussed in anticipation of their essential role in supergravity.

$$\begin{aligned}\mathscr{A}^{kin}[d] = \int_{\mathscr{W}_d} \Big[& \Pi^{\underline{a}}_j V^{\underline{b}} \eta_{\underline{ab}} \wedge \eta^{j i_1} e^{i_2} \wedge \cdots \wedge e^{i_d} \varepsilon_{i_1 \ldots i_d} \\ & - \frac{1}{2d} \big(\Pi^{\underline{a}}_i \Pi^{\underline{b}}_j \eta^{ij} \eta_{\underline{ab}} + d - 2 \big) e^{i_1} \wedge \cdots \wedge e^{i_d} \varepsilon_{i_1 \ldots i_d} \Big]\end{aligned} \quad (7.3.11)$$

The variation of (7.3.11) with respect to $\delta \Pi^{\underline{a}}_j$ yields an equation that admits the unique algebraic solution:

$$V^{\underline{a}}|_{\mathscr{W}_d} = \Pi^{\underline{a}}_i e^i \quad (7.3.12)$$

Hence the 0-form Π^a_i is identified with the intrinsic components along the world-volume vielbein e^i of the bulk vielbein $V^{\underline{a}}$ pulled-back onto the world volume. In other words the field Π^a_i is identified by its own field equation with the field $V^{\underline{a}}_i$ defined in (7.3.7). On the other hand with the chosen numerical coefficients the variation of (7.3.11) with respect to the world-volume vielbein δe^i yields another equation with the unique algebraic solution:

$$\Pi^a_i \Pi^b_j \eta_{\underline{ab}} = \eta_{ij} \quad (7.3.13)$$

which is the flat index transcription of (7.3.10) identifying the world-volume metric with the pull-back of the bulk metric. Hence eliminating all the auxiliary fields via their own equation of motion the first order action (7.3.11) becomes proportional to the 2nd order Nambu-Goto action (7.3.3). The first order form (7.3.11) of the kinetic action is the best suited one to discuss κ-supersymmetry. To illustrate this point we briefly consider the case of the M2-brane

7.3.2 κ-Supersymmetry and the Example of the M2-Brane

In the case of the M2-brane in eleven dimensions the world-volume is three-dimensional and the complete action is simply given by the kinetic action (7.3.11) with $d = 3$ plus the Wess-Zumino term, namely the integral of the 3-form gauge field $\mathbf{A}^{[3]}$. Explicitly we have:

$$\mathscr{A}_{M2} = \mathscr{A}^{kin}[d = 3] - \mathbf{q} \int_{\mathscr{W}_3} \mathbf{A}^{[3]} \quad (7.3.14)$$

where $\mathbf{q} = \pm 1$ is the charge of the M2-brane. As explained in [8], the background fields, namely the bulk elfbein $V^{\underline{a}}$ an the bulk three-form $\mathbf{A}^{[3]}$ are superspace differential forms which are assumed to satisfy the Bianchi consistent rheonomic parameterizations of $D = 11$ supergravity as given in (6.4.8). Hence, although implicitly, the action functional (7.3.14) depends both on 11 bosonic fields, namely the $X^{\underline{\mu}}(\xi)$ coordinates of bulk space-time, and on 32 fermionic fields $\theta^{\underline{\alpha}}(\xi)$, forming an 11-dimensional Majorana spinor. A supersymmetry variation of the background fields is determined by the rheonomic parameterization of the curvatures and has the following explicit form:

$$\begin{aligned}
\delta V^{\underline{a}} &= \mathrm{i}\overline{\varepsilon}\,\Gamma^{\underline{a}}\Psi, \\
\delta\Psi &= \mathscr{D}\varepsilon - \frac{\mathrm{i}}{3}\left(\Gamma^{\underline{b}_1\underline{b}_2\underline{b}_3}F_{\underline{a}\underline{b}_1\underline{b}_2\underline{b}_3} - \frac{1}{8}\Gamma_{\underline{a}\underline{b}_1\cdots\underline{b}_4}F^{\underline{b}_1\cdots\underline{b}_4}\right)\varepsilon V^{\underline{a}},
\end{aligned} \tag{7.3.15}$$

$$\delta\mathbf{A}^{[3]} = -\mathrm{i}\overline{\varepsilon}\,\Gamma^{\underline{ab}}\Psi \wedge V_{\underline{a}} \wedge V_{\underline{b}} \tag{7.3.16}$$

where Ψ is the gravitino 1-form, $F_{\underline{a_1,\ldots,a_4}}$ are the intrinsic components of the $\mathbf{A}^{[3]}$ curvature and ε is a 32-component spinor parameter. Essentially a supersymmetry transformation is a translation of the fermionic coordinates $\theta \mapsto \theta + \varepsilon$, namely at lowest order in θ it is just such a translation. With such an information the κ-supersymmetry invariance of the action (7.3.14) can be established through a two-line computation, using the so called 1.5-order formalism. Technically this consists of the following. In the action (7.3.14) we vary only the background fields V^a, $\mathbf{A}^{[3]}$ with respect to the supersymmetry transformations (7.3.16) and, after variation, we use the first order field equations (7.3.12), (7.3.13). The action is supersymmetric if all the generated terms, proportional to the gravitino 1-form Ψ cancel against each other. This does not happen for a generic 32-component spinor ε but it does if the latter is of the form:

$$\begin{aligned}
\varepsilon &= \frac{1}{2}(1 - \mathbf{q}\mathrm{i}\overline{\Gamma})\kappa, \\
\overline{\Gamma} &\equiv \frac{\varepsilon^{ijk}}{3!}\Gamma_{ijk} = \frac{\varepsilon^{ijk}}{3!}\Pi_i{}^{\underline{a}}\Pi_j{}^{\underline{b}}\Pi_k{}^{\underline{c}}\Gamma_{\underline{abc}},
\end{aligned} \tag{7.3.17}$$

where κ is another spinor. Equation (7.3.17) corresponds to the anticipated projection (7.3.6) which halves the spinor components. It follows that of the 32 fermionic degrees of freedom 16 can be gauged away by κ-supersymmetry. The remaining 16 are further reduced to 8 by their field equation which is implicitly determined by the action (7.3.14). As one sees, once the M2-action is cast into the first order form (7.3.14), κ-supersymmetry invariance can be implemented in an extremely simple and elegant way that requires only a couple of algebraic manipulations with gamma matrices.

The example of the M2-brane is generalized to all other instances of p-branes where the world volume spectrum includes just the scalars (= target space coordinates) and their fermionic partners.

7.3.3 *With Dp-Branes We Have a Problem: The World-Volume Gauge Field* $\mathbf{A}^{[1]}$

It is clear from what we explained above that to deal with κ-supersymmetry in an easy way we need a first order formulation of the action. Yet in the case of *Dp*-

branes there is a new problem intrinsically related to the new structure of the Born-Infeld action (7.3.5) which, differently from the pure Nambu-Goto action (7.3.3) cannot be recast into a first order form of type (7.3.11).[5] The solution of this problem is found through a procedure which is very frequent and traditional in Physics. Indeed, when a certain formulation of a theory cannot be generalized to a wider scenario including additional fields it usually means that there is a second formulation of the same theory which is equivalent to the former in the absence of the new fields, but which, differently from the former, can incorporate them in a natural way. A typical example of this is provided by the relation of Cartan's formulation of General Relativity in terms of vielbein and spin connection with the standard metric formulation. Although the two formulations are fully equivalent in the absence of fermions, yet the former allows the coupling to spinors while the latter does not, as we extensively discussed in Volume 1. The present case is similar. It turns out that there is a new, so far unknown, first order formulation of world-volume actions which, in the absence of world-volume gauge fields is fully equivalent to the formulation of (7.3.11). Yet world-volume 1-forms can be naturally included in the new formalism while they have no place in the old. In full analogy with other examples of the same logical process the new formalism relies on the addition of a new auxiliary field and a new symmetry. The new field is a 0-form rank 2 tensor h_{ij} that is identified with the intrinsic components of the pulled-back bulk metric along a reference world-volume vielbein e^i. The new symmetry is the independence of the action from the choice of the reference vielbein. Explicitly this means the following. Let $K^i{}_j(x)$ be a generic $d \times d$ matrix depending on the world-volume point. The new action we shall construct will be invariant against the local transformation:

$$
\begin{aligned}
e^i &\mapsto K^i{}_j \, e^j \\
h^{ij} &\mapsto \left(K^{-1}\right)^i{}_{i'} \left(K^{-1}\right)^j{}_{j'} \, h^{i'j'} (\det K)
\end{aligned}
\tag{7.3.18}
$$

accompanied by suitable transformation of the other fields. The above symmetry generalizes the local Lorentz invariance of the previously known first order p-brane actions. Indeed, being generic, the matrix K can in particular be an element of the Lorentz group $K \in \mathrm{SO}(1, d-1)$. In this case there is no novelty. However K can also be a representative of a non-trivial equivalence class of the coset $\mathrm{GL}(d, \mathbb{R})/\mathrm{SO}(1, d-1)$. This latter is precisely parameterized by arbitrary symmetric matrices. Hence the additional degrees of freedom introduced by the new auxiliary field h^{ij} are taken away by the enlargement of the local symmetry from $\mathrm{SO}(1, d-1)$ to $\mathrm{GL}(d, \mathbb{R})$.

[5] A partial first order formalism was already introduced in the literature for Dp-branes [31, 32] in the context of the superembedding approach initiated by the Kharkov group and extensively developed also in collaborations with the Padua group and other groups [21–23]. In particular in [33, 34] an action with a partial first order formalism was introduced in the sense that there is an auxiliary F_{ij} field for the gauge degrees of freedom but the action is "second order" in the brane coordinates x and θ, which enter through the pullback of the target space supervielbein E^a.

7.4 The New First Order Formalism

In Sect. 7.4.1 we describe the new formalism as an alternative to the action (7.3.11). Then in Sect. 7.4.2 we show how it allows the inclusion of world volume gauge fields and provides a first order formulation of the Born-Infeld action (7.3.5).

7.4.1 An Alternative to the Polyakov Action for p-Branes

To begin with we consider a world-volume Lagrangian of the following form:

$$\mathscr{L} = \Pi_i^{\underline{a}} V^{\underline{b}} \eta_{\underline{ab}} \eta^{i\ell_1} \wedge e^{\ell_2} \wedge \cdots \wedge e^{\ell_d} \varepsilon_{\ell_1 \ldots \ell_d} + a_1 \Pi_i^{\underline{a}} \Pi_j^{\underline{b}} \eta_{\underline{ab}} h^{ij} e^{\ell_1} \wedge \cdots \wedge e^{\ell_d} \varepsilon_{\ell_1 \ldots \ell_d} + a_2 (\det h)^{-\alpha} e^{\ell_1} \wedge \cdots \wedge e^{\ell_d} \varepsilon_{\ell_1 \ldots \ell_d} \tag{7.4.1}$$

where a_1, a_2, α are real parameters to be determined and the other notations are recalled in (B.1.1) of Appendix B.1.

Performing the $\delta \Pi_i^{\underline{a}}$ variation of the Lagrangian (7.4.1) we obtain:

$$\eta_{\underline{ab}} V_m^{\underline{b}} \eta^{i\ell_1} \varepsilon^{m\ell_2 \ldots \ell_d} \varepsilon_{\ell_1 \ldots \ell_d} + 2(d!) a_1 \eta_{\underline{ab}} \Pi_j^{\underline{b}} h^{ij} = 0 \tag{7.4.2}$$

If we choose:

$$a_1 = -\frac{1}{2d} \tag{7.4.3}$$

then (7.4.2) is solved by:

$$\Pi_m^{\underline{b}} = V_i^{\underline{b}} \eta^{ip} \left(h^{-1}\right)_{pm} \tag{7.4.4}$$

Let us then introduce the following three $d \times d$ matrices:

$$\gamma_{ij} = \Pi_i^a \Pi_j^b \eta_{\underline{ab}}; \qquad G_{ij} = V_i^{\underline{a}} V_j^{\underline{b}} \eta_{\underline{ab}}; \qquad \widehat{G} = \eta G \eta \tag{7.4.5}$$

The solution (7.4.4) of the field equation (7.4.2) implies that:

$$\gamma = \left(h^{-1}\right)^T \eta G \eta h^{-1} = \left(h^{-1}\right)^T \widehat{G} h^{-1} \tag{7.4.6}$$

Next let us consider the variation of the action (7.4.1) with respect to the symmetric matrix h^{ij}. In matrix form such a variational equation reads as follows:

$$a_1 \gamma - a_2 \alpha h^{-1} (\det h)^{-\alpha} = 0 \tag{7.4.7}$$

Setting:

$$a_2 = \frac{a_1}{\alpha} = -\frac{1}{2d\alpha} \tag{7.4.8}$$

(7.4.7) reduces to

$$\gamma = h^{-1}(\det h)^{-\alpha} \tag{7.4.9}$$

which can be solved by the ansatz:

$$h = \gamma^{-1}(\det \gamma)^{\beta} \tag{7.4.10}$$

provided:

$$\beta = \frac{\alpha}{d\alpha + 1} \tag{7.4.11}$$

On the other hand from (7.4.6) we get:

$$\det \gamma = \det G (\det h)^{-2} \tag{7.4.12}$$

so that:

$$h = h\widehat{G}^{-1} h (\det G)^{\beta} (\det h)^{-2\beta} \tag{7.4.13}$$

Equation (7.4.13) can be solved by the ansatz:

$$h = \widehat{G}(\det G)^{p} \tag{7.4.14}$$

provided:

$$p = -\frac{\alpha}{d\alpha - 1} \tag{7.4.15}$$

Combining the last two results we have the final solution for the two auxiliary fields h and γ:

$$h = \widehat{G}(\det G)^{p}; \qquad \gamma = \widehat{G}^{-1}(\det G)^{-2p} \tag{7.4.16}$$

in terms of G which is just the pull-back of the bulk metric onto the world volume, expressed in flat components with respect to an arbitrary reference vielbein e^{ℓ} that lives on $\mathscr{W}$.

Using (7.4.16) we can rewrite the action (7.4.1) in second order formalism. The basic observation is that after implementation of the first order field equations the three terms appearing in (7.4.1) become all proportional to the same term, namely $(\det G)^{-p} \det e \, d^d \xi$, having named ξ the world volume coordinates. Indeed we have:

$$\begin{aligned}
(\det h)^{-\alpha} e^{\ell_1} \wedge \cdots \wedge e^{\ell_d} \varepsilon_{\ell_1 \ldots \ell_d} &= d!(\det G)^{-p} \det e \, d^d \xi \\
\eta_{\underline{ab}} \Pi_i^{\underline{a}} \Pi_j^{\underline{b}} h^{ij} e^{\ell_1} \wedge \cdots \wedge e^{\ell_d} \varepsilon_{\ell_1 \ldots \ell_d} &= d\, d!(\det G)^{-p} \det e \, d^d \xi \\
\Pi_i^{\underline{a}} V^{\underline{b}} \eta_{\underline{ab}} \eta^{i \ell_1} \wedge e^{\ell_2} \wedge \cdots \wedge e^{\ell_d} \varepsilon_{\ell_1 \ldots \ell_d} &= d!(\det G)^{-p} \det e \, d^d \xi
\end{aligned} \tag{7.4.17}$$

Hence the Lagrangian (7.4.1) becomes:

$$\mathscr{L} = (d-1)!(\det G)^{-p} \det e \, d^d \xi = (d-1)! \frac{1}{2p} (\det G_{\mu\nu})^{-p} (\det e)^{2p+1} \, d^d \xi \tag{7.4.18}$$

the second identity following from:

$$G_{ij} = V^{\underline{a}}_{\underline{\mu}} V^{\underline{b}}_{\underline{\nu}} \eta_{\underline{ab}} \partial_\mu X^{\underline{\mu}} \partial_\nu X^{\underline{\nu}} e^\mu_i e^\nu_j = \underbrace{g_{\underline{\mu\nu}} \partial_\mu X^{\underline{\mu}} \partial_\nu X^{\underline{\nu}}}_{G_{\mu\nu}} e^\mu_i e^\nu_j$$
$$\Downarrow \tag{7.4.19}$$
$$\det G_{ij} = (\det G_{\mu\nu})(\det e)^{-2}$$

where $G_{\mu\nu}$ denotes the pull-back of the bulk space-time metric $g_{\underline{\mu\nu}}$ onto the world-volume of the brane.

If we choose:

$$p = -\frac{1}{2} \quad \Rightarrow \quad \alpha = \frac{1}{d-2} \tag{7.4.20}$$

then the original world-volume Lagrangian (7.4.1), already transformed to the second order form (7.4.18) becomes proportional to the Nambu-Goto Lagrangian:

$$\mathscr{L} = (d-1)! \sqrt{\det G_{\mu\nu}}\, d^d\xi \tag{7.4.21}$$

In this way the reference vielbein e^i_μ has disappeared from the Lagrangian. This result is supported by the calculation of the variation in δe^k of the first order action (7.4.1). After variation and substitution of the result for the first order equations $\delta \Pi^{\underline{a}}_i$ and δh_{ij} all terms are already Kronecker deltas proportional to $\det G$. With the choice $p = -1/2$ all terms in this stress energy tensor cancel identically.

Note also that if the transformation (7.3.18) is completed by setting:

$$\Pi^{\underline{a}}_i \mapsto K^i{}_k \eta^{k\ell} \Pi^{\underline{a}}_\ell (\det K)^{-1} \tag{7.4.22}$$

it becomes an exact local symmetry of the action (7.4.1).

In this way we have shown how the standard first order formalism for the Nambu-Goto action can be replaced by a new first order formalism involving the additional field h_{ij}. So far the matrix h was chosen to be symmetric. Including world-sheet vector fields corresponds to the generalization of the above construction to the case where h has also an antisymmetric part.

7.4.2 Inclusion of a World-Volume Gauge Field and the Born-Infeld Action in First Order Formalism

We consider a modification of the first order action (7.4.1) of the following form

$$\begin{aligned}
\mathscr{L} = {} & \Pi^{\underline{a}}_i V^{\underline{b}} \eta_{\underline{ab}} \eta^{i\ell_1} \wedge e^{\ell_2} \wedge \cdots \wedge e^{\ell_d} \varepsilon_{\ell_1 \ldots \ell_d} + a_1 \Pi^{\underline{a}}_i \Pi^{\underline{b}}_j \eta_{\underline{ab}} h^{ij} e^{\ell_1} \wedge \cdots \wedge e^{\ell_d} \varepsilon_{\ell_1 \ldots \ell_d} \\
& + a_2 \big[\det\big(h^{-1} + \mu \mathscr{F}\big)\big]^\alpha e^{\ell_1} \wedge \cdots \wedge e^{\ell_d} \varepsilon_{\ell_1 \ldots \ell_d} \\
& + a_3 \mathscr{F}^{ij} \mathbf{F}^{[2]} \wedge e^{\ell_3} \wedge \cdots \wedge e^{\ell_d} \varepsilon_{ij\ell_3 \ldots \ell_d} + \underbrace{WZT}_{\text{Wess-Zumino terms}}
\end{aligned} \tag{7.4.23}$$

where

$$\mathbf{F}^{[2]} \equiv d\mathbf{A}^{[1]} \tag{7.4.24}$$

is the field strength of a world-volume 1-form gauge field, $\mathscr{F}_{ij} = -\mathscr{F}_{ji}$ is an antisymmetric 0-form auxiliary field and a_3 is a further numerical coefficient to be determined. Furthermore *WZT* denotes the Wess-Zumino terms, *i.e.* the integrals on the world volume of various combinations of the Ramond-Ramond p-forms. These terms depend on the type of Dp-brane considered and will be discussed later in the case of the $D3$-brane.

Performing the $\delta\Pi_i^{\underline{a}}$ variation we obtain:

$$(d-1)!\left[\eta_{\underline{ab}}\, V_l^{\underline{a}} \eta^{il} + 2a_1 d\eta_{\underline{ab}}\, \Pi_j^{\underline{a}} h^{ij}\right] = 0 \tag{7.4.25}$$

that is solved by:

$$\Pi_j^{\underline{a}} = -\frac{1}{2da_1} V_m^{\underline{a}} \left(h^{-1}\right)^m{}_j \tag{7.4.26}$$

and:

$$\gamma = \frac{1}{(2da_1)^2} h^{-1} \widehat{G} h^{-1} \tag{7.4.27}$$

Varying in δh_{ij} we also obtain a result similar to what we had before, namely:

$$a_1\gamma - a_2\alpha h^{-1}\left(h^{-1} + \mu\mathscr{F}\right)_S^{-1} h^{-1}\left[\det\left(h^{-1} + \mu\mathscr{F}\right)\right]^\alpha = 0 \tag{7.4.28}$$

where the suffix S denotes the symmetric part of the matrix to which it is applied.

From the variation in $\delta\mathscr{F}_{ij}$ we obtain instead:

$$-d!a_2\alpha\mu\left(h^{-1} + \mu\mathscr{F}\right)_A^{-1}\left[\det\left(h^{-1} + \mu\mathscr{F}\right)\right]^\alpha + 2(d-2)!a_3 F = 0 \tag{7.4.29}$$

where the suffix A denotes the antisymmetric part of the matrix to which it is applied and where F is the antisymmetric matrix F_{ij} of flat components of the field strength 2-form:

$$\mathbf{F}^{[2]} = F_{ij} e^i \wedge e^j \tag{7.4.30}$$

Hence from δh_{ij} and $\delta\mathscr{F}_{ij}$ we get:

$$\begin{aligned} \frac{2(d-2)!a_3}{d!a_2\alpha\mu} F &= \left(h^{-1} + \mu\mathscr{F}\right)_A^{-1}\left[\det\left(h^{-1} + \mu\mathscr{F}\right)\right]^\alpha \\ \frac{1}{4d^2 a_1 a_2\alpha} \widehat{G} &= \left(h^{-1} + \mu\mathscr{F}\right)_S^{-1}\left[\det\left(h^{-1} + \mu\mathscr{F}\right)\right]^\alpha \end{aligned} \tag{7.4.31}$$

Summing the two equations (7.4.31) together we obtain:

$$\frac{2(d-2)!a_3}{d!a_2\alpha\mu} F + \frac{1}{4d^2 a_1 a_2\alpha} \widehat{G} = \left(h^{-1} + \mu\mathscr{F}\right)^{-1}\left[\det\left(h^{-1} + \mu\mathscr{F}\right)\right]^\alpha \tag{7.4.32}$$

which can be uniquely solved by:

$$h^{-1} + \mu\mathscr{F} = (a\widehat{G} + bF)^{-1}\big[\det(a\widehat{G} + bF)\big]^{\beta}; \qquad \beta = \frac{\alpha}{\alpha\, d - 1} \tag{7.4.33}$$

where:

$$a = \frac{1}{4d^2 a_1 a_2 \alpha}; \qquad b = \frac{2(d-2)!a_3}{d!a_2\alpha\mu} \tag{7.4.34}$$

The coefficients a_1, a_2, a_3 are redundant since they can be reabsorbed into the definition of Π^{a}_{j}, h and $\mathscr{F}$; so we fix them by imposing:

$$a_1 = -\frac{1}{2d}; \qquad a = 1; \qquad b = -\frac{1}{\mu} \tag{7.4.35}$$

Hence using (7.4.34) and (7.4.35) we obtain:

$$a_2 = -\frac{1}{2d\alpha}; \qquad a_3 = -\frac{d!a_2\alpha}{2(d-2)!} = \frac{d-1}{4} \tag{7.4.36}$$

At this point everything proceeds just as in the previous case. Indeed inserting (7.4.27), (7.4.26) back into the action (7.4.23) we obtain:

$$\begin{aligned} &\bigg[\bigg(-\frac{(d-1)!}{2da_1} + a_1\, d!\bigg)\frac{1}{(2da_1)^2}\operatorname{Tr}\big(h^{-1}\widehat{G}\big) + 2a_3(d-2)!\mathscr{F}^{ij}F_{ij} \\ &\quad + a_2 d!\big[\det\big(h^{-1} + \mu\mathscr{F}\big)\big]^{\alpha}\bigg]\det e\, d^d\xi \end{aligned} \tag{7.4.37}$$

Using (7.4.35) and (7.4.36), (7.4.37) becomes:

$$\bigg\{\frac{(d-1)!}{2}\big[\operatorname{Tr}\big(h^{-1}\widehat{G}\big) - \operatorname{Tr}(\mathscr{F}F)\big] - \frac{(d-1)!}{2\alpha}\big[\det\big(h^{-1} + \mu\mathscr{F}\big)\big]^{\alpha}\bigg\}\det e\, d^d\xi \tag{7.4.38}$$

Now we consider the variation δe:

$$\begin{aligned} &\bigg[-\frac{(d-1)!}{4da_1}\operatorname{Tr}\big(Gh^{-1}\big) - 2(d-2)!a_3\operatorname{Tr}(\mathscr{F}F)\bigg]\delta^t_p \\ &\quad - 2\bigg[-\frac{(d-1)!}{4da_1}\big(Gh^{-1}\big)_p{}^t - 2(d-2)!a_3\big(\mathscr{F}^{ti}F_{ip}\big)\bigg] \\ &\quad + a_2 d!\big[\det\big(h^{-1} + \mu\mathscr{F}\big)\big]^{\alpha}\delta^t_p = 0 \end{aligned} \tag{7.4.39}$$

the solution is:

$$\bigg[-\frac{(d-1)!}{4da_1}\big(Gh^{-1}\big)_p{}^t - 2(d-2)!a_3\big(\mathscr{F}^{ti}F_{ip}\big)\bigg] = -\frac{a_2 d!}{d-2}\big[\det\big(h^{-1} + \mu\mathscr{F}\big)\big]^{\alpha}\delta^t_p \tag{7.4.40}$$

Using (7.4.35) and (7.4.36) in matrix form (7.4.40) becomes:

$$h^{-1}\widehat{G} - \mathscr{F} F = \frac{1}{\alpha(d-2)}\bigl[\det\bigl(h^{-1} + \mu \mathscr{F}\bigr)\bigr]^{\alpha}\mathbf{1} \tag{7.4.41}$$

Using the result:

$$\bigl[\det(a\widehat{G}_{ij} + bF_{ij})\bigr] = \bigl[\det(a\widehat{G}_{\mu\nu} + bF_{\mu\nu})\bigr](\det e)^{-2} \tag{7.4.42}$$

and implementing (7.4.41) for δe, we see that (7.4.38) becomes:

$$\begin{aligned}
&(\det e)\, d^d\xi \frac{(d-1)!}{\alpha(d-2)}\bigl[\det\bigl(h^{-1} + \mu\mathscr{F}\bigr)\bigr]^{\alpha} \\
&\quad = (\det e)\, d^d\xi\, \frac{(d-1)!}{\alpha(d-2)}\bigl[\det(a\widehat{G}_{ij} + bF_{ij})\bigr]^{\beta} \\
&\quad = (\det e)^{1-2\beta}\, d^d\xi \frac{(d-1)!}{\alpha(d-2)}\bigl[\det(a\widehat{G}_{\mu\nu} + bF_{\mu\nu})\bigr]^{\beta}
\end{aligned} \tag{7.4.43}$$

Now we take $\beta = 1/2$ and so $\alpha = 1/(d-2)$. The action becomes:

$$S_{BI} = (d-1)! \int_{M_4} d^d\xi \left[\det\left(\widehat{G}_{\mu\nu} - \frac{1}{\mu}F_{\mu\nu}\right)\right]^{1/2} \tag{7.4.44}$$

For $d = 4$, which is the interesting case of the $D3$-brane we obtain:

$$a_1 = -\frac{1}{8}; \qquad a_2 = -\frac{1}{4}; \qquad a_3 = \frac{3}{4}; \qquad \alpha = \beta = \frac{1}{2} \tag{7.4.45}$$

In this way we have shown how the kinetic part of a Dp-brane action, namely the Born-Infeld type of Lagrangian can be written in first order formalism. The new formalism can be applied to all cases except $d = 2$ where the formulae become singular. This is just welcome since for $d = 2$ we have ordinary strings for which the Polyakov formalism is sufficient and no world-volume cosmological term is necessary. For $d = 3$, we are instead in the case of the M2 brane or of its descendant, the D2 brane, for which no Born-Infeld action is necessary either.

7.4.3 Explicit Solution of the Equations for the Auxiliary Fields for $\mathscr{F}$ and h^{-1}

In the transition to second order formalism and in the discussion of κ-supersymmetry through the use of 1.5 order formalism we need the explicit solution of the first order equations and the expression of the auxiliary fields $\mathscr{F}$, h^{-1} in terms of the physical degrees of freedom. This is what we can do most conveniently by fixing the gauge related to the local symmetry (7.3.18) and (7.4.22). Our gauge choice is provided by setting:

$$\widehat{G} = \eta \tag{7.4.46}$$

which is identical with the yield (7.3.13) of the δe^i variation in the old first order formalism. This gauge can certainly be reached by using the degrees of freedom of $\mathrm{GL}(d,\mathbb{R})/\mathrm{SO}(1,d-1)$. Taking (7.4.46) into account let us rewrite our constraint equations into matrix form. Equation (7.4.41) for the δe variation is:

$$h^{-1}\widehat{G} - \mathscr{F}F = \left[\det\left(h^{-1} + \mu\mathscr{F}\right)\right]^{\alpha}\mathbf{1} \tag{7.4.47}$$

and the other equation that we must solve is (7.4.33):

$$\left(h^{-1} + \mu\mathscr{F}\right)\left(\widehat{G} - \frac{1}{\mu}F\right) = \left[\det\left(\widehat{G} - \frac{1}{\mu}F\right)\right]^{1/2}\mathbf{1} \tag{7.4.48}$$

Using our previous result for $[\det(h^{-1} + \mu\mathscr{F})]^{\alpha}$ we conclude that we have the following linear system of matrix equations:

$$\begin{cases} (h^{-1} + \mu\mathscr{F})(\widehat{G} - \frac{1}{\mu}F) = [\det(\widehat{G} - \frac{1}{\mu}F)]^{1/2}\mathbf{1} \\ h^{-1}\widehat{G} - \mathscr{F}F = [\det(\widehat{G} - \frac{1}{\mu}F)]^{1/2}\mathbf{1} \end{cases} \tag{7.4.49}$$

the solution in the gauge (7.4.46) is:

$$\begin{cases} \widehat{G} = \eta \\ \mathscr{F} = \frac{1}{\mu^2}h^{-1}F\eta \\ h\eta = (\mathbf{1} - \frac{1}{\mu^2}F\eta F\eta)[\det(\eta - \frac{1}{\mu}F)]^{-1/2} \end{cases} \tag{7.4.50}$$

Since the η metric just raises and lowers the indices we can just ignore it and write, in more compact form:

$$h = \left(\eta - \frac{1}{\mu^2}F^2\right)\left[\det\left(\eta - \frac{1}{\mu}F\right)\right]^{-1/2} \tag{7.4.51}$$

7.5 The $D3$-Brane Example and κ-Supersymmetry

In this section we focus on the case $d = 4$ and we apply the new first order formalism to the description of the κ-supersymmetric action of a $D3$-brane. As emphasized above, κ-supersymmetry just follows, via a suitable projection, from the bulk supersymmetries as derived from supergravity, the type II B theory, in this case. The latter has a duality symmetry with respect to an $\mathrm{SL}(2,\mathbb{R})$ group of transformations that acts non-linearly on the two scalars of massless spectrum, the dilaton ϕ and the Ramond scalar C_0. Indeed these two parameterize the coset manifold

$\mathrm{SL}(2,\mathbb{R})/\mathrm{O}(2)$ and actually correspond to its solvable parameterization (see (6.8.3) in Sect. 6.8). Hence the $D3$-brane action we want to write, not only should be cast into first order formalism, but should also display manifest covariance with respect to $\mathrm{SL}(2,\mathbb{R})$. This covariance relies on introducing a two component charge vector q_α that transforms in the fundamental representation of $\mathrm{SU}(1,1)$ and expresses the charges carried by the $D3$ brane with respect to the 2-forms $\mathbf{A}^{\alpha}_{[2]}$ of bulk supergravity (both the Neveu Schwarz $\mathbf{B}^{[2]}$ and Ramond-Ramond $\mathbf{C}^{[2]}$). According to the geometrical formulation of type IIB supergravity presented in Sect. 6.8 we set:

$$\begin{aligned} &\mathbf{A}^{\Lambda} = \big(\mathbf{B}^{[2]},\mathbf{C}^{[2]}\big); && \mathbf{A}^{\alpha} = \mathscr{C}^{\alpha}{}_{\Lambda}\mathbf{A}^{\Lambda} \\ &\mathbf{A}^{\alpha=1} = \frac{1}{\sqrt{2}}\big(\mathbf{B}^{[2]} - i\mathbf{C}^{[2]}\big); && \mathbf{A}^{\alpha=2} = \frac{1}{\sqrt{2}}\big(\mathbf{B}^{[2]} + i\mathbf{C}^{[2]}\big) \end{aligned} \tag{7.5.1}$$

and by definition we call $\varepsilon_{\alpha\beta}q^{\beta}$ the orthogonal complement of q_α:

$$q_\alpha q^\alpha = 1; \qquad q_\alpha q_\beta \varepsilon^{\alpha\beta} = 0 \tag{7.5.2}$$

In terms of these objects we write down the complete action of the $D3$-brane as follows:

$$\begin{aligned} \mathscr{L} = {} & \Pi^{\underline{a}}_i V^{\underline{b}}\eta_{\underline{ab}}\eta^{i\ell_1}\wedge e^{\ell_2}\wedge\cdots\wedge e^{\ell_4}\varepsilon_{\ell_1\ldots\ell_4} + a_1\Pi^{\underline{a}}_i\Pi^{\underline{b}}_j\eta_{\underline{ab}}h^{ij}e^{\ell_1}\wedge\cdots\wedge e^{\ell_4}\varepsilon_{\ell_1\ldots\ell_4} \\ & + a_2\big[\det\big(h^{-1}+\mu\mathscr{F}\big)\big]^{\alpha}e^{\ell_1}\wedge\cdots\wedge e^{\ell_4}\varepsilon_{\ell_1\ldots\ell_4} \\ & + a_3\mathscr{F}^{ij}\mathbf{F}^{[2]}\wedge e^{\ell_3}\wedge e^{\ell_4}\varepsilon_{ij\ell_3\ell_4} \\ & + \nu F\wedge F - i a_5 q^{\alpha}\varepsilon_{\alpha\beta}\mathbf{A}^{\beta}\wedge F + a_6\mathbf{C}^{[4]} \end{aligned} \tag{7.5.3}$$

where $\mathbf{C}^{[4]}$ is the 4-form potential, the coefficients

$$\alpha = \frac{1}{2}; \qquad a_1 = -\frac{1}{8}; \qquad a_2 = -\frac{1}{4}; \qquad a_3 = \frac{3}{4} \tag{7.5.4}$$

have already been determined, while a_5, a_6, ν are new coefficients to be fixed by κ-supersymmetry. The first two are numerical, while ν will also depend on the bulk scalars. In the action (7.5.3)

$$\mathbf{F}^{[2]} \equiv d\mathbf{A}^{[1]} + q_\alpha\mathbf{A}^{\alpha} \tag{7.5.5}$$

is the field strength of the world-volume gauge field and depends on the charge vector q^α. The physical interpretation of $\mathbf{F}^{[2]}$ is as follows. By definition a Dp-brane is a locus in space-time where open strings can end or, in the dual picture, boundaries for closed string world-volumes can be located. The type IIB theory contains two kind of strings, the fundamental strings and the D-strings which are rotated one into the other by the $\mathrm{SL}(2,\mathbb{Z})\subset\mathrm{SL}(2,\mathbb{R})$ group. Correspondingly a $D3$ brane can be a boundary either for fundamental or for D-strings or for a mixture of the two. The charge vector q^α just expresses this fact and characterizes the $D3$-brane as a boundary for strings of q-type. Furthermore the definition (7.5.5) of $\mathbf{F}^{[2]}$ encodes the following idea: the world-volume gauge 1-form $\mathbf{A}^{[1]}$ is just the parameter of a gauge transformation for the 2-form $q_\alpha \mathbf{A}^\alpha$, which in a space-time with boundaries can be reabsorbed everywhere except on the boundary itself. Note that if we take $q_\alpha = \frac{1}{\sqrt{2}}(1,1)$ we obtain:

$$q_\alpha \mathbf{A}^\alpha = \mathbf{B}^{[2]}; \qquad -iq^\alpha \varepsilon_{\alpha\beta} \mathbf{A}^\beta = \mathbf{C}^{[2]} \tag{7.5.6}$$

7.5.1 κ-Supersymmetry

Next we want to prove that with an appropriate choice of ν, a_5 and a_6 the action (7.5.3) is invariant against bulk supersymmetries characterized by a projected spinor parameter. For simplicity we do this in the case of the choice $q_\alpha = \frac{1}{\sqrt{2}}(1,1)$. For other choices of the charge type the modifications needed in the prove will be obvious from its details.

To accomplish our goal we begin by writing the supersymmetry transformations of the bulk differential forms $V^{\underline{a}}$, $\mathbf{B}^{[2]}$, $\mathbf{C}^{[2]}$ and $\mathbf{C}^{[4]}$ which appear in the action. From the rheonomic parameterizations (6.8.13), (6.8.14), (6.8.15), (6.8.16) we immediately obtain:

$$\begin{aligned}
\delta V^{\underline{a}} &= \mathrm{i}\frac{1}{2}\big(\overline{\varepsilon}\,\Gamma^{\underline{a}}\psi + \overline{\varepsilon}^*\Gamma^{\underline{a}}\psi^*\big)\\
\delta \mathbf{B}^{[2]} &= -2\mathrm{i}\big[\big(\Lambda^1_+ + \Lambda^2_+\big)\overline{\varepsilon}\,\Gamma^{\underline{a}}\psi^* \wedge V^{\underline{a}} + \big(\Lambda^1_- + \Lambda^2_-\big)\overline{\varepsilon}^*\Gamma^{\underline{a}}\psi \wedge V^{\underline{a}}\big]\\
\delta \mathbf{C}^{[2]} &= 2\big[\big(\Lambda^1_+ - \Lambda^2_+\big)\overline{\varepsilon}\,\Gamma^{\underline{a}}\psi^* \wedge V^{\underline{a}} + \big(\Lambda^1_- - \Lambda^2_-\big)\overline{\varepsilon}^*\Gamma^{\underline{a}}\psi \wedge V^{\underline{a}}\big]\\
\delta \mathbf{C}^{[4]} &= \underbrace{-\frac{1}{6}\big(\overline{\varepsilon}\,\Gamma^{\underline{abc}}\psi - \overline{\varepsilon}^*\Gamma^{\underline{abc}}\psi^*\big)\wedge V^{\underline{a}}\wedge V^{\underline{b}}\wedge V^{\underline{c}}}_{\delta\mathbf{C}^{[4]\prime}}\\
&\quad + \frac{1}{8}\big[\mathbf{B}^{[2]}\wedge\delta\mathbf{C}^{[2]} - \mathbf{C}^{[2]}\wedge\delta\mathbf{B}^{[2]}\big]
\end{aligned} \tag{7.5.7}$$

Note that in writing the above transformations we have neglected all terms involving the dilatino field. This is appropriate since the background value of all fermion fields is zero. The gravitino 1-form ψ is instead what we need to keep track of. Proving κ-supersymmetry is identical with showing that all ψ terms cancel against each other in the variation of the action. Relying on (7.5.7) the variation of the Wess-Zumino term is as follows:

$$\begin{aligned}
&\delta\big(\nu\mathbf{F}^{[2]}\wedge\mathbf{F}^{[2]} + a_5\mathbf{C}^{[2]}\wedge\mathbf{F}^{[2]} + \alpha_6\mathbf{C}^{[4]}\big)\\
&\quad = 2\nu\mathbf{B}^{[2]}\wedge\delta\mathbf{B}^{[2]} + a_5\mathbf{B}^{[2]}\wedge\delta\mathbf{C}^{[2]}\\
&\qquad + \frac{1}{8}a_6\mathbf{B}^{[2]}\wedge\delta\mathbf{C}^{[2]} + a_5 C\wedge\delta\mathbf{B}^{[2]} - \frac{1}{8}a_6\mathbf{C}^{[2]}\wedge\delta\mathbf{B}^{[2]} + a_6\delta\mathbf{C}^{[4]\prime}
\end{aligned} \tag{7.5.8}$$

if we set $a_6 = 8a_5$ the variation (7.5.8) simplifies to:

$$\delta(WZT) = 2\mathbf{B}^{[2]} \wedge \left(\nu\delta\mathbf{B}^{[2]} + a_5\delta\mathbf{C}^{[2]}\right) + 8a_5\delta\mathbf{C}^{[4]\prime} \tag{7.5.9}$$

and with such a choice the complete variation of the Lagrangian under a supersymmetry transformation of arbitrary parameter is:

$$\begin{aligned}
\delta\mathscr{L} &= \delta\mathscr{L}_{\psi} + \delta\mathscr{L}_{\psi^*} \\
\delta\mathscr{L}_{\psi} &= \Big[-3!\mathrm{i}\Pi^{\underline{a},p}\left(\overline{\varepsilon}\,\Gamma^{\underline{b}}\,\psi\right)\eta_{\underline{ab}} + \left(\mu_1\mathscr{F}^{ip} + \mu_2\widetilde{F}^{ip}\right)V_i^{\underline{a}}\left(\overline{\varepsilon}^*\,\Gamma_{\underline{a}}\,\psi\right) \\
&\quad - \frac{4}{3}a_5(\overline{\varepsilon}\,\Gamma_{\underline{abc}}\,\psi)V_i^{\underline{a}}\,V_j^{\underline{b}}\,V_k^{\underline{c}}\,\varepsilon^{ijkp}\Big]\Omega_p^{[3]} \\
\delta\mathscr{L}_{\psi_*} &= \Big[-3!\mathrm{i}\Pi^{\underline{a},p}\left(\overline{\varepsilon}^*\,\Gamma^{\underline{b}}\,\psi^*\right)\eta_{\underline{ab}} + \left(\mu_3\mathscr{F}^{ip} + \mu_4\widetilde{F}^{ip}\right)V_i^{\underline{a}}\left(\overline{\varepsilon}\,\Gamma_{\underline{a}}\,\psi^*\right) \\
&\quad + \frac{4}{3}a_5\left(\overline{\varepsilon}^*\,\Gamma_{\underline{abc}}\,\psi^*\right)V_i^{\underline{a}}\,V_j^{\underline{b}}\,V_k^{\underline{c}}\,\varepsilon^{ijkp}\Big]\Omega_p^{[3]}
\end{aligned} \tag{7.5.10}$$

where:

$$\begin{aligned}
\mu_1 &= -8\mathrm{i}a_3\left(\Lambda_-^1 + \Lambda_-^2\right); \qquad \mu_2 = 8\left[-\mathrm{i}\nu\left(\Lambda_-^1 + \Lambda_-^2\right) + a_5\left(\Lambda_-^1 - \Lambda_-^2\right)\right] \\
\mu_3 &= -8\mathrm{i}a_3\left(\Lambda_+^1 + \Lambda_+^2\right); \qquad \mu_4 = 8\left[-\mathrm{i}\nu\left(\Lambda_+^1 + \Lambda_+^2\right) + a_5\left(\Lambda_+^1 - \Lambda_+^2\right)\right]
\end{aligned} \tag{7.5.11}$$

Recalling (6.8.5) and (6.8.3) of Sect. 6.8 the above (7.5.11) become:

$$\begin{aligned}
\mu_1 &= -6\mathrm{i}e^{\phi/2}; \qquad \mu_2 = 8a_5e^{-\frac{\phi}{2}} \\
\mu_3 &= -6\mathrm{i}e^{\phi/2}; \qquad \mu_4 = -8a_5e^{-\frac{\phi}{2}}
\end{aligned} \tag{7.5.12}$$

where we have chosen:

$$\nu = -a_5\mathbf{C}^{[0} = a_5\,\mathrm{Re}\,\mathscr{N} \tag{7.5.13}$$

In the above equation we have introduced the complex kinetic matrix which would appear in a gauge theory with scalars sitting in $\mathrm{SU}(1,1)/\mathrm{U}(1)$ and determined by the classical Gaillard-Zumino general formula[6] applied to the specific coset:

$$\mathscr{N} = \mathrm{i}\frac{\Lambda_-^1 - \Lambda_-^2}{\Lambda_-^1 + \Lambda_-^2} \quad \Rightarrow \quad \begin{cases} \mathrm{Re}\,\mathscr{N} = -C_0 \\ \mathrm{Im}\,\mathscr{N} = e^{-\phi} \end{cases} \tag{7.5.14}$$

[6]For a general discussion of the Gaillard-Zumino formula see Chap. 8, Sects. 8.3.1–8.3.2.

It is convenient to rewrite the full variation (7.5.10) of the Lagrangian in matrix form in the 2-dimensional space spanned by the fermion parameters $(\varepsilon, \varepsilon^*)$:

$$\delta\mathscr{L} = \delta\mathscr{L}_\psi + \delta\mathscr{L}_{\psi^*} = \left(\overline{\varepsilon}, \overline{\varepsilon}^*\right) A \begin{pmatrix} \psi \\ \psi^* \end{pmatrix} \tag{7.5.15}$$

$$A_k = \begin{pmatrix} -6\mathrm{i}\gamma_k - \frac{4}{3}a_5\gamma_{ijl}\varepsilon^{ijlm}h_{mk} & (\mu_3\mathscr{F}^{lm} + \mu_4\widetilde{F}^{lm})h_{mk}\gamma_l \\ (\mu_1\mathscr{F}^{lm} + \mu_2\widetilde{F}^{lm})h_{mk}\gamma_l & -6\mathrm{i}\gamma_k + \frac{4}{3}a_5\gamma_{ijl}\varepsilon^{ijlm}h_{mk} \end{pmatrix} \tag{7.5.16}$$

where $A = A_k\Omega^k_{[3]}$, and $\Omega^k_{[3]} \equiv \eta^{k\ell}\varepsilon_{\ell ijk}e^i \wedge e^j \wedge e^k$ denotes the quadruplet of three-volume forms.

The matrix A_k is a tensor product of a matrices in spinor space and 2×2 matrices in the space spanned by $(\varepsilon, \varepsilon^*)$. It is convenient to spell out this tensor product structure which is achieved by the following rewriting:

$$A_k = f_1\gamma_k \otimes \mathbf{1} + f_2\tilde{\gamma}^m h_{mk} \otimes \sigma_3 + f_3\Pi_1^m h_{mk} \otimes \sigma_1 + f_4\Pi_2^m h_{mk} \otimes \sigma_2 \tag{7.5.17}$$

where:

$$f_1 = -6\mathrm{i}; \qquad f_3 = -6\mathrm{i}; \qquad f_2 = -\frac{4}{3}a_5; \qquad f_4 = -8\mathrm{i}a_5 \tag{7.5.18}$$

and:

$$\tilde{\gamma}^m \equiv \gamma_{ijl}\varepsilon^{ijlm}; \qquad \Pi_1^m \equiv e^{\phi/2}\mathscr{F}^{lm}\gamma_l; \qquad \Pi_2^m \equiv e^{-\phi/2}\widetilde{F}^{lm}\gamma_l \tag{7.5.19}$$

now using (7.4.50), (7.4.51) we set

$$\begin{aligned} \frac{1}{\mu} &= e^{-\phi/2} = \sqrt{\operatorname{Im}\mathscr{N}} \\ \widehat{F} &\equiv \sqrt{\operatorname{Im}\mathscr{N}}\, F \end{aligned} \tag{7.5.20}$$

and we obtain:

$$\begin{aligned} \Pi_1^m h_{mk} &= e^{\phi/2}\mathscr{F}^{lm}\gamma_l h_{mk} = e^{\phi/2}e^{-\phi}\left(Fh^{-1}\right)^{lm}h_{mk}\gamma_l \equiv \widehat{F}_{lk}\gamma^l \equiv \Pi_k \\ \Pi_2^m &\equiv e^{-\phi/2}\widetilde{F}^{lm}\gamma_l \equiv \widetilde{\widehat{F}}^{lm}\gamma_l \equiv \widetilde{\Pi}^m \end{aligned} \tag{7.5.21}$$

This observation further simplifies the expression of A_k which can be rewritten as:

$$A_k = f_1\gamma_k \otimes \mathbf{1} + f_2\tilde{\gamma}^m h_{mk} \otimes \sigma_3 + f_3\Pi^k \otimes \sigma_1 + f_4\widetilde{\Pi}^m h_{mk} \otimes \sigma_2 \tag{7.5.22}$$

The proof of κ-supersymmetry can now be reduced to the following simple computation. Assume we have a matrix operator Γ with the following properties:

$$\begin{aligned} &[\mathrm{a}] \qquad \Gamma^2 = \mathbf{1} \\ &[\mathrm{b}] \qquad \Gamma A_k = A_k \end{aligned} \tag{7.5.23}$$

It follows that

$$P = \frac{1}{2}(\mathbf{1} - \Gamma) \tag{7.5.24}$$

is a projector since $P^2 = \mathbf{1}$ and that

$$P A_k = \frac{1}{N}(\mathbf{1} - \Gamma) A_k = 0 \tag{7.5.25}$$

Therefore if we use supersymmetry parameters $(\overline{\kappa}, \overline{\kappa}^*) = (\overline{\varepsilon}, \overline{\varepsilon}^*) P$ projected with this P, then the action is invariant and this is just the proof of κ-supersymmetry.

The appropriate Γ is the following [37]:[7]

$$\Gamma = \frac{1}{N}\left[(\omega_{[4]} + \omega_{[0]}) \otimes \sigma_3 + \omega_{[2]} \otimes \sigma_2\right] \tag{7.5.26}$$

where:

$$\begin{aligned}
\omega_{[4]} &= \alpha_4 \varepsilon^{ijkl} \gamma_{ijkl} \\
\omega_{[0]} &= \alpha_0 \varepsilon^{ijkl} \widehat{F}_{ij} \widehat{F}_{kl} \\
\omega_{[2]} &= \alpha_2 \varepsilon^{ijkl} \widehat{F}_{ij} \gamma_{kl} \\
N &= \left[\det(\eta \pm \widehat{F})\right]^{1/2}
\end{aligned} \tag{7.5.27}$$

and the coefficients are fixed to:

$$\alpha_4 = \frac{1}{24}; \qquad \alpha_0 = \frac{1}{8}; \qquad \alpha_2 = \frac{i}{4} \tag{7.5.28}$$

This choice suffices to guarantee property [a] in the above list. Property [b] is also verified if one chooses:

$$a_5 = \frac{3}{4} \tag{7.5.29}$$

The proof of the two properties is given in Appendix B.2. Essential ingredients in the proof are the following identities holding true for any antisymmetric tensor $\widehat{F}$:

$$\det(\eta \pm \widehat{F}) = -1 + \frac{1}{2}\mathrm{Tr}\left(\widehat{F}^2\right) + \left(\frac{1}{8}\varepsilon^{ijkl} \widehat{F}_{ij} \widehat{F}_{kl}\right)^2 \tag{7.5.30}$$

[7]In the paper quoted above the κ-supersymmetry projector presented here was originally introduced within a 2nd order formulation of the theory. It is particularly significant and rewarding that the same projector is valid also in first order formulation. As shown in the appendix the mechanism by means of which it works are very subtle and take advantage of the explicit solutions for the auxiliary fields in terms of the physical ones. In this way one finds an overall non-trivial check of all the algebraic machinery of our new first order formalism.

and

$$\begin{aligned} \widehat{F}\,\widetilde{\widehat{F}} &= -\frac{1}{8}\big(F_{ij}F_{kl}\varepsilon^{ijkl}\big)\mathbf{1} = -\omega_{[0]}\mathbf{1} \\ \widehat{F}^2 + \widetilde{\widehat{F}}^2 &= \frac{1}{2}\,\mathrm{Tr}\big(F^2\big)\mathbf{1} \end{aligned} \tag{7.5.31}$$

7.6 The $D3$-Brane: Summary

In the previous sections we have explained how to construct p-brane world volume actions that allow to reproduce the Born-Infeld second order action via the elimination of a set composed by three auxiliary fields:

- $\Pi^{\underline{a}}_i$,
- h^{ij} (symmetric),
- $\mathscr{F}^{ij}$ (antisymmetric).

Distinctive properties of this formulation are:

1. All fermion fields are implicitly hidden inside the definition of the p-form potentials of supergravity.
2. κ-supersymmetry is easily proven from supergravity rheonomic parameterization.
3. The action is manifestly covariant with respect to the duality group $\mathrm{SL}(2,\mathbb{R})$ of type IIB supergravity.
4. The action functional can be computed on any background which is an exact solution of the supergravity bulk equations.

Of specific interest in applications are precisely the last two properties. Putting together all the partial results we can summarize the $D3$ brane action as it follows:

$$\begin{aligned} \mathscr{L} &= \Pi^{\underline{a}}_i V^{\underline{b}} \eta_{\underline{ab}} \eta^{i\ell_1} \wedge e^{\ell_2} \wedge \cdots \wedge e^{\ell_4} \varepsilon_{\ell_1\ldots\ell_4} \\ &\quad - \frac{1}{8}\Pi^{\underline{a}}_i \Pi^{\underline{b}}_j \eta_{\underline{ab}} h^{ij} e^{\ell_1} \wedge \cdots \wedge e^{\ell_4} \varepsilon_{\ell_1\ldots\ell_4} \\ &\quad - \frac{1}{4}\Big[\det\big(h^{-1} + \sqrt{\mathrm{Im}\,\mathscr{N}}\mathscr{F}\big)\Big]^{1/2} e^{\ell_1} \wedge \cdots \wedge e^{\ell_4} \varepsilon_{\ell_1\ldots\ell_4} \\ &\quad + \frac{3}{4}\mathscr{F}^{ij}\mathbf{F}^{[2]} \wedge e^{\ell_3} \wedge e^{\ell_4} \varepsilon_{ij\ell_3\ell_4} \\ &\quad + \frac{3}{4}\,\mathrm{Re}\,\mathscr{N}\mathbf{F}^{[2]} \wedge \mathbf{F}^{[2]} + \mathrm{i}\frac{3}{4} q^\alpha \varepsilon_{\alpha\beta} \mathbf{A}^\beta \wedge \mathbf{F}^{[2]} + 6\mathbf{C}^{[4]} \\ \mathbf{F}^{[2]} &\equiv d\mathbf{A}^{[1]} + q_\alpha \mathbf{A}^\alpha \\ \mathscr{N} &= \mathrm{i}\frac{\Lambda^1_- - \Lambda^2_-}{\Lambda^1_- + \Lambda^2_-} \end{aligned} \tag{7.6.1}$$

7.7 Supergravity p-Branes as Classical Solitons: General Aspects

We turn next to consider p-branes as classical solitonic solutions of supergravity. Such solutions have the following form:

$$\begin{cases} A^{[D]}_{p\text{-}brane} = \int d^D x \sqrt{-g}[2R[g] + \frac{1}{2}\partial^\mu\phi\partial_\mu\phi + \frac{(-1)^{p+1}}{2(p+2)!} e^{-a\phi}|\mathbf{F}^{[p+2]}|^2] & \text{elec.} \\ A^{[D]}_{\tilde{p}\text{-}brane} = \int d^D x \sqrt{-g}[2R[g] + \frac{1}{2}\partial^\mu\phi\partial_\mu\phi & \\ \qquad + \frac{(-1)^{D-\tilde{p}-3}}{2(D-\tilde{p}-2)!} e^{-a\phi}|\mathbf{F}^{[D-\tilde{p}-2]}|^2] & \text{magn.} \end{cases} \tag{7.7.1}$$

where in both cases $\mathbf{F}^{[n]} \equiv d\mathbf{A}^{[n-1]}$ is the field strength of an $(n-1)$-form gauge potential and a is some real number whose profound meaning will become clear in the later discussion of the solutions. As the reader can notice the two formulae we have written for the p-brane action are actually the same formula since $A^{[D]}_{p\text{-}brane}$ and $A^{[D]}_{\tilde{p}\text{-}brane}$ are mapped into each other by the replacement:

$$\tilde{p} = D - 4 - p; \qquad p = D - 4 - \tilde{p} \tag{7.7.2}$$

The reason why we doubled our writing is that the essentially unique action (7.7.1) admits two classical solutions each of which is interpreted as describing a p-extended and a $\tilde{p}$-extended object respectively. The first solution is driven by an electric $\mathbf{F}^{[p+2]}$ form, while the second is driven by a magnetic $\mathbf{F}^{[D-p+2]}$ form. The role of electric and magnetic solutions of the action $\mathbf{A}^{[D]}_{p\text{-}brane}$ are interchanged as solutions of the dual action $\mathbf{A}^{[D]}_{\tilde{p}\text{-}brane}$ For various values of

$$n = p + 2 \quad \text{and} \quad a \tag{7.7.3}$$

the functional $\mathbf{A}^{[D]}_{p\text{-}brane}$ (or its dual) corresponds to a consistent truncation of some supergravity bosonic action S^{SUGRA}_D in dimension D. This is the reason why the classical configurations we are going to describe are generically named supergravity p-branes. Given that supergravity is the low energy limit of superstring theory, supergravity p-branes are also solutions of superstring theory. They can be approximate or exact solutions, depending whether they do or do not receive quantum corrections. The second case is clearly the most interesting one and occurs, in particular, when the supergravity p-brane is a BPS-state that preserves some amount of supersymmetry. This implies that it is part of a short supersymmetry multiplet and for this reason cannot be renormalized. By consistent truncation we mean that a subset of the bosonic fields have been put equal to zero but in such a way that all solutions of the truncated action are also solutions of the complete one. For instance if we choose:

$$a = 1; \qquad n = \begin{cases} 3 \\ 7 \end{cases} \tag{7.7.4}$$

(7.7.1) corresponds to the bosonic low energy action of $D=10$ heterotic superstring ($\mathcal{N}=1$, supergravity) where the $E_8 \times E_8$ gauge fields have been deleted. The two choices 3 or 7 in (7.7.4) correspond to the two formulations (electric/magnetic) of the theory. Other choices correspond to truncations of the type IIA or type IIB action in the various intermediate dimensions $4 \le D \le 10$. Since the $(n-1)$-form $\mathbf{A}^{[n-1]}$ couples to the world volume of an extended object of dimension:

$$p = n - 2 \tag{7.7.5}$$

namely a p-brane, the choice of the truncated action (7.7.1) is precisely motivated by the search for p-brane solutions of supergravity. According with the interpretation (7.7.5) we set:

$$n = p+2; \qquad d = p+1; \qquad \tilde{d} = D - p - 3 \tag{7.7.6}$$

where d is the world-volume dimension of an electrically charged *elementary* p-brane solution, while $\tilde{d}$ is the world-volume dimension of a magnetically charged *solitonic* $\tilde{p}$-brane with $\tilde{p} = D - p - 4$. The distinction between elementary and solitonic is the following. In the elementary case the field configuration we shall discuss is a true vacuum solution of the field equations following from the action (7.7.1) everywhere in D-dimensional space-time except for a singular locus of dimension d. This locus can be interpreted as the location of an elementary p-brane source that is coupled to supergravity via an electric charge spread over its own world volume. In the solitonic case, the field configuration we shall consider is instead a bona-fide solution of the supergravity field equations everywhere in space-time without the need to postulate external elementary sources. The field energy is however concentrated around a locus of dimension $\tilde{p}$. These solutions have been derived and discussed thoroughly in the literature [41]. Good reviews of such results are [41, 42]. Defining:

$$\Delta = a^2 + 2\frac{d\tilde{d}}{D-2} \tag{7.7.7}$$

it was shown in [41] that the action (7.7.1) admits the following elementary p-brane solution

$$\begin{aligned}
ds^2 &= H(y)^{-\frac{4\tilde{d}}{\Delta(D-2)}}\, dx^\mu \otimes dx^\nu\, \eta_{\mu\nu} - H(y)^{\frac{4d}{\Delta(D-2)}}\, dy^m \otimes dy^n\, \delta_{mn} \\
\mathbf{F}^{[p+2]} &= \frac{2}{\sqrt{\Delta}}(-)^{p+1}\varepsilon_{\mu_1\ldots\mu_{p+1}}\, dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_{p+1}} \wedge d\big[H(y)^{-1}\big] \\
e^{\phi(r)} &= H(y)^{-\frac{2a}{\Delta}}
\end{aligned} \tag{7.7.8}$$

where the coordinates X^M $(M = 0, 1\ldots, D-1)$ have been split into two subsets:

- x^μ, $(\mu = 0, \ldots, p)$ are the coordinates on the p-brane world-volume,
- y^m, $(m = D-d+1, \ldots, D)$ are the coordinates transverse to the brane

and

$$H(y) = \left(1 + \frac{k}{r^{\tilde{d}}}\right) \tag{7.7.9}$$

is a harmonic function $\frac{\partial}{\partial y^m}\frac{\partial}{\partial y^m}H(y) = 0$ in the transverse space to the brane-world volume. By $r \equiv \sqrt{y^m y_m}$ we have denoted the radial distance from the brane and by k the value of its electric charge. The same authors of [41] show that the action (7.7.1) admits also the following solitonic $\tilde{p}$-brane solution:

$$\begin{aligned}
ds^2 &= H(y)^{-\frac{4d}{\Delta(D-2)}}\, dx^\mu \otimes dx^\nu\, \eta_{\mu\nu} - H(y)^{\frac{4\tilde{d}}{\Delta(D-2)}}\, dy^m \otimes dy^n\, \delta_{mn} \\
\widetilde{F}^{[D-p-2]} &= \lambda \varepsilon_{\mu_1\ldots\mu_{\tilde{d}}p}\, dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_{\tilde{d}}} \frac{y^p}{r^{d+2}} \\
e^{\phi(r)} &= H(y)^{\frac{2a}{\Delta}}
\end{aligned} \tag{7.7.10}$$

where the $(D-p-2)$-form $\widetilde{F}^{[D-p-2]}$ is the dual of $\mathbf{F}^{[p+2]}$, k is now the magnetic charge and:

$$\lambda = -2\frac{dk}{\sqrt{\Delta}} \tag{7.7.11}$$

The identification (7.7.11) of the constant λ allows to write the expression of the form $\widetilde{F}^{[D-p-2]}$ in the solitonic solution in the following more compact and inspiring way:

$$\widetilde{F}^{[D-p-2]} = \frac{2}{\sqrt{\Delta}} \star dH(y) \tag{7.7.12}$$

These p-brane configurations are solutions of the second order field equations obtained by varying the action (7.7.1). However, when (7.7.1) is the truncation of a supergravity action it generically happens that both (7.7.8) and (7.7.10) are also the solutions of a *first order differential system of equations* ensuring that they are BPS-extremal p-branes preserving a fraction of the original supersymmetries. The parameter (7.7.7) plays a particularly important role as an intrinsic characterization of the brane solutions since it has the very important property of being invariant under toroidal compactifications. When we step down in dimensions compactifying on a T^x torus each p-brane solution of the D-dimensional supergravity ends up in a p' brane of the $D-x$ supergravity that has the same value of Δ its parent brane had in higher dimension. It also happens that all elementary BPS branes of string or M-theory as the various Dp-branes of the type II A or type II B theory, the M2 and M5 branes, the Neveu Schwarz 5-brane and the elementary type II or heterotic strings are characterized by the property that $\Delta = 4$. Namely we have:

$$\Delta = 4 \quad \Leftrightarrow \quad \text{elementary } p\text{-brane in } D = 10 \text{ or toroidal reduction thereof} \tag{7.7.13}$$

7.8 The Near Brane Geometry, the Dual Frame and the AdS/CFT Correspondence

For the string theory community, the most exciting new development in the last decade of the XXth century was the discovery of the AdS/CFT correspondence [44–46], between the superconformal quantum field theory describing the microscopic degrees of freedom of certain p-branes and classical supergravity compactified on $\mathrm{AdS}_{p+2} \times X^{D-p-2}$. The origin of this correspondence is two-fold. On one side we have the algebraic truth that the AdS_{p+2} isometry group, namely $\mathrm{SO}(2, p+1)$ is also the conformal group in $p+1$ dimensions and, as firstly noticed by the authors of [45, 46], this extends also to the corresponding supersymmetric extensions appropriate to the field theories living on the relevant brane volumes. On the other hand we have the special behavior of those p-branes that are characterized by the conditions:

$$\Delta = 4; \qquad a = 0 \quad \Rightarrow \quad \frac{d\tilde{d}}{D-2} = 2 \tag{7.8.1}$$

In this case the p-brane metric takes the form:

$$ds^2 = \left[H(r)\right]^{-\frac{\tilde{d}}{D-2}} dx^\mu \otimes dx^\nu \eta_{\mu\nu} + \left[H(r)\right]^{\frac{d}{D-2}} \left(dr^2 + r^2 ds^2_{S^{D-p-2}}\right) \tag{7.8.2}$$

where the flat metric $d^m \otimes dy^m$ in the $D-p-1$ dimensions has been written in polar coordinates using the metric $ds^2_{\mathrm{S}^{D-p-2}}$ on an S^{D-p-2} sphere and where the harmonic function is

$$H(r) = \left(1 + \frac{k}{r^{\tilde{d}}}\right) \tag{7.8.3}$$

For large $r \to \infty$ the metric (7.8.2) is asymptotically flat, but for small values of the radial distance from the brane $r \mapsto 0$ the metric becomes a direct product metric:

$$ds^2 \quad \overset{r\to 0}{\Longrightarrow} \quad ds^2_H = \underbrace{(k)^{-\frac{\tilde{d}}{D-2}} r^{\frac{\tilde{d}^2}{D-2}} dx^\mu \otimes dx^\nu \, \eta_{\mu\nu} + (k)^{\frac{d}{D-2}} \frac{dr^2}{r^2}}_{\mathrm{AdS}_{p+2} \text{ metric}} + (k)^{\frac{d}{D-2}} ds^2_{S^{D-p-2}} \tag{7.8.4}$$

We will see shortly from now why the underbraced metric is indeed that of an anti de Sitter space. To this effect it suffices to set:

$$r = (k)^{\tilde{d}/2(D-2)} \exp\left[-(k)^{-d/2(D-2)} \overline{r}\right] \tag{7.8.5}$$

and in the new variable $\overline{r}$ the underbraced metric of (7.8.4) becomes identical to the metric (7.9.14) with

$$\lambda = (k)^{-d/2(D-2)} \frac{\tilde{d}^2}{2(D-2)} \tag{7.8.6}$$

The metric (7.9.14) is indeed the AdS metric in horospherical coordinates. Hence the near brane geometry of the special p-branes satisfying condition (7.8.1) is

$AdS_{p+2} \times S^{D-p-2}$ and this is the very origin of the AdS/CFT correspondence. As it was shown in [43] this mechanism can be extended to the case where the sphere metric is replaced by the metric of other coset manifolds G/H of the same dimensions $D-p-2$ or even more generically by the metric of some Einstein space X^{D-p-2}. This leads to the study of many more non-trivial examples of AdS/CFT correspondence, typically characterized by a reduced non-maximal supersymmetry. [47–55]. The wealth of results obtained in this field is impressive but its review goes much beyond the scope of the present book and we refer the interested reader to the original literature. We just stress that by this token the calculation of exact correlators of certain quantum field theories is reduced to calculations in a classical gravitational theory like supergravity.

7.9 Domain Walls in Diverse Space-Time Dimensions

The generic coupling of a single scalar field to Einstein gravity is described, in space-time dimensions D by the following action

$$A^{[D]}_{grav+scal} = \int d^D x \sqrt{-g}\left[2R[g] + \frac{1}{2}\partial^\mu \phi \partial_\mu \phi - \mathscr{V}(\phi)\right] \tag{7.9.1}$$

where $\mathscr{V}(\phi)$ is the scalar potential. If for this latter we choose the very particular form:

$$\mathscr{V}(\phi) = 2\Lambda e^{-a\phi}; \qquad \begin{cases} 0 < \Lambda \in \mathbb{R} \\ a \in \mathbb{R} \end{cases} \tag{7.9.2}$$

then we have a limiting case of the general p-brane action (7.7.1) we have considered above. Indeed if in the general formulae (7.7.6) we put

$$p = D-2 \quad \Rightarrow \quad \tilde{d} = -1; \qquad d = D-1 \tag{7.9.3}$$

we obtain that the *electric* $(D-2)$-brane couples to a field strength which is a top D-form $\mathbf{F}^{[D]}$, while the *magnetic* solitonic brane couples to a 0-form $\mathbf{F}^{[0]}$, namely to a cosmological constant. Indeed, we can formally set:

$$\mathbf{F}^{[0]} = 2\sqrt{\Lambda} \quad \Rightarrow \quad \widetilde{F}^{[D]} = \text{Volume form on space-time} \tag{7.9.4}$$

and the action (7.9.1) with the potential (7.9.2) is reduced to the general form for an electric $(D-2)$-brane (7.7.1). That $\mathbf{F}^{[0]}$ should be constant and hence could be identified as in (7.9.4) follows from the Bianchi identity that it is supposed to satisfy $d\mathbf{F}^{[0]} = 0$.

Hence we can conclude that the action:

$$A^{[D]}_{D\text{-}Wall} = \int d^D x \sqrt{-g}\left[2R[g] + \frac{1}{2}\partial^\mu \phi \partial_\mu \phi - 2\Lambda e^{-a\phi}\right] \tag{7.9.5}$$

admits a distinguished class of solutions describing $(D-2)$-branes that we name *domain walls* since at each instant of time a brane of this type separates the space manifold into two adjacent non-overlapping regions.

Specializing the general formulae (7.7.8) and (7.7.9) to our particular case we obtain the domain wall solution of (7.9.5) in the following form:

$$ds^2_{DW} = H(y)^{2\alpha}\big(dx^\mu \otimes dx^\nu \eta_{\mu\nu}\big) + H(y)^{2\beta}\, dy^2 \tag{7.9.6}$$

$$e^\phi = H(y)^{-\frac{2a}{\Delta}} \tag{7.9.7}$$

$$H(y) = c \pm Qy \tag{7.9.8}$$

where y is the single coordinate transverse to the wall, c is an arbitrary integration constant and the other parameters appearing in the above formulae have the following values:

$$\alpha = \frac{2}{\Delta(D-2)}; \qquad \beta = 2\frac{D-1}{\Delta(D-2)}; \qquad Q = \sqrt{\Lambda\Delta} \tag{7.9.9}$$

in terms of Δ whose expression (7.7.7) becomes:

$$\Delta = a^2 - 2\frac{D-1}{D-2} \tag{7.9.10}$$

The form (7.9.8) of the function H is easy to understand because in one-dimension a harmonic function is just a linear function. The arbitrariness of the sign in H arises because the equations of motion involve m only quadratically [56]. Since a^2 is a positive quantity, Δ is bounded from below by the special value $\Delta_A dS$ that corresponds to the very simple case of pure gravity with a negative cosmological constant (case $a = 0$ in (7.9.5)):

$$\Delta \geq \Delta_{\mathrm{AdS}} \equiv -2\frac{D-1}{D-2} \tag{7.9.11}$$

The name given to Δ_{AdS} has an obvious explanation. As it was originally shown by Lü, Pope and Townsend in [56], for $a = 0$ the domain wall solution (7.9.6) describes a region of the anti de Sitter space AdS_D. To verify this statement it suffices to insert the value (7.9.11) into (7.9.9) and (7.9.6) to obtain:

$$ds^2_{DW} = H^{-2/(D-1)}(y)\big(dx^\mu \otimes dx^\nu \eta_{\mu\nu}\big) + H(y)^{-2}\, dy^2 \tag{7.9.12}$$

Performing the coordinate transformation:

$$r = \frac{1}{Q}\ln(c \pm Qy) \tag{7.9.13}$$

the metric becomes:

$$ds^2_{DW} = e^{-2\lambda r}\eta_{\mu\nu}\, dx^\mu dx^\nu + dr^2 \tag{7.9.14}$$

where

$$\lambda = \sqrt{\frac{2\Lambda}{(D-1)(D-2)}} = (D-1)Q \tag{7.9.15}$$

In the same coordinates the solution for the dilaton field is:

$$e^{\phi} = \exp\left[-\frac{2a\lambda}{\Delta(D-1)}r\right] \tag{7.9.16}$$

Equation (7.9.14) is the metric of AdS spacetime, in horospherical coordinates. Following [56] we can verify this statement by introducing the $(D+1)$ coordinates (X, Y, Z^{μ}) defined by

$$\begin{aligned} X &= \frac{1}{\lambda}\cosh\lambda r + \frac{1}{2}\lambda\eta_{\mu\nu}x^{\mu}x^{\nu}e^{-\lambda r} \\ Y &= -\frac{1}{\lambda}\sinh\lambda r - \frac{1}{2}\lambda\eta_{\mu\nu}x^{\mu}x^{\nu}e^{-\lambda r} \\ Z^{\mu} &= x^{\mu}e^{-\lambda r} \end{aligned} \tag{7.9.17}$$

They satisfy

$$\eta_{\mu\nu}Z^{\mu}Z^{\nu} + Y^2 - X^2 = -1/\lambda^2 \tag{7.9.18}$$

$$\eta_{\mu\nu}\,dZ^{\mu}\,dZ^{\nu} + dY^2 - dX^2 = e^{-2\lambda r}\eta_{\mu\nu}\,dx^{\mu}\,dx^{\nu} + dr^2 \tag{7.9.19}$$

which shows that (7.9.14) is the induced metric on the algebraic locus (7.9.18) which is the standard hyperboloid corresponding to the *AdS* space-time manifold. The signature of embedding flat space is $(-,+,+,\ldots,+,-)$ and therefore the metric (7.9.14) has the right $\mathrm{SO}(2, D-1)$ isometry of the AdS_D metric.

Still following the discussion in [56] we note that in horospherical coordinates $X+Y = \lambda^{-1}e^{-\lambda r}$ is non-negative if r is real. Hence the region $X+Y<0$ of the full AdS spacetime is not accessible in horospherical coordinates. Indeed this coordinate patch covers one half of the complete AdS space, and the metric describes $\mathrm{AdS}_D/\mathbb{Z}_2$ where $\mathbb{Z}_2$ is the antipodal involution $(X, Y, Z^{\mu}) \to (-X, -Y, -Z^{\mu})$. If D is even, we can extend the metric (7.9.12) to cover the whole anti de Sitter spacetime by setting the integration constant $c=0$ which implies $H = Qy$. So doing the region with $y<0$ corresponds to the previously inaccessible region $X+Y<0$. In odd dimensions, we must restrict H in (7.9.12) to be non-negative in order to have a real metric and thus in this case we have to choose $H = c + Q|y|$, with $c \geq 0$. If the constant c is zero, the metric describes $\mathrm{AdS}_D/\mathbb{Z}_2$, while if c is positive, the metric describes a smaller portion of the complete AdS spacetime. In any dimension, if we set:

$$H = c + Q|y| \tag{7.9.20}$$

the solution can be interpreted as a domain wall at $y=0$ that separates two regions of the anti de Sitter spacetime, with a delta function curvature singularity at $y=0$ if the constant c is positive.

7.9.1 The Randall Sundrum Mechanism

What we have just described is the *anti de Sitter domain wall* that corresponds to $\Delta = \Delta_{\mathrm{AdS}}$. The magic of this solution is that, as shown by Randall and Sundrum in [61, 62], it leads to the challenging phenomenon of *gravity trapping*. These authors have found that because of the exponentially rapid decrease of the factor

$$\exp\bigl[-\lambda|r|\bigr] \quad \text{with } \lambda > 0 \tag{7.9.21}$$

away from the thin domain wall that separates the two asymptotic anti de Sitter regions it happens that gravity in a certain sense is localized near the brane wall. Instead of the D-dimensional Newton's law that gives:

$$\text{force} \sim \frac{1}{R^{D-2}} \tag{7.9.22}$$

one finds the $(D-1)$-dimensional Newton's law

$$\text{force} \sim \frac{1}{R^{D-3}} + \text{small corrections } \mathscr{O}\left(\frac{1}{R^{D-2}}\right) \tag{7.9.23}$$

This can be seen by linearizing the Einstein equations for the metric fluctuations around any domain wall background of the form:

$$ds^2 = W(r)\eta_{\mu\nu}\, dx^\mu\, dx^\nu + dr^2 \tag{7.9.24}$$

that includes in particular the AdS case (7.9.14). In a very sketchy way if one sets:

$$h_{\mu\nu}(x, y) = \exp[\mathrm{i}p \cdot x]\psi_{\mu\nu}(y) \tag{7.9.25}$$

one finds that the linearized Einstein equations translate into an analog Schrödinger equation for the wave-function $\psi(y)$. This problem has a potential that is determined by the warp factor $W(y)$. If in the spectrum of this quantum mechanical problem there is a normalizable zero mode then this is the wave function of a $D-1$ dimensional graviton. This state is indeed a bound state and falls off rapidly when leaving the brane. Since the extra dimension is non-compact the Kaluza Klein states form a continuous spectrum without a gap. Yet $D-1$ dimensional physics is extremely well approximated because the bound state mode reproduces conventional gravity in $D-1$ dimensions while the massive states simply contribute a small correction.

It is clearly of utmost interest to establish which domain walls have this magic trapping property besides the anti de Sitter one. This has been recently done by Cvetič, Lü and Pope in [60] In order to summarize this and other related results we need first to emphasize another aspect of domain walls that puts them into distinguished special class among p-branes.

7.9.2 The Conformal Gauge for Domain Walls

Going back to the general domain wall solution (7.9.6), (7.9.7), (7.9.8), (7.9.9), classified by the value of Δ (7.9.10) we observe that there is still an ambiguity in the powers of the harmonic function (7.9.8) that appear as metric coefficients. This ambiguity is due to coordinate transformations and it is a specific property of $(D-2)$-branes not present in other p-branes, where the harmonic function H is not a linear function. Following a discussion by Bergshoeff and van der Schaar [57] we observe that in the range $y > 0$ we can make the following linear transformation: $y = -\frac{c}{Q} + y' \Rightarrow H(y) = Qy'$ that eliminates the integration constant c. Furthermore we can redefine y' as some other fractional power of a third coordinate $\overline{y}$, namely $y' = -Q^{-\frac{1+\varepsilon}{\varepsilon}}\overline{y}^{-\frac{1}{\varepsilon}}$, then shifting it once again by a constant $\overline{y} = z + \frac{c}{Q}$. Altogether this means that we introduce the coordinate transformation:

$$y = -\frac{c}{Q} - Q^{-\frac{1+\varepsilon}{\varepsilon}}\left(z + \frac{c}{Q}\right)^{-\frac{1}{\varepsilon}} \tag{7.9.26}$$

Under this transformation we have (for positive y):

$$H(y) = -\left[H(z)\right]^{-1/\varepsilon} \tag{7.9.27}$$

and the domain wall metric (7.9.6) becomes:

$$ds^2_{DW} = H(z)^{-\frac{2\alpha}{\varepsilon}}\left(dx^\mu \otimes dx^\nu \eta_{\mu\nu}\right) + H(z)^{-\frac{2\beta+\varepsilon}{\varepsilon}-2}\frac{dz^2}{\varepsilon^2} \tag{7.9.28}$$

This transformation allows for the remarkable possibility of choosing a conformal gauge, namely a coordinate system where it becomes manifest that the domain wall metric is conformally flat. Indeed it suffices to impose that the two powers of the harmonic function appearing in (7.9.28) be equal:

$$-\frac{2\alpha}{\varepsilon} = -\frac{2\beta+\varepsilon}{\varepsilon} - 2 \tag{7.9.29}$$

Using (7.9.9) the solution of (7.9.29) for ε is unique in all cases with the exception of $\Delta = -2$:

$$\varepsilon = -\frac{\Delta+2}{\Delta} \tag{7.9.30}$$

Hence for $\Delta \neq -2$, redefining $z \mapsto \varepsilon z$, $Q \mapsto k|\varepsilon|$ the domain wall solution (7.9.6) can always be rewritten in the following conformally flat way:

$$\begin{aligned} ds^2_{DW/conf} &= \left[H(z)\right]^{\frac{4}{(D-2)(\Delta+2)}}\left(\eta_{\mu\nu}\, dx^\mu \otimes dx^\nu + dz^2\right) \\ e^{\phi(z)} &= H(z)^{-\frac{2a}{\Delta+2}} \\ H(z) &= 1 + k|z| \\ k &= (\Delta+2)\sqrt{\frac{\Lambda}{\Delta+2}} \end{aligned} \tag{7.9.31}$$

Obviously the solution (7.9.31) could have been obtained by directly solving the Einstein equations associated with the action (7.9.5) starting from a conformal ansatz of the type:

$$ds^2_{DW/conf} = \exp\big[A(z)\big]\big(\eta_{\mu\nu}\,dx^\mu\,dx^\nu + dz^2\big) \tag{7.9.32}$$

Yet we preferred to obtain it from the general solution (7.7.8) for supergravity p-branes in order to emphasize its interpretation as a domain wall, namely a $(D-2)$-brane. The direct method of solution can be used to find the conformal representation of the domain wall metric in the exceptional case $\Delta = -2$. As shown in [60] one obtains:

$$\begin{aligned} ds^2 &= e^{-\frac{2k}{d-2}|z|}\big(\eta_{\mu\nu}\,dx^\mu\,dx^\nu + dz^2\big) \\ \phi &= \frac{\sqrt{2}k}{\sqrt{d-2}}|z| \end{aligned} \tag{7.9.33}$$

where k is now given by

$$k^2 = -2\Lambda(d-2) \tag{7.9.34}$$

which is real for negative Λ. There is another important point that we should note. Our starting point, prior to all the subsequent manipulations, has been the form (7.9.6), (7.9.9) which is that of an electric p-brane and not that of a solitonic one (see 7.7.10)). This implies that our domain wall solutions are not exactly bona fide solutions of the action (7.9.5) but require also the coupling to a source term that is the world-volume action of the domain wall, localized at $z = 0$ in the last coordinate frame we have used. Namely the true action is

$$A = \int_{M_D} d^D x\,\sqrt{-g}\Big[2R[g] + \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - 2\Lambda e^{-a\phi}\Big] + \mathscr{T}\int_{WV_{D-1}} d^{D-1}\xi\;\mathscr{L}_{source} \tag{7.9.35}$$

where $\mathscr{L}_{source}$ is world-volume Lagrangian of the $(D-2)$-brane and the parameter $\mathscr{T}$ denotes its tension. An important issue is to relate the wall-tension to the parameters appearing in the classical domain wall solution. This was done in [60] following a standard analysis developed in previous papers [58, 59]. The matching conditions across the singular domain wall source imply that the energy density (tension) of the wall is related to the values of the cosmological constant parameters on either side of the wall, namely the authors of [60] found:

$$\sigma = \mathscr{T} = 2\big(A'_{z=0^-} - A'_{z=0^+}\big) \tag{7.9.36}$$

where the prime denotes a derivative with respect to z. This leads to

$$\begin{aligned} \Delta \neq -2: &\quad \mathscr{T} = -8\,\mathrm{sign}\big[k(\Delta+2)\big]\sqrt{\frac{\Lambda}{\Delta}} \\ \Delta = -2: &\quad \mathscr{T} = \frac{8k}{d-2} \end{aligned} \tag{7.9.37}$$

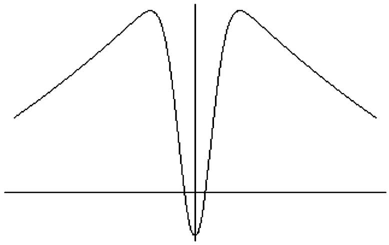

Fig. 7.3 The volcano potential

Thus positive-tension domain-wall solutions exist for $\Delta \leq -2$ with $k > 0$ and for $\Delta > -2$ with $k < 0$. Conversely, negative-tension domain walls arise for $\Delta \leq -2$ with $k < 0$ and for $\Delta > -2$ with $k > 0$. So for our domain walls with $\Delta \leq -2$, we assume the lower bound (7.9.11). To avoid *naked singularities* we also need $k > 0$.

Using the simple conformal gauge (7.9.31) the authors of [60] have analyzed the fluctuations of the metric around such a background and have found that the graviton wave function obeys, as predicted by Randall-Sundrum [61–63] a Schrödinger equation with a potential that is completely fixed by the value of Δ. More precisely one finds that in the conformal gauge the fluctuations of the D-dimensional graviton satisfy the Klein-Gordon equation of a scalar field in the gravitational background namely $\partial_M(\sqrt{-g}g^{MN}\partial_N \Phi) = 0$. Parameterizing:

$$\Phi = \phi(z)e^{\mathrm{i}p\cdot x} = e^{-kz}\psi(z)e^{\mathrm{i}p\cdot x} \tag{7.9.38}$$

where p is the $(D-1)$-dimensional momentum the Klein-Gordon equation becomes the following Schrödinger-type equation,

$$-\frac{1}{2}\psi'' + U\psi = -\frac{1}{2}p^2\psi \tag{7.9.39}$$

where the potential, calculated in [60] is given by

$$\begin{aligned} \Delta \neq -2: &\quad U = -\frac{(\Delta+1)k^2}{2(\Delta+2)^2 H(z)^2} + \frac{k}{\Delta+2}\delta(z) \\ \Delta = -2: &\quad U = \frac{1}{8}k^2 - \frac{1}{2}k\delta(z) \end{aligned} \tag{7.9.40}$$

Such an equation has a normalizable zero-mode wave function if the following condition is satisfied $\Delta \leq -2$. Indeed it is evident from these expressions that for $\Delta \leq -2$, U has a volcano shape as in Fig. 7.3 since the delta function has a negative coefficient, and the "bulk" term is non-negative for all z. Hence the trapping of gravity occurs for positive tension $(D-2)$-branes in the following window:

$$\Delta_{\mathrm{AdS}} \leq \Delta \leq -2 \tag{7.9.41}$$

7.10 Conclusion on This Brane Bestiary

The snapshot survey of the p-brane bestiary we have presented in this chapter was meant to illustrate the wealth of classical solutions of supergravity and their profound interpretation in connection with gauge-theories and many other aspects of quantum field theory. The main message we would like to convey to the student is that supergravity is just a natural extension of General Relativity (the main topic of this book) which stands on its feet independently from string theory. Furthermore the rich park of solutions and mechanisms contributed by supergravity requires attentive consideration and certainly is part of the general theory of gravity.

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Chapter 8
Supergravity: A Bestiary in Diverse Dimensions

Incipit liber de natura quorundam animalium, et lapidum
et quid significetur per eam
from a Medieval Latin Bestiary

8.1 Introduction

In the previous chapter we discussed p-branes as classical solutions of supergravity theories in diverse dimensions, while in Chap. 5 w appreciated the relevance of scalar fields in cosmology. From this viewpoint the basic information one would like to master is the following:

- The scalar field dependence of the kinetic terms of p-forms $\mathscr{N}_{\Lambda\Sigma}(\phi)F^{\Lambda} \wedge \star F^{\Sigma}$ since this latter eventually decides the values of the coefficients a in the exponential factors of the p-brane actions (7.7.1).
- The scalar field potential $\mathscr{V}(\phi)$ which eventually decides the form of the cosmological term in the domain wall actions (7.9.5) and plays a fundamental role in the inflationary scenarios.
- The metric $g_{ij}(\phi)$ appearing in the kinetic term $g_{ij}(\phi)\partial_\mu\phi^i\partial^\mu\phi^j$ of the scalar fields since it is needed as much as the matrix $\mathscr{N}_{\Lambda\Sigma}(\phi)$ to determine the values of a and eventually of Δ.

It turns out that each of the above items involves a wealth of surprisingly sophisticated geometric structures that are skillfully utilized by supergravity, first to stand on its feet at the ungauged level and, secondly, to be gauged producing non-Abelian symmetries and the scalar potential. In the present chapter we survey all these structures and we try to illustrate their meaning in relation with the parent string theory. Obviously the cause that imposes on the theory all such structures is supersymmetry and the presence of the fermions. Yet since the fermions are ugly objects to deal with while their yield, namely the geometric structure of the theory is beautiful, we will only stick to the latter and mention the fermions as seldom as possible. In Chap. 6 we illustrated the general principles that underlie the construction of supergravity theories and we hope that our reader got enough information to understand its logic. In the rest of the present chapter we confine ourselves to a mostly descriptive presentation. Nowhere we pretend to give the proof that the various supergravities are

P.G. Frè, *Gravity, a Geometrical Course*, DOI 10.1007/978-94-007-5443-0_8,

as they are, but we do our best to illustrate their miraculous geometric functioning that eventually governs the p-brane classical physics we are interested in. In view of the advocated correspondences such classical physics is also the quantum physics of the underlying world volume theories. Furthermore the wealth of special geometries involved by supergravity Lagrangians has a value, independently from its supersymmetric origin. The mechanisms unveiled by supergravity concerning dualities, black-hole solutions and the like have a more general range of applications beyond supersymmetric theories.

8.2 Supergravity and Homogeneous Scalar Manifolds G/H

If we consider the whole set of supergravity theories in diverse dimensions we discover an important general property. With the caveat of three noteworthy exceptions in all the other cases the constraints imposed by supersymmetry imply that the scalar manifold $\mathscr{M}_{scalar}$ is necessarily a *homogeneous coset manifold* G/H of the *non-compact type*, namely a suitable non-compact Lie group G modded by the action of a maximal compact subgroup $\mathrm{H} \subset \mathrm{G}$. By $\mathscr{M}_{scalar}$ we mean the manifold parameterized by the scalar fields ϕ^I present in the theory. The metric $g_{IJ}(\phi)$ defining the Riemannian structure of the scalar manifold appears in the supergravity Lagrangian through the scalar kinetic term which is of the σ-model type:

$$\mathscr{L}^{kin}_{scalar} = \frac{1}{2} g_{IJ}(\phi)\partial_\mu \phi^I \partial^\mu \phi^J \tag{8.2.1}$$

The three noteworthy exceptions where the scalar manifold is allowed to be something more general than a coset G/H are the following:

1. $\mathscr{N} = 1$ supergravity in $D = 4$ where $\mathscr{M}_{scalar}$ is simply requested to be a *Hodge Kähler manifold*.
2. $\mathscr{N} = 2$ supergravity in $D = 4$ where $\mathscr{M}_{scalar}$ is simply requested to be the product of a *special Kähler manifold* $\mathscr{SK}_n$[1] containing the n complex scalars of the n vector multiplets with a *quaternionic manifold* $\mathscr{QM}_m$ containing the $4m$ real scalars of the m hypermultiplets.[2]

[1]Special Kähler geometry was introduced in a coordinate dependent way in the first papers on the vector multiplet coupling to supergravity in the middle eighties [2, 14]. Then it was formulated in a coordinate-free way at the beginning of the nineties from a Calabi-Yau standpoint by Strominger [5] and from a supergravity standpoint by Castellani, D'Auria and Ferrara [3, 4]. The properties of holomorphic isometries of special Kähler manifolds, namely the geometric structures of special geometry involved in the gauging were clarified by D'Auria, Frè and Ferrara in [20]. For a review of special Kähler geometry in the setup and notations of the book see [21] and Sect. 8.5 of the present chapter.

[2]The notion of quaternionic geometry, as it enters the formulation of hypermultiplet coupling was introduced by Bagger and Witten in [15] and formalized by Galicki in [17] who also explored the relation with the notion of HyperKähler quotient, whose use in the construction of supersymmetric $\mathscr{N} = 2$ Lagrangians had already been emphasized in [16]. The general problem of clas-

3. $\mathscr{N}=2$ supergravity in $D=5$ where $\mathscr{M}_{scalar}$ is simply requested to be the product of a *very special manifold* $\mathscr{V}\mathscr{S}_n$[3] containing the n real scalars of the n vector multiplets with a *quaternionic manifold* $\mathscr{Q}\mathscr{M}_m$ containing the $4m$ real scalars of the m hypermultiplets.

We shall come back to the case of $\mathscr{N}=2$ supergravity in five dimensions because of its relevance in the quest of domain walls and supersymmetric realizations of the Randall Sundrum scenario and there we shall briefly discuss both very special geometry and quaternionic geometry. Special Kähler geometry and the structure of $\mathscr{N}=2$ supergravity in four dimensions will also be briefly reviewed. Probably the most relevant aspect of special Kähler manifolds is their interpretation as moduli spaces of Calabi-Yau three-folds which connects the structures of $\mathscr{N}=2$ supergravity to superstring theory via the algebraic geometry of compactifications on such three-folds. Here we do not address these topics and we rather focus on the case of homogeneous scalar manifolds which covers all the other types of supergravity Lagrangians and also specific instances of $\mathscr{N}=2$ theories since there exist subclasses of special Kähler and very special manifolds that are homogeneous spaces G/H.

By means of this choice we aim at illustrating some of the very ample collection of supergravity features that encode quite non-trivial aspects of superstring theory and that can be understood in terms of Lie algebra theory and differential geometry of homogeneous coset spaces.

8.2.1 *How to Determine the Scalar Cosets* $\mathrm{G/H}$ *of Supergravities from Supersymmetry*

The best starting point of our discussion is provided by presenting the table of coset structures in four-dimensional supergravities. This is done in the next subsection in Table 8.1 where supergravities are classified according to the number $\mathscr{N}$ of the preserved supersymmetries. Recalling that a Majorana spinor in $D=4$ has four real components the total number of supercharges preserved by each theory is

$$\#\text{ of supercharges} = 4\mathscr{N} \tag{8.2.2}$$

and becomes maximal for the $\mathscr{N}=8$ theory where it is 32.

Here we present a short general discussion that applies to all the diverse dimensions.

There are two ways to determine the scalar manifold structure of a supergravity theory:

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sifying quaternionic homogeneous spaces had been addressed in the mathematical literature by Alekseevski [6].

[3]The notion of very special geometry is essentially due to the work of Günaydin Sierra and Townsend who discovered it in their work on the coupling of $D=5$ supergravity to vector multiplets [9, 10]. The notion was subsequently refined and properly related to special Kähler geometry in four dimensions through the work by de Wit and Van Proeyen [11–13].

- By compactification from higher dimensions. In this case the scalar manifold is identified as the *moduli space* of the internal compact space.
- By direct construction of each supergravity theory in the chosen space-time dimension. In this case one uses all the *a priori* constraints provided by supersymmetry, namely the field content of the various multiplets, the global and local symmetries that the action must have and, most prominently, as we are going to explain, the *duality symmetries*.

The first method makes direct contact with important aspects of superstring theory but provides answers that are specific to the chosen compact internal space $\Omega_{10\text{-}D}$ and not fully general. The second method gives instead fully general answers. Obviously the specific answers obtained by compactification must fit into the general scheme provided by the second method. In the next section we highlight the basic arguments that lead to the construction of Table 8.1. Obviously the table relies on the fact that each of the listed Lagrangians has been explicitly constructed and shown to be supersymmetric[4] but it is quite instructive to see how the scalar manifold, which is the very *hard core* of the theory determining its interaction structure, can be predicted a priori with simple group theoretical arguments.

The first thing to clarify is this: what is classified in Table 8.1 are the *ungauged supergravity theories* where all vector fields are Abelian and the isometry group of the scalar manifold is a global symmetry. *Gauged supergravities* are constructed only in a second time starting from the ungauged ones and by means of a *gauging* procedure which goes beyond the scope of the present book. The interested reader is referred to [28]. We just remark that each ungauged supergravity admits a finite number of different gaugings where suitable subgroups of the isometry group of the scalar manifold are promoted to local symmetries using some or all of the available vector fields of the theory. It is clear that which gaugings are possible is once again determined by the structure of the scalar manifold plus additional constraints that we explain later.

In every space-time dimension D the reasoning that leads to single out the scalar coset manifolds G/H is based on the following elements:

(A) Knowledge of the field content of the various supermultiplets μ_i that constitute irreducible representations of the $\mathscr{N}$-extended supersymmetry algebra in D-dimensions. In particular this means that we know the total number of scalar fields. The scalars pertaining to the various types of multiplets must fill separate submanifolds $\mathscr{M}_i$ of the total scalar manifold which is the direct product of all such subspaces: $\mathscr{M}_{scalar} = \bigotimes_i \mathscr{M}_i$.

(B) Knowledge of the automorphism group $\mathrm{H}_{\mathrm{Aut}}$ of the relevant supersymmetry algebra. This latter acts on the gravitinos and on the other fermion fields as a local symmetry group. The gauge connection for this gauge symmetry is not elementary, rather it is a composite connection derived from the σ-model of the scalar fields:

[4] For a review of supergravity theories both in $D = 4$ and in diverse dimensions the reader is referred to the book [7]. Furthermore for a review of the geometric structure of all supergravity theories in a modern perspective we refer to [1].

$$\Omega_\mu^{\mathrm{Aut}} = \mathbb{P}_{\mathrm{Aut}}\left[g^{-1}(\phi)\frac{\partial}{\partial\phi^I}g(\phi)\right]\partial_\mu\phi^I \tag{8.2.3}$$

where $\mathbb{P}_{\mathrm{Aut}}$ denotes the projection onto the automorphism subalgebra $\mathrm{Aut} \subset \mathbb{H}$ of the isotropy algebra $\mathbb{H}$ of the scalar coset manifold G/H. This is consistent only if the isotropy group has the following direct product structure:

$$\mathrm{H} = \mathrm{Aut} \bigotimes \mathrm{H}' \tag{8.2.4}$$

H' being some other closed Lie group.

(C) Existence of appropriate irreducible representations of G in which we can accommodate each type of $(p+1)$-forms $A^{[p+1]\Lambda}$ appearing in the various supermultiplets. Indeed each $(p+1)$-form sits in some supermultiplet together with fermion fields and with scalars. The transformations of G commute with supersymmetry and must rotate an entire supermultiplet into another one of the same sort. Since the action of G is well defined on scalars we must be able to lift it also to the $(p+1)$-form partners of the scalars. Here we have a bifurcation:

- When the magnetic dual of the (p_i+1)-forms, that are $(D-p_i-3)$-forms have a different degree, namely $D-p_i-3 \neq p_i+1$, then the group G must have irreducible representations D_i of dimensions:

$$\dim(D_i) = n_i \tag{8.2.5}$$

  where n_i is the number of (p_i+1)-forms present in the theory
- When there are $(\overline{p}+1)$-forms, whose magnetic duals have the same degree, namely $D-\overline{p}-3 = \overline{p}+1$, then the group G must have, in addition to the irreducible representations D_i that accommodate the other (p_i+1)-forms as in (8.2.5) also a representation $\overline{D}$ of dimension

$$\dim(\overline{D}) = 2\overline{n} \tag{8.2.6}$$

  which accommodates the $\overline{n}$ forms of degree $\overline{p}$ and has the following additional property. In $D = 6, 10$ it is realized by pseudo-orthogonal matrices in the fundamental of $\mathrm{SO}(\overline{n}, \overline{n})$ while in $D = 4, 8$ it is realized by symplectic matrices in the fundamental of $\mathrm{Sp}(2\overline{n}, \mathrm{R})$. The reason for this apparently extravagant request is that in the case of $(\overline{p}+1)$-forms the lifting of the action of the group G is realized by means of electric/magnetic duality rotations as we explain in Sect. 8.3. Furthermore the reason why, in this discussion, we consider only the even dimensional cases is that self-dual $(\overline{p}+1)$-forms can exist only when $D = 2r$ is even.

8.2.2 The Scalar Cosets of $D = 4$ Supergravities

In four dimensions the only relevant $(p+1)$-forms are the 1-forms that correspond to ordinary gauge vector fields. Indeed 3-forms do not have degrees of freedom and 2-forms can be dualized to scalars. On the contrary $D = 4$ is an even number and 1-forms are self-dual in the sense described in Sect. 8.3 and alluded above in Sect. 8.2.1. Furthermore the automorphism group of the $\mathscr{N}$ extended supersymmetry algebra in $D = 4$ is:[5]

$$\begin{aligned} \mathrm{H}_{\mathrm{Aut}} &= \mathrm{SU}(\mathscr{N}) \times \mathrm{U}(1); \qquad && \mathscr{N} = 1,2,3,4,5,6 \\ \mathrm{H}_{\mathrm{Aut}} &= \mathrm{SU}(8); && \mathscr{N} = 7,8 \end{aligned} \tag{8.2.7}$$

Hence applying the strategy outlined in Sect. 8.2.1 the requests to be imposed on the coset G/H in four-dimensional supergravities are:

1. The total number of spin zero fields must be equal to the dimension of the coset:

$$\#\text{ of spin zero fields} \equiv m = \dim \mathrm{G} - \dim \mathrm{H} \tag{8.2.8}$$

2. The total number of vector fields in the theory $\overline{n}$ must be equal to one half the dimension of a symplectic irreducible representation D_{Sp} of the group G:

$$\#\text{ of spin 1 fields} \equiv \overline{n} = \frac{1}{2} \dim D_{Sp}(\mathrm{G}) \tag{8.2.9}$$

3. The isotropy group H must be of the form:[6]

$$\begin{aligned} \mathrm{H} &= \mathrm{SU}(\mathscr{N}) \times \mathrm{U}(1) \times \mathrm{H}'; \qquad && \mathscr{N} = 3,4 \\ \mathrm{H} &= \mathrm{SU}(\mathscr{N}) \times \mathrm{U}(1); && \mathscr{N} = 5,6 \\ \mathrm{H} &= \mathrm{SU}(8); && \mathscr{N} = 7,8 \end{aligned} \tag{8.2.10}$$

The distinction between the cases $\mathscr{N} = 3,4$ and the cases $\mathscr{N} = 5,6$ comes from the fact that in the former we have both the graviton multiplet plus vector multiplets, while in the latter there is only the graviton multiplet. The vector multiplets can transform non-trivially under the additional group H' for which there is no room in the latter cases. Finally the $\mathscr{N} = 7,8$ supergravities that contain only

---

[5] The role of the $\mathrm{SU}(\mathscr{N})$ symmetry in $\mathscr{N}$-extended supergravity was firstly emphasized in [26, 27].

[6] The difference between the $\mathscr{N} = 7,8$ cases and the others is properly explained in the following way. As far as superalgebras are concerned the automorphism group is always $\mathrm{U}(\mathscr{N})$ for all $\mathscr{N}$, which can extend, at this level also beyond $\mathscr{N} = 8$. Yet for the $\mathscr{N} = 8$ graviton multiplet, which is identical to the $\mathscr{N} = 7$ multiplet, it happens that the U(1) factor in U(8) has vanishing action on all physical states since the multiplet is self-conjugate under CPT-symmetries. From here it follows that the isotropy group of the scalar manifold must be SU(8) rather than U(8). A similar situation occurs for the $\mathscr{N} = 4$ vector multiplets that are also CPT self-conjugate. From this fact follows that the isotropy group of the scalar submanifold associated with the vector multiplet scalars is $\mathrm{SU}(4) \times \mathrm{H}'$ rather than $\mathrm{U}(4) \times \mathrm{H}'$. In $\mathscr{N} = 4$ supergravity, however, the U(1) factor of the automorphism group appears in the scalar manifold as isotropy group of the submanifold associated with the graviton multiplet scalars. This is so because the $\mathscr{N} = 4$ graviton multiplet is not CPT self conjugate.

Table 8.1 Scalar manifolds of extended supergravities in $D = 4$

$\mathscr{N}$	# scal. in scal. m.	# scal. in vec. m.	# scal. in grav. m.	# vect. in vec. m.	# vect. in grav. m.	Γ_{cont}	$\mathscr{M}_{scalar}$
1	2 m			n		$\mathscr{I} \subset \mathrm{Sp}(2n,\mathbb{R})$	Kähler
2	4 m	2 n		n	1	$\mathscr{I} \subset \mathrm{Sp}(2n+2,\mathbb{R})$	Quaternionic $\otimes$ Special Kähler
3		6 n		n	3	$\mathrm{SU}(3,n) \subset \mathrm{Sp}(2n+6,\mathbb{R})$	$\frac{\mathrm{SU}(3,n)}{\mathrm{S}(\mathrm{U}(3)\times\mathrm{U}(n))}$
4		6 n	2	n	6	$\mathrm{SU}(1,1)\otimes\mathrm{SO}(6,n) \subset \mathrm{Sp}(2n+12,\mathbb{R})$	$\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)} \otimes \frac{\mathrm{SO}(6,n)}{\mathrm{SO}(6)\times\mathrm{SO}(n)}$
5			10		10	$\mathrm{SU}(1,5) \subset \mathrm{Sp}(20,\mathbb{R})$	$\frac{\mathrm{SU}(1,5)}{\mathrm{S}(\mathrm{U}(1)\times\mathrm{U}(5))}$
6			30		16	$\mathrm{SO}^\star(12) \subset \mathrm{Sp}(32,\mathbb{R})$	$\frac{\mathrm{SO}^\star(12)}{\mathrm{U}(1)\times\mathrm{SU}(6)}$
7, 8			70		56	$\mathrm{E}_{7(-7)} \subset \mathrm{Sp}(128,\mathbb{R})$	$\frac{\mathrm{E}_{7(-7)}}{\mathrm{SU}(8)}$

the graviton multiplet are indistinguishable theories since their field content and interactions are the same.

Using the above rules and the known list of Lie groups one arrives at the unique solution provided in Table 8.1.

8.2.3 *Scalar Manifolds of Maximal Supergravities in Diverse Dimensions*

In Table 8.1 we have classified supergravities at fixed space-time dimension according to the number of supersymmetries. Another possible classification is according to space time dimensions D at fixed number of supercharges N_Q. In particular one can consider maximal supergravities where $N_Q = 32$ and discuss their structure in the diverse dimensions $3 \le D \le 10$. Such a study is very much rewarding since we can relate it to the alternative way of deriving the scalar manifold of supergravity, namely via compactification. There is indeed a class of hierarchical compactifications that have the distinguished property of preserving the number of supersymmetries at each step of the hierarchy. These are the toroidal compactifications where D-dimensional space-time $\mathscr{M}_D$ is replaced by:

$$\mathscr{M}_D \mapsto \mathscr{M}_{D-x} \times T^x \qquad (8.2.11)$$

T^x denoting an x-dimensional torus and $\mathscr{M}_{D-x}$ being a new space-time in $(D-x)$-dimensions. By means of sequential toroidal compactifications we can reach all maximally extended supergravities in lower dimensions starting from either type IIA or type IIB supergravity in $D = 10$. The result is always the same since su-

Table 8.2 Scalar geometries in maximal supergravities

$D = 9$	$\mathrm{E}_{2(2)} \equiv \mathrm{SL}(2,\mathbb{R}) \otimes \mathrm{O}(1,1)$	$\mathrm{H} = \mathrm{O}(2)$	$\dim_{\mathbb{R}}(\mathrm{G/H}) = 3$
$D = 8$	$\mathrm{E}_{3(3)} \equiv \mathrm{SL}(3,\mathbb{R}) \otimes \mathrm{SL}(2,\mathbb{R})$	$\mathrm{H} = \mathrm{O}(2) \otimes \mathrm{O}(3)$	$\dim_{\mathbb{R}}(\mathrm{G/H}) = 7$
$D = 7$	$\mathrm{E}_{4(4)} \equiv \mathrm{SL}(5,\mathbb{R})$	$\mathrm{H} = \mathrm{O}(5)$	$\dim_{\mathbb{R}}(\mathrm{G/H}) = 14$
$D = 6$	$\mathrm{E}_{5(5)} \equiv \mathrm{O}(5,5)$	$\mathrm{H} = \mathrm{O}(5) \otimes \mathrm{O}(5)$	$\dim_{\mathbb{R}}(\mathrm{G/H}) = 25$
$D = 5$	$\mathrm{E}_{6(6)}$	$\mathrm{H} = \mathrm{Usp}(8)$	$\dim_{\mathbb{R}}(\mathrm{G/H}) = 42$
$D = 4$	$\mathrm{E}_{7(7)}$	$\mathrm{H} = \mathrm{SU}(8)$	$\dim_{\mathbb{R}}(\mathrm{G/H}) = 70$
$D = 3$	$\mathrm{E}_{8(8)}$	$\mathrm{H} = \mathrm{O}(16)$	$\dim_{\mathbb{R}}(\mathrm{G/H}) = 128$

persymmetry allows for unique maximal theories in $D \leq 9$ and there is just one scalar coset manifold, that listed in Table 8.2. Yet this result can be interpreted in two ways depending on whether we look at it from the type IIA or from the type IIB viewpoint. There is indeed a challenging problem that corresponds to retrieving the steps of the two possible chains of sequential toroidal compactifications within the algebraic structure of the isometry groups G_x and identifying which scalar field appears at which step of the sequential chain. Such a problem has a very elegant and instructive solution in terms of a rather simple and classical mathematical theory, namely the *solvable Lie algebra parameterization* of non-compact cosets. This mathematical theory that makes a perfect match with the string theory origin of supergravities plays an important role in the discussion of p-brane solutions. In the Solvable Lie algebra parameterizations the scalar fields are divided into two groups, those that are associated with Cartan generators of the solvable algebra and those that are associated with nilpotent generators. The *Cartan scalars* are those that play the role of generalized *dilatons* and couple to the field strength $(p+2)$-forms as in (7.7.1). Within the algebraic approach, the a parameters appearing in the couplings of type $\exp[-a\phi]|F^{[p+2]}|^2$ have an interpretation in terms of *roots* and *weights* of the G_x Lie algebras which provides a very important insight into the whole matter. The solvable Lie algebra approach, that in maximal supergravities helps so clearly to master the string theory origin of the cosets G/H, can be extended also to the scalar manifolds of theories with a lesser number of supercharges. Indeed, from a mathematical point of view it works for all non-compact cosets.

8.3 Duality Symmetries in Even Dimensions

Generically a p-brane in D-dimensions either carries an electric charge with respect to a $(p+1)$-form gauge field $A^{[p+1]}$ or a magnetic charge with respect to the dual $(D-p-3)$-form $A^{[D-p-3]}_{dual}$. In the general case it cannot be dyonic with respect to the same gauge field since

$$p + 1 \neq D - p - 3 \tag{8.3.1}$$

However, in even dimension $D = 2r$, the Diophantine equation (8.3.1) admits one solution $p = \frac{D-4}{2}$, so that we always have, in this case, a special instance of branes which can be dyonic: they are particles or 0-branes in $D=4$, strings or 1-branes in $D=6$ and 2-branes in $D=8$. The possible presence of such dyonic objects has profound implications on the structure of the even dimensional supergravity Lagrangians. Indeed most of the dualities, T, S and U that relate the five perturbative superstrings have a non-trivial action on p-branes and generically transform them as electric-magnetic duality rotations. Hence, when self-dual $(r-1)$-forms are available, string dualities reflect into *duality symmetries* of the supergravity Lagrangians which constitute an essential ingredient in their construction. By duality symmetry we mean the following: a certain group of transformations G_{dual} acts on the set of *field equations* of supergravity plus the *Bianchi identities* of the $(r-1)$-forms mapping this set into itself. Clearly G_{dual} acts also on the scalar fields ϕ^I and in order to be a symmetry it must respect their kinetic term $g_{IJ}(\phi)\partial_\mu\phi^I\partial^\mu\phi^J$. This happens if and only if G_{dual} is a group of isometries for the scalar metric $g_{IJ}(\phi)$. In other words string dualities are encoded in the isometry group of the scalar manifold of supergravity which is lifted to act as a group of electric-magnetic duality rotations on the $(r-1)$-forms.

The request that these duality symmetries do exist determines the general form of the supergravity Lagrangian and is a key ingredient in its construction. For this reason in the present section we consider the case of even dimensions $D=2r$ and we review the general structure of an Abelian theory containing $\overline{n}$ differential $(r-1)$-forms:

$$A^\Lambda \equiv A^\Lambda_{\mu_1\dots\mu_{r-1}}\, dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_{r-1}}; \quad (\Lambda=1,\dots,\overline{n}) \tag{8.3.2}$$

and $\overline{m}$ real scalar fields ϕ^I. The field strengths of the $(r-1)$-forms and their Hodge duals are defined as follows:

$$\begin{aligned}
F^\Lambda &\equiv dA^\Lambda \equiv \frac{1}{r!}\mathscr{F}^\Lambda_{\mu_1\dots\mu_r}\, dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_r}\\
F^{\Lambda\star} &\equiv \frac{1}{r!}\widetilde{\mathscr{F}}^\Lambda_{\mu_1\dots\mu_r}\, dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_r}\\
\mathscr{F}^\Lambda_{\mu_1\dots\mu_r} &\equiv \partial_{\mu_1}A^\Lambda_{\mu_2\dots\mu_r} + r-2 \text{ terms}\\
\widetilde{\mathscr{F}}^\Lambda_{\mu_1\dots\mu_r} &\equiv \frac{1}{r!}\varepsilon_{\mu_1\dots\mu_r\nu_1\dots\nu_r}\mathscr{F}^{\Lambda|\nu_1\dots\nu_r}
\end{aligned} \tag{8.3.3}$$

Defining the space-time integration volume as:

$$d^Dx \equiv \frac{1}{D!}\varepsilon_{\mu_1\dots\mu_D}\, dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_D} \tag{8.3.4}$$

we obtain:

$$\begin{aligned}
F^\Lambda\wedge F^\Sigma &= \frac{1}{(r!)^2}\varepsilon^{\mu_1\dots\mu_r\nu_1\dots\nu_r}\mathscr{F}^\Lambda_{\mu_1\dots\mu_r}\mathscr{F}^\Sigma_{\nu_1\dots\nu_r}\\
F^\Lambda\wedge F^{\Sigma\star} &= (-)^r\frac{1}{(r!)}\mathscr{F}^\Lambda_{\mu_1\dots\mu_r}\mathscr{F}^{\Sigma|\mu_1\dots\mu_r}
\end{aligned} \tag{8.3.5}$$

The real scalar fields ϕ^I span an $\overline{m}$-dimensional manifold $\mathscr{M}_{scalar}$[7] endowed with a metric $g_{IJ}(\phi)$. Utilizing the above field content we can write the following action functional:

$$
\begin{aligned}
\mathscr{S} &= \mathscr{S}_{tens} + \mathscr{S}_{scal} \\
\mathscr{S}_{tens} &= \int \left[\frac{(-)^r}{2} \gamma_{\Lambda\Sigma}(\phi) F^\Lambda \wedge F^{\Sigma\star} + \frac{1}{2} \theta_{\Lambda\Sigma}(\phi) F^\Lambda \wedge F^\Sigma \right] \\
\mathscr{S}_{scal} &= \int \left[\frac{1}{2} g_{IJ}(\phi) \partial_\mu \phi^I \partial^\mu \phi^J \right] d^D x
\end{aligned}
\tag{8.3.6}
$$

where the scalar field dependent $\overline{n} \times \overline{n}$ matrix $\gamma_{\Lambda\Sigma}(\phi)$ generalizes the inverse of the squared coupling constant $\frac{1}{g^2}$ appearing in ordinary 4D-gauge theories. The field dependent matrix $\theta_{\Lambda\Sigma}(\phi)$ is instead a generalization of the θ-angle of quantum chromodynamics. The matrix γ is symmetric in every space-time dimension, while θ is symmetric or antisymmetric depending on whether $r = D/2$ is an even or odd number. In view of this fact it is convenient to distinguish the two cases, setting:

$$
D = \begin{cases} 4\nu & \nu \in \mathbb{Z} \,|\, r = 2\nu \\ 4\nu + 2 & \nu \in \mathbb{Z} \,|\, r = 2\nu + 1 \end{cases}
\tag{8.3.7}
$$

Introducing a formal operator j that maps a field strength into its Hodge dual:

$$
\left(j \mathscr{F}^\Lambda \right)_{\mu_1 \dots \mu_r} \equiv \frac{1}{(r!)} \varepsilon_{\mu_1 \dots \mu_r \nu_1 \dots \nu_r} \mathscr{F}^{\Lambda | \nu_1 \dots \nu_r}
\tag{8.3.8}
$$

and a formal scalar product:

$$
(G, K) \equiv G^T K \equiv \frac{1}{(r!)} \sum_{\Lambda=1}^{\overline{n}} G^\Lambda_{\mu_1 \dots \mu_r} K^{\Lambda | \mu_1 \dots \mu_r}
\tag{8.3.9}
$$

the total Lagrangian of (8.3.6) can be rewritten as

$$
\mathscr{L}^{(tot)} = \mathscr{F}^T (\gamma \otimes \mathbb{1} + \theta \otimes j) \mathscr{F} + \frac{1}{2} g_{IJ}(\phi) \partial_\mu \phi^I \partial^\mu \phi^J
\tag{8.3.10}
$$

and the essential distinction between the two cases of (8.3.7) is given, besides the symmetry of θ, by the involutive property of j, namely we have:

$$
\begin{aligned}
D = 4\nu \,|\, \theta = \theta^T \quad j^2 = -\mathbb{1} \\
D = 4\nu + 2 \,|\, \theta = -\theta^T \quad j^2 = \mathbb{1}
\end{aligned}
\tag{8.3.11}
$$

[7]Whether the ϕ^I can be arranged into complex fields is not relevant at this level of the discussion.

Introducing dual and antiself-dual combinations:

$$\begin{aligned} D = 4\nu \quad & \begin{cases} \mathscr{F}^{\pm} = \mathscr{F} \mp \mathrm{i} j \mathscr{F} \\ j\mathscr{F}^{\pm} = \pm \mathrm{i} \mathscr{F}^{\pm} \end{cases} \\ D = 4\nu + 2 \quad & \begin{cases} \mathscr{F}^{\pm} = \mathscr{F} \pm j \mathscr{F} \\ j\mathscr{F}^{\pm} = \pm \mathscr{F}^{\pm} \end{cases} \end{aligned} \tag{8.3.12}$$

and the field-dependent matrices:

$$\begin{aligned} D = 4\nu \quad & \begin{cases} \mathscr{N} = \theta - \mathrm{i}\gamma \\ \overline{\mathscr{N}} = \theta + \mathrm{i}\gamma \end{cases} \\ D = 4\nu + 2 \quad & \begin{cases} \mathscr{N} = \theta + \gamma \\ -\mathscr{N}^T = \theta - \gamma \end{cases} \end{aligned} \tag{8.3.13}$$

the tensor part of the Lagrangian (8.3.10) can be rewritten in the following way in the two cases:

$$\begin{aligned} D = 4\nu: \quad & \mathscr{L}_{tens} = \frac{\mathrm{i}}{8}\left[\mathscr{F}^{+T}\mathscr{N}\mathscr{F}^{+} - \mathscr{F}^{-T}\overline{\mathscr{N}}\mathscr{F}^{-}\right] \\ D = 4\nu + 2: \quad & \mathscr{L}_{tens} = \frac{1}{8}\left[\mathscr{F}^{+T}\mathscr{N}\mathscr{F}^{+} + \mathscr{F}^{-T}\mathscr{N}^T\mathscr{F}^{-}\right] \end{aligned} \tag{8.3.14}$$

Introducing the new tensor:

$$\begin{aligned} \widetilde{\mathscr{G}}^{\Lambda}_{\mu_1\dots\mu_r} &\equiv -(r!)\frac{\partial \mathscr{L}}{\partial \mathscr{F}^{\Lambda}_{\mu_1\dots\mu_r}} \quad D = 4\nu \\ \widetilde{\mathscr{G}}^{\Lambda}_{\mu_1\dots\mu_r} &\equiv (r!)\frac{\partial \mathscr{L}}{\partial \mathscr{F}^{\Lambda}_{\mu_1\dots\mu_r}} \quad D = 4\nu + 2 \end{aligned} \tag{8.3.15}$$

which, in matrix notation, corresponds to:

$$j\mathscr{G} \equiv a\frac{\partial \mathscr{L}}{\partial \mathscr{F}^T} = \frac{a}{r!}(\gamma \otimes \mathbb{1} + \theta \otimes j)\mathscr{F} \tag{8.3.16}$$

where $a = \mp$ depending on whether $D = 4\nu$ or $D = 4\nu + 2$, the Bianchi identities and field equations associated with the Lagrangian (8.3.6) can be written as follows:

$$\partial^{\mu_1}\widetilde{\mathscr{F}}^{\Lambda}_{\mu_1\dots\mu_r} = 0; \qquad \partial^{\mu_1}\widetilde{\mathscr{G}}^{\Lambda}_{\mu_1\dots\mu_r} = 0 \tag{8.3.17}$$

This suggests that we introduce the $2\overline{n}$ column vector:

$$\mathbf{V} \equiv \begin{pmatrix} j\mathscr{F} \\ j\mathscr{G} \end{pmatrix} \tag{8.3.18}$$

and that we consider general linear transformations on such a vector:

$$\begin{pmatrix} j\mathscr{F} \\ j\mathscr{G} \end{pmatrix}' = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} j\mathscr{F} \\ j\mathscr{G} \end{pmatrix} \tag{8.3.19}$$

For any matrix $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{GL}(2\overline{n}, \mathbb{R})$ the new vector $\mathbf{V}'$ of *magnetic and electric* field-strengths satisfies the same equations (8.3.17) as the old one. In a condensed notation we can write:

$$\partial \mathbf{V} = 0 \quad \Longleftrightarrow \quad \partial \mathbf{V}' = 0 \tag{8.3.20}$$

Separating the self-dual and antiself-dual parts

$$\mathscr{F} = \frac{1}{2}\left(\mathscr{F}^+ + \mathscr{F}^-\right); \qquad \mathscr{G} = \frac{1}{2}\left(\mathscr{G}^+ + \mathscr{G}^-\right) \tag{8.3.21}$$

and taking into account that for $D = 4\nu$ we have:

$$\mathscr{G}^+ = \mathscr{N} \mathscr{F}^+; \qquad \mathscr{G}^- = \overline{\mathscr{N}} \mathscr{F}^- \tag{8.3.22}$$

while for $D = 4\nu + 2$ the same equation reads:

$$\mathscr{G}^+ = \mathscr{N} \mathscr{F}^+; \qquad \mathscr{G}^- = -\mathscr{N}^T \mathscr{F}^- \tag{8.3.23}$$

the duality rotation of (8.3.19) can be rewritten as:

$$\begin{aligned}
D = 4\nu: \quad & \begin{pmatrix} \mathscr{F}^+ \\ \mathscr{G}^+ \end{pmatrix}' = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \mathscr{F}^+ \\ \mathscr{N}\mathscr{F}^+ \end{pmatrix} \\
& \begin{pmatrix} \mathscr{F}^- \\ \mathscr{G}^- \end{pmatrix}' = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \mathscr{F}^- \\ \overline{\mathscr{N}}\mathscr{F}^- \end{pmatrix} \\
D = 4\nu + 2: \quad & \begin{pmatrix} \mathscr{F}^+ \\ \mathscr{G}^+ \end{pmatrix}' = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \mathscr{F}^+ \\ \mathscr{N}\mathscr{F}^+ \end{pmatrix} \\
& \begin{pmatrix} \mathscr{F}^- \\ \mathscr{G}^- \end{pmatrix}' = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \mathscr{F}^- \\ -\mathscr{N}^T\mathscr{F}^- \end{pmatrix}
\end{aligned} \tag{8.3.24}$$

In both cases the problem is that the transformation rule of $\mathscr{G}^\pm$ must be consistent with the definition of the latter as variation of the Lagrangian with respect to $\mathscr{F}^\pm$ (see (8.3.15)). This request restricts the form of the matrix $\Lambda = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$. As we are just going to show, in the $D = 4\nu$ case Λ must belong to the symplectic subgroup $\mathrm{Sp}(2\overline{n}, \mathbb{R})$ of the special linear group, while in the $D = 4\nu + 2$ case it must be in the pseudo-orthogonal subgroup $\mathrm{SO}(\overline{n}, \overline{n})$:

$$\begin{aligned}
D = 4\nu: \quad & \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2\overline{n}, \mathbb{R}) \subset \mathrm{GL}(2\overline{n}, \mathbb{R}) \\
D = 4\nu + 2: \quad & \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{SO}(\overline{n}, \overline{n}) \subset \mathrm{GL}(2\overline{n}, \mathbb{R})
\end{aligned} \tag{8.3.25}$$

the above subgroups being defined as the set of $2\overline{n} \times 2\overline{n}$ matrices satisfying, respectively, the following conditions:

$$\begin{aligned} \Lambda \in \mathrm{Sp}(2\overline{n}, \mathbb{R}) \to \Lambda^T \begin{pmatrix} \mathbf{0} & \mathbb{1} \\ -\mathbb{1} & \mathbf{0} \end{pmatrix} \Lambda = \begin{pmatrix} \mathbf{0} & \mathbb{1} \\ -\mathbb{1} & \mathbf{0} \end{pmatrix} \\ \Lambda \in \mathrm{SO}(\overline{n}, \overline{n}) \to \Lambda^T \begin{pmatrix} \mathbf{0} & \mathbb{1} \\ \mathbb{1} & \mathbf{0} \end{pmatrix} \Lambda = \begin{pmatrix} \mathbf{0} & \mathbb{1} \\ \mathbb{1} & \mathbf{0} \end{pmatrix} \end{aligned} \tag{8.3.26}$$

To prove the statement we just made, we calculate the transformed Lagrangian $\mathscr{L}'$ and then we compare its variation $\frac{\partial \mathscr{L}'}{\partial \mathscr{F}'^T}$ with $\mathscr{G}^{\pm\prime}$ as it follows from the postulated transformation rule (8.3.24). To perform such a calculation we rely on the following basic idea. While the duality rotation (8.3.24) is performed on the field strengths and on their duals, also the scalar fields are transformed by the action of some diffeomorphism $\xi \in \mathrm{Diff}(\mathscr{M}_{scalar})$ of the scalar manifold and, as a consequence of that, also the matrix $\mathscr{N}$ changes. In other words given the scalar manifold $\mathscr{M}_{scalar}$ we assume that in the two cases of interest there exists a surjective homomorphism of the following form:

$$\iota_\delta : \mathrm{Diff}(\mathscr{M}_{scalar}) \longrightarrow \mathrm{GL}(2\overline{n}, \mathbb{R}) \tag{8.3.27}$$

so that:

$$\begin{aligned} &\forall \xi \in \mathrm{Diff}(\mathscr{M}_{scalar}) : \phi^I \xrightarrow{\xi} \phi^{I\prime} \\ &\exists \iota_\delta(\xi) = \begin{pmatrix} A_\xi & B_\xi \\ C_\xi & D_\xi \end{pmatrix} \in \mathrm{GL}(2\overline{n}, \mathbb{R}) \end{aligned} \tag{8.3.28}$$

Using such a homomorphism we can define the simultaneous action of ξ on all the fields of our theory by setting:

$$\xi : \begin{cases} \phi \longrightarrow \xi(\phi) \\ \mathbf{V} \longrightarrow \iota_\delta(\xi)\mathbf{V} \\ \mathscr{N}(\phi) \longrightarrow \mathscr{N}(\xi(\phi)) \end{cases} \tag{8.3.29}$$

where the notation (8.3.18) has been utilized. In the tensor sector the transformed Lagrangian, is

$$\begin{aligned} \mathscr{L}'_{tens} = \frac{\mathrm{i}}{8}\big[&\mathscr{F}^{+T}(A + B\mathscr{N})^T \mathscr{N}'(A + B\mathscr{N})\mathscr{F}^+ \\ &- \mathscr{F}^{-T}(A + B\overline{\mathscr{N}})^T \overline{\mathscr{N}}'(A + B\overline{\mathscr{N}})\mathscr{F}^- \big] \end{aligned} \tag{8.3.30}$$

for the $D = 4\nu$ case and

$$\begin{aligned} \mathscr{L}'_{tens} = \frac{\mathrm{i}}{8}\big[&\mathscr{F}^{+T}(A + B\mathscr{N})^T \mathscr{N}'(A + B\mathscr{N})\mathscr{F}^+ \\ &- \mathscr{F}^{-T}\big(A - B\mathscr{N}^T\big)^T \mathscr{N}^{T\prime}\big(A - B\mathscr{N}^T\big)\mathscr{F}^- \big] \end{aligned} \tag{8.3.31}$$

for the $D = 4\nu + 2$ case.

In both cases consistency with the definition of $\mathscr{G}^+$ requires, that

$$\mathscr{N}' \equiv \mathscr{N}\big(\xi(\phi)\big) = (C_\xi + D_\xi \mathscr{N})(A_\xi + B_\xi \mathscr{N})^{-1} \tag{8.3.32}$$

while consistency with the definition of $\mathscr{G}^-$ imposes, in the $D = 4\nu$ case the transformation rule:

$$\overline{\mathscr{N}}' \equiv \overline{\mathscr{N}}\big(\xi(\phi)\big) = (C_\xi + D_\xi \overline{\mathscr{N}})(A_\xi + B_\xi \overline{\mathscr{N}})^{-1} \tag{8.3.33}$$

and in the case $D = 4\nu + 2$ the other transformation rule:

$$-\mathscr{N}^{T\prime} \equiv -\mathscr{N}^T\big(\xi(\phi)\big) = \big(C_\xi - D_\xi \mathscr{N}^T\big)\big(A_\xi - B_\xi \mathscr{N}^T\big)^{-1} \tag{8.3.34}$$

It is from the transformation rules (8.3.32), (8.3.33) and (8.3.34) that we derive a restriction on the form of the duality rotation matrix $\Lambda_\xi \equiv \iota_\delta(\xi)$. Indeed, in the $D = 4\nu$ case we have that by means of the fractional linear transformation (8.3.32) Λ_ξ must map an arbitrary *complex symmetric* matrix into another matrix of the *same sort*. It is straightforward to verify that this condition is the same as the first of conditions (8.3.26), namely the definition of the symplectic group $\mathrm{Sp}(2\overline{n}, \mathbb{R})$. Similarly in the $D = 4\nu + 2$ case the matrix Λ_ξ must obey the property that taking the *negative of the transpose* of an arbitrary real matrix $\mathscr{N}$ *before or after* the fractional linear transformation induced by Λ_ξ *is immaterial*. Once again, it is easy to verify that this condition is the same as the second property in (C.1.3), namely the definition of the pseudo-orthogonal group $\mathrm{SO}(\overline{n}, \overline{n})$. Consequently the surjective homomorphism of (8.3.27) specializes as follows in the two relevant cases

$$\iota_\delta : \begin{cases} \mathrm{Diff}(\mathscr{M}_{scalar}) \longrightarrow \mathrm{Sp}(2\overline{n}, \mathbb{R}) \\ \mathrm{Diff}(\mathscr{M}_{scalar}) \longrightarrow \mathrm{SO}(\overline{n}, \overline{n}) \end{cases} \tag{8.3.35}$$

Clearly, since both $\mathrm{Sp}(2\overline{n}, \mathbb{R})$ and $\mathrm{SO}(\overline{n}, \overline{n})$ are finite dimensional Lie groups, while $\mathrm{Diff}(\mathscr{M}_{scalar})$ is infinite-dimensional, the homomorphism ι_δ can never be an isomorphism and actually we always have:

$$\dim \ker \iota_\delta = \infty \tag{8.3.36}$$

What should be clear from the above discussion is that a family of Lagrangians as in (8.3.6) will admit a group of duality-rotations/field-redefinitions that will map one into the other member of the family, as long as a *kinetic matrix* $\mathscr{N}_{\Lambda\Sigma}$ can be constructed that transforms as in (8.3.32). A way to obtain such an object is to identify it with the *period matrix* occurring in problems of algebraic geometry. At the level of the present discussion, however, this identification is by no means essential: any construction of $\mathscr{N}_{\Lambda\Sigma}$ with the appropriate transformation properties is acceptable.

Note also that so far we have used the words *duality-rotations/field-redefinitions* and not the word duality symmetry. Indeed the diffeomorphisms of the scalar manifold we have considered were quite general and, as such had no claim to be symmetries of the action, or of the theory. Indeed the question we have answered is the following: what are the appropriate transformation properties of the tensor gauge fields and of the generalized coupling constants under diffeomorphisms of the scalar manifold? The next question is obviously that of duality symmetries. Suppose that a certain diffeomorphism $\xi \in \mathrm{Diff}(\mathscr{M}_{scalar})$ is actually an *isometry* of the scalar metric g_{IJ}. Naming $\xi^\star : T\mathscr{M}_{scalar} \to T\mathscr{M}_{scalar}$ the push-forward of ξ, this means that

$$\begin{aligned} &\forall X, Y \in T\mathscr{M}_{scalar} \\ &g(X,Y) = g\big(\xi^\star X, \xi^\star Y\big) \end{aligned} \tag{8.3.37}$$

and ξ is an exact global symmetry of the scalar part of the Lagrangian in (8.3.6). The obvious question is: "*can this symmetry be extended to a symmetry of the complete action?*" Clearly the answer is that, in general, this is not possible. The best we can do is to extend it to a symmetry of the field equations plus Bianchi identities letting it act as a duality rotation on the field-strengths plus their duals. This requires that the group of isometries of the scalar metric $\mathrm{G}_{iso}(\mathscr{M}_{scalar})$ be suitably embedded into the duality group (either $\mathrm{Sp}(2\overline{n}, \mathbb{R})$ or $\mathrm{SO}(\overline{n}, \overline{n})$ depending on the case) and that the kinetic matrix $\mathscr{N}_{\Lambda\Sigma}$ satisfies the covariance law:

$$\mathscr{N}\big(\xi(\phi)\big) = \big(C_\xi + D_\xi \mathscr{N}(\phi)\big)\big(A_\xi + B_\xi \mathscr{N}(\phi)\big) \tag{8.3.38}$$

A general class of solutions to this programme can be derived in the case where the scalar manifold is taken to be a homogeneous space G/H. This is the subject of next section.

8.3.1 The Kinetic Matrix $\mathscr{N}$ and Symplectic Embeddings

In our survey of the geometric features of bosonic supergravity Lagrangians that are specifically relevant for p-brane solutions the next important item we have to consider is the kinetic term of the $(p+1)$-form gauge fields. Generically it is of the form:

$$\mathscr{L}^{Kin}_{forms} = \mathscr{N}_{\Lambda\Sigma}(\phi) F^{\Lambda}_{\mu_1 \dots \mu_{p+2}} F^{\Sigma|\mu_1 \dots \mu_{p+2}} \tag{8.3.39}$$

where $\mathscr{N}_{\Lambda\Sigma}$ is a suitable scalar field dependent symmetric matrix. In the case of self-dual $(p+1)$-forms, that occurs only in even dimensions, the matrix $\mathscr{N}$ is completely fixed by the requirement that the *ungauged supergravity theory* should admit duality symmetries. Furthermore as remarked in the previous section, the problem of constructing duality-symmetric Lagrangians of the type (8.3.6) admits general solutions when the scalar manifold is a homogeneous space G/H. Hence we devote the present section to review the construction of the *kinetic period matrix* $\mathscr{N}$ in the case of homogeneous spaces. The case of odd space dimensions where there are no dualities will be addressed in a subsequent section.

The relevant cases of even dimensional supergravities are:

1. In $D=4$ the self-dual forms are ordinary gauge vectors and the duality rotations are symplectic. There are several theories depending on the number of supersymmetries. They are summarized in Table 8.1. Each theory involves a different number $\overline{n}$ of vectors A^Λ and different cosets $\frac{\mathrm{G}}{\mathrm{H}}$ but the relevant homomorphism ι_δ (see (8.3.35)) is always of the same type:

$$\iota_\delta : \mathrm{Diff}\left(\frac{\mathrm{G}}{\mathrm{H}}\right) \longrightarrow \mathrm{Sp}(2\overline{n}, \mathbb{R}) \tag{8.3.40}$$

having denoted by $\overline{n}$ the total number of vector fields that is displayed in Table 8.1.
2. In $D=6$ we have self-dual 2-forms. Also here we have a few different possibilities depending on the number $(\mathcal{N}_+, \mathcal{N}_-)$ of left and right handed supersymmetries with a variable number $\overline{n}$ of 2-forms. In particular for the $(2,2)$ theory that originates from type IIA compactifications the scalar manifold is:

$$\mathrm{G}/\mathrm{H} = \frac{\mathrm{O}(4,n)}{\mathrm{O}(4)\times \mathrm{O}(n)} \times \mathrm{O}(1,1) \tag{8.3.41}$$

while for the $(4,0)$ theory that originates from type IIB compactifications the scalar manifold is the following:

$$\mathrm{G}/\mathrm{H} = \frac{\mathrm{O}(5,n)}{\mathrm{O}(5)\times \mathrm{O}(n)} \tag{8.3.42}$$

Finally in the case of $(\mathcal{N}_+ = 2, \mathcal{N}_- = 0)$ supergravity, the scalar manifold is

$$\mathcal{M}_{scalar} = \frac{\mathrm{O}(1,n)}{\mathrm{O}(n)} \times \mathcal{QM} \tag{8.3.43}$$

the first homogeneous factor $\frac{\mathrm{O}(1,n)}{\mathrm{O}(n)}$ containing the scalars of the tensor multiplets, while the second factor denotes a generic *quaternionic manifold* that contains the scalars of the hypermultiplets. In all cases the relevant embedding is

$$\iota_\delta : \mathrm{Diff}\left(\frac{\mathrm{G}}{\mathrm{H}}\right) \longrightarrow \mathrm{SO}(\overline{n}, \overline{n}) \tag{8.3.44}$$

where $\overline{n}$ is the total number of 2-forms, namely:

$$\begin{cases} \overline{n} = 4+n & \text{for the } (2,2) \text{ theory} \\ \overline{n} = 5+n & \text{for the } (4,0) \text{ theory} \\ \overline{n} = 1+n & \text{for the } (2,0) \text{ theory} \end{cases} \tag{8.3.45}$$

3. In $D=8$ we have self-dual three-forms. There are two theories. The first is maximally extended $\mathcal{N}=2$ supergravity where the number of three-forms is $\overline{n}=3$

and the scalar coset manifold is:

$$\mathrm{G/H} = \frac{\mathrm{SL}(3,\mathbb{R})}{\mathrm{O}(3)} \times \frac{\mathrm{SL}(2,\mathbb{R})}{\mathrm{O}(2)} \tag{8.3.46}$$

The second theory is $\mathcal{N}=1$ supergravity that contains $\overline{n}=1$ three-forms and where the scalar coset is:

$$\mathrm{G/H} = \frac{\mathrm{SO}(2,n)}{\mathrm{SO}(2)\times\mathrm{SO}(n)} \times \mathrm{O}(1,1) \tag{8.3.47}$$

having denoted $n = \#$ of vector multiplets. In the two cases the relevant embedding is symplectic and specifically it is:

$$\iota_\delta : \mathrm{Diff}\left(\frac{\mathrm{G}}{\mathrm{H}}\right) \longrightarrow \begin{cases} \mathrm{Sp}(6,\mathbb{R}) & \text{maximal supergravity} \\ \mathrm{Sp}(2,\mathbb{R}) & \mathcal{N}=1 \text{ supergravity} \end{cases} \tag{8.3.48}$$

8.3.2 *Symplectic Embeddings in General*

Let us begin with the case of symplectic embeddings relevant to $D=4$ and $D=8$ theories.

Focusing on the isometry group of the canonical metric[8] defined on $\frac{\mathrm{G}}{\mathrm{H}}$:

$$\mathrm{G}_{iso}\left(\frac{\mathrm{G}}{\mathrm{H}}\right) = \mathrm{G} \tag{8.3.49}$$

we must consider the embedding:

$$\iota_\delta : \mathrm{G} \longrightarrow \mathrm{Sp}(2\overline{n},\mathbb{R}) \tag{8.3.50}$$

That in (8.3.40) is a homomorphism of finite dimensional Lie groups and as such it constitutes a problem that can be solved in explicit form. What we just need to know is the dimension of the symplectic group, namely the number $\overline{n}$ of $\frac{D-4}{2}$-forms appearing in the theory. Without supersymmetry the dimension m of the scalar manifold (namely the possible choices of $\frac{\mathrm{G}}{\mathrm{H}}$) and the number of vectors $\overline{n}$ are unrelated so that the possibilities covered by (8.3.50) are infinitely many. In supersymmetric theories, instead, the two numbers m and $\overline{n}$ are related, so that there are finitely many cases to be studied corresponding to the possible embeddings of given groups G into a symplectic group $\mathrm{Sp}(2\overline{n},\mathbb{R})$ of fixed dimension $\overline{n}$. Actually taking into account further conditions on the holonomy of the scalar manifold that are also imposed by supersymmetry, the solution for the symplectic embedding problem is

[8] Actually, in order to be true, (8.3.49) requires that the normalizer of H in G be the identity group, a condition that is verified in all the relevant examples.

unique for all extended supergravities as we have already remarked. In $D=4$ this yields the unique scalar manifold choice displayed in Table 8.1, while in the other dimensions gives the results recalled above.

Apart from the details of the specific case considered once a symplectic embedding is given there is a general formula one can write down for the *period matrix* $\mathcal{N}$ that guarantees symmetry ($\mathcal{N}^T = \mathcal{N}$) and the required transformation property (8.3.38). This is the first result we want to present.

The real symplectic group $\mathrm{Sp}(2\overline{n}, \mathbb{R})$ is defined as the set of all *real* $2\overline{n} \times 2\overline{n}$ matrices

$$\Lambda = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{8.3.51}$$

satisfying the first of (C.1.3), namely

$$\Lambda^T \mathbb{C} \Lambda = \mathbb{C} \tag{8.3.52}$$

where

$$\mathbb{C} \equiv \begin{pmatrix} \mathbf{0} & \mathbb{1} \\ -\mathbb{1} & \mathbf{0} \end{pmatrix} \tag{8.3.53}$$

If we relax the condition that the matrix should be real but we still impose (8.3.52) we obtain the definition of the complex symplectic group $\mathrm{Sp}(2\overline{n}, \mathbb{C})$. It is a well known fact that the following isomorphism is true:[9]

$$\mathrm{Sp}(2\overline{n}, \mathbb{R}) \sim \mathrm{USp}(\overline{n}, \overline{n}) \equiv \mathrm{Sp}(2\overline{n}, \mathbb{C}) \cap \mathrm{U}(\overline{n}, \overline{n}) \tag{8.3.54}$$

By definition an element $\mathcal{S} \in \mathrm{USp}(\overline{n}, \overline{n})$ is a complex matrix that satisfies simultaneously (8.3.52) and a pseudounitarity condition, that is:

$$\mathcal{S}^T \mathbb{C} \mathcal{S} = \mathbb{C}; \qquad \mathcal{S}^\dagger \mathbb{H} \mathcal{S} = \mathbb{H}; \quad \mathbb{H} \equiv \begin{pmatrix} \mathbb{1} & \mathbf{0} \\ \mathbf{0} & -\mathbb{1} \end{pmatrix} \tag{8.3.55}$$

The general block form of the matrix $\mathcal{S}$ is:

$$\mathcal{S} = \begin{pmatrix} T & V^\star \\ V & T^\star \end{pmatrix} \tag{8.3.56}$$

and (8.3.55) are equivalent to:

$$T^\dagger T - V^\dagger V = \mathbb{1}; \qquad T^\dagger V^\star - V^\dagger T^\dagger = \mathbf{0} \tag{8.3.57}$$

[9]From the point of view of Lie algebra theory, there are no other independent real sections of the C_n Lie algebra except the non-compact $\mathrm{Sp}(2\overline{n}, \mathbb{R})$ and the compact $\mathrm{USp}(2\overline{n})$. So in mathematical books the Lie group $\mathrm{USp}(\overline{n}, \overline{n})$ does not exist being simply $\mathrm{Sp}(2\overline{n}, \mathbb{R})$. In our present discussion we find it useful to denote by this symbol the realization of $\mathrm{Sp}(2\overline{n}, \mathbb{R})$ elements by means of complex symplectic and pseudounitary matrices as described in the main text.

The isomorphism of (8.3.54) is explicitly realized by the so called Cayley matrix:

$$\mathscr{C} \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbb{1} & \mathrm{i}\mathbb{1} \\ \mathbb{1} & -\mathrm{i}\mathbb{1} \end{pmatrix} \tag{8.3.58}$$

via the relation:

$$\mathscr{S} = \mathscr{C} \Lambda \mathscr{C}^{-1} \tag{8.3.59}$$

which yields:

$$T = \frac{1}{2}(A - \mathrm{i}B) + \frac{1}{2}(D + \mathrm{i}C); \qquad V = \frac{1}{2}(A - \mathrm{i}B) - \frac{1}{2}(D + \mathrm{i}C) \tag{8.3.60}$$

When we set $V = 0$ we obtain the subgroup $\mathrm{U}(\overline{n}) \subset \mathrm{USp}(\overline{n}, \overline{n})$, that in the real basis is given by the subset of symplectic matrices of the form $\left(\begin{smallmatrix} A & B \\ -B & A \end{smallmatrix}\right)$. The basic idea, to obtain the general formula for the period matrix, is that the symplectic embedding of the isometry group G will be such that the isotropy subgroup $\mathrm{H} \subset \mathrm{G}$ gets embedded into the maximal compact subgroup $\mathrm{U}(\overline{n})$, namely:

$$\mathrm{G} \xrightarrow{\iota_\delta} \mathrm{USp}(\overline{n}, \overline{n}); \qquad \mathrm{G} \supset \mathrm{H} \xrightarrow{\iota_\delta} \mathrm{U}(\overline{n}) \subset \mathrm{USp}(\overline{n}, \overline{n}) \tag{8.3.61}$$

If this condition is realized let $\mathbb{L}(\phi)$ be a parameterization of the coset G/H by means of coset representatives. By this we mean the following. Let ϕ^I be local coordinates on the manifold G/H: to each point $\phi \in \mathrm{G/H}$ we assign an element $\mathbb{L}(\phi) \in \mathrm{G}$ in such a way that if $\phi' \neq \phi$, then no $h \in \mathrm{H}$ can exist such that $\mathbb{L}(\phi') = \mathbb{L}(\phi) \cdot h$. In other words for each equivalence class of the coset (labeled by the coordinate ϕ) we choose one representative element $\mathbb{L}(\phi)$ of the class. Relying on the symplectic embedding of (8.3.61) we obtain a map:

$$\mathbb{L}(\phi) \longrightarrow \mathscr{O}(\phi) = \begin{pmatrix} U_0(\phi) & U_1^\star(\phi) \\ U_1(\phi) & U_0^\star(\phi) \end{pmatrix} \in \mathrm{USp}(\overline{n}, \overline{n}) \tag{8.3.62}$$

that associates to $\mathbb{L}(\phi)$ a coset representative of $\mathrm{USp}(\overline{n}, \overline{n})/\mathrm{U}(\overline{n})$. By construction if $\phi' \neq \phi$ *no* unitary $\overline{n} \times \overline{n}$ matrix W *can exist* such that:

$$\mathscr{O}(\phi') = \mathscr{O}(\phi) \begin{pmatrix} W & \mathbf{0} \\ \mathbf{0} & W^\star \end{pmatrix} \tag{8.3.63}$$

On the other hand let $\xi \in \mathrm{G}$ be an element of the isometry group of G/H. Via the symplectic embedding of (8.3.61) we obtain a $\mathrm{USp}(\overline{n}, \overline{n})$ matrix

$$\mathscr{S}_\xi = \begin{pmatrix} T_\xi & V_\xi^\star \\ V_\xi & T_\xi^\star \end{pmatrix} \tag{8.3.64}$$

such that

$$\mathscr{S}_\xi \mathscr{O}(\phi) = \mathscr{O}\big(\xi(\phi)\big) \begin{pmatrix} W(\xi, \phi) & \mathbf{0} \\ \mathbf{0} & W^\star(\xi, \phi) \end{pmatrix} \tag{8.3.65}$$

where $\xi(\phi)$ denotes the image of the point $\phi \in \mathrm{G/H}$ through ξ and $W(\xi,\phi)$ is a suitable $\mathrm{U}(\overline{n})$ compensator depending both on ξ and ϕ. Combining (8.3.65), (8.3.62), with (8.3.60) we immediately obtain:

$$\begin{aligned} U_0^\dagger\big(\xi(\phi)\big) + U_1^\dagger\big(\xi(\phi)\big) &= W^\dagger\big[U_0^\dagger(\phi)\big(A^T + \mathrm{i}B^T\big) + U_1^\dagger(\phi)\big(A^T - \mathrm{i}B^T\big)\big] \\ U_0^\dagger\big(\xi(\phi)\big) - U_1^\dagger\big(\xi(\phi)\big) &= W^\dagger\big[U_0^\dagger(\phi)\big(D^T - \mathrm{i}C^T\big) - U_1^\dagger(\phi)\big(D^T + \mathrm{i}C^T\big)\big] \end{aligned} \tag{8.3.66}$$

Setting:

$$\mathscr{N} \equiv \mathrm{i}\big[U_0^\dagger + U_1^\dagger\big]^{-1}\big[U_0^\dagger - U_1^\dagger\big] \tag{8.3.67}$$

and using the result of (8.3.66) one verifies that the transformation rule (8.3.38) is verified. It is also an immediate consequence of the analogue of (8.3.57) satisfied by U_0 and U_1 that the matrix in (8.3.67) is symmetric:

$$\mathscr{N}^T = \mathscr{N} \tag{8.3.68}$$

Equation (8.3.67) is the master formula derived in 1981 by Gaillard and Zumino [25]. It explains the structure of the gauge field kinetic terms in all $\mathscr{N} \geq 3$ extended supergravity theories and also in those $\mathscr{N} = 2$ theories where, the *special Kähler manifold* $\mathscr{SK}$ is a homogeneous manifold G/H. Similarly it applies to the kinetic terms of the three-forms in $D = 8$. Furthermore, using (8.3.67) we can easily retrieve the structure of $\mathscr{N} = 4$ supergravity.

8.4 General Form of $D = 4$ (Ungauged) Supergravity

What we discussed so far allows us to write the general form of the bosonic Lagrangian of $D = 4$ supergravity without gaugings. It is as follows:

$$\begin{aligned} \mathscr{L}^{(4)} = \sqrt{|\det g|}\bigg[\frac{R[g]}{2} - \frac{1}{4}\partial_\mu\phi^a\partial^\mu\phi^b h_{ab}(\phi) + \operatorname{Im}\mathscr{N}_{\Lambda\Sigma}F^\Lambda_{\mu\nu}F^{\Sigma|\mu\nu}\bigg] \\ + \frac{1}{2}\operatorname{Re}\mathscr{N}_{\Lambda\Sigma}F^\Lambda_{\mu\nu}F^\Sigma_{\rho\sigma}\varepsilon^{\mu\nu\rho\sigma} \end{aligned} \tag{8.4.1}$$

where $F^\Lambda_{\mu\nu} \equiv (\partial_\mu A^\Lambda_\nu - \partial_\nu A^\Lambda_\mu)/2$. In principle the effective theory described by the Lagrangian (8.4.1) can be obtained by compactification on suitable internal manifolds from $D = 10$ supergravity or 11-dimensional M-theory, however, how we stepped down from $D = 10, 11$ to $D = 4$ is not necessary to specify at this level. It is implicitly encoded in the number of residual supersymmetries that we consider. If $N_Q = 32$ is maximal it means that we used toroidal compactification. Lower values of N_Q correspond to compactifications on manifolds of restricted holonomy, Calabi-Yau three-folds, for instance, or orbifolds.

In (8.4.1) ϕ^a denotes the whole set of n_S scalar fields parameterizing the scalar manifold $\mathscr{M}^{D=4}_{scalar}$ which, for $N_Q \geq 8$, is necessarily a coset manifold:

$$\mathscr{M}^{D=4}_{scalar} = \frac{\mathrm{G}}{\mathrm{H}} \tag{8.4.2}$$

For $N_Q \leq 8$ (8.4.2) is not obligatory but it is possible. Particularly in the $\mathscr{N} = 2$ case, *i.e.* for $N_Q = 8$, a large variety of homogeneous special Kähler or quaternionic manifolds fall into the set up of the present general discussion. The fields ϕ^a have σ-model interactions dictated by the metric $h_{ab}(\phi)$ of $\mathscr{M}^{D=4}_{scalar}$. The theory includes also n vector fields $A^{\Lambda}_{\hat{\mu}}$ for which

$$\mathscr{F}^{\pm|\Lambda}_{\mu\nu} \equiv \frac{1}{2}\left[F^{\Lambda}_{\mu\nu} \mp \mathrm{i}\frac{\sqrt{|\det g|}}{2}\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}\right] \tag{8.4.3}$$

denote the self-dual (respectively antiself-dual) parts of the field-strengths. As displayed in (8.4.1) they are non-minimally coupled to the scalars via the symmetric complex matrix

$$\mathscr{N}_{\Lambda\Sigma}(\phi) = \mathrm{i}\,\mathrm{Im}\,\mathscr{N}_{\Lambda\Sigma} + \mathrm{Re}\,\mathscr{N}_{\Lambda\Sigma} \tag{8.4.4}$$

which transforms projectively under G. Indeed the field strengths $F^{\Lambda}_{\mu\nu}$ plus their magnetic duals fill up a $2n$-dimensional symplectic representation of $\mathbb{G}$ which we call by the name of $\mathbf{W}$.

The kinetic matrix is constructed by means of the Gaillard Zumino master formula in all cases where the scalar manifolds is a homogeneous space G/H. In the next section, while discussing special Kähler geometry, we show how to construct $\mathscr{N}_{\Lambda\Sigma}$ also for those $\mathscr{SK}$ manifolds that are not homogeneous. Indeed, as we already explained, when supersymmetry is larger than $\mathscr{N} = 2$ the scalar manifold is always a symmetric coset space. For $\mathscr{N} = 2$, on the other hand, the prediction of supersymmetry is that $\mathscr{M}^{D=4}_{scalar}$ should be a special Kähler manifold $\mathscr{SK}_n$, n being the number of considered vector multiplets.[10] Special Kähler manifolds are a vast category of spaces that typically are not cosets and may admit no continuous group of isometries, as it happens, for instance, in the case of moduli spaces of Kähler structure or complex structure deformations of Calabi-Yau three-folds. Nevertheless there exists a subclass of special Kähler manifolds that are also symmetric spaces. For those manifolds the special Kähler structure and the group structure coexist and are tight together in a specific way.

8.5 Summary of Special Kähler Geometry

As recalled in Table 8.1, special Kähler geometry is that pertaining to the scalars of $\mathscr{N} = 2$ vector multiplets in $D = 4$ supergravity. Its first formulation in spe-

[10] For simplicity we do not envisage the inclusion of hypermultiplets which would span additional quaternionic manifolds.

cial coordinates was introduced in 1984–85 by B. de Wit et al. and E. Cremmer et al. (see pioneering paper [2]), where the coupling of $\mathcal{N}=2$ vector multiplets to $\mathcal{N}=2$ supergravity was fully determined. The more intrinsic definition of special Kähler geometry in terms of symplectic bundles is due to Strominger [5], who obtained it in connection with the moduli spaces of Calabi-Yau compactifications. The coordinate-independent description and derivation of special Kähler geometry in the context of $\mathcal{N}=2$ supergravity is due to Castellani, D'Auria, Ferrara and to D'Auria, Ferrara, Frè (1991) (see Refs. [3, 4, 20]).

Let us summarize the relevant concepts and definitions.

8.5.1 Hodge-Kähler Manifolds

Consider a *line bundle* $\mathcal{L}\xrightarrow{\pi}\mathcal{M}$ over a Kähler manifold. By definition this is a holomorphic vector bundle of rank $r=1$. For such bundles the only available Chern class is the first:

$$c_1(\mathcal{L})=\frac{i}{2\pi}\overline{\partial}\big(h^{-1}\partial h\big)=\frac{i}{2\pi}\overline{\partial}\partial\log h \tag{8.5.1}$$

where the 1-component real function $h(z,\overline{z})$ is some Hermitian fibre metric on $\mathcal{L}$. Let $f(z)$ be a holomorphic section of the line bundle $\mathcal{L}$: noting that under the action of the operator $\overline{\partial}\partial$ the term $\log(\overline{\xi}(\overline{z})\xi(z))$ yields a vanishing contribution, we conclude that the formula in (8.5.1) for the first Chern class can be re-expressed as follows:

$$c_1(\mathcal{L})=\frac{i}{2\pi}\overline{\partial}\partial\log\big\|\xi(z)\big\|^2 \tag{8.5.2}$$

where $\|\xi(z)\|^2=h(z,\overline{z})\overline{\xi}(\overline{z})\xi(z)$ denotes the norm of the holomorphic section $\xi(z)$.

Equation (8.5.2) is the starting point for the definition of Hodge Kähler manifolds. A Kähler manifold $\mathcal{M}$ is a Hodge manifold if and only if there exists a line bundle $\mathcal{L}\longrightarrow\mathcal{M}$ such that its first Chern class equals the cohomology class of the Kähler two-form K:

$$c_1(\mathcal{L})=[\mathrm{K}] \tag{8.5.3}$$

In local terms this means that there is a holomorphic section $W(z)$ such that we can write

$$\mathrm{K}=\frac{i}{2\pi}g_{ij^\star}\,dz^i\wedge d\overline{z}^{j^\star}=\frac{i}{2\pi}\overline{\partial}\partial\log\big\|W(z)\big\|^2 \tag{8.5.4}$$

Recalling the local expression of the Kähler metric in terms of the Kähler potential $g_{ij^\star}=\partial_i\partial_{j^\star}\mathcal{K}(z,\overline{z})$, it follows from (8.5.4) that if the manifold $\mathcal{M}$ is a Hodge manifold, then the exponential of the Kähler potential can be interpreted as the metric $h(z,\overline{z})=\exp(\mathcal{K}(z,\overline{z}))$ on an appropriate line bundle $\mathcal{L}$.

8.5.2 Connection on the Line Bundle

On any complex line bundle $\mathscr{L}$ there is a canonical Hermitian connection defined as:

$$\theta \equiv h^{-1}\partial h = \frac{1}{h}\partial_i h\, dz^i; \qquad \overline{\theta} \equiv h^{-1}\overline{\partial} h = \frac{1}{h}\partial_{i^\star} h\, d\overline{z}^{i^\star} \tag{8.5.5}$$

For the line-bundle advocated by the Hodge-Kähler structure we have

$$[\overline{\partial}\theta] = c_1(\mathscr{L}) = [\mathrm{K}] \tag{8.5.6}$$

and since the fibre metric h can be identified with the exponential of the Kähler potential we obtain:

$$\theta = \partial\mathscr{K} = \partial_i \mathscr{K}\, dz^i; \qquad \overline{\theta} = \overline{\partial}\mathscr{K} = \partial_{i^\star}\mathscr{K}\, d\overline{z}^{i^\star} \tag{8.5.7}$$

To define special Kähler geometry, in addition to the afore-mentioned line-bundle $\mathscr{L}$ we need a flat holomorphic vector bundle $\mathscr{SV} \longrightarrow \mathscr{M}$ whose sections play an important role in the construction of the supergravity Lagrangians. For reasons intrinsic to such constructions the rank of the vector bundle $\mathscr{SV}$ must be $2n_V$ where n_V is the total number of vector fields in the theory. If we have n-vector multiplets the total number of vectors is $n_V = n + 1$ since, in addition to the vectors of the vector multiplets, we always have the graviphoton sitting in the graviton multiplet. On the other hand the total number of scalars is $2n$. Suitably paired into n-complex fields z^i, these scalars span the n complex dimensions of the base manifold $\mathscr{M}$ of the rank $2n + 2$ bundle $\mathscr{SV} \longrightarrow \mathscr{M}$.

In the sequel we make extensive use of covariant derivatives with respect to the canonical connection of the line-bundle $\mathscr{L}$. Let us review its normalization. As it is well known there exists a correspondence between line-bundles and U(1)-bundles. If $\exp[f_{\alpha\beta}(z)]$ is the transition function between two local trivializations of the line-bundle $\mathscr{L} \longrightarrow \mathscr{M}$, the transition function in the corresponding principal U(1)-bundle $\mathscr{U} \longrightarrow \mathscr{M}$ is just $\exp[\mathrm{i}\,\mathrm{Im}\, f_{\alpha\beta}(z)]$ and the Kähler potentials in two different charts are related by: $\mathscr{K}_\beta = \mathscr{K}_\alpha + f_{\alpha\beta} + \overline{f}_{\alpha\beta}$. At the level of connections this correspondence is formulated by setting: U(1)-connection $\equiv \mathscr{Q} = \mathrm{Im}\,\theta = -\frac{\mathrm{i}}{2}(\theta - \overline{\theta})$. If we apply this formula to the case of the U(1)-bundle $\mathscr{U} \longrightarrow \mathscr{M}$ associated with the line-bundle $\mathscr{L}$ whose first Chern class equals the Kähler class, we get:

$$\mathscr{Q} = -\frac{\mathrm{i}}{2}\left(\partial_i \mathscr{K}\, dz^i - \partial_{i^\star}\mathscr{K}\, d\overline{z}^{i^\star}\right) \tag{8.5.8}$$

Let now $\Phi(z,\overline{z})$ be a section of $\mathscr{U}^p$. By definition its covariant derivative is $\nabla\Phi = (d + ip\mathscr{Q})\Phi$ or, in components,

$$\nabla_i\Phi = \left(\partial_i + \frac{1}{2}p\partial_i\mathscr{K}\right)\Phi; \qquad \nabla_{i^*}\Phi = \left(\partial_{i^*} - \frac{1}{2}p\partial_{i^*}\mathscr{K}\right)\Phi \tag{8.5.9}$$

A covariantly holomorphic section of $\mathscr{U}$ is defined by the equation: $\nabla_{i^*}\Phi = 0$. We can easily map each section $\Phi(z,\overline{z})$ of $\mathscr{U}^p$ into a section of the line-bundle $\mathscr{L}$ by setting:

$$\widetilde{\Phi} = e^{-p\mathscr{K}/2}\Phi \tag{8.5.10}$$

With this position we obtain:

$$\nabla_i \widetilde{\Phi} = (\partial_i + p\partial_i \mathscr{K})\widetilde{\Phi}; \qquad \nabla_{i^*}\widetilde{\Phi} = \partial_{i^*}\widetilde{\Phi} \tag{8.5.11}$$

Under the map of (8.5.10) covariantly holomorphic sections of $\mathscr{U}$ flow into holomorphic sections of $\mathscr{L}$ and vice-versa.

8.5.3 *Special Kähler Manifolds*

We are now ready to give the first of two equivalent definitions of special Kähler manifolds:

Definition 8.5.1 A Hodge Kähler manifold is *Special Kähler* (*of the local type*) if there exists a completely symmetric holomorphic 3-index section W_{ijk} of $(T^\star\mathscr{M})^3 \otimes \mathscr{L}^2$ (and its antiholomorphic conjugate $W_{i^*j^*k^*}$) such that the following identity is satisfied by the Riemann tensor of the Levi-Civita connection:

$$\begin{aligned}
&\partial_{m^*} W_{ijk} = 0; \qquad \partial_m W_{i^*j^*k^*} = 0\\
&\nabla_{[m} W_{i]jk} = 0; \qquad \nabla_{[m} W_{i^*]j^*k^*} = 0\\
&\mathscr{R}_{i^*j\ell^*k} = g_{\ell^*j}g_{ki^*} + g_{\ell^*k}g_{ji^*} - e^{2\mathscr{K}}\, W_{i^*\ell^*s^*}W_{tkj}g^{s^*t}
\end{aligned} \tag{8.5.12}$$

In the above equations ∇ denotes the covariant derivative with respect to both the Levi-Civita and the U(1) holomorphic connection of (8.5.8). In the case of W_{ijk}, the U(1) weight is $p = 2$.

Out of the W_{ijk} we can construct covariantly holomorphic sections of weight 2 and -2 by setting:

$$C_{ijk} = W_{ijk}e^{\mathscr{K}}; \qquad C_{i^\star j^\star k^\star} = W_{i^\star j^\star k^\star}e^{\mathscr{K}} \tag{8.5.13}$$

The flat bundle mentioned in the previous subsection apparently does not appear in this definition of special geometry. Yet it is there. It is indeed the essential ingredient in the second definition whose equivalence to the first we shall shortly provide.

Let $\mathscr{L} \longrightarrow \mathscr{M}$ denote the complex line bundle whose first Chern class equals the Kähler form K of an n-dimensional Hodge-Kähler manifold $\mathscr{M}$. Let $\mathscr{SV} \longrightarrow \mathscr{M}$ denote a holomorphic flat vector bundle of rank $2n+2$ with structural group $\mathrm{Sp}(2n+2,\mathbb{R})$. Consider tensor bundles of the type $\mathscr{H} = \mathscr{SV} \otimes \mathscr{L}$. A typical holomorphic section of such a bundle will be denoted by Ω and will have the following structure:

$$\Omega = \begin{pmatrix} X^\Lambda \\ F_\Sigma \end{pmatrix} \quad \Lambda, \Sigma = 0, 1, \dots, n \tag{8.5.14}$$

By definition the transition functions between two local trivializations $U_i \subset \mathscr{M}$ and $U_j \subset \mathscr{M}$ of the bundle $\mathscr{H}$ have the following form:

$$\begin{pmatrix} X \\ F \end{pmatrix}_i = e^{f_{ij}} M_{ij} \begin{pmatrix} X \\ F \end{pmatrix}_j \tag{8.5.15}$$

where f_{ij} are holomorphic maps $U_i \cap U_j \to \mathbb{C}$ while M_{ij} is a constant $\mathrm{Sp}(2n+2, \mathbb{R})$ matrix. For a consistent definition of the bundle the transition functions are obviously subject to the cocycle condition on a triple overlap: $e^{f_{ij}+f_{jk}+f_{ki}} = 1$ and $M_{ij} M_{jk} M_{ki} = 1$.

Let $\mathrm{i}\langle \mid \rangle$ be the compatible Hermitian metric on $\mathscr{H}$

$$\mathrm{i}\langle \Omega \mid \overline{\Omega} \rangle \equiv -\mathrm{i}\Omega^T \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} \overline{\Omega} \tag{8.5.16}$$

Definition 8.5.2 We say that a Hodge-Kähler manifold $\mathscr{M}$ is *special Kähler* if there exists a bundle $\mathscr{H}$ of the type described above such that for some section $\Omega \in \Gamma(\mathscr{H}, \mathscr{M})$ the Kähler two form is given by:

$$\mathrm{K} = \frac{\mathrm{i}}{2\pi} \partial\overline{\partial} \log\bigl(\mathrm{i}\langle \Omega \mid \overline{\Omega} \rangle\bigr) \tag{8.5.17}$$

From the point of view of local properties, (8.5.17) implies that we have an expression for the Kähler potential in terms of the holomorphic section Ω:

$$\mathscr{K} = -\log\bigl(\mathrm{i}\langle \Omega \mid \overline{\Omega} \rangle\bigr) = -\log\bigl[\mathrm{i}\bigl(\overline{X}^\Lambda F_\Lambda - \overline{F}_\Sigma X^\Sigma\bigr)\bigr] \tag{8.5.18}$$

The relation between the two definitions of special manifolds is obtained by introducing a non-holomorphic section of the bundle $\mathscr{H}$ according to:

$$V = \begin{pmatrix} L^\Lambda \\ M_\Sigma \end{pmatrix} \equiv e^{\mathscr{K}/2} \Omega = e^{\mathscr{K}/2} \begin{pmatrix} X^\Lambda \\ F_\Sigma \end{pmatrix} \tag{8.5.19}$$

so that (8.5.18) becomes:

$$1 = \mathrm{i}\langle V \mid \overline{V} \rangle = \mathrm{i}\bigl(\overline{L}^\Lambda M_\Lambda - \overline{M}_\Sigma L^\Sigma\bigr) \tag{8.5.20}$$

Since V is related to a holomorphic section by (8.5.19) it immediately follows that:

$$\nabla_{i^\star} V = \left(\partial_{i^\star} - \frac{1}{2}\partial_{i^\star}\mathscr{K}\right) V = 0 \tag{8.5.21}$$

On the other hand, from (8.5.20), defining:

$$\begin{aligned} U_i &= \nabla_i V = \left(\partial_i + \frac{1}{2}\partial_i \mathscr{K}\right) V \equiv \begin{pmatrix} f_i^{\Lambda} \\ h_{\Sigma|i} \end{pmatrix} \\ \overline{U}_{i^\star} &= \nabla_{i^\star} \overline{V} = \left(\partial_{i^\star} + \frac{1}{2}\partial_{i^\star} \mathscr{K}\right) \overline{V} \equiv \begin{pmatrix} \overline{f}_{i^\star}^{\Lambda} \\ \overline{h}_{\Sigma|i^\star} \end{pmatrix} \end{aligned} \tag{8.5.22}$$

it follows that:

$$\nabla_i U_j = \mathrm{i} C_{ijk} g^{k\ell^\star} \overline{U}_{\ell^\star} \tag{8.5.23}$$

where ∇_i denotes the covariant derivative containing both the Levi-Civita connection on the bundle $\mathscr{T}\mathscr{M}$ and the canonical connection θ on the line bundle $\mathscr{L}$. In (8.5.23) the symbol C_{ijk} denotes a covariantly holomorphic ($\nabla_{\ell^\star} C_{ijk} = 0$) section of the bundle $\mathscr{T}\mathscr{M}^3 \otimes \mathscr{L}^2$ that is totally symmetric in its indices. This tensor can be identified with the tensor of (8.5.13) appearing in (8.5.12). Alternatively, the set of differential equations:

$$\nabla_i V = U_i \tag{8.5.24}$$

$$\nabla_i U_j = \mathrm{i} C_{ijk} g^{k\ell^\star} U_{\ell^\star} \tag{8.5.25}$$

$$\nabla_{i^\star} U_j = g_{i^\star j} V \tag{8.5.26}$$

$$\nabla_{i^\star} V = 0 \tag{8.5.27}$$

with V satisfying (8.5.19), (8.5.20) give yet another definition of special geometry. In particular it is easy to find (8.5.12) as integrability conditions of (8.5.27).

8.5.4 The Vector Kinetic Matrix $\mathscr{N}_{\Lambda\Sigma}$ in Special Geometry

In the bosonic supergravity action (8.4.1) we do not see sections of any symplectic bundle over the scalar manifold but we see the real and imaginary parts of the matrix $\mathscr{N}_{\Lambda\Sigma}$ necessary in order to write the kinetic terms of the vector fields. Special geometry enters precisely at this level, since it is utilized to define such a matrix. Explicitly $\mathscr{N}_{\Lambda\Sigma}$ which, in relation with its interpretation in the case of Calabi-Yau three-folds, is named the *period matrix*, is defined by means of the following relations:

$$\overline{M}_\Lambda = \overline{\mathscr{N}}_{\Lambda\Sigma} \overline{L}^{\Sigma}; \qquad h_{\Sigma|i} = \overline{\mathscr{N}}_{\Lambda\Sigma} f_i^{\Sigma} \tag{8.5.28}$$

which can be solved introducing the two $(n+1) \times (n+1)$ vectors

$$f_I^{\Lambda} = \begin{pmatrix} f_i^{\Lambda} \\ \overline{L}^{\Lambda} \end{pmatrix}; \qquad h_{\Lambda|I} = \begin{pmatrix} h_{\Lambda|i} \\ \overline{M}_{\Lambda} \end{pmatrix} \tag{8.5.29}$$

and setting:

$$\overline{\mathcal{N}}_{\Lambda\Sigma} = h_{\Lambda|I} \circ \left(f^{-1}\right)^{I}{}_{\Sigma} \tag{8.5.30}$$

As a consequence of its definition the matrix $\mathcal{N}$ transforms, under diffeomorphisms of the base Kähler manifold, exactly as it is requested by the rule in (8.3.32). Indeed this is the very reason why the structure of special geometry has been introduced. The existence of the symplectic bundle $\mathcal{H} \longrightarrow \mathcal{M}$ is required in order to be able to pull-back the action of the diffeomorphisms on the field strengths and to construct the kinetic matrix $\mathcal{N}$.

8.6 Supergravities in Five Dimension and More Scalar Geometries

The renewed interest in five-dimensional gauged supergravities stems from two developments. On one hand we have the $\mathrm{AdS}_5/\mathrm{CFT}_4$ correspondence[11] between

(a) superconformal gauge theories in $D = 4$, viewed as the world volume description of a stack of D3-branes,
(b) type IIB supergravity compactified on AdS_5 times a five-dimensional internal manifold X^5 which yields a *gauged* supergravity model in $D = 5$.

On the other hand we have the quest for supersymmetric realizations of the Randall-Sundrum scenarios which also correspond to *domain wall* solutions of appropriate $D = 5$ gauged supergravities. It is, however, noteworthy that five dimensional supergravity has a long and interesting history. The minimal theory ($\mathcal{N} = 2$), whose field content is given by the metric $g_{\mu\nu}$, a doublet of pseudo Majorana gravitinos $\psi_{A\mu}$ ($A = 1, 2$) and a vector boson A_μ was constructed thirty years ago [8] as the first non-trivial example of a rheonomic construction.[12] This simple model remains to the present day the unique example of a perfectly geometric theory where, notwithstanding the presence of a gauge boson A_μ, the action can be written solely in terms of differential forms and wedge products without introducing Hodge duals. This feature puts pure $D = 5$ supergravity into a selective club of few ideal theories whose other members are just pure gravity in arbitrary dimension and pure $\mathcal{N} = 1$ supergravity in four dimensions. The miracle that allows the boson A_μ to propagate

[11] Since 1998 a rich stream of literature has been devoted to the so called AdS/CFT correspondence. In a nut-shell such a correspondence is rooted in the double interpretation of the groups $\mathrm{SO}(2, d-1)$ as anti de Sitter groups in d-dimensions and as conformal groups in $(d-1)$-dimensions. Such double interpretation has very far reaching consequences. At the end of a long chain of arguments it enables to evaluate exactly certain Green functions of appropriate quantum gauge-theories in $d-1$ dimensions by means of classical gravitational calculations in d-dimensions. In more general terms this correspondence is a kind of holography where boundary and bulk calculations can be interchanged.

[12] We leave aside pure $\mathcal{N} = 1$, $D = 4$ supergravity that from the rheonomic viewpoint is a completely trivial case.

without introducing its kinetic term is due to the conspiracy of first order formalism for the spin connection ω^{ab} together with the presence of two Chern-Simons terms. The first Chern Simons is the standard gauge one:

$$CS_{gauge} = F \wedge F \wedge A \tag{8.6.1}$$

while the second is a mixed, gravitational-gauge Chern Simons that reads as follows

$$CS_{mixed} = T^a \wedge F \wedge V_a \tag{8.6.2}$$

where V^a is the vielbein and $T^a = \mathscr{D}V^a$ is its *curvature*, namely the torsion.

The possible matter multiplets for $\mathscr{N} = 2$, $D = 5$ are the *vector/tensor* multiplets and the *hypermultiplets*. The field content of the first type of multiplets is the following one:

$$\left\{ \begin{array}{lll} A^I_\mu & (\mathbf{I} = 1, \ldots, n_V) & \text{vectors} \\ \lambda^i_A & \phi^i \ (i = 1, \ldots, n_V + n_T \equiv n) & (A = 1, 2) \\ B^{\mathscr{M}}_{\mu\nu} & (\mathscr{M} = 1, \ldots, n_T) & \text{tensors} \end{array} \right\} \tag{8.6.3}$$

where by n_V we have denoted the number of vectors or gauge 1-forms A^I_μ, n_T being instead the number of tensors or gauge 2-forms $B^{\mathscr{M}}_{\mu\nu} = -B^{\mathscr{M}}_{\nu\mu}$. In ungauged supergravity, where everything is Abelian, vectors and tensors are equivalent since they can be dualized into each other by means of the transformation:

$$\partial_{[\mu} A_{\nu]} = \varepsilon_{\mu\nu}{}^{\lambda\rho\sigma} \partial_\lambda B_{\rho\sigma} \tag{8.6.4}$$

but in gauged supergravity it is only the 1-forms that can be promoted to non-Abelian gauge vectors while the 2-forms describe massive degrees of freedom. The other members of each vector/tensor multiplet are a doublet of pseudo Majorana spin 1/2 fields:

$$\lambda^i_A = \varepsilon^{AB} \mathscr{C} \left(\overline{\lambda}^{iB}\right)^T; \qquad \overline{\lambda}^{iB} = \left(\lambda^i_B\right)^\dagger \gamma_0; \quad A, B = 1, \ldots, 2 \tag{8.6.5}$$

and a real scalar ϕ^i. The field content of hypermultiplets is the following:

$$\text{hypermultiplets} = \left\{ q^u (u = 1, \ldots, 4m), \zeta^\alpha (\alpha = 1, \ldots 2m) \right\} \tag{8.6.6}$$

where, having denoted m the number of hypermultiplets, q^u are m quadruplets of real scalar fields and ζ^α are m doublets of pseudo Majorana spin 1/2 fields:

$$\zeta^\alpha = \mathbb{C}^{\alpha\beta} \mathscr{C} (\overline{\zeta}_\beta)^T; \qquad \overline{\zeta}_\beta = \left(\zeta^\beta\right)^\dagger \gamma_0; \quad \alpha, \beta = 1, \ldots, 2m \tag{8.6.7}$$

the matrix $\mathbb{C}^T = -\mathbb{C}$, $\mathbb{C}^2 = -\mathbf{1}$ being the symplectic invariant metric of $\mathrm{Sp}(2m, \mathbb{R})$.

In the middle of the eighties Gunaydin, Sierra and Townsend [9, 10] considered the general structure of $\mathscr{N} = 2$, $D = 5$ supergravity coupled to an arbitrary number $n = n_V + n_T$ of vector/tensor multiplets and discovered that this is dictated by a peculiar geometric structure imposed by supersymmetry on the scalar manifold $\mathscr{SV}_n$ that contains the ϕ^i scalars. In modern nomenclature this peculiar geometry is named *very special geometry* and $\mathscr{SV}_n$ are referred to as real *very special manifolds*. The characterizing property of very special geometry arises from the need to reconcile the transformations of the scalar members of each multiplet with those of the vectors in presence of the Chern-Simons term (8.6.1) which generalizes to:

$$\mathscr{L}^{CS} = \frac{1}{8} d_{\Lambda\Sigma\Gamma} F^{\Lambda}_{\mu\nu} F^{\Sigma}_{\rho\sigma} A^{\Gamma}_{\tau} \varepsilon^{\mu\nu\rho\sigma\tau} \tag{8.6.8}$$

the symbol $d_{\Lambda\Sigma\Gamma}$ denoting some appropriate constant symmetric tensor and, having dualized all 2-forms to vectors, the range of the indices Λ, Σ, Γ being:

$$\Lambda = 1, \ldots, n+1 = \{\underbrace{0, I}_{\mathbf{I}}, \mathscr{M}\} \tag{8.6.9}$$

Indeed the total number of vector fields, including the graviphoton that belongs to the graviton multiplet, is always $n+1$, n being the number of vector multiplets. It turns out that very special geometry is completely defined in terms of the constant tensors $d_{\Lambda\Sigma\Gamma}$ that are further restricted by a condition ensuring positivity of the energy. At the beginning of the nineties special manifolds were classified and thoroughly studied by de Wit, Van Proeyen and some other collaborators [11–13] who also explored the dimensional reduction from $D=5$ to $D=4$, clarifying the way *real very special geometry* is mapped into the *special Kähler geometry* featured by vector multiplets in $D=4$ and generically related to Calabi-Yau moduli spaces.

The $4m$ scalars of the hypermultiplet sector have instead exactly the same geometry in $D=4$ as in $D=5$ dimensions, namely they fill a quaternionic manifold $\mathscr{QM}$. These scalar geometries are a crucial ingredient in the construction of both the ungauged and the gauged supergravity Lagrangians. Indeed the basic operations involved by the gauging procedure are based on the specific geometric structures of very special and quaternionic manifolds, in particular the crucial existence of a moment-map. For this reason the present section is devoted to a summary of these geometries and to an illustration of the general form of the bosonic $D=5$ Lagrangians. Yet, before entering these mathematical topics, we want to recall the structure of maximally extended ($\mathscr{N}=8$) supergravity in the same dimensions.

As explained above (see in particular Table 8.2) the scalar manifold of maximal supergravity in five-dimensions is the 42-dimensional homogeneous space:

$$M^{\max}_{scalar} = \frac{\mathrm{E}_{6(6)}}{\mathrm{USp}(8)} \tag{8.6.10}$$

The holonomy subgroup $\mathrm{H} = \mathrm{USp}(8)$ is the largest invariance group of complex linear transformations that respects the pseudo-Majorana condition on the 8 gravitino 1-forms:

$$\psi^A = \Omega^{AB} \mathscr{C} (\overline{\psi}_A)^T; \quad A = 1, \ldots, 8 \tag{8.6.11}$$

where $\Omega^{AB} = -\Omega^{BA}$ is an antisymmetric 8×8 matrix such that $\Omega^2 = -\mathbf{1}$. Using these notations where the capital Latin indices transform in the fundamental 8-representation of USp(8) we can summarize the field content of the $\mathcal{N} = 8$ graviton multiplet as:

1. the graviton field, namely the fünfbein 1-form V^a,
2. eight gravitinos $\psi^A \equiv \psi^A_\mu \, dx^\mu$ in the **8** representation of USp(8),
3. 27 vector fields $A^\Lambda \equiv A^\Lambda_\mu \, dx^\mu$ in the **27** of $\mathrm{E}_{6(6)}$[13],
4. 48 dilatinos χ^{ABC} in the **48** of USp(8),
5. 42 scalars ϕ that parameterize the coset manifold $E_{(6)6}/\mathrm{USp}(8)$. They appear in the theory through the coset representative $\mathbb{L}^{AB}_\Lambda(\phi)$, which is regarded as covariant in the $(\mathbf{27}, \overline{\mathbf{27}})$ of $\mathrm{USp}(8) \times \mathrm{E}_{6(6)}$. This means the following. Since the fundamental **27** (real) representation of $\mathrm{E}_{6(6)}$ remains irreducible under reduction to the subgroup $\mathrm{USp}(8) \subset \mathrm{E}_{6(6)}$ it follows that there exists a constant intertwining 27×27 matrix $\mathscr{I}^{AB}_\Sigma$ that transforms the index $^\Sigma$ running in the fundamental of $\mathrm{E}_{6(6)}$ into an antisymmetric pair of indices ^{AB} with the additional property that $C^{AB}\Omega_{AB} = 0$ which is the definition of the 27 of USp(8). The coset representative we use is to be interpreted as $\mathbb{L}^{AB}_\Lambda(\phi) = \mathbb{L}^\Sigma_\Lambda \mathscr{I}^{AB}_\Sigma$.

The construction of the ungauged theory proceeds then through well established general steps and the basic ingredients, namely the USp(8) connection in the 36 adjoint representation $\mathscr{Q}_A{}^B$ and the scalar vielbein $\mathscr{P}^{ABCD}$ (fully antisymmetric in $ABCD$) are extracted from the left-invariant 1-form on the scalar coset according to:

$$\begin{aligned} \mathbb{L}^{-1}{}_{AB}{}^{\Lambda} \, d\mathbb{L}_\Lambda{}^{CD} &= \mathscr{Q}_{AB}{}^{CD} + \mathscr{P}_{AB}{}^{CD} \\ \mathscr{Q}_{AB}{}^{CD} &= 2\delta^{[C}_{[A} \mathscr{Q}_{B]}{}^{D]} \\ \mathscr{P}_{AB}{}^{CD} &= \Omega_{AE}\Omega_{BF}\mathscr{P}^{EFCD} \end{aligned} \tag{8.6.12}$$

Independently from the number of supersymmetries we can write a general form for the bosonic action of any $D = 5$ *ungauged supergravity*, namely the following one:

$$\begin{aligned} \mathscr{L}^{(ungauged)}_{(D=5)} &= \sqrt{-g}\left(\frac{1}{2}R - \frac{1}{4}\mathscr{N}_{\Lambda\Sigma} F^\Lambda_{\mu\nu} F^{\Sigma|\mu\nu} + \frac{1}{2} g_{ij} \partial_\mu \phi^i \partial^\mu \phi^j\right) \\ &\quad + \frac{1}{8} d_{\Lambda\Sigma\Gamma} \varepsilon^{\mu\nu\rho\sigma\tau} F^\Lambda_{\mu\nu} F^\Sigma_{\rho\sigma} A^\Gamma_\tau \end{aligned} \tag{8.6.13}$$

where g_{ij} is the metric of the scalar manifold $\mathscr{M}_{scalar}$, $\mathscr{N}_{\Lambda\Sigma}(\phi)$ is a positive definite symmetric function of the scalars that under the isometry group G_{iso} of $\mathscr{M}_{scalar}$ transforms in $\bigotimes^2_{sym}\mathbf{R}$, having denoted by $\mathbf{R}$ a linear representation of G_{iso} to which the vector fields A^Γ are assigned. Finally $d_{\Lambda\Sigma\Gamma}$ is a three-index symmetric tensor

[13] In the ungauged theory all two-forms have been dualized to vectors.

invariant with respect to the representation **R**. At this point we invite the reader to compare the above statements with the general discussion of Sect. 8.2.1, in particular points **B** and **C**. As stated in (8.2.4) the automorphism group of $\mathscr{N}$-extended supersymmetry (which in $D = 5$ is $\mathrm{USp}(\mathscr{N})$ due to pseudo Majorana fermions) must be contained as a factor in the holonomy group of the scalar manifold. On the other hand the $(p_i + 1)$-forms must be assigned to linear representations D_i of the isometry group for $\mathscr{M}_{scalar}$. In our case having dualized the two forms we just have vectors, namely $(p + 1 = 1)$-forms and the representation **R** is the only D_i we need to discuss. In the four-dimensional case the construction of the Lagrangian was mainly dictated by the symplectic embedding of (8.3.50). Indeed, since the 1-forms are self-dual in $D = 4$, then the isometries of the scalar manifolds must be realized on the vectors as symplectic duality symmetries, according to the general discussion of Sect. 8.3. In five dimensions, where no such duality symmetry can be realized, the isometry of the scalar manifold has to be linearly realized on the vectors in such a way as to make it an exact symmetry of the Lagrangian (8.6.13). This explains why the kinetic matrix $\mathscr{N}$ must transform in the representation $\bigotimes^2_{sym}\mathbf{R}$.

In maximal $\mathscr{N} = 8$ supergravity the items involved in the construction of the bosonic Lagrangian have the following values:

1. The scalar metric is the $\mathrm{E}_{6(6)}$ invariant metric on the coset (8.6.10), namely:

$$g_{ij} = \frac{1}{6} P_i^{ABCD} P_{ABCD|j} \tag{8.6.14}$$

2. The vector kinetic metric is given by the following quadratic form in terms of the coset representative:

$$\mathscr{N}_{\Lambda\Sigma} = 4\big(\mathbb{L}_\Lambda{}^{AB}\,\mathbb{L}_\Sigma{}^{CD}\,\Omega_{AC}\Omega_{BD}\big) \tag{8.6.15}$$

3. The representation **R** is the fundamental 27 of $\mathrm{E}_{6(6)}$.
4. The tensor $d_{\Lambda\Sigma\Gamma}$ is the coefficient of the cubic invariant of $\mathrm{E}_{6(6)}$ in the 27 representation.

To see how the same items are realized in the case of an $\mathscr{N} = 2$ theory we have to introduce very special and quaternionic geometry. Just before entering this it is worth nothing that also the supersymmetry transformation rule of the gravitino field admits a general form (once restricted to the purely bosonic terms), namely:

$$\delta\psi_{A\mu} = \mathscr{D}_\mu \varepsilon_A - \frac{1}{3}\mathscr{T}^{\rho\sigma}_{AB}\left(g_{\mu\rho}\gamma_\sigma - \frac{1}{8}\varepsilon_{\mu\rho\sigma\lambda\nu}\gamma^{\lambda\nu}\right)\varepsilon^B \tag{8.6.16}$$

where the indices A, B run in the fundamental representation of the automorphism (R-symmetry) group $\mathrm{USp}(\mathscr{N})$ and the tensor $\mathscr{T}^{\rho\sigma}_{AB}$, antisymmetric both in AB and in $\rho\sigma$ and named the graviphoton field strength, is given by:

$$\mathscr{T}^{\rho\sigma}_{AB} = \Phi^\Lambda_{AB}(\phi)\mathscr{N}_{\Lambda\Sigma}F^{\Sigma|\rho\sigma} \tag{8.6.17}$$

the scalar field dependent tensor $\Phi^{\Lambda}_{AB}(\phi)$ being intrinsically defined as the coefficient of the term $\overline{\varepsilon}^{A}\psi^{B}_{\mu}$ in the supersymmetry transformation rule of the vector field A^{Λ}_{μ}, namely:

$$\delta A^{\Lambda}_{\mu} = \cdots + 2\mathrm{i}\Phi^{\Lambda}_{AB}(\phi)\overline{\varepsilon}^{A}\psi^{B}_{\mu} \tag{8.6.18}$$

From its own definition it follows that under isometries of the scalar manifold $\Phi^{\Lambda}_{AB}(\phi)$ must transform in the representation $\mathbf{R}$ of G_{iso} times $\bigwedge^{2}\mathscr{N}$ of the R-symmetry $\mathrm{USp}(\mathscr{N})$. In the case of $\mathscr{N}=8$ supergravity the tensor $\Phi^{\Lambda}_{AB}(\phi)$ is simply the inverse coset representative:

$$\Phi^{\Lambda}_{AB}(\phi) = \left(\mathbb{L}^{-1}\right)_{AB}{}^{\Lambda} \tag{8.6.19}$$

We see in the next subsection how the same object is generally realized in an $\mathscr{N}=2$ theory via very special geometry.

8.6.1 Very Special Geometry

Very special geometry is the peculiar metric and associated Riemannian structure that can be constructed on a very special manifold. By definition a *very special manifold* $\mathscr{V}\mathscr{S}_n$ is a real manifold of dimension n that can be represented as the following algebraic locus in $\mathbb{R}^{n+1}$:

$$1 = \mathrm{N}(X) \equiv \sqrt{d_{\Lambda\Sigma\Delta}X^{\Lambda}X^{\Sigma}X^{\Delta}} \tag{8.6.20}$$

where X^{Λ} ($\Lambda = 1,\ldots,n+1$) are the coordinates of $\mathbb{R}^{n+1}$ while

$$d_{\Lambda\Sigma\Delta} \tag{8.6.21}$$

is a *constant symmetric tensor* fulfilling some additional defining properties that we will recall later on.

A coordinate system ϕ^{i} on $\mathscr{V}\mathscr{S}_n$ is provided by any parametric solution of (8.6.20) such that:

$$X^{\Lambda} = X^{\Lambda}(\phi); \qquad \phi^{i} = \text{free}; \quad i = 1,\ldots,n \tag{8.6.22}$$

The *very special metric* on the very special manifold is nothing else but the pull-back on the algebraic surface (8.6.20) of the following $\mathbb{R}^{n+1}$ metric:

$$ds^{2}_{\mathbb{R}^{n+1}} = \mathscr{N}_{\Lambda\Sigma}\, dX^{\Lambda} \otimes dX^{\Sigma} \tag{8.6.23}$$

$$\mathscr{N}_{\Lambda\Sigma} \equiv -\partial_{\Lambda}\partial_{\Sigma}\ln \mathrm{N}(X) \tag{8.6.24}$$

In other words in any coordinate frame the coefficients of the very special metric are the following ones:

$$g_{ij}(\phi) = \mathscr{N}_{\Lambda\Sigma} f^{\Lambda}_{i} f^{\Sigma}_{j} \tag{8.6.25}$$

where we have introduced the new objects:

$$f_i^{\Lambda} \equiv \partial_i X^{\Lambda} = \frac{\partial}{\partial \phi^i} X^{\Lambda} \tag{8.6.26}$$

If we also define

$$F_{\Lambda} = \frac{\partial}{\partial X^{\Lambda}} \ln \mathrm{N}(X); \qquad h_{\Lambda i} \equiv \partial_i F_{\Lambda} \tag{8.6.27}$$

and introduce the $2(n+1)$-vectors:

$$U = \begin{pmatrix} X^{\Lambda} \\ F_{\Sigma} \end{pmatrix}; \qquad U_i = \partial_i U = \begin{pmatrix} f_i^{\Lambda} \\ h_{\Sigma i} \end{pmatrix} \tag{8.6.28}$$

taking a second covariant derivative it can be shown that the following identity is true:

$$\nabla_i U_j = \frac{2}{3} g_{ij} U + \sqrt{\frac{2}{3}} T_{ijk} g^{k\ell} U_{\ell} \tag{8.6.29}$$

where the world-index symmetric coordinate dependent tensor T_{ijk} is related to the constant tensor $d_{\Lambda\Gamma\Sigma}$ by:

$$d_{\Lambda\Gamma\Sigma} = \frac{20}{27} F_{\Lambda} F_{\Gamma} F_{\Sigma} - \frac{2}{3} \mathscr{N}_{(\Lambda\Gamma} F_{\Sigma)} + \frac{8}{27} T_{ijk} g^{ip} g^{jq} g^{kr} h_{\Lambda p} h_{\Gamma q} h_{\Sigma r} \tag{8.6.30}$$

The identity (8.6.29) is the real counterpart of a completely similar identity that holds true in special Kähler geometry and also defines a symmetric 3-index tensor. In the use of very special geometry to construct a supersymmetric field theory the essential property is the existence of the *section* $X^{\Lambda}(\phi)$. Indeed it is this object that allows the writing of the tensor $\Phi^{\Lambda}_{AB}(\phi)$ appearing in the vector transformation rule (8.6.18). It suffice to set:

$$\Phi^{\Lambda}_{AB}(\phi) = X^{\Lambda}(\phi) \varepsilon_{AB} \tag{8.6.31}$$

Why do we call it a section? Since it is just a section of a *flat vector bundle* of rank $n+1$

$$\mathrm{FB} \xrightarrow{\pi} \mathscr{SV}_n \tag{8.6.32}$$

with base manifold the very special manifold and structural group some subgroup of the $n+1$ dimensional linear group: $\mathrm{G}_{iso} \subset \mathrm{GL}(n+1, \mathbb{R})$. The bundle is flat because the transition functions from one local trivialization to another one are constant matrices:

$$\forall g \in \mathrm{G}_{iso}: \quad X^{\Lambda}(g\phi) = \left(M[g]\right)^{\Lambda}{}_{\Sigma} X^{\Sigma}(\phi); \quad M[g] = \text{constant matrix} \tag{8.6.33}$$

The structural group G_{iso} is implicitly defined as the set of matrices M that leave the $d_{\Lambda\Gamma\Sigma}$ tensor invariant:

$$M \in \mathrm{G}_{iso} \quad \Leftrightarrow \quad M_{\Lambda_1}{}^{\Sigma_1} M_{\Lambda_2}{}^{\Sigma_2} M_{\Lambda_3}{}^{\Sigma_3} d_{\Lambda_1\Lambda_2\Lambda_3} = d_{\Sigma_1\Sigma_2\Sigma_3} \tag{8.6.34}$$

Since the very special metric is defined by (8.6.25) it immediately follows that G_{iso} is also the isometry group of such a metric, its action in any coordinate patch (8.6.22) being defined by the action (8.6.33) on the section X^Λ. This fact explains the name given to this group.

By means of this reasoning we have shown that the classification of very special manifolds is fully reduced to the classification of the constant tensors $d_{\Lambda\Gamma\Sigma}$ such that their group of invariances acts transitively on the manifold $\mathscr{SV}_n$ defined by (8.6.20) and that the special metric (8.6.25) is positive definite. This is the algebraic problem that was completely solved by de Wit and Van Proeyen in [11]. They found all such tensors and the corresponding manifolds. There is a large subclass of very special manifolds that are homogeneous spaces but there are also infinite families of manifolds that are not G/H cosets.

8.6.2 *The Very Special Geometry of the* $\mathrm{SO}(1,1) \times \mathrm{SO}(1,n)/\mathrm{SO}(n)$ *Manifold*

As an example of very special manifold we consider the following class of homogeneous spaces:

$$RT[n] \equiv \mathrm{SO}(1,1) \times \frac{\mathrm{SO}(1,n)}{\mathrm{SO}(n)} \tag{8.6.35}$$

This example is particularly simple and relevant to string theory since reducing it on a circle S^1 from five to four dimensions one finds a supergravity model where the special Kähler geometry is that of

$$ST[2,n] = \frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)} \times \frac{\mathrm{SO}(2,n)}{\mathrm{SO}(2) \times \mathrm{SO}(n)} \tag{8.6.36}$$

which constitutes a primary example with very large applications.

To see that the $RT[n]$ are indeed very special manifolds we consider the following instance of cubic norm:

$$\mathrm{N}(X) = \sqrt{\mathrm{C}(X)} \tag{8.6.37}$$

$$\mathrm{C}(X) = X^0\big(X^+X^- - \mathbf{X}^2\big); \quad \mathbf{X}^2 = \sum_{\ell=1}^{r}\big(X^\ell\big)^2 \tag{8.6.38}$$

It is immediately verified that the infinitesimal linear transformations $X^\Lambda \to X^\Lambda + \delta X^\Lambda$ that leave the cubic polynomial $\mathrm{C}(X)$ invariant are the following ones:

$$\delta_\Delta \begin{pmatrix} X^0 \\ X^+ \\ X^- \\ \mathbf{X} \end{pmatrix} = \left(\begin{array}{c|ccc} -4 & 0 & 0 & 0 \\ \hline 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{array}\right) \begin{pmatrix} X^0 \\ X^+ \\ X^- \\ \mathbf{X} \end{pmatrix} \tag{8.6.39}$$

$$\delta_L \begin{pmatrix} X^0 \\ X^+ \\ X^- \\ \mathbf{X} \end{pmatrix} = \left(\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & -4 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \begin{pmatrix} X^0 \\ X^+ \\ X^- \\ \mathbf{X} \end{pmatrix} \tag{8.6.40}$$

$$\delta_{\mathbf{v}} \begin{pmatrix} X^0 \\ X^+ \\ X^- \\ \mathbf{X} \end{pmatrix} = \left(\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \mathbf{v}^T \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{v} & 0 \end{array}\right) \begin{pmatrix} X^0 \\ X^+ \\ X^- \\ \mathbf{X} \end{pmatrix} \tag{8.6.41}$$

$$\delta_{\mathbf{u}} \begin{pmatrix} X^0 \\ X^+ \\ X^- \\ \mathbf{X} \end{pmatrix} = \left(\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{v}^T \\ 0 & \mathbf{v} & 0 & 0 \end{array}\right) \begin{pmatrix} X^0 \\ X^+ \\ X^- \\ \mathbf{X} \end{pmatrix} \tag{8.6.42}$$

$$\delta_{\mathbf{A}} \begin{pmatrix} X^0 \\ X^+ \\ X^- \\ \mathbf{X} \end{pmatrix} = \left(\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{A} \end{array}\right) \begin{pmatrix} X^0 \\ X^+ \\ X^- \\ \mathbf{X} \end{pmatrix}; \quad \mathbf{A}^T = -\mathbf{A} \in \mathrm{SO}(r) \tag{8.6.43}$$

The transformation δ_Δ generates an SO(1, 1) group that commutes with the SO(1, $r+1$) group generated by the transformations $\delta_L, \delta_{\mathbf{u}}, \delta_{\mathbf{v}}$ and $\delta_{\mathbf{A}}$, hence the symmetry group of the symmetric tensor:

$$d_{\Lambda\Sigma\Gamma} = \begin{cases} d_{0+-} = 1 \\ d_{0\ell m} = -\delta_{\ell m} \\ 0 \quad \text{otherwise} \end{cases} \tag{8.6.44}$$

defined by the cubic polynomial $\mathrm{C}(X)$ is indeed the group $\mathrm{SO}(1,1) \times \mathrm{SO}(1,n)$. This is quite simple and evident. What is important is that the same group has also a transitive action on the manifold defined by the equation $\mathrm{C}(X) = 1$ that can be identified with the product $\mathrm{SO}(1,1) \times \mathrm{SO}(1,n)/\mathrm{SO}(2)$. To verify this statement it suffices to consider that the quadratic equation

$$H^+H^- - \mathbf{H}^2 = 1 \tag{8.6.45}$$

defines the homogeneous manifold $\mathrm{SO}(1,n)/\mathrm{SO}(2)$ on which $\mathrm{SO}(1,n)$ has a transitive action. For instance we can use as independent $r+1$ coordinates the following ones:

$$\phi^0 = H^+; \qquad \phi^\ell = H^\ell \quad (\ell = 1, \dots, r) \quad \Rightarrow \quad H^- = \frac{1+\phi^2}{\phi^0} \tag{8.6.46}$$

and then it suffices to set:

$$X^0[\sigma,\phi] = e^{-2\sigma}; \qquad \left(X^+, X^-, \mathbf{X}\right) = e^\sigma \left(H^+[\phi], H^-[\phi], \mathbf{H}[\phi]\right) \tag{8.6.47}$$

to obtain a parameterization of the section X in terms of coordinates σ, ϕ of the manifold $SO(1, 1) \times SO(1, n)/SO(2) \times SO(n)$. This achieves the desired proof that the group G_{iso} has a transitive action on the special manifold and consequently that the cubic norm (8.6.37), (8.6.38) is admissible as a definition of a very special manifold.

8.6.3 Quaternionic Geometry

Next we turn our attention to the hypermultiplet sector of an $\mathcal{N} = 2$ supergravity. For these multiplets no distinction arises between the $D = 4$ and $D = 5$. Each hypermultiplet contains 4 real scalar fields and, at least locally, they can be regarded as the four components of a quaternion. The locality caveat is, in this case, very substantial because global quaternionic coordinates can be constructed only occasionally even on those manifolds that are denominated quaternionic in the mathematical literature [6, 17]. Anyhow, what is important is that, in the hypermultiplet sector, the scalar manifold $\mathcal{QM}$ has dimension multiple of four:

$$\dim_{\mathbf{R}} \mathcal{QM} = 4m \equiv 4 \text{ \# of hypermultiplets} \tag{8.6.48}$$

and, in some appropriate sense, it has a quaternionic structure.

We name *Hypergeometry* that pertaining to the hypermultiplet sector, irrespectively whether we deal with global or local $\mathcal{N} = 2$ theories. Yet there are two kinds of hypergeometries. Supersymmetry requires the existence of a principal SU(2)-bundle

$$\mathcal{SU} \longrightarrow \mathcal{QM} \tag{8.6.49}$$

The bundle $\mathcal{SU}$ is *flat* in the *rigid supersymmetry case* while its curvature is proportional to the Kähler forms in the *local case*.

These two versions of hypergeometry were already known in mathematics prior to their use [14, 18–20, 22, 23] in the context of $\mathcal{N} = 2$ supersymmetry and are identified as:

$$\begin{aligned} \text{rigid hypergeometry} &\equiv \text{HyperKähler geometry} \\ \text{local hypergeometry} &\equiv \text{quaternionic geometry} \end{aligned} \tag{8.6.50}$$

8.6.4 Quaternionic, Versus HyperKähler Manifolds

Both a quaternionic or a HyperKähler manifold $\mathcal{QM}$ is a $4m$-dimensional real manifold endowed with a metric h:

$$ds^2 = h_{uv}(q)\, dq^u \otimes dq^v; \quad u, v = 1, \ldots, 4m \tag{8.6.51}$$

and three complex structures

$$\left(J^x\right) : T(\mathcal{QM}) \longrightarrow T(\mathcal{QM}) \quad (x = 1, 2, 3) \tag{8.6.52}$$

that satisfy the quaternionic algebra

$$J^x J^y = -\delta^{xy}\mathbb{1} + \varepsilon^{xyz} J^z \tag{8.6.53}$$

and respect to which the metric is Hermitian:

$$\forall \mathbf{X}, \mathbf{Y} \in T\mathscr{Q}\mathscr{M}: \quad h\big(J^x\mathbf{X}, J^x\mathbf{Y}\big) = h(\mathbf{X}, \mathbf{Y}) \quad (x = 1, 2, 3) \tag{8.6.54}$$

From (8.6.54) it follows that one can introduce a triplet of 2-forms

$$K^x = K^x_{uv}\, dq^u \wedge dq^v; \qquad K^x_{uv} = h_{uw}(J^x)^w_v \tag{8.6.55}$$

that provides the generalization of the concept of Kähler form occurring in the complex case. The triplet K^x is named the *HyperKähler* form. It is an SU(2) Lie-algebra valued 2-form in the same way as the Kähler form is a U(1) Lie-algebra valued 2-form. In the complex case the definition of Kähler manifold involves the statement that the Kähler 2-form is closed. At the same time in Hodge-Kähler manifolds (those appropriate to local supersymmetry in $D = 4$) the Kähler 2-form can be identified with the curvature of a line-bundle which in the case of rigid supersymmetry is flat. Similar steps can be taken also here and lead to two possibilities: either HyperKähler or quaternionic manifolds.

Let us introduce a principal SU(2)-bundle $\mathscr{S}\mathscr{U}$ as defined in (8.6.49). Let ω^x denote a connection on such a bundle. To obtain either a HyperKähler or a quaternionic manifold we must impose the condition that the HyperKähler 2-form is covariantly closed with respect to the connection ω^x:

$$\nabla K^x \equiv dK^x + \varepsilon^{xyz}\omega^y \wedge K^z = 0 \tag{8.6.56}$$

The only difference between the two kinds of geometries resides in the structure of the $\mathscr{S}\mathscr{U}$-bundle.

Definition 8.6.1 A HyperKähler manifold is a $4m$-dimensional manifold with the structure described above and such that the $\mathscr{S}\mathscr{U}$-bundle is *flat*.

Defining the $\mathscr{S}\mathscr{U}$-curvature by:

$$\Omega^x \equiv d\omega^x + \frac{1}{2}\varepsilon^{xyz}\omega^y \wedge \omega^z \tag{8.6.57}$$

in the HyperKähler case we have:

$$\Omega^x = 0 \tag{8.6.58}$$

Vice-versa

Definition 8.6.2 A quaternionic manifold is a $4m$-dimensional manifold with the structure described above and such that the curvature of the $\mathscr{S}\mathscr{U}$-bundle is proportional to the HyperKähler 2-form.

Hence, in the quaternionic case we can write:

$$\Omega^x = \lambda K^x \tag{8.6.59}$$

where λ is a non-vanishing real number.

As a consequence of the above structure the manifold $\mathcal{QM}$ has a holonomy group of the following type:

$$\begin{aligned} \mathrm{Hol}(\mathcal{QM}) &= \mathrm{SU}(2) \otimes \mathrm{H} \quad \text{(quaternionic)} \\ \mathrm{Hol}(\mathcal{QM}) &= \mathbb{1} \otimes \mathrm{H} \quad \text{(HyperKähler)} \\ \mathrm{H} &\subset \mathrm{Sp}(2m, \mathbb{R}) \end{aligned} \tag{8.6.60}$$

In both cases, introducing flat indices $\{A, B, C = 1, 2\}\{\alpha, \beta, \gamma = 1, \ldots, 2m\}$ that run, respectively, in the fundamental representation of SU(2) and of $\mathrm{Sp}(2m, \mathbb{R})$, we can find a vielbein 1-form

$$\mathcal{U}^{A\alpha} = \mathcal{U}_u^{A\alpha}(q)\, dq^u \tag{8.6.61}$$

such that

$$h_{uv} = \mathcal{U}_u^{A\alpha} \mathcal{U}_v^{B\beta} \mathbb{C}_{\alpha\beta} \varepsilon_{AB} \tag{8.6.62}$$

where $\mathbb{C}_{\alpha\beta} = -\mathbb{C}_{\beta\alpha}$ and $\varepsilon_{AB} = -\varepsilon_{BA}$ are, respectively, the flat $\mathrm{Sp}(2m)$ and $\mathrm{Sp}(2) \sim \mathrm{SU}(2)$ invariant metrics. The vielbein $\mathcal{U}^{A\alpha}$ is covariantly closed with respect to the SU(2)-connection ω^z and to some $\mathrm{Sp}(2m, \mathbb{R})$-Lie Algebra valued connection $\Delta^{\alpha\beta} = \Delta^{\beta\alpha}$:

$$\nabla \mathcal{U}^{A\alpha} \equiv d\mathcal{U}^{A\alpha} + \frac{i}{2}\omega^x \left(\varepsilon \sigma_x \varepsilon^{-1}\right)^A{}_B \wedge \mathcal{U}^{B\alpha} + \Delta^{\alpha\beta} \wedge \mathcal{U}^{A\gamma} \mathbb{C}_{\beta\gamma} = 0 \tag{8.6.63}$$

where $(\sigma^x)_A{}^B$ are the standard Pauli matrices. Furthermore $\mathcal{U}^{A\alpha}$ satisfies the reality condition:

$$\mathcal{U}_{A\alpha} \equiv \left(\mathcal{U}^{A\alpha}\right)^* = \varepsilon_{AB} \mathbb{C}_{\alpha\beta} \mathcal{U}^{B\beta} \tag{8.6.64}$$

Equation (8.6.64) defines the rule to lower the symplectic indices by means of the flat symplectic metrics ε_{AB} and $\mathbb{C}_{\alpha\beta}$. More specifically we can write a stronger version of (8.6.62) [24]:

$$\left(\mathcal{U}_u^{A\alpha} \mathcal{U}_v^{B\beta} + \mathcal{U}_v^{A\alpha} \mathcal{U}_u^{B\beta}\right) \mathbb{C}_{\alpha\beta} = h_{uv} \varepsilon^{AB} \tag{8.6.65}$$

We have also the inverse vielbein $\mathcal{U}_{A\alpha}^u$ defined by the equation

$$\mathcal{U}_{A\alpha}^u \mathcal{U}_v^{A\alpha} = \delta_v^u \tag{8.6.66}$$

Flattening a pair of indices of the Riemann tensor $\mathcal{R}^{uv}{}_{ts}$ we obtain

$$\mathcal{R}^{uv}{}_{ts}\, \mathcal{U}_u^{\alpha A} \mathcal{U}_v^{\beta B} = -\frac{\mathrm{i}}{2} \Omega_{ts}^x \varepsilon^{AC} (\sigma_x)_C{}^B\, \mathbb{C}^{\alpha\beta} + \mathbb{R}_{ts}^{\alpha\beta}\, \varepsilon^{AB} \tag{8.6.67}$$

Table 8.3 Homogeneous symmetric quaternionic manifolds

m	G/H	m	G/H
m	$\frac{\mathrm{Sp}(2m+2)}{\mathrm{Sp}(2)\times\mathrm{Sp}(2m)}$	2	$\frac{\mathrm{G}_2}{\mathrm{SO}(4)}$
m	$\frac{\mathrm{SU}(m,2)}{\mathrm{SU}(m)\times\mathrm{SU}(2)\times\mathrm{U}(1)}$	7	$\frac{\mathrm{F}_4}{\mathrm{Sp}(6)\times\mathrm{Sp}(2)}$
m	$\frac{\mathrm{SO}(4,m)}{\mathrm{SO}(4)\times\mathrm{SO}(m)}$	10	$\frac{\mathrm{E}_6}{\mathrm{SU}(6)\times\mathrm{U}(1)}$
		16	$\frac{\mathrm{E}_7}{\mathrm{SO}(12)\times\mathrm{SU}(2)}$
		28	$\frac{\mathrm{E}_8}{E_7\times\mathrm{SU}(2)}$

where $\mathbb{R}^{\alpha\beta}_{ts}$ is the field strength of the Sp$(2m)$ connection:

$$d\Delta^{\alpha\beta} + \Delta^{\alpha\gamma} \wedge \Delta^{\delta\beta}\mathbb{C}_{\gamma\delta} \equiv \mathbb{R}^{\alpha\beta} = \mathbb{R}^{\alpha\beta}_{ts}\, dq^t \wedge dq^s \tag{8.6.68}$$

Equation (8.6.67) is the explicit statement that the Levi Civita connection associated with the metric h has a holonomy group contained in SU(2) $\otimes$ Sp$(2m)$. Consider now (8.6.53), (8.6.55) and (8.6.59). We easily deduce the following relation:

$$h^{st} K^x_{us} K^y_{tw} = -\delta^{xy} h_{uw} + \varepsilon^{xyz} K^z_{uw} \tag{8.6.69}$$

that holds true both in the HyperKähler and in the quaternionic case. In the latter case, using (8.6.59), (8.6.69) can be rewritten as follows:

$$h^{st} \Omega^x_{us} \Omega^y_{tw} = -\lambda^2 \delta^{xy} h_{uw} + \lambda \varepsilon^{xyz} \Omega^z_{uw} \tag{8.6.70}$$

Equation (8.6.70) implies that the intrinsic components of the curvature 2-form Ω^x yield a representation of the quaternion algebra. In the HyperKähler case such a representation is provided only by the HyperKähler form. In the quaternionic case we can write:

$$\Omega^x_{A\alpha, B\beta} \equiv \Omega^x_{uv} \mathcal{U}^u_{A\alpha} \mathcal{U}^v_{B\beta} = -i\lambda \mathbb{C}_{\alpha\beta} (\sigma_x)_A{}^C \varepsilon_{CB} \tag{8.6.71}$$

Alternatively (8.6.71) can be rewritten in an intrinsic form as

$$\Omega^x = -\mathrm{i}\lambda \mathbb{C}_{\alpha\beta} (\sigma_x)_A{}^C \varepsilon_{CB} \mathcal{U}^{\alpha A} \wedge \mathcal{U}^{\beta B} \tag{8.6.72}$$

whence we also get:

$$\frac{i}{2} \Omega^x (\sigma_x)_A{}^B = \lambda \mathcal{U}_{A\alpha} \wedge \mathcal{U}^{B\alpha} \tag{8.6.73}$$

The quaternionic manifolds are not requested to be homogeneous spaces, however there exists a subclass of quaternionic homogeneous spaces that are displayed in Table 8.3.

8.7 $\mathcal{N}=2$, $D=5$ Supergravity Before Gauging

Relying on the geometric lore developed in the previous sections it is now easy to state what is the bosonic Lagrangian of a general $\mathcal{N}=2$ theory in five-dimensions. We just have to choose an n-dimensional very special manifold and some quaternionic manifold $\mathcal{QM}$ of quaternionic dimension m. Then recalling (8.6.13) we can specialize it to:

$$\begin{aligned}\mathcal{L}^{(ungauged)}_{(D=5,\mathcal{N}=2)} = \sqrt{-g}\Bigg(&\frac{1}{2}R - \frac{1}{4}\mathcal{N}_{\Lambda\Sigma}(\phi)F^{\Lambda}_{\mu\nu}F^{\Sigma|\mu\nu} \\ &+ \frac{1}{2}g_{ij}(\phi)\partial_\mu\phi^i\partial^\mu\phi^j + \frac{1}{2}h_{uv}(q)\partial_\mu q^u\partial^\mu q^v\Bigg) \\ &+ \frac{1}{8}d_{\Lambda\Sigma\Gamma}\varepsilon^{\mu\nu\rho\sigma\tau}F^{\Lambda}_{\mu\nu}F^{\Sigma}_{\rho\sigma}A^{\Gamma}_{\tau}\end{aligned} \tag{8.7.1}$$

where $h_{uv}(q)$ is the quaternionic metric on the quaternionic manifold $\mathcal{QM}$, while $g_{ij}(\phi)$ is the very special metric on the very special manifold. At the same time the constant tensor $d_{\Lambda\Sigma\Gamma}$ is that defining the cubic norm (8.6.20) while the kinetic metric $\mathcal{N}$ is that defined in (8.6.24). The transformation rule of the gravitino field takes the general form (8.6.16) with the graviphoton defined as in (8.6.17) and the tensor Φ^{Λ}_{AB} given by (8.6.31). In this respect it is noteworthy that gravitino supersymmetry transformation rule does depend only on the vector multiplet scalars and it is independent from the hypermultiplets.

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Chapter 9
Supergravity: An Anthology of Solutions

O tiger's heart wrapped in a woman's hide
William Shakespeare

9.1 Introduction

As we have seen, supergravity theories are just ordinary field theories providing the coupling to Einstein Gravity of a collection of lower spin fields in diverse dimensions, with a variety of self-couplings that are controlled by a web of special geometric structures springing from the scalar sector of the Lagrangian. Ultimately, responsible for the specific choice of these *special geometries* and for the geometric organization of the field-theory are the constraints imposed by supersymmetry, yet the general pattern that, through a historically process, has been unveiled in supergravity, might have a more general validity. There are probably, in the same pattern, further cases of interest that violate supersymmetry but which, without the lesson taught by it, might not have been dreamed of. For instance the class of $D = 4$ theories of the form (8.4.1) is larger than the set obtained as bosonic sectors of supergravity Lagrangians. In supergravity there are special choices for the scalar manifolds that, as recalled in (8.4.2), are symmetric coset manifolds in a large number of cases, yet the list of symmetric cosets is not exhausted by supergravity. As long as the numerator group G can be symplectically embedded in $\mathrm{Sp}(2n_{\mathrm{V}}, \mathbb{R})$ according to (8.3.50), the Gaillard-Zumino master formula (8.3.67) for the kinetic matrix $\mathscr{N}_{\Lambda\Sigma}(\phi)$ holds true and all physical consequences encoded in the duality symmetries follow as well.

Supergravity theories form also an interrelated web. The main connection between Lagrangians in diverse dimensions is provided by compactification and dimensional reduction. A D-dimensional gravitational theory containing p-forms that, as we know, are related to $(p + 1)$-branes, can admit vacuum-solutions of the form:

$$\mathscr{M}_D^{(0)} = \mathscr{M}_d^{(0)} \times \mathscr{M}_{D-d} \tag{9.1.1}$$

where $\mathscr{M}_d^{(0)}$ is a maximally symmetric manifold in d-dimensions and $\mathscr{M}_{D-d}$ is some suitable *compact Einstein manifold* in the complementary dimensions. Typically the splitting $D \Rightarrow d \oplus (D - d)$ and the very existence of the vacuum solution

P.G. Frè, *Gravity, a Geometrical Course*, DOI 10.1007/978-94-007-5443-0_9,

is induced by giving a vacuum expectation value

$$\langle F_{a_1 \ldots a_d} \rangle_0 \neq 0 \tag{9.1.2}$$

to the field strength of a $(d-1)$-form $\mathbf{A}^{(d-1)}$ of the higher dimensional theory. Expanding the higher dimensional theory in modes around such a vacuum and keeping only the lightest ones, one obtains a new gravitational theory in d-dimensions including a variety of new fields, whose interactions are dictated by the geometry of $\mathscr{M}_{D-d}$. In particular the geometry of the scalar manifold $\mathscr{M}_{scalar}$ of the lower dimensional theory which, as we know, controls the entire form of the d-dimensional Lagrangian, is related to $\mathscr{M}_{D-d}$ in the following general way: $\mathscr{M}_{scalar}$ encodes the moduli-space of the structure-deformations of $\mathscr{M}_{D-d}$. Let us explain this deep and general concept. For instance $\mathscr{M}_{D-d}$ is an Einstein manifold. This means that we have a metric $g_{ij}(y)$ defined on it, whose Ricci tensor is proportional to the same metric. That metric can be smoothly deformed by means of parameters that we name moduli and fill a subspace of $\mathscr{M}_{scalar}$. The compact manifold $\mathscr{M}_{D-d}$ can have a more refined geometrical structure, a complex structure for instance, a Kähler structure or in any case a restricted holonomy structure. The deformations of all such structures fill moduli space which are included in $\mathscr{M}_{scalar}$. The special geometry structure of $\mathscr{M}_{scalar}$ follows mathematically from deformation theory.

This scheme, that goes under the name of *flux compactification*, has been unveiled in supergravity but has a more general validity. The key ingredients are:

(a) The existence of p-forms in higher dimensional theories whose fluxes can drive the compactification.
(b) The choice of restricted holonomy manifolds $\mathscr{M}_{D-d}$ for the compact dimensions.

In supersymmetric theories the spectrum of p-forms available in D-dimensions is dictated by the appropriate Free Differential Algebra which, as we learnt in Chap. 6, is ultimately a yield of the super Poincaré Lie algebra cohomology. Without supersymmetry, Free Differential Algebras do exist nonetheless and a more general variety of possibilities is available for the p-form gauge fields.

Similarly, in supersymmetric theories the constraint on the holonomy of the internal manifold $\mathscr{M}_{D-d}$ follows from the request that some of the supersymmetries should be preserved by the compactification. This requires the existence of so named *Killing spinors*, namely of covariantly constant sections of an appropriate spinor bundle on $\mathscr{M}_{D-d}$, whose existence restricts the holonomy. In a more general mathematical set up this is just an instance of the constraints imposed by the existence of some G-structure. Adopting such a language in the context of the more general class of higher dimensional theories postulated above opens a wider spectrum of possibilities.

In such a broader landscape the main mathematical frameworks governing both the construction of the relevant Lagrangians and the search and classification of their solutions will still be the same as in supergravity, namely *deformation theory of special geometrical structures*, *restricted holonomy* and G-*structures*, σ*-model reduction* of duality symmetric Lagrangians (8.4.1).

The recipe to insert almost all of the most advanced aspects of modern differential geometry into Gravity Theory has been discovered by supergravity but certainly will last as an integral part of it even if our own world should turn out to be non-supersymmetric. The same is probably true of the *branes* whose existence and duality with the bulk theories is more general and holds true beyond superstrings and supergravity.

For this reason the last chapter of this book is devoted to *glances* at the *classical solutions* of supergravity. These form an incredibly rich park with many alleys, islands and ponds. There are *vacua solutions*, *brane-solutions*, that were already touched upon, *monopole-solutions*, *instanton solutions*, *cosmological solutions*, *black-hole* and *black-brane* solutions and still several other type of geometrical backgrounds. Each of these categories plays a distinctive important role in superstring/supergravity theory and requires appropriate mathematical techniques to be studied and worked out. Even a simple review of the main features of each category would build up a bestiary too long and too much complicated to be presented within the scope of the present book. Hence we necessarily restricted ourselves to an anthology chosen according to the formative criteria that inspire our writing. Indeed we aim at conveying to the reader some general ideas and some paradigms that, according to the writer's opinion, encapsulate an intrinsically new quality in the understanding of Gravity Theory and introduce new important mathematical structures in its development. Notwithstanding these restrictive conditions, the list of candidate topics and examples came out quite long so that, a little bit arbitrarily, a final short list of three items was drawn, far from being exhaustive, yet providing a very dense conceptual impact.

1. The first addressed topic is that of spherical black solutions in $D = 4$ supergravity. The interest in this class of solutions, whose classification and construction constitutes an active field of current research, is two-fold. From the technical point of view, the most effective approach to the derivation of these solutions, that depend only on one radial coordinate, is provided by the reduction of the supergravity field equations to those of an effective σ-model which, in the case that the scalar manifold $\mathcal{M}_{scalar}$ is equal to a symmetric coset $\frac{\mathrm{G}}{\mathrm{H}}$, were proved to be completely integrable. The same σ-model reduction can be applied also to the case of other *few parameter* solutions, like the cosmological ones, yielding the very interesting phenomenon of *cosmic billiards*, mentioned in Chap. 5. From the conceptual point of view the main new quality encapsulated in these studies is given by the attraction mechanism[1] and by the identification of the extremal black hole entropy with the square root of a *quartic symplectic invariant* constructed with the electromagnetic charges of the hole. This phenomenon goes beyond supersymmetry and is just related with the symplectic structure of the

[1] As we illustrate below the attraction mechanism corresponds to the following notable property of supergravity black holes which was discovered by Ferrara and Kallosh in 1995 [1, 2]: independently from their values at spatial infinity, the scalar fields flow to universal fixed values at the event horizon, dictated solely by the electromagnetic charges of the hole.

duality symmetric theories of type (8.4.1). In a wider contest the charges of the hole can be interpreted in terms of branes and brane-wrappings, thus opening an important window on the statistical interpretation of black-holes. In the next pages we just try to introduce the reader to these fundamental ideas, providing some glimpses of this challenging research field.

2. The second topic addressed is that of *flux vacuum solutions* of M-theory. Supergravity and superstrings impose higher space-time dimensions and $D = 11$ is the maximal one where supergravity takes its simplest and most elegant form. Yet our world is effectively four-dimensional so that any contact with reality can be established only if seven of the extra dimensions are compactified and made observable only at energy scales of the order of the Planck mass. A challenging mechanism is provided by flux compactifications encoded in (9.1.2). Just because M-theory contains a three-form and a six-form, giving a vacuum expectation value to their field strengths splits eleven dimensional space-time into $4 + 7$. At the beginning of the eighties this raised a lot of expectations that produced a large literature going under the name of Kaluza-Klein Supergravity. Although the hope that the standard model of non-gravitational interactions might be retrieved in this way proved too naive, yet the (flux) compactifications of M-theory provide to the present day a very important theoretical laboratory in connection with the gauge/gravity correspondence and with several other aspects of brane physics. From the conceptual and mathematical point of view, the problem of constructing these vacua and classifying their residual supersymmetry brings in the theory of *Killing spinors*, G-structures and *restricted holonomy*. Introducing the reader to these concepts and to their use is the main reason of considering this example. An additional reason resides in the opportunity offered by these examples of deepening our understanding of rheonomy. According to what we explained in Chap. 6, every classical solution of the space-time field equations can be extended to a full superspace solution by integrating the rheonomic conditions. The result is guaranteed but how to do it in practice is a different question. We will show that the integration is immediate and leads to the Maurer Cartan forms of a supercoset manifold in all those θ-directions that correspond to preserved supersymmetries. The θ-integration in the direction of broken supersymmetries is instead more involved and corresponds to some non-trivial fiberings. Our goal is to illustrate this mechanism locating the obstruction both to θ-integration and to supersymmetry preservation, which is the same thing, in a well-defined geometrical datum that is the holonomy tensor.
3. The third addressed topic is similar to the second, dealing with a particular instance of *flux vacuum solution* of type IIA supergravity, namely that on the product manifold $\mathrm{AdS}_4 \times \mathbb{P}^3$. This is done purposely in order to emphasize both the similarities and the essential new features one encounters while solving the same problem in $D = 10$ rather than in $D = 11$. The novelty is provided by a necessary internal flux of the $\mathbf{G}^{[2]}$ Ramond form which pairs with the external flux of the $\mathbf{G}^{[4]}$ Ramond form and which is possible only due to the Kähler structure of the internal manifold $\mathbb{P}^3$. The example of this compactification has an intrinsic value since it corresponds to a situation where we end up with $\mathcal{N} = 6$

gauged-supergravity in $D=4$ from the bulk point of view and we have instead $D2$-branes and Ramond strings from the boundary point of view. Recent work on the AdS/CFT duality with $\mathscr{N}=6$ conformal field theories in three dimensions was indeed centered on this solution of type IIA supergravity.

9.2 Black Holes Once Again

As announced in the introduction the first type of supergravity solutions we consider are the spherical symmetric black holes in $D=4$. The motivations and perspective of this choice were explained above. The technique to obtain such solutions consists in the mapping of the supergravity field equations into those of a σ-model on an appropriate target manifold. This technique allowed to establish a complete integration algorithm that provides all solutions and their full-fledged classification [15–18]. We will not dwell on such integration algorithm and confine ourselves to present the oxidation rules from the σ-model to the actual supergravity configurations. We will also present, without derivation, some examples of exact solutions, our goal being the illustration of the attraction mechanism and the emergence of the quartic invariant as codifier of the black hole entropy.

9.2.1 The σ-Model Approach to Spherical Black Holes

A very powerful token in deriving solutions of supergravity that depend only on one effective parameter, like spherical symmetric black-holes depending only on a radial coordinate r or cosmological configurations depending only on time t, is provided by the reduction of the four-dimensional field equations to those of an effective one-dimensional σ-model. In this section we shortly review such a procedure for the spherical black hole case.

Let us consider the generic form of the bosonic Lagrangian of an ungauged $D=4$ supergravity as given in (8.4.1). Besides the metric field $g_{\mu\nu}(x)$, the theory contains n_s scalar fields and n_{v} vector fields. The geometric data specifying the Lagrangian and hence all interactions are the metric $h_{rs}(\phi)$ of the n_s-dimensional scalar manifold $\mathscr{M}_{scalar}$ which, for $\mathscr{N}>2$ is necessarily a symmetric coset manifold, while for $\mathscr{N}=2$ is any special Kähler manifold $\mathscr{S}\mathscr{K}_n$, and the kinetic $n_{\mathrm{v}}\times n_{\mathrm{v}}$ matrix $\mathscr{N}_{\Lambda\Sigma}(\phi)$ which, for all coset manifold cases is given by the Gaillard-Zumino master formula (8.3.67), while for the generic special Kähler case admits the definition given in (8.5.30). In all $\mathscr{N}=2$ cases the number of vector fields in the theory is $n_{\mathrm{v}}=n+1$ where n is the complex dimension of the scalar manifold ($n_s=2n$), while in the case of other theories the relation between n_{v} and n_s is different. Notwithstanding this difference, we can always introduce a $2n_{\mathrm{v}}\times 2n_{\mathrm{v}}$ field dependent matrix $\mathscr{M}_4$ defined as follows:

$$\mathscr{M}_4=\left(\begin{array}{c|c}\mathrm{Im}\,\mathscr{N}^{-1} & \mathrm{Im}\,\mathscr{N}^{-1}\,\mathrm{Re}\,\mathscr{N}\\ \hline \mathrm{Re}\,\mathscr{N}\,\mathrm{Im}\,\mathscr{N}^{-1} & \mathrm{Im}\,\mathscr{N}+\mathrm{Re}\,\mathscr{N}\,\mathrm{Im}\,\mathscr{N}^{-1}\,\mathrm{Re}\,\mathscr{N}\end{array}\right) \tag{9.2.1}$$

$$\mathscr{M}_4^{-1} = \left(\begin{array}{c|c} \mathrm{Im}\,\mathscr{N} + \mathrm{Re}\,\mathscr{N}\,\mathrm{Im}\,\mathscr{N}^{-1}\,\mathrm{Re}\,\mathscr{N} & -\mathrm{Re}\,\mathscr{N}\,\mathrm{Im}\,\mathscr{N}^{-1} \\ \hline -\mathrm{Im}\,\mathscr{N}^{-1}\,\mathrm{Re}\,\mathscr{N} & \mathrm{Im}\,\mathscr{N}^{-1} \end{array} \right) \tag{9.2.2}$$

and we can introduce the following set of $2+n_s+2n_{\mathrm{v}}$ fields depending on a parameter τ which later will be interpreted as the inverse of the radial coordinate $\tau \propto 1/r$:

	Generic		$\mathscr{N}=2$
Warp factor	$U(\tau)$	1	1
Taub-Nut field	$a(\tau)$	1	1
$D=4$ scalars	ϕ^i	n_s	$2n$
Scalars from vectors	$Z^M(\tau) = (Z^\Lambda(\tau), Z_\Sigma(\tau))$	$2n_{\mathrm{v}}$	$2n+2$
Total		$2+n_s+2n_{\mathrm{v}}$	$4n+4$

the fields $\{U, a, \phi, Z\}$ are interpreted as the coordinates of a new $(2+n_s+2n_{\mathrm{v}})$-dimensional manifold $\mathscr{Q}$, whose metric we declare to be the following:

$$ds^2_{\mathscr{Q}} = \frac{1}{4}\left[dU^2 + h_{rs}\,d\phi^r\,d\phi^s + e^{-2U}\left(da + \mathbf{Z}^T\,\mathbb{C}\,d\mathbf{Z}\right)^2 + 2e^{-U}\,d\mathbf{Z}^T\,\mathscr{M}_4\,d\mathbf{Z}\right] \tag{9.2.3}$$

having denoted by $\mathbb{C}$ the constant symplectic invariant metric in $2n_{\mathrm{v}}$ dimensions that underlies the construction of the matrix $\mathscr{N}_{\Lambda\Sigma}$.

Solutions of the one-dimensional σ-model are just geodesics of the above metric which has the following indefinite signature

$$\mathrm{sign}\left[ds^2_{\mathscr{Q}}\right] = (\underbrace{+,\dots,+}_{2+2n_s},\underbrace{-,\dots,-}_{2n_{\mathrm{v}}+2}) \tag{9.2.4}$$

since the matrix $\mathscr{M}_4$ is negative definite. Hence the geodesics can be time-like, null-like or space-like depending on the three possible cases:

$$\mathscr{L} = \dot{U}^2 + h_{rs}\dot{\phi}^r\dot{\phi}^s + e^{-2U}\left(\dot{a} + \mathbf{Z}^T\mathbb{C}\dot{\mathbf{Z}}\right)^2 + 2e^{-U}\dot{\mathbf{Z}}^T\mathscr{M}_4\dot{\mathbf{Z}} = \begin{cases} v^2 > 0 \\ v^2 = 0 \\ -v^2 < 0 \end{cases} \tag{9.2.5}$$

where the dot denotes derivative with respect to the affine parameter τ.

Every solution of the Euler Lagrangian equations:

$$\begin{aligned} \frac{d}{d\tau}\frac{d\mathscr{L}}{d\dot{\Phi}} &= \frac{d\mathscr{L}}{d\Phi} \\ \Phi(\tau) &\equiv \left\{U, a, \phi^r, Z^M\right\} \end{aligned} \tag{9.2.6}$$

defines a geodesic and provides a solution of the original supergravity field equations according to an oxidation rule that we will specify few lines below. Space-like geodesics correspond to unphysical supergravity solutions with naked singularities and are excluded. Time-like geodesics correspond to non-extremal black-holes while null-like geodesics yield extremal black-holes.

9.2.2 The Oxidation Rules

The $D=4$ solution of supergravity is parameterized in the following way in terms of the σ-model fields. For the metric we have:

$$ds^2_{(4)} = -e^{U(\tau)}(dt + A_{KK})^2 + e^{-U(\tau)}\left[e^{4A(\tau)}\,d\tau^2 + e^{2A(\tau)}\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right] \tag{9.2.7}$$

where $e^{2A(\tau)}$ is a shorthand notation for the following function:

$$e^{2A(\tau)} = \begin{cases} \frac{v^2}{\sinh^2(v\tau)} & \text{if } v^2 > 0 \\ \frac{1}{\tau^2} & \text{if } v^2 = 0 \end{cases} \tag{9.2.8}$$

The parameter v^2 mentioned in the above formula is one of the conserved charges of the dynamical model and it is named the extremality parameter. Its geometrical interpretation within the framework of the σ-model is very simple and clear.

To complete the illustration of the metric (9.2.7) we still have to explain the meaning of the one-form A_{KK}. This latter is the Kaluza-Klein vector, whose field strength $F_{KK} = dA_{KK}$ has just one non-vanishing component $F_{KK} = F_{KK|\theta\varphi}\,d\theta \wedge d\varphi$ given by the following expression:

$$F_{KK|\theta\varphi} = g_{\theta\theta}\,g_{\varphi\varphi}\,F^{\theta\varphi}_{KK} = -\sin\theta\,\underbrace{\left[e^{-2U}\left(\dot{a} + Z^\Lambda \dot{Z}_\Lambda - Z_\Sigma \dot{Z}^\Sigma\right)\right]}_{\mathbf{n}=\text{Taub-NUT charge}} \tag{9.2.9}$$

Actually one can verify that the combination of derivatives under-braced in equation (9.2.9) is a constant of motion of the system defined by the Lagrangian (9.2.5) and is named $\mathbf{n}$, the Taub-NUT charge. The fact that $\mathbf{n}$ is a constant is very important and obligatory in order for the dualization formulae to make sense. Indeed the Kaluza-Klein field strength F_{KK} satisfies the Bianchi identity only in force of the constancy of $\mathbf{n}$. In view of this the Kaluza-Klein vector is easily determined and reads:

$$A_{KK} = 2\mathbf{n}\cos\theta\,d\varphi \tag{9.2.10}$$

The field-strength two-form is instead:

$$F_{KK} = -2\mathbf{n}\sin\theta\,d\theta \wedge d\varphi \tag{9.2.11}$$

This concludes the illustration of the metric.

We still have to describe the parameterization of the gauge fields by means of the σ-model scalar fields. This is done in complete analogy to the case of the Kaluza-Klein vector. The $D=4$ field-strength two-forms are the following ones:

$$\left(\widehat{F}^\Lambda\right)^{\theta\varphi} = \sin\theta\,\underbrace{\left[e^{-2U}\left(\operatorname{Im}\mathcal{N}^{-1}\right)^{\Lambda\Sigma}\left(\dot{Z}_\Sigma + \operatorname{Re}\mathcal{N}_{\Sigma\Gamma}\dot{Z}^\Gamma\right)\right]}_{p^\Lambda=\text{magnetic charges}} \tag{9.2.12}$$

Similarly to the case of the Kaluza-Klein vector, the combinations of derivatives and fields under-braced in the above formula are constants of motion of the dynamical system defined by the Lagrangian (9.2.5) and have the interpretation of magnetic charges. Indeed the magnetic charges are just the upper n_{v} components of the full $2n_{\mathrm{v}}$ vector of magnetic and electric charges. This latter is defined as follows:

$$\mathscr{Q}^M = \sqrt{2}\big[e^{-U}\mathscr{M}_4\dot{\mathrm{Z}} - \mathbf{n}\mathbb{C}\mathrm{Z}\big]^M = \begin{pmatrix} p^\Lambda \\ e_\Sigma \end{pmatrix} \tag{9.2.13}$$

and all of its components are constants of motion.

In view of this the final form of the $D=4$ field-strengths is the following one:

$$F^\Lambda = 2p^\Lambda \sin\theta\, d\theta \wedge d\varphi + \dot{Z}^\Lambda d\tau \wedge (dt + 2\mathbf{n}\cos\theta\, d\varphi) \tag{9.2.14}$$

This concludes the review of the oxidation formulae that allow to write all the fields of $D=4$ supergravity corresponding to a black-hole solution in terms of the fields parameterizing the σ-model defined by (9.2.5).

One very important point to be stressed is that the metric (9.2.3) admits a typically large group of isometries. Certainly it admits all the isometries of the original scalar manifold $\mathscr{M}_{scalar}$ enlarged with additional ones related to the new fields that have been introduced $\{U, a, Z^M\}$. In the case when the $D=4$ scalar manifold is a homogeneous symmetric space:

$$\mathscr{M}_{scalar} = \frac{\mathrm{U}_{D=4}}{\mathrm{H}_{D=4}} \tag{9.2.15}$$

One can show [4, 5, 15], that the manifold $\mathscr{Q}$ with the metric (9.2.3) is a new homogeneous symmetric space

$$\mathscr{Q} = \frac{\mathrm{U}_\sigma}{\mathrm{H}^\star} \tag{9.2.16}$$

whose structure is universal and can be described in general terms.

General Structure of the $\mathbb{U}_\sigma$ Lie Algebra The Lie algebra $\mathbb{U}_\sigma$ of the numerator group always contains, as subalgebra, the duality algebra $\mathbb{U}_{D=4}$ of the parent supergravity theory in $D=4$ and a universal $\mathrm{sl}(2,\mathbb{R})_E$ algebra which is associated with the gravitational degrees of freedom $\{U, a\}$. Furthermore, with respect to this subalgebra $\mathbb{U}_\sigma$ admits the following universal decomposition, holding for all $\mathscr{N}$-extended supergravities:

$$\mathrm{adj}(\mathbb{U}_\sigma) = \mathrm{adj}(\mathbb{U}_{D=4}) \oplus \mathrm{adj}\big(\mathrm{SL}(2,\mathbb{R})_E\big) \oplus W_{(2,\mathbf{W})} \tag{9.2.17}$$

where $\mathbf{W}$ is the *symplectic* representation of $\mathbb{U}_{D=4}$ to which the electric and magnetic field strengths are assigned. Indeed the scalar fields associated with the generators of $W_{(2,\mathbf{W})}$ are just those coming from the vectors in $D=4$. Denoting the generators of $\mathbb{U}_{D=4}$ by T^a, the generators of $\mathrm{SL}(2,\mathbb{R})_E$ by L^x and denoting by

W^{iM} the generators in $W_{(2,\mathbf{W})}$, the commutation relations that correspond to the decomposition (9.2.17) have the following general form:

$$\begin{aligned}
\left[T^a, T^b\right] &= f^{ab}{}_c\, T^c \\
\left[L^x, L^y\right] &= f^{xy}{}_z\, L^z \\
\left[T^a, W^{iM}\right] &= \left(\Lambda^a\right)^M{}_N\, W^{iN} \\
\left[L^x, W^{iM}\right] &= \left(\lambda^x\right)^i{}_j\, W^{jM} \\
\left[W^{iM}, W^{jN}\right] &= \varepsilon^{ij} (K_a)^{MN} T^a + \mathbb{C}^{MN} k^{ij}_x L^x
\end{aligned} \tag{9.2.18}$$

where the 2×2 matrices $(\lambda^x)^i_j$, are the canonical generators of $\mathrm{SL}(2,\mathbb{R})$ in the fundamental, defining representation:

$$\lambda^3 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}; \qquad \lambda^1 = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}; \qquad \lambda^2 = \begin{pmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix} \tag{9.2.19}$$

while Λ^a are the generators of $\mathbb{U}_{D=4}$ in the symplectic representation $\mathbf{W}$. By

$$\mathbb{C}^{MN} \equiv \left(\begin{array}{c|c} \mathbf{0}_{n\times n} & \mathbf{1}_{n\times n} \\ \hline -\mathbf{1}_{n\times n} & \mathbf{0}_{n\times n} \end{array} \right) \tag{9.2.20}$$

we denote the antisymmetric symplectic metric in $2n$ dimensions, $n = n_{\mathrm{v}}$ being the number of vector fields in $D = 4$, as we have already stressed. The symplectic character of the representation $\mathbf{W}$ is asserted by the identity:

$$\Lambda^a \mathbb{C} + \mathbb{C}\left(\Lambda^a\right)^T = 0 \tag{9.2.21}$$

The fundamental doublet representation of $\mathrm{SL}(2,\mathbb{R})$ is also symplectic and by $\varepsilon^{ij} = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ we have denoted the 2-dimensional symplectic metric, so that:

$$\lambda^x \varepsilon + \varepsilon\left(\lambda^x\right)^T = 0 \tag{9.2.22}$$

In (9.2.18) we have used the standard convention according to which symplectic indices are raised and lowered with the appropriate symplectic metric, while adjoint representation indices are raised and lowered with the Cartan-Killing metric.

Orbit of Solutions Using the transformations of the isometry group U_σ every solution of the σ-model generates an entire U_σ orbit of solutions which reflects in a similar U_σ orbit of supergravity solutions. Consequently the black-hole solutions are conveniently organized into U_σ orbits.

9.2.3 General Properties of the $d = 4$ Metric

It is convenient to summarize some general properties of the $d = 4$ metric in (9.2.7). First we consider the case of non-extremal black-holes $v^2 > 0$ and in particular the Schwarzschild solution which, as it was shown in [3, 18] is the unique representative of the whole U_σ orbit of regular black-hole solutions.

The Schwarzschild Case Consider the case where the function $\exp[-U(\tau)]$ and the extremality parameter are the following ones:

$$\exp\bigl[-U(\tau)\bigr] = \exp[-\alpha\tau]; \qquad v^2 = \frac{\alpha^2}{4} \tag{9.2.23}$$

Introducing the following position:

$$\tau = \frac{\log[1 - \frac{2m}{r}]}{2m}; \qquad \alpha = 2m \tag{9.2.24}$$

the reader can immediately verify that the metric (9.2.7) at $A_{KK} = 0$ is turned into the standard Schwarzschild metric:

$$ds^2_{Schw} = -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2\bigl(d\theta^2 + \sin^2\theta\, d\phi^2\bigr) \tag{9.2.25}$$

The Extremal Reissner Nordström Case Consider now the following choices:

$$\exp\bigl[-U(\tau)\bigr] = (1 + q\tau); \qquad v^2 = 0 \tag{9.2.26}$$

Introducing the following position:

$$\tau = \frac{1}{r - q} \tag{9.2.27}$$

by means of elementary algebra the reader can verify that the metric (9.2.7) at $A_{KK} = 0$ is turned into the extremal Reissner Nordström metric:

$$ds^2_{RNext} = -\left(1 - \frac{q}{r}\right)^2 dt^2 + \left(1 - \frac{q}{r}\right)^{-2} dr^2 + r^2\bigl(d\theta^2 + \sin^2\theta\, d\phi^2\bigr) \tag{9.2.28}$$

which follows from the non-extremal one:

$$ds^2_{RN} = -\left(1 - \frac{2m}{r} + \frac{q^2}{r^2}\right) dt^2 + \left(1 - \frac{2m}{r} + \frac{q^2}{r^2}\right)^{-1} dr^2 + r^2\bigl(d\theta^2 + \sin^2\theta\, d\phi^2\bigr) \tag{9.2.29}$$

when the mass is equal to the charge: $m = q$.

It follows from the discussion of this simple example that the extremal black-hole metrics (9.2.7) are all suitable deformations of the extremal Reissner Nordström metric, just as the regular black-hole metrics are suitable deformations of the Schwarzschild one.

Curvature of the Extremal Spaces In order to facilitate the discussion of the various solutions found by means of the σ-model method, it is useful to consider the general form of the Riemann tensor associated with the metrics (9.2.7) in the extremal case. To this effect we introduce the vielbein 1-forms:

$$\begin{aligned}
E^0 &= \exp\left[\frac{U}{2}\right] dt \\
E^1 &= \exp\left[-\frac{U}{2}\right] \frac{d\tau}{\tau^2} \\
E^2 &= \exp\left[-\frac{U}{2}\right] \frac{1}{\tau}\, d\theta \\
E^3 &= \exp\left[-\frac{U}{2}\right] \frac{1}{\tau} \sin[\theta]\, d\phi
\end{aligned} \tag{9.2.30}$$

and the corresponding spin connection:

$$dE^a + \omega^{ab} \wedge E^c \eta_{bc} = 0 \tag{9.2.31}$$

Defining the curvature 2-form in the standard way:

$$\mathfrak{R}^{ab} = d\omega^{ab} + \omega^{ac} \wedge \omega^{db} \eta_{cd} \tag{9.2.32}$$

we find that it is diagonal:

$$\begin{aligned}
\mathfrak{R}^{01} &= \mathscr{C}_1 E^0 \wedge E^1 \\
\mathfrak{R}^{02} &= \mathscr{C}_2 E^0 \wedge E^2 \\
\mathfrak{R}^{03} &= \mathscr{C}_2 E^0 \wedge E^3 \\
\mathfrak{R}^{12} &= \mathscr{C}_3 E^1 \wedge E^2 \\
\mathfrak{R}^{13} &= \mathscr{C}_3 E^1 \wedge E^3 \\
\mathfrak{R}^{23} &= \mathscr{C}_4 E^3 \wedge E^4
\end{aligned} \tag{9.2.33}$$

and involves four independent differential expressions in the function $U(\tau)$, namely

$$\begin{aligned}
\mathscr{C}_1(\tau) &= -\frac{1}{4} e^{U(\tau)} \tau^3 \big(\tau U'(\tau)^2 + 2U'(\tau) + \tau U''(\tau)\big) \\
\mathscr{C}_2(\tau) &= \frac{1}{8} e^{U(\tau)} \tau^3 U'(\tau) \big(\tau U'(\tau) + 2\big) \\
\mathscr{C}_3(\tau) &= \frac{1}{4} e^{U(\tau)} \tau^3 \big(U'(\tau) + \tau U''(\tau)\big) \\
\mathscr{C}_4(\tau) &= -\frac{1}{8} e^{U(\tau)} \tau^3 U'(\tau) \big(\tau U'(\tau) + 4\big)
\end{aligned} \tag{9.2.34}$$

We will consider the behavior of these four independent component of the Riemann tensor in various solutions that we present some pages later.

9.2.4 Attractor Mechanism, the Entropy and Other Special Geometry Invariants

One of the most important features of supergravity black-holes is the attractor mechanism discovered in the nineties by Ferrara and Kallosh for the case of BPS[2] solutions [1, 2] and in recent time extended to non-BPS cases [7–14]. According to this mechanism the evolving scalar fields $\phi^i(\tau)$ flow to fixed values at the horizon of the black-hole ($\tau = -\infty$), which do not depend from their initial values at infinity radius ($\tau = 0$) but only on the electromagnetic charges p, q.

In order to review the attractor mechanism, we must briefly recall the essential items of black hole field equations in the *geodesic potential approach* [6]. In this framework we do not consider all the fields listed in the table after (9.2.2). We introduce only the warp factor $U(\tau)$ and the original scalar fields of $D = 4$ supergravity. The information about vector gauge fields is encoded solely in the set of electric and magnetic charges $\mathscr{Q}$ defined by (9.2.13). Furthermore for the sake of simplicity we focus on the case of an $\mathscr{N} = 2$ theory where the $2n$ scalar fields span a special Kähler manifold and can be organized into n complex combinations z^i. Under these conditions the correct field equations for an $\mathscr{N} = 2$ black-hole are derived from the geodesic one dimensional field-theory described by the following Lagrangian:

$$
\begin{aligned}
S_{eff} &\equiv \int \mathscr{L}_{eff}(\tau)\, d\tau \\
\mathscr{L}_{eff}(\tau) &= \frac{1}{4}\left(\frac{dU}{d\tau}\right)^2 + g_{ij^\star}\frac{dz^i}{d\tau}\frac{dz^{j^\star}}{d\tau} + e^U V_{BH}(z, \bar{z}, \mathscr{Q})
\end{aligned}
\tag{9.2.35}
$$

where the geodesic potential $V(z, \bar{z}, \mathscr{Q})$ is defined by the following formula in terms of the matrix $\mathscr{M}_4$ introduced in (9.2.3):

$$
V_{BH}(z, \bar{z}, \mathscr{Q}) = \frac{1}{4}\mathscr{Q}^t \mathscr{M}_4^{-1}(\mathscr{N})\mathscr{Q} \tag{9.2.36}
$$

The effective Lagrangian (9.2.35) is derived from the σ-model Lagrangian (9.2.1) upon substitution of the first integrals of motion corresponding to the electromagnetic charges (9.2.13) under the condition that the Taub-NUT charge, defined in (9.2.9), vanishes[3] ($\mathbf{n} = 0$). Indeed, when the Taub-NUT charge $\mathbf{n}$ vanishes, which

[2]In the supergravity framework BPS solutions are those that preserve a certain amount of supersymmetry, namely that admit a certain number of so named Killing spinors, *i.e.* of supersymmetry parameters such that supersymmetry transformations along them leave the chosen solution invariant.

[3]In [18] it was shown that every orbit of solutions contains a representative where the Taub-NUT charge is zero. Alternatively from a dynamical system point of view the Taub-NUT charge can be annihilated by setting a constraint which is consistent with the Hamiltonian and which reduces the dimension of the system by one unit. The problem of black hole physics is therefore equivalent to the sigma model based on an appropriate codimension one hypersurface in the $\mathscr{Q}$ manifold.

will be our systematic choice, we can invert the above mentioned relations, expressing the derivatives of the Z^M fields in terms of the charge vector $\mathcal{Q}^M$ and the inverse of the matrix $\mathcal{M}_4$. Upon substitution in the σ-model Lagrangian (9.2.3), we obtain the effective Lagrangian for the $D=4$ scalar fields z^i and the warping factor U given by (9.2.35)–(9.2.37).

The important thing is that, thanks to various identities of special geometry, the effective geodesic potential admits the following alternative representation:

$$V_{BH}(z,\overline{z},\mathcal{Q}) = -\frac{1}{2}\left(|Z|^2+|Z_i|^2\right) \equiv -\frac{1}{2}\left(Z\overline{Z}+Z_i g^{ij^\star}\overline{Z}_{j^\star}\right) \tag{9.2.37}$$

where the symbol Z denotes the complex scalar field valued central charge of the supersymmetry algebra:

$$Z \equiv V^T \mathbb{C}\mathcal{Q} = M_\Sigma p^\Sigma - L^\Lambda q_\Lambda \tag{9.2.38}$$

and Z_i denote its covariant derivatives:

$$\begin{aligned} Z_i &= \nabla_i Z = U_i \mathbb{C}\mathcal{Q}; \qquad & Z^{j^\star} &= g^{j^\star i} Z_i \\ \overline{Z}_{j^\star} &= \nabla_{j^\star} Z = \overline{U}_{j^\star}\mathbb{C}\mathcal{Q}; \qquad & \overline{Z}^i &= g^{ij^\star}\overline{Z}_{j^\star} \end{aligned} \tag{9.2.39}$$

Equation (9.2.37) is a result in special geometry whose proof can be found in several articles and reviews of the late nineties.[4]

9.2.5 *Critical Points of the Geodesic Potential and Attractors*

The structure of the geodesic potential illustrated above allows for a detailed discussion of its critical points, which are relevant for the asymptotic behavior of the scalar fields.

By definition, critical points correspond to those values of z^i for which the first derivative of the potential vanishes: $\partial_i V_{BH} = 0$. Utilizing the fundamental identities of special geometry and (9.2.37), the vanishing derivative condition of the potential can be reformulated as follows:

$$0 = 2Z_i\overline{Z} + \mathrm{i}C_{ijk}\overline{Z}^j\overline{Z}^k \tag{9.2.40}$$

From this equation it follows that there are three possible types of critical points:

$$\begin{array}{llll} Z_i = 0; & Z \neq 0; & & \text{BPS attractor} \\ Z_i \neq 0; & Z = 0; & \mathrm{i}C_{ijk}\overline{Z}^j\overline{Z}^k = 0 & \text{non-BPS attractor I} \\ Z_i \neq 0; & Z \neq 0; & \mathrm{i}C_{ijk}\overline{Z}^j\overline{Z}^k = -2Z_i\overline{Z} & \text{non-BPS attractor II} \end{array} \tag{9.2.41}$$

[4] See for instance the lecture notes [19].

It should be noted that in the case of one-dimensional special geometries, only BPS attractors and non-BPS attractors of type II are possible. Indeed non-BPS attractors of type I are forbidden unless C_{zzz} vanishes identically.

In order to characterize the various type of attractors, the authors of [20] and [21] introduced a certain number of special geometry invariants that obey different and characterizing relations at attractor points of different type. They are defined as follows. Let us introduce the symbols:

$$N_3 \equiv C_{ijk}\overline{Z}^i\overline{Z}^j\overline{Z}^k; \qquad \overline{N}_3 \equiv C_{i^\star j^\star k^\star}Z^{i^\star}Z^{j^\star}Z^{k^\star} \tag{9.2.42}$$

and let us set:

$$\begin{aligned} i_1 &= Z\overline{Z}; & i_2 &= Z_i\overline{Z}_{j^\star}g^{ij^\star} \\ i_3 &= \frac{1}{6}(ZN_3 + \overline{ZN}_3); & i_4 &= \mathrm{i}\frac{1}{6}(ZN_3 - \overline{ZN}_3) \\ i_5 &= C_{ijk}C_{\ell^\star m^\star n^\star}\overline{Z}^j\overline{Z}^k Z^{m^\star}Z^{n^\star}g^{i\ell^\star}; \end{aligned} \tag{9.2.43}$$

An important identity satisfied by the above invariants, that depend both on the scalar fields z^i and the charges (p,q), is the following one:

$$\mathfrak{I}_4(p,q) = \frac{1}{4}(i_1 - i_2)^2 + i_4 - \frac{1}{4}i_5 \tag{9.2.44}$$

where:

$$\mathfrak{I}_4(p,q) = \mathscr{I}_{MNPR}\mathscr{Q}^M\mathscr{Q}^N\mathscr{Q}^P\mathscr{Q}^R \tag{9.2.45}$$

is a quartic polynomial in the electromagnetic charges defined by a symmetric tensor $\mathscr{I}_{MNPR}$ which is invariant with respect to all transformations of the isometry group $\mathrm{U}_{D=4}$ symplectically embedded in $\mathrm{Sp}(2n_v, \mathbb{R})$. This means that in the combination (9.2.44) the dependence on the fields z^i cancels identically.

The generic existence in supergravity models of the quartic invariant (9.2.45) and its relation with the Black-Hole area/entropy is one of the most profound and intriguing contributions of the *supergravity/superstring* studies to Gravity Theory. On one hand it opens a window on the statistical interpretation of the black holes since, in the underlying superstring microscopic interpretation of supergravity, charges are related to branes and to the counting of their wrapping modes, on the other hand it is quite possible that the group-theoretical structures related to the quartic invariant might have a more general validity beyond purely supersymmetric theories. In this book we will not enter the very rich classification of black-holes in the various supergravity models. We will just confine ourselves to an ultra short illustration of the main features of such black holes by means of the simplest $\mathscr{N}=2$ supergravity model containing just one vector multiplet with non-trivial couplings. This is done in the next subsection. After this anticipation we continue with the classification of critical points.

Indeed in [20] it was proposed that the three types of critical points can be characterized by the following relations among the above invariants holding at the attractor point:

At BPS Attractor Points We have:

$$i_1 \neq 0; \qquad i_2 = i_3 = i_4 = i_5 = 0 \tag{9.2.46}$$

At Non-BPS Attractor Points of Type I We have:

$$i_2 \neq 0; \qquad i_1 = i_3 = i_4 = i_5 = 0 \tag{9.2.47}$$

At Non-BPS Attractor Points of Type II We have:

$$i_2 = 3i_1; \qquad i_3 = 0; \qquad i_4 = -2i_1^2; \qquad i_5 = 12i_1^2 \tag{9.2.48}$$

9.2.6 The $\mathcal{N} = 2$ Supergravity S^3-Model

The pedagogical example we consider in this book is the simplest possible case of vector multiplet coupling in $\mathcal{N} = 2$ supergravity: we just introduce one vector multiplet. This means that we have two vector fields in the theory and one complex scalar field z. This scalar field parameterizes a one-dimensional special Kähler manifold which, in our choice, will be the complex lower half-plane endowed with the standard Poincaré metric. In other words:[5]

$$g_{z\bar{z}}\partial^\mu z \partial_\mu \bar{z} = \frac{3}{4}\frac{1}{(\operatorname{Im} z)^2}\partial^\mu z \partial_\mu \bar{z} \tag{9.2.49}$$

is the σ-model part of the Lagrangian (8.4.1). From the point of view of geometry the lower half-plane is the symmetric coset manifold $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2) \sim \mathrm{SU}(1,1)/\mathrm{U}(1)$ which admits a standard solvable parameterization as it follows. Let:

$$L_0 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \qquad L_+ = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}; \qquad L_- = \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{9.2.50}$$

be the standard three generators of the $\mathfrak{sl}(2,\mathbb{R})$ Lie algebra satisfying the commutation relations $[L_0, L_\pm] = \pm L_\pm$ and $[L_+, L_-] = 2L_0$. The coset manifold $\frac{\mathrm{SL}(2,\mathbb{R})}{\mathrm{SO}(2)}$ is metrically equivalent with the solvable group manifold generated by L_0 and L_+. Correspondingly we can introduce the coset representative:

$$\mathbb{L}_4(\phi, y) = \exp[yL_1]\exp[\varphi L_0] = \begin{pmatrix} e^{\varphi/2} & e^{-\varphi/2}y \\ 0 & e^{-\varphi/2} \end{pmatrix} \tag{9.2.51}$$

Generic group elements of $\mathrm{SL}(2,\mathbb{R})$ are just 2×2 real matrices with determinant one:

$$\mathrm{SL}(2,\mathbb{R}) \ni \mathfrak{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}; \quad ad - bc = 1 \tag{9.2.52}$$

[5]The special overall normalization of the Poincaré metric is chosen in order to match the general definitions of special geometry applied to the present case.

and their action on the lower half-plane is defined by usual fractional linear transformations:

$$\mathfrak{A} : z \to \frac{az+b}{cz+d} \tag{9.2.53}$$

The correspondence between the lower complex half-plane $\mathbb{C}_-$ and the solvable-parameterized coset (9.2.51) is easily established observing that the entire set of $\mathrm{Im}\, z < 0$ complex numbers is just the orbit of the number i under the action of $\mathbb{L}(\phi, y)$:

$$\mathbb{L}_4(\phi, y) : \mathrm{i} \to \frac{-e^{\varphi/2}\mathrm{i} + e^{-\varphi/2} y}{e^{-\varphi/2}} = y - \mathrm{i}e^{\varphi} \tag{9.2.54}$$

This simple argument shows that we can rewrite the coset representative $\mathbb{L}(\phi, y)$ in terms of the complex scalar field z as follows:

$$\mathbb{L}_4(z) = \begin{pmatrix} \sqrt{|\,\mathrm{Im}\, z|} & \frac{\mathrm{Re}\, z}{\sqrt{|\,\mathrm{Im}\, z|}} \\ 0 & \frac{1}{\sqrt{|\,\mathrm{Im}\, z|}} \end{pmatrix} \tag{9.2.55}$$

The issue of special Kähler geometry becomes clear at this stage. If we did not have vectors in the game, the choice of the coset metric would be sufficient and nothing more would have to be said. The point is that we still have to define the kinetic matrix of the vector and for that the symplectic bundle is necessary. On the same base manifold $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$ we have different special structures which lead to different physical models and to different σ-model groups U_σ. The special structure is determined by the choice of the symplectic embedding $\mathrm{SL}(2,\mathbb{R}) \to \mathrm{Sp}(4,\mathbb{R})$. The symplectic embedding that defines our pedagogical model and which eventually leads to the σ-model group $\mathrm{U}_\sigma = \mathrm{G}_{2(2)}$ is cubic and it is described in the following subsection.

9.2.6.1 The Cubic Special Kähler Structure on SL(2, ℝ)/SO(2)

The group $\mathrm{SL}(2,\mathbb{R})$ is also locally isomorphic to $\mathrm{SO}(1,2)$ and the fundamental representation of the first corresponds to the spin $J = \frac{1}{2}$ of the latter. The spin $J = \frac{3}{2}$ representation is obviously four-dimensional and, in the $\mathrm{SL}(2,\mathbb{R})$ language, it corresponds to a symmetric three-index tensor t_{abc}. Let us explicitly construct the 4×4 matrices of such a representation. This is easily done by choosing an order for the four independent components of the symmetric tensor t_{abc}. For instance we can identify the four axes of the representation with t_{111}, t_{112}, t_{122}, t_{222}. So doing, the image of the group element $\mathfrak{A}$ in the cubic symmetric tensor product representation is the following 4×4 matrix:

$$\mathscr{D}_3(\mathfrak{A}) = \begin{pmatrix} a^3 & 3a^2 b & 3ab^2 & b^3 \\ a^2 c & da^2 + 2bca & cb^2 + 2adb & b^2 d \\ ac^2 & bc^2 + 2adc & ad^2 + 2bcd & bd^2 \\ c^3 & 3c^2 d & 3cd^2 & d^3 \end{pmatrix} \tag{9.2.56}$$

By explicit evaluation we can easily check that:

$$\mathscr{D}_3^T(\mathfrak{A})\widehat{\mathbb{C}}_4\mathscr{D}_3(A) = \widehat{\mathbb{C}}_4 \quad \text{where } \widehat{\mathbb{C}}_4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -3 & 0 \\ 0 & 3 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \tag{9.2.57}$$

Since $\widehat{\mathbb{C}}_4$ is antisymmetric, (9.2.57) is already a clear indication that the triple symmetric representation defines a symplectic embedding. To make this manifest it suffices to change basis. Consider the matrix:

$$S = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\frac{1}{\sqrt{3}} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{9.2.58}$$

and define:

$$\Lambda(\mathfrak{A}) = S^{-1} D_3(\mathfrak{A}) S \tag{9.2.59}$$

We can easily check that:

$$\Lambda^T(\mathfrak{A})\mathbb{C}_4\Lambda(\mathfrak{A}) = \mathbb{C}_4 \quad \text{where } \mathbb{C}_4 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \tag{9.2.60}$$

So we have indeed constructed a standard symplectic embedding $\mathrm{SL}(2,\mathbb{R}) \mapsto \mathrm{Sp}(4,\mathbb{R})$ whose explicit form is the following:

$$\mathfrak{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \left(\begin{array}{cc|cc} da^2+2bca & -\sqrt{3}a^2c & -cb^2-2adb & -\sqrt{3}b^2d \\ -\sqrt{3}a^2b & a^3 & \sqrt{3}ab^2 & b^3 \\ \hline -bc^2-2adc & \sqrt{3}ac^2 & ad^2+2bcd & \sqrt{3}bd^2 \\ -\sqrt{3}c^2d & c^3 & \sqrt{3}cd^2 & d^3 \end{array}\right) \equiv \Lambda(\mathfrak{A}) \tag{9.2.61}$$

The 2×2 blocks A, B, C, D of the 4×4 symplectic matrix $\Lambda(\mathfrak{A})$ are easily readable from (9.2.61) so that, assuming now that the matrix $\mathfrak{A}(z)$ is the coset representative of the manifold $\mathrm{SU}(1,1)/\mathrm{U}(1)$, we can apply the Gaillard-Zumino formula (8.3.67) and obtain the explicit form of the kinetic matrix $\mathscr{N}_{\Lambda\Sigma}$:

$$\overline{\mathscr{N}} = \begin{pmatrix} -\frac{2ac-ibc+iad+2bd}{a^2+b^2} & -\frac{\sqrt{3}(c+id)(ac+bd)}{(a-ib)(a+ib)^2} \\ -\frac{\sqrt{3}(c+id)(ac+bd)}{(a-ib)(a+ib)^2} & -\frac{(c+id)^2(2ac+ibc-iad+2bd)}{(a-ib)(a+ib)^3} \end{pmatrix} \tag{9.2.62}$$

Inserting the specific values of the entries a, b, c, d corresponding to the coset representative (9.2.55), we get the explicit dependence of the kinetic period matrix on the complex scalar field z:

$$\overline{\mathcal{N}}_{\Lambda\Sigma}(z) = \begin{pmatrix} -\frac{3z+\bar{z}}{2z\bar{z}} & -\frac{\sqrt{3}(z+\bar{z})}{2z\bar{z}^2} \\ -\frac{\sqrt{3}(z+\bar{z})}{2z\bar{z}^2} & -\frac{z+3\bar{z}}{2z\bar{z}^3} \end{pmatrix} \tag{9.2.63}$$

This might conclude the determination of the Lagrangian of our master example, yet we have not yet seen the special Kähler structure induced by the cubic embedding. Let us present it.

The key point is the construction of the required holomorphic symplectic section $\Omega(z)$. As usual the transformation properties of a geometrical object indicate the way to build it explicitly. For consistency we should have that:

$$\Omega\left(\frac{az+b}{cz+d}\right) = f(z)\Lambda(\mathfrak{A})\Omega(z) \tag{9.2.64}$$

where $\Lambda(\mathfrak{A})$ is the symplectic representation (9.2.61) of the considered $\mathrm{SL}(2,\mathbb{R})$ matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $f(z)$ is the associated transition function for that line-bundle whose Chern-class is the Kähler class of the base-manifold. The identification of the symplectic fibres with the cubic symmetric representation provide the construction mechanism of Ω. Consider a vector $\left(\begin{smallmatrix} v_1 \\ v_2 \end{smallmatrix}\right)$ that transforms in the fundamental doublet representation of $\mathrm{SL}(2,\mathbb{R})$. On one hand we can identify the complex coordinate z on the lower half-plane as $z = v_1/v_2$, on the other we can construct a symmetric three-index tensor taking the tensor products of three v_i, namely: $t_{ijk} = v_i v_j v_k$. Dividing the resulting tensor by $(v_2)^3$ we obtain a four vector:

$$\widehat{\Omega}(z) = \frac{1}{v_2^3}\begin{pmatrix} v_1^3 \\ v_1^2 v_2 \\ v_1 v_2^2 \\ v_2^3 \end{pmatrix} = \begin{pmatrix} z^3 \\ z^2 \\ z \\ 1 \end{pmatrix} \tag{9.2.65}$$

Next, recalling the change of basis (9.2.58), (9.2.59) required to put the cubic representation into a standard symplectic form we set:

$$\Omega(z) = S\widehat{\Omega}(z) = \begin{pmatrix} -\sqrt{3}z^2 \\ z^3 \\ \sqrt{3}z \\ 1 \end{pmatrix} \tag{9.2.66}$$

and we can easily verify that this object transforms in the appropriate way. Indeed we obtain:

$$\Omega\left(\frac{az+b}{cz+d}\right) = (cz+d)^{-3}\Lambda(\mathfrak{A})\Omega(z) \tag{9.2.67}$$

The pre-factor $(cz+d)^{-3}$ is the correct one for the prescribed line-bundle. To see this let us first calculate the Kähler potential and the Kähler form. Inserting (9.2.66) into (8.5.18) we get:

$$\begin{aligned} \mathscr{K} &= -\log\bigl(\mathrm{i}\langle \Omega \mid \overline{\Omega}\rangle\bigr) = -\log\bigl(-\mathrm{i}(z-\overline{z})^3\bigr) \\ \mathrm{K} &= \frac{\mathrm{i}}{2\pi}\partial\overline{\partial}\mathscr{K} = \frac{\mathrm{i}}{2\pi}\frac{3}{(\operatorname{Im} z)^2}\, dz \wedge d\overline{z} \end{aligned} \tag{9.2.68}$$

This shows that the constructed symplectic bundle leads indeed to the standard Poincaré metric and the exponential of the Kähler potential transforms with the prefactor $(cz+d)^3$ whose inverse appears in (9.2.67).

To conclude let us show that the special geometry definition of the period matrix $\mathscr{N}$ agrees with the Gaillard-Zumino definition holding true for all symplectically embedded cosets. To this effect we calculate the necessary ingredients:

$$\nabla_z V(z) = \exp\left[\frac{\mathscr{K}}{2}\right]\bigl(\partial_z \Omega(z) + \partial_z \mathscr{K}\,\Omega(z)\bigr) = \begin{pmatrix} \frac{\sqrt{3}z(z+2\overline{z})}{(z-\overline{z})\sqrt{-i(z-\overline{z})^3}} \\ -\frac{3z^2\overline{z}}{(z-\overline{z})\sqrt{-i(z-\overline{z})^3}} \\ -\frac{\sqrt{3}(2z+\overline{z})}{(z-\overline{z})\sqrt{-i(z-\overline{z})^3}} \\ -\frac{3}{(z-\overline{z})\sqrt{-i(z-\overline{z})^3}} \end{pmatrix} \equiv \begin{pmatrix} f_z^{\Lambda} \\ h_{\Sigma z} \end{pmatrix} \tag{9.2.69}$$

Then according to (8.5.29) we obtain:

$$\begin{aligned} f_I^{\Lambda} &= \begin{pmatrix} \frac{\sqrt{3}z(z+2\overline{z})}{(z-\overline{z})\sqrt{-i(z-\overline{z})^3}} & -\frac{2\sqrt{6}\overline{z}^2}{(-i(z-\overline{z}))^{3/2}} \\ -\frac{3z^2\overline{z}}{(z-\overline{z})\sqrt{-i(z-\overline{z})^3}} & \frac{2\sqrt{2}\overline{z}^3}{(-i(z-\overline{z}))^{3/2}} \end{pmatrix} \\ h_{\Lambda|I} &= \begin{pmatrix} -\frac{\sqrt{3}(2z+\overline{z})}{(z-\overline{z})\sqrt{-i(z-\overline{z})^3}} & \frac{2\sqrt{6}\overline{z}}{(-i(z-\overline{z}))^{3/2}} \\ -\frac{3}{(z-\overline{z})\sqrt{-i(z-\overline{z})^3}} & \frac{2\sqrt{2}}{(-i(z-\overline{z}))^{3/2}} \end{pmatrix} \end{aligned} \tag{9.2.70}$$

and applying definition (8.5.30) we exactly retrieve the same form of $\mathscr{N}_{\Lambda\Sigma}$ as given in (9.2.63).

For completeness and also for later use we calculate the remaining items pertaining to special geometry, in particular the symmetric C-tensor. From the general definition (8.5.23) applied to the present one-dimensional case we get:

$$\nabla_z U_z = \mathrm{i} C_{zzz} h^{zz^\star}\overline{U}_{z^\star} \quad \Rightarrow \quad C_{zzz} = -\frac{6\mathrm{i}}{(z-z^\star)^3} \tag{9.2.71}$$

As for the standard Levi-Civita connection we have:

$$\Gamma^z_{zz} = \frac{2}{z-z^\star}; \qquad \Gamma^{z^\star}_{z^\star z^\star} = -\frac{2}{z-z^\star}; \quad \text{all other components vanish} \tag{9.2.72}$$

This concludes our illustration of the cubic special Kähler structure on $\frac{\mathrm{SL}(2,\mathbb{R})}{\mathrm{SO}(2)}$.

9.2.6.2 The Quartic Invariant

In the cubic spin $j=\frac{3}{2}$ of $\mathrm{SL}(2,\mathbb{R})$ there is a quartic invariant which plays an important role in the discussion of black-holes. As it happens for all the other supergravity models, the quartic invariant of the symplectic vector of magnetic and electric charges:

$$\mathscr{Q}=\begin{pmatrix} p^{\Lambda} \\ q_{\Sigma} \end{pmatrix} \tag{9.2.73}$$

is related to the entropy of the extremal black-holes, the latter being its square root. The origin of the quartic invariant is easily understood in terms of the symmetric tensor t_{ijk}. Using the $\mathrm{SL}(2,\mathbb{R})$-invariant antisymmetric symbol ε^{ij} we can construct an invariant order four polynomial in the tensor t_{ijk} by writing:

$$\mathfrak{I}_4 \propto \varepsilon^{ai}\varepsilon^{bj}\varepsilon^{pl}\varepsilon^{qm}\varepsilon^{kr}\varepsilon^{cn} t_{abc}t_{ijk}t_{pqr}t_{lmn} \tag{9.2.74}$$

If we use the standard basis $t_{111}, t_{112}, t_{122}, t_{222}$, we rotate it with the matrix (9.2.58) and we identify the components of the resultant vector with those of the charge vector $\mathscr{Q}$ the explicit form of the invariant quartic polynomial is the following one:

$$\mathfrak{I}_4=\frac{1}{3\sqrt{3}}q_2p_1^3+\frac{1}{12}q_1^2p_1^2-\frac{1}{2}p_2q_1q_2p_1-\frac{1}{3\sqrt{3}}p_2q_1^3-\frac{1}{4}p_2^2q_2^2 \tag{9.2.75}$$

9.2.7 *Fixed Scalars at BPS Attractor Points: The S^3 Explicit Example*

In the case of BPS attractors we can find the explicit expression in terms of the (p,q)-charges for the scalar field fixed values at the critical point.

By means of standard special geometry manipulations the BPS critical point equation

$$\nabla_j Z=0; \qquad \nabla_{j^\star}\overline{Z}=0 \tag{9.2.76}$$

can be rewritten in the following celebrated form which, in the late nineties, appeared in numerous research and review papers (see for instance [19]):

$$p^{\Lambda}=\mathrm{i}\big(Z_{fix}\overline{L}^{\Lambda}_{fix}-\overline{Z}_{fix}L^{\Lambda}_{fix}\big) \tag{9.2.77}$$

$$q_{\Sigma}=\mathrm{i}\big(Z_{fix}\overline{M}^{fix}_{\Sigma}-\overline{Z}_{fix}M^{fix}_{\Sigma}\big) \tag{9.2.78}$$

In all cases where the special Kähler manifold is a homogeneous symmetric space the above formula can be explicitly inverted yielding the fixed values of the scalar fields in terms of the charges. We present such a solution for the S^3-model.

Using the explicit form of the symplectic section $\Omega(z)$ given in (9.2.66), (9.2.78) are solved by the following expressions for the fixed scalars:

$$z_{fixed} = -\frac{p_1 q_1 + 3 p_2 q_2 + \mathrm{i}6\sqrt{\mathfrak{I}_4(p,q)}}{2(q_1^2 + \sqrt{3} p_1 q_2)} \tag{9.2.79}$$

where $\mathfrak{I}_4(p,q)$ is the quartic invariant defined in (9.2.75).

By replacing the fixed values (9.2.79) into the expression (9.2.37) for the potential we find:

$$V_{BH}(z_{fixed}, \overline{z}_{fixed}, \mathscr{Q}) = -\sqrt{\mathfrak{I}_4(p,q)} \tag{9.2.80}$$

The above result implies that the horizon area in the case of an extremal BPS black-hole is proportional to the square root of $\mathfrak{I}_4(p,q)$ and, as such, depends only on the charges. The argument goes as follows.

Consider the behavior of the warp factor $\exp[-U]$ in the vicinity of the horizon, when $\tau \to -\infty$. For regular black-holes the near horizon metric must factorize as follows:

$$ds^2_{\text{near hor.}} \approx \underbrace{-\frac{1}{r_H^2 \tau^2} dt^2 + r_H^2 \left(\frac{d\tau}{\tau}\right)^2}_{\text{AdS}_2 \text{ metric}} + \underbrace{r_H^2 \left(d\theta^2 \sin^2\theta \, d\phi^2\right)}_{\text{S}^2 \text{ metric}} \tag{9.2.81}$$

where r_H is the Schwarzschild radius defining the horizon. This implies that the asymptotic behavior of the warp factor, for $\tau \to -\infty$ is the following one:

$$\exp[-U] \sim r_H^2 \tau^2 \tag{9.2.82}$$

In the same limit the scalar fields go to their fixed values and their derivatives become essentially zero. Hence near the horizon we have:

$$\begin{aligned} (\dot{U})^2 &\approx \frac{4}{\tau^2}; \qquad g_{ij^\star} \frac{dz^i}{d\tau} \frac{dz^{j^\star}}{d\tau} \approx 0 \\ e^U V_{BH}(z, \overline{z}, \mathscr{Q}) &\approx \frac{1}{r_H^2 \tau^2} V(z_{fixed}, \overline{z}_{fixed}, \mathscr{Q}) \end{aligned} \tag{9.2.83}$$

Since for extremal black-holes the sum of the above three terms vanishes (see (9.2.5)), we conclude that:

$$r_H^2 = -V_{BH}(z_{fixed}, \overline{z}_{fixed}, \mathscr{Q}) \tag{9.2.84}$$

which yields

$$\text{Area}_H = 4\pi r_H^2 = 4\pi \sqrt{\mathfrak{I}_4(p,q)} \tag{9.2.85}$$

9.2.7.1 An Explicit Example of Exact Regular BPS Solution

This general mechanism can be illustrated with an explicit example of exact regular solution of the S^3 model. The key identifier of the solution is its vector of electromagnetic charges that in our chosen example is the following one:

$$\begin{pmatrix} p_1 \\ p_2 \\ q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} 0 \\ p \\ -\sqrt{3}q \\ 0 \end{pmatrix}; \quad p, q > 0 \quad \text{or} \quad p, q < 0 \tag{9.2.86}$$

The corresponding explicit solution is given below and depends on three parameters p, q and κ which yields the value of the imaginary part of the scalar field at radial infinity, namely at $\tau = 0$.

The Metric

$$\exp\big[U(\tau)\big] = \frac{\kappa^{3/4}}{\sqrt{-(\kappa^{3/2} - p\tau)(q\sqrt{\kappa}\tau - 1)^3}} \tag{9.2.87}$$

The Scalar Field

$$\operatorname{Im} z(\tau) = -\frac{\sqrt[4]{\kappa}\sqrt{(p\tau - \kappa^{3/2})(q\sqrt{\kappa}\tau - 1)^3}}{(q\sqrt{\kappa}\tau - 1)^2} \tag{9.2.88}$$

$$\operatorname{Re} z(\tau) = 0 \tag{9.2.89}$$

The Electromagnetic Fields

$$Z^1(\tau) = 0 \tag{9.2.90}$$

$$Z^2(\tau) = -\frac{p\tau}{\sqrt{2}\kappa^{3/2}(\kappa^{3/2} - p\tau)} \tag{9.2.91}$$

$$Z_1(\tau) = -\frac{\sqrt{\frac{3}{2}}q\kappa\tau}{q\sqrt{\kappa}\tau - 1} \tag{9.2.92}$$

$$Z_2(\tau) = 0 \tag{9.2.93}$$

The interested reader can verify that the expressions displayed above for all the fields fulfill the variational equations (9.2.6) of the σ-model and hence are bona-fide solutions of the supergravity field theory.

The Fixed Scalars at Horizon and the Entropy Calculating the area of the horizon we find:

$$\frac{1}{4\pi}\text{Area}_H \equiv r_H^2 = \lim_{\tau\to-\infty} \frac{1}{\tau^2}\exp\big[-U(\tau)\big] = \sqrt{pq^3} \tag{9.2.94}$$

which makes sense only as long as $pq^3 > 0$. Inserting (9.2.86) into (9.2.75) we see that $pq^3 = \mathfrak{I}_4$. Hence we conclude that this solution is indeed BPS as expected. The horizon area is:

$$\frac{1}{4\pi}\text{Area}_H \equiv r_H^2 = \sqrt{\mathfrak{I}_4} \tag{9.2.95}$$

9.2.8 The Attraction Mechanism Illustrated with an Exact Non-BPS Solution

Next we illustrate the attraction mechanism with an explicit example of exact regular non-BPS solution of the S^3 model. The vector of electromagnetic charges of the considered solution differs from that of the above BPS-example only by means of a sign, namely it is:

$$\begin{pmatrix} p_1 \\ p_2 \\ q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} 0 \\ p \\ \sqrt{3}q \\ 0 \end{pmatrix}; \quad p, q > 0 \quad \text{or} \quad p, q < 0 \tag{9.2.96}$$

The corresponding explicit solution which is given below depends on four parameters p, q and κ, ξ which yield the values of both the imaginary and the real parts of the scalar field at radial infinity, namely at $\tau = 0$.

The Metric The metric is defined by the function U for which the integration techniques of the σ-model yield the following expression:

$$\begin{aligned} \exp\big[U(\tau)\big] = \kappa^{3/4}/\big(&-q^3\kappa^3\tau^3 - q^3\kappa\xi^2\tau^3 + 3q^2\kappa^{5/2}\tau^2 + 3q^2\sqrt{\kappa}\xi^2\tau^2 \\ &+ p(q\sqrt{\kappa}\tau - 1)^3\tau - 3q\kappa^2\tau - 3q\xi^2\tau + \kappa^{3/2}\big)^{1/2} \end{aligned} \tag{9.2.97}$$

The Scalar Field The complex scalar field $z(\tau)$ has the following form:

$$\begin{aligned} \operatorname{Im} z(\tau) = -\sqrt[4]{\kappa}\big(&-q^3\kappa^3\tau^3 - q^2\kappa\big(q\xi^2 + 3p\big)\tau^3 + 3q^2\kappa^{5/2}\tau^2 + 3q\sqrt{\kappa}\big(q\xi^2 + p\big)\tau^2 \\ &- 3q\kappa^2\tau - \big(3q\xi^2 + p\big)\tau + \kappa^{3/2}\big(pq^3\tau^4 + 1\big)\big)^{1/2}/(q\sqrt{\kappa}\tau - 1)^2 \end{aligned} \tag{9.2.98}$$

$$\operatorname{Re} z(\tau) = \frac{\xi}{(q\sqrt{\kappa}\tau - 1)^2} \tag{9.2.99}$$

The Electromagnetic Fields The explicit form of the two field strengths appearing in the S^3 model is completely determined by (9.2.14). It suffices to know the magnetic charges $(p_1, p_2) = (0, p)$, the Taub-NUT charge $\mathbf{n} = 0$ and the derivatives

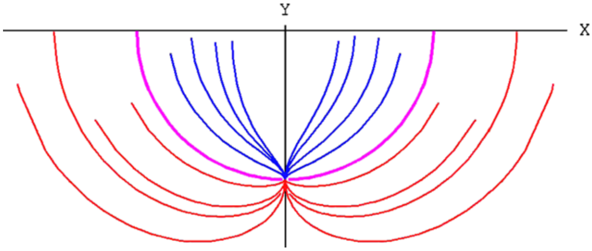

Fig. 9.1 Trajectories of scalar fields from infinity to the horizon in the case of the regular non-BPS black-hole discussed in the main text. The chosen numerical values of the charges are ($p=2$, $q=4$)

of the $Z^{\Lambda}(\tau)$ functions. We have

$$\begin{aligned}\dot{Z}^1(\tau)=&\sqrt{\frac{3}{2}}\xi\big(q\big(2q^3\tau^3\kappa^{7/2}-6q^2\tau^2\kappa^3+6q\tau\kappa^{5/2}-2\kappa^2+2q^3\xi^2\tau^3\kappa^{3/2}\\&-6q^2\xi^2\tau^2\kappa+6q\xi^2\tau\sqrt{\kappa}-3\xi^2\big)-p(q\sqrt{\kappa}\tau-1)^3(3q\sqrt{\kappa}\tau-1)\big)\\&/\big(q^3\kappa^3\tau^3+q^2\kappa\big(q\xi^2+3p\big)\tau^3-3q^2\kappa^{5/2}\tau^2-3q\sqrt{\kappa}\big(q\xi^2+p\big)\tau^2\\&+3q\kappa^2\tau+\big(3q\xi^2+p\big)\tau-\kappa^{3/2}\big(pq^3\tau^4+1\big)\big)^2\end{aligned}\tag{9.2.100}$$

$$\begin{aligned}\dot{Z}^2(\tau)=&-(q\sqrt{\kappa}\tau-1)^2\big(p(q\sqrt{\kappa}\tau-1)^4+3q\xi^2\big)\\&/\big(\sqrt{2}\big(q^3\kappa^3\tau^3+q^2\kappa\big(q\xi^2+3p\big)\tau^3-3q^2\kappa^{5/2}\tau^2-3q\sqrt{\kappa}\big(q\xi^2+p\big)\tau^2\\&+3q\kappa^2\tau+\big(3q\xi^2+p\big)\tau-\kappa^{3/2}\big(pq^3\tau^4+1\big)\big)^2\big)\end{aligned}\tag{9.2.101}$$

The Fixed Scalars at Horizon and the Entropy Calculating the area of the horizon we find:

$$\frac{1}{4\pi}\mathrm{Area}_H\equiv r_H^2=\lim_{\tau\to-\infty}\frac{1}{\tau^2}\exp\big[-U(\tau)\big]=\sqrt{pq^3}\tag{9.2.102}$$

which makes sense only as long $pq^3>0$ namely as long the p, q-charges are both positive or both negative. When this condition, which defines the physical branch of the solution, is fulfilled, (9.2.102) provides the correct expected result for anti-BPS black-holes. Indeed, comparing with the definition of the quartic symplectic invariant in (9.2.75) and with the form of the electromagnetic charges of the present solution we see that:

$$pq^3=-\mathfrak{I}_4\quad\text{if } p,q \text{ have the same sign}\tag{9.2.103}$$

A graphical illustration of the attractor mechanism is given in Fig. 9.1.

9.2.9 *Resuming the Discussion of Critical Points*

In view of the above examples, let us resume the general discussion of critical points of the Black Hole potential applying it to the S^3 model and moreover to a charge vector of the following type:

$$\left\{\begin{array}{c} p_1 \\ p_2 \\ q_1 \\ q_2 \end{array}\right\} = \left\{\begin{array}{c} 0 \\ p \\ \sqrt{3}q \\ 0 \end{array}\right\} \quad \Leftarrow \quad \begin{cases} p > 0, q < 0 \quad \text{or} \quad p < 0, q > 0 & \text{BPS} \\ p > 0, q > 0 \quad \text{or} \quad p < 0, q < 0 & \text{non-BPS} \end{cases} \tag{9.2.104}$$

and let us consider the solution of the attractor equations (9.2.40) with the above charge vector.

Non-BPS Case For p and q having the same sign it is easily verified that there is no solution of the equation $Z_z = 0$ and hence no BPS attractor point. On the other hand there is a solution of the critical point equation (9.2.40) with both $Z_z \neq 0$ and $Z \neq 0$. It corresponds to the following simple fixed value:

$$z_{fixed} = -\mathrm{i}\sqrt{\frac{p}{q}} \tag{9.2.105}$$

With such fixed value the i-invariant take the following values:

$$\{i_1, i_2, i_3, i_4, i_5\} = \left\{\frac{1}{2}\sqrt{\frac{p}{q}}q^2, \frac{3}{2}\sqrt{\frac{p}{q}}q^2, 0, -\frac{pq^3}{2}, 3pq^3\right\} \tag{9.2.106}$$

which satisfy the relations (9.2.48) characterizing a non-BPS attractor point of type II. Furthermore the quartic invariant $\mathfrak{I}_4(p,q) = -pq^3 < 0$ is negative in this case and we expect that the horizon area will be proportional to $\sqrt{-\mathfrak{I}_4}$. This is indeed the case as we verified few lines above. Furthermore if we calculate the limiting value of the scalar field (9.2.98), (9.2.99) at $\tau \to -\infty$ we precisely find the fixed value (9.2.105).

BPS Case If p and q have opposite signs there is just one solution of the equation $Z_z = 0$ with $Z \neq 0$. Hence we a have a BPS attractor. The fixed point is:

$$z_{fixed} = -\mathrm{i}\sqrt{-\frac{p}{q}} \tag{9.2.107}$$

which perfectly fits the general formula (9.2.79). Moreover calculating the i-invariants at the fixed point we obtain:

$$\{i_1, i_2, i_3, i_4, i_5\} = \left\{2\sqrt{-pq^3}, 0, 0, 0, 0\right\} \tag{9.2.108}$$

which fulfills the relations (9.2.47) proper of the BPS attractors.

9.2.10 An Example of a Small Black Hole

Let us consider the exact solution of the σ-model variational equations encoded in the functions displayed below that depend on four parameters y, σ, ξ, κ:

The Metric

$$U(\tau) = -\frac{1}{2}\log\left(2\sigma\tau\left(y^2+1\right)^3+1\right) \tag{9.2.109}$$

The Complex Scalar Field

$$\begin{aligned} z(\tau) = & \frac{\xi(2(y^2+1)\sigma\tau(y^2-1)^2+1) - 4y(y^4-1)\kappa\sigma\tau}{2(y^2+1)\sigma\tau(y^2-1)^2+1} \\ & - \mathrm{i}\frac{\kappa\sqrt{2\sigma\tau(y^2+1)^3+1}}{2(y^2+1)\sigma\tau(y^2-1)^2+1} \end{aligned} \tag{9.2.110}$$

The Electromagnetic Fields

$$\begin{aligned} \dot{Z}^1(\tau) &= \frac{4\sqrt{6}y^2((y^2-1)\kappa+2y\xi)\sigma}{\kappa^{3/2}(2\sigma\tau(y^2+1)^3+1)^2} \\ \dot{Z}^2(\tau) &= \frac{8\sqrt{2}y^3\sigma}{\kappa^{3/2}(2\sigma\tau(y^2+1)^3+1)^2} \end{aligned} \tag{9.2.111}$$

The Charges Using the general formulae discussed in previous pages we find that the Taub-Nut charge is zero:

$$\mathbf{n} = 0 \tag{9.2.112}$$

while for the electromagnetic charges we get:

$$\begin{pmatrix} p_1 \\ p_2 \\ q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} -\frac{2\sqrt{3}(y^2-1)(-\xi y^2+2\kappa y+\xi)^2\sigma}{\kappa^{3/2}} \\ -\frac{2(-\xi y^2+2\kappa y+\xi)^3\sigma}{\kappa^{3/2}} \\ \frac{2\sqrt{3}(y^2-1)^2(\xi y^2-2\kappa y-\xi)\sigma}{\kappa^{3/2}} \\ \frac{2(y^2-1)^3\sigma}{\kappa^{3/2}} \end{pmatrix} \tag{9.2.113}$$

Structure of the Charges and Attractor Mechanism Observing the right hand side of (9.2.113), we realize that in this solution the electromagnetic charges satisfy the following two algebraic constraints:

$$q_1^2 + \sqrt{3}p_1 q_2 = 0 \tag{9.2.114}$$

$$p_1^3 + 3\sqrt{3}p_2^2 q_2 = 0 \tag{9.2.115}$$

which can be solved for q_Λ in terms of p^Σ. Explicitly we have:

$$\{q_1, q_2\} = \left\{\mp\frac{p_1^2}{\sqrt{3}p_2}, -\frac{p_1^3}{3\sqrt{3}p_2^2}\right\} \tag{9.2.116}$$

Only the second branch of the above solution is consistent with (9.2.113) from which the constraints (9.2.115) were derived. Restricting our attention to such a branch, the two magnetic charges p^Σ are identified by (9.2.113) as it follows:

$$\{p_1, p_2\} = \left\{-\frac{2\sqrt{3}(y^2-1)(-\xi y^2+2\kappa y+\xi)^2\sigma}{\kappa^{3/2}}, -\frac{2(-\xi y^2+2\kappa y+\xi)^3\sigma}{\kappa^{3/2}}\right\} \tag{9.2.117}$$

Equation (9.2.117) can now be inverted expressing the parameters y and σ in terms of the charges p^Λ and of the value of the scalar field at infinity κ, ξ. The explicit inversion of the above formulae is quite involved and not relevant for our discussion, so we omit it.

If we calculate the limiting value taken by complex scalar field when $\tau \to -\infty$ we find that it is always real and equal to:

$$\lim_{\tau\to-\infty} z(\tau) = z_{\mathit{fix}} = \left\{\frac{-\xi y^2+2\kappa y+\xi}{1-y^2}\right\} \tag{9.2.118}$$

With a little bit of algebraic work one can verify that this value is actually the same as:

$$z_{\mathit{fix}} = -\frac{\sqrt{3}p_2}{p_1} \tag{9.2.119}$$

This is just the attractor mechanism. Independently from their values at infinity the scalar fields go to a fixed value at the horizon which depends only on the charges. The novelty, however, is that this horizon has a vanishing area. Indeed from the explicit form of the $U(\tau)$ function we obtain:

$$\frac{1}{4\pi}\mathrm{Area}_H = \lim_{\tau\to-\infty}\frac{1}{\tau^2}\exp\bigl[-U(\tau)\bigr] = 0 \tag{9.2.120}$$

This is consistent with the fact that the quartic invariant with such charges as those pertaining to this solution, namely $\{p_1, p_2, \frac{p_1^2}{\sqrt{3}p_2}, -\frac{p_1^3}{3\sqrt{3}p_2^2}\}$, vanishes identically: $\mathfrak{I}_4 = 0$.

9.2.11 Behavior of the Riemann Tensor in Regular Solutions

In order to better appreciate the approach to horizon in regular solutions it is convenient to study more in depth the solution based on the metric (9.2.87) and the scalar field (9.2.88), (9.2.89). As long as we do not mention the accompanying vector functions $Z^\Lambda(\tau)$ we do not know whether (9.2.87), (9.2.88), (9.2.89) describe the non-BPS or the BPS solution. Yet in both cases p, q are restricted to have the same sign which means equal sign for p_2, q_1 in the non-BPS case and opposite sign for the same charges in the BPS one. If we insert the explicit form of the warp factor

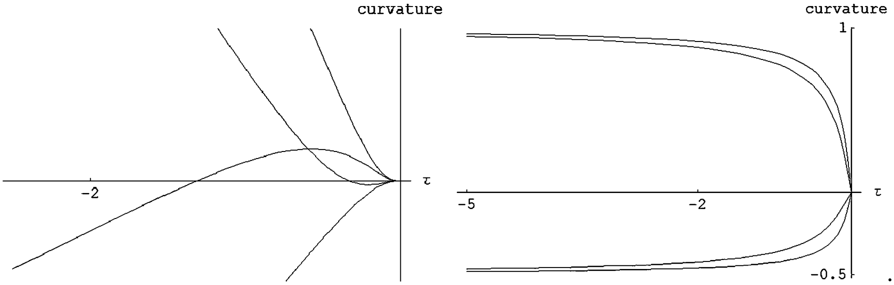

Fig. 9.2 Evolution of the four independent component of the curvature for the non-BPS and BPS solutions with $\xi = 0, \kappa = 1$, $q = 2$, $p = \frac{1}{8}$. In the picture on the *left* we see the behavior of the curvature near $\tau = 0$ namely at asymptotic infinity where they go all to zero. In the picture on the *right* we see the asymptotic behavior for large negative τ, namely near the horizon where the curvatures go to their constant values and the space degenerates into the direct product $\mathrm{AdS}_2 \times \mathrm{S}^2$

(9.2.87) in the expression (9.2.34) for the independent component of the Riemann tensor we can verify the following asymptotic behavior:

$$\lim_{\tau\to 0}\{\mathscr{C}_1(\tau), \mathscr{C}_2(\tau), \mathscr{C}_3(\tau), \mathscr{C}_4(\tau)\} = \{0, 0, 0, 0\} \tag{9.2.121}$$

$$\lim_{\tau\to -\infty}\{\mathscr{C}_1(\tau), \mathscr{C}_2(\tau), \mathscr{C}_3(\tau), \mathscr{C}_4(\tau)\} = \frac{1}{\sqrt{pq^3}}\left\{-\frac{1}{2}, -\frac{1}{2}, 1, 1\right\} \tag{9.2.122}$$

This has a profound meaning. The vanishing of all curvature components at radial infinity corresponds to the condition of *asymptotic flatness* which is a necessary boundary condition for physically meaningful black-holes. On the other hand the characteristic integer values of the Riemann tensor components obtained at the horizon correspond to the factorization of the four-dimensional geometry into the direct product $\mathrm{AdS}_2 \times S^2$. The interpretation of the regular black-holes as an interpolating soliton between two different vacua of supergravity is thus manifest. At $\tau = 0$ we have the vacuum Mink_4. At the horizon we have the vacuum $\mathrm{AdS}_2 \times S^2$ which requires an appropriate form of the electromagnetic fields. In Fig. 9.2 we present the behavior of the four functions in a numerical case-study where the approach to the asymptotic constant values at the horizon can be clearly seen.

9.3 Flux Vacua of M-Theory and Manifolds of Restricted Holonomy

The next instance of supergravity solutions that we consider has been announced in the introduction. We will focus on *compactified vacua* solutions of M-theory. By vacuum it is meant that four of the eleven dimensions of M-theory correspond to those of a maximally symmetric manifold, (Minkowski, de Sitter or anti de Sitter space). Compactification instead occurs if the remaining seven dimensions are rolled up into a compact 7-manifold whose size is fixed by its average curvature radius.

The compactification is flux driven if (9.1.2) holds true for the unique four-index field strength of M-theory.

The construction and classification of such solutions involves the notion of Killing spinors and of manifolds of weak G_2-holonomy that we will explain in the following pages. Furthermore at the heart of the mechanism of symmetry breaking we find the *holonomy tensor* which is a geometrical datum of the full space-time manifold. Its role is encoded in the Bianchi identities of the Free Differential Algebra and our first mission is to single it out in that general context. This is our starting point.

9.3.1 The Holonomy Tensor from $D = 11$ Bianchi Identities

To fulfill our mission it is convenient to rewrite the rheonomic parameterization of M-theory FDA curvatures, that were given in (6.4.8), in a slightly more compact form, namely as follows:

$$\begin{aligned}
\mathfrak{T}^a &\equiv 0 \\
\mathfrak{R}^{ab} &\equiv R^{ab}{}_{mn} V^m \wedge V^n + \overline{\Theta}^{c|ab} \Psi \wedge V_c + \overline{\Psi} \wedge S^{ab} \Psi \\
\rho &\equiv \rho_{ab} V^a \wedge V^b + F_a \Psi \wedge V^a \\
\mathbf{F}^{[4]} &\equiv F_{b_1 \dots b_4} V^{b_1} \wedge \cdots \wedge V^{b_4}
\end{aligned} \tag{9.3.1}$$

In the above equations we have introduced the following spinor and the following matrices:

$$\begin{aligned}
\overline{\Theta}^{c|ab} &= \mathrm{i}\,\overline{\rho}_{mn} \left(\frac{1}{2} \Gamma^{abmnc} - \frac{2}{9} \Gamma^{mn[a} \delta^{b]c} + 2 \Gamma^{ab[m} \delta^{n]c} \right) \\
&= -\mathrm{i}\,\overline{\rho}_{ab} \Gamma_c + 2\mathrm{i}\,\overline{\rho}_{c[a} \Gamma_{b]}
\end{aligned} \tag{9.3.2}$$

$$F_a = T_a{}^{b_1 b_2 b_3 b_4} F_{b_1 b_2 b_3 b_4} \tag{9.3.3}$$

$$S^{ab} = F^{abcd} \Gamma_{cd} + \frac{1}{24} F_{c_1 \dots c_4} \Gamma^{abc_1 \dots c_4} \tag{9.3.4}$$

furthermore we have used the following abbreviation:

$$T_a{}^{b_1 b_2 b_3 b_4} = -\frac{\mathrm{i}}{24} \left(\Gamma^{b_1 b_2 b_3 b_4}{}_a + 8 \delta_a{}^{[b_1} \Gamma^{b_2 b_3 b_4]} \right) \tag{9.3.5}$$

In (9.3.2) the equality of the first with the second line follows from the gravitino field equation, namely the second of (6.4.9). This latter implies that the spinor tensor ρ_{ab} is an irreducible representation $(\frac{3}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ of $\mathrm{SO}(1, 10)$, *i.e*:

$$\Gamma^m \rho_{am} = 0 \tag{9.3.6}$$

As we demonstrate below the most important relations to be extracted from Bianchi identities, besides the rheonomic parameterization, concern the spinor derivatives of the curvature superfield. This latter is determined from the expansion of the inner components of the 4-form field strength $F_{a_1\ldots a_4}$. From the (6.4.6) we obtain:

$$\mathscr{D}_\alpha F_{abcd} = (\Gamma_{[ab}\rho_{cd]})_\alpha \tag{9.3.7}$$

where the spinor derivative is normalized according to the definition:

$$\mathscr{D} F_{abcd} \equiv \overline{\Psi}^\alpha \mathscr{D}_\alpha F_{abcd} + V^m \mathscr{D}_m F_{abcd} \tag{9.3.8}$$

This shows that the gravitino field strength appears at first order in the θ-expansion of the curvature superfield. Next we consider the spinor derivative of the gravitino field strength itself. Using the normalization which streams from the following definition:

$$\mathscr{D}\rho_{ab} = \mathscr{D}_c \rho_{ab} V^c + K_{ab}\Psi \tag{9.3.9}$$

we obtain:

$$K_{ab} = -\frac{1}{4} R^{mn}{}_{ab}\, \Gamma_{mn} + \mathscr{D}_{[a} F_{b]} + \frac{1}{2}[F_a, F_b] \tag{9.3.10}$$

The tensor-matrix K_{ab} is of key importance in the discussion of compactifications. If it vanishes on a given background it means that the gravitino field strength can be consistently put to zero to all orders in θs and on its turn this implies that the 4-field strength can be chosen constant to all orders in θs. This is the case of maximal unbroken supersymmetry. In this case all curvature components of the Free Differential Algebra can be chosen constant and we have a superspace whose geometry is purely described by Maurer Cartan forms of some super coset.

On the other hand if K_{ab}, that we name the *holonomy tensor* does not vanish this implies that both ρ_{ab} and F_{abcd} have some non-trivial θ-dependence and cannot be chosen constant. In this case the geometry of superspace is not described by simple Maurer Cartan forms of some supercoset, since the curvatures of the FDA are not pure constants. This is the case of fully or partially broken SUSY and it is the case we explore. In the $\mathrm{AdS}_4 \times (\mathscr{G}/\mathscr{H})_7$ compactifications it turns out that the matrix K_{ab} is related to the holonomy tensor of the internal manifold $(\mathscr{G}/\mathscr{H})_7$.

Let us finally work out the spinor derivative of the Riemann tensor. Defining:

$$\mathscr{D} R^{ab}{}_{mn} = \mathscr{D}_p R^{ab}{}_{mn}\, V^p + \overline{\Psi}\Lambda^{ab}{}_{mn} \tag{9.3.11}$$

from (6.4.3) we obtain:

$$\Lambda^{ab}{}_{mn} = (\mathscr{D}_{[m} - \overline{F}_{[m})\Theta_{n]}{}^{|ab} + 2S^{ab}\rho_{mn} \tag{9.3.12}$$

where we have introduced the notation:

$$\begin{aligned} \Theta^{n|ab} &= C\big(\overline{\Theta}^{n|ab}\big)^T = \mathrm{i}\Gamma_c\rho_{ab} - 2\mathrm{i}\Gamma_{[a}\rho_{b]c} \\ \overline{F}_a &= C(F_a)^T C^{-1} = \frac{\mathrm{i}}{24}\big(\Gamma^{b_1b_2b_3b_4}{}_a - 8\delta_a{}^{[b_1}\Gamma^{b_2b_3b_4]}\big)F_{b_1b_2b_3b_4} \end{aligned} \tag{9.3.13}$$

The matrix K_{ab} and the spinor $\Lambda^{ab}{}_{mn}$ are the crucial objects we are supposed to compute in each compactification background.

9.3.2 *Flux Compactifications of M-Theory on* $\mathrm{AdS}_4 \times \mathscr{M}_7$ *Backgrounds*

We are interested in compactified backgrounds where the 11-dimensional bosonic manifold is of the form:

$$\mathscr{M}_{11} = \mathscr{M}_4 \times \mathscr{M}_7 \tag{9.3.14}$$

$\mathscr{M}_4$ denoting a four-dimensional maximally symmetric manifold whose coordinates we denote x^μ and $\mathscr{M}_7$ a 7-dimensional compact manifold whose parameters we denote y^I. Furthermore we assume that in any configuration of the compactified theory the eleven dimensional vielbein is split as follows:

$$V^{\underline{a}} = \begin{cases} V^r = E^r(x); & r = 0,1,2,3 \\ V^\alpha = \Phi^\alpha{}_\beta(x)(e^\beta + W^\beta(x)); & \alpha,\beta = 4,5,6,7,8,9,10 \end{cases} \tag{9.3.15}$$

where $E^r(x)$ is a purely x-dependent 4-dimensional vielbein, $W^\alpha(x)$ is an x-dependent 1-form on x-space describing the Kaluza Klein vectors and the purely x-dependent 7×7 matrix $\Phi^\alpha{}_\beta(x)$ encodes part of the scalar fields of the compactified theory, namely the internal metric moduli. From these assumptions it follows that the bosonic field strength is expanded as follows:

$$\begin{aligned}\mathbf{F}^{[4]}_{(Bosonic)} \equiv{}& F^{[4]}(x) + F^{[3]}_\alpha(x) \wedge V^\alpha + F^{[2]}_{\alpha\beta}(x) \wedge V^\alpha \wedge V^\beta \\ &+ F^{[1]}_{\alpha\beta\gamma}(x) \wedge V^\alpha \wedge V^\beta \wedge V^\gamma + F^{[0]}_{\alpha\beta\gamma\delta}(x) \wedge V^\alpha \wedge V^\beta \wedge V^\gamma \wedge V^\delta\end{aligned} \tag{9.3.16}$$

where $F^{[p]}_{\alpha_1\ldots\alpha_{4-p}}(x)$ are x-space p-forms depending only on x.

In bosonic backgrounds with a space-time geometry of the form (9.3.14), the family of configurations (9.3.15) must satisfy the condition that by choosing:

$$E^r = \text{vielbein of a maximally symmetric 4D space-time} \tag{9.3.17}$$

$$\Phi^I{}_J(x) = \delta^I{}_J \tag{9.3.18}$$

$$W^I = 0 \tag{9.3.19}$$

$$F^{[3]}_I(x) = F^{[2]}_{IJ}(x) = F^{[1]}_{IJK}(x) = 0 \tag{9.3.20}$$

$$F^{[4]}(x) = e\varepsilon_{rstu} E^r \wedge E^s \wedge E^t \wedge E^u; \quad (e = \text{const}) \tag{9.3.21}$$

$$F^{[0]}_{\alpha\beta\gamma\delta}(x) = g_{\alpha\beta\gamma\delta} = \text{constant tensor} \tag{9.3.22}$$

we obtain an exact *bona fide* solution of the eleven-dimensional field equations of M-theory.

There are three possible 4-dimensional maximally symmetric Lorentzian manifolds

$$\mathscr{M}_4 = \begin{cases} \mathrm{Mink}_4 & \text{Minkowski space} \\ \mathrm{dS}_4 & \text{de Sitter space} \\ \mathrm{AdS}_4 & \text{anti de Sitter space} \end{cases} \tag{9.3.23}$$

In any case Lorentz invariance imposes (9.3.18), (9.3.19), (9.3.20) while translation invariance imposes that the vacuum expectation value of the scalar fields $\Phi^{\alpha}{}_{\beta}(x)$ should be a constant matrix

$$\left\langle \Phi^{\alpha}{}_{\beta}(x) \right\rangle = \mathscr{A}^{\alpha}_{\beta} \tag{9.3.24}$$

We are interested in 7-manifolds that preserve some residual supersymmetry in $D = 4$. This relates to the holonomy of $\mathscr{M}_7$ which has to be restricted in order to allow for the existence of Killing spinors. In the next subsection we summarize the basic results from the literature on this topic.

9.3.3 M-Theory Field Equations and 7-Manifolds of Weak G_2 Holonomy i.e. Englert 7-Manifolds

In order to admit at least one Killing spinor or more, the 7-manifold $\mathscr{M}_7$ necessarily must have a (weak) holonomy smaller than SO(7): at most G_2. The qualification weak refers to the definition of holonomy appropriate to compactifications on $\mathrm{AdS}_4 \times \mathscr{M}_7$ while the standard definition of holonomy is appropriate to compactifications on Ricci flat backgrounds $\mathrm{Mink}_4 \times \mathscr{M}_7$. To explain in contemporary language these concepts that were discovered in the eighties we have to recall the notion of G-structures. Indeed in the recent literature about flux compactifications the key geometrical notion exploited by most authors is precisely that of G-structures [22].

Following, for instance, the presentation of [22], if $\mathscr{M}_n$ is a differentiable manifold of dimension n, $T\mathscr{M}_n \overset{\pi}{\to} \mathscr{M}_n$ its tangent bundle and $F\mathscr{M}_n \overset{\pi}{\to} \mathscr{M}_n$ its frame bundle, we say that $\mathscr{M}_n$ admits a G-structure when the structural group of $F\mathscr{M}_n$ is reduced from the generic $\mathrm{GL}(n,\mathbb{R})$ to a proper subgroup $\mathrm{G} \subset \mathrm{GL}(n,\mathbb{R})$. Generically, tensors on $\mathscr{M}_n$ transform in representations of the structural group $\mathrm{GL}(n,\mathbb{R})$. If a G-structure reduces this latter to $\mathrm{G} \subset \mathrm{GL}(n,\mathbb{R})$, then the decomposition of an irreducible representation of $\mathrm{GL}(n,\mathbb{R})$, pertaining to a certain tensor t^p, with respect to the subgroup G may contain singlets. This means that on such a manifold $\mathscr{M}_n$ there may exist a certain tensor t^p which is G-invariant, and therefore globally defined. As recalled in [22] existence of a Riemannian metric g on $\mathscr{M}_n$ is equivalent to a reduction of the structural group $\mathrm{GL}(n,\mathbb{R})$ to $\mathrm{O}(n)$, namely to an $\mathrm{O}(n)$-structure. Indeed, one can reduce the frame bundle by introducing orthonormal frames, the vielbein e^I, and, written in these frames, the metric is the $\mathrm{O}(n)$ invariant tensor

δ_{IJ}. Similarly orientability corresponds to an SO(n)-structure and the existence of spinors on spin manifolds corresponds to a Spin(n)-structure.

In the case of seven dimensions, an orientable Riemannian manifold $\mathscr{M}_7$, whose frame bundle has generically an SO(7) structural group, admits a G_2-structure if and only if, in the basis provided by the orthonormal frames $\mathscr{B}^\alpha$, there exists an antisymmetric 3-tensor $\phi_{\alpha\beta\gamma\delta}$ satisfying the algebra of the octonionic structure constants:

$$\begin{aligned} \phi_{\alpha\beta\kappa}\phi_{\gamma\delta\kappa} &= \frac{1}{18}\delta^{\gamma\delta}_{\alpha\beta} - \frac{2}{3}\phi^{\star}_{\alpha\beta\gamma\delta} \\ -\frac{1}{6}\varepsilon_{\kappa\rho\sigma\alpha\beta\gamma\delta}\phi^{\star}_{\alpha\beta\gamma\delta} &= \phi_{\kappa\rho\sigma} \end{aligned} \tag{9.3.25}$$

which is invariant, namely it is the same in all local trivializations of the SO(7) frame bundle. This corresponds to the algebraic definition of G_2 as that subgroup of SO(7) which acts as an automorphism group of the octonion algebra. Alternatively G_2 can be defined as the stability subgroup of the 8-dimensional spinor representation of SO(7). Hence we can equivalently state that a manifold $\mathscr{M}_7$ has a G_2-structure if there exists at least an invariant spinor η, which is the same in all local trivializations of the Spin(7) spinor bundle.

In terms of this invariant spinor the invariant 3-tensor $\phi_{\rho\sigma\kappa}$ has the form:[6]

$$\phi^{\rho\sigma\kappa} = \frac{1}{6}\eta^T\tau^{\rho\sigma\kappa}\eta \tag{9.3.26}$$

and (9.3.26) provides the relation between the two definitions of the G_2-structure.

On the other hand the manifold has not only a G_2-structure, but also G_2-holonomy if the invariant three-tensor $\phi_{\alpha\beta\kappa}$ is covariantly constant, namely:

$$0 = \nabla\phi^{\alpha\beta\gamma} \equiv d\phi^{\alpha\beta\gamma} + 3\mathscr{B}^{\kappa[\alpha}\phi^{\beta\gamma]\kappa} \tag{9.3.27}$$

where the 1-form $\mathscr{B}^{\alpha\beta}$ is the spin connection of $\mathscr{M}_7$. Alternatively the manifold has G_2-holonomy if the invariant spinor η is covariantly constant, namely if:

$$\exists\eta \in \Gamma(\mathrm{Spin}\mathscr{M}_7, \mathscr{M}_7)\backslash 0 = \nabla\eta \equiv d\eta - \frac{1}{4}\mathscr{B}^{\alpha\beta}\tau_{\alpha\beta}\eta \tag{9.3.28}$$

where τ^α ($\alpha = 1, \dots, 7$) are the 8×8 gamma matrices of the SO(7) Clifford algebra (see footnote). The relation between the two definitions (9.3.27) and (9.3.28) of G_2-holonomy is the same as for the two definitions of the G_2-structure, namely it is given by (9.3.26). As a consequence of its own definition a Riemannian 7-manifold with G_2 holonomy is Ricci flat. Indeed the integrability condition of (9.3.28) yields:

$$\mathscr{R}^{\alpha\beta}{}_{\gamma\delta}\tau_{\alpha\beta}\eta = 0 \tag{9.3.29}$$

[6]By τ^α we denote the gamma matrices in 7-dimensions, satisfying the Clifford algebra $\{\tau^\alpha, \tau^\beta\} = -\delta^{\alpha\beta}$. With the symbol $\tau^{\alpha_1\dots\alpha_n}$ we denote, as usual, the antisymmetrized product of n such matrices.

where $\mathscr{R}^{\alpha\beta}{}_{\gamma\delta}$ is the Riemann tensor of $\mathscr{M}_7$. From (9.3.29), by means of a few simple algebraic manipulations one obtains two results:

- The curvature 2-form

$$\mathscr{R}^{\alpha\beta} \equiv \mathscr{R}^{\alpha\beta}{}_{\gamma\delta}\mathscr{B}^{\gamma} \wedge \mathscr{B}^{\delta} \tag{9.3.30}$$

  is G_2 Lie algebra valued, namely it satisfies the condition:

$$\phi^{\kappa\alpha\beta}\mathscr{R}^{\alpha\beta} = 0 \tag{9.3.31}$$

  which projects out the **7** of G_2 from the **21** of SO(7) and leaves with the adjoint **14**.
- The internal Ricci tensor is zero:

$$\mathscr{R}^{\alpha\kappa}{}_{\beta\kappa} = 0 \tag{9.3.32}$$

Next we consider the bosonic field equations of M-theory, namely the first and the last of (6.4.9). We make the compactification ansatz (9.3.14) where $\mathscr{M}_4$ is one of the three possibilities mentioned in (9.3.23) and all of (9.3.18)–(9.3.22) hold true. Then we split the rigid index range as follows:

$$\underline{a}, \underline{b}, \underline{c}, \ldots = \begin{cases} \alpha, \beta, \gamma, \ldots = 4, 5, 6, 7, 8, 9, 10 = \mathscr{M}_7 \text{ indices} \\ r, s, t, \ldots = 0, 1, 2, 3 = \mathscr{M}_4 \text{ indices} \end{cases} \tag{9.3.33}$$

and by following the conventions employed in [23] and using the results obtained in the same paper, we conclude that the compactification ansatz reduces the system of the first and last of (6.4.9) to the following one:

$$R^{rs}{}_{tu} = \lambda \delta^{rs}_{tu} \tag{9.3.34}$$

$$\mathscr{R}^{\alpha\kappa}{}_{\beta\kappa} = 3\nu\delta^{\alpha}_{\beta} \tag{9.3.35}$$

$$F_{rstu} = e\varepsilon_{rstu} \tag{9.3.36}$$

$$g_{\alpha\beta\gamma\delta} = f\mathscr{F}_{\alpha\beta\gamma\delta} \tag{9.3.37}$$

$$\mathscr{F}^{\alpha\kappa\rho\sigma}\mathscr{F}_{\beta\kappa\rho\sigma} = \mu\delta^{\alpha}_{\beta} \tag{9.3.38}$$

$$\mathscr{D}^{\mu}\mathscr{F}_{\mu\kappa\rho\sigma} = \frac{1}{2}e\varepsilon_{\kappa\rho\sigma\alpha\beta\gamma\delta}\mathscr{F}^{\alpha\beta\gamma\delta} \tag{9.3.39}$$

Equation (9.3.35) states that the internal manifold $\mathscr{M}_7$ must be an Einstein space. Equations (9.3.36) and (9.3.37) state that there is a flux of the four-form both on 4-dimensional space-time $\mathscr{M}_4$ and on the internal manifold $\mathscr{M}_7$. The parameter e, which fixes the size of the flux on the four-dimensional space and was already introduced in (9.3.21), is called the Freund-Rubin parameter [24]. As we are going to show, in the case that a non-vanishing $\mathscr{F}^{\alpha\beta\gamma\delta}$ is required to exist, (9.3.38) and (9.3.39), are equivalent to the assertion that the manifold $\mathscr{M}_7$ has weak G_2 holonomy rather than G_2-holonomy, to state it in modern parlance [25]. In paper [26], manifolds admitting such a structure were instead named *Englert spaces* and the underlying notion of weak G_2 holonomy was already introduced there with the different name of *de Sitter* $SO(7)^+$ *holonomy*.

Indeed (9.3.39) which, in the language of the early eighties was named Englert equation [27] and which is nothing else but the first equation of (6.4.9), upon substitution of the Freund Rubin ansatz (9.3.36) for the external flux, can be recast in the following more revealing form: Let

$$\Phi^\star \equiv \mathscr{F}_{\alpha\beta\gamma\delta}\mathscr{B}^\alpha \wedge \mathscr{B}^\beta \wedge \mathscr{B}^\gamma \wedge \mathscr{B}^\delta \tag{9.3.40}$$

be a the constant 4-form on $\mathscr{M}_7$ defined by our non-vanishing flux, and let

$$\Phi \equiv \frac{1}{24}\varepsilon_{\alpha\beta\gamma\kappa\rho\sigma\tau}\mathscr{F}_{\kappa\rho\sigma\tau}\mathscr{B}^\alpha \wedge \mathscr{B}^\beta \wedge \mathscr{B}^\gamma \tag{9.3.41}$$

be its dual. Englert equation (9.3.39) is just the same as writing:

$$\begin{aligned} d\Phi &= 12e\Phi^\star \\ d\Phi^\star &= 0 \end{aligned} \tag{9.3.42}$$

When the Freund Rubin parameter vanishes $e = 0$ we recognize in (9.3.42) the statement that our internal manifold $\mathscr{M}_7$ has G_2-holonomy and hence it is Ricci flat. Indeed Φ is the G_2 invariant and covariantly constant form defining G_2-structure and G_2-holonomy. On the other hand the case $e \neq 0$ corresponds to the weak G_2 holonomy. Just as we reduced the existence of a closed three-form Φ to the existence of a G_2 covariantly constant spinor satisfying (9.3.28) which allows to set the identification (9.3.26), in the same way (9.3.42) can be solved *if and only if* on $\mathscr{M}_7$ there exist a weak Killing spinor η satisfying the following defining condition:

$$\mathscr{D}_\alpha \eta = m e \tau_\alpha \eta \tag{9.3.43}$$

$$\Updownarrow$$

$$D\eta \equiv \left(d - \frac{1}{4}\mathscr{B}^{\alpha\beta}\tau_{\alpha\beta}\right)\eta = me\mathscr{B}^\alpha \tau_\alpha \eta \tag{9.3.44}$$

where m is a numerical constant and e is the Freund-Rubin parameter, namely the only scale which at the end of the day will occur in the solution.

The integrability of the above equation implies that the Ricci tensor be proportional to the identity, namely that the manifold is an Einstein manifold and furthermore fixes the proportionality constant:

$$\mathscr{R}^{\alpha\kappa}{}_{\beta\kappa} = 12m^2e^2\delta^\alpha_\beta \quad \longrightarrow \quad \nu = 12m^2e^2 \tag{9.3.45}$$

In case such a spinor exists, by setting:

$$g_{\alpha\beta\gamma\delta} = \mathscr{F}_{\alpha\beta\gamma\delta} = \eta^T \tau_{\alpha\beta\gamma\delta}\eta = 24\phi^\star_{\alpha\beta\gamma\delta} \tag{9.3.46}$$

we find that Englert equation (9.3.39) is satisfied, provided we have:

$$m = -\frac{3}{2} \tag{9.3.47}$$

In this way Maxwell equation, namely the first of (6.4.9) is solved. Let us also note, as the authors of [26] did many years ago, that the condition (9.3.43) can also be interpreted in the following way. The spin-connection $\mathscr{B}^{\alpha\beta}$ plus the vielbein $\mathscr{B}^{\gamma}$ define on any non-Ricci flat 7-manifold $\mathscr{M}_7$ a connection which is actually SO(8) rather than SO(7) Lie algebra valued. In other words we have a principal SO(8) bundle which leads to an SO(8) spin bundle of which η is a covariantly constant section:

$$0 = \nabla^{\mathrm{SO}(8)}\eta = \left(\nabla^{\mathrm{SO}(7)} - me\mathscr{B}^{\alpha}\tau_\alpha\right)\eta \tag{9.3.48}$$

The existence of η implies a reduction of the SO(8)-bundle. Indeed the stability subgroup of an SO(8) spinor is a well known subgroup $\mathrm{SO}(7)^+$ different from the standard SO(7) which, instead, stabilizes the vector representation. Hence the so named weak G_2 holonomy of the SO(7) spin connection $\mathscr{B}^{\alpha\beta}$ is the same thing as the $\mathrm{SO}(7)^+$ holonomy of the SO(8) Lie algebra valued *de Sitter connection* $\{\mathscr{B}^{\alpha\beta}, \mathscr{B}^{\gamma}\}$ introduced in [26] and normally discussed in the old literature on Kaluza Klein Supergravity.

We have solved Maxwell equation, but we still have to solve Einstein equation, namely the last of (6.4.9). To this effect we note that:

$$\mathscr{F}_{\beta\kappa\rho\sigma}\mathscr{F}^{\alpha\kappa\rho\sigma} = 24\delta^\alpha_\beta \quad \Longrightarrow \quad \mu = 24 \tag{9.3.49}$$

and we observe that Einstein equation reduces to the following two conditions on the parameters (see [23] for details):

$$\begin{aligned} \frac{3}{2}\lambda &= -\left(24e^2 + \frac{7}{2}\mu f^2\right) \\ 3\nu &= 12e^2 + \frac{5}{2}\mu f^2 \end{aligned} \tag{9.3.50}$$

From (9.3.50) we conclude that there are only three possible kind of solutions.

(a) The flat solutions of type

$$\mathscr{M}_{11} = \mathrm{Mink}_4 \otimes \underbrace{\mathscr{M}_7}_{\text{Ricci flat}} \tag{9.3.51}$$

where both $D = 4$ space-time and the internal 7-space are Ricci flat. These compactifications correspond to $e = 0$ and $F_{\alpha\beta\gamma\delta} = 0 \Rightarrow g_{\alpha\beta\gamma\delta} = 0$.

(b) The Freund Rubin solutions of type

$$\mathscr{M}_{11} = \mathrm{AdS}_4 \otimes \underbrace{\mathscr{M}_7}_{\text{Einst. manif.}} \tag{9.3.52}$$

These correspond to anti de Sitter space in 4-dimensions, whose radius is fixed by the Freund Rubin parameter $e \neq 0$ times any Einstein manifold in 7-dimensions with no internal flux, namely $g_{\alpha\beta\gamma\delta} = 0$. In this case from (9.3.50) we uniquely obtain:

$$R^{rs}{}_{tu} = -16e^2 \delta^{rs}_{tu} \tag{9.3.53}$$

$$\mathscr{R}^{\alpha\kappa}_{\beta\kappa} = 12e^2 \delta^{\alpha}_{\beta} \tag{9.3.54}$$

$$F_{rstu} = e\varepsilon_{rstu} \tag{9.3.55}$$

$$F_{\alpha\beta\gamma\delta} = 0 \tag{9.3.56}$$

(c) The Englert type solutions

$$\mathscr{M}_{11} = \mathrm{AdS}_4 \otimes \underbrace{\mathscr{M}_7}_{\substack{\text{Einst. manif.} \\ \text{weak } \mathrm{G}_2 \text{ hol}}} \tag{9.3.57}$$

These correspond to anti de Sitter space in 4-dimensions ($e \neq 0$) times a 7-dimensional Einstein manifold which is necessarily of weak G_2 holonomy in order to support a consistent non-vanishing internal flux $g_{\alpha\beta\gamma\delta}$. In this case combining (9.3.50) with the previous ones we uniquely obtain:

$$\lambda = -30e^2; \qquad f = \pm\frac{1}{2}e \tag{9.3.58}$$

As we already mentioned in the introduction there exist several compact manifolds of weak G_2 holonomy. In particular all the coset manifolds $\mathscr{G}/\mathscr{H}$ of weak G_2 holonomy were classified and studied in the Kaluza Klein supergravity age [23, 26, 28–35] and they were extensively reconsidered in the context of the AdS/CFT correspondence [36–40].

In the present section we present the supergauge completion, namely the extension to a convenient superspace containing all or a subset of the 32 fermionic coordinates θs of the compactifications of the Freund Rubin type, namely on eleven-manifolds of the form:

$$\mathscr{M}_{11} = \mathrm{AdS}_4 \times \frac{\mathscr{G}}{\mathscr{H}} \tag{9.3.59}$$

with no internal flux $g_{\alpha\beta\gamma\delta}$ switched on. As it was extensively explained in [41] and further developed in [36–40], if the compact coset $\mathscr{G}/\mathscr{H}$ admits $\mathscr{N} \leq 8$ Killing spinors η_A, namely $\mathscr{N} \leq 8$ independent solutions of (9.3.43) with $m = 1$, then the isometry group $\mathscr{G}$ is necessarily of the form:

$$\mathscr{G} = \mathrm{SO}(\mathscr{N}) \times \mathrm{G}_{\mathit{flavor}} \tag{9.3.60}$$

where $\mathrm{G}_{\mathit{flavor}}$ is some appropriate Lie group. In this case the isometry supergroup of the considered M-theory background is:

$$\mathrm{Osp}(\mathscr{N}\,|\,4) \times \mathrm{G}_{\mathit{flavor}} \tag{9.3.61}$$

and the spectrum of fluctuations of the background arranges into $\mathrm{Osp}(\mathcal{N}\,|\,4)$ supermultiplets furthermore assigned to suitable representations of the bosonic flavor group.

9.3.4 The SO(8) Spinor Bundle and the Holonomy Tensor

We come next to discuss a very important property of 7-manifolds with a spin structure which plays a crucial role in understanding the supergauge completion. This is the existence of an SO(8) vector bundle whose non-trivial connection is defined by the Riemannian structure of the manifold. To introduce this point and in order to illustrate its relevance to our problem we begin by considering a basis of $D = 11$ gamma matrices well adapted to the compactification on $\mathrm{AdS}_4 \times \mathcal{M}_7$.

9.3.5 The Well Adapted Basis of Gamma Matrices

According to the tensor product representation well adapted to the compactification, the $D = 11$ gamma matrices can be written as follows:

$$\begin{aligned} \Gamma_a &= \gamma_a \otimes \mathbf{1}_{8\times 8} \quad (a = 0, 1, 2, 3) \\ \Gamma_{3+\alpha} &= \gamma_5 \otimes \tau_\alpha \quad (\alpha = 1, \ldots, 7) \end{aligned} \tag{9.3.62}$$

where, following the old Kaluza Klein supergravity literature [26, 30, 41] the matrices τ_α are the real antisymmetric realization of the SO(7) Clifford algebra with negative metric:

$$\{\tau_\alpha, \tau_\beta\} = -2\delta_{\alpha\beta}; \qquad \tau_\alpha = -(\tau_\alpha)^T \tag{9.3.63}$$

In this basis the charge conjugation matrix is given by:

$$C = \mathscr{C} \otimes \mathbf{1}_{8\times 8} \tag{9.3.64}$$

where $\mathscr{C}$ is the charge conjugation matrix in $d = 4$:

$$\mathscr{C}\gamma_a\mathscr{C}^{-1} = -\gamma_a^T; \qquad \mathscr{C}^T = -\mathscr{C} \tag{9.3.65}$$

9.3.6 The $\mathfrak{so}(8)$-Connection and the Holonomy Tensor

Next we observe that using these matrices the covariant derivative introduced in (9.3.48) defines a universal $\mathfrak{so}(8)$-connection on the spinor bundle which is given once the Riemannian structure is given, namely once the vielbein and the spin connection $\{\mathscr{B}^{\alpha}, \mathscr{B}^{\alpha\beta}\}$ are given:

$$\mathbf{U}^{\mathfrak{so}(8)} \equiv -\frac{1}{4}\mathscr{B}^{\alpha\beta}\,\tau_{\alpha\beta} - e\mathscr{B}^{\alpha}\,\tau_{\alpha} \tag{9.3.66}$$

More precisely, let $\zeta_{\widehat{A}}$ be an orthonormal basis:

$$\overline{\zeta}_{\widehat{A}}\zeta_{\widehat{B}} = \delta_{\widehat{A}\widehat{B}} \tag{9.3.67}$$

of sections of the spinor bundle over the Einstein manifold $\mathscr{M}_7$. Any spinor can be written as a linear combination of these sections that are real. Furthermore the bar operation in this case is simply the transposition. Hence, if we consider the $\mathfrak{so}(8)$ covariant derivative of any of these sections, this is a spinor and, as such, it can be expressed as a linear combinations of the same:

$$\nabla^{\mathfrak{so}(8)}\,\zeta_{\widehat{A}} \equiv \big(d + \mathbf{U}^{\mathfrak{so}(8)}\big)\zeta_{\widehat{A}} = \mathbf{U}_{\widehat{A}\widehat{B}}\,\zeta_{\widehat{\mathrm{B}}} \tag{9.3.68}$$

According to standard lore the one-form valued, antisymmetric 8×8 matrix $\mathbf{U}_{\widehat{A}\widehat{B}}$ defined by (9.3.68) is the $\mathfrak{so}(8)$-connection in the chosen basis of sections. If the manifold $\mathscr{M}_7$ admits $\mathscr{N}$ Killing spinors, then it follows that we can choose an orthonormal basis where the first $\mathscr{N}$ sections are Killing spinors:

$$\zeta_A = \eta_A; \qquad \nabla^{\mathfrak{so}(8)}\,\eta_A = 0, \quad A = 1, \ldots, \mathscr{N} \tag{9.3.69}$$

and the remaining $8 - \mathscr{N}$ elements of the basis, whose covariant derivative does not vanish are orthogonal to the Killing spinors:

$$\begin{aligned} &\zeta_{\overline{A}} = \xi_{\overline{A}}; \qquad \nabla^{\mathfrak{so}(8)}\,\xi_{\overline{A}} \neq 0, \quad \overline{A} = 1, \ldots, 8 - \mathscr{N} \\ &\overline{\xi}_{\overline{B}}\,\eta_A = 0 \\ &\overline{\xi}_{\overline{B}}\,\xi_{\overline{C}} = \delta_{\overline{BC}} \end{aligned} \tag{9.3.70}$$

It is then evident from (9.3.69) and (9.3.70) that the $\mathfrak{so}(8)$-connection $\mathbf{U}_{\widehat{A}\widehat{B}}$ takes values only in a subalgebra $\mathfrak{so}(8-\mathscr{N}) \subset \mathfrak{so}(8)$ and has the following block diagonal form:

$$\mathbf{U}_{\widehat{A}\widehat{B}} = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \mathbf{U}_{\overline{AB}} \end{array}\right) \tag{9.3.71}$$

Squaring the SO(8)-covariant derivative, we find

$$\begin{aligned} \nabla^2\zeta_{\widehat{A}} &= \underbrace{(d\mathbf{U}_{\widehat{A}\widehat{B}} - \mathbf{U}_{\widehat{A}\widehat{C}} \wedge \mathbf{U}_{\widehat{C}\widehat{B}})}_{\mathscr{F}_{\widehat{A}\widehat{B}}[\mathbf{U}]}\,\zeta_{\widehat{B}} \\ &= -\frac{1}{4}\underbrace{\big(\mathscr{R}^{\gamma\delta}{}_{\alpha\beta} - 4e^2\delta^{\gamma\delta}{}_{\alpha\beta}\big)}_{\mathscr{C}^{\gamma\delta}{}_{\alpha\beta}}\,\tau_{\gamma\delta}\zeta_{\widehat{A}} \end{aligned} \tag{9.3.72}$$

where $\mathscr{C}^{\gamma\delta}{}_{\alpha\beta}$ is the so called holonomy tensor, essentially identical with the Weyl tensor of the considered Einstein 7-manifold.

9.3.7 The Holonomy Tensor and Superspace

As a further preparation to our subsequent discussion of the gauge completion let us now consider the form taken on the $\mathrm{AdS}_4 \times \mathscr{G}/\mathscr{H}$ backgrounds by the operator K_{ab} introduced in (9.3.9) and governing the mechanism of supersymmetry breaking. We will see that it is just simply related to the holonomy tensor discussed in the previous section, namely to the field strength of the SO(8)-connection on the spinor bundle. To begin with, we calculate the operator $F_{\underline{a}}$ introduced in (9.3.3), (9.3.5). Explicitly using the well adapted basis (9.3.62) for gamma matrices we find:

$$F_{\underline{a}} = \begin{cases} F_a = -2e\gamma_a\gamma_5 \otimes \mathbf{1}_8 \\ F_\alpha = -e\mathbf{1}_4 \otimes \tau_\alpha \end{cases} \tag{9.3.73}$$

Using this input we obtain:

$$K_{\underline{ab}} = \begin{cases} K_{ab} = 0 \\ K_{a\beta} = 0 \\ K_{\alpha\beta} = -\frac{1}{4}\underbrace{(\mathscr{R}^{\gamma\delta}{}_{\alpha\beta} - 4e^2\delta^{\gamma\delta}{}_{\alpha\beta})}_{C^{\gamma\delta}{}_{\alpha\beta}}\tau_{\gamma\delta} \\ \end{cases} \tag{9.3.74}$$

Where the tensor $C^{\gamma\delta}{}_{\alpha\beta}$ defined by the above equation is named the holonomy tensor and it is an intrinsic geometric property of the compact internal manifold $\mathscr{M}_7$. As we see the holonomy tensor vanishes only in the case of $\mathscr{M}_7 = \mathbb{S}^7$ when the Riemann tensor is proportional to an antisymmetrized Kronecker delta, namely, when the internal Einstein 7-manifold is maximally symmetric. The holonomy tensor is a 21×21 matrix which projects the SO(7) Lie algebra to a subalgebra:

$$\mathbb{H}_{hol} \subset \mathrm{SO}(7) \tag{9.3.75}$$

with respect to which the 8-component spinor representation should contain singlets in order for unbroken supersymmetries to survive. Indeed the holonomy tensor appears in the integrability condition for Killing spinors. Indeed squaring the defining equation of Killing spinors with $m = 1$ we get the consistency condition:

$$C^{\gamma\delta}{}_{\alpha\beta}\,\tau_{\gamma\delta}\eta = 0 \tag{9.3.76}$$

which states that the Killing spinor directions are in the kernel of the operators $C^{\gamma\delta}{}_{\alpha\beta}\,\tau_{\gamma\delta}$, namely are singlets of the subalgebra $\mathbb{H}_{hol}$ generated by them.

In view of this we conclude that the gravitino field strength has the following structure:

$$\rho_{\underline{ab}} = \begin{cases} \rho_{ab} = 0 \\ \rho_{a\beta} = 0 \\ \rho_{\alpha\beta} \neq 0; \quad \begin{cases} \text{zero at } \theta = 0 \\ \text{depends only on the broken } \theta\text{s} \end{cases} \end{cases} \tag{9.3.77}$$

Table 9.1 The homogeneous 7-manifolds that admit at least 2 Killing spinors are all Sasakian or tri-Sasakian. This is evident from the fibration structure of the 7-manifold, which is either a fibration in circles $\mathbb{S}^1$ for the $\mathcal{N} = 2$ cases or a fibration in $\mathbb{S}^3$ for the unique $\mathcal{N} = 3$ case corresponding to the N^{010} manifold

$\mathcal{N}$	Name	Coset	Holon. $\mathfrak{so}(8)$ bundle	Fibration
8	$\mathbb{S}^7$	$\frac{\mathrm{SO}(8)}{\mathrm{SO}(7)}$	1	$\begin{cases} \mathbb{S}^7 \overset{\pi}{\Longrightarrow} \mathbb{P}^3 \\ \forall p \in \mathbb{P}^3; \quad \pi^{-1}(p) \sim \mathbb{S}^1 \end{cases}$
2	M^{111}	$\frac{\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)}{\mathrm{SU}(2)\times\mathrm{U}(1)\times\mathrm{U}(1)}$	SU(3)	$\begin{cases} M^{111} \overset{\pi}{\Longrightarrow} \mathbb{P}^2 \times \mathbb{P}^1 \\ \forall p \in \mathbb{P}^2 \times \mathbb{P}^1; \quad \pi^{-1}(p) \sim \mathbb{S}^1 \end{cases}$
2	Q^{111}	$\frac{\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{U}(1)}{\mathrm{U}(1)\times\mathrm{U}(1)\times\mathrm{U}(1)}$	SU(3)	$\begin{cases} Q^{111} \overset{\pi}{\Longrightarrow} \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \\ \forall p \in \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1; \quad \pi^{-1}(p) \sim \mathbb{S}^1 \end{cases}$
2	$V^{5,2}$	$\frac{\mathrm{SO}(5)}{\mathrm{SO}(2)}$	SU(3)	$\begin{cases} V^{5,2} \overset{\pi}{\Longrightarrow} M_a \sim \text{quadric in } \mathbb{P}^4 \\ \forall p \in M_a; \quad \pi^{-1}(p) \sim \mathbb{S}^1 \end{cases}$
3	N^{010}	$\frac{\mathrm{SU}(3)\times\mathrm{SU}(2)}{\mathrm{SU}(2)\times\mathrm{U}(1)}$	SU(2)	$\begin{cases} N^{010} \overset{\pi}{\Longrightarrow} \mathbb{P}^2 \\ \forall p \in \mathbb{P}^2; \quad \pi^{-1}(p) \sim \mathbb{S}^3 \end{cases}$

Table 9.2 The homogeneous 7-manifolds that admit just one Killing spinors are the squashed 7-sphere and the infinite family of N^{pqr} manifolds for $pqr \neq 010$.

$\mathcal{N}$	Name	Coset	Holon. $\mathfrak{so}(8)$ bundle
1	$\mathbb{S}^7_{squashed}$	$\frac{\mathrm{SO}(5)\times\mathrm{SO}(3)}{\mathrm{SO}(3)\times\mathrm{SO}(3)}$	$\mathrm{SO}(7)^+$
1	N^{pqr}	$\frac{\mathrm{SU}(3)\times\mathrm{U}(1)}{\mathrm{U}(1)\times\mathrm{U}(1)}$	$\mathrm{SO}(7)^+$

As a preparation for our next coming discussion it is now useful to remind the reader that the list of homogeneous 7-manifolds $\mathcal{G}/\mathcal{H}$ of Englert type which preserve at least two supersymmetries ($\mathcal{N} \geq 2$) is extremely short. It consists of the Sasakian or tri-Sasakian homogeneous manifolds[7] which are displayed in Table 9.1. For these cases our strategy in order to obtain the supergauge completion will be based on a superextension of the Sasakian fibration. The cases with $\mathcal{N} = 1$ are somewhat more involved since such a weapon is not in our stoke. These cases are also ultra-few and they are displayed in Table 9.2.

[7]The theory of Sasakian manifolds, as applied to supergravity compactifications was discussed in [39]. In short an odd dimensional manifold is named Sasakian if the even dimensional cone constructed over it has vanishing first Chern class. After several manipulations this implies that the Sasakian manifold is an S^1-fibre bundle over a suitable complex base manifold.

9.3.8 Gauged Maurer Cartan **1**-*Forms of* **OSp(8|4)**

A fundamental ingredient in the construction of gauged supergravities is constituted by the gauging of Maurer Cartan forms of the scalar coset manifold $\mathscr{G}/\mathscr{H}$ (see for instance [42] for a survey of the subject). The vector fields present in the supermultiplet, which are 1-forms defined over the space-time manifold $\mathscr{M}_4$, are used to deform the Maurer Cartan 1-forms of the scalar manifold $\mathscr{G}/\mathscr{H}$ that are instead sections of $T^\star(\mathscr{G}/\mathscr{H})$. *Mutatis mutandis*, a similar construction turns out to be quite essential in the problem of gauge completion under consideration. In our case what will be gauged are the Maurer Cartan 1-forms of the supercoset

$$\mathscr{M}^{0|4\mathscr{N}}_{osp} = \frac{\mathrm{Osp}(\mathscr{N}\,|\,4)}{\mathrm{SO}(\mathscr{N})\times \mathrm{Sp}(4,\mathbb{R})} \tag{9.3.78}$$

which contains the fermionic coordinates of the final superspace we desire to construct (for a thorough discussion of the orthosymplectic supergroups and of their cosets we refer the reader to Appendix C). The role of the space-time gauge fields is instead played by the $\mathbf{U}$-connection (9.3.66) of the $\mathfrak{so}(8)$ spinor bundle constructed over the internal 7-manifold $(\mathscr{G}/\mathscr{H})_7$.

Accordingly we define:

$$\widehat{\Lambda} \equiv \mathbb{L}^{-1}\nabla\mathbb{L} = \mathbb{L}^{-1}\big(d\mathbb{L} + [\widehat{\mathbf{U}}, \mathbb{L}]\big) \tag{9.3.79}$$

where $\widehat{\mathbf{U}}$ is the supermatrix defined by the canonical immersion of the $\mathfrak{so}(8)$ Lie algebra into the orthosymplectic superalgebra:

$$\begin{aligned}\widehat{\mathbf{U}} &= \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \mathbf{U}\end{array}\right) = \mathscr{I}(\mathbf{U}) \\ \mathscr{I} &: \mathfrak{so}(8) \mapsto \mathfrak{osp}(8|4)\end{aligned} \tag{9.3.80}$$

As a result of their definition, the gauged Maurer Cartan forms satisfy the following deformed Maurer Cartan equations:

$$\nabla\widehat{\Lambda} + \widehat{\Lambda}\wedge\widehat{\Lambda} = \mathbb{L}^{-1}(\Theta)\big[\widehat{F[\mathbf{U}]}, \mathbb{L}(\Theta)\big] \tag{9.3.81}$$

where

$$\widehat{F[\mathbf{U}]} = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & F[\mathbf{U}]\end{array}\right) \tag{9.3.82}$$

By explicit evaluation, from (9.3.81) we obtain the following deformation of the Maurer Cartan equations (9.4.3):

$$\begin{aligned} d\widehat{\Delta}^{xy} + \widehat{\Delta}^{xz}\wedge\widehat{\Delta}^{ty}\varepsilon_{zt} + 4\mathrm{i}e\widehat{\Phi}^x_A\wedge\widehat{\Phi}^y_{A}, &= -\mathrm{i}\Theta^x_A F_{AB}[\mathbf{U}]\Theta^y_B \\ \nabla\widehat{\mathscr{A}}_{AB} - e\widehat{\mathscr{A}}_{AC}\wedge\widehat{\mathscr{A}}_{CB} - 4\mathrm{i}\widehat{\Phi}^x_A\wedge\widehat{\Phi}^y_B\varepsilon_{xy} &= \mathscr{O}_{AP}(\Theta)F_{PQ}[\mathbf{U}]\mathscr{O}_{QB}(\Theta) \\ &\quad - F_{AB}[\mathbf{U}] \\ d\widehat{\Phi}^x_A + \widehat{\Delta}^{xy}\wedge\varepsilon_{yz}\widehat{\Phi}^z_A - e\widehat{\mathscr{A}}_{AB}\wedge\widehat{\Phi}^x_B &= \Theta^x_P F_{PQ}[\mathbf{U}]\mathscr{O}_{QA}(\Theta)\end{aligned} \tag{9.3.83}$$

The above equations are our main starting point to construct a supergauge completion for compactifications with less preserved supersymmetry.

9.3.9 *Killing Spinors of the* AdS$_4$ *Manifold*

The next main item for the construction of the supergauge completion is given by the Killing spinors of anti de Sitter space. Indeed, in analogy with the Killing spinors of the internal 7-manifold, defined by (9.3.43) with $m = 1$, we can now introduce the notion of Killing spinors of the AdS$_4$ space and recognize how they can be constructed in terms of the coset representative, namely in terms of the fundamental harmonic of the coset.

The analogue of (9.3.43) is given by:

$$\nabla^{\mathrm{Sp}(4)}\chi_x \equiv \left(d - \frac{1}{4}B^{ab}\gamma_{ab} - 2e\gamma_a\gamma_5 B^a\right)\chi_x = 0 \tag{9.3.84}$$

and states that the Killing spinor is a covariantly constant section of the $\mathfrak{sp}(4,\mathbb{R})$ bundle defined over AdS$_4$. This bundle is flat since the vanishing of the $\mathfrak{sp}(4,\mathbb{R})$ curvature is nothing else but the Maurer Cartan equation of $\mathfrak{sp}(4,\mathbb{R})$ and hence corresponds to the structural equations of the AdS$_4$ manifold. We are therefore guaranteed that there exists a basis of four linearly independent sections of such a bundle, namely four linearly independent solutions of (9.3.84) which we can normalize as follows:

$$\overline{\chi}_x\gamma_5\chi_y = \mathrm{i}(\mathscr{C}\gamma_5)_{xy} = \varepsilon_{xy} \tag{9.3.85}$$

Let $\mathrm{L_B}$ the coset representative mentioned in (C.2.6) and satisfying:

$$-\frac{1}{4}B^{ab}\gamma_{ab} - 2e\gamma_a\gamma_5 B^a = \Delta_B = \mathrm{L_B^{-1}}\, d\mathrm{L_B} \tag{9.3.86}$$

It follows that the inverse matrix $\mathrm{L_B^{-1}}$ satisfies the equation:

$$(d + \Delta_B)\mathrm{L_B^{-1}} = 0 \tag{9.3.87}$$

Regarding the first index y of the matrix $(\mathrm{L_B^{-1}})^y{}_x$ as the spinor index acted on by the connection Δ_B and the second index x as the labeling enumerating the Killing spinors, (9.3.87) is identical with (9.3.84) and hence we have explicitly constructed its four independent solutions. In order to achieve the desired normalization (9.3.85) it suffices to multiply by a phase factor $\exp[-\mathrm{i}\frac{1}{4}\pi]$, namely it suffices to set:

$$\chi^y_{(x)} = \exp\left[-\mathrm{i}\frac{1}{4}\pi\right]\left(\mathrm{L_B^{-1}}\right)^y{}_x \tag{9.3.88}$$

In this way the four Killing spinors fulfill the Majorana condition. Furthermore since $\mathrm{L}_{\mathrm{B}}^{-1}$ is symplectic it satisfies the defining relation

$$\mathrm{L}_{\mathrm{B}}^{-1}\mathscr{C}\gamma_5\mathrm{L}_{\mathrm{B}} = \mathscr{C}\gamma_5 \tag{9.3.89}$$

which implies (9.3.85).

9.3.10 Supergauge Completion in Mini Superspace

As it was observed many years ago in [26, 41] and it is reviewed at length in the book [44], given a bosonic Freund Rubin compactification of M-theory on an internal coset manifold $\mathscr{M}_7 = \frac{\mathscr{G}}{\mathscr{H}}$ which admits $\mathscr{N}$ Killing spinors it is fairly easy to extend it consistently to a *mini-superspace* $\mathscr{M}^{11|4\times\mathscr{N}}$ which contains all of the eleven bosonic coordinates but only $4\times\mathscr{N}$ θs, namely those which are associated with unbroken supersymmetries. We review this extension reformulating it in a more inspiring way that treats the two bosonic manifolds in a symmetric way.

In the original formulation, the mini superspace is viewed as the following tensor product

$$\mathscr{M}^{11|4\times\mathscr{N}} \equiv \mathscr{M}^{4|4\mathscr{N}}_{osp} \times \frac{\mathscr{G}}{\mathscr{H}} \tag{9.3.90}$$

and in order to construct the FDA p-forms, in addition to the Maurer Cartan forms of the above coset, we just need to introduce the Killing spinors of the bosonic internal manifold. Let η^A be an orthonormal basis of $\mathscr{N}$ eight component Killing spinors satisfying the Killing spinor condition (9.3.44) and the normalization:

$$\left(\eta^{\underline{A}}\right)^T \eta^{\underline{B}} = \delta^{\underline{AB}} \tag{9.3.91}$$

Next, following [44] and [60], we can now write the complete solution for the background fields in the case of $\mathrm{AdS}_4 \times \frac{\mathscr{G}}{\mathscr{H}}$ Freund-Rubin backgrounds:

$$\begin{aligned}
\widehat{V}^{\underline{a}} &= \begin{cases} \widehat{V}^a = E^a \\ \widehat{V}^\alpha = \mathscr{B}^\alpha - \frac{1}{8}\overline{\eta}_{\underline{A}}\tau^\alpha\eta_{\underline{B}}\mathscr{A}_{AB} \end{cases} \\
\hat{\omega}^{\underline{ab}} &= \begin{cases} \hat{\omega}^{ab} = \omega^{ab} \\ \hat{\omega}^{a b} = 0 \\ \hat{\omega}^{\alpha\beta} = \mathscr{B}^{\alpha\beta} + \frac{e}{4}\overline{\eta}_{\underline{A}}\tau^{\alpha\beta}\eta_{\underline{B}}\mathscr{A}_{\underline{AB}} \end{cases} \\
\widehat{\Psi} &= \eta_{\underline{A}} \otimes \psi_{\underline{A}}
\end{aligned} \tag{9.3.92}$$

where $\{\mathscr{B}^{\alpha\beta}, \mathscr{B}^\alpha\}$ are the spin connection and the vielbein, respectively, of the bosonic seven dimensional coset manifold $\frac{\mathscr{G}}{\mathscr{H}}$.

Let us now observe that in this formulation of the superextension, the fermionic coordinates are actually attached to the space-time manifold AdS_4, which is superextended to a supercoset manifold:

$$\mathrm{AdS}_4 \quad \overset{\text{superextension}}{\Longrightarrow} \quad \frac{\mathrm{Osp}(\mathscr{N}\,|\,4)}{\mathrm{SO}(\mathscr{N}) \times \mathrm{SO}(1,3)} \equiv \mathscr{M}^{4|4\times\mathscr{N}} \tag{9.3.93}$$

At the same time the internal manifold $\mathscr{M}_7 = \frac{\mathscr{G}}{\mathscr{H}}$ is regarded as purely bosonic and it is twisted into the fabric of the Free Differential Algebra through the notion of the Killing spinors η_A, defined as covariantly constant sections of the SO(8) spinor bundle over $\mathscr{M}_7$.

Yet whether supersymmetries are preserved or broken precisely depends on the structure of the SO(8) spinor bundle on $\mathscr{M}_7$. Henceforth it is suggestive to think that the fermionic coordinates should not be attached to either the internal or to external manifold, rather they should live as a fibre over the bosonic manifolds. The first step in order to realize such a programme consists of a reformulation of the superextension in minisuperspace that treats the space-time manifold AdS_4 and the internal manifold $\mathscr{M}_7$ in a symmetric way and in both instances relies on the notion of Killing spinors of the bosonic submanifold as a way of including the fermionic one. This can be easily done in view of (C.2.2) whose precise meaning we have explained in Sect. C.2. Indeed in view of (C.2.2) we can look at (9.3.90) in the following equivalent, but more challenging fashion:

$$\begin{aligned}
\mathscr{M}^{11|4\times\mathscr{N}} &= \mathrm{AdS}_4 \times \mathscr{M}^{0|4\times\mathscr{N}} \times \mathscr{M}_7 \\
&\equiv \underbrace{\frac{\mathrm{Sp}(4,\mathbb{R})}{\mathrm{SO}(1,3)}}_{\mathrm{AdS}_4} \times \underbrace{\frac{\mathrm{Osp}(\mathscr{N}\,|\,4)}{\mathrm{SO}(\mathscr{N}) \times \mathrm{Sp}(4,\mathbb{R})}}_{4\times\mathscr{N}\text{ fermionic manifold}} \times \underbrace{\frac{\mathscr{G}}{\mathscr{H}}}_{\mathscr{M}_7}
\end{aligned} \tag{9.3.94}$$

The above equation simply corresponds to the rewriting of (9.3.92) in the following way

$$\begin{aligned}
\widehat{V}^{\underline{a}} &= \begin{cases} \widehat{V}^a = B^a - \frac{1}{8e}\overline{\chi}_x \gamma^a \chi_y \Delta_F^{xy} \\ \widehat{V}^\alpha = \mathscr{B}^\alpha - \frac{1}{8}\overline{\eta}_{\underline{A}} \tau^\alpha \eta_{\underline{B}} \mathscr{A}_{\underline{AB}} \end{cases} \\
\hat{\omega}^{\underline{ab}} &= \begin{cases} \hat{\omega}^{ab} = B^{ab} + \frac{1}{2}\overline{\chi}_x \gamma_5 \gamma^{ab} \chi_y \Delta_F^{xy} \\ \hat{\omega}^{\alpha b} = 0 \\ \hat{\omega}^{\alpha\beta} = \mathscr{B}^{\alpha\beta} + \frac{e}{4}\overline{\eta}_{\underline{A}} \tau^{\alpha\beta} \eta_{\underline{B}} \mathscr{A}_{\underline{AB}} \end{cases} \\
\widehat{\Psi} &= \eta_{\underline{A}} \otimes \chi_x \Phi^{x|\underline{A}}
\end{aligned} \tag{9.3.95}$$

The next step is that of replacing the Maurer Cartan forms of the fermionic manifold with their gauged counterparts. This should be the order zero solution of the supergauge completion involving also the broken θs. Next order contributions in the broken θs however are necessary. For M-theory this problem was not yet solved while it was solved in the case of the type IIA compactification that we discuss next.

9.3.11 The 3-Form

We have found an explicit expression for the supervielbein $V^{\underline{a}}$, the gravitino 1-form Ψ and an the spin-connection $\omega^{\underline{ab}}$. In order to complete the description of the superextension we need also to provide an expression for the 3-form $A^{[3]}$. According to the general definitions of the FDA curvatures (6.4.2) and the rheonomic parameterization we find that:

$$d\mathbf{A}^{[3]} = \mathbf{F}^{[4]} - \frac{1}{2}\overline{\Psi} \wedge \Gamma_{\underline{ab}}\Psi \wedge V^{\underline{a}} \wedge V^{\underline{b}} \tag{9.3.96}$$

$$\Downarrow$$

$$\begin{aligned} d\mathbf{A}^{[3]} = {} & e\varepsilon_{abcd}E^a \wedge E^b \wedge E^c \wedge E^d + \frac{1}{2}\overline{\chi}_x\gamma_{ab}\chi_y\Phi^x_A \wedge \Phi^y \wedge E^a \wedge E^b \\ & + \frac{1}{2}\overline{\chi}_x\chi_y\zeta_A\tau_{\alpha\beta}\zeta_B\Phi^x_A \wedge \Phi^y_B \wedge \mathscr{B}^\alpha \wedge \mathscr{B}^\beta \\ & + \overline{\chi}_x\gamma_a\gamma_5\chi_y\zeta_A\tau_\beta\zeta_B\Phi^x_A \wedge \Phi^y_B \wedge E^a \wedge \mathscr{B}^\beta \end{aligned} \tag{9.3.97}$$

The expression of $d\mathbf{A}^{[3]}$ as a 4-form is completely explicit in (9.3.97) and by construction it is integrable in the sense that $d^2\mathbf{A}^{[3]} = 0$. One might desire to solve this equation by finding a suitable expression for $\mathbf{A}^{[3]}$ such that (9.3.97) is satisfied. This is not possible in general terms, namely by using only invariant constraints. In order to find explicit solutions, one needs to use some explicit coordinate system and some explicit solution of the constraints. This analysis is not in the spirit we have adopted. Here it is just the constraints what matters, not their explicit solutions.

In the main application one might consider, namely while localizing the action of the supermembrane M2 on such backgrounds, we can easily avoid all such problems. We simply substitute the world volume integral of $\mathbf{A}^{[3]}$ with:

$$\int_{WV_3} \mathbf{A}^{[3]} \mapsto \int_{WV_4} d\mathbf{A}^{[3]} \tag{9.3.98}$$

where the 4-dimensional integration volume WV_4 is such that its boundary is the original supermembrane world-volume:

$$\partial WV_4 = WV_3 \tag{9.3.99}$$

and we circumvent the problem of solving (9.3.97).

With this observation we have concluded our proof that any $\mathrm{AdS}_4 \times \mathscr{G}/\mathscr{H}$ bosonic solution of M-theory field equations can be gauge completed to a solution in the mini-superspace containing all the theta variables associated with unbroken supersymmetries.

9.4 Flux Compactifications of Type IIA Supergravity on $\mathbf{AdS_4} \times \mathbb{P}^3$

As a further example of supergravity exact solution we consider the compactification of type IIA supergravity on the following direct product manifold:

$$\mathscr{M}_{10} = \mathrm{AdS}_4 \times \mathbb{P}^3 \tag{9.4.1}$$

which was constructed in [45].

In this case not only we are able to write an exact solution of the bosonic field equations but we are also able to construct an explicit expression for all the bosonic and fermionic p-forms that close the type IIA differential algebra in such a way that the rheonomic solution of the Bianchi identities is matched. In other words, for this background we possess an explicit and simple integration of the rheonomic conditions just as it is the case for the compactification of M-theory on $\mathrm{AdS}_4 \times \mathbb{S}^7$. This is the main pedagogical reason to present the solution we are going to consider. From the string theory point of view the interest in such backgrounds streams from the recently discovered duality between $\mathscr{N} = 6$ superconformal Chern-Simons theory in three dimensions and superstrings moving on $\mathrm{AdS}_4 \times \mathbb{P}^3$ backgrounds [46–49, 51–55] which has prompted the study of superstrings on $\mathrm{Osp}(\mathscr{N}|4)$ backgrounds [43, 56–58]. Indeed the explicit integration of the rheonomic conditions is obtained through an appropriate use of the isometry supergroup of the background (9.4.1) which is identified with the following supergroup $\mathrm{OSp}(6|4)$. The Maurer Cartan forms of this supergroup on the following super-coset manifold

$$\mathscr{M}^{10|24} = \frac{\mathrm{OSp}(6|4)}{\mathrm{SO}(1,3) \times \mathrm{U}(3)} \tag{9.4.2}$$

allow for an explicit supergauge completion of the solution of the bosonic field equations provided by the geometry of the Riemannian space $\mathrm{AdS}_4 \times \mathbb{P}^3$ in a very similar way to the M-theory cases discussed in the previous section.

To be precise what we will present is not a full gauge completion involving 10 bosonic coordinates and 32 fermionic ones, but only a partial gauge-completion in the *mini-superspace* given by the coset (9.4.2) which contains all the 10 bosonic coordinates but only 24 of the fermionic ones, those associated with the supersymmetries preserved by the background. The gauge completion in the remaining eight θs associated with the broken supersymmetries is a rather difficult problem that, however, has been solved by the authors of [59].

9.4.1 Maurer Cartan Forms of $\mathbf{OSp(6|4)}$

The bosonic subgroup of $\mathrm{OSp}(6|4)$ is $\mathrm{Sp}(4,\mathbb{R}) \times \mathrm{SO}(6)$. The Maurer-Cartan one-forms of $\mathfrak{sp}(4,\mathbb{R})$ are denoted by Δ^{xy} $(x, y = 1, \ldots, 4)$, the $\mathfrak{so}(6)$ one-forms are denoted by $\mathscr{A}_{AB}$ $(A, B = 1, \ldots, 6)$ while the (real) fermionic one-forms are denoted by Φ^x_A and transform in the fundamental representation of $\mathrm{Sp}(4, \mathbb{R})$ and in the fundamental representation of SO(6). These forms satisfy the following OSp(6|4) Maurer-Cartan equations:

$$\begin{aligned} d\Delta^{xy} + \Delta^{xz} \wedge \Delta^{ty}\varepsilon_{zt} &= -4\mathrm{i}e\Phi^x_A \wedge \Phi^y_A \\ d\mathscr{A}_{AB} - e\mathscr{A}_{AC} \wedge \mathscr{A}_{CB} &= 4\mathrm{i}\Phi^x_A \wedge \Phi^y_B\varepsilon_{xy} \\ d\Phi^x_A + \Delta^{xy} \wedge \varepsilon_{yz}\Phi^z_A - e\mathscr{A}_{AB} \wedge \Phi^x_B &= 0 \end{aligned} \tag{9.4.3}$$

where

$$\varepsilon_{xy} = -\varepsilon_{yx} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \tag{9.4.4}$$

The Maurer-Cartan equations are solved in terms of the super-coset representative of (9.4.2). We rely for this analysis on the general discussion in Appendix C. It is convenient to express this solution in terms of the one-forms describing the bosonic submanifolds $\mathrm{AdS}_4 \equiv \frac{\mathrm{Sp}(4,\mathbb{R})}{\mathrm{SO}(1,3)}$, $\mathbb{P}^3 \equiv \frac{\mathrm{SO}(6)}{\mathrm{U}(3)}$ of (9.4.2) and of the one-forms on the fermionic subspace of (9.4.2). Let us denote by B^{ab}, B^a and by $\mathscr{B}^{\alpha\beta}$, $\mathscr{B}^\alpha$ the connections and vielbein on the two bosonic subspaces respectively. The supergauge completion is accomplished by expressing the p-forms satisfying the rheonomic parameterization of the type IIA Free Differential Algebra in terms of the mini-superspace one-forms. The final expression of the $D = 10$ fields will involve not only the bosonic one-forms B^{ab}, B^a, $\mathscr{B}^{\alpha\beta}$, $\mathscr{B}^\alpha$, but also the Killing spinors on the background. The latter play indeed a special role in this analysis since they can be identified with the fundamental harmonics of the cosets $\mathrm{SO}(2,3)/\mathrm{SO}(1,3)$ and $\mathrm{SO}(6)/\mathrm{U}(3)$, respectively. The Killing spinors on the AdS_4 were already discussed. We wills study those on $\mathbb{P}^3$.

9.4.2 Explicit Construction of the $\mathbb{P}^3$ Geometry

The complex three-fold $\mathbb{P}^3$ is Kähler. Indeed the existence of the Kähler 2-form is one of the essential items in constructing the solution ansatz.

Let us begin by discussing all the relevant geometric structures of $\mathbb{P}^3$. We have to construct the explicit form of the internal manifold geometry, in particular the spin connection, the vielbein and the Kähler 2-form. This is fairly easy, since $\mathbb{P}^3$ is a coset manifold:

$$\mathbb{P}^3 = \frac{\mathrm{SU}(4)}{\mathrm{SU}(3) \times \mathrm{U}(1)} \tag{9.4.5}$$

so that everything is defined in terms of the structure constants of the $\mathfrak{su}(4)$ Lie algebra. The quickest way to introduce these structure constants and their chosen normalization is by writing the Maurer-Cartan equations. We do this introducing already the splitting:

$$\mathfrak{su}(4) = \mathbb{H} \oplus \mathbb{K} \tag{9.4.6}$$

between the subalgebra $\mathbb{H} \equiv \mathfrak{su}(3) \times \mathfrak{u}(1)$ and the complementary orthogonal subspace $\mathbb{K}$ which is tangent to the coset manifold. Hence we name H^i $(i = 1, \ldots, 9)$ a basis of one-form generators of $\mathbb{H}$ and K^α $(\alpha = 1, \ldots, 6)$ a basis of one-form generators of $\mathbb{K}$. With these notation the Maurer-Cartan equations defining the structure constants of $\mathfrak{su}(4)$ have the following form:

$$\begin{aligned} dK^\alpha + \mathscr{B}^{\alpha\beta} \wedge K^\gamma \delta_{\beta\gamma} &= 0 \\ d\mathscr{B}^{\alpha\beta} + \mathscr{B}^{\alpha\gamma} \wedge \mathscr{B}^{\delta\beta} \delta_{\gamma\delta} - \mathscr{X}^{\alpha\beta}{}_{\gamma\delta} K^\gamma \wedge K^\delta &= 0 \end{aligned} \tag{9.4.7}$$

where:

1. the antisymmetric one-form valued matrix $\mathscr{B}^{\alpha\beta}$ is parameterized by the 9 generators of the $\mathfrak{u}(3)$ subalgebra of $\mathfrak{so}(6)$ in the following way:

$$\mathscr{B}^{\alpha\beta} = \begin{pmatrix} 0 & H^9 & -H^8 & H^1 + H^2 & H^6 & -H^5 \\ -H^9 & 0 & H^7 & H^6 & H^1 + H^3 & H^4 \\ H^8 & -H^7 & 0 & -H^5 & H^4 & H^2 + H^3 \\ -H^1 - H^2 & -H^6 & H^5 & 0 & H^9 & -H^8 \\ -H^6 & -H^1 - H^3 & -H^4 & -H^9 & 0 & H^7 \\ H^5 & -H^4 & -H^2 - H^3 & H^8 & -H^7 & 0 \end{pmatrix} \tag{9.4.8}$$

2. the symbol $\mathscr{X}^{\alpha\beta}{}_{\gamma\delta}$ denotes the following constant, 4-index tensor:

$$\mathscr{X}^{\alpha\beta}{}_{\gamma\delta} \equiv \left(\delta^{\alpha\beta}_{\gamma\delta} + \mathscr{K}^{\alpha\beta}\mathscr{K}^{\gamma\delta} + \mathscr{K}^{\alpha}{}_{\gamma}\mathscr{K}^{\beta}{}_{\delta}\right) \tag{9.4.9}$$

3. the symbol $\mathscr{K}^{\alpha\beta}$ denotes the entries of the following antisymmetric matrix:

$$\mathscr{K} = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \tag{9.4.10}$$

The Maurer Cartan equations (9.4.7) can be reinterpreted as the structural equations of the $\mathbb{P}^3$ 6-dimensional manifold. It suffices to identify the antisymmetric one-form valued matrix $\mathscr{B}^{\alpha\beta}$ with the spin connection and identify the vielbein $\mathscr{B}^\alpha$ with the coset generators K^α, modulo a scale factor λ

$$\mathscr{B}^\alpha = \frac{1}{\lambda} K^\alpha \tag{9.4.11}$$

With these identifications the first of (9.4.7) becomes the vanishing torsion equation, while the second singles out the Riemann tensor as proportional to the tensor $\mathscr{X}^{\alpha\beta}{}_{\gamma\delta}$ of (9.4.9). Indeed we can write:

$$\begin{aligned}\mathscr{R}^{\alpha\beta} &= d\mathscr{B}^{\alpha\beta} + \mathscr{B}^{\alpha\gamma} \wedge \mathscr{B}^{\delta\beta}\delta_{\gamma\delta} \\ &= \mathscr{R}^{\alpha\beta}{}_{\gamma\delta}\mathscr{B}^{\gamma} \wedge \mathscr{B}^{\delta}\end{aligned} \tag{9.4.12}$$

where:

$$\mathscr{R}^{\alpha\beta}{}_{\gamma\delta} = \lambda^2 \mathscr{X}^{\alpha\beta}{}_{\gamma\delta} \tag{9.4.13}$$

Using the above Riemann tensor we immediately retrieve the explicit form of the Ricci tensor:

$$\mathrm{Ric}_{\alpha\beta} = 4\lambda^2 \eta_{\alpha\beta} \tag{9.4.14}$$

For later convenience in discussing the compactification ansatz it is convenient to rename the scale factor as follows:

$$\lambda = 2e \tag{9.4.15}$$

In this way we obtain:

$$\mathrm{Ric}_{\alpha\beta} = 16e^2 \eta_{\alpha\beta} \tag{9.4.16}$$

which will be recognized as one of the field equations of type IIA supergravity.

Let us now come to the interpretation of the matrix $\mathscr{K}$. This matrix is immediately identified as encoding the intrinsic components of the Kähler 2-form. Indeed $\mathscr{K}$ is the unique antisymmetric matrix which, within the fundamental 6-dimensional representation of the $\mathfrak{so}(6) \sim \mathfrak{su}(4)$ Lie algebra, commutes with the entire subalgebra $\mathfrak{u}(3) \subset \mathfrak{su}(4)$. Hence $\mathscr{K}$ generates the $\mathrm{U}(1)$ subgroup of $\mathrm{U}(3)$ and this guarantees that the Kähler 2-form will be closed and coclosed as it should be. Indeed it is sufficient to set:

$$\widehat{\mathscr{K}} = \mathscr{K}_{\alpha\beta}\mathscr{B}^{\alpha} \wedge \mathscr{B}^{\beta} \tag{9.4.17}$$

namely:

$$\widehat{\mathscr{K}} = -2\big(\mathscr{B}^1 \wedge \mathscr{B}^4 + \mathscr{B}^2 \wedge \mathscr{B}^5 + \mathscr{B}^3 \wedge \mathscr{B}^6\big) \tag{9.4.18}$$

and we obtain that the 2-form $\widehat{\mathscr{K}}$ is closed and coclosed:

$$d\widehat{\mathscr{K}} = 0, \qquad d^{\star}\widehat{\mathscr{K}} = 0 \tag{9.4.19}$$

Let us also note that the antisymmetric matrix $\mathscr{K}$ satisfies the following identities:

$$\begin{aligned}\mathscr{K}^2 &= -1_{6\times 6} \\ 8\mathscr{K}_{\alpha\beta} &= \varepsilon_{\alpha\beta\gamma\delta\tau\sigma}\mathscr{K}^{\gamma\delta}\mathscr{K}^{\tau\sigma}\end{aligned} \tag{9.4.20}$$

Using the $\mathfrak{so}(6)$ Clifford Algebra defined in Appendix C.3.1 we define the following spinorial operators:

$$\mathscr{W} = \mathscr{K}_{\alpha\beta}\tau^{\alpha\beta}; \qquad \mathscr{P} = \mathscr{W}\tau_7 \tag{9.4.21}$$

and we can verify that the matrix $\mathscr{P}$ satisfies the following algebraic equations:

$$\mathscr{P}^2 + 4\mathscr{P} - 12 \times \mathbf{1} = 0 \tag{9.4.22}$$

whose roots are 2 and -6. Indeed in the chosen τ-matrix basis the matrix $\mathscr{P}$ is diagonal with the following explicit form:

$$\mathscr{P} = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -6 \end{pmatrix} \tag{9.4.23}$$

Let us also introduce the following matrix valued one-form:

$$\mathscr{Q} \equiv \left(\frac{3}{2}\mathbf{1} + \frac{1}{4}\mathscr{P}\right)\tau_\alpha \mathscr{B}^\alpha \tag{9.4.24}$$

whose explicit form in the chosen basis is the following one:

$$\mathscr{Q} = \begin{pmatrix} 0 & 2\mathscr{B}^3 & -2\mathscr{B}^2 & 0 & -2\mathscr{B}^6 & 2\mathscr{B}^5 & -2\mathscr{B}^4 & 2\mathscr{B}^1 \\ -2\mathscr{B}^3 & 0 & 2\mathscr{B}^1 & 2\mathscr{B}^6 & 0 & -2\mathscr{B}^4 & -2\mathscr{B}^5 & 2\mathscr{B}^2 \\ 2\mathscr{B}^2 & -2\mathscr{B}^1 & 0 & -2\mathscr{B}^5 & 2\mathscr{B}^4 & 0 & -2\mathscr{B}^6 & 2\mathscr{B}^3 \\ 0 & -2\mathscr{B}^6 & 2\mathscr{B}^5 & 0 & -2\mathscr{B}^3 & 2\mathscr{B}^2 & 2\mathscr{B}^1 & 2\mathscr{B}^4 \\ 2\mathscr{B}^6 & 0 & -2\mathscr{B}^4 & 2\mathscr{B}^3 & 0 & -2\mathscr{B}^1 & 2\mathscr{B}^2 & 2\mathscr{B}^5 \\ -2\mathscr{B}^5 & 2\mathscr{B}^4 & 0 & -2\mathscr{B}^2 & 2\mathscr{B}^1 & 0 & 2\mathscr{B}^3 & 2\mathscr{B}^6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{9.4.25}$$

and let us consider the following Killing spinor equation:

$$\mathscr{D}\eta + e\mathscr{Q}\eta = 0 \tag{9.4.26}$$

where, by definition:

$$\mathscr{D} = d - \frac{1}{4}\mathscr{B}^{\alpha\beta}\tau_{\alpha\beta} \tag{9.4.27}$$

denotes the $\mathfrak{so}(6)$ covariant differential of spinors defined over the $\mathbb{P}^3$ manifold. The connection $\mathscr{Q}$ is closed with respect to the spin connection

$$\Omega = -\frac{1}{4}\mathscr{B}^{\alpha\beta}\tau_{\alpha\beta} \tag{9.4.28}$$

since we have:

$$\mathscr{D}\mathscr{Q} \equiv d\mathscr{Q} + e^2\Omega \wedge \mathscr{Q} + \mathscr{Q} \wedge \Omega = 0 \tag{9.4.29}$$

as it can be explicitly checked. The above result follows because the matrix $\mathscr{K}_{\alpha\beta}$ commutes with all the generators of $\mathfrak{u}(3)$. In view of (9.4.29) the integrability of the Killing spinor equation (9.4.26) becomes the following one:

$$\mathrm{Hol}\,\eta = 0 \tag{9.4.30}$$

where we have defined the holonomy 2-form:

$$\mathrm{Hol} \equiv \left(\mathscr{D}^2 + e^2\mathscr{Q} \wedge \mathscr{Q}\right) = \left(-\frac{1}{4}\mathscr{R}^{\alpha\beta}\tau_{\alpha\beta} + e^2\mathscr{Q} \wedge \mathscr{Q}\right) \tag{9.4.31}$$

and $\mathscr{R}^{\alpha\beta}$ denotes the curvature 2-form (9.4.12). Explicit evaluation of the holonomy 2-form yields the following result.

$$\mathrm{Hol} = e^2 \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 8[\mathscr{B}^2 \wedge \mathscr{B}^6 - \mathscr{B}^3 \wedge \mathscr{B}^5] & 8\mathscr{B}^5 \wedge \mathscr{B}^6 - 8\mathscr{B}^2 \wedge \mathscr{B}^3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 8\mathscr{B}^3 \wedge \mathscr{B}^4 - 8\mathscr{B}^1 \wedge \mathscr{B}^6 & 8[\mathscr{B}^1 \wedge \mathscr{B}^3 - \mathscr{B}^4 \wedge \mathscr{B}^6] \\ 0 & 0 & 0 & 0 & 0 & 0 & 8[\mathscr{B}^1 \wedge \mathscr{B}^5 - \mathscr{B}^2 \wedge \mathscr{B}^4] & 8\mathscr{B}^4 \wedge \mathscr{B}^5 - 8\mathscr{B}^1 \wedge \mathscr{B}^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 8[\mathscr{B}^2 \wedge \mathscr{B}^3 - \mathscr{B}^5 \wedge \mathscr{B}^6] & 8[\mathscr{B}^2 \wedge \mathscr{B}^6 - \mathscr{B}^3 \wedge \mathscr{B}^5] \\ 0 & 0 & 0 & 0 & 0 & 0 & 8\mathscr{B}^4 \wedge \mathscr{B}^6 - 8\mathscr{B}^1 \wedge \mathscr{B}^3 & 8\mathscr{B}^3 \wedge \mathscr{B}^4 - 8\mathscr{B}^1 \wedge \mathscr{B}^6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 8[\mathscr{B}^1 \wedge \mathscr{B}^2 - \mathscr{B}^4 \wedge \mathscr{B}^5] & 8[\mathscr{B}^1 \wedge \mathscr{B}^5 - \mathscr{B}^2 \wedge \mathscr{B}^4] \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -8\widehat{\mathscr{K}} \\ 0 & 0 & 0 & 0 & 0 & 0 & 8\widehat{\mathscr{K}} & 0 \end{pmatrix} \tag{9.4.32}$$

It is evident by inspection that the holonomy 2-form vanishes on the subspace of spinors that belong to the eigenspace of eigenvalue 2 of the operator $\mathscr{P}$. In the chosen basis this eigenspace is spanned by all those spinors whose last two components are zero and on such spinors the operator Hol vanishes.

Let us now connect these geometric structures to the compactification ansatz.

9.4.3 The Compactification Ansatz

As usual we denote with Latin indices those in the direction of 4-space and with Greek indices those in the direction of the internal 6-space. Let us also adopt the notation: B^a for the AdS_4 vielbein just as $\mathscr{B}^\alpha$ is the vielbein of the Kähler threefold described in the previous section. With these notations the Kaluza-Klein ansatz is the following one:

$$\mathscr{G}_{\underline{ab}} = \begin{cases} 2e\exp[-\varphi_0]\mathscr{K}_{\alpha\beta} \\ 0 \quad \text{otherwise} \end{cases}$$

$$
\begin{aligned}
\mathscr{G}_{\underline{a_1 a_2 a_3 a_4}} &= \begin{cases} -e \exp[-\varphi_0] \varepsilon_{a_1 a_2 a_3 a_4} \\ 0 \quad \text{otherwise} \end{cases} \\
\mathscr{H}_{\underline{a_1 a_2 a_3}} &= 0 \\
\varphi &= \varphi_0 = \text{const} \\
V^a &= B^a \\
V^\alpha &= \mathscr{B}^\alpha \\
\omega^{ab} &= B^{ab} \\
\omega^{\alpha\beta} &= \mathscr{B}^{\alpha\beta}
\end{aligned}
\tag{9.4.33}
$$

where B^a, B^{ab} respectively denote the vielbein and the spin connection of AdS_4, satisfying the following structural equations:

$$
\begin{aligned}
0 &= dB^a - B^{ab} \wedge B^c \eta_{bc} \\
dB^{ab} - B^{ac} \wedge B^{db} \eta_{cd} &= -16 e^2 B^a \wedge B^b \\
&\Downarrow \\
\mathrm{Ric}_{ab} &= -24 e^2 \eta_{ab}
\end{aligned}
\tag{9.4.34}
$$

while $\mathscr{B}^\alpha$ and $\mathscr{B}^{\alpha\beta}$ are the analogous data for the internal $\mathbb{P}^3$ manifold:

$$
\begin{aligned}
0 &= d\mathscr{B}^\alpha - \mathscr{B}^{\alpha\beta} \wedge \mathscr{B}^\gamma \eta_{\beta\gamma} \\
d\mathscr{B}^{\alpha\beta} - \mathscr{B}^{\alpha\gamma} \wedge \mathscr{B}^{\delta\beta} \eta_{\gamma\delta} &= -R^{\alpha\beta}{}_{\gamma\delta} \mathscr{B}^\gamma \wedge \mathscr{B}^\delta \\
&\Downarrow \\
\mathrm{Ric}_{\alpha\beta} &= 16 e^2 \eta_{\alpha\beta}
\end{aligned}
\tag{9.4.35}
$$

whose geometry we described in the previous section.

With these normalizations we can check that the dilaton equation (6.7.44) and the Einstein equation (6.7.39), are satisfied upon insertion of the above Kaluza Klein ansatz. All the other equations are satisfied thanks to the fact that the Kähler form $\widehat{\mathscr{K}}$ is closed and coclosed.

9.4.4 Killing Spinors on $\mathbb{P}^3$

The next task we are faced with is to determine the equation for the Killing spinors on the chosen background, which by construction is a solution of supergravity equations.

Following a standard procedure we recall that the vacuum has been defined by choosing certain values for the bosonic fields and setting all the fermionic ones equal to zero:

$$\begin{aligned} \psi_{L/R|\underline{\mu}} &= 0 \\ \chi_{L/R} &= 0 \\ \rho_{L/R|\underline{ab}} &= 0 \end{aligned} \tag{9.4.36}$$

The equation for the Killing spinors will be obtained by imposing that the parameter of supersymmetry preserves the vanishing values of the fermionic fields once the specific values of the bosonic ones is substituted into the expression for the susy rules, namely into the rheonomic parameterizations.

To implement these conditions we begin by choosing a well adapted basis for the $d = 11$ gamma matrices. This is done by setting:

$$\Gamma^{\underline{a}} = \begin{cases} \Gamma^a = \gamma^a \otimes \mathbf{1} \\ \Gamma^\alpha = \gamma^5 \otimes \tau^\alpha \\ \Gamma^{11} = \mathrm{i}\gamma^5 \otimes \tau^7 \end{cases} \tag{9.4.37}$$

Next we consider the tensors and the matrices introduced in (6.7.20), (6.7.22) and (6.7.23), (6.7.24). In the chosen background we find:

$$\begin{aligned} &\mathscr{M}_{\alpha\beta} = \frac{1}{4} e \mathscr{K}_{\alpha\beta}; \qquad \mathscr{M}_{abcd} = \frac{1}{16} e \varepsilon_{abcd} \\ &\mathscr{N}_0 = 0; \qquad \mathscr{N}_{\alpha\beta} = \frac{1}{2} e \mathscr{K}_{\alpha\beta}; \qquad \mathscr{N}_{abcd} = -\frac{1}{24} e \varepsilon_{abcd} \end{aligned} \tag{9.4.38}$$

all the other components of the above matrices being zero. Hence in terms of the operators introduced in the previous section we find:

$$\begin{aligned} \mathscr{M}_\pm &= \mathrm{i}e\left(\mp\frac{1}{4}\mathbf{1}\otimes\mathscr{W} - \frac{3}{2}\mathrm{i}\gamma_5\otimes\mathbf{1}\right) \\ \mathscr{N}_\pm^{(even)} &= e\left(\frac{1}{2}\mathbf{1}\otimes\mathscr{W} \mp \mathrm{i}\gamma_5\otimes\mathbf{1}\right) \\ \mathscr{N}_\pm^{(odd)} &= 0 \end{aligned} \tag{9.4.39}$$

It is now convenient to rewrite the Killing spinor condition in a non-chiral basis introducing a supersymmetry parameter of the following form:

$$\varepsilon = \varepsilon_L + \varepsilon_R \tag{9.4.40}$$

In this basis the matrices $\mathscr{M}$ and $\mathscr{N}^{(even)}$ read

$$\begin{aligned} \mathscr{M} &= \mathscr{M}_+ \frac{1}{2}\left(\mathbb{1} + \Gamma^{11}\right) + \mathscr{M}_- \frac{1}{2}\left(\mathbb{1} - \Gamma^{11}\right) \\ &= -\frac{i}{8} e^\varphi G_{\underline{ab}} \Gamma^{\underline{ab}} \Gamma^{11} - \frac{i}{16} e^\varphi G_{\underline{abcd}} \Gamma^{\underline{abcd}} = \\ &= \frac{e}{4} \gamma_5 \otimes (\mathscr{W}\tau_7 + 6\mathbb{1}) \end{aligned} \tag{9.4.41}$$

$$
\begin{aligned}
\mathscr{N}^{(even)} &= \mathscr{N}_+^{(even)} \frac{1}{2}\left(\mathbb{1} + \Gamma^{11}\right) + \mathscr{N}_-^{(even)} \frac{1}{2}\left(\mathbb{1} - \Gamma^{11}\right) \\
&= \frac{1}{4} e^{\varphi} G_{\underline{ab}} \Gamma^{\underline{ab}} + \frac{1}{24} e^{\varphi} G_{\underline{abcd}} \Gamma^{\underline{abcd}} \\
&= \frac{e}{2} \mathbb{1} \otimes (\mathscr{W} + 2\tau_7)
\end{aligned} \tag{9.4.42}
$$

Upon use of this parameter the Killing spinor equation coming from the gravitino rheonomic parameterization (6.7.32) takes the following form:

$$
\mathscr{D}\varepsilon = -\mathscr{M} \Gamma_{\underline{a}} V^{\underline{a}} \varepsilon \tag{9.4.43}
$$

while the Killing spinor equation coming from the dilatino rheonomic parameterization is as follows:

$$
0 = \mathscr{N}^{(even)} \varepsilon \tag{9.4.44}
$$

Let us now insert these results into the Killing spinor equations and let us take a tensor product representation for the Killing spinor:

$$
\varepsilon = \varepsilon \otimes \eta \tag{9.4.45}
$$

where ε is a 4-component $d = 4$ spinor and η is an 8-component $d = 6$ spinor.

With these inputs (9.4.43) becomes:

$$
\begin{aligned}
0 = {} & \mathscr{D}_{[4]} \varepsilon \otimes \eta - e \gamma_a \gamma_5 B^a \varepsilon \otimes \left(\frac{3}{2} + \frac{1}{4}\mathscr{P}\right)\eta \\
& + \varepsilon \otimes \left[\mathscr{D}_{[6]} + e\left(\frac{3}{2} + \frac{1}{4}\mathscr{P}\right)\tau_\alpha \mathscr{B}^\alpha\right]\eta
\end{aligned} \tag{9.4.46}
$$

while (9.4.44) takes the form:

$$
0 = \varepsilon \otimes \left(\frac{1}{2}\mathscr{W} + \tau_7\right)\eta \tag{9.4.47}
$$

Let us now recall that (9.4.26) is integrable on the eigenspace of eigenvalue 2 of the $\mathscr{P}$-operator. Then (9.4.46) is satisfied if:

$$
\begin{aligned}
\left(\mathscr{D}_{[4]} - 2e\gamma_a \gamma_5 B^a\right)\varepsilon &= 0 \\
\mathscr{P}\eta &= 2\eta \\
(\mathscr{D}_{[6]} + e\mathscr{Q})\eta &= 0
\end{aligned} \tag{9.4.48}
$$

The first of the above equation is the correct equation for Killing spinors in AdS_4. It emerges if the eigenvalue of $\mathscr{P}$ is 2. The second and the third are the already studied integrable equation for six Killing spinors out of eight. It should now be checked that the dilatino equation (9.4.47) is satisfied on the eigenspace of eigenvalue 2, which is indeed the case:

$$\mathscr{P}\eta = 2\eta \quad \Rightarrow \quad \left(\frac{1}{2}\mathscr{W} + \tau_7\right)\eta = 0 \tag{9.4.49}$$

9.4.5 Gauge Completion in Mini Superspace

As a necessary ingredient of our construction let η_A $(A = 1, \dots, 6)$ denote a complete and orthonormal basis of solutions of the internal Killing spinor equation, namely:

$$\begin{aligned} \mathscr{P}\eta_A &= 2\eta_A \\ (\mathscr{D}_{[6]} + e\mathscr{Q})\eta_A &= 0 \\ \eta_A^T \eta_B &= \delta_{AB}; \quad A, B = A = 1, \dots, 6 \end{aligned} \tag{9.4.50}$$

On the other hand let χ_x denote a basis of solutions of the Killing spinor equation on AdS_4-space, namely (9.3.84), normalized as in (9.3.85). Furthermore let us recall the matrix $\mathscr{K}$ defining the intrinsic components of the Kähler 2-form.

In terms of these objects we can satisfy the rheonomic parameterizations of the one-forms spanning the $d = 10$ super-Poincaré subalgebra of the FDA with the following position:[8]

$$\Psi = \chi_x \otimes \eta_A \Phi^{x|A} \tag{9.4.51}$$

$$V^a = B^a - \frac{1}{8e}\overline{\chi}_x \gamma^a \chi_y \Delta^{xy} \tag{9.4.52}$$

$$V^\alpha = \mathscr{B}^\alpha - \frac{1}{8}\eta_A^T \tau^\alpha \eta_B \mathscr{A}^{AB} \tag{9.4.53}$$

$$\omega^{ab} = B^{ab} + \frac{1}{2}\overline{\chi}_x \gamma^{ab} \gamma_5 \chi_y \Delta^{xy} \tag{9.4.54}$$

$$\omega^{\alpha\beta} = \mathscr{B}^{\alpha\beta} + \frac{e}{4}\eta_A^T \tau^{\alpha\beta} \eta_B \mathscr{A}^{AB} - \frac{e}{4}\mathscr{K}^{\alpha\beta}\mathscr{K}_{AB}\mathscr{A}^{AB} \tag{9.4.55}$$

The proof that the above ansatz satisfies the rheonomic parameterizations is by direct evaluation upon use of the following crucial spinor identities.

Let us define

$$\mathscr{U} = \left(\frac{3}{2}\mathbf{1} + \frac{1}{4}\mathscr{P}\right) \tag{9.4.56}$$

[8] With respect to the results obtained for the mini superspace extension of M-theory configuration everything is identical in (9.4.51)–(9.4.54) except the obvious reduction of the index range of $(\alpha, \beta, \dots)$ from 7 to 6-values. The only difference is in (9.4.55) where the last contribution proportional to the Kähler form is an essential novelty of this new type of compactification.

We can verify that:

$$\left(\eta_A \tau^\alpha \mathscr{U} \tau^\alpha \eta_B - \eta_A \tau^{\alpha\beta} \eta_B\right) \mathscr{A}^{AB} = \mathscr{K}^{\alpha\beta} \mathscr{K}_{AB} \mathscr{A}^{AB} \tag{9.4.57}$$

Furthermore, naming:

$$\Delta \mathscr{B}^\alpha = -\frac{1}{8} \eta_A^T \tau^\alpha \eta_B \mathscr{A}^{AB} \tag{9.4.58}$$

$$\Delta \omega^{\alpha\beta} = \frac{e}{4} \eta_A^T \tau^{\alpha\beta} \eta_B \mathscr{A}^{AB} - \frac{e}{4} \mathscr{K}^{\alpha\beta} \mathscr{K}_{AB} \mathscr{A}^{AB} \tag{9.4.59}$$

we obtain:

$$-\Delta \omega^{\alpha\beta} \wedge \Delta \mathscr{B}^\beta = \frac{e}{8} \eta_A^T \tau^\alpha \eta_B \mathscr{A}^{AC} \wedge \mathscr{A}^{CB} \tag{9.4.60}$$

These identities together with the $d = 4$ spinor identities (C.3.11), (C.3.12) suffice to verify that the above ansatz satisfies the required equations.

9.4.6 Gauge Completion of the $\mathbf{B}^{[2]}$ *Form*

The next task is the derivation of the explicit expression for the $\mathbf{B}^{[2]}$ form. Differently from the case of the 3-form this is possible and has a great value since it allows an explicit expression for the Green Schwarz σ-model describing string propagation in this background.

There is an ansatz for $\mathbf{B}^{[2]}$ which is the following one:

$$\mathbf{B}^{[2]} = \alpha \overline{\chi}_x \chi_y \overline{\eta}_A \tau_7 \eta_B \Phi_A^x \wedge \Phi_B^y \tag{9.4.61}$$

By explicit evaluation we verify that with

$$\alpha = \frac{1}{4e} \tag{9.4.62}$$

The rheonomic parameterization of the H-field strength is satisfied, namely:

$$d\mathbf{B}^{[2]} = -\mathrm{i}\overline{\psi} \wedge \Gamma_{\underline{a}} \Gamma_{11} \psi \wedge V^{\underline{a}} \tag{9.4.63}$$

9.4.7 Rewriting the Mini-Superspace Gauge Completion as Maurer Cartan Forms on the Complete Supercoset

Next we can rewrite the mini-superspace extension of the bosonic solution solely in terms of Maurer Cartan forms on the supercoset (9.4.2). Let the graded matrix

$\mathbb{L} \in \mathrm{Osp}(6|4)$ be the coset representative of the coset $\mathscr{M}^{10|24}$, such that the Maurer Cartan form Σ can be identified as:

$$\Sigma = \mathbb{L}^{-1}\, d\mathbb{L} \tag{9.4.64}$$

Let us now factorize $\mathbb{L}$ as follows:

$$\mathbb{L} = \mathbb{L}_F \mathbb{L}_B \tag{9.4.65}$$

where $\mathbb{L}_F$ is a coset representative for the coset:

$$\frac{\mathrm{Osp}(6|4)}{\mathrm{SO}(6) \times \mathrm{Sp}(4,\mathbb{R})} \ni \mathbb{L}_F \tag{9.4.66}$$

while $\mathbb{L}_B$ rather than being the $\mathrm{Osp}(6|4)$ embedding of a coset representative of just AdS_4, is the embedding of a coset representative of $\mathrm{AdS}_4 \times \mathbb{P}^3$, namely:

$$\mathbb{L}_B = \left(\begin{array}{c|c} \mathrm{L}_{\mathrm{AdS}_4} & 0 \\ \hline 0 & \mathrm{L}_{\mathbb{P}^3} \end{array}\right); \qquad \frac{\mathrm{Sp}(4,\mathbb{R})}{\mathrm{SO}(1,3)} \ni \mathrm{L}_{\mathrm{AdS}_4}; \qquad \frac{\mathrm{SO}(6)}{\mathrm{U}(3)} \ni \mathrm{L}_{\mathbb{P}^3} \tag{9.4.67}$$

In this way we find:

$$\Sigma = \mathbb{L}_B^{-1}\, \Sigma_F \mathbb{L}_B + \mathbb{L}_B^{-1}\, d\mathbb{L}_B \tag{9.4.68}$$

Let us now write the explicit form of Σ_F:

$$\Sigma_F = \left(\begin{array}{c|c} \Delta_F & \Phi_A \\ \hline 4ie\overline{\Phi}_A \gamma_5 & -e\widetilde{\mathscr{A}}_{AB} \end{array}\right) \tag{9.4.69}$$

where Φ_A is a Majorana-spinor valued fermionic one-form and where Δ_F is an $\mathfrak{sp}(4,\mathbb{R})$ Lie algebra valued one-form presented as a 4×4 matrix. Both Φ_A as Δ_F and $\widetilde{\mathscr{A}}_{AB}$ depend only on the fermionic θ coordinates and differentials.

On the other hand we have:

$$\mathbb{L}_B^{-1}\, d\mathbb{L}_B = \left(\begin{array}{c|c} \Delta_{\mathrm{AdS}_4} & 0 \\ \hline 0 & \mathscr{A}_{\mathbb{P}^3} \end{array}\right) \tag{9.4.70}$$

where the Δ_{AdS_4} is also an $\mathfrak{sp}(4,\mathbb{R})$ Lie algebra valued one-form presented as a 4×4 matrix, but it depends only on the bosonic coordinates x^μ of the anti de Sitter space AdS_4. In the same way $\mathscr{A}_{\mathbb{P}^3}$ is an $\mathfrak{su}(4)$ Lie algebra element presented as an $\mathfrak{so}(6)$ antisymmetric matrix in 6-dimensions. It depends only on the bosonic coordinates y^α of the internal $\mathbb{P}^3$ manifold. According to (C.1.5) we can write:

$$\Delta_{\mathrm{AdS}_4} = -\frac{1}{4} B^{ab} \gamma_{ab} - 2e\gamma_a \gamma_5 B^a \tag{9.4.71}$$

where $\{B^{ab}, B^a\}$ are respectively the spin-connection and the vielbein of AdS_4.

Similarly, using the inversion formula (C.4.3) presented in appendix we can write:

$$\mathscr{A}_{\mathbb{P}^3} = \left(-2\mathscr{B}^{\alpha}\overline{\tau}_{\alpha} + \frac{1}{4e}\mathscr{B}^{\alpha\beta}\overline{\tau}_{\alpha\beta} - \frac{1}{4e}\mathscr{B}^{\alpha\beta}\mathscr{K}_{\alpha\beta}K\right) \tag{9.4.72}$$

where $\{\mathscr{B}^{\alpha\beta}, \mathscr{B}^{\alpha}\}$ are the connection and vielbein of the internal coset manifold $\mathbb{P}^3$.

Relying once again on the inversion formulae discussed in Appendix C.4 we conclude that we can rewrite (9.4.51)–(9.4.55) as follows:

$$\Psi^{x|A} = \Phi^{x|A} \tag{9.4.73}$$

$$V^{a} = E^{a} \tag{9.4.74}$$

$$V^{\alpha} = E^{\alpha} \tag{9.4.75}$$

$$\omega^{ab} = E^{ab} \tag{9.4.76}$$

$$\omega^{\alpha\beta} = E^{\alpha\beta} \tag{9.4.77}$$

where the objects introduced above are the Maurer Cartan forms on the supercoset (9.4.2) according to:

$$\begin{aligned} \Sigma &= \mathbb{L}^{-1}\,d\mathbb{L} \\ &= \left(\begin{array}{c|c} -\frac{1}{4}E^{ab}\gamma_{ab} - 2e\gamma_a\gamma_5 E^a & \Phi \\ \hline 4\mathrm{i}e\overline{\Phi}\gamma_5 & 2eE^{\alpha}\overline{\tau}_{\alpha} - \frac{1}{4}\mathscr{B}^{\alpha\beta}\overline{\tau}_{\alpha\beta} + \frac{1}{4}E^{\alpha\beta}\mathscr{K}_{\alpha\beta}K \end{array}\right) \end{aligned} \tag{9.4.78}$$

Consequently the gauge completion of the $\mathbf{B}^{[2]}$ form becomes:

$$\mathbf{B}^{[2]} = \frac{1}{4e}\overline{\Phi}(1 \otimes \overline{\tau}_7) \wedge \Phi \tag{9.4.79}$$

9.5 Conclusions

As we stressed in the introduction to the present very long chapter, the topics we might still address in this context are both numerous, relevant and challenging. We might discuss instanton solutions, compactifications on Calabi-Yau manifolds, the generic strategy of harmonic analysis to derive the spectra of given compactifications, toroidal compactifications with brane wrapping, D-brane solutions with conifolds sitting in the transverse dimensions to the brane and much more. Obviously there is neither room nor enough mathematical background in order to develop such topics and therefore it is time to stop.

We just hope that our reader has been able to follow our arguments up to this point. If this has happened, starting from the first intuitions about Lorentz symmetry in the first chapter of the first volume he has made a long and adventurous trip to the frontiers of current research in gravitational theories, little by little absorbing a remarkable wealth of geometrical lore and of consciousness about its physical meaning.

Hopefully our reader should by now be convinced that the geometrical seeds first implanted in the XIXth century by Gauss and Riemann, not only inspired Einstein and, through his mind, produced a beautiful and so far fully verified theory, but have still a lot to say about Gravity. Whether supersymmetric or not, Gravitation is certainly the most fundamental interaction among the fundamental ones, it governs the structure of the Universe and it is a manifestation of Geometry. Which geometry the humans will still debate for a long time.

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Chapter 10
Conclusion of Volume 2

10.1 The Legacy of Volume 1

In the first volume we have presented the theory of General Relativity comparing it at all times with the other gauge theories that describe non-gravitational interactions. We have also followed the complicated historical development of the ideas and of the concepts underlying both of them. In particular we have traced back the origin of our present understanding of all fundamental interactions as mediated by connections on principal fibre-bundles and emphasized the special status of gravity within this general scheme. While recalling the historical development we have provided a, hopefully rigorous, exposition of all the mathematical foundations of gravity and gauge theories in a contemporary geometrical approach. In the last two chapters of Volume 1 we also considered relevant astrophysical applications of General Relativity that provide some of the most accurate tests of its predictions.

10.2 The Story Told in Volume 2

Volume 2's mission was the historical and mathematical analysis of the further conceptual developments in the Theory of Gravity up to contemporary times. The first part of the Volume concentrated on two main issues:

1. Black Holes,
2. Cosmology.

Black Holes are probably the most profound implication of General Relativity. They occupy a distinguished and outstanding position both in Mathematics and in Physics, being an endless source of inspiring views and of challenging problems, like that of the information loss and the thermodynamical formulation of their dynamical laws which leads to the question of the statistical interpretation of their entropy. The historical/conceptual path going from the interpretation of the Schwarzschild radius as an event horizon to the proper definitions of causal structures and Penrose diagrams was carefully described and the necessary mathematics

P.G. Frè, *Gravity, a Geometrical Course*, DOI 10.1007/978-94-007-5443-0_10,

was step by step developed. With rotating black holes and the area-entropy law we opened the first window on a more profound level of gravitational theory that leads to superstring and supergravity.

Cosmology is equally important in the development of gravitational theory since gravity is the only relevant interaction at very large scales. Einstein theory leads necessarily to the view of an expanding universe, notwithstanding the original philosophical enmity of Einstein himself to this very idea. The mathematical formulation of the Cosmological Principle requires the notion of homogeneous coset manifolds and that was the occasion to develop this important chapter of Differential Geometry whose uses are ubiquitous in Theoretical Physics and in particular in Supergravity. The resolution of the conceptual problems raised by the Standard Cosmological Model leads to the paradigm of the inflationary universe whose predictions seem to be generically in agreement with the observations of the anisotropies in the Cosmic Microwave Background radiation. The simplest mechanism of inflation requires scalar fields that are abundant in all versions of the supergravity/superstring modeling of the fundamental laws of physics. Scalar fields are anyhow required by symmetry breaking and their experimental revelation is by now overdue.[1]

The second part of the Volume was conceived as an introduction to Supergravity, Branes and Strings. Once again our approach was conceptual and historical. The goal was explicitly that of continuing the logical development initiated in Volume 1 introducing supergravity as the dynamical theory of the super-Poincaré connection, in the same way as General Relativity is the dynamical theory of the Poincaré connection. From the mathematical point of view, new concepts enter the stage: the extension of (super-)Lie algebras to Free Differential Algebras and the generalized principle of analyticity named rheonomy. On one hand Free Differential Algebras are canonically induced by their normal Lie subalgebra, by means of its cohomology, on the other hand the new generators of Free Differential Algebras are p-forms that naturally couple to the world volume of $(p-1)$-branes. This is the core of the Bulk-Brane dualism which we tried to illustrate in some detail in a dedicated chapter. The other important legacy of the supersymmetric extension of General Relativity is the incredible wealth of new geometrical structures that are contributed by the scalar sector of the various supergravities. Without any claim to completeness we tried to provide the reader with an overview of this richness and with and an introduction to the Supergravity Bestiary.

From the point of view of Gravity Theory, which is that adopted in this book, the main interest of supergravity, namely of the *Beyond GR World*, is its contribution of an impressive variety of new classical solutions of quite different type and with quite different interesting properties. A taste of that variety is provided to the reader in Chap. 9 through the presentation of a series of examples.

Repeating what was already stressed at the end of that chapter, the author considers his mission fulfilled if the reader could follow his arguments from the first intuitions about Lorentz symmetry at the end of XIXth century to some of the frontiers

[1]By the publication date of this book the Higgs boson, that is a scalar particle, has already been discovered at CERN with very high likehood.

of current research in Gravitational Theory at the beginning of the XXIst century. The author will also be satisfied of his own work if, while traveling along this path which was rich in conceptual developments, inclusion of complex mathematical structures and discovery of new physical phenomena, the reader has strengthened his belief in that the Universe is Gravitation, Gravitation is Geometry but Geometry is enormously variegated and full of yet undiscovered surprises.

Appendix A: Spinors and Gamma Matrix Algebra[2]

A.1 Introduction to the Spinor Representations of $\mathrm{SO}(1, D-1)$

The spinor representations of the orthogonal and pseudo-orthogonal groups have different structure in various dimensions. Starting from the representation of the Dirac gamma matrices one begins with a complex representation whose dimension is equal to the dimension of the gammas. A vector in this complex linear space is named a *Dirac spinor*. Typically Dirac spinors do not form irreducible representations. Depending on the dimensions, one can still impose $\mathrm{SO}(1, D-1)$ invariant conditions on the Dirac spinor that separate it into irreducible parts. These constraints can be of two types:

(a) A reality condition which maintains the number of components of the spinor but relates them to their complex conjugates by means of linear relations. This reality condition is constructed with an invariant matrix $\mathscr{C}$, named the *charge conjugation matrix* whose properties depend on the dimensions D.
(b) A chirality condition constructed with a *chirality matrix* Γ_{D+1} that halves the number of components of the spinor. The chirality matrix exists only in even dimensions.

Depending on which conditions can be imposed, besides Dirac spinors, in various dimensions D, one has Majorana spinors, Weyl spinors and, in certain dimensions, also Majorana-Weyl spinors. In this appendix we discuss the properties of gamma matrices and we present the various types of irreducible spinor representations in all relevant dimensions from $D = 4$ to $D = 11$. The upper bound $D = 11$ is dictated by supersymmetry since supergravity, *i.e.* the supersymmetric extension of Einstein gravity, can be constructed in all dimensions up to $D = 11$, which is maximal in this respect.

A.2 The Clifford Algebra

In order to describe spinors one needs the Dirac gamma matrices. These form the Clifford algebra:

$$\{\Gamma_a, \Gamma_b\} = 2\eta_{ab} \qquad \text{(A.2.1)}$$

[2]This appendix is present also in Volume 1. It is repeated in Volume 2 for reader's convenience.

where η_{ab} is the invariant metric of SO$(1, D-1)$, that we always choose according to the mostly minus conventions, namely:

$$\eta_{ab} = \mathrm{diag}(+, -, -, \dots, -) \tag{A.2.2}$$

To study the general properties of the Clifford algebra (A.2.1) we use a direct construction method.

We begin by fixing the following conventions. $\Gamma^0 = \Gamma_0$ corresponding to the time direction is Hermitian:

$$\Gamma_0^\dagger = \Gamma_0 \tag{A.2.3}$$

while the matrices $\Gamma_i = -\Gamma^i$ corresponding to space directions are anti-Hermitian:

$$\Gamma_i^\dagger = -\Gamma_i \tag{A.2.4}$$

In the study of Clifford algebras it is necessary to distinguish the case of even and odd dimensions.

A.2.1 Even Dimensions

When $D = 2\nu$ is an even number the representation of the Clifford algebra (A.2.1) has dimension:

$$\dim \Gamma_a = 2^{\frac{D}{2}} = 2^\nu \tag{A.2.5}$$

In other words the gamma matrices are $2^\nu \times 2^\nu$. The proof of such a statement is easily obtained by iteration. Suppose that we have the gamma matrices γ_a corresponding to the case $\nu' = \nu - 1$, satisfying the Clifford algebra (A.2.1) in $D-2$ dimensions and that they are $2^{\nu'}$-dimensional. We can write down the following representation for the gamma matrices in D-dimension by means of the following $2^\nu \times 2^\nu$ matrices:

$$\begin{aligned}
&\Gamma_{a'} = \left(\begin{array}{c|c} 0 & \gamma_{a'} \\ \hline \gamma_{a'} & 0 \end{array}\right); \qquad \Gamma_{D-2} = \left(\begin{array}{c|c} \mathrm{i} & 0 \\ \hline 0 & -\mathrm{i} \end{array}\right) \\
&\Gamma_{D-1} = \left(\begin{array}{c|c} 0 & \mathbf{1} \\ \hline -\mathbf{1} & 0 \end{array}\right); \qquad a' = 0, 1, \dots, D-3
\end{aligned} \tag{A.2.6}$$

which satisfy the correct anticommutation relations and have the correct hermiticity properties specified above. This representation admits the following interpretation in terms of matrix tensor products:

$$\Gamma_{a'} = \gamma_{a'} \otimes \sigma_1; \qquad \Gamma_{D-2} = \mathbf{1} \otimes \mathrm{i}\sigma_3; \qquad \Gamma_{D-1} = \mathbf{1} \otimes \mathrm{i}\sigma_2 \tag{A.2.7}$$

where $\sigma_{1,2,3}$ denote the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \qquad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}; \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{A.2.8}$$

To complete the proof of our statement we just have to show that for $\nu = 2$, corresponding to $D = 4$ we have a 4-dimensional representation of the gamma matrices. This is well established. For instance we have the representation:

$$\gamma_0 = \left(\begin{array}{c|c} 0 & \mathbf{1} \\ \hline \mathbf{1} & 0 \end{array}\right); \qquad \gamma_{1,2,3} = \left(\begin{array}{c|c} 0 & \sigma_{1,2,3} \\ \hline -\sigma_{1,2,3} & 0 \end{array}\right) \tag{A.2.9}$$

In $D = 2\nu$ one can construct the *chirality matrix* defined as follows:

$$\Gamma_{D+1} = \alpha_D \Gamma_0 \Gamma_1 \Gamma_2 \ldots \Gamma_{D-1}; \qquad |\alpha_D|^2 = 1 \tag{A.2.10}$$

where α_D is a phase-factor to be fixed in such a way that:

$$\Gamma_{D+1}^2 = \mathbf{1} \tag{A.2.11}$$

By direct evaluation one can verify that:

$$\{\Gamma_a, \Gamma_{D+1}\} = 0 \quad a = 0, 1, 2, \ldots, D-1 \tag{A.2.12}$$

The normalization α_D is easily derived. We have:

$$\Gamma_0 \Gamma_1 \ldots \Gamma_{D-1} = (-)^{\frac{1}{2}D(D-1)} \Gamma_{D-1} \Gamma_{D-2} \Gamma_{D-1} \tag{A.2.13}$$

so that imposing (A.2.11) results into the following equation for α_D:

$$\alpha_D^2 (-)^{\frac{1}{2}D(D-1)} (-)^{(D-1)} = 1 \tag{A.2.14}$$

which has solution:

$$\begin{aligned} \alpha_D &= 1 \quad \text{if } \nu = 2\mu + 1 \sim \text{odd} \\ \alpha_D &= \mathrm{i} \quad \text{if } \nu = 2\mu \sim \text{even} \end{aligned} \tag{A.2.15}$$

With the same token we can show that the chirality matrix is Hermitian:

$$\Gamma_{D+1}^{\dagger} = \alpha^{\star} (-)^{\frac{1}{2}D(D-1)} (-)^{(D-1)} \Gamma_0 \Gamma_1 \Gamma_2 \ldots \Gamma_{D-1} = \Gamma_{D+1} \tag{A.2.16}$$

A.2.2 Odd Dimensions

When $D = 2\nu + 1$ is an odd number, the Clifford algebra (A.2.1) can be represented by $2^\nu \times 2^\nu$ matrices. It suffices to take the matrices $\Gamma_{a'}$ corresponding to the even case $D' = D - 1$ and add to them the matrix $\Gamma_D = \mathrm{i}\Gamma_{D'+1}$, which is anti-Hermitian and anti-commutes with all the other ones.

A.3 The Charge Conjugation Matrix

Since Γ_a and their transposed Γ_a^T satisfy the same Clifford algebras it follows that there must be a similarity transformation connecting these two representations of the same algebra on the same carrier space. Such statement relies on Schur's lemma and it is proved in the following way. We introduce the notation:

$$\Gamma_{a_1\ldots a_n} \equiv \Gamma_{[a_1}\Gamma_{a_2}\ldots\Gamma_{a_n]} = \frac{1}{n!}\sum_P (-)^{\delta_P}\Gamma_{a_{P(a_1)}}\ldots\Gamma_{a_{P(a_n)}} \tag{A.3.1}$$

where $\sum_P$ denotes the sum over the $n!$ permutations of the indices and δ_P the parity of permutation P, *i.e.* the number of elementary transpositions of which it is composed. The set of all matrices $1, \Gamma_a, \Gamma_{a_1a_2}, \ldots, \Gamma_{a_1\ldots a_D}$ constitutes a finite group of $2^{[D/2]}$-dimensional matrices. Furthermore the groups generated in this way by Γ_a, $-\Gamma_a$ or Γ_a^T are isomorphic. Hence by Schur's lemma two irreducible representations of the same group, with the same dimension and defined over the same vector space, must be equivalent, that is there must be a similarity transformation that connects the two. The matrix realizing such a similarity is called the *charge conjugation matrix*. Instructed by this discussion we define the charge conjugation matrix by means of the following equations:

$$\begin{aligned} \mathscr{C}_-\Gamma_a\mathscr{C}_-^{-1} &= -\Gamma_a^T \\ \mathscr{C}_+\Gamma_a\mathscr{C}_+^{-1} &= \Gamma_a^T \end{aligned} \tag{A.3.2}$$

By definition $\mathscr{C}_\pm$ connects the representation generated by Γ_a to that generated by $\pm\Gamma_a^T$. In even dimensions both $\mathscr{C}_-$ and $\mathscr{C}_+$ exist, while in odd dimensions only one of the two is possible. Indeed in odd dimensions Γ_{D-1} is proportional to $\Gamma_0\Gamma_1\ldots\Gamma_{D-2}$ so that the $\mathscr{C}_-$ and $\mathscr{C}_+$ of $D-1$ dimensions yield the same result on Γ_{D-1}. This decides which $\mathscr{C}$ exists in a given odd dimension.

Another important property of the charge conjugation matrix follows from iterating (A.3.2). Using Schur's lemma one concludes that $\mathscr{C}_\pm = \alpha\mathscr{C}_\pm^T$ so that iterating

Table A.1 Structure of charge conjugation matrices in various space-time dimensions

Charge conjugation matrices		
D	$\mathscr{C}_+^\star = \mathscr{C}_+$ (real)	$\mathscr{C}_-^\star = \mathscr{C}_-$ (real)
4	$\mathscr{C}_+^T = -\mathscr{C}_+;\ \mathscr{C}_+^2 = -\mathbf{1}$	$\mathscr{C}_-^T = -\mathscr{C}_-;\ \mathscr{C}_-^2 = -\mathbf{1}$
5	$\mathscr{C}_+^T = -\mathscr{C}_+;\ \mathscr{C}_+^2 = -\mathbf{1}$	
6	$\mathscr{C}_+^T = -\mathscr{C}_+;\ \mathscr{C}_+^2 = -\mathbf{1}$	$\mathscr{C}_-^T = \mathscr{C}_-;\ \mathscr{C}_-^2 = \mathbf{1}$
7		$\mathscr{C}_-^T = \mathscr{C}_-;\ \mathscr{C}_-^2 = \mathbf{1}$
8	$\mathscr{C}_+^T = \mathscr{C}_+;\ \mathscr{C}_+^2 = \mathbf{1}$	$\mathscr{C}_-^T = \mathscr{C}_-;\ \mathscr{C}_-^2 = \mathbf{1}$
9	$\mathscr{C}_+^T = \mathscr{C}_+;\ \mathscr{C}_+^2 = \mathbf{1}$	
10	$\mathscr{C}_+^T = \mathscr{C}_+;\ \mathscr{C}_+^2 = \mathbf{1}$	$\mathscr{C}_-^T = -\mathscr{C}_-;\ \mathscr{C}_-^2 = -\mathbf{1}$
11		$\mathscr{C}_-^T = -\mathscr{C}_-;\ \mathscr{C}_-^2 = -\mathbf{1}$

again we obtain $\alpha^2 = 1$. In other words $\mathscr{C}_+$ and $\mathscr{C}_-$ are either symmetric or antisymmetric. We do not dwell on the derivation which can be obtained by explicit iterative construction of the gamma matrices in all dimensions and we simply collect below the results for the properties of $\mathscr{C}_\pm$ in the various relevant dimensions (see Table A.1).

A.4 Majorana, Weyl and Majorana-Weyl Spinors

The Dirac conjugate of a spinor ψ is defined by the following operation:

$$\overline{\psi} \equiv \psi^\dagger \Gamma_0 \tag{A.4.1}$$

and the charge conjugate of ψ is defined as:

$$\psi^c = \mathscr{C}\overline{\psi}^T \tag{A.4.2}$$

where $\mathscr{C}$ is the charge conjugation matrix. When we have such an option we can either choose $\mathscr{C}_+$ or $\mathscr{C}_-$. By definition a Majorana spinor λ satisfies the following condition:

$$\lambda = \lambda^c = C\Gamma_0^T \lambda^\star \tag{A.4.3}$$

Equation (A.4.3) is not always self-consistent. By iterating it a second time we get the consistency condition:

$$\mathscr{C}\Gamma_0^T \mathscr{C} = \Gamma_0 \tag{A.4.4}$$

There are two possible solutions to this constraint. Either $\mathscr{C}_-$ is antisymmetric or $\mathscr{C}_+$ is symmetric. Hence, in view of the results displayed above, Majorana spinors exist only in

$$D = 4, 8, 9, 10, 11 \tag{A.4.5}$$

In $D = 4, 10, 11$ they are defined using the $\mathscr{C}_-$ charge conjugation matrix while in $D = 8, 9$ they are defined using $\mathscr{C}_+$.

Weyl spinors, on the contrary, exist in every even dimension; by definition they are the eigenstates of the Γ_{D+1} matrix, corresponding to the $+1$ or -1 eigenvalue. Conventionally the former eigenstates are named *left-handed*, while the latter are named *right-handed* spinors:

$$\Gamma_{D+1}\psi_{\binom{L}{R}} = \pm\psi_{\binom{L}{R}} \tag{A.4.6}$$

In some special dimensions we can define Majorana-Weyl spinors which are both eigenstates of Γ_{D+1} and satisfy the Majorana condition (A.4.3). In order for this to be possible we must have:

$$\mathscr{C}\Gamma_0^T \Gamma_{D+1}^\star \psi^\star = \Gamma_{D+1}\psi \tag{A.4.7}$$

which implies:

$$\mathscr{C}\Gamma_0^T\,\Gamma_{D+1}^{\star}\Gamma_0^T\mathscr{C}^{-1} = \Gamma_{D+1} \tag{A.4.8}$$

With some manipulations the above condition becomes:

$$\mathscr{C}\Gamma_{D+1}\mathscr{C}^{-1} = -\Gamma_{D+1}^T \tag{A.4.9}$$

which can be checked case by case, using the definition of Γ_{D+1} as product of all the other gamma matrices. In the range $4 \le D \le 11$ the only dimension where (A.4.9) is satisfied is $D = 10$ which is the critical dimensions for superstrings. This is not a pure coincidence.

Summarizing we have:

Spinors in $4 \le D \le 11$				
D	Dirac	Majorana	Weyl	Majora-Weyl
4	Yes	Yes	Yes	No
5	Yes	No	No	No
6	Yes	No	Yes	No
7	Yes	No	No	No
8	Yes	Yes	Yes	No
9	Yes	Yes	No	No
10	Yes	Yes	Yes	Yes
11	Yes	Yes	No	No

A.5 A Particularly Useful Basis for $D = 4$ γ-Matrices

In this section we construct a $D = 4$ gamma matrix basis which is convenient for various purposes. Let us first specify the basis and then discuss its convenient properties.

In terms of the standard matrices (A.2.8) we realize the $\mathfrak{so}(1,3)$ Clifford algebra:

$$\{\gamma_a, \gamma_b\} = 2\eta_{ab}; \qquad \eta_{ab} = \mathrm{diag}(+,-,-,-) \tag{A.5.1}$$

by setting:

$$\begin{aligned} \gamma_0 &= \sigma_1 \otimes \sigma_3; & \gamma_1 &= \mathrm{i}\sigma_2 \otimes \sigma_3 \\ \gamma_2 &= \mathrm{i}\mathbf{1} \otimes \sigma_2; & \gamma_3 &= \mathrm{i}\sigma_3 \otimes \sigma_3 \\ \gamma_5 &= \mathbf{1} \otimes \sigma_1; & \mathscr{C} &= \mathrm{i}\sigma_2 \otimes \mathbf{1} \end{aligned} \tag{A.5.2}$$

where γ_5 is the chirality matrix and $\mathscr{C}$ is the charge conjugation matrix. In this basis the generators of the Lorentz algebra $\mathfrak{so}(1,3)$, namely γ_{ab} are particularly simple and nice 4×4 matrices. Explicitly we get:

$$\gamma_{01} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}; \qquad \gamma_{02} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$
$$\gamma_{03} = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}; \qquad \gamma_{12} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \tag{A.5.3}$$
$$\gamma_{13} = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}; \qquad \gamma_{23} = \begin{pmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \end{pmatrix}$$

Let us mention some relevant formulae that are easily verified in the above basis:

$$\gamma_0\gamma_1\gamma_2\gamma_3 = \mathrm{i}\gamma_5 \tag{A.5.4}$$

and if we fix the convention:

$$\varepsilon_{0123} = +1 \tag{A.5.5}$$

we obtain:

$$\frac{1}{24}\varepsilon^{abcd}\gamma_a\gamma_b\gamma_c\gamma_d = -\mathrm{i}\gamma_5 \tag{A.5.6}$$

Appendix B: Auxiliary Tools for p-Brane Actions

In this appendix we collect some auxiliary calculations and algebraic tools relevant to the discussion of p-brane world volume actions presented in Chap. 7.

B.1 Notations and Conventions

General adopted notations for first order world volume actions are the following ones:

$$d = \text{dimension of the world-volume } \mathscr{W}_d$$
$$D = \text{dimension of the bulk space-time } \mathscr{M}_D$$
$$V^{\underline{a}} = \text{vielbein 1-form of bulk space-time}$$

$$
\begin{aligned}
\Pi_i^{\underline{a}} &= D \times d \text{ matrix. 0-form auxiliary field} \\
h^{ij} &= d \times d \text{symmetric matrix. 0-form auxiliary field} \\
e^{\ell} &= \text{vielbein 1-form of the world-volume} \\
\eta_{\underline{ab}} &= \text{diag}\{+, \underbrace{-, \ldots, -}_{D-1 \text{ times}}\} = \text{flat metric on the bulk} \\
\eta^{ij} &= \text{diag}\{+, \underbrace{-, \ldots, -}_{d-1 \text{ times}}\} = \text{flat metric on the world-volume}
\end{aligned}
\tag{B.1.1}
$$

B.2 The κ-Supersymmetry Projector for $D3$-Branes

In this appendix we present the derivation of the κ-supersymmetry projector utilized in Sect. 7.5 to establish the κ-susy invariance of the $D3$-brane action. In particular we refer to (7.5.23).

Let us begin with property (a) and consider the ansatz in (7.5.26). By direct calculation we find:

$$
\begin{aligned}
\omega_{[4]}^2 &= \alpha_4^2 (4!)^2 \\
\omega_{[2]}^2 &= \frac{(\alpha_2)^2 \omega_{[0]} \omega_{[4]}}{3! \alpha_0 \alpha_4} + 8 (\alpha_2)^2 \operatorname{Tr}\big(\widehat{F}^2\big)
\end{aligned}
\tag{B.2.1}
$$

so that we get:

$$
\Gamma^2 = \frac{1}{N^2} \left[(4! \alpha_4)^2 + \omega_{[0]} \omega_{[4]} \left(\frac{(\alpha_2)^2}{3! \alpha_0 \alpha_4} + 2 \right) + 8 (\alpha_2)^2 \operatorname{Tr}\big(\widehat{F}^2\big) \right]
\tag{B.2.2}
$$

so we obtain $\Gamma^2 = \mathbf{1}$ if the normalization factor N is chosen as in (7.5.27) and if the coefficients are chosen as in (7.5.28). This conclusion is easily reached using the identity (7.5.30) of the main text.

Let us now turn to property (b), namely to the condition

$$
\Gamma A_k = A_k
\tag{B.2.3}
$$

To implement it we need to calculate some γ matrix products:

$$
\begin{aligned}
\omega_{[4]} \gamma_k &= \frac{1}{6} \tilde{\gamma}_k \\
\omega_{[4]} \tilde{\gamma}_k &= 6 \gamma_k \\
\omega_{[4]} \Pi_k &= -\frac{1}{2} \widetilde{\widehat{F}}^{ij} \gamma_{ijk} \equiv -\frac{1}{2} \widetilde{\Delta}_k \\
\omega_{[4]} \widetilde{\Pi}_k &= -\frac{1}{2} \widehat{F}^{ij} \gamma_{ijk} \equiv -\frac{1}{2} \Delta_k
\end{aligned}
\tag{B.2.4}
$$

$$
\begin{aligned}
\omega_{[2]}\gamma_k &= \frac{i}{2}\widetilde{\Delta}_k + i\widetilde{\Pi}_k \\
\omega_{[2]}\tilde{\gamma}_k &= -3i\Delta_k - 6i\Pi_k \\
\omega_{[2]}\Pi_k &= -\frac{i}{6}\big(\widehat{F}^2\big)_{kl}\tilde{\gamma}^l - i\omega_{[0]}\gamma_k \\
\omega_{[2]}\widetilde{\Pi}_k &= i\big(\widehat{\widetilde{F}}{}^2\big)_{kl}\gamma^l + \frac{i}{6}\omega_{[0]}\tilde{\gamma}_k
\end{aligned}
\tag{B.2.5}
$$

Now we impose (B.2.3) and we obtain the following equations.

- The contributions from Δ_k and $\widetilde{\Delta}_k$ are:

$$
\begin{aligned}
\Delta^m h_{mk}\left(\frac{i}{2}f_4 + 3f_2\right)\otimes\sigma_1 &= 0 \\
\widetilde{\Delta}_k\left(-\frac{i}{2}f_3 + \frac{i}{2}f_1\right)\otimes\sigma_2 &= 0
\end{aligned}
\tag{B.2.6}
$$

- For the contributions with γ_k we have two equations, one proportional to σ_3 and one proportional to $\mathbf{1}$, namely:

$$
\gamma_k\omega_{[0]}(f_1 - f_3)\otimes\sigma_3 = 0 \tag{B.2.7}
$$

  and

$$
\frac{1}{N}\big(6f_2\mathbf{1}_{4\times4} + if_4\widehat{\widetilde{F}}{}^2\big)h\otimes\mathbf{1}_{2\times2} = \mathrm{i}f_1\mathbf{1}_{4\times4}\otimes\mathbf{1}_{2\times2} \tag{B.2.8}
$$

  For:

$$
\begin{aligned}
6f_2 &= \mathrm{i}f_1 \\
\mathrm{i}f_1 &= -if_4
\end{aligned}
\tag{B.2.9}
$$

  and using the property (7.5.31) we obtain:

$$
\begin{aligned}
N^{-1}\big(\mathbf{1} - \widehat{\widetilde{F}}{}^2\big)h &= \mathbf{1} \\
N^{-1}\big(\mathbf{1} - \widehat{\widetilde{F}}{}^2\big)N^{-1}\big(\mathbf{1} - \widehat{F}^2\big) &= \mathbf{1} \\
\big(\mathbf{1} - \widehat{F}^2 - \widehat{\widetilde{F}}{}^2 + \widehat{\widetilde{F}}{}^2\widehat{F}^2\big) &= N^2 \\
\left[\mathbf{1} - \frac{1}{2}\mathrm{Tr}\big(\widehat{F}^2\big) + \left(\frac{1}{8}F_{ij}F_{kl}\varepsilon^{ijkl}\right)\mathbf{1}\right] &= N^2
\end{aligned}
\tag{B.2.10}
$$

- For the contributions with $\tilde{\gamma}_k$ we get the following equations:

$$
\begin{aligned}
\tilde{\gamma}_m h^m{}_k\omega_{[0]}\left(f_2 + \frac{i}{6}f_4\right)\otimes\mathbf{1}_{2\times2} &= 0 \\
N^{-1}\tilde{\gamma}_m\left[\frac{1}{6}f_1\delta^m{}_k - \frac{1}{6}f_3\big(\widehat{F}^2\big)^m{}_k\right]\otimes\sigma_3 &= f_2\tilde{\gamma}_m h^m{}_k\otimes\sigma_3
\end{aligned}
\tag{B.2.11}
$$

Then if:

$$
\begin{aligned}
f_2 &= -\frac{i}{6} f_4 \\
f_1 &= f_3 \\
f_1 &= 6 f_2
\end{aligned}
\tag{B.2.12}
$$

we obtain that the second of equations (B.2.11) as a matrix equation becomes:

$$N^{-1}\big[\mathbf{1} - \big(\widehat{F}^2\big)\big] = h \tag{B.2.13}$$

and just coincides with the solution (7.4.51) for the auxiliary field h in terms of the physical ones.

- Now we consider Π and $\widetilde{\Pi}$.

  The equation proportional to σ_1 is:

$$
\begin{aligned}
\alpha\omega_{[0]}\widetilde{\Pi}^m h_{mk} + \beta \Pi^m h_{mk} &= N\gamma \Pi^k \\
\alpha\omega_{[0]}\widetilde{\widehat{F}}^{lm} h_{mk}\gamma_l + \beta \widehat{F}^{lm} h_{mk}\gamma_l &= \gamma N \widehat{F}^l{}_k \gamma_l
\end{aligned}
\tag{B.2.14}
$$

in matrix form we have:

$$
\begin{aligned}
\alpha\omega_{[0]}(\widetilde{\widehat{F}} h) + \beta(\widehat{F} h) &= \gamma N \widehat{F} \\
\alpha\omega_{[0]}\widetilde{\widehat{F}}\big[\mathbf{1} - \big(\widehat{F}^2\big)\big]N^{-1} + \beta \widehat{F}^2\big[\mathbf{1} - \big(\widehat{F}^2\big)\big]N^{-1} &= \gamma N \widehat{F} \\
\alpha\omega_{[0]}\widetilde{\widehat{F}}\big[\mathbf{1} - \big(\widehat{F}^2\big)\big] + \beta \widehat{F}^2\big[\mathbf{1} - \big(\widehat{F}^2\big)\big] &= \gamma N^2 \widehat{F} \\
\alpha\omega_{[0]}\widetilde{\widehat{F}}\big[\mathbf{1} - \big(\widehat{F}^2\big)\big] + \beta \widehat{F}^2\big[\mathbf{1} - \big(\widehat{F}^2\big)\big] &= \gamma\left[1 - \frac{1}{2}\operatorname{Tr}\big(\widehat{F}^2\big) + \omega_{[0]}^2\right]\widehat{F} \\
\alpha\omega_{[0]}\widetilde{\widehat{F}} - \alpha\omega_{[0]}(\widetilde{\widehat{F}}\widehat{F})\widehat{F} + \beta\widehat{F} - \beta\widehat{F}^2\widehat{F} &= \gamma\widehat{F} - \frac{\gamma}{2}\operatorname{Tr}\big(\widehat{F}^2\big)\widehat{F} + \gamma\omega_{[0]}^2\widehat{F}
\end{aligned}
\tag{B.2.15}
$$

if:

$$\beta = \gamma \tag{B.2.16}$$

and using (7.5.31), than (B.2.15) become:

$$\alpha\omega_{[0]}\widetilde{\widehat{F}} + \alpha\omega_{[0]}^2\widehat{F} - \beta\widehat{F}^2\widehat{F} = -\frac{\gamma}{2}\operatorname{Tr}\big(\widehat{F}^2\big)\widehat{F} + \gamma\omega_{[0]}^2\widehat{F} \tag{B.2.17}$$

if:

$$\alpha = \gamma \tag{B.2.18}$$

$$\begin{aligned} \alpha\omega_{[0]}\widehat{\widetilde{\widetilde{F}}} - \beta\widehat{F}^2\widehat{F} &= -\frac{\gamma}{2}\,\mathrm{Tr}\big(\widehat{F}^2\big)\widehat{F} \\ \alpha\omega_{[0]}\widehat{\widetilde{\widetilde{F}}} - \alpha\widehat{F}^2\widehat{F} &= -\alpha\big(\widehat{F}^2 + \widehat{\widetilde{\widetilde{F}}}^2\big)\widehat{F} \\ \alpha\omega_{[0]}\widehat{\widetilde{\widetilde{F}}} &= -\alpha\widehat{\widetilde{\widetilde{F}}}^2\widehat{F} \end{aligned} \tag{B.2.19}$$

and it is correct by (7.5.31).

The equation proportional to σ_2 is:

$$\begin{aligned} \mu\omega_{[0]}\Pi_k + \nu\widetilde{\Pi}_k &= N\rho\widetilde{\Pi}^m h_{mk} \\ \mu\omega_{[0]}\widehat{F}_{lk}\gamma^l + \nu\widehat{\widetilde{\widetilde{F}}}_{lk}\gamma^l &= N\rho\widehat{\widetilde{\widetilde{F}}}_l^{\ m} h_{mk}\gamma^l \\ \mu\omega_{[0]}\widehat{F}_{lk} + \nu\widehat{\widetilde{\widetilde{F}}}_{lk} &= N\rho\widehat{\widetilde{\widetilde{F}}}_{lm}\big[\delta^m{}_k - \big(\widehat{F}^2\big)^m{}_k\big]N^{-1} \end{aligned} \tag{B.2.20}$$

for:

$$\begin{aligned} \nu &= \rho \\ \mu &= \rho \end{aligned} \tag{B.2.21}$$

we obtain the first of the relations (7.5.31).

Where:

$$\begin{array}{lll} \alpha = -if_4 & \beta = 6if_2 & \gamma = f_3 \\ \mu = if_3 & \nu = if_1 & \rho = f_4 \end{array} \tag{B.2.22}$$

Using the fact that $a_5 = \frac{3}{4}$ and (7.5.18) we have that (B.2.6), (B.2.7), (B.2.9), (B.2.12), and (B.2.16), (B.2.18), (B.2.21) are automatically satisfied. This concludes the proof of property (b) and hence of κ supersymmetry.

Appendix C: Auxiliary Information About Some Superalgebras

C.1 The OSp($\mathscr{N}$|4) *Supergroup, Its Superalgebra and Its Supercosets*

In this appendix we provide some explicit information and a collection of very useful formulae relative to the very important class of supergroups OSp($\mathscr{N}$|4) which appears in the compactification of superstrings and of M-theory on anti de Sitter backgrounds. The presented material closely follows two sections of paper [1].

C.1.1 The Superalgebra

The real form $\mathfrak{osp}(\mathscr{N}|4)$ of the complex $\mathfrak{osp}(\mathscr{N}|4,\mathbb{C})$ Lie superalgebra which is relevant for the study of $\mathrm{AdS}_4 \times \mathscr{G}/\mathscr{H}$ compactifications is that one where the ordinary Lie subalgebra is the following:

$$\mathfrak{sp}(4,\mathbb{R}) \times \mathfrak{so}(\mathcal{N}) \subset \mathfrak{osp}(\mathcal{N}|4) \tag{C.1.1}$$

This is quite obvious because of the isomorphism $\mathfrak{sp}(4,\mathbb{R}) \simeq \mathfrak{so}(2,3)$ which identifies $\mathfrak{sp}(4,\mathbb{R})$ with the isometry algebra of anti de Sitter space. The compact algebra $\mathfrak{so}(\mathcal{N})$ is instead the R-symmetry algebra acting on the supersymmetry charges.

The superalgebra $\mathfrak{osp}(\mathcal{N}|4)$ can be introduced as follows: consider the two graded $(4+\mathcal{N}) \times (4+\mathcal{N})$ matrices:

$$\widehat{C} = \left(\begin{array}{c|c} C\gamma_5 & 0 \\ \hline 0 & -\frac{\mathrm{i}}{4e}\mathbf{1}_{\mathcal{N}\times\mathcal{N}} \end{array}\right); \qquad \widehat{H} = \left(\begin{array}{c|c} \mathrm{i}\gamma_0\gamma_5 & 0 \\ \hline 0 & -\frac{1}{4e}\mathbf{1}_{\mathcal{N}\times\mathcal{N}} \end{array}\right) \tag{C.1.2}$$

where C is the charge conjugation matrix in $D=4$. The matrix $\widehat{C}$ has the property that its upper block is antisymmetric while its lower one is symmetric. On the other hand, the matrix $\widehat{H}$ has the property that both its upper and lower blocks are Hermitian. The $\mathfrak{osp}(\mathcal{N}|4)$ Lie algebra is then defined as the set of graded matrices Λ satisfying the two conditions:

$$\Lambda^T \widehat{C} + \widehat{C}\Lambda = 0 \tag{C.1.3}$$

$$\Lambda^\dagger \widehat{H} + \widehat{H}\Lambda = 0 \tag{C.1.4}$$

Equation (C.1.3) defines the complex osp$(\mathcal{N}|4)$ superalgebra while (C.1.4) restricts it to the appropriate real section where the ordinary Lie subalgebra is (C.1.1). The specific form of the matrices $\widehat{C}$ and $\widehat{H}$ is chosen in such a way that the complete solution of the constraints (C.1.3), (C.1.4) takes the following form:

$$\Lambda = \left(\begin{array}{c|c} -\frac{1}{4}\omega^{ab}\gamma_{ab} - 2e\gamma_a\gamma_5 E^a & \psi_A \\ \hline 4\mathrm{i}e\overline{\psi}_B\gamma_5 & -e\mathscr{A}_{AB} \end{array}\right) \tag{C.1.5}$$

and the Maurer-Cartan equations

$$d\Lambda + \Lambda \wedge \Lambda = 0 \tag{C.1.6}$$

read as follows:

$$\begin{aligned} d\omega^{ab} - \omega^{ac} \wedge \omega^{db}\eta_{cd} + 16e^2 E^a \wedge E^b &= -\mathrm{i}2e\overline{\psi}_A \wedge \gamma^{ab}\gamma^5\psi_A \\ dE^a - \omega^a{}_c \wedge E^c &= \mathrm{i}\frac{1}{2}\overline{\psi}_A \wedge \gamma^a\psi_A \\ d\psi_A - \frac{1}{4}\omega^{ab} \wedge \gamma_{ab}\psi_A - e\mathscr{A}_{AB} \wedge \psi_B &= 2eE^a \wedge \gamma_a\gamma_5\psi_A \\ d\mathscr{A}_{AB} - e\mathscr{A}_{AC} \wedge \mathscr{A}_{CB} &= 4\mathrm{i}\overline{\psi}_A \wedge \gamma_5\psi_B \end{aligned} \tag{C.1.7}$$

Interpreting E^a as the vielbein, ω^{ab} as the spin connection, and ψ^a as the gravitino 1-form, (C.1.7) can be viewed as the structural equations of a supermanifold

$\mathrm{AdS}_{4|\mathscr{N}\times 4}$ extending anti de Sitter space with $\mathscr{N}$ Majorana supersymmetries. Indeed the gravitino 1-form is a Majorana spinor since, by construction, it satisfies the reality condition

$$C\overline{\psi}_A^T = \psi_A, \qquad \overline{\psi}_A \equiv \psi_A^\dagger \gamma_0 \tag{C.1.8}$$

The supermanifold $\mathrm{AdS}_{4|\mathscr{N}\times 4}$ can be identified with the following supercoset:

$$\mathscr{M}_{\mathrm{osp}}^{4|4\mathscr{N}} \equiv \frac{\mathrm{Osp}(\mathscr{N}|4)}{\mathrm{SO}(\mathscr{N}) \times \mathrm{SO}(1,3)} \tag{C.1.9}$$

Alternatively, the Maurer Cartan equations can be written in the following more compact form:

$$\begin{aligned} d\Delta^{xy} + \Delta^{xz} \wedge \Delta^{ty}\varepsilon_{zt} &= -4\mathrm{i}e\Phi_A^x \wedge \Phi_A^y, \\ d\mathscr{A}_{AB} - e\mathscr{A}_{AC} \wedge \mathscr{A}_{CB} &= 4\mathrm{i}\Phi_A^x \wedge \Phi_B^y \varepsilon_{xy} \\ d\Phi_A^x + \Delta^{xy} \wedge \varepsilon_{yz}\Phi_A^z - e\mathscr{A}_{AB} \wedge \Phi_B^x &= 0 \end{aligned} \tag{C.1.10}$$

where all 1-forms are real and, according to our conventions, the indices x, y, z, t are symplectic and take four values. The real symmetric bosonic 1-form $\Omega^{xy} = \Omega^{yx}$ encodes the generators of the Lie subalgebra $\mathfrak{sp}(4,\mathbb{R})$, while the antisymmetric real bosonic 1-form $\mathscr{A}_{AB} = -\mathscr{A}_{BA}$ encodes the generators of the Lie subalgebra $\mathfrak{so}(\mathscr{N})$. The fermionic 1-forms Φ_A^x are real and, as indicated by their indices, they transform in the fundamental 4-dim representation of $\mathfrak{sp}(4,\mathbb{R})$ and in the fundamental $\mathscr{N}$-dim representation of $\mathfrak{so}(\mathscr{N})$. Finally,

$$\varepsilon_{xy} = -\varepsilon_{yx} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \tag{C.1.11}$$

is the symplectic invariant metric.

The relation between the formulation (C.1.7) and (C.1.10) of the same Maurer Cartan equations is provided by the Majorana basis of $d = 4$ gamma matrices discussed in Appendix C.3.2. Using (A.5.2), the generators γ_{ab} and $\gamma_a\gamma_5$ of the anti de Sitter group SO(2, 3) turn out to be all given by real symplectic matrices, as is explicitly shown in (A.5.3) and the matrix $\mathscr{C}\gamma_5$ turns out to be proportional to ε_{xy} as shown in (C.3.7). On the other hand a Majorana spinor in this basis is proportional to a real object times a phase factor $\exp[-\pi\mathrm{i}/4]$.

Hence (C.1.7) and (C.1.10) are turned ones into the others upon the identifications:

$$\begin{aligned} \Omega^{xy}\varepsilon_{yz} \equiv \Omega^x{}_z &\leftrightarrow -\frac{1}{4}\omega^{ab}\gamma_{ab} - 2e\gamma_a\gamma_5 E^a \\ \mathscr{A}_{AB} &\leftrightarrow \mathscr{A}_{AB} \\ \psi_A^x &\leftrightarrow \exp\left[\frac{-\pi\mathrm{i}}{4}\right]\Phi_A^x \end{aligned} \tag{C.1.12}$$

As it is always the case, the Maurer Cartan equations are just a property of the (super) Lie algebra and hold true independently of the (super) manifold on which the 1-forms are realized: on the supergroup manifold or on different supercosets of the same supergroup.

C.2 The Relevant Supercosets and Their Relation

Let us also consider the following pure fermionic coset:

$$\mathscr{M}^{0|4\mathscr{N}}_{\text{osp}} = \frac{\text{Osp}(\mathscr{N}|4)}{\text{SO}(\mathscr{N}) \times \text{Sp}(4,\mathbb{R})} \tag{C.2.1}$$

There is an obvious relation between these two supercosets that can be formulated in the following way:

$$\mathscr{M}^{4|4\mathscr{N}}_{\text{osp}} \sim \text{AdS}_4 \times \mathscr{M}^{0|4\mathscr{N}}_{\text{osp}} \tag{C.2.2}$$

In order to explain the actual meaning of (C.2.2) we proceed as follows. Let the graded matrix $\mathbb{L} \in \text{Osp}(\mathscr{N}|4)$ be the coset representative of the coset $\mathscr{M}^{4|4\mathscr{N}}_{\text{osp}}$, such that the Maurer Cartan form Λ of (C.1.5) can be identified as:

$$\Lambda = \mathbb{L}^{-1}\, d\mathbb{L} \tag{C.2.3}$$

Let us now factorize $\mathbb{L}$ as follows:

$$\mathbb{L} = \mathbb{L}_F \mathbb{L}_B \tag{C.2.4}$$

where $\mathbb{L}_F$ is a coset representative for the coset:

$$\frac{\text{Osp}(\mathscr{N}|4)}{\text{SO}(\mathscr{N}) \times \text{Sp}(4,\mathbb{R})} \ni \mathbb{L}_F \tag{C.2.5}$$

and $\mathbb{L}_B$ is the $\text{Osp}(\mathscr{N}|4)$ embedding of a coset representative of AdS_4, namely:

$$\mathbb{L}_B = \left(\begin{array}{c|c} \text{L}_B & 0 \\ \hline 0 & \mathbf{1}_{\mathscr{N}} \end{array}\right); \qquad \frac{\text{Sp}(4,\mathbb{R})}{\text{SO}(1,3)} \ni \text{L}_B \tag{C.2.6}$$

In this way we find:

$$\Lambda = \mathbb{L}_B^{-1} \Lambda_F \mathbb{L}_B + \mathbb{L}_B^{-1}\, d\mathbb{L}_B \tag{C.2.7}$$

Let us now write the explicit form of Λ_F in analogy to (C.1.5):

$$\Lambda_F = \left(\begin{array}{c|c} \Delta_F & \Theta_A \\ \hline 4ie\overline{\Theta}_A\gamma_5 & -e\widetilde{\mathscr{A}}_{AB} \end{array}\right) \tag{C.2.8}$$

where Θ_A is a Majorana-spinor valued fermionic 1-form and where Δ_F is an $\mathfrak{sp}(4,\mathbb{R})$ Lie algebra valued 1-form presented as a 4×4 matrix. Both Θ_A as Δ_F and $\widetilde{\mathscr{A}}_{AB}$ depend only on the fermionic θ coordinates and differentials.

On the other hand we have:

$$\mathbb{L}_B^{-1}\, d\mathbb{L}_B = \left(\begin{array}{c|c} \Delta_B & 0 \\ \hline 0 & 0 \end{array}\right) \tag{C.2.9}$$

where the Ω_B is also an $\mathfrak{sp}(4,\mathbb{R})$ Lie algebra valued 1-form presented as a 4×4 matrix, but it depends only on the bosonic coordinates x^μ of the anti de Sitter space AdS_4. Indeed, according to (C.1.5) we can write:

$$\Delta_B = -\frac{1}{4} B^{ab} \gamma_{ab} - 2e \gamma_a \gamma_5 B^a \tag{C.2.10}$$

where $\{B^{ab}, B^a\}$ are respectively the spin-connection and the vielbein of AdS_4, just as $\{\mathscr{B}^{\alpha\beta}, \mathscr{B}^\alpha\}$ are the connection and vielbein of the internal coset manifold $\mathscr{M}_7$.

Inserting now these results into (C.2.7) and comparing with (C.1.5) we obtain:

$$\begin{aligned} \psi_A &= \mathrm{L}_B^{-1} \Theta_A \\ \mathscr{A}_{AB} &= \widetilde{\mathscr{A}}_{AB} \\ -\frac{1}{4}\omega^{ab}\gamma_{ab} - 2e\gamma_a\gamma_5 E^a &= -\frac{1}{4} B^{ab}\gamma_{ab} - 2e\gamma_a\gamma_5 B^a + \mathrm{L}_B^{-1}\Delta_F \mathrm{L}_B \end{aligned} \tag{C.2.11}$$

The above formulae encode an important information. They show how the super-vielbein and the superconnection of the supermanifold (C.1.9) can be constructed starting from the vielbein and connection of AdS_4 space plus the Maurer Cartan forms of the purely fermionic supercoset (C.2.1). In other words formulae (C.2.11) provide the concrete interpretation of the direct product (C.2.2). This will also be our starting point for the actual construction of the supergauge completion in the case of maximal supersymmetry and for its generalization to the cases of less super-symmetry.

C.2.1 Finite Supergroup Elements

We studied the $\mathfrak{osp}(\mathscr{N}|4)$ superalgebra but for our purposes we cannot confine ourselves to the superalgebra, we need also to consider finite elements of the corresponding supergroup. In particular the supercoset representative. Elements of the supergroup are described by graded matrices of the form:

$$M = \left(\begin{array}{c|c} A & \Theta \\ \hline \Pi & D \end{array}\right) \tag{C.2.12}$$

where A, D are submatrices made out of even elements of a Grassmann algebra while Θ, Π are submatrices made out of odd elements of the same Grassmann algebra. It is important to recall, that the operations of transposition and Hermitian conjugation are defined as follows on graded matrices:

$$M^T = \left(\begin{array}{c|c} A^T & \Pi^T \\ \hline -\Theta^T & D^T \end{array}\right)$$
$$M^\dagger = \left(\begin{array}{c|c} A^\dagger & \Pi^\dagger \\ \hline \Theta^\dagger & D^\dagger \end{array}\right) \tag{C.2.13}$$

This is done in order that the supertrace should preserve the same formal properties enjoyed by the trace of ordinary matrices:

$$\begin{aligned} \mathrm{Str}(M) &= \mathrm{Tr}(A) - \mathrm{Tr}(D) \\ \mathrm{Str}(M_1 M_2) &= \mathrm{Str}(M_2 M_1) \end{aligned} \tag{C.2.14}$$

Equations (C.2.13) and (C.2.14) have an important consequence. The consistency of the equation:

$$M^\dagger = \left(M^T\right)^\star \tag{C.2.15}$$

implies that the complex conjugate operation on a super matrix must be defined as follows:

$$M^\star = \left(\begin{array}{c|c} A^\star & -\Theta^\star \\ \hline \Pi^\star & D^\star \end{array}\right) \tag{C.2.16}$$

Let us now observe that in the Majorana basis which we have adopted we have:

$$\widehat{C} = \mathrm{i}\left(\begin{array}{c|c} \varepsilon & 0 \\ \hline 0 & -\frac{1}{4e}\mathbf{1}_{\mathcal{N}\times\mathcal{N}} \end{array}\right) = \mathrm{i}\hat{\varepsilon}$$
$$\widehat{H} = \left(\begin{array}{c|c} \mathrm{i}\varepsilon & 0 \\ \hline 0 & -\frac{1}{4e}\mathbf{1}_{\mathcal{N}\times\mathcal{N}} \end{array}\right) \tag{C.2.17}$$

where the 4×4 matrix ε is given by (C.3.7). Therefore in this basis an orthosymplectic group element $\mathbb{L} \in \mathrm{OSp}(\mathcal{N}|4)$ which satisfies:

$$\mathbb{L}^T \widehat{C} \mathbb{L} = \widehat{C} \tag{C.2.18}$$
$$\mathbb{L}^\dagger \widehat{H} \mathbb{L} = \widehat{H} \tag{C.2.19}$$

has the following structure:

$$\mathbb{L} = \left(\begin{array}{c|c} \mathcal{S} & \exp[-\mathrm{i}\frac{\pi}{4}]\Theta \\ \hline \exp[-\mathrm{i}\frac{\pi}{4}]\Pi & \mathcal{O} \end{array}\right) \tag{C.2.20}$$

where the bosonic sub-blocks $\mathcal{S}$, $\mathcal{O}$ are respectively 4×4 and $\mathcal{N} \times \mathcal{N}$ and real, while the fermionic ones Θ, Π are respectively $4 \times \mathcal{N}$ and $\mathcal{N} \times 4$ and also real.

The orthosymplectic conditions (C.2.18) translate into the following conditions on the sub-blocks:

$$\begin{aligned}\mathscr{S}^T\varepsilon\mathscr{S} &= \varepsilon - \mathrm{i}\frac{1}{4e}\Pi^T\Pi \\ \mathscr{O}^T\mathscr{O} &= \mathbf{1} + \mathrm{i}4e\Theta^T\varepsilon\Theta \\ \mathscr{S}^T\varepsilon\Theta &= -\frac{1}{4e}\Pi^T\mathscr{O}\end{aligned} \tag{C.2.21}$$

As we see, when the fermionic off-diagonal sub-blocks are zero the diagonal ones are respectively a symplectic and an orthogonal matrix.

If the graded matrix $\mathbb{L}$ is regarded as the coset representative of either one of the two supercosets (C.1.9), (C.2.1), we can evaluate the explicit structure of the left-invariant one form Λ. Using the $\mathscr{M}^{0|4\times\mathscr{N}}$ style of the Maurer Cartan equations (C.1.10) we obtain:

$$\Lambda \equiv \mathbb{L}^{-1}d\mathbb{L} = \left(\begin{array}{c|c}\Delta & \exp[-\mathrm{i}\frac{\pi}{4}]\Phi \\ \hline -4e\exp[-\mathrm{i}\frac{\pi}{4}]\Phi^T\varepsilon & -e\mathscr{A}\end{array}\right) \tag{C.2.22}$$

where the 1-forms Δ, $\mathscr{A}$ and Φ can be explicitly calculated, using the explicit form of the inverse coset representative:

$$\mathbb{L}^{-1} = \left(\begin{array}{c|c}-\varepsilon\mathscr{S}^T\varepsilon & \exp[-\mathrm{i}\frac{\pi}{4}]\frac{1}{4e}\varepsilon\Pi^T \\ \hline -\exp[-\mathrm{i}\frac{\pi}{4}]4e\Theta^T\varepsilon & \mathscr{O}^T\end{array}\right) \tag{C.2.23}$$

$$\begin{aligned}e\mathscr{A} &= -\mathscr{O}^T d\mathscr{O} - \mathrm{i}4e\Theta^T\varepsilon\, d\Theta \\ \Omega &= -\varepsilon\mathscr{S}^T\varepsilon\, d\mathscr{S} - \mathrm{i}\frac{1}{4e}\Pi^T d\Pi \\ \Phi &= -\varepsilon S^T\varepsilon\, d\Theta + \frac{1}{4e}\varepsilon\Pi^T d\mathscr{O}\end{aligned} \tag{C.2.24}$$

C.2.2 The Coset Representative of OSp($\mathscr{N}$|4)/SO($\mathscr{N}$) × Sp(4)

It is fairly simple to write an explicit form for the coset representative of the fermionic supermanifold

$$\mathscr{M}^{0|4\times\mathscr{N}} = \frac{\mathrm{OSp}(\mathscr{N}|4)}{\mathrm{Sp}(4,\mathbb{R})\times\mathrm{SO}(\mathscr{N})} \tag{C.2.25}$$

by adopting the upper left block components Θ of the supermatrix (C.2.20) as coordinates. It suffices to solve (C.2.21) for the sub blocks $\mathscr{S}$, $\mathscr{O}$, Π. Such an explicit solution is provided by setting:

$$\begin{aligned}\mathscr{O}(\Theta) &= \left(\mathbf{1} + 4\mathrm{i}e\Theta^T\varepsilon\Theta\right)^{1/2} \\ \mathscr{S}(\Theta) &= \left(\mathbf{1} + 4\mathrm{i}e\Theta\Theta^T\varepsilon\right)^{1/2} \\ \Pi &= 4e\left(\mathbf{1} + 4\mathrm{i}e\Theta^T\varepsilon\Theta\right)^{-1/2}\Theta^T\varepsilon\left(\mathbf{1} + 4\mathrm{i}e\Theta\Theta^T\varepsilon\right)^{1/2} \\ &= 4e\Theta^T\varepsilon\end{aligned} \tag{C.2.26}$$

In this way we conclude that the coset representative of the fermionic supermanifold (C.2.25) can be chosen to be the following supermatrix:

$$\mathbb{L}(\Theta) = \left(\begin{array}{c|c} (\mathbf{1} + 4\mathrm{i}e\Theta\Theta^T\varepsilon)^{1/2} & \exp[-\mathrm{i}\frac{\pi}{4}]\Theta \\ \hline -\exp[-\mathrm{i}\frac{\pi}{4}]4e\Theta^T\varepsilon & (\mathbf{1} + 4\mathrm{i}e\Theta^T\varepsilon\Theta)^{1/2} \end{array}\right) \tag{C.2.27}$$

By straightforward steps from (C.2.23) we obtain the inverse of the supercoset element (C.2.27) in the form:

$$\mathbb{L}^{-1}(\Theta) = \mathbb{L}(-\Theta) = \left(\begin{array}{c|c} (\mathbf{1} + 4\mathrm{i}e\Theta\Theta^T\varepsilon)^{1/2} & -\exp[-\mathrm{i}\frac{\pi}{4}]\Theta \\ \hline -\exp[-\mathrm{i}\frac{\pi}{4}]4e\Theta^T\varepsilon & (\mathbf{1} + 4\mathrm{i}e\Theta^T\varepsilon\Theta)^{1/2} \end{array}\right) \tag{C.2.28}$$

Correspondingly we work out the explicit expression of the Maurer Cartan forms:

$$\begin{aligned} e\mathscr{A} &= \left(\mathbf{1} + 4\mathrm{i}e\Theta^T\varepsilon\Theta\right)^{1/2} d\left(\mathbf{1} + 4\mathrm{i}e\Theta^T\varepsilon\Theta\right)^{1/2} - \mathrm{i}4e\Theta^T\varepsilon\, d\Theta \\ \Phi &= \left(\mathbf{1} + 4\mathrm{i}e\Theta\Theta^T\varepsilon\right)^{1/2} d\Theta + \Theta\, d\left(\mathbf{1} + 4\mathrm{i}e\Theta^T\varepsilon\Theta\right)^{1/2} \\ \Delta &= \left(\mathbf{1} + 4\mathrm{i}e\Theta\Theta^T\varepsilon\right)^{1/2} d\left(\mathbf{1} + 4\mathrm{i}e\Theta\Theta^T\varepsilon\right)^{1/2} - \mathrm{i}4e\Theta\, d\Theta^T\,\varepsilon \end{aligned} \tag{C.2.29}$$

C.3 *$D = 6$ and $D = 4$ Gamma Matrix Bases*

In the discussion of the $\mathrm{AdS}_4 \times \mathbb{P}^3$ compactification we need to consider the decomposition of the $D = 10$ gamma matrix algebra into the tensor product of the $\mathfrak{so}(6)$ Clifford algebra times that of $\mathfrak{so}(1,3)$. In this section we discuss an explicit basis for the $\mathfrak{so}(6)$ gamma matrix algebra using that of $\mathfrak{so}(7)$. Conventionally we identify the 7-matrix τ_7 with the chirality matrix in $d = 6$.

C.3.1 $D = 6$ Clifford Algebra

In this section, the indices $\alpha, \beta, \dots$ run on six values and denote the vector indices of $\mathfrak{so}(6)$. In order to discuss the gamma matrix basis we introduce $\mathfrak{so}(7)$ indices

$$\overline{\alpha} = \alpha, 7 \tag{C.3.1}$$

which run on seven values and we define the Clifford algebra with negative metric:

$$\{\tau_{\overline{\alpha}}, \tau_{\overline{\beta}}\} = -\delta_{\overline{\alpha\beta}} \tag{C.3.2}$$

This algebra is satisfied by the following, real, antisymmetric matrices:

$$
\tau_1 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} \; ; \quad
\tau_2 = \begin{pmatrix}
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

$$
\tau_3 = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} \; ; \quad
\tau_4 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0
\end{pmatrix}
\tag{C.3.3}
$$

$$
\tau_5 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0
\end{pmatrix} \; ; \quad
\tau_6 = \begin{pmatrix}
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0
\end{pmatrix}
$$

$$
\tau_7 = \begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0
\end{pmatrix}
$$

C.3.2 $D = 4$ γ-Matrix Basis and Spinor Identities

In this section we construct a basis of $\mathfrak{so}(1,3)$ gamma matrices such that it explicitly realizes the isomorphism $\mathfrak{so}(2,3) \sim \mathfrak{sp}(4,\mathbb{R})$ with the conventions used in the main text. Naming σ_i the standard Pauli matrices:

$$
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ; \qquad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} ; \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{C.3.4}
$$

we realize the $\mathfrak{so}(1,3)$ Clifford algebra:

$$\{\gamma_a, \gamma_b\} = 2\eta_{ab}; \qquad \eta_{ab} = \mathrm{diag}(+,-,-,-) \tag{C.3.5}$$

by setting:

$$\begin{aligned} \gamma_0 &= \sigma_2 \otimes \mathbf{1}; & \gamma_1 &= \mathrm{i}\sigma_3 \otimes \sigma_1 \\ \gamma_2 &= \mathrm{i}\sigma_1 \otimes \mathbf{1}; & \gamma_3 &= \mathrm{i}\sigma_3 \otimes \sigma_3 \\ \gamma_5 &= \sigma_3 \otimes \sigma_2; & \mathscr{C} &= \mathrm{i}\sigma_2 \otimes \mathbf{1} \end{aligned} \tag{C.3.6}$$

where γ_5 is the chirality matrix and $\mathscr{C}$ is the charge conjugation matrix. Making now reference to (C.1.2) and (C.1.3) of the main text we see that the antisymmetric matrix entering the definition of the orthosymplectic algebra, namely $\mathscr{C}\gamma_5$ is the following one:

$$\mathscr{C} = \mathrm{i}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \qquad \mathscr{C}\gamma_5 = \varepsilon = \mathrm{i}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \tag{C.3.7}$$

namely it is proportional, through an overall i-factor, to a real completely off-diagonal matrix. On the other hand all the generators of the $\mathfrak{so}(2,3)$ Lie algebra, *i.e.* γ_{ab} and $\gamma_a\gamma_5$ are real, symplectic 4×4 matrices. Indeed we have

$$\begin{aligned}
\gamma_{01} &= \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}; & \gamma_{02} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \\
\gamma_{12} &= \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}; & \gamma_{13} &= \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \\
\gamma_{23} &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}; & \gamma_{34} &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \\
\gamma_0\gamma_5 &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}; & \gamma_1\gamma_5 &= \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
\gamma_2\gamma_5 &= \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}; & \gamma_3\gamma_5 &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}
\end{aligned} \tag{C.3.8}$$

On the other hand we find that $\mathscr{C}\gamma_0 = \mathrm{i}\mathbf{1}$. Hence the Majorana condition becomes:

$$\mathrm{i}\psi = \psi^\star \tag{C.3.9}$$

so that a Majorana spinor is just a real spinor multiplied by an overall phase $\exp[-i\frac{\pi}{4}]$.

These conventions being fixed let χ_x $(x = 1, \dots, 4)$ be a set of (commuting) Majorana spinors normalized in the following way:

$$\begin{aligned} \chi_x &= \mathscr{C}\overline{\chi}_x^T; \quad \text{Majorana condition} \\ \overline{\chi}_x\gamma_5\chi_y &= \mathrm{i}(\mathscr{C}\gamma_5)_{xy}; \quad \text{symplectic normal basis} \end{aligned} \tag{C.3.10}$$

Then by explicit evaluation we can verify the following Fierz identity:

$$\frac{1}{2}\gamma^{ab}\chi_z\overline{\chi}_x\gamma_5\gamma_{ab}\chi_y - \gamma_a\gamma_5\chi_z\overline{\chi}_x\gamma_a\chi_y = -2\mathrm{i}\big[(C\gamma_5)_{zx}\chi_y + (C\gamma_5)_{zy}\chi_x\big] \tag{C.3.11}$$

Another identity which we can prove by direct evaluation is the following one:

$$\begin{aligned} &\overline{\chi}_x\gamma_5\gamma_{ab}\chi_y\overline{\chi}_z\gamma^b\chi_t - \overline{\chi}_z\gamma_5\gamma_{ab}\chi_t\overline{\chi}_x\gamma^b\chi_y \\ &\quad = \mathrm{i}\big(\overline{\chi}_x\gamma_a\chi_t(\mathscr{C}\gamma_5)_{yz} + \overline{\chi}_y\gamma_a\chi_t(\mathscr{C}\gamma_5)_{xz} + \overline{\chi}_x\gamma_a\chi_z(\mathscr{C}\gamma_5)_{yt} + \overline{\chi}_y\gamma_a\chi_z(\mathscr{C}\gamma_5)_{xt}\big) \end{aligned} \tag{C.3.12}$$

Finally let us mention some relevant formulae for the derivation of the $\mathrm{AdS}_4 \times \mathbb{P}^3$ compactification. With the above conventions we find:

$$\gamma_0\gamma_1\gamma_2\gamma_3 = \mathrm{i}\gamma_5 \tag{C.3.13}$$

and if we fix the convention:

$$\varepsilon_{0123} = +1 \tag{C.3.14}$$

we obtain:

$$\frac{1}{24}\varepsilon^{abcd}\gamma_a\gamma_b\gamma_c\gamma_d = -\mathrm{i}\gamma_5 \tag{C.3.15}$$

C.4 An $\mathfrak{so}(6)$ Inversion Formula

In order to discuss the conversion of supergravity forms into MC forms of the super-coset a key role is played by an inversion formula which we utilize in the main text and we discuss in this appendix. Let us define the following set of 6×6 matrices:

$$\begin{aligned} \overline{\tau}^\alpha_{AB} &\equiv \eta_A^T\tau^\alpha\eta_B \\ \overline{\tau}^{\alpha\beta}_{AB} &= \eta_A^T\tau^{\alpha\beta}\eta_B \end{aligned} \tag{C.4.1}$$

$$K_{AB} = \mathscr{K}_{AB} = \frac{1}{2}\mathscr{K}_{\alpha\beta}\overline{\tau}^{\alpha\beta}_{AB}.$$

where η_A are the 6 internal Killing spinors and τ denote the 1-index and 2-index $\mathfrak{so}(6)$ gamma-matrices. By construction the barred $\overline{\tau}$s are antisymmetric 6×6 matrices, hence $\mathfrak{so}(6)$ generators in the fundamental representation just as the Kähler form K. Counting these matrices we find that they are $6+15+1$, namely 22, which is too much as a set of independent generators of $\mathfrak{so}(6)$. This means that there must be linear dependences. By calculating traces of these matrices we find that the 6 matrices $\overline{\tau}^{\alpha}$ are linear independent and orthogonal to the 15, $\overline{\tau}^{\alpha\beta}$, and to the unique K while among these latter 16 matrices only 9 are linear independent.

This observation is important for the following reason. When we write the following formulae:

$$\begin{aligned} \Delta\mathscr{B}^{\alpha} &= -\frac{1}{8}\overline{\tau}^{\alpha}_{AB}\mathscr{A}^{AB} \\ \Delta\mathscr{B}^{\alpha\beta} &= \frac{e}{4}\overline{\tau}^{\alpha\beta}_{AB}\mathscr{A}^{AB} - \frac{e}{4}\mathscr{K}^{\alpha\beta}K_{AB}\mathscr{A}^{AB} \end{aligned} \tag{C.4.2}$$

we are actually decomposing the $\mathfrak{so}(6)$ connection $\mathscr{A}^{AB}$ along an over-complete basis of $15+6=21$ generators of $\mathfrak{so}(6)$, which is obviously a well defined operation.

It is interesting to establish the inverse formula, namely to express the original connection $\mathscr{A}^{AB}$ in terms of the over complete set of objects ΔB^{α} and $\Delta B^{\alpha\beta}$. The inverse formula can be established by means of direct calculation in the explicit τ-matrix basis we have chosen and we find what follows:

$$\mathscr{A}_{AB} = \left(-2\Delta\mathscr{B}^{\alpha}\overline{\tau}_{\alpha} + \frac{1}{4e}\Delta\mathscr{B}^{\alpha\beta}\overline{\tau}_{\alpha\beta} - \frac{1}{4e}\Delta B^{\alpha\beta}\mathscr{K}_{\alpha\beta}K\right)_{AB} \tag{C.4.3}$$

Appendix D: MATHEMATICA Package NOVAMANIFOLDA

In this section we describe the MATHEMATICA Package *NOVAMANIFOLDA* that can be downloaded as supplementary material form the Springer distribution site.

This notebook contains various packages for the calculation of the spin connection and the curvatures of various manifolds, both homogeneous (= cosets) and also non-homogeneous. It is divided in various sections.

Coset Manifolds (Euclidian Signature)

Instructions for the Use

This notebook has the following purpose, that of calculating the Riemann tensor and the connection of the several coset manifolds. In particular:

(1) The manifold:
$\frac{\mathrm{SU}(3)}{\mathrm{SU}(2)\times\mathrm{U}(1)} = \mathrm{CP}_2$

(2) The spheres:
$\frac{\mathrm{SO}(m+1)}{\mathrm{SO}(m)} = S_m$

(3) The manifold:
$\frac{\mathrm{SU}(3)}{U(1)} = N^{010}$

The calculation is done using the RUNCOSET package constructed by Prof. Leonardo Castellani. The input are the structure constants of the corresponding group that are calculated by suitable routines inserted in this package.
First read the two sections of PROGRAMME
and then start by the command
start
If you want to calculate the structure constants for CP2, spheres or N010 you just type:
cp2stru, spheres or n010stru
and then initialize the RUNCOSET programme by the command
initial
then supply the file cc=fff
and you can calculate with the commands of RUNCOSET
that are described in the section below

Description of the Main Commands of RUNCOSET

The available commands one can use at this point are the following ones
1.
doriemann2
This command generates as an output a tensor **Rie[[a,b,c,d]]** = $(R^{\mathrm{ab}})_{\mathrm{cd}}$ where $(R^{\mathrm{ab}})_{\mathrm{cd}}$ is the Riemann tensor in the conventions of the old Kaluza-Klein literature, namely Universal mass relations Ann. of Phys. 162, (1985) 372 by D'Auria and Frè.
2.
doconnection
This command generates as an output a tensor **connten[[a,b]]** = B^{ab} where B^{ab} is the spin connection 1-form in the conventions of the old Kaluza-Klein literature, namely Universal mass relations Ann. of Phys. 162, (1985) 372 by D'Auria and Frè.
3.
doconcomp
This command generates as an output a tensor **contor[[c,a,b]]** = $(B^{\mathrm{ab}})_c$ where $(B^{\mathrm{ab}})_c$ is the torsion part of the spin connection 1-form in the conventions of the old Kaluza-Klein literature, namely Universal mass relations Ann. of Phys. 162, (1985) 372 by D'Auria and Frè.

4.
doricci
This command generates as an output a tensor **ricten[[a,b]]** $=R^{\text{ca}}_{\text{cb}}$ where $R^{\text{ca}}_{\text{cb}}$ is the Ricci tensor in the conventions of the old Kaluza-Klein literature, namely Universal mass relations Ann. of Phys. 162, (1985) 372 by D'Auria and Frè.
5.
docurvaform
This command generates the curvature 2-form once the Riemann tensor has been generated
Index ordering
The index ordering is as follows:
A=(a,i); a=1,.....,dim G/H; i=dim G/H+1,........,dim G
The index a enumerates the coset directions, while the index i enumerates the H subalgebra directions.

Structure Constants for CP2

This programme calculates the generators and the structure constants of SU(3) in such a way that the first 4 generators are those of the coset:
$\frac{\text{SU}(3)}{\text{SU}(2)\times U(1)}$

Spheres

This programme calculates the generators and the structure constants of the group SO($n+1$) and orders them in such a way that the first n generators are those of the coset:
$\frac{\text{SO}(n+1)}{\text{SO}(n)}$

N010 Coset

This programme generates the structure constants of SU(3) but in such an order that the first 7 generators are those of the coset
$\frac{\text{SU}(3)}{U(1)}$
U(1) being generated by the 8th Gell Mann matrix

RUNCOSET Package (Euclidian Signature)

This is a new package based on the package "Cosets" that was written by Leonardo Castellani and which computes the Riemann tensor, Weyl tensor and the spin connection for G/H manifolds. The package has been modified and adapted to the way we want to perform our calculations.

Geometry of Quasi-Homogeneous Manifolds (Euclidian, or Lorentzian) and of General Manifolds (General Signature)

This routine is devised to calculate the geometry in the following very common situation where the vielbein of a space in dimension
n=1+r
is given in the following form:
e^1=dμ
$e^i=f_i\,(\mu)^i\,\sigma^i$
where $f_i(\mu)$ are functions of the coordinate μ and σ^i are vielbeins of an r-dimensional space for which the contorsion is already known:
$d\sigma^i = - t^i_{\text{jk}}\;\sigma^j\;\wedge\sigma^k$
We name such manifolds quasi-homogeneous

Main

You start this programme by typing **mainspin**

Spin Connection and Curvature Routines

This routine is devised to calculate the intrinsic components of the spin connection once the contorsion tensor as already been calculated
$de^i = c^i_{\text{jk}}\;e^i \wedge e^k$
There are two versions of the programme one for quasi-homogeneous manifolds called **spinpack** and one for general manifolds called **spinpackgen** in the second case you will be prompted to supply also the signature of space-time a n vector of plus or minus 1s = **signat**

Routine Curvapack

This routine is devised to calculate the curvature of a manifold when the intrinsic components of the spin connection depend only on one coordinate μ and the first vielbein is
e_1= A(μ)*dμ

Routine Curvapackgen

This routine is devised to calculate the curvature two form and the Riemann tensor in a general situation for an arbitrary dimensional manifold and with the vielbein depending on all the coordinates. You can start this programme only after having computed the spin connection via the package **spinpack**

Contorsion Routine for Mixed Vielbeins

This routine is devised to calculate the contorsion tensor in the following very common situation where the vielbein of a space in dimension
n=1+r
is given in the following form:
e^1=A(μ)dμ
$e^i = f_i(\mu)\,\sigma^i$
where $f_i(\mu)$ are functions of the coordinate μ and σ^i are vielbeins of an r-dimensional space for which the contorsion is already known:
$d\sigma^i = -\,t^i_{\mathrm{jk}}\,\sigma^j \wedge \sigma^k$
So that we find:
$c^1_{\mathrm{AB}}=0$
$c^i_{1i}=\frac{1}{2}\,\frac{1}{A(\mu)}\,\partial_\mu\,\mathrm{Log}[f_i(\mu)]$
$c^i_{\mathrm{jk}}= -\,\frac{f_i(\mu)}{f_j(\mu)\,f_k(\mu)}\,t^i_{\mathrm{jk}}$
TO START THE ROUTINE TYPE **contors**

Calculation of the Contorsion for General Manifolds

This routine is devised to calculate the contorsion for general manifolds. The inputs are
1) the dimension n
2) the set of coordinates a n vector = **coordi**
3) the set of differentials, a nnn vector = **diffe**
4) the set of vielbein 1-forms a nnn vector = **fform**
TO START this programme you type **contorgen** and then you follow instructions

Calculation for Cartan Maurer Equations and Vielbein Differentials (Euclidian Signature)

This package is devised to calculate the exterior differential of a set of 1-forms, for instance vielbeins or Cartan Maurer 1-forms.
This part is initialized by typing **extdiff** then you follow the computer instructions.

SO(3), SO(4) t' Hooft Matrices and Euler Angles

This programme is devised to calculate the differential 1 forms on the 3 sphere in terms of the Euler angles, introducing the self-dual and antiself-dual generators of the SO(4) group, namely the 't Hooft matrices. This calculation relies on the routine **spheres** belonging to another section of this notebook.

YOU START THIS PROGRAMME by TYPING **eulerus**. When you will prompted for the sphere dimension you have to type 3.
The output 1-forms are encoded in two 3-vectors named **sigmap** and **sigmam**, respectively.

Routine Thoft

Running **thoft** one generates the 't Hooft matrices. The self-dual ones are named Jp1, Jp2, Jp3, the antiself-dual ones are named Jm1, Jm2, Jm3.

AdS Space in Four Dimensions (Minkowski Signature)

The routines of this section are deviced to calculate the algebra of SO(2, 3) the solvable parameterization of anti de Sitter space in four dimensions, to construct its Killing vectors and make various other checks. If you want to run the entire package and see what it does you can just type: **mainads4**

Lie Algebra of SO(2, 3) and Killing Metric

In this section one defines the 10 generators of the SO(2, 3) Lie algebra
K0, K1, K2, K3, N1, N2, N3, J1, J2, J3
where J are the SO(3) rotations, N the three Lorentz boosts and K0, K1, K2, K3 the 4 translations generators.
The routine is activated by typing: **algso23**

Solvable Subalgebra Generating the Coset and Construction of the Vielbein

This routine constructs the coset representative in solvable parameterization, the vierbein and the structure constants of the full Lie algebra.
The routine is activated by typing: **cosetto**.
To run this routine you need first to run algso23

Killing Vectors

This programme verifies the Killing vectors in the solvable parameterization and then exhibits them explicitly.
The routine is initialized by typing: **verkilling**
To run this routine you need to run first also23 and cosetto

Trigonometric Coordinates

In this section we turn to trigonometric coordinates in which the metric of AdS space has the following form:
ds^2= - dτ^2 +Cos[τ]^2[dλ^2+Sinh[λ]^2(dα^2+Sin[α]^{2}dβ^2)]

Test of Killing Vectors

This routine is devised to test whether a set of vector fields are Killing vectors for a given metric.
The inputs to be given before running the routine are:
ggmunu = *metric as n x n matrix*;
xmu = set of coordinates as an n-vector;
dxmu = set of coordinate differentials as an n-vector;
derdxmu = set of coordinate derivatives as an n-vector;
killus = set of Killing vectors to be tested
dim = dimension of manifold
kilnu = number of Killing vectors;
the routine is than activated by typing **testkillus**

Appendix E: Examples of the Use of the Package NOVAMANIFOLDA

In this appendix we describe some applications of the package *Novamanifolda*. The MATHEMATICA notebook file with these examples can be downloaded as supplementary material from the Springer distribution site.

MANIFOLDPROVA

In this Notebook we display some examples of the use of the package NOVAMANIFOLDA. Obviously you have to evaluate first the NoteBook Novamanifolda.

The 4-Dimensional Coset CP^2

We initialize the programme

start
{Null}

We calculate the structure constant of the SU(3) Lie algebra

cp2stru
{Null}

The result of this calculation is a tensor named **fff** and stored in the computer memory (if you wanted another group, you had to calculate the structure constants of its Lie algebra and store them in a tri-tensor named also **fff**. It is important that in ordering the generators the first dim G/H should correspond to the coset generators, while the late dim H should correspond to the stability subgroup H generators)

Next we initial the RUNCOSET programme

initial
===
Welcome to RUNCOSET, a new package built by Petrus
on Leonardus technology
It computes various geometric quantities of G/H cosets
Please insert the dimensions of the group G and
of the coset G/H

Now you need to provide the structure constants of the group
and the rescalings
The structure constants must be given as a tensor cc[[A,B,C]];
The rescalings must be given as r[1]=?, r[2]=?...

{Null}

We supply the calculated SU(3) structure constants
cc = fff;

we calculate the spin connection one-form for this coset.

doconnection
12 non-zero
13 non-zero
14 non-zero
21 non-zero
23 non-zero
24 non-zero
31 non-zero
32 non-zero
34 non-zero
41 non-zero
42 non-zero
43 non-zero
I have finished the calculation
The tensor connten[[a,b]] giving the formal expression
of the spin connection B[a,b] as a 1-form

is ready for storing on hard disk
Store it in your preferred directory with the name you choose

———————————-

{Null}

We display the result of this calculation. In the formula below om[i], (i=1,. . . ,8) denote the Maurer Cartan one-forms of the SU(3) group ordered and normalized according to the conventions used for the structure constants.

MatrixForm[connten]

$$\begin{pmatrix} 0 & \frac{\omega_7}{2} & \frac{1}{2}(\sqrt{3}\omega_5+\omega_6) & \frac{\omega_8}{2} \\ -\frac{\omega_7}{2} & 0 & \frac{\omega_8}{2} & \frac{1}{2}(\sqrt{3}\omega_5-\omega_6) \\ \frac{1}{2}(-\sqrt{3}\omega_5-\omega_6) & -\frac{\omega_8}{2} & 0 & \frac{\omega_7}{2} \\ -\frac{\omega_8}{2} & \frac{1}{2}(-\sqrt{3}\omega_5+\omega_6) & -\frac{\omega_7}{2} & 0 \end{pmatrix}$$

Let us now insert the rescaling factors r[i]. These are as many as there are irreducible representations of H in the complementary subspace in the decomposition G=H⊕K. For CP2 the 4 coset generators span just one irreducible representation of the su(2)×u(1) Lie algebra, hence there is only one scaling factor.

$\boldsymbol{r[1]=\lambda; r[2]=\lambda; r[3]=\lambda; r[4]=\lambda}$

Next we compute the torsion of the coset

doconncomp

————————————-

Now I calculate the torsion part of the spin connection
I have finished the calculation
The tensor contor[a,b]] giving the torsion part B[c,a,b]
of the spin connection B[a,b]
is ready for storing on hard disk
Store it in your preferred directory with the name you choose

———————————-

{Null}

The CP2 coset is symmetric and torsionless and this is indeed verified by the computer

contor

{{{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}},
{{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}},
{{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}},
{{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}}}

Then we calculate the Riemann tensor

doriemann2

Now I calculate the Riemann tensor Rie(a,b,c,d)
121 2 non-zero
122 1 non-zero
123 4 non-zero
124 3 non-zero
131 3 non-zero
132 4 non-zero
133 1 non-zero
134 2 non-zero
141 4 non-zero
142 3 non-zero
143 2 non-zero
144 1 non-zero
211 2 non-zero
212 1 non-zero
213 4 non-zero
214 3 non-zero
231 4 non-zero
232 3 non-zero
233 2 non-zero
234 1 non-zero
241 3 non-zero
242 4 non-zero
243 1 non-zero
244 2 non-zero
311 3 non-zero
312 4 non-zero
313 1 non-zero
314 2 non-zero
321 4 non-zero
322 3 non-zero
323 2 non-zero
324 1 non-zero
341 2 non-zero
342 1 non-zero
343 4 non-zero
344 3 non-zero
411 4 non-zero
412 3 non-zero
413 2 non-zero
414 1 non-zero
421 3 non-zero
422 4 non-zero

423 1 non-zero
424 2 non-zero
431 2 non-zero
432 1 non-zero
433 4 non-zero
434 3 non-zero
I have finished the calculation
The tensor Rie(a,b,c,d) is ready for storing on hard disk
Store it in your preferred directory with the name you choose

---

Now I evaluate the curvature 2-form of your space
I find the following answer
R[12] = $2(\frac{1}{8}\lambda^2 e_1{**}e_2 + \frac{1}{8}\lambda^2 e_3{**}e_4)$
R[13] = $2(\frac{1}{2}\lambda^2 e_1{**}e_3 + \frac{1}{4}\lambda^2 e_2{**}e_4)$
R[14] = $2(\frac{1}{8}\lambda^2 e_1{**}e_4 + \frac{1}{8}\lambda^2 e_2{**}e_3)$
R[23] = $2(\frac{1}{8}\lambda^2 e_1{**}e_4 + \frac{1}{8}\lambda^2 e_2{**}e_3)$
R[24] = $2(\frac{1}{4}\lambda^2 e_1{**}e_3 + \frac{1}{2}\lambda^2 e_2{**}e_4)$
R[34] = $2(\frac{1}{8}\lambda^2 e_1{**}e_2 + \frac{1}{8}\lambda^2 e_3{**}e_4)$
The result is encoded in a tensor RR[i,j]
Its components are encoded in a tensor Rie[i,j,a,b]
{Null}

and we calculate the explicit form of the curvature two-form

docurvaform

I evaluate the curvature 2-form of your coset
I find the following answer
R[12] = $\frac{1}{8}\lambda^2 V_1{**}V_2 - \frac{1}{8}\lambda^2 V_2{**}V_1 + \frac{1}{8}\lambda^2 V_3{**}V_4 - \frac{1}{8}\lambda^2 V_4{**}V_3$
R[13] = $\frac{1}{2}\lambda^2 V_1{**}V_3 + \frac{1}{4}\lambda^2 V_2{**}V_4 - \frac{1}{2}\lambda^2 V_3{**}V_1 - \frac{1}{4}\lambda^2 V_4{**}V_2$
R[14] = $\frac{1}{8}\lambda^2 V_1{**}V_4 + \frac{1}{8}\lambda^2 V_2{**}V_3 - \frac{1}{8}\lambda^2 V_3{**}V_2 - \frac{1}{8}\lambda^2 V_4{**}V_1$
R[23] = $\frac{1}{8}\lambda^2 V_1{**}V_4 + \frac{1}{8}\lambda^2 V_2{**}V_3 - \frac{1}{8}\lambda^2 V_3{**}V_2 - \frac{1}{8}\lambda^2 V_4{**}V_1$
R[24] = $\frac{1}{4}\lambda^2 V_1{**}V_3 + \frac{1}{2}\lambda^2 V_2{**}V_4 - \frac{1}{4}\lambda^2 V_3{**}V_1 - \frac{1}{2}\lambda^2 V_4{**}V_2$
R[34] = $\frac{1}{8}\lambda^2 V_1{**}V_2 - \frac{1}{8}\lambda^2 V_2{**}V_1 + \frac{1}{8}\lambda^2 V_3{**}V_4 - \frac{1}{8}\lambda^2 V_4{**}V_3$
Now choose a value for the rescaling parameters
writing rullina = {....}
Then type redisplay
{Null}

Finally we calculate the Ricci tensor

doricci

Now I calculate the Ricci tensor
11 non-zero
22 non-zero
33 non-zero
44 non-zero
I have finished the calculation
The tensor ricten[[a,b]] giving the Ricci tensor
is ready for storing on hard disk
Store it in your preferred directory with the name you choose

---

{Null}
MatrixForm[ricten]

$$\begin{pmatrix} \frac{3\lambda^2}{4} & 0 & 0 & 0 \\ 0 & \frac{3\lambda^2}{4} & 0 & 0 \\ 0 & 0 & \frac{3\lambda^2}{4} & 0 \\ 0 & 0 & 0 & \frac{3\lambda^2}{4} \end{pmatrix}$$

The above expression of the Ricci tensor is provided in the flat indices. Indeed the Ricci tensor is the trace of the Riemann tensor calculated in the flat basis as components of the curvature two-form along the vielbein basis.

Calculation of the (Pseudo-)Riemannian Geometry of a Kasner Metric in Vielbein Formalism

As an example of calculation of the pseudo-Riemannian geometry in vielbein formalism we consider in this section the case of a Kasner cosmological metric

We initialize this general package by typing contorgen

contorgen
Give me the dimension of your space
Your space has dimension n = 4
Now I stop and you give me three vectors of dimension 4
vector fform = vector of 1-form vielbeins
vector coordi = vector of coordinates
vector diffe = vector of differentials
Then resume the calculation typing: contorgenresume
If you already have the contorsion type
spinpackgen
{Null}

Next we supply the information required by the computer, namely, vielbein, coordinates and coordinate differentials

fform $= \{\mathrm{dt}, s_1[t]\mathrm{dx}_1, s_2[t]\mathrm{dx}_2, s_3[t]\mathrm{dx}_3\}$;
coordi $= \{t, x_1, x_2, x_3\}$;
diffe $= \{\mathrm{dt}, \mathrm{dx}_1, \mathrm{dx}_2, \mathrm{dx}_3\}$;

We proceed the calculation evaluating the external differential of the vielbein

contorgenresume

I calculate the exterior differentials of the vielbeins

I finished!
Next I calculate the inverse vielbein
Done!
I resume the calculation of the contorsion
I calculate the contorsion c[i,j,k] for
i = 1
i = 2
i = 3
i = 4
I have finished!
The result, encoded in a vector dE[i] is the following:
dE[1] = 0
dE[2] = $\frac{e_1{**}e_2 s_1'[t]}{s_1[t]}$
dE[3] = $\frac{e_1{**}e_3 s_2'[t]}{s_2[t]}$
dE[4] = $\frac{e_1{**}e_4 s_3'[t]}{s_3[t]}$

The contorsion is encoded in tensor named contens

Now you can begin the calculation of the spin connection by typing spinpackgen
{Null}

We initialize the calculation of the spin connection

spinpackgen

I start
now give me the contorsion tensor
by writing cont = ?
and give me the signature a vector of +/- 1
by writing signat = ?
then resume the calculation by typing spinresumegen
{Null}

Requested by the computer we indicate the file containing the contorsion and we specify the signature

cont = contens;
signat $= \{-1, 1, 1, 1\}$;

We conclude the calculation of the spin connection

spinresumegen
I resume the calculation of the spin connection

---

the result is
$\omega[12] = \frac{e_2 s_1'[t]}{s_1[t]}$
$\omega[13] = \frac{e_3 s_2'[t]}{s_2[t]}$
$\omega[14] = \frac{e_4 s_3'[t]}{s_3[t]}$
$\omega[23] = 0$
$\omega[24] = 0$
$\omega[34] = 0$
Task finished
The result is encoded in a tensor omega[i,j]
Its components are encoded in a tensor ometen[i,j,m]
If you want the curvature, type curvapack for quasi-homogeneous manifolds
Otherwise, type curvapackgen for general manifolds
{Null}

Next we calculate the curvature two form and the Ricci tensor

curvapackgen

I calculate the Riemann tensor
I tell you my steps:
a = 1
b = 1
b = 2
b = 3
b = 4
a = 2
b = 1
b = 2
b = 3
b = 4
a = 3
b = 1
b = 2
b = 3
b = 4
a = 4
b = 1
b = 2
b = 3
b = 4
Finished

Now I evaluate the curvature 2-form of your space
I find the following answer
R[12] = $\frac{e_1{**}e_2 s_1''[t]}{s_1[t]}$
R[13] = $\frac{e_1{**}e_3 s_2''[t]}{s_2[t]}$
R[14] = $\frac{e_1{**}e_4 s_3''[t]}{s_3[t]}$
R[23] = $\frac{e_2{**}e_3 s_1'[t]s_2'[t]}{s_1[t]s_2[t]}$
R[24] = $\frac{e_2{**}e_4 s_1'[t]s_3'[t]}{s_1[t]s_3[t]}$
R[34] = $\frac{e_3{**}e_4 s_2'[t]s_3'[t]}{s_2[t]s_3[t]}$
The result is encoded in a tensor RR[i,j]
Its components are encoded in a tensor Rie[i,j,a,b]
————————————————-
Now I calculate the Ricci tensor
11 non-zero
22 non-zero
33 non-zero
44 non-zero
I have finished the calculation
The tensor ricten[a,b]] giving the Ricci tensor
is ready for storing on hard disk
We display the Ricci tensor

ricten

$\Big\{\Big\{\frac{1}{2}\Big(\frac{s_1''[t]}{s_1[t]}+\frac{s_2''[t]}{s_2[t]}+\frac{s_3''[t]}{s_3[t]}\Big),0,0,0\Big\},$
$\Big\{0,\frac{s_2[t]s_1'[t]s_3'[t]+s_3[t](s_1'[t]s_2'[t]+s_2[t]s_1''[t])}{2s_1[t]s_2[t]s_3[t]},0,0\Big\},$
$\Big\{0,0,\frac{s_1[t]s_2'[t]s_3'[t]+s_3[t](s_1'[t]s_2'[t]+s_1[t]s_2''[t])}{2s_1[t]s_2[t]s_3[t]},0\Big\},$
$\Big\{0,0,0,\frac{s_1[t]s_2'[t]s_3'[t]+s_2[t](s_1'[t]s_3'[t]+s_1[t]s_3''[t])}{2s_1[t]s_2[t]s_3[t]}\Big\}\Big\}$

The above result presents the Ricci tensor for a Kasner like metric with three independent scale factors for each of the three Euclidian axes.

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