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Felix V. Dolzhansky

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Felix V. Dolzhansky

Fundamentals of Geophysical Hydrodynamics

Translated by Boris Khesin

Springer

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(1937–2008)

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Preface

This book has its origins in courses for graduate and undergraduate students at the Moscow Institute of Physics and Technology (MPhTI). It is the result of over forty years of the author's work in the Laboratory of Geophysical Hydrodynamics founded by A.M. Obukhov in the Institute of Atmospheric Physics (IAP) of the Russian Academy of Science. The first four sections (Part I) are devoted to the basic principles and laws of motion of ideal incompressible and compressible fluids. Particular attention is given to the first integrals of the hydrodynamic equations, and in particular, to the Kelvin invariant and its analogs in stratified incompressible and compressible fluids, the Rossby–Obukhov and Ertel invariants.

In Part II, we discuss what geophysical fluid dynamics is with its unusual properties of fluid motion along isobars and with its suppressed vertical velocity component (the Proudman–Taylor theorem). In deriving the quasi-geostrophic equations of motion we used the axiomatic approach to streamline the presentation. Namely, the whole space of inviscid global geophysical flows is in a sense spanned by the four-dimensional Obukhov–Charney basis, which is defined by the conditions of quasi-hydrostatic and quasi-geostrophic equilibria, as well as the Lagrangian invariance of potential temperature and potential vorticity. The problem of filtering fast motions and adaptation of meteorological fields to the above mentioned equilibria is discussed merely by illustrations and physical reasoning. We introduce the concepts of Rossby waves, their resonant interaction, thermal wind, available potential energy, singular Helmholtz and Obukhov vortices, we derive the Kirchhoff equations for such vortices and demonstrate their applications in geophysical setting.

Part III discusses the problem of the barotropic and baroclinic stability of global geophysical flows. It is preceded by a summary of the classical results in the theory of hydrodynamical stability, which is easily generalized to the case of a rotating fluid. When choosing the material I enjoyed a short and very intelligible book *Hydrodynamical stability and atmosphere dynamics* by L.A. Dikii whose line I followed with minor changes in this book. Our exposition is supplemented by the consideration (following E. Lorenz) of oversimplified nonlinear equations of atmospheric dynamics, which illustrate the evolution of motion after the stability loss by the primary flow. We draw the reader's attention to the mechanism of baroclinic instability and its relation to the available potential energy. In my opinion, the best physical interpretation of this complex phenomenon was given by its discoverer Eady, and a modern account of his work is included in that section.

Viscous geophysical flows and the general atmospheric circulation are discussed in Part IV; its content and exposition differ most of all from the traditional approach to these issues. After deriving the Navier–Stokes equations the reader's attention is focused on describing the geophysical boundary layers, namely, the Ekman and Proudman–Stewartson layers, responsible for the dissipation of kinetic energy of global flows. We derive the quasi-geostrophic equation for the transformation of the potential vorticity of a quasi-two-dimensional barotropic atmosphere taking into account the planetary boundary layer. Starting with this equation we study in detail the linear stability of the Kolmogorov zonal flow (with the sine velocity profile) on the infinite f-plane. In this exceptional case the linear stability problem can be solved analytically. This clearly illustrates the crucial role of exterior friction created by the planetary boundary layer for the stability characteristics of global atmospherical flows. The results of solving this problem are formulated in terms of internal and external Reynolds numbers and then extended to flows with arbitrary velocity profiles. These results point to the structural instability of strictly two-dimensional flows with respect to the inclusion of friction. On the other hand, omitting internal friction has practically no effect on the results of quasi-two-dimensional theory. Thus, it becomes apparent that the crucial parameter of the barotropic atmosphere is the Reynolds number defined by the external friction, rather than the usual Reynolds number reaching astronomical values for global motions. Self-similarity with respect to the internal Reynolds number and the relatively low supercritical response of global movements on the exterior Reynolds number is exactly what explains the relatively quiet nature of general atmospheric circulation, which is not captured by the developed large-scale turbulence.

The final Part V treats the general circulation of the atmosphere with the help of its mechanical prototype, the Euler–Poisson equations of motion of a rigid body, generalized to the case of a rotating system. It is worth mentioning yet one more distinctive feature of this course. Based on the concept of a generalized rigid body introduced by Arnold and the concept of a generalized heavy rigid body introduced by the author, we draw the following analogy. On the one hand, one has the hydrodynamical Euler equations of motion of an ideal fluid and the Oberbeck–Boussinesq equations of motion of a heavy fluid, while on the other hand we deal with the Euler equations of a classical gyroscope and the Euler–Poisson equations for motion of a heavy top. As a consequence, such mechanical invariants as the square of angular momentum and projection of angular momentum to the direction of gravity are treated as the Kelvin and Ertel invariants, respectively. The mechanical prototype of general atmospheric circulation, which has the fundamental symmetry properties of the original, is obtained by introducing linear friction, simulating the effect of the planetary boundary layer and the Newton heat sources, which are proportional to temperature deviations from the background. As a result, based upon analytical calculations and numerical integration of few-component dynamical systems, one can view atmospheric action as a heat engine with its Hadley and Rossby fundamental modes, its reverse meridional circulation cell, its characteristic energy cycle, few-component stochastization, and therefore the weather unpredictability for long periods, as well as its transitions from one long-term metastable weather state to another that are not externally motivated.

For better understanding, each chapter is equipped with exercises, hints, and usually solutions. The literature citation in each chapter is minimal and focuses mainly on the widely available publications where the reader can find more details. The references in the text are given only by the author's name and the publication year and listed at the end of the corresponding section. We give more details on difficult-to-find publications, which are not (yet) included in the well-known editions. The book contains both theoretical and experimental results obtained over years in the Laboratory of Geophysical Hydrodynamics of the IAP RAS by A. Batchaev, V.A. Dovzhenko, V.A. Krymov, D.Yu. Manin, and Yu.L. Chernousko.

I am using this opportunity to express my sincere gratitude to V.P. Dymnikov for his encouragement of both my teaching activity and faster completion of the manuscript. Numerous consultations and the accompanying constructive criticism of my colleagues G.S. Golitsyn, V.P. Goncharov, V.I. Klyatskin, and I.G. Yakushkin were invaluable in preparing this material. I am particularly indebted to A.E. Gledzer, E.B. Gledzer, and V.M. Ponomarev for our productive discussions, proofreading of the manuscript and the preparation of illustrations.

F. Dolzhansky

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Boris Khesin

[1] The present version of the book was finalized by June 2009 (e-mail: lgg@ifaran.ru).

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Part I
Main Principles and Laws of Motion of an Ideal Fluid

Chapter 1
Equations of Motion of an Ideal Incompressible Fluid; Kelvin's Circulation Theorem

Fluid motion as a physical process is associated with the Euler or Navier–Stokes equations in hydrodynamics. These equations describe an immense set of qualitatively different phenomena—from the simplest small oscillations of a continuum, such as the propagation of sound in a homogeneous fluid (gas), to the mysterious phenomenon of turbulence observed in a vast majority of natural and technological flows. This "comprehensive nature" of the equations means that it is impossible (at least, as of today) to construct their general solutions. Consequently, it also means that there is a need for an appropriate reduction of these equations based on both observations and on the physical nature of the class of motions under study. For this reason, by now individual areas of hydrodynamics, such as the theory of sound, vortex dynamics, hydrodynamical stability theory, magnetohydrodynamics, convection theory, aerodynamics and many others, have all taken on the status of independent domains of science with their own physical features, applications, and often with a specially developed mathematical toolbox, as is the case in nonlinear wave theory.

Geophysical hydrodynamics has also become an independent science, encompassing a rather wide range of phenomena observed in rotating fluids. These include, in particular, the oceans and Earth's atmosphere, their laboratory counterparts, atmospheres of other revolving planets, the Sun and other stars, and even galaxies whose evolution takes place under general rotation.

In order to better understand what geophysical hydrodynamics is and what place it occupies among other hydrodynamical sciences, it is worth recalling the fundamental principles and laws of motion in fluids, as it is forbidden to violate them in any kind of simplified formulation of problems.

1.1 What is an Incompressible Fluid?

Let us start with an ideal, i.e., inviscid and non-thermo-conducting, incompressible fluid. Incompressibility means that during motion the density $\rho = \rho(t, \mathbf{x})$ of an arbitrarily isolated fluid parcel, regarded as a function of time t and location $\mathbf{x}$, remains

F.V. Dolzhansky, *Fundamentals of Geophysical Hydrodynamics*,
Encyclopaedia of Mathematical Sciences 103,
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constant. Since the radius-vector $\mathbf{x} = \mathbf{x}(t)$ of the location of a moving particle in turn depends on time, the constancy condition of ρ is expressed mathematically by the equation

$$\frac{d\rho}{dt} \equiv \frac{\partial \rho}{\partial t} + (\mathbf{u}\nabla)\rho = 0, \tag{1.1}$$

where $\mathbf{u}(t, \mathbf{x}) = \dot{\mathbf{x}}(t)$ is by definition the velocity of the flow at a point $\mathbf{x}$ at the moment t. The operator

$$\frac{d}{dt} \doteq \frac{\partial}{\partial t} + \mathbf{u}\nabla \tag{1.2}$$

is called the *substantial* (or *individual) derivative* and it reflects the fact that a change of some characteristics of a moving fluid parcel is not only caused by its explicit time dependency, but also by the spatial nonhomogeneity of this quantity. Its rate of change in the direction of the flow is given by the derivative $\mathbf{u}\nabla$ in the direction of the velocity vector $\mathbf{u}$. A symbol $\doteq$ stands for "by definition is equal to".

We shall call the quantities characterizing the state of a fluid, e.g. its weight and velocity, at any point of the space occupied by the fluid as *field characteristics* or simply *fields*. Scalar field characteristics satisfying Eq. (1.1) are called *Lagrangian invariants*. As we shall see below, they play a crucial role in geophysical fluid dynamics. Let us emphasize yet one more time the main property of a Lagrangian invariant, which follows from its definition: it is passively transported by fluid motion, i.e., its value remains constant for each individual particle, and it is only its location that changes.

1.2 Equations of Motion of an Ideal Incompressible Fluid

The single physical characteristic describing an ideal incompressible fluid at a complete rest is its density, a local measure of inertia of a continuum. Therefore, the motion of such a system has to be governed exclusively by mechanical principles. As a source, one can use any two independent principles, for instance, the law of conservation of mass and Newton's second law, which provide a foundation for formulating the equations of balance for energy, momentum, and angular momentum. For an arbitrary fixed fluid volume V, mass conservation is expressed by the equation

$$\frac{\partial}{\partial t} \int_V \rho dV = -\oint_{\partial V} \rho \mathbf{u} \cdot d\boldsymbol{\sigma} = -\int_V \operatorname{div}(\rho \mathbf{u}) dV. \tag{1.3}$$

It expresses the fact that the rate of change of mass of the fixed volume is equal to the mass flux through the closed surface ∂V which bounds it. (The minus sign on the right-hand side of Eq. (1.3) indicates that the direction of the exterior normal $\mathbf{n}$ is chosen to be the positive direction of the fluid flux through an element $d\boldsymbol{\sigma} = \boldsymbol{n} d\sigma$ of the surface ∂V. As applied to an individual fluid parcel of a unit volume this means that

$$\frac{\partial\rho}{\partial t} + \operatorname{div}(\rho\mathbf{u}) \equiv \frac{d\rho}{dt} + \rho\operatorname{div}\mathbf{u} = 0. \tag{1.3$'$}$$

Due to (1.1), for an incompressible medium Eq. (1.3′) splits into two equations:

$$\frac{d\rho}{dt} \doteq \frac{\partial\rho}{\partial t} + (\mathbf{u}\nabla)\rho = 0, \qquad \operatorname{div}\mathbf{u} = 0, \tag{1.4}$$

i.e., the motion of an incompressible fluid is described by divergence-free velocity vector fields. Such fields are often called *solenoidal*.

Newton's second law, as applied to an individual fluid parcel of the unit volume, can be written as follows:

$$\rho\frac{d\mathbf{u}}{dt} \doteq \rho\left(\frac{\partial\mathbf{u}}{\partial t} + (\mathbf{u}\nabla)\mathbf{u}\right) = \mathbf{F}, \tag{1.5}$$

where $\mathbf{F}$ is a constraint reaction, i.e., the total of all forces of the surrounding medium acting on the particle.

However, the system of (1.4) and (1.5) is not yet closed, as the number of the unknowns ρ, $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{F} = (F_1, F_2, F_3)$ exceeds the number of equations by two. Sometimes to close up the system (1.4) and (1.5) one takes a somewhat inconsistent step that involves using the notion of pressure. Pressure, strictly speaking, belongs to the realm of thermodynamics and thus moves us outside of the scope of a purely mechanical point of view of the medium being considered. This can be avoided by the following reasoning.

For the sake of simplicity let us assume that fluid's density is constant throughout and equals ρ_0. In this case we can transform the system (1.4) and (1.5) into the form

$$\operatorname{div}\mathbf{u} = 0, \tag{1.6}$$

$$\frac{\partial\mathbf{u}}{\partial t} = -(\mathbf{u}\nabla)\mathbf{u} + \frac{\mathbf{F}}{\rho_0}. \tag{1.7}$$

Since there are no other independent principles of motion for studying a fluid within a strictly mechanical framework besides those mentioned above, one has to assume that there exists a scalar function $p = p(t, \mathbf{x})$ determined by the equations of motion. This function defines $\mathbf{F}$ in the only invariant way by the relation

$$\mathbf{F} = -\nabla p, \tag{1.8}$$

where the logic behind the choice of the minus sign will become clear below.

The quantity p can be regarded as a gauge function that provides the divergence-free (solenoidal) property of the right-hand side of Eq. (1.7) and satisfies the Poisson equation

$$\Delta p = -\rho_0 \sum_{i,k} \frac{\partial u_i}{\partial x_k}\frac{\partial u_k}{\partial x_i}, \quad (i, k = 1, 2, 3). \tag{1.9}$$

(Apply the operator div to (1.7) while keeping in mind (1.8).)

Further, take into account that the force acting on an individual particle of volume V is equal to

$$\int_V \mathbf{F} dV \equiv -\int_V \nabla p \, dV = -\oint_{\partial V} p \, d\boldsymbol{\sigma}$$

($d\boldsymbol{\sigma}$ is an area element of the closed surface ∂V which bounds the volume V). We conclude that p is numerically equal to the force acting at point $\mathbf{x}$ on a unit area of arbitrary orientation, i.e., the value p can be identified with the pressure arising in the fluid due to its motion. Now the choice of the minus sign in (1.8) becomes clear: positive acceleration of the fluid should be pointing in the direction of the decrease in pressure, i.e., opposite to its gradient.

The above arguments remain valid for an incompressible fluid of variable density, with the only difference that instead of (1.9), the pressure in this case satisfies the equation

$$\Delta p - \nabla p \cdot \nabla(\ln \rho) = -\rho \sum_{i,k} \frac{\partial u_i}{\partial x_k} \frac{\partial u_k}{\partial x_i}. \tag{1.9'}$$

Thus, the equations of motion of an ideal incompressible fluid assume the form

$$\frac{d\mathbf{u}}{dt} \doteq \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p, \tag{1.10}$$

$$\operatorname{div} \mathbf{u} = 0, \qquad \frac{d\rho}{dt} \doteq \frac{\partial \rho}{\partial t} + (\mathbf{u}\nabla)\rho = 0. \tag{1.11}$$

The first equation is called *the Euler equation of motion of an ideal fluid* (L. Euler, 1755). By using the well-known formula of vector calculus

$$\tfrac{1}{2}(\nabla \mathbf{u})^2 = \mathbf{u} \times \operatorname{rot} \mathbf{u} + (\mathbf{u}\nabla)\mathbf{u}, \tag{1.12}$$

the Euler equation is often more conveniently rewritten in the Bernoulli or Gromeko–Lamb form:

$$\frac{\partial \mathbf{u}}{\partial t} - \mathbf{u} \times \operatorname{rot} \mathbf{u} = -\frac{1}{\rho}\nabla p - \frac{1}{2}\nabla(\mathbf{u})^2. \tag{1.10'}$$

Note an important distinctive feature of the Euler equation for a fluid of constant density: its right-hand side is the gradient of a scalar function. In the case of variable density this could only take place if $p = p(\rho)$, i.e., if the pressure were a function of density (the case of a so-called barotropic fluid). For an incompressible medium, this would mean overdetermined equations of motion, as well as the fact that the pressure would move along with the fluid. The latter is impossible because of the physical nature of this quantity. This distinction is rather essential and, as we will see below, it affects the properties of solutions of hydrodynamical equations, especially when the fluid is placed in an external potential field.

1.3 Kelvin's Circulation Theorem

The notion of vorticity and Kelvin's theorem (Lord Kelvin, 1869) are fundamental in understanding and describing the motion of a fluid of constant density ($\rho = \rho_0$).

A curve L is said to be *liquid* if it moves along with the fluid, i.e., any point on the curve L moves together with the fluid particle located at that point initially. The circulation of velocity along a closed liquid contour C is defined by the integral

$$K \doteq \oint_C \mathbf{u}\delta\mathbf{r}, \tag{1.13}$$

where $\delta\mathbf{r}$ is an infinitesimal element of the contour C, equal to the difference of radius vectors of its endpoints. Since the contour C is deformed while moving with the fluid, the substantial derivative of the quantity K can be found from the equality

$$\frac{dK}{dt} = \oint_C \frac{d\mathbf{u}}{dt}\delta\mathbf{r} + \oint_C \mathbf{u}\delta\left(\frac{d\mathbf{r}}{dt}\right), \tag{1.14}$$

where by definition $d\mathbf{r}/dt = \mathbf{u}$. Hence

$$\oint_C \mathbf{u}\delta\left(\frac{d\mathbf{r}}{dt}\right) = \oint_C \mathbf{u}\delta\mathbf{u} = \frac{1}{2}\oint_C d\left(\mathbf{u}^2\right) = 0 \tag{1.15}$$

as an integral of a complete differential over a closed contour. Now substituting (1.10) into (1.14) and employing the constancy of $\rho = \rho_0$ and (1.15), for the same reason we find that

$$\frac{dK}{dt} = \oint_C \frac{d\mathbf{u}}{dt}\delta\mathbf{r} = -\frac{1}{\rho_0}\oint_C \nabla p\delta\mathbf{r} = -\frac{1}{\rho_0}\oint_C dp = 0. \tag{1.14$'$}$$

This proves Kelvin's circulation theorem. According to this theorem K is a Lagrangian invariant, i.e.,

$$\frac{dK}{dt} \doteq \frac{d}{dt}\oint_C \mathbf{u}\delta\mathbf{r} = 0. \tag{1.16}$$

Another formulation of Kelvin's theorem is based on the application of the Stokes theorem

$$\oint_C \mathbf{A}\delta\mathbf{r} = \int_S \operatorname{rot}\mathbf{A}d\boldsymbol{\sigma}, \tag{1.17}$$

which holds for an arbitrary sufficiently smooth vector field $\mathbf{A}$ and an arbitrary *contractible* liquid contour C. Here $d\boldsymbol{\sigma}$ is an infinitesimal area element of a surface S bounded by a closed contour C. A contour is called *contractible* if it can be smoothly deformed into a point without leaving the domain occupied by the fluid.

According to the Stokes theorem,

$$K \doteq \oint_C \mathbf{u}\delta\mathbf{l} = \int_S \boldsymbol{\Omega}d\boldsymbol{\sigma}, \quad \boldsymbol{\Omega} \doteq \operatorname{rot}\mathbf{u},$$

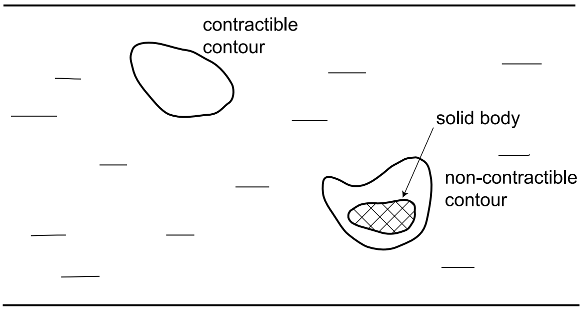

Fig. 1.1 Contractible and non-contractible contours in a two-dimensional fluid filling a non-simply connected domain around a solid body

while for an infinitesimal closed contour C the value of K can be written as

$$K \doteq \oint_C \mathbf{u}\delta\mathbf{l} = \boldsymbol{\Omega} d\boldsymbol{\sigma}. \tag{1.18}$$

This shows that velocity circulation along a closed contractible liquid contour C can be interpreted as a vorticity flux given by the vector field $\boldsymbol{\Omega}$ across the surface bounded by the contour C.

The equalities (1.16) and (1.18) imply that the above-mentioned vorticity flux is a Lagrangian invariant, and so in that regard one can consider a transfer of vorticity by the fluid. As it becomes clear from the above proof, the velocity circulation is preserved along any closed liquid contour, including non-contractible ones, for example, the one encompassing a solid body in a multiply-connected region (see Fig. 1.1). Therefore one needs to remember that the velocity circulation is a broader concept than the vorticity one, and it is used, for example, in the theory of flow of an ideal homogeneous incompressible fluid around solids.

The Lagrangian invariance of the quantity (1.18) can be also interpreted in terms of vortex tubes constructed as follows. Let us introduce the notion of a *vortex line*, which by definition is tangent to the vorticity vector $\boldsymbol{\Omega}$ at any point. Vortex lines are defined by a family of solutions of differential equations

$$\frac{dx}{\Omega_x} = \frac{dy}{\Omega_y} = \frac{dz}{\Omega_z}. \tag{1.19}$$

A set of vortex lines passing through points of a closed contractible curve forms a cylindrical surface that is called a *vortex tube* (see Fig. 1.2(a)). This tube, by virtue of its construction, is such that on its surface $\boldsymbol{\Omega} \cdot \mathbf{n} = 0$ ($\mathbf{n}$ is the unit normal to the surface). Consequently, the vorticity flux through any of its cross-sections $\delta\boldsymbol{\sigma}$ is constant along the tube and is called the intensity of the vortex tube. Therefore, the vorticity transport by the fluid now means that the vortex tube is liquid, i.e., it moves along with the fluid. Indeed, according to the Kelvin theorem, the intensity of a moving tube remains constant.

For better understanding that a vorticity field is stationary relative to the fluid or, equivalently, that its force lines are liquid themselves apply the rotor operator to the

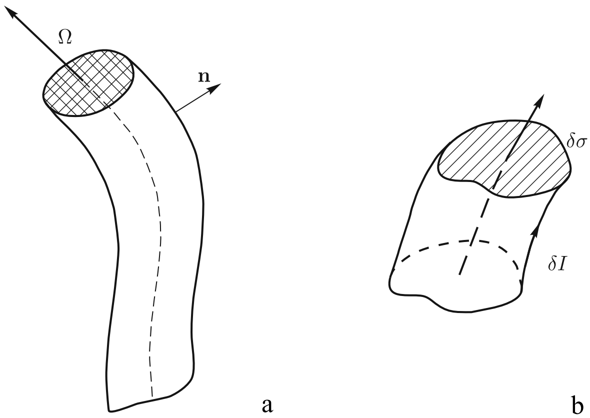

Fig. 1.2 A vortex tube (**a**) and an infinitesimal volume element (**b**)

Eq. (1.10′) using the following formulas of vector calculus

$$\operatorname{rot}(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\nabla)\mathbf{A} - (\mathbf{A}\nabla)\mathbf{B} + \mathbf{A}\operatorname{div}\mathbf{B} - \mathbf{B}\operatorname{div}\mathbf{A}, \tag{1.20}$$

$$\operatorname{rot}(\varphi\mathbf{A}) = (\nabla\varphi\times\mathbf{A}) + \varphi\operatorname{rot}\mathbf{A}. \tag{1.21}$$

Taking into account the divergence-free property of the velocity field $\mathbf{u} = \mathbf{u}(t,\mathbf{x})$, we obtain the vortex equation

$$\frac{\partial\boldsymbol{\Omega}}{\partial t} - \operatorname{rot}(\mathbf{u}\times\boldsymbol{\Omega}) \equiv \frac{\partial\boldsymbol{\Omega}}{\partial t} + (\mathbf{u}\nabla)\boldsymbol{\Omega} - (\boldsymbol{\Omega}\nabla)\mathbf{u} = 0, \tag{1.22}$$

named after Helmholtz (1858). The Helmholtz equation can be reformulated as

$$\frac{d\boldsymbol{\Omega}}{dt} = (\boldsymbol{\Omega}\nabla)\mathbf{u}. \tag{1.22′}$$

According to Cochin, Kibel and Rose (1955) *for the force lines of a divergence-free vector field to be liquid, it is necessary and sufficient that the vector field itself be governed by the Helmholtz equation.*

Proving the *necessity* is straightforward. Let $\delta\mathbf{l}$ be an infinitesimal element of a liquid line (i.e., its tangent element), that is, it connects the vectors $\mathbf{l}$ and $\mathbf{l}+\delta\mathbf{l}$. Then the rate of change of this element is equal to the length's difference of that element at close moments, divided by the corresponding difference in time:

$$\frac{d\delta\mathbf{l}}{dt} = \frac{\delta\mathbf{l}(t+dt) - \delta\mathbf{l}(t)}{dt}.$$

A fluid parcel with vector $\mathbf{l}$ at time dt will travel into vector $\mathbf{l} + dt\cdot\mathbf{u}$. Accordingly, a fluid parcel with vector $\mathbf{l}+\delta\mathbf{l}$ in time dt will travel into vector $\mathbf{l}+\delta\mathbf{l}+dt\cdot(\mathbf{u}+\delta\mathbf{u})$. Therefore, one obtains the new value of the vector $\delta\mathbf{l}(t+dt)$ which connects new positions of the same fluid parcels:

$$\delta\mathbf{l}(t+dt) = \delta\mathbf{l}(t) + dt\cdot(\mathbf{u}+\delta\mathbf{u}) - dt\cdot\mathbf{u} = \delta\mathbf{l}(t) + dt\cdot\delta\mathbf{u}.$$

And since the velocity difference $\delta\mathbf{u}$ at time t of two fluid parcels distanced by vector $\delta\mathbf{l}$ is defined as $(\delta\mathbf{l}\nabla)\mathbf{u}$, we obtain

$$\frac{d\delta\mathbf{l}}{dt} = (\delta\mathbf{l}\nabla)\mathbf{u} \quad \text{or} \quad \frac{\partial\delta\mathbf{l}}{\partial t} = (\delta\mathbf{l}\nabla)\mathbf{u} - (\mathbf{u}\nabla)\delta\mathbf{l}, \tag{1.23}$$

which is equivalent to (1.22′) or (1.22).

To prove *sufficiency*, note that the incompressibility of the fluid implies Lagrangian invariance of the volume element $\delta\mu = \delta\mathbf{l}\cdot\delta\boldsymbol{\sigma}$, where $\delta\mathbf{l}$ is a linear fluid element and $\delta\boldsymbol{\sigma}$ is the area of an element of an oriented surface transversal to $\delta\mathbf{l}$ (Fig. 1.2(b))

$$\frac{d\delta\mu}{dt} = \frac{d}{dt}(\delta\mathbf{l}\cdot\delta\boldsymbol{\sigma}) = 0. \tag{1.24}$$

Being a linear liquid element, $\delta\mathbf{l}$ satisfies the Helmholtz equation. Hence from (1.23) and (1.24) it follows that $\delta\boldsymbol{\sigma}$ is described by the equation

$$\frac{d\delta\boldsymbol{\sigma}}{dt} = -\delta\boldsymbol{\sigma}\frac{\partial\mathbf{u}}{\partial\mathbf{r}},$$

or, in tensor notation by

$$\frac{d\delta\sigma_i}{dt} = -\delta\sigma_k\frac{\partial u_k}{\partial x_i}. \tag{1.25}$$

Indeed, substituting (1.23) into (1.24) and using the tensor notation we obtain the equality

$$\delta l_i\left(\delta\sigma_k\frac{\partial u_k}{\partial x_i} + \frac{d\delta\sigma_i}{dt}\right) = 0,$$

which is equivalent to (1.25) due to the arbitrary nature of $\delta\mathbf{l}$.

Now it is easy to prove that the force lines of a divergence-free vector field are liquid, provided that the vector field satisfies the Helmholtz equation. Indeed, by the definition of force lines (1.19) a length element $\delta\mathbf{l}$ is also governed by the Helmholtz equation. At some initial moment consider a volume element $\delta\mu = \delta\mathbf{l}\cdot\delta\boldsymbol{\sigma}$, where $\delta\boldsymbol{\sigma}$ is an element of a *liquid* surface transversal to the element $\delta\mathbf{l}$ at their intersection point. Then since $\delta\mathbf{l}$ and $\delta\boldsymbol{\sigma}$ satisfy (1.23) and (1.25) respectively, one has that $\delta\mu$ is a Lagrangian invariant, i.e., it is a liquid volume element, and hence $\delta\mathbf{l}$ is a liquid length element.

In passing, note that *a necessary and sufficient condition for a selected surface to stay at rest relative to the fluid (i.e., that the surface is also liquid) is the condition that every element of the surface $\delta\boldsymbol{\sigma}$ satisfies equation* (1.25).

1.4 Exercises

1. Show that the vorticity at a point $\mathbf{x}$ is equal to the double angular velocity of the local rotation of the fluid at the point $\mathbf{x}$.
2. Consider a two-dimensional irrotational (i.e., everywhere $\boldsymbol{\Omega} = 0$) flow of an incompressible homogeneous fluid in an infinite plane containing a solid body. Assume that the circulation of the flow velocity along the boundary of the solid is nonzero and equals Γ_0. Show that the velocity circulation over an arbitrary contour encompassing the solid is also nonzero and equals Γ_0.

3. Let a fluid be in a potential field Φ (where $\nabla\Phi$ is a force acting on a unit mass) and rotate as a whole with a constant angular velocity $\mathbf{\Omega_0}$. Show that in the coordinate system rotating with angular velocity $\mathbf{\Omega_0}$ the Euler equations can be written in the form

$$\frac{d\mathbf{u}}{dt} + 2\mathbf{\Omega_0} \times \mathbf{u} = -\frac{1}{\rho}\nabla p + \nabla\left(\frac{1}{2}(\mathbf{\Omega_0} \times \mathbf{r})^2 + \Phi\right).$$

   Hint: Use the transformation formula for the time derivatives

$$\frac{d\mathbf{A}}{dt} = \left(\frac{d\mathbf{A}}{dt}\right)_R + \mathbf{\Omega_0} \times \mathbf{A},$$

   where the index R stands for the time derivative in the rotating coordinate system.
4. Assume that we do not know the equations of motion for an ideal incompressible homogeneous fluid, but we do know that Kelvin's circulation theorem holds (e.g., as an empirical fact). Show that this theorem and the condition of the medium incompressibility together imply the Euler equations of fluid motion. (Newton's Second Law could have been discovered using Kelvin's theorem.)

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Chapter 2
Potential Vorticity and the Conservation Laws of Energy and Momentum for a Stratified Incompressible Fluid

2.1 Potential Vorticity of a Stratified Incompressible Fluid

For an incompressible but stratified fluid, for which $\rho = \rho(t, \mathbf{x}) \neq \rho_0$, the Kelvin theorem is not valid in general. The reason is that the right-hand side $-\rho^{-1}\nabla p$ of the Euler equation (1.10) might not be the gradient of a scalar function any longer, and hence $\rho^{-1}\nabla p \delta \mathbf{r}$ might not be a complete differential. The latter condition was exactly what provided the vanishing of the right-hand side of (1.14′).

Note, however, an important feature of the motion of a stratified fluid: it is fibered into surfaces of constant density (isopycnic or iso-density surfaces) and remains such in the process of evolution: every fluid particle belonging to such a surface at the initial moments remains on the same surface due to the Lagrangian invariance of density. (This is why stratified fluid is also often called fibered.) In turn, the motion along any iso-density surface $\rho(t, \mathbf{x}) = \rho_0 = \text{const}$ is the motion of a homogeneous incompressible fluid, for which the Kelvin theorem holds. In particular,

$$K_0 \doteq \oint_{C_0} \mathbf{u}\delta\mathbf{l} = \mathbf{\Omega} d\boldsymbol{\sigma}_0 \tag{2.1}$$

is a Lagrangian invariant ($dK_0/dt = 0$), where C_0 is an infinitesimal closed contour on the iso-density surface, while $d\boldsymbol{\sigma}_0$ is an element of this surface bounded by the contour C_0 (Fig. 2.1).

Consider a fluid tube intersecting surfaces of constant density, such that one of the tube sections is a marked contour C_0. Two such surfaces corresponding to close density values ρ_0 and $\rho_0 + \delta\rho$ cut out a cylindrical element of the tube, whose volume is equal to

$$\delta\mu = d\boldsymbol{\sigma}_0 \cdot \mathbf{n}\delta h, \tag{2.2}$$

where δh is the height of the cut-out cylinder and $\mathbf{n}$ is the unit normal to the iso-density surface. The normal direction coincides with the density gradient $\nabla\rho = |\nabla\rho| \cdot \mathbf{n}$. The value of $\delta\rho$ in turn can be written as

$$\delta\rho = |\nabla\rho| \cdot \delta h. \tag{2.3}$$

F.V. Dolzhansky, *Fundamentals of Geophysical Hydrodynamics*,
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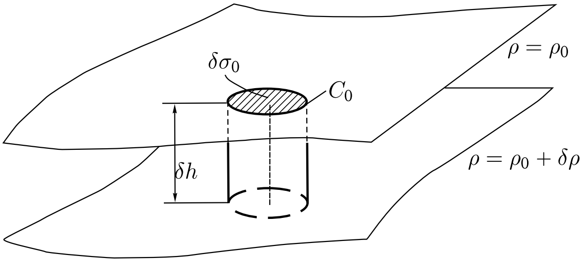

Fig. 2.1 The ends of a liquid cylinder situated on isopycnic (or iso-density) surfaces $\rho = \rho_0$ and $\rho = \rho_0 + \delta\rho$ at the initial moment will remain on them during the evolution

Now by comparing (2.2) and (2.3) we find that

$$d\boldsymbol{\sigma}_0 = \frac{\delta\mu}{\delta h}\mathbf{n} = \frac{\delta\mu}{\delta\rho}\cdot\frac{\delta\rho}{\delta h}\cdot\mathbf{n} = \frac{\delta\mu}{\delta\rho}\nabla\rho,$$

while

$$K_0 = \boldsymbol{\Omega}\cdot d\boldsymbol{\sigma}_0 = \frac{\delta\mu}{\delta\rho}\boldsymbol{\Omega}\cdot\nabla\rho.$$

The invariance of K_0 implies the invariance of the value of

$$\Pi \equiv \boldsymbol{\Omega}\cdot\nabla\rho \tag{2.4}$$

(and vice versa) when taking into account that the values of $\delta\mu$ and $\delta\rho$ are conserved during the evolution process, respectively, by our construction and in view of the incompressibility of the medium. The quantity Π is called the *potential vorticity* (*PV*) *of an incompressible fluid*. In other words, *the conservation of the potential vorticity exactly means applicability of the Kelvin theorem to contractible liquid contours lying on surfaces of constant density.* (The notion of potential vorticity was independently introduced to hydrodynamics by C.-G. Rossby (1939) in relation to the ocean, by H. Ertel (1942) in its most general form, and by Obukhov (1949) in relation to the atmosphere. In the following chapters we are going to return repeatedly to various manifestations of this notion or, more precisely, of the Kelvin invariance theorem.)

Formally one can prove the invariance of Π in the following way. Let us apply the curl operator to the Euler equation (1.10′) in the Bernoulli form and use the formulas (1.20) and (1.21). As a result we obtain the vorticity equation for a stratified incompressible fluid, which is named after A.A. Fridman. By taking incompressibility into account, this equation assumes the form

$$\frac{\partial\boldsymbol{\Omega}}{\partial t} - \mathrm{rot}(\mathbf{u}\times\boldsymbol{\Omega}) \equiv \frac{\partial\boldsymbol{\Omega}}{\partial t} + (\mathbf{u}\nabla)\boldsymbol{\Omega} - (\boldsymbol{\Omega}\nabla)\mathbf{u} = \frac{1}{\rho^2}\nabla\rho\times\nabla p. \tag{2.5}$$

For a fluid of constant density it becomes the Helmholtz equation (1.22) or (1.22′). We would like to make the following two remarks in this regard. First, even for motion of a homogeneous incompressible fluid, the vorticity tubes undergo stretching and squeezing, since the elements of the vortex lines are governed by the same Helmholtz equation $d\delta\mathbf{l}/dt = (\delta\mathbf{l}\nabla)\mathbf{u}$ and hence are not preserved in time. Secondly, by comparing (2.5) and (1.22) we see that inhomogeneous density of the fluid generates fluid vorticity. Therefore, a potential vorticity-free flow (described by a scalar function φ, $\mathbf{u} = \nabla\varphi$) of a stratified medium is an exceptional rather than a typical phenomenon. On the contrary, in a homogeneous fluid if its vorticity vanishes at the initial moment, such a flow remains potential forever.

Now let us consider the inner product with $\nabla\rho$ of the hydrodynamical equations in Fridman's form (2.5):

$$\frac{\partial\mathbf{\Omega}}{\partial t} + (\mathbf{u}\nabla)\mathbf{\Omega} - (\mathbf{\Omega}\nabla)\mathbf{u} = \frac{\nabla\rho\times\nabla p}{\rho^2}. \tag{2.6}$$

The right-hand side vanishes in this case. For the left-hand side we use straightforward transformations

$$\nabla\rho\cdot\big[(\mathbf{u}\nabla)\mathbf{\Omega}\big] = (\mathbf{u}\nabla)\big[(\mathbf{\Omega}\cdot\nabla\rho)\big] - \mathbf{\Omega}\cdot\big[(\mathbf{u}\nabla)\nabla\rho\big],$$
$$\nabla\rho\cdot\big[(\mathbf{\Omega}\nabla)\mathbf{u}\big] = (\mathbf{\Omega}\nabla)\big[(\mathbf{u}\nabla)\rho\big] - \mathbf{u}\cdot\big[(\mathbf{\Omega}\nabla)\nabla\rho\big],$$

and hence their difference gives the second and third terms on the left-hand side:

$$(\mathbf{u}\nabla)(\mathbf{\Omega}\cdot\nabla\rho) - \mathbf{\Omega}\cdot(\mathbf{u}\nabla)\nabla\rho - (\mathbf{\Omega}\nabla)(\mathbf{u}\nabla)\rho + \mathbf{u}\cdot(\mathbf{\Omega}\nabla)\nabla\rho.$$

The incompressibility condition implies that $(\mathbf{u}\nabla)\rho = -\partial\rho/\partial t$, and hence the third term can be transformed further as follows:

$$-(\mathbf{\Omega}\nabla)(\mathbf{u}\nabla)\rho = (\mathbf{\Omega}\nabla)\frac{\partial\rho}{\partial t} = \mathbf{\Omega}\cdot\frac{\partial\nabla\rho}{\partial t}.$$

Thus we come to the equation

$$+\nabla\rho\cdot\frac{\partial\mathbf{\Omega}}{\partial t} + (\mathbf{u}\nabla)(\mathbf{\Omega}\cdot\nabla\rho) + \mathbf{\Omega}\cdot\frac{\partial\nabla\rho}{\partial t} = +\mathbf{\Omega}(\mathbf{u}\nabla)\nabla\rho - \mathbf{u}(\mathbf{\Omega}\nabla)\nabla\rho.$$

The right-hand side of the latter equality vanishes. Indeed, the simplest way to see it is to rewrite it in tensor notation:

$$+\Omega_i u_k \frac{\partial^2\rho}{\partial x_k\partial x_i} - u_k\Omega_i\frac{\partial^2\rho}{\partial x_i\partial x_k} = 0$$

(where the same indices assume summation).

Furthermore,

$$\nabla\rho\cdot\frac{\partial\mathbf{\Omega}}{\partial t} + \mathbf{\Omega}\cdot\frac{\partial\nabla\rho}{\partial t} \equiv \frac{\partial}{\partial t}(\mathbf{\Omega}\cdot\nabla\rho).$$

Hence by introducing the potential vorticity of an incompressible fluid $\Pi = \mathbf{\Omega}\cdot\nabla\rho$, one can pass from Fridman's hydrodynamical equations to the following conservation law:

$$\frac{d\Pi}{dt} \equiv \frac{d}{dt}(\mathbf{\Omega}\cdot\nabla\rho) = 0. \tag{2.7}$$

2.2 The Bernoulli Equation

One of the most curious integrals of motion of an ideal incompressible fluid is the Bernoulli integral (Daniel Bernoulli, 1738). It is also called the Bernoulli equation and it characterizes steady flows along trajectories of its fluid particles. One should emphasize that solutions of the Euler equation describe not trajectories, but rather *stream lines* defined as such curves, whose tangents at any moment coincide with the velocity directions of points on the curves. Stream lines at a moment t are defined as solutions of the equations

$$\frac{dx}{u(\mathbf{x},t)} = \frac{dy}{v(\mathbf{x},t)} = \frac{dz}{w(\mathbf{x},t)} \tag{2.8}$$

($\mathbf{u} = (u, v, w)$ are velocity components in the directions of the x-, y-, and z-axes respectively) and they coincide with the trajectories of motion of the fluid particles only in the stationary case. Indeed, *tangents to stream lines coincide with the velocity directions of different particles at a fixed moment in time, while tangents to the trajectory coincide with the velocity directions for the motion of a fixed particle at different moments: the equation for the trajectory of a fluid particle which at $t = 0$ was located at the point $\mathbf{x} = \mathbf{a}$ has the form* $\frac{d\mathbf{X}(t,\mathbf{a})}{dt} = \mathbf{u}(\mathbf{X}(t,\mathbf{a}),t)$. In the stationary case all particles on one and the same stream line move along one and the same trajectory.

The observation that trajectories and stream lines coincide in steady fluid flows allows one to integrate the Euler equation along its trajectories. First, we would like to stress the following important fact. If an incompressible fluid is stratified, then during its motion a fluid particle never leaves the corresponding iso-density surface. In steady flows iso-density surfaces do not change their location in space as time goes by. Therefore for each fluid particle on an iso-density surface the normal (to the surface) component of the particle velocity must vanish. This implies that in stationary flows of a stratified incompressible fluid the trajectory of each fluid particle, and hence the corresponding stream line, entirely belongs to the surface of constant density.

Now it is not difficult to find a first integral for a steady Euler equation along the stream line. For this we use the Bernoulli form (1.10′) of the equation, and take its inner product with an arbitrary element of the arc $d\mathbf{l}$ of a stream line

$$(\mathbf{u} \times \operatorname{rot} \mathbf{u}) \cdot d\mathbf{l} = \left(\frac{1}{\rho}\nabla p + \frac{1}{2}\nabla \mathbf{u}^2\right) \cdot d\mathbf{l}. \tag{2.9}$$

The left-hand side of (2.9) vanishes, since the directions of $d\mathbf{l}$ and $\mathbf{u}$ coincide. The right-hand side is a complete differential in the direction of $d\mathbf{l}$, since the integration is carried over a curve belonging to an iso-density surface, and ρ^{-1} can be taken under the gradient sign as a constant. Hence, along the trajectory of a fluid particle the following equality holds:

$$\frac{p}{\rho} + \frac{\mathbf{u}^2}{2} = \operatorname{const}(\mathbf{x}_0), \tag{2.10}$$

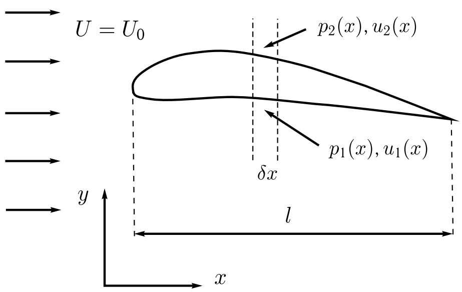

Fig. 2.2 A constant flow of an incompressible fluid past a narrow wing

where the integration constant depends on the trajectory and it is determined uniquely by any point $\mathbf{x}_0$ on it.

The equality (2.10) is called the *Bernoulli equation* and it holds for an arbitrary steady flow of an incompressible stratified fluid (and in particular, even when $\operatorname{rot}\mathbf{u} \neq 0$). In the case of a potential flow of a stratified fluid ($\mathbf{u} = \nabla\varphi$) the left-hand side of (2.9) vanishes for any element $d\mathbf{l}$ (not necessarily tangent to the stream line). Therefore the integration constant in the Bernoulli equation should be replaced by $\text{const}(\rho)$, i.e., by a constant depending on the choice of an iso-density surface only, but not on the trajectory of the fluid particle. Finally, for stationary potential flows of a homogeneous fluid, the integration constant is universal for the whole fluid domain.

2.3 Why do Planes Fly?

This is a classical example for an application of the Bernoulli integral. Let a narrow wing have a profile as depicted in Fig. 2.2. Suppose that it is immersed in a fluid which has constant density $\rho = \rho_0$ and is approaching this wing with constant velocity $\mathbf{u} = \mathbf{u}_0$. Assume also that at the initial moment the velocity circulation

$$K = \oint_C \mathbf{u} \cdot \mathbf{dl}$$

over a closed fluid contour C encompassing the wing arbitrarily close to its boundary is nonzero. Since the normal component of velocity to the wing surface is zero, the fluid contour C will preserve the above properties for an arbitrary long time. The value of K at any moment will be equal to its initial value due to its Lagrangian invariance.[1]

Employing the Bernoulli integral, one obtains

$$p_1 - p_2 = \frac{1}{2}\rho_0\big(u_2^2 - u_1^2\big) = \frac{1}{2}\rho_0(u_1 + u_2)(u_2 - u_1),$$

[1] In a viscous fluid the velocity circulation around the wing is forced, e.g., by a propeller.

where the values with indices 1 and 2 refer to the bottom and top edges of the wing respectively. Since the wing is narrow (the ratio of its width to its length is much smaller than 1) and due to the conservation of mass one has $(u_1+u_2)/2 = u_0$. Hence

$$p_1 - p_2 = \rho_0 u_0 (u_2 - u_1),$$

while the vertical component of the resulting force is equal to

$$F = \rho_0 u_0 \int_0^l (u_2 - u_1) dx,$$

where l is the wing's length. The above-mentioned circulation is

$$K = \oint_C \mathbf{u} \cdot \mathbf{dl} = \int_0^l u_1 dx + \int_l^0 u_2 dx = -\int_0^l (u_2 - u_1) dx$$

(here the positive direction is counterclockwise). Whence one has

$$F = -\rho_0 u_0 K, \tag{2.11}$$

which is the Kutta–Zhukovsky theorem on the lifting force on the wing. The force is indeed lifting provided that $K < 0$, i.e., if the circulation happens to be clockwise. In the opposite case the wing is pressed to the ground. In the latter case it is called anti-wing and it is used, e.g., in Formula One racing cars to strengthen their traction with the road. The required direction of circulation is achieved by the attack angle and the wing profile in the direction along the flow.

2.4 Conservation Laws for the Momentum and Energy of an Incompressible Fluid

For an ideal incompressible fluid the Lagrangian invariance of a quantity $\theta = \theta(t, \mathbf{x})$, i.e., $\frac{d\theta}{dt} = 0$, implies conservation of the integral

$$\Theta(t) = \int_V \theta d\mu, \qquad \frac{d\Theta}{dt} = 0, \tag{2.12}$$

provided that the fluid fills a bounded volume V and there are no exterior forces acting on it. Indeed, the Lagrangian invariance of θ along with the incompressibility property $\operatorname{div} \mathbf{u} = 0$ can be written in the form

$$\frac{\partial \theta}{\partial t} = -\mathbf{u} \nabla \theta = -\operatorname{div}(\mathbf{u}\theta). \tag{2.13}$$

Integrate (2.13) over the fluid domain and use the Gauss–Ostrogradskii formula:

$$\frac{\partial}{\partial t}\int_V \theta d\mu \equiv \frac{d\Theta}{dt} = -\int_V \operatorname{div}(\mathbf{u}\theta)d\mu = -\oint_{\partial V} \theta\mathbf{u}\cdot d\boldsymbol{\sigma}, \tag{2.14}$$

where ∂V is the closed surface which bounds the domain V, and $d\boldsymbol{\sigma}$ is an oriented area element of this surface. This implies the statement of the conservation law (2.12), since there is no flux across the boundary ∂V ($\mathbf{u}\cdot d\boldsymbol{\sigma} = 0$) and hence the right-hand side of (2.14) vanishes.

The converse statement is not true in general. For instance, under the above assumptions the total kinetic energy of an ideal incompressible fluid must be conserved due to the absence of dissipation and other types of energy. However, simple physical reasoning shows that this is not true locally because an individual fluid particle interacts with the surrounding medium and can lose or gain energy as a result of the constraint reaction forces, i.e., the hydrodynamical pressure.

The following question arises: what is the form of local conservation laws for quantities whose total integrals are preserved? In this relation it is interesting to derive local conservation laws for momentum and energy directly from the equations of motion (1.10) and (1.11) and to verify with their help the invariance of the corresponding integral quantities. This is all the more so since a priori nothing indicates that the Euler equations would imply the conservation of total momentum and total energy.

2.4.1 The Local Momentum Conservation Law

In this case it is convenient to write the equations of motion (1.10), (1.11) in tensor notation:

$$\rho\frac{\partial u_i}{\partial t} + \rho u_k\frac{\partial u_i}{\partial x_k} = -\frac{\partial p}{\partial x_i}, \tag{2.15}$$

$$\frac{\partial u_k}{\partial x_k} = 0, \qquad \frac{\partial \rho}{\partial t} + u_k\frac{\partial \rho}{\partial x_k} = 0. \tag{2.16}$$

Now, based on (2.15) and (2.16), we find the rate of change for the momentum of an individual fluid particle of unit volume to be

$$\begin{aligned}\frac{\partial \rho u_i}{\partial t} &= -u_i u_k\frac{\partial \rho}{\partial x_k} - \rho u_k\frac{\partial u_i}{\partial x_k} - \frac{\partial p}{\partial x_i} = -u_k\left(u_i\frac{\partial \rho}{\partial x_k} + \rho\frac{\partial u_i}{\partial x_k}\right) - \frac{\partial p}{\partial x_i}\\ &= -u_k\frac{\partial \rho u_i}{\partial x_k} - \frac{\partial p}{\partial x_i} = -\frac{\partial \rho u_i u_k}{\partial x_k} - \frac{\partial p}{\partial x_i},\end{aligned}$$

where the last equality is due to $\frac{\partial u_k}{\partial x_k} = 0$. It follows that the local momentum conservation law can be written in the form

$$\frac{\partial \rho u_i}{\partial t} = -\frac{\partial \Gamma_{ik}}{\partial x_k}, \tag{2.17}$$

where

$$\Gamma_{ik} = \rho u_i u_k + \delta_{ik} p \tag{2.18}$$

is the density tensor for the momentum flux, whose components describe the flux of the ith component of the momentum of a fluid parcel of unit volume in the direction of the kth component of the velocity. The latter becomes evident if one integrates (2.17) over any closed fluid domain V and applies the Gauss–Ostrogradskii formula

$$\frac{\partial}{\partial t} \int\limits_V \rho u_i d\mu = - \int\limits_V \frac{\partial \Gamma_{ik}}{\partial x_k} d\mu = - \oint\limits_{\partial V} \Gamma_{ik} n_k d\sigma. \tag{2.19}$$

Here $d\mu$ is a volume element, ∂V is the closed surface which bounds the volume V, $n_k d\sigma$ is an area element of the surface, which is oriented by the exterior normal n_k. (*The application of the Gauss–Ostrogradskii formula consists of the replacement* $d\mu\, \partial/\partial x_k \to n_k d\sigma$.) Now equality (2.19) complemented by (2.18) can be rewritten in the vector form

$$\frac{\partial}{\partial t} \int\limits_V \rho \mathbf{u} d\mu = - \oint\limits_{\partial V} \big[\rho \mathbf{u}(\mathbf{u} \cdot \mathbf{n}) + p\mathbf{n}\big] d\boldsymbol{\sigma}. \tag{2.20}$$

It follows that the vector whose components are $\Gamma_{ik} n_k$ and which is equal to

$$\boldsymbol{\Gamma} = \rho \mathbf{u}(\mathbf{u} \cdot \mathbf{n}) + p\mathbf{n} \tag{2.21}$$

defines the momentum flux in the direction $\mathbf{n}$, i.e., across a surface element of the unit area orthogonal to $\mathbf{n}$.

Note that it is convenient to use the vector $\boldsymbol{\Gamma}$ when posing the boundary conditions, since according to (2.21) the momentum flux in the velocity direction is equal to $\rho \mathbf{u}^2 + p$, while this flux in any normal to the velocity direction is merely p. Therefore, in particular, on the boundary between two non-mixing media the pressure must be continuous, while the velocity may have a jump.

Now let V be the whole fluid domain. Then the right-hand side of (2.20) vanishes since there is no fluid flux across any surface element ∂V and since the total pressure acting on ∂V vanishes. Thus the invariance of the total integral of the momentum follows from the local momentum conservation law (2.17), (2.18).

2.4.2 The Local Energy Conservation Law

The local energy conservation law can be derived in a similar way by computing the rate of change for the density of the kinetic energy $\rho \mathbf{u}^2/2$ with the help of the equations of motion (1.10) and (1.11) (the reader may want to do it as a useful exercise). Here we present the result based on physical reasoning, which is often useful to undertake before a formal mathematical consideration.

Let V be an arbitrary fluid domain bounded by a closed surface ∂V. Then the rate of change for the kinetic energy of the fluid inside this domain is composed of the flux of the energy through the surface ∂V and the work done by the constraint reaction forces, i.e., by the pressure, over the given fluid volume in the unit time. Mathematically this is expressed as the following equality:

$$\frac{\partial}{\partial t}\int_V \frac{1}{2}\rho\mathbf{u}^2 d\mu = -\oint_{\partial V}\left(\frac{1}{2}\rho\mathbf{u}^2\right)\mathbf{u}\cdot d\boldsymbol{\sigma} - \oint_{\partial V} p\mathbf{u}\cdot d\boldsymbol{\sigma}. \tag{2.22}$$

The minuses on the right-hand side of (2.22) mean that $d\boldsymbol{\sigma}$ is directed as the exterior normal to the surface ∂V.

By applying the Gauss–Ostrogradskii formula to the right-hand side of (2.22) and making equal the integrands of the left-hand and right-hand sides, we conclude that the local energy conservation law is given by the equality

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho\mathbf{u}^2\right) = -\mathrm{div}\left[\mathbf{u}\left(\frac{1}{2}\rho\mathbf{u}^2 + p\right)\right]. \tag{2.23}$$

This implies the conservation of the total kinetic energy of the flow. One can easily see this by setting ∂V in (2.22) to be the boundary of the whole flow domain, and using that the normal component of the velocity vanishes on ∂V.

2.5 Exercise

1. Derive the local energy conservation law (2.23) directly from the equations of motion (1.10) and (1.11).

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Chapter 3
Helicity; Equations of Gas Dynamics; The Ertel Invariant

3.1 The Helicity Invariant

The notion of helicity

$$\chi \doteq \mathbf{u} \cdot \boldsymbol{\Omega} \quad (\boldsymbol{\Omega} \doteq \operatorname{rot} \mathbf{u}), \tag{3.1}$$

although being less widely known, is rather important for the description of such phenomena as tornadoes and typhoons. Unlike stream lines, the vortex lines are frozen into the fluid, according to the Kelvin theorem. Hence for non-stationary processes, the mutual location of vortex and stream lines, i.e., the structure and topology of the flow, change in time. The value of χ serves as a measure of this local structure change. On the other hand, intuition suggests that if the vortex lines are knotted or linked, the total number of such linkings should not change during the evolution, at least for an unbounded volume of fluid, since according to the Kelvin theorem the vortex lines are never born and never disappear. This is why it is interesting to derive the evolution equation for helicity to resolve the question on the existence of an integral topological invariant. For this purpose we are going to use the Euler equation in the Bernoulli form (1.10′):

$$\frac{\partial \mathbf{u}}{\partial t} = \mathbf{u} \times \boldsymbol{\Omega} - \frac{\nabla p}{\rho} - \nabla\left(\frac{\mathbf{u}^2}{2}\right), \tag{3.2}$$

while the vorticity equation can be put in the following form:

$$\frac{\partial \boldsymbol{\Omega}}{\partial t} = \operatorname{rot}(\mathbf{u} \times \boldsymbol{\Omega}) - \operatorname{rot} \frac{\nabla p}{\rho}. \tag{3.3}$$

By multiplying (3.2) and (3.3) by $\boldsymbol{\Omega}$ and $\mathbf{u}$ respectively and adding them together we get on the left-hand side:

$$\partial(\mathbf{u} \cdot \boldsymbol{\Omega})/\partial t = \partial \chi/\partial t, \quad \chi = \mathbf{u} \cdot \boldsymbol{\Omega}.$$

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On the right-hand side we make a note of the term $(\mathbf{u} \times \boldsymbol{\Omega})\boldsymbol{\Omega}$, which vanishes identically. Then the expression $\mathbf{u} \cdot \mathrm{rot}(\mathbf{u} \times \boldsymbol{\Omega})$ can be rewritten by using the formulas

$$\mathrm{div}(\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot \mathrm{rot}\,\mathbf{A} - \mathbf{A} \cdot \mathrm{rot}\,\mathbf{B}, \tag{3.4}$$

$$(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}). \tag{3.5}$$

(Here $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ are arbitrary sufficiently smooth vector fields.) Thus we obtain the following expression: $\mathbf{u} \cdot \mathrm{rot}(\mathbf{u} \times \boldsymbol{\Omega}) = \mathrm{div}[\boldsymbol{\Omega}(\mathbf{u} \cdot \mathbf{u}) - \mathbf{u}(\mathbf{u} \cdot \boldsymbol{\Omega})]$.

The right-hand side of the equation for the helicity evolution assumes the form

$$\mathrm{div}\big[\boldsymbol{\Omega}(\mathbf{u} \cdot \mathbf{u}) - \mathbf{u} \cdot \chi\big] - \mathrm{div}\left[\left(\frac{p}{\rho} + \frac{(\mathbf{u} \cdot \mathbf{u})}{2}\right) \cdot \boldsymbol{\Omega}\right] - \boldsymbol{\Omega} \cdot \frac{\nabla p}{\rho} + \nabla\left(\frac{p}{\rho}\right) \cdot \boldsymbol{\Omega} - \mathbf{u} \cdot \mathrm{rot}\left(\frac{\nabla p}{\rho}\right). \tag{3.6}$$

Further, we use the vector identity

$$\mathrm{rot}(\alpha \mathbf{A}) = \nabla\alpha \times \mathbf{A} + \alpha\,\mathrm{rot}\,\mathbf{A}, \tag{3.7}$$

(where α is any scalar field) to transform the term $(\mathbf{u} \cdot \mathrm{rot}(\frac{\nabla p}{\rho}))$.

Then the right-hand side simplifies to

$$-\mathrm{div}\left[\mathbf{u} \cdot \chi + \boldsymbol{\Omega} \cdot \left(\frac{p}{\rho} - \frac{(\mathbf{u} \cdot \mathbf{u})}{2}\right)\right] - \frac{p}{\rho^2} \cdot (\boldsymbol{\Omega} \cdot \nabla\rho) + \mathbf{u} \cdot \left(\frac{\nabla\rho}{\rho^2} \times \nabla p\right). \tag{3.8}$$

We see that the right-hand side contains the by-now-familiar to us potential vorticity $\Pi = \boldsymbol{\Omega} \cdot \nabla\rho$ of an incompressible stratified fluid. We now obtain the evolution equation for helicity χ:

$$\frac{\partial\chi}{\partial t} = -\mathrm{div}\left[\mathbf{u} \cdot \chi + \boldsymbol{\Omega} \cdot \left(\frac{p}{\rho} - \frac{(\mathbf{u} \cdot \mathbf{u})}{2}\right)\right] - \frac{p}{\rho^2} \cdot \Pi + \mathbf{u} \cdot \left(\frac{\nabla\rho}{\rho^2} \times \nabla p\right).$$

By taking into account the easily derivable relation

$$+\mathrm{div}\left[\frac{p}{\rho^2} \cdot (\nabla\rho \times \mathbf{u})\right] = -\frac{p}{\rho^2} \cdot \Pi + \frac{\nabla p}{\rho^2} \cdot (\nabla\rho \times \mathbf{u}),$$

one obtains the evolution equation for local helicity:

$$\frac{\partial\chi}{\partial t} = -\mathrm{div}\left[\mathbf{u} \cdot \chi + \boldsymbol{\Omega} \cdot \left(\frac{p}{\rho} - \frac{(\mathbf{u} \cdot \mathbf{u})}{2}\right) + \frac{p}{\rho^2} \cdot (\nabla\rho \times \mathbf{u})\right] - \frac{2p}{\rho^2}\Pi + \frac{\nabla p}{\rho^2} \cdot (\nabla\rho \times \mathbf{u}) + \mathbf{u} \cdot \frac{\nabla\rho}{\rho^2} \times \nabla p.$$

Using the formulas

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$$

one has

$$\nabla p \cdot (\nabla \rho \times \mathbf{u}) = \mathbf{u} \cdot (\nabla p \times \nabla \rho) = -\mathbf{u} \cdot (\nabla \rho \times \nabla p),$$

which eliminates the last two terms from the right-hand side of the expression for the helicity evolution and brings it to the divergent form

$$\frac{\partial \chi}{\partial t} = -\mathrm{div}\left\{\mathbf{u} \cdot \chi + \boldsymbol{\Omega} \cdot \left(\frac{p}{\rho} - \frac{(\mathbf{u} \cdot \mathbf{u})}{2}\right) + \frac{p}{\rho^2} \cdot (\nabla \rho \times \mathbf{u})\right\} - \frac{2p}{\rho^2} \Pi. \tag{3.9}$$

This expression establishes that the source of helicity is the potential vorticity of the fluid. If the latter is absent, i.e., in the case of a homogeneous medium, the right-hand side of (3.9) has the divergent form. In turn, this means that the total helicity

$$\mathrm{H} = \int_V \chi d\mathrm{V} \tag{3.10}$$

satisfies

$$\begin{aligned} \frac{d\mathrm{H}}{dt} &= -\int_V \mathrm{div}\left[\mathbf{u} \cdot \chi + \boldsymbol{\Omega}\left(\frac{p}{\rho_0} - \frac{1}{2}\mathbf{u}^2\right)\right] d\mu \\ &= -\oint_{\partial V} \left[\mathbf{u} \cdot \chi + \boldsymbol{\Omega}\left(\frac{p}{\rho_0} - \frac{1}{2}\mathbf{u}^2\right)\right] d\boldsymbol{\sigma} \end{aligned}$$

and is preserved, provided that not only the stream lines but also the vortex lines are tangent to the surface ∂V which bounds the flow domain V.

The value of H, which is called the *helicity invariant* (discovered by J.J. Moreau in 1961 and rediscovered by H.K. Moffatt in 1969) characterizes the degree of knottedness or linking of the vortex lines. Note here without proof that, for instance, in the case of two simply linked vortex rings with strengths Γ_1 and Γ_2 (see Fig. 8.3a) the value of helicity is $H = \pm|\Gamma_1 \Gamma_2|$, where the sign depends on whether or not each of the rings moves in the direction of the vorticity of the other. This statement will be proved in Chap. 8, where we introduce the notion of singular vortex lines and discuss their behavior.

3.2 Equations of Gas Dynamics or Equations of an Ideal Compressible Fluid

Recall that an ideal incompressible fluid in a steady state is described by the only physical quantity, i.e., density, which measures the degree of inertia of the medium. This is why the equations of motion of an incompressible fluid can be formulated solely within the mechanical framework, where the pressure in such a medium is understood as the constraint reaction and it vanishes when there is no relative motion.

In the case of a compressible fluid the situation changes drastically. The density alone is not enough to describe the local physical state of the fluid, since under uniform stretching or squeezing the density and pressure can assume various values even when there is no relative motion. The momentum and energy conservation laws form an incomplete system of equations, and to close it up one needs additional considerations.

Taking the above into account, an ideal compressible fluid can be thought of as a thermodynamical system. More precisely, it is a collection of thermo-isolated thermodynamical systems that are fluid parcels whose macroscopic state is described by two independent parameters, for instance, pressure p and density ρ. (This is true provided that there are no chemical reactions or phase transitions. Otherwise the chemical potential must be added to the list of independent parameters.) Such a consideration is possible if the time required to come to a thermodynamic equilibrium is much smaller than the characteristic time for medium macroscopic changes. This is one of the fundamental assumptions on which N.N. Bogolyubov based his derivation of the hydrodynamical equations from the Boltzmann equation, see e.g., (Uhlenbeck and Ford, 1965). In the latter case the laws of equilibrium thermodynamics are applicable locally and to close up the equations of motion one can use the second law of thermodynamics. According to this law, the entropy of a thermo-isolated system remains constant (and the energy exchange between particles takes place only due to the work of the forces of pressure). This means that the entropy of an individual particle of an ideal compressible fluid is a Lagrangian invariant. Then the equations of motion can be written in the form

$$\frac{d\mathbf{u}}{dt} \doteq \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p, \tag{3.11}$$

$$\frac{d\rho}{dt} \doteq \frac{\partial \rho}{\partial t} + (\mathbf{u}\nabla)\rho = -\rho\,\mathrm{div}\,\mathbf{u}, \tag{3.12}$$

$$\frac{ds}{dt} \doteq \frac{\partial s}{\partial t} + (\mathbf{u}\nabla)s = 0, \tag{3.13}$$

$$s = s(\rho, p). \tag{3.14}$$

Here $s = s(\rho, p)$ is the specific entropy, i.e., the entropy of an individual fluid parcel of the unit mass. It is assumed to be a known function of ρ and p as a function of the state of a thermodynamic equilibrium system. The system (3.11)–(3.14) is also called the equations of an ideal gas dynamics.

Sometimes it is convenient to write the adiabatic motion condition (3.13) in the form of the local mass conservation law

$$\frac{\partial \rho s}{\partial t} + \mathrm{div}(\rho s\,\mathbf{u}) \equiv \frac{d\rho s}{dt} + \rho s\,\mathrm{div}\,\mathbf{u} = 0, \tag{3.13$'$}$$

while the Euler equation (3.11) can be written in the Bernoulli form (see (1.12) and (1.10′))

$$\frac{\partial \mathbf{u}}{\partial t} - \mathbf{u}\times\operatorname{rot}\mathbf{u} = -\frac{1}{\rho}\nabla p - \nabla\left(\frac{1}{2}\mathbf{u}^2\right). \qquad (3.11')$$

In general, it is useful to remember that the quantity $\varphi = \varphi(t, \mathbf{x})$ is a Lagrangian invariant of the gas dynamics if and only if the equation for the quantity $\Phi = \rho\varphi$ can be written as the local mass conservation law.

3.3 Isentropic Motion of a Compressible Fluid

One of the important cases of motion of a compressible fluid is its *isentropic motion*, where by definition the entropy has the same value at all points of the fluid. This process is physically possible, since if at the initial moment the entropy has the same value for all fluid particles, so will it be for the whole motion due to entropy's Lagrangian invariance (3.13). The condition of entropy's constancy

$$s(\rho, p) = \text{const} \qquad (3.15)$$

determines the density ρ, and hence any other thermodynamical quantity, as a function of pressure p only. (Recall that we do not consider here processes related to phase transitions or chemical reactions.) The motion for which the density is a function of the pressure only is called *barotropic*. Thus isentropic motion is barotropic. The converse is also true. Indeed, the dependence $\rho = \rho(p)$ implies the dependence $s = s(p)$, and according to (3.13)

$$\frac{ds}{dt} = \frac{ds}{dp}\cdot\frac{dp}{dt} = 0 \Rightarrow \frac{ds}{dp} = 0 \Rightarrow s(p) = \text{const},$$

since p is physically not a passive scalar, and hence $dp/dt \neq 0$.

Now for the function $f(p) \equiv 1/\rho(p)$, we introduce its primitive

$$W(p) = \int f(p)dp = \int \frac{dp}{\rho}, \qquad \frac{dW}{dp} = \frac{1}{\rho}, \qquad (3.16)$$

which allows us to represent the right-hand side of the Euler equation (3.11) in the form

$$-\frac{1}{\rho}\nabla p = -\frac{dW}{dp}\nabla p = -\nabla W.$$

Therefore for an isentropic or barotropic motion of a compressible fluid the equations of gas dynamics assume a much simpler form:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\nabla)\mathbf{u} = -\nabla W, \qquad \frac{\partial \rho}{\partial t} + \operatorname{div}(\rho\mathbf{u}) = 0, \quad W = W(p), \qquad (3.17)$$

where W is supposed to be a known function of the pressure p and is called the *specific enthalpy* or the *heat function* of the fluid unit mass. (Recall that, generally speaking, enthalpy of a thermodynamical system of the unit mass is defined by the equality $dW = Tds + \rho^{-1}dp$, which for an isentropic process coincides with the definition of the primitive (3.16). Here T is the absolute temperature of the medium.)

Equations (3.17) are called the *equations of motion for a barotropic fluid* and they possess a number of remarkable properties. In particular, the Kelvin theorem on the conservation of circulation over any closed fluid contour holds for them, as well as does the Bernoulli equation hold for stationary trajectories of fluid particles, which can be written in this case in the following form:

$$\frac{1}{2}\mathbf{u}^2 + W(p) = \text{const.} \tag{3.18}$$

(Let us remind the reader that if the flow is not potential, the choice of the constant in (3.18) is determined by the trajectory along which we integrate.)

The above mentioned properties directly follow from the fact that the right-hand side of Eq. (3.17) becomes a complete differential after taking its inner product with an element $\delta\mathbf{r}$ of the integration curve (see Chap. 2, Sect. 2.2). From this point of view a barotropic fluid is analogous to a homogeneous incompressible fluid. Note that the latter formally satisfies the definition of a barotropic fluid, for which $W = p/\rho_0$, and it is the *only incompressible barotropic fluid* (why?).

3.4 The Kelvin Theorem and the Bernoulli Integral in Gas Dynamics

For non-isentropic, also called *baroclinic*, *motion* of an ideal compressible fluid the Kelvin theorem does not hold in general. However, similar to the fibration of an ideal nonhomogeneous fluid into isopycnal surfaces, an ideal compressible fluid is fibered into non-intersecting isentropic surfaces. Indeed, a fluid particle belonging initially to one of such surfaces remains sitting on it during the whole evolution due to the Lagrangian invariance of the quantity s. For the same reason the same happens with a closed fluid contour initially belonging to a surface of constant entropy. By taking into account that the motion along such surfaces is barotropic, according to the preceding section, one can apply to this motion the Kelvin theorem and the Bernoulli equation. In other words, the following statements hold.

I. During a baroclinic motion of an ideal compressible fluid the velocity circulation over a closed fluid contour lying on an isentropic surface is preserved.

II. For stationary flows of an ideal compressible fluid the quantity $B = \frac{1}{2}u^2 + W(p, s_0)$ is preserved along the trajectories of fluid particles, where the value of $W(p, s_0) = \int dp/\rho(p, s_0)$ depends on $s(\rho, p) = s_0$ as a parameter defining the surface to which this trajectory belongs.

Being applied to an infinitesimally small closed contour C_0, belonging to an isentropic surface $s(\rho, p) = s_0$, the statement I means that the quantity

$$\mathrm{K}_0 = \oint_{C_0} \mathbf{u} \cdot d\mathbf{r} = \operatorname{rot} \mathbf{u} \cdot d\boldsymbol{\sigma}_0 \tag{3.19}$$

is a Lagrangian invariant. Here $d\boldsymbol{\sigma}_0$ is an oriented area element of the isentropic surface bounded by the contour C_0. In addition to the surface $s(\rho, p) = s_0$, let us consider infinitesimally close to it the isentropic surface $s(\rho, p) = s_0 + \delta s$. Through the contour C_0, we draw a liquid cylindrical surface, whose intersection with the additional isentropic surface is also a closed contour (see Fig. 2.1). The ends of the constructed cylindrical element will remain on the corresponding isentropic surfaces during the evolution. Then the mass of the fluid parcel under consideration can be found according to the formula

$$\mathrm{M} = \rho h \, \mathbf{n} \cdot d\boldsymbol{\sigma}_0, \tag{3.20}$$

where $\mathbf{n}$ is the normal to the surface s_0 whose direction coincides with ∇s and h is the cylinder height. Because of the infinitesimal proximity of the isentropic surfaces, the value of δs can be written as

$$\delta s = \nabla s \cdot \mathbf{n} \, h. \tag{3.21}$$

By comparing (3.20) and (3.21) we conclude that

$$d\boldsymbol{\sigma}_0 = \frac{\mathrm{M}}{\delta s} \cdot \frac{\nabla s}{\rho}. \tag{3.22}$$

Plug (3.22) into (3.19) and recall that δs and M do not change during the evolution: the former is preserved according to the definition of δs, while the latter does not change as the mass of the given fluid volume which consists of the same particles at any moment. This allows us to conclude that the invariance of K_0 implies the invariance of the quantity

$$\Pi_E = \frac{\operatorname{rot} \mathbf{u} \cdot \nabla s}{\rho}. \tag{3.23}$$

The latter quantity is called the *potential vorticity of the equations of gas dynamics* or the *Ertel invariant*. H. Ertel (1942) proved the invariance of Π_E for the first time directly from the equations of motion (3.11)–(3.14) by vector calculus methods (see Exercise 2). The discussed above elegant and physically transparent derivation of the invariance of Π_E, as well as its direct relation to the Kelvin theorem, was proposed by F. Moran (1942) the same year, and then, apparently independently, reconstructed by J.G. Charney (1948) in his famous work on the dynamics of global geophysical flows.

3.5 Exercises

1. Why can a barotropic fluid of nonhomogeneous density not be incompressible?
2. The balance equation for the helicity of a homogeneous rotating fluid in a potential field Φ has the form

$$\frac{\partial}{\partial t}\left[\mathbf{u}(\operatorname{rot}\mathbf{u}+4\boldsymbol{\Omega}_0)\right] = -\operatorname{div}\left\{\mathbf{u}\left[\mathbf{u}(\operatorname{rot}\mathbf{u}+2\boldsymbol{\Omega}_0)\right]+\left(\frac{p}{\rho_0}+\Phi-\frac{1}{2}\mathbf{u}^2\right)\operatorname{rot}\mathbf{u}+4\boldsymbol{\Omega}_0\left(\frac{p}{\rho_0}+\Phi\right)\right\}.$$

Show this by using the equations of motion written in a rotating coordinate system (see Sect. 1.4 in Kurgansky 1993)

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Chapter 4
The Rossby–Obukhov Potential Vortex; Energy and Momentum of a Compressible Fluid; Hydrodynamic Approximation of Equations of Gas Dynamics

4.1 The Rossby–Obukhov Potential Vortex in Shallow-Water Theory

The notion of a potential vortex is of crucial importance for geophysical fluid dynamics and dynamic meteorology. So in addition to its formal expressions $\Pi \equiv \mathbf{\Omega} \cdot \nabla \rho$ (see (2.4)) and $\Pi_E = \frac{\mathrm{rot}\,\mathbf{u} \cdot \nabla s}{\rho}$ (see (3.23)), we would like to provide the reader with its expression for the *equations of shallow-water theory*. The latter are equations describing a two-dimensional motion of a thin layer of an ideal incompressible fluid of constant density ρ_0 with a free surface in a gravitational field (Fig. 4.1). The layer thinness condition means that the characteristic horizontal scale L of the flow is much greater than the layer thickness H. This allows one to neglect vertical accelerations of particles in comparison with both the gravity acceleration g and the dependence of the horizontal components of velocity on the vertical coordinate z. In this case, the pressure p satisfies the quasi-hydrostatic relation

$$\frac{\partial p}{\partial z} + g\rho_0 = 0, \qquad p = \rho_0 g\big(H(x, y, t) - z\big), \tag{4.1}$$

while the equations of motion assume the form

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -g\frac{\partial H}{\partial x}, \tag{4.2}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -g\frac{\partial H}{\partial y}, \tag{4.3}$$

$$\frac{dH}{dt} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \equiv \frac{\partial H}{\partial t} + \frac{\partial (Hu)}{\partial x} + \frac{\partial (Hv)}{\partial y} = 0. \tag{4.4}$$

Here $u = u(x, y, t)$, $v = v(x, y, t)$ are z-independent velocity components in the directions of x- and y-axes respectively. Equation (4.4) is obtained by integrating in z the divergence-free condition for the three-dimensional velocity field $\partial u/\partial x +$

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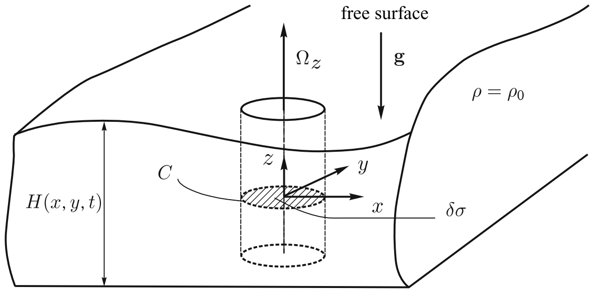

Fig. 4.1 The shallow-water theory describes a two-dimensional motion of vertical liquid columns in a layer of fluid of constant density $\rho = \rho_0$ with a free surface in the field of gravity

$\partial v/\partial y + \partial w/\partial z = 0$, given that $w(z = 0) = 0$ (the layer rests on an impermeable surface) and that $w(z = H) = dH/dt$ by definition.

Equations (4.2)–(4.4) have a beautiful interpretation in terms of gas dynamics. Namely, they describe barotropic motion of a two-dimensional compressible fluid whose pressure and density are constrained by the polynomial relation (polytropic gas)

$$p = \frac{1}{2}\alpha g \rho^2,$$

where α is a dimensional constant. Substituting this relation into (3.11) and making the change $\alpha\rho = H$ in (3.11) and (3.12), one obtains the shallow-water equations.

Now one can proceed with a formal approach and apply the Kelvin theorem to the barotropic motion of a two-dimensional gas. According to this theorem, the value

$$K = \Omega_z d\sigma$$

is a Lagrangian invariant. Here $\Omega_z \doteq \partial v/\partial x - \partial u/\partial y$ is the vertical component of vorticity (all other components vanish because of the two-dimensionality of the motion) and $d\sigma$ is the area of any horizontal cross-section of a liquid column. This area is bounded by an infinitesimal closed contour C (see Fig. 4.1) contained entirely in the plane (x, y). Note that $d\sigma = m/\rho_0 H$, where the mass m of the column comprised of the same particles remains unchanged during the motion. Then the Lagrangian invariance of K implies Lagrangian invariance of the quantity

$$\Pi_{RO} = \Omega_z / H. \tag{4.5}$$

This quantity is called *the potential vorticity for shallow-water equations*. It was in this form that the concept of potential vorticity was introduced into geophysical hydrodynamics by Rossby (1939) for the ocean and, independently, by Obukhov (1949) for the atmosphere. The reader can easily verify the invariance of this quantity by means of a direct calculation (Exercise 1).

Recall (see Exercise 1 in Chap. 1) that $\Omega_z = 2\omega$, where ω is the angular velocity of local rotation of the fluid. One should note that the invariance of Π_{RO} in the shallow-water theory is equivalent to the conservation of angular momentum of the liquid cylinder, whose base is the element $d\sigma$, and H is its height (Fig. 4.1). Indeed, the momentum of inertia of the cylinder is

$$I = \frac{1}{2}mr^2 = \frac{1}{2\pi}md\sigma = \frac{1}{2\pi}m\frac{V}{H} = \frac{1}{2\pi}\frac{m^2}{\rho_0}\frac{1}{H},$$

where r is the cylinder radius and $V = d\sigma H$ is its volume. Then its angular momentum is

$$M = I\omega = \frac{1}{4\pi}\frac{m^2}{\rho_0}\frac{\Omega_z}{H}. \tag{4.6}$$

This implies that the singled out cylindrical element behaves similarly to a ballerina or figure skater who, by stretching upwards and raising their arms can speed up their rotation and, vice versa, by "flattening", i.e., by squatting and spreading their arms can slow it down.

The theorem on conservation of potential vorticity also provides some insight into the behavior of an inhomogeneous fluid, whether it is a gas or an incompressible inhomogeneous fluid, which splits into disjoint isentropic or isopycnic surfaces that are intersected by vortex tubes. Besides, due to relation (4.5), areas of higher concentration of the above-mentioned surfaces (i.e., area of smaller values of H) are characterized by lower vorticity (smaller values of Ω_z), and vice versa. This results in alternating higher and lower local twists of the fluid. Areas of higher vorticity can cause turbulence to appear. Therefore, it is possible that turbulent spots observed in both the atmosphere and in the ocean, and chaotically arranged at different heights or depths are formed under the influence of the irregular stratification of compressible or inhomogeneous incompressible fluid.

4.2 Conservation Laws and Fluxes of Energy and Momentum in Compressible Fluids

There is a fundamental distinction between a gas and an incompressible fluid. As a thermodynamical object any gas particle, along with the kinetic energy of its macroscopic motion, possesses an internal energy that serves as a measure of the total kinetic energy of all the molecules forming that particle. The following considerations point to the possibility that while a particle is in motion one type of energy transforms into another. The first law of thermodynamics for a fixed mass system is expressed as follows:

$$\delta\varepsilon = T\delta s - p\delta V = T\delta s + \frac{p}{\rho^2}\delta\rho, \tag{4.7}$$

where ε, T, s, p and V are, respectively, internal energy, absolute temperature, entropy, pressure and volume of the medium considered.

According to (3.12) and (3.13), in the application to an adiabatic motion δs of an individual gas particle of a unit mass, the formula (4.7) can be rewritten as

$$\frac{d\varepsilon}{dt} = \frac{p}{\rho^2}\frac{d\rho}{dt} = -\frac{p}{\rho}\operatorname{div}\mathbf{u}. \tag{4.8}$$

It follows that if $\operatorname{div}\mathbf{u} \neq 0$, then the macroscopic motion of a fluid can cause changes in its internal energy, and vice versa, a change in internal energy inevitably generates a macroscopic current. Therefore, for gases the equation of balance of local energy has to be written for the value

$$E = \rho\left(\frac{1}{2}\mathbf{u}^2 + \varepsilon\right), \tag{4.9}$$

which expresses the total energy per unit volume of the fluid.

Now let us compute $\partial E/\partial t$ using the equations of gas dynamics (3.11)–(3.14). According to (4.7)

$$\delta(\rho\varepsilon) = \varepsilon\delta\rho + \rho T\delta s + \frac{p}{\rho}\delta\rho = \left(\varepsilon + \frac{p}{\rho}\right)\delta\rho + \rho T\delta s.$$

This implies that

$$\frac{\partial\rho\varepsilon}{\partial t} = \left(\varepsilon + \frac{p}{\rho}\right)\frac{\partial\rho}{\partial t} + \rho T\frac{\partial s}{\partial t}.$$

Employing now (3.12) and (3.13) to calculate $\partial\rho/\partial t$ and $\partial s/\partial t$, one obtains

$$\frac{\partial\rho\varepsilon}{\partial t} = -\left(\varepsilon + \frac{p}{\rho}\right)\operatorname{div}(\rho\mathbf{u}) - \rho T(\mathbf{u}\nabla)s. \tag{4.10}$$

Again apply the first law of thermodynamics (4.7) and with $\mathbf{u}\nabla$ replacing a rather arbitrary differential operator δ in front of the quantities ε, s and ρ on both the left- and right-hand sides of the equation. This way we find $T(\mathbf{u}\nabla)s$ and plug it into the expression (4.10). Finally, we have

$$\frac{\partial\rho\varepsilon}{\partial t} = -\operatorname{div}\left\{\rho\mathbf{u}\left(\varepsilon + \frac{p}{\rho}\right)\right\} + \mathbf{u}\nabla p. \tag{4.11}$$

Similarly, (3.11) and (3.12) can be used to compute

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{1}{2}\rho\mathbf{u}^2\right) = \rho\mathbf{u}\frac{\partial\mathbf{u}}{\partial t} + \frac{1}{2}\mathbf{u}^2\frac{\partial\rho}{\partial t} &= -\rho\mathbf{u}(\mathbf{u}\nabla)\mathbf{u} - \frac{1}{2}\mathbf{u}^2\operatorname{div}(\rho\mathbf{u}) - (\mathbf{u}\nabla)p\\ &= -\operatorname{div}\left(\mathbf{u}\frac{1}{2}\rho\mathbf{u}^2\right) - (\mathbf{u}\nabla)p.\end{aligned}$$

Adding the latter equality with (4.11) we can express the energy conservation law for a compressible fluid by the equality

$$\frac{\partial}{\partial t}\left[\rho\left(\frac{1}{2}\mathbf{u}^2+\varepsilon\right)\right]=-\operatorname{div}\left\{\rho\mathbf{u}\left(\frac{1}{2}\mathbf{u}^2+\varepsilon+\frac{p}{\rho}\right)\right\}, \tag{4.12}$$

where $\varepsilon + p/\rho = W$ is an enthalpy per unit mass of the fluid. Its appearance on the right-hand side of (4.12) is due to the work of the pressure forces (cf. (2.23)).

As for the local law of momentum conservation, it assumes the same form for a gas as for an incompressible fluid, i.e., by formulas (2.17) and (2.18) (*why?*). This can be easily verified by a direct calculation using the equations of motion (3.11)–(3.14). (We encourage the reader to do this as a useful exercise.)

4.3 The Speed of Sound

We observed that a gas medium accumulates two kinds of energy, internal and kinetic, along with the possibility of their transformation into each other. This points to the fact that a gas, unlike an incompressible fluid, can perform free, or so-called *eigen*, *oscillations*, i.e., oscillations in the absence of any external influence. These oscillations can be induced by initially creating a domain of higher (or lower) density by a local adiabatic compression (stretching). Because of this work, an excess (or a deficit) of pressure that forms in such a domain prevents further compression (or stretching) and creates, in turn, a reciprocal force. As a result, the areas of compression and stretching will spread throughout the fluid (like circles on the water surface) generating longitudinal (in contrast to the water surface) oscillations of the fluid particles in the vicinity of their original location. This process is known as sound, and its propagation speed is one of the fundamental physical characteristics of a compressible fluid. In particular, this characteristic allows one to formulate sufficient conditions in order to ignore medium's compressibility, thus materializing the seemingly rather abstract concept of an ideal incompressible fluid. In this regard, it is worth recalling the formula for the speed of sound.

If the equilibrium state of a compressible medium is defined by the constant values $\mathbf{u}=0$, $p=p_0$, $\rho=\rho_0$ and $s(\rho_0, p_0)=s_0$, then with pinpoint accuracy, small oscillations of this medium can be described by linear hydrodynamic equations:

$$\frac{\partial \mathbf{u}'}{\partial t}=-\frac{1}{\rho_0}\nabla p', \tag{4.13}$$

$$\frac{\partial \rho'}{\partial t}=-\rho_0\nabla\mathbf{u}', \tag{4.14}$$

where $\mathbf{u}'$, p' and ρ' are small perturbations with zero mean value, and besides

$$s\left(\rho_0+\rho', p_0+p'\right)=s_0, \tag{4.15}$$

because the motion of an ideal compressible medium is adiabatic. (We ignore non-linear terms as quantities of a higher order of magnitude in comparison with perturbations.)

Now regarding the pressure as a function of thermodynamical quantities s and ρ and keeping in mind (4.15), it is easy to see that p' and ρ' are connected by the following relation:

$$p' = \left(\frac{\partial p}{\partial \rho}\right)_{s=s_0} \rho'. \tag{4.16}$$

Obviously, $(\partial p/\partial \rho)_{s=s_0} = \text{const} > 0$ since the excess/deficit of the pressure leads to an increase/decrease of the density. Let us then set

$$\left(\frac{\partial p}{\partial \rho}\right)_{s=s_0} = c^2. \tag{4.17}$$

By substituting expressions (4.16) and (4.17) into Eq. (4.13) we get

$$\frac{\partial \mathbf{u}'}{\partial t} = -\frac{c^2}{\rho_0}\nabla \rho'. \tag{4.18}$$

Now eliminating $\nabla \mathbf{u}'$ from (4.14) and (4.18) we obtain the wave equation for ρ'

$$\frac{\partial^2 \rho'}{\partial t^2} - c^2 \Delta \rho' = 0, \tag{4.19}$$

where $c = \sqrt{(\partial p/\partial \rho)_{s=s_0}}$ is the propagation speed of acoustic waves.

Essentially, the speed of sound characterizes the degree of compressibility of the medium: the higher the c, the less compressible is the medium. As a rule, variations in entropy are small compared to its background value $s_0 = \langle s \rangle$, so the quantity $\sqrt{(\partial p/\partial \rho)_s}$ depends rather weakly on the space-time coordinates. Taking that into account, the speed of sound should be regarded as *a fundamental physical parameter of a continuum, although it does not explicitly enter the equations of motion.*

Expression (4.17) determines the speed of sound by the adiabatic compressibility of the medium. Since in applications one uses absolute temperature T instead of entropy s to describe a thermodynamical system, it is more convenient to rewrite the expression for the speed of sound in terms of isothermal compressibility, applying the following well-known thermodynamic relation:

$$\left(\frac{\partial p}{\partial \rho}\right)_s = \frac{C_p}{C_v}\left(\frac{\partial p}{\partial \rho}\right)_T, \tag{4.20}$$

where C_p and C_v are specific heat capacities of the medium at constant pressure and constant volume, respectively.

In particular, for an ideal gas (from a thermodynamical point of view), whose state is described by the Mendeleev–Clapeyron equation

$$p = \rho R_0 T/\mu = \rho R T \tag{4.21}$$

(where R_0 is the universal gas constant, μ is the molecular weight, and R is the universal gas constant), the speed of sound is given by

$$c = \sqrt{\gamma RT} \quad (\text{where} \gamma = C_p/C_v). \tag{4.22}$$

In this formula the value of γ is usually weakly dependent on the temperature.

Consequently, the speed of sound in a gas is $c \sim \sqrt{T}$ and it *is almost pressure-independent at a fixed temperature.*

4.4 Hydrodynamic Approximation of the Equations of Gas Dynamics

From a formal thermodynamical perspective the incompressibility of a medium means that

$$c^2 = \left(\frac{\partial p}{\partial \rho}\right)_s = \infty.$$

Indeed, let us fix an arbitrary volume V of the medium bounded by a closed surface S, and subject it to an adiabatic compression by a fixed excessive pressure δp. The less compressible the medium, the smaller is the change of volume δV, and hence, the less is the density change $\delta\rho \propto -\rho\delta V/V$. Thus, in the limit for an incompressible fluid one has $(\delta p/\delta\rho)_s = \infty$. From the physics point of view it makes no sense to assume the speed of sound to be arbitrarily large, as that would mean the existence of such media in which perturbations (or information) propagate instantaneously. Intuitively it is clear that the physical interpretation of incompressibility of a medium should be related to describing such flows whose velocities satisfy $u \ll c$, or, as it is called, whose *Mach numbers* satisfy

$$\text{Ma} \doteq u/c \ll 1.$$

Indeed, for stationary flows ($\partial/\partial t = 0$) according to either the Euler or the Bernoulli equation a characteristic pressure change is $\delta p \propto \rho u^2$. On the other hand, from relation (4.16) we obtain

$$\delta\rho = \frac{1}{(\partial p/\partial\rho)_s}\delta p \sim \rho\frac{u^2}{c^2}, \tag{4.23}$$

i.e., the condition of weak compressibility $\delta\rho/\rho \ll 1$ is equivalent to $\text{Ma}^2 \ll 1$. In this case for a stationary flow, from the equation

$$\text{div}\,\rho\mathbf{u} = \rho\,\text{div}\,\mathbf{u} + (\mathbf{u}\nabla)\rho = 0$$

it follows that

$$\text{div}\,\mathbf{u} = -u\frac{\nabla\rho}{\rho} \sim \frac{U}{L}\frac{\delta\rho}{\rho} = \frac{U}{L}O\big(\text{Ma}^2\big), \tag{4.24}$$

where U and L are the characteristic velocity and linear scale of the flow.

Let τ be a characteristic time for changes of a field $\phi(\mathbf{x}, t)$ at a fixed point $\mathbf{x}$ in space, let U be the transport velocity, and let L be a typical linear scale of the field change. Then the condition $|\frac{\partial \phi}{\partial t}| \ll |\mathbf{u} \cdot \nabla \phi|$ assumes the form $1/\tau \ll U/L$, that is

$$\tau \geq \frac{L}{U} \gg \frac{L}{c}. \tag{4.24'}$$

The latter inequality in (4.24′) means that the time L/c required by an acoustic signal to travel the distance L is small compared to the time τ of a noticeable change in the fluid's motion. In that sense, the signal propagation takes place almost instantaneously, while *slow, substantially subsonic flows of a compressible fluid are asymptotically precisely described by the equations of an incompressible fluid up to the order of magnitude* $O(\mathrm{Ma}^2)$. We would like to emphasize yet again that *this is the one and only meaning in which a real fluid is idealized as an incompressible medium*.

4.5 Exercises

1. Show by a direct calculation that the quantity Ω_z/H is a Lagrangian invariant for the shallow-water equations of motion.
2. Formulate a local law for energy conservation for the equations of motion of shallow-water and provide its physical interpretation.

   Answer:

$$\frac{\partial}{\partial t}\left[H\left(\frac{1}{2}\mathbf{u}^2 + \frac{1}{2}gH\right)\right] = -\operatorname{div}\left[H\mathbf{u}\left(\frac{1}{2}\mathbf{u}^2 + gH\right)\right], \qquad \operatorname{div} = \partial/\partial x + \partial/\partial y. \tag{4.25}$$

3. What is the speed of sound in shallow-water theory? Formulate the conditions of a hydrodynamical approximation of Eqs. (4.2)–(4.4), i.e., conditions under which the motion of thin layers of a fluid with a free surface can be considered two-dimensionally divergence-free.

   Answer:

$$c = \sqrt{gH}; \qquad |\mathbf{u}| \ll c. \tag{4.26}$$

   Solution: Formally, based on a gas-dynamical interpretation of equations (4.2)–(4.4), for a two-dimensional barotropic gas with $p = \frac{1}{2}\alpha g \rho^2$ and $\alpha\rho = H$, "the speed of sound" by definition is $c = \sqrt{dp/d\rho} = \sqrt{g\alpha\rho} = \sqrt{gH}$. In fact, however, it is the speed of gravitational waves related to oscillations of the free surface of a fluid in a gravity field. Indeed, the small oscillations of a fluid's free surface are described by the linearized equations (4.2)–(4.4)

$$\frac{\partial \mathbf{u}'}{\partial t} = -g\nabla h, \qquad \frac{\partial h}{\partial t} + H_0 \nabla \mathbf{u}' = 0,$$

where h is a deviation of the free surface from the equilibrium state $H = H_0$ for which $\mathbf{u} = 0$. Eliminating $\mathbf{u}'$, one obtains a wave equation on h:

$$\frac{\partial^2 h}{\partial t^2} - c_0^2 \Delta h = 0,$$

where $c_0 = \sqrt{gH_0}$.

Next, let $H = H_0 + h$ and rewrite (4.2)–(4.4) in vector form:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\nabla)\mathbf{u} = -g\nabla h, \tag{4.27}$$

$$\frac{\partial h}{\partial t} + (\mathbf{u}\nabla)h = -H\nabla\mathbf{u}. \tag{4.28}$$

Assume that a characteristic time for a flow change is $\tau \gtrsim L/U$ (where U is a typical speed and L is a typical geometric scale). This means that $\partial/\partial t \lesssim \mathbf{u}\nabla$, i.e., the result of application of the first operator is less than or comparable in the order of magnitude with the application result of the second one. Then, according to (4.27) and (4.28),

$$gh \propto U^2, \qquad \operatorname{div}\mathbf{u} \propto \frac{U}{L}\cdot\frac{h}{H} = \frac{U}{L}\cdot\frac{gh}{gH} = \frac{U}{L}\cdot\frac{U^2}{c^2} = \frac{U}{L}\,\mathrm{Ma}^2\,.$$

When $\mathrm{Ma} \ll 1$ the motion of thin layers of a fluid with a free surface is two-dimensionally divergence-free up to the square of "the Mach number", which is determined by the propagation speed of gravitational waves. In hydrodynamics this number is known as the Froude number.

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Part II
Quasi-geostrophic Approximations of the Equations of Motion of Rotating Barotropic and Baroclinic Fluids

Chapter 5
Equations of Motion of a Rotating Fluid; The Notion of a Geophysical Flow

5.1 Preliminary Remarks

Before going further let us summarize the preceding material and clarify the terminology. The division of continuous media into incompressible fluid and gas (compressible fluid) discussed above is physically natural since the two substances consist of different physical elements. Recall that parcels of an incompressible fluid are characterized just by their mechanical property, inertia, measured by their density. This is why the equations of motion of an incompressible fluid can be formulated solely within a mechanical framework by interpreting pressure as the constraint reaction coming from the interaction of a fluid particle with the surrounding medium. On the other hand, the gas elements are thermodynamic equilibrium systems whose description requires one to use entropy in addition to density. As a consequence, the corresponding equations of motion for a compressible fluid are formulated using the laws of thermodynamics.

From the hydrodynamical point of view it is more natural and more convenient to divide continuous media into barotropic and baroclinic which, to a large extent, differ by their vorticity dynamics. Strictly speaking, these terms should be used not for the media, but for their motions, since, e.g., motions of the same gas could be both isentropic (barotropic) and non-isentropic (baroclinic). Taking into account this reservation, we recall that a fluid is called barotropic if its density is a function of only pressure. A homogeneous (constant density) incompressible fluid whose flow is divergence-free also belongs to this class and this is the only exception among barotropic fluids. In all other cases a barotropic fluid is compressible since otherwise, as we mentioned above, the pressure would be transported as a passive scalar, while the system of equations of motion would be overdetermined.

The main feature of the vorticity dynamics of a barotropic fluid is that the Kelvin theorem holds for an arbitrary closed liquid contour, i.e., vorticity tubes can have any configuration and can be arbitrarily located in space. On the other hand, the fluid itself does not have its own sources of vorticity because of the alignment of isobaric and isopycnic (i.e., iso-density) surfaces. Indeed, in this case $\nabla\rho\times\nabla p\equiv 0$

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(see Friedman's equation (2.5)). This is why a potential flow of a barotropic fluid can continue for an indefinitely long time.

A characteristic feature of a baroclinic fluid, compressible or not, is its stratification into isentropic or isopycnic surfaces along which the particles flow. A closed liquid contour belonging to an isentropic or isopycnic surface at the initial moment will remain on it for all time. This allows the Kelvin theorem to hold for such a contour and, as a consequence, the potential vorticity is a Lagrangian invariant. This in turn creates the stratification according to the potential vorticity, i.e., a baroclinic fluid has a natural time-dependent tube structure, formed by intersections of isentropic (isopycnic) surfaces and surfaces of constant potential vorticity. The mass confined inside each tube between a pair of isentropic (isopycnic) surfaces which is cut by a pair of surfaces of constant potential vorticity will remain constant during the evolution, since fluid particles lying on such surfaces never leave them. Note also that according to the Friedman equation (2.5), deviation of isobaric and isopycnic surfaces is a generator of vorticity for a barotropic fluid. Such a deviation implies that $\nabla\rho\times\nabla p \neq 0$.

Apparently the reader will already have noted that in the discussion of the main principles of fluid motion I mostly concentrated on, so to speak, the "genetic" properties of its dynamics, not taking into account the impact of the "social environment," that is exterior fields, boundary conditions, etc. We devoted our main attention to conservation laws for the following reasons. First, motion invariants reveal fundamental symmetry properties of fluid motion and point to processes that are impossible. For instance, it is impossible to create perpetual motion or to break the momentum or mass conservation laws. However, in spite of the evidence above, such violations occasionally happen in some studies constructing reductions or discrete analogs of hydrodynamical equations. Secondly, during numerical modeling of inviscid hydrodynamical equations, the conservation laws allow one to formulate important criteria for obtaining precise numerical algorithms. And finally, thirdly, in many cases which we discuss below conservation laws help find solutions to hydrodynamical equations and formulate fundamental stability criteria for fluid motions.

5.2 Equations of Motion for a Rotating Fluid

Historically, geophysical hydrodynamics has its origins in dynamical meteorology, the science of the Earth's weather. It was gradually realized that the laws formulated in the latter science have wider applications. So it naturally turned out that the main objects of study in geophysical fluid dynamics, at its initial development stage, were large-scale motions of the atmosphere, the ocean, Earth's inner liquid core and their laboratory counterparts. Later these directions were complemented by the atmospheric motions of other planets, circulations on the Sun and other stars, and even by the evolution of galaxies. The common element for all these objects which greatly influences their behavior is the rotation of the whole system. Therefore, it is appropriate to begin the study of such flows with the formulation of hydrodynamical equations in a rotating frame, which is natural for these objects. Denote the angular velocity of this rotation by $\boldsymbol{\Omega}_0$. Both its magnitude (angular speed) and its direction can vary. The unit basis vectors defining the axes of this rotating rectangular coordinate system also rotate with this angular velocity. Their time derivatives are defined by the formulas

$$\frac{d\mathbf{i}}{dt} = \boldsymbol{\Omega}_0 \times \mathbf{i}, \qquad \frac{d\mathbf{j}}{dt} = \boldsymbol{\Omega}_0 \times \mathbf{j}, \qquad \frac{d\mathbf{k}}{dt} = \boldsymbol{\Omega}_0 \times \mathbf{k}.$$

To find the time derivative of any vector $\mathbf{A}$ in its representation via vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, we need to differentiate both the coordinates and these basis vectors. Then we obtain the well-known relation

$$\frac{d\mathbf{A}}{dt} = \left(\frac{d\mathbf{A}}{dt}\right)_r + \boldsymbol{\Omega}_0 \times \mathbf{A}. \tag{5.1}$$

Here $\boldsymbol{\Omega}_0$ is the angular velocity of the whole system, while the index r stands for the time derivative in the rotating frame. Since $\mathbf{u} \doteq d\mathbf{r}/dt$, according to (5.1) we have

$$\mathbf{u} = \mathbf{u_r} + \boldsymbol{\Omega}_0 \times \mathbf{r}, \tag{5.2}$$

where $\mathbf{u}$ and $\mathbf{u_r}$ are respectively velocities of the motion relative to the inertial and rotating coordinate systems. Now differentiating equality (5.2) and again applying formula (5.1), we arrive at the well-known result of classical mechanics:

$$\frac{d\mathbf{u}}{dt} = \frac{d\mathbf{u_r}}{dt} + \boldsymbol{\Omega}_0 \times \mathbf{u} = \left(\frac{d\mathbf{u_r}}{dt}\right)_r + 2\boldsymbol{\Omega}_0 \times \mathbf{u_r} + \boldsymbol{\Omega}_0 \times (\boldsymbol{\Omega}_0 \times \mathbf{r}), \tag{5.3}$$

according to which the "absolute" acceleration of a fluid particle is a combination of its relative acceleration (the first term on the right-hand side of (5.3)), the Coriolis acceleration (the second term), and centripetal acceleration (the last term).

Furthermore, it is easy to show (see Exercise 1) that *the individual time derivative of the scalar quantity is invariant with respect to the inertial and rotating coordinate systems, i.e.*,

$$\frac{d\alpha}{dt} = \left(\frac{d\alpha}{dt}\right)_r. \tag{5.4}$$

For instance, the temperature of a person on a moving carousel is the same as the person's temperature in a queue to that carousel if we exclude the adrenaline effect coming from overload.

Now plugging (5.3) and (5.4) into (3.11)–(3.14) and omitting the index r, one can write the equations of motion for a rotating ideal compressible fluid in the form

$$\frac{d\mathbf{u}}{dt} + 2\boldsymbol{\Omega}_0 \times \mathbf{u} \doteq \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\nabla)\mathbf{u} + 2\boldsymbol{\Omega}_0 \times \mathbf{u} = -\frac{1}{\rho}\nabla p - \nabla \Phi, \tag{5.5}$$

$$\frac{d\rho}{dt} \doteq \frac{\partial \rho}{\partial t} + (\mathbf{u}\nabla)\rho = -\rho \operatorname{div} \mathbf{u}, \tag{5.6}$$

$$\frac{ds}{dt} \doteq \frac{\partial s}{\partial t} + (\mathbf{u}\nabla)s = 0, \tag{5.7}$$

$$s = s(\rho, p). \tag{5.8}$$

Here $\Phi = \Phi_c + \Phi_g$ is the total potential of centrifugal forces and the exterior gravitational field. (After substituting (5.3) into (3.11), the last term in (5.3) moved to the right-hand side of (5.5) changes sign and hence can be interpreted as the centrifugal acceleration.) From this moment on the gravitational field is going to be included in our consideration as one of the main factors defining properties of geophysical flows. The potential of centrifugal forces is defined by the equality (see Exercise 2)

$$\Phi_c = -\frac{1}{2}(\boldsymbol{\Omega}_0 \times \mathbf{r})^2, \qquad -\nabla\Phi_c = (\boldsymbol{\Omega}_0 \times \mathbf{r}) \times \boldsymbol{\Omega}_0.$$

For rotating planets the quantity $\mathbf{g}' = -\nabla\Phi = \mathbf{g} + (\boldsymbol{\Omega}_0 \times \mathbf{r}) \times \boldsymbol{\Omega}_0$ (where $\mathbf{g} = -\nabla\Phi_g$ is the gravitational acceleration) is sometimes called *effective gravitational acceleration*, which does not coincide with the vertical direction. We recall the Foucault pendulum (J.B.L. Foucault, Pantheon in Paris, 1851) in this relation.

Strictly speaking, the expression "the equation of motion of a rotating fluid" employed above is not quite appropriate since both systems (3.11)–(3.14) and (5.5)–(5.8) describe one and the same motion but in different coordinate frames. But according to the well-known relativity principle the latter should not matter. However, from the point of view of a mathematical description of physical phenomena and their interpretation, the choice of a natural coordinate system is of utmost importance. The classical illustration of this is the coordinate systems of Ptolemy and Copernicus. On the one hand, the system of Claudius Ptolemy (2nd century AD) was in effect just short of one and a half thousand years. On the other hand, the system of Nicolaus Copernicus (Nicolai Copernici, "On rotation of the celestial spheres," Nuremberg, 1543) overturned fundamental concepts of the universe and encouraged the construction of fundamentally new physical concepts.

Einstein pointed out that although the question of whether the Sun revolves around the Earth or vice versa is not solvable from the formal position of the relativity principle, it is not a problem from a physical point of view, since the center of gravity for the Sun–planets system belongs to the Sun. Therefore, the above mentioned coordinate system only emphasizes that the adopted form of equations is most natural for the observer who is rotating along with the fluid, and knows about this (for example, for all of us, located on the ground and watching the weather).

Let us write the main law of geophysical hydrodynamics, the conservation of potential vorticity for an isentropic fluid motion, that is for Eqs. (5.5)–(5.8). By definition of enthalpy W, we have (see Chap. 3):

$$T \cdot ds = dW - \frac{dp}{\rho}.$$

This implies that

$$-\frac{\nabla p}{\rho} = -\nabla W + T\nabla s.$$

First, let us obtain the vorticity equation from the equations of fluid motion in a rotating frame. For this we apply the operator rot to Eq. (5.5) in the Bernoulli form

$$\frac{\partial \mathbf{u}}{\partial t} - \mathbf{u} \times \operatorname{rot} \mathbf{u} + 2\boldsymbol{\Omega}_0 \times \mathbf{u} = -\frac{1}{\rho}\nabla p - \nabla \Phi - \nabla\left(\frac{\mathbf{u}^2}{2}\right).$$

As the result we obtain

$$\begin{aligned} &\frac{\partial(\boldsymbol{\Omega} + 2\cdot\boldsymbol{\Omega}_0)}{\partial t} + (\mathbf{u}, \nabla)(\boldsymbol{\Omega} + 2\cdot\boldsymbol{\Omega}_0) \\ &\quad - (\boldsymbol{\Omega} + 2\cdot\boldsymbol{\Omega}_0, \nabla)\mathbf{u} + (\boldsymbol{\Omega} + 2\cdot\boldsymbol{\Omega}_0)\cdot\nabla\mathbf{u} \\ &= \nabla T \times \nabla s. \end{aligned}$$

The way to derive this law is similar to the one we used in Chap. 2 for a stratified incompressible fluid, but now the role of the quantity conserved along a trajectory is played by the entropy density s. Take the inner product of this equation with ∇s, unlike (2.6). All other analytical calculations described in Chap. 2 remain the same. Note only the presence of the term $\operatorname{div}\mathbf{u}$, which did not appear in Chap. 2 in view of the divergence-free property of the velocity field. Eventually, we obtain

$$\begin{aligned} &\frac{\partial(\boldsymbol{\Omega} + 2\cdot\boldsymbol{\Omega}_0)\nabla s}{\partial t} + (\mathbf{u}, \nabla)\big[(\boldsymbol{\Omega} + 2\cdot\boldsymbol{\Omega}_0)\nabla s\big] \\ &\quad = +(\boldsymbol{\Omega} + 2\cdot\boldsymbol{\Omega}_0)\big[(\mathbf{u}\nabla)\nabla s\big] - \mathbf{u}\big[(\boldsymbol{\Omega} + 2\cdot\boldsymbol{\Omega}_0, \nabla)\nabla s\big] \\ &\qquad - (\boldsymbol{\Omega} + 2\cdot\boldsymbol{\Omega}_0)\nabla s\cdot\operatorname{div}\mathbf{u} + (\nabla T \times \nabla s)\cdot\nabla s. \end{aligned}$$

The same computations as in Chap. 2 allow us to conclude that the first two terms on the right-hand side give zero. Then by using the expression (5.6) for $\operatorname{div}\mathbf{u}$ instead of the incompressibility equation, i.e., $\operatorname{div}\mathbf{u} = -\frac{1}{\rho}\cdot\frac{d\rho}{dt}$, we conclude that

$$\frac{d}{dt}\left[\frac{(\boldsymbol{\Omega} + 2\boldsymbol{\Omega}_0)\cdot\nabla s}{\rho}\right] = 0.$$

5.3 Notion of a Geophysical Flow as a Hydrodynamical Object

However paradoxical it may look, in the theory of rotating fluids, as well as in any other part of hydrodynamics, the basic equations of fluid motion are used merely for preliminary analysis of fundamental properties of solutions rather than for their direct integration in specific problems, even in the corresponding numerical simulations. The set of exact nontrivial solutions of nonlinear hydrodynamic equations (see, for example, textbooks by H. Lamb, *Hydrodynamics* (1932, reprinted in 1993), L.M. Milne-Thompson, *Theoretical Hydrodynamics* (1968, reprinted in 1996), G.K. Batchelor, *An Introduction to Fluid Dynamics* (1973), etc.) is quite scarce and the discovery of a new exact solution so far is perceived, albeit with some skepticism, as a serious scientific achievement. The point is that when posing precise initial and boundary conditions which provide the uniqueness of a solution to the original equations of motion, such a solution usually contains a vast variety of different scales of motion. Any attempt to describe in detail all these scales is somewhat similar to an attempt to describe the state of a gas by means of all the trajectories of its particles.

In fact, this means that it is necessary to reduce the initial hydrodynamical equations. Such a reduction is often possible to implement, based on the fundamental properties of the desired class of solutions, which are known from observations or experimental results. In our case, we are going to take an even more radical step. Taking into account not only observational data, but also the results of the above theoretical studies, we introduce the concept of a geophysical flow and formulate axiomatically its main properties. In the future this will allow us to bypass preliminary work and immediately begin deriving simplified equations of motion which constitute the mathematical basis of geophysical fluid dynamics. We first introduce the notion of a geophysical flow for a barotropic atmosphere.

A relative motion of an inviscid rotating barotropic fluid in a gravitational field is called geophysical if the following conditions are satisfied:

$$\begin{aligned} &\frac{\partial}{\partial t} \leq \mathbf{u}\nabla, \qquad Ma \doteq \frac{U}{c} \ll 1, \\ &\varepsilon \doteq \frac{U}{2\Omega_0 L} = O\left(\frac{\omega}{2\Omega_0}\right) \ll 1, \qquad \delta \doteq \frac{\Omega_0^2 L}{g} \ll 1. \end{aligned} \tag{5.9}$$

Here c is the speed of sound, U and L are the characteristic speed and typical geometric scale of the flow, and g is the gravitational acceleration. The smallness of the dimensionless parameter δ means that centrifugal forces entering the equations of motion in the same way as gravitational forces have virtually no effect on the behavior of geophysical flows, i.e., *with respect to this parameter the motions under consideration are almost self-similar.*

The quantity ε is equal to the typical value of the ratio of the advective acceleration $|(\mathbf{u}\nabla)\mathbf{u}|$ to the Coriolis acceleration $|2\boldsymbol{\Omega}_0 \times \mathbf{u}|$ or, which is the same, the ratio of relative vorticity to the double velocity of the total rotation. This quantity is called *the Rossby–Kibel number and it is the main small parameter used for the expansion of the original equations of motion.* The smallness of the Rossby–Kibel number implies, in particular, that the characteristic scale of a geophysical flow is $L \gg U\tau/4\pi$, where τ is the rotation time for one revolution of the system, one day. It means that this scale substantially exceeds the typical distance covered by a fluid particle over the time of about one tenth of a day. For the Earth's atmosphere where typical wind speed is about 10 m/s this distance is of the order of magnitude of about 100 km and more. Therefore the Earth's geophysical flows (pardon the necessary tautology) have the characteristic scale of order 1000 km and more, comparable with the planet's radius.

Now note that among the external physical parameters Ω_0, g and c which determine the behavior of the rotating fluid (one has to add the speed of sound c to this list as the measure of medium compressibility, see Chap. 4) one can construct two natural geometric scales:

$$L_c = \frac{g}{\Omega_0^2}, \qquad L_0 = \frac{c}{2\Omega_0}. \tag{5.10}$$

The former is of no interest to us, since it is the flow size in which the centrifugal accelerations are comparable to gravitational ones (substitute L_c into the expression for δ), which contradicts the conditions (5.9). We also note in passing that in the Earth's conditions, L_c attains the "astronomical" value $\approx 10 \times 10^8$ m $= 10^6$ km ($\Omega_0 = 2\pi/(24 \times 3600)$ s$^{-1} \approx 7.3 \times 10^{-5}$ s^{-1}).

The scale of L_0 is called the *Rossby–Obukhov radius*, and it is among the fundamental parameters of geophysical hydrodynamics. Jumping ahead we mention that this is a typical scale of vortices, cyclones and anticyclones that are observed in fast rotating fluids and satisfy condition (5.9), and this scale is usually comparable with the exterior sizes of corresponding geophysical objects. For the Earth's atmosphere, for instance, $L_0 \approx 1500$ km, which is of a scale comparable to the Earth's radius $a \approx 6378$ km. Summarizing the above we come to the conclusion that *inviscid geophysical flows are large-scale, slowly changing in time, and noticeably subsonic motions of rotating fluids that are characterized by small values of the Rossby–Kibel number.*

The concept of a geophysical flow as a hydrodynamical object should not be identified with the physical understanding of geophysical flows, that covers a much wider class of motions. The latter includes, in particular, such mesoscale processes as Rayleigh convection and the formation of cumulus clouds, virtually the entire spectrum of wave motions, including internal, surface, and tidal waves, the appearance and propagation of fronts, as well as dangerous but intriguing for scientists phenomena such as storms and tornadoes. The total rotation of the fluid certainly imposes its own features on all these processes, but they can exist as well without it, and therefore they should be regarded as general hydrodynamical objects rather than geophysical ones.

The situation is fundamentally different for the flows defined by (5.9), which constitute the main object of study in geophysical fluid dynamics. The expedience of singling them out is necessitated for the following reasons. First, they are the main atmospheric elements determining the weather since they carry the lion's share of the energy and vorticity of the rotating fluid. Secondly, this distinguished class of flows in the hydrodynamical realm forms some kind of "state within a state", some kind of Vatican if you like, which develops according to its own laws that are binding only for the priests, and it exists only as long as the Earth rotates.

Finally, most importantly, as a consequence of the second item above, this chosen class of flows is governed by special equations of motion, which can be obtained from the original system of asymptotically sharp hydrodynamical equations by filtering out the corresponding fast components. In contrast to (5.5)–(5.8), these equations are applicable to the description of geophysical flows only and do not allow taking the limit as $\Omega_0 \to 0$. This means that we are not talking about fluid flows that are well-known from classical hydrodynamics and that are modified by the rotation of the fluid as a whole, but rather about new features of solutions to hydrodynamical equations generated by this rotation, and this justifies the introduction of this key concept of geophysical fluid dynamics.

5.4 Exercises

1. Try and prove without reference to the solution below that

$$\frac{d\alpha}{dt} = \left(\frac{d\alpha}{dt}\right)_r.$$

Proof: The relation between the individual time derivatives of a scalar field in inertial and rotating frames can be written in the following form by using (5.2):

$$\frac{d\alpha}{dt} = \left(\frac{\partial\alpha}{\partial t}\right)_{\mathbf{r}} + \mathbf{u}\nabla\alpha = \left(\frac{\partial\alpha}{\partial t}\right)_{\mathbf{r_r}} + \left(\frac{\partial\mathbf{r_r}}{\partial t}\right)_{\mathbf{r}}\nabla\alpha + (\mathbf{u_r} + \boldsymbol{\Omega}_0 \times \mathbf{r})\nabla\alpha$$

or

$$\frac{d\alpha}{dt} = \left(\frac{\partial\alpha}{\partial t}\right)_{\mathbf{r_r}} + \mathbf{u_r}\nabla\alpha + \left[\left(\frac{\partial\mathbf{r_r}}{\partial t}\right)_{\mathbf{r}} + \boldsymbol{\Omega}_0 \times \mathbf{r}\right]\nabla\alpha.$$

Here the indices $\mathbf{r}$ and $\mathbf{r_r}$ denote, respectively, the time derivatives for constant space coordinates in the laboratory and rotating frames. Now, according to (1), $(\partial\mathbf{r_r}/\partial t)_{\mathbf{r}} + \boldsymbol{\Omega}_0 \times \mathbf{r} = 0$.

2. Prove that $\mathbf{g}_c = -\nabla\Phi_c = (\boldsymbol{\Omega}_0 \times \mathbf{r}) \times \boldsymbol{\Omega}_0$ if $\Phi_c = -\frac{1}{2}(\boldsymbol{\Omega}_0 \times \mathbf{r})^2$.

Hint: Use the formula

$$\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A} \times \operatorname{rot}\mathbf{B} + \mathbf{B} \times \operatorname{rot}\mathbf{A} + (\mathbf{A}\nabla)\mathbf{B} + (\mathbf{B}\nabla)\mathbf{A}, \tag{5.11}$$

which at some point everyone should derive on one's own.

References

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R. Salmon, *Lectures on Geophysical Fluid Dynamics*, Oxford University Press, Oxford, 1998.

Chapter 6
What is Geophysical Hydrodynamics?

6.1 The Obukhov–Charney Basis

In this chapter I will attempt to describe the main features and characteristics of global geophysical flows, which are sometimes rather unusual for a classical hydrodynamist, and which constitute the peculiarity of the general atmospheric circulation. Below we confine ourselves to those geophysical systems in which one of the two conditions holds. Either the rotation direction of the fluid as a whole coincides with the direction of the gravity acceleration $\mathbf{g} = -\nabla \Phi_g$, as is the case in laboratory experiments on modeling atmospheric and ocean flows. Or one can neglect the influence of the components of the vector $\boldsymbol{\Omega}_0$ that are orthogonal to the vector $\mathbf{g}$, by virtue of certain physical reasons. The latter, for example, holds for the Earth's atmosphere in areas remote from the equator, because of the smallness of the vertical velocity components in comparison with its horizontal components and because of the smallness of the vertical components of the Coriolis acceleration as compared to the gravity acceleration. In this case, the vector $\boldsymbol{\Omega}_0$ in the equations of motion is replaced by $\mathbf{k}\Omega_0 \sin\varphi$ (where $\mathbf{k}$ is the unit vector in the direction opposite to $\mathbf{g}$, while φ is the latitude of the observation point, see Fig. 6.1). In fact, here one is dealing with the so-called *beta-effect*, that is the differential rotation of the fluid, whose angular velocity depends on spatial coordinates.

Recall that the Rossby–Kibel number $\varepsilon = U/(2\Omega_0 L)$ (5.9) is small for global geophysical flows. In such circumstances, the principal feature of large-scale dynamics of an ideal rotating fluid is that this dynamics is almost completely determined by only four fundamental properties. This was first noted in relation to atmospherical motions of synoptic scale by outstanding meteorologists A.M. Obukhov and J.G. Charney. These properties are the following.

(I) *Geophysical flows are quasi-hydrostatic, and the hydrostatic relation for them holds up to $o(\varepsilon)$:*

$$\frac{\partial p}{\partial z} + g\rho = o(\varepsilon). \tag{6.1}$$

F.V. Dolzhansky, *Fundamentals of Geophysical Hydrodynamics*,
Encyclopaedia of Mathematical Sciences 103,
DOI 10.1007/978-3-642-31034-8_6,

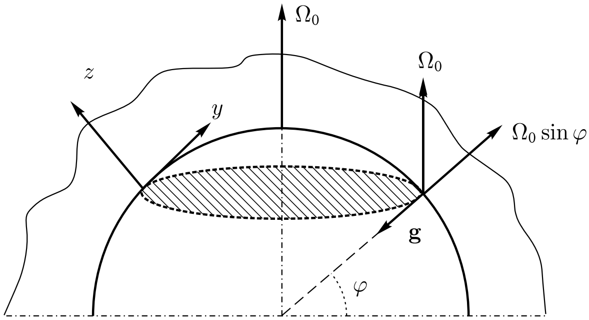

Fig. 6.1 A schematic representation of the meridian section of the Earth's northern hemisphere. The coordinates x, y and z are measured, respectively, to the east, north and up, φ is the latitude. The horizontal component Ω_{0y} in the expression for the Coriolis force is not taken into account

Here z is the vertical coordinate, measured in the direction opposite to the vector of the gravity acceleration.

(II) *The currents are in the so-called quasi-geostrophic equilibrium, i.e., the Coriolis force with accuracy of order ε is balanced by the horizontal 2D-gradient of pressure* (recall that the vertical component of the Coriolis force is not taken into account):

$$2\mathbf{\Omega}_0 \times \mathbf{v} = -\frac{1}{\rho}\nabla p + O(\varepsilon) \qquad \left(\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y}\right), \tag{6.2}$$

where $\mathbf{v} = u\mathbf{i} + v\mathbf{j}$ is the horizontal wind.

(III) *The potential vorticity of individual fluid parcels is preserved. In the rotating reference frame it can be written as*

$$\Pi = \frac{(\mathbf{\Omega} + 2\mathbf{\Omega}_0)\cdot \operatorname{grad}\Theta}{\rho} \left(\mathbf{\Omega} \doteq \operatorname{rot}\mathbf{u},\ \operatorname{grad} = \nabla + \mathbf{k}\frac{\partial}{\partial z},\ \frac{d\Pi}{dt} = 0\right), \tag{6.3}$$

where $\mathbf{u} = u\mathbf{i} + v\mathbf{j} + \mathbf{k}w = \mathbf{v} + \mathbf{k}w$, Θ is the potential temperature, playing the role of the specific entropy (see below).

(IV) *The potential temperature of individual fluid parcels is preserved*:

$$\Theta \doteq T\left(\frac{p_0}{p}\right)^k, \quad k = \frac{R}{C_p}, \quad \frac{d\Theta}{dt} = 0. \tag{6.4}$$

Here p, ρ and T are, respectively, the pressure, density, and absolute temperature of the fluid parcels, R is the gas constant entering the equation of state $p = \rho RT$ (the Mendeleev–Clapeyron equation), C_p is the specific heat capacity at constant pressure. For many reasons in geophysical hydrodynamics the thermodynamical characteristic Θ of an individual fluid parcel is used more often than the specific entropy s, related to Θ by a one-to-one (up to an additive constant) correspondence $s = C_p \ln\Theta$. According to formula (6.4), the *value* of Θ *coincides with the temperature, which the fluid parcel would have under its adiabatic compression* *to the value of the "surface" pressure* $p = p_0$ (hence the term of the *potential temperature*).

Conditions (6.1), (6.2), rewritten as a single vector equation

$$\mathbf{G} \doteq 2\boldsymbol{\Omega}_0 \times \mathbf{u} + \rho^{-1}\,\mathrm{grad}\,p - \mathbf{g} = O(\varepsilon), \tag{6.5}$$

can be treated, with the use of physical terminology, as adiabatic invariance, an approximate "persistence" of the zero value of the vector **G**. Recall that *in physics an adiabatic invariant* is a quantity $I(t)$, for which the difference $I(t) - I(0)$ remains small for all considered t. *In hydrodynamics the term of an adiabatic invariant* stands for first integrals of the equations of motion of an ideal compressible fluid, i.e., for quantities that are preserved in the absence of dissipation and external sources of energy. Examples of such are, in particular, the potential vorticity and the potential temperature.

In this relation it is appropriate to emphasize that although the properties (III) and (IV) hold for any flow of an ideal compressible fluid, it is especially for geophysical flows that the potential vorticity and potential temperature along with the approximate invariance of the zero value of the vector **G** are of particular importance as comprehensive characteristics of the motion. Therefore it is of no coincidence that in recent years successful attempts were carried out to diagnose the atmospheric processes of synoptic scale in terms of the above-mentioned fundamental adiabatic invariants (see, for example, Kurgansky, 1993). Figuratively speaking, *properties* (I)–(IV) *form a peculiar "four-dimensional basis," which spans the entire space of geophysical flows and which can be used as a launching pad for studying their structure and reductions of the original 3D equations.* Therefore we are not going to dwell on the mathematical rigor related to such delicate issues of the rotating fluid theory as to why the system is in the vicinity of a geostrophic balance, how the hydrodynamical fields adapt to it, what the role is of small scales and various wave processes that do not satisfy the conditions (I)–(IV), etc. These, and if necessary, other related issues will be touched upon only at the level of simple illustrations, analogies, and the physical interpretation to help with their understanding, but without providing the proofs. For the latter I address the reader to the vast traditional literature in geophysical fluid dynamics, mentioned in the bibliography to chapters.

6.2 Fundamental Properties of Geophysical Flows

Already the first two frames in the Obukhov–Charney basis, expressed by the equality (6.5), allow one to emphasize distinctive features of geophysical flows, which at first glance contradict the classical picture of fluid behavior.

(1) Indeed, let us start with the question *Where does the wind blow?* To answer this question, we write down the horizontal components of Eq. (6.5), bearing in

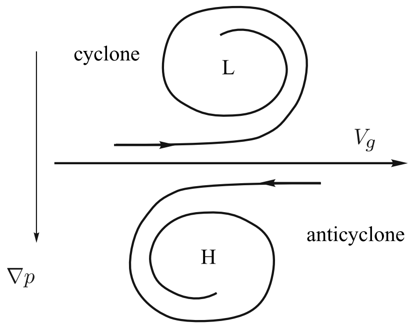

Fig. 6.2 The geostrophic wind, contrary to common sense, blows not across, but along the isobars. L and H are centers of low- and high-pressure areas, respectively

mind that $g_x = g_y = 0$:

$$-2\Omega_0 v + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0,$$

$$+2\Omega_0 u + \frac{1}{\rho}\frac{\partial p}{\partial y} = 0,$$

where $\Omega_0 = \Omega_z$ is the vertical component of the vector $\mathbf{\Omega}_0$, as shown in Fig. 6.1. It implies that

$$u = -\frac{1}{2\Omega_0 \rho}\frac{\partial p}{\partial y} + O(\varepsilon), \qquad v = +\frac{1}{2\Omega_0 \rho}\frac{\partial p}{\partial x} + O(\varepsilon). \tag{6.6}$$

The precise relation, equivalent to (6.6),

$$\mathbf{v_g} = \frac{1}{2\Omega_0 \rho}\mathbf{k} \times \nabla p \tag{6.7}$$

is called the *geostrophic wind.* It follows that the wind is blowing, at first glance, contrary to common sense (Fig. 6.2), not in the direction of pressure deficit, i.e., not across but along the isobars. *If you turn your back to the wind, then to your left there will be the low pressure area, while on the right will be the high one*. (In practice, this rule should be applied by orienting yourself via the cloud movement because as we shall see below, in the vicinity of the Earth's surface the wind direction is significantly different from the geostrophic one due to the influence of surface friction.) The air masses in the Earth's atmosphere near the center of a low (or high) pressure revolve around the center along helices converging to (or diverging from) it, and form a large-scale vortex, cyclone (or anticyclone).

(2) *The criterion of incompressibility of a rotating fluid.* In Chap. 4 using the formula

$$\delta\rho = \frac{1}{(\partial p/\partial \rho)_s}\delta p = \frac{\delta p}{c^2}, \tag{6.8}$$

it was shown that the criterion of weak compressibility $\delta\rho/\rho \ll 1$ holds for essentially subsonic flows. Moreover, relatively slowly evolving flows ($\partial/\partial t \leqslant \mathbf{u}\nabla$) are divergence-free up to the square of the Mach number. The situation is different in a fluid rotating as a whole. According to (6.2) or (6.5) the variations in pressure caused by the relative fluid motion $\delta p \sim 2\Omega_0 U L \rho$, where U and L are the characteristic wind velocity and the horizontal scale of its change. (Pressure variations due to vertical currents can be safely neglected since the hydrostatic relation (6.1) is satisfied with high accuracy.) Then, making the substitution of this estimate in (6.8), we have

$$\frac{\delta\rho}{\rho} \sim \frac{2\Omega_0 U L}{c^2} = \frac{2\Omega_0 L}{c}\frac{U}{c} = \frac{L}{c/2\Omega_0} Ma = \frac{L}{L_0} Ma,$$

where $L_0 = c/2\Omega_0$ is the Rossby–Obukhov scale introduced above. It follows (see Chap. 4) that for geophysical flows the 3D-divergence satisfies

$$\operatorname{Div}\mathbf{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \frac{U}{L} O\left(\frac{L}{L_0} Ma\right). \tag{6.9}$$

Since the characteristic linear scale of geophysical flows is of the order L_0, as was already mentioned, then according to (6.9), a *rotating fluid is less incompressible to the next order of magnitude in comparison with a non-rotating fluid.*

(3) *Quasi-two-dimensionality of geophysical flows.* For a barotropic atmosphere, the formula (6.6) can be rewritten in the form

$$\mathbf{v} = \frac{1}{2\Omega_0}\mathbf{k} \times \nabla W + O(\varepsilon), \quad W(p) = \int \frac{dp}{\rho(p)}, \tag{6.10}$$

where $W(p)$ is a primitive of the function $f(p) = \rho^{-1}(p)$ (see Chap. 3). Then according to the formula $\operatorname{div}(\mathbf{A} \times \mathbf{B}) = \mathbf{B}\operatorname{rot}\mathbf{A} - \mathbf{A}\operatorname{rot}\mathbf{B}$ (or simply differentiate the first equation of (6) in x, differentiate the second in y, and add them together)

$$\operatorname{div}\mathbf{v} = O(\varepsilon). \tag{6.11}$$

Comparing (6.11) to (6.9), we find that $\partial w/\partial z = O(\varepsilon)$ and since one has $w(z = 0) = 0$ on the bottom solid boundary of the atmosphere,

$$w(x, y, z) = O(\varepsilon). \tag{6.12}$$

In Chap. 5, we wrote down the result of acting on (6.5) by rot. Since $\mathbf{\Omega}_0 = (0, 0, \Omega_0)$ we have

$$-2\Omega_0 \frac{\partial \mathbf{u}}{\partial z} + 2\mathbf{\Omega}_0 \operatorname{Div}\mathbf{u} = O(\varepsilon).$$

Whence, by taking into account (6.9),

$$\frac{\partial \mathbf{u}}{\partial z} = O(\varepsilon) \Longrightarrow \frac{\partial \mathbf{v}}{\partial z} = O(\varepsilon), \qquad \frac{\partial w}{\partial z} = O(\varepsilon), \tag{6.13}$$

because as a rule, the number Ma is somewhat less than the Rossby number ε (for example, for the atmosphere $Ma \approx 1/30$, while $\varepsilon \approx 0.1$).[1] The statements (6.12) and (6.13) are sometimes referred to as the *Proudman–Taylor theorem, according to which the general rotation suppresses the vertical velocity of barotropic geophysical flows, as well as the dependence of their horizontal velocity component on the vertical coordinate.* Later we will show that statement (6.12) under certain conditions holds for baroclinic geophysical flows as well. In other words, *geophysical flows are quasi-two-dimensional.*

(4) *The thermal wind*. For baroclinic geophysical flows the application of the operation rot to (6.5) gives

$$-2\Omega_0 \frac{\partial \mathbf{u}}{\partial z} + 2\mathbf{\Omega}_0 \mathrm{Div}\, \mathbf{u} + \mathrm{rot}\left(\frac{1}{\rho}\,\mathrm{grad}\, p\right) = O(\varepsilon),$$

or, given (6.9) and $\mathrm{rot}(\varphi \mathbf{A}) = (\mathrm{grad}\, \varphi \times \mathbf{A}) + \varphi\, \mathrm{rot}\, \mathbf{A}$,

$$2\Omega_0 \frac{\partial \mathbf{u}}{\partial z} + \frac{1}{\rho^2}(\mathrm{grad}\, \rho \times \mathrm{grad}\, p) = O(\varepsilon).$$

Now take into account that the characteristic vertical scale of a geophysical flow is much smaller than the horizontal one, due to its quasi-two-dimensionality. Then according to (6.1), grad $p = \mathbf{g}\rho + O(\varepsilon)$ and the latter formula can be written as

$$\frac{\partial \mathbf{u}}{\partial z} = -\frac{1}{2\Omega_0 \rho}(\mathrm{grad}\, \rho \times \mathbf{g}) + O(\varepsilon) \equiv -\frac{1}{2\Omega_0}(\mathrm{grad}\, \ln \rho \times \mathbf{g}) + O(\varepsilon), \tag{6.14}$$

or in the coordinate form

$$\frac{\partial u}{\partial z} = \frac{g}{2\Omega_0 \rho}\frac{\partial \rho}{\partial y} + O(\varepsilon) = \frac{g}{2\Omega_0}\frac{\partial \ln \rho}{\partial y} + O(\varepsilon), \tag{6.14$'$}$$

$$\frac{\partial v}{\partial z} = -\frac{g}{2\Omega_0 \rho}\frac{\partial \rho}{\partial x} + O(\varepsilon) = -\frac{g}{2\Omega_0}\frac{\partial \ln \rho}{\partial x} + O(\varepsilon). \tag{6.14$''$}$$

In laboratory experiments of modeling baroclinic geophysical flows, one typically uses the Oberbeck–Boussinesq fluid, in which the distributions of density and temperature are given by the equalities $\rho = \rho_0 + \rho'(x, y, z, t)$ $T = T_0 + T'(x, y, z, t)$, where ρ_0 and T_0 are the average values of density and temperature independent of the coordinates and time, while the deviations are related by

$$\frac{\rho'}{\rho_0} = -\frac{T'}{T_0}. \tag{6.15}$$

Then (6.14) in the component form is written as

$$\frac{\partial u}{\partial z} = -\frac{g}{2\Omega_0 T_0}\frac{\partial T}{\partial y} + O(\varepsilon), \qquad \frac{\partial v}{\partial z} = \frac{g}{2\Omega_0 T_0}\frac{\partial T}{\partial x} + O(\varepsilon). \tag{6.16}$$

---

[1] In (6.11), (6.12), and (6.13) we omitted the corresponding dimensional factors. This is also done below if it does not cause confusion.

Here the prime is omitted, since T_0 does not depend on the coordinates. Formulas (6.16) are applied to the ocean. Below it will be shown that for a baroclinic atmosphere formulas (6.16) are replaced by the following:

$$\frac{\partial u}{\partial z} = -\frac{g}{2\Omega_0 \Theta_s}\frac{\partial \Theta}{\partial y} + O(\varepsilon), \qquad \frac{\partial v}{\partial z} = \frac{g}{2\Omega_0 \Theta_s}\frac{\partial \Theta}{\partial x} + O(\varepsilon), \tag{6.17}$$

where $\Theta_s = \Theta_s(z)$ is the vertical distribution of potential temperature for the so-called standard atmosphere, obtained at each level by averaging over the horizontal coordinates. In fact, these formulas reveal one of the main mechanisms of external energy drive, which feeds on the general circulation of the atmosphere and ocean.

Relations (6.16) and (6.17), called the *thermal wind*, show that the vertical wind shear is induced by the horizontal temperature gradient, which, in turn, is generated by the pole-equator temperature difference created by the uneven solar heating of the atmosphere and the Earth’s surface.

Remark on Further Notation Given the quasi-two-dimensional specifics of geophysical flows, we shall repeatedly switch between the 3D and 2D descriptions. Therefore, to avoid any confusion we adopt the following rule to refer to commonly used differential operators. We will denote the three-dimensional individual time derivative, divergence, and gradient operators, respectively, by D/Dt, Div and grad, while their two-dimensional counterparts by d/dt, div and ∇. So in the Cartesian coordinate system

$$\frac{d}{dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}, \qquad \operatorname{div}\mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y}, \qquad \nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y};$$

$$\frac{D}{Dt} = \frac{d}{dt} + w\frac{\partial}{\partial z}, \qquad \operatorname{Div}\mathbf{A} = \operatorname{div}\mathbf{A} + \frac{\partial A_z}{\partial z}, \qquad \operatorname{grad} = \nabla + \mathbf{k}\frac{\partial}{\partial z}.$$

We will also agree that the vertical z-axis always points upwards, i.e., opposite to the gravity vector, and that the horizontal axes x and y together with z form a right-hand coordinate system. So if the x-axis points to the east, the y-axis faces north (see Fig. 6.1).

6.3 “Shallow-Water” Theory for a Rotating Ideal Fluid of Constant Density

One way to simplify the description of large-scale dynamics of a rotating fluid is to employ expansion in a small parameter of the original 3D equations of motion. It was already mentioned above that the main small parameter used in such an expansion is the Rossby–Kibel number $\varepsilon = U/2\Omega_0 L$. The zero approximation corresponds to the strictly zero value of the vector $\mathbf{G}$, i.e., to the precise hydrostatic

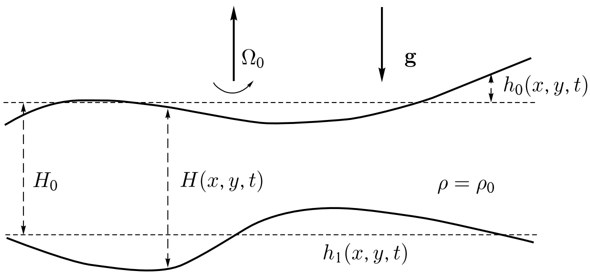

Fig. 6.3 A layer of a constant-density fluid with a free surface and uneven bottom, rotating in a gravity field with constant angular velocity

equilibrium and the strict relation of the geostrophic wind. These relations are unified by the following equality (cf. (6.5)):

$$2(\mathbf{\Omega}_0 \times \mathbf{u}) = -\frac{1}{\rho}\operatorname{grad} p + \mathbf{g}. \tag{6.18}$$

Equality (6.18) replaces the Euler equation. At first glance it might seem that the zero approximation (6.18) along with the divergence-free property of the 2D velocity field (6.11) and $w = 0$ describe a certain stationary climate state of a barotropic atmosphere, which could be taken as a basis for simplifying the initial equations of motion. However, this is not the case, since zero divergence of the wind field, as stated earlier, directly follows from (6.18) and consequently, the above-mentioned system is not closed. Vector **G**, in physics terminology, is merely an adiabatic invariant that can correspond to various solutions of hydrodynamical equations, including different climate states of the atmosphere.

However, the original formulation of the problem based on 3D hydrodynamical equations can be substantially simplified by using the Obukhov–Charney basis. This can be illustrated by an example of a layer of an ideal fluid of constant density $\rho = \rho_0$, of variable depth, and with a free surface that rotates about the vertical axis z with constant angular velocity Ω_0 (Fig. 6.3). The bottom relief is defined by a smooth function $z = h_1(x, y) \ll H_0$ of horizontal coordinates.

Using (I) and (II), one can obtain a nondegenerate approximation as follows.

First, based on (6.12) and (6.13) in the Euler equation

$$\frac{\partial \mathbf{v}}{\partial t} + u\frac{\partial \mathbf{v}}{\partial x} + v\frac{\partial \mathbf{v}}{\partial y} + w\frac{\partial \mathbf{v}}{\partial z} + 2\mathbf{\Omega}_0 \times \mathbf{v} = -\frac{1}{\rho}\nabla p \tag{6.19}$$

for the horizontal velocity $\mathbf{v}$ we can neglect the term $w\partial\mathbf{v}/\partial z$ as a quantity of second order in ε, while keeping only terms of order $O(1)$ and $O(\varepsilon)$.

Secondly, note that relation (6.1) implies a more precise validity of the quasi-statics condition, as compared to the condition of quasi-geostrophic equilibrium (6.2). This is indeed the case, and thus it is not by chance that the quasi-static equilibrium is used even for mesoscale (of about 100 km) processes with the help of so-called primitive equations of motion. The latter differ from the original ones in that they use a sharp equality (6.1) instead of the equation on the vertical velocity component. We will do the same and in the context of our system rewrite the above-mentioned condition as

$$p = \rho_0 g\big(H(x, y, t) - z\big). \tag{6.20}$$

According to this equation the hydrodynamic pressure component is given by the deviation $h(x, y, z, t) = H(x, y, z, t) - H_0 + h_1(x, y)$ of the height of the free surface from its equilibrium value (Fig. 6.3). Now after substituting (6.20) into (6.19), the equation for the horizontal velocity $\mathbf{v} = u\mathbf{i} + v\mathbf{j}$ of the flow takes the form

$$\frac{d\mathbf{v}}{dt} \doteq \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\nabla\mathbf{v}) + 2\boldsymbol{\Omega}_\mathbf{0} \times \mathbf{v} = -g\nabla H(x, y, t). \tag{6.21}$$

We should, however, keep the continuity equation in its original form, corresponding to the 3D incompressibility of the medium:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0,$$

since, according to our agreement, such small terms of the order $O(\varepsilon)$ as $\partial w/\partial z$ are not excluded from consideration.

A way to close the resulting system of equations comes out of the observation that the right-hand side of (6.21) does not depend on z (recall the Proudman–Taylor theorem). Therefore assuming $u = u(x, y, t)$ and $v = v(x, y, t)$ and integrating the last equation along the layer height with $w(z = h_1) = 0$ and $w(z = H) \doteq dH/dt$, one can rewrite the equation of mass conservation as follows:

$$\frac{dH}{dt} + H\,\mathrm{div}\mathbf{v} \equiv \frac{\partial H}{\partial t} + \mathrm{div}\,(H\mathbf{v}) = 0. \tag{6.22}$$

Let us write the condition of conservation of the potential vorticity for Eqs. (6.21)–(6.22). The simplest way to do this is to break the system (6.21) into two equations. Then, by differentiating the first equation in y and the second one in x and taking the difference, we get an equation for vorticity Ω_z. Then we use Eq. (6.22). Finally, we obtain

$$\frac{d}{dt}\left(\frac{\Omega_z + 2\Omega_0}{H}\right) = 0, \quad \left(\Omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right). \tag{6.23}$$

Thus, the fundamental properties (I) and (II) of geophysical flows allow one to reduce the problem of describing three-dimensional motions of an incompressible fluid to the study of two-dimensional motions of a barotropic gas with the help of equations (6.21) and (6.22), which are called the *equations of a rotating shallow water* (cf. (4.2)–(4.4)). In this regard, it is worth noting that, in contrast to the classical shallow-water theory (see Chap. 4), application of a shallow-water approximation of a rotating fluid is not limited by the condition $H/L \ll 1$ (where L is a typical horizontal scale of the flow). The reason is that in this case the two-dimensionality of the motion is a consequence of rotation rather than of the layer thinness. In other words, Equations (6.21) and (6.22) "work" for deep water as well, which is particularly important for laboratory modeling of geophysical flows. A special note is in order for the atmosphere of the Earth and of other revolving planets. For spherical layers the restriction mentioned above is rather significant because of the smallness of the Coriolis parameter $f = 2\Omega_0 \sin\varphi$ in the vicinity of the equator.

6.4 Exercises

1. Why does the wind twist inwards in a cyclone and outwards in an anticyclone?
2. Explain why the weather mostly comes from the west by using the formulas for thermal wind?
3. Why does the wind of Atlantic cyclones arrive at the Moscow region from the south?

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Chapter 7
The Obukhov–Charney Equation; Rossby Waves

The equations of rotating shallow-water (6.21) and (6.22) should not be considered as a completed next step in the expansion of the original equations in parameter ε with respect to the initial approximation (6.18). In fact they exceed the precision $O(\varepsilon)$, which was basic for this expansion, as can be seen already from the fact that these equations describe the propagation of long gravitational-gyroscopic waves for which the smallness condition for the Mach and Rossby–Kibel numbers is not valid. For the Earth's atmosphere, for instance, the group velocity of their propagation nearly coincides with the speed of sound, which corresponds to $\varepsilon \approx 1$ already for the wave length of one and a half thousand kilometers (see Sect. 7.4 below).

For the further reduction according to Sect. 6.1 we need to turn to the property (III) of geophysical flows, i.e., the conservation equation for the potential vorticity, which singles out the vorticity component of the motion. This will allow us to get rid of "extra" fast processes (e.g., gravitational-gyroscopic waves), for which the potential vorticity vanishes and concentrate on slow geophysical flows, which are naturally vortical (we discuss this question in more detail in Sect. 7.4).

7.1 Quasi-geostrophical Approximation of the Conservation Equation for Potential Vorticity

For a rotating shallow-water the conservation equation for the potential vorticity can be written in the form (see (6.23)):

$$\frac{d\Pi}{dt} = 0, \quad \Pi = \frac{\Omega_z + f_0}{H} \quad \left(\Omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right). \tag{7.1}$$

Here we use the notations $f_0 = 2\Omega_0,\ H = H_0 + h(x, y, t) - h_1(x, y)$. Recall that the function $h_1(x, y)$ defines the surface orography and it is assumed that there is no orographic impact at the zero approximation for the velocity field. Thus the relations

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for the geostrophic wind for Eqs. (6.21) and (6.22) are defined by the formulas

$$u = -\frac{g}{f_0}\frac{\partial h}{\partial y} + O(\varepsilon), \qquad v = +\frac{g}{f_0}\frac{\partial h}{\partial x} + O(\varepsilon). \tag{7.2}$$

Then one obtains the following estimate on the values of h for a free surface:

$$\frac{h}{H_0} \propto \frac{f_0 U L}{g H_0} = \frac{U}{f_0 L}\frac{L^2}{g H_0/f_0^2} = \varepsilon \frac{L^2}{L_0^2}, \quad L_0 \doteq \frac{\sqrt{g H_0}}{f_0}, \tag{7.3}$$

where the Rossby–Obukhov radius L_0 is determined by the propagation speed of gravitational waves, which plays the role of the speed of sound in shallow-water theory.

Now one can see that the smallness of the Rossby–Kibel number implies the smallness of the same order for the ratio h/H_0, provided that the linear size of the flow L varies in a neighborhood of the characteristic scale of global vorticity formations (cyclones and anticyclones), as was mentioned above.

Take into account that $\Omega_z/f_0 \propto U/2\Omega_0 L = \varepsilon$, and now for the first approximation of the potential vorticity one can write

$$\frac{H_0}{f_0}\Pi = \frac{H_0}{f_0}\left(\frac{\Omega_z + f_0}{H}\right) = 1 + \frac{\Omega_z}{f_0} - \frac{h}{H_0} + \frac{h_1}{H_0} + O\big(\varepsilon^2\big), \tag{7.4}$$

and the remainder is estimated by using the equality (7.3).

By renormalizing, i.e., by taking the quantity $\tilde{\Omega} = H_0 \Pi$ for the potential vorticity and introducing the notation $\psi \doteq gh/f_0$, one can note that $\Omega_z = \Delta\psi$, $\frac{f_0}{H_0}h = \frac{\psi}{L_0^2}$. Combining this with (7.2)–(7.4) we obtain the so-called quasi-geostrophic expression for the potential vorticity

$$\tilde{\Omega} = f_0\left(1 + \frac{h_1}{H_0}\right) + \Delta\psi - L_0^{-2}\psi + O\big(\varepsilon^2\big), \tag{7.5}$$

where $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$. We would like to emphasize that the term $L_0^{-2}\psi$ is implied by the term h/H_0, entering the formula (7.4). Further, taking into account that according to (7.2)

$$u = -\frac{\partial\psi}{\partial y}, \qquad v = \frac{\partial\psi}{\partial x}, \tag{7.2a}$$

the conservation equation for the potential vorticity in a quasi-geostrophic approximation is written in the form

$$\frac{d\tilde{\Omega}}{dt} = \frac{\partial}{\partial t}\big(\Delta\psi - L_0^{-2}\psi\big) + [\psi, \Delta\psi] + \beta\frac{\partial\psi}{\partial x} - \gamma\frac{\partial\psi}{\partial y} = 0, \tag{7.6}$$

$$\beta = \frac{f_0}{H_0}\frac{\partial h_1}{\partial y}, \qquad \gamma = \frac{f_0}{H_0}\frac{\partial h_1}{\partial x}. \tag{7.7}$$

This equation was first introduced by Charney (1948) and, independently by Obukhov (1949).

The nonlinear terms in Eq. (7.6) are present in the form of the term $[\psi, \Delta\psi]$, where $[A, B] = \frac{\partial A}{\partial x}\frac{\partial B}{\partial y} - \frac{\partial A}{\partial y}\frac{\partial B}{\partial x}$.

The linear terms in Eq. (7.6) with coefficients β and γ describe the so-called beta-effect that is responsible for dispersion of large-scale wave processes. These terms arise from the term $f_0\frac{h_1}{H_0}$, i.e., the first term of the right-hand side of (5) after the action of the operator $\frac{d}{dt}$. A similar effect also occurs in the case of a differential rotation of the system. Recall that for the Earth's atmosphere the crucial local parameter is the doubled projection of the angular velocity of the planet rotation to the normal to its surface, i.e., the quantity $f = 2\Omega_0 \sin\varphi$ (where φ is the latitude, see Chap. 6), which is called the *Coriolis parameter*. In this case the coordinates x and y, used in the derivation of the equation for conservation of the potential vorticity in quasi-geostrophic approximation, are measured in longitude and latitude to the east and north, respectively. The Coriolis parameter, varying only in latitude, has the derivative in the y coordinate only. The expression for $\tilde{\Omega}$ itself assumes the form $f + \Delta\psi - L_0^{-2}\psi + O(\varepsilon^2)$, where $f = f_0(1+\beta y)$, $\beta = df/dy$.

In any case the quasi-geostrophic approximation for equations of the theory of a rotating shallow-water can be written in the form

$$\frac{d}{dt}\left(f + \Delta\psi - L_0^{-2}\psi\right) = 0, \tag{7.8}$$

$$u = -\frac{\partial\psi}{\partial y}, \qquad v = \frac{\partial\psi}{\partial x}. \tag{7.9}$$

Being applied to the spherical Earth, they correspond to the shallow-water approximation, similar to (6.21) and (6.22), with the only difference that $2\mathbf{\Omega}_0$ is replaced by $\mathbf{k}f$.

One should remember that in the definition $\psi \doteq gh/f_0$ the quantity f_0 cannot be already replaced by f since the beta-effect is a quantity of the order of magnitude $O(\varepsilon)$. For the atmosphere, one usually takes for f_0 the value of f at $\varphi = 45°$.

7.2 Generalization to the Case of a Barotropic Fluid

One needs a somewhat more subtle approach to reduce the 3D equations of motion to a two-dimensional conservation equation for the potential vorticity of a barotropic fluid, in which by definition, the pressure, and therefore other thermodynamic features, depend only on the density. The upshot is that in such a system the surfaces of constant pressure, density, and temperature coincide. This is exactly the reason why in order to simplify the equations of motion one does not need to invoke the fourth "frame" of the Obukhov–Charney basis, the conservation equation for potential temperature (6.4). By averaging over verticals the problem can be reduced to a description of a two-dimensional motion governed by Eqs. (7.8) and (7.9). This motion takes place over a certain effective surface which can be assigned fixed values of density, pressure, and temperature. In this case, for calculating the parameter $L_0 = \sqrt{gH_0}/f_0$ entering the Obukhov–Charney equation, one takes the so-called *height of the homogeneous atmosphere* $H_0 = p_0/g\rho_0$ as the effective thickness of the layer. Here p_0 and ρ_0 are the standard (i.e., related to the static state of the atmosphere) "near-surface" values of the pressure and density. The propagation speed of gravitational waves $c_g = \sqrt{gH_0}$ corresponding to this altitude coincides with the speed of sound $c_s = \sqrt{\gamma p_0/\rho_0}$ up to a numerical factor of order one (where γ is the ratio of heat capacities, see Chap. 4, (4.22)). This speed of sound, of course, had to enter determining the scale L_0. For the Earth's atmosphere, for example, $p_0 \approx 1$ bar, $\rho_0 \approx 1.3$ kg/m^3 and the effective height of its barotropic model is $H_0 \approx 8$ km, while the speed of sound is $c_s \approx 280$ m/s.

A detailed derivation of the Obukhov–Charney motion of a barotropic atmosphere can be found in any textbook cited in the bibliography. However, I recommend the reader at some point to turn to the originals, Charney (1948) and Obukhov (1949), an acquaintance with which is not only useful but also brings aesthetic pleasure to follow and compare the reasoning of the two classics.

For a baroclinic fluid, in which the thermodynamical quantities are related only by the Mendeleev–Clapeyron relation ($p = \rho RT$), the above mentioned procedure is impossible. Otherwise, it would mean the loss of so-called available potential energy of the fluid, that is that part of the internal energy which is accumulated in the fluid due to uneven temperature distribution and which has the ability to transform into the kinetic energy of large-scale flows. The angle between the isobars and isotherms serves as a local measure of this available potential energy. The problem of describing baroclinic geophysical flows is discussed in Chap. 9.

7.3 Fundamental Invariants of Motion

One of the criteria that the reduction of the hydrodynamical equations is well-defined is, as we mentioned above, the existence of first integrals related to the fundamental conservation laws. It is not difficult to show that the Obukhov–Charney equation satisfies this criterion, i.e., it has the following invariants of motion in the integral form. The invariant

$$\frac{1}{2}\iint\left[(\nabla\psi)^2 + L_0^{-2}\psi^2\right]dxdy = \text{const} \tag{7.10}$$

corresponds to the energy conservation law (here the first term under the integral sign of (7.10) is the density of the fluid kinetic energy, while the second term is the density of its potential energy);

$$\iint\left(\Delta\psi - L_0^{-2}\psi\right)dxdy = \text{const} \tag{7.11}$$

means that the total potential vorticity of the fluid is invariant;

$$\int\!\!\int \psi\, dx dy = \text{const} \tag{7.12}$$

stands for the mass conservation law;

$$\int\!\!\int \left(\mathbf{i}\frac{\partial \psi}{\partial x} + \mathbf{j}\frac{\partial \psi}{\partial y}\right) dx dy = \text{const} \tag{7.13}$$

corresponds to the conservation of the total momentum of the medium.

7.4 Rossby Waves

The relations (6.1) and (6.2), used in the derivation of the Obukhov–Charney equation, serve as some kind of filter, which catches fast sound and gravity waves. These waves virtually do not affect the development of global processes, but significantly impede the "big game hunting." To get a feeling how it works, consider the problem of small oscillations of a rotating fluid in a shallow approximation. The complete formulation of this problem reduces to a solution of the tidal Laplace equation, see its detailed study in e.g., L.A. Dikii (1969).

After linearization with respect to the steady state (in the rotating coordinate system) Eqs. (6.21) and (6.22) of the shallow-water theory are written in the form

$$\frac{\partial \mathbf{v}}{\partial t} + f_0(\mathbf{k} \times \mathbf{v}) = -g\nabla h, \tag{7.14}$$

$$\frac{\partial h}{\partial t} + H_0 \mathrm{div}\mathbf{v} = 0, \tag{7.15}$$

where, for the sake of simplicity, the Coriolis parameter is assumed to be constant and equal to f_0, and the fluid depth is $H(x, y, t) = H_0 + h(x, y, t)$.

Instead of (7.14) and (7.15) it is more convenient to consider the system of equations with respect to $\Omega_z = \partial v/\partial x - \partial u/\partial y$ and h:

$$\frac{\partial \tilde{\Omega}}{\partial t} = 0, \qquad \tilde{\Omega} \doteq \Omega_z - f_0\frac{h}{H_0}, \tag{7.16}$$

$$\frac{\partial^2 h}{\partial t^2} + H_0 f_0 \Omega_z = g H_0 \Delta h. \tag{7.17}$$

By using the formula for rot$(\mathbf{A} \times \mathbf{B})$, it is easy to derive Eq. (7.16) by excluding div$\mathbf{v}$ from (7.15) and from the vorticity equation, where the latter is obtained by applying the operator rot$_z$ to (14). For the derivation of (7.17), apply the operator div to (14), differentiate Eq. (7.15) in time, and eliminate div $\mathbf{v}$.

In fact, $\tilde{\Omega}$ can be interpreted as the potential vorticity (cf. formulas (7.4) and (7.5)) of the linear problem (7.14) and (7.15), and in the case under consideration this vorticity becomes an invariant. From (7.16) and (7.17) one immediately obtains the existence of two types of solutions to the linear problem:

(a) The motions for which the potential vorticity vanishes, $\tilde{\Omega} = 0$, i.e., $\Omega_z = f_0 h/H_0$. This gives the telegraph equation for h:

$$\frac{\partial^2 h}{\partial t^2} + f_0^2 h = c_g^2 \Delta h, \quad c_g = \sqrt{g H_0}, \tag{7.18}$$

which describes the propagation of fast gravitational-gyroscopic waves in the rotating medium (plug in $h \propto \exp[i(\mathbf{k}\cdot\mathbf{r} - \omega t)]$, $\mathbf{k} = (k_x, k_y)$, $\mathbf{r} = (x, y)$, where i is the imaginary unit, to (7.18)). These waves have the dispersion relation

$$\omega^2 = f^2 + k^2 c_g^2, \quad k^2 = k_x^2 + k_y^2. \tag{7.19}$$

Their group velocity

$$C \doteq \frac{d\omega}{dk} = \frac{c_g}{\sqrt{1 + (f/kc_g)^2}} \tag{7.20}$$

in the Earth's conditions already for the wave length $L = 2\pi/k = 1500$ km almost coincides with $c_g \approx 280$ m/s.

Note that in the linear problem framework the wave motions Ω, $h_0 \propto \exp(-i\omega t)$ are possible only for $\tilde{\Omega} \equiv 0$, as follows from substituting $\Omega \propto \exp(-i\omega t)$ into (7.16), which gives $i\omega\tilde{\Omega} \equiv 0$.

(b) The motions whose field $\tilde{\Omega} \neq 0$ is nonvanishing and depends on the space coordinates only. Then the height of the free surface h will also be stationary, and hence $\operatorname{div} \mathbf{v} = 0$ (see (7.15)). This means that one can introduce the stream function

$$\psi = \frac{gh}{f_0}, \qquad u = -\frac{\partial \psi}{\partial y}, \qquad v = \frac{\partial \psi}{\partial x}. \tag{7.21}$$

Then from Eq. (7.8) we have

$$\Delta\psi - L_0^{-2}\psi = \tilde{\Omega}_0(x, y). \tag{7.22}$$

The velocity field (7.21) satisfies the relation of geostrophical wind.

In the nonlinear formulation of the problem the case (b) corresponds to the Obukhov–Charney equation (7.6) with zero beta-effect ($\beta = \gamma = 0$). Suppose that the beta-effect is different from zero, by setting, for example, $\gamma = 0$ and $\beta = \text{const} > 0$ (this case is called the beta-plane approximation, which is obtained from (7.8) for $f = f_0 + \beta y$). Then by direct substitution into (7.6) it is easy to verify that the functions

$$\psi_{kl} = A \exp\{i(kx + ly - \omega t)\}, \tag{7.23}$$

(where A is an arbitrary constant, k and l are the longitudinal and transverse wave numbers, respectively) *are exact particular solutions of the Obukhov–Charney* *equation describing the dispersing waves that propagate to the west* (*i.e., in the direction opposite to the fluid rotation as a whole*) *with the phase velocity*

$$c_R = \frac{\omega}{k} = -\frac{\beta}{k^2 + l^2 + L_0^{-2}}. \tag{7.24}$$

The wave solutions with such a dispersion relation are called the *Rossby waves* (Rossby, 1939), and sometimes, in relation to the spherical Earth, *Rossby–Haurwitz waves* (B. Haurwits). In this case they can be expressed in terms of spherical functions, see Comment to Exercise 2. To estimate the magnitude of their phase velocity it is useful to recall that by definition the beta-effect is of the order ε, i.e., $L/R = O(\varepsilon)$, where L is the flow characteristic size and R is the characteristic linear scale for which there is a noticeable change in the Coriolis parameter f. Therefore

$$|c_R| = \frac{f}{R}\frac{L_0^2 L^2}{L^2 + L_0^2} \propto \frac{1}{2} f \frac{L_0^2}{R} = \frac{1}{2} f \frac{\sqrt{gH_0}}{f}\frac{L_0}{R} = \frac{1}{2} c_g \frac{L_0}{R} = c_g O(\varepsilon), \tag{7.25}$$

since $L = O(L_0)$. It follows that the stationary solutions of the linear problem with nonvanishing $\tilde{\Omega}$ correspond to slow processes in the nonlinear case. The propagation speed of slow processes is much less than the propagation speed of gravitational waves, which are not described by Eq. (7.6).

In studying the interaction of Rossby waves (see, for example, Longuet-Higgins, Gill, 1967) it is convenient to divide them into the "short" planetary waves for which $L < L_0$ ($L^{-1} \doteq \sqrt{k^2 + l^2}$) and "long" planetary waves with $L > L_0$. According to (10) the density of the kinetic and potential energies of the Rossby waves (7.23) are equal to $A^2/4L^2$ and $A^2/4L_0^2$, respectively. So in short planetary waves the kinetic energy dominates, while in longer waves the energy is concentrated mainly in the potential component. This component can be measured by the deviation of the free surface of the fluid from its equilibrium level.

The Rossby waves or, as they are often called, planetary waves are typical representatives of geophysical flows which have no analogues in the non-rotating fluid. They constitute an important element of the general circulation of the ocean and atmosphere and have a significant effect on characteristics of the large-scale turbulence and instability of global motions. In this relation it is worth mentioning the so-called Rossby solitons. They belong to the family of Rossby waves, but are not covered by the Obukhov–Charney equation because their typical size exceeds L_0 by the order of magnitude. Their role is apparently important in the cyclogenesis processes in the atmospheres of giant planets, whose radius greatly exceeds the Rossby–Obukhov scale. This issue was studied in an article by M.V. Nezlin (1986), in which, in particular, one can find a detailed description of the methods and results of laboratory simulations of such vortex structures, as well as a comparison with observations from nature.

The scale L_0 serves, as we mentioned before, as a natural "watershed" between Rossby solitons and the waves and vortex structures considered here. The point is that when the flow scale exceeds L_0 in the order of magnitude the estimate (7.3) becomes invalid. (Recall the trend noted above: the longer the Rossby wave, the greater the deviation h_0 of the free surface from its equilibrium level.) The latter necessitates taking into account subsequent terms in the expansion in (7.4). This in turn leads to the appearance of an additional nonlinear term in the approximation equation, generalizing the Obukhov–Charney equation for the conservation of the potential vorticity. This additional term comes from a more precise control of the "horizontal compressibility" related to the change of height of the free surface. This nonlinearity can compensate for the dispersing impact of the beta-effect on the wave packet. As a result, solitary anticyclonic vortices, the Rossby solitons, can form in the fluid (compensating for the dispersion by non-linearity is impossible in cyclonic vortices, which implies the cyclone-anticyclone asymmetry observed at such scales).

From the point of view of problems discussed below, this limitation of possible applicability of the equations of dynamic meteorology is not fundamental, but it allows one to avoid additional technical difficulties.

7.5 Exercises

1. Prove that the expressions defined by the left-hand sides of equalities (7.10)–(7.13) are indeed invariants of the Obukhov–Charney equation. For the sake of simplicity, put $\beta = \text{const} \neq 0$, $\gamma = 0$ in (7.6) (this is the beta-plane approximation, widely used in dynamic meteorology).

   Solution: To prove the equality (7.10) one can multiply it by ψ and group the terms. One obtains

$$\frac{1}{2}\frac{\partial}{\partial t}\left[\left(\frac{\partial\psi}{\partial x}\right)^2+\left(\frac{\partial\psi}{\partial y}\right)^2+L_0^{-2}\psi^2\right]$$
$$=\frac{\partial}{\partial x}\left(\psi\frac{\partial^2\psi}{\partial x\partial t}+\psi\frac{\partial\psi}{\partial y}\Delta\psi+\frac{\beta}{2}\psi^2\right)+\frac{\partial}{\partial y}\left(\psi\frac{\partial^2\psi}{\partial y\partial t}-\psi\frac{\partial\psi}{\partial x}\Delta\psi\right). \quad (7.26)$$

   Now integrate (7.26) over some bounded domain S filled by the fluid and apply the Gauss–Ostrogradskii formula

$$\iint\limits_S \operatorname{div}\mathbf{A}\,dxdy=\int\limits_{\partial S}\mathbf{A}\cdot\mathbf{n}\,dl$$

   to the right-hand side (here $\mathbf{n}$ is the exterior normal to the boundary ∂S). Then taking into account regular behavior of the function ψ, send S to infinity. When integrating over such a large domain, the integral of the right-hand side increases with the same rate as the boundary perimeter of the domain S, while the integral of the left-hand side increases as the area of S. Hence we obtain

$$\iint\limits_S\left[\frac{1}{2}\left(\frac{\partial\psi}{\partial x}\right)^2+\frac{1}{2}\left(\frac{\partial\psi}{\partial y}\right)^2+\frac{1}{2}L_0^{-2}\psi^2\right]dxdy=0. \quad (7.27)$$

Here the first two terms under the integral sign represent the density of the kinetic energy, while the third one stands for the density of the potential energy.

Proofs for (7.12) and (7.13), as well as the conservation law for the total potential vorticity $\iint_S [\Delta\psi - L_0^{-2}\psi]\, dx dy = 0$ can be given by a similar reasoning; we refer the interested reader to the paper by Longuet-Higgins, Gill (1970).

2. Show that the planetary waves on a spherical surface in a two-dimensional incompressible medium ($L_0^{-1} = 0$) propagate westward with the angular velocity

$$\omega_R = \frac{2\Omega_0}{n(n+1)}, \tag{7.28}$$

where n is any positive integer (see Longuet-Higgins, Gill (1970) and Comment below).

Comment to Exercise 2: On a spherical surface the Obukhov–Charney equation in the dimensionless form is written as follows:

$$\frac{\partial}{\partial t}\left(\Delta\psi - L_{0s}^{-1}\psi\right) + \frac{1}{\sin\theta}\left(\frac{\partial\psi}{\partial\theta}\frac{\partial\Delta\psi}{\partial\lambda} - \frac{\partial\psi}{\partial\lambda}\frac{\partial\Delta\psi}{\partial\theta}\right) + 2\Omega_0\frac{\partial\psi}{\partial\lambda} = 0, \tag{7.29}$$

$$v_\theta = -\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\lambda}, \qquad v_\lambda = \frac{\partial\psi}{\partial\theta}. \tag{7.30}$$

Here $\theta = \frac{1}{2}\pi - \varphi$ is the complement to the latitude φ, the parameter λ is the longitude, L_{0s} is the Rossby–Obukhov parameter, and

$$\Delta\psi = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2\psi}{\partial\lambda^2}.$$

Recall that for flows on the sphere the Coriolis parameter is $f = 2\Omega_0\cos\theta$. Furthermore, local Cartesian coordinates in a neighborhood of the observation point are related to spherical coordinates by $dx = a\sin\theta$, $dy = -a d\vartheta$, where a is the Earth's radius. Therefore, the linear term $\beta\partial\psi/\partial x$ in the Obukhov–Charney equation preserves its sign and is replaced by $2\Omega_0\partial\psi/\partial\lambda$.

If the value of L_{0s} is less than or equal to one by the order of magnitude (i.e., the effect of two-dimensional compressibility of the medium is not small, as, for example, in the Earth's atmosphere), planetary waves are approximately described by spheroidal wave functions (see the literature cited in Longuet-Higgins, Gill, 1967). Otherwise (in the ocean, for example) one can neglect the value of $L_{0s}^{-2}\psi$ in comparison with $\Delta\psi$ in the Obukhov–Charney equation. Then *the spherical harmonic is a spherical analog of a plane Rossby wave*

$$\psi = AY_n^m(\theta,\lambda) = AP_n^m(\cos\theta)\cos(m\lambda + \omega t), \tag{7.31}$$

where m and n are integers related to the wave numbers by $m = ak\sin\theta$, $n = a\sqrt{k^2 + l^2}$. This implies, in particular, that m is the number of wavelengths which fit the latitude circle; $P_n^m(z)$ is the associated Legendre function of the first kind of degree n and order m.

Show that for $\omega = \omega_R$ (formula (7.28)) *function* (7.31) *satisfies the nonlinear equation*

$$\frac{\partial \Delta \psi}{\partial t} + 2\Omega_0 \frac{\partial \psi}{\partial \lambda} = 0. \tag{7.32}$$

The exact solution to the nonlinear equation (7.29) for $L_{0s}^{-1} = 0$, which describes the wave propagation westward is expressed via the spherical function of degree n

$$\psi(\theta, \lambda, t) = Y_n(\theta, \lambda + \omega t), \tag{7.33}$$

$$Y_n(\theta, \lambda) = A_0 P_n(\cos\theta) + \sum_{m=1}^{n} A_n^m P_n^m(\cos\theta)\cos\left(m\lambda + \lambda_n^m\right), \tag{7.34}$$

where A_0, A_n^m, λ_n^m are constants.

Show that *function* (7.33) *is an exact solution to the nonlinear vorticity equation only for* $\omega = \omega_R$.

Hint: The spherical function (7.34) is the eigenfunction of the Laplace operator, i.e., $\Delta Y_n = -n(n+1)Y_n$.

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Chapter 8
Resonant Interaction of Rossby Waves; Helmholtz and Obukhov Singular Vortices; The Kirchhoff Equations

8.1 Group Velocity of Rossby Waves

The theory of wave processes tells us that the energy of waves of any nature propagates not with the phase velocity but with the group velocity

$$\mathbf{C}_{gr} = \nabla_{\mathbf{k}}\omega, \quad \omega = \omega(\mathbf{k}), \tag{8.1}$$

where $\omega(\mathbf{k})$ is the dispersion relation for waves of this nature, and $\nabla_{\mathbf{k}}$ is the gradient operation in the $\mathbf{k}$-space of wave numbers. The waves whose phase velocity does not coincide with the group velocity are called *dispersion waves*. So, for instance, are gravitational-gyroscopic waves with the dispersion relation (7.20), according to which they isotropically propagate in space.

This is not the case for planetary waves because the dispersion relation

$$c_R = \frac{\omega}{k} = -\frac{\beta}{k^2 + l^2 + L_0^{-2}} \tag{8.2}$$

implies that

$$C_{RX} = \frac{\partial \omega}{\partial k} = c_R\left(1 - \frac{2k^2}{k^2 + l^2 + L_0^{-2}}\right), \tag{8.3}$$

$$C_{RY} = \frac{\partial \omega}{\partial l} = -c_R\frac{2kl}{k^2 + l^2 + L_0^{-2}}, \tag{8.4}$$

where C_{RX} and C_{RY} are the components of the group velocity $\mathbf{C}_R$.

Thus, in contrast to the phase velocity which is always directed to the west, the group velocity has two non-vanishing components, and its meridional component is always directed to the north, while the zonal component changes sign at $k^2/(l^2 + L_0^{-2}) = 1$ (Fig. 8.1). It is important to emphasize that, unlike the energy which can propagate by Rossby waves both in the western and the eastern directions, the

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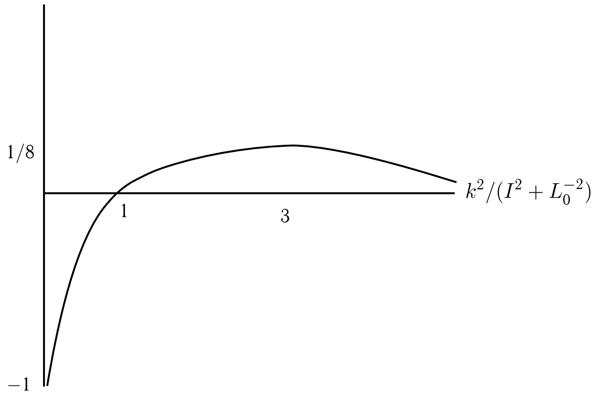

Fig. 8.1 The group velocity of Rossby waves in the zonal direction

vorticity

$$\Delta\psi \sim \exp\{i(kx+ly-\omega t)\}$$

is transported by these waves westwards only, i.e., in the direction opposite to the total rotation.

8.2 Resonant Interaction of Planetary Waves

The resonant interaction of planetary waves is the following important element of the general atmospheric circulation. Write the Obukhov–Charney equation on the beta-plane in the form

$$\frac{\partial}{\partial t}\left(\Delta\psi - L_0^{-2}\psi\right) + \beta\frac{\partial\psi}{\partial x} = [\Delta\psi,\psi] = \frac{\partial\Delta\psi}{\partial x}\cdot\frac{\partial\psi}{\partial y} - \frac{\partial\Delta\psi}{\partial y}\cdot\frac{\partial\psi}{\partial x}. \tag{8.5}$$

Assume that at the initial moment the flow is described by two Rossby waves:

$$\psi_1 = a_1\cos(\mathbf{k}_1\mathbf{x}-\omega_1 t), \qquad \psi_2 = a_2\cos(\mathbf{k}_2\mathbf{x}-\omega_2 t),$$

where $\mathbf{k}_1 = k_1\mathbf{i}+l_1\mathbf{j}$, $\mathbf{k}_2 = k_2\mathbf{i}+l_2\mathbf{j}$, and each pair $(\mathbf{k}_i,\omega_i)$ $(i=1,2)$ satisfies relation (8.2).

Plug in the expressions for the stream function $\psi = \psi_1+\psi_2$ to the Obukhov–Charney equation. It is not difficult to see that each of the waves separately makes the Jacobian vanish. Then the right-hand side of the equation assumes the form

$$a_1a_2\sin\{k_1x+l_1y-\omega_1t\}\sin\{k_2x+l_2y-\omega_2t\}\cdot(k_1l_2-k_2l_1)$$
$$\times\left[-\left(k_1^2+l_1^2\right)+\left(k_2^2+l_2^2\right)\right].$$

The left-hand side of Eq. (8.5) vanishes due to the fact that each wave satisfies the dispersion relation. The product of two sines can be expressed as the difference of cosines of the sum and the difference of the sine arguments. Thus, the right-hand side of (8.5) can be regarded as a periodic external force acting on the linear system.

The system response to this forcing will be small as long as there is no resonance, i.e., until the wave vector and frequency of the external force do not coincide with the wave vector $\mathbf{k}_3$ and frequency ω_3 of any eigenmode of the linear operator on the left-hand side of Eq. (8.5). In fact, the following equalities can be considered as conditions for the resonant interaction:

$$\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 = 0, \qquad \omega_1 + \omega_2 + \omega_3 = 0. \tag{8.6}$$

To find the equations for the resonant interaction of three waves, following M.S. Longuet-Higgins and A.E. Gill (1970), we will be looking for a solution to Eq. (8.5) in the form

$$\psi = a_1 \cos\theta_1 + a_2 \cos\theta_2 + a_3 \cos\theta_3, \tag{8.7}$$

where the a_i $(i = 1, 2, 3)$ are slowly changing functions of time, while

$$\theta_i = k_i x + l_i y - \omega_i t + \phi_i \quad (i = 1, 2, 3). \tag{8.8}$$

Here the waves numbers and frequences satisfy the dispersion relations

$$\omega_i \left(k_i^2 + l_i^2 + L_0^{-2}\right) + \beta \cdot k_i = 0 \quad (i = 1, 2, 3) \tag{8.9}$$

and the resonance conditions

$$k_1 + k_2 + k_3 = 0, \qquad l_1 + l_2 + l_3 = 0, \qquad \omega_1 + \omega_2 + \omega_3 = 0, \tag{8.10}$$

while the initial phases are related by

$$\varphi_1 + \varphi_2 + \varphi_3 = 0. \tag{8.11}$$

Again recalling the formula for sine products, we write

$$\sin\theta_1 \cdot \sin\theta_2 \equiv -\frac{\cos(\theta_1 + \theta_2) - \cos(\theta_1 - \theta_2)}{2}.$$

Substitute (8.7) into Eq. (8.5) by taking into account the dispersion relations (8.9). The terms containing $\sin\theta_i$ $(i = 1, 2, 3)$ cancel out thanks to the dispersion relations. As to the remaining terms, we already mentioned that the right-hand side is represented by the sine product, while the left-hand side contains (time) derivatives of a_i $(i = 1, 2, 3)$ as well. So we have

$$\begin{aligned}
&\left(\varkappa_1^2 + L_0^{-2}\right)\dot{a}_1 \cos\theta_1 + \left(\varkappa_2^2 + L_0^{-2}\right)\dot{a}_2 \cos\theta_2 + \left(\varkappa_3^2 + L_0^{-2}\right)\dot{a}_3 \cos\theta_3 \\
&\quad = C_1 a_2 a_3 \big[+\cos(\theta_2 + \theta_3) - \cos(\theta_2 - \theta_3)\big] \\
&\qquad + C_2 a_3 a_1 \big[+\cos(\theta_3 + \theta_1) - \cos(\theta_3 - \theta_1)\big] \\
&\qquad + C_3 a_1 a_2 \big[+\cos(\theta_1 + \theta_2) - \cos(\theta_1 - \theta_2)\big],
\end{aligned} \tag{8.12}$$

where $\varkappa_i^2 = k_i^2 + l_i^2$ $(i = 1, 2, 3)$, while the interaction coefficient is

$$C_1 = \frac{1}{2}\left(\varkappa_2^2 - \varkappa_3^2\right)(k_2 l_3 - k_3 l_2) = \frac{1}{2}\left(\kappa_2^2 - \kappa_3^2\right) \cdot (\mathbf{z} \cdot \mathbf{k_2} \times \mathbf{k_3}), \tag{8.13}$$

and $\mathbf{z}$ is the unit vector directed vertically upwards. The other coefficients are obtained by cyclic index permutations. Making use of the resonance conditions (8.10) one easily obtains

$$+k_3 l_1 - k_1 l_3 = +k_1 l_2 - k_2 l_1, \qquad +k_2 l_3 - k_3 l_2 = +k_1 l_2 - k_2 l_1. \tag{8.14}$$

Then in view of (8.11) we have $\theta_1 + \theta_2 + \theta_3 = 0$, while the three terms on the left-hand side of (8.12) are balanced by the three terms on the right-hand side provided that

$$\begin{aligned} \left(L_0^{-2} + \varkappa_1^2\right)\dot{a}_1 &= \Gamma\left(\varkappa_2^2 - \varkappa_3^2\right)a_2 a_3, \\ \left(L_0^{-2} + \varkappa_2^2\right)\dot{a}_2 &= \Gamma\left(\varkappa_3^2 - \varkappa_1^2\right)a_3 a_1, \\ \left(L_0^{-2} + \varkappa_3^2\right)\dot{a}_3 &= \Gamma\left(\varkappa_1^2 - \varkappa_2^2\right)a_1 a_2, \end{aligned} \tag{8.15}$$

up to non-resonant terms on the right-hand side of (8.12), which, as one could expect, have a weak response. The quantity Γ stands for the following:

$$\mathbf{z} \cdot \mathbf{k_2} \times \mathbf{k_3} = \mathbf{z} \cdot \mathbf{k_3} \times \mathbf{k_1} = \mathbf{z} \cdot \mathbf{k_1} \times \mathbf{k_2} = 2\Gamma. \tag{8.16}$$

A comparison of the amplitudes of non-resonant waves that are generated by the right-hand side of (8.12) with the amplitudes of the resonant waves shows (see Exercise 4) that this is indeed the case, if the nonlinearity is weak, i.e.,

$$\varkappa^2 a \ll \omega \tag{8.17}$$

(here $\varkappa$, a and ω are typical values of the wave number, amplitude, and frequency of the waves involved in the resonant interaction). In this case, the nonlinear terms in (8.5) are small when compared with the linear ones, and Eqs. (8.15) can be viewed as the result of an expansion in a small parameter $\varepsilon = \varkappa^2 a/\omega$, which is called the *degree of interaction*. Its smallness means that $\varkappa a \ll \omega/\varkappa$, i.e., that the velocity of a fluid particle is small in comparison with the phase velocity of the Rossby wave. Another equivalent interpretation of the resulting approximation is that it can also be seen as the result of averaging Eq. (8.12) over the "fast" time, on the assumption that the wave amplitudes are functions of slow time.

By an appropriate change of time t and coefficients $\varkappa_i$ $(i = 1, 2, 3)$, the system (8.15) can be reduced to the Euler equations for the motion of the classical gyroscope (see Chap. 12). Then one can write the following two quadratic invariants of the system (8.15):

$$E = \frac{1}{2}\left[\left(L_0^{-2} + \varkappa_1^2\right)a_1^2 + \left(L_0^{-2} + \varkappa_2^2\right)a_2^2 + \left(L_0^{-2} + \varkappa_3^2\right)a_3^2\right], \tag{8.18}$$

$$\Pi^2 = \left(L_0^{-2} + \varkappa_1^2\right)^2 a_1^2 + \left(L_0^{-2} + \varkappa_2^2\right)^2 a_2^2 + \left(L_0^{-2} + \varkappa_3^2\right)^2 a_3^2, \tag{8.19}$$

which correspond to the kinetic energy of a gyroscope and the square of its angular momentum. In our case, (8.18) corresponds to the full, i.e., the kinetic plus potential energy

$$\frac{1}{2}\iint \left[\left(\frac{\partial \psi}{\partial x}\right)^2 + \left(\frac{\partial \psi}{\partial y}\right)^2 + L_0^{-2}\psi^2\right] dxdy, \tag{8.20}$$

while (8.19) is related to the integral of the square of the potential vorticity

$$\iint \left[\Delta\psi - L_0^{-2}\psi\right]^2 dxdy \tag{8.21}$$

of the flow governed by Eq. (8.5).

8.3 The Helmholtz Singular Vortex and the Obukhov Geostrophic Vortex

In classical hydrodynamics, strictly two-dimensional vortex flows of an incompressible fluid are described by a stream function ψ. In terms of this function the vorticity equation has the form (cf. (8.5))

$$\frac{\partial \Delta\psi}{\partial t} + [\psi, \Delta\psi] = 0. \tag{8.22}$$

In this case the velocity components are expressed by the equalities

$$u = -\frac{\partial \psi}{\partial y}, \qquad v = \frac{\partial \psi}{\partial x}, \tag{8.23}$$

which automatically provide the divergence-free property of the velocity field, whose vorticity is $\Omega = \mathrm{rot}_z\, \mathbf{v} = \partial v/\partial x - \partial u/\partial y = \Delta\psi$. Therefore the velocity field of a two-dimensional flow can be reconstructed from its vorticity field by solving the Poisson equation

$$\Delta\psi = \Omega(x, y, t). \tag{8.24}$$

Its Green function for the unbounded integration domain and regular boundary conditions at infinity is given by

$$\psi_H = \kappa \ln r, \qquad \Delta\psi_H = 2\pi\kappa\delta(\mathbf{r}). \tag{8.25}$$

Here $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$ is the radius vector of the observation point in the plane of the flow, $r = \sqrt{x^2 + y^2}$, $\delta(\mathbf{r})$ is the two-dimensional Dirac delta-function.

In other words, the Green function (8.25) describes a two-dimensional fluid flow induced by a singular vortex tube whose intensity is $2\pi\kappa$ and whose vorticity is concentrated on the infinite straight line, orthogonal to the plane of motion and passing through the origin. This Green function is called the *Helmholtz singular vortex*, after Helmholtz who first introduced this concept in fluid dynamics.

A singular vortex tube is sometimes called a *vortex filament* which, given the two-dimensional nature of the motion, can be identified with a point on the plane and assigned a characteristic κ to this point. The quantity κ is called the strength of the singular vortex and is determined as follows. For an infinitesimal vortex tube $\kappa = \Omega\pi a^2/2\pi = \frac{1}{2}\Omega a^2$, where a is the tube's radius, while $\Omega\pi a^2$ is its intensity, which is an invariant of motion by the Kelvin theorem. Now by sending a to zero and Ω to infinity so that the product Ωa^2 remain constant, we find the strength of the vortex filament. In the presence of N such vortex filaments in the fluid, each of them moves along with the fluid at the speed induced at its location by the other $N-1$ singular vortices. Taking into account that this speed is equal to the vector sum of the velocities induced by the $N-1$ vortices at the given point, we derive the equations of motion for the N-vortex system.

Indeed, first consider two singular vortices with strengths κ_1 and κ_2, which are located at points $\mathbf{r} = \mathbf{r}_1(t) = x_1(t)\mathbf{i} + y_1(t)\mathbf{j}$ and $\mathbf{r} = \mathbf{r}_2(t) = x_2(t)\mathbf{i} + y_2(t)\mathbf{j}$, and which induce the velocity fields described by the stream functions $\psi_1 = \kappa_1 \ln|\mathbf{r}-\mathbf{r}_1|$ and $\psi_2 = \kappa_2 \ln|\mathbf{r}-\mathbf{r}_2|$, respectively. It is easy to check by a direct calculation that the movement of the first vortex in the velocity field of the second vortex is described by the equations

$$u_1 \doteq \dot{x}_1(t) = -\frac{\partial\psi_2}{\partial y}(\mathbf{r}=\mathbf{r}_1) = -\kappa_2\frac{\partial \ln|\mathbf{r}-\mathbf{r}_2|}{\partial y}(\mathbf{r}=\mathbf{r}_1)$$
$$= -\frac{1}{\kappa_1}\left(\kappa_1\kappa_2\frac{\partial \ln|\mathbf{r}_1-\mathbf{r}_2|}{\partial y_1}\right),$$
$$v_1 \doteq \dot{y}_1(t) = \frac{\partial\psi_2}{\partial x}(\mathbf{r}=\mathbf{r}_1) = \kappa_2\frac{\partial \ln|\mathbf{r}-\mathbf{r}_2|}{\partial x}(\mathbf{r}=\mathbf{r}_1)$$
$$= \frac{1}{\kappa_1}\left(\kappa_1\kappa_2\frac{\partial \ln|\mathbf{r}_1-\mathbf{r}_2|}{\partial x_1}\right).$$

Similarly,

$$u_2 \doteq \dot{x}_2(t) = -\frac{1}{\kappa_2}\left(\kappa_1\kappa_2\frac{\partial \ln|\mathbf{r}_2-\mathbf{r}_1|}{\partial y_2}\right),$$
$$v_2 \doteq \dot{y}_2(t) = \frac{1}{\kappa_2}\left(\kappa_1\kappa_2\frac{\partial \ln|\mathbf{r}_2-\mathbf{r}_1|}{\partial x_2}\right).$$

In terms of the function

$$\Psi = \kappa_1\kappa_2 \ln|\mathbf{r}_1-\mathbf{r}_2|,$$

these equations are written in the Hamiltonian form

$$\dot{x}_i = -\frac{1}{\kappa_i}\frac{\partial\Psi}{\partial y_i}, \qquad \dot{y}_i = +\frac{1}{\kappa_i}\frac{\partial\Psi}{\partial x_i}, \tag{8.26}$$

where $i = 1, 2$.

Now it is easy to foresee that in the case of N vortices, the Hamiltonian function has the form

$$\Psi = \sum_{i \neq j} \kappa_i \kappa_j \ln |\mathbf{r}_i - \mathbf{r}_j|, \tag{8.27}$$

where in both equations (8.26) and expression (8.27) the indices i and j assume values from 1 to N.

The system (8.26) and (8.27) is called the *Kirchhoff equations* (G.R. Kirchhoff, 1824–1887) *of the motion of N singular vortices.* It is worth mentioning that Kirchhoff wrote these equations in terms of complex variables, introducing complex coordinates $z_n = x_n + i y_n$ (where i is the imaginary unit) and the complex Hamiltonian

$$H_K = i \sum_{m \neq n} \kappa_m \kappa_n \ln(z_n - z_m) \quad (\Psi = \mathrm{Im} H_K). \tag{8.28}$$

Then the equations of motion assume the form

$$\dot{z}^* = \dot{x}_n - \dot{y}_n = \frac{1}{\kappa_n} \frac{\partial H_K}{\partial z_n}, \tag{8.26$'$}$$

where star means the complex conjugate.

It is easy to see that due to the Kirchhoff equations the following equalities hold:

$$\sum_{i=1}^{N} \kappa_i \dot{x}_i = 0, \qquad \sum_{i=1}^{N} \kappa_i \dot{y}_i = 0, \qquad \frac{d\Psi}{dt} = 0, \tag{8.29}$$

i.e., the "mass center", which is the vorticity center of the system

$$\mathbf{r}_0 = \sum_{i=1}^{N} \kappa_i \mathbf{r}_i \Big/ \sum_{i=1}^{N} \kappa_i, \tag{8.30}$$

does not move, while the Hamiltonian Ψ is a first integral of the system.

The system of two vortices (Fig. 8.2) is already interesting enough and, as we shall see, has a geophysical application. In particular, the vortex pair ($\kappa_1 = \kappa_2 = \kappa$, i.e., the vortex strengths are the same in sign and magnitude) rotates around its vorticity center with constant angular velocity (Fig. 8.2a)

$$\boldsymbol{\omega} = \frac{2\kappa}{L^2} \mathbf{z}, \tag{8.31}$$

where L is the constant in time (*why?*) distance between the vortices, and $\mathbf{z}$ is the unit vector in the direction of the z-axis. This shows that the pair of cyclones of positive vorticity rotates counterclockwise, while a pair of anticyclones rotates clockwise.

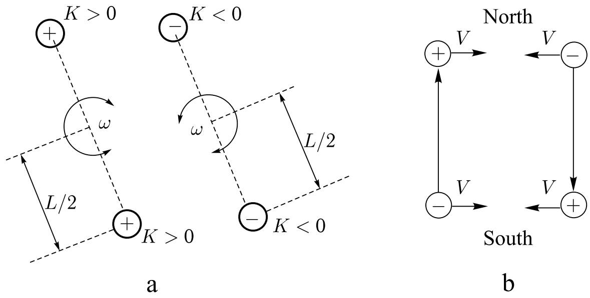

Fig. 8.2 Systems of two vortices: (**a**) a pair of singular vortices of equal strengths rotates about the center with angular velocity $\omega = 2\kappa/L^2$; (**b**) a dipole consisting of a cyclone and anticyclone moves with velocity $V = \kappa/L$ in the zonal direction

The vortex dipole ($\kappa_1 = \kappa = -\kappa_2$, i.e., the vortex strengths are equal in magnitude but of opposite signs) is moving along a straight line with constant velocity

$$\mathbf{V} = \frac{\kappa}{L^2}\mathbf{L} \times \mathbf{z}, \tag{8.32}$$

where $\mathbf{L} = \mathbf{r}_+ - \mathbf{r}_-$ (Fig. 8.2b).

In geophysical hydrodynamics the two-dimensional field of geostrophic wind can be reconstructed not from its vorticity but from its potential vorticity, which in the quasi-geostrophic approximation and after subtracting the Coriolis parameter is equal to $\tilde{\Omega} = \Delta\psi - L_0^{-2}\psi$. Hence, the *geostrophic singular vortex*, which was first introduced by Obukhov, is defined as the Green function for the equation

$$\Delta\psi - L_0^{-2}\psi = \tilde{\Omega}(x, y, t) \tag{8.33}$$

with the regular boundary conditions at infinity. This fundamental solution has the form

$$\psi_O = -\kappa K_0(r/L_0), \qquad \Delta\psi_O - L_0^{-2}\psi_O = 2\pi\kappa\delta(\mathbf{r}). \tag{8.34}$$

Here $K_0(x)$ is the Macdonald function, whose values for small and large arguments are expressed by the following asymptotic formulas:

$$K_0(r/L_0) \approx -\ln(r/L_0), \quad \text{for } r/L_0 \ll 1, \tag{8.35}$$

$$K_0(r/L_0) \approx -\frac{1}{2}\frac{\kappa}{\sqrt{2\pi r/L_0}}\exp\left(-\frac{r}{L_0}\right). \tag{8.36}$$

The motion of N singular geostrophic vortices, as can be easily seen, is also governed by the Kirchhoff equations (8.26) (or (8.26′)) with the only difference that in the Hamiltonian (8.27) (or (8.28)), the logarithmic function is replaced by the Macdonald function, for instance,

$$\Psi = \sum_{i \neq j} \kappa_i \kappa_j K_0\left(\frac{|\mathbf{r}_i - \mathbf{r}_j|}{L_0}\right). \tag{8.37}$$

The principal difference between the geostrophic vorticity filaments and the Helmholtz vortices is that the former are screened, i.e., due to (8.36) they have a finite interaction radius L_0, beyond which they virtually do not interact. This imposes certain features for the motion of such vortices. For example, if at the initial moment, the collection of vortices can be divided into clusters, the distances between which are much greater than L_0, then within the framework of applicability of inviscid hydrodynamical equations, each such cluster will evolve virtually independently. On the other hand, at short distances geostrophic singular vortices essentially do not differ from the Helmholtz vortices (see (8.35)).

Consider in this regard the behavior of the geophysical vortex dipole cyclone-anticyclone. In the northern hemisphere, the cyclonic circulation, i.e., the rotation around a low-pressure zone, has the same direction as the Earth's rotation and therefore is positive. The anticyclone vorticity is negative. Imagine that such a dipole is located in the zonal atmospheric flow, which is directed from west to east (recall that according to meteorological forecasts, the weather usually comes from the west). In the typical situation, where the cyclone is located to the north of the anticyclone, according to (8.32), the dipole moves to the east, ahead of the zonal transport. Otherwise, which is a rare phenomenon, the dipole is moving against the flow. Under certain conditions, the velocities of the zonal flow and dipole movement may be close with respect to the magnitude and opposite with respect to the direction. Then, for an observer on the Earth's surface, a vortex dipole is almost motionless, and the weather service does not change its short-term weather forecast for a long time. This phenomenon is called *blocking*, and in various regions of the world it is related to the setting of prolonged droughts in the summer and steady frosts in the winter. (Anticyclone is accompanied by fair weather.) It is no accident, therefore, that despite the tentative character of the application of the theory of dipolar geostrophic vortices to the description of real atmospheric processes, the appearance of the pair "a cyclone in the south and an anticyclone in the north" on the synoptic chart is for the weather forecaster one of the precursors of long-term fair weather.

It is worth noting that for over a century the theory of singular vortices has been attracting the attention of specialists, including such outstanding hydrodynamists as N.E. Zhukovsky, Th. von Karman, etc. First, this is related to the purity in formulating the problem, the rigor and elegance of mathematical approaches used to address specific related tasks, and secondly, with the ability of such a theory to explain a number of important hydrodynamical phenomena, such as the behavior of vortex structures in the trails behind the bodies moving in a fluid. (Recall Karman vortex street, the vorticity breakaway from the edge of the wing, vortex shedding past a cylinder, etc.). This theory also explains features of two-dimensional turbulence simulated with a large number of singular vortices. In particular, based on this approach to the geophysical hydrodynamics, there have been attempts to describe trajectories of tornadoes and even tropical cyclones. The reader can find a fairly complete picture of the progress in this area in the review by Aref et al. 1988, see Bibliography.

8.4 Exercises

1. What is the phase velocity of the Rossby wave evolving on the beta-plane in the presence of zonal flow $\mathbf{u} = (U_0, 0)$, $U_0 = \text{const}$? For what values of U_0 is the Rossby wave stationary with respect to the Earth or does it propagate westwards? We set the characteristic scale for the variation of the Coriolis parameter and length of the Rossby wave equal to (5–10) $\times 10^3$ km and 3×10^3 km, respectively.

   Answer:

$$c_R = U_0 - \frac{\beta + U_0 L_0^{-2}}{k^2 + l^2 + L_0^{-2}}, \qquad U_0 \geq 5\text{–}10 \text{ m/s}.$$

   Hint: Look for the solution of the equation

$$\frac{\partial}{\partial t}\left(\Delta\psi - L_0^{-2}\psi\right) + [\psi, \Delta\psi] + \beta\frac{\partial\psi}{\partial x} = 0$$

   in the form

$$\psi = \Psi_0 + \varphi(x, y, t),$$

   where $\Psi_0 = -U_0 y$, $\varphi = A\exp\{i(kx + ly - \omega t)\}$.

2. Calculate the maximum and minimum of the zonal group velocity as a function of k, and find the corresponding values of k^2 for which they are assumed.

   Answer:

$$\min C_{RX} = -\frac{\beta}{l^2 + L_0^{-2}} = c_R(k = 0), \quad \text{for } k = 0,$$

$$\max C_{RY} = \frac{\beta}{8(l^2 + L_0^{-2})} = -\frac{1}{8}c_R(k = 0), \quad \text{for } \frac{k^2}{l^2 + L_0^{-2}} = 3.$$

3. The term on the right-hand side of (8.12) proportional to $\cos(\theta_1 - \theta_3)$, induces the non-resonant wave $a_1' \sin(\theta_1 - \theta_3)$. Show by a direct computation that

$$a_1' = \frac{C_1}{(k_2 - k_3) - (1 - \varkappa'^2)(\omega_2 - \omega_3)} a_2 a_3,$$

   where $\varkappa' = |\mathbf{k}_2 - \mathbf{k}_3|$, while C_1 is given by formula (8.13). Starting with this formula, and assuming that the condition (8.17) holds, verify that $a_1'/a \ll 1$ for short ($\varkappa \gg 1$), as well as for long ($\varkappa \ll 1$), planetary waves.

4. Use the Euler theorem on the instability of the rigid body rotation around the middle axis of the inertia tensor in order to specify the conditions under which

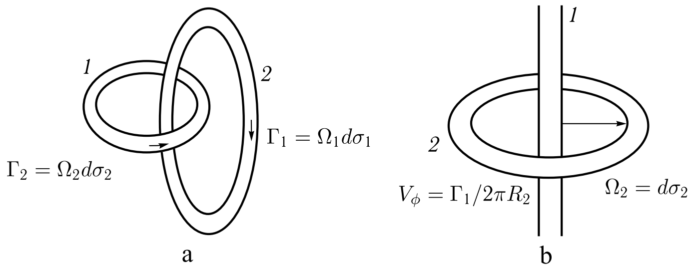

Fig. 8.3 Two simply linked closed vortex filaments

the Rossby wave with amplitude a_1 and the wave number $\mathbf{k}_1$ splits into two waves that are resonant to the initial one.

Answer: $\varkappa_3 \gtrless \varkappa_1 \gtrless \varkappa_2$.

5. Describe the behavior of two vortices with strengths equal and opposite in sign by using the Kirchhoff equations and their integrals of motion. Prove the formulas (8.31) and (8.32). How do the solutions change if $|\kappa_1| \neq |\kappa_2|$? What can one say about the behavior of N singular vortices of equal strength initially located at the vertices of a regular polyhedron? (Start with three vortices.) This problem has practical application. It is known that in nature and laboratory experiments, one can observe regular vortex formations whose vorticity centers are located on a circle. However, more than seven vortices are not observed in practice. It is no coincidence, since it is proved that if $N > 7$ then the corresponding vortex structure is unstable.

6. By using the concept of a singular vortex, try and calculate the helicity of two simply linked closed infinitesimally narrow vortex tubes–filaments (see Fig. 8.3).

   Answer: In this case we present the helicity as

$$H = \iiint \mathbf{u} \cdot \operatorname{rot} \mathbf{u} \cdot d\mu = 2\Gamma_1 \Gamma_2,$$

where $d\mu$ is the volume element, $\Gamma_i = 2\pi\kappa_i$ $(i = 1, 2)$ are the filament strengths.

Solution: The helicity characterizes knottedness, or the degree of entanglement of the vorticity lines, and it cannot change under a smooth deformation of the vortex filaments. By using stretching and squeezing we deform the configuration shown in Fig. 8.3a, in such a way that the first filament would be a straight line, i.e., it would close at infinity, while the second filament would surround the first along a circle of radius $r = R_2$ which is lying in the plane orthogonal to the line 1 (Fig. 8.3b). According to (8.25), line 1 creates an azimuthal velocity field in the space

$$v_\varphi = \frac{\partial \psi_1}{\partial r} = \frac{\kappa_1}{r} = \frac{\Gamma_1}{2\pi r},$$

parallel to this plane. The vorticity outside of the filaments is zero, while $\Gamma_2 \doteq \mathrm{rot}\,\mathbf{u}\cdot\delta\boldsymbol{\sigma}_2$, where $\delta\boldsymbol{\sigma}_2$ is the area of the oriented cross-section of the filament 2. Therefore, the contribution into the integral H of the configuration shown in Fig. 8.3b is

$$\frac{\Gamma_1}{2\pi R_2}\cdot 2\pi\,\Gamma_2 R_2 = \Gamma_1\Gamma_2.$$

The configuration in which vortices are interchanged gives exactly the same contribution.

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Chapter 9
Equations of Quasi-geostrophic Baroclinic Motion

9.1 Equilibrium State of a Rotating Baroclinic Medium

As we mentioned above, in a moving baroclinic medium isobaric and isopycnic (or iso-density) surfaces usually do not match. Recall that in the case of an incompressible baroclinic fluid its density and pressure are independent quantities, while the density of the baroclinic gas depends not only on pressure, but on yet one more thermodynamical quantity, for instance on the potential temperature Θ, i.e., $\rho = \rho(p, \Theta)$. (I would like to emphasize yet again that for the sake of simplicity, the possibility of phase transitions in the medium is not considered here, so there are only two independent thermodynamical variables.) Denote by the index s equilibrium distributions of thermodynamical quantities, which describe the medium state in the absence of relative motions, and use them as the background characteristics of the medium, while its motion is a deviation from this background.

It is worth mentioning that the choice of the background state of the medium is a rather delicate question, which, strictly speaking, should be resolved based on the spatial and time scales of the studied processes and by taking into account the influence of motions excluded from consideration. The latter remark is important, in particular, in order to compare specific calculations with data from observations. At this stage, it is sufficient to formulate the following typical constraints for the Earth's atmosphere and oceans that are imposed on the background characteristics selected above.

1. The equilibrium values of the thermodynamical quantities depend only on the height z and precisely satisfy the Mendeleev–Clapeyron equation

$$p_s = R\rho_s T_s \tag{9.1}$$

and the hydrostatic relation

$$\frac{dp_s}{dz} + g\rho_s = 0. \tag{9.2}$$

F.V. Dolzhansky, *Fundamentals of Geophysical Hydrodynamics*, Encyclopaedia of Mathematical Sciences 103,
DOI 10.1007/978-3-642-31034-8_9,

2. The background state of the medium corresponds to a statically stable vertical distribution of the equilibrium potential temperature for the gas, i.e.,

$$\Theta_s = T_s \left(\frac{p_0}{p_s} \right)^{R/C_p} \tag{9.3}$$

$$\frac{d\Theta_s}{dz} > 0, \tag{9.4a}$$

and density for the fluid

$$\frac{d\rho_s}{dz} < 0. \tag{9.4b}$$

3. The dimensionless parameter

$$\eta \doteq \frac{N^2 H_0}{g} \ll 1. \tag{9.5}$$

Here g is the acceleration of gravity, H_0 is the typical depth of the medium, while N is the D. Brunt–V. Väisälä frequency, given by the formulas:

$$\text{for baroclinic gas (atmosphere)} \quad N = \left(\frac{g}{\Theta_s} \frac{d\Theta_s}{dz} \right)^{1/2}, \tag{9.6a}$$

$$\text{for stratified fluid (ocean)} \quad N = \left(-\frac{g}{\rho_s} \frac{d\rho_s}{dz} \right)^{1/2}. \tag{9.6b}$$

The physical meaning of N is that this is the frequency of small oscillations of a fluid parcel about its equilibrium position in a statically stable environment. Indeed, suppose, for example, that the atmosphere is in a static equilibrium state, described by the distributions of density $\rho = \rho_s(z)$ and pressure $p = p_s(z)$. At an arbitrary level $z = z_0$ we choose a fluid parcel of mass δm and density $\rho_s(z_0)$ and which is also under pressure $p_s(z_0)$, and displace this particle *adiabatically* to a level $z = z_0 + \delta z$ (Fig. 9.1). Let $\rho(z_0 + \delta z)$ be the density of the displaced fluid parcel to this new level. Then the volume displaced by this particle is equal to $\delta m/\rho(z_0 + \delta z)$ and the displaced mass of the surrounding atmosphere is $\rho_s(z_0 + \delta z)\delta m/\rho(z_0 + \delta z)$. Therefore the total of the gravity force directed downwards and the Archimedes force directed upwards is

$$F = -g\delta m + \frac{\rho_s(z_0 + \delta z)}{\rho(z_0 + \delta z)} g\delta m.$$

Then the acceleration $\frac{d^2\delta z}{dt^2}$ of the displaced mass δm can be found from Newton's law

$$\delta m \frac{d^2\delta z}{dt^2} = F.$$

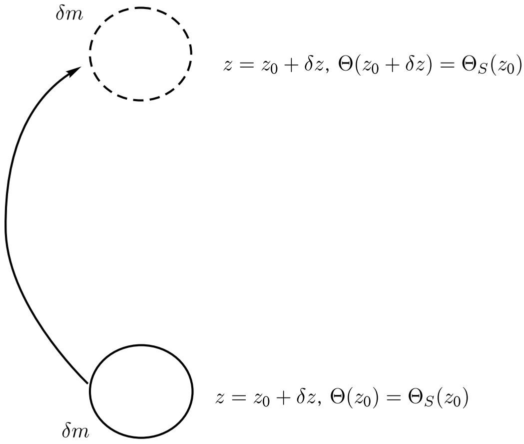

Fig. 9.1 A fluid parcel of mass δm moves adiabatically from level $z = z_0$ to an infinitesimally close level $z = z_0 + \delta z$. Its potential temperature at this level is $\Theta(z_0+\delta z) = \Theta(z_0) = \Theta_s(z_0) \neq \Theta_s(z_0 + \delta z)$; here Θ_s is the potential temperature of the surrounding medium

Whence, after the expansion $\rho_s(z)$ and $\rho(z)$ in a neighborhood of $z = z_0$ and taking into account that $\rho(z_0) = \rho_s(z_0)$ by definition, the resulting equality can be written as follows, modulo terms $O(\delta z^2)$:

$$\frac{d^2\delta z}{dt^2} = -g\left[+\frac{1}{\rho}\frac{d\rho}{dz}\bigg|_{z=z_0} \cdot \delta z - \frac{1}{\rho_s}\frac{d\rho_s}{dz}\bigg|_{z=z_0} \cdot \delta z\right]. \tag{9.7}$$

Now compute $d\rho/dz$, given that the density $\rho(z)$ of the chosen particle changes only with pressure. The formula

$$\Theta = T\left(\frac{p_0}{p}\right)^{R/C_p} \tag{9.8}$$

can be rewritten in terms of ρ and p (where $C_v = C_p - R$ is the specific thermocapacity for constant volume) by the Mendeleev–Clapeyron relation $p = R\rho T$:

$$\Theta = \frac{p_0}{R\rho}\left(\frac{p}{p_0}\right)^{C_v/C_p}. \tag{9.9}$$

Then

$$\rho(z) = \frac{p_0}{R\Theta(z_0)}\left(\frac{p(z)}{p_0}\right)^{C_v/C_p}.$$

Differentiating this equality in z and using (9.8) we obtain

$$\frac{d\rho}{dz} = \frac{C_v}{C_p}\frac{1}{R\Theta(z_0)}\left(\frac{p_0}{p(z)}\right)^{R/C_p}\frac{dp}{dz} = \frac{C_v}{C_p}\frac{1}{RT(z)}\frac{\Theta(z)}{\Theta(z_0)}\frac{dp}{dz}.$$

Setting $z = z_0$ and using the Mendeleev–Clapeyron formula once more, we have

$$\frac{d\rho}{dz} = \frac{C_v}{C_p}\frac{\rho_s}{p_s}\frac{dp_s}{dz} \quad \text{at the point } z = z_0,$$

and Eq. (9.7) is written in the form

$$\frac{d^2\delta z}{dt^2} = -g\left[\frac{C_v}{C_p}\frac{1}{p_s}\frac{dp_s}{dz} - \frac{1}{\rho_s}\frac{d\rho_s}{dz}\right]_{z=z_0}\delta z.$$

But, according to (9.9), the expression in square brackets is the derivative in z of $\ln\Theta_s = -\ln\rho_s + (C_v/C_p)\ln p_s + \text{const}$. Therefore, the equation of motion of the chosen particle can be written as

$$\frac{d^2\delta z}{dt^2} + \frac{g}{\Theta_s}\frac{d\Theta_s}{dz}\delta z = 0. \tag{9.10}$$

This implies that for

$$N^2 = \frac{g}{\Theta_s}\frac{d\Theta_s}{dz} > 0, \tag{9.10$'$}$$

the equilibrium state of the atmosphere is stable, while the particle makes harmonic oscillations with frequency (9.10′).

It is also pertinent to note that equalities (9.1)–(9.3) imply the relation

$$\frac{d\Theta_s}{dz} = \frac{\Theta_s}{T_s}\left(\frac{dT_s}{dz} + \frac{g}{C_p}\right), \tag{9.11}$$

according to which the medium preserves statistical stability even for $dT_s/dz < 0$ if

$$-\frac{dT_s}{dz} < \frac{g}{C_p}. \tag{9.12}$$

The quantity $\gamma_a \doteq g/C_p$ is called the *dry adiabatic gradient of temperature*, that for the Earth's atmosphere, for example, assumes the value 10 deg/km ($C_p \approx$ 1000 J/(kg deg)), whereas the real fall in the absolute temperature with height in the troposphere is approximately 6 deg/km.

The smallness of the parameter η becomes evident when we note that the quantity $g' = N^2 H_0$ can be interpreted as the effective acceleration of gravity of a fluid parcel under the influence of the total for the buoyancy and gravitational forces. For the Earth's atmosphere, for example, $\eta = O(0.1)$, whereas for the ocean $\eta = O(10^{-3})$. This distinction is due to the fact that the stratification of the ocean environment is caused not by its compressibility, but by the density stratification due to the nonhomogeneous salinity of water.

9.2 Quasi-geostrophic Approximation of the Equations of Motion of a Baroclinic Fluid

With regard to the baroclinic geophysical flows, in addition to the smallness of the Rossby–Kibel parameter and of η:

$$\varepsilon \doteq \frac{U}{f_0 L} \ll 1 \quad \text{and} \quad \eta \doteq \frac{N^2 H_0}{g} = O(\varepsilon), \tag{9.13}$$

we will also assume that the following parameter is small:

$$\xi \doteq \frac{f_0^2 L^2}{g H_0} = O(\varepsilon). \tag{9.14}$$

Generally speaking, the parameters ε, ξ and η are independent and the restrictions are typical, for instance, for the Earth's atmosphere. One can also use other constraints without altering structural properties of the final result. This is the case for the ocean, where $\eta = o(\varepsilon)$ (see the above estimates for η). Furthermore, the analysis below can be easily modified or generalized to the case of less restrictive conditions than (9.13) and (9.14), provided that the above-mentioned parameters remain small.

The smallness of the parameter ξ means that the linear velocity induced by the absolute vorticity of the medium f_0 is also noticeably "subsonic", i.e., it is at least half of the order of magnitude less than the propagation speed of long gravitational or internal waves. In fact this is the condition of weak 3D-compressibility of a rotating baroclinic gas.

Up until now we used a shortened Obukhov–Charney basis for deriving simplified "inviscid" equations of motion and we ignored the equation of conservation of potential temperature. According to the discussion above, for a baroclinic fluid both of the conserved quantities Π and Θ should be taken into account. The exact formulation of their Lagrangian invariance can be written in the form

$$\frac{D\Pi}{Dt} = \frac{D}{Dt} \frac{(\mathbf{\Omega} + 2\mathbf{\Omega}_0) \cdot \nabla \Theta}{\rho} = 0, \tag{9.15}$$

$$\frac{D\Theta}{Dt} = 0 \quad \left(\frac{D}{Dt} = \frac{d}{dt} + w \frac{\partial}{\partial z} \right). \tag{9.16}$$

It is convenient to represent nonequilibrium thermodynamic variables of the moving medium as follows:

$$p = p_s(z) + p'(x, y, z, t), \qquad \rho = \rho_s(z) + \rho'(x, y, z, t),$$

$$\Theta = \Theta_s(z) + \theta(x, y, z, t), \qquad T = T_s(z) + \vartheta(x, y, z, t),$$

where the second terms on the right-hand side of each of the equations describe *small deviations* of the corresponding quantity from its equilibrium value. Let us estimate the order of their smallness.

According to the geostrophic wind equation (6.7) and hydrostatic relations (6.1) and (9.2), $[p'] = f_0 L[\rho_s]U$, and $[p_s] = [\rho_s] g H_0$, where the square brackets stand for typical values of the encompassed quantities. Hence by (9.14)

$$\frac{p'}{p_s} \sim \frac{f_0 L U}{g H_0} = \frac{U}{f_0 L} \cdot \frac{f_0^2 L^2}{g H_0} = O(\varepsilon \xi) = O\left(\varepsilon^2\right). \tag{9.17}$$

According to the assumption (6.1) the hydrostatic relation approximately holds for the quantity $p = p_s + p'$ as well. Therefore $[p'] = [\rho'] g H_0$ and

$$\frac{\rho'}{\rho_s} = O\left(\frac{p'}{p_s}\right) = O\left(\varepsilon^2\right). \tag{9.18}$$

By using the equation of state (9.1) and the formula for potential temperature (9.9), which also hold for both equilibrium and nonequilibrium quantities, it is easy to show that the relative deviations ϑ / T_s and θ / Θ_s are connected with p'/p_s and ρ'/ρ_s by the following approximate relations:

$$\frac{\vartheta}{T_s} \approx \frac{p'}{p_s} - \frac{\rho'}{\rho_s}, \qquad \frac{\theta}{\Theta_s} \approx \frac{C_v}{C_p} \frac{p'}{p_s} - \frac{\rho'}{\rho_s}. \tag{9.19}$$

Hence, according to (9.18),

$$\frac{\vartheta}{T_s} = O\left(\varepsilon^2\right), \qquad \frac{\theta}{\Theta_s} = O\left(\varepsilon^2\right). \tag{9.20}$$

Now we move to the direct derivation of the quasi-geostrophic equation for potential vorticity. Single out in the expression for potential vorticity the main part associated with the vertical derivatives, since the vertical scale of global geophysical flows is much less than the horizontal one. Consequently, it is natural to assume that the derivatives related to the vertical scale will prevail. In addition, take into account the smallness of the Rossby number Ro and obtain that the relative vorticity Ω_z is much smaller than the Coriolis parameter $f = 2\Omega_0 \sin\phi$. Thus,

$$\begin{aligned} \Pi &= (\vec{\Omega} + 2\vec{\Omega}_0) \cdot \frac{\nabla \Theta}{\rho} \approx \frac{(\Omega_z + f) \cdot (\frac{d\Theta_s}{dz} + \frac{\partial \theta}{\partial z})}{\rho_s} \\ &= \frac{(\Omega_z + f)}{\rho_s} \cdot \frac{d\Theta_s}{dz} + \frac{f}{\rho_s} \cdot \frac{\partial \theta}{\partial z}. \end{aligned} \tag{9.21}$$

Since $|\Omega_z| \ll |f|$ and $|\theta'| \ll |\Theta_s|$, we can approximately write down the main part of the substantial derivative of the potential vorticity:

$$\begin{aligned} 0 = \frac{d\Pi}{dt} &= \frac{\partial \Pi}{\partial t} + u \cdot \frac{\partial \Pi}{\partial x} + v \cdot \frac{\partial \Pi}{\partial y} + w \cdot \frac{\partial \Pi}{\partial z} \\ &= \left(\frac{\partial}{\partial t} + u \cdot \frac{\partial}{\partial x} + v \cdot \frac{\partial}{\partial y}\right)\left[\frac{(\Omega_z + f)}{\rho_s} \cdot \frac{d\Theta_s}{dz} + \frac{f}{\rho_s} \cdot \frac{\partial \theta}{\partial z}\right] + w \cdot \frac{d}{dz}\left[\frac{f}{\rho_s} \cdot \frac{d\Theta_s}{dz}\right]. \end{aligned}$$

Note that the part $w\frac{\partial}{\partial z}$ of the substantial derivative which comes from the vertical velocity is taken into account here.

The equation for conservation of the potential temperature has the form

$$\left(\frac{\partial}{\partial t}+u\cdot\frac{\partial}{\partial x}+v\cdot\frac{\partial}{\partial y}\right)\theta+w\cdot\frac{d\Theta_s}{dz}=0. \tag{9.22}$$

Since the quantity $\frac{1}{\rho_s}\cdot\frac{d\Theta_s}{dz}$ is a function of variable z only, the operator $(\frac{\partial}{\partial t}+u\cdot\frac{\partial}{\partial x}+v\cdot\frac{\partial}{\partial y})$ does not act on it. Therefore, transform the derivative $\frac{d\Pi}{dt}$ as follows:

$$\begin{aligned}\frac{d\Pi}{dt}=\frac{1}{\rho_s}\cdot\frac{d\Theta_s}{dz}\cdot\left(\frac{\partial}{\partial t}+u\cdot\frac{\partial}{\partial x}+v\cdot\frac{\partial}{\partial y}\right)\left[\Omega_z+f+\frac{f}{d\Theta_s/dz}\cdot\frac{\partial\theta}{\partial z}\right]\\ +w\cdot\frac{d}{dz}\left[\frac{f}{\rho_s}\cdot\frac{d\Theta_s}{dz}\right]=0.\end{aligned}$$

Now recall the value of the frequency N^2 defined in (9.10′), and use the following identity, whose validity is easy to verify:

$$\begin{aligned}&\frac{g}{f\cdot\rho_s}\cdot\frac{\partial}{\partial z}\left[\frac{f^2\cdot\rho_s}{N^2}\cdot\frac{\theta}{\Theta_s}\right]\\ &\equiv\frac{f}{\rho_s}\cdot\frac{\partial}{\partial z}\left[\frac{\rho_s}{d\Theta_s/dz}\cdot\theta\right]\\ &=\frac{f}{\rho_s}\cdot\frac{\rho_s}{d\Theta_s/dz}\cdot\frac{\partial\theta}{\partial z}-\frac{f}{\rho_s}\cdot\frac{\rho_s\cdot\theta}{(d\Theta_s/dz)^2}\cdot\frac{d}{dz}\frac{d\Theta_s}{dz}+\frac{f}{\rho_s}\cdot\frac{\theta}{d\Theta_s/dz}\cdot\frac{d\rho_s}{dz}\\ &=\frac{f}{\rho_s}\cdot\frac{\rho_s}{d\Theta_s/dz}\cdot\frac{\partial\theta}{\partial z}-\frac{\rho_s\cdot\theta}{(d\Theta_s/dz)^2}\cdot\frac{d}{dz}\left[\frac{f}{\rho_s}\cdot\frac{d\Theta_s}{dz}\right]\\ &=\frac{f}{d\Theta_s/dz}\cdot\frac{\partial\theta}{\partial z}-\frac{\rho_s\cdot\theta}{(d\Theta_s/dz)^2}\cdot\frac{d}{dz}\left[\frac{f}{\rho_s}\cdot\frac{d\Theta_s}{dz}\right].\end{aligned}$$

Here we substitute the expression $\frac{f}{d\Theta_s/dz}\cdot\frac{\partial\theta}{\partial z}$ into that for $\frac{d\Pi}{dt}$, which we divide by $\frac{1}{\rho_s}\cdot\frac{d\Theta_s}{dz}$ in advance:

$$\begin{aligned}&\left(\frac{\partial}{\partial t}+u\cdot\frac{\partial}{\partial x}+v\cdot\frac{\partial}{\partial y}\right)\left[\Omega_z+f+\frac{g}{\rho_s\cdot f}\cdot\frac{\partial}{\partial z}\left(\frac{f^2}{N^2}\cdot\rho_s\cdot\frac{\theta}{\Theta_s}\right)\right]\\ &+\left(\frac{\partial}{\partial t}+u\cdot\frac{\partial}{\partial x}+v\cdot\frac{\partial}{\partial y}\right)\left[\frac{\rho_s\cdot\theta}{(d\Theta_s/dz)^2}\cdot\frac{d}{dz}\left(\frac{f}{\rho_s}\cdot\frac{d\Theta_s}{dz}\right)\right]\\ &+\frac{w\cdot\rho_s}{d\Theta_s/dz}\cdot\frac{d}{dz}\left[\frac{f}{\rho_s}\cdot\frac{d\Theta_s}{dz}\right]=0.\end{aligned} \tag{9.23}$$

Now introduce the *constant* Coriolis parameter $f_0=2\Omega_0\sin\phi_0$ and use it to replace the variable quantity f in the equation above. We use the equation for the conservation of potential temperature (9.22), and this allows one to cancel out the two last terms, since the quantity $\frac{\rho_s}{(d\Theta_s/dz)^2}\cdot\frac{d}{dz}(\frac{f_0}{\rho_s}\cdot\frac{d\Theta_s}{dz})$ depends on z only, and the operator $(\frac{\partial}{\partial t}+u\cdot\frac{\partial}{\partial x}+v\cdot\frac{\partial}{\partial y})$ does not act on it. Finally, we have

$$\left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right)\left[\Omega_z+f_0+\frac{g}{\rho_s\cdot f_0}\frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\rho_s\frac{\theta}{\Theta_s}\right)\right]=0. \tag{9.24}$$

In order to formulate this equation in terms of only one unknown function, one needs to have a relation of θ with the hydrodynamical component of the pressure $p'=p'(x,y,z,t)$ which defines the geostrophic velocity field through the relation (6.7). For this purpose use the second formula (9.19), which is transformed into the form

$$\frac{\theta}{\Theta_s}=\frac{1}{g\rho_s}\frac{\partial p'}{\partial z}-\left(\frac{C_v}{C_p}\frac{1}{p_s}\frac{dp_s}{dz}\right)\frac{p'}{g\rho_s} \tag{9.25}$$

by the hydrostatic relations (9.2) and (6.1) for the background and perturbed flows. To compare the terms on the right-hand side of (9.25), it is more convenient to formulate the obtained relation in terms of the frequency N (9.10′), which serves as the main measure of medium stratification in the theory of motion of stratified fluids. By using the equation of state (9.1), formula (9.3) for the potential temperature can be rewritten in terms of ρ_s and p_s:

$$\Theta_s=\frac{p_0}{R\rho_s}\left(\frac{p_s}{p_0}\right)^{C_v/C_p}. \tag{9.26}$$

Its logarithmic derivative gives the equality

$$\frac{C_v}{C_p}\frac{1}{p_s}\frac{dp_s}{dz}=\frac{1}{\Theta_s}\frac{d\Theta_s}{dz}+\frac{1}{\rho_s}\frac{d\rho_s}{dz}. \tag{9.27}$$

After substituting (9.27) into (9.25) the desired relation assumes the form

$$\frac{\theta}{\Theta_s}=\frac{1}{g}\frac{\partial}{\partial z}\left(\frac{p'}{\rho_s}\right)-\frac{N^2}{g^2}\frac{p'}{\rho_s},$$

which allows one to compare the terms on the right-hand side of the latter equality with the help of (9.13):

$$\frac{N^2}{g^2}\frac{p'}{\rho_s}=\frac{N^2H_0}{g}O\left(\frac{1}{gH_0}\frac{p'}{\rho_s}\right)=\eta O\left(\frac{1}{g}\frac{\partial}{\partial z}\left(\frac{p'}{\rho_s}\right)\right)\lesssim\frac{1}{g}\frac{\partial}{\partial z}\left(\frac{p'}{\rho_s}\right)O(\varepsilon).$$

As a result, the required relation between θ and p' with the desired degree of accuracy is given by the formula

$$\frac{\theta}{\Theta_s}=\frac{1}{g}\frac{\partial}{\partial z}\left(\frac{p'}{\rho_s}\right). \tag{9.28}$$

The latter formula, together with the geostrophic wind relations (6.7), which in this case can be written as

$$u = -\frac{1}{f_0}\frac{\partial}{\partial y}\left(\frac{p'}{\rho_s}\right), \qquad v = +\frac{1}{f_0}\frac{\partial}{\partial x}\left(\frac{p'}{\rho_s}\right), \tag{9.29}$$

allows one to close up Eq. (9.24). In terms of the geostrophic stream function

$$\psi(x, y, z, t) = \frac{p'}{f_0 \rho_s} \tag{9.30}$$

the quasi-geostrophic equation of the potential vorticity can be written as

$$\frac{d}{dt}\left[\Delta\psi + f + \frac{1}{\rho_s}\frac{\partial}{\partial z}\left(\frac{f_0^2 \rho_s}{N^2}\frac{\partial \psi}{\partial z}\right)\right] = 0 \quad \left(\Delta\psi = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right), \tag{9.31}$$

$$u = -\frac{\partial \psi}{\partial y}, \qquad v = +\frac{\partial \psi}{\partial x}. \tag{9.32}$$

It is easy to show (do it as a useful exercise) that for the ocean, for which in contrast with the atmosphere $\eta \ll \varepsilon$, while the role of Θ is played by ρ, the quasi-geostrophic equation of vorticity has a similar form. Here one merely needs to use the formula (9.6b) to calculate N.

One should note that the information on the baroclinic stream function $\psi = \psi(x, y, z, t)$ allows one to recover the vertical component of the flow velocity as well. Indeed, by excluding the pressure pulsation p' from (9.28) and (9.30), we obtain the relation

$$\frac{\theta}{\Theta_s} = \frac{f_0}{g}\frac{\partial \psi}{\partial z}. \tag{9.33}$$

After its substitution into (9.22), the vertical velocity can be written in the form

$$w = -\frac{f_0}{N^2}\frac{d}{dt}\left(\frac{\partial \psi}{\partial z}\right). \tag{9.34}$$

Remark We emphasize that, unlike (9.32), the formula (9.34) is valid only for adiabatic processes, i.e., in the absence of thermal conductivity and external sources of heat. Otherwise, the right-hand side of (9.22) will be different from zero, and denoting it by Q, we obtain

$$w = \frac{f_0}{N^2}\left[-\frac{d}{dt}\left(\frac{\partial \psi}{\partial z}\right) + \frac{g}{f_0}\frac{Q}{\Theta_s}\right] \tag{9.35}$$

instead of (9.34).

Thus, geophysical flows of a baroclinic fluid in the quasi-geostrophic approximation are also described in terms of the stream function, although in this case, it explicitly depends on the vertical coordinate z. The latter is due to the presence in a baroclinic fluid of the so-called available potential energy. Such an energy comes from horizontally inhomogeneous distribution of buoyancy forces, or, which is practically the same, of the potential temperature. We will discuss more details of this later. In the meantime, we note that the influence of the horizontal inhomogeneity in the distribution of potential temperature on the dynamics of geophysical flows can be seen directly from *relations of the thermal wind*

$$\frac{\partial u}{\partial z} = -\frac{g}{f_0 \Theta_s}\frac{\partial \theta}{\partial y}, \qquad \frac{\partial v}{\partial z} = +\frac{g}{f_0 \Theta_s}\frac{\partial \theta}{\partial x}, \tag{9.36}$$

which are easily obtained from (9.28) and (9.29) by eliminating the pressure. The relation of the thermal wind can be written as a single vector equality (cf. (6.16) and (6.17)):

$$\frac{\partial \mathbf{v}}{\partial z} = \frac{g}{f_0 \Theta_s}\mathbf{k} \times \nabla \Theta, \tag{9.37}$$

where $\mathbf{k}$ is the vertical unit vector, while the deviation θ is replaced by Θ, since the equilibrium quantities depend only on the vertical coordinate.

According to (9.37), the horizontal gradient of the potential temperature causes a systematic vertical wind shear. We recall (see Chap. 6) that for the Earth's atmosphere this means that the temperature difference between the poles and the equator is one of the main reasons for the motion instability of synoptic scale (the vertical velocity shear causes vortex formation). Therefore, in geophysical hydrodynamics, instability and cyclogenesis induced by the vertical wind shear are referred to as baroclinic processes, thus emphasizing their convective origin. This goes in contrast to the barotropic vortex formation, which is caused by a purely hydrodynamical instability of horizontal shear flows.

9.3 Exercises

1. Specify the stability conditions for a static equilibrium of a stratified incompressible fluid in the gravitational field. Show that in a stably stratified medium a fluid parcel that is slightly deviated from its equilibrium position will perform harmonic oscillations with the Brunt–Väisälä frequency N, given by the formula (9.6b).
2. Derive the quasi-geostrophic equation for the potential vorticity of the ocean, assuming the parameters ε, η and ξ small, but independent.

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Chapter 10
The Energy Balance, Available Potential Energy, and Rossby Waves in a Baroclinic Atmosphere

10.1 The Energy Conservation Law and the Concept of Available Potential Energy

In order to compare barotropic and baroclinic flows it is worth recalling the formulation of the local energy conservation law (7.26) for the Obukhov–Charney equation, which in the beta-plane approximation ($f = f_0 + \beta y$, $\beta = \text{const}$) can be written as follows:

$$\frac{\partial}{\partial t}\left[\frac{1}{2}(\nabla\psi)^2 + \frac{1}{2}\frac{\psi^2}{L_0^2}\right] = -\operatorname{div}\mathbf{S}_{bt}(x, y, t) \quad \left(\psi = \psi(x, y, t) \doteq \frac{gh}{f_0}\right), \tag{10.1}$$

$$\mathbf{S}_{bt} = \mathbf{i}\left(-\psi\frac{\partial^2\psi}{\partial x\partial t} - \frac{1}{2}\beta\psi^2 + \psi\frac{\partial\psi}{\partial y}\tilde{\Omega}_{bt}\right) + \mathbf{j}\left(-\psi\frac{\partial^2\psi}{\partial y\partial t} - \psi\frac{\partial\psi}{\partial x}\tilde{\Omega}_{bt}\right), \tag{10.2}$$

where $\tilde{\Omega}_{bt} \doteq \Delta\psi - L_0^{-2}\psi$ is the relative quasi-geostrophic potential vorticity for barotropic flows. Unlike the formula (7.26), here instead of $\Delta\psi$ one has $\Delta\psi - L_0^{-2}\psi$. This is justified, because the contribution of the additional term into the divergence is equal to zero:

$$+\frac{\partial}{\partial x}\left[-\psi\frac{\partial\psi}{\partial y}L_0^{-2}\psi\right] + \frac{\partial}{\partial y}\left[-\psi\frac{\partial\psi}{\partial x}L_0^{-2}\psi\right] = 0.$$

Having completed Exercise 1 in Chap. 7, the reader will easily derive that for baroclinic flows on the beta-plane governed by Eqs. (9.31) and (9.32), the local conservation law for the energy is formulated in the following form (multiply (9.31) by $-\rho_s\psi$ and regroup the terms):

$$\frac{\partial}{\partial t}\left[\frac{1}{2}\rho_s(\nabla\psi)^2 + \frac{1}{2}\rho_s\frac{f_0^2}{N^2}\left(\frac{\partial\psi}{\partial z}\right)^2\right] = -\operatorname{div}\mathbf{S}_{bc}(x, y, t), \tag{10.3}$$

F.V. Dolzhansky, *Fundamentals of Geophysical Hydrodynamics*,
Encyclopaedia of Mathematical Sciences 103,
DOI 10.1007/978-3-642-31034-8_10,

$$\mathbf{S}_{bc} = \mathbf{i}\left(-\rho_s\psi\frac{\partial^2\psi}{\partial x\partial t} - \frac{1}{2}\rho_s\beta\psi^2 + \rho_s\psi\frac{\partial\psi}{\partial y}\tilde{\Omega}_{bc}\right)$$
$$+\mathbf{j}\left(-\rho_s\psi\frac{\partial^2\psi}{\partial y\partial t} - \rho_s\psi\frac{\partial\psi}{\partial x}\tilde{\Omega}_{bc}\right) + \mathbf{k}\left(-\rho_s\frac{f_0^2}{N^2}\psi\frac{\partial^2\psi}{\partial z\partial t}\right). \quad (10.4)$$

Here

$$\psi = \psi(x, y, z, t) \doteq \frac{p'}{f_0\rho_s},$$
$$\tilde{\Omega}_{bc} = \Delta\psi + \frac{1}{\rho_s}\frac{\partial}{\partial z}\left(\rho_s\frac{f_0^2}{N^2}\frac{\partial\psi}{\partial z}\right)$$

is the relative quasi-geostrophic potential vorticity of the baroclinic flow.

If there is no mass flux across the boundary of the domain occupied by a baroclinic fluid, i.e., the velocity normal component on the boundary vanishes, one can show (see Chap. 11) that (10.3) and (10.4) imply the invariance of the total energy

$$E_{bc} \doteq \iiint_V \left[\frac{1}{2}\rho_s(\nabla\psi)^2 + \frac{1}{2}\rho_s\frac{f_0^2}{N^2}\left(\frac{\partial\psi}{\partial z}\right)^2\right]\delta\mu \quad \left(\frac{dE_{bc}}{dt} = 0\right), \quad (10.5)$$

where $\delta\mu = dxdydz$ is a volume element of a fluid parcel and V is the total fluid volume.

The expression for the total energy can be rewritten in terms of velocity and thermodynamical characteristics of the fluid by using (9.33):

$$E_{bc} \doteq \iiint_V \left[\frac{1}{2}\rho_s(\nabla\psi)^2 + \frac{1}{2}\rho_s\frac{g^2}{N^2}\frac{\theta^2}{\Theta_S^2}\right]\delta\mu. \quad (10.5')$$

The physical meaning of the second term on the right-hand side of (10.5′) is easy to understand by recalling that we are discussing those motions whose impact on thermodynamical characteristics of fluid parcels, and hence on their vertical coordinates, produces only slight deviations from their equilibrium values. Recall (see Chap. 9) that for a fluid parcel of unit volume, deviating along the vertical by an infinitesimal distance δz from its static equilibrium position, the total of the gravitational and buoyant forces which acts on this particle is equal to $F = -\rho_s N^2\delta z$. Therefore the potential energy (relative to the equilibrium position) corresponding to this force is

$$\delta P_{bc} = \frac{1}{2}\rho_s N^2(\delta z)^2. \quad (10.6)$$

The potential temperature of a displaced fluid parcel differs from the equilibrium potential temperature at its new position by the value $\theta = (d\Theta_s/dz)\delta z$. Now expressing δz via θ and substituting it into (10.6) subject to (9.6a), we find that the second term of the integrand on the right-hand side of (10.5′) (and hence of (10.5)) coincides with the *potential energy of a fluid parcel of unit volume relative to its static equilibrium position*, i.e.,

$$\delta P_{bc} = \frac{1}{2}\rho_s \frac{g^2}{N^2}\frac{\theta^2}{\Theta_s^2} = \frac{1}{2}\rho_s \frac{f_0^2}{N^2}\left(\frac{\partial\psi}{\partial z}\right)^2. \tag{10.7}$$

The quantity

$$P_{bc} \doteq (APE)_{bc} = \iiint\limits_V \frac{1}{2}\rho_s \frac{g^2}{N^2}\frac{\theta^2}{\Theta_s^2}\delta\mu = \iiint\limits_V \frac{1}{2}\rho_s \frac{f_0^2}{N^2}\left(\frac{\partial\psi}{\partial z}\right)^2 \delta\mu, \tag{10.8}$$

equal to the potential energy of the entire fluid minus the potential energy corresponding to its equilibrium state, is called the *available potential energy*. This concept was first introduced by a prominent American meteorologist E. Lorenz (1955). Its meaning is that it is this fraction of the potential energy that can be converted into kinetic energy of the geophysical motions, whereas the potential energy corresponding to the equilibrium state of the fluid turns out to be inaccessible for generating motions of the scale studied and therefore it is excluded from consideration.

From the point of view of this concept it is useful to compare the energy invariants for equations of the shallow-water theory (6.21) and (6.22) and their quasi-geostrophic approximation, that is the Obukhov–Charney equation (7.8). According to the local laws of energy conservation (4.25) (see Exercise 2 in Chap. 4, taking into account that the Coriolis forces do not perform work) and (10.1), the corresponding integral energy invariants can be represented in the form:

$$E_{sw} = \iint\limits_D \rho_0\left(\frac{1}{2}H\mathbf{v}^2 + \frac{1}{2}gH^2\right)\delta\sigma, \tag{10.9}$$

$$E_{bt} = \iint\limits_D \rho_0\left[\frac{1}{2}(\nabla\psi)^2 + \frac{1}{2}\frac{\psi^2}{L_0^2}\right]\delta\sigma = \iint\limits_D \rho_0\left(\frac{1}{2}\mathbf{v}^2 + \frac{1}{2}gh^2\right)\delta\sigma, \tag{10.10}$$

where $H = H(x, y, t)$ is the current height of the free surface of the fluid, H_0 is the equilibrium thickness of the "shallow water" layer, $h = H(x, y, t) - H_0$, $\delta\sigma = dxdy$ is an area element of the two-dimensional integration domain D and ρ_0 is the constant density of the fluid. (Here sw stands for *shallow water*.)

The quantity

$$P_{bt} = \frac{1}{2}\iint\limits_D \rho_0 g h^2\delta\sigma = \frac{1}{2}\iint\limits_D \rho_0 g(H - H_0)^2\delta\sigma, \tag{10.11}$$

that is equal to

$$\frac{1}{2}\iint\limits_D \rho_0 g H^2\delta\sigma - \frac{1}{2}\iint\limits_D \rho_0 g H_0^2\delta\sigma$$

(the integral of the linear in h term vanishes due to the mass conservation (7.12)), is the *available potential energy of barotropic geophysical flows*, according to the above definition. Indeed, the potential energy $\rho_0 H_0 d\sigma \cdot g \cdot \frac{1}{2} \cdot H_0$ of a fluid column of the height H_0 and of the area $d\sigma$ cannot be converted into fluid motion (unless, of course, one allows it to flow out somewhere). The portion of this available energy from the full potential energy of the layer, is measured by the ratio $\langle h^2\rangle/\langle H^2\rangle$ (where $\langle\ldots\rangle$ is the average over the fluid volume) and according to (7.3) is equal by the order of magnitude to the quantity $\varepsilon^2(L^2/L_0^2)^2$. When applied to the Earth's atmosphere, for example, this value has the order of 10^{-2}. A similar estimate also holds for the baroclinic component of the available potential energy. This characterizes the efficiency of the atmospheric heat engine, i.e., essentially coincides with the estimate of its efficiency coefficient equal to a few percent.

This analysis allows one to answer the question that naturally arises when comparing the barotropic and baroclinic quasi-geostrophical models of geophysical flows. The point is that the result of the limiting procedure in Eq. (9.31) as $N \to 0$, corresponding to the transition from the motion description of a stratified fluid to the motion description of a homogeneous fluid, *does not* coincide with the Obukhov–Charney equation (7.8). (This limit can be found in the book by L.A. Dikii.) This is easy to show by the following reasoning without resorting to formal mathematical calculations. Suppose the contrary, that is, assume that in the limit $N = 0$ Eq. (9.31) coincides with (7.8). This would mean that as $N \to 0$ one has

$$\frac{1}{\rho_s}\frac{\partial}{\partial z}\left(\frac{f_0^2\rho_s}{N^2}\frac{\partial\psi}{\partial z}\right) \longrightarrow -L_0^{-2}\psi.$$

But then according to (10.1), which is a consequence of (7.8), as $N \to 0$ the available potential energy of an individual fluid parcel of unit volume (see (10.3), (10.7)) would be

$$\delta P_{bc} = \frac{1}{2}\rho_s\frac{f_0^2}{N^2}\left(\frac{\partial\psi}{\partial z}\right)^2 \longrightarrow \frac{1}{2}\rho_0\frac{\psi^2}{L_0^2}.$$

However, according to (10.6) $\delta P_{bc} \longrightarrow 0$ as $N \to 0$. This means that in deriving (9.31), unlike (7.8), one did not take into account the barotropic component of the available potential energy, related to height changes of the fluid's free surface, cf. (10.11). This deficiency can be corrected by formulating the baroclinic model, for instance, in the so-called p-coordinates, in which one uses the pressure p instead of z for the vertical coordinate, based on the quasi-hydrostatic relation. This approach along with its advantages and disadvantages is discussed in the next chapter.

We note one more important conclusion which follows from comparison of the energy relations. The ratio of the available potential energy to the kinetic one has the order of magnitude L^2/L_0^2 for barotropic flows and L^2/L_R^2 for baroclinic ones, where the quantity

$$L_R \doteq \frac{NH_0}{f} = \frac{\sqrt{g'H_0}}{f} \quad \left(g' = N^2 H_0\right) \tag{10.12}$$

of the length dimension is called the *internal Rossby deformation radius*. In the stability theory of baroclinic flows this fundamental parameter plays the role which is as important as that of $L_0 = \sqrt{gH_0}/f$ in the stability theory of barotropic motions. In particular, L_R is the typical size of cyclones and anticyclones generated by instability of the vertical shear of the wind, i.e., by the instability of an inhomogeneous horizontal distribution of buoyancy forces, according to the thermal wind relation (9.37). Note that for $L = L_R$, the kinetic and potential energies of the vortex have the same order of magnitude. For the Earth's atmosphere (or ocean) L_R and L_0 are quantities of the same order of magnitude, equal to 1000 km (or 100 km respectively), although the square of their ratio $(L_R/L_0)^2$ is usually assumed to be equal to 0.1 with some reservations. This circumstance is related to one of the main difficulties in constructing an analytical theory of the general atmosphere and ocean circulation.

10.2 Baroclinic Rossby Waves

The baroclinic property of the environment affects the behavior of planetary waves as well. The background density changes at the characteristic scale given by the formula

$$D^{-1} = -\frac{1}{\rho_s}\frac{d\rho_s}{dz} \tag{10.13}$$

for the ocean and isothermal atmosphere, and one can set it to be constant. According to (9.6b) this also holds for the Brunt–Väisälä frequency in the ocean, but to a much lesser extent in the atmosphere. Note that for the ocean one has $H_0/D \ll 1$ because of the substantial smallness of the parameter $\eta = N^2 H_0/g$ (see Eq. (9.13)), whereas in the atmosphere, this ratio is of order one. Assuming values N and D to be constant as a reasonable approximation in both cases, Eq. (9.31) for the beta-plane ($f = f_0 + \beta y$) can be written in the form

$$\frac{d}{dt}\left(\Delta\psi + \frac{f_0^2}{N^2}\left(\frac{\partial^2\psi}{\partial z^2} - \frac{1}{D}\frac{\partial\psi}{\partial z}\right)\right) + \beta\frac{\partial\psi}{\partial z} = 0. \tag{10.14}$$

For the operator $\partial^2/\partial z^2 - D^{-1}\partial/\partial z$ the eigenfunction corresponding to the wave-like perturbation along the z axis with nodes at the distances that are multiples of the value H_0/m (m is an arbitrary multiple of π), has the form

$$\psi_m = \exp\left(\frac{z}{2D}\right)\Psi(x, y, t)\exp\left(\frac{imz}{H_0}\right), \tag{10.15}$$

where $\Psi(x, y, t)$ is so far an arbitrary function of horizontal coordinates and time. The eigenvalue corresponding to the function (10.15) is equal to

$$\lambda_m = -\left(\frac{1}{4D^2} + \frac{m^2}{H_0^2}\right), \tag{10.16}$$

i.e., ψ_m satisfies the equation

$$\frac{\partial^2 \psi_m}{\partial z^2} - \frac{1}{D}\frac{\partial \psi_m}{\partial z} = -\left(\frac{1}{4D^2} + \frac{m^2}{H_0^2}\right)\psi_m. \tag{10.17}$$

Now by making the substitution (10.17) and (10.15) into (10.14) for finding the function $\Psi(x, y, t)$, we obtain the Obukhov–Charney equation

$$\frac{d}{dt}\left[\Delta\Psi - \left(\frac{H_0^2}{4D^2} + m^2\right)\frac{\Psi}{L_R^2}\right] + \beta\frac{\partial \Psi}{\partial x} = 0, \tag{10.18}$$

in which the role of L_0, up to a positive multiplicative constant, is played by the Rossby inner radius of deformation $L_R = NH_0/f_0$. Therefore, the *family of exact partial solutions of Eq.* (10.14), *describing baroclinic modes of planetary-scale gyroscopic waves in a layer of stratified fluid of thickness* H_0, *is described by the functions*

$$\psi_{klm} = A\exp\left(\frac{z}{2D}\right)\exp\left\{i\left(kx + ly + \frac{mz}{H_0} - \sigma t\right)\right\}, \tag{10.19}$$

where A is an arbitrary constant, while

$$\sigma = -\frac{\beta k}{k^2 + l^2 + (m^2 + H_0^2/4D^2)L_R^{-2}}. \tag{10.20}$$

Hence, in particular, it follows that for $H_0/D \ll 1$ (for instance, in the ocean) and $m = 0$ solutions (10.19) and (10.20) degenerate into the family of barotropic modes, since according to (9.34) the vertical velocity almost vanishes and, consequently, equilibrium isopycnic surfaces remain unperturbed due to the lack of buoyancy forces.

In this regard, it is appropriate to mention the misunderstanding that could have arisen from the energy analysis that the baroclinic model (9.31) and (9.32) does not describe barotropic effects. In fact, the specified model does not take into account only the barotropic component of the available potential energy which is involved in the energy balance, while it maintains a mutual exchange of the kinetic energy between different barotropic modes, and this exchange is a key element of the mechanism of barotropic instability of geophysical flows.

10.3 Exercises

1. Derive the local conservation law of energy (10.3) and (10.4) for a baroclinic atmosphere by using the experience in deriving such a law for a barotropic medium.
2. Try and prove the energy invariance (10.5) using the conditions of fluid impermeability across the boundary of the integration domain and conservation of the

velocity circulation over any horizontal closed contour belonging entirely to the vertical part of the boundary. Assume that the integration domain is a cylinder with the vertical axis.

3. By using the dispersion relation (10.20) calculate the group velocity of the baroclinic Rossby wave and show that the *wave energy and phase propagate in opposite vertical directions*.

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Chapter 11
Important Remarks on the Description of Baroclinic Geophysical Flows

11.1 On p-Coordinates

Using the quasi-hydrostatic relation, atmospheric motion can be described in a coordinate system where the pressure p is chosen to be an independent vertical coordinate, while the height $z = z(x, y, p, t)$ of an isobaric surface $p = \text{const}$ becomes a dependent variable. Without going into technicalities of calculations (see Thompson, 1962), let us write the equations of a rotating compressible baroclinic fluid in these new independent variables x, y, p, t:

$$\frac{d\mathbf{v}}{dt} + w^* \frac{\partial \mathbf{v}}{\partial p} + \mathbf{k} \times f\mathbf{v} = g\nabla_p z, \tag{11.1}$$

$$\frac{\partial z}{\partial p} = -\frac{1}{g\rho}, \tag{11.2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w^*}{\partial p} = 0, \tag{11.3}$$

$$\frac{d\Theta}{dt} + w^* \frac{\partial \Theta}{\partial p} = 0, \tag{11.4}$$

which are closed up by relation (9.9)

$$\Theta = \frac{p_0}{R\rho}\left(\frac{p}{p_0}\right)^{C_v/C_p}. \tag{11.5}$$

Here $w^* = dp/dt$ plays the role of the vertical velocity, $\mathbf{v} = u\mathbf{i} + v\mathbf{j}$, $d/dt = \partial/\partial t + u\partial/\partial x + v\partial/\partial y$, $\nabla_p = \mathbf{i}\partial/\partial x + \mathbf{j}\partial/\partial y$, and all the derivatives with respect to horizontal coordinates and time are taken at constant pressure p.

This shows that the advantage of p-coordinates is in that Eq. (11.1) for horizontal velocity and the continuity equation (11.3) are written in the form as if the atmosphere were incompressible. The geostrophic wind in the p-coordinates is given by

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the formula

$$\mathbf{v} = \frac{g}{f_0}\mathbf{k} \times \nabla_p z, \tag{11.6}$$

or, in the coordinate form,

$$u = -\frac{g}{f_0}\frac{\partial z}{\partial y}, \qquad v = +\frac{g}{f_0}\frac{\partial z}{\partial x}, \tag{11.6$'$}$$

while the quasi-geostrophic equation for conservation of potential vorticity can be written as follows:

$$\frac{d}{dt}\left[\Delta\psi + f + \frac{\partial}{\partial p}\left(\frac{p^2}{L_R^2}\frac{\partial\psi}{\partial p}\right)\right] = 0. \tag{11.7}$$

Here $\psi = gz/f_0$, L_R is the inner radius of the Rossby deformation, defined above by formula (10.12), and according to (11.6$'$)

$$u = -\frac{\partial\psi}{\partial y}, \qquad v = +\frac{\partial\psi}{\partial x}. \tag{11.8}$$

It is worth noting that the simplification of the equations of motion do not come for free: difficulties hidden in the change of variables arise in setting the boundary conditions because, for example, the vertical velocity

$$w = \frac{dz}{dt} + w^*\frac{\partial z}{\partial p} \tag{11.9}$$

vanishes at the Earth's surface, which is not isobaric. Therefore, precise boundary conditions are replaced by approximate ones, which are posed on the "boundary" isobaric surfaces $p = 0$ and $p = p_0$, simulating the upper and lower horizontal boundaries of the atmosphere, respectively. One can show (see, for example, Kurgansky 1993) that for the quasi-geostrophic approximation (11.7) and (11.8), the vanishing of w^* on the upper horizontal boundary of the atmosphere and of w on the lower one is approximately expressed by the relations

$$\frac{d}{dt}\left(p^2\frac{\partial\psi}{\partial p}\right) \longrightarrow 0 \quad \text{as } p \longrightarrow 0, \tag{11.10}$$

$$\frac{d}{dt}\left(p\frac{\partial\psi}{\partial p} + \alpha^2\psi\right) = 0 \quad \text{for } p = p_0. \tag{11.11}$$

Here α^2 is the so-called *baroclinicity parameter* defined by the equality

$$\alpha^2 = \frac{R(\gamma_a + \gamma_s)}{g}\frac{T_s}{T_{0s}}, \tag{11.12}$$

where $\gamma_a = g/C_p$ is the dry-adiabatic temperature gradient defined above, γ_s and T_s are the background or the equilibrium temperature gradient and the temperature corresponding to the level of the isobaric surface in question, while T_{0s} is the average temperature over the Earth's surface.

Lateral vertical boundaries ∂D of the integration domain D are usually equipped with impermeability conditions, i.e., vanishing of the horizontal velocity component normal to the boundary:

$$v_n = -\frac{\partial \psi}{\partial l} = 0 \quad \text{on } \partial D \tag{11.13}$$

and the circulation conservation

$$\Gamma \doteq \oint_{d_p} v_l \delta l = \oint_{d_p} \frac{\partial \psi}{\partial n} \delta l \quad \left(\frac{d\Gamma}{dt} = 0 \right), \tag{11.14}$$

for each pressure level, i.e., for each closed contour d_p formed by intersection of the lateral border ∂D with an isobar $p = \text{const}$. Here $\partial/\partial l$ and $\partial/\partial n$ denote differentiation in the directions of the horizontal tangent and normal to ∂D respectively (cf. (11.8)).

Subject to the conditions (11.10), (11.11) and (11.13), (11.14), the energy invariant of the system (11.7), (11.8) can be expressed as

$$E_{bc} = \frac{1}{2} \iiint_D \left[(\nabla \psi)^2 + L_R^{-2} \left(p \frac{\partial \psi}{\partial p} \right)^2 \right] dx dy dp$$
$$+ \frac{1}{2} \iint_S \frac{\alpha^2}{L_R^2} p_0 \psi^2 dx dy, \tag{11.15}$$

where S is a two-dimensional area on the isobaric surface $p = p_0$ bounded by the contour d_{p_0} (derivation of this formula can be found in the book by A. Monin and A. Yaglom, p. 90).

This shows that in p-coordinates the energy integral includes both baroclinic and barotropic components of the available potential energy, the latter being described by the double integral in (11.15). Compare it with the expression (10.10) for the available potential energy of a barotropic atmosphere. Expression (11.12) for the baroclinicity parameter α^2 using the already known formulas

$$\frac{d\Theta_s}{dz} = \frac{\Theta_s}{T_s}(\gamma_a + \gamma_s), \qquad L_R^2 = \frac{N^2 H_0^2}{f_0^2}, \qquad L_0^2 = \frac{g H_0}{f_0^2},$$

$$N^2 = \frac{g}{\Theta_s} \frac{d\Theta_s}{dz}, \qquad p_s = \rho_s R T_s$$

(recall that H_0 is the height of a homogeneous atmosphere, i.e., of the same weight as the baroclinic atmosphere in question) can be rewritten as

$$\alpha^2 = k\frac{L_R^2}{L_0^2} \quad \left(k = \frac{p_s}{g\rho_s H_0}\frac{T_s}{T_{0s}} = O(1)\right). \tag{11.16}$$

The coefficient k can be regarded as an adjusting parameter. Its unit value provides precise consistency of (10.10) with the double integral in the formula (11.15). Therefore, it makes sense to interpret the *baroclinicity parameter as a squared ratio of the internal radius of the Rossby deformation to the Rossby–Obukhov scale of the barotropic atmosphere*, which is also handy for memorizing.

Thus, while the limit in Eq. (11.7) as $N \longrightarrow 0$ is not at all obvious, the energy integral demonstrates that this equation, unlike (9.31), with a well-defined approximation of the boundary conditions does describe the energy cycle involving both types of available potential energy. Actually, it is for this reason that I have chosen the p-coordinates. We will not employ them further to avoid complications of the analysis by having to adjust the boundary conditions and by comparing the results of the theory of the hydrodynamic stability of classical and geophysical flows. All the more so in that the properties of the former in many cases can be easily extended to the latter.

11.2 On Computing Integral Invariants

In the third part of this course we will use integral invariants to study stability of geophysical flows. In this regard, let us consider an example of their calculations and show how *the local energy conservation law* (10.3) *and* (10.4) *for baroclinic flows implies invariance of the total energy* (10.5), *provided that the normal component of the velocity is zero on the boundaries of the fluid domain*. As the integration domain V we take a cylindrical annular tank with flat horizontal ends positioned at fixed heights $z = z_1, z_2$. (This is the most common domain of integration that is used to study the stability of zonal atmospherical flows and their laboratory counterparts.)

Use the relation of geostrophic wind $\mathbf{v} = \mathbf{k} \times \nabla\psi$, i.e., $u = -\frac{\partial\psi}{\partial y}$, $v = +\frac{\partial\psi}{\partial x}$ to express the conditions for the vanishing of the velocity normal component on the side walls of the annular channel and for conservation of circulation Γ in terms of the stream function:

$$\mathbf{v}\cdot\mathbf{n} = 0 \Longrightarrow \mathbf{n}\times\nabla\psi = 0 \quad \text{on } \partial V_S, \tag{11.17}$$

$$\frac{d}{dt}\oint_C \mathbf{v}\cdot\delta\mathbf{l} = 0 \Longrightarrow \frac{d}{dt}\oint_{C_0} \mathbf{v}\cdot\delta\mathbf{l} = 0. \tag{11.18}$$

Here ∂V_S stands for the lateral boundary of integration, $\mathbf{n}$ and $\mathbf{k}$ are, respectively, the unit normal to ∂V_S and the vertical unit vector, while the integration is performed over the closed contours C and C_0 entirely belonging to the lateral boundary.

Let us integrate (10.3) over the volume V taking (10.4) into account. To convert the right-hand side of (10.3) we use the Gauss–Ostrogradskii formula

$$-\iiint\limits_V \operatorname{div} \mathbf{S}_{bc}\delta\mu = -\iint\limits_{\partial V_S} \mathbf{S}_{bc}\cdot\delta\boldsymbol{\sigma}$$
$$= \iint\limits_{\partial V_S} \rho_S\psi\frac{\partial^2\psi}{\partial y\partial t}dxdz + \iint\limits_{\partial V_S} \rho_S\psi\frac{\partial\psi}{\partial x}\tilde{\Omega}_{bc}dxdz$$
$$+ \iint\limits_{\partial V_H} \frac{f_0^2}{N^2}\rho_S\psi\frac{\partial^2\psi}{\partial z\partial t}dxdy. \tag{11.19}$$

Contribution of the **i**-component of (10.4) into (11.19) is zero due to the periodicity in the x coordinate directed along the parallel. In (11.19) ∂V_H is the horizontal boundary of the domain of integration.

The second term in (11.19) is zero due to the boundary condition (11.17): $v = +\frac{\partial\psi}{\partial x}|_{\partial V_S} = 0$. For the same reason the function ψ on the lateral boundary ∂V_S does not depend on x. Hence, for the first term in (11.19), the integral over the x coordinate can be written as follows:

$$\oint \rho_S\psi\frac{\partial}{\partial t}\frac{\partial\psi}{\partial y}dx = \rho_S\psi\frac{\partial}{\partial t}\oint\frac{\partial\psi}{\partial y}dx = -\rho_S\psi\frac{d}{dt}\oint\limits_{\partial V_S} u\cdot dx = 0. \tag{11.20}$$

The last expression is zero, assuming the no-slip condition on the boundaries. Another way to make the right-hand side of (11.20) vanish is to use the conservation of velocity circulation over the boundary ∂V_S, $\Gamma = \oint_{\partial V_S} u dx$, $\frac{d\Gamma}{dt} = 0$ (see Appendix A).

Employing the relations (9.10′) and (9.33) the last term on the right-hand side of (11.19) can be written as

$$\iint\limits_{\partial V_H} \frac{f_0^2}{N^2}\rho_S\psi\frac{\partial^2\psi}{\partial z\partial t}dxdy = (d\Theta_s/dz)^{-1} f_0\rho_s \iint\limits_{\partial V_H} \psi\frac{\partial\theta}{\partial t}dxdy. \tag{11.21}$$

Let us now turn to Eq. (9.22). Multiplying it by ψ and making some straightforward transformations with the zero divergence of geostrophic wind $u = -\frac{\partial\psi}{\partial y}$, $v = +\frac{\partial\psi}{\partial x}$ in mind, we get

$$\psi\frac{\partial\theta}{\partial t} = -\operatorname{div}(\psi\theta\mathbf{v}) - \psi w\frac{d\Theta_s}{dz}.$$

Substituting the latter equation into the integrand of the right-hand side of (11.21) and again using the Gauss–Ostrogradskii formula we obtain

$$\iint\limits_{\partial V_H} \psi\frac{\partial\theta}{\partial t}dxdy = -\oint \psi\theta\mathbf{v}\cdot\mathbf{n}dl - \frac{d\Theta_s}{dz}\iint\limits_{\partial V_H} \psi w\, dxdy. \tag{11.22}$$

The right-hand side of this equation equals zero due to the vanishing of the velocity normal component on ∂V. That proves the invariance of the total energy given by formula (10.5). It is useful, however, to bear in mind that in those cases when the vertical velocity on the horizontal boundaries is non-zero, according to (11.22) the last term on the right-hand side of (11.19) can be expressed in the form

$$\iint\limits_{\partial V_H} \frac{f_0^2}{N^2}\rho_s\psi\frac{\partial^2\psi}{\partial z\partial t}dxdy = -f_0\rho_s\iint\limits_{\partial V_H}\psi w\,dxdy = -\iint\limits_{\partial V_H} p' w\,dxdy. \quad (11.23)$$

The latter describes the work of the pressure forces at the ends of the annular channel. As we shall see below, in the case of a viscous fluid this mechanism of the energy loss provides an effective inhibition of the atmosphere against the Earth's surface due to the formation of Ekman boundary layers in its vicinity. We will elaborate on this inhibition mechanism in the study of viscous geophysical flows.

11.3 Exercise

1. Show that under assumptions of Sect. 11.2 *the total potential vorticity is a first integral of motion of the system* (9.31), (9.32), *i.e.*,

$$\frac{d}{dt}\iiint\limits_V\left[\Delta\psi + f + \frac{1}{\rho_s}\frac{\partial}{\partial z}\left(\frac{f_0^2\rho_s}{N^2}\frac{\partial\psi}{\partial z}\right)\right]dxdydz = 0. \quad (11.24)$$

   Hint: Using (9.31), write the time partial derivative of the integrand of (11.24) in the divergence form and integrate the resulting equation over the entire volume taking into account the boundary conditions (11.17) and (11.18).

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Part III
Hydrodynamic Stability and Atmospheric Dynamics

Chapter 12
The Notion of Dynamical Stability via the Example of Motion of a Rigid Body with a Fixed Point

12.1 Statement of the Problem

The scale of vortex motions ranges from regular laboratory flows to irregular leakage of water from the tap, to turbulent fluid flows in industrial machines and fast rivers, to circulations in the ocean, in planetary atmospheres, on the stars and even in the formation of galaxies. Emergence of such flows is usually associated with the loss of stability of the so-called primary flow. Its configuration is specified by the initial conditions in case of an ideal fluid or by external energy sources that sustain the motion of a viscous fluid.

The known mechanisms of instability are very few. For instance, the global atmospheric and oceanic currents are mainly formed under the influence of barotropic instability (i.e., due to the presence of horizontal shear of velocity), of baroclinic or convective instability caused by the excess of potential energy of a stratified fluid flow, and of orographic instability evoked by the underlying surface topography. It is also possible that the resonant interaction of planetary waves mentioned in Part II and so-called parametric instability may play a certain role here.

The stability of a solution can be understood in several ways. For example, Lyapunov stability means that for any $\varepsilon > 0$, there is a $\delta > 0$ such that if the initial conditions change by no more than δ, then the solution at any moment will change by no more than ε, i.e., *this is stability with respect to perturbations of initial data.* However, this approach can be rigorously used only on the basis of the unreduced (i.e., nonlinear) hydrodynamic equations and therefore it has limited analytical applicability, especially when dealing with viscous flows (see Part IV).

The main method of studying stability is based on the use of linearized hydrodynamic equations with respect to the considered state under the assumption that the initial perturbations of this state are infinitesimal. In this case, instability implies the existence of infinitely growing solutions of linear equations. As time increases, perturbation reaches a certain critical value in which the linear equations cease to be "working". At this value, according to the exact equations, under the influence of nonlinearity the system shifts into a new qualitatively different state, or, as physicists say, there occurs a phase transition.

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Now let us consider a cloud of initial data in a small vicinity of an unstable state. Every such perturbation will inevitably reach a value beyond which one has to take into account the nonlinear terms, and the system will move into the same secondary state regardless of the chosen initial conditions. However, each point of the cloud reaches that critical value at a different moment. So, if we now make each of the chosen initial perturbations tend to zero, the corresponding solutions for any moment in time will also tend to zero, but *nonuniformly in time*. This implies that *there is no Lyapunov stability. In other words, a proof of instability in linear approximation gives actual instability as well.*

The converse is generally not true. In this case it is necessary to solve the problem in a nonlinear formulation. For an ideal fluid an effective method is that of Lyapunov–Arnold, based solely on using first integrals of motion of hydrodynamic equations rather than on their solutions. In order not to burden the reader with unnecessary technical difficulties that inevitably arise when solving specific problems of hydrodynamic stability and obscure the fundamental idea, at this initial stage both approaches are illustrated below by the example of motion of a rigid body with a fixed point.

12.2 Linear Theory

The choice of such an example is not accidental, but it is made for several reasons. Firstly, the Euler equations of motion of a rigid body with a fixed point have two hydrodynamical interpretations: they describe both the flow of a homogeneous incompressible fluid within a heteraxial ellipsoid in the class of spatially linear velocity fields (see Part V below) and the resonant interaction of three dispersing waves, such as Rossby waves (see Sect. 8.2 of Chap. 8). Therefore, the Euler theorems proved below also have hydrodynamical applications. We shall see that again in studying global atmospheric motions and the Kolmogorov flow. Secondly, the Euler theorems turn out to be a mechanical analog of the hydrodynamic Rayleigh theorem on the stability of a plane shear flow of a homogeneous incompressible fluid, as discussed in Chap. 16.

The Euler equations (see Landau and Lifschitz 1973) in a mechanical interpretation are expressed in the coordinate system frozen into the body and take the form

$$\dot{\mathbf{m}} = \mathbf{m} \times \boldsymbol{\omega}, \quad \mathbf{m} = I\boldsymbol{\omega}. \tag{12.1}$$

Here $\boldsymbol{\omega}$ is the vector of angular velocity of the rotation, $\mathbf{m}$ is the angular momentum, I is the inertia tensor, i.e., the tensor of inertia momenta, which in the principal axes has a diagonal matrix with the diagonal elements I_1, I_2, I_3.

In the coordinate form Eq. (12.1) becomes

$$\begin{aligned} I_1\dot{\omega}_1 &= (I_3 - I_2)\omega_2\omega_3, \\ I_2\dot{\omega}_2 &= (I_1 - I_3)\omega_1\omega_3, \\ I_3\dot{\omega}_3 &= (I_2 - I_1)\omega_1\omega_2. \end{aligned} \tag{12.2}$$

Let us study the stability of the following fixed points

$$\omega_i = \omega_0 = \text{const} \neq 0, \qquad \omega_j = \omega_k = 0 \quad (i \neq j \neq k \neq i),$$

i.e., stationary fluid rotations about the principal axes of the ellipsoid. Let ω_i' $(i = 1, 2, 3)$ be an infinitesimal perturbation, say of the state $\boldsymbol{\omega}_0 = (0, 0, \omega_0)$.

We will look for a solution of the linearized equations of motion

$$\begin{aligned} I_1 \frac{d\omega_1'}{dt} &= (I_3 - I_2)\omega_0 \omega_2', \\ I_2 \frac{d\omega_2'}{dt} &= (I_1 - I_3)\omega_0 \omega_1', \\ I_3 \frac{d\omega_3'}{dt} &= O\big(\omega_1'\omega_2'\big) \end{aligned} \tag{12.3}$$

in the form of harmonic oscillations

$$\omega_1' = Ae^{-i\lambda t}, \qquad \omega_2' = Be^{-i\lambda t}.$$

This substitution results in the following system of linear algebraic equations on the coefficients A and B:

$$\begin{aligned} iI_1\lambda A + (I_3 - I_2)\omega_0 B &= 0, \\ (I_1 - I_3)\omega_0 A + iI_2\lambda B &= 0. \end{aligned}$$

This system has nonzero solutions provided that its determinant vanishes:

$$\begin{vmatrix} iI_1\lambda & (I_3 - I_2)\omega_0 \\ (I_1 - I_3)\omega_0 & iI_2\lambda \end{vmatrix} = 0.$$

Therefore

$$\lambda^2 = \omega_0^2 (I_3 - I_1)(I_3 - I_2)/I_1 I_2.$$

Since the roots of the characteristic equation are complex conjugate, the stability conditions imply that $\operatorname{Im}\lambda = 0$. Otherwise, there is a root with a positive imaginary part corresponding to the exponentially growing solution. Hence, *in linear approximation* the considered solution is stable, provided that

$$\begin{aligned} &\text{(a)} \quad I_3 > I_1, I_2 \quad \text{or} \\ &\text{(b)} \quad I_3 < I_1, I_2. \end{aligned}$$

These results can be expressed as the Euler theorems.

Theorem 1 *Rotations of a rigid body with a fixed point about the minor and major axes of the inertia tensor are stable.*

Theorem 2 *Rotations of a rigid body with a fixed point about the middle axis of the inertia tensor are unstable.*

Remark The linear system (12.3) can be rewritten in the form

$$\frac{d\boldsymbol{\omega}'}{dt} = L(\boldsymbol{\omega}_0)\boldsymbol{\omega}',$$

where $\boldsymbol{\omega}' = (\omega_1', \omega_2', \omega_3')$, $\boldsymbol{\omega}_0 = (0, 0, \omega_0)$ and

$$L(\boldsymbol{\omega}_0) = \begin{pmatrix} 0 & \frac{I_3 - I_2}{I_1}\omega_0 & 0 \\ \frac{I_1 - I_3}{I_2}\omega_0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then $-i\lambda_{1,2} = \sigma_{1,2}$ are eigenvalues of the matrix $L(\boldsymbol{\omega}_0)$ corresponding to the eigenmodes (eigenvectors) $\boldsymbol{\omega}_1' = (\exp(-i\lambda_1 t), 0, 0)$ and $\boldsymbol{\omega}_2' = (0, \exp(-i\lambda_2 t), 0)$ in the vicinity of the dynamic equilibrium of the system, i.e.,

$$L(\boldsymbol{\omega}_0)\boldsymbol{\omega}_{1,2}' = \sigma_{1,2}\boldsymbol{\omega}_{1,2}'.$$

Thus, the above procedure for studying linear stability of stationary solutions is reduced to finding eigenvalues and eigenvectors of the linear operator obtained by linearizing the nonlinear equations at the equilibrium position. This also holds for fixed points of an arbitrary dynamical system $\dot{\mathbf{x}} = N(\mathbf{x})$ (where $N(\mathbf{x})$ is a nonlinear operator and $\mathbf{x} = (x_1, x_2, \ldots, x_n)$) of order n, which after linearization at its fixed point $\mathbf{x} = \mathbf{X}_0$ ($\dot{\mathbf{X}}_0 = 0$) reduces to a system of linear equations $\frac{d\mathbf{x}'}{dt} = L(\mathbf{X}_0)\mathbf{x}'$ for infinitesimal perturbations $\mathbf{x}' = \mathbf{x} - \mathbf{X}_0$. At this stage it is already worth noting that as opposed to dynamical systems with a finite number of degrees of freedom, in hydrodynamics, the linearized operator has not only discrete but also a continuous spectrum of eigenvalues. This gives rise to an algebraic (polynomial) instability, often substantially complicating the study.

12.3 Nonlinear Theory: The Lyapunov–Arnold Method

The Euler equations (12.1) have two positive definite first integrals of motion. Those are the kinetic energy and the square of the angular momentum:

$$E = \frac{1}{2}\boldsymbol{\omega} \cdot \mathbf{m} = \frac{m_1^2}{2I_1} + \frac{m_2^2}{2I_2} + \frac{m_3^2}{2I_3}, \tag{12.4}$$

$$m^2 = \mathbf{m} \cdot \mathbf{m} = m_1^2 + m_2^2 + m_3^2. \tag{12.5}$$

In terms of m_i $(i = 1, 2, 3)$ the Euler equations assume that form:

$$\dot{m}_1 = \left(\frac{1}{I_2} - \frac{1}{I_3}\right)m_2 m_3,$$

$$\dot{m}_2 = \left(\frac{1}{I_3} - \frac{1}{I_1}\right) m_3 m_1, \tag{12.3'}$$

$$\dot{m}_3 = \left(\frac{1}{I_1} - \frac{1}{I_2}\right) m_1 m_2.$$

The idea of the Lyapunov–Arnold method is to find such a linear combination of invariants of motion that would serve as a Lyapunov function F, with its vanishing first variation at the fixed point $\mathbf{m}_0 = (0, 0, m_0)$ $(m_0 = I_3\omega_0)$. Then if $\delta F(\mathbf{m}_0) = 0$ and $\delta^2 F(\mathbf{m}_0)$ is a positive definite quadratic form of variations δm_i $(i = 1, 2, 3)$ (the negative definiteness can always be changed to the positive one by changing the sign of the Lyapunov function). This implies that the point $\mathbf{m}_0$ is a minimum of the Lyapunov function, and the second variation itself can be taken as a measure of deviation of the solution from the stationary solution by setting $\|\mathbf{m} - \mathbf{m}_0\|^2_{(1)} = \delta^2 F(\mathbf{m}_0)$ by definition. As the second measure, one can take the quantity $\|\mathbf{m} - \mathbf{m}_0\|^2_{(2)} = F(\mathbf{m}) - F(\mathbf{m}_0) = \delta^2 F(\mathbf{m}_0) + o(\delta \mathbf{m}^2)$. Due to the latter equality the two measures are equivalent, i.e., there are such positive constants C_1 and C_2 that

$$C_1 \|\mathbf{m} - \mathbf{m}_0\|_{(1)} \le \|\mathbf{m} - \mathbf{m}_0\|_{(2)} \le C_2 \|\mathbf{m} - \mathbf{m}_0\|_{(1)}. \tag{12.6}$$

Suppose now that initially the deviation $\delta\mathbf{m}$ from the fixed point $\mathbf{m}_0$ is small. Then, the measure $\|\mathbf{m} - \mathbf{m}_0\|_{(1)}$ is also small, and because of the second inequality of (12.6), so is the measure $\|\mathbf{m} - \mathbf{m}_0\|_{(2)}$. However, this second measure is an invariant of motion that will remain small at all times. According to the first inequality of (12.6), the measure $\|\mathbf{m} - \mathbf{m}_0\|_{(1)}$ will remain small as well, its positive definiteness implies smallness of the deviation $\delta\mathbf{m}$ at all times. Let us illustrate the above by specific calculations.

We will look for a Lyapunov function using the method of Lagrange multipliers, assuming

$$F = E + \lambda m^2 = \frac{m_1^2}{2I_1} + \frac{m_2^2}{2I_2} + \frac{m_3^2}{2I_3} + \lambda\left(m_1^2 + m_2^2 + m_3^2\right),$$

where λ is a constant determined by the condition $\delta F(\mathbf{m}_0) = 0$.

First, let us compute the first and second variations of F:

$$\delta F = \left(\frac{1}{I_1} + 2\lambda\right) m_1 \delta m_1 + \left(\frac{1}{I_2} + 2\lambda\right) m_2 \delta m_2 + \left(\frac{1}{I_3} + 2\lambda\right) m_3 \delta m_3,$$

$$\delta^2 F = \left(\frac{1}{I_1} + 2\lambda\right)(\delta m_1)^2 + \left(\frac{1}{I_2} + 2\lambda\right)(\delta m_2)^2 + \left(\frac{1}{I_3} + 2\lambda\right)(\delta m_3)^2.$$

The requirement $\delta F(\mathbf{m}_0) = 0$ implies that $2\lambda = -1/I_3$. Hence,

$$F = \frac{1}{2}\left(\frac{1}{I_1} - \frac{1}{I_3}\right) m_1^2 + \frac{1}{2}\left(\frac{1}{I_2} - \frac{1}{I_3}\right) m_2^2,$$

$$\delta^2 F = \left(\frac{1}{I_1} - \frac{1}{I_3}\right)(\delta m_1)^2 + \left(\frac{1}{I_2} - \frac{1}{I_3}\right)(\delta m_2)^2.$$

Then the stability conditions (the definiteness of $\delta^2 F$) mean that

$$\text{(a)} \quad I_3 > I_1, I_2 \quad \text{or}$$

$$\text{(b)} \quad I_3 < I_1, I_2,$$

which gives the same results as those of the linear theory.

Obviously, whenever $I_1 > I_3 > I_2$ or $I_1 < I_3 < I_2$, the quadratic form $\delta^2 F$ is sign-indefinite. In the context of nonlinear theory this just means a necessary condition for instability, and hence the proof of instability in general requires further study. However, in this case the instability of the rotation around the middle axis is proved by the linear theory, which as mentioned above implies actual instability.

12.4 Geometric Interpretation

In the space of kinetic moments m_i $(i = 1, 2, 3)$ trajectories of the top are obtained as the intersections of the "energy" ellipsoids (12.4), whose major axes are $\sqrt{2EI_i}$ $(i = 1, 2, 3)$, and the "circulation" spheres (12.5) of radius $|\mathbf{m}|$ centered at the origin. This is illustrated in Fig. 51 in Landau and Lifschitz (1973), which we refer to for more details. The intersection of the sphere and the ellipsoid is nonempty provided that the quantity $|\mathbf{m}|^2$ lies between the minimum and maximum values of $2EI_i$. When $|\mathbf{m}|$ is only slightly greater than the shortest semi-axis of the ellipsoid, the sphere intersects it along two small closed curves surrounding the corresponding ellipsoid axis. As the value of $|\mathbf{m}|$ (i.e., the sphere radius) increases, these curves expand, and when this radius coincides with the middle semi-axis of the ellipsoid, the corresponding curves degenerate into two ellipses that intersect each other at the points of the ellipsoid's middle axis. As $|\mathbf{m}|$ increases further, there again two separate closed curves appear, but this time they are surrounding the longest axis of the ellipsoid. Thus, in a neighborhood of the fixed points belonging to the shortest and longest principal axes of the ellipsoid, the top is doing small rotational oscillations. In other words, these fixed points are of the center type and are stable.

On the other hand, for the intersection of the ellipsoid with the sphere whose radius is equal to the middle semi-axis, one has a saddle point. It is the intersection point of two separatrices, such that nearby trajectories are attracted to the fixed point along one of the separatrices and are repelled from it along the other.

12.5 Exercises

1. The equations of motion of a rigid body with a fixed point in the field of the Coriolis forces can be written as follows (their derivation is given in Chap. 24,

Eq. (24.18)):

$$\dot{\mathbf{m}} = (\mathbf{m} + \mathbf{m}_0) \times \boldsymbol{\omega}, \quad \mathbf{m} = I\boldsymbol{\omega}, \quad \mathbf{m}_0 = I\boldsymbol{\omega}_0. \tag{12.7}$$

By replacing here $\boldsymbol{\omega}_0 \longrightarrow 2\boldsymbol{\omega}_0$ and formally setting $\boldsymbol{\omega} \longrightarrow -\boldsymbol{\omega}$ in Eq. (12.7) we obtain

$$\dot{\mathbf{m}} = \boldsymbol{\omega} \times (\mathbf{m} + \mathbf{2m}_0), \quad \mathbf{m} = I\boldsymbol{\omega}, \quad \mathbf{m}_0 = I\boldsymbol{\omega}_0. \tag{12.8}$$

Let the rotational motion of a rigid body be around its principal axis x_3 with constant angular velocity $\boldsymbol{\omega}_0 = (0, 0, \omega_0)$, while $\boldsymbol{\omega} = (0, 0, \Omega)$ is a stationary solution of Eq. (12.8). Using the methods of linear and Lyapunov–Arnold theory, try to formulate conditions for stability of this solution for each (shortest, middle, and longest) of the principal axes of the ellipsoid in terms of the Rossby number $Ro = \Omega/2\omega_0$.

Hint: The roots of the characteristic equation are determined from the expression

$$\lambda^2 = \left(\frac{I_3 - I_1}{I_2}\Omega + 2\frac{I_3}{I_2}\omega_0\right)\left(\frac{I_3 - I_2}{I_1}\Omega + 2\frac{I_3}{I_1}\omega_0\right).$$

Sketch the results graphically in the plane of parameters (Ro, λ^2) and show that the Coriolis forces have a dual effect: they destabilize stable solutions and stabilize unstable ones. Here the situation resembles the behavior of the Kapitza pendulum, the standard physical pendulum with a vibrating suspension point. Under certain conditions, the upper equilibrium position of the pendulum becomes stable, while the bottom one becomes unstable. The Coriolis force has a similar effect on global geophysical flows.

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Chapter 13
Stating the Linear Stability Problem for Plane-Parallel Flows of Ideal Homogeneous and Nonhomogeneous Fluids

13.1 Choosing the Initial Model

In the previous chapter we used the motion of a rigid body with a fixed point to get acquainted with two methods of studying linear and nonlinear stability of fixed points of a system. Those methods were based on linearization of the equations of motion and on the use of the first integrals of motion, respectively. The subject of the next few chapters will be the development and use of these methods in studying the stability of two-dimensional stationary flows of an ideal fluid. According to the theorem of H.B. Squire (see Lin, 1958) the most unstable modes develop along the plane of the flow, so one can ignore the three-dimensional perturbations and remain within the two-dimensional formulation of the problem.

We have already mentioned that stratification of a fluid rather than its compressibility plays the decisive role in the formation of baroclinic global atmospheric flows. So let us first study the linear stability of two-dimensional flows of an incompressible stratified fluid in a gravity field without taking into account the Coriolis forces. The results, as we shall see below, can be easily generalized to the global baroclinic geophysical flows. As a bonus, assuming the fluid density to be constant, we will obtain results on barotropic flows, and also generalize them to global geophysical flows.

The equations of motion of an incompressible stratified fluid can be written as follows:

$$\mathrm{R}\frac{d\mathbf{u}}{dt} = -\nabla P + \mathrm{R}\mathbf{g} \quad \left(\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{u}\nabla\right), \tag{13.1}$$

$$\frac{d\mathrm{R}}{dt} = 0, \qquad \operatorname{div}\mathbf{u} = 0. \tag{13.2}$$

Such a non-traditional notation R for the density is chosen because we are going to use a further simplification, assuming a weakly stratified fluid, i.e.,

$$\mathrm{R}(\mathbf{r}, t) = \rho_0 + \rho(\mathbf{r}, t), \qquad P(\mathbf{r}, t) = P_0(z) + p(r, t),$$

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where $\rho_0 = \text{const}$, $P_0(z)$ is the hydrostatic pressure corresponding to the density ρ_0:

$$\frac{dP_0(z)}{dz} + \rho_0 g = 0$$

and $\rho/\rho_0 = O(p/P_0) \ll 1$.

Make the substitution of the last three equations into (13.1) and (13.2). Now we omit the excess of the dynamic pumping $\rho\, d\mathbf{u}/dt$ as compared to the buoyancy forces $\rho\,\mathbf{g}$, since we are talking about convective flows, for which $|d\mathbf{u}/dt| \ll |\mathbf{g}|$. This way we obtain the equations named after A. Oberbeck (1879) and J. Boussinesq (1903):

$$\frac{d\mathbf{u}}{dt} = -\frac{1}{\rho_0}\nabla p + \frac{\rho}{\rho_0}\mathbf{g}, \qquad \frac{d\rho}{dt} = 0, \qquad \operatorname{div}\mathbf{u} = 0. \tag{13.3}$$

Such an approximation does not qualitatively affect the results and conclusions of the linear stability theory, but it somewhat simplifies the cumbersome formulas in formulating the problem precisely.

13.2 Linearization of the Equations of Motion

In the vertical plane (x, z), Eq. (13.3) can be written in the form

$$\begin{gathered}\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x},\\ \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} - g\frac{\rho}{\rho_0},\\ \frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + w\frac{\partial \rho}{\partial z} = 0, \qquad \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0.\end{gathered} \tag{13.4}$$

Recall that here ρ and p are deviations of density and pressure from their background values ρ_0 and P_0 defined above.

We study for stability the steady motion of a fluid of density $\rho_0 + \bar\rho(z)$ with the horizontal velocity $u = U(z)$, $w = 0$. According to (13.4), pressure $\bar p(z)$ in this case satisfies the hydrostatic relation $d\bar p/dz + g\bar\rho = 0$. We impose infinitesimal perturbations on this steady motion assuming $u = U(z) + u'(x, z, t)$, $w = w'(x, z, t)$, $\rho = \rho_0 + \bar\rho(z) + \rho'(x, z, t)$ and $p = p_0 + \bar p(z) + p'(x, z, t)$ and linearize the system (13.4). In other words, when substituting these expressions into the equations of motion we omit products of small quantities. Then the linear system of equations of motion with respect to perturbations u', w', ρ', and p' takes the form

$$\begin{gathered}\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)u' + \frac{dU}{dz}w' = -\frac{1}{\rho_0}\frac{\partial p'}{\partial x},\\ \left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)w' = -\frac{1}{\rho_0}\frac{\partial p'}{\partial z} - g\frac{\rho'}{\rho_0},\end{gathered} \tag{13.5}$$

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\rho'_x+\frac{d\bar{\rho}}{dz}w'=0, \qquad \frac{\partial u'}{\partial x}+\frac{\partial w'}{\partial z}=0.$$

According to the last equation of the system (13.5), we introduce the stream function ψ for velocity perturbations

$$u'=-\frac{\partial\psi}{\partial z}, \qquad w'=\frac{\partial\psi}{\partial x}.$$

Then the only nonzero component of vorticity of the perturbed component of motion, which is normal to the plane (x,z), is equal to

$$\frac{\partial w'}{\partial x}-\frac{\partial u'}{\partial z}=\Delta\psi.$$

Now applying the operation rot to the first two equations, i.e., differentiating the first equation in z, the other in x, and subtracting the first from the second, we obtain

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\Delta\psi-U''\psi_x=-g\frac{\rho'_x}{\rho_0}.$$

To eliminate the ρ'_x we apply the operator $\partial/\partial t+U\partial/\partial x$ to the last equality and use the third equation of system (13.5). As a result, we obtain the *main equation of linear stability theory of plane-parallel flows of a heavy stratified fluid, written only in terms of the stream function of velocity perturbations*:

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)^2\Delta\psi-U''\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\psi_x=-g\beta\psi_{xx} \quad \left(\beta=-\frac{1}{\rho_0}\frac{d\bar{\rho}}{dz}\right). \tag{13.6}$$

The parameter β is defined in such a way that its positive values correspond to a stable stratification of the initial state, i.e., to density decreasing with altitude.

13.3 Reduction of Boundary Conditions

If a fluid's motion is bounded by a solid horizontal wall, then one has $w'=\partial\psi/\partial x=0$ or $\psi=\text{const}$ on this wall. For smooth profiles of velocity and density for the primary flow this completes the formulation of the boundary value problem, except for the regularity conditions at infinity if the fluid occupies a half-space or even the whole space.

A more complicated case is when the primary flow has a discontinuous velocity or density at some level of $z=z_0$, separating, for example, two immiscible liquids of different densities. Perturbations of the primary flow induce fluctuations of the surface of discontinuity, or interface, whose equation $z=z_0+\zeta(x,t)$ now involves an unknown function $\zeta(x,t)$. This function, in general, is determined by solving the boundary problem: *two conditions have to be satisfied in this interface. One of them* *is kinematical: continuity of the velocity component normal to the interface, which is equal to the velocity $d\zeta/dt$ of the interface itself. The other condition is dynamical: continuity of the pressure. Note that both conditions are imposed on the surface of discontinuity, whose shape is unknown in advance*. Because of the infinitesimal character of the initial perturbations of velocity and density, it is natural to assume that deviations $\zeta(x,t)$ from the level $z = z_0$ are small. Then the situation can be greatly simplified by linearization of the conditions at the surface of discontinuity, by expanding them in powers of ζ in the vicinity of the level of $z = z_0$. As a result, conditions at the surface of discontinuity will be replaced by approximate conditions at $z = z_0$. In this case distortions, which are inevitably introduced into the motion in the vicinity of the discontinuity because of the linearization procedure, will only slightly manifest themselves in the regions that are remote from the interface. For the vertical velocity we have

$$w'(x, z_0 + \zeta, t) = w'(x, z_0, t) + O(\zeta),$$

where the second term on the right-hand side has the next order of smallness, since the vertical velocity itself is infinitesimally small. Therefore, the value w' at the discontinuity will be replaced by its value at $z = z_0$.

It is not a good idea to replace the pressure $P(x, z_0 + \zeta, t) = P_0(z_0 + \zeta) + \bar{p}(z_0 + \zeta) + p'(x, z_0 + \zeta, t)$ at the discontinuity by $P(x, z_0, t)$, due to the loss of the buoyancy forces arising from the invasion of the heavy fluid into the region of the light fluid, or vice versa. In fact, one can write the equation of the interface motion in a rigorous formulation of the problem, i.e., without resorting to the Oberbeck–Boussinesq approximation (recall that at the surface of discontinuity $w' = d\zeta/dt$):

$$\mathrm{R}\frac{d^2\zeta}{dt^2} = -\frac{\partial P}{\partial z} - \mathrm{R}g.$$

Multiply both sides of this equation by ζ. Since ζ is infinitesimal, the equation can be rewritten in the form

$$P(x, z_0 + \zeta, t) = P(x, z_0, t) - g\mathrm{R}\zeta - \mathrm{R}\zeta d^2\zeta/dt^2. \tag{13.7}$$

This means that the pressure at level z_0 differs from the pressure at level $z_0 + \zeta$ by the sum of two terms. The first is the weight of the fluid column concluded between the levels and whose cross-section is of the unit area, while the second is the dynamic pumping equal to the product of the mass of this column and the acceleration. It should be emphasized, however, that *in the presence of perturbations, the discontinuity interface is no longer the level $z = z_0$. On the other hand, in the absence of perturbations the pressure on the unperturbed boundary $z = z_0$ is equal to the hydrostatic one, i.e., $P(z_0) = P_0(z_0) + \bar{p}(z_0)$. Therefore, in order to find the relation between the pressure perturbations at the perturbed interface, and perturbations at level $z = z_0$, from both sides of the equality* (13.7) *one should subtract $P(z_0)$ and linearize the product $g\mathrm{R}\zeta$*. As a result, we find that for these pressure perturbations the following estimate holds:

$$p'(x, z_0 + \zeta, t) = p'(x, z_0, t) - g\bar{\mathrm{R}}(z_0)\zeta + O\big(\zeta^2\big), \tag{13.8}$$

$$\big(\bar{\mathrm{R}}(z_0) = \rho_0 + \bar{\rho}(z_0)\big),$$

because the last term on the right-hand side of (13.7) is a product of quantities of the same order of smallness. Thus, *in linear stability theory the dynamic condition on the perturbed interface $z = z_0 + \zeta$ of two fluids is replaced by the continuity condition for the quantity $p'(x, z, t) - g\bar{\mathrm{R}}(z)\zeta$ at level $z = z_0$, while when setting the kinematic conditions, the value of the vertical velocity $w'(x, z_0 + \zeta, t)$ on the perturbed boundary is replaced by its value $w'(x, z_0, t)$ at level $z = z_0$.*

The kinematic condition means that the fluid particle located at the discontinuity will always remain on it. Therefore one has $z = z_0 + \zeta(x, t)$ for this particle. Hence at the discontinuity $w' = dz/dt = d\zeta/dt$. Linearizing this we find that on both sides of the interface

$$w'^{\pm} = \left(\frac{\partial}{\partial t} + U^{\pm}\frac{\partial}{\partial x}\right)\zeta(x, t), \tag{13.9}$$

where the indices $+$ and $-$ refer to the upper and lower fluids, respectively. By excluding $\zeta(x, t)$ from the top and bottom kinematic conditions (13.9), we obtain

$$\left(\frac{\partial}{\partial t} + U^{-}\frac{\partial}{\partial x}\right)w'^{+} = \left(\frac{\partial}{\partial t} + U^{+}\frac{\partial}{\partial x}\right)w'^{-},$$

or

$$\left(\frac{\partial}{\partial t} + U^{-}\frac{\partial}{\partial x}\right)\psi_x^{+} = \left(\frac{\partial}{\partial t} + U^{+}\frac{\partial}{\partial x}\right)\psi_x^{-}. \tag{13.10}$$

This is the desired form of the kinematic condition at the discontinuity surface for the linear stability problem. In the absence of jump of the horizontal velocity U, the condition (13.10) implies the continuity of w' or

$$\psi_x^{+} = \psi_x^{-}. \tag{13.10$'$}$$

In order to formulate the dynamic condition in terms of ψ, we first apply the operator $-\partial/\partial x$ to the identity

$$p'^{+}(x, z_0, t) - g\bar{\mathrm{R}}^{+}(z_0)\zeta = p'^{-}(x, z_0, t) - g\bar{\mathrm{R}}^{-}(z_0)\zeta$$

and employ the first equation of the system (13.5) to eliminate the pressure. Then we obtain

$$\rho_0^{+}\left[\left(\frac{\partial}{\partial t} + U^{+}\frac{\partial}{\partial x}\right)u'^{+} + \frac{dU^{+}}{dz}w'^{+} + g\bar{\mathrm{R}}^{+}\frac{\partial \zeta}{\partial x}\right]$$
$$= \rho_0^{-}\left[\left(\frac{\partial}{\partial t} + U^{-}\frac{\partial}{\partial x}\right)u'^{-} + \frac{dU^{-}}{dz}w'^{-} + g\bar{\mathrm{R}}^{-}\frac{\partial \zeta}{\partial x}\right].$$

Denote by the symbol $\{\}_-^+$ the corresponding jump at the discontinuity. Then introducing the stream function one can rewrite the last equality as follows:

$$\left\{\rho_0\left[-\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\psi_z+\frac{dU}{dz}\psi_x\right]\right\}_-^+ +\{g\bar{\mathrm{R}}\}_-^+\frac{\partial\zeta}{\partial x}=0.$$

Now, to exclude the ζ we apply to this equation the operator $\partial/\partial t+U^+\partial/\partial x$. As a result, in view of (13.9) the dynamic condition at the discontinuity is expressed in terms of one unknown function ψ:

$$\left(\frac{\partial}{\partial t}+U^+\frac{\partial}{\partial x}\right)\left\{\rho_0\left[\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\psi_z-\frac{dU}{dz}\psi_x\right]\right\}_-^+ -\{g\bar{\mathrm{R}}\}_-^+\psi_{xx}^+=0. \quad (13.11)$$

Remark 1 One should mention that conditions (13.10) and (13.11) are weaker than the original ones because of the application of one more differentiation. Therefore, in order to satisfy the original conditions one should take care while recovering the velocity field from the stream function ψ. In addition, the condition equivalent to (13.11) can be obtained by applying the operator $\partial/\partial t+U^-\partial/\partial x$ to exclude ζ, but they both are asymmetrical with respect to the interface:

$$\left(\frac{\partial}{\partial t}+U^-\frac{\partial}{\partial x}\right)\left\{\rho_0\left[\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\psi_z-\frac{dU}{dz}\psi_x\right]\right\}_-^+ -\{g\bar{\mathrm{R}}\}_-^+\psi_{xx}^-=0. \quad (13.11')$$

Condition (13.11) can be symmetrized by additionally applying the operator $\partial/\partial t+U^-\partial/\partial x$ to it, but then it will become even weaker.

For homogeneous fluids ($\mathrm{R}=\mathrm{const},\ \beta=0$) Eq. (13.6) is replaced by the equation of lower order:

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\Delta\psi-U''\psi_x=0, \quad (13.12)$$

while instead of (13.11) at the interface, one assumes continuity of pressure perturbations, i.e., $p'^+(x,z_0,t)=p'^-(x,z_0,t)$. After differentiation in x and taking into account the first equation in (13.5), this gives the relation at the discontinuity:

$$\left\{\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\psi_z-\frac{dU}{dz}\psi_x\right\}_-^+=0. \quad (13.13)$$

13.4 Exercises

1. Formulate the linear boundary value problem of stability for a stationary plane-parallel flow of an inhomogeneous fluid on the basis of the exact Eqs. (13.1), (13.2).

Answer: The equation

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)^2(\Delta\psi-\beta\psi_z)-\left(U''-\beta U'\right)\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\psi_x=-g\beta\psi_{xx} \tag{13.14}$$

($\beta=-\frac{1}{\bar{\mathrm{R}}}\frac{d\bar{\mathrm{R}}}{dz}$) with the boundary conditions formulated above (see Dikii 1976).

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Chapter 14
The Method of Normal Modes and Its Simplest Applications in the Theory of Linear Stability of Plane-Parallel Flows

14.1 Reduction of the Problem by the Method of Normal Modes

Let us sum up the essence of the previous chapter. Namely, the boundary value problem for linear stability of a stationary plane-parallel flow of an Oberbeck–Boussinesq heavy fluid is defined by the equation

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)^2\Delta\psi - U''\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\psi_x = -g\beta\psi_{xx}\left(\beta=-\frac{1}{\rho_0}\frac{d\bar{\rho}}{dz}\right) \quad (14.1)$$

with boundary conditions

(a) on the horizontal solid wall

$$\frac{\partial\psi}{\partial x}=0 \quad (\psi=\text{const}), \quad (14.2)$$

(b) on the surface of discontinuity of velocity and/or density

$$\left(\frac{\partial}{\partial t}+U^-\frac{\partial}{\partial x}\right)\psi_x^+ = \left(\frac{\partial}{\partial t}+U^+\frac{\partial}{\partial x}\right)\psi_x^-, \quad (14.3)$$

$$\left(\frac{\partial}{\partial t}+U^+\frac{\partial}{\partial x}\right)\left\{\rho_0\left[\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\psi_z-\frac{dU}{dz}\psi_x\right]\right\}_-^+ - \{g\bar{\mathrm{R}}\}_-^+\psi_{xx}^+ = 0. \quad (14.4)$$

Here $\psi=\psi(x,z,t)$ is the stream function for velocity perturbations $u'=-\partial\psi/\partial z$ and $w'=\partial\psi/\partial x$, $U=U(z)$ is the velocity of the primary horizontal flow, and $\bar{\mathrm{R}}(z)=\rho_0+\bar{\rho}(z)$ is the density corresponding to the hydrostatic state.

Coefficients of Eq. (14.1) and of boundary conditions (14.2)–(14.4) are independent of the horizontal coordinate x and of time t. Thus one can look for a solution in the normal mode form, namely, as a running wave

$$\psi(x,z,t)=\psi(z)\exp\{i\alpha(x-ct)\}, \quad (14.5)$$

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which is a hydrodynamical analog of the harmonic oscillations of a mechanical system. Here α is a longitudinal wave number and c is the phase velocity.

After substituting (14.5) into (14.2)–(14.4) and keeping in mind the mappings $\partial/\partial t \Longrightarrow -i\alpha c$, $\partial/\partial x \Longrightarrow i\alpha$, $(\partial/\partial t + U\partial/\partial x) \Longrightarrow -i\alpha(U-c)$, the boundary value problem for the amplitude $\psi(z)$ is defined by equation

$$(U-c)^2\big(\psi'' - \alpha^2\psi\big) - U''(U-c)\psi = -g\beta\psi \tag{14.1$'$}$$

with boundary conditions

(a) on the horizontal solid wall

$$i\alpha\psi = 0, \tag{14.2$'$}$$

(b) on the surface of discontinuity

$$\big(U^- - c\big)\psi^+ = \big(U^+ - c\big)\psi^-, \tag{14.3$'$}$$

$$\big(U^+ - c\big)\big\{\rho_0\big[(U-c)\psi_z - U'\psi\big]\big\}_-^+ - g\{\bar{R}\}_-^+\psi^+ = 0. \tag{14.4$'$}$$

Now the problem is stated as follows. For each α find the value of c for which there exists a solution to the reduced boundary value problem. Real values of c provide stability. Otherwise the flow is unstable, because the eigenvalue spectrum of c consists of complex conjugate pairs (a consequence of the absence of viscosity) one of which has a positive imaginary part, which corresponds to the exponential growth of perturbations. A natural question is whether the unstable modes that are found exhaust the entire set of unstable perturbations. In other words, can any solution of the original problem be represented as a Fourier series

$$\psi(x,z,t) = \sum_{k,l} A_{kl} \exp\big\{i\alpha_k(x - c_{kl}t)\big\}\psi_{kl}(z), \tag{14.6}$$

i.e., do eigenfunctions form a complete system of solutions?

In our case the situation is complicated by the fact that for $U = c$ the equation has a singularity point (the main term vanishes). The analysis shows (see Dikii 1976) that the solution has a discontinuity at the point z, where $U(z) = c$, and one requires some additional considerations to glue the pieces of the solution on both sides of the interface. But even after the gluing conditions are defined, these solutions are too few. As a rule, the set of eigenvalues c consists of a finite number of elements. Therefore, the assumption that $\psi(x,z,t)$ can be expanded into a Fourier series is not true.

The reason is that so far we discussed only the discrete part of the spectrum of eigenvalues of the linear operator of stability which, similar to the quantum mechanical operators, also has a continuous spectrum. This can be most easily illustrated by the Rayleigh equation, which is valid for flows of homogeneous fluids ($\beta = 0$, $\mathrm{R} = \rho_0 = \text{const}$):

$$(U-c)\big(\psi'' - \alpha^2\psi\big) - U''\psi = 0. \tag{14.7}$$

Let us denote by $\tilde{\Delta}$ the operator $d^2/dz^2 - \alpha^2$ with boundary conditions $\psi = 0$ at the endpoint of the interval $a \le z \le b$ at which the solid walls are positioned. Then for $\varphi = \tilde{\Delta}^{-1}\psi$ we obtain equation

$$\left(U - U''\tilde{\Delta}^{-1}\right)\varphi = c\varphi$$

with zero boundary conditions. In that case c must be an eigenvalue of the operator that represents the sum of the so-called compact operator $-U''\tilde{\Delta}^{-1}$ and the operator of multiplication by the function $U(z)$. The latter has a continuous spectrum comprised of all the values $c = U(\alpha)$ contained in the interval $[U(a), U(b)]$. These spectral values correspond to eigenfunctions $\varphi_\alpha(z) = \delta(z - \alpha)$ $(\alpha \in [a, b])$ satisfying the "orthogonality" condition (recall the properties of quantum-mechanical operators corresponding to continuously varying physical quantities)

$$\int \varphi_\alpha(z)\varphi_\beta(z)dz = \delta(\alpha - \beta).$$

From the theory of linear operators it is well known that the addition of a compact operator does not affect the continuous spectrum but only changes the discrete spectrum. Therefore, the continuous spectrum of the Rayleigh equation also fills the entire segment $[U(a), U(b)]$. Unfortunately, unlike the quantum-mechanical operators, the Rayleigh operator is not self-adjoint. This circumstance does not allow one to use the theory of spectral decomposition to represent the solution as a sum of the Fourier series over the discrete spectrum and the Fourier integral over the continuous spectrum instead of (14.6).

And yet, in certain important cases, the method of normal modes turns out to be exhaustive. The reason is that the integration is performed along the real axis of variable z. So a singular point occurs for real values of c, i.e., for neutral oscillations. The continuous spectrum filling interval $[U_{\min}, U_{\max}]$ also belongs to the real axis, and thus it does not give rise to unstable oscillations. Real eigenvalues c of the discrete spectrum can cause instability only if they are repeated. Then there are "secular" perturbations, linearly increasing in time, as a consequence of the non-self-adjoint property of the operator of linear stability. Let us state without proof the following theorem (Dikii 1976).

A two-dimensional plane-parallel flow of a homogeneous incompressible fluid with a monotonic velocity profile, whose boundary values $U(a)$ and $U(b)$ are not eigenvalues of the reduced operator of stability can be unstable only if the problem has either non-real eigenvalues in the discrete spectrum or repeated ones.

However, one should not get carried away by the method of normal modes. Returning to the original formulation of the problem, we can encounter solutions that are not covered by the reduced problem and that grow over time not exponentially but polynomially (the so-called algebraic instability). This may either modify or supplement the conclusions drawn on the basis of the reduction. Such an example is studied in the next chapter.

14.2 Examples

Example 14.1 Let the half-spaces $z < 0$ and $z > 0$ be occupied by an incompressible homogeneous fluid of densities $\mathrm{R}_1 = \rho_{01}$ and $\mathrm{R}_2 = \rho_{02}$, respectively.

In this case $U = 0$, $\beta = 0$ and the boundary value problem is defined by the equation

$$c^2\left(\psi'' - \alpha^2\psi\right) = 0$$

with regularity conditions at $\pm\infty$, and on the interface $z = 0$

$$c\left(\psi^+ - \psi^-\right) = 0,$$

$$c^2\left(\rho_{02}\psi_z^+ - \rho_{01}\psi_z^-\right) = g(\rho_{02} - \rho_{01})\psi^+,$$

(a) $c = 0$ is an eigenvalue of infinite multiplicity: for $\psi(z)$ one can take any function satisfying the condition $\psi^+(0) = 0$.
(b) Let $c \neq 0$. Then according to the regularity conditions at $\pm\infty$ there is a unique eigenfunction $\psi(z)$ equal to

$$\psi^+(z) = A\exp(-\alpha z) \quad \text{at } z > 0$$

$$\psi^-(z) = B\exp(\alpha z) \quad \text{at } z < 0.$$

The kinematic condition at $z = 0$ implies $A = B$, and according to the dynamic condition,

$$c = \pm\sqrt{\frac{g}{\alpha}\frac{\rho_{01} - \rho_{02}}{\rho_{01} + \rho_{02}}}. \tag{14.8}$$

Thus, for any α there are two corresponding nonzero eigenvalues c. Physically the obvious result is that when $\rho_{01} > \rho_{02}$ (the heavy fluid is on the bottom) fluctuations are neutral, and such a state is stable. Otherwise, one of the eigenvalues has a positive imaginary part and instability develops. Note that the instability caused by the density increasing with altitude is called the *Taylor instability*.

Example 14.2 A steady Oberbeck–Boussinesq fluid layer is bounded by solid horizontal walls at the levels of $z = 0$ and $z = H$. Fluid density is distributed in a linear fashion $\bar{\rho} = -\rho_0\beta z$ $(\beta = -\rho_0^{-1}d\bar{\rho}/dz)$. It is worth noting that in the rigorous formulation of the problem (see (13.14)) this distribution corresponds to the exponential distribution $\bar{\mathrm{R}} = \rho_0\exp(-\beta z)$.

The boundary problem is given by equation

$$\psi'' + \left(\frac{g\beta}{2} - \alpha^2\right)\psi = 0$$

with zero boundary conditions, since $c = 0$ is not an eigenvalue. Hence

$$\psi = A\exp(\lambda_1 z) + B\exp(\lambda_2 z),$$

where $\lambda_{1,2}$ are the roots of the equation

$$\lambda^2 + \left(\frac{g\beta}{2} - \alpha^2\right) = 0.$$

According to the boundary conditions

$$A + B = 0, \qquad \exp(\lambda_1 H) - \exp(\lambda_2 H) = 0 \quad \text{or} \quad \exp(\lambda_1 - \lambda_2)H = 1.$$

This last equality implies that

$$(\lambda_1 - \lambda_2)H = i2\pi N \quad (N \text{ is any nonzero integer}).$$

By Vieta's theorem

$$\lambda_1 + \lambda_2 = 0, \qquad \lambda_1\lambda_2 = \frac{g\beta}{2} - \alpha^2,$$

whence

$$\lambda_1 = -\lambda_2 = \frac{i\pi N}{H}, \qquad \lambda_1\lambda_2 = \left(\frac{\pi N}{H}\right)^2 = \frac{g\beta}{c^2} - \alpha^2$$

and therefore

$$c^2 = \frac{g\beta}{(\frac{\pi N}{H})^2 + \alpha^2},$$

where the sign of c^2 coincides with the sign of β.

Thus, there is a countable set of eigenvalues c, which are real for positive β (density decreases with altitude) and purely imaginary for negative β (Taylor instability). Neutral oscillations at $\beta > 0$ are called cellular waves.

Example 14.3 A fluid of constant density $\mathrm{R} = \rho_0$ ($\beta = 0$) occupies the entire space and moves with constant velocity U_1 for $z < 0$ and with constant velocity $U_2 > U_1$ for $z > 0$.

Passing to a coordinate system moving with the average velocity of the flow, we can define

$$U = U_0 = \frac{U_2 - U_1}{2} \quad \text{for } z > 0,$$

$$U = -U_0 = \frac{U_1 - U_2}{2} \quad \text{for } z < 0.$$

Then Eq. (14.1′) for ψ_+ ψ_- assumes the form

$$(U_0 - c)^2\left(\psi_+'' - \alpha^2\psi_+\right) = 0 \quad \text{for } z > 0,$$

$$(U_0+c)^2\big(\psi_-''-\alpha^2\psi_-\big)=0 \quad \text{for } z<0$$

with conditions at the interface $z=0$

$$(U_0+c)\psi_+ = -(U_0-c)\psi_-,$$

$$(U_0-c)\big[(U_0-c)\psi_+' + (U_0+c)\psi_-'\big]=0,$$

(a) $c=\pm U_0$ are eigenvalues of infinite multiplicity with the corresponding eigenfunctions

- for $c=U_0$:

for $z\geq 0$ function $\psi=\psi_+(z)$ is an arbitrary function vanishing at $z=0$, for $z<0$ function $\psi=\psi_-(z)\equiv 0$;

- for $c=-U_0$:

for $z>0$ function $\psi=\psi_+(z)\equiv 0$, at $z\leq 0$ function $\psi=\psi_-(z)$ is an arbitrary function vanishing at $z=0$.

(b) For $c\neq \pm U_0$ we get

$$\psi_+ = A\exp(-\alpha z), \qquad \psi_- = B\exp(\alpha z).$$

According to the jump conditions

$$(U_0+c)A+(U_0-c)B=0, \qquad (U_0-c)A-(U_0+c)B=0.$$

Equations are compatible provided that

$$(U_0+c)^2+(U_0-c)^2=0.$$

Hence

$$c=\pm iU_0.$$

The flow is unstable, but not because of the inhomogeneous density distribution, but because of the velocity shear. Such instability is called *the Helmholtz instability*.

14.3 Exercises

1. How will the results of Example 2 change, if the problem is addressed in the rigorous formulation (13.14)?

   Answer:

$$c^2=\frac{g\beta}{(\frac{\pi N}{H})^2+\alpha^2+\frac{\beta^2}{4}}.$$

2. Consider the combined case, i.e., a flow with a jump in both density and velocity.

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Chapter 15
The Taylor Problem of Stability of Motion of a Stratified Fluid with a Linear Velocity Profile

This problem deserves special attention as it demonstrates incompleteness of the study using the normal modes method, while to achieve completion of the study, it is necessary to resort to the original equations.

15.1 Solution of the Reduced Problem

A fluid of density $\bar{\mathrm{R}}(z) = \rho_0 + \bar{\rho}$, $\bar{\rho} = -\rho_0 \beta z$ ($\beta = -\rho_0^{-1} d\bar{\rho}/dz$) occupies the half-space $z > 0$ bounded by a solid wall, and moves with velocity $U = kz$. Recall that the chosen vertically linear distribution of density in a precise formulation of the problem (see (13.14)) corresponds to the exponential distribution.

The reduced problem is given by equation

$$(kz - c)^2 \left(\psi'' - \alpha^2 \psi\right) + g\beta\psi = 0 \tag{15.1}$$

with the boundary conditions of $\psi = 0$ at $z = 0$ and of regularity at ∞.

Let us make a substitution

$$\xi = \left(z - \frac{c}{k}\right)\alpha.$$

Now Eq. (15.1) takes the form

$$\psi_{\xi\xi} + \left(\frac{g\beta}{k^2\xi^2} - 1\right)\psi = 0$$

with the boundary condition of $\psi = 0$ at $\xi = -c\alpha/k$. The dimensionless parameter

$$Ri = \frac{g\beta}{k^2} = -\frac{g\,d\bar{\rho}/dz}{\rho_0 (dU/dz)^2}$$

is called the *Richardson number* and characterizes the degree of stratification of the fluid.

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Let us make another substitution $\psi = \xi^{1/2} f$ and introduce a parameter $\nu^2 = 1/4 - Ri$ instead of Ri. Then f satisfies the differential equation for the modified Bessel functions (Abramowitz and Stegun 1979):

$$f_{\xi\xi} + \frac{f_\xi}{\xi} - \left(1 + \frac{\nu^2}{\xi^2}\right) f = 0.$$

Its fundamental solutions are cylindrical functions $I_\nu(\xi)$ and $K_\nu(\xi)$ of index ν. However only the latter one, the Macdonald function, is regular at $+\infty$, which is precisely what we need from the function $\psi(\xi)$. Therefore,

$$\psi(\xi) = C\sqrt{\xi} K_\nu(\xi) \quad (C \text{ is an arbitrary constant}).$$

Attention: $\psi(\xi) = 0$ at $\xi = -c\alpha/k$. Hence, in order to solve the problem one has to find zeros ξ_n of the function $\psi(\xi)$. Those zeros determine the eigenvalues $c_n = -k\xi_n/\alpha$. The Macdonald function $K_\nu(\xi)$ has a branch point at the origin. The cut should be done along the half-line $(-\infty, 0)$, i.e., one should look for the zeros on the leaf $|\arg \xi| < \pi$ in order for the function $K_\nu(\xi)$ to decay along the half-line from the root to infinity. When passing to the original variable, this ray becomes the semi-axis $[0, \infty)$.

Let ξ_n be one of such roots; then

$$\psi_n = \sqrt{\alpha\left(z - \frac{c_n}{k}\right)} K_\nu\left(\alpha\left(z - \frac{c_n}{k}\right)\right)$$

is an eigenfunction corresponding to the eigenvalue $c_n = -k\xi_n/\alpha$.

Let us first consider the case

$$\text{(a)} \quad Ri > \frac{1}{4}, \qquad \nu = \left(\frac{1}{4} - Ri\right)^{\frac{1}{2}} = i\mu \quad \textit{is purely imaginary.}$$

In the area $|\arg \xi| < \pi$ the Macdonald function of imaginary index has a countable set of zeros on the real axis, and no other zeros. The asymptotic behavior of these zeros in the neighborhood of $\xi = 0$ can be obtained by using the principal terms in the power expansion of ξ. We need the following properties of the fundamental solutions of the modified Bessel equation (see, e.g., Abramowitz and Stegun 1979; Whittaker and Watson, 1963):

$$K_\nu(\xi) = K_{-\nu}(\xi), \quad K_\nu(\xi) = \frac{\pi}{2} \frac{I_{-\nu}(\xi) - I_\nu(\xi)}{\sin(\nu\xi)}.$$

For $|\xi| < 1$ one has

$$I_\nu(\xi) = \frac{(\xi/2)^\nu}{\Gamma(1+\nu)}, \quad \nu \neq -1, -2, \ldots.$$

Recall that on the complex half-plane $\operatorname{Re} z > 0$ the function $\Gamma(\nu)$ is defined by the Euler integral

$$\Gamma(\nu) = \int_0^\infty t^{\nu-1} e^{-t} dt$$

and analytically continues onto the entire complex plane, except for $\nu = 0, -1, -2, \dots$, where it has simple poles.

Therefore for $|\xi| \ll 1$,

$$K_\nu(\xi) \approx \frac{\pi}{2} \frac{1}{\sin(\nu\xi)} \left[\left(\frac{(\xi/2)^{-\nu}}{\Gamma(1-\nu)} - \frac{(\xi/2)^{\nu}}{\Gamma(1+\nu)} \right) \right].$$

We use the following property of the gamma function:

$$\Gamma(1+i\mu)\Gamma(1-i\mu) = \left|\Gamma(1 \pm i\mu)\right|^2.$$

By purely formal calculations,

$$\begin{aligned} K_{i\mu}(\xi) &\propto \frac{(\xi/2)^{-i\mu}}{\Gamma(1-i\mu)} - \frac{(\xi/2)^{i\mu}\Gamma(1-i\mu)}{|\Gamma(1-i\mu)|^2} \\ &= \exp\bigl\{-i\mu\ln(\xi/2) - \ln\bigl|\Gamma(1-i\mu)\bigr| - i\arg\Gamma(1-i\mu)\bigr\} \\ &\quad - \exp\bigl\{i\mu\ln(\xi/2) + \ln\bigl|\Gamma(1-i\mu)\bigr| + i\arg\Gamma(1-i\mu) - 2\ln\bigl|\Gamma(1-i\mu)\bigr|\bigr\}. \end{aligned}$$

Hence zeros of the function

$$K_{i\mu}(\xi) \propto -\bigl|\Gamma(1-i\mu)\bigr|^{-1} \sin\bigl[\mu\ln(\xi/2) + \arg\Gamma(1-i\mu)\bigr]$$

are roots of the equation

$$\mu\ln(\xi/2) + \arg\Gamma(1-i\mu) = -\pi n.$$

This yields a countable sequence of positive roots

$$\xi_n = 2\exp\bigl\{-\bigl[\pi n + \arg\Gamma(1-i\mu)\bigr]/\mu\bigr\},$$

and the eigenvalues of the problem are given by the asymptotic formula

$$c_n \sim -\frac{2k}{\alpha} \exp\bigl\{-\bigl[\pi n + \arg\Gamma(1-i\mu)\bigr]/\mu\bigr\}.$$

They are all real. Consequently, the flow is stable, but as $Ri \to 1/4$ ($\mu \to 0$) the eigenvalues contract to zero, which indicates a decrease in stability. Note that as α increases, all c_n also shrink to zero, while frequencies $\alpha \cdot c$ remain approximately constant. The eigenfunctions ψ_n significantly differ from zero only in the neighborhood of $z = 0$ of order $1/\alpha$, i.e., with an increase of α they are pressed against the solid wall.

Now consider the case

$$\text{(b)} \quad Ri < \frac{1}{4}, \qquad \nu = \left(\frac{1}{4} - Ri\right)^{\frac{1}{2}} \quad \textit{is real and} < 1/2.$$

In the latter case one can show (Whittaker and Watson, 1963) that the function $K_\nu(\xi)$ has no roots in the area $|\arg\xi| < \pi$, i.e., there exists no such c for $\varphi(kz - c)$ to satisfy the equation and the boundary conditions of the problem. In other words, *at* $Ri < 1/4$ *the problem has no solution in the form of normal modes*.

15.2 An Approximate Solution of the Nonreduced Problem

Stability of the Taylor flow diminishes with a decrease of the Richardson number, so the latter can be regarded as a measure of stability. This also naturally suggests that for $Ri < 1/4$ instability will occur. Prandtl was inclined to think this way. Strictly speaking, to clarify this question one should refer to the definition of Lyapunov stability (see Chap. 12) and to investigate stability of the Cauchy problem with respect to perturbations of the initial data. This rather complex investigation of the Taylor flow was made on the basis of the Cauchy problem using the Laplace transform. It turned out that for $Ri < 1/4$ the flow is stable. Below we confine ourselves to less rigorous reasoning pointing in favor of this result. The lack of rigor is related to our consideration of the problem over the whole two-dimensional space and to discarding the boundary condition at $z = 0$. But the further considerations are rigorous.

Since the method of normal modes does not work here, one needs to return to the original Eq. (13.6), which, for $U = kz$, can be written as follows:

$$\left(\frac{\partial}{\partial t} + kz\frac{\partial}{\partial x}\right)^2 \Delta\psi = -g\beta\psi_{xx} \quad \left(\beta = -\frac{1}{\rho_0}\frac{d\bar{\rho}}{dz}\right). \tag{15.2}$$

The coefficients of this equation depend on z, which does not allow separation of the variables. But for a linear velocity profile this difficulty can be avoided by passing to the semi-Lagrangian coordinate system:

$$t \to t, \qquad z \to z, \qquad x_1 = x - kzt.$$

In the new variables, Eq. (15.2) assumes the form in which the coefficients depend on t instead of z:

$$\frac{\partial^2}{\partial t^2}\left[\frac{\partial^2}{\partial x^2} + \left(\frac{\partial}{\partial z} - kt\frac{\partial}{\partial x_1}\right)^2\right]\psi + g\beta\frac{\partial^2\psi}{\partial x_1^2} = 0.$$

So now one can use harmonic dependence on the spatial variables

$$\psi(x_1, z, t) = \varphi(t)\exp(lx_1 + nx).$$

Then $\varphi(t)$ satisfies the equation

$$\frac{d^2}{dt^2}\left[l^2 + (n - lkt)^2\right]\varphi + g\beta l^2\varphi = 0.$$

Since we are interested in the behavior of solutions to this equation as $t \to \infty$, the constants in the first coefficient can be neglected, i.e., asymptotics of solutions are described by the equation

$$t^2\varphi_{tt} + 4t\varphi_t + \left(2 + \frac{g\beta}{k^2}\right)\varphi = 0.$$

Its solution is

$$\varphi(t) = C_1 t^{m_1} + C_2 t^{m_2},$$

where m_1 and m_2 are the roots of the algebraic equation

$$m(m-1) + 4m + (2 + Ri) = 0,$$

whence

$$m_{1,2} = -\frac{3}{2} \pm \sqrt{\frac{1}{4} - Ri}.$$

For $Ri \leq 1/4$ the roots are real and negative, while for $Ri > 1/4$ their real part is negative, i.e., in all cases, the solutions are damped. In relation to the results obtained it is worth comparing the above with the following problem.

15.3 On Stability of a Flow of a Homogeneous Fluid with a Linear Velocity Profile

By setting $\beta = 0$ in (15.2), in order to solve the problem one can use the equation

$$\left(\frac{\partial}{\partial t} + kz\frac{\partial}{\partial x}\right)\Delta\psi = 0$$

with regularity conditions as $z \to \pm\infty$. By keeping the harmonic dependence on x, i.e., $\psi(x, z, t) = \varphi(z, t)\exp(i\alpha x)$, we get the Cauchy problem defined by the equation

$$\left(\frac{\partial}{\partial t} + i\alpha kz\right)\left(\varphi_{zz} - \alpha^2\varphi\right) = 0 \qquad (15.3)$$

with the initial condition $\varphi(z, 0) = \varphi_0(z)$ and the boundedness conditions for $\varphi(z, t)$ as $z \to \pm\infty$. After integration over t we obtain

$$\varphi_{zz} - \alpha^2\varphi = f(z)\exp(-i\alpha kzt),$$

where $f(z)$ is defined by the initial condition

$$f(z) = (\varphi_0)_{zz} - \alpha^2 \varphi_0.$$

It remains to solve the second-order linear differential equation with a prescribed right-hand side. The general solution can be found using variation of parameters:

$$\varphi(z,t) = -\int_z^\infty \frac{1}{\alpha} f(\zeta) \sinh\{\alpha(z-\zeta)\} \exp(-i\alpha k \zeta t) d\zeta$$
$$+ C_1 \exp(\alpha z) + C_2 \exp(-\alpha z).$$

Taking into account the conditions at $\pm\infty$, the constants $C_1 = 0$ and $C_2 = 0$, and the solution have the form

$$\varphi(z,t) = -\int_z^\infty \frac{1}{\alpha} f(\zeta) \sinh\{\alpha(z-\zeta)\} \exp(-i\alpha k \zeta t) d\zeta.$$

Under natural physical assumptions on the finiteness of the energy of initial perturbations, e.g., $f(z) \neq 0$ only for a finite domain of z, it is obvious that $\varphi(z,t)$ is bounded as $t \to \infty$ since

$$\left|\varphi(z,t)\right| \leq \int_z^\infty \left|\frac{1}{\alpha} f(\zeta)\right| \cdot \left|\sinh\{\alpha(z-\zeta)\}\right| d\zeta.$$

The exceptional simplicity of the solution of this problem is due to a degenerate form of Eq. (15.2). In the general case ($U'' \neq 0$) the equation gains the terms proportional to φ, which are natural to deal with using the Laplace transform method.

15.4 Exercises

1. Solve the Taylor problem for $Ri < 1/4$ in the framework of the rigorous problem formulation (13.14), assuming exponential (rather than linear) density distribution with altitude ($\bar{\mathrm{R}} = \mathrm{R}_0 \exp(-\beta z)$), and show that the result remains the same.

   Hint: After passing to the semi-Lagrangian coordinate system make the substitution $\psi = \varphi \exp(\beta z/2)$.
2. Solve the stability problem for a flow of a homogeneous fluid with a linear velocity profile, assuming that the fluid occupies the half-plane $z > 0$ and is bounded by a solid wall at the level $z = 0$.

   Answer (see also Dikii 1976):

$$\varphi(z,t) = -\int_z^\infty \frac{1}{\alpha} f(\zeta) \sinh\{\alpha(z-\zeta)\} \exp(-i\alpha k \zeta t) d\zeta$$

$$-\int_0^\infty \frac{1}{\alpha} f(\zeta)\sinh\{\alpha\zeta\}\exp\left\{-(i\alpha k\zeta t+\alpha z)\right\}d\zeta.$$

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Chapter 16
Applications of Integral Relations and Conservation Laws in the Theory of Hydrodynamic Stability

16.1 General Theorems Based on Integral Relations

In order to describe the spectrum (or eigenvalues) of the reduced operator of stability (14.1′), (14.2′) in the complex plane, it is usually useful to study the properties of its quadratic form, a well-known method in the theory of linear operators in a Hilbert space. In the framework of linear theory we begin with a rigorous proof of the Miles stability criterion for a flow of a stratified fluid.

Theorem 16.1 (Miles 1961) *A plane-parallel flow of a stratified fluid with its Richardson number*

$$Ri = -\frac{g d\bar{\rho}/dz}{\rho_0 (dU/dz)^2} \equiv \frac{g\beta}{(U')^2} \quad \textit{everywhere} > \frac{1}{4},$$

is stable.

Recall that the value of Ri gives a local criterion. Hence the requirement *everywhere*. In addition, in the rigorous formulation of the problem (13.14) both quantities $\bar{\rho}$ and ρ_0 in the definition of the Richardson number are to be replaced by $\bar{\mathrm{R}}$.

Proof We consider the boundary value problem given by the equation

$$(U-c)^2\left(\psi'' - \alpha^2\psi\right) - U''(U-c)\psi = -g\beta\psi \quad \left(\beta = -\frac{1}{\rho_0}\frac{d\bar{\rho}}{dz}\right)$$

with conditions $\psi = 0$ on the solid walls or the regularity at $\pm\infty$. Since the eigenvalues of the boundary value problem are complex conjugate, one has to prove that nonreal eigenvalues do not exist. Assume the contrary, i.e., that $c = c_r + ic_i$ $(c_i \neq 0)$ and make the change

$$W(z) = U(z) - c, \qquad \psi(z) = \sqrt{W}\varphi(z).$$

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Function φ satisfies the equation

$$(W\varphi')' - \left[\frac{1}{2}U'' + \alpha^2 W + \frac{1}{W}\left(\frac{1}{4}(U')^2 - g\beta\right)\right]\varphi = 0.$$

Multiply by φ^* (that is the function complex conjugate to φ) and integrate over the entire cross-section of the flow taking into account the boundary conditions:

$$\int_a^b (W\varphi')'\varphi^* dz = (W\varphi')\varphi^*\big|_{z=b} - (W\varphi')\varphi^*\big|_{z=a} - \int_a^b W\varphi'\varphi^{*\prime} dz$$

$$= -\int_a^b W|\varphi'|^2 dz.$$

Then the integrated equation can be written as follows:

$$\int_a^b \left\{W\left(|\varphi'|^2 + \alpha^2|\varphi|^2\right) + \frac{1}{2}U''|\varphi|^2\right\}dz$$

$$+\int_a^b W^*\left(\frac{1}{4}(U')^2 - g\beta\right)\frac{|\varphi|^2}{|W|^2}dz = 0.$$

Its imaginary part gives the equality $(U - c = U - c_r - ic_i,\ c_i \neq 0)$

$$c_i \int_a^b \left\{|\varphi'|^2 + \alpha^2|\varphi|^2 - \left(\frac{1}{4}(U')^2 - g\beta\right)\frac{|\varphi|^2}{|W|^2}\right\}dz = 0.$$

However for $(g\beta - \frac{1}{4}(U')^2) > 0$, i.e., for $Ri > 1/4$, this is impossible. The theorem is proved. □

Physical Interpretation of the Miles Criterion (*Heuristic Derivation*) At first glance the mysterious critical value of the Richardson number equal to 1/4 becomes clear after the following reasoning. First, consider a physical pendulum (Fig. 16.1a) of mass m, oscillating in a gravity field $\mathbf{g}$. Let l be the distance from its suspension point O to the pendulum center of mass, and let V be its speed in the lower equilibrium position O'. Then the pendulum keeps oscillating while its kinetic energy $E = mV^2/2$ in the lower equilibrium position does not exceed the excess $\Delta\Pi = 2mgl$ of the potential energy at the upper equilibrium O'' relative to the lower one, i.e., provided that

$$\frac{2mgl}{mV^2/2} > 1 \quad \text{or} \quad \frac{gl}{V^2} > \frac{1}{4}.$$

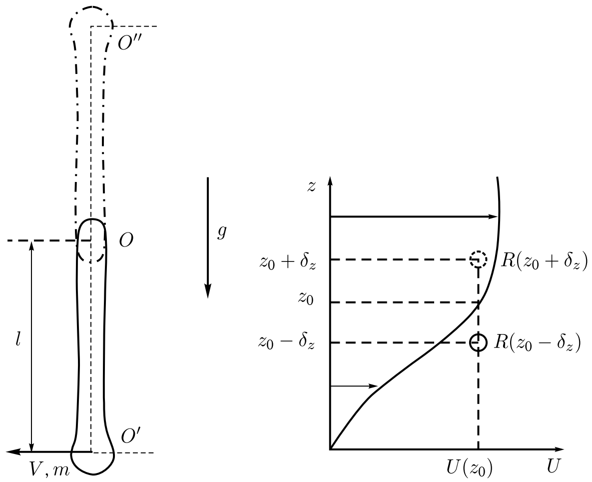

Fig. 16.1 (**a**) A physical pendulum oscillates relative to its lower equilibrium position, provided that $4gl/V^2 > 1$. (**b**) The shear flow of a stratified fluid in a gravity field. For each particle located at the level $z = z_0 - \delta z$ in the coordinate system moving with constant velocity $V = U(z_0)$, the role of l is played by δz, the role of g belongs to $g' = N^2\delta z$, while $V = U(z_0 - \delta z) - U(z_0) = \delta U$. The flow is stable provided that $4g'\delta z/(\delta U)^2 = 4N^2/(\delta U/\delta z)^2 \doteq 4Ri > 1$

Otherwise the oscillations of the pendulum turn into its rotating movements.

Now let $U = U(z)$ be the velocity profile of a fluid flow with density stratification $\bar{R} = \bar{R}(z)$ (Fig. 16.1b). Make the coordinate change to a system moving with constant velocity $U(z_0)$ corresponding to an arbitrary level $z = z_0$. Then in this coordinate system the kinetic energy of a fluid parcel of the unit volume located at the level $z = z_0 - \delta z$ is equal to

$$K = \frac{1}{2}\bar{R}(z_0 - \delta z)\big[U(z_0 - \delta z) - U(z_0)\big]^2.$$

The potential energy of this particle shifted to the level $z = z_0 + \delta z$ is equal to the product of the total of gravity and buoyancy forces by δz, i.e.,

$$\Delta\Pi = g\big[\bar{R}(z_0 - \delta z) - \bar{R}(z_0 + \delta z)\big]\delta z.$$

Therefore, the condition that the motion of the fixed particle will not become rotational relative to the particle located at the level $z = z_0$ (i.e., by the arbitrariness of the choice of $z = z_0$ the condition that none of the particles can destroy the density stratification of the fluid due to the appearance of local vorticity) can be written as follows:

$$\frac{\Delta\Pi}{K} = \frac{g[\bar{R}(z_0 - \delta z) - \bar{R}(z_0 + \delta z)]\delta z}{\bar{R}(z_0 - \delta z)[U(z_0 - \delta z) - U(z_0)]^2/2} > 1.$$

Now dividing the numerator and denominator by δz^2 and letting δz tend to zero, we obtain a sufficient condition for stability of the flow of a stratified fluid

$$-\frac{4g\,d\bar{R}/dz}{\bar{R}(dU/dz)^2} = \frac{4N^2}{(dU/dz)^2} \doteq 4Ri > 1, \tag{16.1}$$

which must be fulfilled at any level of z. Recall that

$$-\frac{g}{\bar{R}}\frac{d\bar{R}}{dz} = N^2$$

is the square of the Brunt–Väisälä frequency for an incompressible stratified fluid (see Chap. 9).

In other words, $4Ri(z)$ *is the ratio of the potential and kinetic energies of the fluid layer, defined by an infinitesimal neighborhood of the level* z, *calculated in a coordinate system moving with constant velocity* $U(z)$ *with respect to the laboratory coordinate system.* This way the Miles criterion becomes physically transparent.

Theorem 16.2 (Howard 1961) *All nonreal eigenvalues of the boundary value problem for* $\beta > 0$ *and all real ones at* $\beta = 0$ *lie inside a circle whose diameter is given by the interval* $[U_{\min}, U_{\max}]$.

It is convenient to do our consideration in a coordinate system moving with the mean velocity $U_{\text{mean}} = (U_{\min} + U_{\max})/2$. Then the above-mentioned segment is $[-U_0, U_0]$, where $U_0 = (U_{\max} - U_{\min})/2$. This transformation is equivalent to substituting $U(z) = U_{\text{mean}} + U_1(z)$, $c = U_{\text{mean}} + c_1$ into (16.1).

Assume the contrary to the assertion of Theorem 16.2 and introduce the substitution $\psi(z) = (U - c)f(z)$. By assumption, the function $(U - c)$ does not vanish in the interval of integration, while f satisfies the equation

$$\left[(U-c)^2 f'\right]' - \alpha^2 (U-c)^2 f = -g\beta f.$$

Multiply this by f^* and integrate over z. Then, after integrating the first term by parts and taking into account the boundary conditions, we obtain

$$\int (U-c)^2 \left\{\left|f'\right|^2 + \alpha^2 |f|^2\right\} dz = g \int \beta |f|^2 dz. \tag{16.2}$$

This shows that for $\beta = 0$, i.e., for a homogeneous fluid, c cannot be real. For nonreal values of c we first consider the imaginary part of Eq. (16.2):

$$c_i \int (U - c_r)\left\{\left|f'\right|^2 + \alpha^2 |f|^2\right\} dz = 0. \tag{16.3}$$

Since $c_i \neq 0$, this equality is only possible provided that $-U_0 < c_r < U_0$.

The real part of the equality (16.2) gives

$$\int \left[(U-c_r)^2 - c_i^2\right]\left\{\left|f'\right|^2 + \alpha^2 |f|^2\right\} dz = g \int \beta |f|^2 dz. \tag{16.4}$$

Since $(U - c_r)^2 - c_i^2 = [U^2 - (c_r^2 + c_i^2)] - 2c_r(U - c_r)$, then from (16.4) and by using (16.3) it follows that

$$\int \left(U^2 - |c|^2\right)\left\{\left|f'\right|^2 + \alpha^2 |f|^2\right\} dz \geq 0.$$

Hence $|c|^2 \le U_0^2$, which completes the proof of the theorem.

Theorem 16.3 (Rayleigh 1880) *A plane-parallel flow of a homogeneous ideal incompressible fluid is stable if its velocity profile has no inflection points.*

In this case the boundary value problem for the eigenvalues is given by the Rayleigh equation

$$(U-c)\big(\psi''-\alpha^2\psi\big)-U''\psi=0$$

with zero conditions on solid boundaries. Again we assume that $c=c_r+c_i$ ($c_i \ne 0$). Divide the equation by $U-c$, multiply it by ψ^* and integrate over the flow cross-section. Taking into account the boundary conditions, one has

$$\int \psi''\psi^* dz = -\int |\psi'|^2 dz.$$

As a result we obtain

$$\int\left(|\psi'|^2+\alpha^2|\psi|^2+\frac{U''}{U-c}|\psi|^2\right)dz=0. \tag{16.5}$$

The imaginary part of the latter equation gives

$$c_i\int\frac{U''}{(U-c)^2}|\psi|^2dz=0. \tag{16.6}$$

But this is possible if U'' changes its sign, i.e., if the flow has an inflection point (a necessary condition for instability).

Since in this case the quantity $U'=dU/dz$ ($W=0$) is the vorticity of the two-dimensional flow, the Rayleigh theorem can be reformulated as follows.

For stability of the above-mentioned flow it suffices that its vorticity changes monotonically from one wall to the other.

Theorem 16.4 (Fjortoft 1950) *A plane-parallel flow of a homogeneous incompressible fluid is stable if there exists such a constant K that $(U-K)U'' \ge 0$.*

As before, we assume $c=c_r+c_i$ ($c_i \ne 0$). The real part of Eq. (16.5) gives

$$\int\frac{U''(U-c_r)}{|U-c|^2}|\psi|^2dz=-\int\big(|\psi'|^2+\alpha^2|\psi|^2\big)dz<0.$$

By (16.6) one can take any constant K for c_r, since c_r is a coefficient at the zero integral. Then

$$\int\frac{U''(U-K)}{|U-c|^2}|\psi|^2dz<0,$$

which is possible if there exists a point at which $U''(U-K)<0$. Hence, the assumption that $c_i \ne 0$ is incorrect. The theorem is proved.

The Fjortoft theorem includes two special cases. Namely, it includes (a) *the Rayleigh theorem if we assume that* U'' *does not change sign, and we take* $K > |U_{\max}|$, *while the sign of* K *coincides with the sign of the quantity* $-U''$. The other special case is where (b) *the flow has one inflection point, i.e.*, $U''(z_c) = 0$ *and* $[U(z) - U(z_c)]U'' > 0$. *Then* $K = U(z_c)$.

16.2 Proof of the Rayleigh Theorem by the Lyapunov–Arnold Method

Up until now we studied hydrodynamic stability in a linear formulation. Following the program outlined in Chap. 12, we now use the first integrals of motion to construct a Lyapunov function and study the stability of the shear plane-parallel flows of a homogeneous fluid.

Recall that in terms of stream function $\psi = \psi(x, z, t)$, the equations for a two-dimensional flow of an ideal homogeneous incompressible fluid can be written as follows:

$$\frac{\partial \Delta\psi}{\partial t} + u\frac{\partial \Delta\psi}{\partial x} + w\frac{\partial \Delta\psi}{\partial z} \equiv \frac{\partial \Delta\psi}{\partial t} + [\psi, \Delta\psi] = 0, \tag{16.7}$$

$$u = -\frac{\partial \psi}{\partial z}, \qquad w = \frac{\partial \psi}{\partial x}. \tag{16.8}$$

Under the conditions that at the boundary C of the flow domain

$$\psi|_{C=\text{const}} \quad \text{(an impermeable boundary)}, \tag{16.9}$$

$$\oint\limits_C \mathbf{u}\delta\mathbf{l} = \oint\limits_C \frac{\partial \psi}{\partial n}\delta l = \text{const} \quad (\mathbf{n} \text{ is an exterior normal}), \tag{16.10}$$

Eqs. (16.7) and (16.8) have the two first integrals of motion (see solutions of Exercises 2 and 3)

$$E = \frac{1}{2}\iint (\nabla\psi)^2 dxdz, \qquad I = \iint \Phi(\Delta\psi)dxdz. \tag{16.11}$$

They express the conservation of energy and total vorticity, respectively (here Φ is an arbitrary function of one variable). Note that the latter boundary condition follows from Kelvin's theorem applied to the liquid contour directly adjacent to the border. Any fluid particle of such a contour would never leave the border due to the vanishing normal component of velocity. Therefore, the above-mentioned contour can be regarded as coinciding with the boundary itself.

Recall the main idea of the Lyapunov–Arnold method. Let $\psi = \psi_0(x, z)$ be the stream function of a stationary flow whose stability we are testing. Choose an arbitrary function $\Phi(\Delta\psi)$ entering the functional I in such a way that the conserved functional

$$F[\psi] = E[\psi] + I[\psi] = \iint \left[\frac{1}{2}(\nabla\psi)^2 + \Phi(\Delta\psi)\right] dxdz$$

has an extremal value at $\psi = \psi_0$ as compared with its values at all other ψ close to ψ_0. Then the sign-definiteness of its second variation at $\psi = \psi_0$ implies stability of the flow considered.

As noted above, the Rayleigh condition, which is the absence of inflection points in the velocity profile $U(z)$ of the flow, means a monotonic change of its vorticity U' from one wall to the other, i.e., from one streamline $\psi_0 = \text{const}$ to another. We therefore assume that the vorticity $\Delta\psi_0$ is a monotone function of ψ_0. Since a monotonic function is invertible, the stream function $\psi_0 = \Psi(\Delta\psi_0)$ is a monotonic function of $\Delta\psi_0$.

Now let $\psi(x,z,t) = \psi_0 + \delta\psi(x,z,t)$ be the stream function of a perturbed flow. We find the difference $F[\psi] - F[\psi_0]$ up to second-order variations:

$$F[\psi] - F[\psi_0] = \iint \left[\nabla\psi_0 \nabla\delta\psi + \Phi'(\Delta\psi_0)\delta\Delta\psi\right] dxdz$$
$$+ \frac{1}{2}\iint \left[(\nabla\delta\psi)^2 + \Phi''(\Delta\psi_0)(\delta\Delta\psi)^2\right] + \cdots .$$

Integrate by parts the first term on the right-hand side:

$$\iint \nabla\psi_0 \nabla\delta\psi\, dxdz = \nabla\delta\psi| \oint_C \frac{\partial\psi_0}{\partial n}\delta l - \iint \psi_0 \Delta\delta\psi\, dxdz.$$

Therefore, in view of the boundary conditions, the first variation

$$\delta F[\psi_0] = \iint \left[-\psi_0 + \Phi'(\Delta\psi_0)\right]\Delta\delta\psi\, dxdz$$

vanishes for $\psi_0 = \Psi(\Delta\psi_0) = \Phi'(\Delta\psi_0)$ (a necessary condition for an extremum). Then

$$F[\psi] - F[\psi_0] = \frac{1}{2}\iint \left[(\nabla\delta\psi)^2 + \Psi'(\Delta\psi_0)(\delta\Delta\psi)^2\right] dxdz + \cdots .$$

Furthermore,

$$U = -\frac{\partial\psi_0}{\partial z} = -\frac{\partial}{\partial z}\Psi(\Delta\psi_0),$$

and a for plane-parallel flow $\Delta\psi_0 = -U'$. Hence $\Psi' = U/U''$ and

$$F[\psi] - F[\psi_0] = \frac{1}{2}\iint \left[(\nabla\delta\psi)^2 + \frac{U}{U''}(\delta\Delta\psi)^2\right] dxdz + \cdots .$$

So far nothing implies that the integral on the right-hand side is positive definite. Therefore we use the following argument. Note that our consideration can be done in any coordinate system moving with respect to the original one with constant velocity K. If ψ is the stream function in this new coordinate system, the latter formula can be rewritten as

$$F[\psi] - F[\psi_0] = \frac{1}{2}\iint \left[(\nabla\delta\psi)^2 + \frac{U(z) - K}{U''}(\delta\Delta\psi)^2 \right] dxdz + \cdots \tag{16.12}$$

The value of K can always be chosen to make the coefficient $(U(z) - K)/U''$ positive. In this case, $\delta^2 F[\psi_0]$ is a positive definite quadratic form of $\delta\psi$, which can be taken as a measure of deviation of ψ from ψ_0:

$$\|\psi - \psi_0\|_1^2 = \frac{1}{2}\iint \left[(\nabla\delta\psi)^2 + \frac{U(z) - K}{U''}(\delta\Delta\psi)^2 \right] dxdz.$$

As a second measure one can take

$$\|\psi - \psi_0\|_2^2 = F[\psi] - F[\psi_0].$$

Since in (16.12) dots stand for the remainder terms of an order higher than $(\delta\psi)^2$, then both measures are equivalent, i.e., there exist constants $C_1 > 0$ and $C_2 > 0$ such that

$$C_1\|\psi - \psi_0\|_1 \le \|\psi - \psi_0\|_2 \le C_2\|\psi - \psi_0\|_1. \tag{16.13}$$

Let the velocity deviation $\nabla\delta\psi$ and vorticity deviation $\delta\Delta\psi$ be small at the initial moment, and hence the norm $\|\psi - \psi_0\|_1$ is small. Then, by (16.13), at the initial moment, the measure $\|\psi - \psi_0\|_2$ is small as well. But the latter is an invariant, which will remain this small during the entire motion. Again, by (16.13) the measure $\|\psi - \psi_0\|_1$ will remain small too. Because of the positive definiteness of the latter the values of the velocity $\nabla\delta\psi$ and vorticity $\Delta\delta\psi$ will also remain small in the mean square sense. Thus, now *the Rayleigh theorem is proved here in the nonlinear formulation of the problem, i.e., for finite perturbations*.[1]

16.3 Exercises

1. For which of the flow profiles shown in Fig. 16.2 is the Fjortoft theorem applicable or non-applicable?

[1] V.P. Dymnikov drew the author's attention to the following circumstance. In fact, the so-called formal stability of a shear flow proved above is based on the formal application of the stability criterion for finite-dimensional dynamical systems to systems with infinite number of degrees of freedom. The vanishing of the first variation and the positivity of the second variation, generally speaking, do not provide the minimum of the Lyapunov function (A.N. Filatov, *Stability Theory*, INM RAS, Moscow, 2002).

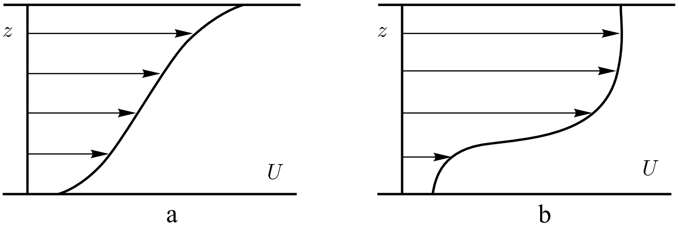

Fig. 16.2 (**a**) Is the Fjortoft criterion applicable? (**b**) Is the Fjortoft criterion non-applicable?

2. Prove that under the conditions (16.9) and (16.10) the kinetic energy E (see (16.11)) is a first integral of Eqs. (16.7) and (16.8).

Proof:

$$\frac{dE}{dt} = \iint \nabla\psi\nabla\psi_t dxdz$$
$$= \psi|_C \oint_C \frac{\partial\psi_t}{\partial n}\delta l - \iint \psi\Delta\psi_t dxdz \quad \text{(integration by parts)}$$
$$= \psi|_C \frac{d}{dt}\oint_C \frac{\partial\psi}{\partial n}\delta l - \iint \psi\Delta\psi_t dxdz \quad \text{(since the contour } C \text{ does not move)}$$
$$= \iint \psi[\psi, \Delta\psi]dxdz \quad \big(\text{according to (16.10) and the equations of motion}\big)$$
$$= \iint \big\{(\psi\Delta\psi\psi_x)_z - (\psi\Delta\psi\psi_z)_x\big\}dxdz \quad \text{(straightforward verification)}$$
$$= \oint_C \psi\Delta\psi\frac{\partial\psi}{\partial l}\delta l \quad (\textit{by the Stokes theorem}) = \oint_C \psi\Delta\psi d\psi$$
$$= 0 \ \big(\text{according to (16.9)}\big).$$

3. Prove that under the conditions (16.9) and (16.10) the quantity I (see (16.11)) is a first integral of Eqs. (16.7) and (16.8).

Proof:

$$\frac{dI}{dt} = \iint \Phi'(\Delta\psi)\Delta\psi_t dxdz = -\iint \Phi'(\Delta\psi)[\psi, \Delta\psi]dxdz$$
$$= \iint [\Phi(\Delta\psi), \psi]dxdz = \iint \big\{\big(\Phi(\Delta\psi)\psi_x\big)_z - \big(\Phi(\Delta\psi)\psi_z\big)_x\big\}dxdz$$
$$= \oint_C \Phi(\Delta\psi)\frac{\partial\psi}{\partial l}\delta l = \oint_C \Phi(\Delta\psi)d\psi = 0.$$

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Chapter 17
Stability of Zonal Flows of a Barotropic Atmosphere; The Notion of Barotropic Instability

17.1 The Kuo Theorem

Now we turn to the study of stability of global atmospheric flows. It is worth recalling the filtered equations of motion of a rotating fluid, which allow for an easy generalization of the classical stability theory of strictly two-dimensional flows of a nonrotating fluid to the case at hand. The motion of a barotropic atmosphere in quasi-geostrophic approximation is described by the Obukhov–Charney equation

$$\frac{d}{dt}\left(f + \Delta\psi - L_0^2\psi\right) \equiv \frac{\partial}{\partial t}\left(\Delta\psi - L_0^2\psi\right) + [\psi, \Delta\psi] + \beta\frac{\partial\psi}{\partial x} = 0, \quad (17.1)$$

$$u = -\frac{\partial\psi}{\partial y}, \qquad v = +\frac{\partial\psi}{\partial x}. \quad (17.2)$$

Here the coordinates x and y are measured in the east and north directions, respectively. The geostrophic stream function $\psi = gh/f_0$, where $h = z(x, y, t) - z_p$ is the deviation of the height $z(x, y, t)$ of an arbitrary isobaric surface $p(x, y, z, t) = \text{const}$ from its equilibrium hydrostatic value $z(p) = z_p$ (recall that in a barotropic atmosphere isobars, coinciding with isochores, vary in a similar way to one another). The value f_0 is the average value of the Coriolis parameter $f = 2\Omega_0 \sin\varphi$ (φ is latitude), $L_0 = c/f_0$ is the Rossby–Obukhov scale and $\beta = df/dy$. In the approximation of shallow water theory h is the height of deviation of a free surface from its equilibrium level $H_*(x, y) = H_0 - h_1(x, y)$, $L_0 = \sqrt{gH_0}/2\Omega_0$, while the beta-effect $\beta = 2\Omega_0 H_0^{-1} dh_1/dy$ is provided by nonuniform bottom topography (see Fig. 6.3 in Chap. 6).

The system (17.1) and (17.2) has three first integrals of motion (see Chap. 7)

$$E = \frac{1}{2}\iint \left[(\nabla\psi)^2 + L_0^{-2}\psi^2\right] dx\,dy \quad (17.3)$$

$$I = \iint \Phi(\Pi)\,dx\,dy, \qquad \langle\psi\rangle = \iint \psi\,dx\,dy, \quad (17.4)$$

F.V. Dolzhansky, *Fundamentals of Geophysical Hydrodynamics*, Encyclopaedia of Mathematical Sciences 103,
DOI 10.1007/978-3-642-31034-8_17,

where $\Phi(\Pi)$ is an arbitrary function of the quasi-geostrophic potential vorticity $\Pi = f + \Delta\psi - L_0^{-2}\psi$.

Kuo Theorem (Hsiao-Lan Kuo, 1949) *A zonal flow of an ideal barotropic atmosphere is stable if the potential vorticity changes monotonically from one pole to the other.*

Proof Let $\psi_0 = \Psi_0(y)$ be the stream function of such a stationary zonal flow, so that the corresponding potential vorticity $\Pi_0 = f + \Delta\psi_0 - L_0^{-2}\psi_0$ is a monotonic function of the latitude coordinates y. Then Π_0 can be taken for a new latitude coordinate, i.e., one can assume that $\psi_0 = \Psi_0(\Pi_0)$.

We calculate the variation of the functional $F = E + I$:

$$\delta F = \iint \bigl\{\nabla\psi_0\delta\nabla\psi + L_0^{-2}\psi_0\delta\psi + \Phi'(\Pi_0)\delta\Pi\bigr\}dxdy$$

(after integration by parts)

$$= \iint \bigl\{-\psi_0\delta\nabla\psi + L_0^{-2}\psi_0\delta\psi + \Phi'(\Pi_0)\delta\Pi\bigr\}dxdy$$

$$= \iint \bigl[-\Psi_0(\Pi_0) + \Phi'(\Pi_0)\bigr]\delta\Pi dxdy,$$

since $\psi_0\delta\nabla\psi - L_0^{-2}\psi_0\delta\psi = \delta\Pi$.

A necessary condition for an extremum is $\Phi'(\Pi_0) = \Psi_0(\Pi_0)$. The second variation is

$$\delta^2 F = \iint \bigl\{(\delta\nabla\psi)^2 + L_0^{-2}(\delta\psi)^2 + \Phi''(\Pi_0)(\delta\Pi)^2\bigr\}dxdy$$

$$= \iint \bigl\{(\delta\nabla\psi)^2 + L_0^{-2}(\delta\psi)^2 + \Psi'(\Pi_0)(\delta\Pi)^2\bigr\}dxdy.$$

For positive definiteness of $\delta^2 F$ one needs to have $\Psi'(\Pi_0) > 0$. In the Earth's conditions, exactly the opposite is true. The zonal velocity $U = -\partial\psi_0/\partial y$ is directed from west to east, i.e., ψ_0 increases from north to south (recall that the axes $0x$ and $0y$ are directed, respectively, to the east and north). But the potential vorticity, whose sign is determined by a transported vorticity f, on the contrary, increases from south to north. Hence $\Psi'(\Pi_0) < 0$. It is important, however, that this quantity be sign-definite. Therefore one can do the following. Pass to a coordinate system that rotates relative to the original one with angular velocity Ω_1. In this coordinate system the new stream function $\psi_1 = \psi - \phi$, where $\phi = a^2\Omega_1 \sin\varphi$ is the time-independent stream function corresponding to the rotation with constant angular velocity Ω_1 (φ is the latitude). In terms of ψ_1 the evolution equation for the potential vorticity can be written as follows:

$$\frac{d}{dt}\bigl(f_1 + \Delta\psi_1 - L_0^{-2}\psi_1 - L_0^{-2}\phi\bigr) = 0. \tag{17.5}$$

Here $f_1 = 2(\Omega + \Omega_1)\sin\varphi$. Thus, in Eq. (17.5), as compared to (17.1), there appears the parameter $f_{eff} = f_1 - L_0^{-2}\phi$ instead of the Coriolis parameter f, which does not violate the law of energy conservation. The above expressions for variations remain valid. Therefore, all the above considerations hold in this coordinate system as well. In the coordinate system, moving ahead of the atmospheric motion, the wind is blowing from east to west, and then $\Psi'(\Pi_0) > 0$. The theorem is proved. □

17.2 The Barotropic Instability Mechanism via an Example of the Utmost Simplistic Equations of Atmospheric Dynamics

According to the Kuo theorem, a necessary condition for the instability of a zonal flow is latitudinal nonmonotonicity of its vorticity. To understand possible consequences that this may cause and to describe, at least qualitatively, the development of instability, if any, we subject the equations of barotropic atmospheric dynamics to the maximum simplification, following E. Lorenz (1960). This will allow us to avoid numerical integration and to perform an analytical study of the mechanism of nonlinear instability.

In the first place, the simplification is that we do not take into account the two-dimensional compressibility of the medium ($L_0^{-2} = 0$), while a two-dimensional doubly-periodic flow is considered on the f-plane instead of the sphere, i.e., without taking the beta-effect into account. Then the vortex dynamics of the system is described by the usual two-dimensional vorticity equation

$$\frac{\partial(\Delta\psi)}{\partial t} + [\psi, \Delta\psi] = 0 \tag{17.6}$$

with periodicity conditions

$$\psi(x + 2\pi/k, y + 2\pi/l) = \psi(x, y). \tag{17.7}$$

Here k and l are certain fixed nonzero real numbers that specify the maximum spatial periods in the directions of the axes x and y, respectively.

The trigonometric functions $\cos(mkx + nly)$ and $\sin(mkx + nly)$, where m and n are nonzero integers, are eigenfunctions of the Laplace operator in this geometry. Expansion in these eigenfunctions gives the following Fourier representation for the vorticity:

$$\Delta\psi = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\left[A_{mn}\cos(mkx + nly) + B_{mn}\sin(mkx + nly)\right],$$
$$(A_{00} = B_{00} = 0). \tag{17.8}$$

It can be rewritten in the form

$$\Delta\psi = \sum_{\mathbf{M}} C_{\mathbf{M}} \exp\{i\mathbf{MR}\}, \tag{17.9}$$

where $\mathbf{M} = \mathbf{i}mk + \mathbf{j}nl$, $\mathbf{R} = \mathbf{i}x + \mathbf{j}y$, while $C_\mathbf{M} = \frac{1}{2}(A_{mn} - iB_{mn})$, $\mathbf{i}$ and $\mathbf{j}$ are unit vectors in the directions x and y.

The Fourier series for ψ corresponding to the expansion (17.9) has the form

$$\psi = \sum_\mathbf{M} (\mathbf{M} \cdot \mathbf{M})^{-1} C_\mathbf{M} \exp\{i\mathbf{MR}\}. \tag{17.10}$$

Replacing the summation index $\mathbf{M}$ by $\mathbf{L}$ in (17.9), and similarly $\mathbf{M}$ by $\mathbf{H}$ in (17.10), and then substituting (17.9) and (17.10) in the Jacobian of $[\psi, \Delta\psi]$, we obtain its Fourier representation:

$$[\psi, \Delta\psi] = \sum_{\mathbf{H},\mathbf{L}} (\mathbf{k} \cdot \mathbf{H} \times \mathbf{L})(\mathbf{H} \cdot \mathbf{H})^{-1} C_\mathbf{H} C_\mathbf{M} \exp\big\{i(\mathbf{H} + \mathbf{L}) \cdot \mathbf{R}\big\}. \tag{17.11}$$

After replacing in (17.11) the summation index $\mathbf{M}$ by $\mathbf{H} + \mathbf{L}$ and substituting (17.9) and (17.11) into (17.6) we obtain an infinite system of ordinary differential equations for the coefficients $C_\mathbf{M}$, the Fourier representation (17.6) in the wavenumber space:

$$\frac{dC_{-\mathbf{M}}}{dt} = \sum_\mathbf{H} \frac{\mathbf{k} \cdot \mathbf{H} \times \mathbf{M}}{\mathbf{H} \cdot \mathbf{H}} C_\mathbf{H} C_{\mathbf{M}-\mathbf{H}}. \tag{17.12}$$

Similar equations are obtained for a sphere, if one does the expansion in spherical harmonics, eigenfunctions of the spherical Laplace operator.

Equation (17.6) is known to have two first integrals of motion, the kinetic energy $E = \frac{1}{2} \iint (\nabla\psi)^2 dxdy$ and the total vorticity squared $I = \iint (\Delta\psi)^2 dxdy$. In the space of wavenumbers these integrals are written as follows:

$$E = \frac{1}{2} \sum_\mathbf{M} \frac{C_\mathbf{M} C_{-\mathbf{M}}}{\mathbf{M} \cdot \mathbf{M}}, \qquad I = \sum_\mathbf{M} C_\mathbf{M} C_{-\mathbf{M}}. \tag{17.13}$$

Now note that each of the time derivatives of E and I is the sum of a series whose elements are the products of $C_\mathbf{H} C_{\mathbf{M}-\mathbf{H}} C_{-\mathbf{M}}$ with the corresponding coefficients. The sum of such a series can be represented as a sum of blocks, each of which is the sum of six terms obtained by permutations of the vector indices $\mathbf{H}$, $\mathbf{M} - \mathbf{H}$, and $-\mathbf{M}$. Any such block is identically equal to zero. Therefore, by applying the Galerkin method, the system can be maximally simplified so that in the nonlinear interaction only modes corresponding to the fixed three vectors $\mathbf{H}$, $\mathbf{M} - \mathbf{H}$, and $-\mathbf{M}$ were involved. If we assume that at the initial moment all the coefficients $C_\mathbf{M}$ vanish except for those corresponding to the above-mentioned vectors, we find that the values of E and I, represented by finite sums of pairwise products of these nonzero coefficients, are first integrals of the reduced system. Of course, due to the convergence of the Galerkin method, this approach would be the more accurate, the bigger the number of fixed triples of vectors is taken into account.

In the simplest case, we can confine ourselves to a single triple, in which each index m and n takes the value $1, 0, -1$, which corresponds to considering the nonlinear interaction of modes of the largest scale. In spite of the crudeness of this approach, this example, first of all, allows one to describe the behavior of the system at some initial stage if we use the above initial conditions, and secondly and most importantly, allows one to clarify the instability mechanism in terms of energy.

For chosen values of m and n, the expansion (17.8) assumes the form:

$$\Delta\psi = A_{10}\cos kx + A_{01}\cos ly + A_{11}\cos(kx+ly) + A_{1-1}\cos(kx-ly)$$
$$+ B_{10}\sin kx + B_{01}\sin ly + B_{11}\sin(kx+ly) + B_{1-1}\sin(kx-ly).$$

Further simplification is related to the following observation. If at the initial moment $A_{1-1} = -A_{11}$, while $B_{10} = B_{01} = B_{11} = B_{1-1} = 0$, then they will remain so at any time t. Now, once we set $A_{01} = \sqrt{2}A_1$, $A_{10} = \sqrt{2}A_2$ and $A_{1-1} = A_3$, the maximally shortened expansions for the vorticity and the stream function can be written as

$$\Delta\psi = \sqrt{2}A_1\cos ly + \sqrt{2}A_2\cos kx + 2A_3\sin ly\sin kx,$$
$$\psi = -\frac{\sqrt{2}A_1}{l^2}\cos ly - \frac{\sqrt{2}A_2}{k^2}\cos kx - \frac{2A_3}{k^2+l^2}\sin ly\sin kx.$$

They correspond to the following maximally simplified equations for the dynamics of global geophysical flows:

$$\begin{aligned}
\dot{A}_1 &= -\left(\frac{1}{k^2} - \frac{1}{k^2+l^2}\right)klA_2A_3,\\
\dot{A}_2 &= +\left(\frac{1}{l^2} - \frac{1}{k^2+l^2}\right)klA_3A_1,\\
\dot{A}_3 &= \left(\frac{1}{k^2} - \frac{1}{l^2}\right)klA_1A_2.
\end{aligned} \tag{17.14}$$

They have two positive definite quadratic first integrals of motion:

$$E = \frac{1}{2}\left(\frac{A_1^2}{l^2} + \frac{A_2^2}{k^2} + \frac{A_3^2}{k^2+l^2}\right),$$
$$I = A_1^2 + A_2^2 + A_3^2.$$

According to the Obukhov theorem, any quadratically nonlinear dynamical system of the third order, which has two positive definite quadratic first integrals of motion, is equivalent to the Euler equations of motion of a rigid body with a fixed point. In this particular case this is easy to verify by making a formal change of variables $d\tau = kldt$, $l^2 = I_1$, $k^2 = I_2$, $k^2 + l^2 = I_3$. Then the dynamical system (17.14) can be rewritten in the form of the above-mentioned Euler equations in terms of the components of the angular momentum $M_i \equiv A_i$, $i = 1, 2, 3$ (see Chap. 12):

$$\dot{M}_1 = \left(\frac{1}{I_3} - \frac{1}{I_2}\right)M_2M_3,$$
$$\dot{M}_2 = \left(\frac{1}{I_1} - \frac{1}{I_3}\right)M_3M_1,$$

$$\dot{M}_3 = \left(\frac{1}{I_2} - \frac{1}{I_1}\right) M_1 M_2.$$

From the point of view of dynamic meteorology, Eqs. (17.14) describe evolution of the shear zonal flow with velocity profile $U = -\partial\psi/\partial y \propto \sin ly$ and nonmonotonic vorticity $\Delta\psi \propto \cos ly$, which is subjected to wave-like large-scale nonzonal perturbations with latitudinal wavenumber k. In the absence of perturbations, the zonal flow is described by a stationary solution ($A_1 = \text{const}$, $A_2 = A_3 = 0$) of the system (17.14). According to the analysis in Chap. 14, such a zonal flow is unstable for $k^2 < l^2$, i.e., with respect to perturbations whose scale exceeds the linear dimension of the shift. From the physical point of view, this is not surprising since in the limiting case of very small-scale nonzonal perturbations ($k^2 \gg l^2$) they will not "notice" the presence of shear at all.

Solutions of the Euler equations are known to be given by the corresponding elliptic functions, describing nonlinear oscillations of all three components of the system. In this connection it is instructive to note that the exponential growth of nonzonal perturbations, observed at the initial stage, is due to the kinetic energy of the zonal flow. Later this growth slows down because of the nonlinearity effect. The resulting motion presents a strictly periodic interchange of the kinetic energies between perturbations and the zonal flow. It is with this process one associates fluctuations of the circulation index observed in the atmosphere, which are fluctuations of angular velocity relative to the zonal rotation of air in the middle latitudes.

17.3 Exercises

1. Try to derive the maximally simplified equations for the dynamics of a barotropic atmosphere, taking into account the beta-effect described by the equation

$$\frac{\partial}{\partial t}\left(\Delta\psi - \alpha^2\psi\right) + [\psi, \Delta\psi] + \beta\frac{\partial\psi}{\partial x} = 0, \tag{17.15}$$

where $\alpha^2 = L_0^{-2}$.

Hint: Following Lorenz, in the expansion of ψ over trigonometric functions keep the following terms:

$$\begin{aligned}\psi(x, y, t) = {} & A_{10}(t)\cos(kx) + A_{01}(t)\cos(ly) + B_{10}(t)\sin(kx) \\ & + B_{01}(t)\sin(ly) + A_{11}(t)\cos(kx + ly) + A_{1-1}(t)\cos(kx - ly) \\ & + B_{11}(t)\sin(kx + ly) + B_{1-1}(t)\sin(kx - ly).\end{aligned}$$

After substituting this expansion into (17.15) and equating coefficients at the same harmonics, introduce the notations

$$A_{1m} = A_{11} - A_{1-1}, \qquad A_{1p} = A_{11} + A_{1-1},$$

$$B_{1m} = B_{11} - B_{1-1}, \qquad B_{1p} = B_{11} + B_{1-1}.$$

Finally, one finds out that the resulting dynamical system admits partial solutions for $A_{1p} = 0$, $B_{1p} = 0$, $B_{01}(0) = 0$, which are described by the following equations (A.E. Gledzer, unpublished):

$$\begin{aligned}
\dot{A}_{10} &= \sigma_k B_{10} + N_k A_{01} A_{1m}, \qquad \dot{A}_{01} = -N_l(A_{10}A_{1m} + B_{10}B_{1m}), \\
\dot{A}_{1m} &= \sigma B_{1m} + N A_{10} A_{01}, \\
\dot{B}_{10} &= -\sigma_k A_{10} + N_k A_{01} B_{1m}, \qquad \dot{B}_{1m} = -\sigma A_{1m} + N A_{01} B_{10}.
\end{aligned} \tag{17.16}$$

Here

$$\sigma = \frac{\beta k}{k^2 + l^2 + \alpha^2}, \qquad \sigma_k = \frac{\beta k}{k^2 + \alpha^2},$$

$$N = \frac{kl(l^2 - k^2)}{k^2 + l^2 + \alpha^2}, \qquad N_k = \frac{(1/2)k^3 l}{k^2 + \alpha^2}, \qquad N_l = \frac{(1/2)kl^3}{l^2 + \alpha^2}.$$

Thus, the *maximally simplified equations of motion for a barotropic atmosphere with the beta-effect taken into account are defined by the dynamical system of order 5.*

References

L.A. Dikii, *Hydrodynamic Stability and the Dynamics of the Atmosphere*, Gidrometeoizdat, Leningrad, 1976.

E.N. Lorenz, *Maximum simplification of the dynamical equations*, Tellus, Vol. 12, p. 3, 1960.

H.-L. Kuo, *Dynamic instability of heterogeneous shear flows*, J. Met., Vol. 6, p. 2, 1949.

Chapter 18
The Concept of Baroclinic Instability; The Eady Model

18.1 Stating the Problem

Recall that global movements of a baroclinic atmosphere is described by the quasi-geostrophic equation for potential vorticity (see Chap. 9):

$$\frac{d}{dt}\left[f + \Delta\psi + \frac{1}{\rho_s}\frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\rho_s\frac{\partial\psi}{\partial z}\right)\right] = 0,$$
$$\left(\frac{d}{dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}, \quad \Delta\psi = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right). \tag{18.1}$$

Here the expression in square brackets is the baroclinic quasi-geostrophic potential vorticity, $\psi = \psi(x, y, z) = p'/f_0\rho_s$ is the quasi-geostrophic stream function, p' is the pressure deviation from its background hydrostatic value, $\rho_s = \rho_s(z)$ is the background density distribution, $N^2 = g \cdot d(\ln\Theta_s)/dz$ is the square of the Brunt–Väisälä frequency, $\Theta_s = \Theta_s(z)$ is the background distribution of potential temperature, $f = 2\Omega\sin\varphi$ is the Coriolis parameter (φ stands for the latitude) and f_0 is its average. Finally, u and v are zonal and meridional components of the geostrophic wind, which in terms of the stream function are given by the equalities

$$u = -\frac{\partial\psi}{\partial y}, \qquad v = \frac{\partial\psi}{\partial x}. \tag{18.2}$$

The condition of impermeability of the lower boundary can be written in terms of the stream function as

$$w = -\frac{f_0}{N^2}\frac{d}{dt}\frac{\partial\psi}{\partial z} = 0 \quad \text{for } z = 0. \tag{18.3}$$

Recall also two important relations:

$$\frac{\theta}{\Theta_s} = \frac{f_0}{g}\frac{\partial\psi}{\partial z}, \tag{18.4}$$

F.V. Dolzhansky, *Fundamentals of Geophysical Hydrodynamics*,
Encyclopaedia of Mathematical Sciences 103,
DOI 10.1007/978-3-642-31034-8_18,

establishing the relationship between the variation $\theta = \theta(x, y, z, t)$ of the potential temperature and the stream function, and the thermal wind:

$$\frac{\partial \mathbf{v}}{\partial z} = \frac{g}{f_0 \Theta_s} \mathbf{k} \times \nabla \Theta \quad \left(\Theta = \Theta_s + \theta, \ \nabla = \mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} \right), \tag{18.5}$$

or in the coordinate form

$$\frac{\partial u}{\partial z} = -\frac{g}{f_0 \Theta_s} \frac{\partial \theta}{\partial y}, \qquad \frac{\partial v}{\partial z} = \frac{g}{f_0 \Theta_s} \frac{\partial \theta}{\partial x}. \tag{18.5'}$$

According to the latter relations the vertical shear of horizontal velocity is generated by the horizontal gradient of the potential temperature. Instability induced by vertical shear of the velocity is called *baroclinic instability*, because, unlike the barotropic instability, in this case the source of the kinetic energy of perturbations is the baroclinic available potential energy, whose local measure is the horizontal inhomogeneity of temperature. In this relation it is pertinent to recall first integrals of motion for Eq. (18.1), which will be used in formulating the stability conditions of baroclinic zonal flows. Here we are talking on the energy invariant

$$E = \frac{1}{2} \iiint\limits_V \rho_s \left((\nabla \psi)^2 + \frac{f_0^2}{N^2} \left(\frac{\partial \psi}{\partial z} \right)^2 \right) dx dy dz \tag{18.6}$$

and the generalized integral potential vorticity

$$I = \iiint\limits_V \Phi(\Pi, z) dx dy dz, \tag{18.7}$$

where $\Phi(\Pi, z)$ is an arbitrary function of the potential vorticity Π and z. One can add here the invariant

$$G = \iint\limits_{z=0} \Gamma \left(\frac{\partial \psi}{\partial z} \right) dx dy \tag{18.8}$$

(Γ is an arbitrary function of one argument), which follows from the lower boundary condition (18.3).

The energy integral with the help of (18.2) and (18.4) can be written in terms of the horizontal velocity and potential temperature (see Chap. 10):

$$E = \frac{1}{2} \iiint\limits_V \rho_s \mathbf{v}^2 dx dy dz + \frac{1}{2} \iiint\limits_V \rho_s \frac{g^2}{N^2} \frac{\theta^2}{\Theta_s^2} dx dy dz, \tag{18.6'}$$

where the second term, denoted by $P_{bc} \doteq (APE)_{bc}$ is exactly the baroclinic available potential energy of the atmosphere. Recall for comparison that the available potential energy of a barotropic atmosphere is given by

$$(APE)_{bt} = \frac{1}{2} \int \int L_0^{-2} \psi^2 dx dy.$$

Both types of energy, as discussed in Chap. 11, can be taken into account within one model by passing to the p-coordinates.

It should be noted, however, that $(APE)_{bc}$ in the form in which it is defined is not a universal expression for the baroclinic available potential energy, but, in general, depends on the particular formulation of the problem. Such an expression, in particular, is not suitable if the background is taken to be an equilibrium state corresponding to a steady neutrally-stratified atmosphere. For example, this is justified if we are discussing quasi-geostrophic motions of a weakly-stratified incompressible fluid. In this case $N^2 \approx 0$, while the available potential energy is determined by the horizontal stratification of the medium. The simplest example of this kind is discussed in one of chapters below. A rigorous approach to the calculation of APE can be found in the monograph by Kurgansky (1993). Here it is only important for us to show the role of $(APE)_{bc}$ in the mechanism of baroclinic instability.

18.2 The Charney–Stern Theorem

The Charney–Stern Theorem (1962) *If the near-surface temperature is constant, then for the stability of a baroclinic geostrophic flow it is sufficient that for any fixed level* $z =$ const, *the potential vorticity of the zonal motion decreased monotonically in the pole-equator direction.*

Remark It is worth noting that Charney and Stern, unlike us, were using the p-coordinate representation of the equations of motion and therefore required the constant near-surface density rather than temperature. Both assumptions are rather rough and to some extent they are valid only in the mid-latitudes, which are sufficiently remote from the equator and poles.

Proof The value of G (see (18.8)) is constant by virtue of the theorem's assumptions. Therefore as a Lyapunov function one can use the functional $F = E + I$, i.e.,

$$F = \iiint\limits_V \left[\frac{1}{2}\rho_s(\nabla\psi)^2 + \frac{1}{2}\rho_s\frac{f_0^2}{N^2}\left(\frac{\partial\psi}{\partial z}\right)^2 + \Phi(\Pi, z)\right]dxdydz.$$

Then

$$\delta F = \iiint\limits_V \left[\rho_s\nabla\psi_0\delta\nabla\psi + \rho_s\frac{f_0^2}{N^2}\frac{\partial\psi_0}{\partial z}\delta\frac{\partial\psi}{\partial z} + \frac{\partial\Phi}{\partial\Pi}\delta\Pi\right]dxdydz$$

(after integration by parts)

$$= \iiint\limits_V \left\{-\rho_s\psi_0\delta\Delta\psi - \psi_0\delta\left[\frac{\partial}{\partial z}\left(\rho_s\frac{f_0^2}{N^2}\frac{\partial\psi}{\partial z}\right)\right]\right\}dxdydz$$

$$+\iiint\limits_V \left\{ \frac{\partial \Phi}{\partial \Pi} \delta \Pi(y,z) \right\} dxdydz$$

$$= \iiint\limits_V \left(-\rho_s \psi_0 + \frac{\partial \Phi}{\partial \Pi} \right) \delta \Pi dxdydz,$$

since

$$\delta \Delta \psi + \delta \left[\frac{1}{\rho_s} \frac{\partial}{\partial z} \left(\rho_s \frac{f_0^2}{N^2} \frac{\partial \psi}{\partial z} \right) \right] = \delta \Pi.$$

Here $\psi_0 = \psi_0(y,z)$ is the stream function of the zonal baroclinic flow that is studied for stability.

By virtue of the theorem's assumptions one can take Π to be the latitudinal coordinate instead of y, i.e., one can assume that $\psi_0(y,z) = \Psi(\Pi, z)$. Then $\delta F = 0$ implies $\rho_s \Psi = \partial \Phi / \partial \Pi$. Calculate the second variation:

$$\delta^2 F = \iiint\limits_V \rho_s \left[(\delta \nabla \psi)^2 + \frac{f_0^2}{N^2} \left(\delta \frac{\partial \psi}{\partial z} \right)^2 + \frac{\partial \Psi}{\partial \Pi} (\delta \Pi)^2 \right] dxdydz.$$

Positive definiteness of $\delta^2 F$ requires that $\partial \Psi / \partial \Pi > 0$. Further arguments are similar to those used in the proof of the Kuo theorem, i.e., one needs to consider the motion in the frame, rotating ahead of the wind. □

18.3 The Eady Model

I would like to complete the description of the elements of inviscid stability theory of global geophysical flows by presenting the work of Eady. Published back in 1949 in the famous Swedish meteorological journal *Tellus*, this work has long since become classic and is described in almost every textbook on geophysical fluid dynamics or dynamic meteorology. The reason is that Eady not only for the first time revealed the mechanism of baroclinic instability, i.e., the cause of the birth of vertical vorticity (cyclones and anticyclones) because of *inhomogeneous horizontal distributions of density or entropy*. But he also explained this mechanism in such a physically transparent way that all subsequent work on the subject, even in a nonlinear formulation of the problem, only quantitatively develops this idea without introducing any new qualitative explanations. The Eady model is the result of Eady's doctoral thesis, and during its preparation he could not yet have known the fundamental quasi-geostrophic equations of global motions. Apparently for this reason, there is no list of references in his article, thus emphasizing the lack of precursors, even though in conclusion he mentions the work of Charney (1947), "...which in many (but not all) respects is consistent with his own." For the same reason, Eady formulates the boundary value problem not for the equation of potential vorticity, but for the evolution equation of vertical velocity, where in order to simplify it he uses essentially the same assumptions as in the derivation of quasi-geostrophic equations mentioned above. Below we present a modern exposition of the work by Eady, i.e., in terms of potential vorticity, which is more concise than the original.

18.3.1 Formulation of the Problem

For this problem we assume that the quantity $f_0^2 N^{-2} \rho_s$ is a slowly varying function of the vertical coordinate, which allows one to take it out of the differentiation in z in Eq. (18.1). It is pertinent to note that this assumption, although being valid for the Oberbeck–Boussinesq fluid, is rather arguable for the real atmosphere. This procedure is usually used when working with the p-coordinates (in the latter case under the vertical differentiation one has the quantity $L_R^{-2} p^2$, where $L_R = NH/f_0$ is the internal radius of deformation, see (11.7)), but even then it does not look more convincing.

However, from a physical point of view, we have not committed a big sin by slightly changing the spatial distribution of the available potential energy. Therefore the results obtained below are suitable for estimates of the real characteristics of linear stability. It is also important to note that in the vertical direction the medium is assumed to be stably stratified ($N > 0$). A three-dimensional wind field at the middle and high latitudes of the Earth's atmosphere is approximately described by formulas (18.2) and (18.3), where the horizontal components of the wind velocity u and v are measured in the directions toward east and north, respectively, while a positive vertical velocity is directed upward.

The problem is considered on the f-plane ($df/dy = 0$) in the region bounded only by vertical solid surfaces at the levels $z = 0, H$. We are studying stability of a strongly zoned (along circles of latitude) motion of the atmosphere, which has a vertical shear or, equivalently, vorticity in the direction opposite to the latitudinal gradient of potential temperature:

$$u = U(z) = -\frac{\partial \Psi}{\partial y}, \qquad v = w = 0. \tag{18.9}$$

18.3.2 Solution Using the Method of Normal Modes

The stream function and potential vorticity corresponding to the ground state (18.9) are, respectively, equal to

$$\Psi(x, y) = -U(z)y + F(z), \tag{18.10}$$

$$\Pi(x, y) = -\frac{f_0^2}{N^2}\left(U''y + F''\right) + f, \tag{18.11}$$

since $\partial^2 \Psi/\partial z^2 = -U''y + F''$, $\Delta\psi = 0$.

Then Eq. (18.1), linearized with respect to the primary flow, and by taking into account the above assumptions, can be written as follows:

$$\frac{d}{dt}(\Pi + \chi) = \left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\chi + \frac{\partial \psi}{\partial x}\frac{\partial \Pi}{\partial y} = 0,$$

where $\psi = \psi(x, y, z, t)$ is the stream function of infinitesimal perturbations u, v and w of the primary flow, while

$$\chi = \Delta\psi + \frac{f_0^2}{N^2}\frac{\partial^2\psi}{\partial z^2}$$

represents the potential vorticity corresponding to them.

Further, recalling the Squire theorem one can assume that the perturbations do not depend on the coordinate y orthogonal to the plane (x, z), i.e., $\psi = \psi(x, z, t)$. The thermal wind $\Omega_y = dU/dz$ for simplicity is set to be constant along the vertical. Then $U'' = 0$, $\partial\Pi/\partial y = 0$, and the linear stability problem is defined by the equation

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\left(\frac{\partial^2\psi}{\partial x^2} + \frac{f_0^2}{N^2}\frac{\partial^2\psi}{\partial z^2}\right) = 0 \tag{18.12}$$

with the boundary conditions

$$w = -\frac{f_0}{N^2}\frac{d}{dt}\frac{\partial(\Psi + \psi)}{\partial z} = 0 \quad \text{for } z = 0, H.$$

After linearization the latter can be written in the form

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\frac{\partial\psi}{\partial z} - \Omega_y\frac{\partial\psi}{\partial x} = 0 \quad \text{for } z = 0, H. \tag{18.13}$$

Coefficients in Eq. (18.12) and boundary conditions (18.13) depend only on z. Therefore, turning to the method of normal modes, we seek the solution in the form

$$\psi(x, z, t) = \psi(z)\exp\{ik(x - ct)\}.$$

After substituting this in (18.12) and (18.13) and taking into account $\partial/\partial x \Longrightarrow ik$, $\partial/\partial t \Longrightarrow -ikc$ and $(\partial/\partial t + U\partial/\partial x) \Longrightarrow ik(U - c)$, we obtain the following eigenvalue problem:

$$(U - c)\left(-k^2\psi + \frac{f_0}{N^2}\psi''\right) = 0, \tag{18.14}$$

$$(U - c)\psi' - \Omega_y\psi = 0 \quad \text{for } z = 0, H. \tag{18.15}$$

It is clear that the continuous spectrum of the problem belongs to the interval $[U_{\min}, U_{\max}]$ and it does not contribute to instability. In what follows we assume that $U_{\min} = U(0) = 0$, and $U_{\max} = U(H) = \Omega_y H = U_0$. Then for $c \neq U$ the general solution of Eq. (18.14) can be written as follows:

$$\psi(z) = A\cosh\{\lambda z\} + B\sinh\{\lambda z\}, \quad \lambda = \frac{Nk}{f_0}.$$

Here, according to the boundary conditions (18.15), the coefficients A and B satisfy the following homogeneous system of linear equations:

$$\begin{aligned}
&\Omega_y A + c\lambda B = 0,\\
&\big[\lambda(U_0 - c)\sinh\{\lambda H\} - \Omega_y \cosh\{\lambda H\}\big]A\\
&\quad + \big[\lambda(U_0 - c)\cosh\{\lambda H\} - \Omega_y \sinh\{\lambda H\}\big]B = 0.
\end{aligned}$$

Its solvability condition means that c must be a root of the quadratic equation:

$$\begin{aligned}
&c^2 - U_0 c + \left(U_0 \frac{\Omega_y}{\lambda}\coth\{\lambda H\} - \frac{\Omega_y^2}{\lambda^2}\right)\\
&\quad \equiv c^2 - U_0 c + \left(\frac{U_0^2}{\lambda H}\coth\{\lambda H\} - \frac{U_0^2}{\lambda^2 H^2}\right) = 0. \qquad (18.16)
\end{aligned}$$

The discriminant of this equation

$$D = U_0^2\big[1 - 4\alpha^{-2}\big(\alpha\coth(\alpha) - 1\big)\big], \quad \alpha = \lambda H = NkH/f_0,$$

by the identity

$$\coth(\alpha) = \frac{1}{2}\left(\tanh\frac{\alpha}{2} + \coth\frac{\alpha}{2}\right)$$

can be presented in the form

$$D = \frac{4}{\alpha^2}U_0^2\left(\frac{\alpha}{2} - \coth\frac{\alpha}{2}\right)\left(\frac{\alpha}{2} - \tanh\frac{\alpha}{2}\right).$$

And then $2c = U_0 \pm \sqrt{D}$. Since $\alpha/2 \geq \tanh(\alpha/2)$ for all α, then the critical value of α is the root of the equation

$$\frac{\alpha}{2} = \coth\frac{\alpha}{2}, \qquad \alpha_{\mathrm{cr}} \approx 2.399.$$

For $\alpha > \alpha_{\mathrm{cr}}$ one has $D < 0$ and the zonal flow is stable (the solution described by a superposition of two neutral modes). It is interesting to estimate the wavelength of the critical mode, by setting for this that $N \approx 10^{-2}\ \mathrm{s}^{-1}$, $H \approx 10$ km and $f_0 \approx 10^{-4}\ \mathrm{s}^{-1}$. Then

$$L_{\mathrm{cr}} = \frac{2\pi}{k_{\mathrm{cr}}} = \frac{2\pi NH}{f_0\alpha_{\mathrm{cr}}} \approx \frac{2\pi\, 10^{-2}\cdot 10}{10^{-4}\cdot 2.4} \approx 2.6\times 10^3\ \mathrm{km}.$$

Thus, the baroclinic instability is of a long-wave nature: the modes with wavelength $L > L_{cr}$ $(\alpha < \alpha_{cr})$ exponentially grow at a rate

$$k\,\mathrm{Im}\,c = \frac{k}{\alpha}U_0\left[\left(\frac{\alpha}{2} - \coth\frac{\alpha}{2}\right)\left(\frac{\alpha}{2} - \tanh\frac{\alpha}{2}\right)\right]^{1/2},$$

where the maximum of the growth rate is for the value $\alpha_m \approx 1.75$. It corresponds to the wave length $L_m = \frac{2\pi NH}{f_0\alpha_m} \approx 3.6 \times 10^3$ km, whose quarter is exactly comparable with the characteristic size of cyclones and anticyclones observed in the atmosphere. With regard to this problem, the characteristic size of cyclones or anticyclones should match exactly a quarter of the above length since the discussed solutions divide the entire integration domain into alternating subregions of cyclonic and anticyclonic vorticity each of size $L/2$. It is also easy to show that the minimum time of development of vertical vorticity (the time during which the amplitude of a maximally unstable perturbation increases e-fold) is equal to

$$\tau_m \approx \frac{L_m}{U_0}\frac{\alpha_m}{0.3 \cdot 2\pi} \approx 4\,\text{days}.$$

This is also consistent with the data observed.

From this analysis it becomes clear that the source of the instability under consideration is precisely the available potential energy, entering as the second term in the expression for the total energy (18.6). Indeed, according to (18.16), in the absence of thermal wind ($\Omega_y = 0$) unstable modes do not exist, while subject to the simplification made, the general solution of Eq. (18.1) satisfying the boundary conditions $w = 0$ at $z = 0, H$ can be written in the form

$$\Psi(x, y, z, t) = \psi(x, y, t) + F(z).$$

Here $\psi(x, y, t)$ is a solution of the equation

$$\frac{d}{dt}(f + \Delta\psi) = 0,$$

and $F(z)$ is an arbitrary function of z independent of time and being a symbol of the inaccessible potential energy (e.g., the potential energy of the background distribution of potential temperature). Indeed, the quantity $\partial\Psi/\partial z = dF/dz$ only formally corresponds to nonvanishing potential energy in Eq. (18.6). Being an invariant of motion, the latter is not converted into the kinetic energy and, consequently, it can be excluded from our consideration. Therefore it is exactly the nonvanishing thermal wind $\Omega_y = dU/dz$ that provides a vertical shear of the stream function. This vertical shear corresponds to potential energy which is converted into kinetic energy and is generating a large-scale vertical vorticity, i.e., which is one of the principal reasons for cyclogenesis. Eady gives the following explanation for baroclinic instability: as in the Rayleigh–Bernard convection, the negative vertical gradient of temperature causes the fluid to turn over in the vertical plane ("vertical overturning"), one has a similar effect in the atmosphere. Namely, the difference in the atmosphere temperatures in the pole-equator direction generates the air overturning in the horizontal plane ("horizontal overturning"), and in both cases the process proceeds in the direction of decreasing the system's potential energy.

From the analysis in Chaps. 16–18 it is useful to extract, understand, and memorize the following important feature of global geophysical flows. However paradoxical it sounds, both types of shifts of horizontal velocity, i.e., the horizontal shear and vertical shear, due to their instability generate large-scale vertical vorticity. The difference is that in the first case the source of cyclogenesis is the kinetic energy of the main flow, whereas in the second case that is its available potential energy. Both mechanisms, the barotropic and baroclinic instabilities, play a decisive role in shaping the general circulation of the Earth's atmosphere. The latter circumstance is one of the principal obstacles in constructing a general circulation theory.

18.4 Exercises

1. Complete the proof of the Charney–Stern theorem.
2. Show that the above estimate for τ_m indeed holds.

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Part IV
Friction in Geophysical Boundary Layers and Their Models

Chapter 19
Equations of Motion of a Viscous Fluid; The Boundary Conditions

So far we considered the motions of an ideal fluid which reflect, so to speak, genetic features of a real fluid's behavior and are not aggravated by the influence of irreversible thermodynamic processes. The latter, however, are always present due to internal friction (viscosity) and thermal conductivity of the medium. With regard to global geophysical flows, the situation is complicated by the fact that the role of irreversible non-adiabatic factors is assumed not only (or rather to a much lesser degree) by molecular viscosity and thermal conductivity, but rather by small-scale motions that are not taken into account by quasi-geostrophic approximation.

As we shall see below, the Earth's surface has a very special and crucial influence on the formation of general atmospheric circulation. Without friction on this surface the weather and climate on the Earth would have been totally unsuitable for human civilization. Irreversible diabatic processes start to noticeably affect the behavior of global atmospheric motions already on the third day after observations begin. Therefore, one cannot avoid including these irreversible processes in weather predictions for longer terms and for climate descriptions. We are going to start our consideration with a derivation of the equations of motion of a viscous fluid.

19.1 Derivation of the Navier–Stokes Equations

The content of this chapter is based on the material of Sections 15, 16 and 49, 50 of the textbook by Landau and Lifschitz (1986). Recall that the Euler equations of motion of an ideal fluid in terms of the specific impulse and tensor notations can be written as follows:

$$\frac{\partial \rho u_i}{\partial t} = -\frac{\partial \Pi_{ik}}{\partial x_k}, \tag{19.1}$$

where the density tensor of the momentum flux

$$\Pi_{ik} = \delta_{ik} p + \rho u_i u_k \tag{19.2}$$

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describes the completely reversible transfer of momentum caused by the influence of pressure and by the movements of various parts of the fluid from one place to another.

Viscosity, or internal friction, manifests itself in a fluid by the presence of an additional, irreversible transfer of momentum in the direction of decreasing velocity. It can be taken into account in (19.2) with an additional term σ'_{ik}, which irreversibly consumes a fraction of the "ideal" momentum flux:

$$\Pi_{ik} = \delta_{ik} p + \rho u_i u_k - \sigma'_{ik} = \rho u_i u_k - \sigma_{ik}. \quad (19.3)$$

The tensor σ'_{ik} is called the *viscous stress tensor*, while the tensor

$$\sigma_{ik} = -\delta_{ik} p + \sigma'_{ik} \quad (19.4)$$

is the *stress tensor* distinguishing that portion of the momentum flux, which is not related to the transfer of momentum of the fluid's moving mass.

The following considerations help establish the general form of the tensor σ'_{ik}.

(a) Internal friction in the fluid occurs only when there is movement of portions of the fluid relative to one another. Therefore, σ'_{ik} should depend on the derivatives of velocity in spatial coordinates. If the velocity gradients are not very large, then one can confine oneself to only the first derivatives in this relationship, while the very dependence of σ'_{ik} on $\partial u_i/\partial x_k$ in this approximation can be assumed to be linear.
(b) Terms independent of $\partial u_i/\partial x_k$ should not appear in the expression for σ'_{ik}, since $\sigma'_{ik} = 0$ for $\mathbf{u}(\mathbf{x}, t) \equiv \text{const}$.
(c) Evidently, $\sigma'_{ik} = 0$ for an eddy rotation of the fluid with constant angular velocity $\boldsymbol{\Omega}$, which corresponds to the velocity field $\mathbf{u} = \boldsymbol{\Omega} \times \mathbf{r}$. Linear combinations of the derivatives $\partial u_i/\partial x_k$, vanishing for $\mathbf{u} = \boldsymbol{\Omega} \times \mathbf{r}$, are $\partial u_i/\partial x_k + \partial u_k/\partial x_i$, which must determine σ'_{ik}.

The most general form of a rank-two tensor satisfying (a)–(c) is

$$\sigma'_{ik} = \eta \left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} - \frac{2}{3} \delta_{ik} \frac{\partial u_s}{\partial x_s} \right) + \zeta \delta_{ik} \frac{\partial u_s}{\partial x_s}, \quad (19.5)$$

with coefficients η and ζ independent of the velocity. (The last statement follows from the fluid's isotropy as a medium whose properties are described merely by scalar quantities, in our case by η and ζ.) Terms in (19.5) are grouped in such a way that the convolution of the tensor appearing in the brackets, i.e., the sum of its diagonal terms (the tensor trace), vanishes. The quantities η and ζ are called the *viscosity coefficients*, and ζ is often called the *coefficient of the second viscosity*. Below we show that they are both positive.

The equations of motion of a viscous fluid are now obtained by adding expression $\partial \sigma'_{ik}/\partial x_k$ to the right-hand side of (19.1), which, given the continuity equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_s}{\partial x_s} \equiv \frac{d\rho}{dt} + \rho \frac{\partial u_s}{\partial x_s} = 0, \quad (19.6)$$

can be written in the form

$$\rho\left(\frac{\partial u_i}{\partial t}+u_s\frac{\partial u_i}{\partial x_s}\right)=-\frac{\partial p}{\partial x_i}+\frac{\partial}{\partial x_k}\eta\left(\frac{\partial u_i}{\partial x_k}+\frac{\partial u_k}{\partial x_i}-\frac{2}{3}\delta_{ik}\frac{\partial u_s}{\partial x_s}\right)+\frac{\partial}{\partial x_i}\left(\zeta\frac{\partial u_s}{\partial x_s}\right). \tag{19.7}$$

This is the most general form of the equations of motion of a viscous fluid, in which the quantities η and ζ, in general, depend on pressure and temperature, and therefore they cannot be taken out of the differentiation. However, in most cases one can neglect this dependency. Then (19.7) can be rewritten in the vector form

$$\rho\left[\frac{\partial \mathbf{u}}{\partial t}+(\mathbf{u}\nabla)\mathbf{u}\right]=-\nabla p+\eta\Delta\mathbf{u}+\left(\zeta+\frac{\eta}{3}\right)\nabla\,\mathrm{div}\,\mathbf{u}. \tag{19.8}$$

This is the famous *Navier–Stokes equation*, which is used in an overwhelming number of cases to describe the motion of a viscous fluid (C.R. Navier, 1827, and G.G. Stokes, 1845).

In the description of essentially subsonic flows, the fluid can be regarded as incompressible, and then the last term on the right-hand side of (19.8) can be neglected. In this case, the Navier–Stokes equation is usually written as

$$\frac{\partial \mathbf{u}}{\partial t}+(\mathbf{u}\nabla)\mathbf{u}=-\frac{1}{\rho}\nabla p+\nu\Delta\mathbf{u}, \tag{19.9}$$

where the quantity $\nu=\eta/\rho$ is called *kinematic viscosity*. To a large extent the kinematic viscosity determines the dissipation rate of the kinetic energy of the fluid, i.e., the efficiency of its internal friction. The quantity η itself is described as the *dynamic viscosity*. In this relation we present the comparison table of the quantities η and ν for some liquids and gases (at the temperature of 20 °C):

	η (g/s·cm)	ν (cm^2/s)
Water	0.010	0.01
Air	1.8×10^{-4}	0.15
Alcohol	0.018	0.022
Glycerin	8.5	6.8
Mercury	0.0156	0.0012

In particular, this shows that although the dynamic viscosity of water is greater than the dynamic viscosity of air by almost two orders of magnitude, the effective internal friction of water is over an order of magnitude smaller than the internal friction of air, all other things being equal. Note also that the dynamic viscosity of gases at a given temperature is independent of pressure, while the kinematic viscosity, respectively, is inversely proportional to pressure.

The Poisson equation for a homogeneous incompressible ideal fluid ($\rho=\rho_0=$ const)

$$\Delta p=-\rho_0\frac{\partial u_i}{\partial x_k}\frac{\partial u_k}{\partial x_i}=-\rho_0\frac{\partial^2(u_iu_k)}{\partial x_k\partial x_i} \tag{19.10}$$

is used to reconstruct the pressure from the velocity field. It remains valid for a viscous fluid as well since it is obtained by applying the operator div to (19.9) for $\rho = \rho_0 = \text{const}$.

19.2 Formulation of Boundary Conditions

We confine ourselves to considering boundary conditions for flows of an incompressible fluid ($\operatorname{div} \mathbf{u} = 0$). In this case the viscous stress tensor and the stress tensor itself take the following simple form:

$$\sigma'_{ik} = \eta \left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} \right), \tag{19.11}$$

$$\sigma_{ik} = -\delta_{ik} p + \sigma'_{ik} = -\delta_{ik} p + \eta \left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} \right). \tag{19.12}$$

The most typical examples of boundary conditions are those of (a) a solid wall, (b) an interface between two immiscible liquids, and (c) free surface of a fluid. In the case (a) the fluid sticks to the wall due to the forces of molecular cohesion. Therefore, the fluid velocity at the solid wall is equal to the velocity of the wall itself, and on an immobile wall one has $\mathbf{u} = 0$. Thus, unlike the case of an ideal fluid, in this case not only the normal, but also the tangential components of the fluid velocity on the wall must vanish. This is related to an increase in the order of spatial derivatives in the equations of motion. Consider these boundary conditions in more detail.

(a) The force acting on a surface element $d\mathbf{s}$ is nothing but the momentum flux across this element:

$$\Pi_{ik} = (\rho u_i u_k - \sigma_{ik}) ds_k,$$

where $ds_k = n_k ds$, while $\mathbf{n}$ is the unit normal vector to the surface. To determine the force exerted by the fluid on the solid wall, one needs to pass to a coordinate system in which the wall is at rest: the force is merely equal to the momentum flux only in the case of an immobile wall. Therefore, setting $\mathbf{u} = 0$ in the last formula, we find that the force $\mathbf{F}$ applied to a unit area of the solid surface is

$$F_i = -\sigma_{ik} n_k = p n_i - \sigma'_{ik} n_k. \tag{19.13}$$

The first term is the usual pressure of the fluid, while the second is the force of friction over the surface due to the fluid's viscosity. It should be emphasized that the vector $\mathbf{n}$ in (19.13) is the unit normal to the surface that is *external relative to the fluid*, i.e., internal relative to the solid surface.

(b) At the interface between two immiscible fluids, the velocities of both fluids must be equal, while the forces with which they act upon each other are equal in magnitude and opposite in direction. The second condition means that

$$\sigma_{ik}^{(1)} n_k^{(1)} + \sigma_{ik}^{(2)} n_k^{(2)} = 0.$$

The normal vectors $\mathbf{n}^{(1)}$ and $\mathbf{n}^{(2)}$ to the surfaces of fluids 1 and 2 have mutually opposite directions. Therefore, setting $\mathbf{n}^{(1)} = -\mathbf{n}^{(2)} = \mathbf{n}$, the second boundary condition can be rewritten in the form

$$\sigma_{ik}^{(1)} n_k = \sigma_{ik}^{(2)} n_k. \tag{19.14}$$

(c) At the free surface the tension vanishes, i.e.,

$$\sigma_{ik} n_k = \sigma'_{ik} n_k - p n_i = 0. \tag{19.15}$$

19.3 Dissipation of Kinetic Energy in an Incompressible Fluid

The total kinetic energy of an incompressible fluid ($\operatorname{div} \mathbf{u} = 0$) is equal to

$$E_{\text{kin}} = \frac{1}{2} \int_V \rho u^2 dV,$$

where V is the total volume occupied by the fluid. If the fluid occupies an unbounded space, it is assumed that the fluid is at rest at infinity. Using the Navier–Stokes equation in the form

$$\frac{\partial u_i}{\partial t} = -u_k \frac{\partial u_i}{\partial x_k} - \frac{1}{\rho} \frac{\partial p}{\partial x_i} + \frac{1}{\rho} \frac{\partial \sigma'_{ik}}{\partial x_k}$$

and the equation of mass conservation

$$\frac{\partial \rho}{\partial t} + u_k \frac{\partial \rho}{\partial x_k} = 0,$$

we calculate the time derivative of the kinetic energy per unit volume of the fluid:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2\right) = \rho u_i \frac{\partial u_i}{\partial t} + \frac{1}{2} u^2 \frac{\partial \rho}{\partial t}.$$

As a result we obtain (complete Exercise 1)

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2\right) = -\operatorname{div}\left[\rho \mathbf{u}\left(\frac{1}{2}u^2 + \frac{p}{\rho}\right) - \mathbf{u} * \sigma'\right] - \sigma'_{ik} \frac{\partial u_i}{\partial x_k}, \tag{19.16}$$

where $\mathbf{u} * \boldsymbol{\sigma}'$ stands for the vector with components $u_i \sigma'_{ik}$.

The first term in the square brackets coincides with the energy flux of an ideal fluid (see Chap. 2), created by the usual mass transfer of the fluid during its motion.

But the second term $\mathbf{u} * \boldsymbol{\sigma}'$ is the energy flux due to the internal friction, since the presence of viscosity leads to the appearance of momentum flux σ'_{ik} (see (19.3)), while the momentum transfer entails the transfer of energy which is equal to the product of the momentum flux by the velocity. And finally, the last term in the right-hand side of (19.16) describes the *kinetic energy dissipation per unit volume, i.e., its conversion into heat.*

Integrating (19.16) over the entire volume of the fluid and employing the boundary conditions (a) or vanishing of the velocity at infinity, we obtain the dissipation rate of the entire kinetic energy of the fluid or, equivalently up to sign, the rate of heat generation due to viscosity:

$$\dot{E}_{\text{kin}} = -\int\limits_V \sigma'_{ik} \frac{\partial u_i}{\partial x_k} dV = -\frac{1}{2} \int\limits_V \sigma'_{ik} \left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} \right) dV$$

(the tensor σ'_{ik} is symmetric). Substituting (19.11) into the latter formula we find that the dissipation rate of kinetic energy of all the fluid is given by the formula

$$\dot{E}_{\text{kin}} = -\frac{\eta}{2} \int\limits_V \left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} \right)^2 dV. \tag{19.17}$$

This implies that $\eta > 0$, *since the dissipation* $\dot{E}_{\text{kin}} < 0$.

19.4 Heat Transfer in a Compressible Fluid

The system consisting of the Navier–Stokes equation (19.8) for a compressible fluid and the continuity equation

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{u}) \equiv \frac{d\rho}{dt} + \rho \operatorname{div} \mathbf{u} = 0 \tag{19.18}$$

closes up by the heat transfer equation. The latter can be derived from the following considerations. Let $dQ = \rho T ds$ be the heat influx per unit volume of fluid during the time dt, where s is the specific entropy, i.e., the entropy per unit mass of the fluid, T is the absolute temperature. Then the equation of heat transfer can be written in the form

$$\rho T \frac{ds}{dt} = \frac{dQ}{dt},$$

where on the right-hand side one has the heat source generated by the dissipation of kinetic energy together with the molecular thermal conductivity of the fluid itself. The dissipation term, according to (19.16), is equal to $\sigma'_{ik} \partial u_i / \partial x_k$. The density of the heat flow transferred via thermal conductivity according to the Fourier law is equal to $\mathbf{q} = -\varkappa \nabla T$. The positive quantity $\varkappa$ is called the *thermal conductivity* *coefficient* or simply *thermal conductivity*. Then $\operatorname{div}\mathbf{q} = -\operatorname{div}(\varkappa\nabla T)$ is the heat outflux of a fixed unit volume per unit time, and the heat balance equation becomes

$$\rho T\frac{ds}{dt} = \sigma'_{ik}\frac{\partial u_i}{\partial x_k} - \operatorname{div}\mathbf{q}$$

or

$$\rho T\left(\frac{\partial s}{\partial t} + (\mathbf{u}\nabla)s\right) = \sigma'_{ik}\frac{\partial u_i}{\partial x_k} + \operatorname{div}(\varkappa\nabla T). \tag{19.19}$$

Equation (19.19) is called the *general equation of heat transfer.*

The heat generation from internal friction with the help of (19.5) can be written as

$$\sigma'_{ik}\frac{\partial u_i}{\partial x_k} = \eta\frac{\partial u_i}{\partial x_k}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} - \frac{2}{3}\delta_{ik}\frac{\partial u_s}{\partial x_s}\right) + \zeta\frac{\partial u_i}{\partial x_k}\delta_{ik}\frac{\partial u_s}{\partial x_s}.$$

It is easy to check that the first term on the right-hand side of this equation is identical to the expression

$$\frac{\eta}{2}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} - \frac{2}{3}\delta_{ik}\frac{\partial u_s}{\partial x_s}\right)^2,$$

while the second one is

$$\zeta\frac{\partial u_i}{\partial x_k}\delta_{ik}\frac{\partial u_s}{\partial x_s} = \zeta\left(\frac{\partial u_s}{\partial x_s}\right)^2 = \zeta(\operatorname{div}\mathbf{u})^2.$$

As a result, Eq. (19.19) takes the form

$$\rho T\left(\frac{\partial s}{\partial t} + (u\nabla)s\right) = \operatorname{div}(\varkappa\nabla T) + \frac{\eta}{2}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} - \frac{2}{3}\delta_{ik}\frac{\partial u_s}{\partial x_s}\right)^2 + \zeta(\operatorname{div}\mathbf{u})^2. \tag{19.20}$$

With the help of (19.18) and (19.20) it is easy to show (do Exercise 3) that

$$\frac{\partial(\rho s)}{\partial t} = -\operatorname{div}(\rho s\mathbf{u}) + \frac{1}{T}\operatorname{div}(\varkappa\nabla T) + \frac{\eta}{2T}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} - \frac{2}{3}\delta_{ik}\frac{\partial u_s}{\partial x_s}\right)^2 + \frac{\zeta}{T}(\operatorname{div}\mathbf{u})^2. \tag{19.21}$$

Let us consider an unbounded volume of fluid at rest ($\mathbf{u} = 0$) and uniformly heated ($\nabla T = 0$) at infinity. Integrating the latter equation over the entire volume of the fluid, after passing to the integral over the surface at infinity, we find that the contribution of the first term on the right-hand side vanishes. The integral of the second term is transformed as follows:

$$\int\frac{1}{T}\operatorname{div}(\varkappa\nabla T)dV = \int\operatorname{div}\left(\frac{\varkappa\nabla T}{T}\right)dV + \int\frac{\varkappa(\nabla T)^2}{T^2}dV,$$

and by the second condition at infinity the first term vanishes. As a result we obtain

$$\frac{\partial}{\partial t}\int \rho s dV = \int \frac{\varkappa(\nabla T)^2}{T^2}dV + \int \frac{\eta}{2T}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} - \frac{2}{3}\delta_{ik}\frac{\partial u_s}{\partial x_s}\right)^2 dV$$
$$+ \int \frac{\zeta}{T}(\operatorname{div}\mathbf{u})^2 dV. \qquad (19.22)$$

The total entropy of the system which does not undergo any exterior influence can only increase, i.e., the right-hand side of (19.22) must be positive. In addition, in each term of this sum the integrand may be different from zero even when the other two integrals vanish. Therefore, each of these integrals must always be positive. Hence, this implies positivity of the second viscosity coefficient, as well as the positivity of η and $\varkappa$ already known to us.

19.5 Heat Transfer in an Incompressible Fluid

If the fluid velocity is small compared to the speed of sound, then the pressure changes resulting from the fluid's movement are so small that one can neglect changes in density (and other thermodynamical quantities) caused by them, unless we are talking about global flows of a rotating fluid. However, the nonuniformly heated fluid is not quite incompressible at the same time. One cannot ignore the density changes due to changes in temperature, even if the speed of a nonuniformly heated fluid is small, and, consequently, the density cannot be assumed to be constant. Therefore, in this case, the pressure must be regarded as constant in determining the derivatives of the thermodynamic quantities.

Then

$$\frac{\partial s}{\partial t} = \left(\frac{\partial s}{\partial T}\right)_p \frac{\partial T}{\partial t}, \qquad \nabla s = \left(\frac{\partial s}{\partial T}\right)_p \nabla T,$$

and since $T(\partial s/\partial T)_p = C_p$ is the specific heat at constant pressure, then

$$T\frac{\partial s}{\partial t} = C_p\frac{\partial T}{\partial t}, \qquad T\nabla s = C_p \nabla T.$$

Equation (19.19) now assumes the form

$$\rho C_p\left(\frac{\partial T}{\partial t} + (\mathbf{u}\nabla)T\right) = \operatorname{div}(\varkappa\nabla T) + \sigma'_{ik}\frac{\partial u_i}{\partial x_k}. \qquad (19.23)$$

Assume further that $\rho = \rho_0 + \rho'$, $T = T_0 + T'$, where ρ_0 and T_0 are average values of density and temperature, while their fluctuations, which are small compared with the mean values, are connected by the linear relation $\rho'/\rho_0 = -T'/T_0$. The quantity ρ in (19.22) can be replaced by ρ_0 and the fluid is assumed incompressible ($\operatorname{div}\mathbf{u} = 0$, see the Oberbeck–Boussinesq approximation in Chap. 13). Now by substituting formula (19.11) into (19.23), we obtain the heat transfer equation in an incompressible fluid:

$$\frac{\partial T}{\partial t} + (\mathbf{u}\nabla)T = k\Delta T + \frac{\nu}{2C_p}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right)^2, \tag{19.24}$$

where $\nu = \eta/\rho_0$ is the kinematic viscosity, and instead of $\varkappa$ it involves the *temperature conductivity coefficient* $k = \varkappa/\rho_0 C_p$. Since the temperature appears in (19.23) and (19.24) only under the differentiation, T is not replaced by T'. In practical calculations the last term in (19.24) is generally not accounted for because of its smallness in comparison with the effect of molecular heat transfer.

In conclusion, it is worth noting that the Navier–Stokes equations were derived by Bogolyubov by expanding the Liouville equation, the basic equation of statistical mechanics (more precisely, the system of Bogolyubov equations equivalent to it). First, at the kinetic stage it was done by expansion in the small parameter $\varepsilon = \tau/t_0$ (where τ is the interaction time of molecules, t_0 is the time spent by a molecule for the mean free path). Later, at the hydrodynamical stage, this was done by expansion in the parameter $\theta = t_0/t_*$ (where t_* is the characteristic time for changes in the macroscopic motion). In the first-order expansion in θ, one obtains the Euler equations of motion of an ideal fluid, while in the second-order one gets the Navier–Stokes equations with two viscosity coefficients and the Fourier thermal conductivity law $\mathbf{q} = -\varkappa\nabla T$. These issues are discussed in detail in the monograph (Uhlenbeck and Ford, 1963).

19.6 Exercises

1. Derive (19.16) from the equations of motion of an incompressible fluid.
2. Formulate the local energy conservation law for a compressible fluid through its equations of motion (19.6), (19.7) and (19.19), by using the thermodynamic relation (the first law of thermodynamics) $d\varepsilon = Tds - pdV = Tds + (p/\rho^2)d\rho$, where ε is the internal energy per unit mass of the fluid.

   Hint: see Section 49 of the above-mentioned book by Landau and Lifschitz (1986).

   Answer:

$$\frac{\partial}{\partial t}\left(\frac{\rho u^2}{2} + \rho\varepsilon\right) = -\mathrm{div}\left[\rho\mathbf{u}\left(\frac{1}{2}u^2 + \frac{p}{\rho}\right) - \mathbf{u} * \sigma' - \varkappa\nabla T\right]. \tag{19.25}$$

3. Derive (19.21) from (19.18) and (19.20).

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Chapter 20
Friction Mechanisms in Global Geophysical Flows; Quasi-geostrophic Equation for Transformation of Potential Vorticity

One of the main features of the dynamics of viscous global geophysical flows is that the dissipation of their kinetic energy is mainly due to friction of free atmosphere over so-called geophysical boundary layers (GBL). Under free atmosphere one understands the atmosphere's areas that are remote from solid boundaries or from sharp fluctuations of horizontal velocity. The friction is caused by the exchange of momentum between these boundary layers and the free atmosphere. In the GBL themselves velocity gradients are intensified, and hence the internal friction is strengthened. This leads to disruption in quasi-hydrostatic and quasi-geostrophic equilibria. Moreover, the effects of internal friction manifest differently in neighborhoods of the horizontal and vertical GBLs. Based on this it is convenient to adopt the following definition of GBLs. *A geophysical boundary layer (GBL)* is a region in which the conditions of quasi-hydrostatic and quasi-geostrophic equilibria are violated under the influence of viscosity forces.

This allows one to avoid a somewhat cumbersome procedure of expanding the original hydrodynamical equations in the Rossby parameter, which is usually used for reduction of the problem, and to immediately formulate the required equations of motion. The GBL effect manifests itself in the most transparent way in the motion of a homogeneous incompressible fluid. Therefore, our further considerations are related to laboratory analogues of global geophysical flows that are modeled in rotating annular tanks filled with water. In this case the equations of motion are formulated in vector notation as follows:

$$2\mathbf{\Omega} \times \mathbf{u} = -\frac{\nabla p}{\rho} + \mathbf{f}_\nu, \tag{20.1}$$

$$\nabla \mathbf{u} \equiv \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \tag{20.2}$$

Here $\mathbf{\Omega}$ is the angular velocity of the general rotation of a fluid of constant density ρ, $\mathbf{u}$ is the velocity field of the flow in the GBL, p is a pressure deviation from its hydrostatic value, and $\mathbf{f}_\nu = \nu \Delta \mathbf{u}$ are viscosity forces ($\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$), where ν is the kinematic viscosity of the fluid.

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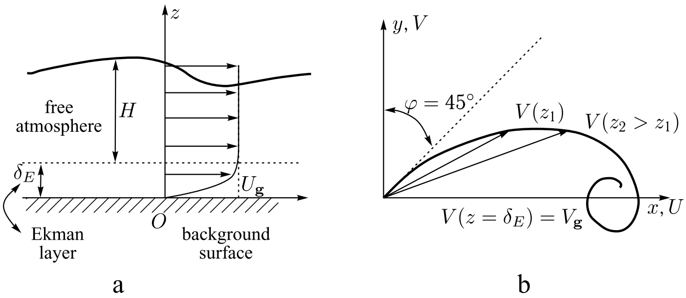

Fig. 20.1 (**a**) Vertical cross-section of the "laboratory atmosphere" and the profile of the longitudinal velocity component U in the plane (x, z), where the x axis coincides with the direction of the geostrophic wind $\mathbf{V}_g$. (**b**) The wind rotation with the altitude: the Ekman spiral in the horizontal plane

20.1 Ekman Planetary Boundary Layer

In the vicinity of the horizontal boundary of a laboratory atmosphere (Fig. 20.1a), the characteristic horizontal scale for velocity changes is much greater than its vertical scale. Therefore, in the expression for the viscous forces $\mathbf{f}_\nu = \nu(\partial^2\mathbf{u}/\partial x^2 + \partial^2\mathbf{u}/\partial y^2 + \partial^2\mathbf{u}/\partial z^2)$ one can neglect the first two terms. In addition, as we shall see below, since the thickness of the horizontal GBL is much less than the atmosphere's height, the geostrophic wind $\mathbf{v}_g(x, y, t) = u_g(x, y, t)\mathbf{i} + v_g(x, y, t)\mathbf{j}$, satisfying the relation

$$2\boldsymbol{\Omega} \times \mathbf{v_g} = -\frac{\nabla p}{\rho} \quad \left(\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y}\right),$$

can be assumed to be independent of the vertical coordinate. Then, by subtracting the last equation from (1), for the horizontal velocity component $\mathbf{v}(x, y, z, t) = u(x, y, z, t)\mathbf{i} + v(x, y, z, t)\mathbf{j}$, we obtain the equation

$$2\boldsymbol{\Omega} \times (\mathbf{v} - \mathrm{v_g}) = \nu\frac{\partial^2\mathbf{v}}{\partial z^2} \tag{20.3}$$

with the boundary conditions of adhesion at the lower boundary of GBL and turning $\mathbf{v}$ into the geostrophic wind at the upper boundary:

$$\mathbf{v}|_{z=0} = 0, \qquad \mathbf{v}|_{z=\infty} = \mathbf{v_g}. \tag{20.4}$$

The top condition is posed at infinity, since the main contribution to the integral $\int_0^\infty \operatorname{div}\mathbf{v}\,dz$, which we are interested in below, comes from the actual thickness of the horizontal boundary layer.

By taking into account $\mathbf{\Omega} = (0, 0, \Omega)$, the solution of the boundary problem (20.3) and (20.4) can be written in the form (do Exercise 1):

$$\mathbf{v}(x, y, z, t) = \mathbf{v_g}(x, y, t)\big[1 - \exp(-z/\delta_E)\cos(z/\delta_E)\big] \\ + \mathbf{k} \times \mathbf{v_g}(x, y, t)\exp(-z/\delta_E)\sin(z/\delta_E), \tag{20.5}$$

where $\mathbf{k}$ is the vertical unit vector, while $\delta_E = \sqrt{\nu/\Omega}$ is the actual thickness of the horizontal GBL, called the *Ekman layer*, or *planetary boundary layer* (PBL). Note that wind in the free atmosphere controls the time dependence of the horizontal wind in the Ekman layer.

Finally, make the substitution (20.5) in the condition of zero divergence of the three-dimensional flow in the Ekman layer which is integrated over the height, i.e.,

$$w_E = -\int_0^\infty \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) dz.$$

We obtain the expression for the velocity vertical component at the upper boundary of the PBL:

$$w_E = \delta_E \left(\frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y}\right). \tag{20.6}$$

Precisely because of the presence of such a nonzero vertical flux at the upper boundary of PBL there is an exchange of horizontal momentum between the PBL and the free atmosphere, which leads to slowing down the motion of the latter. We discuss this in more detail in Sect. 20.3. Now we only mention two points.

First, in laboratory experiments with water as the working fluid $\nu = 0.01$ cm^2/s, while $\Omega = 0.1$–1.0 s^{-1}. Therefore, the thickness of the laboratory Ekman layers has a range within a few mm. Formally putting for the real atmosphere that $\nu = 0.15$ cm^2/s and $\Omega \sim 10^{-4}$ s^{-1}, we obtain $\delta_E \approx 40$ cm. This is the thickness of the so-called *laminar Ekman layer*, which is observed in laboratory experiments. The PBL of a real atmosphere is dominated by fully developed three-dimensional small-scale turbulence, i.e., chaotic vortex motions whose scale ranges from centimeters to tens of meters. These vortices are fed by the kinetic energy of large-scale currents, whose inhibition is far more effective than under the influence of molecular viscosity. The simplest, although somewhat naive but natural way of accounting for such an inhibition is to replace the molecular viscosity ν by the so-called coefficient of turbulent viscosity ν_{turb}, which in the atmosphere exceeds ν by six to seven orders of magnitude, according to various empirical estimates. In this case, depending on the intensity of the small-scale turbulence in the planetary boundary layer, its thickness ranges from several hundred meters to one kilometer.

Secondly, as shown in Fig. 20.1b, which presents the Ekman spiral, i.e., the hodograph of the velocity vector (20.5), in a PBL there is a wind rotation with height, and where rotation angle $\alpha = 45°$ provided that $\nu_{\text{turb}} = \text{const}$, i.e., it does not depend on the height (do Exercise 2).

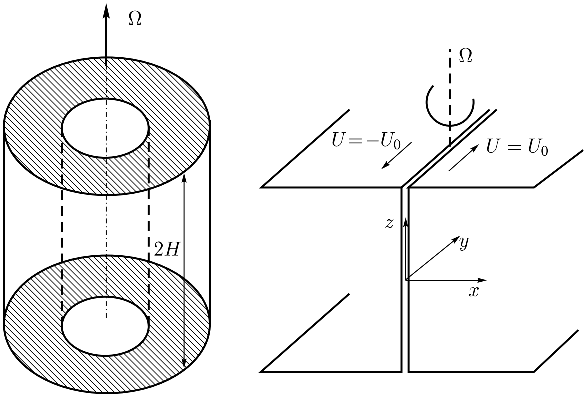

Fig. 20.2 A schematic presentation of the experimental setup for the simulation of vertical GBLs. The cylindrical tank with a fluid rotates with constant angular velocity Ω around the vertical axis of symmetry. The interior and exterior ends rotate in the opposite directions

20.2 The Praudman–Stewartson Layers

In the vicinity of the vertical walls, Eqs. (20.1) and (20.2) describe a different mechanism for the dissipation of kinetic energy. Following Stewartson (see Praudman 1956; Stewartson 1957; Greenspan 1968; Dolzhansky 1999), in order to understand this, let us consider the problem of motion of a viscous incompressible homogeneous rotating fluid between two horizontal planes, which is induced by the strictly antisymmetric motion of each of the half-planes towards each other (Fig. 20.2). It is assumed that the horizontal half-planes, located on the same side of the vertical plane of symmetry $y = 0$, move in the same direction with the same velocity $u_0 = u_0(y)$. In this case, the flow characteristics are independent of the longitudinal (azimuthal) coordinate x, and the boundary value problem is given by equations

$$-2\Omega v = \nu\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right), \tag{20.1$'$}$$

$$2\Omega u = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right), \tag{20.1$''$}$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right), \tag{20.1$'''$}$$

$$0 = \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \tag{20.2$'$}$$

with the conditions of sticking to the horizontal walls and regularity at $y = \pm\infty$, p is the deviation from the hydrostatic pressure.

Obviously, the problem is antisymmetric with respect to the vertical plane $y = 0$ and symmetric with respect to the horizontal plane $z = 0$, located midway between the moving planes. Let the latter distance be equal to $2H$. Then exclude p from

(20.1″) and (20.1‴), and introduce in (20.2′) the "vertical" stream function Ψ according to the relations

$$v = \frac{\partial \Psi}{\partial z}, \qquad w = -\frac{\partial \Psi}{\partial y}.$$

Now by making the equation dimensionless with the help of the length scale H and the characteristic velocity U, the boundary value problem for the unknown functions $u(y, z)$ and $\Psi(y, z)$ can be defined by equations

$$2E^{-1}\frac{\partial u}{\partial z} = \Delta^2 \Psi, \tag{20.7}$$

$$-2E^{-1}\frac{\partial \Psi}{\partial z} = \Delta u \tag{20.8}$$

with the boundary conditions

$$\frac{\partial \Psi}{\partial y} = \frac{\partial \Psi}{\partial z} = 0 \quad \text{for } z = \pm 1, \tag{20.9}$$

$$u = u_0(y) \quad \text{for } z = \pm 1. \tag{20.10}$$

Here we do not introduce new notation for the dimensionless quantities, while the constant $E = \nu/\Omega H^2 = \delta_E^2/H^2$, called the Ekman number, is a small parameter due to the above estimates.

To simulate the vertical wall located in the plane $y = 0$, take as an external drive the step-type velocity $u_0(y) = y/|y|$ $(0 < |y| < \infty)$. The problem is solved by the method of discrete-continuous Fourier expansions in trigonometric and hyperbolic functions, which requires a rather cumbersome calculation procedure. We present below without proof Stewartson's formulas, describing the flow away from the horizontal walls and which are of the main interest to us:

$$u(y, z) = F_{St}(y, z), \qquad \Psi(y, z) = \Psi_{St}(y, z), \tag{20.11}$$

where

$$F_{St}(y, z) = \operatorname{sign}(y)\big[1 - \exp\big(-|y|/E^{1/4}\big) - E^{1/6}\Phi(y, z)\big], \tag{20.12}$$

$$\Phi(y, z) = \frac{2}{3}\sum_{n=1}^{\infty} \frac{(-1)^n \cos(n\pi z)}{(2\pi n)^{2/3}} \times \left[\exp\big(-\gamma_n |y|\big) - 2\exp\left(\frac{-\gamma_n |y|}{2}\right)\cos\left(\frac{\sqrt{3}}{2}\gamma_n |y| - \frac{\pi}{3}\right)\right], \tag{20.13}$$

$$\Psi_{St}(y, z) = \operatorname{sign}(y) E^{1/2}\left[\frac{1}{2} z \exp\big(-|y|/E^{1/4}\big) + \Gamma(y, z)\right], \tag{20.14}$$

$$\Gamma(y, z) = \frac{1}{3\pi}\sum_{n=1}^{\infty} \frac{(-1)^n \sin(n\pi z)}{n}$$

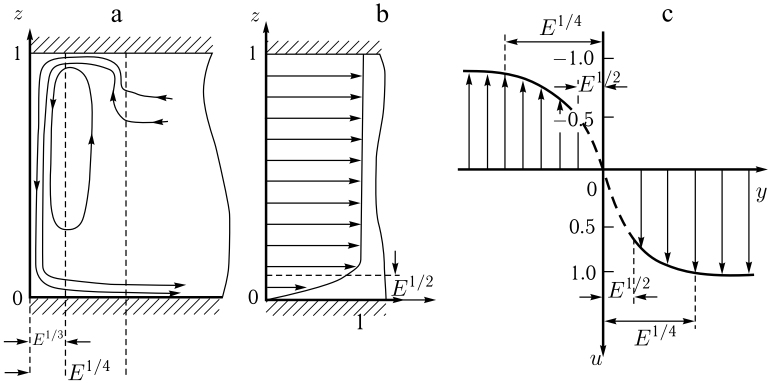

Fig. 20.3 The transverse circulation (**a**), the vertical profile of the longitudinal component of the velocity u far away from the discontinuity plane $y = 0$ (**b**), and the horizontal profile of the longitudinal component of the velocity away from the horizontal walls (**c**)

$$\times\left[\exp\bigl(-\gamma_n|y|\bigr) + 2\exp\bigl(-\gamma_n|y|/2\bigr)\cos\left(\frac{\sqrt{3}}{2}\gamma_n|y|\right)\right]. \qquad (20.15)$$

Here $\gamma_n = (2\pi n/E)^{1/3}$.

The pattern for transverse and longitudinal flows is shown in Fig. 20.3. Cross-linking of the horizontal and vertical GBL is made in Dolzhansky (1999). The formulas presented indicate that in the vicinity of the vertical wall two Praudman–Stewartson layers are formed: the outer one of thickness $\delta_{St} = E^{1/4}H = \sqrt{\delta_E H}$ and the inner one of thickness $\delta_{in} = E^{1/3}H = \sqrt[3]{\delta_E^2 H}$. In these layers there are intensive transverse circulation cells of opposite directions, providing smoothing of the step shear of the longitudinal velocity component. In essence, this result means that the *shear width of the horizontal viscous geophysical flows cannot be smaller than the thickness of the Praudman–Stewartson outer layer*. In the atmosphere, the Praudman–Stewartson layers are formed in the vicinity of fronts that are boundaries of large-scale air masses with very different dynamical and thermodynamical characteristics. The formation of strong vertical flows in such areas makes it rather difficult to describe their dynamics since the motion is no longer quasi-two-dimensional. Therefore, it is also inadmissible to consider fronts as singular solutions of two-dimensional hydrodynamic equations, as it is occasionally done.

20.3 The Quasi-geostrophic Equation for Transformation of Potential Vorticity of a Barotropic Viscous Atmosphere

Now, having learned the features of kinetic energy dissipation in a viscous atmosphere, we can easily derive the equations for its global barotropic motions. To describe the behavior of the free atmosphere we again use the shallow water approximation of the hydrodynamic equations of a rotating fluid, which after including the viscosity can be written as

$$\frac{du}{dt} - fv = -g\frac{\partial H}{\partial x} + \nu\Delta u, \tag{20.16}$$

$$\frac{dv}{dt} + fu = -g\frac{\partial H}{\partial y} + \nu\Delta v, \tag{20.17}$$

$$\frac{dH}{dt} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = w_E. \tag{20.18}$$

Recall that here u and v are the longitudinal (zonal) and transverse (meridional) components of horizontal velocity of the free atmosphere, $f = f(y)$ is the Coriolis parameter, and $H = H(x, y, t)$ is the height of the free surface of a barotropic atmosphere, measured from the top of the Ekman layer. Equation (20.18) of mass conservation for the free atmosphere is written by taking into account the fact that at its lower boundary the vertical velocity is $w_E \neq 0$.

Subtracting now Eq. (20.16) differentiated in y from Eq. (20.17) differentiated in x, we obtain the equation for the vertical vorticity $\omega = \partial v/\partial x - \partial u/\partial y$:

$$\frac{d\omega}{dt} + (f + \omega)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \frac{df}{dy}v = \nu\Delta\omega.$$

After getting rid of the divergence in the horizontal velocity by using (20.18) we have

$$\frac{d\omega}{dt} - \frac{(f + \omega)}{H}\frac{dH}{dt} + \frac{df}{dy}v = \nu\Delta\omega - \frac{w_E}{H}. \tag{20.19}$$

Now, use the relations of the geostrophic wind

$$v = \frac{g}{f_0}\frac{\partial H}{\partial x} \doteq \frac{\partial \psi}{\partial x}, \qquad u = -\frac{g}{f_0}\frac{\partial H}{\partial y} \doteq -\frac{\partial \psi}{\partial y} \quad (\psi \doteq gH/f_0), \tag{20.20}$$

where f_0 is the average value of the Coriolis parameter. Then by formula (20.6) and using (20.20) we have

$$w_E = \delta_E \Delta\psi, \tag{20.21}$$

and find that the quasi-geostrophic equation of evolution (or transformation) of the potential vorticity of a viscous barotropic atmosphere is written as (cf. (7.6)):

$$\frac{d}{dt}\left(\Delta\psi - \frac{f_0^2}{gH_0}\psi\right) + \beta\frac{\partial\psi}{\partial x} = \nu\Delta^2\psi - \frac{f_0\delta_E}{H_0}\Delta\psi.$$

Here $\beta = df/dy$. In addition, for deriving this equation we have also taken into account that $\delta_E \ll H_0$, where H_0 is the average height of the atmosphere, so that if H is not differentiated, it can be replaced by H_0.

Now recalling that $\sqrt{gH_0}/f_0 = L_0$ is the Obukhov scale, $\delta_E H_0 = \delta_{St}^2$ and $\delta_E = \sqrt{2\nu/f_0}$, we obtain the traditional form of the required equation:

$$\frac{\partial}{\partial t}\left(\Delta\psi - L_0^{-2}\psi\right) + [\psi, \Delta\psi] + \beta\frac{\partial\psi}{\partial x} = \nu\Delta^2\psi - \lambda\Delta\psi, \tag{20.22}$$

where $[a, b] = \partial a/\partial x \cdot \partial b/\partial y - \partial a/\partial y \cdot \partial b/\partial x$ is the Jacobian of the scalar functions $a = a(x, y)$ and $b = b(x, y)$, while $\lambda = 2\nu/\delta_{St}^2$ is called the effective friction coefficient of the atmosphere over the underlying surface. It is worth mentioning that in its most general form the equation for transformation of potential vorticity was derived in Obukhov (1962).

Thus, in the framework of the assumptions made, *by means of the Ekman boundary layer the Earth's surface inhibits the quasi-two-dimensional geophysical motions according to the dry friction law. The coefficient of this friction is linear in velocity and is determined by the thickness of the Praudman–Stewartson outer layer*. As we shall see below, this very friction exhibits a decisive influence not only on dissipation of the kinetic energy, but also on the mechanisms of barotropic instability and formation of cyclones and anticyclones in the atmosphere. Note that the effect of the Ekman layer on a baroclinic atmosphere can be accounted for by using formula (20.21) as the lower boundary condition for the quasi-geostrophic equation of the potential vorticity of the baroclinic atmosphere (see (9.31)) and taking into account the internal viscosity of the environment.

20.4 Exercises

1. Solve the boundary value problem (20.3) and (20.4) taking into account that $\mathbf{\Omega} = (0, 0, \Omega)$.

   Hint: introduce a complex dependent variable $W = u + iv$ and point the axis Ox in the direction of the geostrophic wind.
2. Construct the hodograph of the horizontal wind velocity within the Ekman layer and show that the wind turns with height up to 45°. In what direction is this rotation going on?

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Chapter 21
Kolmogorov Flow and the Role of Surface Friction

21.1 Formulation of a Linear Stability Problem

The influence of planetary boundary layers (PBLs) on barotropic instability of global atmospheric flows in its purest form can be identified by considering an extremely idealized stability problem for a spatially periodic plane flow of an incompressible fluid with a sinusoidal velocity profile. This problem was proposed by A.N. Kolmogorov in 1960 in his seminar and was already solved the next year in the paper (Meshalkin and Sinai, 1961) in the strictly two-dimensional setting.

Following those authors, we will consider this problem in the quasi-two-dimensional setting. In other words, we will take into account external friction that is linear in velocity and simulates the influence of the bottom on the motion of shallow water, while in atmospheric applications it simulates the PBL impact on quasi-geostrophic flows. (This generalization was done in the work of Bondarenko et al. (1979). Later it turned out to be fundamental and it is described in detail in Gledzer et al. (1981) and Dolzhansky et al. (1990).) To avoid extra complications in dealing with the problem, we will not take into account the Coriolis force, since on the f-plane, i.e., in the absence of the beta-effect, the vorticity flows that we are interested in are described by the usual two-dimensional vorticity equation. In this case the following equations can be taken as the initial ones:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \nu\Delta u - \lambda u + f, \tag{21.1}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} + \nu\Delta v - \lambda v, \tag{21.2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{21.3}$$

Here, the fluid density is set to be equal to one, ν and λ are the coefficients of internal and external tensions, respectively, and $f = a\sin(y/l)$ is an external force creating

F.V. Dolzhansky, *Fundamentals of Geophysical Hydrodynamics*,
Encyclopaedia of Mathematical Sciences 103,
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the initial sinusoidal velocity profile. Others notations are standard. The periodicity conditions in the y-coordinate with period $2\pi l$ and the absence of the total mass transport in the longitudinal direction, i.e.,

$$\int_{-\infty}^{\infty} u(x, y, t)dy = 0 \tag{21.4}$$

are taken as the boundary conditions.

After making the variable dimensionless by using the natural scales of length l and velocity $U = \sqrt{al}$, as well as by introducing the stream function according to the equations

$$u = -\frac{\partial \psi}{\partial y}, \qquad v = \frac{\partial \psi}{\partial x}, \tag{21.5}$$

the vorticity equation for $\Delta\psi = \partial v/\partial x - \partial u/\partial y$, corresponding to the system (1)–(3) with the above-mentioned force f can be written as

$$\frac{\partial \Delta\psi}{\partial t} + [\Delta\psi, \psi] = \widehat{\nu}\Delta^2\psi - \widehat{\lambda}\Delta\psi - \cos y. \tag{21.6}$$

For the dependent and independent dimensionless variables we do not introduce new notations. Note that the inverse dimensionless kinematic viscosity and the coefficient of external friction

$$\widehat{\nu}^{-1} = \frac{Ul}{\nu} \doteq R_\nu, \qquad \widehat{\lambda}^{-1} = \frac{U}{\lambda l} \doteq R_\lambda \tag{21.7}$$

are well-known similarity criteria characterizing the ratios between the nonlinear terms and the forces of internal and external friction, respectively, and are called the *Reynolds numbers of internal and external friction*. The higher the values of the Reynolds numbers, the closer the flow of a viscous fluid to an ideal fluid flow. Note also by comparing (21.6) with (20.22) that Eq. (21.6) can be interpreted as the quasi-geostrophic vorticity equation for an incompressible ($L_0^{-1} = 0$) viscous atmosphere on the f-plane ($\beta = 0$), where the atmosphere undergoes the action of the zonal force of the sinusoidal profile.

The primary flow of the system at hand is given by the equalities:

$$u_0 = \frac{1}{\widehat{\nu} + \widehat{\lambda}} \sin y, \qquad \Delta\psi_0 = -\frac{1}{\widehat{\nu} + \widehat{\lambda}} \cos y, \qquad \psi_0 = \frac{1}{\widehat{\nu} + \widehat{\lambda}} \cos y. \tag{21.8}$$

Now let $\psi = \psi_0 + \varphi(x, y, t)$, where φ is an infinitesimal perturbation of the primary flow. Then the result of the linearization of Eq. (21.6) can be written as

$$\frac{\partial \Delta\varphi}{\partial t} + \frac{1}{\widehat{\nu} + \widehat{\lambda}} \sin y \frac{\partial (\Delta\varphi + \varphi)}{\partial x} = \widehat{\nu}\Delta^2\varphi - \widehat{\lambda}\Delta\varphi. \tag{21.9}$$

The periodicity condition of φ in y allows one to use the Fourier expansion, i.e., to seek a solution in the form

$$\varphi(x, y, t) = e^{\sigma t} \sum_{-\infty}^{\infty} c_n \exp\{i(\alpha x + ny)\}. \tag{21.10}$$

After substituting (21.10) into (21.9) and grouping similar terms for the coefficients c_n, we obtain the system of equations

$$\frac{2(\widehat{\nu} + \widehat{\lambda})}{\alpha}(\alpha^2 + n^2)\left[\sigma + \widehat{\lambda} + \widehat{\nu}(\alpha^2 + n^2)\right]c_n \\ + \left[\alpha^2 - 1 + (n-1)^2\right]c_{n-1} - \left[\alpha^2 - 1 + (n+1)^2\right]c_{n+1} = 0. \tag{21.11}$$

We are interested in the sign of the real part of those values of σ, for which there exists a nontrivial solution of system (21.11) decaying to zero as $|n| \to \infty$. A study of system (21.11) and a description of its critical curves of stability are given in Appendix A. One of approximate expressions for the critical stability curve will be used in the next chapter.

21.2 Application to a Stability Study of Rossby Waves

In connection with barotropic atmospheric instability, Lorenz (1972) examined the stability problem of the elementary Rossby wave

$$\psi_0 = A_0 \cos k_0(x + c_0 t), \tag{21.12}$$

satisfying the quasi-geostrophic equation of barotropic potential vorticity without taking into account two-dimensional compressibility ($L_0^{-1} = 0$):

$$\frac{\partial \Delta \psi}{\partial t} + [\psi, \Delta \psi] + \beta \frac{\partial \psi}{\partial x} = 0. \tag{21.13}$$

To clarify the effect of the Ekman layer on such instability in the work of Dolzhansky (1985) this problem was considered for Eq. (20.22) describing transformation of the potential vorticity

$$\frac{\partial \Delta \psi}{\partial t} + [\psi, \Delta \psi] + \beta \frac{\partial \psi}{\partial x} = \nu \Delta^2 \psi - \lambda \Delta \psi + q, \tag{21.14}$$

in the same approximation in which $\lambda = 2\nu/\delta_{St}^2$, while $q = q_0 \cos k_0(x + ct)$ is the source of vorticity moving in the negative direction of the x-axis with speed c. If $c = c_0$, the source generates the wave (21.12) with amplitude $A_0 = q_0/(\lambda k_0^2 + \nu k_0^4)$.

Passing to the coordinate system moving with the speed $-c_0$, i.e., making the change $X = x + c_0 t$, and then making Eq. (21.14) dimensionless, we obtain the

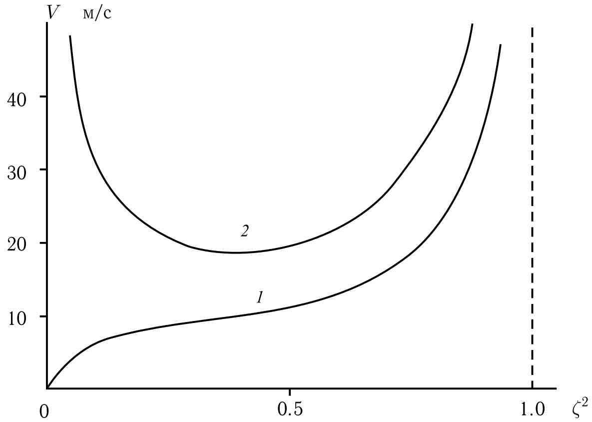

Fig. 21.1 The critical curves for the planetary wave with longitudinal wavenumber $n_0 = 6$: *1* is for the inviscid Lorenz theory and *2* is for the influence of the Ekman layer taken into account

Kolmogorov problem, but already for the equation

$$\frac{\partial \Delta\psi}{\partial t} + [\psi, \Delta\psi] + C_0 \frac{\partial}{\partial X}(\Delta\psi + n_0\psi) = \varkappa\Delta^2\psi - \gamma\Delta\psi + R\cos n_0 X. \quad (21.15)$$

Here $C_0 = c_0/\lambda r$, r should be interpreted as the average radius of a circle of latitude, then $n_0 = k_0 r$ is the number of wavelengths of the function ψ_0 which fit into the average latitudinal circumference $2\pi r$, $\varkappa = \nu/\lambda r^2$, $R = q_0/\lambda^2$, while $\gamma = 1$ or $\gamma = 0$ depending on whether or not the Ekman layer is taken into account.

As described in Appendix A, the procedure for finding the critical curve can be applied to Eq. (21.15) as well. The necessary convergence criteria were formulated and proved in Dolzhansky (1985). The critical curve determined from the first approximation is given by the formula

$$A_{\text{cr}}^2 = \frac{2(\gamma + \varkappa l^2)[(l^2 + n_0^2)^2(\gamma + \varkappa(l^2 + n_0^2))^2 + C_0 n_0^2 l^2]}{n_0^2 l^2 (n_0^4 - l^4)(\gamma + \varkappa(l^2 + n_0^2))}, \quad (21.16)$$

where l is the meridional wavenumber of the infinitesimal perturbation

$$\varphi = e^{i\sigma t} \sum_{n=-\infty}^{\infty} c_n e^{i(nX+ly)},$$

imposed on the main flow.

Critical curves of the "inviscid" and "viscous" Rossby waves when the influence of the PBL is taken into account are shown in Fig. 21.1 in the parameter plane $\varsigma^2 = l^2/n_0^2$ and $V_{\text{cr}} = n_0 r \lambda A_{\text{cr}}$. This figure shows the drastic change in shape of the critical curve once the external friction is taken into account (see also Fig. B.1 in Appendix B). In particular, according to Lorenz (1972) a typical wave with $n_0 = 6$ and amplitude $V = 12$ m/s is unstable. Furthermore, the perturbation with meridional wavenumber $l = 0.5n_0$ has the largest increment. According to the inviscid

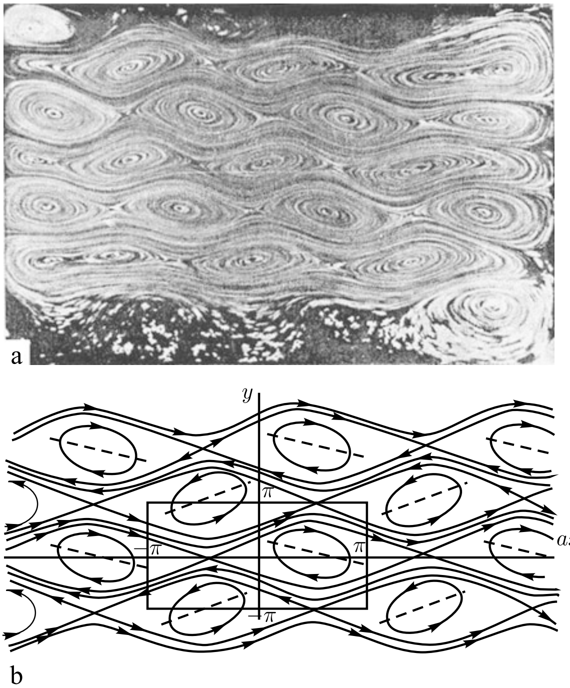

Fig. 21.2 Snapshot of a supercritical regime (**a**) and the streamlines calculated by the Galerkin method (**b**) for the Kolmogorov flow

and viscous theories for such a value of l

$$6.6\text{ m/s} \approx V_L < 12\text{ m/s} < V_D \approx 17.7\text{ m/s},$$

where V_L and V_D are the critical values of velocity according to the inviscid and viscous theories. Note also that the inclusion of only internal viscosity does not stabilize the wave under consideration, although it gives the *finite nonzero threshold for the amplitude value which still does not change the shape of the critical curve*.

21.3 Conclusions

A comparison of the critical curves of the Kolmogorov flow and planetary waves for $\lambda = 0$ and $\lambda \neq 0$ implies that the external friction suppresses the most unstable long wavelength modes of a strictly two-dimensional flow, which results in a fundamental change in the shape of the critical curve. Moreover, according to the nonlinear theory (Klyatskin 1972; Yudovich 1973; Nepomniaschy 1976) all stationary and oscillatory modes that mathematically exist in the supercritical region of a strictly two-dimensional flow turn out to be unstable. This indicates the birth of a chaotic turbulent regime immediately after the loss of stability of the primary regime.

The situation is different in the quasi-two-dimensional Kolmogorov flow. The most unstable mode with a nonvanishing wavenumber α_0, corresponding to the minimum of the critical curve, at low and moderate supercritical values form secondary steady flows, shown in Fig. 21.2 as a result of both nonlinear theory and laboratory experiments. In the next chapter we will see that these conclusions are of a general nature, i.e., they are valid for a two-dimensional flow of an arbitrary velocity profile.

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Chapter 22
Stability of Quasi-two-dimensional Shear Flows with Arbitrary Velocity Profiles

22.1 New Interpretation of the Results in Linear Stability Theory for the Kolmogorov Flow

Equation (B.10) of Appendix B for the critical curve of the Kolmogorov flow, written in terms of the inverse Reynolds numbers $\widehat{\nu} = R_\nu^{-1}$ and $\widehat{\lambda} = R_\lambda^{-1}$,

$$\widehat{\nu} = \sqrt{\frac{\alpha^2(1-\alpha^2)}{2(1+\widehat{\lambda}/\widehat{\nu}+\alpha^2)(\widehat{\lambda}/\widehat{\nu}+\alpha^2)(1+\alpha^2)}} \tag{22.1}$$

in the parameter space $(\alpha, \widehat{\lambda}, \widehat{\nu})$ can be regarded as a critical surface, or the surface of neutral stability of the Kolmogorov flow on which the increments σ of infinitesimal perturbations (21.10) vanish. It is easy to construct such a surface, as shown in Fig. 22.1, by taking into account that the critical curves in the planes $\widehat{\lambda} = 0$ and $\widehat{\nu} = 0$ are defined, respectively, by equations

$$\widehat{\nu} = \frac{1}{\sqrt{2}} \frac{\sqrt{1-\alpha^2}}{1+\alpha^2}, \tag{22.2}$$

$$\widehat{\lambda} = \frac{\alpha}{\sqrt{2}} \sqrt{\frac{1-\alpha^2}{1+\alpha^2}}. \tag{22.3}$$

The figure shows that the critical curve (22.2) is not a uniform limit of function (22.1) as $\widehat{\lambda} \to 0$, since each plane section of the critical surface has a horseshoe shape with the exception of the plane section $\widehat{\lambda} = 0$. On the other hand, for the same reason the curve (22.3) is the uniform limit of the function $\widehat{\lambda} = \widehat{\lambda}(\alpha, \widehat{\nu})$, implicitly defined by (22.1) as $\widehat{\nu} \to 0$.

Thus one can conclude the following: the *linear stability theory of the strictly two-dimensional Kolmogorov flow, i.e., constructed without taking into account external friction, is structurally unstable with respect to the inclusion of the latter and, conversely, the linear theory of the quasi-two-dimensional flow, i.e., constructed by taking into account external friction, qualitatively is not sensitive to*

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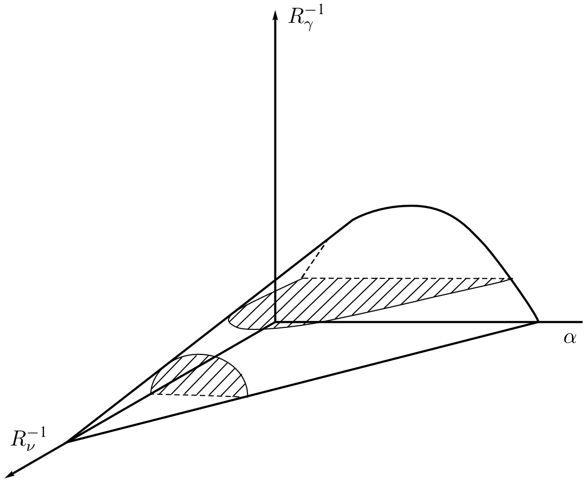

Fig. 22.1 The surface of neutral stability of a quasi-two-dimensional Kolmogorov flow in the space of parameters $(\alpha, 1/Re_\lambda, 1/Re_\nu)$

the inclusion or exclusion of internal viscosity. Moreover, it is easy to see that for $\lambda_0 = \widehat{\lambda}/\widehat{\nu} = R_\nu/R_\lambda \gg 1$, the results in the quasi-two-dimensional linear theory are almost self-similar in R_ν. What is most important is that this conclusion holds for the nonlinear stability theory, as we shall see below.

Is the situation described above an exclusive feature of the Kolmogorov flow, or is it typical for shear flows with arbitrary profiles? To answer this seemingly very difficult question, we will give a new interpretation of the results in the preceding chapter, which will allow us to draw certain conclusions regarding the stability of quasi-two-dimensional shear flows, using the well-developed stability theory of strictly two-dimensional flows.

The critical surface (22.1) is the result of solving system (21.11), in which, by virtue of the Lin stability principle, the value of σ was assumed to be zero. Now set $\widehat{\lambda} = 0$ and try to solve the problem of finding the dependence of the growth rate σ of the perturbation (21.10) on the wavenumber α for an arbitrarily given positive value of $\widehat{\nu}$. In other words, we are interested in finding the dispersion relation $\sigma = \sigma(\alpha, \widehat{\nu})$. To do this, one should set $\widehat{\lambda}$ instead of σ equal to zero in (21.11). But the problem (21.11) at $\widehat{\lambda} = 0$ and $\sigma \neq 0$ up to replacing $\widehat{\lambda}$ with σ coincides with (21.11) at $\sigma = 0$ and $\widehat{\lambda} \neq 0$. Therefore, the desired solution is implicitly given by the expression

$$\widehat{\nu} = \sqrt{\frac{\alpha^2(1-\alpha^2)}{2(1+\sigma/\widehat{\nu}+\alpha^2)(\sigma/\widehat{\nu}+\alpha^2)(1+\alpha^2)}}. \tag{22.4}$$

In particular, as $\widehat{\nu} \to 0$ we obtain the dispersion relation for an inviscid Kolmogorov flow:

$$\sigma = \frac{\alpha}{\sqrt{2}}\sqrt{\frac{1-\alpha^2}{1+\alpha^2}}. \tag{22.4$'$}$$

Thus, *if one knows the dispersion relation* $\sigma = \sigma(\alpha, \widehat{\nu})$ *for the linear problem of the strictly two-dimensional Kolmogorov flow, then the critical curve of the quasi-two-dimensional Kolmogorov flow is defined by the relation*

$$\widehat{\lambda} = \sigma(\alpha, \widehat{\nu}), \tag{22.5}$$

which essentially means *the equality of the increment of the growing mode of the strictly two-dimensional flow and its decrement related to the influence of the external friction*.

22.2 Results in Linear Stability Theory for Strictly Two-Dimensional Shear Flows and Their New Interpretation

It is obvious from physical considerations that the construction principle for the critical surface of the Kolmogorov flow is applicable to an arbitrary shear flow. To make this statement convincing, we formulate the classical problem of linear stability, which we considered in Chap. 14, for the equation of transformation of potential vorticity (20.22):

$$\frac{\partial}{\partial t}\left(\Delta\psi - L_0^{-2}\psi\right) + [\psi, \Delta\psi] + \beta\frac{\partial\psi}{\partial x} = \nu\Delta^2\psi - \lambda\Delta\psi + q, \tag{22.6}$$

where the source of potential vorticity q depends only on the transverse (meridional) coordinate y. Then (22.6) has a stationary solution $\psi = \Psi(y)$, which describes the main flow $U(y) = -d\Psi/dy$ in the direction of the x-axis and depends only on y. It is required to study the stability of this solution with respect to infinitesimal perturbations.

In the absence of the beta-effect and two-dimensional compressibility ($\beta = 0$, $L_0^{-1} = 0$), whose role is not crucial, the dimensionless linear stability problem reduced by the method of normal modes narrows down to the eigenvalue problem for the Orr–Sommerfeld equation (see Lin, 1958):

$$\begin{aligned}\left\{U - \left[c_r + i\left(c_i + \frac{\widehat{\lambda}}{\alpha}\right)\right]\right\}\left(\varphi'' - \alpha^2\varphi\right) - U''\varphi \\ = i\frac{\widehat{\nu}}{\alpha}\left(\varphi^{IV} - 2\alpha^2\varphi'' + \alpha^4\varphi\right)\end{aligned} \tag{22.7}$$

with the boundary conditions of adhesion at the lateral boundaries $\alpha\varphi = \varphi' = 0$ at $y = y_1,\ y_2$ and the regularity at $|y| \to \infty$, if the domain of integration is not bounded. Here φ is the dimensionless amplitude of the harmonic perturbation

$$\psi = \varphi(y)\exp\left\{i\alpha(x - ct)\right\}, \quad c = c_r + ic_i, \tag{22.8}$$

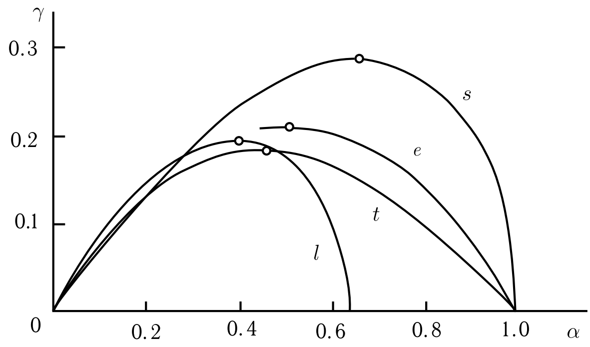

Fig. 22.2 The dispersion relation $\sigma = \sigma(\alpha)$ for $U = \sin(y)$ (curve s), $\text{erf}(y)$ (curve e), $\tanh(y)$ (curve t) and a piecewise-linear profile (curve l) in the inviscid case

α is the dimensionless longitudinal wavenumber, $\alpha c_i = \sigma$ is the growth rate that assumes a positive value for unstable modes. Recall that the Squire theorem allows one to be confined to the two-dimensional problem setting. According to this theorem the most dangerous perturbations are located in the plane of the main flow.

Now we see that problem (22.7) with these boundary conditions and $\widehat{\lambda} = 0$ is equivalent to the same problem, but at $c_i = 0$ and $\widehat{\lambda} \neq 0$ and, consequently, *formula* (22.5) *holds, where* $\sigma = \sigma(\alpha, \widehat{\nu})$ *is the dispersion dependence of the increment on the wavenumber at different* $\widehat{\nu}$ *for strictly two-dimensional flows.*

Consider in this regard the results in classical linear stability theory of two-dimensional shear flows, most of which are related to the special case of inviscid motions ($\nu = 0$), i.e., to the Rayleigh equation (16.7):

$$\big[U - (c_r + i c_i)\big]\big(\varphi'' - \alpha^2 \varphi\big) - U'' \varphi = 0. \tag{22.9}$$

Recall that equation (22.9), unlike (22.7), is invariant with respect to the operation of complex conjugation up to the sign change of c_i. Therefore, the existence of solutions with negative c_i implies the existence of a complex conjugate solution with positive c_i. Consequently, any $c_i \neq 0$ implies instability in the inviscid theory, a property which the Orr–Sommerfeld solutions do not have. In this connection it is appropriate to mention the important result of Vazov and Lin (see Lin, 1958), according to which among growing ($c_i \neq 0$) solutions of the Rayleigh equation, only the ones with $c_i > 0$ are the limits of solutions of the Orr–Sommerfeld equation as $\nu \to 0$.

Typical dispersion curves obtained by different authors for the inviscid problem are shown in Fig. 22.2. Curve (l) in this figure corresponds to a piecewise linear profile

$$U = y \quad \text{for } |y| \le 1, \qquad U = \frac{y}{|y|} \quad \text{for } |y| \ge 1, \tag{22.10}$$

and it is described by the formula whose derivation goes back to Rayleigh, see Betchov and Kriminale (1971), Dolzhansky et al. (1990)

$$\sigma = \frac{1}{2}\big[e^{-4\alpha} - (1 - 2\alpha)^2\big]^{1/2}. \tag{22.11}$$

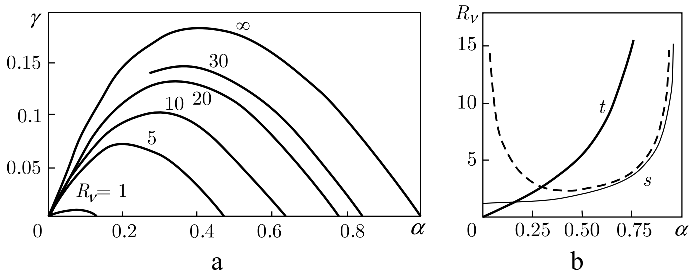

Fig. 22.3 (**a**) The dispersion relation $\gamma = \sigma(\alpha, R_\nu)$ for $U = \tanh(y)$. (**b**) Critical curves for $U = \sin(y)$ (curve s) and $\tanh(y)$ (curve t) without an external friction

The curve (s) refers to the Kolmogorov flow and it is described by formula (22.4′). Dispersion curves (e) and (t) for profiles $U = \text{erf}(y)$ and $U = \tanh y$ are obtained by numerical integration (see Drazin and Reid 1981). Note that with rare exceptions, the Rayleigh equation does not admit an exact analytical integration. Asymptotics for the dispersion relations and eigenfunctions in the vicinity of the neutral points $\alpha = 0$ and $\alpha = \alpha_s \neq 0$ (at which $\sigma = 0$) can be found in (Betchov and Kriminale 1971) and (Drazin and Reid 1981).

Figure 22.3a shows dispersion curves of the viscous problem in the plane $(\alpha, \sigma(\alpha, R_\nu))$ that are obtained by numerical integration at different Reynolds numbers (see Betchov and Kriminale 1971). The pattern shown is typical for the different profiles of shear flows, i.e., invariant with respect to changes in the shape of the profile, which do not significantly affect the magnitude and location of maxima and zeros of the dispersion curves. Critical curves for viscous shear flows with profiles $U = \sin y$ and $U = \tanh y$ are presented for comparison in Fig. 22.3b. Their shapes are also typical for the different profiles. However, focus on the difference in the stability thresholds equal to zero and $\sqrt{2}$ for $\tanh y$ and $\sin y$, respectively. The profiles of (e) and (l) also have zero stability thresholds. This is easy to understand, since in the vicinity of $\alpha = 0$ the long wavelength modes perceive the current as a step-shaped Helmholtz flow $U = y/|y|$ (the ratio of shear width to the wavelength tends to zero as $\alpha \to 0$), which is exponentially unstable with respect to any wave-like perturbation. Its dispersion curve in dimensional values of the increment γ and the wavenumber k is shown in Fig. 22.4a (see Chap. 14, Example 3). For comparison, Fig. 22.4b presents the dependence $\gamma = \gamma(k)$ for a "smeared" jump of width D, which shows that the elimination of the velocity jump leads to stabilization of the current relative to small-scale perturbations.

22.3 Surface of Neutral Stability of Typical Quasi-two-dimensional Shear Flows

Now using formula (22.5), we replace in Fig. 22.3a the letter σ by $\widehat{\lambda}$. To plot the values of $\widehat{\nu} = R_\nu^{-1}$, we direct the corresponding coordinate axis orthogonally to the

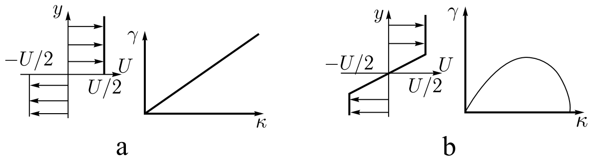

Fig. 22.4 (**a**) The velocity profile and dispersion curve of a tangential discontinuity. (**b**) Same as in (**a**) but for a "smeared" discontinuity

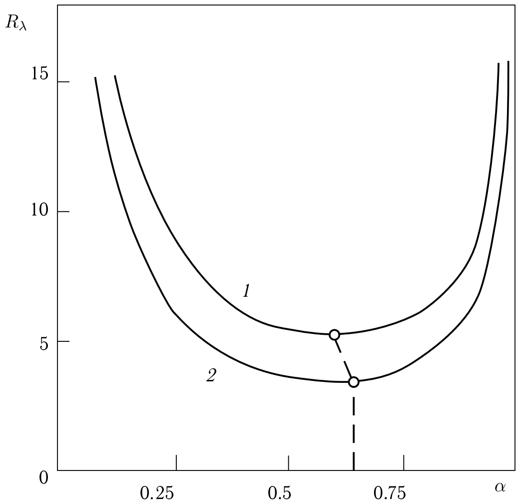

Fig. 22.5 The curves of neutral stability for $U = \sin(y)$ at $Re_\nu/Re_\lambda = 1$ (*curve 1*) and at $Re_\nu/Re_\lambda = \infty$ (*curve 2*)

plane of the drawing. As a result, in the space $(\alpha, \widehat{\lambda}, \widehat{\nu})$ we obtain the same surface of neutral stability of the quasi-two-dimensional flow with velocity profile $U = \tanh y$, as we saw in Fig. 22.1. This surface is typical for all such viscous flows, subjected to external friction. Consequently, all the conclusions drawn about the Kolmogorov flow are carried over to almost arbitrary quasi-two-dimensional shear flows.

We would like to mention two important points. First, when discussing geophysical boundary layers (GBLs, see Chap. 20) we emphasized that one of the applicability criteria for the quasi-two-dimensional approximation is that the characteristic horizontal scale of the flow, the width of its velocity shear should be much larger than the thickness of the outer Praudman–Stewartson layer. The latter, on the contrary, has large vertical velocities, and hence the motion is three-dimensional. For geophysical flows this condition is satisfied by definition, and since $\delta_{St} \ll H \ll L_0$ and $\lambda = \nu/\delta_{St}^2$ $(\delta_{St}^2 = H\delta_E)$, then $R_\nu/R_\lambda = (UL_0/\nu)/(U/L_0\lambda) = L_0^2/\delta_{St}^2 = L_0^2/\delta_E H = O(10^5) \gg 1$. But in the latter case, as follows from the results obtained, at least the linear stability theory is self-similar in R_ν. Compare, for example, critical curves in Fig. 22.5. This conclusion is quite nontrivial and it indicates the possibility of neglecting internal viscosity, i.e., the term with the highest derivative in the equations of motion, which, generally speaking, might be rather dangerous without an adequate reason for doing so.

Secondly, self-similarity in R_ν makes it possible to simulate global atmospheric flows in laboratory conditions, in which the huge real values of R_ν are not achievable. Note that this unattainability of very large values of R_ν was often used by skeptics as one of the main arguments that a comparison of natural and laboratory flows is not proper. On the contrary the self-similarity conditions are easily achieved in the laboratory. In particular, in both the experiments of Hide and Mason (1975) and the Laboratory of geophysical hydrodynamics of IAP RAS, the width of the rotating annular channels and the depth of the fluid (water) contained therein was about 10 cm, while the thickness of the Ekman layer was about 1 mm. These experiments reproduce the fundamental regimes of general atmospheric circulation. (Recall that the quasi-two-dimensionality is provided not by the shallow depth of the fluid, but by the rotation of the system, see the Praudman–Taylor theorem, Chap. 6). Therefore, $R_\nu/R_\lambda = O(10^2)$, which is quite sufficient for self-similarity in R_ν.

Here is another example: in experiments on modeling, the instability of shear flows in thin layers of electrically conducting fluid by the MHD method (see Dolzhansky at al. 1990), the characteristic horizontal (L) and vertical (H) scales are of the order of a few centimeters and millimeters, respectively. In the shallow-water approximation of a viscous fluid, the bottom friction coefficient is $\lambda = 2\nu/H^2$, and then again $R_\nu/R_\lambda = 2L^2/H^2 = O(10^2)$.

22.4 On Nonlinear Stability Theory of Quasi-two-dimensional Shear Flows

Given the limited scope of this book, we would not like a discussion of the physical nature of phenomena that accompany the formation of supercritical regimes to be overshadowed by other topics, such as a detailed description of technical difficulties in constructing the nonlinear stability theory for flows and the mathematical methods that allow one to overcome these difficulties. Therefore, I will only mention briefly the key points of the problem considered.

In exceptional but important cases the problem can be reduced by the Galerkin method to studying a nonlinear dynamical system of low order. The success of this approach is associated, first of all, with a good choice of basis functions used in the expansion of the required solutions, which provides the method's fast convergence. Secondly, often the data observed or the results of laboratory measurements indicate that secondary flows are generated by only a very few modes. The canonical example of this kind is the Kolmogorov flow, whose instability and secondary modes are described with a high degree of precision by a dynamical system of the third order. In particular, the critical curve corresponding to this model coincides with (22.4). I strongly recommend verifying this by doing Exercise 1. The basic principles for constructing finite-dimensional and discrete analogs of the hydrodynamic equations are formulated in the monograph (Gledzer et al. 1981).

An effective approach to the study of soft regimes of stability loss, which are observed in these flows, is associated with the Stewart–Watson method (see Drazin and Reid 1981). The following idea of this method goes back to the works of Poincaré and Landau. The linear stability problem is the first term of the expansion in powers of perturbations of small amplitude A (the soft loss of stability means the smallness of A for weak supercriticalities). Continue this expansion and obtain the Landau equation for the amplitude

$$\dot{A} = \sigma A + K_L |A|^2 A, \quad A \ll 1, \tag{22.12}$$

where the constant K_L is called the *Landau constant*.

The linear part of this equation describes the growth of perturbations due to linear instability, i.e., due to the interaction of perturbations with the main flow, while the nonlinear term describes self-interaction of perturbations that slows down or enhances the growth of their amplitude depending on the sign of K_L. Physically, self-interaction arises because of the nonlinearity of hydrodynamic equations: a harmonic perturbation generates its second harmonic, distorts the average profile of longitudinal velocity, and then interacts with the deviation from this average profile.

Technically, the problem reduces to deriving and solving the equations for perturbations of up to the second order of smallness inclusively. The linear part of the operator of these equations for $R_\nu \gg 1$ coincides with the operator of the Rayleigh equation. After that, the evolution equation for amplitude is derived from the solvability condition of the third-order equation. Its coefficients, including the Landau constant, are expressed as certain integrals of perturbations of the first and second orders. It is interesting to note that according to the studies of Romanova and Annenkov (2005), the linear instability of shear flows under certain conditions is inhibited by quadratic, rather than cubic, nonlinearity.

The main technical problem of the Stuart–Watson method in its classical form ($\lambda = 0$) is related to regularization of the critical layer. From a mathematical point of view, the critical layer is a neighborhood of the singular point $y = y_c$ of the Rayleigh equation

$$\left\{ U - \left[c_r + i \left(c_i + \frac{\widehat{\lambda}}{\alpha} \right) \right] \right\} (\varphi'' - \alpha^2 \varphi) - U'' \varphi = 0,$$

in which the coefficient at the highest derivative vanishes. At $\widehat{\lambda} = 0$ in a neighborhood of the curve of neutral stability ($\alpha c_i = \sigma = 0$), the critical point is close to the real axis. And although for traditionally considered antisymmetric velocity profiles $U(y) = -U(-y)$ the quantity U'' also vanishes, the singularity still appears in the following orders of the expansion. In order to glue the solutions of the Rayleigh equation on the left and on the right from the singularity, one constructs special expansions in the critical layer, whose form depends on which of the terms dominates the equation: the viscous, nonlinear, or nonstationary term.

External friction eliminates the problem of the critical layer, first, because $\widehat{\lambda} \neq 0$ and the singular point shifts to the complex plane. Secondly, the minimum of the curve of neutral stability $R_\lambda = R_\lambda\ (\alpha, R_\nu^{-1} = 0)$, at which one takes the expansion, is assumed at the value $\alpha = \alpha_0 \neq 0$. This method was used in calculating the secondary modes for different velocity profiles of the main flow. As it turned out, the nature of supercritical regimes is extremely sensitive to small changes in the profile of the primary flow.

So we found that for $R_\nu/R_\lambda \gg 1$ the linear and weakly nonlinear stability theories of quasi-two-dimensional shear flows are self-similar in R_ν. Taking into account that under natural and laboratory conditions the supercriticality of shear flows in R_λ is rather small, we arrive at the following conclusions in relation to general atmospheric circulation.

(1) One of the defining criteria of self-similarity of the general circulation of the atmosphere is the Reynolds number that is defined by external friction, and not by internal viscosity.
(2) A relatively quiet nature of general atmospheric circulation is due to a moderate supercriticality in R_λ and self-similarity in R_ν, assuming in the atmosphere the astronomical values ($R_\nu = UL_0/\nu = O(10^{12})$ for molecular viscosity $\nu = 0.15$ cm^2/s and $R_\nu = O(10^6)$ for turbulent viscosity, whereas the transition to turbulence in the absence of friction occurs at $R_\nu = O(10^3)$).
(3) One of the reasons for unpredictability of weather over long periods may be related to the barotropic instability of global shear flows due to the strong sensitivity of the secondary modes to small changes in the profile of the main flow.

Finally, we mention that the conclusion about the structural instability of the results in the classical theory of strictly two-dimensional flows is of a general nature. A similar situation occurs if the influence of exterior friction on the fluid is replaced, for instance, by that of a stratification, a magnetic field (in the case of an electrically conducting fluid), or rotation of the system as a whole.

22.5 Exercises

1. Applying the usual Galerkin procedure to Eq. (21.6) and confining the expansion of the stream function to the most unstable modes $n = 0, \pm 1$, i.e., setting

$$\psi = \Psi(t)\cos y + \left[\exp(i\alpha_0 x)\sum_{-1}^{1}\varphi_n \exp(iny) + \text{compl. conjugate}\right],$$

derive the nonlinear dynamical system for the variables $\Psi(t)$, $z_0 = \varphi_0(t)$ and $z_1 = [\varphi_1(t) - \varphi_{-1}(t)]/2\alpha_0$, where α_0 is the value of the wavenumber, which corresponds to the imaginary one for the critical curve of the quasi-two-dimensional Kolmogorov flow at a fixed value of R_ν. Find stationary solutions and show that stability conditions of the primary mode coincide with the first approximation of the stability criterion for the quasi-two-dimensional Kolmogorov flow, obtained from its linear stability theory.
2. Try to solve the stability problem for the flow with a piecewise linear velocity profile (22.10) by the Stuart–Watson method. It is a very instructive task, showing that nonlinearity is not always able to suppress linear instability. (If your attempt fails, refer to the work Manin 1989.)

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Chapter 23
Friction in a Turbulent Boundary Layer

We return now to the equations of motion of a viscous fluid described in Chap. 19, and consider briefly an approach to the description of turbulent flows in the framework of an incompressible fluid. The basis of a statistical approach to the theory of turbulence is the passage from considering a single turbulent flow (an implementation) to considering a statistical ensemble of possible implementations for fixed external conditions. In other words, velocity, temperature and other characteristics of a turbulent flow are now to be considered as random fields. Because of the uncertainty in the probability distribution in the space of realizations, we approach the very delicate issue of calculating the average values in a way that is common in turbulence theory. Under the average value $\langle f(\mathbf{r},t)\rangle$ of a random field $f(\mathbf{r},t)$ we mean the average over the set of possible implementations (or, in other words, "the ensemble average"), which in practical applications is replaced by the average over time, based on the ergodic hypothesis. In this case the quantity $f(\mathbf{r},t)$ itself can be written as $f(\mathbf{r},t)=\langle f(\mathbf{r},t)\rangle + f'(\mathbf{r},t)$, where $f'(\mathbf{r},t)$ are fluctuations, pulsations, deviations from the mean-field, $\langle f'(\mathbf{r},t)\rangle = 0$. Taking into account the above definitions and using the medium incompressibility ($\rho=\rho_0=\text{const}$) the averaged Navier–Stokes equations are written as follows:

$$\frac{\partial\langle u_i\rangle}{\partial t}+\langle u_j\rangle\frac{\partial\langle u_i\rangle}{\partial x_j}=-\frac{1}{\rho_0}\frac{\partial\langle p\rangle}{\partial x_i}-\frac{\partial\langle u_i'u_j'\rangle}{\partial x_j}+\nu\Delta\langle u_i\rangle, \tag{23.1}$$

where $\langle u_i'\rangle = 0$ $(i=1,2,3)$, while the quantity

$$\tau_{ij}=\left\langle u_i'u_j'\right\rangle \tag{23.2}$$

is called the *Reynolds stress tensor*.

Let us follow the analogy by introducing the concepts of the viscous stress tensor and viscosity coefficients in the derivation of the Navier–Stokes equations (Chap. 19). The traditional phenomenological method of closing up Eq. (23.1), subject to the dissipative effect of velocity fluctuations u_i' on the average flow $\langle u_i\rangle$, consists in interpreting the Reynolds stress tensor as the stress tensor for turbulent

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viscosity:

$$\tau_{ij} = -\nu_{\text{turb}}\left(\frac{\partial\langle u_i\rangle}{\partial x_j} + \frac{\partial\langle u_j\rangle}{\partial x_i}\right), \tag{23.3}$$

where turbulent viscosity $\nu_{\text{turb}} \sim \Delta U L_0$ is defined by the dimension of the outer scale L_0 and the velocity amplitude ΔU.

Obviously, in the practical use of the closure hypothesis (23.3), the choice of values for turbulent viscosity depends on spatial and time scales over which the motion is averaged. As a rule, ν_{turb} exceeds the kinematic viscosity by several orders of magnitude, which allows us to neglect the last term on the right-hand side of (23.1). For example, in describing the atmospheric Ekman layer, based on the observations that the height of the planetary atmospheric boundary layer $\delta_E = (\nu_{\text{turb}}/\Omega)^{1/2}$ is of the order of several hundred meters, the value of the turbulent viscosity coefficient is assumed to be 10^5–10^6 cm^2 s^{-1}, exceeding the value of the kinematic viscosity of air by six to seven orders of magnitude.

One of the noticeable features of atmospheric turbulence is that regions of influx and outflux of kinetic energy are separated not only in spaces of scales or wavenumbers, but also in the real physical space. Dissipation of the kinetic energy takes place mainly in the planetary boundary layer, including a surface layer of several tens of meters in height, which is dominated by friction with the underlying surface which spreads its influence to the whole layer because of turbulent mixing. We consider these two layers from the point of view of turbulence theory.

23.1 Turbulence in the Atmospheric Surface Layer

For the sake of simplicity we confine ourselves to studying a neutrally stratified layer, free from the influence of buoyancy forces. Since near the Earth's surface, the pressure and Coriolis forces are small as compared to the forces of friction, the neutrally stratified surface boundary layer is defined as the region in which the friction stresses are balanced by the Reynolds stresses $\tau_{ij} = \langle u_i' u_j' \rangle$.

It is obvious that the characteristic horizontal scale of averaged motions in the surface layer is much greater than its height. Hence for lack of a specified horizontal direction, all averaged characteristics of motion can be assumed to depend only on the height z, while the turbulent momentum flux is assumed to be directed vertically.

Let the x-axis coincide with the direction of the mean wind velocity vector $\langle \mathbf{u} \rangle$, so that $\langle u_x \rangle \doteq u(z)$, $\langle u_y \rangle \doteq v = 0$, $\langle u_z \rangle \doteq w = 0$. *Then, by virtue of the above assumptions and symmetry considerations, one has* $\tau_{12} = \tau_{21} = \tau_{23} = \tau_{32} = 0$. *Moreover, the vertical flux of the horizontal momentum* $\langle w'u' \rangle = \tau_{31}$ *is equal to the horizontal flux of the vertical momentum* $\langle u'w' \rangle = \tau_{13}$ (why?). *These fluxes balance the friction tension and are constant.* It is clear that the value of this constant is negative (the atmosphere gives away its momentum to the Earth), which we denote by

$$-u_*^2 = \left\langle u'w' \right\rangle = \tau_{13}, \tag{23.4}$$

bearing in mind its dimension of velocity squared.

The velocity derivative, as well as the velocity itself, in the surface layer depends only on z, and for dimensional reasons it is uniquely expressed in terms of the parameters z and u_*:

$$\frac{du}{dz} = \frac{u_*}{\varkappa z}, \tag{23.5}$$

where the numerical constant $\varkappa$ is called the *Karman constant*, and its experimental value is approximately equal to 0.4.

According to formula (23.3), where we set $i = 1$ and $j = 3$,

$$-u_*^2 = -\nu_{\text{turb}} \frac{u_*}{\varkappa z}.$$

This implies the expression for the turbulent viscosity coefficient:

$$\nu_{\text{turb}} = \varkappa u_* z, \tag{23.6}$$

depending on the altitude. This example shows that while using the closure hypothesis (23.3) in turbulent boundary layers, it is not sufficient to know the value of the turbulence coefficient at one point. One should keep in mind its possible dependence on the spatial coordinates, which is defined by the specific balance of forces inherent in the boundary layer under consideration.

By making the substitution (23.3) into (23.1) it is easy to calculate (see the derivation of formula (19.17)) the specific (per unit mass) dissipation rate of the kinetic energy

$$\varepsilon = \nu_{\text{turb}} \left(\frac{\partial \langle u_i \rangle}{\partial x_j} + \frac{\partial \langle u_j \rangle}{\partial x_i} \right) \frac{\partial \langle u_i \rangle}{\partial x_j} = \frac{1}{2} \nu_{\text{turb}} \left(\frac{\partial \langle u_i \rangle}{\partial x_j} + \frac{\partial \langle u_j \rangle}{\partial x_i} \right)^2. \tag{23.7}$$

In this case, taking into account $\langle \mathbf{u} \rangle = (u(z_*), 0, 0)$ and formulas (23.5) and (23.6), one has

$$\varepsilon = \frac{u_*^3}{\varkappa z}. \tag{23.8}$$

Sometimes, using the estimate $\nu_{\text{turb}} \sim L_0 \Delta U = L_0^2 \Delta U / L_0$, the turbulence coefficient is written in the form

$$\nu_{\text{turb}} = L_0^2 \left| \frac{du}{dz} \right|,$$

that allows one to estimate the outer scale of turbulence in the surface layer:

$$L_0 = \varkappa z. \tag{23.9}$$

Integrating Eq. (23.5), we obtain the logarithmic law of the wind distribution in height, known from observations in the surface layer:

$$u(z) = \frac{u_*}{\varkappa} \ln \frac{z}{z_0}. \tag{23.10}$$

The integration constant z_0 is interpreted as the height of roughness, which depends on the structure of the underlying surface. It was not taken into account in formulating Eq. (23.5), and therefore the law obtained is valid only for $z \gg z_0$. Formula (23.10) is employed, in particular, to determine u_* and, hence the Reynolds stresses in the surface layer by using relatively simple measurements of the mean wind velocity at different altitudes.

23.2 Turbulent Planetary Boundary Layer (PBL) and Its Impact on Motions of Global Scale

The influence of the planetary boundary layer on large-scale motions of the atmosphere is carried out by the vertical flows at its upper border.[1] Such flows materialize the exchange of horizontal momentum between the free atmosphere and the boundary layer, which ultimately leads to slowing down the global flows. This mechanism of inhibition in its most explicit form manifests itself in the derivation of the quasi-geostrophic equation of transformation of the potential vorticity of a viscous barotropic atmosphere. It can be written (see Chap. 20) in the form

$$\begin{aligned} &\frac{\partial}{\partial t}\left(\Delta\Psi - L_0^{-2}\Psi\right) + [\Psi, \Delta\Psi] + \beta\frac{\partial\Psi}{\partial x} = -\frac{f_0}{H_0}w_E + q, \\ &U_x = -\frac{\partial\Psi}{\partial y}, \qquad U_y = +\frac{\partial\Psi}{\partial x}. \end{aligned} \tag{23.11}$$

(Indeed, use Eq. (20.19) and pass to the description in terms of the stream function Ψ, while keeping intact the term containing the vertical velocity w_E at the upper boundary of the Ekman layer.) Here q denotes the sum of the external source of vorticity and its dissipation $\nu\Delta^2\Psi$ due to viscosity. Unlike in Chap. 20, the velocity components and the stream function related to the free atmosphere are denoted here by the capital letters U_x, U_y and Ψ, respectively. The expression for the vertical velocity

$$w_E = \delta_E \Delta\Psi, \tag{23.12}$$

valid in the case of a laminar (laboratory) Ekman layer was also extended in Chap. 20 to the case of a turbulent atmospheric planetary boundary layer upon replacing the kinematic viscosity ν by the turbulent viscosity ν_{turb}. As was already mentioned above, the latter can be estimated from the condition that the height of the PBL (planetary boundary layer), known from observations and ranging from a few hundred meters to one kilometer, is set to be equal to $\delta_{E\text{turb}} = (2\nu_{\text{turb}}/f_0)^{1/2}$. Practically, this means that the estimates can vary by more than an order of magnitude.

[1] In this section we follow Dolzhansky and Manin (1993), Danilov et al. (1993).

This already indicates that the straightforward replacement of kinematic viscosity by the empirical coefficient of turbulent viscosity in order to describe the real planetary layer and its impact on global motions is a rather naive oversimplification. To convince ourselves about this, we consider a less idealized, but still a rather remote from a reality situation in which the movement of the PBL is a developed turbulence according to the Richardson–Kolmogorov–Obukhov scenario (see Monin and Yaglom 1992). More precisely, we are talking about satisfying the following conditions.

1. The planetary boundary layer is a domain of a developed three-dimensional turbulence.
2. The characteristic linear scale λ of vortices located in the PBL is much smaller than the height $\delta_{E\text{turb}}$ of the PBL ($\lambda \ll \delta_{E\text{turb}}$).
3. The characteristic times τ for vortices in the PBL are much smaller than the characteristic time of the weather change ($\tau \ll f_0^{-1}$).
4. The characteristic scale of inhomogeneities in the PBL in horizontal directions greatly exceeds the scale of its inhomogeneities in the vertical direction.
5. Planetary boundary layer (PBL) is neutrally stratified.

The proposed idealization takes into account neither the vertical temperature stratification of a real PBL nor the presence of so-called coherent structures in it. Such structures consist of ordered systems of vortices whose size is of the order of the layer thickness and swirled around the horizontally-oriented axes. *Our goal is to clarify the influence of "fine-seed" three-dimensional turbulence with vortex scales of the order of* 0.1–10 *m on the dynamics of global flows whose size is of order* 1000 *kilometers or more*.

These assumptions allow us to consider turbulence in the PBL as a steady horizontally homogeneous process, almost instantly adjusting to changes in the motion of the free atmosphere. Dependence on time and the horizontal coordinates x and y is taken into account only parametrically through the horizontal wind velocity $U(x, y, t)$ on the upper boundary of the PBL and the Coriolis parameter $f(x, y)$. Here the *average characteristics of the PBL under consideration are self-similar in* ν_{turb}. This drastically distinguishes the turbulent planetary boundary layer from the laminar Ekman layer or its turbulent "opponent" with the turbulent viscosity coefficient ν_{turb}.

It follows that the vertical dependence of the characteristics of the boundary layer is reduced to the functions of the dimensionless vertical coordinate $\zeta = zf/U$, uniquely determined by dimensionality considerations for U and f, the two external parameters of the PBL. Then in the coordinate system whose x-axis coincides with the direction of the quasi-geostrophic wind $\mathbf{U}(x, y, t)$ on the upper boundary of the layer, the averaged horizontal components of the velocity inside the PBL can be written as follows:

$$u = U(x, y, t)\varphi_1(\zeta), \qquad v = U(x, y, t)\varphi_2(\zeta), \quad U = |\mathbf{U}|. \tag{23.13}$$

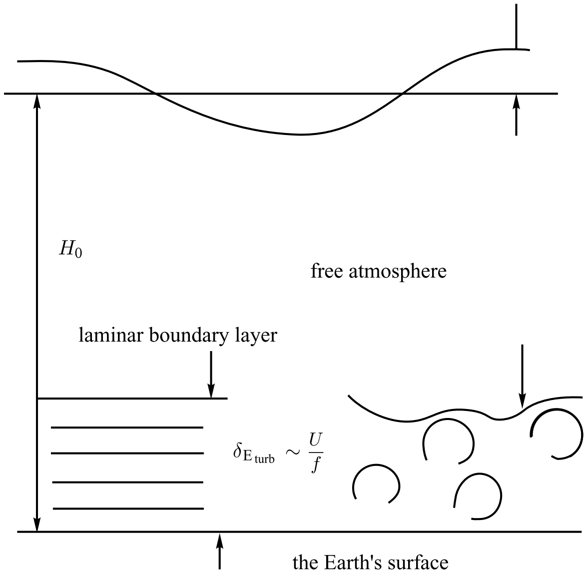

Fig. 23.1 Schematic representation of laminar and turbulent planetary boundary layers

Here $\varphi_1(\zeta)$ and $\varphi_2(\zeta)$ are universal dimensionless functions of the dimensionless vertical coordinate ζ, satisfying the natural boundary conditions

$$\varphi_1(0) = \varphi_2(0) = \varphi_2(\infty) = 0, \quad \varphi_1(\infty) = 1. \tag{23.14}$$

In the more general case, when the wind $\mathbf{U}$ has two components $\mathbf{U} = (U_1, U_2)$, we have

$$u = U_1\varphi_1 - U_2\varphi_2, \qquad v = U_1\varphi_2 + U_2\varphi_1. \tag{23.13$'$}$$

As it is usually done in the theory of boundary layers, the upper boundary is conventionally moved to infinity, and $\varphi_2(\zeta)$ does not vanish identically because of the wind turning with height, already familiar to us (see Chap. 20). Note also that the influence of the surface layer on the vertical distribution of the horizontal wind is not marked here explicitly. However it is taken into account in further estimates of the function values $\varphi_1(\zeta)$ and $\varphi_2(\zeta)$ (which cannot be determined from dimensional or similarity considerations) by using measurements in independent laboratory experiments.

One of the major and fundamental differences between a turbulent PBL and a laminar one, schematically illustrated in Fig. 23.1, is that even for $f = f_0 = \mathrm{const}$ its thickness $\delta_{E\mathrm{turb}} \sim U/f$ defined by two, rather than three (ν, U and f), external parameters is a function of coordinates and time. As we shall see, this has quite unexpected consequences. In what follows, for the sake of simplicity we consider PBL without taking into account the dependence of f on the coordinates.

23.2.1 Vertical Velocity at the Upper Boundary of a Turbulent PBL

We assume the medium to be incompressible and homogeneous in the boundary layer. Integrating the continuity equation over the height of this boundary layer we obtain

$$w_E = -\int_0^{\delta_{E\mathrm{turb}}} \operatorname{div}\mathbf{u}\cdot dz, \qquad \left(\operatorname{div}\mathbf{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right). \tag{23.15}$$

In subsequent arguments the unknown quantity $\delta_{E\mathrm{turb}}$ is required only to be much less than the height of the atmosphere.

Formulas (23.13′) for the horizontal wind field can be rewritten in the form

$$\mathbf{u} = A(\zeta)\mathbf{U}(x,y,t), \quad A(\zeta) = \begin{pmatrix} \varphi_1(\zeta) & -\varphi_2(\zeta) \\ \varphi_2(\zeta) & \varphi_1(\zeta) \end{pmatrix}, \tag{23.16}$$

whence

$$\operatorname{div}\mathbf{u} = A_{ik}\frac{\partial U_k}{\partial x_i} - \frac{\zeta}{U}\frac{\partial U}{\partial x_i}A'_{ik}(\zeta)U_k. \tag{23.17}$$

The second term on the right-hand side appeared as a result of differentiation of the quantity $\zeta = zf/U(x,y,t)$ in x, y. Here A_{ik} are matrix entries of A, U_k are components of the vector field $\mathbf{U}$, whose modulus is denoted by $U = \sqrt{U_1^2 + U_2^2}$, the prime denotes differentiation in ζ, while the same indices imply summation.

After substituting (23.17) into (23.15) and integration by parts we have

$$w_E = \frac{1}{f}B_{ik}\frac{\partial(UU_k)}{\partial x_i}, \quad B = \begin{pmatrix} \varkappa_1 & \varkappa_0 \\ -\varkappa_0 & \varkappa_1 \end{pmatrix}, \tag{23.18}$$

where

$$\varkappa_0 = \int_0^{\infty} \varphi_2(\zeta)d\zeta, \qquad \varkappa_1 = \int_0^{\infty}[1-\varphi_1(\zeta)]d\zeta \tag{23.19}$$

are positive constants (see below). Now using the easily verifiable identities

$$\operatorname{div}(U\mathbf{U}) = \operatorname{rot}_z(\mathbf{k}\times U\mathbf{U}), \qquad \operatorname{rot}_z(U\mathbf{U}) = \operatorname{div}(\mathbf{k}\times U\mathbf{U}), \tag{23.20}$$

we obtain the following dual representation for w_E:

$$\begin{aligned} w_E &= \frac{\varkappa_0}{f}\operatorname{rot}_z(U\mathbf{U}) + \frac{\varkappa_1}{f}\operatorname{rot}_z(\mathbf{k}\times U\mathbf{U}) \\ &= \frac{\varkappa_0}{f}\operatorname{div}(\mathbf{k}\times U\mathbf{U}) + \frac{\varkappa_1}{f}\operatorname{div}(U\mathbf{U}). \end{aligned} \tag{23.21}$$

Here $\mathbf{k}$ is the vertical unit vector and $\operatorname{rot}_z(U\mathbf{U}) = \partial(UU_y)/\partial x - \partial(UU_x)/\partial y$. In the derivation of (23.21) we used the boundary conditions (14) under the assumption that the functions $[1-\varphi_1(\zeta)]$ and $\varphi_2(\zeta)$ decay at infinity faster than $1/\zeta$. The hydrodynamic meaning of the dual representation (23.21) becomes obvious if we recall that w_E in Eq. (23.11) plays the role of a source (or sink) of vorticity, while in Eq. (20.18) of the mass conservation of the free atmosphere it plays the role of the source of mass.

Positivity of the constants $\varkappa_0$ and $\varkappa_1$ follows from the fact that the wind in the PBL turns in the direction of the pressure deficit, and therefore, the components of the wind velocity in the above special coordinate system inside the layer are strictly positive. In addition, one can show that the constants $\varkappa_0$ and $\varkappa_1$ are small and have the same order of magnitude. Therefore, one can set $\varkappa_1 = \alpha\varkappa_0$, where $\alpha = O(1)$ and $\varkappa_0 \ll 1$. Note also that since the functions φ_1 and φ_2 are of order one, then $\varkappa_0$ can be regarded as the dimensionless thickness of the PBL, i.e., measured in units of U/f.

23.2.2 Equations of Global Flows Under the Influence of the Turbulent Planetary Boundary Layer

Formally, Eq. (23.11) for $L_0^{-1} = 0$ can be interpreted as the vorticity equation for a strictly two-dimensional atmosphere with an external source of vorticity, which is obtained by applying rot_z to the two-dimensional hydrodynamic equations written in terms of velocity. Then

$$-\frac{f}{H_0} w_E = \text{rot}_z\, \mathbf{F},$$

where $\mathbf{F}$, according to the first formula in (23.21) can be written as

$$\mathbf{F} = -\frac{1}{D_0}(\mathrm{U}\mathbf{U} + \alpha\mathbf{k}\times \mathrm{U}\mathbf{U}) + \nabla\Phi. \tag{23.22}$$

Here $\Phi = \Phi(x, y, t)$ is an arbitrary scalar function of horizontal coordinates and time, while $D_0 = H_0/\varkappa_0$.

Restoring now the equation for the velocity field of the hypothetically strictly two-dimensional atmosphere under consideration, we obtain the hydrodynamic equations with quadratic friction and differential rotation:

$$\begin{aligned} &\frac{d\mathbf{U}}{dt} + \left(1 + \alpha\frac{\mathrm{U}}{D_0 f}\right) f\mathbf{k}\times\mathbf{U} = -\frac{1}{\rho}\nabla p - \frac{1}{D_0}\mathrm{U}\mathbf{U} + \mathbf{F}_q, \\ &\frac{d}{dt} = \frac{\partial}{\partial t} + U_x\frac{\partial}{\partial x} + U_y\frac{\partial}{\partial y}. \end{aligned} \tag{23.23}$$

Here $\mathbf{F}_q$ is the force that creates an external source of vorticity $q = \text{rot}_z\,\mathbf{F}_q$, while Φ is included in the pressure. For $L_0^{-1} \neq 0$, substituting (23.21) into (23.11) and taking into account the identities

$$\text{rot}_z(\mathrm{U}\mathbf{U}) \equiv \text{div}(\mathbf{k}\times\mathrm{U}\mathbf{U}) \equiv |\nabla\Psi|\Delta\Psi + \nabla\Psi\nabla|\nabla\Psi|, \tag{23.24}$$

$$\mathrm{rot}_z(\mathbf{k}\times \mathrm{U}\mathbf{U}) \equiv \mathrm{div}(\mathrm{U}\mathbf{U}) \equiv \left[\Psi, |\nabla\Psi|\right] \tag{23.25}$$

gives the quasi-geostrophic equation for transformation of the potential vorticity for a barotropic atmosphere with a turbulent PBL:

$$\begin{aligned} &\frac{\partial}{\partial t}\left(\Delta\Psi - L_0^{-2}\Psi\right) + \left[\Psi, \left(\Delta\Psi + f + \frac{\alpha}{D_0}|\nabla\Psi|\right)\right] \\ &\quad = -\frac{1}{D_0}\left(|\nabla\Psi|\Delta\Psi + \nabla\Psi\nabla|\nabla\Psi|\right) + q. \end{aligned} \tag{23.26}$$

The gyroscopic term in (23.26) can be rewritten in the form

$$f + \frac{\alpha}{D_0}|\nabla\Psi| = f\left(1 + \frac{\alpha\varkappa_0|\nabla\Psi|}{H_0 f}\right) = f\left(1 + \alpha\frac{\delta_E}{H_0}\right).$$

Here $\delta_{E\mathrm{turb}} = \varkappa_0|\nabla\Psi|/f = \varkappa_0 U/f$ is the above-mentioned evolving height of the turbulent PBL, creating a nonlinear orographic beta-effect defined by the α-term in Eq. (23.26).

Thus, *in contrast to the laminar Ekman layer, a turbulent planetary boundary layer inhibits the free atmosphere according to the nonlinear friction law, creating an additional nonlinear orographic beta-effect.* The physical interpretation of this phenomenon is quite simple: in contrast to a laminar PBL, in a turbulent PBL not only translational but also rotational degrees of freedom are excited, while the thickness of the PBL, which varies in space and time, is equivalent to the orographic effect.

In the bibliography below one can find further developments of the material presented here, including taking into account specifics of the atmospheric turbulence.

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Part V
Mechanical Prototypes of Equations of Motion of a Rotating Stratified Fluid and a Toy Model of Atmospheric Circulation

Chapter 24
Hydrodynamic Interpretation of the Euler Equations of Motion of a Classical Gyroscope and Their Invariants

24.1 A Hydrodynamical Top

In 1879, a prominent English hydrodynamist A.G. Greenhill made an observation whose theoretical value was recognized almost a century later (see Sect. 24.3 below). He observed that the Euler equations for a rigid body with a fixed point describe the flow of an ideal homogeneous incompressible fluid (whose equations of motion are also named after Euler) inside a triaxial ellipsoid within the class of linear velocity fields. This discovery was used, in particular, by such classics of science as N.E. Zhukovskii, S.S. Hough, and H. Poincaré to study the motions of solids with cavities filled with a fluid (see Moiseev and Rumyantsev 1965).

Consider the motion of an ideal homogeneous incompressible fluid inside a triaxial ellipsoid

$$S \equiv \frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \frac{x_3^2}{a_3^2} - 1 = 0,$$

assuming that the coordinate axes coincide with the directions of its principal axes, while the center of the ellipsoid is at the origin. Generally speaking, such a motion is described by the Euler equations:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\nabla)\mathbf{u} = -\frac{1}{\rho_0}\nabla p, \qquad \operatorname{div}\mathbf{u} = 0, \tag{24.1}$$

or the Helmholtz equation

$$\frac{\partial \boldsymbol{\Omega}}{\partial t} = \{\boldsymbol{\Omega}, \mathbf{u}\} = (\boldsymbol{\Omega}\boldsymbol{\nabla})\mathbf{u} - (\mathbf{u}\boldsymbol{\nabla})\boldsymbol{\Omega}, \tag{24.2}$$

where

$$\{\mathbf{A}, \mathbf{B}\} \doteq (\mathbf{A}\nabla)\mathbf{B} - (\mathbf{B}\nabla)\mathbf{A}$$

F.V. Dolzhansky, *Fundamentals of Geophysical Hydrodynamics*,
Encyclopaedia of Mathematical Sciences 103,
DOI 10.1007/978-3-642-31034-8_24,

is the Poisson bracket of vector fields **A** and **B**. In this case we are talking about strictly solenoidal flows satisfying the impermeability condition on the boundary

$$(\mathbf{u}\nabla)S = 0 \quad \text{for } S = 0. \tag{24.3}$$

Here the density is $\rho_0 = \text{const}$, and $\boldsymbol{\Omega} = \operatorname{rot} \mathbf{u}$.

The divergence-free vector fields

$$\begin{aligned} \mathbf{W}_1 &= -\frac{a_2}{a_3}x_3\mathbf{j} + \frac{a_3}{a_2}x_2\mathbf{k}, \\ \mathbf{W}_2 &= -\frac{a_3}{a_1}x_1\mathbf{k} + \frac{a_1}{a_3}x_3\mathbf{i}, \\ \mathbf{W}_3 &= -\frac{a_1}{a_2}x_2\mathbf{i} + \frac{a_2}{a_1}x_1\mathbf{j} \end{aligned} \tag{24.4}$$

(here **i**, **j**, **k** are vectors in the directions of coordinate axes) are particular stationary solutions of the differential equations (24.1) and (24.2) satisfying the boundary conditions (24.3), and they describe fluid "elliptical" rotations around the corresponding principal axes of the ellipsoid. In the space of vector fields where the metric is given by inner product

$$\langle \mathbf{A}, \mathbf{B} \rangle = \iint_D \mathbf{A} \cdot \mathbf{B}\, dx\, dy\, dz$$

(here the symbol $\cdot$ means the standard local inner product, D is the volume bounded by the ellipsoid), the above vector fields are orthogonal to each other

$$\langle \mathbf{W}_i, \mathbf{W}_j \rangle = 0 \quad \text{for } i \neq j. \tag{24.5}$$

Therefore, in the space of linear divergence-free vector fields tangent to the boundary of D, the set $\mathbf{W}_k$ $(k = 1, 2, 3)$ can be regarded as an orthogonal basis. One can look for a general nonstationary solution of the hydrodynamical equations in such a space in the form

$$\mathbf{u}(\mathbf{r}, t) = \sum_{k=1}^{3} \omega_k(t)\mathbf{W}_k(\mathbf{r}). \tag{24.6}$$

The coefficients $\omega_k(t)$ $(k = 1, 2, 3)$ depending on time only are called the Poincaré parameters and are expressed via components of the vorticity $\boldsymbol{\Omega}$ by the formulas:

$$\omega_k = \frac{a_1 a_2 a_3}{a_k I_k}\Omega_k \quad (k = 1, 2, 3). \tag{24.7}$$

Here $I_k = \sum_{s=1}^{3} a_s^2 - a_k^2$ $(k = 1, 2, 3)$ are nonvanishing entries of the diagonal matrix I (see (24.9)).

Now making the substitution (24.6) in the first Eq. (24.1), multiplying it by each of the basis vectors $\mathbf{W}_k$ and integrating over the volume D, and also taking into account the orthogonality (24.5) and boundary conditions (24.3), we obtain the following system of equations for the vector components $\boldsymbol{\omega} = \omega_1 \mathbf{i} + \omega_2 \mathbf{j} + \omega_3 \mathbf{k}$:

$$\begin{aligned} I_1 \dot{\omega}_1 &= (I_3 - I_2)\omega_2\omega_3, \\ I_2 \dot{\omega}_2 &= (I_1 - I_3)\omega_3\omega_1, \\ I_3 \dot{\omega}_3 &= (I_2 - I_1)\omega_1\omega_2. \end{aligned} \tag{24.8}$$

In vector form this can be rewritten as

$$\dot{\mathbf{m}} = \boldsymbol{\omega} \times \mathbf{m}, \quad \mathbf{m} = I\boldsymbol{\omega}. \tag{24.8$'$}$$

Here I is the above-mentioned diagonal matrix whose components are expressed through the principal axes of the ellipsoid:

$$I = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} = \begin{pmatrix} a_2^2 + a_3^2 & 0 & 0 \\ 0 & a_3^2 + a_1^2 & 0 \\ 0 & 0 & a_1^2 + a_2^2 \end{pmatrix}. \tag{24.9}$$

Equations (24.8) or (24.8′) coincide with the Euler equations of motion of the classical gyroscope (or rigid body with a fixed point, see (12.2)) and, therefore, have two quadratic positive first integrals of motion:

$$E_m = \frac{1}{2}\left(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2\right) \equiv \frac{1}{2}\left(\frac{m_1^2}{I_1} + \frac{m_2^2}{I_2} + \frac{m_3^2}{I_3}\right), \tag{24.10}$$

$$\mathbf{m}^2 = I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2 \equiv m_1^2 + m_2^2 + m_3^2. \tag{24.11}$$

The origin of the first invariant (24.10) is straightforward: by making the substitution (24.6) in the expression for the kinetic energy of the fluid contained inside the ellipsoid, we obtain

$$E = \frac{1}{2}\rho_0 \iint_D u^2 dxdydz = \frac{1}{5}\mu E_m, \quad \mu = \frac{4}{3}\rho_0 \pi a_1 a_2 a_3.$$

In other words, the invariance of E_m, as in the case of a mechanical gyroscope, means the conservation of the kinetic energy of the fluid top.

Consider the origin of the second invariant (24.11). The vorticity satisfying the Helmholtz equation (24.2) is the vector field that characterizes the motion of an ideal incompressible homogeneous fluid. As a consequence, one obtains the Kelvin theorem, whose infinitesimal formulation can be written as follows:

$$\frac{d}{dt}(\boldsymbol{\Omega} \cdot \delta\boldsymbol{\sigma}) \equiv \frac{d\boldsymbol{\Omega}}{dt}\delta\boldsymbol{\sigma} + \boldsymbol{\Omega}\frac{d\delta\boldsymbol{\sigma}}{dt} = 0 \quad \left(\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{u}\nabla\right), \tag{24.12}$$

where $\mathbf{\Omega} \cdot \delta\boldsymbol{\sigma} = K$ is the Kelvin invariant, while $\delta\boldsymbol{\sigma}$ is the area of an element of an oriented surface bounded by a contractible closed liquid contour. Recall (see (1.25)) that the motion of this element is described by the equation

$$\frac{d\delta\boldsymbol{\sigma}}{dt} = -\delta\boldsymbol{\sigma}\frac{\partial \mathbf{u}}{\partial \mathbf{r}} \quad \left(\left(\delta\boldsymbol{\sigma}\frac{\partial \mathbf{u}}{\partial \mathbf{r}}\right)_i = \sum_{k=1}^{3} \delta\sigma_k \frac{\partial u_k}{\partial x_i}\right). \tag{24.13}$$

For the flow under consideration, take $\delta\boldsymbol{\sigma}$ to be an element of the plane P, passing through the origin, the ellipsoid's center (see Dolzhansky 2005). Since we are dealing with a flow which does not move the fluid particle located at the origin, the chosen element, regarded as a liquid surface, will only deform and rotate in space without changing its center location. This means that $\delta\boldsymbol{\sigma} = \delta\boldsymbol{\sigma}(t)$ is a function of time only and it does not depend on spatial coordinates. Then, by substituting (24.6) and (24.7) into (24.12) and (24.13), we obtain

$$\frac{d}{dt}(\mathbf{m} \cdot \mathbf{l}) = 0, \qquad l_i = \frac{a_i}{a_1 a_2 a_3}\delta\sigma_i \quad (i = 1, 2, 3), \tag{24.14}$$

$$\dot{\mathbf{l}} = \boldsymbol{\omega} \times \mathbf{l}. \tag{24.15}$$

From this it follows that the Kelvin invariant for this class of flows can be written in the form

$$K = \mathbf{m} \cdot \mathbf{l},$$

where $\mathbf{l}$ satisfies the Poisson equation (24.15).

Since $\mathbf{m}$ is described by Eq. (24.8′), which is formally identical to Eq. (24.15), then substituting $\mathbf{m}$ instead of $\mathbf{l}$ in (24.14) we get:

The invariant $\mathbf{m}^2$ *for the hydrodynamic gyroscope is a direct consequence of Kelvin's theorem, and its mechanical prototype is the conservation law of angular momentum.*

24.2 Mechanical and Fluid Gyroscopes in the Field of Coriolis Forces

From the point of view of geophysical fluid dynamics, the influence of Coriolis forces on the motion of fluid and mechanical tops is of particular interest. With regard to the mechanical top one only has to specify which rotating coordinate system is actually under consideration. Since the Euler equations for the hydrodynamic gyroscope are written relative to the space, while for the mechanical top relative to the body, then the rotation axis of the new coordinate system for the mechanical top must be chosen to be stationary relative to the body, and not relative to the space. Otherwise, the equations of the mechanical and fluid gyroscopes will not be equivalent because the total angular velocity, measured relative to the body will already depend on time.

Thus, let $\boldsymbol{\Omega}_0$ be the constant angular velocity for the rotation of a new coordinate system relative to the body. Employ the well-known formula for transformation of the time derivatives for an arbitrary vector $\mathbf{A}$ relative to the stationary and rotating coordinate systems (see Landau and Lifschitz 1973):

$$\frac{d\mathbf{A}}{dt} = \left(\frac{d\mathbf{A}}{dt}\right)_R + \boldsymbol{\Omega}_0 \times \mathbf{A}, \tag{24.16}$$

where the index R stands for the time derivative relative to the rotating coordinate system.

Let $\boldsymbol{\omega}$ and $\mathbf{m}$ be the angular velocity and kinetic moment of the body relative to the space, while $\boldsymbol{\omega}_\mathbf{r}$ and $\mathbf{m}_\mathbf{r}$ are the angular velocity and angular momentum for the new coordinate system, where $\boldsymbol{\omega} = \boldsymbol{\omega}_\mathbf{r} + \boldsymbol{\Omega}_0$, $\mathbf{m} = \mathbf{m}_\mathbf{r} + \mathbf{m}_0$ $(\mathbf{m}_0 = I\boldsymbol{\Omega}_0)$. Applying formula (24.16) and taking into account that $d\mathbf{m}/dt = 0$ (conservation of the angular momentum), we have

$$\left(\frac{d(\mathbf{m}_\mathbf{r} + \mathbf{m}_0)}{dt}\right)_R + (\boldsymbol{\omega}_\mathbf{r} + \boldsymbol{\Omega}_0) \times (\mathbf{m}_\mathbf{r} + \mathbf{m}_0) = 0. \tag{24.17}$$

We now consider the coordinate system fixed relative to the body, which rotates relative to the originally chosen one with the angular velocity $-\boldsymbol{\Omega}_0$. Then, according to (24.16),

$$\left(\frac{d(\mathbf{m}_\mathbf{r} + \mathbf{m}_0)}{dt}\right)_R = \left(\frac{d(\mathbf{m}_\mathbf{r} + \mathbf{m}_0)}{dt}\right)_C - \boldsymbol{\Omega}_0 \times (\mathbf{m}_\mathbf{r} + \mathbf{m}_0),$$

where the index of C denotes the time derivative in a coordinate system that is fixed relative to the body. After substituting the last formula in (24.17) and by taking into account that in the chosen coordinate system one has $\dot{\mathbf{m}}_0 = 0$, the equations of motion for a rigid body with a fixed point in a field of Coriolis forces can be written in the form (the indexes R and C are omitted):

$$\dot{\mathbf{m}} = (\mathbf{m} + \mathbf{m}_0) \times \boldsymbol{\omega}, \quad \mathbf{m} = I\boldsymbol{\omega}, \quad \mathbf{m}_0 = I\boldsymbol{\Omega}_0, \tag{24.18}$$

where $\boldsymbol{\omega}$ and $\mathbf{m}$ stand for the angular velocity of the body and its kinetic momentum relative to the rotating coordinate system, but measured relative to the system fixed in the body.

The equations of motion of the fluid gyroscope in the field of Coriolis forces can be easily obtained by applying to the equations of motion of a rotating fluid

$$\frac{d\mathbf{u}}{dt} + 2\boldsymbol{\Omega}_0 \times \mathbf{u} = -\frac{1}{\rho_0}\nabla p, \qquad \operatorname{div}\mathbf{u} = 0, \tag{24.19}$$

the same operation that was used above for Eq. (24.1). As a result, we obtain that within the class of linear velocity fields, the motion of an ideal homogeneous incompressible fluid inside an ellipsoid rotating with constant angular velocity $\boldsymbol{\Omega}_0$ is described by the equations:

$$\dot{\mathbf{m}} = \boldsymbol{\omega} \times (\mathbf{m} + 2\mathbf{m}_0), \quad \mathbf{m} = I\boldsymbol{\omega}, \quad \mathbf{m}_0 = I\boldsymbol{\omega}_0. \tag{24.20}$$

Here, $\boldsymbol{\omega}$ and $\boldsymbol{\omega}_0$ are the vorticity $\boldsymbol{\Omega}$ and the angular velocity of the total rotation $\boldsymbol{\Omega}_0$ (see (24.7)) upon the affine transformation. The equations (24.20) up to trivial changes $\boldsymbol{\omega} \longrightarrow -\boldsymbol{\omega}$ and $2\boldsymbol{\omega}_0 \longrightarrow -\boldsymbol{\Omega}_0$ coincide with (24.18). The need for such changes is related, first to the fact that in the Eulerian description the hydrodynamic equations are written not with respect to "the body" (fluid), but relative to the space, and secondly, Eq. (24.20) are written in terms of vorticity, equal to the doubled angular velocity of the local rotation of the fluid.

Equations (24.20) have two quadratic first integrals of motion:

$$E = \frac{1}{2}\boldsymbol{\omega} \cdot \mathbf{m}, \qquad K^2 = (\mathbf{m} + 2\mathbf{m}_0)^2, \tag{24.21}$$

which correspond to the kinetic energy and the Kelvin invariant. In particular, this means that Eq. (24.20), as well as the mechanical Euler equations, are integrable in quadratures.

24.3 A Historical Note

The Greenhill result mentioned earlier in this chapter suggests that the mechanical and hydrodynamical Euler equations share common fundamental symmetry properties, which should be manifest in common characteristics of their solutions. The decisive step in this direction was made by V.I. Arnold, who formulated a group-theoretical notion of a rigid body whose configuration space (the space of generalized coordinates) is an arbitrary Lie group. This construction, called the *generalized rigid body* (GRB) by Arnold, includes both the mechanical and hydrodynamical Euler equations as special cases. They are obtained if for a configuration space one takes, respectively, SO(3), the group of isometric rotations of the three-dimensional Euclidean space, and SDiff(D), the group of smooth mappings of a bounded domain D of the three-dimensional Euclidean space into itself that preserve the volume element. Thus, the Euler equations of motion of the classical gyroscope can be regarded as mechanical prototypes of the Euler equations of motion of an ideal incompressible fluid. In addition, Arnold discovered that mechanical analogs of the Kelvin circulation theorem and Rayleigh's theorem (see Chap. 16) on the stability of motion of an ideal fluid, whose velocity profile has no inflection points are, respectively, the conservation laws for angular momentum and Euler's theorem on the stability of the gyroscope rotations about the minor and major axes of its inertia tensor.

The next step was made by the author of this book, who generalized Arnold's construction to the cases of the GRB motion in external force fields with a scalar or vector potential. As a result, the group-theoretic concepts of a generalized heavy top (GHT) and the generalized magnetohydrodynamical system (GMHD) were introduced. These concepts include the Euler–Poisson equations of motion of a heavy top and the Oberbeck–Boussinesq equations of motion of an incompressible stratified fluid in the gravity field and respectively the equations of motion of an ideal solid conductor in a magnetic field and the magnetohydrodynamics equations. On this basis, mechanical prototypes for other known fundamental hydrodynamical invariants, such as the potential vorticity and the MHD Woltjer invariants were found. Some of these results related to homogeneous and stratified incompressible fluids are illustrated here in an elementary way (while the rigorous proof is possible only on the basis of the group-theoretic approach) by using the hydrodynamical treatment of the mechanical Euler and Euler–Poisson equations. This will help us construct a toy model of the general atmospheric circulation that reproduces the fundamental properties of the global motions of the real atmosphere. More details on these issues and rather accessible treatment can be found in Dolzhansky (2005).

24.4 Exercises

1. Try to draw the phase portrait of a mechanical or fluid gyroscope in the space of angular momentum or of its hydrodynamical counterpart, using the invariants (24.10), (24.11). Find the fixed points (stationary solutions). How do the phase trajectories behave in the vicinity of fixed points?
2. Express the pressure inside the fluid gyroscope via the Poincaré parameters.

   Answer:

$$p(r,t) = \rho_0 \sum_{i=1}^{3} \sum_{j=1}^{3} P_{ij}\omega_i\omega_j,$$

   where

$$P_{ii} = \frac{1}{2}\sum_{s=1}^{3} x_s^2 - x_i^2, \qquad P_{ij} = -\frac{a_i a_j}{a_i^2 + a_j^2} x_i x_j.$$

3. By using the invariants (24.21) try to sketch the phase portrait of a gyroscope in the field of Coriolis forces, depending on the number $Ro = |\boldsymbol{\omega}/2\boldsymbol{\omega}_0|$, provided that the total rotation occurs around one of the principal axes of the ellipsoid.

   Hint: Trajectories are the intersection of the “energy ellipsoid”

$$\frac{m_1^2}{2EI_1} + \frac{m_2^2}{2EI_2} + \frac{m_3^2}{2EI_3} = 1$$

   with the “circulation sphere” of radius K and centered at $-2\mathbf{m}_0$. In general, the solution of this problem should be computer-assisted, but for small Rossby numbers the result is easy to imagine. For small Rossby numbers describe the motion in an analytical form. What does it remind you of?

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Chapter 25
Mechanical Interpretation of the Oberbeck–Boussinesq Equations of Motion of an Incompressible Stratified Fluid in a Gravitational Field

25.1 A Baroclinic Top

The Oberbeck–Boussinesq equations are of particular interest in connection with their extensive use in the studies of convection of an incompressible fluid, including convection of a rotating fluid, and mechanisms of baroclinic instability. In Part I we already noted that stratification of the fluid, rather than its compressibility, plays a decisive role in the mechanism of baroclinic instability. That is why in theoretical studies it does not make sense to complicate the problem by taking compressibility into account, if one does not consider near- or super-sonic motions. The Oberbeck–Boussinesq equations are written in the form

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\nabla)\mathbf{u} = -\frac{1}{\rho_0}\nabla p + \frac{\rho}{\rho_0}\mathbf{g}, \tag{25.1}$$

$$\frac{\partial \rho}{\partial t} + (\mathbf{u}\nabla)\rho = 0, \qquad \operatorname{div}\mathbf{u} = 0. \tag{25.2}$$

Here $\mathbf{g}$ is the acceleration of gravity, $\rho = \rho(\mathbf{r}, t)$ is the density deviation from the average background value $\rho_0 = \text{const}$, p is the deviation of pressure from the equilibrium hydrostatic distribution $P_0 = P_0(z)$ $(dP_0/dz + g\rho_0 = 0)$. In deriving (25.1) one ignores the excess of hydrodynamical pressure $\rho d\mathbf{u}/dt$ as compared with the Archimedean forces, while $(\rho/\rho_0)\mathbf{g}$ is the total of both gravity and Archimedes forces (see Landau and Lifschitz 1986).

In terms of the vorticity $\mathbf{\Omega}$ and $\mathbf{q} = \nabla\rho/\rho_0$, the equations of motion assume the form

$$\frac{\partial \mathbf{\Omega}}{\partial t} - \{\mathbf{\Omega}, \mathbf{u}\} = -\mathbf{g}\times\mathbf{q}, \tag{25.3}$$

$$\frac{\partial \mathbf{q}}{\partial t} + (\mathbf{u}\nabla)\mathbf{q} = -\mathbf{q}\frac{\partial \mathbf{u}}{\partial \mathbf{r}}. \tag{25.4}$$

F.V. Dolzhansky, *Fundamentals of Geophysical Hydrodynamics*,
Encyclopaedia of Mathematical Sciences 103,
DOI 10.1007/978-3-642-31034-8_25,

They conserve the full energy of the fluid

$$E = \frac{1}{2}\rho_0 \int\!\!\int_D u^2 dxdydz - \int\!\!\int_D \rho \mathbf{g} \cdot \mathbf{r} dxdydz \qquad (25.5)$$

and have two Lagrangian invariants: potential vorticity

$$\Pi = \boldsymbol{\Omega} \cdot \nabla\rho \qquad (25.6)$$

and density ρ (by definition).

An elliptic rotation of such a stratified fluid inside an ellipsoid, arbitrarily oriented in space, can be described in the class of spatially linear velocity fields (24.4), (24.6) and density

$$\rho(r,t) = \mathbf{r} \cdot \nabla\rho = \left.\frac{\partial \rho}{\partial x_1}\right|_0 x_1 + \left.\frac{\partial \rho}{\partial x_2}\right|_0 x_2 + \left.\frac{\partial \rho}{\partial x_3}\right|_0 x_3, \quad \rho(0,t) = 0, \qquad (25.7)$$

where $\nabla\rho = \nabla\rho(t)$ depends only on time. Substituting (24.6) and (24.7) into (25.3) and (25.4), one obtains the system

$$\dot{\mathbf{m}} = \boldsymbol{\omega} \times \mathbf{m} + g\boldsymbol{\sigma} \times \mathbf{l}_0, \qquad (25.8)$$

$$\dot{\boldsymbol{\sigma}} = \boldsymbol{\omega} \times \boldsymbol{\sigma}, \qquad \mathbf{m} = \mathrm{I}\boldsymbol{\omega}, \qquad (25.9)$$

where components of the vector $\boldsymbol{\sigma}$ are relative differences in densities on the major semiaxes of the ellipsoid:

$$\boldsymbol{\sigma} = \frac{1}{\rho_0}\left(a_1 \left.\frac{\partial \rho}{\partial x_1}\right|_0 \mathbf{i} + a_2 \left.\frac{\partial \rho}{\partial x_2}\right|_0 \mathbf{j} + a_3 \left.\frac{\partial \rho}{\partial x_3}\right|_0 \mathbf{k} \right).$$

The vector $\mathbf{l}_0$ is a constant vector having the dimension of length. This vector is defined by the ellipsoid's orientation in space:

$$\mathbf{l}_0 = a_1 \cos\alpha_1 \mathbf{i} + a_2 \cos\alpha_2 \mathbf{j} + a_3 \cos\alpha_3 \mathbf{k},$$

where quantities $\cos\alpha_i$ $(i = 1,2,3)$ are cosines of the corresponding angles of the gravity acceleration vector $\mathbf{g}$ with the principal ellipsoid axes.

According to (25.9), $\boldsymbol{\sigma}^2 = \text{const}$. Therefore, by introducing the unit vector $\boldsymbol{\gamma} = \boldsymbol{\sigma}/\sigma$ and making the changes $\boldsymbol{\omega} \to -\boldsymbol{\omega}$ and $\boldsymbol{\sigma} \to -\boldsymbol{\sigma}$, the system (25.8) and (25.9) can be rewritten in the form

$$\dot{\mathbf{m}} = \mathbf{m} \times \boldsymbol{\omega} + g\sigma\boldsymbol{\gamma} \times \mathbf{l}_0, \qquad (25.10)$$

$$\dot{\boldsymbol{\gamma}} = \boldsymbol{\gamma} \times \boldsymbol{\omega}, \qquad \mathbf{m} = \mathrm{I}\boldsymbol{\omega}. \qquad (25.11)$$

And this is precisely the Euler–Poisson equations of motion of a heavy top, written in a coordinate system that is fixed relative to the body. In this case $\mathbf{m}$ and $\boldsymbol{\omega}$ are the angular momentum and angular velocity of the body, σ is the mass of the top,

$\boldsymbol{\gamma}$ is the unit vector in the gravity direction, and $\mathbf{l}_0$ is the radius vector of the body's center of mass. The Euler–Poisson equations have three first integrals of motion:

$$E_m = \frac{1}{2}\mathbf{m}\cdot\boldsymbol{\omega} + g\sigma\mathbf{l}_0\cdot\boldsymbol{\gamma}, \tag{25.12}$$

$$\Pi_m = \mathbf{m}\cdot\boldsymbol{\gamma}, \qquad \boldsymbol{\gamma}^2 = \gamma_1^2+\gamma_2^2+\gamma_1^2. \tag{25.13}$$

The former integral is the total kinetic and potential energy of the mechanical system. The second one is the projection of angular momentum in the direction of the gravitational field, which, according to E. Noether's theorem, is preserved because of invariance of the Hamiltonian (i.e., energy) with respect to rotations around the vertical axis. The invariance of $\boldsymbol{\gamma}^2$ is a consequence of gravity's immobility relative to the space.

From the hydrodynamical point of view, E_m remains the energy, whereas Π_m can now be regarded as the potential vorticity of flows. The latter can be easily verified by a direct substitution of (24.7) and (25.7) into (25.6). It is remarkable, however, that the invariance of potential vorticity is also a consequence of E. Noether's theorem. Indeed, in the dynamics of an incompressible stratified fluid, the role of equipotential surfaces is played not by horizontal levels, as in the mechanical case, but by surfaces of constant density: any map of such a surface into itself does not change the total potential energy of a stratified fluid. Therefore, to obtain a hydrodynamical analogue of the mechanical invariant Π_m, one has to project not in the vertical direction, but in the direction that is normal to the surface of constant density, i.e., in the direction of $\nabla\rho$. Thus, there is almost a literal analogy between the mechanical and hydrodynamical invariants Π_m and Π.

The described analogy between the equations of motion for a heavy fluid and a heavy top and between their invariants remains valid for motions in the field of Coriolis forces, provided that in the case of a mechanical system, the reference frame is rotated relative to the body rather than relative to the space. In this case, one has mechanical prototypes for the equations of motion of a rotating stratified fluid,

$$\frac{\partial\mathbf{u}}{\partial t} + (\mathbf{u}\nabla)\mathbf{u} + 2\boldsymbol{\Omega}_0\times\mathbf{u} = -\frac{1}{\rho_0}\nabla p + \frac{\rho}{\rho_0}\mathbf{g}, \tag{25.14}$$

$$\frac{\partial\rho}{\partial t} + (\mathbf{u}\nabla)\rho = 0, \qquad \operatorname{div}\mathbf{u} = 0 \tag{25.15}$$

with the integral invariant

$$E = \frac{1}{2}\rho_0\iint\limits_D \mathbf{u}^2\,dxdydz - \iint\limits_D \rho\mathbf{g}\cdot\mathbf{r}\,dxdydz \tag{25.16}$$

and Lagrangian invariants

$$\Pi = (\boldsymbol{\Omega}+2\boldsymbol{\Omega}_0)\cdot\nabla\rho \quad \text{and} \quad \rho. \tag{25.17}$$

The equations of these mechanical prototypes are

$$\dot{\mathbf{m}} = \boldsymbol{\omega} \times (\mathbf{m} + 2\mathbf{m}_0) + g\boldsymbol{\sigma} \times \mathbf{l}_0, \tag{25.18}$$

$$\dot{\boldsymbol{\sigma}} = \boldsymbol{\omega} \times \boldsymbol{\sigma}, \qquad \mathbf{m} = \mathrm{I}\boldsymbol{\omega}, \qquad \mathbf{m}_0 = \mathrm{I}\boldsymbol{\omega}_0. \tag{25.19}$$

By substituting $\boldsymbol{\omega} \to -\boldsymbol{\omega}$, $2\boldsymbol{\omega}_0 \to -\boldsymbol{\omega}_0$ and $\boldsymbol{\sigma}/\sigma \to -\boldsymbol{\gamma}$ these equations can be reduced to the heavy top equations in the Coriolis force field,

$$\dot{\mathbf{m}} = (\mathbf{m} + \mathbf{m}_0) \times \boldsymbol{\omega} + g\sigma\boldsymbol{\gamma} \times \mathbf{l}_0, \tag{25.20}$$

$$\dot{\boldsymbol{\gamma}} = \boldsymbol{\gamma} \times \boldsymbol{\omega}, \qquad \mathbf{m} = \mathrm{I}\boldsymbol{\omega}, \qquad \mathbf{m}_0 = \mathrm{I}\boldsymbol{\omega}_0 \tag{25.21}$$

with the first integrals of motion

$$E_m = \frac{1}{2}\mathbf{m} \cdot \boldsymbol{\omega} + g\sigma\mathbf{l}_0 \cdot \boldsymbol{\gamma}, \tag{25.22}$$

$$\Pi_m = (\mathbf{m} + \mathbf{m}_0) \cdot \boldsymbol{\gamma}, \quad \boldsymbol{\gamma}^2 = \gamma_1^2 + \gamma_2^2 + \gamma_1^2 = 1. \tag{25.23}$$

Below we shall use the following terminology taking into account the hydrodynamical interpretation (24.20) for equations of the classical gyroscope. We will call a *barotropic top* the equations of motion for the classical gyroscope in the Coriolis force field (24.18), while the name *baroclinic top* will stand for Eqs. (25.18) and (25.19) taking into account the stratification of the fluid medium.

25.2 Quasi-geostrophic Approximation of a Baroclinic Top

Having in mind the above analogies, from the point of view of geophysical hydrodynamics it is of special interest to construct a mechanical prototype of quasi-geostrophic equations of motion of a baroclinic atmosphere and to understand its hydrodynamical interpretation. To do this we have the perfect tool, a baroclinic top with its invariants, emphasizing fundamental symmetry properties of the equations of a rotating baroclinic fluid. First, we note that the atmospheric circulation and its laboratory analogues are convective processes. For their description, the Oberbeck–Boussinesq equations are written in terms of temperature fluctuations that are related to density fluctuations having the ratio $T/T_0 = -\rho/\rho_0$. In this case, one needs to replace the quantity $\boldsymbol{\sigma}$ in Eqs. (25.18) and (25.19) by

$$-\boldsymbol{\sigma} = \mathbf{q} = \frac{1}{T_0}\left(a_1 \frac{\partial T}{\partial x_1}\bigg|_0 \mathbf{i} + a_2 \frac{\partial T}{\partial x_2}\bigg|_0 \mathbf{j} + a_3 \frac{\partial T}{\partial x_3}\bigg|_0 \mathbf{k}\right), \tag{25.24}$$

in terms of which the invariants assume the form

$$E_m = \frac{1}{2}\mathbf{m} \cdot \boldsymbol{\omega} + g\mathbf{l}_0 \cdot \mathbf{q}, \tag{25.25}$$

$$\Pi_m = (\mathbf{m} + 2\mathbf{m}_0) \cdot \mathbf{q}, \quad \mathbf{q}^2 = q_1^2 + q_2^2 + q_1^2. \tag{25.26}$$

To derive the desired approximation, we use exactly the same scheme which was used in Part II with respect to the equations of motion of the baroclinic atmosphere. Recall that our approach was as follows.

I. The Rossby number $\varepsilon = U/f_0 L = \Omega_z/f_0$, together with the dimensionless parameters

$$\xi = \frac{f_0^2 L^2}{gH} = O(\varepsilon), \qquad \eta = \frac{N^2 H}{g} = O(\varepsilon) \tag{25.27}$$

are assumed to be small. Note that the same order of smallness is not necessary, and it was used only to simplify the reasoning. Here f_0 is the averaged Coriolis parameter, L and H are typical horizontal and vertical scales of the global atmospherical flows, U and Ω_z are their characteristic horizontal velocity and vertical vorticity, while $N^2 = -g\rho_0^{-1}\partial\rho/\partial z = gT_0^{-1}\partial T/\partial z$ is the square of the Brunt–Väisälä frequency, provided that $\partial T/\partial z > 0$.

II. The motion is assumed to be quasi-hydrostatic and quasi-geostrophic, i.e., relations for the thermal wind are satisfied up to $O(\varepsilon)$.

III. The desired approximation is obtained by expanding the equations for conservation of potential vorticity and temperature transport in parameter ε with accuracy up to the terms $O(\varepsilon^2)$.

Let $\mathbf{g}$ be directed in the negative direction of the axis x_3, around which the ellipsoid rotates with angular velocity $\boldsymbol{\Omega}_0$. For the system (25.18)–(25.19) the parameters ε, L^2 and H are defined by

$$\varepsilon = \frac{\omega}{2\omega_0}\left(\omega = \sqrt{\omega_1^2 + \omega_2^2 + \omega_3^2}\right), \qquad 2L^2 = a_1^2 + a_2^2, \qquad H = a_3. \tag{25.28}$$

Then

$$\xi = \frac{2\omega_0^2(a_1^2 + a_2^2)}{ga_3} = O(\varepsilon), \tag{25.29}$$

$$N^2 = \frac{g}{T_0}\frac{\partial T}{\partial x_3} = \frac{gq_3}{a_3}, \qquad \eta = \frac{N^2 a_3}{g} = q_3 = O(\varepsilon). \tag{25.30}$$

For the hydrodynamical equations (25.14) and (25.15) the thermal wind is defined by

$$-(2\boldsymbol{\Omega}_0\nabla)\mathbf{u} = \frac{1}{T_0}\mathbf{g} \times \nabla T + O(\varepsilon) \tag{25.31}$$

or in the coordinate form

$$\frac{\partial u}{\partial z} = -\frac{g}{2\Omega_0 T_0}\frac{\partial T}{\partial y} + O(\varepsilon), \qquad \frac{\partial v}{\partial z} = +\frac{g}{2\Omega_0 T_0}\frac{\partial T}{\partial x} + O(\varepsilon). \tag{25.32}$$

The model equations (25.18) and (25.19) are associated with the following vector relation for the thermal wind, which follows from (25.18) and (25.24):

$$\boldsymbol{\omega} \times 2\mathbf{m}_0 + g\mathbf{l}_0 \times \mathbf{q} = O(\varepsilon) \tag{25.33}$$

or in the coordinate form $\mathbf{l}_0 = (0, 0, -a_3)$ and

$$\omega_2 = -\frac{a_3 g q_2}{2I_3\omega_0} + O(\varepsilon), \qquad \omega_1 = -\frac{a_3 g q_1}{2I_3\omega_0} + O(\varepsilon). \tag{25.32$'$}$$

With the help of (24.4), (24.6), (25.32) and (25.32′) it is not difficult to show that $\omega_2 \propto \partial u/\partial z \propto -\partial T/\partial y$ and $\omega_1 \propto -\partial v/\partial z \propto -\partial T/\partial x$. Therefore ω_2 and ω_1 can be regarded as affine transformed components of the thermal wind.

According to (25.29), (25.30), and (25.32′),

$$\frac{\omega_2}{\omega_0} \propto \frac{\omega_1}{\omega_0} \propto O(\varepsilon) \propto \frac{q_2}{O(\varepsilon)} \propto \frac{q_1}{O(\varepsilon)},$$

and hence

$$q_1 \propto q_2 \propto O\big(\varepsilon^2\big). \tag{25.34}$$

The model equations (25.18) and (25.19) can be represented in the coordinate form

$$\begin{aligned} I_1\dot{\omega}_1 &= (I_3 - I_2)\omega_2\omega_3 + 2I_3\omega_0\omega_2 + ga_3\sigma_2, \\ I_2\dot{\omega}_2 &= (I_1 - I_3)\omega_1\omega_3 - 2I_3\omega_0\omega_1 - ga_3\sigma_1, \end{aligned} \tag{25.35}$$

$$\begin{aligned} I_3\dot{\omega}_3 &= (I_2 - I_1)\omega_2\omega_3, \\ \dot{\sigma}_1 &= \omega_2\sigma_3 - \omega_3\sigma_2, \\ \dot{\sigma}_2 &= \omega_3\sigma_1 - \omega_1\sigma_3, \end{aligned} \tag{25.36}$$

$$\dot{\sigma}_3 = \omega_1\sigma_2 - \omega_2\sigma_1, \tag{25.37}$$

where in comparison with (25.18) and (25.19) one replaced $\boldsymbol{\sigma} \to -\boldsymbol{\sigma}$, i.e., instead of $\mathbf{q}$ in (25.33) and (25.32′) one uses $\boldsymbol{\sigma} = -\mathbf{q}$.

The system (25.35)–(25.37) has, in particular, the following families of fixed points describing the stationary states of rotations about the principal axes:

(i) $\omega_1 = \omega_2 = 0, \quad \sigma_1 = \sigma_2 = 0, \quad \omega_3 = \omega_{30}, \quad \sigma_3 = \sigma_{30};$

(ii) $\omega_1 = \omega_3 = 0, \quad \sigma_1 = \sigma_3 = 0, \quad \omega_2 = \omega_{20}, \quad \sigma_2 = \sigma_{20},$

$$2I_3\omega_0\omega_{20} + ga_3\sigma_{20} = 0;$$

(iii) $\omega_2 = \omega_3 = 0, \quad \sigma_2 = \sigma_3 = 0, \quad \omega_1 = \omega_{10}, \quad \sigma_1 = \sigma_{10},$

$$2I_3\omega_0\omega_{10} + ga_3\sigma_{10} = 0.$$

The variables marked by the index 0 can assume arbitrary real values (these variables are not to be confused with the external parameter ω_0). It is easy to see that any representative of the family (ii) or (iii) is a *nontrivial strictly geostrophic stationary regime* of motion for any $\omega_0 \neq 0$. From Eq. (25.18), according to estimate (25.33) and relations of thermal wind (25.32′), as well as (25.29), it follows that $\dot{\sigma}_3 = o(\varepsilon^3)$. Consequently, $\sigma_3 = \sigma_{30}$ is constant with a high degree of accuracy, and the last two equations of system (25.36) with the required accuracy can be rewritten as follows:

$$\dot{\sigma}_1 = \omega_2\sigma_{30} - \omega_3\sigma_2, \qquad \dot{\sigma}_2 = \omega_3\sigma_1 - \omega_1\sigma_{30}. \tag{25.38}$$

Now eliminating from (25.36) the quantities σ_1 and σ_2 and using (25.32′), we obtain the system

$$\dot{\sigma}_1 = -\left(\frac{ga_3\sigma_{30}}{2I_3\omega_0} + \omega_3\right)\sigma_2, \qquad \dot{\sigma}_2 = \left(\frac{ga_3\sigma_{30}}{2I_3\omega_0} + \omega_3\right)\sigma_1, \tag{25.39}$$

which can be interpreted as an analogue of the equation for "potential" temperature (more precisely, the equation for its gradient, see Chap. 9), written in terms of the components of thermal wind and reduced by expansion in the parameter ε.

Now it remains to find out what the potential vorticity is in quasi-geostrophic approximation. By the above estimates, the expression for potential vorticity (see (25.26))

$$\Pi = (\mathbf{m} + 2\mathbf{m}_0)\cdot\boldsymbol{\sigma} = I_1\omega_1\sigma_1 + I_2\omega_2\sigma_2 + I_3\omega_3\sigma_3 + 2I_3\omega_0\sigma_3$$

can be rewritten in the form

$$\Pi = I_3(2\omega_0 + \omega_3)\sigma_{30} + O\left(\varepsilon^3\right).$$

Therefore, the quasi-geostrophic potential vorticity is

$$\Pi_G = I_3(2\omega_0 + \omega_3)\sigma_{30}, \qquad \dot{\Pi}_G = I_3\sigma_{30}\dot{\omega}_3, \tag{25.40}$$

and its evolution is described by the first equation of system (25.36).

Thus, the *quasi-geostrophic approximation of system* (25.35)–(25.37) *of the sixth order describing the motion of a baroclinic top is reduced to the dynamical system of order three*:

$$\begin{aligned} I_3\dot{\omega}_3 &= (I_2 - I_1)\omega_1\omega_2, \\ \dot{\omega}_1 &= -\left(\frac{ga_3\sigma_{30}}{2I_3\omega_0} + \omega_3\right)\omega_2, \\ \dot{\omega}_2 &= \left(\frac{ga_3\sigma_{30}}{2I_3\omega_0} + \omega_3\right)\omega_1, \end{aligned} \tag{25.41}$$

in which one employs equations (25.39) and, for uniformity of notation, one makes a formal substitution $\sigma_1 \to \omega_1$, $\sigma_2 \to \omega_2$. System (25.41) corresponds to equations for slow variables in the theory of relaxation oscillations (see, e.g., Arnold et al. 1986), and in this case it describes the slow evolution of the principal components of global geophysical flows, namely, the vertical vorticity ω_3 and the thermal wind (ω_1, ω_2).

The system is written in terms of the defining characteristics of global geophysical flows: namely, the vertical vorticity, the components of the thermal wind, and the vertical stratification. Note that the latter is invariant in this approximation and it enters the equations of motion as an a priori given parameter. This is similar to the case of the quasi-geostrophic approximation for the equations of motion for the real baroclinic atmosphere.

After dividing each of Eqs. (25.41) by ω_0^2 and introducing slow time and new dependent variables

$$\tau = \omega_0 t, \qquad X = \frac{\omega_1}{\omega_0}, \qquad Y = \frac{\omega_2}{\omega_0}, \qquad Z = S + \frac{\omega_3}{\omega_0},$$

system (25.41) can be written in the exceptionally simple form:

$$\dot{X} = -YZ, \qquad \dot{Y} = ZX, \qquad \dot{Z} = \Gamma XY, \tag{25.42}$$

$$\Gamma = \frac{I_2 - I_1}{I_3} = \frac{a_1^2 - a_2^2}{a_1^2 + a_2^2}, \qquad S = \frac{g a_3 \sigma_{30}}{2 I_3 \omega_0^2}. \tag{25.43}$$

Here S is nothing but the stratification parameter S known in geophysical fluid dynamics (see Pedlosky, 1987) and it is related to the parameter of baroclinicity $\alpha^2 = L_R^2 / L_0^2$ (see Chap. 11) as follows:

$$S = \frac{N^2 H^2}{f_0^2 L^2} = \frac{L_R^2}{L^2} = \alpha^2 \frac{L_0^2}{L^2}. \tag{25.44}$$

Here, L_0 is the Rossby–Obukhov scale and L_R is the Rossby internal deformation radius.

Below, without loss of generality, one can set $a_1 > a_2$. System (25.42) has two quadratic first integrals of motion:

$$E_G = \frac{1}{2}\left(\Gamma X^2 + Z^2\right), \qquad \Theta_G = X^2 + Y^2, \tag{25.45}$$

which can be treated as full energy and entropy when $S = 0$ (neutral stratification). As we shall see below, $S \neq 0$ characterizes the degree of deviation from a quasi-geostrophic motion. According to the Obukhov theorem (Gledzer et al. 1981), system (25.42) having two quadratic positive invariants is equivalent to the Euler equations of motion for the classical gyroscope.

Thus, we obtained the following somewhat unexpected result:

The quasi-geostrophic approximation of the equations of motion for a heavy fluid top in the field of Coriolis forces is the Euler equation of motions of a rigid body with a fixed point, formulated in terms of the defining characteristics of global geophysical flows, i.e., in terms of its vertical vorticity and components of thermal wind.

25.3 Exercises

1. Find the fixed points of system (25.35)–(25.37). How do they correspond to the fixed points of system (25.41)? What stationary motions do they describe?
2. Sketch the phase portrait of system (25.42) in the space (X, Y, Z), using the invariants (25.45).
3. Study stability of the regimes (i)–(iii) in the framework of the original and truncated systems (25.35)–(25.37) and (25.41), respectively. What is the impact of the vertical stratification σ_{30}?
4. Find the first integrals of the system (25.41). What is their physical meaning and how do they relate to the first integrals of the quasi-geostrophic equations of motion of a baroclinic atmosphere?

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Chapter 26
Motion of Barotropic and Baroclinic Tops as Mechanical Prototypes for the General Circulation of Barotropic and Baroclinic Inviscid Atmospheres

26.1 Motion of a Barotropic Top

The equation of motion of a barotropic top (24.20)

$$\dot{\mathbf{m}} = \boldsymbol{\omega} \times (\mathbf{m} + 2\mathbf{m}_0) \tag{26.1}$$

in terms of $\mathbf{m_a} = \mathbf{m} + 2\mathbf{m}_0$ can be written in the form

$$\dot{\mathbf{m}}_\mathbf{a} = \boldsymbol{\omega} \times \mathbf{m_a}. \tag{26.2}$$

Taking the scalar product of (26.1) with $\boldsymbol{\omega}$ and of (26.2) with $\mathbf{m_a}$, we obtain two first integrals of motion:

$$2E = \boldsymbol{\omega} \cdot \mathbf{m} = I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2, \tag{26.3}$$

$$\mathbf{m}_\mathbf{a}^2 = (m_1 + 2m_{01})^2 + (m_2 + 2m_{02})^2 + (m_3 + 2m_{03})^2. \tag{26.4}$$

Their existence for the considered class of solutions means the conservation of kinetic energy and validity of the Kelvin circulation theorem, respectively (see Chap. 24).

Using the invariants (26.3) and (26.4), similar to the case $\boldsymbol{\Omega}_0 = 0$ (see Chap. 12), one can get an idea about the behavior of a barotropic top without integrating its equations of motion. In the space of angular momenta, the top trajectories are obtained as the intersections of the "energy" ellipsoids

$$\frac{m_1}{2EI_1} + \frac{m_2}{2EI_2} + \frac{m_3}{2EI_3} = 1$$

with the "circulation" spheres

$$\frac{(m_1 + 2m_{01})^2}{m_a^2} + \frac{(m_2 + 2m_{02})^2}{m_a^2} + \frac{(m_3 + 2m_{03})^2}{m_a^2} = 1$$

of radius $m_a = |\mathbf{m} + 2\mathbf{m}_0|$ centered at the point $-2\mathbf{m}_0$.

F.V. Dolzhansky, *Fundamentals of Geophysical Hydrodynamics*,
Encyclopaedia of Mathematical Sciences 103,
DOI 10.1007/978-3-642-31034-8_26, © Springer-Verlag Berlin Heidelberg 2013

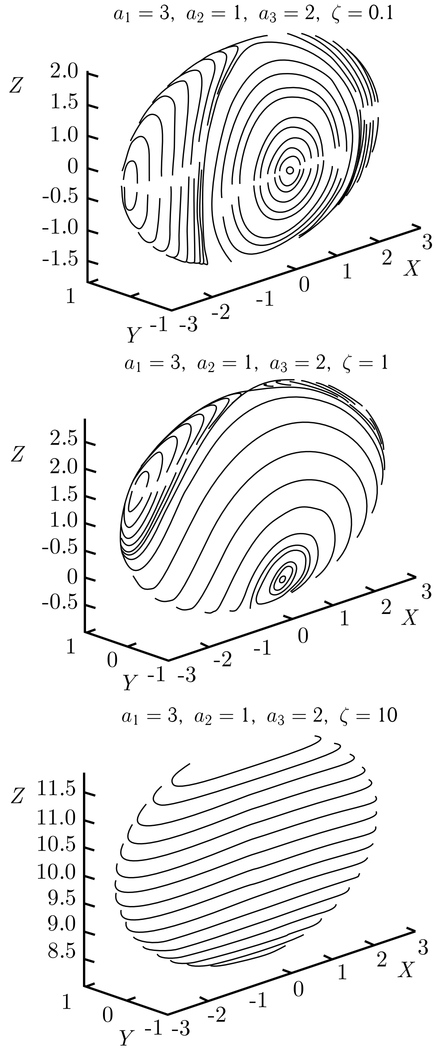

Fig. 26.1 Phase portraits of a barotropic top in the space of angular momenta for various Rossby numbers

Figure 26.1 shows typical phase portraits of the dynamical system (26.1) for various values of the Rossby parameter $\varepsilon = |\mathbf{m}|/|2\mathbf{m}_0|$. These pictures are of interest from the hydrodynamical point of view because they illustrate the process of gradually disappearing complicated elements of motion as the Coriolis force increases. One can see that as ε decreases starting with $\varepsilon = \infty$, hyperbolic points disappear in a consecutive way one after another. Global geophysical flows correspond to small Rossby numbers, for which the trajectories of a barotropic top become almost the intersections of the energy ellipsoid with a family of planes orthogonal to the vector $\mathbf{m}_0$.

This leads to two conclusions: (a) *the phase portrait of geophysical motions of a barotropic top consists of closed elliptic trajectories and does not contain hyper-*

bolic points, (b) *at small Rossby numbers the projection of angular momentum of the barotropic top to the direction of* $\mathbf{m}_0$ *is almost preserved* (up to $O(\varepsilon^2)$).

Motion along the closed trajectories can be easily described under the assumption that the direction of $\mathbf{m}_0$ coincides with the direction of one of the ellipsoid's principal axes, for example, the z-axis or, equivalently, x_3-axis. In this case, (26.1) in the coordinate form becomes (see (25.35)–(25.37)):

$$\begin{aligned} I_1\dot{\omega}_1 &= (I_3 - I_2)\omega_2\omega_3 + 2I_3\omega_0\omega_2, \\ I_2\dot{\omega}_2 &= (I_1 - I_3)\omega_1\omega_3 - 2I_3\omega_0\omega_1, \\ I_3\dot{\omega}_3 &= (I_2 - I_1)\omega_1\omega_2. \end{aligned} \tag{26.5}$$

For $\varepsilon \ll 1$ one has that $\dot{m}_3 = O(\varepsilon^2)$, $m_3 = m_{30} + O(\varepsilon^2)$, $m_{30} = \text{const} = O(\varepsilon)$. Then the equations of motion in the variables $X = \sqrt{I_2}m_1$ and $Y = \sqrt{I_1}m_2$ with an accuracy of order ε^2 can be written as follows:

$$\dot{X} = +2\frac{I_3}{\sqrt{I_1 I_2}}\omega_0 Y, \qquad \dot{Y} = -2\frac{I_3}{\sqrt{I_1 I_2}}\omega_0 X. \tag{26.6}$$

This implies that the endpoint of the vector $\mathbf{m}$, or equivalently $\mathbf{m_a}$, rotates along an elliptical trajectory $m_1^2/I_1 + m_2^2/I_2 = \text{const}$ with the angular velocity

$$\sigma = -2\frac{I_3}{\sqrt{I_1 I_2}}\omega_0 = -2\frac{a_1 a_2}{\sqrt{I_1 I_2}}\Omega_0, \tag{26.7}$$

i.e., in the direction opposite to the rotation of the reference frame.

Given the dual interpretation of the equations of motion of a rigid body with a fixed point, these precessions of a barotropic top can be regarded as a *mechanical prototype of the process of propagation of planetary waves that carry away the angular momentum of the atmosphere in the direction opposite to the Earth's rotation.* In turn, the approximate invariance of the projection of its angular momentum to the direction of $\mathbf{m}_0$ can be thought of as a *mechanical prototype of the approximate Lagrangian invariance of the vertical vorticity of global atmospheric movements, expressed by the Obukhov–Charney equation.*

26.2 Motion of a Baroclinic Top

In Chap. 25 we obtained the simplest three-mode system (25.41):

$$\dot{X} = -YZ, \qquad \dot{Y} = ZX, \qquad \dot{Z} = \Gamma XY, \tag{26.8}$$

where $\Gamma = (I_2 - I_1)/I_3 = (a_1^2 - a_2^2)/(a_1^2 + a_2^2)$ with two positive definite first integrals of motion:

$$2E_G = \Gamma X^2 + Z^2, \qquad \Theta_G = X^2 + Y^2. \tag{26.9}$$

Recall that according to the Obukhov theorem (see Gledzer et al. 1981), this means that the *quasi-geostrophic approximation for the equations of motion of a baroclinic top is equivalent to the Euler equations of motion for a classical gyroscope, written in terms of the main characteristics of general atmospheric circulation, namely, the vertical vorticity, the components of thermal wind, and the stratification parameter.* It is worth mentioning in this relation that the quasi-geostrophic approximation of the reduced equations of motion of a rotating shallow water also coincides with the mechanical Euler equations and it describes the slow evolution of the Rossby waves (Lorenz, 1980).

The families (i)–(iii) (see Chap. 25) of stationary solutions of the complete system of equations (25.35)–(25.37) exhaust the set of fixed points of the reduced system (26.8) and in the new variables they can be written as follows:

$$\begin{aligned} &\text{(i)} \quad X = Y = 0, \qquad Z = Z_0; \\ &\text{(ii)} \quad X = Z = 0, \qquad Y = Y_0; \\ &\text{(iii)} \quad Y = Z = 0, \qquad X = X_0. \end{aligned}$$

Here the quantities with the zero subscript can assume any real value. Note that the trivial solution $X = Y = Z = 0$ for $S \neq 0$ describes circulation around the vertical axis, and in fact it also is a nontrivial representative of the family (i).

The above-mentioned first integrals of motion (26.9) should be interpreted as the total energy and an analogue of the Lagrangian invariance for the potential temperature. The second term in the energy expression for $S = 0$ is the kinetic energy of the vertical vorticity, whereas the first term, *defined by one of the components of the thermal wind*, should be interpreted as a *measure of the available potential energy* of the system. And here is why.

The kinetic energy of the horizontal vorticity, which is not taken into account in a quasi-geostrophic approximation, is generated by the horizontal inhomogeneity of the potential temperature and therefore it is attributed to the potential energy of the quasi-geostrophic system. (This statement is a general provision, which is relevant for any global geophysical flows.) A steady state of system (26.8)

$$X = X_0 \neq 0, \qquad Y = Z = 0 \tag{26.A}$$

with a nonvanishing X-component of the thermal wind corresponds, in dimensional variables, to the horizontal vorticity $\omega_{10} \propto -\partial T/\partial x$. Then the kinetic energy of this vorticity is

$$K_X = \frac{1}{2} I_1 \omega_{10}^2 = \frac{1}{2}\left(a_2^2 + a_3^2\right)\omega_{10}^2.$$

The same vorticity in Y-direction corresponds to the stationary solution

$$X = 0, \qquad Y = X_0 \neq 0, \qquad Z = 0, \tag{26.B}$$

but with the kinetic energy

$$K_Y = \frac{1}{2} I_2 \omega_{10}^2 = \frac{1}{2}\left(a_1^2 + a_3^2\right)\omega_{10}^2.$$

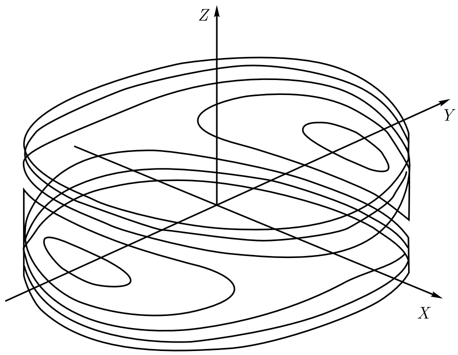

Fig. 26.2 Phase portrait of a quasi-geostrophic motion of a baroclinic top in the space of the dimensionless coordinates X, Y, Z, illustrating the mechanism of the Eady baroclinic instability

Now note that the integrals of motion do not forbid the transition from state (26.A) to the nonstationary state

$$X = 0, \qquad Y = X_0 \neq 0, \qquad Z = \sqrt{\Gamma} X_0,$$

in which the components of X and Y are exchanging roles, but prohibit the reverse transition due to the violation of the energy conservation law. *The difference between the kinetic energies of states* (26.A) *and* (26.B)

$$\Delta K = K_Y - K_X = \frac{1}{2}\left(a_1^2 - a_2^2\right)\omega_{10}^2$$

is exactly the measure of the excess of potential energy, which state (26.A) *has with respect to state* (26.B), *i.e., the available potential energy, which generates vertical vorticity. In dimensionless variables this energy is equal to*

$$P_{bc} = \frac{1}{2}\frac{(a_1^2 - a_2^2)\omega_{10}^2}{I_3\omega_0^2} = \frac{1}{2}\Gamma X^2. \tag{26.10}$$

The phase portrait of system (26.8) is shown in Fig. 26.2. It follows from the portrait that the fixed point (ii) is stable, while (iii), being a hyperbolic point, is unstable. In fact, Fig. 26.2 illustrates Eady's pioneering result, which we discussed in Chap. 18. It was shown that a flow with vanishing vertical vorticity and nonzero thermal wind $\partial u/\partial z = -(g\beta/2\Omega_0)\partial T/\partial y$ turns out to be unstable due to an excess of available potential energy that having been converted into the kinetic energy of vertical vorticity generates atmospheric cyclogenesis. This is exactly the mechanism described by the model (26.8).

In the case of $S \neq 0$ the quantity Z^2 already cannot be interpreted as a measure of the kinetic energy of the system. Therefore, as we shall see below, it is no accident that accurate and quasi-geostrophical solutions achieve the greatest consistency for $S = 0$: the more S differs from zero, the greater the discrepancy between the exact and quasi-geostrophic trajectories. Therefore, the *stratification parameter, in a sense, can be regarded as a measure of deviation of trajectories of the original model from the slow manifold described by the system* (26.8).

Remark It is worth mentioning that solutions of quasi-geostrophical and original atmospheric equations of motion reach the best agreement for $S = O(1)$, but not at zero. In this relation we recall (see Chap. 9) that quasi-geostrophic atmospheric equations of motion are formulated for deviations from the static equilibrium with a stable vertical profile of potential temperature, which corresponds to a positive value of N^2. In our case, however, we started from an equilibrium state with $N^2 = 0$, which was related to the choice of the Oberbeck–Boussinesq approximation for deviations from the state of static equilibrium with a uniform profile of the mean temperature $T_0 = \beta^{-1} = \text{const}$.

26.3 Comparison of Quasi-geostrophic and Exact Motions of a Baroclinic Top Depending on the Stratification Parameter at Small Initial Rossby Numbers

In the case of spherical ($a_1 = a_2 = a_3$) or cylindrical (the Lagrange top) symmetries, the analytical solutions of the quasi-geostrophic triplet could be compared with known analytical solutions of the initial model equations. For a spherically symmetric top, such a comparison was done in the paper by Glukhovsky and Dolzhansky (1980). However, these examples are less interesting, since they completely or partially excluded the hydrodynamical mechanism of the nonlinear interaction of ω-components, which is responsible for generating the ageostrophic component of the motion (see (25.35)). To avoid such a simplification, I shall present the results of numerical integration of the approximate and original model equations, which were performed by A.E. Gledzer and V.M. Ponomarev, and published in the papers by Dolzhansky and Ponomarev (2002), A.E. Gledzer (2003), Dolzhansky (2005). (The proof of non-integrability of an asymmetric heavy top goes back to S. Kowalewski. A modern proof is presented in Ziglin 1980.) All numerical experiments regarding the comparison of solutions to quasi-geostrophic and original model equations were carried out for the ellipsoid with semiaxes $a_1 = 3, a_2 = 1, a_3 = 2$. For the initial data one used the unstable steady state of the quasi-geostrophic triplet $\boldsymbol{\omega}/\omega_0 = (0.1, 0, 0)$ with initial perturbation $\boldsymbol{\omega}'/\omega_0 = (0, 0, 10^{-5})$, while the corresponding values of the components σ_1 and σ_2 were calculated using the thermal wind formulas (25.32′). From one variant to another one altered only the stratification parameter S.

The results of computation are presented in Fig. 26.3 in the form of the projections of phase trajectories of the approximate and original models to the two-dimensional (X, Y) subspace for various positive and negative initial values of S (recall that S is an invariant for the quasi-geostrophic model only). The parameter S was varied within the range $0 \le |S| \le 1$.

First, let us note that for $|S| \ll 1$ the phase portraits of the approximate and original models are almost the same. This points to the existence of the so-called slow manifold, an invariant set in the space of solutions to the inviscid equations of motion of a rotating fluid in whose neighborhood the solutions are quasi-geostrophic, or in physical terms, the set defined as the hypersurface of the adiabatic invariant

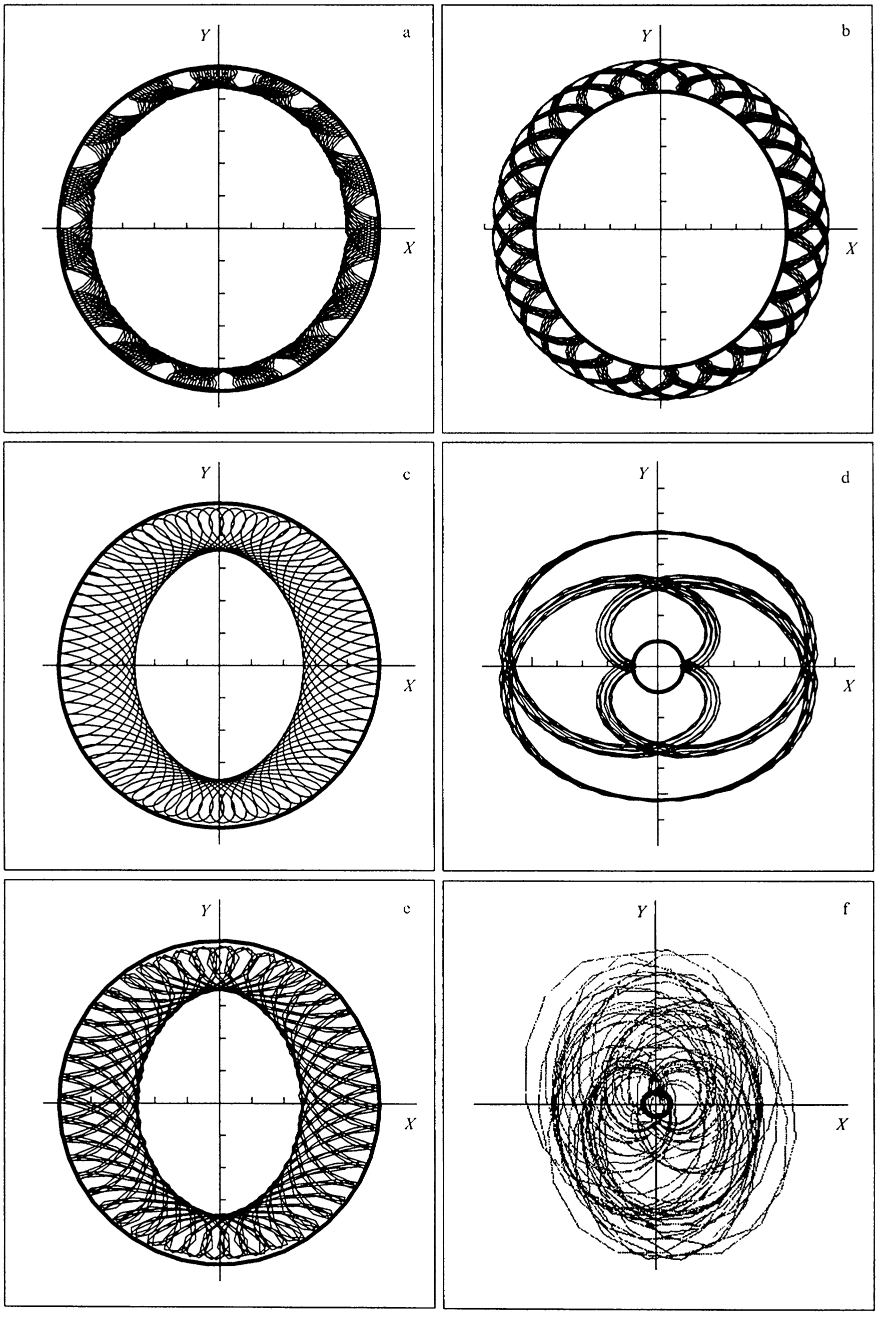

Fig. 26.3 Phase portraits of quasi-geostrophic and "exact" motions of a baroclinic top in the plane X, Y for different values of the parameter S: the numbering is left-to-right and top-to-bottom for $S = 0.2, -0.2, 0.6, -0.6, 0.65, -0.65$. *Boldfaced curves* correspond to quasi-geostrophic trajectories, while *thin lines* correspond to exact trajectories

(6.5). The problem of existence and stability of a quasi-geostrophic manifold, first mentioned by A.M. Obukhov (1949), is intensely discussed in modern hydrodynamical literature (see, e.g., Lorenz 1980, 1986; Lorenz and Krishnamurthy 1987, and references therein).

For small and moderate values of $|S|$ the "exact" trajectories are reflected from the slow manifold either inside or outside of it depending on the sign of S, as if it were some kind of distorting mirror. The higher the value of $|S|$, the greater the amplitude of deviation. We emphasize that this feature is observed even when the ageostrophic amplitude becomes in magnitude comparable to (or even exceeding) the geostrophical component for positive changes in the stratification parameter up to $S = 1$ and $|S| \approx 0.64$ for its negative changes. The breakdown of the "mirror" happens from the outside with $S \approx -0.65$, and a trajectory fills the previously inaccessible space within a finite time interval $\Delta\tau \sim 10^2$, which is apparently accompanied by an appearance of chaos.

A hierarchy of models occupying an intermediate position between the quasi-geostrophic and original equations of motion of a baroclinic top and which allow an analytic description of its behavior at moderate values of S was constructed by A.E. Gledzer (2003). In this paper, in particular, one considered a system of equations for the next approximation after that of thermal wind (25.32′). In this approximation one sets the sum of the second and third terms on the right-hand sides of Eq. (25.35) to zero (the geostrophic balance). The system, which includes both slow and fast oscillations, demonstrated in Fig. 26.3, is obtained from (25.35) and (25.36) by neglecting the nonlinear terms in the equations for ω_1, ω_2 and setting $\omega_3 = 0$, $\sigma_3 = \text{const} = \sigma_{30}$ in the equations for σ_1, σ_2:

$$\begin{aligned} I_1\dot{\omega}_1 &= +2I_3\omega_0\omega_2 + ga_3\sigma_2, \\ I_2\dot{\omega}_2 &= -2I_3\omega_0\omega_1 - ga_3\sigma_1, \\ \dot{\sigma}_1 &= +\omega_2\sigma_3, \qquad \dot{\sigma}_2 = -\omega_1\sigma_3 \end{aligned} \tag{26.11}$$

where

$$I_1 = a_2^2 + a_3^2, \qquad I_2 = a_1^2 + a_3^2, \qquad I_3 = a_1^2 + a_2^2.$$

We transform this system to the dimensionless time

$$\tau = \omega_0 \frac{a_1^2 + a_2^2}{a_1 a_2} t$$

and new thermal variables instead of σ_1, σ_2, σ_3:

$$q_i = -\frac{ga_3}{2(a_1^2 + a_2^2)\omega_0}\sigma_i, \quad i = 1, 2, \qquad q_3 = -\sigma_3 \frac{ga_3 a_1 a_2}{2[(a_1^2 + a_2^2)\omega_0]^2}. \tag{26.12}$$

Then Eqs. (26.11) assume the form

$$\frac{d\omega_1}{dt} = +\frac{2a_1a_2}{a_2^2 + a_3^2}(\omega_2 - q_2),$$

$$\frac{d\omega_2}{dt} = -\frac{2a_1a_2}{a_1^2 + a_3^2}(\omega_1 - q_1), \tag{26.13}$$

$$\frac{dq_1}{dt} = +\omega_2 q_3, \qquad \frac{dq_2}{dt} = -\omega_1 q_3.$$

From (26.12) and (25.42) it follows that

$$S = -\frac{a_1^2 + a_2^2}{a_1a_2} q_3.$$

Thermal wind approximation (25.32′) in (26.13) corresponds to the equalities $\omega_1 = q_1$, $\omega_2 = q_2$.

The solution of linear system (26.13) for $\omega_1|_{t=0} = \omega_{10} \ll 1$, $\omega_2|_{t=0} = \omega_{20} \ll 1$, $q_1(0) = \omega_{10}$, $q_2(0) = \omega_{20}$, $q_3 = A_3 = \text{const}$ has the form

$$\begin{aligned}
\omega_1(t) &= \omega_{10}\cos(\lambda_1 t) + \frac{\omega_{10}}{\lambda_2^2 - \lambda_1^2}\left(\frac{A_3}{r_1} + \lambda_1^2\right)\big(\cos(\lambda_1 t) - \cos(\lambda_2 t)\big) \\
&\quad + \frac{\omega_{20}}{\lambda_2^2 - \lambda_1^2}\frac{A_3}{r_2 r_1}\left(\frac{\sin(\lambda_1 t)}{\lambda_1} - \frac{\sin(\lambda_2 t)}{\lambda_2}\right), \\
\omega_2(t) &= \omega_{20}\cos(\lambda_1 t) + \frac{\omega_{20}}{\lambda_2^2 - \lambda_1^2}\left(\frac{A_3}{r_2} + \lambda_1^2\right)\big(\cos(\lambda_1 t) - \cos(\lambda_2 t)\big) \\
&\quad - \frac{\omega_{10}}{\lambda_2^2 - \lambda_1^2}\frac{A_3}{r_2 r_1}\left(\frac{\sin(\lambda_1 t)}{\lambda_1} - \frac{\sin(\lambda_2 t)}{\lambda_2}\right),
\end{aligned} \tag{26.14}$$

where

$$r_1 = \frac{b^2 + c^2}{2ab}, \qquad r_2 = \frac{a^2 + c^2}{2ab}.$$

For the chosen parameters $a_1 = 3$, $a_2 = 1$, $a_3 = 2$, $r_1 = 5/6$, $r_2 = 13/6$. Solution (26.14) describes the trajectories shown in Fig. 26.3. In formulas (26.14) there are two frequencies λ_1 and λ_2:

$$\lambda_{1,2}^2 = \frac{(1 - (r_1 + r_2)A_3) \mp \sqrt{(1 - (r_1 + r_2)A_3)^2 - 4A_3^2 r_1 r_2}}{2r_1r_2}, \tag{26.15}$$

where the frequency $\lambda_1 \approx (A_3^2 + (r_1 + r_2)A_3^3)^{1/2}$ is slow.

The oscillating nonincreasing solution, according to (26.15), exists for

$$A_3 < \frac{1}{(\sqrt{r_1} + \sqrt{r_2})^2},$$

which for the selected parameters r_1 and r_2 in terms of the stratification parameter S corresponds to the values of $S > -0.65$ discussed above. Using formulas (26.14) one can also prove (see Gledzer 2003) the “mirror” property of the slow manifold:

for $S < 0$ ($A_3 > 0$) trajectories lie outside of the circles corresponding to the thermal wind (see the right images in Fig. 26.3), while for $S > 0$ ($A_3 < 0$) they lie inside the circles (the left images in Fig. 26.3).

It is interesting to note that the results described in this chapter are very similar to those of E. Lorenz (1980), who compared the exact solutions of the Galerkin-reduced atmospheric equations of motion with the solutions to their quasi-geostrophic approximation. In the Lorentz model the roles of slow and fast motions are respectively played by the planetary and *inertia-gravity waves*. In our case, with $S \neq 0$ the slow evolutions of vertical vorticity and thermal wind are accompanied by high-frequency *inertia-gravity fluctuations*, which periodically move the phase trajectories away from the slow manifold (see Fig. 26.3).

Let us summarize. The *simplest dynamical system, having the principal symmetries of the equations of motion for an ideal rotating fluid reflects the fundamental elements of general atmospheric circulation. These elements include the Rossby waves that carry angular momentum (vorticity) to the west, an approximate invariance of the vorticity projection to the rotation direction, the baroclinic mechanism of the Eady instability, the slow quasi-geostrophic manifold, and its influence on ageostrophic motions. I emphasize that the model is indeed simplest in the sense that any of its reductions will imply a loss of at least one of these fundamental symmetries*.

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Chapter 27
Toy Model for General Circulation of a Viscous Atmosphere

27.1 Consideration of Friction and External Heating

The reader already knows (see Chap. 20) that in geophysical hydrodynamical systems dissipation of kinetic energy mostly occurs in the planetary boundary layer. This layer slows down the motion of a free atmosphere according to an approximately linear friction law. External heating can be accounted for by Newton's formula, as it is often done in theoretical studies. According to this formula, specific heat fluxes are directly proportional to the temperature deviation from its background value. The background temperature distribution is usually taken to be the temperature field which is settled in a stationary fluid due to external nonuniform heating and thermo-conductivity of the medium. Then, following the above assumptions, the viscous motion of a baroclinic top is described by the equations

$$\dot{\mathbf{m}} = \boldsymbol{\omega} \times (\mathbf{m} + 2\mathbf{m}_0) + g\mathbf{l}_0 \times \boldsymbol{\sigma} - \lambda \mathbf{m}, \qquad \dot{\boldsymbol{\sigma}} = \boldsymbol{\omega} \times \boldsymbol{\sigma} + \mu(\boldsymbol{\sigma}_{\boldsymbol{B}} - \boldsymbol{\sigma}). \tag{27.1}$$

Equations (27.1) correspond to (25.18) and (25.19) after replacing $\boldsymbol{\sigma} \to -\boldsymbol{\sigma}$, which is related to transition to the temperature (25.24). In (27.1) $\boldsymbol{\sigma}_{\boldsymbol{B}}$ corresponds to the spatially linear distribution of the background temperature, while the quantities λ and μ should be interpreted as the effective viscosity and thermo-conductivity coefficients. They have dimension reciprocal to time and are determined by the physical parameters of the medium. Recall that, for example, for the atmosphere one has $\lambda \simeq 2\nu/\delta_{St}^2$ (see Chap. 20, after formula (20.22)) and we use this approximation for making quantitative estimates. Other notations are the same as above.

27.2 Toy Circulation of Hadley and Rossby

Consider a typical geophysical situation in which the motion of a viscous baroclinic top occurs under the influence of horizontally inhomogeneous heating. Let us assume, as before, that $a_1 > a_2$, while vectors $\mathbf{g}$ and $\mathbf{\Omega}_0$ are in opposite directions and parallel to the x_3-axis. In order to activate the mechanism of baroclinic instability,

F.V. Dolzhansky, *Fundamentals of Geophysical Hydrodynamics*,
Encyclopaedia of Mathematical Sciences 103,
DOI 10.1007/978-3-642-31034-8_27,

we direct the gradient of the background temperature along the axis x_1. In this case, $\boldsymbol{\sigma_B} = (\sigma_{B1}, 0, 0)$.

Using the procedure described in Chap. 25, we obtain the quasi-geostrophic approximation of system (27.1):

$$\begin{gathered} I_3\dot{\omega}_3 = (I_2 - I_1)\omega_1\omega_2 - \lambda I_3\omega_3, \\ \dot{\omega}_1 = -\omega_2\omega_3 - \mu\omega_1 - \mu g a_3 \sigma_{B1}/2I_3\omega_0, \\ \dot{\omega}_2 = \omega_3\omega_1 - \mu\omega_2. \end{gathered} \tag{27.2}$$

Note that the quantity σ_3 does not enter system (27.2), either parametrically (cf. (25.41)) or in the form of an equation for it, since σ_3 eventually decays because of the homogeneity of the vertical distribution of the background temperature (this is the result of numerical integration).

In dimensionless variables

$$X = \frac{\omega_1}{\mu}, \qquad Y = \frac{\omega_2}{\mu}, \qquad Z = \frac{\omega_3}{\mu}, \qquad \tau = \mu t, \qquad \zeta = \frac{\lambda}{\mu}$$

system (27.2) assumes the form

$$\dot{X} = -YZ - X - D, \qquad \dot{Y} = ZX - Y, \qquad \dot{Z} = \Gamma XY - \zeta Z. \tag{27.3}$$

This choice of the slow time is dictated by the fact that for geophysical systems one usually has $\mu^{-1} \gtrapprox \lambda^{-1} \gg \Omega_0^{-1}$ (for example, the characteristic time of radiative cooling of the Earth's atmosphere is about 10 days, while you can try to estimate the value of λ^{-1} yourself by setting $\nu = \nu_T = 10^5$ cm^2/s, see Chap. 23). The quantity $\zeta = \lambda/\mu$ can be interpreted as an effective Prandtl number and $D = g a_3 \sigma_{B1}/2I_3\omega_0\mu$ as a dimensionless thermal drive.

Let the vector $\boldsymbol{\sigma_B}$ be directed towards the negative side of the axis x_1, so that $D = -|D|$. In this case, the natural convection excited by external heat sources corresponds to positive values of ω_2. System (27.3) has two types of stationary solutions:

$$(H) \quad X = |D|, \qquad Y = Z = 0, \tag{27.4}$$

$$\begin{gathered} (R_{+,-}) \quad X = D_0 \equiv (\zeta/\Gamma)^{1/2}, \qquad Y = \pm D_0^{1/2}\big(|D| - D_0\big)^{1/2}, \\ Z = \pm D_0^{-1/2}\big(|D| - D_0\big)^{1/2}. \end{gathered} \tag{27.5}$$

The meaning of these solutions becomes clear if we compare them with regimes of general atmospheric circulation according to the action mechanisms of toy and real systems as heat engines. First, we note that according to the analysis of solutions in the initial model (see Dolzhansky and Pleshanova 1980; Gledzer et al. 1981) in the regime H both quantities Y and Z are negligible, but strictly positive for any $D \neq 0$ which does not violate the quasi-geostrophic balance. The smallness and positivity of Y means that the natural convection that occurs in cross-sections

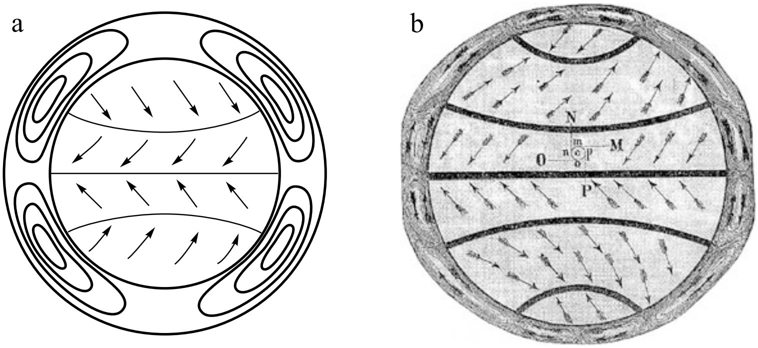

Fig. 27.1 Schemes of general atmospheric circulation: according to (**a**) Hadley (1735) and (**b**) Ferrel (1859)

orthogonal to the axis x_2 is very ineffective for the heat transfer from heater to cooler. An intense circulation around the axis x_1 does not increase the effectiveness of this regime. As a result, the temperature distribution that settled in the fluid is almost identical with the background one (according to the thermal wind relations, $X = -D$ and $Y = 0$ means that $\nabla T = \nabla T_B$). From an energy point of view, the Hadley regime, which is observed in nature, as well as in laboratory and numerical modeling of *general atmospheric circulation* (GAC), is characterized by similar properties. Namely, it features a powerful but ineffective zonal flow, orthogonal to the pole-equator direction, and a feeble natural convection in the meridional (radial) plane (see Figs. 27.1a and 27.2a). The intensity of the meridional circulation is two orders of magnitude weaker than that of the zonal flow (Lorenz, 1967).

In the regimes R the situation changes drastically. Now the intensity of the circulation around the axis x_1 and, according to thermal wind relations, the temperature difference on this axis already does not depend on D, i.e., on the power of an external heat source. If ΔT is this temperature difference, one has $\Delta T/\Delta T_B = |D_0/D| < 1$, and hence in the R regimes the considered heat engine becomes significantly more efficient. The intensity of fluid rotations around the axes orthogonal to ∇T_B increases with an increase of D according to the law $\sqrt{|D| - D_0}$. However, while in the mode R_+, both of these rotations contribute to the heat transfer from heater to cooler, in the mode R_- the circulation around the axis x_2 goes in the direction opposite to the natural convection. This phenomenon, which is observed in both natural and laboratory conditions (see Figs. 27.1b and 27.2b), was once linked to the effect of so-called negative viscosity (Starr, 1968). Again, from an energy point of view, the situation described is similar to the atmospheric and laboratory Rossby regimes. Indeed, have a look at Fig. 27.2b, which schematically presents the results of laboratory experiments on modeling of GAC (see Lorenz 1967; Hide and Mason 1975). It shows that the negative effect of "unnatural" convection in the radial (meridional) plane is compensated by an intense horizontal jet current. The latter comes in contact alternately with the heater and the cooler and enacts the heat transfer in the correct direction. Large-scale vortices enveloped by the jet current

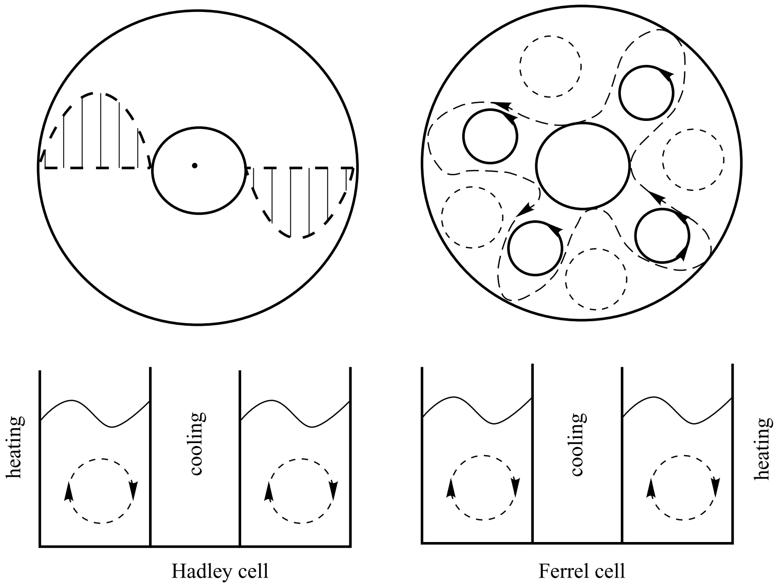

Fig. 27.2 A schematic illustration of the Hadley and Rossby regimes observed in experiments with rotating annuli filled with a fluid which is heated in the periphery and cooled in the center. The top and side views are presented. In radial sections not shown are the compensatory cells which provide a near-surface zonal counter-flow to comply with the conservation law of angular momentum

also facilitate it. In toy Rossby regimes the vertical vorticity ω_3 plays the role of such large-scale vortices. A similar pattern is observed in the atmosphere.

27.2.1 Stability Diagram of the Hadley and Rossby Regimes

In relation to the above-mentioned properties of the H and R regimes, it is of interest to study their regions of existence and stability for comparison with similar regions of the corresponding regimes for nature or laboratory global geophysical flows. According to (27.5), the value of $|D| = D_0$ is the lower boundary for the existence of R modes. In geophysical hydrodynamics, a convection of a rotating fluid is usually described in terms of the thermal Rossby number and the Taylor number

$$Ro_T = \frac{1}{2}\frac{gH\beta\Delta T}{\Omega_0^2 L^2}, \qquad Ta = 4\Omega_0^2 H^4 \nu^{-2},$$

where H is the fluid depth, L is the characteristic horizontal scale (the width of the annular channel in laboratory experiments or the Earth's radius for natural flows), β is the coefficient of thermal expansion, and ΔT is the difference of the heater and cooler temperatures (either of the equator and poles temperatures for the atmosphere or those for the inner and outer walls of the annular channel in laboratory experiments). For the value of ν one takes the molecular kinematic viscosity in laboratory experiments or the turbulent viscosity for atmospherical currents, which is greater than the molecular one by six to seven orders of magnitude.

Being applied to the model under consideration, it is natural to define these quantities by the following formulas:

$$Ro_T = \frac{ga_3|\sigma_{B1}|}{2I_3\omega_0^2}, \qquad Ta = \frac{\omega_0^2}{\lambda^2}. \tag{27.6}$$

In terms of these parameters

$$|D| = Ro_T Ta^{1/2}\zeta. \tag{27.7}$$

Then in the plane of external similarity criteria (Ta, Ro_T) the above-mentioned lower boundary is given by the curve:

$$Ro_T = (\zeta\Gamma Ta)^{-1/2}. \tag{27.8}$$

In the quasi-geostrophic approximation this curve coincides with the lower bound of stability of the R regimes, which is not surprising. What is actually surprising is that *this curve coincides with the asymptotic of the lower bound for the existence and stability of Rossby regimes in annular channels, which was theoretically obtained by Lorenz (1962) on the basis of a truncated two-layer model of a baroclinic flow.*

A detailed study carried out in the works by Dolzhansky and Pleshanova (1980) and E.B. Gledzer et al. (1981), based on the unreduced equations (27.1) and adjusted in Dolzhansky (2005) by additional calculations, shows that the regions of existence and stability of Rossby regimes have a shape similar to that shown in Fig. 27.3, where the upper branch behaves asymptotically as $Ro_T \sim Ta^{1/2}$. This does not mean that the regions of existence and stability for the R_+ and R_- regimes coincide. In particular, the solid curve in Fig. 27.3 reproduces the stability boundary of the regime R_-, whereas the regime R_+ is stable not only within the area bounded by this curve, but also in its outer neigbourhood.

The stability curve of Rossby regimes in annuli which was found in the cited paper by Lorenz (1962) differs from the one presented in Fig. 27.3 by the behavior of its upper branch. In the Lorenz case it asymptotically tends to a constant. It would be somewhat daring to give a rigorous explanation of this distinction, since the models studied are different. Just note that in contrast to Lorenz, in our case construction of the upper branch did not involve the quasi-geostrophic approximation.

Now if we compare theoretical stability boundaries for Rossby regimes with the experimental critical curve shown in Fig. 27.4, we see that, although all such curves have the anvil shape, the lower branch of the experimental curve behaves as $Ro_T \sim Ta^{-1}$, rather than as $Ro_T \sim Ta^{-1/2}$. (One does not have sufficient experimental data with respect to the upper branch of the curve.) The reason for this discrepancy remained unclear until recently. We will return to this issue in Sect. 27.3.

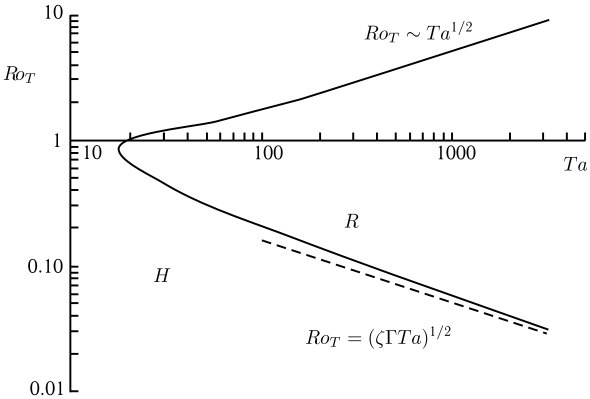

Fig. 27.3 The critical curve of the R_- regime in the plane of external similarity parameters (Ta, Ro_T); the asymptotics of the lower branch coincides with the asymptotics for the lower bound of stability of Rossby modes in annuli, which was theoretically obtained by Lorenz (1962)

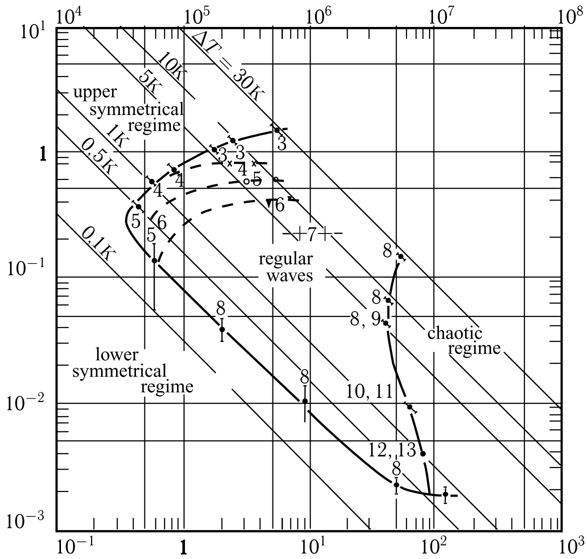

Fig. 27.4 Experimental diagram of instability of convection regimes for a horizontally nonuniformly heated fluid in rotating channels (Hide and Mason, 1975). The square of the angular velocity Ω^2 is proportional to the Taylor number, while the parameter θ, equal to the ratio of the horizontal density difference to Ω^2, is proportional to the thermal Rossby number

Note the following two points in conclusion of this section. First, the critical curve of the mode R_+ was not built because of difficulty in distinguishing this regime from the regime H in the vicinity of the upper branch. Secondly, although in the quasi-geostrophic approximation the $R_\pm$ regimes are equal from the stability point of view, the transition $H \longrightarrow R_+$ dominates over $H \longrightarrow R_-$ in the framework of the original model.

27.2.2 *Efficiency of Toy Regimes of Hadley and Rossby*

Now it is interesting to compare quantitatively the effectiveness of the Hadley and Rossby regimes in terms of the heat transfer from heater to cooler and the production of kinetic energy. In the absence of rotation ($\Omega_0 = 0$) the H regime is a common room convection, whose effectiveness can be evaluated as follows.

(a) Efficiency of the room convection of a nonrotating fluid. If one points $\boldsymbol{\sigma_B}$ in the positive direction of the axis x_1, then for the adopted configuration ($\mathbf{l}_0 = (0, 0, -a_3)$) the room convection according to (27.1) (see also (25.35)–(25.37)) is described by the system of equations:

$$\begin{aligned} I_2\dot{\omega}_2 &= -ga_3\sigma_1 - \lambda I_2\omega_2, \\ \dot{\sigma_1} &= \omega_2\sigma_3 - \mu\sigma_1 + \mu\sigma_{B1}, \\ \dot{\sigma_3} &= -\omega_2\sigma_1 - \mu\sigma_3, \\ \omega_1 &= \omega_3 = 0, \qquad \sigma_2 = 0. \end{aligned} \tag{27.9}$$

Its stationary solutions in terms of dimensionless variables

$$x = \frac{\sigma_1}{\sigma_{B1}}, \qquad y = \frac{\omega_2}{\mu}, \qquad z = \frac{\sigma_3}{\sigma_{B1}}$$

satisfy the equations

$$R_a x + y = 0, \qquad yz - x + 1 = 0, \qquad xy + z = 0, \tag{27.10}$$

$$R_a = \frac{ga_3\sigma_{B1}}{I_2\lambda\mu}. \tag{27.11}$$

Here R_a is the Rayleigh number (in terms of effective coefficients of viscosity and thermal conductivity), which is usually used to characterize the convection of a nonrotating fluid.

It is easy to show that the system (27.10), which in the variable y can be reduced to the incomplete cubic equation

$$y^3 + y + R_a = 0,$$

has a unique real solution. For $R_a \gg 1$, typical for natural and laboratory flows, this solution with good accuracy can be represented in the form

$$x = R_a^{-2/3}, \qquad y = -R_a^{1/3}, \qquad z = R_a^{-1/3}. \tag{27.12}$$

The convection efficiency is the ratio

$$\eta = \frac{\text{speed of the KE generation} = \dot{K}}{\text{solar radiation, i.e., pumping} = Q}, \tag{27.13}$$

where KE is the kinetic energy. Now we need to find out what Q is in this model. The energy conservation law for system (27.9) can be written as follows:

$$\dot{E} = -\lambda I_2\omega_2^2 + \mu g a_3\sigma_3, \tag{27.14}$$

where the total energy is

$$E = \frac{1}{2} I_2 \omega_2^2 - g a_3 \sigma_3. \tag{27.15}$$

The quantity Q is a constant, independent of the dynamical variables of the problem. Therefore, the energy conservation law in the form of (27.14) gives us no information on Q. However, in terms of $\gamma = \sigma_{B1} - \sigma_3$, ω_2, and σ_{B1}, it can be rewritten as

$$\dot{E} = -\lambda I_2 \omega_2^2 - \mu g a_3 \gamma + \mu g a_3 \sigma_{B1}, \tag{27.14$'$}$$

where for the total energy one can now take the value

$$E = \frac{1}{2} I_2 \omega_2^2 + g a_3 \gamma. \tag{27.15$'$}$$

By making the above substitution we have explicitly introduced the temperature difference between heater and cooler as a measure of the external heat pump:

$$Q = \mu g a_3 \sigma_{B1}. \tag{27.16}$$

Since in (27.13) for stationary regimes the value of $\dot{K}$ is equal to minus the dissipation rate of the kinetic energy, then according to (27.11) and (27.12) we have

$$\eta = \frac{\dot{K}}{Q} = \frac{\lambda I_2 \omega_2^2}{\mu g a_3 \sigma_{B1}} = \left(\frac{g a_3 \sigma_{B1}}{I_2 \lambda \mu} \right)^{-1} \frac{\omega_2^2}{\mu^2} = R_a^{-1/3}. \tag{27.17}$$

A straightforward computation shows that the rate of conversion of potential energy $P = g a_3 \gamma$ into kinetic energy measured in Q units, i.e., $\dot{P}/Q$, is also equal to $R_a^{-1/3}$.

For the atmosphere we assume the vertical scale to be $H = a_3 = 10$ km, the horizontal scale $L = \sqrt{I_2} = 5 \times 10^3$ km, $\lambda^{-1} = \mu^{-1} = 10$ days, and $\sigma_{B1} = \Delta T / T_0 = 60/300$ (here 60 K is the temperature difference between the equator and pole), we obtain the estimate

$$\eta \approx 12\ \%. \tag{27.18}$$

This quantity is clearly overestimated, but considering reservations in the use of formula (27.17) to the real atmosphere, it can be found suitable for comparison with the efficiency of toy regimes of Hadley and Rossby, other conditions being the same.

(b) The efficiency of the Hadley and Rossby regimes. Let us now consider the effectiveness of Hadley and Rossby regimes in the quasi-geostrophic approximation. In this case the energy conservation law for system (27.3) has the form

$$\dot{E} = -\zeta Z^2 - \Gamma X^2 + \Gamma |D| X, \tag{27.19}$$

and the sum of the kinetic (KE) and available potential energy (APE) of the toy general atmospheric circulation is

$$E = \frac{1}{2} Z^2 + \frac{1}{2} \Gamma X^2. \tag{27.20}$$

To find the pump we rewrite the energy balance (27.19) in the form

$$\dot{E} = -\zeta Z^2 - \Gamma\left(X^2 - |D|X + \frac{1}{4}|D|^2 - \frac{1}{4}|D|^2\right)$$
$$= -\zeta Z^2 - \Gamma\left(X - \frac{1}{2}|D|\right)^2 + \frac{1}{4}\Gamma|D|^2,$$

whence

$$Q_G = \frac{1}{4}\Gamma|D|^2, \tag{27.21}$$

while the energy and its balance in variables Z, $\Theta = X - |D|/2$ assume the form

$$E = \frac{1}{2}Z^2 + \frac{1}{2}\Gamma\left(\Theta + \frac{1}{2}|D|\right)^2, \qquad (27.20')$$
$$\dot{E} = -\zeta Z^2 - \Gamma\Theta^2 + Q_G.$$

Now, using (27.4) and (27.5), one can find the entire energy cycle:

$$\frac{\text{production of APE}}{Q_G} = \frac{\text{conversion rate of APE into KE}}{Q_G}$$
$$= \frac{\text{generation rate of KE}}{Q_G} = \frac{\zeta Z^2}{Q_G} = \eta_G,$$

where in the regimes (27.4) and (27.5)

$$(H) \quad \eta_G = 0; \qquad (R) \quad \eta_G = \frac{4}{\Lambda}\left(1 - \frac{1}{\Lambda}\right), \tag{27.22}$$
$$\Lambda = \frac{|D|}{D_0} = (\zeta\Gamma Ta)^{1/2} Ro_T.$$

Thus, in the framework of quasi-geostrophic approximation, the effectiveness of the regime H is zero (the incoming radiation energy converts into ageostrophic component and reradiates into space). On the other hand, in the regimes R, the Oort cycle is realized (see Lorenz 1967) with the efficiency

$$\eta_G \approx 17\ \%,$$

if we take for the atmosphere $\zeta = \lambda/\mu \approx 1$, and typical values for $Ro_T \approx 0.03$ and $Ta = 10^6$, adjusted for the fact that the Taylor number, which we defined above, is about four times less than the traditional one. On the diagram shown in Fig. 27.3, the point with the chosen values of Ta and Ro_T is located in the region of the existence and stability of Rossby regimes. Furthermore, in the computation of Γ, the temperature difference $\Delta T = 60$ K in the pole-equator direction is set at the distance of a quarter of the equatorial circumference. The characteristic horizontal scale in the longitude direction corresponds to the most unstable baroclinic mode, for which the number of wavelengths fitting into the average latitudinal circle is equal to 6. This corresponds to $\Gamma \approx 0.6$ and $\Lambda \approx 24$.

The zero efficiency of regime H in practice means that the efficiency of the Hadley regime does not exceed one percent, since in the quasi-geostrophic approximation the kinetic energy is calculated up to the Rossby number squared. Thus, the transition from regime H to regime R the effectiveness of a heat engine considered increases almost by an order of magnitude. However, one should pay attention to one important distinction in the energy cycles of the toy and real general circulations. The toy Rossby mode depletes the reservoir of the APE almost completely:

$$\frac{\mathrm{APE}}{\mathrm{KE}} = \Gamma \frac{X^2}{Z^2} = \zeta [\Lambda - 1]^{-1} \approx 4\ \%,$$

whereas in the real atmosphere this ratio is approximately equal to 3.7 % (Lorenz, 1967).

Apparently, this discrepancy can be explained by the lack of the zonal component of the APE in the interpretation of the model under consideration. Therefore, this discrepancy should be viewed as inevitable retribution for simplifications made.

It is noteworthy that this estimate of "quasi-geostrophic" efficiency is close to the maximal thermodynamically permissible value $(T_E - T_P)/T_E \approx 20\ \%$ (where T_E and T_P are temperatures of the equator and poles, respectively). Moreover, according to (27.22) on the curve $\Lambda = 2 = (\zeta \Gamma Ta)^{1/2} Ro_T$ one has $\eta_G = 1$. In this case this value is not forbidden by thermodynamical laws, since by its very construction the quasi-geostrophic approximation selects only the part Q_G of the external drive power. This is the part which produces only the available potential energy (see the first equation of (27.3)), whose measure is the quantity $\Gamma X^2/2$ in the model considered. In dimensionless variables we accepted above the energy balance equation of the unreduced model can be written in the form

$$\dot{E} = -\zeta \left(\frac{I_1}{I_3} X^2 + \frac{I_2}{I_3} Y^2 + Z^2 \right) - \frac{g a_3}{I_3 \mu^2} \gamma + Q,$$

where $\gamma = |\sigma_{B1}| - \sigma_3$, while the dimensionless power of the external drive is

$$Q = \frac{g a_3 |\sigma_{B1}|}{I_3 \mu^2} = 2\zeta^2 Ta Ro_T.$$

According to the formula (27.21)

$$Q_G = \frac{1}{4} \Gamma |D|^2 = \frac{1}{4} \Gamma \zeta^2 Ta Ro_T^2.$$

It follows that the actual efficiency of the vertical vorticity

$$\eta = \frac{\zeta Z^2}{Q} = \frac{Q_G}{Q} \eta_G = \frac{1}{8} \Gamma Ro_T \eta_G$$

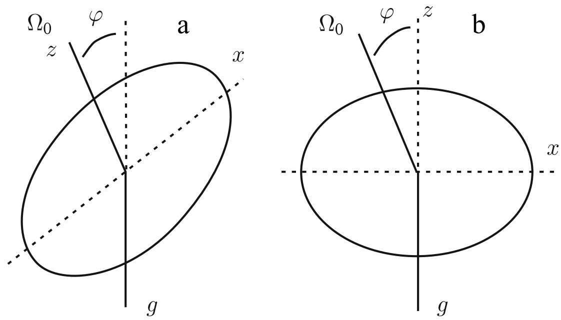

Fig. 27.5 Two orientations of the ellipsoid relative to nonparallel directions of gravity and general rotation, which create the same sloping effect in quasi-geostrophic approximation

constitutes only a fraction of a percent.

27.3 Influence of the Inclination Angle of the Axis of General Rotation Relative to the Gravity Direction

The experimental diagram in Fig. 27.4 shows that in the region bounded by the critical curve of Rossby regimes, along with stationary modes one also observes strictly periodic and irregular oscillations. However, within the above problem formulation, no oscillations were found in system (27.1). The reason is that we missed an important factor, which is the beta-effect. Global baroclinic geophysical flows are nothing but an inclined convection of a rotating fluid, which emerges under the condition when the axis of general rotation is not parallel to the total gravity force, and which is what leads to the emergence of the beta-effect.

Not being able to mimic the beta-effect, let us restrict ourselves to the convection “inclination”, i.e., its sloping effect (or, sloping convection). Let $\boldsymbol{\omega}_0$ and $-\mathbf{g}$ form an angle φ, as shown in Fig. 27.5 for two orientations of the ellipsoid with respect to gravity and general rotation. The angle is assumed to be sufficiently small in order not to affect thermal wind relations (25.32′).

Given the smallness of φ, the quasi-geostrophic approximation of system (27.1) for both of the orientations in terms of X, Y and Z can be written as follows:

$$\dot{X} = -YZ - X - D, \qquad \dot{Y} = XZ - Y, \qquad \dot{Z} = \Gamma XY - \beta Y - \zeta Z, \tag{27.23}$$

where $\beta = \beta_0 Ta^{1/2}\zeta$ and $\beta_0 = 2(a_1/a_3)\varphi$ or $\beta_0 = 2(I_1/I_3)\varphi$ depending on the orientation (a) or (b) in Fig. 27.5.

For $\Gamma = 0$ ($a_1 = a_2$) by changing the variables

$$X = (\zeta/\beta)z - D, \qquad Y = -(\zeta/\beta)y, \qquad Z = x,$$

system (27.23) reduces to the well-known stochastic dynamical Lorenz system of the third-order (Lorenz, 1963b):

$$\dot{x} = \zeta(y - x), \qquad \dot{y} = -xz - y + rx, \qquad \dot{z} = yx - bz, \tag{27.24}$$

where $b = 1$ and $r = (\beta/\zeta)D = \text{sign}(\sigma_{B1})\beta_0\zeta Ro_T Ta$.

This means that under certain conditions discussed below, system (27.23) describes stochastic regimes of toy global geophysical flows. Note that in the geophysical interpretation, Eqs. (27.24) describe the slow quasi-geostrophical manifold in the reduced system of interacting planetary and inertia-gravity waves (Lorenz, 1980).

Systems (27.23) and (27.24) are invariant under the changes $D \to -D$, $\beta \to -\beta$ ($\sigma_{B1} \to -\sigma_{B1}$, $\varphi \to -\varphi$). Therefore, we define the sloping effect to be positive (respectively, negative) if $D\beta$ or $\sigma_{B1}\varphi > 0$ (respectively, $D\beta$ or $\sigma_{B1}\varphi < 0$). Since the sloping effect breaks the symmetry of the initial configuration of forces, there are two types of Rossby regimes depending on the sign of this effect. As before, we assume that $\sigma_{B1} < 0$. So for a positive sloping effect, i.e., for $D < 0$ and $\beta < 0$, stationary Hadley and Rossby regimes are described by the formulas:

$$(H) \quad X = |D|, \qquad Y = Z = 0,$$

$$(R_{+,-}) \quad X = D_1 \equiv \sqrt{\beta^2/4\Gamma^2 + \zeta/\Gamma} - |\beta|/2\Gamma,$$

$$Y = \pm D_1^{1/2}\big(|D| - D_1\big)^{1/2}, \qquad Z = \pm D_1^{-1/2}\big(|D| - D_1\big)^{1/2}.$$

For a negative sloping effect, i.e., for $D < 0$ and $\beta > 0$,

$$(H) \quad X = |D|, \qquad Y = Z = 0,$$

$$(R_{+,-}) \quad X = D_2 \equiv \sqrt{\beta^2/4\Gamma^2 + \zeta/\Gamma} + |\beta|/2\Gamma,$$

$$Y = \pm D_2^{1/2}\big(|D| - D_2\big), \qquad Z = \pm D_2^{-1/2}\big(|D| - D_2\big)^{1/2}.$$

It follows that the equations of lower branches of the existence region of Rossby modes are given by equalities $|D| = D_{1,2}$. In the plane of external similarity criteria (Ta, Ro_T) they are described by the following curves (use (27.6) and the corresponding expression for β):

$$Ro_T = \sqrt{\beta_0^2/4\Gamma^2 + 1/\zeta\Gamma Ta} \mp |\beta_0|/2\Gamma, \tag{27.25}$$

which asymptotically for $Ta \to \infty$ behave as $Ro_T = (\zeta|\beta_0|Ta)^{-1}$ and $Ro_T = \beta_0/\Gamma = \text{const}$ respectively, for positive and negative sloping effects. For more details about these asymptotes, see A.E. Gledzer et al. (2006).

27.4 Conclusions

Thus, under the influence of the sloping effect, the behavioral asymptotics of the lower branch of the critical curve change from $Ro_T \sim Ta^{-1/2}$ to $Ro_T \sim Ta^{-1}$ or $Ro_T = const$, depending on positivity or negativity of the sloping effect, respectively. These changes appear at an angle $|\varphi| = 1°$ and at quite realistic values of the Taylor number, i.e., typical for most laboratory experiments discussed in the papers by Lorenz (1967) and by Hide and Mason (1975) (see also the diagram in Fig. 27.4). Experiments were carried out with rotating annular tanks with a free-surface fluid filling them. The distortion of the free surface under the influence of centrifugal forces creates a positive sloping effect, corresponding to small but finite values of φ. It is likely that the smallness of this angle led Lorenz (1962; 1963a) to neglect the beta-effect created by centrifugal forces while constructing the 8- and 12-component models, which gave the asymptotics $Ro_T \sim Ta^{-1/2}$. It was shown above, that *accounting for the small inclination angle* $\mathbf{\Omega}_0$ *relative to* $-\mathbf{g}$ *yields a result consistent with the experiment* (see Gledzer et al. 2006; Gledzer 2008).

The sloping effect is also a cause of regular and chaotic oscillations in the considered model, which are really observed in laboratory experiments. A number of results obtained via numerical experiments, including the diagram with the asymptotics (27.25), are presented in the author's paper (Dolzhansky, 2005), published in the UFN.

Thus, a heavy top in the field of Coriolis forces can be regarded as a mechanical prototype of the atmospheres of rotating planets: it reproduces the baroclinic mechanism of the Eady instability, the energy framework and stability regions of the fundamental regimes of Hadley and Rossby, the reverse convective Ferrel cell, the few-component turbulence and unpredictability of global geophysical flows. It also reproduces the coexistence of fundamentally different modes of motion and nonzero probability of mutual transitions from one metastable dynamical state to another, which are unmotivated from outside (the latter phenomenon is of particular interest to climatologists).

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Appendix A
On a Certain Boundary Condition

When integrating strictly two-dimensional hydrodynamical equations on a finite region D (Fig. A.1a) one uses the following conservation of the velocity circulation along the boundary ∂D as one of the boundary conditions:

$$\frac{d\Gamma}{dt} = 0, \qquad \Gamma \doteq \oint_{\partial D = C} \mathbf{v}\delta\mathbf{l}. \tag{A.1}$$

This condition is precise, since the liquid contour C which initially belonged to ∂D will remain so at any time due to vanishing of the velocity normal component to the boundary. According to the Kelvin theorem, the value of Γ is preserved since C is a liquid contour.

The integration region V of baroclinic geophysical flows is three-dimensional. For instance, it is shown in Fig. A.1b in the form of a vertical cylinder with horizontal solid ends and solid lateral surface ∂V_S. The liquid contour C belonging to it initially will also remain on ∂V, because of the impenetrability condition. Hence the Kelvin theorem is also applicable to it. However, in this case condition (A.1) is not constructive, since C does not remain at rest and it occupies an unknown position on the surface ∂V. The help comes from the quasi-two-dimensionality property of baroclinic geophysical flows satisfying (9.13) and (9.14). It turns out that in this case the circulation

$$\Gamma_0 \doteq \oint_{C_0} \mathbf{v}\delta\mathbf{l}_0 \tag{A.2}$$

along any contour C_0 formed by the intersection of the horizontal plane $z = z_0$ with the lateral cylinder surface ∂V_S is preserved up to $O(\varepsilon)$.

Indeed, consider the isentropic surface $\Theta(x, y, z, t) = \Theta_s(z_0)$, where $z = z_0$ is the level at which the contour C_0 is located. The circulation along the contour C, formed by the intersection of this surface with ∂V is precisely conserved. The equality $\Theta(x, y, z, t) = \Theta_s(z_0)$ can be rewritten as

$$\theta(x, y, z, t) = \Theta_s(z_0) - \Theta_s(z), \tag{A.3}$$

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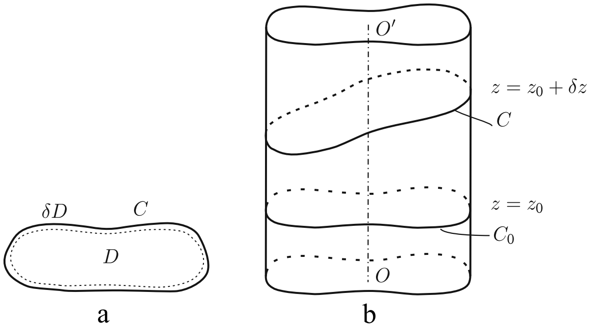

Fig. A.1 (**a**) The liquid contour C, adjacent to the boundary δD of a two-dimensional integration domain D at the initial moment will always remain adjacent because of impenetrability conditions. (**b**) A three-dimensional integration domain: C_0 is a contour formed by the intersection of a horizontal surface $z = z_0$ with the lateral surface of the cylinder, C is a contour formed by an isentropic surface and the lateral surface of the cylinder

where $\theta(x, y, z, t) = \Theta(x, y, z, t) - \Theta_s(z)$ is a deviation of potential temperature from its equilibrium value at level z, induced by the fluid motion. Expanding (A.3) in powers of $\delta z = z - z_0$ and taking into account (9.20), we obtain the following estimate on vertical distances between points of the contour C from the corresponding points of the contour C_0:

$$\delta z = O\left[\left(\frac{1}{\Theta_s}\frac{d\Theta_s}{dz}\right)^{-1}_{z=z_0}\frac{\theta(x, y, z_0, t)}{\Theta_s(z_0)}\right] = \frac{g}{N^2}O\left(\varepsilon^2\right). \tag{A.4}$$

It is convenient to rewrite this formula in accordance with (9.13) in the form

$$\delta z = H_0\frac{g}{N^2 H_0}O\left(\varepsilon^2\right) = H_0\frac{O(\varepsilon^2)}{\eta} = H_0 O(\varepsilon). \tag{A.5}$$

Then the angle φ between the elements $\boldsymbol{\delta}\mathbf{l}$ and $\boldsymbol{\delta}\mathbf{l}_0$ of contours C and C_0 is estimated by the equality

$$\varphi = O\left(\frac{\delta z}{L}\right) = \frac{H_0}{L}O(\varepsilon), \tag{A.6}$$

where L is the characteristic horizontal scale of the baroclinic flow. Consequently,

$$\boldsymbol{\delta}\mathbf{l} = \frac{\boldsymbol{\delta}\mathbf{l}_0}{\cos\varphi} = \boldsymbol{\delta}\mathbf{l}_0\left(1 + \frac{H_0^2}{L^2}O\left(\varepsilon^2\right)\right). \tag{A.7}$$

Furthermore, $\mathbf{v}(z) = \mathbf{v}(z_0 + \delta z) = \mathbf{v}(z_0) + (\partial\mathbf{v}/\partial z)_{z=z_0}\delta z + O[(\delta z)^2]$, and according to the thermal wind relations (9.36)

$$\left|\frac{\partial\mathbf{v}}{\partial z}\right| = O\left(\frac{g}{f_0 L}\frac{\theta}{\Theta_s}\right) = \frac{g}{f_0 L}O\left(\varepsilon^2\right).$$

Therefore, given the relations (9.13) and (9.14) we have

$$\left|\frac{\partial \mathbf{v}}{\partial z}\delta z\right| = \frac{gH_0}{f_0 L}O\big(\varepsilon^3\big) = \frac{gH_0}{f_0^2 L^2}\frac{f_0 L}{U}U\,O\big(\varepsilon^3\big) = \xi^{-1}\varepsilon^{-1}U\,O\big(\varepsilon^3\big) = U\,O(\varepsilon).$$

As a result, we obtain the estimate $\mathbf{v}(z) = \mathbf{v}(z_0 + \delta z) = \mathbf{v}(z_0) + U\,O(\varepsilon)$, the substitution of which into $\Gamma = \oint_C \mathbf{v}\delta\mathbf{l}$ gives

$$\Gamma = \Gamma_0\big(1 + O(\varepsilon)\big). \tag{A.8}$$

This fact allows one to use the preservation of Γ_0 as a boundary condition in integrating the quasi-geostrophic equations of baroclinic motions in the ocean, since in this case, the flow velocities are determined with the same accuracy.

Note that for barotropic geophysical flows the invariance of Γ_0 is precise because in this case there is no vertical velocity shear and fluid particles move along horizontal surfaces. Verify this yourself by integrating the shallow water equations written in the Gromeka–Lamb form along a horizontal closed contour lying on the boundary of the flow domain.

Appendix B
Stability of the Kolmogorov Flow with an External Friction

B.1 Derivation of the Equation for σ

Introduce the notation:

$$a_n = a_n(\widehat{\nu}, \widehat{\lambda}, \sigma) = \frac{2(\widehat{\nu} + \widehat{\lambda})}{\alpha} \cdot \frac{(\alpha^2 + n^2)[\sigma + \widehat{\lambda} + \widehat{\nu}(\alpha^2 + n^2)]}{\alpha^2 - 1 + n^2},$$
$$d_n = d_n(\widehat{\nu}, \widehat{\lambda}, \sigma) = \left(\alpha^2 - 1 + n^2\right) c_n.$$

Then system (21.11) can be rewritten as

$$a_n d_n + d_{n-1} - d_{n+1} = 0. \tag{B.1}$$

Suppose that system (B.1) has a solution satisfying the above requirements. Then there is no value k, for which $d_k = 0$. Indeed, if $d_k = 0$ and $k > 0$, then for $k' \neq k$ one has $d_{k'} \neq 0$. Otherwise the solution according to (B.1) would be trivial because of the regularity condition at infinity. Therefore, for $n > k + 1$ one can introduce the quantity $\rho_n = d_n / d_{n-1}$ and rewrite (B.1) as follows:

$$a_n + \frac{1}{\rho_n} = \rho_{n+1}, \quad n > k + 1. \tag{B.2}$$

A solution of system (B.2) can be written as follows:

$$\rho_n = a_{n-1} + \cfrac{1}{a_{n-2} + \cfrac{1}{a_{n-3} + \cfrac{1}{a_{n-4} + \cfrac{}{\ddots \; a_{k+1} + \cfrac{1}{\rho_{k+1}}}}}}.$$

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Assume that σ is real and $\sigma > -\widehat{\lambda}$. Then $a_n > 0$ for $n > 0$ and $\rho_n > a_{n-1} \to \infty$, which contradicts the requirement that c_n tends to zero as $n \to \infty$. If σ is complex, system (B.2) can be considered separately for real and imaginary coefficients a_n. For the real part of a_n, the same arguments give the same result. The case $d_k = 0$ for $k < 0$ is similar.

So, $d_k \neq 0$ for any k, and for arbitrary n one can set

$$\begin{aligned} \rho_n &= \rho_n(\widehat{\nu}, \widehat{\lambda}, \sigma) = \frac{d_n}{d_{n-1}} \quad (n > 0), \\ \rho_n^* &= \frac{d_{n-1}}{d_n} \quad (n \leq 0). \end{aligned} \tag{B.3}$$

Introduce the following notation for the infinite continued fraction

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots \, a_k + \cfrac{1}{a_{k+1}}}}} \doteq [a_0; a_1, a_2, \ldots, a_k, \ldots].$$

The key statement is as follows. *If*

$$\rho_1 = -[0; a_1, a_2, \ldots, a_k, \ldots], \tag{B.4}$$

then $\rho_n \to 0$ *for* $n \to \infty$. *If equality* (B.4) *does not hold, then* $|\rho_n| \to \infty$.

Indeed, from (B.2) it follows that for $\operatorname{Re}\sigma > -\widehat{\lambda}$, if $\operatorname{Re}\rho_n \geq 0$, then also $\operatorname{Re}\rho_{n+k} \geq 0$ for $k > 0$, or $\operatorname{Re}\rho_{n+k} > \operatorname{Re}a_n \to \infty$ as $n \to \infty$, which is impossible. It is therefore necessary to have $\operatorname{Re}\rho_n < 0$ for all positive n. For any fixed σ there exists such a value k, that $\operatorname{Re}a_n > 1$ for $n > k$, since $\operatorname{Re}a_n \to \infty$ as $n \to \infty$. For $n > k$ the condition $\operatorname{Re}\rho_{n+1} < 0$ along with Eq. (B.2) means that ρ_n in the complex plane is located inside a circle of radius $1/\operatorname{Re}a_n$, tangent to the imaginary axis and lying in the left half-plane (complete Exercise 1). By using Eq. (B.2) again one can show that ρ_{n-1} is located inside a certain circle of radius less than $1/\operatorname{Re}a_n$, and lying in the left half-plane, and so on. It follows that ρ_k lies in a circle in the left half-plane, having a radius less than $1/\operatorname{Re}a_n$. The intersection of circles constructed for ρ_k for various n cannot contain more than one point, since their the radii tend to zero as $n \to \infty$ (why this point does not belong to the imaginary axis?).

One can show (do Exercise 2) that the value ρ_k, given by the formula

$$\rho_k = -[0; a_k, a_{k+1}, \ldots, a_{k+l}, \ldots],$$

lies inside of all these circles and therefore it is the only possible value for ρ_k. Using (B.2) we obtain formula (B.4).

For $n \le 0$ using similar arguments we find that the only value of ρ_0^* that is meaningful for this problem is given by the formula

$$\rho_0^* = [0; a_{-1}, a_{-2}, \dots, a_{-n}, \dots] = [0; a_1, a_2, \dots, a_n, \dots]. \tag{B.5}$$

From (B.1) for $n = 0$ it follows that

$$a_0 + \rho_0^* = \rho_1,$$

whence, due to (B.4) and (B.5),

$$-\frac{a_0}{2} = [0; a_1, a_2, \dots, a_k, \dots]. \tag{B.6}$$

Thus we have proved the following theorem (Meshalkin, Sinai, 1961): *For system* (B.1) *in order to have a solution that tends to zero as $n \to \infty$, it is necessary and sufficient for σ to satisfy* Eq. (B.6).

B.2 Critical Curves

An analysis of Eq. (B.6) allows one to draw some conclusions about stability conditions of the Kolmogorov flow. Denote the right-hand side of (B.6) by $D = [0; a_1, a_2, \dots, a_k, \dots]$, while $D_k = [0; a_1, a_2, \dots, a_k]$ stands for the continued fraction D, truncated at the k-th term. First, we note that for $\alpha > 1$ Eq. (B.6) has no solutions σ, for which $\operatorname{Re}\sigma > 0$, i.e., the Kolmogorov flow is stable with respect to perturbations with this longitudinal wavenumber. Indeed, for $\alpha > 1$ the value $\operatorname{Re}(-a_0/2) < 0$. On the other hand, for any positive k and $\operatorname{Re}\sigma > -\widehat{\lambda}$, the quantity D_k has positive real part and therefore equality (B.6) is impossible.

A refinement of this argument shows that for $\alpha < 1$, Eq. (B.6) with $\operatorname{Re}\sigma > -\widehat{\lambda}$ has only real solutions. Indeed, to fix ideas, suppose that $\operatorname{Im}\sigma \ge 0$. Then

$$\arg\left(-\frac{a_0}{2}\right) \ge \arg a_1, \tag{B.7}$$

and $\arg a_n \ge \arg a_{n+1}$ for any $n \ge 1$, where the equalities are possible only for $\arg\sigma = 0$. The latter inequalities imply that $|\arg D_{k+1}| \le |\arg D_k|$ for any $k \ge 1$ and therefore

$$|\arg D_k| \le |\arg D_{k-1}| \le \dots \le |\arg D_1| = |\arg a_1|.$$

Hence the equality

$$-\frac{a_0}{2} = D_k \tag{B.8}$$

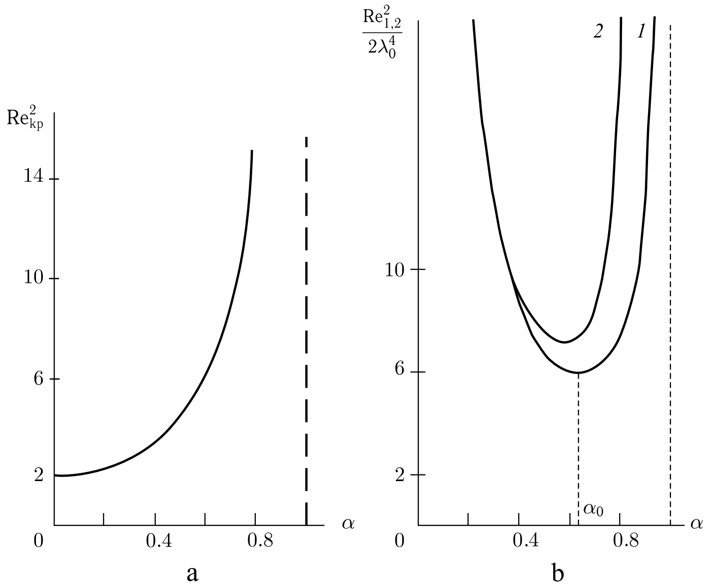

Fig. B.1 (**a**) The critical curve of a strictly two-dimensional Kolmogorov flow. (**b**) The critical curve of a quasi-two-dimensional flow (when taking an external friction into account) is located between curves 1 and 2, corresponding to the first ($k = 1$) and the second ($k = 2$) approximations according to formula (B.8)

for any $k \geq 1$ is possible only provided that $|\arg(-a_0/2)| \leq |\arg a_1|$. Comparing this inequality with (B.7), we conclude that $\mathrm{Im}\,\sigma = 0$. By similar arguments we obtain the same result under the assumption that $\mathrm{Im}\,\sigma \leq 0$. It follows that the *roots of Eq.* (B.6) *for which* $\mathrm{Re}\,\sigma > -\widehat{\lambda}$ *are necessarily real. In this case the principle of the Lin stability change is valid, according to which the critical curve can be found from the condition* $\sigma = 0$.

Approximate critical curves that are the stability boundaries for $\lambda = 0$ and $\lambda \neq 0$ are given in Fig. B.1. The graph plotting is based on the inequalities

$$D_2 < D < D_1. \tag{B.9}$$

Here it is not difficult to show that for $\lambda = 0$ the critical Reynolds number $\widehat{\nu}_{\mathrm{cr}}^{-1} = R_{\nu\mathrm{cr}} \to \infty$ as $\alpha \to 1$ and $R_{\nu\mathrm{cr}} \approx \sqrt{2}$ for $\alpha = 0$, while for $\lambda \neq 0$ the value $R_{\nu\mathrm{cr}} \to \infty$ for $\alpha \to 1$ and $\alpha \to 0$. In particular, an approximate critical curve defined by equality (B.8) for $k = 1$, is described by the formula

$$R^2_{\nu\mathrm{cr}} \approx \frac{2(1+\lambda_0+\alpha^2)(\lambda_0+\alpha^2)(1+\alpha^2)}{\alpha^2(1-\alpha^2)}, \tag{B.10}$$

$$\lambda_0 = \frac{\lambda l^2}{\nu} = \frac{R_\nu}{R_\lambda} = \frac{\widehat{\lambda}}{\widehat{\nu}},$$

which is used in the beginning of Chap. 22.

B.3 Exercises

1. Prove that for $n > k$ the condition $\operatorname{Re}\rho_{n+1} < 0$, together with Eq. (B.2), means that ρ_n in the complex plane is located inside a circle of radius $1/\operatorname{Re}a_n$, tangent to the imaginary axis and lying in the left half-plane.
   Hint: According to the condition $\operatorname{Re}\rho_{n+1} < 0$ and Eq. (B.2)
   $$\operatorname{Re}a_n + \operatorname{Re}\left(\frac{1}{\rho_n}\right) < 0.$$
   Let $\rho_n = r_n e^{i\varphi}$. Then $\operatorname{Re}a_n + r_n^{-1}\cos\varphi < 0$ $(-\pi/2 \le \varphi \le \pi/2)$, while $r_n = -\cos\varphi$ is a circle of radius 1 tangent to the imaginary axis.
2. Show that the value ρ_k, given by the formula
   $$\rho_k = -[0; a_k, a_{k+1}, \ldots, a_{k+l}, \ldots],$$
   lies inside of all the circles having radius less than $1/\operatorname{Re}a_n$, and located in the left half-plane.
3. Try and derive the first approximation (21.16) for the critical curve of the Rossby wave, without trying to prove the convergence of continued fractions in the complex plane.

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