

NONLINEAR
PHYSICAL
SCIENCE

Vladimir V. Uchaikin

Fractional Derivatives for Physicists and Engineers

Volume I Background and Theory



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NONLINEAR PHYSICAL SCIENCE

NONLINEAR PHYSICAL SCIENCE

Nonlinear Physical Science focuses on recent advances of fundamental theories and principles, analytical and symbolic approaches, as well as computational techniques in nonlinear physical science and nonlinear mathematics with engineering applications.

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Fractional Derivatives for Physicists and Engineers

Volume I Background and Theory

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*To my friends
who made my life longer*

Preface

“God made the integers; all else is the work of man”¹. For centuries, the ancients were satisfied with using natural numbers called simply “numbers”. What we call *irrational numbers* was not included into this notion by the Greeks. Not even rational fractions were called numbers.

Hence, *numbers* were conceived as discontinuous, while *magnitudes* were continuous. The two notations appeared, therefore, entirely distinct. The transfer from numbers to magnitudes (to lengths, for example) was a difficult and important step. Perhaps, the most dramatic confrontation of the notions exhibited in *Zeno’s paradoxes*. One of them says: “Achilles cannot overtake a tortoise. Why? Achilles must first reach the place from which the tortoise started. By that time, the tortoise will have moved on a little way. Achilles must then traverse that, and still the tortoise will be ahead. He is always nearer, yet never makes up to it”. The paradox resolution became possible only after extending the concept “number” over the whole real axis.

The real numbers have formed a basis of classical analysis whose major concept is the *continuity*. In frame of this conception, the set of natural and even rational numbers is vanishing (in cardinality) as compared with the continuum of real numbers. In numerical calculations, we use rational numbers as approximations to real ones, but namely irrational numbers reflect the real world. Nobody can make a rod with rational length and nobody can check that the length is rational.

Of course, when we are dealing with a set of isolated objects we use the natural numbers for counting the objects. However, if the objects are numbers and we consider the sum of the numbers, the situation may change. Thus, the number n in the expressions

$$y_n = x_1 + x_2 + \cdots + x_n = \sum_{j=1}^n x_j$$

and

¹ I’m very grateful to Prof. V. Kiryakova for her remark that this phrase is due to Leopold Kronecker (1886).

$$z_n = x_1 \cdot x_2 \cdots x_n = \prod_{j=1}^n x_j$$

needs no comments, it is natural. But if the numbers x_i are identical (and positive), these operations can be easily continued on the whole real axis: $n = \{1, 2, 3, \dots\} \rightarrow v \in (-\infty, \infty)$:

$$y_v = vx, \quad z_v = x^v, \quad v \in \mathbb{R}.$$

The famous Euler invention called Euler's Gamma function

$$\Gamma(v) = \int_0^\infty e^{-x} x^{v-1} dx, \quad v > 0,$$

$$\Gamma(n+1) = n!, \quad n = 1 \cdot 2 \cdot 3 \cdots$$

has played a crucial role in extending the concept “number of operations” on the noninteger values. With this function, there was made a scientific breakthrough in the differential calculus, which enriched it with differentiation and integration of fractional orders called shortly *fractional calculus*. The foundation of fractional calculus is connected with the names of Riemann, Liouville, Weyl, Grünwald, Letnikov and others. Though the first works in this direction were made of about two centuries ago, these ideas had not found any practical applications for a long time. However, the situation has been changed dramatically during a couple of last decades, while about 3 thousand works were published on the subject.

For better understanding of what the extension of this concept can bring, let us consider popular differential equations of theoretical physics of the form

$$a \frac{\partial^m f(x, t)}{\partial t^m} + b \frac{\partial^n f(x, t)}{\partial x^n} = F, \quad (0.1)$$

where x, t are space-time variables, a, b , and F are given functions of x and t , and $m, n = 0, 1, 2, \dots$ are integer numbers. If one of the numbers, say n , is zero, the corresponding variable x becomes a plain parameter. Omitting it, we arrive at the ordinary differential equation:

$$a \frac{d^m f(t)}{dt^m} + b f(t) = F.$$

Putting $n = 1$ in Eq. (0.1) and interpreting $f(t)$ as a velocity of a material point performing one-dimensional motion under action of the force $F - bf$, we recognize here the simplest version of the Newton equation. If $a, b > 0$, then the term $-bf$ can be interpreted as a friction force, and we meet the relaxation problem. When a denotes mass and f means the coordinate of a particle, we again see the Newton equation describing one-dimensional motion along the x -axis. This time, the term $-bf$ means the elastic force and the equation describes the harmonic oscillator driven by the force F (assuming $b = \text{const}$).

Choosing $m = n = 1$, we obtain a one-dimensional continuity equation. This is the simplest equation of partial differential equations of mathematical physics. The

other popular versions of equations of mathematical physics are represented in Table 0.1. They are well-known and do not need any comments. In the last line, you can see the equations containing time- and space-derivatives of fractional orders. These operations are significantly less familiar to the majority of physicists and engineers. One could pay no serious attention to such exotic mathematical construction, but only one glance at Fig. 0.1 may shake the scepticism: we see that the set of

Table 0.1

m, n	1D-equations	3D-equations	Phys. sense	Math. type
1, 0	$ a \frac{dv}{dt} + bv = F$	$ a \frac{d\mathbf{v}}{dt} + b\mathbf{v} = F$	Damped motion	–
2, 0	$ a \frac{d^2x}{dt^2} + bx = F$	$ a \frac{d^2\mathbf{r}}{dt^2} + b\mathbf{r} = F$	Oscillation	–
1, 1	$ a \frac{\partial f}{\partial t} + \frac{\partial(bf)}{\partial x} = F$	$ a \frac{\partial f}{\partial t} + \nabla(\mathbf{b}f) = F$	Continuity	–
1, 2	$ a \frac{\partial f}{\partial t} - b \frac{\partial^2 f}{\partial x^2} = F$	$ a \frac{\partial f}{\partial t} - b \nabla^2 f = F$	Diffusion	Parabolic
2, 2	$ a \frac{\partial^2 f}{\partial t^2} - b \frac{\partial^2 f}{\partial x^2} = F$	$ a \frac{\partial^2 f}{\partial t^2} - b \nabla^2 f = F$	Waves	Hyperbolic
0, 2	$af + b\frac{\partial^2 f}{\partial x^2} = F$	$af + b\nabla^2 f = F$	Static fields	Elliptic
μ, ν non-integers	$a\frac{\partial^\mu f}{\partial t^\mu} + b\left(\frac{\partial^2}{\partial x^2}\right)^{\nu/2}f = F$ $t > 0, -\infty < x < \infty$	$a\frac{\partial^\mu f}{\partial t^\mu} + b\Delta^{\nu/2}f = F$	?	No classified yet

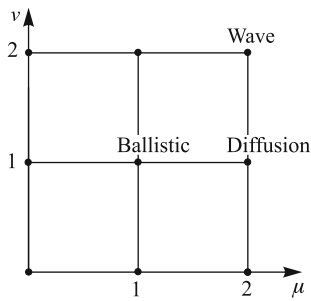


Fig. 0.1 Continuous manifold of fractional partial equations.

well-known and well-investigated differential equations of mathematical physics is represented by only a few points on (μ, ν) diagram, while the continuous set of all other points of the whole (μ, ν) -plane is a *terra incognita*, which can not but attract attention of graduate and postgraduate students, promising scientists and young engineers. This is the readership the book is addressed in the first place. Nevertheless, I hope that it can attract the attention of more experienced researchers, both physicists and mathematicians, as comparatively new tools for investigating and modeling of complex natural processes.

I was 55 years old when I went on a trip for years over this “fractional” land and found many treasures there: about three thousand articles using the tools for solving different problems in physics and engineering have been published during the last two decades. They include

- inverse mechanical problems
- stochastic kinetics and dynamical chaos
- motion in viscous fluid
- heat flow spreading
- electrochemistry of electrodes
- percolation through porous media
- rheology of viscoelastic materials
- electrical and radio engineering
- plasma physics
- quantum optics and nanophysics
- astrophysics and cosmology
- biophysics and medicine

Fortunately, our group managed to participate in developing fractional approach to description of anomalous (dispersive) transport in disordered semiconductors, non-Debye relaxation in solid dielectrics, penetration of light beam through a turbulent medium, transport of resonance radiation in plasma, blinking fluorescence of quantum dots, subrecoil laser cooling of atoms, penetration and acceleration of cosmic ray in the Galaxy, large-scale statistical cosmography and solving some other problems. These investigations allowed us to become aware of deep links between fractional calculus, non-Gaussian Lévy-stable statistics and stochastic fractals. The presence of a time-fractional derivative in the equation is interpreted as a special property of the process under consideration called the *memory*, the *after-effect*, or, when we handle with a stochastic process, the *non-Markovian property*. Fractional derivatives with respect to coordinates reflect a medium with inhomogeneities of some special kind called *selfsimilar inhomogeneities* or *fractals*. We meet such structures in turbulent flows, plasma, and interstellar media.

Like many my colleagues, I'm convinced that the fractional derivative, or, as it is often called, the *fractional differential* given by the expression

$${}_a f^{(\nu)}(x) = \frac{1}{\Gamma(n-\nu)} \frac{d^n}{dx^n} \int_0^x (x-\xi)^{n-\nu-1} f(\xi) d\xi$$

with $n - 1 \leq \nu < n$ if $\nu \geq 0$ and $n = 0$ if $\nu < 0$, is much more than merely a sequence of differential and special integral operators². Important properties of Nature underlie this mathematical concept. Starting to write this book, I decided to begin it with discussion of these properties, enveloping, in my opinion, *heredity*, *nonlocality*, *selfsimilarity*, and *stochasticity*. This is why the first three chapters of the book are united into the first part “Background” which contains description of various natural phenomena demonstrating such properties.

Chapter 1 contains a modern exposition of the Volterra heredity concept whose main tool is the integral operation

$$f(x) \mapsto \int_0^x K(x, \xi) f(\xi) d\xi$$

with the kernel $K(x, \xi)$ interpreted as a memory in case x and ξ are time, or as nonlocality in case the variables are spatial coordinates. The variety of physical processes including mechanical, molecular, hydrodynamical, thermodynamical phenomena demonstrating hereditary properties are described in this chapter.

In Chapter 2 we review physical processes characterized by power-type memory functions and basic mechanisms generated this property. This list may shake the opinion that the exponential function is the queen-function of theoretical physics and show that without power functions like without maids of honor the queen court would be more tiresome.

Chapter 3 opens a wide panorama of stochastic processes which shows that probabilistic long tails of power type results from selfsimilarity of the processes and the latter is connected with existence of limit distributions, namely with the Lévy-stable laws. The reader will go into the fractional Brownian motion continuous-time random walk processes, fractional Poisson process and walking on fractals.

The second part of the book, “Theory”, contains the elements of fractional calculus theory with review of various fractional equations, and their analytical and numerical solutions.

Chapter 4 serves as a mathematical introduction to fractional calculus containing basic definitions of fractional operators, their properties and rules of applications. Readers can recognize many of them as corresponding generalization of well-known analogous from integer-order calculus such as the Leibnitz rule or the Taylor formula.

Chapter 5 shows how equations with fractional derivatives are solved. The reader will meet the description of some analytical methods of solution and many examples of their applications to ordinary and partial fractional equations. Of course, this review can not pretend on a strict and exhaustive exposition, but it will be useful for physicists and engineers as a first acquaintance.

Chapter 6 contains an introduction to numerical methods of solving fractional equation. Starting with the fractional difference operators and based on this con-

² Here are two remarkable quotations: “The fractional calculus is the calculus of the XXI century” (K. Nishimoto, 1989) and “We may say that Nature works with fractional time derivatives” (S. Westerlund, 1991).

cept the Grünwald-Letnikov definition of fractional derivatives, the reader is acquainted with the finite-difference methods of computing fractional integrals, fractional derivatives, and fractional equations of various kinds. The last section of this chapter is devoted to some aspects of Monte Carlo techniques.

The third part, opening the second volume of the book, exposes a wide field of applications of fractional calculus in modern physics including mechanics, hydrodynamics, viscoelasticity, thermodynamics, electrodynamics, plasma physics, quantum physics, and cosmic ray physics.

Finally, the fourth, last part of the book contains various auxiliary materials (special functions, notation of fractional derivatives, main formulas of fractional calculus, tables and graphs of some functions, which are playing a special role in the solution of fractional equations).

Ulyanovsk (Russia), March 2012

Vladimir V. Uchaikin

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Part I

Background

Chapter 1

Heredity and Nonlocality

Most physical processes are described in terms of differential equations containing time-derivatives of not high orders. First-order equations are called the *evolution equations*. They describe such systems whose evolution $f(t)$ is fully determined by their dynamical state at the initial moment t_0 , and doesn't depend on the prehistory $f(t')$, $t' < t$. These systems are referred to as the *systems without memory*. If it is not the case, that is the future of a system, being at a fixed state $f(t_0)$ at time t_0 , depends on its prehistory $f(t')$, $t' < t_0$, then we call it the *system with memory*. This property is often named as *heredity*, *retardation*, or *after-effect* (for stochastic processes, we use the terms *Markovian* and *non-Markovian* for processes without and with memory, respectively). This chapter is devoted to consideration of such processes. At the end we shall discuss the property of nonlocality and some physical systems where it arises.

1.1 Heredity

1.1.1 Concept of heredity

We start with talking about a cause-effect link between two processes $f(t)$ and $g(t)$. In many cases, it is admissible to suppose that the temporal connection between these processes is instantaneous, that is the state $g(t)$ is completely determined by the state $f(t)$ at the same instant:

$$g(t) = F(f(t), t). \quad (1.1)$$

Actually, such instantaneous effect of one process to the other is usually a result of some approximation. A real process requires a certain time for realization of such effect and the only question is whether we should take this retardation into account for the problem under consideration or not. The situation is analogous to the concept of the material particle: we disregard its dimensions while they do not play any role

in our problem, but we make some corrections or abandon the concept at all if the role of the dimensions becomes very essential.

Generally, the mathematical expression of the heredity is realized by replacement of the function $F(f(t), t)$ by some *functional* (*hereditary operator*) $\mathcal{F}(f_t(\cdot); t)$ of the process prehistory $\{f_t(\tau); \tau < t\}$:

$$g(t) = \mathcal{F}[f_t(\cdot); t]. \quad (1.2)$$

Note, that $g(t)$ can be the function $f(t)$ itself and also its first or higher-order derivative:

$$\frac{df(t)}{dt} = \mathcal{F}[f_t(\cdot); t],$$

and so on. We call $\mathcal{F}$ the *functional of influence* of the process $f(t)$ on $g(t)$. If $g(t)$ is also a characteristic of the process $f(t)$ (as before), then the functional can be called the *memory functional*.

1.1.2 A short excursus in history

Apparently, the first description of a mechanical system with retardation was given by L. Boltzmann (1874). Considering an isotropic viscoelastic medium, he assumed that the stress at time t depends on the strains not only at this time, but also on the whole prehistory of the process ($t' < t$), and that the longer the interval (t', t) is, the smaller is the corresponding contribution of the element dt' to the stress at time t . Now, this assumption is called the *memory fading principle*.

The second Boltzmann assumption is the *linear superposition principle*. In his footnote, L. Boltzmann underlined that principle is valid only for small deformations and will not hold for large ones. Some experiments with glass fibers in torsional vibration show good agreement well with Boltzmann's calculations made on these assumptions.

His publication, however, did not point out clearly the need knowing past states for prediction of elasticity effects. Moreover, this point of view stood in contradiction with the Newtonian conception claiming that the knowledge of the present state of a system suffices for the prediction. Thirty years later, E. Picard (1907) came to the conclusion that there exist some systems for prediction of whose behavior in the future ($t > t_0$) one should know not only the present their state (at $t = t_0$) but their past history ($t < t_0$). E. Picard believed that the cause of this was in the presence of *hidden variables*. He brought as an example the pendulum clock whose descending weight is encased. As long as we cannot observe the present position of the weight and its rate of descent, a prediction of the future motion of the clock hands requires the knowledge of when the clock was last wound.

The outstanding Italian mathematician and physicist Vito Volterra devoted a series of papers and several chapters in his book (Volterra, 1928, 1930) to the development of the heredity concept and its application to physical and other problems. His

previous interests were mostly in mechanics, including irreversible phenomena and elasticity. The latter had led him to develop the theory of functionals and integro-differential equations. N. Minorsky (1947) studied the problem of ship stabilization and automatic steering, and pointed out the importance of the delay considerations in the feedback mechanism. A more important role belongs to delaying neutrons in nuclear reactors: the control of the latter would be impossible without this process providing the necessary feedback mechanism.

It is worth to point out the work (Gerasimov, 1948), where the hereditary approach to the viscoelasticity was first formulated in terms of fractional derivatives. Another Russian scientist A. Myshkis (1949) gave the first correct mathematical formulation of the initial value problem for a system with retardation. Furthermore, he introduced a general class of equations with delayed arguments and laid the foundation for a general theory of linear systems (Myshkis, 1955).

The next logical step in the development of the viscoelastic fluid theory was undertaken by Green, Rivlin, Spencer (1957—1960), Coleman and Noll (1960, 1963), and Pipkin (1966) with the use of tensor-valued functionals, Fréchet derivatives and certain formal continuity requirements. This has led to the development of so-called *simple fluid* or more general *simple material model* covering both mechanical and thermodynamical aspects of viscoelasticity.

Viscoelastic properties have been recovered in the special case of thin liquid films under high pressure. In addition to liquids, solids can also behave viscoelastically under mechanical stresses. Polymeric materials, in particular, can show viscoelastic behavior because of their molecular chain structure.

However, there is no need to believe that the memory effect appears only in some specific material or under some special conditions. Let us consider Brownian motion. Its mathematical model assumes that a tracer (heavy particle) displacement during each time interval Δt is a random vector variable independent of preceding and future displacements and distributed according to the Gaussian law. The model prescribes to the process the Markovian property, fixing the lack of memory. One can notice that the model contradicts our intuition. Indeed, if the displacement during Δt_n occurred notably greater than usual, we expect that the next displacement in Δt_{n+1} will also be greater, what means the presence of positive correlations. Our expectation is based on the concept of inertia. But it is not the case: the process of Brownian motion as it is understood by mathematicians doesn't possess this property. Some people may have the opposite meaning: the next displacement should be smaller than the expected in order to compensate the previous displacement (negative correlations). This also contradicts to Markovian property: the next displacement of Brownian particle is independent of the previous one.

From physical point of view, the Markovian property may be assigned to material particle only in the approximate sense. Indeed, the collision time which is supposed to be zero in the model of rigid balls, in reality is finite, and during this time the changes of velocities in the nearest past give some information about the near future. If the tracer mass is commensurable with the mass of surrounding molecules, then the Markovian property can be broken as a result of trapping of molecules colliding with the tracer. At last, the movement of the tracer particle caused perturbations

in the surrounding fluid which affect the further particle motion. The surrounding fluid, like in the dynamic case mentioned above, acts as a memory accumulator breaking the Markovian property of the process. Numerical calculations (Alder and Wainwright, 1967; Wood, 1975) have confirmed that the motion of a single molecule is not a Markovian process namely for these reasons. The theory describing fluids with account of memory effect is often called the *generalized hydrodynamics*.

At the end of this short excursion, I'd like to make the following comment. The relativistic principle claiming boundedness of any signal speed in nature makes cause-effect retardation a common law of all natural phenomena. Consider, for example, the electrical dipole moment induced in an atom by a charged particle flying near by it. In this case, $f(t')$ is the flying particle coordinate at time t' , $g(t)$ is the induced electrical dipole moment at time $t > t'$. The effect of the particle on the atom is carried out through the electric field which propagation requires a some finite time $t - t' = r/c$. However, if the particle velocity is negligibly small in comparison with the field propagation velocity (the light speed), then the effect may be considered as practically instantaneous, and can be described by Eq. (1.1). Otherwise, the retardation should be taken into account, and the argument of the function $f(t)$ should be shifted for r/c , where r is the distance from the particle to the atom at time t , and c is the speed of light. Here, we meet an elementary case of the heredity which is expressed only by a shift of argument (observe that this is only the first-order approximation: the exact solution of the problem requires evaluation of the field generated by the entire prehistory of the particle).

1.2 Volterra's heredity theory

1.2.1 Volterra's heredity laws

For the development of an informal theory and its effective utilization in practice, it is necessary to impose some constraints on the functional $\mathcal{F}$. The first step in this direction was made by V. Volterra who formulated the next principles named by him the *general laws of heredity* (Volterra, 1930, Sect. 148).

1. The *linearity* principle:

$$\mathcal{F}(f_t(\cdot); t) = \int_{-\infty}^t \phi(t, \tau) f(\tau) d\tau$$

(V. Volterra singled the δ -component out of the influence function, complementing the integral by the addend $\sigma(t)f(t)$). According to this, Equation (1.2) takes the form:

$$g(t) = \int_{-\infty}^t \phi(t, \tau) f(\tau) d\tau. \quad (1.3)$$

2. The *fading memory* principle:

$$\phi(t, \tau) \rightarrow 0, \tau \rightarrow -\infty.$$

3. The *invariance* principle:

$$\phi(t, \tau) = \phi(t - a, \tau - a) = \phi(t - \tau, 0) \equiv \Phi(t - \tau).$$

Heredity is called *bounded* if there is such a time $\tau_0 < \infty$, that $\phi(\tau) = 0$ for all $\tau > \tau_0$. The boundedness of heredity allows to expand the concept of hereditary system state: two identical hereditary systems are in the same state at t if their dynamic variables coincide in the interval $(t - \tau_0, t)$.

Actually, these principles determine only the simplest class of hereditary phenomena. In Sect. 153 of the above-cited book, referring to some contemporary works V. Volterra notes that the linear heredity is not enough to explain some electrodynamic phenomena.

1.2.2 Hereditary string

V. Volterra (1930) demonstrate the application of hereditary idea by example of the viscoelastic string one end of which is anchored and the second one is subjected to the torque M which causes this end to turn through angle Θ . If the rotation is slow and the angle is small (the quasistatic case), then the connection between Θ and M is approximated by the linear instantaneous relation

$$\Theta(t) = KM(t), \quad (1.4)$$

where K is a material constant. If the angle is not small enough, we are outside the linear approximation and observe the more complicated dependence:

$$\Theta(t) = F(M(t)), \quad (1.5)$$

where $F(M)$ is a nonlinear material function. Formula (1.4) thus represents the first non-vanishing term of the Maclaurin series:

$$\Theta = F(0) + F'(0)M + \dots \approx KM, \quad F(0) = 0, \quad F'(0) = K.$$

A common property of Eqs. (1.4) and (1.5) is that both variables M and Θ are taken at the same moment: the influence of M on Θ is supposed to be instantaneous, independent of the prehistory of the process. However, more precise measurements show that $\Theta(t)$ depends not only on $M(t)$ at the measurment instant but also on the previous history $M(\tau)$, $\tau < t$. This means that the string has the property of heredity. So, instead of Eq. (1.4), it is necessary to write a functional relationship:

$$\Theta(t) = \mathcal{F}[M_t(\cdot)].$$

Similarly to its predecessor (1.5), the relationship can also be linearized via expansion into the *functional* series up to the first non-vanishing term:

$$\Theta(t) = \mathcal{F}[0] + \int_{-\infty}^t \frac{\delta \mathcal{F}(0)}{\delta M_t(\tau)} M(\tau) d\tau + \dots \approx \int_{-\infty}^t \phi(t, \tau) M(\tau) d\tau. \quad (1.6)$$

It is not difficult to understand the physical sense of the function $\phi(t, \tau)$. Assuming $M(\tau) = \delta(\tau - \tau')$,

$$\phi(t, \tau') = \Theta(t)|_{M(\tau)=\delta(\tau-\tau')},$$

we recognize that this is the rotation angle of the string end at the moment t when the end is subjected to an impulse torque at time $\tau' < t$.

Like the material constants, the material function $\phi(t, \tau)$ cannot be obtained from “first principles” and is deduced either from simulations or from experimental data. According to the invariance principle:

$$\phi(t, \tau) = \Phi(t - \tau),$$

so instead of (1.6), we have

$$\Theta(t) = \int_{-\infty}^t \Phi(t - \tau) M(\tau) d\tau = \int_0^{\infty} \Phi(\tau) M(t - \tau) d\tau.$$

The solution of the equation written as

$$M(t) = \int_{-\infty}^t \Psi(t - \tau) \Theta(\tau) d\tau = \int_0^{\infty} \Psi(\tau) \Theta(t - \tau) d\tau, \quad (1.7)$$

can be considered, in its turn, as the equation for $\Theta(t)$ with $M(t)$ given.

V. Volterra (1930) wrote that if the hereditary effect may be neglected up to a certain time $t_0 < t$ (for the sake of simplicity, t_0 can be considered as the starting time: $t_0 = 0$), then (1.7) takes the form:

$$M(t) = \int_0^t \Psi(t - \tau) \Theta(\tau) d\tau,$$

where both integral limits are finite.

We note that the above neglecting is not obligatory since the condition $M(\tau) = 0$ for $\tau < 0$ can be given as an *initial problem statement*. On the other hand, if the hereditary is present, and we are going to neglect the effect of negative semi-axis on the process at time t , then t should be implied large enough.

Passage to dynamical problems (e.g. torsional vibrations of a string) is performed according to d'Alembert principle by substituting $M(t) - \mu \ddot{\Theta}(t)$ for $M(t)$, where μ is a positive constant. As a result, we get the integro-differential equation for the dynamical hereditary problem:

$$M(t) - \mu \ddot{\Theta}(t) = \int_{-\infty}^t \Psi(t - \tau) \Theta(\tau) d\tau.$$

For finding $\Theta(t)$ caused by torque $M(t)$, it is enough to integrate this equation twice with respect to time from 0 to t and then solve the obtained Volterra integral equation of the second kind in $\Theta(t)$.

1.2.3 Hereditary oscillator

The second section of Chapter VI in Volterra's book ("Hereditary dynamics") is devoted to hereditary generalization of a one-dimensional harmonic oscillator. He replaces the differential equation:

$$m\ddot{q} + kq = 0, \quad m, k > 0,$$

by the integro-differential one:

$$m\ddot{q} + k \int_{-\infty}^t \Phi(t - \tau)q(\tau)d\tau = 0.$$

Without any detailed reasoning on the problem as well as on physical sense of such replacement, he found the following expression for "potential energy":

$$U(q) = \frac{k}{2} \left\{ \int_0^\infty \Phi(\tau)d\tau \cdot q^2(t) - \int_0^\infty \Phi(\tau)[q(t) - q(t - \tau)]^2 d\tau \right\},$$

so that

$$\frac{\partial U}{\partial q} = k \int_0^\infty \Phi(\tau)q(t - \tau)d\tau.$$

Introducing an external force $Q(t)$ and separating the singular (non-hereditary) part from the regular (hereditary) component in $\Phi(\tau)$,

$$k\Phi(\tau) = K\delta(\tau) + \varphi(\tau),$$

Volterra comes to the equation:

$$\begin{aligned} d \left\{ \frac{m\dot{q}^2(t)}{2} + \frac{Kq^2(t)}{2} - \frac{1}{2} \int_0^\infty \varphi(\tau)[q(t) - q(t - \tau)]^2 d\tau \right\} \\ + \frac{dt}{2} \int_0^\infty \dot{\varphi}(\tau)[q(t) - q(t - \tau)]^2 d\tau = Qdq. \end{aligned} \quad (1.8)$$

In absence of hereditary effects, $\varphi(\tau) = 0$ and Eq. (1.8) takes the form usual for non-hereditary dynamics:

$$d \left\{ \frac{m\dot{q}^2}{2} + \frac{Kq^2}{2} \right\} = Qdq.$$

This means that the work done by external forces is equal to an augment of total mechanical energy of the system. The account of heredity leads to modification of the *total mechanical energy* formula:

$$E_M = \frac{m\dot{q}^2}{2} + \frac{Kq^2}{2} - \frac{1}{2} \int_0^\infty \varphi(\tau)[q(t) - q(t - \tau)]^2 d\tau,$$

and also to the appearance of additional (as can be proved, positive) term:

$$W dt = \frac{1}{2} \int_0^\infty \dot{\varphi}(\tau)[q(t) - q(t - \tau)]^2 d\tau \cdot dt,$$

corresponding to the energy dissipation in the hereditary process. As a result, we have

$$dE_M + W dt = Q dq,$$

or, in an integral form,

$$E_M - E_M^0 + \int_{t_0}^t W dt = A. \quad (1.9)$$

Here E_M and E_M^0 are the values of total mechanical energy at the moments t and t_0 , respectively, and A is the work by external forces during this period.

1.2.4 Energy principle

Following Volterra, we specify the term *state of a hereditary system* more precisely: the hereditary system is in the same state at moments t_1 and t_2 if its dynamical variables coincide not only at these instants but also during the preceding intervals $(t_1 - \tau_0, t_1)$ and $(t_2 - \tau_0, t_2)$ of length τ_0 characterizing the heredity (memory) decay.

Volterra theorem. *If a hereditary system returns to its initial state at the end of some time interval, then the work of external forces is positive.*

Since, from a dynamical point of view, the state of the system is the same, then the total mechanical energy does not change and the positive work

$$E_D = \int_{t_0}^t W dt$$

should pass into other forms of energy and, first of all, into the thermal one. This principle is assumed as a basis of the whole mainstream in researches of thermodynamical media with memory (i.e., the *hereditary thermodynamics*).

1.2.5 Hereditary electrodynamics

Heredity property is observed in the connection between strengths and inductions of rapidly varying electric and magnetic fields in matters expressed by *hereditary*

material or *constitutive equations*. Instead of quasi-static non-hereditary relations:

$$\mathbf{D}(t) = \varepsilon \mathbf{E}(t)$$

and

$$\mathbf{B}(t) = \mu \mathbf{H}(t),$$

we have

$$\mathbf{D}(t) = \varepsilon \mathbf{E}(t) + \int_0^\infty \phi(\tau) \mathbf{E}(t - \tau) d\tau \quad (1.10)$$

and

$$\mathbf{B}(t) = \mu \mathbf{H}(t) + \int_0^\infty \psi(\tau) \mathbf{H}(t - \tau) d\tau. \quad (1.11)$$

These equations form a basis of Volterras hereditary electrodynamics. He used them when deriving an equation for the electromagnetic energy density and obtained

$$\frac{\dot{\mathbf{D}} \cdot \mathbf{E} + \dot{\mathbf{B}} \cdot \mathbf{H}}{4\pi} = \frac{d}{dt}(W_e + W_m) + w_d,$$

where

$$W_e = \frac{1}{8\pi} \left(\varepsilon \mathbf{E}^2(t) + \int_0^\infty \phi(\tau) \mathbf{E}^2(t - \tau) d\tau \right),$$

$$W_m = \frac{1}{8\pi} \left(\mu \mathbf{H}^2(t) + \int_0^\infty \psi(\tau) \mathbf{H}^2(t - \tau) d\tau \right),$$

and

$$w_d = -\frac{1}{8\pi} \int_0^\infty \{ \dot{\phi}(\tau) [\mathbf{E}(t - \tau) - \mathbf{E}(t)]^2 + \dot{\psi}(\tau) [\mathbf{H}(t - \tau) - \mathbf{H}(t)]^2 \} d\tau$$

is the *electromagnetic energy dissipation density* conditioned by the heredity property. All the three densities are positive.

We should note that besides the “fast” heredity being an intrinsic property of electrodynamic processes, Equations (1.10) and (1.11) also represent slow processes observed in some solids after shutdown of the field sources (Westerlund, 1991).

1.3 Hereditary kinetics

1.3.1 Mechanical origin of heredity

Traditions and principles of theoretical physics provoke us to look for the hereditary origin in the molecular nature of matter. The crucial role in bridging these two banks belongs to the method of projection operators. R. Zwanzig was the first who used the technique as a powerful tool for the derivation of formally exact hereditary equations for the projected part of a phase space distribution function (psdf) from a

Markovian equation for psdf itself (Zwanzig, 1961). A few years later Mori (1965) put forward the projection operator technique in the Heisenberg picture, which leads to hereditary Langevin equations. Their initiative called the *Zwanzig-Mori formalism* was relatively soon appreciated and taken up by research community (Balescu, 1975; Résibois and De Leener, 1977; Zubarev et al., 1996; Mokshin and Yulmetyev, 2006; etc.).

The Zwanzig-Mori method was to separate the distribution function into a “relevant” and an “irrelevant” part by means of a projection operator; to solve the equation for the irrelevant part, formally, in terms of the relevant part; and to substitute the solution back into the equation for the relevant part. Let us make an outline of the idea.

Consider a Hamiltonian system with n degrees of freedom. Let $A(t)$ be some dynamical variable differentiable with respect to all coordinates $x_i, p_i, i = 1, \dots, n$ of the phase state vector x . This variable obeys the Liouville equation:

$$\frac{\partial A(t)}{\partial t} = \mathbf{L}A(t),$$

where $\mathbf{L}$ is a linear differential operator acting on the state vector A in an infinite-dimensional Hilbert space with respect to the phase space variables (the *Liouville operator*).

The inner product of two phase state vectors A and B is defined by using the equilibrium distribution density $f_{eq}(x)$,

$$(A, B) \equiv \int dx f_{eq}(x) A(x) B^*(x) = \langle AB^* \rangle_{eq},$$

where $*$ denotes the complex conjugate. Now, we are in a position to choose a set of normalized and mutually orthogonal functions $\phi_j(x)$,

$$(\phi_j, \phi_k) = \delta_{ij}.$$

and to expand $A(x, t)$ in this orthonormal set,

$$A(x, t) = \sum_m a_m(t) \phi_m(x),$$

where the coefficients are

$$a_m(t) = (A(t), \phi_m).$$

As a result, the Liouville equation takes on the form:

$$\frac{\partial a_m(t)}{\partial t} = \sum_n L_{mn} a_n(t),$$

where

$$L_{mn} = (\phi_m, \mathbf{L}\phi_n).$$

The simplest nontrivial problem is the motion of a two-dimensional system. The Liouville equation becomes a pair of differential equations for the two-component vector (a_1, a_2) :

$$\begin{aligned}\frac{\partial a_1}{\partial t} &= L_{11}a_1 + L_{12}a_2, \\ \frac{\partial a_2}{\partial t} &= L_{21}a_1 + L_{22}a_2.\end{aligned}$$

Suppose that we are interested in the relevant coefficient a_1 and do not care for the irrelevant coefficient a_2 . Solving the second of these equations,

$$a_2(t) = \exp(L_{22}t)a_2(0) + \int_0^t dt' \exp[L_{22}(t-t')]L_{21}a_1(t')$$

and inserting the solution into the first equation, we arrive at the hereditary equation for $a_1(t)$:

$$\frac{\partial a_1(t)}{\partial t} = L_{11}a_1(t) + L_{12} \int_0^t dt' \exp[L_{22}(t-t')]L_{21}a_1(t') + L_{12} \exp(L_{22}t)a_2(0).$$

Rewriting the initial Liouville equation in a vector-matrix form,

$$\frac{\partial \mathbf{a}}{\partial t} = \mathbf{L}\mathbf{a},$$

one can obtain the same result by using matrix projection operators:

$$\mathbf{P} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{Q} = 1 - \mathbf{P} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

possessing the properties:

$$\mathbf{P}^2 = \mathbf{P}, \quad \mathbf{P}\mathbf{Q} = \mathbf{Q}\mathbf{P} = 0.$$

For a system described by the Hamiltonian:

$$H(x) = \frac{1}{2} \sum_{i=1}^n p_i^2 + V(x_i),$$

we should use the projection operator in the form:

$$\mathbf{P}A = \sum_{ij} (A, \phi_i) \phi_i.$$

Then, the projected equations of an arbitrary dynamic variable A , which is defined within the projected subspace, are given by the hereditary equation (Eq. (8.29) in (Zwanzig, 2001)):

$$\frac{\partial A(t)}{\partial t} = i\Omega A(t) - \int_0^t dt' K(t-t')A(t') + F(t), \quad (1.12)$$

where

$$F(t) = \exp(t\mathbf{QL})\mathbf{QL}A, \quad K(t) = -(F(t), \mathbf{L}A), \quad i\Omega = \mathbf{PL}, \quad (A, A) = 1.$$

Equation (1.12) is the generalized Langevin equation. It separates dynamics of a system into three parts:

- a) the term with Ω corresponding to instantaneous oscillations and describing the motion of $A(t)$ due to its inner dynamics; $\Omega = 0$ if this degree of freedom changes sign under time reversal;
- b) the term with the memory function $K(t)$ describing the influence of prehistory of $A(t)$ on its present rate; this term arises due to links of A with all other degrees of freedom and determines a flow of motion state information to heat bath and back;
- c) the driving force term $F(t)$ determining effect of inner bath dynamics on behavior of $A(t)$; it is a microscopic force which progresses in accordance with modified Liouville operator $\mathbf{QLQ}$ with excluded heat fluctuations of $A(t)$.

One should stress that the dynamical variable $A(t)$ and the force $F(t)$ in a complex system are rapidly oscillating functions of time and for this reason are considered as random functions. It is often enough to confine oneself by knowing only average function $\langle A(t) \rangle$ and autocorrelation function

$$C_{AA}(t, t') \equiv C_A(t - t') = \langle A(t)A^*(t') \rangle.$$

These functions obey the equations:

$$\frac{\partial \langle A(t) \rangle}{\partial t} = i\Omega \langle A(t) \rangle - \int_0^t dt' K(t-t') \langle A(t') \rangle + \langle F(t) \rangle$$

and

$$\frac{\partial C_A(t)}{\partial t} = i\Omega C_A(t) - \int_0^t dt' K(t-t') C_A(t') + \langle F(t)A^*(0) \rangle.$$

These equations are usually simplified on the assumption that $F(t)$ is a zero-mean random function noncorrelated with $A(0)$, so that

$$\langle F(t)A^*(0) \rangle = \langle F(t) \rangle \langle A^*(0) \rangle = 0.$$

An obvious case of physical reasons of the memory arising can be found in molecular dynamics. According to the classical representations, the collision chain for each molecule forms a sequence of independent events, its prehistory has no influence upon the characteristics of molecules entering in collisions. However, computer molecular dynamics simulations of hard sphere systems (Alder and Wainwright, 1967) discovered that at intermediate densities, a particle, after having suffered more than ten collisions, still showed preference to move in its initial direction

in contradiction to the classical Boltzmann-Enskog transport theory predicting much faster relaxation. It was observed that the particles in front and in back of the central particle all tend to acquire velocity the same direction as the central particle. Thus, the diffusing particle on the average would push the particles ahead as it moves and the particles behind would follow in its wake. In this way a vortex pattern develops in the velocity field after about ten collision times. This provided a physical explanation of the persistent velocity correlation. The velocity autocorrelation function does not decay exponentially at long times, instead, it decays like $t^{-d/2}$, where $d = 2$ or 3 is the dimensionality of the system. Theoretical derivations inspired by these simulations have also confirmed such a long-time behavior of correlation.

Various models have been proposed to compute the velocity autocorrelation function (VCF) by introducing an ansatz for the memory function. Such a form is usually chosen as a compromise resulting from the combination of mathematical convenience and physical intuition. Descriptions of some of them can be found in (Boon and Yip, 1980). We will touch here upon one of them, *itinerant-oscillator model* (Sears, 1965), i.e., a body of mass M moves in fluid and contains in its interior a damped oscillator of mass m . In this model the Laplace transform of $K(t)$ is given by

$$\hat{K}(\lambda) = \hat{\gamma}_1(\lambda) + \frac{\omega_1^2}{\lambda + \omega_0^2/(\lambda + \hat{\gamma}_0(\lambda))} \quad (1.13)$$

where $\hat{\gamma}_1(\lambda)$ and $\hat{\gamma}_0(\lambda)$ are memory functions and ω_1^2 and ω_0^2 are coupling constants defined from the following equations of motion:

$$\frac{d\mathbf{v}_1(t)}{dt} + \int_0^t \gamma_1(t-t')\mathbf{v}_1(t')dt' + \omega_1^2 \int_0^t [\mathbf{v}_1(t') - \mathbf{v}_0(t')]dt' = \mathbf{F}_1(t), \quad (1.14)$$

$$\frac{d\mathbf{v}_0(t)}{dt} + \int_0^t \gamma_0(t-t')\mathbf{v}_0(t')dt' + \omega_0^2 \int_0^t [\mathbf{v}_0(t') - \mathbf{v}_1(t')]dt' = \mathbf{F}_0(t). \quad (1.15)$$

Each of the equations is of the same type as those describing a damped stochastic oscillator with time-dependent friction coefficient γ_1 or γ_0 . The system describes a model based on the idea that, when a molecule moves, it drags along with it the surrounding fluid. Here the surroundings are replaced by a fictitious center, labeled with subscript 0, whereas the molecule under consideration is denoted with subscript 1. The memory function (1.13) is obtained from the solution of the system (1.14)–(1.15) for the correlation function $\langle \mathbf{v}_1(t)\mathbf{v}_1(0) \rangle$, using the formal analogy with the master equation. The computation of the VAF then requires the specification of time dependence of the frictional coefficients. Recall that according to numerical simulations, in case of a simple dense fluid the behavior of $K(t)$ should involve two relaxation times: one characteristic of the short time dynamics and one describing the long-time behavior.

Quite a different type of model was introduced by Zwanzig and Bixon (1970). The physical picture is that of a single tagged particle moving through a viscoelastic continuum; its motion is described by the Langevin equation with the frequency-dependent friction coefficient determined from hydrodynamic equations. The ana-

lytical form of the memory function is rather complicated and the reader is referred to the original literature.

1.3.2 Hereditary Boltzmann equation

Now, we consider approach based on the BBGKY (Bogolubov-Born-Green-Kirkwood-Yvon) chain of kinetic equations following from the Liouville equation and leading to the Boltzmann equation. In spatially homogeneous case the latter is of the form:

$$\frac{\partial f(\mathbf{v}, t)}{\partial t} = \mathbf{B}_0[\mathbf{v}, f(\cdot, t)]. \quad (1.16)$$

The bilinear Boltzmann *collision operator* $\mathbf{B}_0$ acts on the single-particle velocity distribution density $f(\mathbf{v}, t)$ as follows:

$$\mathbf{B}_0[\mathbf{v}, f(\cdot, t)] \equiv \int d\mathbf{u} \int \sigma(\Theta; g) [f(\mathbf{v}', t)f(\mathbf{u}', t) - f(\mathbf{v}, t)f(\mathbf{u}, t)] d\Omega.$$

Here $\mathbf{v}$ and $\mathbf{u}$ are velocities of colliding molecules before and after (primed) their collision, $g = |\mathbf{v} - \mathbf{u}|$ is their relative velocity, Θ is the scattering angle, $\sigma(\Theta; g)$ denotes the differential scattering cross-section, $d\Omega$ is an elementary space angle. The energy and momentum conservation laws definitely connect $\mathbf{v}'$, $\mathbf{u}'$, $\mathbf{v}$, $\mathbf{u}$, and Θ .

The derivation of the Boltzmann equation is based on assumptions which essentially simplify the solution of hydrodynamic problems, but they are considered to be still poorly justified for dense gases and liquids. Further development of kinetic theory associated with names of W. Pauli, R. Balescu, N. Bogolubov, R. Brout, L. van Hove, E. Montroll, I. Prigogine, P. Resibois, D. Zubarev, R. Zwanzig, and other outstanding theorists has led to the *generalized Boltzmann equation*. Like the Boltzmann equation, it describes irreversible processes of relaxation to equilibrium state, but being derived under weaker restrictions, it describes the processes more accurately, especially in case of dense gases and liquids. The main mathematical distinctive feature of the generalized equations is the retardation (hereditary) character of the collision operator:

$$\frac{\partial f(\mathbf{v}, t)}{\partial t} = \int_0^t \mathbf{B}_\tau[\mathbf{v}, f(\cdot, t - \tau)] d\tau + F(\mathbf{v}, t).$$

At equilibrium state, the solution of the equation doesn't depend on time, $f(\mathbf{v}, t) = f_{eq}(\mathbf{v})$ and the retardation effect disappears.

The process of derivation of the generalized equation consists of the following stages (see for details (R sibois and De Leener, 1977)):

1. Derivating the *Liouville equation* from the Hamilton equations for a system of N interacting molecules without any additional assumptions ($\rho^N \equiv \rho^N(\{\mathbf{r}, \mathbf{v}\}; t)$):

$$\frac{\partial \rho^N}{\partial t} = \mathbf{L}^N \rho^N,$$

where $\mathbf{L}^N$ is the *Liouville operator*;

2. Introducing the *projection operator* $\mathbf{P}$,

$$\mathbf{P}\rho^N = \frac{1}{V^N} \int_V \dots \int_V \rho^N(\{\mathbf{r}, \mathbf{v}\}; t) \prod_{j=1}^N d\mathbf{r}_j \equiv \varphi^N(\{\mathbf{v}\}; t),$$

and transforming the Liouville equation into the *formal kinetics equation*:

$$\frac{\partial \varphi^N}{\partial t} = \int_0^t \mathbf{C}_\tau^N [\varphi^N(\{\mathbf{v}\}|t - \tau)] d\tau + \phi^N(\{\mathbf{v}\}, t),$$

where $\mathbf{C}_\tau^N$ is the operator of molecular interaction. The time-integral appears as a result of dividing ρ^N into two components — “relevant” $\varphi^N = \mathbf{P}\rho^N$ and “irrelevant” $(1 - \mathbf{P})\rho^N$ — and further elimination of the last component from the obtained system of two equations. No new assumptions or restrictions are introduced in the course of these transformations. Every Markovian process possesses this property: averaging over a part of variables involves the time-integration that confirms non-Markovian character of *reduced* process (the opposite statement is also true: a non-Markovian process can be transformed into a Markovian one via appropriate extension of phase space). The free term of equation ϕ^N depends only on the beginning value of “irrelevant” part. The number of independent variables having two times decreased is still too large.

3. Introducing the *molecular chaos hypothesis* allows to reduce the N -particle distribution to the single-particle one and to close the obtained equation. The hypothesis says that *random velocities of interacting molecules are mutually independent* and has the following mathematical expression:

$$\varphi^N(\{\mathbf{v}\}; t - \tau) = \prod_{j=1}^N f(\mathbf{v}_j; t - \tau),$$

where $f(\mathbf{v}_j; t - \tau)$ is the single-particle density in velocity space. As a result, we get the *generalized kinetic equation*:

$$\frac{\partial f(\mathbf{v}, t)}{\partial t} = \int_0^t \mathbf{B}_\tau[\mathbf{v}, f(\cdot, t - \tau)] d\tau + \phi(\mathbf{v}, t), \quad (1.17)$$

where

$$\mathbf{B}_\tau[\mathbf{v}, f(\cdot|t - \tau)] \equiv \mathbf{TL} \int \dots \int \mathbf{C}_\tau^N \left[\prod_{j=1}^N f(\mathbf{v}_j|t - \tau) \right] \prod_{j=2}^N d\mathbf{v}_j,$$

integration is supposed to hold over all numbered velocities (index 1 is omitted), and $\mathbf{TL}$ stands for the thermodynamic limit.

4. Introducing the *model of instantaneous binary collisions* being applicable for gases and plasmas diluted enough. The gas is diluted enough so we can deal with molecule interaction within the framework of instantaneous binary collisions. In this case

$$\mathbf{B}_\tau[\mathbf{v}, f(\cdot, t - \tau)] = \mathbf{B}_0[\mathbf{v}, f(\cdot, t - \tau)]\delta(\tau)$$

and we arrive at the ordinary *Boltzmann equation*.

On abandoning the last limitation and using a continuous approximation to the time-dependence of the collision operator:

$$\mathbf{B}_\tau[\mathbf{v}, f(\cdot, t - \tau)] = \mathbf{B}_0[\mathbf{v}, f(\cdot, t - \tau)]b(\tau), \quad b(\tau) \geq 0, \quad \int_0^\infty b(\tau)d\tau = 1,$$

we obtain the *generalized Boltzmann equation*:

$$\frac{\partial f(\mathbf{v}, t)}{\partial t} = \int_0^t \mathbf{B}_0[\mathbf{v}, f(\cdot, t - \tau)]b(\tau)d\tau + \phi(\mathbf{v}, t) \quad (1.18)$$

forming the mathematical basis of hereditary *generalized kinetics*.

The second term in the right part of generalized kinetic equations (1.17)–(1.18) describes the decay of space correlations which can exist at the initial time. One can show that if the correlations are initially absent, then they will always be absent, $\phi(\mathbf{v}; t) = 0$ for all $t > 0$. Suppose now that at the initial instant the correlations stretch to about interaction radius r_0 . This means that the molecules tend to form spatial clusters, but chaotically directed velocities destroy the clusters, and the particles flying out of each other stop to interact after the time $\tau_{col} = r_0/\bar{v}$. It then should be expected that

$$\phi(\mathbf{v}; t) \approx 0, \quad t \gg \tau_{col}.$$

Calculations confirm this for gases with weakly exponentially interacting molecules but a strict proof of this property in general case hasn't been found.

A similar situation takes place with the operator term. It is usually believed that

$$\mathbf{B}_\tau[f] \simeq 0, \quad t \gg \tau_{col}.$$

1.3.3 Fokker-Planck equation

If a particle under consideration differs somehow from others (for example by mass) and other molecules form an equilibrium ensemble then nonlinear kinetic equations become linear for the function $f(\mathbf{v}, t)$ taking now the sense of a probability density function for the particle in the velocity space:

$$\frac{\partial f(\mathbf{v}, t)}{\partial t} = \int_0^t \mathbf{B}'_0[\mathbf{v}, f(\cdot, t - \tau)]d\tau,$$

$$\mathbf{B}'_0[\mathbf{v}, f(\cdot, t)] \equiv \int d\mathbf{v}' \overline{\sigma}(\Theta; \mathbf{v}') [f(\mathbf{v}', t) - f(\mathbf{v}, t)] d\mathbf{v}'.$$

and

$$\frac{\partial f(\mathbf{v}, t)}{\partial t} = \int_0^t \mathbf{B}'_\tau[\mathbf{v}, f(\cdot, t - \tau)] d\tau + \phi(\mathbf{v}, t),$$

$$\mathbf{B}'_\tau[\mathbf{v}, f(\cdot, t)] \equiv \int_0^t d\tau b(\tau) \int d\mathbf{v}' \overline{\sigma}(\Theta; \mathbf{v}') [f(\mathbf{v}', t) - f(\mathbf{v}, t)] d\mathbf{v}'.$$

If we are interested not only in the particle velocity but also in its location in space, we have to introduce the joint coordinate and velocity distribution density $p(\mathbf{r}, \mathbf{v}, t)$. Assuming initial conditions $\mathbf{r}(0) = 0$ and $\mathbf{v}(0) = \mathbf{v}_0$, we arrive at the equation:

$$\frac{\partial p(\mathbf{r}, \mathbf{v}, t)}{\partial t} + \mathbf{v} \cdot \nabla p(\mathbf{r}, \mathbf{v}, t) = \int_0^t \mathbf{B}'_\tau[\mathbf{v}, p(\mathbf{r}, \cdot, t - \tau)] d\tau + \delta(\mathbf{r}) \delta(\mathbf{v} - \mathbf{v}_0) \delta(t).$$

If we actually have in mind a Brownian particle that is observed under a microscope, we should take into account that its mass M is much greater than masses of molecules pushing it and every push slightly changes its velocity. In this case

$$\mathbf{B}'_0[\mathbf{v}, p(\mathbf{r}, t)] \approx \zeta \nabla_{\mathbf{v}} \cdot (\mathbf{v} + MkT \nabla_{\mathbf{v}}) p(\mathbf{r}, \mathbf{v}, t)$$

and the above equation turns into the hereditary Fokker-Planck equation:

$$\begin{aligned} \frac{\partial p(\mathbf{r}, \mathbf{v}, t)}{\partial t} + \mathbf{v} \cdot \nabla p(\mathbf{r}, \mathbf{v}, t) \\ = \zeta \nabla_{\mathbf{v}} \cdot (\mathbf{v} + MkT \nabla_{\mathbf{v}}) \int_0^t d\tau b(\tau) p(\mathbf{r}, \mathbf{v}, t - \tau) + \delta(\mathbf{r}) \delta(\mathbf{v} - \mathbf{v}_0) \delta(t), \end{aligned}$$

where k is the Boltzmann constant, T is the absolute temperature of the medium, ζ is the friction coefficient. When $b(\tau) = \delta(\tau)$, it takes the form of the ordinary Fokker-Planck equation:

$$\frac{\partial p(\mathbf{r}, \mathbf{v}, t)}{\partial t} + \mathbf{v} \cdot \nabla p(\mathbf{r}, \mathbf{v}, t) = \zeta \nabla_{\mathbf{v}} \cdot (\mathbf{v} + MkT \nabla_{\mathbf{v}}) p(\mathbf{r}, \mathbf{v}, t) + \delta(\mathbf{r}) \delta(\mathbf{v} - \mathbf{v}_0) \delta(t).$$

1.3.4 Pauli and Van Hove equations

A quantum analogue for the Boltzmann equation is the Pauli kinetic equation for the probability $P(a, t)$ of finding a system in quantum state a :

$$\frac{dP(a, t)}{dt} = \sum_b [W(a, b)P(b, t) - W(b, a)P(a, t)].$$

Here $W(a, b)$ is the probability of transition $b \rightarrow a$ per unit of time. For coarse-grained probability density function $p(a, t)$, the equation takes the form:

$$\frac{dp(a,t)}{dt} = 2\pi\lambda^2 \int db [w(a,b)p(b,t) - w(b,a)p(a,t)].$$

It is necessary to combine it with an initial condition, for example $p(a,0) = \delta(a - a_0)$. The Pauli equation is derived for a weakly interacting system ($\lambda^2 \tau_{rel} \simeq 1$) on assumption that phases of coefficients $c(a,t)$ in decomposition:

$$|\varphi_t\rangle = \int da c(a,t) |a\rangle$$

at *all times* are random, independent of each other and uniformly distributed over all possible values. This presumption is similar to the Boltzmann hypothesis of molecular chaos and, strictly speaking, is valid for any system in the state of the absolute thermodynamic equilibrium. The random time intervals $\Delta T_j = T_j - T_{j-1}$ between consecutive interactions are independent and exponentially distributed so that the random sequence $T_1, T_2, T_3, \dots$ forms the Poissonian process.

Van Hove derived a hereditary analogue of this equation on the assumption that all mentioned properties of random phases take place only at *the initial time* (Van Hove, 1957). The kinetic equation, generalized in this way has the form:

$$\begin{aligned} \frac{dp(a,t)}{dt} = & 2\pi\lambda^2 \int_0^t d\tau \int db [w(a,b;\tau)p(b,t-\tau) - w(b,a;\tau)p(a,t-\tau)] \\ & + \delta(a - a_0)\phi(a;t). \end{aligned}$$

Its main distinction from the Pauli equation is the integration over the retarding time argument in all summands describing transitions and in the presence of the last non-homogeneous term. This makes the process non-Markovian and the sequence $T_1, T_2, T_3, \dots$ non-Poissonian. Analyzing the passage from exact non-Markovian (hereditary) kinetic equation to its low-order approximation being Markovian equation, van Hove notes that the hereditary nature of the general exact equation is caused by interference of wave perturbations. These effects are demonstration of some determined phase correlations. In case of small perturbations, these correlations become negligibly small and namely for this reason the low-order equation is a Markovian equation.

1.3.5 Hybrid kinetic equations

An interesting approach combining advantages of both methods is offered in (Balescu et al., 1994) and developed in the consecutive papers of Balescu et al. It is based on the *hybrid kinetic equation*, which less rigorous dynamical foundation is compensated by greater simplicity. This equation is represented in the form of a continuity equation:

$$\frac{\partial f(\mathbf{r},t)}{\partial t} + \mathbf{v}(\mathbf{r},t) \cdot \nabla f(\mathbf{r},t) = 0 \quad (1.19)$$

describing the transport of particles in a random velocity field given by a random potential $\phi(\mathbf{r}, t)$,

$$\mathbf{v}(\mathbf{r}, t) \propto [\nabla \phi(\mathbf{r}, t), \mathbf{e}_z].$$

The potential is determined as a Gaussian stationary homogeneous and isotropic process with zero average and double-point Euler correlation function:

$$\langle \phi(\mathbf{r}_1, t_1) \phi(\mathbf{r}_1 + \mathbf{r}, t_1 + t) \rangle = \varepsilon^2 E(\mathbf{r}/\lambda_c) h(t/\tau_c),$$

where $E = E(\mathbf{r}, t)$ and $h = h(\mathbf{r}, t)$ are dimensionless functions decreasing down to zero when arguments increase, ε is the measure of fluctuation potential intensity, λ_c and τ_c are space and time correlation scales (radiuses).

Decomposing the solution f into an average $n = \langle f \rangle$ and a fluctuation f' , $\langle f' \rangle = 0$,

$$f(\mathbf{r}, t) = n(\mathbf{r}, t) + f'(\mathbf{r}, t), \quad (1.20)$$

and combining the results with Eq. (1.19) under the incompressible condition $\nabla \cdot \mathbf{v} = 0$. Balescu found a system of coupled equations for the density profile and for the fluctuation:

$$\begin{aligned} \frac{\partial n}{\partial t} &= -\nabla \cdot \langle \mathbf{v}(\mathbf{r}, t) f'(\mathbf{r}, t) \rangle, \\ \frac{\partial f'}{\partial t} &= -\mathbf{v}(\mathbf{r}, t) \cdot \nabla f'(\mathbf{r}, t) = S(\mathbf{r}, t) \end{aligned}$$

with the source term:

$$S(\mathbf{r}, t) = -\mathbf{v}(\mathbf{r}, t) \cdot \nabla n(\mathbf{r}, t) + \nabla \cdot \langle \mathbf{v}(\mathbf{r}, t) f'(\mathbf{r}, t) \rangle.$$

Finding the solution of Eq. (1.20) and substituting it into Eq. (1.19) yield hereditary diffusion equation after some simplifying manipulations yields the hereditary master equation for anomalous diffusion in plasmas:

$$\frac{\partial n(\mathbf{r}, t)}{\partial t} = \int_0^t d\tau \Lambda(t - \tau) \Delta n(\mathbf{r}, \tau) \quad (1.21)$$

with

$$\Lambda(\tau) = \langle \mathbf{v}(\mathbf{r}, \tau) \cdot \mathbf{v}(\mathbf{R}(\tau|0), 0) \rangle,$$

where $\mathbf{R}(t|t')$ is the time-inverted trajectory of a particle which was situated at $\mathbf{r}$ at time t . Here is also assumed, that the initial condition of the distribution function is deterministic, for example, $n(\mathbf{r}, 0) = \delta(\mathbf{r})$.

This is evidently a non-Markovian equation since the velocity of a density variation at time t depends on its all previous values. One of the most important mechanisms underlying the non-Markovian property of the process is trapping. "Trapping" means that some particles in the fluid move along closed trajectories holding in a bounded part of phase space. This process was first predicted in the work (Bernstein et al., 1957) devoted to analytical construction of equilibrium electrostatic systems possessing such traps. The existence of such areas in phase space was confirmed by

numerical modeling (Berk and Roberts, 1967; Morse and Nielson, 1969). Further development of the plasma traps concept and its current state is well described in the review (Luque and Schamel, 2005).

The non-Markovian influence rate on the turbulent diffusion process is determined by the Kubo number:

$$Ku = \frac{\varepsilon \tau_c}{\lambda_c^2}.$$

In case of a weak turbulence ($Ku \ll 1$), one can neglect the retardation in the master equation ($n(\mathbf{x}, t - \tau) \approx n(\mathbf{x}, t)$) and extend integration over τ up to infinity. The process becomes Markovian. Finally, we come to the ordinary diffusion equation with the right-hand part describing sources of diffusing particles

$$\frac{\partial n(\mathbf{x}, t)}{\partial t} = D \Delta n(\mathbf{x}, t) + S(\mathbf{x}, t).$$

1.4 Hereditary hydrodynamics

1.4.1 Physical motivation

The next step on the way to simplify the process description is to do the transition from integro-differential kinetic equation (the Boltzmann equation) to differential hydrodynamic equation (the Navier-Stokes equation). For a long time the hydrodynamic approximation to kinetic problems has been considered as applicable at times of many orders greater than “kinetic” times, i.e., the collision time ($\sim 10^{-15}$ s) and the inter-collision time (for typical liquids, for example, water $\sim 10^{-14}$ s). However, numerical modeling has shown that in some cases the ordinary hydrodynamics works even at far smaller time scales — approximately to 30 collisions that is about 10^{-13} s (notice that the number 30 is considered in statistics as such a sample size which is large enough for the central limit theorem be applicable).

Nevertheless, the expanding hydrodynamic approach from macroscopic to mesoscopic scales required additional information on the molecular structure of the medium. This is the first reason for generalizing classical hydrodynamics. The second reason is the long time tails of autocorrelation functions discovered in molecular dynamic simulations and confirmed by experiments with light and gamma ray scattering on fluids. A consequence of the phenomena is that none of the Taylor series expansions of the transport coefficients, such as in density, gradients, temperature, exist.

The reason for the divergences is that, because of the long time memories which the tail represents, the distribution of distances a particle travels in a given time is not Gaussian except in the infinite time limit. In that limit, the second moment of that Gaussian, or the half width, determines the diffusion coefficient. As an illustration of the divergences, the linear Burnett coefficients for the Lorentz gas diffusion can serve. The Burnett coefficients measure the dependence of the transport coefficients

on the size of the relevant gradient, that is, on the wavelength of the fluctuation. The Burnett coefficients are the coefficients that appear in the Chapman-Enskog expansion for the Boltzmann equation and in the higher order corrections to Fick's law for diffusion:

$$\frac{\partial f(x,t)}{\partial t} = D_0 \frac{\partial^2 f(x,t)}{\partial x^2} + D_2 \frac{\partial^4 f(x,t)}{\partial x^4} + \dots,$$

where f is the distribution function, and D_0 , D_2 are two first linear Burnett coefficients. The proper generalization of Fick's law that leads to a transformation of the Burnett coefficients into Burnett integral operators gives the hereditary property (Alder and Alley, 1981):

$$\frac{\partial f(x,t)}{\partial t} = D_0 \int_0^t \rho_0(t-t') \frac{\partial^2 f(x,t')}{\partial x^2} dt' + D_2 \int_0^t \rho_2(t-t') \frac{\partial^4 f(x,t')}{\partial x^4} dt' + \dots$$

Generalization of the Navier-Stokes equations keeps its form but changes the transport coefficients which become space-time-dependent (Alder and Alley, 1984). The generalized viscosity $\eta = \eta(\mathbf{r} - \mathbf{r}', t - t')$ reveals two properties – *nonlocality* and *memory*: a signal emitted at time t' from a point $\mathbf{r}'$ will reach a point $\mathbf{r}$ at later time t . This retardation is caused rather by the diffusive character of fluctuation spreading than by inertial properties of a liquid. In any case, molecules need some time for rearrangement of their mutual configuration. Thus, the above equations can be considered as reduced forms of a more general equation:

$$\frac{\partial f(x,t)}{\partial t} = D \int_0^t dt' \int_{-\infty}^{\infty} dx' \rho(x-x', t-t') \frac{\partial^2 f(x',t')}{\partial x'^2},$$

combining heredity and nonlocality.

In addition, the molecular dynamics simulations have clarified the necessity of account of correlations with preceding collision partners. Earlier, the correlations were supposed to decay exponentially and therefore seemed inessential at hydrodynamic scales (the *molecular chaos hypothesis*). Numerical modeling have shown the power decay of correlations: they were observed after hundreds of collisions. In the macroscopic hydrodynamics this fact is known as far back as Stokes, who established it when studying a ball motion through a fluid. However, before numerical experiments nobody supposed that this conclusion remains valid for balls of atomic sizes.

The second origin of memory phenomena is finite duration of collisions: under some conditions collisions can not be regarded as instantaneous. The typical collision time τ_{col} becomes now commensurable with relaxation time τ_{rel} , and this needs to be considered at high densities. Indeed at low densities τ_{col} is determined by interaction radius-average velocity-ratio,

$$\tau_{col} \approx r_0 / \bar{v},$$

while the relaxation time is estimated by reducing the kinetic equation to the relaxation type

$$\frac{\partial(\varphi - \varphi_{eq})}{\partial t} \approx -nr_0^2\bar{v}(\varphi - \varphi_{eq}),$$

and takes the form:

$$\tau_{rel} \approx 1/(nr_0^2\bar{v}).$$

Thus, we can see that

$$\tau_{col}/\tau_{rel} \approx nr_0^3,$$

and the collision time τ_{col} is negligibly small in comparison with relaxation time only at low densities n . This result can be also obtained on the basis of inverse time expansion into series with respect to molecules density (concentration) n (R  sibois and Leener, 1977)

$$\tau_{rel}^{-1} = r_0^2 n \bar{v} [1 + r_0^3 n + (r_0^3 n)^2 + \dots].$$

Ratio of the times reads

$$\tau_{col}/\tau_{rel} = r_0^3 n + (r_0^3 n)^2 + \dots,$$

so if we wish to hold on the term $(r_0^3 n)^2$, we don't have a right to consider collisions as instantaneous events.

Alder (1985) writes, "The purpose of generating these generalized linear transport coefficients is to replace the constant transport coefficients in the Navier-Stokes equations by them so as to introduce a length and time scale into the hydrodynamic equation to learn to what small distance and short time scales hydrodynamics can be extended. We already know from the hydrodynamic model for the tail that hydrodynamics applies on a submicroscale and at less than picosecond times. We also know that in order to explain the negative feature in the velocity autocorrelation function near solidification densities by a hydrodynamic model, a nonlocal wavelength and frequency dependent viscosity is required. The primary cause of this nonlocality in time of the viscosity is the molasses tail. Moreover, we know that such viscosity is required to explain shear mode propagation observed in the fluid phase near solidification in the neutron scattering function. The only hydrodynamic problem actually solved so far with generalized transport coefficients is the Stokes problem of an infinitely massed sphere falling in a fluid, in order to find out what the corrections to Stokes law would be if the sphere was of atomic dimensions. Indeed, it was found that the corrections were small, only about 30% for the friction coefficient compared to the macroscopic sized sphere. In general, it is extremely difficult to solve the Navier-Stokes equation with nonlocal transport coefficients".

1.4.2 Polymeric liquids

Polymeric liquids reveal the most striking difference from ordinary hydrodynamics. They consist of huge polymeric molecules — long tangled close chains of large

number of recurring units — monomers (molecular weight equals $10^5 \sim 10^8$). Polymeric liquids can represent both polymeric solution and undiluted polymers, also called *melts*.

The polymeric liquids display a number of amazing phenomena that are qualitatively different from Newtonian liquids behavior in such situations (Bird and Curtiss, 1984). For example, when a revolving rod is immersed vertically into a liquid, the liquid surface near the rod is lifted up if the liquid is Newtonian and is drawn down if it is polymeric. A rotation of a disk at the bottom of a cylindrical vessel leads the Newtonian liquid to move out of the center, while the polymeric one moves towards the center. When flowing out of a vertical tube, the Newtonian liquid flow converges gradually at the diameter, while polymeric liquid quickly extends (more than two times) at first, and only after then it starts converging. To pump Newtonian fluid out of a vessel by suction tube, one should immerse the tube into the fluid while polymeric liquid can be pumped out even if the tube is at a distance of several centimeters from the surface. Should we say that ordinary hydrodynamics can't predict such peculiarities?

Look at the next experiment described by Bird and Curtiss (1984) (Fig. 1.1). A pump leaks a fluid through a tube. At the beginning of the experiment a section of the fluid is marked with a paint. During the stream process the marked surface takes the parabolic form typical for the Poiseuille flow. When the pump is turned off the fluid stops. Herewith the Newtonian liquid keeps being motionless while polymeric liquid streams some distance back, though it doesn't take its first position. The back motion process reveals the “memory” of polymeric liquid and the fact that the liquid doesn't take its beginning condition, as a spring does, is the evidence of memory attenuation.

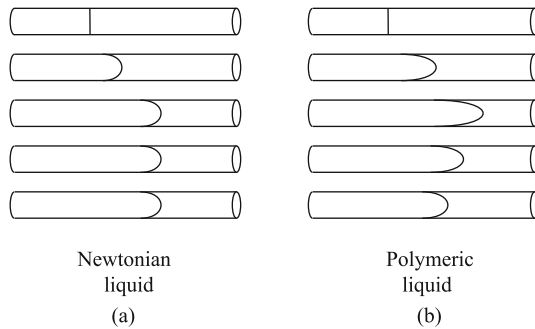


Fig. 1.1 Hereditary behavior of a Newtonian liquid (a) versus Polymeric liquid (b).

One of the main problems in polymeric dynamics is to find the appropriate expression for stress tensor, which would replace the Newtonian law for viscous liquid. Authors of the paper (Bird and Curtiss, 1984) proposed the next formula for polymeric melt of molecules in the form of freely connected chains consisting of N balls and $N - 1$ links of length a :

$$p_{ij} = p\delta_{ij} + nNkT \left[\frac{1}{3}\delta_{ij} - \int_{-\infty}^t \mu(t-t')A_{ij}(t')dt' - \varepsilon \int_{-\infty}^t v(t-t')B_{ij}(t')dt' \right],$$

here A_{ij} and B_{ij} are tensors dependent on the finite deformation tensor and its velocity, n is polymeric molecules concentration,

$$v(t) = (16/\pi^2\lambda) \sum_l l^{-2} \exp(-\pi l^2 t/\lambda),$$

and

$$\mu(t) = -\frac{\lambda}{2} \frac{dv}{dt}.$$

The constant λ having the dimension of time is connected with model parameters via the relation:

$$\lambda = N^{3+\beta} \zeta a^2 / 2kT,$$

where β is an empirical constant.

It is appropriate to cite here the theoretical work (Neogi, 1993) where the effective diffusivity and a memory-dependent source/sink term are obtained for describing the overall transport in heterogeneous media (glassy polymers with microvoids) under local volume averaging. The author starts with the conservation equation for the solute in the polymer (excluding holes):

$$\frac{\partial}{\partial t} \int_{V_p} c_p dV = \int_S \mathbf{n} \cdot D_p \nabla c_p dS + N \int_{S_i} \mathbf{n} \cdot D_p \nabla c_p dS,$$

where V_p is the total volume of the polymer, c_p is the concentration of the polymer, S is the outer bounding surface, and S_i is the surface of a hole. The unit normal vectors point outwards from the polymer. N is the number of holes. It is assumed that the surface S does not cut through a hole, which is easily imagined here as the number of holes are low. Defining $c_p = 0$ in a hole, and dividing with the total volume V , one has

$$\frac{\partial}{\partial t} \langle c_p \rangle = D_p \Delta \langle c_p \rangle + \frac{N}{V} \int_{S_i} \mathbf{n} \cdot D_p \nabla c_p dS$$

on applying Green's theorem. Phase volumes and diffusivity D_p have been assumed to be constants, the last term is m_T , the total mass transferred to the holes. Averaging the equation after some simplifying assumptions, the author reduces it to the hereditary form:

$$\frac{\partial \langle c \rangle}{\partial t} = \int_0^t \mu(t-t') \Delta \langle c(\mathbf{r}, t') \rangle dt'.$$

The memory function $\mu(t)$ is represented through its Laplace image (Eq. (50) of the cited work).

Although some simplifications (such as replacement $\langle \nabla c \rangle$ by $\nabla \langle c \rangle$) can be criticized, the satisfactory agreement with experimental data seems to be convincing.

1.4.3 Turbulent diffusion

The turbulent diffusion of a passive particle in an incompressible fluid is usually described by the parabolic advection-diffusion equation, proposed by Boussinesq (1877) and Taylor (1915),

$$\frac{dn}{dt} - \nabla \cdot [(K + D)\nabla n(\mathbf{r}, t)] = S(\mathbf{r}, t). \quad (1.22)$$

Here, $n(\mathbf{r}, t)$ is the concentration, K is the molecular diffusion coefficient, $D(\mathbf{r}, t)$ is the turbulent diffusion coefficient, $S(\mathbf{r}, t)$ is the strength of the source of particles,

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla$$

is the material (total) derivative and $\mathbf{u}(\mathbf{r}, t)$ stands for the mean velocity of the fluid. However, Batchelor and Townsend were not satisfied with this equation and suggested that “a description of the diffusion by some kind of integral equation is more to be expected” (Batchelor and Townsend, 1956, p. 360). One of ways to derive such an equation was offered in (Ghosal and Keller, 2000). Let us trace this way.

The authors start with rewriting Eq. (1.22) in the form:

$$\left[\frac{\partial}{\partial t} + u_i \frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_i} K_{ij} \frac{\partial}{\partial x_j} \right] n(\mathbf{r}, t) = S(\mathbf{r}, t) \quad (1.23)$$

and assuming that $\mathbf{u}(\mathbf{r}, t)$ and $S(\mathbf{r}, t)$ are random functions and consequently $f(\mathbf{r}, t)$ is also random (repeated indices are summed). Then, they decompose each of the random function f into the average $\langle f(\mathbf{r}, t) \rangle$ and the fluctuation $f'(\mathbf{r}, t)$ (they write it also in terms of a projection operator $\mathbf{P}$, $f' = \mathbf{P}f$) and write:

“We begin by averaging Eq. (1.23) and writing

$$\frac{\bar{d}}{dt} = \frac{\partial}{\partial t} + \langle u_i(\mathbf{r}, t) \rangle \frac{\partial}{\partial x_i}$$

to obtain

$$\left[\frac{\bar{d}}{dt} - \frac{\partial}{\partial x_i} K_{ij} \frac{\partial}{\partial x_j} \right] n(\mathbf{r}, t) + \left\langle u'_i \frac{\partial f'}{\partial x_j} \right\rangle = \langle S(\mathbf{r}, t) \rangle. \quad (1.24)$$

Then we subtract (1.24) from (1.23) to obtain the following equation for f' :

$$\left[\frac{\bar{d}}{dt} - \frac{\partial}{\partial x_i} K_{ij} \frac{\partial}{\partial x_j} + \mathbf{P} u'_i \frac{\partial}{\partial x_i} \right] f'(\mathbf{r}, t) = S'(\mathbf{r}, t) - u'_i \frac{\partial n}{\partial x_j}. \quad (1.25)$$

Because of the incompressibility condition, the molecular diffusion tensor K_{ij} can be considered as non-fluctuating, and therefore the transformation is valid.

Representing the solution of Eq. (1.25) by means of the inverse operator,

$$f'(\mathbf{r}, t) = \left[\frac{\bar{d}}{dt} - \frac{\partial}{\partial x_i} K_{ij} \frac{\partial}{\partial x_j} + \mathbf{P} u'_i \frac{\partial}{\partial x_i} \right]^{-1} \left(S'(\mathbf{r}, t) - u'_i \frac{\partial n}{\partial x_j} \right) + f'(\mathbf{r}, 0), \quad (1.26)$$

and substituting the result into (1.24) one obtains

$$\begin{aligned} & \left\{ \frac{\bar{d}}{dt} - \frac{\partial}{\partial x_i} K_{ij} \frac{\partial}{\partial x_j} - \left\langle u'_i \frac{\partial}{\partial x_i} \left[\frac{\bar{d}}{dt} - \frac{\partial}{\partial x_k} K_{km} \frac{\partial}{\partial x_m} + \mathbf{P} u'_k \frac{\partial}{\partial x_k} \right]^{-1} u'_j \right\rangle \frac{\partial}{\partial x_j} \right\} n(\mathbf{r}, t) \\ &= \langle S \rangle - \left\langle u'_k \frac{\partial}{\partial x_k} \left[\frac{\bar{d}}{dt} - \frac{\partial}{\partial x_i} K_{ij} \frac{\partial}{\partial x_j} + \mathbf{P} u'_i \frac{\partial}{\partial x_i} \right]^{-1} S' \right\rangle - \left\langle u'_i(\mathbf{r}, t) \frac{\partial f'(\mathbf{r}, 0)}{\partial x_i} \right\rangle. \end{aligned}$$

Referring to some atmospheric and wind tunnel experiments, Ghosal and Keller show that in both cases the term $u'_k \partial / \partial x_k$ is small compared with $\bar{d}/dt$ and can be neglected. The resulting operator $\bar{d}/dt - \partial / \partial x_k K_{km} \partial / \partial x_m$ is then the diffusion operator, which can be inverted by using its Green's function. For simplicity, they assume that the molecular diffusive part $\partial / \partial x_k K_{km} \partial / \partial x_m$ is small compared with the advective part $\bar{d}/dt$. Introducing the characteristic curve or particle path $\mathbf{y}(s, t, \mathbf{r})$ which passes through $\mathbf{r}$ at $s = t$, one can write the explicit expression for the inverse operator:

$$\left(\frac{d}{dt} \right)^{-1} f = \int_0^t f[\mathbf{y}(s, t, \mathbf{r}), s] ds.$$

After introducing the correlation function:

$$C_{ij}(\mathbf{r}, t, \mathbf{x}, s) = \langle u'_i(\mathbf{r}, t) u'_j(\mathbf{y}, s) \rangle$$

and assuming that u'_i is independent of S' and $f'(\mathbf{r}, 0)$, authors of the cited work arrive at its main *nonlinear hereditary* turbulent diffusion equation:

$$\left(\frac{\bar{d}}{dt} - \frac{\partial}{\partial x_i} K_{ij} \frac{\partial}{\partial x_j} \right) n(\mathbf{r}, t) - \frac{\partial}{\partial x_i} \int_0^t C_{ij}[\mathbf{r}, t, \mathbf{y}(s, t, \mathbf{r}), s] \frac{\partial n[\mathbf{y}(s, t, \mathbf{r}), s]}{\partial y_j} ds = \langle S(\mathbf{r}, t) \rangle.$$

1.4.4 Coarse-grained diffusion models

Very few classical laws of physics are as ubiquitous as Ficks law of diffusion. Known by different names in various disciplines ranging from electromagnetism (Ohm's law) to heat conduction (Fourier's law) to flow in porous media (Darcy's law), it postulates a direct, instantaneous relationship between the cause (the concentration gradient $\nabla n(\mathbf{r}, t)$) and effect (the flux density vector $\mathbf{J}(\mathbf{r}, t)$)

$$\mathbf{J}(\mathbf{r}, t) = -D(\mathbf{r}) \nabla n(\mathbf{r}, t).$$

Despite apparent successes of Fick's law of diffusion, many examples of the non-Fickian behavior of diffusive transport in heterogeneous environments can be found

in almost every discipline in the natural sciences ranging from biology to atmospheric physics and oceanography. A special field of such kind is the transport in porous media modeling contaminant diffusion in groundwater flows (Dentz and Tartakovsky, 2006). As supposed in (Horsthemke, 1999; Fort and Méndez, 2002), averaging Fick's law over a porous media ensemble leads to the non-Fickian delayed interrelation:

$$\bar{\mathbf{J}}(\mathbf{r}, t) \equiv \langle \mathbf{J}_d(\mathbf{r}, t) \rangle = - \int_0^t D(t-t') \nabla \bar{n}(\mathbf{r}, t') dt'.$$

In other words, the non-homogeneous (porous) media without memory is approximately equivalent to some hypothetic homogeneous media with memory. As a result, the initial advection-diffusion equation (in the absence of sources and sinks):

$$\frac{\partial n(\mathbf{r}, t)}{\partial t} + \nabla \cdot [\mathbf{u} - D(\mathbf{r}) \nabla] n(\mathbf{r}, t) = 0$$

takes the form:

$$\frac{\partial \bar{n}(\mathbf{r}, t)}{\partial t} + \nabla \cdot \int_0^t [\mathbf{u} q_a(t-t') - D q_d(t-t') \nabla] \bar{n}(\mathbf{r}, t') dt' = 0,$$

where $q_a(t)$ and $q_d(t)$ are the advection and diffusion kernels, respectively.

1.5 Hereditary viscoelasticity

1.5.1 Boltzmann's viscoelasticity model

We begin this section with the phenomenological Boltzmann approach considering one-dimensional behavior of a viscoelastic medium.

Let $\psi(t)$, $t > 0$ be the strain evolution caused by the unit jump of stress,

$$\sigma(t) = 1(t) \equiv \begin{cases} 0, & t < 0; \\ 1, & t > 0, \end{cases}$$

called the *creep function*. Approximating an arbitrary sectionally continuous stress-function $\sigma(t)$ by a multistep function with jumps $\Delta\sigma(t_j)$ at times t_j and passing to the limit as $\Delta t_j = t_j - t_{j-1} \rightarrow 0$, we obtain the *creep integral*:

$$\varepsilon(t) = \sum_{t_j < t} \psi(t - t_j) \Delta\sigma(t_j) \rightarrow \int_{-\infty}^t \psi(t - t') d\sigma(t').$$

Using the symbol $\varphi(t)$, $t > 0$ for the *relaxation function* which represents the stress evolution caused by the unit deformation $\varepsilon(t) = 1(t)$, we write down the *relaxation integral* in the form:

$$\sigma(t) = \sum_{t_j < t} \varphi(t - t_j) \Delta \varepsilon(t_j) \rightarrow \int_{-\infty}^t \varphi(t - t') d\varepsilon(t').$$

Representation of heritable effects with the help of the integrals reflects the assumed linearity and causality principles.

Integrating by parts, one can reduce the above relations to the form:

$$\varepsilon(t) = \psi(0)\sigma(t) + \int_{-\infty}^t \dot{\psi}(t - t') \sigma(t') dt'$$

and

$$\sigma(t) = \varphi(0)\varepsilon(t) + \int_{-\infty}^t \dot{\varphi}(t - t') \varepsilon(t') dt',$$

where the dots above the creep function and relaxation function symbols denote their derivatives. This representation separates the instantaneous (non-hereditary) components $\psi(0)\sigma(t)$ and $\varphi(0)\varepsilon(t)$ from the retardation hereditary components.

1.5.2 Elastic solid: a mesoscopic approach

The similar system of hereditary equations is obtained for elastic solid when it is divided into mesoscopic scale elements (*finite-size particles*) in such a way that the location and velocity of their mass center are considered as new *coarse-grained* variables:

$$\mathbf{R}_i = \frac{1}{M_i} \sum_{n=1}^{N_i} m_n \mathbf{r}_n,$$

$$\mathbf{V}_i = \frac{1}{M_i} \sum_{n=1}^{N_i} m_n \dot{\mathbf{r}}_n,$$

where sums are taken over all N_i particles belonging to the i th element of the coarse-grained system, and

$$M_i = \sum_{n=1}^{N_i} m_i$$

is their total mass.

With the use of Mori's projection operators formalism (Zwanzig, 2001), the following system of equations was derived in (Cubero and Yaliraki, 2005a, b)

$$\frac{d\mathbf{R}_i}{dt} = \mathbf{V}_i(t),$$

$$M_i \frac{d\mathbf{V}_i(t)}{dt} = - \sum_j \left[\Lambda_{ij} \mathbf{R}_j(t) + \int_0^t d\tau \Phi_{ij}(\tau) \mathbf{V}_j(t - \tau) \right] + \mathbf{F}_i(t), \quad (1.27)$$

where Λ_{ij} is a renormalized matrix force, $\Phi_{ij}(t)$ is a memory kernel related to dissipative forces, and $\mathbf{F}_i(t)$ is a force which contains the information of the non-

relevant variables and can be interpreted as a random force. They satisfy the condition $\langle \mathbf{F}_i(t) \rangle_{eq} = 0$ and the *non-Markovian fluctuation-dissipation theorem*:

$$\langle \mathbf{F}_i(t) \cdot \mathbf{F}_j(0) \rangle_{eq} = kT \Phi_{ij}(t)$$

guaranteeing that the system admits the thermodynamic equilibrium solution.

Equation (1.27) is the generalized Langevin equation for the coarse-grained system with the noise-term $\mathbf{F}_i$. Acting in a usual way, that is multiplying this equation by $\mathbf{V}_j(0)$ and taking equilibrium averages, one obtains a set of integro-differential equations without noise:

$$M_i \frac{dC_{ij}(t)}{dt} = - \sum_k \int_0^t dt' [\Lambda_{ik} + \Phi_{ik}(t-t')] C_{kj}(t'),$$

which together with the initial condition $C_{ij}(0) = \delta_{ij} M_i k_B T$ determine the velocity autocorrelation function:

$$C_{ij}(t) = \langle \mathbf{V}_i(t) \cdot \mathbf{V}_j(0) \rangle_{eq}.$$

Alternatively, if one knows VAF $C_{ij}(t)$ one can derive the memory kernel $\Phi_{ij}(t)$ from these equations. As shown in Appendix B of the cited paper, if $C_{ij}(t)$ presents a discontinuity in its derivative at times t_r , then $\Phi_{ij}(t)$ presents Dirac-delta singularities at the same times,

$$\Phi_{ij}(t) = - \left[\frac{\dot{C}_{ij}(0)}{C_{jj}(0)} \delta(t) + \sum_r \frac{\Delta \dot{C}_{ij}(t_r)}{C_{jj}(0)} \delta(t - t_r) \right] + \varphi_{ij}(t),$$

where $\Delta \dot{C}_{ij}(t_r) = \dot{C}_{ij}(t_r + 0) - \dot{C}_{ij}(t_r - 0)$, and $\varphi_{ij}(t)$ is a smooth function.

The advantage of the coarse-graining approach is that it provides us with equations of motion where the forces are decomposed in three parts. The first two are functions of the relevant variables $\mathbf{R}_i(t)$ and $\mathbf{V}_i(t)$ alone, and can be identified with conservative and dissipative type force terms, respectively. The force $\mathbf{F}_i(t)$ can be considered as a random force. Therefore, the equations can be cast inside the theory of stochastic processes. While the physical interpretation of these equations is clear (the authors interpret the memory phenomenon as a consequence of perturbations that are transferred by sound waves inside enlarged particles), the main problem resides in finding both closed forms of these expressions for Λ_{ij} and $\Phi_{ij}(t)$ as well as ways of actually computing them in practice.

1.5.3 One-dimensional harmonic lattice

A very instructive model of the hereditary process is due to R. J. Rubin (1960). The model is a one-dimensional harmonic lattice in which one particle is much heavier than the rest ($M \gg m$). The heavy particle appears to behave like a freely

moving Brownian particle with a frictional force proportional to its velocity. The coordinate and velocity of the j th particle are x_j and v_j where j goes from 0 to $N - 1$. Periodic boundary conditions are used, $x_N = x_0$, and the limit of very large $N \rightarrow \infty$ is taken. The nearest neighbor particles are connected by harmonic springs so that the equations of motions are

$$[m + (M - m)\delta_{0j}]\ddot{x}_j = K(x_{j+1} - 2x_j + x_{j-1}).$$

The equations of motion are solved by taking the Laplace transform:

$$\hat{x}_0(\lambda) \equiv \int_0^\infty e^{-\lambda t} x_0(t) dt = \hat{\phi}(\lambda) \left[\frac{M}{m} v_0(0) - \frac{M - m}{m} \lambda^2 \hat{x}_0(\lambda) \right], \quad (1.28)$$

where

$$\hat{\phi}(\lambda) = \frac{1}{N} \sum_{k=0}^{N-1} \frac{1}{\lambda^2 + \omega_k^2}$$

and

$$\omega_k^2 = \frac{2k}{m} \left(1 - \cos \frac{2\pi k}{N} \right)$$

are the normal mode frequencies. Changing variables from k to θ , replacing the sum over k by integral over θ , and taking the limit of large N , one obtains

$$\hat{\phi}(\lambda) \sim \frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{\lambda^2 + (2K/m)(1 - \cos \theta)} = \frac{1}{\lambda \sqrt{\lambda^2 + 4K/m}},$$

and

$$\hat{x}_0(\lambda) = \frac{(1 + Q)\hat{\phi}(\lambda)}{1 + Q\lambda^2\hat{\phi}(\lambda)} v_0(0)$$

with

$$Q = M/m - 1.$$

The Laplace transform of the velocity is

$$\hat{v}_0(\lambda) = \int_0^\infty e^{-\lambda t} \frac{dx_0}{dt} dt = \lambda \hat{x}_0(\lambda),$$

so the transform of the velocity correlation function, normalized to the unity at $t = 0$,

$$C(t) = \frac{\langle v_0(t)v_0(0) \rangle_{eq}}{\langle v_0^2(0) \rangle_{eq}},$$

is

$$\hat{C}(\lambda) = \frac{1 + Q}{1 + Q\lambda^2\hat{\phi}(\lambda)} \sim \frac{1 + Q}{Q\lambda + \sqrt{\lambda^2 + 4K/m}}, \quad N \rightarrow \infty.$$

After some algebraic rearrangement, the transform of the velocity correlation function may be written in the memory function form,

$$\widehat{C}(\lambda) = \frac{1}{\lambda + (m/M)\widehat{k}(\lambda)}, \quad \widehat{k}(\lambda) = \sqrt{\lambda^2 + 4K/m} - \lambda. \quad (1.29)$$

On inverting the transforms, this is equivalent to the hereditary equation:

$$\frac{dC(t)}{dt} = -\frac{m}{M} \int_0^t dt' k(t-t') C(t') + \delta(t) \quad (1.30)$$

with the memory function:

$$k(t) = \frac{4K}{m} \frac{J_1(2t\sqrt{K/m})}{t}.$$

Equations similar to Eq. (1.30) occur frequently in the nonequilibrium statistical mechanics.

1.5.4 Axiomatic approach to continuum mechanics

The classical theories of continuum mechanics are based on simple analytical representation of functional stress-strain relations obtained mostly from experiments. In the general theory of material behavior the constitutive equations are incorporated into a general framework. With regard to a systematic construction of material models by means of constitutive equations, symmetry properties and constraint conditions, it makes sense to take note of general ideas and principles.

Three following principles play a crucial role in dynamics of materials with memory (Noll, 1958):

1. The principle of determinism, which asserts that the current state of stress in a material point is uniquely determined by the past motion of the entire material body.
2. The principle of local action, which asserts that the present stress at a particle is determined by the history of an arbitrarily small neighborhood of that particle and not by the motion of all body particles.
3. The principle of material frame-indifference (objectivity), which asserts that the response of a material is the same for all observers.

A general constitutive law for such material coupling the stress tensor $\mathbf{S}$ with the strain tensor $\mathbf{C}$ has a functional representation:

$$\mathbf{S}(t) = \mathcal{F}[\mathbf{C}(\cdot); t].$$

In (Pioletti and Rakotomanana, 2000), there was proposed to decompose the functional into three parts according to the time scale:

$$\mathbf{S}(t) = \mathbf{F}(\mathbf{C}(t)) + {}_{t-\delta}\mathcal{F}_t[\mathbf{C}(\cdot); t] + {}_{-\infty}\mathcal{F}_{t-\delta}[\mathbf{C}(\cdot); t]. \quad (1.31)$$

The constitutive law enables us to consider the different mechanical behaviors based on the time scale of their effects. The first term is the immediate contribution (elastic behavior) as it supports the contribution of the deformation at the actual time t . The second term in Eq. (1.31) represents the almost immediate contribution, i.e., the short time memory and can be approximated by a differential type material as δ is close to zero,

$$\mathbf{S}(t) = \mathbf{F}_0(\mathbf{C}(t)) + \mathbf{F}_\delta(\dot{\mathbf{C}}(t), \mathbf{C}(t)) + {}_{-\infty}\mathcal{F}_{t-\delta}[\mathbf{C}(\cdot); t].$$

Finally, the third term in Eq. (1.31) represents the delayed contribution, i.e., the long time memory. The long time memory contribution requires the history of the strain tensor $\mathbf{C}$ at all the past times to calculate the stress $\mathbf{S}$.

To be consistent with the observation that memories are imperfect, the principle of fading memory is introduced: deformations which occurred in the distant past have a smaller effect on the present forces than have recent deformations (Coleman and Noll, 1961). Fading memory expresses the intuitive idea that the recent rather than the remote history of deformation of a material body should have a greater effect on the present stress. Mathematical interpretation of the fading memory principle asserts that constitutive functionals have continuous Fréchet derivatives. In the frame of first-order approximation, one obtains

$$\mathbf{S}(t) = \mathbf{F}_0(\mathbf{C}(t)) + \mathbf{F}_\delta(\dot{\mathbf{C}}(t), \mathbf{C}(t)) + \int_{-\infty}^{t-\delta} \Sigma(\mathbf{C}(t'); t) dt'.$$

The tensor-valued function Σ was transformed in the product of two functions of strain and time only (Pioletti and Rakotomanana, 2000) and applied in this work to soft biological tissues.

1.6 Hereditary thermodynamics

1.6.1 Mechanical approach

Cultivation of hereditary methods for describing mechanical processes in condensed matter, initiated by L. Boltzmann and V. Volterra, have inspired the development of hereditary thermodynamics but essentially later. Up to now thermodynamics remains the most phenomenological branch of theoretical physics because many people prefer to consider it rather based on axiomatic foundation than derived from statistical physics. Nevertheless we start our discussion with the latter approach.

It is worth mentioning that R. Zwanzig (1961) started developing his approach with the familiar laws of irreversible thermodynamics having the characteristic form:

$$d\alpha_i(t)/dt = \sum_{k=1}^n L_{ik} F_k(\alpha_1(t), \dots, \alpha_n(t)), \quad (1.32)$$

where α_i is the deviation of the i th state variable from its value at thermal equilibrium, F_k is a thermodynamic force, and L_{ij} are transport coefficients, symmetrical with respect to permutation of i, j . According to these equations, the response of a system to an applied force is simultaneous with the application of the force. This is an important property of Onsager's theory. But such simultaneity in a microscopic theory turns out to be an approximation to a *causal behavior*, where the response to a force comes *after* the application of the force. R. Zwanzig takes account this effect by passing from Markovian equations (1.32) to hereditary integrodifferential ones,

$$d\alpha_i(t)/dt = \sum_{k=1}^n \int_0^1 dt' K_{ik}(t-t') F_k(\alpha_1(t'), \dots, \alpha_n(t')),$$

and remarks that such a causal behavior is always associated with ignoring certain molecular variables while a complete specification of the state of a microscopic system leads to Markovian equations but requires knowledge of an “astronomically” large number of molecular variables.

1.6.2 Hereditary heat-transfer

The standard phenomenological theory of the heat transfer postulates that the heat propagates through the matter with the infinite speed. This found its reflection in the Fourier law connecting the flux of heat with the temperature gradient at the same place at the same time,

$$\mathbf{q}(\mathbf{r}, t) = -\lambda \nabla T(\mathbf{r}, t), \quad (1.33)$$

Under ordinary conditions, this is a good approximation, but it breaks down when the perturbation is very fast or when the response time is very long as observed in non-stationary high-intensity processes in rare gases. In these situations, a heat flux $\mathbf{q}(t)$ delays in time with regard to related to its respective conjugate force ∇T . It was P. Vernotte (1961) who drawn attention to this fact, but twenty years earlier Russian scientists A. V. Lykov (1941) asserted the need for taking into account a finite speed of heat- and moisture-transfer in capillary-porous media.

The retardation is expressed through the hereditary integral:

$$\mathbf{q}(t) = - \int_{-\infty}^t M(t-t') \nabla T(t') dt'$$

with the memory function $M(t-t')$. When

$$M(t-t') = \lambda \delta(t-t')$$

one recovers Eq. (1.33), but the next simplest case

$$M(t-t') = (\mu/\tau) \exp[(t-t')/\tau]$$

leads to a generalized transport equation of the form:

$$\tau \frac{\partial \mathbf{q}}{\partial t} = -(\mathbf{q} + \mu \nabla T), \quad (1.34)$$

which is generally referred to as the *Maxwell-Cattaneo* equation. Introducing Eq. (1.34) into the balance equation for the specific internal energy density $u(\mathbf{r}, t)$,

$$\rho \frac{\partial u}{\partial t} = -\nabla \cdot \mathbf{q}, \quad (1.35)$$

where $du = c_V dT$, with c_V being the heat capacity per unit mass at constant volume. Combining Eqs. (1.34) and (1.35) one obtains the heat transfer hereditary equation of the *telegraph equation form*:

$$\tau \frac{\partial^2 T}{\partial t^2} + \frac{\partial T}{\partial t} - \chi \Delta T = 0, \quad (1.36)$$

where $\chi = \mu / \rho c_V$ designates the heat diffusivity, wherein the coefficients τ and μ are assumed to be constant. The solutions of Eq. (1.36) propagate with finite speed $\sqrt{\chi/\tau}$, and the amplitude of the pulse decreases in time as a consequence of the irreversible term.

We considered the case when the memory decays exponentially, but it could also decay in other fast ways according to the Coleman and Noll fading memory principle (Coleman and Noll, 1960, 1963; Wang, 1964).

1.6.3 Extended irreversible thermodynamics

Extended Irreversible Thermodynamics (EIT) provides generalized transport equations incorporating memory and nonlocal effects. Reducing to the classical transport equations for low frequency and short mean-free paths, it is able to describe the high-frequency behavior of dissipative signals as, for instance, a finite speed of propagation of heat pulses, and general features of ballistic behavior for long mean-free paths. Thus, EIT provides not only small corrections to the usual transport equations, but it turns out to be useful even under extreme non-equilibrium situations where the classical thermodynamics completely fail.

EIT describes a rich phenomenology in heat transport, including thermal waves and phonon hydrodynamics, by introducing a generalized entropy and entropy flux depending not only on the classical variables but also on the fluxes. The behavior of small systems of micro- and nano-scales is strongly influenced by non-linear effects, because even a small difference in temperature over a small distance generates very high gradients. Nowadays, the nano-systems form a new area of technological applications of EIT.

In the classical transport theory, the fluxes $\mathbf{J}$ and the conjugated thermodynamic forces $\mathbf{X}$ are related by linear transport laws of the form:

$$\mathbf{J}(t) = \mathbf{L}\mathbf{X}(t)$$

with $\mathbf{L}$ being suitable phenomenological operator. Assuming an instantaneous response of the system, this approach is in general a good approximation, but it breaks down when the perturbation is very fast or when the response time is very long. In these situations, a flux $\mathbf{J}(t)$ at a given time t is related to its respective conjugate force $\mathbf{X}(t')$ at all preceding times $t' < t$:

$$\mathbf{J}(t) = \mathcal{J}[\mathbf{X}_t(\cdot)].$$

The mostly developed version of EIT is based on the linear approximation of this dependence,

$$\mathbf{J}(t) = \int_{-\infty}^t \mathbf{K}(t-t')\mathbf{X}(t')dt'.$$

When

$$\mathbf{K}(t) = (\mathbf{L}/\tau) \exp[-(t-t')/\tau],$$

one recovers a generalized transport equation of the form:

$$\tau \frac{\partial \mathbf{J}}{\partial t} = -(\mathbf{J} - \mathbf{L}\mathbf{X})$$

which can be referred to as the generalized *Maxwell-Cattaneo equation*. The case of heat transport was considered above.

Recall, that the Laplace transform $t \mapsto \lambda$ leads to replacements $\partial/\partial t \mapsto \lambda$, $\partial^2/\partial t^2 \mapsto \lambda^2$. Because according to Tauberian theorems, short times relate to large values of λ and vice versa, we obtain two different asymptotic equations for these different regions:

$$\tau \frac{\partial^2 T}{\partial t^2} + \frac{\partial T}{\partial t} = \chi \Delta T \sim \begin{cases} \tau \partial^2 T / \partial t^2 = \chi \Delta T, & \text{at short times;} \\ \partial T / \partial t = \chi \Delta T, & \text{at long times.} \end{cases}$$

At short times the transport equation (1.36) is reversible and heat propagates in a ballistic way with a finite speed, whereas at longer times the process becomes irreversible and heat is diffused throughout the system. In the context of chaotic deterministic systems, τ may be interpreted as Lyapunov time beyond which predictivity is lost (Nicolis and Prigogine, 1989). Thus, from a conceptual perspective, Eq. (1.36) is more suitable than the Fourier equation (corresponding to $\tau = 0$), as it describes the transition from reversible to irreversible behavior.

If the heat flux relaxation is relatively small,

$$|\tau \partial \mathbf{q} / \partial t| \cong |\delta \mathbf{q}| \ll |\mathbf{q}|,$$

then Eq. (1.36) reduces to Fourier's law

$$\mathbf{q} = -\lambda \nabla T.$$

Probably, the most compact introducing the hereditary principle into the heat transport problem can be found in very instructive article (Jou et al., 2008) (see also review (Sobolev, 1997)).

The problem of propagation of thermal signals was first deal with by Cattaneo (1958) and Vernotte (1958, 1961) who based their analysis on kinetic theory arguments and added a relaxation term to the Fourier equation. The Cattaneo-Maxwell equation (telegraph equation) has been applied to several phenomena, such as waves in thermoplastic media, fast explosions, and second sound in solids. However, all is not well than this equation: it does not describe accurately the ballistic regime (except one-dimensional process, see (Uchaikin and Saenko, 2001)), nor the second sound signal.

1.6.4 Axiomatic approach

The axiomatic approach mentioned in Sect. 1.5.4 turned out to be very appropriate for introducing hereditary ideas into *thermodynamics of continuous systems with memory* (Coleman and Noll, 1960, 1963; Coleman, 1964; Wang, 1964; Coleman and Owen, 1970; Kolpashchikov and Shnipp, 1978, 1984, 1995; etc.).

In equilibrium thermodynamics problems when processes under consideration run very slowly, interrelations between active (x_j) and passive (y_i) variables do not contain time in an explicit way,

$$y_i(t) = \phi^i(x_j(t)).$$

When the processes run faster so that time of influence of x_j on y_i becomes comparable with a typical time of system state change then the list of active arguments of ϕ^i is supplemented by their time-derivatives:

$$y_i(t) = \phi^i(x_j(t), \dot{x}_j(t)).$$

One can imagine that further acceleration of the process will require involving of higher-order derivatives. In general case functions ϕ^i are replaced by functionals Φ^i :

$$y_i(t) = {}_{-\infty}\Phi_t^i[x_j(\cdot)].$$

Limits $(-\infty, t)$ indicate that the memory functional determining y_i at time t depends on the prehistory of active variables $x_j(t')$, $t' < t$ but not on their future values (the *causality principle*). Let us consider the basic concepts of this approach rather more detailed.

A substance for which the stress tensor $\mathbf{T}$ at time t is determined by the history of the strain is called a *simple material*; for such a material one writes

$$\mathbf{T}(t) = \mathcal{T}[\mathbf{F}_t(\cdot); t]. \quad (1.37)$$

Here

$$\mathbf{F}_t(t') = \begin{cases} \mathbf{F}(t'), & t' \leq t; \\ 0, & t' > t, \end{cases}$$

$\mathbf{F}(t')$ is the deformation gradient at time t' , and $\mathcal{T}$ is a functional mapping functions $\mathbf{F}_t$ into tensors $\mathbf{T}$. Much can be done in the theory of simple materials without arbitrarily specifying the form of $\mathcal{T}$, but rather treating $\mathcal{T}$ as a general functional subject only to the requirements of material symmetry, the principle of material objectivity, and the principle of fading memory. The latter principle is a smoothness postulate which makes $\mathbf{T}(t)$ more sensitive to values of $\mathbf{F}_t(t')$ involving small $s \equiv t - t'$ (the recent past) than to values involving large s (the distant past). The hereditary theory based on Eq. (1.37) was called by Coleman the *mechanical theory of simple materials* to distinguish it from a more general *thermodynamic theory* of simple materials.

In a more general version of the theory, the stress $\mathbf{T}(t)$ would depend not only on each past strain but also on each past temperature $\theta(t')$; in place of Eq. (1.37) one would write

$$\mathbf{T}(t) = \mathcal{T}[\mathbf{F}_t(\cdot), \theta_t(\cdot); t]. \quad (1.38)$$

Further, one must set down a constitutive equation for the heat flux $\mathbf{q}$ which depends on the temperature gradient $\mathbf{g} \equiv \nabla\theta$:

$$\mathbf{q}(t) = \mathcal{Q}[\mathbf{F}_t(\cdot), \theta_t(\cdot); \mathbf{g}(t)]. \quad (1.39)$$

It seems also reasonable to assume that the specific internal energy u , like $\mathbf{T}$, is determined, when the pre-histories $\mathbf{F}_t(t')$, $\theta_t(t')$, $t' \leq t$ are known:

$$u(t) = \mathcal{U}[\mathbf{F}_t(\cdot), \theta_t(\cdot); \mathbf{g}(t)]. \quad (1.40)$$

Assuming the process to be either isothermal ($\theta_t(t') = \text{const}$) or adiabatic ($\mathbf{q} = 0$) we return to pure mechanical theory. In the adiabatic approximation $\theta(t')$ is imagined to be determined as an integral of the energy balance equation, which reduces to (Coleman, 1964)

$$\rho \dot{u} = \text{Tr}\{\mathbf{T} \dot{\mathbf{F}} \mathbf{F}^{-1}\},$$

here ρ is the density, and the dot indicates the material derivative.

For the further development of the thermodynamics, we have to supplement Eqs. (1.38)–(1.40) with an expression for the specific entropy:

$$\eta(t) = \mathcal{H}[\mathbf{F}_t(\cdot), \theta_t(\cdot); \mathbf{g}(t)].$$

The functionals $\mathcal{T}$, $\mathcal{Q}$, $\mathcal{U}$, and $\mathcal{H}$ are not assumed linear in any of their variables $\mathbf{F}_t(\cdot)$, $\theta_t(\cdot)$, or $\mathbf{g}(t)$, nor are these functionals assumed to have any special form or representation. It is, however, assumed that they obey the *principle of fading memory* (Wang, 1964).

Unfortunately, we have no possibility for further description of the generalized thermodynamics because it would take too long time, so we restrict ourselves referring to the article (Breuer and Onat, 1964). It concerns the study of the free energy F

and the entropy production θ in a linear viscoelastic solid defined by the stress-strain law in the form:

$$\sigma(t) = \int_{-\infty}^t G(t-t') d\varepsilon(t'), \quad (1.41)$$

where $G(t)$ is the relaxation modulus approximated by a sum of exponential functions:

$$G(t) = \sum_{i=1}^N c_i e^{-a_i t}, \quad t \geq 0, \quad c_i > 0, \quad a_{i+1} > a_i > 0. \quad (1.42)$$

The starting point is the equation combining the first law of thermodynamics and the entropy balance equation:

$$\sigma(t) \dot{\varepsilon}(t) = \dot{F} + T_0 \dot{\theta}. \quad (1.43)$$

The deformations are supposed to be isothermal, so that the temperature T_0 of the element is a constant. On assumption that F and θ are functionals of the strain rate history to which the material element under consideration has been subjected, the authors inquire whether the knowledge of the relaxation modulus (1.42) will enable them to determine these two functionals, with the help of the constitutive relation (1.41), the thermodynamic equation (1.43) and the second law of thermodynamics $\dot{\theta} \geq 0$. The free energy was presented as a non-negative bilinear functional of the given strain rate prehistory with the continuous, symmetric, and differentiable with respect to both variables kernel $K(t_1, t_2)$:

$$-_{\infty}F_t = \int_{-\infty}^t dt_1 \int_{-\infty}^t dt_2 K(t-t_1, t-t_2) \dot{\varepsilon}(t_1) \dot{\varepsilon}(t_2). \quad (1.44)$$

Differentiating Eq. (1.44) with respect to t and making use of Eq. (1.43) yield:

$$\begin{aligned} \sigma(t) \dot{\varepsilon}(t) &= 2 \dot{\varepsilon}(t) \int_{-\infty}^t dt_1 K(0, t-t_1) \dot{\varepsilon}(t_1) \\ &+ \int_{-\infty}^t dt_1 \int_{-\infty}^t dt_2 \frac{\partial K(t-t_1, t-t_2)}{\partial t} \dot{\varepsilon}(t_1) \dot{\varepsilon}(t_2) + T_0 \dot{\theta}. \end{aligned}$$

It was shown by (Breuer and Onat, 1964) that the knowledge of $G(t)$ supplemented by equations following from the above conditions upon the unknowing kernel $K(x, y)$:

$$\begin{aligned} -_{\infty}F_t &= \int_{-\infty}^t dt_1 \int_{-\infty}^t dt_2 K(t-t_1, t-t_2) \dot{\varepsilon}(t_1) \dot{\varepsilon}(t_2) \geq 0, \\ T_0 \dot{\theta} &= - \int_{-\infty}^t dt_1 \int_{-\infty}^t dt_2 \frac{\partial K(t-t_1, t-t_2)}{\partial t} \dot{\varepsilon}(t_1) \dot{\varepsilon}(t_2) \geq 0, \end{aligned}$$

and

$$K(0, t) = \frac{1}{2} G(t),$$

is not sufficient to determine the general form of $K(x, y)$ and hence of F . For more detailed acquaintance with the problem the reader may be referred to (Fabrizio et al., 1994; Coleman and Owen, 1974, 1975; Golden, 2004) and others.

1.6.5 Ecology and climatology

A presently much debated question in climate dynamics pertains the role of greenhouse gases in the global atmospheric warming and, specifically, the possibility that deforestation may act as a triggering factor of an irreversible climate change. Some pertinent contributions have been recently given by Eshleman (2000), who proposes a simple linear response model to describe the flux of nitrate from a forested watershed subjected to a large-scale disturbance of vegetation:

$$N(t) = B + \int_0^t dt' U(t - t') D(t'),$$

where N is the nitrogen export from watershed, U is the unit nitrogen-export response function, D is the proportion of forested watershed disturbed, and B is the baseline nitrogen export from watershed in the absence of disturbance. In practice, a discretized version of this equation is used in (Eshleman, 2000). Moreover, Eshleman has proposed to extend this approach to the study of the carbon budget using data from satellite remote sensing (<http://al.umces.edu/fiscus/research/nasa-propfinal.doc>). Such an approach would yield further insight into the polluting effects of deforestation, elucidating the main qualitative features of the phenomenon and allowing for an appreciation of parameter sensitivity; general properties valid for all watersheds might be obtained (Cavallini, 2002).

1.7 Nonlocal models

1.7.1 Many-electron atoms

In this section we review some nonlocal model developed in physics mainly for describing collective effects. One of the first fruitful application of this idea is connected with the multielectron atom structure.

A stationary state $|n\rangle$ of the electron in a one-electron atom is described by Schrödinger's equation

$$\left[-\frac{\hbar^2}{2m} \Delta + eV(\mathbf{r}) \right] \psi_k(\mathbf{r}) = E_k \psi_k(\mathbf{r}).$$

Here, $V(\mathbf{r})$ is the potential at point $\mathbf{r}$ created by the atomic nucleus being at the origin ($\mathbf{r} = 0$). In case of a Z -electron atom, this equation may approximately be considered

as an equation for one of these electrons if we take the influence of other electrons into account. As Hartree offered in due course, this may be done by including into the Hamiltonian an addition term $W_k(\mathbf{r})$:

$$\left[-\frac{\hbar^2}{2m}\Delta + eV(\mathbf{r}) + W_k(\mathbf{r}) \right] \psi_k(\mathbf{r}) = E_k \psi_k(\mathbf{r}). \quad (1.45)$$

This term called the *Hartree self-consistent potential* represents the aggregate effect of the rest of electrons averaged over their possible positions:

$$W_k(\mathbf{r}) \equiv \sum_{n \neq k}^Z \int \psi_k^*(\mathbf{r}') eV_{ee}(\mathbf{r} - \mathbf{r}') \psi_n(\mathbf{r}') d\mathbf{r}' = \int V_{ee}(\mathbf{r} - \mathbf{r}') \rho_k(\mathbf{r}') d\mathbf{r}', \quad (1.46)$$

where

$$\rho_k(\mathbf{r}) = \sum_{j \neq k}^Z \psi_j^*(\mathbf{r}) e\psi_j(\mathbf{r})$$

is the total average charge distribution density of $Z - 1$ electrons.

Fock (1930) obtained the system (1.45)–(1.46) by using the variational principle

$$\delta \int \psi^*(\mathbf{r}) H \psi(\mathbf{r}) d\mathbf{r} = 0$$

with the constrain

$$\int \psi^*(\mathbf{r}) \psi(\mathbf{r}) d\mathbf{r} = 1$$

and on assumption that

$$\psi(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_Z) = \psi(\mathbf{r}_1) \psi_2(\mathbf{r}_2) \cdots \psi_Z(\mathbf{r}_Z).$$

The choice of the trial function in a plane form of the one-electron wave functions product implies mutually independent motion of atomic electrons. This function does not meet the claim of permutation symmetry and therefore does not take the exchange interaction into account. The wave function improved by using the Slater determinant (*Slater orbital*) obeys the *Fock-Dirac integrodifferential equation*:

$$\left[-\frac{\hbar^2}{2m}\Delta + eV(\mathbf{r}) + W_k(\mathbf{r}) \right] \psi_k(\mathbf{r}) - \int W_k^{exc}(\mathbf{r}, \mathbf{r}') \psi_k(\mathbf{r}') d\mathbf{r}' = E_k \psi_k(\mathbf{r}).$$

Now, Hamiltonian consists of four parts: the kinetic energy $(-\hbar^2/2m)\Delta$, the potential of nucleus $eV(\mathbf{r})$, the total potential of $Z - 1$ electrons, and the fourth nonlocal term. The term represents the effect of exchange interactions between the k th state and other $Z - 1$ occupied states (see for details (Messiah, 1962)) and namely this gives the nonlocal character to the equation.

1.7.2 Electron correlation in metals

Following (Kudrin, 1974), we consider the simplest model of appearance of electron correlations in a metal. Let k be as before a set of quantum numbers characterizing the state $|n\rangle$ and $\psi(x)$, $x = (\mathbf{r}, \sigma, t)$ be the field operator in the Heisenberg representation. The matrix elements $\langle m|\psi(x)|k\rangle$ describe the motion of electrons while $\langle m|\psi^+(x)|k\rangle$ relate to holes motion. Both one-particle amplitudes obey the *Schwinger equation* (Schwinger, 1951):

$$\left[i\hbar \frac{\partial}{\partial t} - \mathbf{H} \right] \varphi_k(x) - \int dx' \mathbf{M}(x', x) \varphi_k(x') = 0,$$

where $\mathbf{H}$ is a Hamiltonian of the Hartree type complemented by a self-energy,

$$\mathbf{H} = h(x) + \int dx' eV_{ee} \langle \psi^+(x') \psi(x') \rangle = h(x) + \int dx' V_{ee} \sum_j n_j \varphi_j^*(x') \varphi(x'),$$

where n_j are occupation numbers, $\mathbf{M}(x', x)$ is the mass operator of the self-energy. In the absence of nonlocal integral term, $\varphi(x)$ oscillates with frequency $E_k/\hbar$, where E_k is the averaged one-particle energy including exchange interactions and electron correlations. If $\mathbf{M}$ is homogeneous in time, that is $\mathbf{M}(x', x) = \mathbf{M}(\mathbf{r}', \mathbf{r}, t' - t)$, then the equation can be reduced to

$$[E_k - \mathbf{H}] \varphi_k(\mathbf{r}) - \int d\mathbf{r}' \mathbf{M}(\mathbf{r}', \mathbf{r}, E_k/\hbar) \varphi_k(\mathbf{r}') = 0.$$

This is an exact one-particle equation which is unsolvable in the analytical way, because $\mathbf{M}$ is a nonlocal and generally speaking a non-hermitian operator.

1.7.3 Plasma

The ordinary Boltzmann equation describes transport processes in systems of particles with short-range interactions. Such interaction vanishes as soon as the collision comes to an end and both particles move freely. But in case of plasma, we meet with long-range Coulomb's interactions which don't stop when the particles ways part. Accounting this collective effect is implemented by adding a special term in the right-hand side of the equation:

$$\frac{\partial f(\mathbf{r}, \mathbf{v}, t)}{\partial t} + \mathbf{v} \cdot \nabla f(\mathbf{r}, \mathbf{v}, t) = \mathbf{B}_0[\mathbf{v}, f(\mathbf{r}, \cdot, t)] + \mathbf{V}[\mathbf{r}, f(\cdot, \cdot, t)],$$

where

$$\mathbf{V}[\mathbf{r}, f(\cdot, \cdot, t)] = - \left[\nabla \int d\mathbf{r}' \int d\mathbf{v}' V(\mathbf{r} - \mathbf{r}') f(\mathbf{r}', \mathbf{v}', t) \right] \cdot \frac{\partial f(\mathbf{r}, \mathbf{v}, t)}{m \partial \mathbf{v}}$$

(see for details (Balescu, 1975, Sect. 11.7)). This term contains a non-linear non-local operator integrating the product $V(\mathbf{r} - \mathbf{r}')f(\mathbf{r}', \mathbf{v}', t)$ over a whole interaction domain. Consequently, in case of short-range interactions, the $\mathbf{B}$ -term dominates, but in case of long-range interactions the $\mathbf{V}$ -term becomes principal. Neglecting the former term, we arrive at the equation

$$\frac{\partial f(\mathbf{r}, \mathbf{v}, t)}{\partial t} + \mathbf{v} \cdot \nabla f(\mathbf{r}, \mathbf{v}, t) = \mathbf{V}[\mathbf{r}, f(\cdot, \cdot, t)], \quad (1.47)$$

called the *Vlasov equation*. In case of Coulomb's interactions, it can be split into two coupled equations,

$$\frac{\partial f(\mathbf{r}, \mathbf{v}, t)}{\partial t} + \mathbf{v} \cdot \nabla f(\mathbf{r}, \mathbf{v}, t) - \nabla U(\mathbf{r}, t) \cdot \frac{\partial f(\mathbf{r}, \mathbf{v}, t)}{m \partial \mathbf{v}} = 0 \quad (1.48)$$

and

$$\Delta U(\mathbf{r}, t) = -4\pi e \rho(\mathbf{r}, t), \quad (1.49)$$

where $\rho(\mathbf{r}, t)$ is the total charge density at point $\mathbf{r}$ at time t . Observe that this equation relates to a system of particles with the equal charge e . In a more real case, we should write a system of equations each of which corresponds to particles with a specified charge.

It is clearly seen that the particle represented by Eq. (1.48) moves in the field (1.49) created by surrounding particles. This field depends on their instantaneous spatial distribution which is connected to the solution of Eq. (1.48). Thus we deal again with a self-consistent field being the cause of nonlocality and nonlinearity of the equations under consideration.

One more example of a nonlocal equation relating to plasma describes the transport of excitations by resonance radiation transfer. By virtue of proportionality of linear absorption coefficient $k(\omega)$ to radiation coefficient in spectral line, the majority of emitted photons are immediately absorbed, reradiated again, absorbed and so on until the random frequency of an emitted photon happens to be far away from the line center on its wing. The probability of the latter is very small but at the same time the probability of such photon to be absorbed is small too, and whereas it has got such frequency, then its passage to the next interaction will be anomalously large. In the frame of most popular in these calculations *model of total redistribution* of frequencies, the reradiated photon frequency doesn't depend on that of absorbed one, and equation for concentration $N \equiv N(\mathbf{x}, t)$ of excited atoms in plasma, known as the *Biberman-Holstein equation*, has the form:

$$\frac{\partial N(\mathbf{x}, t)}{\partial t} = \gamma \int d\mathbf{x}' K(\mathbf{x} - \mathbf{x}') [N(\mathbf{x}', t) - N(\mathbf{x}, t)],$$

where γ is the atom's radiation (deexcitation) rate, $K(\mathbf{x})$ is the averaged over frequency photon transfer kernel:

$$K(\mathbf{x}) = \left\langle k(\omega) \frac{e^{-k(\omega)|\mathbf{x}|}}{4\pi|\mathbf{x}|^2} \right\rangle.$$

Details of this approach can be found in monograph (), review (Uchaikin) and numerous articles.

1.7.4 Vlasov's nonlocal statistical mechanics

The well-known Russian scientist, professor of Moscow State University A. A. Vlasov has made an essential contribution into the statistical physics. During the last years of his life, he worked on a new approach to statistical mechanics based on the distribution function as an initial concept of the theory. Professor Vlasov failed to complete this work (see his last monograph (Vlasov, 1978) published after his death). Here are some initial concepts introduced at the beginning of the monograph.

Any physical system is characterized by an infinite sequence of distribution functions $f_0(\mathbf{r}, t)$, $f_1(\mathbf{r}, \dot{\mathbf{r}}, t)$, $f_2(\mathbf{r}, \dot{\mathbf{r}}, \ddot{\mathbf{r}}, t)$, ... connected with each other via relations:

$$\begin{aligned} f_0(\mathbf{r}, t) &= \int f_1(\mathbf{r}, \dot{\mathbf{r}}, t) d\dot{\mathbf{r}}, \\ f_1(\mathbf{r}, \dot{\mathbf{r}}, t) &= \int f_2(\mathbf{r}, \dot{\mathbf{r}}, \ddot{\mathbf{r}}, t) d\ddot{\mathbf{r}}, \end{aligned}$$

and so on. Further, it is assumed that the system satisfies the conservation laws in the form of integro-differential relations connecting the densities with correspondent average currents:

$$\begin{aligned} \frac{\partial f_0(\mathbf{r}, t)}{\partial t} + \text{div}_{\mathbf{r}} \int \dot{\mathbf{r}} f_1(\mathbf{r}, \dot{\mathbf{r}}, t) d\dot{\mathbf{r}}, \\ \frac{\partial f_1(\mathbf{r}, \dot{\mathbf{r}}, t)}{\partial t} + \text{div}_{\mathbf{r}}(\dot{\mathbf{r}} f_1(\mathbf{r}, \dot{\mathbf{r}}, t)) + \text{div}_{\dot{\mathbf{r}}} \int \ddot{\mathbf{r}} f_2(\mathbf{r}, \dot{\mathbf{r}}, \ddot{\mathbf{r}}, t) d\ddot{\mathbf{r}}, \end{aligned}$$

and so on. Each line contains two unknown functions and for this reason the conservation laws don't form a closed system for solving the problem. Naturally, one should supplement them with some information about forces acting in the system. Vlasov offered to do this by involving the averaged acceleration:

$$\langle \ddot{\mathbf{r}} \rangle = \frac{\int \ddot{\mathbf{r}} f_2(\mathbf{r}, \dot{\mathbf{r}}, \ddot{\mathbf{r}}, t) d\ddot{\mathbf{r}}}{\int f_2(\mathbf{r}, \dot{\mathbf{r}}, \ddot{\mathbf{r}}, t) d\ddot{\mathbf{r}}} = -\frac{1}{m} \nabla_{\mathbf{r}} U(\mathbf{r}, t).$$

The potential function U includes collective interactions between elements of the continuum and on the simplest linearity assumption is expressed through the integral

$$U(\mathbf{r}, t) = \int K(\mathbf{r}, \mathbf{r}', t) f_0(\mathbf{r}', t) d\mathbf{r}' = \int \int K(\mathbf{r}, \mathbf{r}', t) f_1(\mathbf{r}', \dot{\mathbf{r}}, t) d\mathbf{r}' d\dot{\mathbf{r}}. \quad (1.50)$$

The kernel $K(\mathbf{r}, \mathbf{r}', t)$ should be determined from molecular interactions.

Thus, the second line of the conservation law system becomes

$$\frac{\partial f_1(\mathbf{r}, \dot{\mathbf{r}}, t)}{\partial t} + \dot{\mathbf{r}} \cdot \nabla_{\mathbf{r}} f_1(\mathbf{r}, \dot{\mathbf{r}}, t) - \frac{1}{m} \nabla_{\mathbf{r}} U(\mathbf{r}, t) \cdot \nabla_{\dot{\mathbf{r}}} f_1(\mathbf{r}, \dot{\mathbf{r}}, t) = 0. \quad (1.51)$$

The couple of Eqs. (1.50)–(1.51) forms a nonlinear nonlocal equation for $f_1(\mathbf{r}, \dot{\mathbf{r}}, t)$ on the base of self-consistency concept.

Vlasov applied this approach not only to plasmas problems but to crystalline structures as well. Taking into account of external forces is realized by adding a special local term.

The Vlasov approach continues to progress and an impressive example of such kind is the paper of Kozlov (2010). Introducing the velocity field,

$$\mathbf{u}(\mathbf{r}, t) \equiv \langle \dot{\mathbf{r}} \rangle = \frac{\int \dot{\mathbf{r}} f_1(\mathbf{r}, \dot{\mathbf{r}}, t) d\dot{\mathbf{r}}}{\int f_1(\mathbf{r}, \dot{\mathbf{r}}, t) d\dot{\mathbf{r}}},$$

he has derived nonlocal equations of continuum motion generalizing the Euler local equations:

$$\frac{\partial f_0}{\partial t} + \frac{\partial(f_0 u_i)}{\partial x_i} = 0$$

and

$$\frac{\partial(f_0 u_i)}{\partial t} + \frac{\partial(f_0 u_i u_j)}{\partial x_j} = -\frac{\partial P_{ij}}{\partial x_j} - f_0 \frac{\partial U}{\partial x_i},$$

where

$$P_{ij}(\mathbf{r}, t) = \int (\dot{x}_i - u_i(\mathbf{r}, t))(\dot{x}_j - u_j(\mathbf{r}, t)) f_1(\mathbf{r}, \dot{\mathbf{r}}, t) d\dot{\mathbf{r}}$$

is the stress tensor. Kozlov has also proved the *nonlocal energy conservation law*: if $K(\mathbf{r}, \mathbf{r}') = K(\mathbf{r}', \mathbf{r})$ then the total energy of a system is preserved:

$$T(t) + W(t) = \text{const},$$

where

$$T(t) = \frac{1}{2} \int \int f_1(\mathbf{r}, \dot{\mathbf{r}}, t) |\dot{\mathbf{r}}|^2 d\mathbf{r} d\dot{\mathbf{r}}$$

is the kinetic equation, and

$$W(t) = \frac{1}{2} \int \int \int \int f_0(\mathbf{r}, t) f_0(\mathbf{r}', t) d\mathbf{r} d\mathbf{r}'$$

is the potential energy. Observe, Kozlov notices, that the nonlocal kinetic energy is greater than its local counterpart $(1/2) \int v^2(\mathbf{r}, t) f_0(\mathbf{r}, t) d\mathbf{r}$. Indeed, according to the Cauchy-Bunyakowsky inequality

$$\begin{aligned} & \int \int |\dot{\mathbf{r}}|^2 f_1(\mathbf{r}, \dot{\mathbf{r}}, t) d\mathbf{r} d\dot{\mathbf{r}} - \int \left[\int \dot{\mathbf{r}} f_1(\mathbf{r}, \dot{\mathbf{r}}, t) d\dot{\mathbf{r}} / f_0(\mathbf{r}, t) \right]^2 f_0(\mathbf{r}, t) d\mathbf{r} \\ &= \int \frac{1}{f_0(\mathbf{r}, t)} \left\{ \int |\dot{\mathbf{r}}|^2 f_1(\mathbf{r}, \dot{\mathbf{r}}, t) d\dot{\mathbf{r}} \int f_1(\mathbf{r}, \dot{\mathbf{r}}, t) d\dot{\mathbf{r}} - \left[\int \dot{\mathbf{r}} f_1(\mathbf{r}, \dot{\mathbf{r}}, t) d\dot{\mathbf{r}} \right]^2 \right\} d\mathbf{r} > 0. \end{aligned}$$

1.7.5 Turbulence

Apparently, the first implementation of nonlocal concept into turbulence modeling was made by Monin (1955, 1956) in connection with description of the relative diffusion of tracers in a turbulent medium. Introducing the corresponding probability density function $f(\mathbf{r}, t)$ for the relative distance vector $\mathbf{r}$, he assumed the Markovian character of the process and wrote the equation for $f(\mathbf{r}, t)$ in the form:

$$\frac{\partial f(\mathbf{r}, t)}{\partial t} = \mathbf{A}f(\mathbf{r}, t),$$

where $\mathbf{A}$ is a linear operator. Passing to the Fourier variable space yields

$$\frac{\partial \tilde{f}(\mathbf{k}, t)}{\partial t} = \tilde{A}(\mathbf{k}) \tilde{f}(\mathbf{k}, t). \quad (1.52)$$

For an isotropic turbulent medium, the Fourier transform of $\mathbf{A}$ depends only on absolute value of $\mathbf{k}$,

$$\tilde{A}(\mathbf{k}) = \tilde{a}(k).$$

Acting in frame of basic Kolmogorov's hypotheses, it is logical to admit a power type dependence of $\tilde{a}(k)$,

$$\tilde{a}(k) = -a_1 k^\alpha, \quad a_1 > 0, \quad 0 < \alpha \leq 2. \quad (1.53)$$

The restriction on the values of a_1 and α has been dictated by necessity for the solution of Eq. (1.52) to be a probability density function. Inverting Eq. (1.52), we arrive at the integro-differential equation,

$$\frac{\partial f(\mathbf{r}, t)}{\partial t} = \int A(\mathbf{r} - \mathbf{r}') f(\mathbf{r}', t) d\mathbf{r}', \quad (1.54)$$

with the kernel

$$A(\mathbf{r}) = \frac{1}{(2\pi)^3} \int e^{-i\mathbf{k} \cdot \mathbf{r}} \tilde{A}(\mathbf{k}) d\mathbf{k}.$$

As we will see in the Chapter 3, under conditions (1.53) with $\alpha = 2$, Eq. (1.54) becomes an ordinary *local* diffusion equation:

$$\frac{\partial f(\mathbf{r}, t)}{\partial t} = a_1 \Delta f(\mathbf{r}, t),$$

while in case $\alpha < 2$ we deal with nonlocal equation. Its solution is expressed through the stable density, and the width of the diffusion packet grows as $t^{1/\alpha}$. This law coincides with the Richardson law on choosing $\alpha = 2/3$.

One cannot say that this rather formal result had attracted much attention of “turbulent community”, but with the course of time, the interest to nonlocal ideas in the turbulence phenomenology was growing and more physical arguments were found.

For planar unidirectional laminar flow, the velocity $u(x, t)$ is governed by the Navier-Stokes equation

$$\frac{\partial u(x, t)}{\partial t} = \nu \frac{\partial^2 u(x, t)}{\partial x^2}, \quad (1.55)$$

having the form of a diffusion equation. The second spatial derivative term expresses influence of local interaction between immediate neighbors. A turbulent flow, however, bring into contact fluid parcels that would otherwise not be neighbors. This is why the fluctuations are no longer local. This idea is taken as a new principle of the *integral closure*. Using the direct interaction approximation Kraichnan (1964) showed that the nonlocal eddy diffusivity can be approximated using the averaged Green’s function and the velocity correlation.

Berkowicz and Prahm(1980), Stull(1984), Fiedler (1984), Nakayama and Ven-gadesan(1993), Hamba (2005), Cushman-Roisin and Jenkins (2006), and other authors developed and are developing the nonlocal models of such a complex phenomena as turbulence. One of them called *transilient turbulence theory* describes the nonlocal transport using a matrix of mixing coefficients (Stull, 1984), it was applied to large-eddy simulation of the convective boundary layer (Ebert et al., 1989), to stratospheric mixing and stellar convection. Fiedler (1984) proposed the *integral closure model* in which the spatial derivative is replaced by an integral over the entire domain in the form:

$$\frac{\partial u(y, t)}{\partial t} = \int_0^h a(y, y') u(y', t) dy',$$

or

$$\frac{\partial u(y, t)}{\partial t} = \int_0^h a(y, y') [u(y', t) - u(y, t)] dy',$$

with kernel $a(y, y')$ being a positive weighting factor measuring the contribution of the momentum exchange between levels y' and y .

Berkowicz and Prahm (1980) developed the approach expressing the vertical scalar flux

$$\langle u' \theta \rangle(y) = -\kappa \frac{\partial \langle \theta(y) \rangle}{\partial y}$$

by a spatial integral of the scalar gradient,

$$\langle u'\theta \rangle(y) = - \int_0^h \kappa^{NL}(y, y') \frac{\partial \langle \theta(y') \rangle}{\partial y'} dy',$$

where the prime indicates the fluctuating components and κ^{NL} stands for a coefficient representing a nonlocal contribution (the *nonlocal eddy diffusivity*). Kraichnan (1987) considers the three-dimensional anisotropic case including heredity effect

$$\langle u'\theta \rangle(\mathbf{r}, t) = - \sum_{j=1}^3 \int d\mathbf{r}' \int_{-\infty}^t dt' \kappa_{ij}^{NL}(\mathbf{r}, t; \mathbf{r}', t') \frac{\partial \langle \theta(\mathbf{r}', t') \rangle}{\partial x'_i},$$

and expresses the *nonlocal hereditary eddy diffusivity* through the Green function $g(\mathbf{r}, t; \mathbf{r}', t')$ via relation:

$$\kappa_{ij}^{NL}(\mathbf{r}, t; \mathbf{r}', t') = \langle g(\mathbf{r}, t; \mathbf{r}', t') \rangle \langle u'_i(\mathbf{r}, t) u'_j(\mathbf{r}', t') \rangle. \quad (1.56)$$

The Green function satisfies the equation:

$$\left[\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla - \kappa_0 \Delta \right] g(\mathbf{r}, t; \mathbf{r}', t') = \delta(\mathbf{r} - \mathbf{r}', t - t'),$$

where κ_0 is the molecular diffusivity for the scalar.

Hamba (2005) derived an exact explicit expression for the scalar flux using the modified Green's function, investigated limitation of the local approximation and showed the importance of the nonlocal effect, especially near the wall where the local approximation overestimate the scalar flux and shift the peak location in the streamwise direction, meanwhile the nonlocal approximation is accurate in all considered cases.

It is worth to stress that in accordance with the detail analysis the nonlocality in description of this process arises as a result of the ensemble averaging (see Eq. (1.56)).

1.7.6 Aggregation equations

During the last decade, we observe the heightened interest to the following nonlinear nonlocal equation:

$$\frac{\partial f(x, t)}{\partial t} = \frac{\partial}{\partial x} \left[f(x, t) \int_{-\infty}^{\infty} V'(x - x') f(x', t) \right], \quad (1.57)$$

arising in problems of diffusion through porous media, in biophysics as an individual cell-based model, and in some other problems (Aronson, 1969, 1986; Aronson and Vázquez, 1987; Bates and Fusco, 2000; Bodnar and Velazquez, 2005; Morale et al., 2005; Oelschläger, 1990).

For better understanding the equation, we touch upon main points of its derivation following the paper (Bodnar and Velazquez, 2006). Let $X_k(t)$ be positions

of particles, and $[x - h, x + h]$ a “microscopic” interval containing a large number $N(X_k \in [x - h, x + h])$ of particles. Then the coarse-grained particle density can be defined as

$$f(x, t) \approx \frac{N(X_k \in [x - h, x + h])}{2h}$$

with h satisfying $\varepsilon \ll h \ll 1$ where ε is a typical average distance between particles. The particles move according to equations:

$$\frac{dX_k}{dt} = - \sum_{i \neq k} V'(X_k(t) - X_i(t)),$$

where $V(x)$ is the potential in (1.57), and $V'(x) \equiv dV/dx$. In the continuum representation, the coarse-grained velocity:

$$u(x, t) \approx - \sum_i V'(x - X_i(t))$$

takes the form:

$$u(x, t) = - \int V'(x - x') f(x', t) dx'.$$

On substituting this function into the continuity equation

$$\frac{\partial f(x, t)}{\partial t} + \frac{\partial [u(x, t) f(x, t)]}{\partial x} = 0,$$

we arrive at Eq. (1.57). Its multidimensional analog is

$$\frac{\partial f(\mathbf{x}, t)}{\partial t} = \nabla \cdot \left[f(\mathbf{x}, t) \int_{\mathbb{R}^d} \nabla V(\mathbf{x} - \mathbf{x}') f(\mathbf{x}', t) d\mathbf{x}' \right].$$

This aggregation equation shares some features with the classical Patlak-Keller-Segel scheme modeling of animal collective behavior: flocks, schools, or swarms formed by insects, fishes, and birds, including *chemotaxis* that is the motion of microorganisms and living cells under action of chemical substances. The main similarity is the possible formation of a finite time point concentration and the main difference is the strong singularity of the potential in the PKS system.

A more rigorous derivation of the equation can be found in the paper (Grosskin-sky et al., 2003).

1.7.7 Nonlocal models in nano-plasticity

As noted above, although the classical (local) plasticity theory is able to interpret the material behavior in a large number of applications, it becomes inadequate to model phenomena such as the experimentally observed size-dependence of the plastic response of micro-sized solids or the appearance of localization bands of finite width

in the presence of softening or very large strains. At the same time, experiments performed at the micro- or nano-scale such as nano-indentation (Nix, 1989; Atkinson, 1995; Ashby et al., 1996; Nix and Gao, 1998), bending of thin metallic beams (Haque and Saif, 2003; Stolken and Evans, 1998) or micro-torsion of thin copper wires (Fleck et al., 1994) have provided empirical evidence of strain gradient hardening, which makes the response dependent on the scale of the structure. Hardness and strength increase as the specimen size is decreased; this size effect, which is negligible for macro-specimen, becomes important at very small scales and cannot be captured by local models. Inspired by the works of (Comi and Driemeier, 2005) and Picard et al. (2005), I give a brief outline of these approaches.

The constitutive model of an elastoplastic body Ω is expressed by the following state equations, loading-unloading conditions, and evolution equations:

$$\begin{aligned}\sigma &= E : (\varepsilon - \varepsilon^p), \quad \chi = \chi(\eta), \\ f(\sigma, \chi) &\leq 0, \quad \dot{\lambda} \geq 0, \quad f\dot{\lambda} = 0, \\ \dot{\varepsilon}^p &= \frac{\partial g(\sigma, \chi)}{\partial \sigma} \dot{\lambda}, \quad \dot{\eta} = -\frac{\partial g(\sigma, \chi)}{\partial \chi} \dot{\lambda},\end{aligned}$$

where σ is the stress tensor, ε and ε^p are the total and plastic strain tensors, E is the elasticity tensor, η is the set of internal variables describing hardening, χ are the thermodynamic forces conjugate to η , f is the yield function, $\dot{\lambda}$ is the plastic multiplier and g is the inelastic potential. The generalization accounting for size effect at the microscale is realized by introducing an internal material length l in the model and a nonlocal total strain measure

$$\bar{\varepsilon}(\mathbf{x}) = \int_{\Omega} W(\mathbf{x} - \mathbf{x}') \varepsilon(\mathbf{x}') d\mathbf{x}',$$

W being a weight function. This function is proposed to be chosen as

$$W(\mathbf{x} - \mathbf{x}') = \frac{1}{W_0(\mathbf{x})} \exp\left(-\frac{|\mathbf{x} - \mathbf{x}'|^2}{2l^2}\right)$$

with

$$W_0(\mathbf{x}) = \int_{\Omega} \exp\left(-\frac{|\mathbf{x} - \mathbf{x}'|^2}{2l^2}\right) d\mathbf{x}'.$$

The nonlocal strain tensor $\bar{\varepsilon}$ is only used in the corresponding yield function $\bar{f}$, in isotropic hardening/softening case being written as

$$\bar{f}(\sigma, \chi) = \varphi(E : (\varepsilon - \varepsilon^p)) - \chi(\eta),$$

where φ is an equivalent stress. The state equations remain in a local form. Comi and Driemeier(2005) discussed three simple examples of application of this approach: an elastoplastic softening bar subject to an imposed axial displacement, bending of ultra-thin beams under plain strain conditions, and the torsion tests of the size effect at the microscale in ductile materials.

In the latter example, a circular wire of radius R subject to a torque M . The only nonzero stress and strain components are the tangential stress τ and the shear strain γ , which varies with radius r from the axis x_3 of twist

$$\gamma = \phi r,$$

ϕ is the twist per unit length of the wire:

$$M = 2\pi \int_0^R \tau r^2 dr.$$

Taking into account the axial symmetry of the system, one obtains

$$\begin{aligned} \bar{\gamma}(r) &= \int_0^R W(r-s) \gamma(s) ds \\ &= \phi \frac{2l \left(r e^{-\frac{r^2}{2l^2}} - (R+r) e^{-\frac{(R-r)^2}{2l^2}} \right) + \sqrt{2\pi} (l^2 + r^2) \left(\operatorname{erf} \left(\frac{R-r}{\sqrt{2}l} \right) + \operatorname{erf} \left(\frac{r}{\sqrt{2}l} \right) \right)}{2l \left(-e^{-\frac{(R-r)^2}{2l^2}} + e^{-\frac{r^2}{2l^2}} \right) + \sqrt{2\pi} r \left(\operatorname{erf} \left(\frac{R-r}{\sqrt{2}l} \right) + \operatorname{erf} \left(\frac{r}{\sqrt{2}l} \right) \right)}, \end{aligned}$$

$\operatorname{erf}[r/(\sqrt{2}l)]$ is the error function. Fig. 8 in the cited article shows the $\bar{\gamma}(r)$ for different wire size R .

A nonlocal mesoscopic model for elastoplastic behavior proposed by Picard et al. (2005) was constructed on the bases of two minimal and generic ingredients: localized plastic events associated with a microscopic yield stress, and the resulting elastic relaxation of the stress over the system. Considering a two-dimensional material to which an average shear rate $\dot{\gamma}$ is applied macroscopically (to a z -dependent displacement in the x -direction), the authors describe it at a coarse-grained level, intermediate between microscopic (molecular) and macroscopic scale. The quantity of interest is the xz -component of the time-dependent local shear stress $\sigma(x, z; t)$. Four basic rules are state for this model: (i) below a locally defined yield stress σ_Y , the system responds elastically to the imposed deformation; (ii) above σ_Y , plastic events may occur in the system; (iii) plastic events take the form of a localized shear strain; (iv) such a plastic event induces a long-range elastic perturbation of the shear stress field in the material. Involving the Green function $G(x, x'; z, z')$, relating the stress variation $\delta\sigma$ at any point of the system, to the xz -component of the plastic strain $e^{pl}(x', z'; t)$ associated with the plastic event localized at (x', z') , yields

$$\delta\sigma(x, z; t) = 2\mu \int d\mathbf{r}' G(x, x'; z, z') e^{pl}(x', z'; t),$$

where μ is the shear modulus. Summing up at this point, the evolution of the stress field results from the global elastic loading $\dot{\gamma}$ plus the perturbations induced by the localized plastic events,

$$\frac{\partial\sigma(x, z; t)}{\partial t} = \mu \dot{\gamma} + 2\mu \int d\mathbf{r}' G(x, x'; z, z') \dot{e}^{pl}(x', z'; t).$$

The rules assumed as a basis of the model lead to a complex spatiotemporal behavior of the system at small shear rates. In particular, a correlation length is exhibited which diverges at small shear rates, corresponding to intermittent collective events (correlated bursts of plastic events) leading to the creation of long-lived fragile zones where the deformation takes place.

1.7.8 Nonlocal wave equations

For acquaintance with nonlocal wave equations, the reader may be recommended to refer to the monograph (Naumkin and Shishmarev, 1994) where he can meet the following equations.

The Benjamin-Ono equation

$$\frac{\partial f(x,t)}{\partial t} + f(x,t) \frac{\partial f(x,t)}{\partial x} + \text{p.v.} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\partial^2 f(x',t)}{\partial x'^2} \frac{dx'}{x-x'} = 0$$

describes long internal waves in a stratified fluid of infinite depth.

The Leibovitz equation

$$\frac{\partial f(x,t)}{\partial t} + a(x,t) \frac{\partial f(x,t)}{\partial x} + f(x,t) \frac{\partial f(x,t)}{\partial x} + A \frac{\partial^2}{\partial x^2} \int_{-\infty}^{\infty} \frac{\partial f(x',t)}{\partial x'} \frac{dx'}{\sqrt{(x-x')^2 + B^2}} = 0$$

describes axially-symmetric long waves with a low amplitude in a nonviscous incompressible rotating fluid being infinite in radial directions.

The equation

$$\frac{\partial f(x,t)}{\partial t} + f(x,t) \frac{\partial f(x,t)}{\partial t} + a \frac{\partial^2 f(x,t)}{\partial x^3} + b f(x,t) + c \int_0^x \frac{\partial f(x',t)}{\partial x'} \frac{dx'}{\sqrt{x-x'}} = 0$$

is proposed by Nakoryakov and Shreiber and by Kobelev and Ostrovskii for the acoustic pressure f inside a bubble (a, b , and c are constants).

Rudenko and Soluyan proposed the equation

$$\frac{\partial f(x,t)}{\partial t} + f(x,t) \frac{\partial f(x,t)}{\partial x} + a \frac{\partial^2 f(x,t)}{\partial x^2} + b \frac{\partial}{\partial x} \int_{-\infty}^x \frac{\partial f(x',t)}{\partial x'} e^{-(x-x')/c} dx' = 0$$

for the velocity of particles in the ocean wave with a glance of dissipation and relaxation due to molecular viscosity.

The Langmuir waves in plasma are governed by the equation:

$$\frac{\partial f(x,t)}{\partial t} + i f^2(x,t) f^*(x,t) + i \frac{\partial^2 f(x,t)}{\partial x^2} + \int_{-\infty}^{\infty} K(x-x') f(x',t) dx' = 0.$$

Klimontovich proposed the following magnetohydrodynamic equation for a non-isothermal collisionless plasma:

$$\frac{\partial f(x,t)}{\partial t} + f(x,t) \frac{\partial f(x,t)}{\partial x} + c \int_{-\infty}^{\infty} K_0(x-x') \frac{\partial f(x',t)}{\partial x'} dx' = 0,$$

where f is the ion velocity, K_0 is the zero-order MacDonald function, and c is a constant.

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Chapter 2

Selfsimilarity

Volterra heredity principles could form a basis for intensive development of a new direction in physics. Unfortunately they didn't: those times physicists went in for the revolutionary *fundamental* stream – relativity theory, quantum mechanics, nuclear physics, and elementary particles (gold age of physics! – will it ever recur?). However, last decades are characterized by growing concentration of skills in the direction of research of complex systems and processes which mostly can be described only at *phenomenological* level.

The influence function (memory function) plays a key role in the hereditary theory. Usually, it is introduced on the base of some general assumptions, among which an important position is occupied by the self similarity property being a generic feature of power functions.

2.1 Power functions

2.1.1 Standard power function

Without a doubt, the exponent is the most popular function in theoretical physics. A great number of non-equilibrium processes are described by using exponent (or superposition of exponents). However as far back as 150 years ago, it was written that the voltage decreasing with time of the first electric capacitor, the Leiden jar, follows not exponential but power law $t^{-\gamma}$, where $\gamma \approx 1$ (Kohlrausch, 1854). Since then, numerous phenomena have been discovered which follow power laws (named also Zipf laws and Pareto distribution). The origin of power laws can have deeper causes in the natural processes that display such kind of behavior. In physics, for example, phase transitions in thermodynamic systems are associated with the emergence of power-law distributions of certain quantities, whose exponents are referred to as the critical exponents of the system. Diverse systems with the same critical

exponents can be shown, via renormalization group theory, to share the same fundamental dynamics. For instance, the behavior of water and CO_2 at their boiling points fall in the same universality class because they have identical critical exponents. In fact, almost all material phase transitions are described by a small set of universality classes. Similar observations have been made for various self-organized critical systems, where the critical point of the system is an attractor. Formally, this sharing of dynamics is referred to as universality, and systems with precisely the same critical exponents are said to belong to the same universality class (see Wikipedia, Power Law).

A large collection of power regularities in physics is gathered in books (Sessler, 1980; Jonscher, 1983, 1996; Uchaikin and Zolotarev, 1999; Sornette, 2006).

Many remarkable examples of power laws that appeared in other realms of human activities (music, architecture, painting, linguistics, psychology, acoustics, even without mentioning such fields as geometry, probability theory, geophysics) are collected in the book (M. Schroeder, 1992). Schroeder considers that wherever we should look at or whatever we should listen to we shall see or hear that behavior occurs much more often than it is expected.

Power functions play an important role in theoretical and applied research. Particularly in mathematical analysis they found the integro-differential calculus of fractional order. In complex analysis, with the help of power function it is introduced the notion of poles, built the residue theory, developed integration methods and the asymptotic expansion theory is formulated. In probability theory the transition to “wide” distributions with power law asymptotics marked the whole age of *Lévy stable laws*, which radically changed the central limit theorem and extended its application area.

In theoretical physics, power functions underlines the theory of electromagnetism and gravitation (laws of Coulomb, Newton and others). In hydrodynamics they help to formulate selfsimilar laws of turbulence and describe phenomena of percolation in porous media. In elementary particles physics they found the Regge poles theory. Power functions are the most important elements of quantum chromodynamics, theory of similarity in mechanics and thermodynamics of continuous media. They are signs of fractal structures characterizing in particular the large-scale structure of the Universe. Below we shall discuss a number of concrete examples.

Following (Gelfand and Shilov, 1964) we introduce the particularly normalized power function

$$\Phi_\mu(t) = \frac{1}{\Gamma(\mu)} t_+^{\mu-1},$$

and call it *the standard power function* (see Fig. 2.1). This expression defines functions Φ_μ for all real indexes $\mu \in (-\infty, \infty)$ except zeros of gamma-function: $\mu = 0, -1, -2, \dots$, standing in the denominator. At these points, they are defined as generalized functions, expressed via *the delta function and its derivatives*.

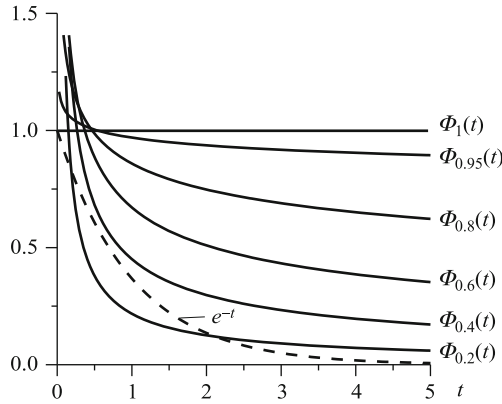


Fig. 2.1 Standard power functions $\Phi_\mu(t)$. Dashed line shows e^{-t} .

2.1.2 Properties of power functions

In a certain sense, the functions Φ_μ are more interesting than the exponential ones due to following properties.

1. They are *selfsimilar*: $\Phi_\mu(ct) = c^{\mu-1}\Phi_\mu(t)$.
2. They possess *semi-group property*:

$$\Phi_\mu \star \Phi_\nu(t) = \int_0^t \Phi_\mu(t-\xi)\Phi_\nu(\xi)d\xi = \Phi_{\mu+\nu}(t).$$

3. At $\mu = -m$, $m = 0, 1, 2, \dots$ functions $\Phi_\mu(t)$ are defined as *generalized functions*, expressed via *delta-function and its derivatives* (Gelfand and Shilov, 1964):

$$\Phi_{-m}(t) = (-1)^m \delta^{(m)}(t).$$

4. *Derivatives and integrals* of power functions belong to the same family:

$$\frac{d\Phi_\mu(t)}{dt} = \Phi_{\mu-1}(t), \quad t > 0;$$

$$\int_0^t \Phi_\mu(t')dt' = \Phi_{\mu+1}(t), \quad t > 0, \quad \mu > 0.$$

5. The *Laplace transforms* of power functions up to a constant factor *belong to the same family*:

$$\begin{aligned} \widehat{\Phi}_\mu(\lambda) &= \mathcal{L}\{\Phi_\mu(t)\}(\lambda) \equiv \int_0^\infty e^{-\lambda t} \Phi_\mu(t) dt \\ &= \frac{1}{\Gamma(\mu)} \int_0^\infty e^{-\lambda t} t^{\mu-1} dt = \lambda^{-\mu}. \end{aligned}$$

The inverse transform formula reads

$$\mathcal{L}^{-1}\{\lambda^{-\mu}\}(t) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} e^{\lambda t} \lambda^{-\mu} d\lambda = \frac{1}{\Gamma(\mu)} t^{\mu-1}.$$

The same statement is valid with respect to power function derivatives and integrals.

6. *Tauberian theorems.* Above formulas can be extended to *asymptotically power functions*:

$$f(t) \sim At^{\mu-1}, t \rightarrow \infty \Leftrightarrow \widehat{f}(\lambda) \sim A\Gamma(\mu)\lambda^{-\mu}, \quad \lambda \rightarrow 0, \quad \mu > 0.$$

Statements connecting a function behavior at large arguments with its Laplace transform at small arguments are called *Tauberian theorems*. The detailed description of these theorems can be found in (Feller, 1971). We'll put here another formulation useful for solving probabilistic problems. Let $p(t)$, $t > 0$ be the probability distribution density of some positive random variable, and $\alpha \in (0, 1)$. Then

$$\int_t^\infty p(t') dt' \sim At^{-\alpha}, t \rightarrow \infty \Leftrightarrow 1 - \widehat{p}(\lambda) \sim A\Gamma(1-\alpha)\lambda^\alpha, \quad \lambda \rightarrow 0.$$

7. *The Fourier transform is given as:*

$$\widetilde{\Phi}_\mu(k) = \mathcal{F}\{\Phi_\mu(t)\}(k) \equiv \int_0^\infty e^{ikt} \Phi_\mu(t) dt = (-ik)^{-\mu} = |k|^{-\mu} e^{i\mu(\pi/2)\text{sign } k}.$$

2.1.3 Memory

Exponential functions $f(t) = Ae^{-\mu t}$ lie in the base of description of majority of process including Debye relaxation, particles decay, optical absorption of light beam, etc. The equation

$$\frac{df(t)}{dt} + \mu f(t) = 0,$$

leading to exponential functions or their modifications can be recognized in many more complicated equations of theoretical and mathematical physics.

The most important property of exponential functions is their shift invariance (up to a constant factor):

$$f(t) = f(t_0 + t') = Ae^{-\mu(t_0+t')} = A'e^{-\mu t'},$$

where $A' = Ae^{-\mu t_0}$ is a constant again. Starting observation at t_0 , we see the same exponential evolution in shifted time $t' = t - t_0$ regardless of the prehistory. If we know the state of the system at t_0 , we know A' and there is no need to know anything else, for example, t_0 itself.

All other functions do not possess this property. If we take, for example, the power function

$$f(t) = At^{-\alpha}$$

and perform the same manipulations as above

$$f(t) = A \cdot (t_0 + t')^{-\alpha} \neq A't'^{-\alpha},$$

we can not find any constant A' to get the same law for the shifted function. The process “keeps in mind” its starting moment t_0 , the knowledge of a number $f(t_0)$ is not enough to predict the future of the process, and its evolution depends also on t_0 .

It is worth noticing that recent investigations in percolation phenomena, anomalous Brownian motion, fractal structures, chaotic dynamics of Hamiltonian systems, turbulence and semiconductors, quantum optics and cosmic processes unambiguously demonstrate the presence of memory. This becomes apparent in non-exponential type of functions describing such processes.

Considering the memory aspect from the stochastic point of view, we suppose that $f(t)$ is a probability density function $p_T(t)$ describing a non-negative random time variable T ,

$$\int_0^\infty p_T(t)dt = 1.$$

The product $p_T(t)dt$ can be interpreted as a probability of event $\{T \in (t, t + dt)\}$. Let $p_{T-t_0}(t'|t_0)dt'$, $t' = t - t_0$ be the conditional probability of the event $\{T - t_0 \in (t', t' + dt')\}$ conditioning on $\{T \geq t_0\}$:

$$p_{T-t_0}(t'|t_0)dt' = \left[\int_{t_0}^\infty p_T(t)dt \right]^{-1} p_T(t_0 + t)dt.$$

Thus, in case of exponential distribution, $p_T(t) = \mu e^{-\mu t}$, we obtain

$$p_{T-t_0}(t'|t_0) = \mu e^{-\mu t'} = p_T(t').$$

In the second, in power case, we take pdf

$$p_T(t) = \begin{cases} At^{-\alpha-1}, & t > (A/\alpha)^{1/\alpha}; \\ 0, & t < (A/\alpha)^{1/\alpha}. \end{cases}$$

Then, for $t_0 > (A/\alpha)^{1/\alpha}$ we obtain

$$p_{T-t_0}(t'|t_0) = \alpha t_0^\alpha (t + t_0)^{-\alpha-1}.$$

It is easy to see that in the first case the conditional probability does not depend on t_0 while in the second case it does. In other words, exponential functions do not possess the memory, but power functions do.

2.1.4 Fractals

The power-law distribution is the only distribution which does not include a characteristic length scale and for this reason it is a natural tool for describing scale-invariant phenomena. The scale invariance (fractality) provides a rational basis for the applicability of the power law. The fractal concept which is introduced and developed by Mandelbrot (1982) has become an effective tool for generating synthetic landscapes that look remarkably similar to actual landscapes (see the remarkable book (Peitgen and Richter, 1986)).

The selfsimilarity is not a special property of some exclusive systems and processes, it is an intrinsic property of many natural objects and phenomena. Relating to geometrical structures, the *strict selfsimilarity* means the scaling invariance (*scaling*, for brevity), that is the invariance with respect to the coordinate map $x \mapsto ax$, $a > 0$. Points, straight lines, and planes are scaling invariant because the scaling transformation maps each of them into itself: a point turns into a point, a straight line turns into a straight line, a plane turns into a plane, and so on.

Wide sense selfsimilarity is determined as such property that “a rough or fragmented geometric shape that can be split into parts, each of which is (at least approximately) a reduced-size copy of the whole” (Mandelbrot, 1982). Roots of mathematically rigorous treatment of fractals can be traced back to functions studied by Karl Weierstrass, Georg Cantor, and Felix Hausdorff in studying functions that were continuous but not differentiable; However, the term “fractal” was coined by Benoit Mandelbrot in 1975 and was derived from the Latin *fractus* meaning “broken” or “fractured.”

There are two main approaches to generate a fractal structure. One is growing from a unit object and the other is to construct the subsequence divisions of an original structure.

Natural objects that are approximated by fractals to a degree include clouds, mountain ranges, lightning bolts, coastlines, snow flakes, various vegetables (cauliflower and broccoli), and animal coloration patterns. However, not all selfsimilar objects are fractals. For example, the real line (a straight Euclidean line) is formally selfsimilar but fails to have other fractal characteristics; in particular, it is regular enough to be described in Euclidean terms.

An important role in the analysis of selfsimilar structures belongs to the conception of *dimension*. In physics, this term is used in various senses. In fractal analysis, many various definitions of the fractal dimension were introduced. The simplest of them is the *scaling dimension* determined by the formula:

$$d_f = \frac{\ln N}{\ln n}, \quad (2.1)$$

where N is the number of identical parts of the fractal object, being similar to the original one but n times smaller in linear size. It is easy to see, that the dimension of a straight line segment, a plane square, and a three-dimensional cube determined by means of the formula which is equal to 1, 2, and 3, respectively. In each of these

cases, d_f is an integer number coinciding with the dimension of a corresponding space. Nevertheless, one can construct such objects which have fractional dimension, namely they are called *fractals*.

Take a segment of length L and remove its central third (Fig. 2.2). Repeating this operation with the secondary segments infinite number of times, we obtain in the limit some selfsimilar (in the wide sense) set. As one can see from the construction itself, $N = 2$ and $n = 3$, therefore we obtain a fractal, embedded in one-dimensional space and having the fractal dimension

$$d_f = \frac{\ln 2}{\ln 3} \approx 0.63093. \quad (2.2)$$

The fractal is called the *Cantor set*.



Fig. 2.2 The Cantor set.

Now, we load the fractal with the *mass* that is a positive function $m(x)$ which for each continuous segment is proportional to its length. So, if the segment L is divided in three subsegments one of them occurs empty and two others are similar to the original, then

$$m(L) = 2m(L/3).$$

It is easy to verify that the solution of the equation is a power function

$$m(x) = m(1) x^{d_f}, \quad (2.3)$$

where d_f is again given by Eq. (2.2). We obtain the set which is already not a point ($d_f > 0$) but is not yet a line ($d_f < 1$).

Equation (2.3) can be considered as one more definition of the fractal dimension, which is more handy for physicists and engineers, especially if to express it in terms of the radius R of a sphere centered at one of points of the fractal:

$$m(R) = m(1) R^{d_f}. \quad (2.4)$$

The exponent d_f determined in such a way can be called the *mass fractal dimension*. If the mass is uniformly distributed over the volume of the sphere, we find that $m(R) = m(1)R^3$ and the mass dimension is 3.

Let us build some other set on a plane. This time, we will not just remove each central third of the segments but replace it by two sides of an equilateral triangle built on this empty third (Fig. 2.3). The evident interrelation

$$m(L) = 4m(L/3),$$

leads to the mass-length relation

$$m(x) = m(1)x^{d_f},$$

saying that we obtain fractal set with the dimension

$$d_f = \ln 4 / \ln 3 \approx 1.26186.$$

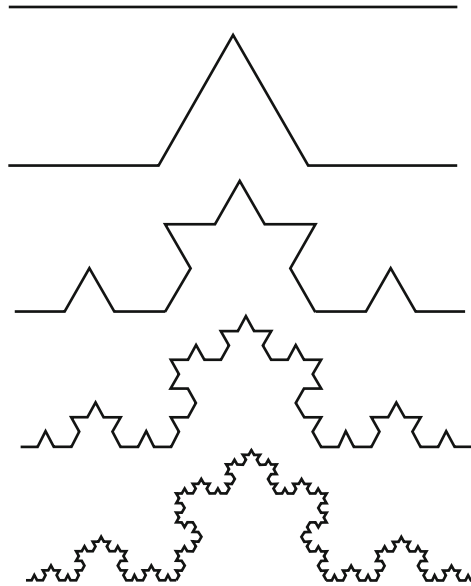


Fig. 2.3 The Koch curve.

This time, the fractal dimension lies between 1 and 2. One can say, we deal now with an object called the *Koch curve* which is not already a line but not yet a plane. Fractals can be constructed in any dimension space. A fractal property can be spatial, as in the fixed geometry of the mathematical examples above; and it can be temporal, as in a series of data taken from a system over an interval of time.

Generally speaking, two ways for generating fractals exist: from outside inwards and from inside outwards. The initiator defines an upper cutoff length of the fractal or a lower cutoff length respectively. Fig. 2.4 (b) shows both ways of generating the most popular fractal called the Sierpinski gasket. Its fractal dimension $d_f = \ln 3 / \ln 2 \approx 1.585$.

Of course, these elementary structures serve only as the simplest examples of realization of fractal structures and can not be considered as some natural objects. As is usual in the application of mathematical models to nature, natural fractals are

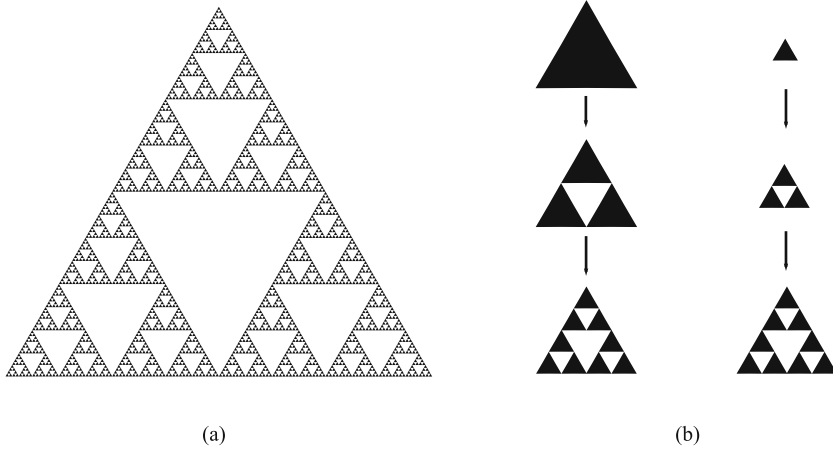


Fig. 2.4 (a) Sierpinski gasket. (b) Two ways for generating the Sierpinski gasket.

more restricted than mathematical ones. The ideal elements (infinite lines, smooth planes, etc.) of Euclidean geometry are never realized in nature, and neither are the ideal elements of fractal geometry, although the latter may be closer to nature than the former. For more close approximation to many natural selfsimilar systems and processes, random elements should be involved. As we will see below, these *random fractals* occur very useful for statistical description of complex systems.

2.2 Hydrodynamics

2.2.1 Newtonian fluids

Newtonian fluid dynamics is governed by the Navier–Stokes equations, which for incompressible fluids has the form

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla p + \nu \Delta \mathbf{u} + \mathbf{F},$$

$$\nabla \cdot \mathbf{u} = 0. \quad (2.5)$$

Here the vector $\mathbf{u} = \mathbf{u}(\mathbf{r}, t)$ denotes the velocity field at position $\mathbf{r}$ at moment t , $p = p(\mathbf{r}, t)$ the pressure, $\mathbf{F} = \mathbf{F}(\mathbf{r}, t)$ the force, $\rho = \rho(\mathbf{r}, t)$ is the density and $\nu = \eta/\rho$ is the kinematic viscosity linked to the dynamic viscosity η by means of relation $\nu = \eta/\rho$.

The most known example of hereditary dependence of power type demonstrates the motion of a sphere in a viscous fluid. Let a sphere of radius a stay motionless in a liquid of density ρ and viscosity η up to some time t_0 . At time t_0 it is impulsively to

put into the progressive steady rectilinear motion with the velocity V_0 . The resistance force acting on it at time $t > t_0$ is (Sljoskin, 1955)

$$F(t) = 6\pi\eta aV_0 \left[1 + \sqrt{\frac{\rho a^2}{\pi\eta(t-t_0)}} \right], \quad t > t_0. \quad (2.6)$$

The body moves with the constant velocity but the fluid “remembers” when the motion started, and this memory comes out in the dependence of fluid resistance on the t_0 . Eventually the memory decreases, and the medium “forgets” when the motion began. The resistance force fails getting close to the value, giving by the Stokes formula:

$$F(\infty) = 6\pi\eta aV_0.$$

2.2.2 Turbulence

2.2.2.1 Turbulent flows

In fluid dynamics, turbulence or turbulent flow is a fluid regime characterized by chaotic and stochastic changes of properties. This includes low momentum diffusion, high momentum convection, and rapid variation of pressure and velocity in space and time. Nobel Laureate R. Feynman describes turbulence as “the most important unsolved problem of classical physics”. There exists a special parameter which is of the utmost importance for understanding conditions for emergence of turbulence. This is the *Reynolds number*, a dimensionless number that gives a measure of the ratio of inertial forces $\rho v^2/L$ to viscous forces $\eta v^2/L^2$. Although laminar-turbulent transition is not governed by Reynolds number, the same transition occurs if the size of the object is gradually increased, or the viscosity of the fluid is decreased, or if the density of the fluid is increased.

In principle, the equations that govern turbulent flows are the same that describe laminar flows, the Navier–Stokes equations for incompressible fluids (Eq. (2.5)).

Given an initial state of the flow field, Equation (2.5) suggests that the evolving field $u(\mathbf{r}, t)$ is deterministic. However, we are uncertain about the uniqueness of the solution and therefore cannot characterize the phenomena as a deterministic process. Moreover, the number of degrees of freedom of a turbulent flow is extremely large, which warrants a statistical rather than a deterministic description.

2.2.2.2 Kolmogorov’s hypothesis

The basis for statistical description is formed by three Kolmogorov’s hypothesis (Kolmogorov, 1941a, 1941b; Frisch, 1995; Davidson, 2004).

Hypothesis 1 *In the limit of infinite Reynolds numbers, all the possible symmetries of the Navier-Stokes equation, usually broken by the mechanism producing the turbulent flow, are restored in a statistical sense at small scales and away from boundaries.*

Hypothesis 2 *On the same assumptions as in H1, the turbulent flow is selfsimilar at small scales.*

Hypothesis 3 *On the same assumptions as in H1, the turbulent flow has a finite non-vanishing mean rate of dissipation ε per unit mass.*

The fruitfulness of the Kolmogorov theory following from these hypothesis is that it does provide a simple scaling for hydrodynamic motions. If the velocity at a scale l from the inertial range is u_l , the Kolmogorov theory states that the kinetic energy ($\rho u_l^2 \sim u_l^2$ as the density $\rho = \text{const}$ in case of an incompressible fluid) is transferred to next scale within one eddy turnover time l/u_l . Thus within the Kolmogorov theory the energy transfer rate $u_l^2/(l/u_l)$ is scale-independent,

$$\frac{u_l^2}{l/u_l} = \text{const},$$

one gets the famous *Kolmogorov scaling*

$$u_l \propto l^{1/3}$$

and so-called *Kolmogorov's 4/5th law*

$$\langle (\Delta u_r)^3 \rangle = -\frac{4}{5} \langle \varepsilon \rangle r, \quad (2.7)$$

where $\Delta u_r = u(x+r) - u(x)$, $u(x)$ is the x -component of the velocity field, and r is measured along x . The result (2.7) is obtained by dropping the viscous term in the full equation valid for all scales $r \ll l$.

The one-dimensional energy spectrum $E(k)$ is the amount of energy between the wavenumber k and $k+dk$ divided by dk . When $E(k)$ is a power function, $kE(k)$ is the energy near the wavenumber $k \sim 1/l$. Since $u_l^2 \approx kE(k)$, Kolmogorov's scaling implies

$$E(k) \propto k^{-5/2}.$$

2.2.2.3 Intermittency

Consider a dynamical system. Let x be observed variable. If x plotted as a function of time, exhibits segments of relative constant values interspersed by erratic bursts, we say the system dynamics is intermittent. The intermittency in this case is called *on-off intermittency*.

Intermittency is an intrinsic property of turbulent flows.

The question whether inertial-range intermittency is contained in Eq. (2.7) was investigated in (Vainstein et al., 1994) using *generalized structural functions*

$$S_q(r) = \langle |\Delta u_r|^q \rangle = C_q u_l^q (r/l)^{\zeta_q} \quad (2.8)$$

and

$$S'_q(r) = \langle |\Delta u_r|^q \text{sign}(\Delta u_r) \rangle = C'_q u_l^q (r/l)^{\zeta'_q}, \quad (2.9)$$

where q is a real number, and u_l is the root-mean-square velocity. Authors of the cited work have shown that the nonzero value of the generalized third-order structure function implies that either negative or positive part of Δu_r possesses scaling at all scales of the inertial range and that the scaling exponents are nontrivial. In a formal sense, this is intermittency. The authors give also a physical interpretation of the result and draw an instructive picture of what is going on in the flow.

2.2.2.4 Turbulence decay

The microscopic fluctuations observed in turbulent flows are also characterized by power laws. The isotropic turbulence obeying the condition $\langle \mathbf{u} \rangle = 0$ can be imagined as motion of liquid after power short stirring. Inspired by (Landau and Lifschitz, 1970), we consider the law of decay of the turbulence being guided by a specific conservation law (Loytsianskii, 1973):

$$\Lambda \equiv \int \langle \mathbf{u}(\mathbf{x}, t) \mathbf{u}(\mathbf{x} + \mathbf{r}, t) \rangle r^2 d\mathbf{r} = \text{const},$$

following from the angular momentum conservation law for a turbulent medium. The integrand rapidly decreases out of a region of size l (the main scale of turbulent motion) and approximately may take

$$u^2 l^5 \simeq \text{const}. \quad (2.10)$$

Until the Reynolds number is large, the energy dissipation rate in a unit volume, ε , is estimated as $\rho u^3/l$. Equating it to estimate of the kinetic energy rate $\rho u^2/t$, we find $l \sim ut$, whence the interrelation

$$u \simeq \frac{\text{const}}{t^{5/7}}$$

appears. Thus, at this stage of the process, the speed of isotropic turbulent motion decays inversely $t^{5/7}$, while a main turbulence scale grows according to the Kolmogorov law

$$l \propto t^{2/7}.$$

However, the Reynolds number $\text{Re} = ul/\nu$ falls proportionally to $t^{-3/7}$ and some period later viscosity begin to play an essential role in the turbulent motion of the liquid. At this stage

$$\varepsilon = \frac{\rho \nu}{2} \left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} \right)^2 \sim \frac{\rho \nu u^2}{l^2},$$

and comparison of this quantity with the rate of kinetic energy loss leads to formula:

$$l \sim \sqrt{\nu t}.$$

Comparing it with expression (2.10), we arrive at the law determining decay of the isotropic turbulence at the last stage of its existing:

$$u \propto t^{-5/4}.$$

The corresponding turbulent velocity autocorrelation function is of the form

$$\langle \mathbf{u}(\mathbf{x}, t) \mathbf{u}(\mathbf{x} + \mathbf{r}, t) \rangle \propto t^{-5/2} \exp(-r^2/[8\nu t]).$$

2.2.3 Microscopic fluctuations

When we are dealing with laminar motion of a liquid, we don't see any irregularities and fluctuations. Nevertheless, they exist on the molecular level and play a very important role in observed properties of liquids. The major characteristics of these microscopic fluctuations in gas and liquid determine light scattering, critical opalescence and some other phenomena are space- and time-correlation (auto-correlation) functions. As R. Balescu noted in his book (Balescu, 1975, Sect. 21.5), for a long time most specialists believed that the correlations decay exponentially. This opinion was supported by investigation of such relatively simple systems like the ideal gas. However, the further inquiry has revealed that for systems with long scale interactions, high density and so on, this is not the case: the correlations decay significantly more slowly. The first who discovered long tail behavior of the correlations was Guernsey (1962). He had shown that the correlations with small wave vectors decay proportionally to t^{-1} . But a common interest to this problem was attracted thanks to works (Alder and Wainwright, 1967, 1969). Using the method of molecular dynamics, they performed numerical simulations of systems consisting of hard discs and of hard spheres and found out that the correlations in a d_s -dimensional system decay according to law $t^{-d_s/2}$. As Balescu wrote, this result turned out to be utterly unexpected and shook preconceived ideas. Not long after, Dorfman and Cohen (1970) showed that to get this result theoretically one should go beyond the Boltzmann approximation. Under this condition, divergent terms in the expansion in density powers become finite and provide the power asymptotics of correlation functions. Follow-up calculations of different authors confirmed supposition that the inverse power correlation decay

$$C_\tau(t) = At^{-d_s/2} = A\Gamma(1 - d_s/2)\Phi_{1-d_s/2}(t)$$

should be considered as a common feature in behavior of long-wave hydrodynamic perturbations (Balescu, 1975). In Balescu's opinion, the power function appears as a result of exponential function superposition describing long-distance propagation

of perturbations according to microscopic hydrodynamic laws. This is a very slow process observed for a long time after exponentially fast transition processes have decayed.

The decay of the velocity autocorrelation function at intermediate fluid densities can be interpreted in the following way. The velocity autocorrelation function, whose integral is the diffusion coefficient, shows a positive persistence of velocity that can be observed for some 100 collisions. That positive tail can be quantitatively accounted for by a hydrodynamic model in which a sphere, surrounded by a continuum fluid, characterized by compressibility and viscosity, is given an initial velocity or momentum. Those initial conditions lead to a positive pressure in the fluid ahead of the sphere that generates a sound wave and a corresponding rarefaction wave behind. In addition, the positive pressure in front and the negative pressure behind can equalize by creating a double vortex structure around the sphere. This phenomena is verified by investigating the velocity field $\sum_j^n \mathbf{u}_j(t) \mathbf{u}_i(0)$ surrounding the sphere, by molecular dynamics. The velocity field is the correlation between the original velocity of the sphere, i , and the net velocity of all the particles, j , in a surrounding volume element at some time, t , later. The agreement between this molecular dynamically determined velocity field and the corresponding hydrodynamic one is quantitative after some 10 collisions, or about 10^{-13} seconds. The remarkable validity of hydrodynamics at this time scale and distance scale of only a few molecular diameters will be referred to again later. A dimensional analysis of the hydrodynamic model leads to a quantitative description of the tail. After first observing that at long times only the vortex mode contributes since it spreads diffusively as $(\nu t)^{1/2}$ while the sound wave spreads faster, namely to further distances as ct , where ν is the kinematic viscosity (the shear viscosity divided by the density) and c is the speed of sound. Since momentum mu is conserved, $m \int_V \mathbf{u} dV$ must be a constant, where V is the volume element, which must be integrated over the entire space to account for all the momentum. Since V spreads as $(\nu t)^{d/2}$, where d is the dimensionality of the space, u must decay as $(\nu t)^{-d/2}$ to keep the momentum constant. The more difficult but readily doable task is to determine the coefficient A in the power law decay of the velocity, $A/(\nu t)$, since that requires knowing how much of the momentum is carried away by the sound wave and how much of the rest is fed back via the medium to the moving sphere. This same power law has also been derived by graph theoretical methods involving summation of ring diagrams (Dorfman and Cohen, 1970).

2.2.4 Non-Newtonian fluids

2.2.4.1 Shear viscosities

A Newtonian fluid is a fluid whose stress is proportional to strain rate

$$\tau = \eta \dot{\gamma}.$$

In common terms, this means the fluid continues to flow, regardless of the forces acting on it. For example, water is Newtonian, because it continues to exemplify fluid properties no matter how fast it is stirred or mixed. Other examples may be aqueous solutions, emulsions. Contrast this with a non-Newtonian fluid, in which stirring can either leave a “hole” behind (that gradually fills up over time — this behavior is seen in materials such as pudding and oobleck (a fluid made from 2 parts corn starch to 1 part water), or, to a less rigorous extent, sand), or climb the stirring rod (the Weissenberg effect) because of shear thinning, the drop in viscosity causing it to flow more. The central role in investigation of non-Newtonian fluid belongs to molecular-dynamics simulations. In particular, it was demonstrated that the energy and pressure of simple Lennard-Jones and Barker-Fisher-Watts fluids can be described by simple power-law scaling, that the power exponents are linear functions of temperature and density; a useful consequence of this simple scaling behavior is that it can be used to predict the solid-liquid phase coexistence curve at equilibrium (Todd, 2005). In the cited work, the viscosities were computed for the Lennard-Jones fluid across a range of densities and temperatures that span the liquid state. It was shown as well that the exponents of the viscosity versus strain rate curves have a remarkably simple analogous linear dependence on the temperature and density in the liquid phase. For each value of temperature and density, a viscosity profile was computed and a simple power law of the form:

$$\eta = \eta_0 - \eta_1 \dot{\gamma}^\beta$$

was found to fit the data, where η_0 , η_1 , and β are positive constants. The exponent β can have any value between 0.2 and 1.6.

2.2.4.2 Sound waves

It is known that viscosity and thermal conductivity cause the dissipation of sound waves energy. Consequently the amplitude (velocity, pressure) of the sound wave, propagating along the z axis in homogeneous media, exponentially decreases

$$A(z) = Ae^{-\gamma z},$$

where attenuation coefficient is proportional to the square of frequency

$$\gamma = a\omega^2,$$

only if the amplitude decreasing is small within the distances of the wave length order. This is true for gases and fluids if internal friction forces are not large. However, there exists media in which the power form of γ dependence is held over wide frequency range but the power α is different from 2 (Chen, 2005).

$$\gamma(\omega) = a\omega^\alpha, \quad \alpha \in [0, 2].$$

2.2.4.3 Sheared granular liquids

The systems considered in (Otsuki and Hayakava, 2010) are three-dimensional sheared granular liquids consisting of N identical smooth and inelastic hard spherical particles in the volume V , where the mass and the diameter of each grain are respectively given by m and σ . The granular systems often cause shear-bands near the boundary. The authors focused on the uniformly sheared granular liquids characterized by α component of the velocity field $c_\alpha(\mathbf{r}) = \dot{\gamma} \delta_{\alpha,x}$ with the shear rate $\dot{\gamma}$. It is known that such uniform flow can be realized if the system size is not extremely large and granular assemblies are nearly elastic under Lees-Edwards boundary condition. It is also assumed that the particles collide instantaneously with each other.

Because of the uniformly sheared case, the temperature T only appears through the time evolution equation of the velocity fields $\mathbf{u}(\mathbf{r}, t)$. Thus, the generalized hydrodynamic equations can be taken in the form:

$$\frac{\partial n}{\partial t} + \nabla \cdot (\mathbf{u}n) = 0,$$

$$\frac{\partial u_\alpha}{\partial t} + u_\alpha \nabla_\beta u_\beta + \frac{1}{m} \nabla_\alpha \mu + \frac{1}{m} \nabla_\beta (\Sigma_{\alpha\beta}^D + \Sigma_{\alpha\beta}^R) = 0,$$

where $n = n(\mathbf{r}, t)$ is the number density,

$$\mu = \mu(\mathbf{r}, t) = T \left[\ln n(\mathbf{r}, t) - \int d\mathbf{r}' C(\mathbf{r} - \mathbf{r}', e, \dot{\gamma}) \delta n(\mathbf{r}', t) + \dots \right]$$

is the generalized chemical potential or effective pressure, $C(\mathbf{r} - \mathbf{r}', e, \dot{\gamma})$ is the direct correlation function,

$$\begin{aligned} \Sigma_{\alpha\beta}^D(\mathbf{r}, t) = & - \int d\mathbf{r}' \{ \eta(\mathbf{r} - \mathbf{r}', e) \dot{\epsilon}_{\alpha\beta}(\mathbf{r}', t) \\ & - [2\eta(\mathbf{r} - \mathbf{r}', e)/3 - \zeta(\mathbf{r} - \mathbf{r}', e)] \dot{\epsilon}_{\gamma\gamma}(\mathbf{r}', t) \delta_{\alpha,\beta} \} \end{aligned}$$

with

$$\dot{\epsilon}_{\alpha\beta}(\mathbf{r}', t) \equiv (\nabla_\alpha u_\beta + \nabla_\beta u_\alpha)/2$$

is the viscous stress tensor, and $\Sigma_{\alpha\beta}^R(\mathbf{r}, t)$ is the random part of the stress tensor satisfying the fluctuation-dissipation relation

$$\langle \Sigma_{\alpha,\beta}^R(\mathbf{r}, t) \Sigma_{\gamma,\delta}^R(\mathbf{r}', t') \rangle + 2T \delta(t - t') [\eta(\mathbf{r} - \mathbf{r}', e) \Delta_{\alpha\beta\gamma\delta} + \zeta(\mathbf{r} - \mathbf{r}', e) \delta_{\alpha\beta} \delta_{\gamma\delta}]$$

with $\Delta_{\alpha\beta\gamma\delta} \equiv \delta_{\alpha\gamma} \delta_{\beta\delta} + \delta_{\alpha\delta} \delta_{\beta\gamma} - 2\delta_{\alpha\beta} \delta_{\gamma\delta}/3$. Here, we have used Einstein's sum rule on the Greek subscripts.

At the end of the article (Otsuki and Hayakava, 2010) its authors obtain the steady-state spatial correlation functions for the fluctuations of the number density δn and the momentum density $\delta \mathbf{p} = m n \delta \mathbf{u} \equiv m n (\mathbf{u}(\mathbf{r}, t) - \mathbf{c}(\mathbf{r}))$:

$$C_{nn}(\mathbf{r}) \equiv \langle \delta n(\mathbf{r} + \mathbf{r}', 0) \delta n(\mathbf{r}', 0) \rangle \propto \left(\frac{r}{l_c} \right)^{-11/2}$$

and

$$C_{pp}(\mathbf{r}) \equiv \langle \delta \mathbf{p}(\mathbf{r} + \mathbf{r}', 0) \cdot \delta \mathbf{p}(\mathbf{r}', 0) \rangle \propto \left(\frac{r}{l_c} \right)^{-5/2},$$

for $r \gg l_c \equiv \sigma/\dot{\gamma}$. The validity of the existence of long-range correlation has been verified by the numerical simulation in (Otsuki and Hayakawa, 2009), where the theory can be used, at least, until $\phi \leq 0.50$ with the volume fraction ϕ .

A wide class of non-Newtonian fluids is formed by polymers and their solutions.

2.3 Polymers

2.3.1 The Nutting law

In 1921, Nutting reported about his observations that the mechanical stress in viscoelastic materials caused by instantaneous strain decays according to the inverse power law:

$$\sigma(t) \simeq \text{const} \cdot \Delta \varepsilon \cdot t^{-\alpha},$$

where $\alpha \in (0, 1)$. Later, the Gemant investigation of viscoelastic materials (Gemant, 1936) confirmed such their behavior under action of a harmonic load. For many materials, this exponent is close to 0.5 (Bagley and Torvik, 1983). This conclusion came into conflict with the classical conception of exponential character of the relaxation.

2.3.2 Relaxation of polymer chains

A theoretical model of the algebraic hereditary in viscoelasticity was offered by Rouse (1953). He represented a polymer molecule as a chain of point monomers linked by elastic bonds (massless springs with stiffness K). Its spatial configuration is given by a set of monomers radius-vectors $\{\mathbf{r}_n\}$, $n = 0, 1, 2, \dots, N-1$. The segments $[\mathbf{r}_n, \mathbf{r}_{n+1}]$ of mean length a can free rotate around their ends. The molecule is immersed into a viscous Newtonian fluid at the temperature T , so that the n th monomer is under action of random force $\mathbf{f}(n, t)$. Rouse considered it as the Gaussian heat noise

$$\langle \mathbf{f}(n, t) \rangle = 0, \quad \langle f_i(m, s) f_j(n, t) \rangle = 2\zeta kT \delta_{ij} \delta_{mn} \delta(s - t), \quad (2.11)$$

where $i, j = 1, 2, 3$ indicate cartesian projections of the force and k is the Boltzmann constant.

Suppose that a force jump $\Delta F(t_0)$ acts on an extreme monomer $\mathbf{r}_0$ at the moment t_0 in x -direction. Statistical calculations (Doi and Edwards, 1995) show that the monomer begins to move according to law

$$\langle x_0(t) \rangle \approx \frac{a\Delta F(t_0)}{\sqrt{\zeta kT}}(t - t_0)^{1/2}, \quad t > t_0, \quad (2.12)$$

while the motion of a single free monomer in a viscous fluid is uniform, $\langle x_0(t) \rangle \propto (t - t_0)$. Departure the power law (2.12) from linear one is explained by collective character of motion: the monomers begin the motion not all immediately, but consecutively. The number n_τ of monomers drawn into the motion during time $\tau = t - t_0$ is proportional to $\tau^{1/2}$:

$$n_\tau \approx \sqrt{\frac{kT}{a^2\zeta}}\tau^{1/2}, \quad \tau = t - t_0.$$

This is in case of infinite chain. In case of a molecule of a finite length N , all its atoms come to the directed motion during time

$$\tau_N = \zeta(aN)^2/(kT)$$

and the motion of its center of mass becomes uniform.

Schiessel et al. (2000) investigated a more dense macromolecular structure, then a linear chain, namely, *fractal net* with fractal dimension d_f . Considering the momentum propagation through the system as a random walk on the fractal, the authors found that for infinite structure

$$n_\tau \approx \left(\frac{kT}{a^2\zeta}\right)^{d_f/2} \tau^{d_f/2}, \quad 1 < d_f < 2,$$

and

$$\langle x_0(t) \rangle \approx \frac{a^{d_f}\Delta F(t_0)}{\zeta^{1-d_f/2}(kT)^{d_f/2}}(t - t_0)^{1-d_f/2}, \quad t > t_0. \quad (2.13)$$

Passage from these influence functions to the case of an arbitrary force $F(t)$ is easily realized by the linearity principle:

$$\langle x_0(t) \rangle \approx A \int_{-\infty}^t \Phi_{2-d_f/2}(t - t') dF(t') \equiv A \Phi_{2-d_f/2} * dF/dt.$$

Here $A = a^{d_f}\Gamma(2 - d_f/2)/[\zeta^{1-d_f/2}(kT)^{d_f/2}]$, k is the Boltzmann constant, T is absolute temperature.

Following the seminal work (De Gennes, 1979), polymer solution dynamics in the dilute and the semidilute regime have been extensively studied over the past three decades. Due to the overlap of polymer chains, different concentration dependent dynamical regimes were predicted. Experimental efforts have a large extent focused on semidilute solutions, for which the distance between the contact points, or the

dynamical screening length for the hydrodynamic interaction, ξ_h , is anticipated to follow a simple power law

$$\xi_h \sim \phi^{-\alpha}.$$

Here α is 0.75, 0.5, and 1 in a good, a marginal, and a θ solvent, respectively, and ϕ is the volume fraction of the polymer. Experimentally, the validity of this law has been examined with photon correlation spectroscopy by probing the collective diffusion coefficient D_c , related to ξ_h via the Stokes-Einstein relation $D_c = k_B T / (6\pi\eta_s \xi_h)$, where η_s is the viscosity of the pure solvent. Thereby, an analogous scaling prediction is obtained for D_c : $D_c \sim \phi^\alpha$.

The detailed investigation of the problem can be found in (Uematsu et al., 2003).

2.3.3 Interpenetrating polymer networks

As we know now, the interpenetrating polymer network with its two phases and complex morphology can be represented quite well by a power law relaxation over an extremely wide time or frequency range. Power law relaxation implies that the material is in a critical state. We cannot say whether the material has a longest relaxation time which is finite. Such behavior would be outside our experimental range. Winter and Chambon (1986) were possibly the first who hypothesized that the material at the gel point exhibits a power law relaxation over the entire time scale and have proposed a two parameter constitutive equation

$$\tau(t) = S \int_{-\infty}^t (t-t')^{-\alpha} \dot{\gamma}(t') dt'$$

to describe the behavior of the material. The two parameters are the relaxation exponent α ($0 < \alpha < 1$) and the front factor S (a measure of the rigidity of the material).

Antonietti et al. (1989) found that microgels synthesized by endlinking of polystyrene molecules in dilute solution, exhibit power law relaxation over wide ranges of frequency. Conventional polymer melts might also exhibit power law relaxation, however, only at intermediate frequencies (or times) since they have a longest relaxation time which is finite.

We end this subsection referring to the article (Mani et al., 1989) considering emulsion polymerized interpenetrating polymer networks (IPN) of polyacrylate and polystyrene, which exhibit a power law relaxation over 5 decades of frequency and correspondingly of time. The experimental samples were compression-molded from a latex interpenetrating polymer network of polyacrylate (copolymer of 70 % butyl and 30% ethyl acrylates) and polystyrene. The polyacrylate is crosslinked with butadiene and the polystyrene is crosslinked with divinylbenzene. Six different compositions are investigated and the correspondent values of relaxation exponent γ are found; they are in between 0.07 and 0.16.

2.4 Reaction-diffusion

2.4.1 Diffusion

Recall two classical experiments on measuring the diffusion coefficient. In the first, one takes a long cylindric vessel with the cross-section area S and cross plate in the plane $x = 0$. In the right part of the cylinder there is placed a pure solvent, and in the left part, a solution with the concentration c_0 . At time $t = 0$, the plate is put out, starting the diffusion process which is described by the equation

$$\frac{\partial c}{\partial t} = K \frac{\partial^2 c(x, t)}{\partial x^2}$$

with the initial condition

$$c(x, 0) = \begin{cases} c_0, & x < 0; \\ 0, & x > 0. \end{cases}$$

Its solution has the form:

$$c(x, t) = \frac{c_0}{2K\sqrt{\pi t}} \int_{-\infty}^x \exp(-\xi^2/(4Kt)) d\xi.$$

During the elementary time-interval $d\tau$, the quantity

$$dq = K[\partial c(x, \tau)/\partial x]_{x=0} S d\tau$$

of a diffusing substance passes through the plane $x = 0$. At the moment t , the plate put in the cylinder and the total quantity of that substance in its right half,

$$Q(t) = \int_0^t K[\partial c(x, \tau)/\partial x]_{x=0} S d\tau = c_0 S \sqrt{K/\pi t}^{1/2}.$$

This expression is used for finding the diffusivity K , but we are interested in appearance of function Φ_μ :

$$Q(t) = A\Phi_{3/2}(t), \quad A = c_0 S \sqrt{K}/2.$$

In the other test, the baffle is not put out but replaced (at $t = 0$) by a tacky plate and t seconds later, the plate is taken out and the quantity of the substance settled on it is measured. Brillouin believed that the result should be the same but in reality it takes the doubled value

$$M(t) = 2A\Phi_{3/2}(t).$$

The reason for the doubling is that the diffusing particles coming to the tacky plate can not come back while in the first test a half of the diffusing substance goes back to the left half of the vessel.

Now, let the diffusing substance initially be concentrated in plane $x = -s$, $s > 0$: $c(x, 0) = C\delta(x + s)$. Then the rate of setting of the substance on the unit area of the

plate is

$$m(t) = \dot{M}(t) = \frac{sC}{2\sqrt{\pi Kt^3}} \exp\left(-\frac{s^2}{4Kt}\right) \sim A\Phi_{-1/2}(t), \quad t \rightarrow \infty.$$

With $C = 1$, it can be interpreted as a probability density function of the first reaching time of plane $x = 0$ being at a distance s of it at the initial time.

2.4.2 Polymerization

Self-avoiding walk is a kinetic process asymptotically mimicking the equilibrium statistics of linear polymers in a good solvent. In the course of the process, a particle is allowed to migrate over a simple d -dimensional lattice sites but forbidden to visit sites repeatedly: the particle should avoid the sites already visited before. The corresponding coarse-graining probability density function for the particle started from the origin is given by the normalized product

$$p(\mathbf{r}, t) = A(t)S(\mathbf{r}, t)p_0(\mathbf{r}, t), \quad A(t) = \left[\int S(\mathbf{r}, t)p_0(\mathbf{r}, t)d\mathbf{r} \right]^{-1},$$

of d -dimensional Gaussian density $p_0(\mathbf{r}, t)$ and *survival probability* $S(\mathbf{r}, t)$ resulting from intersection prohibition. Qualitatively, this probability can be estimated as follows (Peliti, 1986). Let q be the probability of the second visiting the site during t (estimated as $At/|\mathbf{r}|^d$) and Δt be a time interval between independent consecutive jumps. Then $S(\mathbf{r}, t)$ can be written as the probability that during $t = N\Delta t$ this event will occur not once:

$$S(\mathbf{r}, t) = (1 - q)^N.$$

Passage to the limit under $N \rightarrow \infty$ yields:

$$S(\mathbf{r}, t) \sim \exp(-Nq) = \exp(-gt^2/|\mathbf{r}|^d),$$

where g is a positive constant. Inserting this function into expression for pdf $p(\mathbf{r}, t)$ and computing the mean square of displacement during t leads to integral

$$\langle |\mathbf{r}(t)|^2 \rangle = A(t) \int \exp[-F(\mathbf{r}, t)] |\mathbf{r}|^2 d^d \mathbf{r}, \quad A(t) = \left[\int \exp[-F(\mathbf{r}, t)] d^d \mathbf{r} \right]^{-1},$$

with

$$F(\mathbf{r}, t) = \frac{|\mathbf{r}|^2}{4Kt} + \frac{gt^2}{|\mathbf{r}|^d}.$$

Using saddle-point approximation, we arrive at the following asymptotic expression:

$$\sqrt{\langle |\mathbf{r}(t)|^2 \rangle} \sim (gKt)^{VF}, \quad t \rightarrow \infty,$$

where the exponent ν_F is given by the *Flory formula*:

$$\nu_F = \frac{3}{2+d}.$$

2.4.3 Coagulation and fragmentation

The *diffusion-limited aggregation model* (DLAM) was introduced by Witten and Sander (1981). DLAM is a mathematical model, or, to be more precise, a family of such models, meant for description of growth processes: coagulation and polymerization, aerosol formation, colloid grows, red corpuscle aggregation, nucleation in phase transitions, colloidal solutions and cosmic dust evolution etc. (Family and Landau, 1984).

These models are based on the Smoluchowsky *mean field approximation* approach which consider the cluster coalescence as a result of binary collisions of the clusters without taking into account fluctuations in their spatial distribution. Evolution of k -particle cluster concentration, $n_k(t)$, under condition that m particles are generated in unit volume per unit time, is governed by the differential equations

$$\frac{dn_k(t)}{dt} = m\delta_{k1} - \left[n_k(t) \sum_{j=1}^{\infty} K_{kj}n_j(t) - \frac{1}{2} \sum_{i+j=k} K_{ij}n_i(t)n_j(t) \right].$$

Kernels K_{ij} approximately describe dependence collision cross-sections and clusters mobility on their size. After change of variables $\tilde{t} = \sqrt{mt}$, $\tilde{n}_k(\tilde{t}) = n_k(t)/\sqrt{m}$, multiplier m in front of delta-symbol vanishes and in long time asymptotics when the system forgets its initial state, its solution takes a selfsimilar form:

$$n_k(t; m) \sim \sqrt{m} f_k(\sqrt{mt}).$$

Thus the process possesses the relaxation time scale $\tau_k \sim 1/\sqrt{m}$ and stationary size scale $\bar{n}_k \sim \sqrt{m}$, while in case $m = 0$ the relaxation law is non-exponential:

$$n_k(t; 0) \sim t^{-1}, \quad t \rightarrow \infty.$$

A more general approach going beyond the mean field approximation uses the homogeneous representation of the right side of the equation, in the spirit of scaling postulates of critical phenomena theory:

$$\frac{dn_k(t)}{dt} = m\delta_{k1} - G_k(n_1, n_2, \dots), \quad G_k(\lambda n_1, \lambda n_2, \dots) = \lambda^\delta G_k(n_1, n_2, \dots).$$

Asymptotical behavior of the equation solution (Rácz, 1986)

$$n_k(t; m) \sim m^{1/\delta} f_k(m^{1-1/\delta} t)$$

differs from written above due to an arbitrary exponent δ . In a similar way, the relaxation law exponent changes:

$$n_k(t;0) \sim t^{-1/(\delta-1)}, \quad t \rightarrow \infty.$$

As noticed in (Safronov, 1969) and (Trubnikov, 1971), the inverse power mass spectrum, obtained from the equations:

$$n(m) \propto m^{-3/2},$$

is in a qualitative agreement with observation data about masses of small bodies in the solar system such as meteors, comets, and asteroids.

Investigation of a backward process, the fragmentation, also leads to power-law distributions of masses

$$n(m) \propto m^{-\mu}$$

with exponents μ between 1.9 and 2.7 clustering around 2.4 (Turcotte, 1986). Figure 3.11 in (Sornette, 2006) illustrates the power law approximation which also applies to the distribution of meteorite sizes orbiting Earth at two different altitudes. Several models have been proposed to rationalize the observations of power law distributions but there is no accepted theoretical description (Sornette, 2006).

2.5 Solids

2.5.1 Dielectrics

The voltage jump applied to an ideal (vacuum) capacitor at time t_0 ,

$$U(t) = \begin{cases} U = \text{const}, & t < t_0; \\ U + \Delta U(t_0) = \text{const}, & t > t_0 \end{cases}$$

generates the delta pulse of current

$$I_0(t) = C_0 \Delta U(t_0) \delta(t - t_0),$$

where C_0 is a “geometrical capacity” (Westerlund, 1991). Filling the inter-electrode space with a dielectric implies an additional current component, $I_\chi(t)$, depending on the dielectric properties and on time:

$$I(t) = I_0(t) + I_\chi(t).$$

As opposed to the first, geometrical component, the second, material component monotonically decreases because of energy losses. The simplest, Debye model takes this effect by means of introducing a resistor into its equivalent circuit (De-

bye, 1912). This leads to the popular formula

$$I_{\chi}(t) = I_{\chi}(\tau)e^{-(t-t_0)/\tau}, \quad \tau > 0,$$

confirmed by experiments with gases and many fluids and written in all textbooks. On the contrary, most of solid materials have turned out of the range of applicability of Debye's theory. Numerous experiments have shown that the observed dependence $I_{\chi}(t)$ falls more rapidly at small times and more slowly at large times as the exponential function. Analytically, it expresses by Curie-von Schweidler law (Curie, 1889a,b; von Schweidler, 1907), established even earlier than the Debye's:

$$I_{\chi}(t) = B\Delta U(t_0)(t - t_0)^{\mu-1} = A\Delta U(t_0)\Phi_{\mu}(t - t_0),$$

$$B = \text{const}, \quad A = B\Gamma(\mu), \quad 0 < \mu < 1, \quad t > t_0.$$

Discussing physical grounds of non-Debye relaxation in solids Jonscher notes among other factors that the relaxing elements are charges, dipoles, and other aggregates jumping from state to state interact with each other and with their environment. Relaxation processes change the medium and this has an influence upon the process itself. While the Debye theory describes relaxation as a set of independent parallel processes, it seems to be more likely that the relaxation in solids is a hierarchical sequence described similarly to turbulence by power type patterns.

One can meet the following realizations of spectral relaxation functions concordant with Jonscher's asymptotics:

Cole-Cole formula (Cole and Cole, 1941)

$$\tilde{\chi}(\omega) = \frac{1}{1 + (i\omega\tau)^{\alpha}}, \quad 0 < \alpha \leq 1;$$

Cole-Davidson formula (Davidson and Cole, 1951)

$$\tilde{\chi}(\omega) = \frac{1}{(1 + i\omega\tau)^{\beta}}, \quad 0 < \beta \leq 1;$$

Havriliak-Negami formula (Havriliak and Negami, 1966)

$$\tilde{\chi}(\omega) = \frac{1}{[1 + (i\omega\tau)^{\alpha}]^{\beta}}, \quad 0 < \alpha\beta \leq 1.$$

For $\alpha = \beta = 1$, all of them take the form

$$\tilde{\chi}(\omega) = \frac{1}{1 + i\omega\tau}$$

corresponding to Debye relaxation law.

The work (Cole and Cole, 1941) contains a rich list of approximation parameters for various dielectrics.

2.5.2 Semiconductors

The universal relaxation law is observed in some semiconductors as well.

Let us consider a classical time-of-flight experiment on the determination of the drift mobility of charge carriers (see Fig. 2.5). Electrons and holes are usually generated in the sample by a laser light pulse from the side of a semitransparent electrode. A voltage applied to the electrodes is such that the corresponding electric field inside the sample significantly exceeds the field of the nonequilibrium charge carriers. The electrons (or holes, in dependence on the voltage sign) pass to the semitransparent electrode, whereas the holes (or, correspondingly, electrons) drift to the opposite electrode. In the case of normal transport, carriers drifting without capture in the field E form a square photocurrent pulse

$$I(t) \propto \begin{cases} \text{const}, & t < t_T, \\ 0, & t > t_T, \end{cases} \quad (2.14)$$

where the time-of-flight t_T is expressed in terms of the drift velocity v and sample length L as $t_T = L/v$. The scattering of delocalized carriers in the drift process, capture to the localized states, and the thermal release of carriers leads to the smearing of a real packet. This packet has the Gaussian form with the mean value $\langle x(t) \rangle \propto t$ and width $\Delta x(t) \propto \sqrt{t}$. In this case, the transient current $I(t)$ remains constant until the leading edge of the Gaussian pulse reaches the other edge of the sample. The current decreases for time $\Delta x/v$. As a result, we observe a smeared right edge of the photocurrent pulse. Such a pattern is observed in most ordered materials.

However, in certain disordered semiconductors (in amorphous, porous, disordered organic, strongly doped, etc.), a specific signal of the transient current $I(t)$, which consists of two regions with the power law behavior of $I(t)$ and the intermediate region, is observed:

$$I(t) \propto \begin{cases} t^{-1+\alpha}, & t < t_T, \\ t^{-1-\alpha}, & t > t_T, \end{cases} \quad \alpha < 1. \quad (2.15)$$

The exponent α , called the dispersion parameter, is a function of the parameters of the medium and temperature. By analogy with a normal transient processes, the parameter t_T is called the time of flight, although its physical meaning is somewhat different. It was found experimentally (Scher and Montroll, 1975) that the dispersive transport is characterized by the relation

$$t_T \propto (L/U)^{1/\alpha}, \quad (2.16)$$

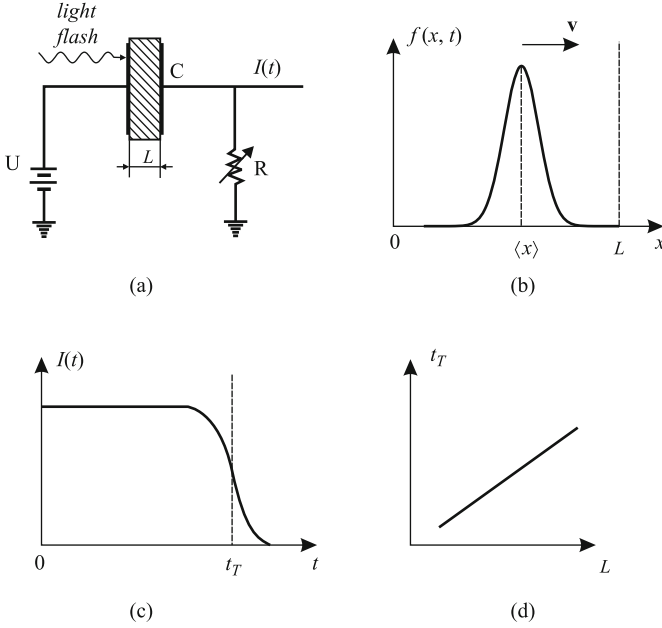


Fig. 2.5 Experimental method on the determination of drift mobility: **(a)** electrical scheme of the experiment, **(b)** charge carrier distribution at the normal transport, **(c)** transient current curve at the normal transport, **(d)** linear dependence of transient time on the sample thickness.

where U is the voltage.

The transient current signal shape in the reduced coordinates $\lg[I(t)/I(t_T)] - \lg(t/t_T)$ is almost independent of the applied voltage and sample sizes (for more details, see (Madan and Shaw, 1988)). This property is inherent in many materials and is called the property of the universality of the shape of the transient current curve. The prevalence of these features for various disordered materials confirms the universality of the transport properties.

If dependences (2.15)–(2.16) are observed, the transient current curves automatically have the asymptotic universality properties. Indeed, let us rewrite Eq. (2.15) in the form

$$I(t) \sim \begin{cases} A(L, E, \alpha, \dots) t^{-1+\alpha}, & t < t_T, \\ B(L, E, \alpha, \dots) t^{-1-\alpha}, & t > t_T, \end{cases} \quad \alpha < 1. \quad (2.17)$$

The time-of-flight t_T is determined from the intersection of the asymptotic curves:

$$I_T = A(L, E, \alpha, \dots) t_T^{-1+\alpha} = B(L, E, \alpha, \dots) t_T^{-1-\alpha},$$

so $t_T = (B/A)^{1/2\alpha}$. The asymptotic universality property means that the function $I(\tau t_T)/I_T$ for $\tau \ll 1$ $\tau \gg 1$ is independent of t_T . It is easily seen that the functions with the asymptotic behavior given by Eq. (2.17) satisfy this property:

$$I(\tau t_T)/I_T \sim \begin{cases} \tau^{-1+\alpha}, & \tau < 1, \\ \tau^{-1-\alpha}, & \tau > 1, \end{cases} \quad \alpha < 1.$$

Note that $I(t_T) \neq I_T$. The point (t_T, I_T) is determined from the intersection of the asymptotic curves of the transient current at short and long terms.

Both phenomena described above are easily interpreted in terms of trapping concept, when the motion of a carrier looks like a consequence of instantaneous jumps with independent identically distributed random waiting times. There are a few mechanisms leading to power type distributions of waiting times in traps. We discuss three of them.

2.5.3 Spinglasses

A lot of researchers show interest in spinglasses as materials in which interaction between neighboring Ising's spins can be both ferromagnetic and antiferromagnetic. Classical example of such kind is Fe-Au alloy containing only a few percent of gold. This alloy displays transition to the frozen spin state under temperature $T_f \approx 5 \div 15$ K depending on concentration. Spins of Fe are randomly distributed among non-magnetic ions Au and interact with each other due to the non-direct exchange mechanism. At large distances, this interaction is described by the interaction parameter $J(r) \propto r^{-3} \cos(2k_F r)$, where k_F is the fermionic wave vector of conducting electrons in Au.

Monte Carlo simulations of spin glasses dynamic (Binder and Stauffer, 1984) show that if initially all spins have identical direction then magnetization

$$M(t) = \langle \sigma_i(t) \rangle,$$

being 1 at the beginning falls in such a way that residual magnetization decreases very slow: it goes to zero not exponentially but rather by inverse power law:

$$M(t) \propto t^{-\alpha}, \quad \alpha \approx (1/2)kT/\Delta J.$$

The same phenomenon was observed in experiments too (Holtzberg et al., 1977).

2.5.4 Jonscher's universal relaxation law

All the above non-Debye relaxation models satisfy the criteria of Jonscher's universal response:

$$f(t) = A[(t/\tau)^{\mu+1} + (t/\tau)^{\nu}]^{-1} \sim \begin{cases} A(t/\tau)^{-\mu-1}, & t \gg \tau, \quad 0 \leq \mu \leq 1; \\ A(t/\tau)^{-\nu}, & t \ll \tau, \quad 0 \leq \nu \leq 1. \end{cases}$$

$$\tilde{\chi}(\omega) \sim \begin{cases} B(\omega\tau)^\mu, & \omega\tau \ll 1, \\ B(\omega\tau)^{\nu-1}, & \omega\tau \gg 1. \end{cases}$$

Here, the term “universality” means that relaxation types of many processes can be characterized by single points on the α, β plane. Indeed, the experimental data analysis of relaxation of polarization in polar and non-polar dielectrics, magnetics in magnetic field, hopping conductivity in semiconductors, ion conductivity, slow decrease of luminescence, surface conductivity of isolators, chemical reactions kinetics, mechanical relaxation of elastic-plastic materials generally confirms the presumption (Jonscher, 1983, 1996; Sornette, 2006).

We have formerly discussed the physical reasons of such behavior of complex systems. In the work (Jonscher et al., 2003) this picture is complemented with the concept of correlated dipole clusters and looks as follows.

At microscopic level: one has a random number of active dipoles, which follow an external field variation; Their individual relaxation rates are defined by interaction with passive neighbors, forming random size clusters around them.

At mesoscopic level: there appears clusters, containing molecules connected with each other; group relaxation rate in such mesoscopic regions (domains) correlates with the number of active dipoles in a domain and with stochastic properties of individual behavior, representing both space and time correlation scales.

At macroscopic level: averaging in the process of summation of all domains' contributions of a system leads to the universal power law behavior of relaxation of the system.

Within the Jonscher's concept, effective screening depends on the relation between density of thermally activated dipoles with energy W ,

$$n_d \propto \exp\{-W/k_B T\}$$

and the “extremal” density

$$n_c \propto k_B T / \mu^2,$$

where μ is the dipole moment of a screened molecule. Accordingly to this concept, if $n_d \ll n_c$ then the number of dipoles in every dipole environment rises almost exponentially, so that their behavior is collective and defines universal type of relaxation of the system as a whole. The contrary case, when $n_d \gg n_c$ is characterized by effective screening which makes dipoles behave independently on their neighbors, thus defining the Debye type of relaxation (see Table. 2.1).

Table 2.1 Debye and non-Debye relaxations (Sornette, 2006)

Characteristics	Debye law	Universal law
Concentrations	$n_d \gg n_c$	$n_d \ll n_c$
Screening	effective	ineffective
Interaction	weak	strong
Behavior	individual	collective

2.6 Optics

2.6.1 Luminescence decay

To estimate kinetics of spontaneous luminescence decay, one often uses the exponential law as in case of the Debye relaxation:

$$I = I_0 e^{-t/\tau}.$$

However, this law is rarely observed in reality, when radiative and nonradiative transitions, trapping electrons and holes, and such complex processes play an essential role. In this case, the luminescence decays according to the *Becquerel hyperbola*

$$I = I_0(1 + t/\tau)^{-\alpha}, \quad 0.5 < \alpha \leq 2,$$

where τ as above is a positive constant with time-dimensionality. The constant is often so small, that experimental data are approximated by the plain inverse power function

$$I(t) = At^{-\alpha} = A\Gamma(1 - \alpha)\Phi_{1-\alpha}(t)$$

in a wide region of times. Thus, Figure 2.6 (Jonscher, 1983) shows how the curve $At^{-1.34}$ approximates the experimental data obtained for the luminescence decay in GaAs (*p*-type) at the temperature 4.2 after optical excitation with the 7 ns pulse: it does this very well in between $5 \cdot 10^{-8}$ and 10^{-3} seconds (see Fig. 2.6). The luminescence in the biphenyl glass at 77 K, excited by a short pulse of fast electrons

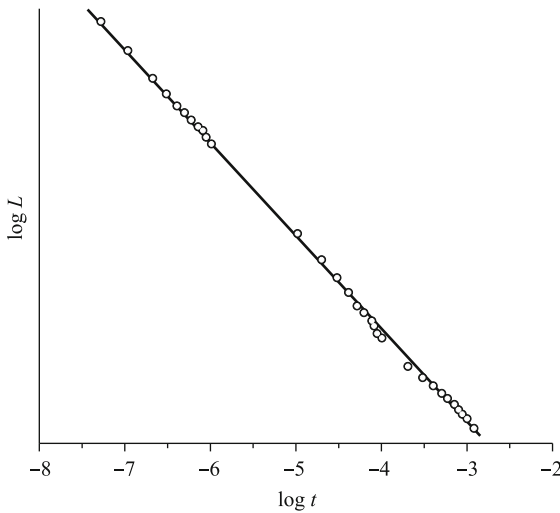


Fig. 2.6 The luminescence decay in GaAs (*p*-type) at 4.2 K after optical excitation with the 7 ns (points are experimental data, the line is the power approximation $At^{-1.34}$) (Jonscher, 1983).

and a long period of gamma radiation decays according to the same power law with the exponent $\alpha = 1.06$ in the diapason 10^{-6} - 10^{-3} s.

2.6.2 Anomalous exciton kinetics

Anomalous exciton kinetics is observed in isotopic naphthalene alloys and in naphthalene samples with complex geometrical constrains: naphthalene in channels of porous glasses, of porous polymer membranes, in voids of polymer glasses, or at grain boundaries in naphthalene films at low temperatures. A common property of the systems is unusual behavior of the exciton annihilation process: the annihilation rate k is not a constant but falls with time by the power law

$$k(t) \propto t^{-\alpha}$$

with the exponent α , $0 \leq \alpha < 1$, vanishing for homogeneous samples (Kopelman, 1986).

2.6.3 Blinking fluorescence of quantum dots

Fluorescence of single semiconducting colloidal nanocrystals, for example, quantum dots CdSe with ZnS shell displays intermittency: being under laser illumination, nanocrystals are blinking, that is on-states when a nanoparticle emits fluorescent photons alternate with off-states when it doesn't emit photons (see for details (Osad'ko, 2006)). Experimental investigations (Kuno et al., 2001; Shimizu et al., 2001) show that one can approximately consider on- and off-intervals are random, independent and distributed according to asymptotically inverse power laws:

$$P(T_{\text{on}} > t) \propto t^{-\alpha}, \quad P(T_{\text{off}} > t) \propto t^{-\beta}.$$

The exponents $\alpha, \beta \in (0, 1)$ practically do not vary under change of experiment conditions: temperature, intensity of laser radiation, and quantum dots size. This is why determination of the intermittency mechanism is a complicated problem. Physical origin of the phenomenon is not absolutely clear yet, although there exist a few plausible hypotheses. Efros and Rosen (1997) explain the fluorescence blinking as a result of Auger-ionizations of quantum dots with consecutive neutralization. However, this assumption leads to exponentially distributed on- and off-interval duration. As possible mechanisms, leading to power type distributions, different authors offer thermally activated ionization (Kuno et al., 2001), tunneling through fluctuating walls (Kuno et al., 2001), resonant tunneling between nucleus and charged localized states (Shimizu et al., 2001).

2.7 Geophysics

2.7.1 Atmosphere and ocean turbulence

Geophysical turbulent flows are characterized by rather large Reynolds numbers. Therefore, it has been a common expectation that universal relations such as energy spectrum $E(k) \propto k^{-5/3}$ and others should be valid in such flows. Richardson (1926) initiated the modern approach to the subject of turbulent diffusion (Taylor, 1959), stressing the importance of relative diffusion rather than single-particle diffusion. Richardson found out that the experimentally determined coefficient of eddy diffusivity K varies with scale l proportionally to $l^{4/3}$ ranging from 0.05 to 10^8 cm. Taylor wrote that the equation

$$K = 0.2 l^{4/3}$$

is a very good approximation to the experimental data between $l = 10^2$ and $l = 10^6$ cm. The Richardson law was claimed to be confirmed in a large number of experiments (Monin and Yaglom, 1973, 1975). Nevertheless, there exist numerous examples of turbulent flows in the atmosphere, ocean, and laboratory do not follow the Richardson law. Moreover, Henschel and Procaccia (1983) show that even the data of Richardson's original paper are best fitted by a relation

$$K \propto l^{4/3+2\mu/3}$$

with the non-zero intermittency exponent $\mu = 0.36$. One of explanations of the deviation from the Richardson law is the phenomenon of spontaneous breaking of statistical isotropy (rotational and/or reflexional) symmetry — locally and globally (Tsinober, 1994). Bershadskii et al. (1993) argued that regions with large fluctuations of turbulent energy are characterized by strong anisotropy and a local cascade of angular momentums, i.e., of a quantity of the type of Loytsianskii's invariant

$$\Lambda = \int_{\Omega_r} \langle \mathbf{u}(\mathbf{x}) \mathbf{u}(\mathbf{x} + \mathbf{r}) \rangle r^2 d\mathbf{r}^{D_\infty},$$

where D_∞ characterizes the subregions Ω_r with large fluctuations of turbulent energy

$$\int_{\Omega_r} \langle \mathbf{u}^2(\mathbf{x}) \rangle d\mathbf{x} \propto r^{D_\infty}, r \rightarrow \infty.$$

From the dimensional arguments

$$\int_{\Omega_r} \langle \mathbf{u}^2(\mathbf{x}) \rangle d\mathbf{x} \propto L^{2/3} r^{13/15}, \quad L = \left| \frac{d(\Lambda/V)}{dt} \right|,$$

and $D_\infty = 13/5$. These arguments are supported by laboratory and numerical data on asymptotic values of intermittency exponent μ which give a value of $D_\infty = 2.6 \pm 0.05$.

The cited article (Tsinober, 1994) contains experimental results obtained from observations of diffusion phenomena in the troposphere and in the ocean. Numerous observations of cloud evolution with time show that horizontal standard deviation of diffusing tracer as a function of downwind travel time in the troposphere can approximately be fitted by the relation $l \propto t^{7/6}$ that corresponds to the diffusivity l -dependence $K \propto l^{8/7}$. Evidently, it differs from the Richardson 4/3-law. This is why, Tsinober considers, that Kolmogorov (homogeneous) turbulence is unstable in respect to local states: *fractons* (see (Alexander, 1986)). They appear to be sub-regions with high helicity and arise spontaneously in turbulent flow. The governing dimensional parameter for helical fractons is so called *renormalized dissipation rate* $\bar{\epsilon}$ (Bershadskii et al., 1993), which has the dimensionality

$$[\bar{\epsilon}] = [l]^2 [t]^{-1-D_f}$$

with fracton dimension $D_f = 4/3$. The diffusivity K in fractons follows the relation

$$K \propto \bar{\epsilon}^{3/7} l^{8/7},$$

consistent with troposphere observation data (Gifford, 1983). The same approximation fits also the ocean observation data (Okubo, 1971).

2.7.2 Groundwater

The *hyporheic zone* is the region of near-stream aquifers containing exchange flows of stream water that originates from, and is returned to the stream channel. The hyporheic zone may extend for tens of meters horizontally from the channel and a meter or more beneath it and is an important location for biologically-mediated reactions that modify the chemistry of stream water (see (Haggerty et al., 2002) and references therein). Exchange flows bring stream water into close contact with biofilms on sediment surfaces where reactions tend to occur. However, the extent to which transformations occur is dependent, in part, on the residence time of stream water in the subsurface. Further, because the hyporheic zone may have a large volume with long residence times, it can retain solutes present in stream water and later release them back to the stream channel, thus delaying or attenuating chemical signals produced by precipitation and watershed processes. Differences in residence time distributions, then, will determine the relative importance of biogeochemical transformations and the transport of contaminants and other solutes in streams and rivers.

Haggerty et al. (2002) inform about measurements of the hyporheic residence time distribution in a 2nd-order mountain stream. The measurements show that the distribution follows a power-law $t^{-1.28}$ over at least 1.5 orders of magnitude in time (1.5 hr to 3.5 d) and is poorly characterized by an exponential model. Because of the small power-law exponent, efforts to characterize the mean hyporheic residence

time (t_s) in this system result in estimates that are scale invariant, increasing with the characteristic advection time within the stream channel (t_{ad}). The distribution implies the hyporheic zone has a very large range of exchange timescales, with significant quantities of water and solutes stored over timescales very much longer than t_{ad} . The hyporheic zone in such streams may contribute to short-time fractal scaling in time series of solute concentrations observed in small-watershed studies.

2.7.3 Earthquakes

The spatiotemporal complexity of earthquakes is often invoked as an illustration of the phenomenon of critical self-organization with scale-invariant properties (Helmstetter and Sornette, 2002). This concept points to the importance of developing a system approach in which large scale properties can emerge from the repeating interactions occurring at smaller scales. Such approach reflects the most solidly established facts in the phenomenology of earthquakes, in particular, the three most important laws.

Law 1 (the Gutenberg-Richter law). *The cumulative distribution of earthquake magnitudes m sampled over broad regions and large time intervals is proportional to 10^{-bm} , with $b \approx 1$.* In terms of energy, this leads to probability distribution $P(\mathcal{E} > E) \propto E^{-B}$, $B \approx 2/3$.

Law 2 (the Omori law). *The rate of earthquakes triggered by a mainshock decays with time according to an inverse power t^{-p} of time with an exponent $p \geq 1$.*

Law 3. *The earthquakes are clustered in space along hierarchical fault structures and their spatial distribution over long times can be approximately described by a fractal dimension close to 2.2 (in three dimensions).* In other words, the mean number of aftershocks in the sphere of radius R is proportional to R^q , $q \approx 2.2$.

These phenomenological laws were assumed as basis of the ETAS (epidemic-time aftershocks) model introduced by (Kagan and Knopoff, 1987).

2.7.4 Tsunami

Many natural systems have cumulative number-size distributions that follow a power law or an upper-truncated power law. A well-known example is the Gutenberg-Richter scaling relationship for earthquakes. Tsunamis are a natural hazard for many coastal regions and are often triggered by earthquakes. Tsunami size is measured as run-up height, which may be measured either on a tide gauge or by the vertical height that the wave reaches on land.

Like earthquakes and other natural hazards, the probability of tsunami occurrence at a particular location depends on both the size distribution of events and the distribution of inter-event or recurrence times. Given sufficient observation data,

the probability can be determined empirically. As noted in (Burroughs and Tebbens, 2005), the size distribution tends to follow a power law similar to the Gutenberg-Richter law (see Kagan, 2002). The sequence of the event time was usually assumed to be that of a stationary Poisson process. This means in particular that inter-event times are distributed exponentially. Geist and Parsons (2008) and other researchers examined a few catalogues of tsunami source times and came to the conclusion that observed realizations deviate from Poissonian process: both short and long inter-event times occur more frequently than in Poissonian process, which is more typical for inverse power inter-event time distributions. This analogy to the Omory law indicates to direct link between tsunami and earthquake.

2.7.5 Fractal approach

Geophysical objects and processes are usually characterized by highly irregular forms and behavior. Clouds in the sky and mountain landscapes have shapes which are very far from classical geometrical figures. The continuous motion of atmospheric and oceanic masses are often interrupted by turbulence zones and hurricanes. The earth's topography is created by various tectonic processes like faulting, folding, and flexure. It is modified and destroyed by erosion and sedimentation and often appears to be complex and chaotic. The motion of plates of Earth's crust is attended by seismic impacts, earthquakes, and volcanic explosions. All these facts stimulate application of the fractal concept to describing and solving geophysical problems.

Let us see how Tsinober (1994) uses the fractal approach to illustrate the difference between three-dimensional (Kolmogorov) and quasi-two-dimensional (stratosphere) turbulence. Considering the expansion of a cloud with an effective radius $R(t)$, he approximates the perimeter of its central cross-section by a broken line consisting of sections of length r and suppose that the number N of these sections

$$N \propto (R/r)^{d_p}.$$

The positive constant d_p is the fractal dimension of the perimeter, which is connected with the fractal dimension of the cloud surface d_σ by the relation

$$d_\sigma = d_p + 1.$$

The effective rate of increase of the cross-section area is estimated as follows:

$$\frac{dS}{dt} = \frac{r(\delta u_r dt)N}{dt} \propto r^{1-d_p} \delta u_r R^{d_p},$$

where δu_r is the velocity of the section normal to it (Townsend, 1966). Using the well-known relations

$$\delta u_r \propto r^\alpha, \quad \alpha(\gamma - 1)/2, \quad E(k) \propto k^{-\gamma}$$

(Monin and Yaglom, 1971, 1975), Tsinober find that

$$\frac{dS}{dt} \propto r^{1-d_p+\alpha}.$$

Since the rate dS/dt is independent of r ,

$$d_p = 1 + \alpha = (1 + \gamma)/2.$$

Two important cases should be noted here: Kolmogorov three-dimensional turbulence,

$$\gamma = 5/3, \quad d_p = 4/3,$$

and quasi-two-dimensional turbulence, when

$$\gamma = 7/3, \quad d_p = 5/3.$$

As shown in the cited article, these numbers are in good agreement with measurements of area versus perimeter of rain and cloud areas, determined from radar and satellite data.

2.8 Astrophysics and cosmology

2.8.1 *Solar wind*

Solar wind (see for review (Goldstein and Roberts, 1995)) flows nearly radially away from the Sun, at up to about 700 km/s. This is much faster than both spacecraft motions and the Alfvén speed. Therefore, the turbulence is “frozen” and the fluctuations at frequency f are directly related to fluctuations at the scale k in the direction of the wind, as $k = 2\pi f/v$, where v is the solar wind velocity (Horbury, 1999).

Usually two types of solar wind are distinguished, one is the fast wind which originates in coronal holes, and the slower is bursty wind. Both of them show, however, $f^{-5/3}$ scaling on small scales. The turbulence is strongly anisotropic with the ratio of power in motions perpendicular to the magnetic field to those parallel to the magnetic field being around 30. The intermittency of the solar wind turbulence is very similar to the intermittency observed in hydrodynamic flows (Horbury and Balogh, 1997).

2.8.2 *Interstellar magnetic fields*

Interstellar medium (ISM) is filled with magnetic field, plasma, gas, dust, and cosmic rays. Magnetic field statistics are the most poorly constrained aspect of ISM

turbulence. The polarization of starlight and of the Far-Infrared Radiation (FIR) from aligned dust grains is affected by the ambient magnetic fields. Assuming that dust grains are always aligned with their longer axes perpendicular to magnetic field, one gets the 2D distribution of the magnetic field directions in the sky. Note that the alignment is a highly non-linear process in terms of the magnetic field and therefore the magnetic field strength is not available.

The statistics of starlight polarization is rather rich for the Galactic plane and it allows to establish the spectrum $E(K) \sim K^{-1.5}$, where K is a two dimensional wave vector describing the fluctuations over sky patch.

For uniformly sampled turbulence $E(K) \sim K^\alpha$ for $K < K_0$ and K^{-1} for $K > K_0$, where K_0^{-1} is the critical angular size of fluctuations which is proportional to the ratio of the injection energy scale to the size of the turbulent system along the line of sight. For Kolmogorov turbulence $\alpha = -11/3$ (see for more details (Cho et al., 2002)).

2.8.3 Scintillation statistics

Studies of turbulence statistics of ionized media (see (Spangler and Gwinn, 1990)) have provided information on the statistics of plasma density at scales 108-1015 cm. This was based on a clear understanding of processes of scintillations and scattering achieved by theorists. A peculiar feature of the measured spectrum is the absence of the slope change at the scale at which the viscosity by neutrals becomes important. Scintillation measurements are the most reliable data in the big power law (Armstrong et al., 1995). However there are intrinsic limitations to the scintillations technique due to the limited number of sampling directions, its relevance only to ionized gas at extremely small scales, and the impossibility of getting velocity (the most important!) statistics directly. Therefore with the data one faces the problem of distinguishing actual turbulence from static density structures. Moreover, the scintillation data does not provide the index of turbulence directly, but only shows that the data are consistent with the Kolmogorov turbulence.

2.8.4 Velocity and density statistics from spectral lines

Spectral line data cubes are unique sources of information on interstellar turbulence. Doppler shifts due to supersonic motions contain information on the turbulent velocity field which is otherwise difficult to obtain. Moreover, the statistical samples are extremely rich and not limited to discrete directions. In addition, line emission allows us to study turbulence at large scales, comparable to the scales of star formation and energy injection. In particular, the power law asymptote of the emissivity fluctuations changes when the dispersion of the velocity at the scale under study becomes of the order of the velocity slice thickness (the integrated width of the channel map).

These results are the foundation of the Velocity-Channel Analysis (VCA) technique which provides velocity and density statistics using spectral line data cubes. Within the VCA some observation results can be interpreted as implying that on scales less than 1 pc the Galactic and Small Magellanic Cloud atomic hydrogen velocity is suppressed by ambipolar drag and the spectrum of density fluctuations is shallow $P(k) \sim k^{-2.8}$ (Cho et al., 2002).

2.8.5 Large-scale structure

The correlation analysis of the galaxy distribution is a widely approach to describe statistical properties of spatial distribution of visible matter in the Universe. During the 1970th, the availability of extended angular galaxy samples made possible the realization of accurate correlation analysis, mainly pursued by Peebles and his collaborators (Peebles, 1980). The analysis has shown that the angular 2-point correlation function is well represented by the power law

$$w_g(\vartheta) = A_g \vartheta^{1-\gamma},$$

with $\gamma = 1.77 \pm 0.04$ and amplitude $A_g \propto D_s^{-1}$ decreasing with the depth D_s of the sample, with a break from the power law behavior at large angular separations.

On providing a method for deprojecting angular data, the angular function $w_g(\vartheta)$ was converted into the spatial 2-point function $\xi_g(r)$. The latter turned out to be

$$\xi_g(r) = \left(\frac{r_{0,g}}{r} \right)^\gamma,$$

with the same exponent γ and the clustering length $r_{0,g}$, depending on the amplitude A_g of the angular function and on the depth D_s :

$$r_{0,g} \propto D_s A_g^{1/\gamma}.$$

The power law of spatial correlations reflects the cluster character of observed matter distribution. Each galaxy is nothing but a huge cluster of stars, therefore, a cluster of galaxies is the cluster of clusters of stars. This way of thinking may be continued, and one may suppose that galaxy clusters form in their turns clusters, called *superclusters*. Astronomical investigations have not only confirmed this supposition but discovered that the reach cluster correlation function can also be fitted by the power law expression

$$\xi_c(r) = \left(\frac{r_{0,c}}{r} \right)^\gamma,$$

with $r_{0,c} \simeq 25h^{-1}$ Mpc and $\gamma \simeq 1.8$, in the distant range $5 < r < 150h^{-1}$ Mpc (here h is the Hubble constant). This function exhibits the same slope as the galaxy function, but with a remarkably larger correlation length (see for details (Borgani, 1995)).

2.8.6 Stochastic selfsimilarity

The work (Iovane et al., 2004) develops Sakharov's (1965) idea about expanding quantum primordial fluctuations towards the present epoch leading to the observed galaxies, clusters, and superclusters of galaxies. The authors have used the Brownian motion relation that was firstly offered by Eddington, according to which structures formation of the Universe appears as if it was a classically selfsimilar random process at all astrophysical scales. Following this line they arrived at a selfsimilar universe and demonstrated an agreement for the present hypotheses of segregation with a size of astrophysical structures by using a comparison between quantum quantities and astrophysical ones. It appears that the Universe has a memory of its quantum origin as suggested by R. Penrose with respect to quasi-crystal. A more accurate analysis shows that the present theory can be extended from the astrophysical to the nuclear scale by using generalized (stochastically) selfsimilar random process. In particular, (Iovane et al., 2004) analyzed the scale invariant law

$$R(N) = (h/Mc)N^\alpha, \quad (2.18)$$

where R is the radius of the astrophysical structures, h is the Planck constant, $M = Nm_n$ is the total mass of the self-gravitating system, c is the speed of light, N is the number of nucleons into the structures. The choice $\alpha = 3/2$ leads to the Random Walk interrelation

$$R(N) = l\sqrt{N}, \quad l = h/m_nc,$$

firstly used by Eddington and Weinberg. Basing on observation data the authors managed to find out that for globular clusters, galaxies, galaxy clusters, and superclusters $\alpha \in (1.50, 1.55)$, for solar system objects (including Moon and Sun itself) $\alpha \in (1.41, 1.43)$. Moreover, for the periodic table of elements they obtained $\alpha \in (1.35, 1.42)$. The conclusion made at the end of this article says that the fractal power law suggest a fractal Universe, and that "the nature uses the language of a relativistic, quantum, and fractal geometry". Analysis of power type laws at molecular, atomic, and nuclear scales is performed in (Paar et al., 2002).

2.9 Some statistical mechanisms

2.9.1 Three simple examples

Let us discuss a few statistical mechanisms often involved to explanations of power law origin. We shall denote random variables by capital letters X, Y and so on and write

$$X \stackrel{d}{=} Y$$

if the variables X and Y are defined at the same probability space and have the same probability distribution:

$$\mathbf{P}(X \in dx) = \mathbf{P}(Y \in dx).$$

As usual, $F(x)$ will stand for cumulative distribution function

$$F_X(x) = \mathbf{P}(X < x),$$

and $p_X(x)$ for probability density function (pdf):

$$F_X(x) = \int_{-\infty}^x p_X(x') dx'.$$

The lower subscript can be omitted if it does not lead to misunderstanding.

Neither probability density functions $p(x)$ nor cumulative distribution function $F(x)$ can be of pure power type because this would violate the normalizing condition

$$\int_{-\infty}^{\infty} p(x) dx = 1, \quad F(-\infty) = 0, \quad F(\infty) = 1.$$

This is why talking about power type distributions we bear in mind their tails called heavy tails or long tails. Consider three simple examples of such distributions.

Let X be a random variable with the standard normal (Gaussian) distribution:

$$p_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}. \quad (2.19)$$

The inverse random variable $Y_1 = 1/X$ has the cumulative distribution function

$$F_{Y_1}(y) \equiv \mathbf{P}(Y_1 < y) = \frac{1}{\sqrt{2\pi}} \int_{1/y}^{\infty} e^{-x^2/2} dx,$$

and probability density function

$$p_{Y_1}(y) = F'_{Y_1}(y) = \frac{1}{\sqrt{2\pi}y^2} e^{-1/2y^2} \sim \frac{1}{\sqrt{2\pi}y^2}, \quad y \rightarrow \infty.$$

Similarly to Gaussian distribution, the distribution is unimodal and symmetric with respect to the origin but its moments do not exist.

The random variable $Y_2 = 1/X^2$ is positive, its pdf:

$$p_{Y_2}(y) = \frac{d\mathbf{P}(Y^2 < y)}{dy} = \frac{d}{dy} \sqrt{\frac{2}{\pi}} \int_{1/\sqrt{y}}^{\infty} e^{-x^2/2} dx = \frac{1}{\sqrt{2\pi}y^3} e^{-1/2y},$$

called the *Lévy-Smirnov density* and has a heavy tail $p_{Y_2}(y) \propto y^{-3/2}$, $y \rightarrow \infty$.

Let X_1 and X_2 are independent identically distributed with the density (2.19) random variables and $Y_3 = X_1/X_2$. The distribution of this random variable

$$p_{Y_3}(y) = \frac{1}{\pi(1+y^2)}$$

is called the *Cauchy distribution*.

2.9.2 Activation mechanism

Consider an ensemble of particles being in localized states with activation energy ε , which will be called *traps*. According to the standard approach, the waiting time T , that is duration of a particle dwell in the trap, is a random variable Θ with exponential probability distribution

$$\mathbf{P}(\Theta > t|\varepsilon) = e^{-t/\tau(\varepsilon)}.$$

The mean value of the time is determined by the Arrhenius equation

$$\tau(\varepsilon) = \tau_0 \exp(\varepsilon/kT).$$

When a medium is inhomogeneous different traps can have different activated energies, thus considering a randomly chosen trap we deal with the random variable ε . Taking for its distribution the exponential law,

$$p(\varepsilon) = (1/\varepsilon_0) \exp(-\varepsilon/\varepsilon_0),$$

we get

$$\begin{aligned} \mathbf{P}(\Theta > t) &= \int_0^\infty \mathbf{P}(\Theta > t|\varepsilon) p(\varepsilon) d\varepsilon \\ &= \int_0^\infty \exp(-(t/\tau_0) e^{-\varepsilon/kT}) (1/\varepsilon_0) e^{-\varepsilon/\varepsilon_0} d\varepsilon = \alpha \left(\frac{t}{\tau_0} \right)^{-\alpha} \int_0^{t/\tau_0} e^{-z} z^{\alpha-1} dz, \end{aligned}$$

where $\alpha = kT/\varepsilon_0$. This probability goes to 1 as $t \rightarrow 0$, and has a power tail

$$\mathbf{P}(\Theta > t) \sim \Gamma(1+\alpha) \left(\frac{t}{\tau_0} \right)^{-\alpha},$$

in the long-time asymptotics ($t \rightarrow \infty$). For probability distribution function we obtain:

$$p_\Theta(t) \sim \alpha \Gamma(1+\alpha) \tau_0^\alpha t^{-1-\alpha}, \quad t \rightarrow \infty.$$

2.9.3 Tunneling

A similar but somewhat different situation takes place in the problem discussed in (Tunaley, 1972). Carriers considered there execute a one-dimensional motion along the x -axis coincident with applied field direction. The axis contains randomly distributed identical traps – point potential wells. Transitions between neighboring wells occur through tunneling. If the traps form Poissonian ensemble, all random distances X between neighboring traps are independent and distributed exponentially:

$$\mathbf{P}(X > x) = e^{-\mu x}.$$

Neglecting all other sources of waiting time fluctuations except fluctuations in X , that is using the deterministic interrelation between Θ and X (Harper, 1967),

$$\Theta = \beta(e^{X/x_0} - 1)$$

with β being inversely the potential gradient, one arrives at a power tail distribution:

$$\begin{aligned} \mathbf{P}(\Theta > t) &= \mathbf{P}(\beta(e^{X/x_0} - 1) > t) = \mathbf{P}(X > x_0 \ln(1 + t/\beta)) \\ &= \frac{1}{(1 + t/\beta)^{\mu x_0}} \sim (t/\beta)^{-\mu x_0}, \quad t \rightarrow \infty. \end{aligned}$$

2.9.4 Multiple trapping

Third model of origin of power time distribution is introduced in (Blumen et al., 1986). Let t be dimensionless time and

$$\psi_n(t) = \mu^n \exp(-\mu^n t), \quad t \geq 0, \quad \mu > 0, \quad n = 1, 2, 3, \dots \quad (2.20)$$

be the probability distribution function of random interval between two consecutive events of Poisson flow with the rate μ^n . Replace the natural n by a random variable N , distributed according to geometrical law:

$$\mathbf{P}(N = n) = (1 - q)q^{n-1}, \quad 0 < q < 1, \quad n = 1, 2, 3, \dots \quad (2.21)$$

Averaging distribution (2.20) over random number N ,

$$\psi(t) = \langle \psi_N(t) \rangle = \frac{1 - q}{q} \sum_{n=1}^{\infty} (\mu q)^n \exp(-\mu^n t),$$

and replacing t by μt , we bring the expression to the form

$$\psi(\mu t) = \frac{1}{\mu q} \psi(t) - \frac{1 - q}{q} e^{-\mu t}.$$

It is not difficult to make sure that

$$\psi(\mu t) \sim \frac{1}{\mu q} \psi(t), \quad t \rightarrow \infty \sim \frac{1}{\mu q} t^{-\gamma-1}, \quad t \rightarrow \infty,$$

where $\gamma = \ln q / \ln \mu$.

Physically, this is a simplified model of as noted above, the rates of transitions from low levels to carrier bands in activation processes depend exponentially on the energy. Equidistance of the energy levels, $E_n = n\Delta E$ yields the transition rates being proportional to

$$\exp(-E_n/kT) = [\exp(-\Delta E/kT)]^n.$$

The state density at a tail of band often falls with energy like

$$\exp(-E_n/kT_0) = [\exp(-\Delta E/kT_0)]^n,$$

where T_0 is some constant. Thus,

$$\mu = \exp(-\Delta E/kT),$$

$$q = \exp(-\Delta E/kT_0),$$

and we arrive at the power type distribution with the exponent $\gamma = T/T_0$.

2.9.5 Averaging over a parameter

Specific features of resonance radiation transfer in plasma result from close link between absorption and the emission atomic spectra. On the assumption of local thermodynamic equilibrium and completely redistribution of frequency, the absorption coefficient $k(\omega)$ is proportional to the emission coefficient $\varepsilon(\omega)$,

$$k(\omega) = C\varepsilon(\omega), \quad C > 0.$$

Consequently, photons with a fixed frequency ω have free path R distribution of exponentially type,

$$\mathbf{P}(R > x|\omega) = e^{-k(\omega)x}, \quad x > 0,$$

in case of a random frequency has the average (unconditional) free path distribution

$$\mathbf{P}(R > x) = \langle \mathbf{P}(R > x|\omega) \rangle = \int_0^\infty \varepsilon(\omega) e^{-k(\omega)x} d\omega.$$

The following processes have an influence on shaping of the spectral line profile.

First of all, this is electrodynamics itself (Jackson, 1975) giving the Lorentz spectrum:

$$\varepsilon(\omega) = \varepsilon_0(\omega) \equiv \frac{\gamma}{2\pi[(\omega - \omega_0)^2 + (\gamma/2)^2]}, \quad \gamma > 0, \quad \gamma/2 \ll \omega_0.$$

Passing to the new integration variable $s = 2(\omega - \omega_0)/\gamma$, we find

$$P(R > x) = \int_0^\infty \varepsilon_0(\omega) e^{-k(\omega)x} d\omega \approx \frac{1}{2\pi} e^{-k_0 x/2} \int_{-\pi}^{\pi} e^{-\frac{1}{2} k_0 x \cos \varphi} d\varphi = e^{-k_0 x/2} I_0(k_0 R/2).$$

At large x , this function has the asymptotics

$$P(R > x) \sim (\pi k_0)^{-1/2} x^{-\alpha}, \quad R \rightarrow \infty$$

with $\alpha = 1/2$.

The foregoing result relates to a system of fixed noninteracting neutral atoms. In reality, each atom is under an action of a surrounded atom system. Because of random positions of surrounded atoms relatively to the radiating atom we should carry the additional averaging over the entire ensemble of the systems. As a result, we obtain again a free path distribution with power type asymptotics.

Thermal motion of atoms leads to shift of the frequency (Doppler effect). This has an influence upon the central region only, and the tail of the distribution remains of the power type as before (see for details (Molisch and Oehry, 1998) and Ref. therein).

2.9.6 Fermi acceleration

After cosmic rays were discovered and the inverse power type of their energy spectrum was established the question had arisen: how such spectrum emerged, or in a more general statement of the question: how the cosmic rays originates. Perhaps this is the most important problem in cosmic ray physics. Although various possible mechanisms have been offered and tested this problem can not be considered as completely solved yet. We recall here the first mechanism called *Fermi-mechanism* after its author—the great Italian physicist Enrico Fermi (Fermi, 1949).

Using simple mathematical tools Fermi displayed that interaction of a charged particle with chaotically moving magnetic clouds in interstellar space brings to its acceleration. Neglecting statistical fluctuations, one can obtain that the energy increment dw is proportional to energy itself w and consequently the function $w(t)$ is exponential, namely

$$w(t) = mc^2 e^{b^2 t / \tau},$$

where m is the particle mass, c is speed of light, $b = u/c$, u is the characteristic speed of a magnetic cloud, and τ is the mean time between consecutive collisions of the particle with magnetic clouds. The variable t means here the time of acceleration. Obviously, it is a random quantity. Denoting its random value by T and assuming following Fermi that its distribution is exponential,

$$P(T > t) = e^{-t/\theta},$$

we arrive at his well-known formula:

$$\begin{aligned}
P(w(T) > E) &= P(mc^2 e^{b^2 T/\tau} > E) = P\left(T > \frac{\tau}{b^2} \ln\left(\frac{E}{mc^2}\right)\right) \\
&= \exp\left(-\frac{\tau}{b^2 \theta} \ln\left(\frac{E}{mc^2}\right)\right) = \left(\frac{E}{mc^2}\right)^{-\gamma}, \quad \gamma = \frac{\tau}{b^2 \theta}.
\end{aligned}$$

Obviously, $P(w(T) > E) = 1$ for $E < mc^2$.

This elegant result did not become a solution of the problem mainly because of one reason: the life times θ for different cosmic particles (protons, alpha-particles, other nuclei) are very different, meanwhile the observed exponents γ are close to each other.

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Chapter 3

Stochasticity

There exist two kinds of motion in classical physics: the deterministic motion with smooth trajectories obeying Newtonian law and the stochastic one with highly irregular broken trajectories described in terms of random processes theory. An ocular demonstration of the latter processes gives us Brownian motion. This chapter is devoted to stochastic processes. For simplicity, we will mainly focus on the one-dimensional case, but most of statements have higher-dimensional analogues.

3.1 Brownian motion

3.1.1 Two kinds of motion

The standard Hamiltonian dynamic system is characterized by absence of memory. This means that if we know the state $x(t_0)$ of the system at the present instant t_0 , we are able to predict its motion in future ($t > t_0$) without knowing its past ($t < t_0$). When repeating the observation of the system with the same initial condition $x(t_0) = x_0$, we obtain the same trajectory of motion $x(t)$, $t > t_0$.

Stochastic systems differ from dynamic ones by randomness: repeating the experiment with the same initial condition x_0 generates all the ensemble trajectories and we can talk about the coordinate at instant t as a random variable or random vector $X(t)$ with conditional probability distribution $\mathbf{P}(X(t) \in dx | X(t_0) = x_0)$. This is a stochastic analogue of the dynamic *causality principle*, leading to the concept of *Markov processes*. In this case, we must know the probability density function (pdf $p_0(x, t_0)$) at an initial moment t_0 and the transition pdf $p(x', t_0 \rightarrow x, t)$ in order to predict the pdf $p(x, t)$ at observation moment t :

$$p(x, t) = \int_{-\infty}^{\infty} p_0(x', t_0) p(x', t_0 \rightarrow x, t) dx'.$$

The Markov process homogeneous with respect to space and time is called the *Lévy process* (*L-process*) (Lévy, 1965; Bertoin, 1966), its transition pdf depends only on differences of arguments:

$$p(x', t' \rightarrow x, t) = p(0, 0 \rightarrow x - x', t - t') \equiv p(x - x', t - t').$$

At first glance, there are no common features between the Newtonian (deterministic) kind of motion when a particle, for example, a planet moves along a smooth line in space and the Brownian (random) kind of motion when the trajectory of a particle is a highly irregular everywhere broken random line. However, such a common feature exists at least at small scales: both motions are *selfsimilar*.

3.1.2 Dynamic selfsimilarity

The *selfsimilarity* (*scaling*) is a special kind of symmetry such that a change in scale of some variables (i.e., spatial co-ordinates) can be compensated by rescaling others. The order of the selfsimilarity is determined by a constant H called the *Hurst exponent*.

A deterministic process $x(t)$, $t \geq 0$ is said to be the *selfsimilar process of order H* (in short *H-ss-process*) *with respect to point x_0 and time t_0* if there exists such a positive H that

$$x(at) - x(at_0) = a^H [x(t) - x(t_0)]$$

for any $a > 0$ and $t > 0$. When a body falls from height $x_0 = x(t_0)$ in homogeneous gravitational field g , then

$$x(t) - x_0 = -g(t - t_0)^2$$

and

$$x(at) - x(at_0) = -g(at - at_0)^2 = a^2 [x(t) - x(t_0)].$$

Therefore, the motion is a 2-ss process with respect to the initial coordinate x_0 and time t_0 when the body has zero-velocity. Taking another point on the trajectory as an initial one, say $t_1 > t_0$, when the velocity is $-v_1$ and

$$x(t) - x(t_1) = -v_1(t - t_1) - g(t - t_1)^2,$$

we arrive at the relation

$$x(at) - x(at_1) = -av_1(t - t_1) - a^2g(t - t_1)^2.$$

Thus the point $(t_1, x(t_1))$ is not a center of selfsimilarity, because there is no such H satisfying the condition:

$$-ag(t - t_1) - a^2g(t - t_1)^2 = a^H [-g(t - t_1) - g(t - t_1)^2].$$

However, if we consider the motion of a free body, then

$$x(t) - x(t_0) = v_0(t - t_0),$$

and the condition of selfsimilarity holds for any point of the trajectory and any time with $H = 1$.

It is convenient to choose $t_0 = 0$, $x_0 = 0$ and write the selfsimilarity property in the form:

$$x(at) = a^H x(t).$$

The solution of the equation is

$$x_H(t) = x_H(1)t^H.$$

When H is an integer number, the following equation takes place

$$\frac{d^H x_H}{dt^H} = H! x_H(1).$$

Extending this on non-integer values of H we arrive at the simplest fractional differential equation

$${}_0D_t^H x_H(t) = \Gamma(H + 1)x_H(1).$$

3.1.3 Stochastic selfsimilarity

The next representation is inspired from monographs (Bertoin, 1996; Samorodnitsky and Taqqu, 1994). A stochastic (random) process $\{X(t)\}$ is called *selfsimilar with index $H > 0$ in the strict sense* if for any $a > 0$ and any $n \geq 1$, $t_1, t_2, \dots, t_n$ the joint distribution of the random variables $X(at_1), \dots, X(at_n)$ is identical with the joint distribution of $a^H X(t_1), \dots, a^H X(t_n)$:

$$(X(at_1), \dots, X(at_n)) \stackrel{d}{=} (a^H X(t_1), \dots, a^H X(t_n)).$$

In many cases, it is more suitable to use the following modification of this criterion: for any positive $t_1, \dots, t_n, a$ the relation

$$\sum_{j=1}^n \theta_j X(at_j) \stackrel{d}{=} \sum_{j=1}^n \theta_j a^H X(t_j)$$

should be fulfilled. This relation will be expressed succinctly as follows:

$$\{X(at)\} \stackrel{d}{=} \{a^H X(t)\}.$$

We will write

$$X(at) \stackrel{d}{=} a^H X(t)$$

to mean that the random processes $X(at)$ and $a^H X(t)$ have identical one-dimensional distributions. This property will be termed the *selfsimilarity in the wide sense*.

For an L-process to be selfsimilar, it is enough that

$$X(at) \stackrel{d}{=} a^H X(t)$$

or equivalently

$$X(t) \stackrel{d}{=} t^H X(1).$$

In terms of pdf $p(x, t)$, this property is characterized by the relation

$$p(x, t) = t^{-H} p(xt^{-H}, 1)$$

following from the chain of evident equalities

$$\int_{-\infty}^x p(x', t) dx' = \mathbf{P}(X(t) < x) = \mathbf{P}(t^H X(1) < x) = \mathbf{P}(X(1) < xt^{-H}) = \int_{-\infty}^{xt^{-H}} p(x', 1) dx'$$

after differentiating it with respect to x .

Formally, the Newtonian motion of a free particle with some constant velocity v can be described as well in terms of δ -pdf $p(x, t) = \delta(x - vt)$ and we have

$$p(x, t) = \delta(x - vt) = \delta((x/t - v)t) = t^{-1} \delta(x/t - v) = t^{-H} p(xt^{-H}, 1), \quad H = 1$$

as before.

We will call the process $X(t)$ *selfsimilar on the average* if the ss property takes place only for the average value:

$$\langle X(at) \rangle \stackrel{d}{=} a^H \langle X(t) \rangle.$$

3.1.4 Selfsimilarity and stationarity

A non-degenerate stochastic H -ss process can not be stationary (Samorodnitsky and Taqqu, 1994). Really, if it were, we would have for any $a > 0$ and $t > 0$

$$X(at) \stackrel{d}{=} a^H X(t)$$

from one side and

$$X(at) \stackrel{d}{=} X(t)$$

from the other. However, there is no such an $H > 0$ to equality

$$a^H X(t) \stackrel{d}{=} X(t)$$

be fulfilled for any $t > 0$.

There exists the following important interrelation between selfsimilar and stationary processes: if $\{X(t), t > 0\}$ is H -ss, then the process $\{Y(t) = e^{-Ht}X(e^t), -\infty < t < \infty\}$ is stationary. Conversely, if $\{Y(t), -\infty < t < \infty\}$ is stationary, then $\{X(t) = t^H Y(\ln t), t > 0\}$ is H -ss. Really, using the second criterion of stationarity we can satisfy ourselves in validity of these propositions: if $\{X(t), t > 0\}$ is H -ss, then for any $\theta_1, \dots, \theta_n$ and $\theta > 0$

$$\sum_{j=1}^n \theta_j Y(t_j + \tau) = \sum_{j=1}^n \theta_j e^{-Ht_j} e^{-H\tau} X(e^\tau e^{t_j}) \stackrel{d}{=} \sum_{j=1}^n \theta_j e^{-Ht_j} X(e^{t_j}) = \sum_{j=1}^n \theta_j Y(t_j)$$

and if $\{Y(t), -\infty < t < \infty\}$ is stationary, then for any positive $t_1, \dots, t_n$ and a

$$\sum_{j=1}^n \theta_j X(at_j) = \sum_{j=1}^n \theta_j a^H t_j^H Y(\ln a + \ln t_j) \stackrel{d}{=} \sum_{j=1}^n \theta_j a^H t_j^H Y(\ln t_j) = \sum_{j=1}^n \theta_j a^H X(t_j).$$

Notice, however, that every stationary process is 0-ss in medium.

3.1.5 Brownian motion

The most known selfsimilar L-process is Brownian motion that is the L-process with transition pdf

$$p(x, t) = \frac{1}{\sqrt{2\pi t} \sigma} \exp \left\{ -\frac{x^2}{2\sigma^2 t} \right\}.$$

It can also be determined as follows:

Definition of Bm. The random process $\{X(t), t \geq 0\}$ is called (standard) Brownian motion (Bm) if

- 1) $X(0) = 0$ almost certainly;
- 2) $\{X(t), t \geq 0\}$ is a process with independent increments;
- 3) $X(t + \tau) - X(t) \stackrel{d}{=} \tau^{1/2} G$ at any t and τ where G is a normally distributed random variable with the variance 2:

$$p_G(x) = \frac{1}{2\sqrt{\pi}} e^{-x^2/4}.$$

Figure 3.1 shows a typical Brownian motion trajectory.

Recall that a random process is called *Gaussian process* if for any $n \geq 1$ and $t_n, t_{n-1}, \dots, t_1$, the density $p_n(x_n, t; \dots; x_1, t_1)$ is an n -dimensional normal (Gaussian) distribution. Consequently, the Bm is a homogeneous Gaussian process with independent increments, starting from the origin and having no drift.

Let us list main properties of Bm.

1. Finite-dimensional densities

The joint pdf's of $B(t_1), \dots, B(t_n)$ for $0 < t_1 < \dots < t_n$ are expressed by the products

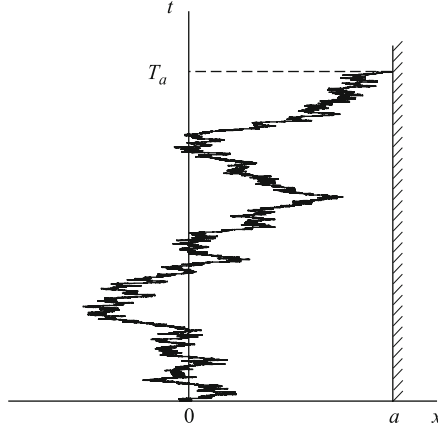


Fig. 3.1 A typical Brownian motion trajectory. T_a is the first passage time.

$$\begin{aligned}
 & f(x_n, t_n; \dots; x_2, t_2; x_1, t_1) \\
 &= f(x_n, t_n | x_{n-1}, t_{n-1}) \cdots f(x_2, t_2 | x_1, t_1) f(x_1, t_1) \\
 &= f(x_n - x_{n-1}, t_n - t_{n-1}) \cdots f(x_2 - x_1, t_2 - t_1) f(x_1, t_1) \\
 &= \frac{\exp\{-(1/2\sigma_0^2)[(x_n - x_{n-1})^2/(t_n - t_{n-1}) + \cdots + (x_2 - x_1)^2/(t_2 - t_1) + x_1^2/t_1]\}}{\sigma_0^2 (2\pi)^{n/2} [(t_n - t_{n-1}) \cdots (t_2 - t_1) t_1]^{1/2}}.
 \end{aligned}$$

One can directly verify that the finite-dimensional distributions of $\{B(at), t \geq 0\}$ are identical to those of $\{a^{1/2}B(t), t \geq 0\}$ and consequently the process is self-similar with $H = 1/2$.

2. Covariance function

As a Gaussian process the Bm-process is completely defined by its mean values $\langle B(t) \rangle = 0$ and covariance function $\text{Cov}(B(t_1), B(t_2))$. The latter can be computed directly from the definition. Letting $0 < t_1 < t_2$ and taking into account $\langle B(t) \rangle = 0$, we get

$$\begin{aligned}
 \text{Cov}(B(t_1), B(t_2)) &= \langle B(t_1)B(t_2) \rangle = \langle B(t_1)[B(t_1) + B(t_2) - B(t_1)] \rangle = \langle B(t_1)B(t_1) \rangle \\
 &+ \langle B(t_1)[B(t_2) - B(t_1)] \rangle = \langle B^2(t_1) \rangle + \langle B(t_1) \rangle \langle [B(t_2) - B(t_1)] \rangle = \sigma_0^2 t_1.
 \end{aligned}$$

Consequently, the correlation of Bm coordinates at an arbitrary pair of times t_1, t_2 is described by the autocovariance function

$$\text{Cov}(B(t_1), B(t_2)) = \langle B(t_1)B(t_2) \rangle = \begin{cases} \sigma_0^2 \min\{t_1, t_2\}, & \text{if } t_1 t_2 > 0; \\ 0, & \text{if } t_1 t_2 < 0. \end{cases}$$

The function can be rewritten in a form more convenient for algebra:

$$\text{Cov}(B(t_1), B(t_2)) = [\sigma_0^2/2](|t_1| + |t_2| - |t_1 - t_2|).$$

When $t_1 = t_2 = t$, then the covariance becomes the variance linearly increasing with time:

$$\text{Cov}(B(t), B(t)) = \langle B^2(t) \rangle = \sigma_0^2 |t|.$$

Since the function $|t_1| + |t_2| - |t_1 - t_2|$ is non-negative, the correlations between $B(t_1)$ and $B(t_2)$ are always positive. This results from the independence of Bm increments: the more $B(t_1)$, the more (in average) $B(t_2) = B(t_1) + [B(t_2) - B(t_1)]$, because the increment in square brackets does not depend on the first summand.

3. First passage time

Let T_a be the first passage time taken to Bm traverse from the origin to the point $a > 0$. Its probability distribution $P(T_a \leq t)$ can be computed by considering the event $\{B(t) \geq a\}$ and conditioning on whether or not $T_a \leq t$:

$$\mathbf{P}(B(t) \geq a) = \mathbf{P}(B(t) \geq a | T_a \leq t) \mathbf{P}(T_a \leq t) + \mathbf{P}(B(t) \geq a | T_a > t) \mathbf{P}(T_a > t).$$

Taking into account that

$$\mathbf{P}(|B(t)| \geq a | T_a \leq t) = 1, \quad \mathbf{P}(B(t) \geq a | T_a \leq t) = 1/2,$$

we obtain for the cumulative distribution function

$$\begin{aligned} F_{T_a}(t) &= \mathbf{P}(T_a \leq t) = 2\mathbf{P}(B(t) \geq a) = \frac{2}{\sigma_0 \sqrt{2\pi t}} \int_a^\infty \exp(-x^2/2\sigma_0^2 t) dx \\ &= \sqrt{\frac{2}{\pi}} \int_{a/\sigma_0 \sqrt{t}}^\infty \exp(-z^2/2) dz \end{aligned}$$

and for the probability density

$$p_{T_a}(t) = \frac{a}{\sigma_0 \sqrt{2\pi}} \exp[-a^2/(2\sigma_0^2 t)] t^{-3/2}, \quad a > 0, \quad t > 0.$$

This is the *Smirnov-Lévy stable distribution density*. Because of the symmetry of Bm-process this formula can be extended to the a of an arbitrary sign:

$$p_{T_a}(t) = \frac{|a|}{\sigma_0 \sqrt{2\pi}} \exp[-a^2/(2\sigma_0^2 t)] t^{-3/2}, \quad |a| > 0, \quad t > 0.$$

The integral of this density converges to 1 for any $a \neq 0$, this means that the Brownian particle sooner or later hits any point $x \in (-\infty, \infty)$.

The Laplace transform of the Smirnov-Lévy density is of the form:

$$\hat{p}_{T_a}(\lambda) = \frac{|a|}{\sigma_0 \sqrt{2\pi}} \int_0^\infty \exp[-\lambda t - a^2/(2\sigma_0^2 t)] t^{-3/2} dt = \exp[-(a/\sigma_0) \sqrt{2\lambda}],$$

which shows that the case $a = 0$ is characterized by degenerated δ -distribution:

$$\hat{p}_{T_0}(\lambda) = 1 \Leftrightarrow p_{T_0}(t) = \delta(t).$$

4. Brownian Sample Paths

- (1) The Brownian particle being at point $x = a$ at time t with unit probability visits both regions x_a and $x < a$ during any small time interval $(t, t + h)$.
- (2) With a unit probability, the Brownian particle sooner or later hits any point $x \in (-\infty, \infty)$.
- (3) With a unit probability, the Bm-trajectories are continuous. Mathematically, it is expressed in terms of the *Lindeberg condition*:

$$\lim_{\tau \rightarrow 0} \mathbf{P}(|X(t + \tau) - X(t)| \geq \Delta) / \tau = 0 \text{ for all } \Delta > 0.$$

Indeed, on substituting here the Bm pdf and applying the rule of L'Hospital

$$\begin{aligned} \lim_{\tau \rightarrow 0} \frac{\mathbf{P}(|B(t + \tau) - B(t)| \geq \Delta)}{\tau} &= \lim_{\tau \rightarrow 0} \frac{1}{\sqrt{2\pi}} \frac{d}{d\tau} \int_{\Delta/(\sigma_0\sqrt{\tau})}^{\infty} \exp\left(-\frac{z^2}{2}\right) dz \\ &= \lim_{\tau \rightarrow 0} \frac{\Delta \exp[-\Delta^2/(2\sigma_0^2\tau)]}{2\sqrt{2\pi}\tau^{3/2}} = 0, \end{aligned}$$

we verify the Lindeberg condition.

3.1.6 Bm in a nonstationary nonhomogeneous environment

3.1.6.1 Three types of the stochastic integral

Let us consider the stochastic equation

$$\frac{dX(t)}{dt} = b(X(t), t) \xi(t),$$

where $b(x, t)$ is a non-random function of the $-\infty < x < \infty$. Following (Van der Broeck, 1997), we take the shock model of the noise

$$\xi(t) = \sum_j \delta(t - T_j),$$

where T_j are random times generated by the Poisson process with the unit rate. At time T_j , the variable X will jump from one value to another, and each time a new term appears in the right side of the stochastic equation. However, the amplitude of the jump, b , depends on x . Which x -value should be taken then: the value, before the jump, after the jump, or something else? There exist three different interpretations of stochastic integral of the equation.

The first of them, the *Ito interpretation*, corresponds to taking x -value *before the jump* and leads to the following deterministic equation for probability density (as before, we restrict ourselves by the one-dimensional case)

$$\frac{\partial f}{\partial t} = \frac{1}{2} \frac{\partial^2}{\partial x^2} \{b^2(x, t) f(x, t)\}. \quad (3.1)$$

This interpretation is more suitable for theoretical investigations (with the use, for example, of the martingale formalism) and also desirable when simulating the process. But the price is that the normal rules of calculus become not valid and must be replaced by the less familiar Ito stochastic calculus.

The second approach is based on the *Stratonovich interpretation*, where the *average of the value prior to and after the jump* is taken. It leads to equation

$$\frac{\partial f}{\partial t} = \frac{1}{2} \frac{\partial}{\partial x} \left\{ b(x, t) \frac{\partial}{\partial x} [b(x, t) f(x, t)] \right\}. \quad (3.2)$$

The argument in favor of this rule is that one can show that it is a correct interpretation if $\xi(t)$ is in fact the limit of a shot noise with very small correlation time.

The third approach is tightly connected with the physical way of thinking based on the fundamental *continuity equation*

$$\frac{\partial f}{\partial t} = - \frac{\partial J}{\partial x}$$

added by the empirical *Fick law*:

$$J(x) = -K(x, t) \frac{\partial f}{\partial x}.$$

Here, $J(x)$ denotes the current density (more strictly, the x -projection of its vector) and $K(x, t)$ is the coefficient called *diffusivity*. As a result, we have

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial x} \left\{ K(x, t) \frac{\partial f}{\partial x} \right\}. \quad (3.3)$$

An advantage of the latter equation is that the spatial operator $\frac{\partial}{\partial x} \left\{ K(x, t) \frac{\partial}{\partial x} \right\}$ is a self-adjoint in the Lagrange sense differential operator. An additional argument in favor of this approach is that namely the latter form appears as a reduced equation from the more detailed Boltzmann kinetic equation.

The diffusion equation (3.3) with variable diffusivity in space and/or in time has been considered by several authors. Richardson (1926) considered the case $K(x, t) \propto |x|^{4/3}$ in order to describe turbulent diffusion. Batchelor (1953) used $K(x, t) \propto t^2$ for the same problem. Later, Okubo (1962) and Hentschel and Procaccia (1984) solved the equation with mixed algebraic form $K(x, t) = |x|^{-\theta} t^\beta$. Following (Fa and Lenzi, 2003, 2005a,b), we bring below recent results in this area obtained in frame of the three approaches.

3.1.6.2 Separation of variables

Assuming the factorized diffusivity

$$K(x, t) = K(x) \Theta(t)$$

and introducing a new variable

$$\tau = \int_0^t \Theta(t) dt,$$

we reduce the three equations listed in the preceding section to the case of time-independent diffusivity $K(x)$:

$$\frac{\partial f(x, \tau)}{\partial \tau} = \frac{\partial^2}{\partial x^2} \{K(x)f(x, \tau)\}, \quad (3.4)$$

$$\frac{\partial f(x, \tau)}{\partial \tau} = \frac{\partial}{\partial x} \left\{ \sqrt{K(x)} \frac{\partial [\sqrt{K(x)}f(x, \tau)]}{\partial x} \right\}, \quad (3.5)$$

and

$$\frac{\partial f(x, \tau)}{\partial \tau} = \frac{\partial}{\partial x} \left\{ K(x) \frac{\partial f(x, \tau)}{\partial x} \right\}. \quad (3.6)$$

Evidently, the three equations coincide with each other in case of homogeneous medium $K(x) = \text{const}$ and have the same fundamental solution

$$f(x, t) = \frac{1}{2\sqrt{\pi K \tau}} \exp \left\{ -\frac{x^2}{4K\tau} \right\}, \quad f(x, 0) = \delta(x).$$

We will consider their solutions for power-type diffusivity

$$K(x) = |x|^{-\theta} \quad (3.7)$$

under the same initial condition

$$f(x, 0) = \delta(x).$$

3.1.6.3 Three analytical solutions

Solution to Eq. (3.4). Inserting Eq. (3.7) into Eq. (3.4) yields

$$\frac{\partial f_1(x, \tau)}{\partial \tau} = \frac{\partial^2}{\partial x^2} \left\{ |x|^{-\theta} f_1(x, \tau) \right\}.$$

By representing the spatial variable in the form

$$z = \frac{x}{g(\tau)}$$

and setting

$$f_1(x, \tau) = \frac{1}{g(\tau)} \phi \left(\frac{x}{g(\tau)} \right),$$

we obtain

$$-\frac{1}{g^2(\tau)} \frac{d\phi(\tau)}{d\tau} \frac{d[z\phi(z)]}{dz} = \frac{1}{g^{3+\theta}(\tau)} \frac{d^2[|z|^{-\theta}\phi(z)]}{dz^2}.$$

Separating t and z yields two equations

$$g^{1+\theta}(\tau) \frac{dg(\tau)}{d\tau} = C$$

and

$$-C \frac{[z\phi(z)]}{dz} = K_1 \frac{d^2[|z|^{-\theta}\phi(z)]}{dz^2},$$

where C is a constant of separation. Their solutions are of the form:

$$g(\tau) = [C(2 + \theta)\tau]^{1/(2+\theta)}$$

and

$$\phi(z) = C_1 |z|^\theta \exp \left\{ -\frac{C|z|^{2+\theta}}{(2 + \theta)} \right\}$$

respectively, where C_1 is one of the integration constants and the other is put to be zero. As a result, we obtain the distribution

$$f_1(x, \tau) = \frac{|x|^\theta \tau^{-(1+\theta)/(2+\theta)}}{2(2 + \theta)^{\theta/(2+\theta)} \Gamma[(1 + \theta)/(2 + \theta)]} \exp \left\{ -\frac{|x|^{2+\theta}}{(2 + \theta)^2 \tau} \right\}$$

with

$$\theta \in (-\infty, -2) \cup (-1, \infty).$$

Its moments are

$$\langle X^{2n}(t) \rangle_1 \equiv \int_{-\infty}^{\infty} x^{2n} f(x, t) dx = \frac{(2 + \theta)^{4n/(2+\theta)} \Gamma((2n + 1 + \theta)/(2 + \theta))}{2^{n/(2+\theta)} \Gamma((1 + \theta)/(2 + \theta))} t^{2n/(2+\theta)}.$$

In particular, the variance is given by

$$\langle X^2(t) \rangle_1 = \frac{(2 + \theta)^{4/(2+\theta)} \Gamma((3 + \theta)/(2 + \theta))}{2^{1/(2+\theta)} \Gamma((1 + \theta)/(2 + \theta))} t^{2/(2+\theta)}.$$

Solution to Eq. (3.5). By introducing a new spatial variable

$$y = \int_0^x \frac{dx}{\sqrt{K(x)}} = \frac{1}{\theta/2 + 1} |x|^{\theta/2+1} \text{sign}(x)$$

the equation

$$\frac{\partial f_2(x, \tau)}{\partial \tau} = \frac{\partial}{\partial x} \left\{ |x|^{-\theta/2} \frac{\partial [|x|^{-\theta/2} f_2(x, \tau)]}{\partial x} \right\}$$

is reduced to the standard diffusion equation with a unit diffusivity. As a result we get

$$f_2(x, t) = \frac{|x|^{\theta/2}}{2\sqrt{\pi\tau}} \exp \left\{ -\frac{|x|^{2+\theta}}{(2+\theta)^2\tau} \right\}, \quad \tau > 0.$$

Spatial moments related to this process are given by

$$\langle X^{2n}(t) \rangle_2 = (1/\sqrt{\pi})(2+\theta)^{4n/(2+\theta)} \Gamma((4n+2+\theta)/(4+2\theta)) \tau^{2n/(2+\theta)}, n = 1, 2, \dots$$

In particular, the second moment yields

$$\langle X^2(t) \rangle_2 = (1/\sqrt{\pi})(2+\theta)^{4/(2+\theta)} \Gamma((6+\theta)/(4+2\theta)) \tau^{2/(2+\theta)}.$$

Solution to Eq. (3.6) The solution of Eq. (3.6) with $K(x) = |x|^{-\theta}$,

$$\frac{\partial f_3(x, \tau)}{\partial \tau} = \frac{\partial}{\partial x} \left\{ |x|^{-\theta} \frac{\partial f_3(x, \tau)}{\partial x} \right\}$$

was found in works (Hentschel and Procaccia, 1984), (Banavar and Willemssen, 1984). It reads

$$f_3(x, t) = \frac{(2+\theta)^{\theta/(2+\theta)}}{2\Gamma(1/(2+\theta))\tau^{1/(2+\theta)}} \exp \left\{ -\frac{|x|^{2+\theta}}{(2+\theta)^2\tau} \right\}, \quad \theta > -2.$$

The moments of the distributions are given by

$$\langle X^{2n}(t) \rangle_3 = \frac{(2+\theta)^{4n/(2+\theta)} \Gamma((2n+1)/(2+\theta))}{\Gamma(1/(2+\theta))} \tau^{2n/(2+\theta)}, \quad n = 1, 2, 3, \dots$$

3.1.6.4 Selfsimilarity of the distributions

Observe that all the three solutions for a power-type diffusivity possess the selfsimilarity property:

$$f_n(x, t) dx = \phi_n(\xi) d\xi, \quad \xi = \frac{\sqrt{2}}{|2+\theta|} \frac{|x|^{1+\theta/2}}{\sqrt{\tau}} \text{sign}(x),$$

where

$$\begin{aligned} \phi_1(\xi) &= \frac{2^{-(1+\theta)/(2+\theta)}}{\Gamma((1+\theta)/(2+\theta))} e^{-\xi^2/2} |\xi|^{(2+\theta)/(2+\theta)-1}, \\ \phi_2(\xi) &= \frac{1}{\sqrt{2\pi}} e^{-\xi^2/2}, \end{aligned}$$

and

$$\phi_3(\xi) = \frac{2^{-1/(2+\theta)}}{\Gamma(1/(2+\theta))} e^{-\xi^2/2} |\xi|^{-\theta/(2+\theta)}.$$

The variance of the distributions increases with time, and the law of increasing is determined by function $\tau(t)$. Taking $\tau = K_1 t^\theta$, that is assuming

$$K(x, t) = K_1 |x|^\gamma \theta t^{\theta-1}, \quad t > 0, \quad \theta > 0,$$

we recover the family of OHP (Okubo-Hentschel-Procaccia) solutions

$$f(x, t) = \frac{2-\gamma}{2\Gamma(1/(2-\gamma))} \left[\frac{t^{-\theta}}{(2-\gamma)^2 K_1} \right]^{1/(2-\gamma)} \exp \left\{ -\frac{|x|^{2-\gamma} t^{-\theta}}{(2-\gamma)^2 K_1} \right\}, \quad t > 0,$$

to the equation

$$\frac{\partial f}{\partial t} = K_1 \theta t^{\theta-1} \frac{\partial}{\partial x} \left\{ |x|^\gamma \frac{\partial f}{\partial x} \right\}$$

with mean squared displacements $\langle X^2(t) \rangle \propto t^{2\theta/(2-\gamma)}$. This family contains the known results of Richardson ($\gamma = 4/3$, $\theta = 1$):

$$f_R(x, t) = \frac{9}{4} \frac{1}{\sqrt{\pi}(K_1 t)^{3/2}} \exp \left\{ -\frac{9}{4} \frac{|x|^{2/3}}{K_1 t} \right\}, \quad \langle X^2(t) \rangle_R = (280/243)(K_1 t)^3,$$

and those of Bachelor ($\gamma = 0$, $\theta = 3$):

$$f_B(x, t) = \frac{1}{2\sqrt{\pi K_1 t^3}} \exp \left\{ -\frac{x^2}{4K_1 t^3} \right\}, \quad \langle X^2(t) \rangle_B = 2K_1 t^3.$$

Observe that these two distributions, derived to describe the relative diffusion of a pair of particles in a turbulent media, are different in the form, although their mean squared replacements follows the power type law with the same exponent 3.

In (Fa and Lenzi, 2005a), Eq. (3.6) has been solved at the presence of absorbing boundaries and the first passage time distribution has been derived for a stationary diffusivity ($\theta = 1$):

$$p_{T_a}(t) = \frac{a^{1-\gamma}}{t\Gamma((1-\gamma)/(2-\gamma))} \frac{\exp\{-a^{2-\gamma}/(2-\gamma)^2 K_1 t\}}{[(2-\gamma)^2 K_1 t]^{(1-\gamma)/(2-\gamma)}}, \quad t > 0.$$

In a homogeneous medium, when $\gamma = 1$, this density becomes the Lévy-Smirnov density with $\sigma_0^2 = 2K_1$.

3.2 One-dimensional Lévy motion

3.2.1 Stable random variables

Brownian motion is a selfsimilar Lévy process with $H = 1/2$. Let's now consider an arbitrary selfsimilar Lévy process $\{X(t), t > 0\}$. The random coordinates of a

particle performing motion of such kind at times t_1 and $t_1 + t_2$ are linked via the relation

$$X(t_1 + t_2) = X(t_1) + X(t_2).$$

Provided that

$$X(0) = 0,$$

the random variables $X(t_1)$ and $X(t_2)$ are increments of the process in non-intersecting intervals $(0, t_1)$ and $(t_1, t_1 + t_2)$ and, hence, are independent. The probability density function of their sum is given by convolution of densities:

$$p(x, t_1 + t_2) = \int_{-\infty}^{\infty} p(x - x', t_1) p(x', t_2) dx' \equiv p(x, t_1) * p(x, t_2).$$

Let us pass from densities to characteristic functions:

$$\tilde{p}(k, t) = \langle e^{ikX(t)} \rangle = \int_{-\infty}^{\infty} e^{ikx} p(x, t) dx.$$

In this case, the condition of the selfsimilarity becomes

$$\tilde{p}(k, t) = \int_{-\infty}^{\infty} e^{ikx} p(xt^{-H}, 1) t^{-H} dx = \tilde{p}(kt^H, 1),$$

and the convolution is transformed to the product of characteristic functions

$$\tilde{p}(k, t_1 + t_2) = \tilde{p}(k, t_1) \tilde{p}(k, t_2) = \tilde{p}(kt_1^H, 1) \tilde{p}(kt_2^H, 1).$$

Combining two last formulas leads to the functional equation

$$\tilde{p}(k(t_1 + t_2)^H, 1) = \tilde{p}(kt_1^H, 1) \tilde{p}(kt_2^H, 1).$$

Let $S^{(\alpha)}$, $\alpha = 1/H$ be a random variable described by the characteristic function

$$g^{(\alpha)}(k) \equiv \tilde{p}(k, 1),$$

so

$$\tilde{p}(k, t) = \tilde{p}(kt^{1/\alpha}, 1) = g^{(\alpha)}(kt^{1/\alpha})$$

As one can see from above, the random variable satisfies the relation

$$(t_1 + t_2)^{1/\alpha} S^{(\alpha)} \stackrel{d}{=} t_1^{1/\alpha} S_1^{(\alpha)} + t_2^{1/\alpha} S_2^{(\alpha)},$$

where $S_1^{(\alpha)}$ and $S_2^{(\alpha)}$ are independent copies of the random variable $S^{(\alpha)}$. This relation is the definition property of α -stable random variables with a characteristic index α . The term covers its characteristic function $\tilde{g}^{(\alpha)}(k)$ and the related probability density function $g^{(\alpha)}(x)$.

3.2.2 Stable characteristic functions

To find an explicit formula for the stable characteristic functions, we shall introduce “second characteristics”:

$$\psi^{(\alpha)}(k) = \ln \tilde{g}^{(\alpha)}(k),$$

for which the property of stability takes the form

$$\psi^{(\alpha)}(t_1^{1/\alpha}k) + \psi^{(\alpha)}(t_2^{1/\alpha}k) = \psi^{(\alpha)}((t_1 + t_2)^{1/\alpha}k).$$

Extending this relation on the sum of arbitrary number n of identically distributed ($t_1 = t_2 = \dots = t_n = 1$) terms, we shall obtain

$$n\psi^{(\alpha)}(k) = \psi^{(\alpha)}(n^{1/\alpha}k).$$

According to the property

$$\psi^{(\alpha)}(-k) = [\psi^{(\alpha)}(k)]^*,$$

it is enough to determine the function $\psi^{(\alpha)}(k)$ for positive arguments. Taking into account its continuity in a neighborhood of the origin of coordinates and the conditions resulting from the definition of characteristic functions

$$\psi^{(\alpha)}(0) = 0,$$

we discover that

$$|\psi^{(\alpha)}(k)| = \text{const} \cdot k^\alpha \quad (k > 0, \alpha > 0)$$

and

$$\psi^{(\alpha)}(k) = -k^\alpha [c_0 - ic_1].$$

Since the characteristic function satisfies the requirement

$$|\tilde{g}(k)| \leq 1,$$

then

$$\text{Re } \psi^{(\alpha)}(k) \leq 0$$

and the real constant c_0 should be positive. On the other hand, from the condition

$$\tilde{g}''(0) = -\langle Y^2 \rangle$$

it follows that

$$\psi^{(\alpha)''}(0) = -\langle S^2 \rangle + \langle S \rangle^2 \equiv -\sigma_S^2 \leq 0.$$

Computing the second derivative

$$[\psi^{(\alpha)}(k)]'' = -[c_0 - ic_1]\alpha(\alpha - 1)k^{\alpha-2},$$

and assuming $k \rightarrow 0$, one can find out that the variance is finite if and only if $\alpha = 2$. Moreover, the variance is a real number, thus c_1 should be equal to 0. When $\alpha < 2$ the variance is infinite (in this case the value c_1 doesn't play any roles). Finally, for $\alpha > 2$ the limit value of the derivative is zero and

$$\int_{-\infty}^{\infty} x^2 g(x) dx = 0.$$

This condition can only be satisfied by δ -distribution concentrated at the origin:

$$g(x) = \delta(x).$$

A non-degenerated function should oscillate to have such a second moment and for this reason it can not serve as a probability density function. Therefore, the domain of admissible values of parameter α is $(0, 2]$.

The constants c_0 and c_1 can be chosen in various ways. This leads to various forms of characteristic functions. Form A is obtained by choosing

$$c_0 = 1, \quad c_1 = \beta \tan(\alpha\pi/2),$$

where the second real parameter $\beta \in [-1, 1]$ characterizes the degree of the distribution asymmetry (the skew parameter). Thus, the characteristic function of one-dimensional stable density on a positive semi-axis is of the form:

$$\tilde{g}^{(\alpha, \beta)}(k) = \exp\{-k^\alpha [1 - i\beta \tan(\alpha\pi/2)]\}, \quad k > 0.$$

Similar calculations for the negative semi-axes yield

$$\tilde{g}^{(\alpha, \beta)}(k) = \exp\{-(-k)^\alpha [1 + i\beta \tan(\alpha\pi/2)]\}, \quad k < 0.$$

Combining the two formulas, we obtain

$$\tilde{g}^{(\alpha, \beta)}(k) = \exp\{-|k|^\alpha [1 - i\beta \tan(\alpha\pi/2) \text{sign } k]\}, \quad -\infty < k < \infty. \quad (\text{A})$$

This is a *standard* representation of the stable characteristic function in Form A. There is often used the so-called Form C with a different choice of the skew parameter

$$\tilde{g}(k; \alpha, \theta) = \exp\{-|k|^\alpha \exp\{-i(\theta\alpha\pi/2) \text{sign } k\}\}. \quad (\text{C})$$

In the above expression, $\alpha \in (0, 2]$ is the characteristic exponent of the stable law, $\beta \in [-1, 1]$ and $\theta \in [-\theta_\alpha, \theta_\alpha]$, $\theta_\alpha = \min\{1, 2/\alpha - 1\}$ — are skew parameters in Forms A and C respectively. In what follows, we will use the notations shown in Table 3.1. Observe that the characteristic exponent α is the same in both Forms A and C, the skewness parameters β and θ are linked by the formula

$$\beta = \tan(\theta\alpha\pi/2)/\tan(\alpha\pi/2),$$

and the random variables themselves obey the interrelation

$$S(\alpha, \theta) \stackrel{d}{=} [\cos(\theta\alpha\pi/2)]^{1/\alpha} S^{(\alpha, \beta)}.$$

We will denote the positive random variable $S(\alpha, 1)$, $0 < \alpha \leq 1$ by $S_+(\alpha)$ and call it the subordinator.

Table 3.1

Characteristics	Form A	Form C
Probability density function	$g^{(\alpha, \beta)}(x)$	$g(x; \alpha, \theta)$
Characteristic function	$\tilde{g}^{(\alpha, \beta)}(k)$	$\tilde{g}(k; \alpha, \theta)$
Random variable	$S^{(\alpha, \beta)}$	$S(\alpha, \theta)$

3.2.3 Stable probability densities

The details of computing stable pdf's by inverting Fourier transform:

$$g(x; \alpha, \theta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} \tilde{g}(k; \alpha, \theta) dk$$

can be found in books (Zolotarev, 1986; Lukacs, 1960; Uchaikin and Zolotarev, 1999).

We restrict ourselves by listing the most important properties of stable distributions (in Form C).

1. All stable densities are unimodal.
2. Variances of all stable distributions except Gaussian ($\alpha = 2$) are infinite.
3. Mean values of stable distributions with exponents $\alpha \leq 1$ do not exist.
4. The stable densities satisfy the inversion relation:

$$g(x; \alpha, \theta) = g(-x; \alpha, -\theta).$$

5. The stable densities satisfy the duality relation:

$$g(x; \alpha, \theta) = x^{-1-\alpha} g(x^{-\alpha}; \alpha', \theta'), \quad \text{for } \alpha \geq 1$$

where $\alpha' = 1/\alpha$, $\theta' = \alpha(1 + \theta) - 1$.

6. At the origin, the cumulative distribution function

$$G(x; \alpha, \theta) = \int_{-\infty}^x g(x'; \alpha, \theta) dx',$$

pdf and its derivative have the following values:

$$G(0; \alpha, \theta) = (1 - \theta)/2,$$

$$g(0; \alpha, \theta) = \pi^{-1} \Gamma(1 + 1/\alpha) \cos(\theta\pi/2),$$

$$g'(0; \alpha, \theta) = (2\pi)^{-1} \Gamma(1 + 2/\alpha) \sin(\theta\pi).$$

7. If the pdf $g(x; \alpha, \theta)$ is not extreme (i.e., $\theta \neq \pm\theta_\alpha$), both of its tails fall according to inverse power laws $|x|^{-\alpha-1}$ (“heavy” or “long” tails):

$$g(\pm|x|; \alpha, \theta) \sim \frac{\Gamma(1 + \alpha)}{\pi} \sin[\alpha(1 \pm \theta)\pi/2] |x|^{-1-\alpha}, \quad x \rightarrow \infty.$$

8. For extreme densities, when

$$\theta = \pm 1, \quad \text{when } \alpha < 1,$$

$$\theta = \pm(2/\alpha - 1), \quad \text{when } \alpha > 1,$$

one of the tails falls exponentially (“short” tail).

9. As follows from the formula for $G(0; \alpha, \theta)$, the most part of probability is distributed over the positive semiaxis in case $\theta > 0$ and over the negative semiaxis in case $\theta < 0$. When $\alpha \leq 1$, $G(0; \alpha, 1) = 0$ and $G(0; \alpha, -1) = 1$, i.e., the extreme distributions become *one-sided*, distributed exclusively over one half of the axis. In the limit case $\alpha = 1$,

$$g(x; 1, \pm 1) = \delta(x \mp 1).$$

We will use for the positive one-sided pdf the same term as for the corresponding random variable, i.e., the subordinator, and use the notation

$$g_+(x; \alpha) = g(x; \alpha, 1).$$

10. The following stable densities can be represented in terms of elementary functions: the Gaussian distribution

$$g(x; 2, 0) = \frac{1}{2\sqrt{\pi}} \exp\left\{-\frac{x^2}{4}\right\},$$

the Cauchy distribution

$$g(x; 1, 0) = \frac{1}{\pi(1+x^2)}.$$

and the Lévy-Smirnov distribution

$$g_+(x; 1/2) = \frac{1}{2\sqrt{\pi x^3}} \exp\{-(4x)^{-1}\}.$$

Some examples of the stable probability densities are depicted in Fig. 3.2.

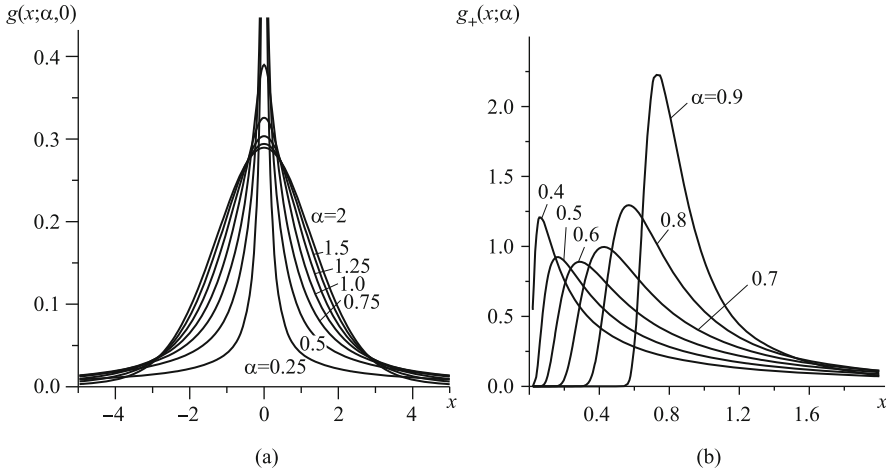


Fig. 3.2 Symmetric (a) and one-sided (b) one-dimensional stable densities.

3.2.4 Discrete time Lévy motion

Considering random states of Lévy motion (Lm) at equally spaced times $t = 0, 1, 2, \dots$ we are coming to the problem of summation of identically distributed independent standard stable variables:

$$\Sigma_n = \sum_{j=1}^n S_j^{(\alpha)}.$$

As shown above, for any stable random variable the following relation takes place

$$(t_1 + t_2)^{1/\alpha} S^{(\alpha)} \stackrel{d}{=} t_1^{1/\alpha} S_1^{(\alpha)} + t_2^{1/\alpha} S_2^{(\alpha)}.$$

Step by step, it can be extended to an arbitrary integer number of terms. Making this and putting $t_1 = t_2 = \dots = t_n = 1$, we obtain the general relation of equivalence for sums Σ_n of the independent identically distributed stable random variables S_j :

$$\Sigma_n = \sum_{j=1}^n S_j^{(\alpha)} \stackrel{d}{=} n^{1/\alpha} S^{(\alpha)}. \quad (3.8)$$

This relation is valid for any forms of representation A and C. The correspondent pdf of the random variable $X(n) \equiv \Sigma_n$ is

$$p(x, n) = n^{-1/\alpha} g(n^{-1/\alpha} x).$$

To emphasize the intrinsic features of the DTLM, we refer to Feller (1971) who noted important and unexpected consequences of the result. Let's consider, for example, a stable distribution with $\alpha < 1$. The arithmetic mean Σ_n/n has the same dis-

tribution as $S_1 n^{-1+1/\alpha}$. Meanwhile, the factor $n^{-1+1/\alpha}$ tends to infinity as n grows. Without claiming to be strict, one can say, that the arithmetic mean turns out to be considerably greater than almost any summand. This is possible only in the case where the largest term

$$M_n = \max\{S_1, \dots, S_n\} \quad (3.9)$$

grows extremely fast and gives a leading contribution to the sum S_n .

We do not perform a detailed analysis here and restrict ourselves only by simple arguments of rather heuristic nature.

The cumulative distribution function for the maximum term (3.9) can readily be written. Indeed, the event $\{M_n < x\}$ implies $\{S_1 < x, \dots, S_n < x\}$ and vice versa, therefore, the cumulative probability function

$$F_{M_n}(x) = P\{M_n < x\} = P\{S_1 < x, \dots, S_n < x\}.$$

Since S_j are independent, the right-hand side of this expression can be transformed to the product of probabilities of individual events $\{S_j < x\}$. In view of coincidence of the distributions of S_j , it is the mere n th power of this probability:

$$F_{M_n}(x) = P^n\{S_j < x\} = F_Y^n(x). \quad (3.10)$$

If the maximum term gives a primary contribution to the sum, the following relation should be satisfied (if $\beta \neq -1$)

$$\bar{F}_{\Sigma_n}(x) \sim \bar{F}_{M_n}(x), \quad x \rightarrow \infty, \quad (3.11)$$

where

$$\bar{F}(x) \equiv 1 - F(x).$$

By virtue of (3.8),

$$\bar{F}_{\Sigma_n}(x) = P\{\Sigma_n \geq x\} = P\{S \geq n^{-1/\alpha}x\} = \bar{F}_S(n^{-1/\alpha}x). \quad (3.12)$$

Substituting (3.12) in the left-hand side of (3.11) and (3.10) in the right-hand one, we obtain

$$\bar{F}_S(n^{-1/\alpha}x) \sim 1 - F_S^n(x).$$

Changing the variable $n^{-1/\alpha}x = y$, we arrive at the equation

$$\bar{F}_S(y) \sim 1 - [1 - \bar{F}_S(yn^{1/\alpha})]^n \sim n\bar{F}_S(yn^{1/\alpha}), \quad y \rightarrow \infty, \quad (3.13)$$

solution of which, provided that $\beta \neq -1$, is

$$1 - F_S(x) = \bar{F}_S(x) \sim cx^{-\alpha}, \quad x \rightarrow \infty. \quad (3.14)$$

Similarly, we obtain

$$F_S(x) \sim d|x|^{-\alpha}, \quad x \rightarrow -\infty. \quad (3.15)$$

Differentiating (3.14) and (3.15) with respect to x yields the following asymptotic expressions for the density:

$$\begin{aligned} p_S(x) &= -\bar{F}_S'(x) \sim \alpha c x^{-\alpha-1}, \quad x \rightarrow \infty, \\ p_S(x) &= F_S'(x) \sim \alpha d |x|^{-\alpha-1}, \quad x \rightarrow -\infty. \end{aligned}$$

It is clear that this does not correspond to the normal law ($\alpha = 2$), and that there are no $\alpha > 2$: *the normal law is the only stable law with finite variance*. Thus, the characteristic exponent $\alpha < 2$ takes values from the interval $(0, 2]$; for $\alpha = 2$ we have the Gaussian law; with $\alpha < 2$ we have stable laws whose probabilities of large deviations are power functions (“long” tails). In extreme cases, only one of the tails may be long.

3.2.5 Generalized limit theorem

The results discussed above admit a very important generalization known as the **generalized limit theorem**:

Let $X_1, \dots, X_n$ be independent identically distributed random variables with the distribution function $F_X(x)$ obeying the conditions

$$1 - F_X(x) \sim c x^{-\mu}, \quad x \rightarrow \infty, \quad (3.16)$$

$$F_X(x) \sim d |x|^{-\mu}, \quad x \rightarrow -\infty$$

with $\mu > 0$. Then there exist such sequences a_n and $b_n > 0$ that the distribution of the centered and normalized sums

$$Z_n = \left(\sum_{i=1}^n X_i - a_n \right) / b_n \quad (3.17)$$

weakly converges to the reduced stable distribution with parameters

$$\alpha = \begin{cases} \mu, & \mu \leq 2, \\ 2, & \mu > 2, \end{cases} \quad (3.18)$$

and

$$\beta = \frac{c - d}{c + d}, \quad (3.19)$$

as $n \rightarrow \infty$,

$$F_{Z_n}(x) \Rightarrow G^{(\alpha, \beta)}(x). \quad (3.20)$$

The coefficients a_n and b_n can be taken in the form given in Table 3.2:

Certainly, the last case ($\mu > 2$) is covered by the central limit theorem which holds true for any random variables X_i with finite variance but not for power-type distributions (3.16) and (3.17) with $\mu < 2$.

Table 3.2 Centering and normalizing coefficients a_n and b_n (form A).

μ	α	a_n	b_n
$0 < \mu < 1$	μ	0	$[\pi(c+d)]^{1/\alpha} [2\Gamma(\alpha) \sin(\alpha\pi/2)]^{-1/\alpha} n^{1/\alpha}$
$\mu = 1$	μ	$\beta(c+d)n \ln n$	$(\pi/2)(c+d)n$
$1 < \mu < 2$	μ	$n\langle X \rangle$	$[\pi(c+d)]^{1/\alpha} [2\Gamma(\alpha) \sin(\alpha\pi/2)]^{-1/\alpha} n^{1/\alpha}$
$\mu = 2$	2	$n\langle X \rangle$	$(c+d)^{1/2} [n \ln n]^{1/2}$
$\mu > 2$	2	$n\langle X \rangle$	$[(1/2)\sigma_X^2]^{1/2} n^{1/2}$

We will not prove this theorem but cite some heuristic reasoning following the work (Bouchaud and Georges, 1990) for the case of non-negative summands with the power asymptotics

$$\bar{F}_X(x) \sim cx^{-\alpha}, \quad 0 < \alpha < 1, \quad x \rightarrow \infty \quad (3.21)$$

leading to one-sided stable distributions. According to (3.10), the probability density function of a maximum term M_n in the sum Σ_n has the form

$$p_{M_n}(x) = \frac{dF_{M_n}(x)}{dx} = n \left[1 - \int_x^\infty p_X(x') dx' \right]^{n-1} p_X(x). \quad (3.22)$$

We denote the most probable value M_n by x_n :

$$\left. \frac{dp_{M_n}(x)}{dx} \right|_{x=x_n} = 0. \quad (3.23)$$

Differentiating (3.22) with respect to x and substituting the result in (3.23), we obtain

$$\left[1 - \int_{x_n}^\infty p_X(x') dx' \right] \frac{dp_X(x_n)}{dx_n} + (n-1)p_X^2(x_n) = 0. \quad (3.24)$$

As $x \rightarrow \infty$

$$\begin{aligned} \int_x^\infty p_X(x') dx' &\sim cx^{-\alpha}, \\ p_X(x) &\sim c\alpha x^{-\alpha-1}, \\ dp_X(x)/dx &\sim -c\alpha(\alpha+1)x^{-\alpha-2}. \end{aligned} \quad (3.25)$$

and if $n \gg 1$ Eq. (3.24) yields the asymptotic relation:

$$\alpha + 1 \sim nc\alpha x_n^{-\alpha},$$

whence

$$x_n \sim Cn^{1/\alpha}, \quad C = \left(\frac{c\alpha}{\alpha+1} \right)^{1/\alpha}.$$

Estimating a characteristic value of the sum

$$\Sigma_n = \sum_{i=1}^n X_i$$

with the help of the “truncated average”

$$\langle X \rangle_{x_n} = \int_0^{x_n} xp_X(x)dx,$$

one obtains

$$\Sigma_n \sim \langle \Sigma_n \rangle_{x_n} = n \int_0^{x_n} xp_X(x)dx = \int_0^{Cn^{1/\alpha}} xp_X(x)dx.$$

Let $A > 0$ be such that as $x > A$ the density $p_X(x)$ follows its asymptotic expression (3.25). Then

$$\langle \Sigma_n \rangle_{x_n} = n \left\{ \int_0^A xp_X(x)dx + \int_A^{Cn^{1/\alpha}} xp_X(x)dx \right\} = n\langle X \rangle_A + \langle \Sigma'_n \rangle_{x_n},$$

where

$$\langle \Sigma'_n \rangle_{x_n} = \alpha cn \int_A^{Cn^{1/\alpha}} x^{-\alpha} dx.$$

From here it can be seen, that as $n \rightarrow \infty$

$$\Sigma_n \sim \langle \Sigma_n \rangle_{x_n} \sim \begin{cases} c_0 n^{1/\alpha}, & \alpha < 1; \\ c_1 n \ln n, & \alpha = 1, \end{cases} \quad (3.26)$$

where c_0 and c_1 are positive constants.

Estimating similarly a truncated variance of the sum Σ_n , with $\alpha > 1$ we obtain

$$\langle (\Sigma_n - \langle \Sigma'_n \rangle)^2 \rangle_{x_n} \sim \int_0^{x_n} (x - \langle X \rangle)^2 p_X(x)dx \sim \begin{cases} c_3 n^{1/\alpha}, & \alpha < 2, \\ c_4 n \ln n, & \alpha = 2. \end{cases} \quad (3.27)$$

Conclusions (3.26) and (3.27) show a qualitative agreement with the generalized limit theorem.

Those readers who are interested in applications of stable laws rather than in their mathematical aspects have a possibility now to pass readily to the second part of our book (Uchaikin and Zolotarev, 1999).

3.2.6 Continuous time Lévy motion

A random process $\{X(t), t \geq 0\}$ is called the (standard) Continuous Time Lévy Motion (CTLTM), if

- 1) $X(0) = 0$ almost certainly;
- 2) $\{X(t), t \geq 0\}$ is a process with independent increments;
- 3) $X(t + \tau) - X(t) \stackrel{d}{=} \tau^{1/\alpha} S$ at any t and τ , $0 < \alpha \leq 2$.

As one can directly see from this definition the CTLTM pdf is given by the formula

$$p(x, t) = t^{-1/\alpha} g^{(\alpha, \beta)}(t^{-1/\alpha} x).$$

In order to make the difference between sample paths of Lévy motion and Brownian motion more clear, let us consider the behavior of function

$$Q(\tau, \Delta) \equiv P(|X(t + \tau) - X(t)| \geq \Delta) / \tau$$

as $\tau \rightarrow 0$. For Bm-process

$$Q_W(\tau, \Delta) = \frac{1}{\sqrt{\pi\tau^{3/2}}} \int_{\Delta}^{\infty} e^{-x^2/4\tau} dx = \frac{1}{\sqrt{\pi\tau}} \int_{\Delta/\sqrt{\tau}}^{\infty} e^{-z^2/4} dz.$$

Applying L'Hospital's rule, we get the expression

$$\lim_{\tau \rightarrow 0} Q_W(\tau, \Delta) = 0,$$

reflecting the property of *continuity* of Bm's sample paths. In the case of Lm with $\alpha < 2$

$$Q_W(\tau, \Delta) = P(|S^{(\alpha, \beta)}| \geq \Delta \tau^{-1/\alpha}) / \tau.$$

It is known that

$$P(|S^{(\alpha, \beta)}| \geq x) \propto x^{-\alpha}, \quad x \rightarrow \infty,$$

therefore

$$\lim_{\tau \rightarrow 0} Q_W(\tau, \Delta) = \text{const } \Delta^{-\alpha} > 0, \quad \tau \rightarrow 0.$$

When $\alpha < 2$, sample paths of $L^{(\alpha, \beta)}$ m process are *not continuous* anymore and have jump-like form. This is their main difference. The jump amplitudes are independent random variables depending on α : the less α , the longer jumps.

The width of a diffusion packet grows with time proportionally $t^{1/\alpha}$. When $\alpha < 2$, the speed of widening of Lm packet is greater than Bm and is not of the Gaussian form. Its variance is infinite now, and we should use some other measure of the packet width. Some typical sample paths are shown at Fig. 3.3.

Bm is not the only non-degenerate Lm process possessing continuous sample paths. Besides, two degenerate Lm processes with continuous paths exist, repre-

senting ballistic non-random motion with a constant velocity to both directions of the real axis ($\beta = \pm 1$).

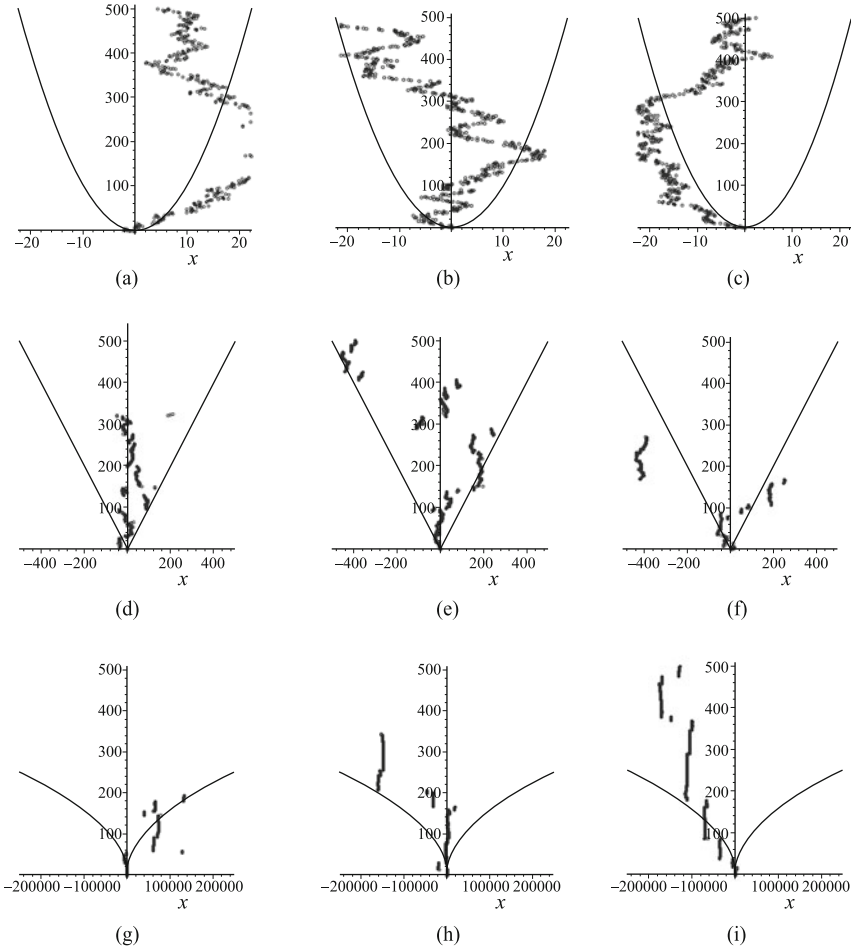


Fig. 3.3 Examples of random trajectories of Gauss ($\alpha = 2$, (a), (b), and (c)), Cauchy ($\alpha = 1$, (d), (e), and (f)), and Lévy-Smirnov ($\alpha = 1/2$, (g), (h), and (i)) processes.

3.3 Multidimensional Lévy motion

3.3.1 Multivariate symmetric stable vectors

The concept of stable laws can be naturally extended to the case of spaces with arbitrary dimensions. A d -dimensional random vector $\mathbf{S} = \{S_1, S_2, \dots, S_d\}$ is said to

be a *stable random vector*¹ if for any positive numbers t_1 and t_2

$$t_1^{1/\alpha} \mathbf{S}_1 + t_2^{1/\alpha} \mathbf{S}_2 \stackrel{d}{=} (t_1 + t_2)^{1/\alpha} \mathbf{S},$$

where $\mathbf{S}_1$ and $\mathbf{S}_2$ are independent copies of the random vector $\mathbf{S}$. The stable vector $\mathbf{S}$ is called *symmetric stable* if it satisfies the relation

$$\mathbf{S} \stackrel{d}{=} -\mathbf{S}.$$

The characteristic function of a symmetric stable vector can be written in the form

$$\langle e^{i\mathbf{k} \cdot \mathbf{S}} \rangle \equiv \tilde{g}_d(\mathbf{k}; \alpha, \Gamma(\cdot)) = \exp \left\{ - \int_{U_d} |\mathbf{k} \cdot \mathbf{u}|^\alpha \Gamma(d\mathbf{u}) \right\},$$

where $\alpha \in (0, 2]$ and $\Gamma(d\mathbf{u})$, $d\mathbf{u} \subset U_d = \{\mathbf{x}/|\mathbf{x}| : \mathbf{x} \in \mathbb{R}^d\}$, is the *spectral measure* (see (Samorodnitsky and Taqqu, 1994)). It is a symmetric finite measure given on the unite d -dimensional sphere U_d :

$$\Gamma(d\mathbf{u}) = \Gamma(-d\mathbf{u}).$$

We will designate the generated by this measure random vector by

$$\mathbf{S}(\alpha; \Gamma(\cdot)) = \{S_1(\alpha; \Gamma(\cdot)), \dots, S_d(\alpha; \Gamma(\cdot))\}.$$

Understanding the sense of the spectral measure is revealed by virtue of the following statement proved by Araujo and Giné (1980):

$$\begin{aligned} & \lim_{r \rightarrow \infty} \mathbf{P}(\mathbf{S}/S \in d\Omega \mid |\mathbf{S}(\alpha; \Gamma(\cdot))| > r) \\ & \equiv \lim_{r \rightarrow \infty} \frac{\mathbf{P}(\mathbf{S}/S \in d\Omega \mid |\mathbf{S}(\alpha; \Gamma(\cdot))| > r)}{\mathbf{P}(|\mathbf{S}(\alpha; \Gamma(\cdot))| > r)} \sim \frac{M(d\Omega)}{\Gamma(U_d)}, \quad r \rightarrow \infty. \end{aligned}$$

In other words, the mass that M assigns to the cone $d\Omega$ determines the tail behavior of $\mathbf{S}$ in the direction $d\Omega$.

To make the new objects be more clear, we take a look at the one-dimensional case. In this case U_d consists of only two points $+1$ and -1 with equal measures $\Gamma\{1\} = \Gamma\{-1\} = 1/2$. Writing $\Gamma_1^\pm(du_1)$ for this measure, we obtain a known expression

$$\tilde{g}_1(k_1; \alpha, \Gamma_1^\pm(\cdot)) = \exp(-|k_1|^\alpha).$$

In a multidimensional case, two types of the symmetric stable distributions are of special interest because of their simplicity: isotropic distributions, the spectral measure of which is uniformly distributed over the sphere, $\Gamma(d\mathbf{u}) = \Gamma^0(d\mathbf{u}) \equiv \text{const } d\mathbf{u}$, and distributions of vectors with independent components, when the measure $\Gamma(d\mathbf{u})$ is concentrated only on the intersection of the sphere with the coordinate axis,

¹ Strictly stable.

$$\Gamma(d\mathbf{u}) = \Gamma^1(du_1, du_2, \dots, du_d) \equiv \Gamma_1^\pm(du_1) + \Gamma_2^\pm(du_2) + \dots + \Gamma_d^\pm(du_d)$$

The corresponding characteristic functions are

$$\tilde{g}_d(\mathbf{k}; \alpha, \Gamma^0(\cdot)) = e^{-\gamma_0 |\mathbf{k}|^\alpha}$$

and

$$\tilde{g}_d(\mathbf{k}; \alpha, \Gamma^1(\cdot)) = \prod_{j=1}^d e^{-\gamma_j |k_j|^\alpha},$$

where $\gamma_0, \gamma_1, \dots, \gamma_d$ are appropriate positive constants (Samorodnitsky and Taqqu, 1994).

Observe that only one of isotropic distributions describes a vector with independent components. This is the normal distribution ($\alpha = 2$).

The projection of a d -dimensional symmetric stable vector on an n -dimensional subspace is again symmetric stable. In particular, each its component is a symmetric stable variable. The converse is not true: not any set of d symmetric stable variables form a symmetric stable d -vector.

3.3.2 Sub-Gaussian random vectors

Let $\mathbf{G} = \{G_1, G_2, \dots, G_d\}$ be a d -dimensional zero-mean Gaussian vector with covariance matrix $C_{ij} = \langle G_i G_j \rangle$, and $S_+(\alpha/2)$ be the subordinator of order $\alpha/2$. Then, the random vector

$$\mathbf{X} = \sqrt{S_+(\alpha/2)} \mathbf{G}$$

called *sub-Gaussian vector* has characteristic function

$$\langle \exp \{i\mathbf{k} \cdot \mathbf{X}\} \rangle = \exp \left\{ -(1/2) \left| \sum_{i,j=1}^d C_{ij} k_i k_j \right|^{\alpha/2} \right\}.$$

The statement can be easily proved by conditioning on $S_+(\alpha/2)$:

$$\begin{aligned} \langle \exp \{i\mathbf{k} \cdot \mathbf{X}\} \rangle &= \langle \langle \exp \{i\mathbf{k} \cdot \mathbf{X}\} \rangle_{S_+(\alpha/2)} \rangle \\ &= \left\langle \exp \left\{ i \sqrt{S_+(\alpha/2)} \mathbf{k} \cdot \mathbf{G} \right\}_{S_+(\alpha/2)} \right\rangle \\ &= \exp \left\{ -(1/2) \left| \sum_{i,j=1}^d C_{ij} k_i k_j \right|^{\alpha/2} \right\}. \end{aligned}$$

The Fourier inverse transformation leads to multivariate probability densities.

3.3.3 Isotropic stable distributions as limit distributions

The family of d -dimensional stable laws can be obtained as limit distributions of normalized sums of independent random vectors. We demonstrate this by way of example of isotropic stable distribution making use of the d -dimensional analogue of the d -dimensional Zipf–Pareto symmetric distribution:

$$\mathbf{P}\{|\mathbf{X}| > r\} = \begin{cases} Ar^{-\alpha}, & r > \varepsilon; \\ 1, & r < \varepsilon. \end{cases}$$

The characteristic function of this random vector is

$$\varphi_d(\mathbf{k}; \alpha) = 2^{d/2-1} \alpha A \Gamma(d/2) |\mathbf{k}|^\alpha \int_{\varepsilon|\mathbf{k}|}^{\infty} s^{-\alpha-d/2} J_{d/2-1}(s) ds.$$

Integrating by parts, in view of the relation

$$\frac{d}{ds} [s^{-(d/2-1)} J_{d/2-1}(s)] = -s^{-d/2+1} J_{d/2}(s),$$

we obtain

$$\begin{aligned} \varphi_d(\mathbf{k}; \alpha) &= 2^{d/2-1} A \Gamma(d/2) |\mathbf{k}|^\alpha \left\{ (\varepsilon|\mathbf{k}|)^{-d/2-\alpha+1} J_{d/2-1}(\varepsilon|\mathbf{k}|) - \int_{\varepsilon|\mathbf{k}|}^{\infty} s^{-d/2-\alpha+1} J_{d/2}(s) ds \right\}. \end{aligned}$$

As $\mathbf{k} \rightarrow 0$,

$$\{\dots\} \sim (\varepsilon|\mathbf{k}|)^{-d/2-\alpha+1} [(\varepsilon|\mathbf{k}|/2)^{d/2-1} / \Gamma(d/2) - \dots] - \int_0^{\infty} s^{-d/2-\alpha+1} J_{d/2}(s) ds,$$

and therefore

$$1 - \varphi_d(\mathbf{k}; \alpha) \sim \frac{A \Gamma(d/2) \Gamma(1 - \alpha/2)}{\Gamma((d + \alpha)/2)} (|\mathbf{k}|/2)^\alpha.$$

Now set

$$\mathbf{Z}_n = (\mathbf{X}_1 + \dots + \mathbf{X}_n) / b_n$$

and

$$b_n = b_1 n^{1/\alpha}.$$

As $n \rightarrow \infty$, the characteristic function of the normalized sum

$$\begin{aligned} \langle \exp \{i\mathbf{k} \cdot \mathbf{Z}_n\} \rangle &= \varphi_d^n(\mathbf{k}/b_n; \alpha) \sim \left\{ 1 - \frac{A \Gamma(d/2) \Gamma(1 - \alpha/2)}{\Gamma((d + \alpha)/2)} |\mathbf{k}/(2b_n)|^\alpha \right\}^n \\ &\rightarrow \exp(-|\mathbf{k}|^\alpha) = \tilde{g}_d(\mathbf{k}; \alpha) \end{aligned}$$

with

$$b_1 = \frac{1}{2} \left[\frac{A\Gamma(d/2)\Gamma(1-\alpha/2)}{\Gamma((d+\alpha)/2)} \right]^{1/\alpha}.$$

3.3.4 Isotropic stable densities

The multivariate isotropic stable densities are expressed through the inverse Fourier transformation of characteristic functions

$$g_d(\mathbf{x}; \alpha) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i\mathbf{k} \cdot \mathbf{x}} \tilde{g}(\mathbf{k}; \alpha) d\mathbf{k} = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i\mathbf{k} \cdot \mathbf{x} - |\mathbf{k}|^\alpha} d\mathbf{k},$$

which can be reduced to a one-fold integral

$$g_d(\mathbf{x}; \alpha) = (2\pi)^{-d/2} r^{1-d/2} \int_0^\infty e^{-k^\alpha} J_{d/2-1}(rk) k^{d/2} dk, \quad r = |\mathbf{x}|$$

(here and in what follows by J_ν we denote the Bessel function of order ν),

In two-dimensional case,

$$g_2(\mathbf{x}; \alpha) = \frac{1}{2\pi} \int_0^\infty e^{-k^\alpha} J_0(kr) k dk.$$

Setting here $\alpha = 1$ and $\alpha = 2$, we find the bivariate Cauchy and Gaussian distribution densities, respectively:

$$g_2(\mathbf{x}; 1) = \frac{1}{2\pi(1+r^2)^{3/2}},$$

$$g_2(\mathbf{x}; 2) = \frac{1}{4\pi} e^{-r^2/4}.$$

These functions are plotted in Figs. 3.4 and 3.5.

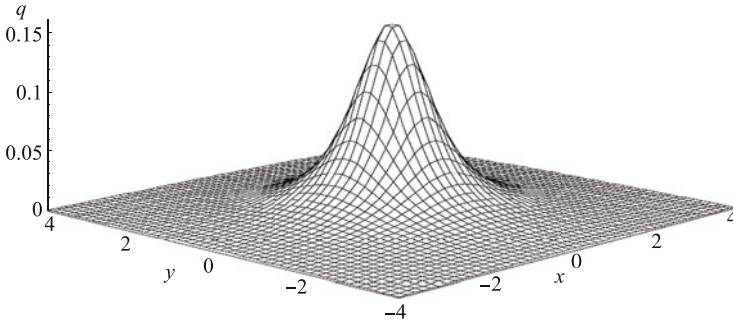


Fig. 3.4 Bivariate Cauchy distribution density $g_2(\mathbf{x}; 1)$.

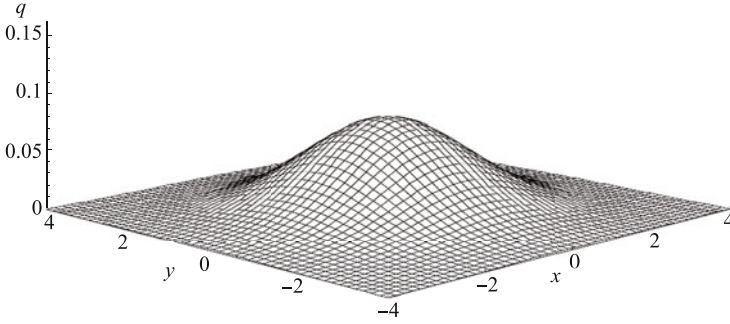


Fig. 3.5 Bivariate Gauss distribution density $g_2(\mathbf{x}; 2)$.

In case $d = 3$,

$$g_3(\mathbf{x}; \alpha) = \frac{1}{2\pi^2 r} \int_0^\infty e^{-k\alpha} \sin(kr) k dk.$$

For $\alpha = 1$ and $\alpha = 2$, we obtain the trivariate Cauchy and Gauss distributions, respectively:

$$g_3(\mathbf{x}; 1) = \frac{1}{\pi^2(1+r^2)^2},$$

$$g_3(\mathbf{x}; 2) = \frac{1}{(4\pi)^{3/2}} e^{-r^2/4}.$$

Expanding the exponential or the Bessel function into a series, we obtain two expansions of the isotropic stable density of an arbitrary dimension:

$$g_d(\mathbf{x}; \alpha) = \frac{1}{\pi(r\sqrt{\pi})^d} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!} \Gamma((n\alpha+d)/2) \Gamma(n\alpha/2+1) \sin(\alpha n\pi/2) (r/2)^{-n\alpha}$$

and

$$g_d(\mathbf{x}; \alpha) = \frac{2}{\alpha(2\sqrt{\pi})^d} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{\Gamma((2n+d)/\alpha)}{\Gamma(n+d/2)} (r/2)^{2n}.$$

As in the univariate case, the first series is convergent for $\alpha < 1$ and asymptotical for $\alpha \geq 1$, whereas the second converges for $\alpha \geq 1$ and is asymptotic for $\alpha < 1$.

Multiplying the Bessel function representation of $g(\mathbf{x}; \alpha)$ by $|\mathbf{x}|^s d|\mathbf{x}|$ and integrating along the semiaxis, we obtain the Mellin transform of the density:

$$\bar{g}_d(s; \alpha) \equiv \int_0^\infty g_d(\mathbf{x}; \alpha) |\mathbf{x}|^s d|\mathbf{x}| = \frac{2^{1+s}}{\alpha(4\pi)^{d/2}} \frac{\Gamma((d-s-1)/\alpha) \Gamma((1+s)/2)}{\Gamma((d-s-1)/2)}.$$

This formula allows us to represent the absolute moment of isotropic stable vector $\mathbf{S}(\alpha)$ in an explicit form:

$$\langle |\mathbf{S}(\alpha)|^s \rangle \equiv \int_{\mathbb{R}^d} g_d(\mathbf{x}; \alpha) |\mathbf{x}|^s d\mathbf{x} = 2^s \frac{\Gamma(1-s/\alpha) \Gamma((s+d)/2)}{\Gamma(1-s/2) \Gamma(d/2)}.$$

The obtained expression can be regarded as an analytical function in the s plane excluding the points of the form $s = k\alpha$ and $s = -d - k + 1$ ($k = 1, 2, \dots$) where this function possesses simple poles. Hence it follows, in particular, that $\langle |\mathbf{S}|^s \rangle$ admits the Taylor expansion in a power series of s , in the circle $|s| < \min(d, \alpha)$.

A more detailed discussion of multivariate symmetric distributions called also the *Lévy-Feldheim distributions* can be found in (Nolan, 1998; Abdul-Hamid and Nolan, 1998; Uchaikin and Zolotarev, 1999).

3.3.5 Lévy-Feldheim motion

The natural generalization of the one-dimensional Lévy motion to d -dimensional Lévy-Feldheim (LF) motion is realized by the equation

$$\frac{\partial p(\mathbf{k}, t)}{\partial t} = -CL(\mathbf{k}; \alpha, \Gamma_d(\cdot))p(\mathbf{k}, t) + \delta(t), \quad \mathbf{k} \in \mathbb{R}^d$$

with

$$L(\mathbf{k}; \alpha, \Gamma_d(\cdot)) = \int_{U_d} |\mathbf{k} \cdot \mathbf{u}|^\alpha \Gamma_d(d\mathbf{u}).$$

Its solution reads

$$p(\mathbf{k}, t) = \exp\{-CL(\mathbf{k}; \alpha, \Gamma_d(\cdot))t\} = g_d((Ct)^{1/\alpha} \mathbf{k}; \alpha, \Gamma_d(\cdot)),$$

Fourier inversion leads to the equation

$$\frac{\partial p(\mathbf{x}, t)}{\partial t} = -CL(\alpha, \Gamma_d(\cdot))p(\mathbf{x}, t) + \delta(\mathbf{x})\delta(t),$$

where

$$L(\alpha, \Gamma_d(\cdot))p(\mathbf{x}, t) = \mathcal{F}^{-1}\{L(\mathbf{k}; \alpha, \Gamma_d(\cdot))p(\mathbf{k}, t)\}(\mathbf{x}) = \mathcal{F}^{-1}\{L(\mathbf{k}; \alpha, \Gamma_d(\cdot))\}\tilde{p}(\mathbf{x}).$$

Its solution is the d -dimensional Lévy-Feldheim density:

$$g_d(\mathbf{x}; \alpha, \Gamma_d(\cdot)) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \exp\left\{-i\mathbf{k} \cdot \mathbf{x} - \int_{U_d} |\mathbf{k} \cdot \mathbf{u}|^\alpha \Gamma_d(d\mathbf{u})\right\} d\mathbf{k}, \quad 0 < \alpha \leq 2.$$

This equation describes the Markovian process generalizing multidimensional Brownian motion ($\alpha = 2$) to arbitrary characteristic exponents $\alpha \in (0, 2]$ and spectral measures Γ_d . When the spectral measure is uniformly distributed over the unit sphere U_d , $\Gamma_d(d\mathbf{u}) = \Gamma_d^0(d\mathbf{u}) = \gamma_d^0 d\mathbf{u}$, we observe a d -dimensional isotropic LF motion pdf of which depends only on $|\mathbf{x}|$.

If $\alpha = 2$, different components of isotropically distributed stable random vector are independent of each other. However, this property takes place only in this case. When $\alpha < 2$, the components are not independent and vice versa, if the components are independent, then the distribution is not isotropic. In the last case, the spectral measure is concentrated at points of intersection of the sphere U_d with coordinate

axis, $\Gamma_d = \Gamma_d^1 = \Gamma_1^{(1)} + \Gamma_1^{(2)} + \dots + \Gamma_1^{(d)}$:

$$g_d(x_1, x_2, \dots, x_d; \alpha, \Gamma_d^1) = g_1(x_1; \alpha, \Gamma_1^{(1)}) g_1(x_2; \alpha, \Gamma_1^{(2)}) \dots g_1(x_d; \alpha, \Gamma_1^{(d)}).$$

These distributions can describe diffusion process in such anisotropic structures as crystals and others.

3.4 Fractional Brownian motion

3.4.1 Differential Brownian motion process

As shown above, the correlations of Bm coordinates at an arbitrary pair of times t_1 , t_2 are described by the covariance function

$$\text{Cov}(B(t_1), B(t_2)) = \langle B(t_1)B(t_2) \rangle = [\sigma_0^2/2](|t_1| + |t_2| - |t_1 - t_2|).$$

Consider the *differential Bm process* (dBm), i.e., the process of Bm increments

$$dB(t) \equiv B(t + dt) - B(t), \quad dt = \text{const.}$$

Evidently,

$$dB(t) \stackrel{d}{=} B(dt),$$

and therefore,

$$\langle dB(t) \rangle = 0, \quad \sigma_{dB}^2 = \sigma_0^2 dt.$$

Autocorrelations in $dB(t)$ are described by the covariance function $\text{Cov}(dB(t_1), dB(t_2))$ which can easily be calculated from the correspondent expression for Bm:

$$\begin{aligned} \text{Cov}(dB(t_1), dB(t_2)) &= \langle dB(t_1)dB(t_2) \rangle = \frac{\partial^2 \langle B(t_1)B(t_2) \rangle}{\partial t_1 \partial t_2} dt_1 dt_2 \\ &= \frac{\sigma_0^2}{2} \frac{\partial^2 (|t_1| + |t_2| - |t_1 - t_2|)}{\partial t_1 \partial t_2} dt_1 dt_2 \\ &= \sigma_0^2 \frac{\partial 1(t_1 - t_2)}{\partial t_1} dt_1 dt_2 = \sigma_0^2 \delta(t_1 - t_2) dt_1 dt_2. \end{aligned}$$

The differential Bm process $dB(t)$ is an example of *stochastic differentials* $dX(t)$. Many authors prefer to write

$$dB(t) = \xi(t)dt$$

or even

$$\frac{dB(t)}{dt} = \xi(t), \quad (3.28)$$

and call equations of such kind *stochastic equations*, and the “functions” $\xi(t)$ *random noises*. In the case $B(t)$ represents the Brownian motion, the noise $\xi(t)$ is called the *white noise*. We shall use for $\xi(t)$ the notation $B^{(1)}(t)$.

As follows from above, the dBm process possesses the following properties.

1) Its mean value is zero:

$$\langle B^{(1)}(t) \rangle = 0.$$

2) It is delta-correlated:

$$\text{Cov}(B^{(1)}(t_1), B^{(1)}(t_2)) = \langle B^{(1)}(t_1)B^{(1)}(t_2) \rangle = \sigma_0^2 \delta(t_1 - t_2).$$

3) The white noise is a stationary stochastic process.

4) Its *stochastic integral*

$$B(t) = \int_0^t dB(t') \equiv \lim_{\max \Delta t \rightarrow 0} \sum_n \Delta B(t_i) = \int_0^t B^{(1)}(t') dt'$$

is a Gaussian random variable:

$$\int_0^t B^{(1)}(t') dt' \stackrel{d}{=} G(0, \sigma_0^2 t).$$

The latter property can be generalized to integration of any arbitrary integrable function, namely the integral

$$I = \int_a^b f(t) B^{(1)}(t) dt$$

is a *Gaussian random variable* with the mean

$$\langle I \rangle = \left\langle \int_a^b f(t) B^{(1)}(t) dt \right\rangle = \int_a^b f(t) \langle B^{(1)}(t) \rangle dt = 0$$

and the variance

$$\begin{aligned} \sigma_I^2 &= \left\langle \left[\int_a^b f(t) B^{(1)}(t) dt \right]^2 \right\rangle \\ &= \int_a^b dt_1 \int_a^b dt_2 f(t_1) f(t_2) \langle B^{(1)}(t_1) B^{(1)}(t_2) \rangle = \sigma_0^2 \int_a^b f^2(t) dt. \end{aligned}$$

3.4.2 Integral Brownian motion process

A stochastic process

$$B^{(-1)}(t) = \int_0^t B(t') dt' \tag{3.29}$$

is called the *integral Bm (iBm)*. The iBm process is also a Gaussian process. One can easily verify it by representing the integral as a limit of approximation sums

$$S_n(t) = \sum_{j=1}^n B(t_j) \Delta t_j = \sum_{j=1}^n \left[\sum_{k=1}^j \Delta B(t_k) \right] \Delta t_j$$

$$= \sum_{k=1}^n \left[\sum_{j=k}^n \Delta t_j \right] \Delta B(t_k) = \sum_{k=1}^n (t_n - t_{k-1}) \Delta B(t_k) \quad (3.30)$$

and taking into account that any set of linear superpositions of independent normally distributed random variables $\Delta B(t_k) \equiv B(t_k) - B(t_{k-1})$ is jointly normal. At the limit, we have

$$\int_0^t B(t') dt' = \lim_{\max \Delta t_k \rightarrow 0} \sum_{k=1}^n (t_n - t_{k-1}) \Delta B(t_k) \equiv \int_0^t (t - t') dB(t').$$

Since $\{B^{(-1)}, t \geq 0\}$ is Gaussian, its distribution is completely determined by its mean value and covariance function. They are easily computed and have the form

$$\langle B^{(-1)}(t) \rangle = \left\langle \int_0^t B(t') dt' \right\rangle = \int_0^t \langle B(t') \rangle dt' = 0,$$

for $t_1 < t_2$,

$$\begin{aligned} \text{Cov}[B^{(-1)}(t_1), B^{(-1)}(t_2)] &= \langle B^{(-1)}(t_1) B^{(-1)}(t_2) \rangle = \int_0^{t_1} dt' \int_0^{t_2} dt'' \langle B(t') B(t'') \rangle \\ &= \int_0^{t_1} dt' \int_0^{t_2} dt'' \sigma_0^2 \min\{t', t''\} = \sigma_0^2 t_1^2 \left(\frac{t_2}{2} - \frac{t_1}{6} \right). \end{aligned}$$

Note that the process $\{B^{(-1)}(t), t \geq 0\}$ is not a Markov process, however, the vector process $\{\{B^{(-1)}(t), B(t)\}, t \geq 0\}$ is again a zero-mean Markov process of Gaussian type with components having the covariance

$$\text{Cov}\left(B^{(-1)}(t), B(t)\right) = \sigma_0^2 t^2 / 2.$$

The concepts of stochastic integrals and differentials are generalized to operating with arbitrary (in some sense) random functions $X(t)$, $Y(t)$, $Z(t)$, $W(t)$:

$$I = \int_a^b Y(t) dX(t),$$

$$dW(t) = X(t)dt + Y(t)dZ(t).$$

The simplest (after the Bm) example of such an equation is

$$\frac{dX(t)}{dt} = -\mu X(t) + \xi(t).$$

Interpreting X as the velocity of a Brownian particle, we can recognize in $-\mu X$ the Stokes viscous force. The solution of the equation under condition $X(0) = x_0$

$$X(t) = x_0 e^{-\mu t} + \int_0^t e^{-\mu(t-t')} \xi(t') dt', \quad t \geq 0$$

is the Gaussian process with the mean

$$\langle X(t) \rangle = x_0 e^{-\mu t}$$

and the variance

$$\text{Var } X(t) = \left\langle \left[\int_0^t e^{-\mu(t-t')} \xi(t') dt' \right]^2 \right\rangle = \sigma_0^2 \int_0^t e^{-2\mu(t-t')} dt' = (1 - e^{-2\mu t}) \frac{\sigma_0^2}{2\mu}.$$

At each t , $X(t)$ has the normal distribution. In the limit $t \rightarrow \infty$ it takes on the sense of an *equilibrium distribution*

$$f_{\text{eq}}(x) = \frac{\sqrt{\mu}}{\sqrt{\pi} \sigma_0} e^{-\mu x^2 / \sigma_0^2}, \quad -\infty < x < \infty.$$

A three-dimensional analog of the distribution is known in statistical physics as the *Maxwell distribution*.

If we define a process by the integral

$$X(t) = \int_{-\infty}^t e^{-\mu(t-t')} \xi(t') dt', \quad -\infty < t < \infty,$$

which is remove the initial moment to $-\infty$, we obtain the *Ornstein-Uhlenbeck process*. Note that for any fixed $t_1 < t$

$$X(t) - e^{-\mu(t-t_1)} X(t_1) = \int_{t_1}^t e^{-\mu(t-t')} \xi(t') dt'$$

is independent of $X(t'_1)$, $t'_1 < t_1$. This implies that the Ornstein-Uhlenbeck process is, in fact, a Markov process, moreover it is the only stationary Gaussian process possessing the Markovian property. The covariance function of the process is easily computed:

$$\begin{aligned} \text{Cov}(X(t_1), X(t_2)) &= \langle X(t_1) X(t_2) \rangle = \int_{-\infty}^{t_2} dt'_2 \int_{-\infty}^{t_1} dt'_1 e^{-\mu(t_2-t'_2) - \mu(t_1-t'_1)} \langle \xi(t'_1) \xi(t'_2) \rangle \\ &= \sigma_0^2 \int_{-\infty}^{t_1} e^{-\mu(t_1+t_2-2t')} dt' = \frac{\sigma_0^2}{2\mu} e^{-\mu(t_2-t_1)}, \quad t_2 > t_1. \end{aligned}$$

Typical samples of the processes are shown in Fig. 3.6.

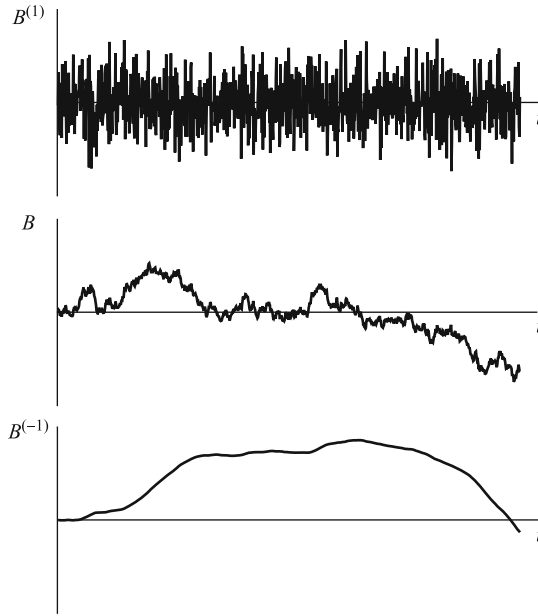


Fig. 3.6 Samples of three stochastic processes $B^{(1)}$, B , and $B^{(-1)}$.

3.4.3 Fractional Brownian motion

Using properties of power functions $\Phi_\mu(t)$ (Sect. 2.1.2), one can represent three processes considered above as the Laplace convolutions

$$\begin{aligned} B^{(-1)}(t) &= \int_0^t dt' \int_0^{t'} \xi(t'') dt'' = \Phi_2 \star \xi(t), \\ B(t) &= \int_0^t \xi(t') dt' = \Phi_{-1} \star \xi(t), \\ B^{(1)}(t) &= \xi(t) = \Phi_0 \star \xi(t). \end{aligned}$$

These formulas provoke us to introduce a general kind of *fractional Brownian motion* (fBm)

$$B^{(\nu)}(t) = \Phi_{1-\nu} \star \xi(t) = {}^v_0 D_t B(t) = \frac{1}{\Gamma(1-\nu)} \int_0^t (t-t')^{-\nu} dB(t'),$$

where ν can now take noninteger values as well.

It is easy to see that fBm is a selfsimilar Gaussian process with the Hurst exponent

$$H = 1/2 - \nu,$$

and the exponent is usually used in notation of fBm:

$$B_H(t) \equiv B^{(1/2-H)}(t) = \frac{1}{\Gamma(H+1/2)} \int_0^t (t-t')^{H-1/2} dB(t').$$

Though the process is selfsimilar, its increments are stationary only when $H = 1/2$ and it becomes the ordinary Bm:

$$B_{1/2}(t) = \int_0^t dB(t') = B(t).$$

Mandelbrot and Van Ness (1968) gave the now widely accepted version of fBm using a modified fractional integral of Weyl type

$$B_H(t) = \frac{1}{\Gamma(H+1/2)} \left\{ \int_{-\infty}^0 \left[(t-t')^{H-1/2} - (-t')^{H-1/2} \right] dB(t') \right. \\ \left. + \int_0^t (t-t')^{H-1/2} dB(t') \right\},$$

where for negative t the notation $\int_0^t$ should be interpreted as $-\int_t^0$. Another shorter form of the expression is

$$B_H(t) = \frac{1}{\Gamma(H+1/2)} \int_{-\infty}^{\infty} \left[(t-t')_+^{H-1/2} - (-t')_+^{H-1/2} \right] dB(t').$$

By direct computation one can find the following representation for the autocovariance function:

$$\begin{aligned} \text{Cov}(B_H(t_1), B_H(t_2)) &= \langle B_H(t_1) B_H(t_2) \rangle = \frac{1}{[\Gamma(H+1/2)]^2} \\ &\times \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[(t_1-t'_1)_+^{H-1/2} - (-t'_1)_+^{H-1/2} \right] \\ &\left[(t_2-t'_2)_+^{H-1/2} - (-t'_2)_+^{H-1/2} \right] \langle dB(t_1) dB(t_2) \rangle \\ &= \frac{\sigma_H^2}{2} [|t_1|^{2H} + |t_2|^{2H} - |t_1 - t_2|^{2H}], \end{aligned}$$

where

$$\sigma_H^2 = \langle B_H^2(1) \rangle = \frac{\Gamma(1-2H) \cos(H\pi)}{H\pi} \sigma_0^2.$$

By definition, the Hurst exponent is a selfsimilarity index which should be positive. From the other side, if $H < 1$, the fBm is the only selfsimilar Gaussian process with stationary increments (Samorodnitsky and Taqqu, 1994). For these reasons, the Hurst exponent values are bounded by the region $0 < H \leq 1$ and the fBm is defined as follows:

A Gaussian H -ss process $\{X(t)\}$ with $\langle X(t) \rangle = 0$, $0 < H \leq 1$ and stationary increments is called *fractional Brownian motion* (fBm), and *standard fBm* if, in addition, $\sigma_0 = 1$.

When $H = 1/2$, fBm becomes the ordinary Bm: $\{B_{1/2}(t)\} = \{B(t)\}$. The case $1/2 < H < 1$ relates to *persistent* or *fractional superdiffusion* (enhanced diffusion), the process with $H < 1/2$ describes *antipersistent* or *fractional subdiffusion*. Note that all these processes are characterized by Gaussian one-dimensional distribution:

$$p(x, t) = \frac{1}{2\sqrt{\pi}\sigma t^H} \exp\left\{-\frac{x^2}{4\sigma^2 t^{2H}}\right\}.$$

3.4.4 Fractional Gaussian noises

The sequence of stationary increments of fBm $Y_j = B_H(j+1) - B_H(j)$, $\dots, -1, 0, 1, \dots$ forms the *fractional Gaussian noise* (fGn), or the *standard fGn* if, in addition, $\langle Y_j^2 \rangle = 1$. Direct calculations yield the following integral representation of the fGn:

$$Y_j = \frac{\sigma_0}{\Gamma(H+1/2)} \int_{-\infty}^{j+1} \left[(j+1-t)_+^{H-1/2} - (j-t)_+^{H-1/2} \right] dB(t).$$

From the foregoing, some remarkable properties of the fGn process follow.

1) The fGn is a stationary Gaussian sequence with mean zero and variance $\langle Y_j^2 \rangle = \langle B_H^2(1) \rangle = \sigma_0^2$.

2) The covariance function of fGn $R(j) = \langle Y_0 Y_j \rangle$ is

$$R(j) = \frac{\sigma_0^2}{2} [|j+1|^{2H} - 2|j|^{2H} + |j-1|^{2H}].$$

If $H < 1/2$, the Y_j are negatively correlated; in case of $H > 1/2$, they are positively correlated. When $H = 1/2$, we have a sequence of independent random variables.

3) The fGn spectral density $S(f)$, $-\pi < f < \pi$ connected to $R(j)$ via relations

$$S(f) = \frac{1}{2\pi} \sum_{j=-\infty}^{\infty} e^{-ifj} R(j), \quad R(j) = \int_{-\pi}^{\pi} e^{ifj} S(f) df,$$

is of the form (Samorodnitsky and Taqqu, 1994)

$$S(f) = \sigma_0^2 C^2(H) |e^{if} - 1|^2 \sum_{j=-\infty}^{\infty} |f + 2\pi j|^{-2H-1}, \quad -\pi \leq f \leq \pi,$$

where

$$C^2(H) = \frac{H(1-2H)}{2\Gamma(2-2H)\cos(H\pi)}$$

is a normalizing constant. Notice that $C^2(1/2) = (2\pi)^{-1}$.

It is not hard to conclude that both functions $R(j)$ and $S(f)$ have power asymptotics in corresponding regions (see (Samorodnitsky and Taqqu, 1994)):

$$R(j) \sim \sigma_0^2 H(2H-1)j^{2H-1}, \quad j \rightarrow \infty$$

and

$$S(f) \sim \sigma_0^2 C^2(H)|f|^{1-2H}, \quad f \rightarrow 0.$$

When $H = 1/2$, we deal with the white noise, the case $H = 1$ reveals the $1/f$ -flicker noise.

Notice that $R(j)$ goes to 0 for all values of $H \in (0, 1)$, but when $H > 1/2$, it goes so slowly that the sum $\sum_{j=-\infty}^{\infty} R(j)$ diverges. Such a behavior of Y_j is interpreted as *long-range dependence*. The case $H \neq 1/2$ provides a counterexample to the central limit theorem. Although the variance of Y_j is finite, a non-degenerated limit distribution of

$$Z_n = \frac{1}{\sqrt{n}} \sum_{j=1}^n Y_j, \quad n \rightarrow \infty,$$

does not exist. To get a non-trivial limit one must take normalized factors n^{-H} instead of $n^{-1/2}$, because Y_j are dependent.

In terms of hereditary concept, one can say that the H parameter regulates the presence or absence of memory: long-memory for $1/2 < H < 1$, no memory at $H = 1/2$, and short-time memory if $0 < H < 1/2$.

3.4.5 Barnes-Allan model

Barnes and Allan (1966) have developed another model able to characterize the $1/f$ noise (see also (Magre and Guglielmi, 1997)). They consider the filtering of a white Gaussian noise by the system described by its impulse response

$$h(t) = \Phi_{\mu+1/2}(t).$$

The output is

$$Z(t) = h \star \xi(t) = \frac{1}{\Gamma(H+1/2)} \int_0^t (t-t')^{H-1/2} dB(t').$$

This model possesses the following properties.

1) Selfsimilarity. For all $a > 0$, and for all t , we observe

$$Z(at) \stackrel{d}{=} a^H Z(t).$$

2) Non-stationarity. The autocorrelation function is

$$\langle Z(t_2)Z(t_1) \rangle = \frac{1}{[\Gamma(H+1/2)]^2} \left[|t_1 - t_2|^{2H} \int_0^{t_2/t_1} u^{H-1/2} (1-u)^{-2H-1} du \right].$$

3.4.6 Fractional Lévy motion

Further generalization of the way of inserting hereditary into selfsimilar processes is based on using stochastic integrals with respect to the random measure

$$dL^{(\alpha)}(t) = L^{(\alpha)}(t+dt) - L^{(\alpha)}(t),$$

which describes the random increment of the Lévy motion process in $(t, t+dt)$ and

$$X(t) = \int_0^t dL^{(\alpha)}(\tau), \quad t > 0,$$

$$X(t+\tau) - X(t) = \int_t^{t+\tau} dL^{(\alpha)}(\tau') \stackrel{d}{=} \tau^{1/\alpha} S^{(\alpha, \beta)}.$$

Samples of the Cauchy process

$$C^{(1)}(t) = \frac{dL^{(1)}(t)}{dt}, \quad C(t) = L^{(1)}(t), \quad \text{and} \quad C^{(-1)} = \int_0^t L^{(1)}(t) dt$$

are shown in Fig. 3.7.

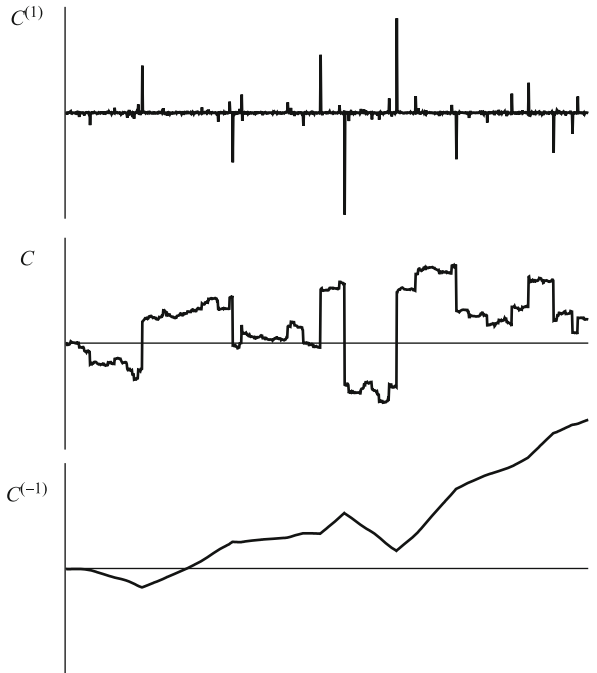


Fig. 3.7 Samples of three stochastic processes $C^{(1)}$, C , and $C^{(-1)}$.

Here, the heredity is introduced using the function $h(t, \tau)$, which determines the contribution of a unite measure at time τ to the state of the process at time t :

$$X(t) = \int_{-\infty}^{\infty} h(t, \tau) dL^{(\alpha)}(\tau).$$

If the function $h(t, \tau)$ is invariant with respect to shift in time,

$$h(t, \tau) = h(t - \tau),$$

such a process is referred to as a *moving-average process* (MA process).

The Ornstein-Uhlenbeck-Lévy process can serve as an example of MA process:

$$X(t) = \int_{-\infty}^t e^{-\lambda(t-\tau)} dL^{(\alpha)}(\tau).$$

Constructed on the same principle, the process

$$X_H^\alpha(t) = \int_{-\infty}^{\infty} (|t - \tau|^{H-1/\alpha} - |\tau|^{H-1/\alpha}) dL^{(\alpha)}(\tau)$$

with $0 < H < 1$ and $H \neq 1/\alpha$, is called the *fractional Lévy motion* (fLm), since it is obtained from the Lévy motion process by fractional-order integration. Note two important properties of the process $\{X_H^\alpha(t)\}$. Firstly, it is selfsimilar with the exponent H , i.e., for any $a > 0$ and $t_1, \dots, t_n$

$$(X_H^\alpha(at_1), \dots, X_H^\alpha(at_n)) \stackrel{d}{=} (a^H X_H^\alpha(t_1), \dots, a^H X_H^\alpha(t_n)).$$

Secondly, its increments are stationary

$$X_H^\alpha(t) - X_H^\alpha(0) \stackrel{d}{=} X_H^\alpha(t + \tau) - X_H^\alpha(\tau).$$

In the particular case $\alpha = 2$, $H = 1/2$, fLm turns to an ordinary Brownian motion. In the case $\alpha = 2$, $H \neq 1/2$, we deal with *fractional Brownian motion*. Its mean value is zero, the variance is

$$\langle [X_2^H(t)]^2 \rangle = t^{2H} \sigma^2, \quad \sigma^2 = \langle [X_2^H(1)]^2 \rangle,$$

and the covariation function is

$$C_2^H(t_1, t_2) = \langle X_2^H(t_1) X_2^H(t_2) \rangle = (|t_1|^{2H} + |t_2|^{2H} - |t_1 - t_2|^{2H}) \sigma^2 / 2.$$

The case of $H = 1/2$ and

$$C_2^{1/2}(t_1, t_2) = \begin{cases} \sigma^2 \min(t_1, t_2), & \text{if } t_1 \text{ and } t_2 \text{ are of the same sign,} \\ 0, & \text{if } t_1 \text{ and } t_2 \text{ are of opposite signs.} \end{cases}$$

corresponds to the ordinary Brownian motion. Since the Bm has stationary increments, the sequence

$$\{Z_j = X_2^H(j+1) - X_2^H(j), \quad j = \dots, -1, 0, 1, \dots\}$$

is stationary its auto-covariance function is

$$R_j = (|j+1|^{2H} - 2|j|^{2H} + |j-1|^{2H})\sigma_0^2/2 \sim \sigma_0^2 H(2H-1)j^{2H-2}, \quad j \rightarrow \infty, \quad H \neq 1/2.$$

3.5 Fractional Poisson motion

3.5.1 Renewal processes

The above scheme of the anomalous diffusion process is based on the selfsimilar generalization of Brownian motion. Historically, it was developed in a different way using asymptotic analysis of jump processes. The groundwork for this approach was laid by Montroll and Weiss (1965), and none of the review articles on anomalous diffusion has avoided making reference to their study. We note here the main milestones on this avenue using the terminology of the renewal theory (Cox, 1967). Being less formal, this way is more ocular and more productive for physical interpretations in various problems.

Let T called the *waiting time* or *interarrival time* be a positive random variable with pdf $q(t)$ and $T_1, T_2, \dots$ be a sequence of its independent copies. The new sequence

$$T(n) = \sum_{j=1}^n T_j, \quad T(0) = 0$$

will be referred to as the *renewal times* or *arrival times*. In physical processes, some transitions from one state of a system to another, collisions of particles, emission or absorbtion of photons, etc., take such a short time that can be considered as instant transitions. The registered transitions of this kind generate in a measuring electric device a correspondent sequence of current pulses of a very short duration. In many cases, they can be considered as zero-duration pulses. We will call these zero-duration phenomena *events* or *jumps*. Let $N(t)$ denote a random number of the events in the interval $(0, t]$. In this case, the difference $N(t_2) - N(t_1)$ means the number of events in the interval $(t_1, t_2]$.

The random process $\{N(t), t \geq 0\}$ is said to be a *counting process* if it satisfies:

- 1) $N(t)$ is integer valued;
- 2) $N(t_1) \leq N(t_2)$ if $t_1 < t_2$.

The function $N(t)$ jump-like increasing at each arrival time is called a *counting function*. Thus, $T_{N(t)}$ denotes the arrival time of the last event before t , while $T_{N(t)+1}$ is the first arrival time after t . In these terms, $N(t)$ can be determined as a largest value of n for which the n th event occurs before or at time t :

$$N(t) = \max\{n : T_n \leq t\}.$$

In other words, the number of events by time t is greater than or equal to n if and only if the n th event occurs before or at time T :

$$N(t) \geq n \iff T_n \leq t.$$

Feller noted that considering renewal processes we deal merely with sums of independent identically distributed random variables, and the only reason for introducing a special term is using such a power analytic tool as the renewal equation. Let us call the mean number of events by time t $\langle N(t) \rangle$ the *renewal function*. It can be represented in the form

$$\langle N(t) \rangle = \sum_{n>0} \mathbf{P}(T(n) < t) = \int_0^t q^{*n}(t') dt', \quad q^{*0}(t) = \delta(t).$$

The renewal function is a non-decreasing, finite-valued, non-negative, and semi-additive function:

$$\langle N(t+s) \rangle \leq \langle N(t) \rangle + \langle N(s) \rangle, \quad t, s \geq 0.$$

It obeys the *renewal equation*

$$\langle N(t) \rangle = \int_0^t [1 + \langle N(t-t') \rangle] q(t') dt'.$$

Its interpretation is very clear: the mean number of events within $(0, t)$ is equal to the contribution of the first event plus the mean number of subsequent events. For the mean frequency of the events

$$f(t) = \frac{d\langle N(t) \rangle}{dt},$$

we obtain from here the similar equation:

$$f(t) = q(t) + \int_0^t f(t-t') q(t') dt'. \quad (3.31)$$

Equation (3.31) has a solution in the form of series of multifold convolutions

$$f(t) = \sum_{n=1}^{\infty} q^{*(n)}(t),$$

n -th of them describes the probability density function of the n th event arrival time:

$$q^{*(n)}(t) dt = \mathbf{P}(T_1 + T_2 + \dots + T_n \in (t, t + dt)).$$

Taking into account that

$$p_n(t) \equiv \mathbf{P}(N(t) = n) = \int_0^t Q(t-t')q^{*(n)}(t')dt', \quad Q(t) = \int_t^\infty q(\tau)d\tau,$$

one can easily derive the equation system for these probabilities

$$p_n(t) = \delta_{n0} \int_t^\infty q(\tau)d\tau + [1 - \delta_{n0}] \int_0^t q(t-\tau)p_{n-1}(\tau)d\tau, \quad n = 0, 1, 2, \dots$$

3.5.2 Selfsimilar renewal processes

Let us try to answer the following question: what form should have transition pdf $q(t)$ for the process $N(t)$ to be ν -ss in medium? In other words, we want to find such $q(t) \equiv \psi_\nu(t)$ that

$$\langle N_\nu(t) \rangle = \mu_\nu t^\nu, \quad f_\nu(t) = \mu_\nu \nu t^{\nu-1}, \quad \mu_\nu > 0, \quad 0 < \nu \leq 1. \quad (3.32)$$

Following B. Mandelbrot (1968), we will call the ensemble of random points on t -axis the *fractal dust* and the pdf $\psi_\nu(x)$ the *fractal dust generator* (fdg). As follows from above, it is linked with the mean fractal dust density $f_\nu(x)$ via equation:

$$\psi_\nu(t) = f_\nu(t) - \int_0^t f_\nu(t-t')\psi_\nu(t')dt'. \quad (3.33)$$

Applying the Laplace transform

$$f(t) \mapsto \widehat{f}(\lambda) = \mathcal{L}\{f(t)\}(\lambda) \equiv \int_0^\infty e^{-\lambda t} f(t) dt$$

yields the expression (Repin and Saichev, 2000)

$$\widehat{\psi}_\nu(\lambda) = \frac{\widehat{f}_\nu(\lambda)}{1 + \widehat{f}_\nu(\lambda)} = \frac{\mu}{\mu + \lambda^\nu}, \quad \mu = \mu_\nu \nu \Gamma(\nu), \quad (3.34)$$

which for $\nu = 1$ coincides with the corresponding expression for the ordinary Poisson process:

$$\mathcal{L}\{\psi_1(t)\}(\lambda) = \frac{\mu}{\mu + \lambda}, \quad \psi_1(t) = \mu e^{-\mu t}.$$

Wang and Wen (2003) used formula (3.34) for introducing a fractional Poisson processes (fPp) and derived the fractional integral equation for this density

$$\psi_\nu(t) + \frac{\mu}{\Gamma(\nu)} \int_0^t [\mu(t-\tau)]^{\nu-1} \psi_\nu(\tau) d\tau = \frac{\mu^\nu}{\Gamma(\nu)} t^{\nu-1}.$$

3.5.3 Three forms of fractal dust generator

The solution of the above equations was represented in two forms (Repin and Saichev, 2000; Wang and Wen, 2003; Wang et al., 2006). The first of them is obtained by performing the backward transformation

$$\psi_v(t) = \mathcal{L}^{-1} \{ \hat{\psi}_v(\lambda) \} (t) = \mathcal{L}^{-1} \left\{ \frac{\mu}{\mu + \lambda^v} \right\} (t)$$

with the use of the geometrical progression formula

$$\lambda^{-v} \frac{1}{1 + a\lambda^{-v}} = \sum_{j=0}^{\infty} (-a)^j \lambda^{-\gamma}, \quad \gamma = jv + v$$

and the relation

$$\mathcal{L}^{-1} \{ \lambda^{-\gamma} \} (t) = \frac{t^{\gamma-1}}{\Gamma(\gamma)}.$$

This leads to the two-parameter Mittag-Leffler function:

$$\psi_v(t) = \mu t^{v-1} \sum_{j=0}^{\infty} \frac{(-\mu t^v)^j}{\Gamma(vj + v)} = \mu t^{v-1} E_{v,v}(-\mu t^v). \quad (3.35)$$

In particular,

$$\psi_{1/2}(t) = \frac{\mu}{\sqrt{\pi t}} - \mu^2 e^{\mu^2 t} \text{Erfc}(\mu \sqrt{t}), \quad (3.36)$$

where $\text{Erfc}(t)$ is the complementary error function: $\text{Erfc}(t) = \frac{2}{\sqrt{\pi}} \int_t^{\infty} e^{-z^2} dz$. By making use of the formula:

$$\int_0^{\infty} e^{-u} E_{v,v}(u^v x) u^{v-1} du = \frac{1}{1-x},$$

one can verify that the density (3.36) really has the Laplace transform (3.34) with $v = 1/2$.

The second form

$$\psi_v(t) = \frac{1}{t} \int_0^{\infty} e^{-x} \phi_v(\mu^v t/x) dx, \quad (3.37)$$

$$\phi_v(\xi) = \frac{\sin(v\pi)}{\pi[\xi^v + \xi^{-v} + 2\cos(v\pi)]}$$

allows us with easy to find asymptotical expressions for short and long times:

$$\psi_v(t) \sim \begin{cases} \frac{\mu}{\Gamma(v)} t^{v-1}, & t \rightarrow 0, \\ \frac{v\mu^{-1}}{\Gamma(1-v)} t^{-v-1}, & t \rightarrow \infty. \end{cases}$$

As shown in (Repin and Saichev, 2000; Jumarie, 2001; Laskin, 2003) by making use of the fractional equations, the transition from $\psi(t) = \mu e^{-\mu t}$ to $\psi_v(t)$, $v < 1$ leads to changes in the process characteristics represented in Table 3.3. Results of numerical calculations of the density $\psi_v(t)$ are shown in Fig. 3.8.

Table 3.3 Properties of FPP compared with those of the Poisson process.

	Poisson process ($v = 1$)	Fractional Poisson Process ($v < 1$)
$P(0, t)$	$e^{-\mu t}$	$E_v(-\mu t^v)$
$\psi(t)$	$\mu e^{-\mu t}$	$\mu t^{v-1} E_{v,v}(-\mu t^v)$
$P(n, t)$	$\frac{(\mu t)^n}{n!} e^{-\mu t}$	$\frac{(\mu t^v)^n}{n!} \sum_{k=0}^{\infty} \frac{(k+n)!}{k!} \frac{(-\mu t^v)^k}{\Gamma(v(k+n)+1)}$
$\langle N(t) \rangle$	μt	$\frac{\mu t^v}{\Gamma(v+1)}$
$\sigma_{N(t)}^2$	μt	$\frac{\mu t^v}{\Gamma(v+1)} \left\{ 1 + \frac{\mu t^v}{\Gamma(v+1)} \left[\frac{vB(v, 1/2)}{2^{2v-1}} - 1 \right] \right\}$

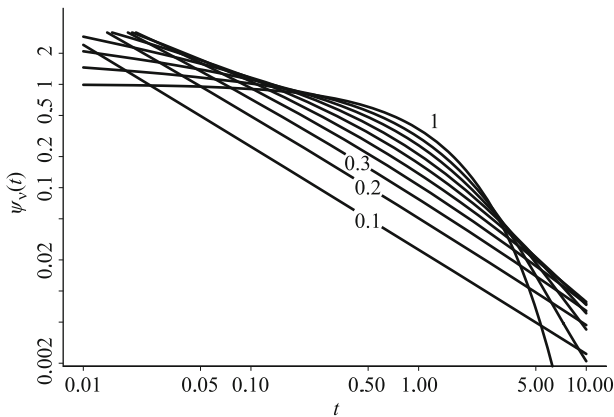


Fig. 3.8 The densities $\psi_v(t)$, $v = 0.1$ (0.1) 1.

The third form is given by the next Lemma proved in (Uchaikin et al., 2008):
Lemma. *The complement cumulative distribution function*

$$P(T > t) = \int_t^{\infty} \psi_v(t') dt'$$

can be represented in the form

$$P(T > t) = E_v(-\mu t^v) = \int_0^{\infty} e^{-\mu t^v / \tau^v} g_+(\tau; v) d\tau, \quad (3.38)$$

where $g_+(\tau; v) \equiv g(\tau; v, 1)$ is the one-sided v -stable density.

3.5.4 The n th arrival time distribution

For the standard Poisson process, the pdf of the n th arrival time is given by Gamma distribution

$$\psi^{*n}(t) = \mu \frac{(\mu t)^{n-1}}{(n-1)!} e^{-\mu t}. \quad (3.39)$$

According to the Central Limit Theorem

$$\Psi^{(n)}(t) \equiv (\sqrt{n}/\mu) \psi^{*n}(n/\mu + t\sqrt{n}/\mu) \Rightarrow \frac{1}{\sqrt{2\pi}} e^{-t^2/2}, \quad n \rightarrow \infty.$$

As numerical calculations show, $\Psi^{(n)}(t)$ practically reaches its limit form already by $n = 10$ (Fig. 3.9).

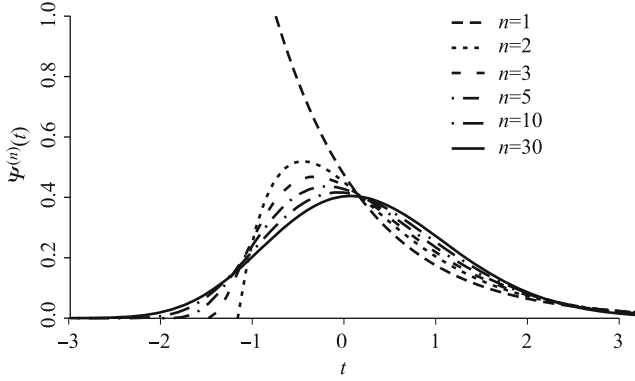


Fig. 3.9 Rescaled arrival time distributions for the standard Poisson process ($v=1$, $n=1, 2, 3, 5, 10$, and 30).

In case of the fPp,

$$\langle T \rangle = \int_0^\infty \psi_v(t) t dt = \infty$$

and the Central Limit Theorem is not applicable. Applying the Generalized Limit Theorem (see for example (Uchaikin & Zolotarev, 1999)), we obtain:

$$\Psi_v^{(n)}(t) \equiv \left(\frac{n}{\mu}\right)^{1/v} \psi_v^{*n}\left(t \left(\frac{n}{\mu}\right)^{1/v}\right) \Rightarrow g^{(v)}(t), \quad n \rightarrow \infty,$$

where

$$\overset{\circ}{\Psi}_v(t) = \psi_v(t)|_{\mu=1} = t^{v-1} E_{v,v}(-t^v).$$

Computing this multiple integrals can be performed by Monte Carlo technique. Taking $\mu = 1$ and observing that $\Psi_v^{(n)}(t)$ is the probability density of the renormalized sum $(T_1 + T_2 + \dots + T_n)/n^{1/v}$ of independent random variable, distributed according to $\psi_v(t)$, we could directly simulate this sum by making use of the algorithm given in Sect. 6.3.4 and construct the corresponding histogram. However, the

left tail of the densities is too steep for this method, and we applied some modification of Monte Carlo method based on the partial analytical averaging of the last term.

By making use of this modification, we computed the distributions $\Psi_v^{(n)}(t)$ for various n and v . An example of these results is represented in Fig. 3.10.

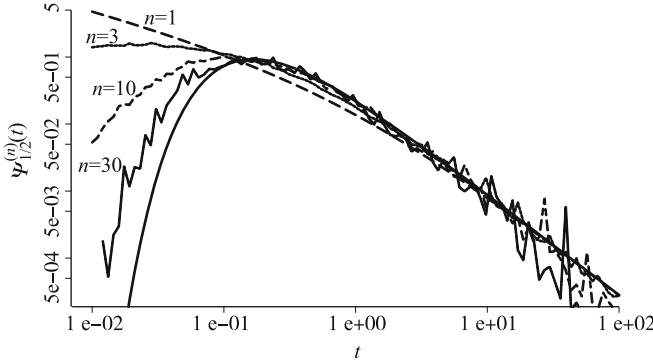


Fig. 3.10 Rescaled arrival time distributions for the fractional Poisson process ($v=1/2$; $n=1, 3, 10$, and 30).

3.5.5 Limit fractional Poisson distributions

In case of the standard Poisson process, the probability distribution for random number $N(t)$ of events follows the Poisson law with $\langle N(t) \rangle = \mu t = \bar{n}$ which approaches to the normal one at large $\bar{n}$. Introducing normalized random variable $Z = N(t)/\bar{n}$ and quasicontinuous variable $z = n/\bar{n}$, one can express the last fact as follows:

$$f(z; \bar{n}) = \bar{n} \frac{\bar{n}^{\bar{n}z}}{\Gamma(\bar{n}z + 1)} e^{-\bar{n}}$$

$$\sim \sqrt{\frac{\bar{n}}{2\pi}} \exp \left\{ -\frac{(z-1)^2}{2/\bar{n}} \right\}$$

as $\bar{n} \rightarrow \infty$. In the limit case $\bar{n} \rightarrow \infty$, the distribution of Z becomes degenerated one:

$$\lim_{\bar{n} \rightarrow \infty} f(z; \bar{n}) = \delta(z - 1).$$

Considering the case of fPp, we pass from the generating function to the Laplace characteristic function

$$g(u, t) = E_v(\mu t^v (u - 1)) = E_v(\bar{n} \Gamma(v + 1) (u - 1)).$$

Introducing a new parameter $\lambda = -\bar{n} \ln u$ we get

$$\mathbf{E}u^{N(t)} = \mathbf{E}e^{-\lambda Z} = E_{\nu}(\bar{n}\Gamma(\nu+1)(e^{-\lambda/\bar{n}} - 1)).$$

At large $\bar{n}$ relating to large time t ,

$$\begin{aligned} \mathbf{E}e^{-\lambda Z} &\equiv \int_0^{\infty} e^{-\lambda z} f_{\nu}(z) dz \sim E_{\nu}(-\lambda'), \\ \lambda' &= \lambda \Gamma(\nu+1). \end{aligned}$$

Comparison of this equation with formula (6.9.8) of the book (Uchaikin and Zolotarev, 1999)

$$\begin{aligned} E_{\nu}(-\lambda') &= \nu^{-1} \int_0^{\infty} \frac{e^{-\lambda'x}}{x^{1+1/\nu}} g_+(x^{-1/\nu}; \nu) dx \\ &= \int_0^{\infty} e^{-\lambda z} \frac{[\Gamma(\nu+1)]^{1/\nu}}{\nu z^{1+1/\nu}} g_+\left(\frac{z^{-1/\nu}}{[\Gamma(\nu+1)]^{-1/\nu}}; \nu\right) dz \end{aligned}$$

shows that the random variable Z has the non-degenerated limit distribution at $t \rightarrow \infty$ (see also (Uchaikin, 1999)):

$$f_{\nu}(z; \bar{n}) \rightarrow f_{\nu}(z) = \frac{[\Gamma(\nu+1)]^{1/\nu}}{\nu z^{1+1/\nu}} g_+\left(\frac{z^{-1/\nu}}{[\Gamma(\nu+1)]^{-1/\nu}}; \nu\right)$$

with moments

$$\langle Z^k \rangle = \frac{[\Gamma(1+\nu)]^k \Gamma(1+k)}{\Gamma(1+k\nu)}.$$

By making use of series for $g^{(\nu)}$, we obtain

$$f_{\nu}(z) = \sum_{k=0}^{\infty} \frac{(-z)^k}{k! \Gamma(1 - (k+1)\nu) [\Gamma(\nu+1)]^{k+1}}.$$

When $z \rightarrow 0$,

$$f_{\nu}(z) \rightarrow f_{\nu}(0) = \frac{1}{\Gamma(1+\nu)\Gamma(1-\nu)} = \frac{\sin(\nu\pi)}{\nu\pi}.$$

It is also worth to note, that $\langle Z^0 \rangle = 1$, $\langle Z^1 \rangle = 1$, and $\langle Z^2 \rangle = 2\nu B(\nu, 1+\nu)$, so that the limit relative fluctuations are given by

$$\delta_\nu \equiv \sigma_{N(t)} / \langle N \rangle = \sqrt{2\nu B(\nu, 1 + \nu) - 1}.$$

In particular cases

$$\delta_0 = 1, \quad \delta_1 = 0, \quad \delta_{1/2} = \sqrt{\pi/2} - 1.$$

For $\nu = 1/2$, one can obtain an explicit expression for $f_\nu(z)$:

$$f_{1/2}(z) = \frac{2}{\pi} e^{-z^2/\pi}, \quad z \geq 0.$$

The densities for various ν are plotted in Fig. 3.11.

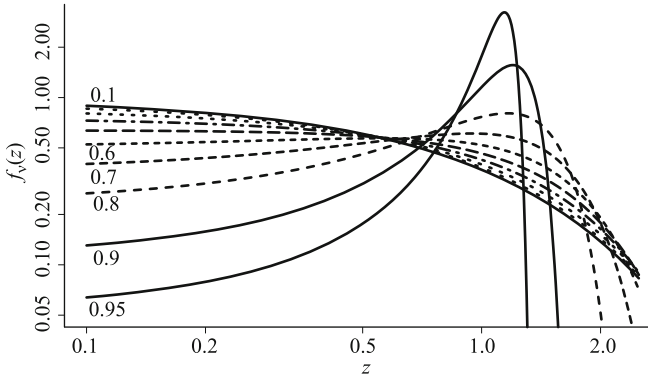


Fig. 3.11 Limit distributions $f_\nu(z)$ for $\nu = 0.1$ (0.1) 0.9 and 0.95.

3.5.6 An alternative models of fPp

The model of FPP considered above is not unique.

Another way of generalization is based on analogy with fractional Brownian motion instead of stochastic equation

$${}_0B^{(\nu)}(t) = \Phi_{1-\nu} \star \xi(t),$$

where $\xi(t)$ is a white Gaussian noise, the equation

$${}_0P^{(\nu)}(t) = \Phi_{1-\nu} \star \eta(t) = \frac{1}{\Gamma(1-\nu)} \int_0^t (t-t')^{-\nu} dN(t') \quad (3.40)$$

is considered with $\eta(t)$ denoting the standard Poisson flow

$$\eta(t) \equiv \frac{dN(t)}{dt} = \sum_{j=1}^{\infty} \delta(t - T^{(j)}),$$

where $T^{(j)} = T_1 + T_2 + \dots + T_j$, and $T_1, T_2, \dots, T_j$ are independent random variables with common density

$$\psi(t) = \mu e^{-\mu t}, \quad t \geq 0, \mu > 0.$$

Computing the convolution (3.40) yields

$${}_0P^{(\nu)}(t) = \frac{1}{\Gamma(\nu)} \int_0^t \frac{\eta(\tau) d\tau}{(t-\tau)^{1-\nu}} = \sum_{j=1}^{N(t)} \frac{1}{\Gamma(\nu)} \frac{1}{(t-T^{(j)})_+^{1-\nu}}.$$

It is easy to see that for $\nu = 1$ the process becomes the standard Poisson process $N(t)$.

The process can be interpreted as a resulting signal generated by the Poisson flow of pulses, each of which gives the contribution

$$A_j \equiv A(t - T^{(j)}) = \frac{1}{\Gamma(\nu)(t - T^{(j)})_+^{1-\nu}}. \quad (3.41)$$

It is well known that under the condition that $N(t) = n$, the times $T^{(1)}, T^{(2)}, \dots, T^{(n)}$ at which events occur, considered as unordered random variables, are distributed independently and uniformly in the interval $(0, t)$. Therefore,

$$Y_\nu(t)|_{N(t)=n} = \sum_{j=1}^n A_j,$$

where A_j is determined by making use of Eq. (3.41). Now,

$$\begin{aligned} \mathbf{P}(A_j > y) &= \mathbf{P}\left(\Gamma(\nu)(t - T^{(j)})^{1-\nu} < y^{-1}\right) = \mathbf{P}\left(t - T^{(j)} < [\Gamma(\nu)y]^{-1/(1-\nu)}\right) \\ &= \mathbf{P}\left(T^{(j)} < [t\Gamma(\nu)y]^{-1/(1-\nu)}\right) = \frac{1}{t[\Gamma(\nu)y]^{1/(1-\nu)}}. \end{aligned}$$

When $\nu > 0$, expectation of A_j exists, and according to the law of large numbers, the limit distribution of the scaled random variable Z has the degenerate limit distribution $f_\nu(z) = \delta(z - 1)$.

To make our results clearer, we present in Fig. 3.12 some typical realizations of Poisson process discussed above. Panel (a) shows a realization of the standard Poisson process which goes almost along a straight line at large times. Panel (b) represents an fPp with $\nu = 1/2$: one can clearly see large intervals with constant $N(t)$ corresponding to empty bins. Panel (c) represents the realization of fPp of the second type, generated by stochastic fractional equation (3.40).

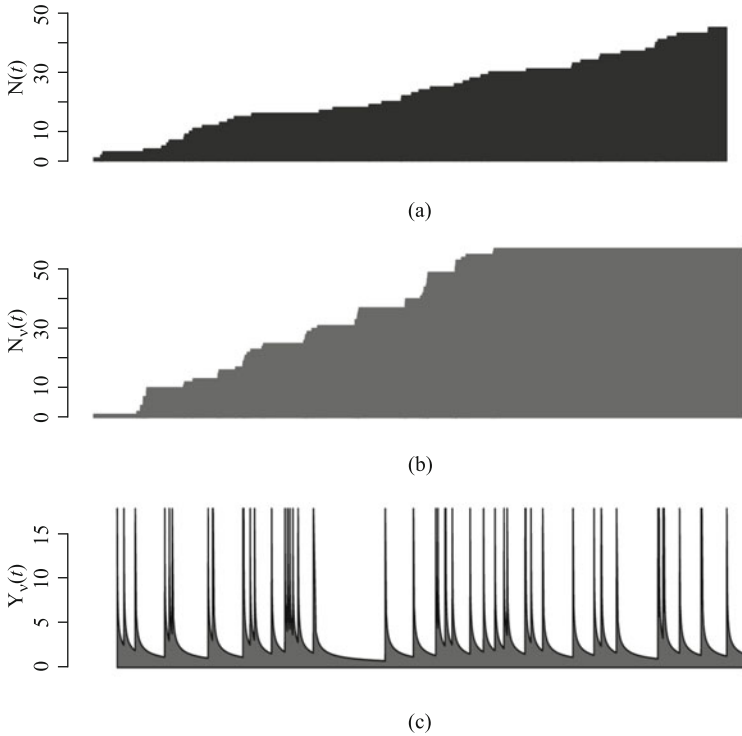


Fig. 3.12 Sample trajectories of (a) standard Poisson process, (b) fPp, and (c) the alternative FPP generated by stochastic fractional differential equation (3.40), with $\nu = 1/2$.

Other fractional generalizations of the Poisson process can be found in (Jumarie, 2001; Repin and Saichev, 2000; Wang et al., 2006).

3.5.7 Compound Poisson process

The Poisson process admits a very simple but productive generalization, called *compound Poisson process*. The idea of this generalization is based on replacing unit jumps at random arrival times by jumps of random length $X^{(j)}$, $j = 1, 2, 3, \dots$ at the same time. The random variables are independent of each other and of arrival times. Consequently, instead of random function

$$N(t) = \sum_{j=1}^{N(t)} 1$$

for the Poisson process we have

$$X(t) = \sum_{j=1}^{N(t)} X^{(j)}$$

for the compound Poisson process. Let $N(t)$ be the Poisson process with the rate μ and $p(x)$, $-\infty < x < \infty$, denotes the pdf for $X^{(j)}$, then pdf $f(x, t)$ for $X(t)$ is represented in the form:

$$f(x, t) = e^{-\mu t} \sum_{j=0}^{\infty} \frac{(\mu t)^j}{j!} p^{*j}(x), \quad t > 0.$$

This density obeys the integro-differential *Feller-Kolmogorov equation*

$$\frac{\partial f}{\partial t} = -\mu f(x, t) + \mu \int_{-\infty}^{\infty} p(x - x') f(x', t) dx' = 0$$

with the initial condition

$$f(x, 0+) = \delta(x)$$

or, equivalently, the equation

$$\frac{\partial f}{\partial t} = -\mu f(x, t) + \mu \int_{-\infty}^{\infty} p(x - x') f(x', t) dx' = \delta(x) \delta(t) \quad (3.42)$$

with the condition

$$f(x, 0) = \delta(x) \delta(t).$$

Passing to characteristic functions

$$\frac{\partial \tilde{f}(k, t)}{\partial t} = -\mu [1 - \tilde{p}(k)] \tilde{f}(k, t) + \delta(t)$$

and developing the coefficient in square brackets as series in k

$$\mu [1 - \tilde{p}(k)] \sim -iCk + Dk^2, \quad k \rightarrow 0,$$

we get the *Kolmogorov type equation*:

$$\left[\frac{\partial}{\partial t} + C \frac{\partial}{\partial x} - D \frac{\partial^2}{\partial x^2} \right] f(x, t) = \delta(x) \delta(t). \quad (3.43)$$

However, this approximation is valid only in case when $p(x)$ belongs to the domain of attraction of the Lévy stable laws, that is when the condition of the Central Limit Theorem is fulfilled:

$$\int_{-\infty}^{\infty} x^2 p(x) dx < \infty.$$

If it is not the case but $p(x)$ satisfies the condition of the Generalized Limit Theorem, which is

$$\int_{|x'| < x} p(x') dx' \propto x^\alpha, \quad x \rightarrow \infty, \quad \alpha \in (0, 2),$$

the generalized Kolmogorov equation contains the fractional operator $\left[\frac{\partial^2}{\partial x^2}\right]^{\alpha/2}$ instead of $\frac{\partial^2}{\partial x^2}$.

On the other hand, Eq. (3.42) can be transformed into Volterra's integral equation without loss of accuracy:

$$f(x, t) = e^{-\mu t} 1_+(t) \delta(x) + \mu \int_0^t e^{-\mu(t-t')} \int_{-\infty}^{\infty} p(x-x') f(x', t') dx'. \quad (3.44)$$

It is not hard to see that a time-differentiable solution of Eq. (3.44) satisfies Eq. (3.42) as well. Indeed, on differentiating equation

$$e^{\mu t} f(x, t) = 1_+(t) \delta(x) + \int_0^t dt' e^{\mu t'} \int_{-\infty}^{\infty} p(x-x') f(x', t') dx'$$

with respect to time, we arrive at the equation

$$e^{\mu t} \left[\frac{\partial}{\partial t} + \mu \right] f(x, t) = \delta(x) \delta(t) + e^{\mu t} \int_{-\infty}^{\infty} p(x-x') f(x', t) dx',$$

which coincides with Eq. (3.42) because

$$e^{-\mu t} \delta(t) = \delta(t).$$

Replacing the ordinary exponential function in Eq. (3.44) by its fractional generalization $\psi_\nu(t)$, we obtain an integral equation for the fractional compound Poisson process:

$$f(x, t) = E_\nu(-\mu t^\nu) 1_+(t) \delta(x) + \mu \int_0^t dt' E_{\nu\nu}[-\mu(t-t')^\nu] (t-t')^{\nu-1} \int_{-\infty}^{\infty} p(x-x') f(x', t') dx'. \quad (3.45)$$

3.6 Lévy flights and Lévy walks

3.6.1 Lévy Flights

Equations (3.44) and (3.45) are special cases of a more general equation

$$f(\mathbf{x}, t) = Q(t) \delta(\mathbf{x}) + \int_0^t dt' q(t-t') \int_{\mathbb{R}^d} p(\mathbf{x}-\mathbf{x}') f(\mathbf{x}', t') d\mathbf{x}' \quad (3.46)$$

describing a random process in $\mathbb{R}^d$ called the *Continuous Time Random Flights* (CTRF). This process (walker or walking particle) starts its history at time $t = 0$ at

point $\mathbf{x} = 0$. A random time T_1 later, the particle jumps to a random point $\mathbf{X}_1 = \mathbf{R}_1$, where it stays during a random time T_2 and then jumps to point $\mathbf{X}_2 = \mathbf{R}_1 + \mathbf{R}_2$ at time $T_1 + T_2$ and so on. The random variables $T_1, \mathbf{R}_1, T_2, \mathbf{R}_2, T_3, \mathbf{R}_3, \dots$ are assumed to be mutually independent, T_j are distributed with a common density $q(t) = -dQ/dt$, $Q(t) = P(T > t)$, and $\mathbf{R}_j$ have a common d -dimensional density $p(\mathbf{x})$.

Equation (3.46) supplies us a probability density function $f(\mathbf{x}, t)$ for the process

$$\mathbf{X}(t) = \sum_{j=1}^{N(t)} \mathbf{R}_j, \quad (3.47)$$

where $N(t)$ is the number of jumps to t , obeying the relation

$$T_1 + \dots + T_{N(t)} < t < T_1 + \dots + T_{N(t)} + T_{N(t)+1}.$$

The Fourier-Laplace transform

$$p(\mathbf{k}, \lambda) = \int_{\mathbb{R}^d} d\mathbf{x} \int_0^\infty dt e^{i\mathbf{k} \cdot \mathbf{x} - \lambda t} p(\mathbf{x}, t), \quad \mathbf{k} \in \mathbb{R}^d$$

of the pdf $p(\mathbf{x}, t)$ is easily expressed in terms of the Laplace transform of the waiting-time pdf and the Fourier transform of the jump vector pdf:

$$p(\mathbf{k}, \lambda) = \frac{1 - q(\lambda)}{\lambda [1 - p(\mathbf{k})q(\lambda)]}. \quad (3.48)$$

The desired solution is found by inverse transformations:

$$p(\mathbf{x}, t) = i^{-1} (2\pi)^{-d-1} \int_{\mathbb{R}^d} d\mathbf{k} \int_L d\lambda e^{i\mathbf{k} \cdot \mathbf{x} + \lambda t} p(\mathbf{k}, \lambda). \quad (3.49)$$

At long time, when the particle performs many jumps and the spatial distribution of probability becomes wide, the density is determined by the behavior of the transform $p(k, \lambda)$ in the region of small k and λ . Formulae (3.48)–(3.49) represent a complete mathematical formulation of the multidimensional CTRF problem called also the *Montroll-Weiss problem*.

If we are interested in asymptotic selfsimilar solutions of the centrosymmetrical type, we should put

$$1 - \tilde{q}(\lambda) \sim B\lambda^\nu, \quad \lambda \rightarrow 0 \quad (3.50)$$

and

$$1 - \tilde{p}(\mathbf{k}) \sim A \int_{U_d} |\mathbf{k} \cdot \mathbf{u}|^\alpha \Gamma(d\mathbf{u}), \quad |\mathbf{k}| \rightarrow 0, \quad (3.51)$$

where α, ν, A and B are positive constants, U_d is the unit sphere in $\mathbb{R}^d$, and $\Gamma(d\mathbf{u})$ is a spectral measure. The ranges of exponents α and ν are determined from the conditions that $p(\mathbf{x})$ is a symmetrical d -variate and $q(t)$ is the pdf of a positive

random variable T . Really, as $T > 0$

$$[dq(\lambda)/d\lambda]_0 = - \int_0^\infty tq(t)dt$$

must be negative, but inserting here Eq. (3.50),

$$[dq(\lambda)/d\lambda]_0 = -[Bv\lambda^{v-1}]_0,$$

we see that

$$\int_0^\infty tq(t)dt = \begin{cases} \infty, & v < 1; \\ B, & v = 1; \\ 0, & v > 1. \end{cases}$$

In the same way, positivity of the second jump length moment bounds the exponent α by the interval $(0, 2]$.

We will call such CTRF's the *Lévy flights* (Lf).

3.6.2 Asymptotic solution of the LF problem

Confining ourselves to investigating just the long-time asymptotics of $p(\mathbf{x}, t)$, we can use the correspondent (in the Tauber sense) asymptotics of the Fourier-Laplace transform

$$p(\mathbf{k}, \lambda) \sim \frac{\lambda^{v-1}}{\lambda^v + CL(\mathbf{k}; \alpha, \Gamma(\cdot))}, \quad \lambda \rightarrow 0, \quad \mathbf{k} \rightarrow 0, \quad |\mathbf{k}|^\alpha / \lambda^v \sim \text{const}, \quad (3.52)$$

where $C = A/B$ and $\alpha \in (0, 2)$, $v \leq 1$. In order to invert the transform (3.52), we reduce it to the form

$$p(\mathbf{k}, \lambda) = \lambda^{v-1} \int_0^\infty \exp\left\{-\left[\lambda^v + CL(\mathbf{k}; \alpha, \Gamma(\cdot))\right]y\right\} dy.$$

On inverting the Laplace transformation

$$p(\mathbf{k}, t) = \int_0^\infty dy \exp\{-CL(\mathbf{k}; \alpha, \Gamma(\cdot))y\} (2\pi i)^{-1} \int_{\mathbb{C}} d\lambda \lambda^{v-1} \exp\{\lambda t - \lambda^v y\},$$

taking the inner integral by parts, and making the change of variable $s = y^{1/v} \lambda$, we obtain

$$p(\mathbf{k}, t) = \beta^{-1} t \int_0^\infty dy \exp\{-CL(\mathbf{k}; \alpha, \Gamma(\cdot))y\} y^{-1-1/v} \cdot \left[(2\pi i)^{-1} \int_{\mathbb{C}'} \exp\{s y^{-1/v} t - s^v\} ds \right]. \quad (3.53)$$

The function

$$\widehat{g}_+(\lambda; \nu) \equiv e^{-\lambda^\nu}, \quad \nu \leq 1$$

is the Laplace transform of the subordinator

$$g_+(t; \nu) = (2\pi i)^{-1} \int_{\mathbb{C}'} e^{\lambda t - \lambda^\nu} d\lambda.$$

Thus, Equation (3.53) is represented in the following form

$$p(\mathbf{k}, t) = \int_0^\infty \exp\{-CL(\mathbf{k}; \alpha, \Gamma(\cdot))(t/\tau)^\nu\} g_+(\tau; \nu) d\tau. \quad (3.54)$$

The Fourier inversion yields

$$p(\mathbf{x}, t) = (Ct^\nu)^{-d/\alpha} \Psi_d\left(\mathbf{x}(Ct^\nu)^{-1/\alpha}; \alpha, \nu, \Gamma\right), \quad (3.55)$$

where

$$\Psi_d^{(\alpha, \nu)}(\mathbf{x}; \Gamma) = \int_0^\infty g_d\left(\mathbf{x}\tau^{\nu/\alpha}; \alpha, \Gamma\right) g_+(\tau, \nu) \tau^{d\nu/\alpha} d\tau. \quad (3.56)$$

In case $\nu = 1$ and $t \rightarrow \infty$, the Lévy flights process becomes the Lévy-Feldheim motion process (see Sect. 3.3.5.) When $\nu < 1$, we meet a process called the *subordinated LF-motion*. This term means a random process with intrinsic time $T(t)$, $t \geq 0$ being a non decreasing random function of observation time t in the LFm (Feller, 1971)

$$\mathbf{X}(t) = \mathbf{Y}(T(t)).$$

In order to clarify the validity of this term, we note that choosing $N(t)$ for the intrinsic time we obtain

$$\mathbf{P}(N(t) = n) = Q^{*(n+1)}(t) - Q^{*(n)}(t),$$

where

$$Q^{*(n)}(t) = \int_t^\infty q^{*(n)}(t') dt'$$

and

$$q^{*(n)}(t) = \int_0^t q^{*(n-1)}(t-t') q(t') dt', \quad q^{*(0)}(t) = \delta(t), \quad q^{*(1)}(t) = q(t).$$

With such choice, $\mathbf{Y}(N)$ is a sum of independent random vectors $\mathbf{S}'_j(\alpha; \Gamma)$ belonging to the domain of attraction of a centrosymmetric Lévy-Feldheim law,

$$\mathbf{Y}(n) = \sum_{j=1}^n \mathbf{S}'_j(\alpha; \Gamma) \stackrel{d}{\sim} n^{1/\alpha} \mathbf{S}(\alpha; \Gamma), \quad n \rightarrow \infty.$$

Consequently,

$$p(\mathbf{x}|N = n) \sim n^{-d/\alpha} g_d(\mathbf{x}n^{-1/\alpha}; \alpha, \Gamma), \quad n \rightarrow \infty.$$

Averaging this distribution over $N(t)$ under a fixed t ,

$$p(\mathbf{x}, t) = \langle p(\mathbf{x} | N(t)) \rangle = \sum_{n=0}^{\infty} p(\mathbf{x} | n) \mathbf{P}(N(t) = n)$$

leads to Eq. (3.55)

The pdf (3.56) is called the *fractional stable density* because describes the distribution of the ratio

$$\mathbf{Z} = \frac{\mathbf{S}(\alpha, \Gamma)}{S_+^{v/\alpha}(v)},$$

where numerators and denominators are independent of each other. Really, computing the characteristic function of this vector, one can verify that

$$\begin{aligned} & \int_{\mathbb{R}^d} e^{i\mathbf{k} \cdot \mathbf{x}} \Psi_d^{(\alpha, v)}(\mathbf{x}; \Gamma) d\mathbf{x} \\ &= \int_0^\infty \left(\int_{\mathbb{R}^d} e^{i\mathbf{k} \cdot \mathbf{x}} g_d(\mathbf{x} \tau^{v/\alpha}; \alpha, \Gamma) d\mathbf{x} \right) g_+(\tau; v) \tau^{dv/\alpha} d\tau \\ &= \int_0^\infty \exp\left(-\tau^{-v} \int_{U_d} |\mathbf{k} \cdot \mathbf{u}|^\alpha \Gamma(d\mathbf{u})\right) g_+(\tau; v) d\tau \\ &= \left\langle \exp\left(i\mathbf{k} \cdot \mathbf{S}(\alpha, \Gamma) / S_+^{v/\alpha}(v)\right) \right\rangle. \end{aligned}$$

Observe that the LFM is a particular case of the SLFM when $v = 1$. As one can see from Eq. (3.55), the law of the diffusion packet spreading is determined by the ratio v/α : the process reveals superdiffusive behavior if $v > \alpha/2$ and subdiffusive behavior if $v < \alpha/2$. When $v = \alpha/2$, the width of the diffusion packet grows in time as in the normal case, but its form differs from Gaussian and depends on α . This regime can be called the *quasinormal diffusion*. When $v = \alpha$, the diffusion packet expands linearly in time. We call this regime the *quasiballistic diffusion*.

3.6.3 Continuous time random walk

Now we modify the random process replacing instantaneous jumps between traps by continuous motion with a finite speed. The modified process called the *continuous time random walk* (CTRW) is defined in the following way.

(1) There exist two possible states of a particle: the state of rest (trapping) labeled $i = 0$ and the state of moving ($i = 1$) in $\mathbb{R}^d$ with a constant speed $v \equiv |\mathbf{v}|$.

(2) The external source produces one particle at the origin of coordinates $\mathbf{x} = 0$ at the time $t = 0$. The particle occurs in the state of rest with the probability p_0 and in the state of moving with the probability p_1 .

(3) In the course of time, the particle changes its state at random points of time.

(4) A random direction $\Omega \equiv \mathbf{v}/v$ of the particles leaving the source or traps is characterized by the same probability density $W(\Omega)$ independently of the previous

direction:

$$\int_{U_d} W(\Omega) d\Omega = 1.$$

(5) The free path distribution for the state of moving and the waiting time distribution for the state of rest are independent of each other and among themselves and given by the densities $p(\xi)$ and $q(\tau)$ respectively.

Denoting the probabilities of appearance of the particle in state i , $j = 0, 1$ in the element $d\mathbf{x}dt$ by $F_j(\mathbf{x}, t)d\mathbf{x}dt$, we can write (Uchaikin, 1998):

$$p(\mathbf{x}, t) = \int_0^t d\tau Q(\tau) F_0(\mathbf{x}, t - \tau) + (1/v) \int d\mathbf{x}' P(\mathbf{x}') F_1(\mathbf{x} - \mathbf{x}', t - \xi/v), \quad (3.57)$$

$$F_0(\mathbf{x}, t) = \int d\mathbf{x}' p(\mathbf{x}') F_1(\mathbf{x} - \mathbf{x}', t - \xi/v) + p_0 \delta(\mathbf{r}) \delta(t), \quad (3.58)$$

$$F_1(\mathbf{x}, t) = \int_0^t d\tau q(\tau) F_0(\mathbf{x}, t - \tau) + p_1 \delta(\mathbf{x}) \delta(t), \quad (3.59)$$

where

$$P(\mathbf{x}') = P(\xi) W(\mathbf{x}'/\xi)/\xi^2, \quad p(\mathbf{x}') = p(\xi) W(\mathbf{x}'/\xi)/\xi^2, \\ \xi = |\mathbf{x}'|, \quad P(\xi) = \int_{\xi}^{\infty} p(\xi') d\xi',$$

and $F_j(\mathbf{x}, t)$ are equal to zero when $t < 0$.

For better understanding what the equations mean, we consider their one-dimensional version:

$$p(x, t) = \int_0^t d\tau Q(\tau) F_0(x, t - \tau) \quad (3.60)$$

$$+ (2v)^{-1} \int_0^{vt} d\xi P(\xi) [F_1(x - \xi, t - \xi/v) + F_1(x + \xi, t - \xi/v)],$$

$$F_0(x, t) = (1/2) \int_0^{vt} d\xi p(\xi) [F_1(x - \xi, t - \xi/v) + F_1(x + \xi, t - \xi/v)] \quad (3.61) \\ + p_0 \delta(x) \delta(t),$$

$$F_1(x, t) = \int_0^t d\tau q(\tau) F_0(x, t - \tau) + p_1 \delta(x) \delta(t). \quad (3.62)$$

Every random trajectory is made up of a set of segments each of them is parallel to one of the straight lines $x = 0$, $x = vt$, $x = -vt$ (Fig. 3.13).

According to this the density $p(x, t)$ falls into the sum

$$p(x, t) = p^{(0)}(x, t) + p^{(+)}(x, t) + p^{(-)}(x, t).$$

Taking into account that $F_0(x, t)dxdxdt$ is the probability for the particle to fall into the trap in the space-time region $dxdxdt$ and $Q(\tau)$ is the probability to wait here longer than τ , one can write

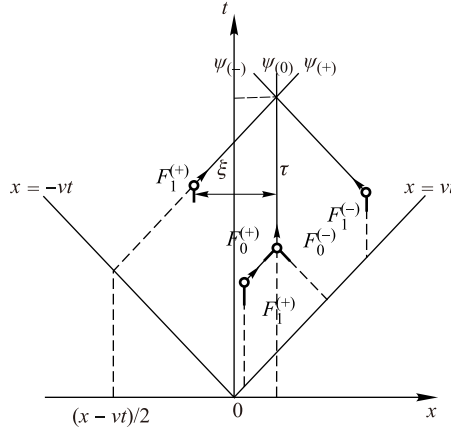


Fig. 3.13 Space-time diagram for a one-dimensional walk process with trapping.

$$p^{(0)}(x, t) = \int_0^t d\tau Q(\tau) F_0(x, t - \tau).$$

This is the first term of the sum (3.60). Reasoning in much the same way for a moving particle one can obtain the rests.

To explain Eq. (3.61) we refer to the diagram again. As one can see from it, the density $F_0(x, t)$ by $t > 0$ in its turn falls to two parts: $F_0^{(+)}(x, t)$ and $F_0^{(-)}(x, t)$ relating to the particles moving before collision to the right and to the left correspondingly. With this one can readily write

$$F_0^{(+)}(x, t) = (1/2) \int_0^{vt} d\xi p(\xi) F_1(x - \xi, t - \xi/v),$$

$$F_0^{(-)}(x, t) = (1/2) \int_0^{vt} d\xi p(\xi) F_1(x + \xi, t - \xi/v),$$

where $p(\xi)d\xi$ is the probability that the random free path falls into the segment $(\xi, \xi + d\xi)$. Similar reasoning explains Eq. (3.62) as well.

3.6.4 Some special cases

In this section, we consider some special cases of Eqs. (3.57)–(3.59).

Let $v = \infty$, $p_0 = 1$, and $q(\tau) = \delta(\tau - 1)$. This means that the walker initially placed at the origin performs instantaneous jumps at instants $t = 1, 2, 3, \dots$. Eqs. (3.57)–(3.59) take the form

$$\begin{aligned}
 p(\mathbf{x}, t) &= \int_0^1 d\tau F_0(\mathbf{x}, t - \tau), \\
 F_0(\mathbf{x}, t) &= \int d\mathbf{x}' p(\mathbf{x}') F_0(\mathbf{x} - \mathbf{x}', t - 1) + \delta(\mathbf{x}) \delta(t).
 \end{aligned}
 \tag{3.63}$$

The latter equation has the solution

$$F_0(\mathbf{r}, t) = \sum_{n=0}^{\infty} p^{*(n)}(\mathbf{x}) \delta(t - n), \tag{3.64}$$

where

$$p^{*(n)}(\mathbf{x}) = \int_{-\infty}^{\infty} p^{*(n-1)}(\mathbf{x} - \mathbf{x}') p(\mathbf{x}') d\mathbf{x}', \quad p^{*(0)}(\mathbf{x}) = \delta(\mathbf{x}), \quad p^{*(1)}(\mathbf{x}) = p(\mathbf{x}).$$

Substitution of Eq. (3.64) in Eq. (3.63) yields

$$p(\mathbf{x}, t) = \int_{t-1}^t dt' F_0(\mathbf{x}, t') = p^{*[t]}(\mathbf{x}), \tag{3.65}$$

where $[t]$ is an integer part of t . This result is merely the distribution density function of the sum of a fixed number $[t]$ of independent random vectors $\mathbf{R}_j$:

$$\mathbf{X}(t) = \sum_{j=1}^{[t]} \mathbf{R}_j.$$

It is easy to see, that

$$p(\mathbf{x}, t + 1) = \int d\mathbf{x}' p(\mathbf{x}') p(\mathbf{x} - \mathbf{x}', t)$$

is merely the discrete-time version of the Chapman-Kolmogorov equation (Feller, 1971). It can also be rewritten in the form

$$p(\mathbf{x}, t + 1) - p(\mathbf{x}, t) = \int d\mathbf{x}' p(\mathbf{x}') [p(\mathbf{x} - \mathbf{x}', t) - p(\mathbf{x}, t)],$$

which is in agreement with Eq. (5) from the work (Chukbar, 1995), where the one-dimensional case has been considered only.

Taking now $q(t) = \mu e^{-\mu t}$ instead of the delta-function, we obtain

$$p(\mathbf{x}, t) = \int_0^t d\tau e^{-\mu \tau} F_0(\mathbf{x}, t - \tau) = p(\mathbf{x}) e^{-\mu t} + \mu \int_0^t d\tau e^{-\mu \tau} \int d\mathbf{x}' p(\mathbf{x}') p(\mathbf{x} - \mathbf{x}', t - \tau).$$

By differentiation with respect to t , we arrive at the continuous-time Feller-Kolmogorov equation

$$\frac{\partial p(\mathbf{x}, t)}{\partial t} = -\mu p(\mathbf{x}, t) + \mu \int d\mathbf{x}' p(\mathbf{x}') p(\mathbf{x} - \mathbf{x}', t), \quad p(\mathbf{x}, 0) = \delta(\mathbf{x}),$$

describing the distribution of the sum of a random Poisson's number $N(t)$ of independent random vectors $\mathbf{R}_j$ distributed according to the density $p(\mathbf{x})$:

$$\mathbf{X}(t) = \sum_{j=1}^{N(t)} \mathbf{R}_j.$$

Let us come back to an arbitrary waiting time distribution and $p_0 = 1$. Combining Eqs. (3.58) and (3.59) yields

$$\begin{aligned} F_0(\mathbf{x}, t) &= \int d\mathbf{x}' \int dt' w(\mathbf{x}', t') F_0(\mathbf{x} - \mathbf{x}', t - t') + \delta(\mathbf{x}) \delta(t), \\ F_1(\mathbf{x}, t) &= \int d\tau q(\tau) F_0(\mathbf{x}, t - \tau), \end{aligned} \quad (3.66)$$

where

$$w(\mathbf{x}, t) = p(\mathbf{x}) q(t - |\mathbf{x}|/v). \quad (3.67)$$

The density $p(\mathbf{x}, t)$ can be divided into two components: $p^{(0)}(\mathbf{x}, t)$ related to the quiescence state and $p^{(*)}(\mathbf{x}, t)$ related to the motion state. It is easy to see that the first of them

$$p^{(0)}(\mathbf{x}, t) = \int d\tau Q(\tau) F_0(\mathbf{x}, t - \tau) \quad (3.68)$$

satisfies the closed integral equation

$$p^{(0)}(\mathbf{x}, t) = \int d\mathbf{x}' \int dt' w(\mathbf{x}', t') p^{(0)}(\mathbf{x} - \mathbf{x}', t - t') + Q(t) \delta(\mathbf{x}). \quad (3.69)$$

For literal perception of this equation, one should refer the probability $w(\mathbf{x}', t') d\mathbf{x}' dt'$ to the trajectory origin, interpreting Eq. (3.69) as the total probability theorem, namely, if the first trappings of the walker occurred at point $\mathbf{x}_0 = 0$ at time $t_0 = 0$, then the next one will occur in $d\mathbf{x}' dt'$ with probability $w(\mathbf{x}', t') d\mathbf{x}' dt'$ and the walker should reach the observation point $\mathbf{x}$ from $\mathbf{x}'$ during the rest period $t - t'$. The latter term of Eq. (3.69) relates to the case when the walker has been stayed in a trap during the whole period $[0, t]$.

Equations (3.67), (3.68), and (3.69) are identical with Eqs. (17), (18), and (19) of the article (Klafter et al., 1987) respectively, devoted to continuous-time random-walk (CTRW) models with coupled memories. As it is shown in (Compte, 1996) and (Compte et al., 1997), the long-time (or long-distance) limit behavior of the distribution is described in terms of fractional equations.

The Fourier-Laplace counterpart of Eq. (3.69)

$$p^{(0)}(\mathbf{k}, \lambda) = w(\mathbf{k}, \lambda) p^{(0)}(\mathbf{k}, \lambda) + Q(\lambda)$$

has the solution

$$p^{(0)}(\mathbf{k}, \lambda) = \frac{Q(\lambda)}{1 - w(\mathbf{k}, \lambda)} = \frac{1 - q(\lambda)}{\lambda(1 - w(\mathbf{k}, \lambda))}, \quad (3.70)$$

where

$$w(\mathbf{k}, \lambda) = \int d\mathbf{x} \int dt e^{i\mathbf{k} \cdot \mathbf{x} - \lambda t} p(\mathbf{x}) q(t - |\mathbf{x}|/v)$$

is an indication of the space-time *coupled memory* of the process. This means that the random time- and space-variables characterizing each transition from one trapping to the next one, are partially linked with each other through the velocity: $T_{\text{tot}} = T_{\text{trap}} + |\mathbf{R}|/v$, where $\mathbf{R}$ doesn't depend of T_{trap} . In absence of traps, we observe the total one-to-one correspondence between these variables, $T_{\text{tot}} = |\mathbf{R}|/v$ taking place in ordinary kinetic processes. In the limit $v \rightarrow \infty$, we get another extreme case characterized by *decoupled memory*

$$w(\mathbf{k}, \lambda) = p(\mathbf{k})q(\lambda)$$

and leading to the Montroll-Weiss result (Montroll and Weiss, 1965):

$$p(\mathbf{k}, \lambda) = \frac{1 - q(\lambda)}{\lambda[1 - p(\mathbf{k})q(\lambda)]} \quad (3.71)$$

(see for details (Montroll and West, 1976)). Asymptotic calculations of the corresponding pdf

$$p(\mathbf{x}, t) = i^{-1} (2\pi)^{-d-1} \int_{\mathbb{R}^d} d\mathbf{k} \int_L d\lambda e^{i\mathbf{k} \cdot \mathbf{x} + \lambda t} p(\mathbf{k}, \lambda)$$

under conditions

$$1 - \widehat{q}(\lambda) \sim B\lambda^v, \quad \lambda \rightarrow 0$$

and

$$1 - \widetilde{p}(\mathbf{k}) \sim A \int_{U_d} |\mathbf{k} \cdot \mathbf{u}|^\alpha \Gamma(d\mathbf{u}), \quad |\mathbf{k}| \rightarrow 0$$

have been executed in Sect. 3.6.2. Below we consider the finite speed effect.

3.6.5 Speed limit effect

We start with one-dimensional symmetrical walking described by Eqs. (3.60)–(3.62). In terms of Fourier-Laplace variables, we have

$$p(k, \lambda) = \frac{1 - q(\lambda)}{\lambda} F_0(k, \lambda) + (1/v) P(k, \lambda/v) F_1(k, \lambda),$$

$$F_0(k, \lambda) = p(k, \lambda/v) F_1(k, \lambda) + p_0,$$

$$F_1(k, \lambda) = \frac{p_1 + q(\lambda)p_0}{1 - q(\lambda)p(k, \lambda/v)},$$

where

$$p(k, \lambda/v) = \int_0^\infty d\xi p(\xi) e^{-(\lambda/v)\xi} \cos k\xi$$

and so is $P(k, \lambda/v)$.

Choosing the free path probability density $p(\xi)$ in the form

$$p(\xi) = \alpha \xi_0^\alpha \xi^{-\alpha-1}, \quad \xi > \xi_0, \quad \alpha < 1$$

we have (in the asymptotic region $k \rightarrow 0$)

$$p(k) \sim 1 - A|k|^\alpha, \quad A = \xi_0^\alpha \Gamma(1 - \alpha) \cos(\pi\alpha/2),$$

$$p(k, z) \sim 1 - \xi_0^\alpha u_\alpha(z/|k|)|k|^\alpha,$$

$$P(k, z) \sim \xi_0^{\alpha-1} U_\alpha(z/|k|)|k|^{\alpha-1},$$

where

$$u_\alpha(z) = \int_0^\infty x^{-\alpha} [\sin x + z \cos x] e^{-zx} dx$$

and

$$U_\alpha(z) = \int_0^\infty x^{-\alpha} \cos x e^{-zx} dx.$$

Using the formulae together with

$$q(\lambda) \sim 1 - B\lambda^v, \quad \lambda \rightarrow 0, \quad v < 1$$

we obtain

$$p(x, t) = i^{-1} (2\pi)^{-2} \int_{-\infty}^\infty dk \int_{\mathbb{C}} d\lambda e^{-ikx + \lambda t} p(k, \lambda) \sim t^{-v/\alpha} \Psi_v(xt^{-v/\alpha}, t^{v/\alpha-1}), \quad (3.72)$$

where

$$\Psi_v^{(\alpha, v)}(x, t) = \frac{1}{(2\pi)^2 i} \int_{-\infty}^\infty dk \int_{a-i\infty}^{a+i\infty} d\lambda \frac{B\lambda^{\beta-1} + (1/v)|k\xi_0|^{\alpha-1} U_\alpha(\lambda t/|vk|)t}{B\lambda^\beta + |k\xi_0|^\alpha u_\alpha(\lambda t/|vk|)} e^{\lambda - ikx}. \quad (3.73)$$

It is easy to see from here, the self-similar (scaling) behavior of the solution is observed in the case $v = \alpha$ only:

$$p(x, t) \sim (1/m) \Psi_v^{(\alpha, \alpha)}(x/m, 1), \quad m = t.$$

In case ($v \neq \alpha$), the solution has no such a property because of the finite speed.

It should be noted that the insertion of the limited speed into a normal diffusion model does not change the selfsimilar property of the asymptotics. Taking into account, that in this case

$$p(k, \lambda/v) \sim 1 - \langle \xi/v \rangle \lambda - \langle \xi^2/2 \rangle k^2, \quad P(k, \lambda/v) \sim \langle \xi \rangle, \quad \lambda \rightarrow 0,$$

and consequently

$$p(x, t) \sim (1/\sqrt{t}) \Phi(x/\sqrt{t}),$$

we find

$$\begin{aligned} \Phi(x) &= \frac{1}{(2\pi)^2 i} \int_{-\infty}^{\infty} dk \int_{a-i\infty}^{a+i\infty} d\lambda \frac{\langle \tau + \xi/v \rangle}{\langle \tau + \xi/v \rangle + \langle \xi^2/2 \rangle k^2} e^{-ikx + \lambda} \\ &= \frac{1}{\sqrt{2\pi \langle \xi^2 \rangle / \langle \tau + \xi/v \rangle}} \exp \left\{ -\frac{x^2}{2 \langle \xi^2 \rangle / \langle \tau + \xi/v \rangle} \right\}. \end{aligned}$$

This result generalizes the ordinary diffusion distribution to the case of normal trapping when

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} x^2 \psi(x, t) dx \sim \frac{\langle \xi^2 \rangle}{\langle \tau \rangle + \langle \xi \rangle/v} t.$$

3.6.6 Moments of spatial distribution

Let us come back to the system of Eqs. (3.57)–(3.59) describing the CTRW model. The most important difference of the model from the CTRF model is the boundedness of spatial distribution: at any time t , the distribution is bounded by a sphere of radius $R = vt$. For this reason, the method of spatial moments being absolutely inapplicable to CTRF problems becomes an effective numerical method of solution of CTRW problems.

Let us define even spatial moments of the distribution $p(\mathbf{x}, t)$ as

$$m_j^{(2n)}(t) = \int_{\mathbb{R}^d} |\mathbf{x}|^{2n} p_j(\mathbf{x}, t) d\mathbf{x}, \quad j = 0, 1, \quad n = 1, 2, 3, \dots,$$

where $j = 0$ if the walker history begins with trapping, and $j = 1$ if it begins with leaving the trap. Equations for the moments can be derived from Eqs. (3.57)–(3.59) for the corresponding densities (Uchaikin, 1998) but it seems to be more instructive to derive them directly from probabilistic reasoning (Uchaikin, 1998a)

$$m_j^{(2n)}(t) = \langle |\mathbf{X}_j(t)|^{2n} \rangle, \quad j = 0, 1, \quad n = 0, 1, 2, 3, \dots$$

It is convenient to use this derivation for the stochastic relation:

$$\begin{aligned} \mathbf{X}_0(t) &= \begin{cases} 0, & \text{with probability } Q(t); \\ \mathbf{X}_1(t - \tau), & \text{with probability } q(\tau) d\tau, \quad 0 < \tau \leq t; \end{cases} \\ \mathbf{X}_1(t) &= \begin{cases} vt\Omega, & \text{with probability } P(vt)d\Omega/S_d; \\ \xi\Omega + \mathbf{X}_0(t - \xi/v), & \text{with probability } p(\xi)d\xi d\Omega/S_d, \quad 0 < \xi \leq vt, \end{cases} \end{aligned}$$

where Ω is the unite vector with isotropic distribution, S_d is the area of d -dimensional sphere surface with unite radius. Raising the random vectors to power $2n$,

$$|\mathbf{X}_0(t)|^{2n} = \begin{cases} \delta_{n0}, & \text{with probability } Q(t); \\ |\mathbf{X}_1(t - \tau)|^{2n}, & \text{with probability } q(\tau)d\tau, \quad 0 < \tau \leq t; \end{cases}$$

$$|\mathbf{X}_1(t)|^{2n} = \begin{cases} (vt)^{2n}, & \text{with probability } P(vt)d\Omega/S_d; \\ |\xi\Omega + \mathbf{X}_0(t - \xi/v)|^{2n}, & \text{with probability } p(\xi)d\xi d\Omega/S_d, \quad 0 < \xi \leq vt, \end{cases}$$

and averaging the result, we obtain

$$m_0^{(2n)}(t) = \delta_{n0}Q(t) + \int_0^t m_1^{(2n)}(t - \tau)q(\tau)d\tau, \quad (3.74)$$

$$m_1^{(2n)}(t) = P(vt)(vt)^{2n} + \sum_{j=0}^n \binom{2n}{2j} \int_0^{vt} \xi^{2j} m_0^{(2(n-j))}(t - \xi/v)p(\xi)d\xi. \quad (3.75)$$

In particular,

$$m_0^{(2)}(t) = \int_0^t m_1^{(2)}(t - \tau)q(\tau)d\tau, \quad (3.76)$$

and

$$m_1^{(2)}(t) = P(vt)(vt)^2 + \int_0^{vt} \xi^2 p(\xi)d\xi + \int_0^{vt} m_0^{(2)}(t - \xi/v)p(\xi)d\xi. \quad (3.77)$$

The second moments are of special interest because they characterize the width of the diffusion packet as a function of time.

The equations for the moments can be solved by the Laplace transform method which gives

$$\widehat{m}_0^{(2)}(\lambda) = \widehat{q}(\lambda)\widehat{m}_1^{(2)}(\lambda)$$

and

$$\widehat{m}_1^{(2)}(\lambda) = \frac{2h(\lambda/v)}{\lambda[1 - \widehat{q}(\lambda)p(\lambda/v)]}, \quad (3.78)$$

where $\widehat{q}(\lambda)$, $\widehat{p}(\lambda)$, and $h(\lambda)$ are Laplace transforms of functions $q(t)$, $p(\xi)$, and $\xi P(\xi)$ correspondingly.

In case of exponential distributions

$$p(\xi) = \sigma e^{-\sigma\xi}, \quad q(\tau) = \mu e^{-\mu\tau},$$

the Laplace transforms of the moments are

$$\widehat{m}_0^{(2)}(\lambda) = \frac{\mu}{\mu + \lambda} \widehat{m}_1^{(2)}(\lambda)$$

and

$$\widehat{m}_1^{(2)}(\lambda) = \frac{2v^2(\lambda + \mu)}{\lambda^2(\lambda + \sigma v)(\lambda + \mu + \sigma v)}.$$

Inverting the Laplace transform by using the residue technique, we find:

$$m_1^{(2)}(t) = \frac{2\mu vt}{\sigma(\mu + \sigma v)} - \frac{2(\mu - \sigma v)}{\sigma^2 \mu} (1 - e^{-\sigma vt}) - \frac{2\sigma v^3}{(\mu + \sigma v)^2 \mu} \left[1 - e^{-(\mu + \sigma v)t} \right].$$

When there are no traps ($\mu \rightarrow \infty$, $\mu e^{-\mu t} \rightarrow \delta(t)$), this equation becomes the known Monin result (Monin and Yaglom, 1971)

$$m^{(2)}(t) = 2(v/\sigma)t - (2/\sigma^2) (1 - e^{-\sigma vt})$$

whose asymptotic behavior

$$m^{(2)}(t) \sim 2Dt, \quad t \rightarrow \infty$$

is consistent with ordinary diffusion process with diffusivity $D = v/\sigma$. Exponential traps do not change the asymptotic law

$$m^{(2)}(t) \sim 2D^*t, \quad t \rightarrow \infty$$

affecting only the diffusivity

$$D^* = \frac{\mu v / \sigma}{\mu + \sigma v}.$$

In case of power kind distributions $p(\xi)$ and $q(\tau)$, only asymptotic parts of the solutions can be represented in a simple analytical form. They are obtained by means of the Tauberian theorem (Feller, 1971) saying that Laplace transform

$$\widehat{u}(\lambda) = \int_0^\infty e^{-\lambda t} u(t) dt$$

of some monotonic function $u(t)$ has the asymptotical form

$$\widehat{u}(\lambda) \sim c\lambda^{-\rho}, \quad \lambda \rightarrow 0$$

if and only if

$$u(t) \sim ct^{\rho-1}/\Gamma(\rho), \quad \rho > 0, \quad t \rightarrow \infty.$$

At first we consider the case when the mean free path exists,

$$p(\lambda) \sim 1 - \langle \xi \rangle \lambda + C\lambda^\delta, \quad 1 < \delta \leq 2, \quad \lambda \rightarrow 0. \quad (3.79)$$

Taking the time distribution $q(\tau)$ in the form

$$q(\tau) \sim vB\tau^{-v-1}, \quad v < 1, \quad \tau \rightarrow \infty$$

we obtain its Laplace transform asymptotic

$$q(\lambda) \sim 1 - B\Gamma(1 - \nu)\lambda^\nu, \quad \lambda \rightarrow 0. \quad (3.80)$$

Substitution of Eqs. (3.79)–(3.80) and

$$h(\lambda) \sim C(\delta - 1)\lambda^{-(2-\delta)}$$

into Eq. (3.78) yields

$$\hat{m}_1^{(2)}(\lambda) \sim \frac{2C(\delta - 1)\nu^{2-\delta}}{B\Gamma(1 - \nu)}\lambda^{-(3-\delta+\nu)}, \quad \lambda \rightarrow \infty.$$

It follows from here

$$m_1^{(2)}(t) \sim \frac{2C(\delta - 1)\nu^{2-\delta}}{B\Gamma(1 - \nu)\Gamma(3 - \delta + \nu)}t^{2-\delta+\nu}, \quad t \rightarrow \infty. \quad (3.81)$$

Thus, by $\nu = \delta - 1$, we observe the normal diffusion regime. In the case $\nu < \delta - 1$, the width of the diffusion packet grows slower than $t^{1/2}$ (subdiffusion) and in the case $\nu > \delta - 1$, the picture is reversed: we observe superdiffusion. If the variance of the free path is finite ($\delta = 2$), the subdiffusion takes place with the diffusivity

$$D = \frac{C}{B\Gamma(1 - \nu)}$$

being independent of the velocity ν . When both distributions $p(\xi)$ and $q(\tau)$ have infinite variances, which is $\alpha < 1$ and $\nu < 1$, we have got

$$p(\lambda) \sim 1 - A\Gamma(1 - \alpha)\lambda^\alpha, \quad \lambda \rightarrow 0,$$

and application of the Tauberian theorem yields

$$\hat{m}_1^{(2)}(\lambda) \sim \frac{2A\Gamma(2 - \alpha)\nu^2\lambda^{-3}}{A\Gamma(1 - \alpha) + B\Gamma(1 - \nu)\nu^{-\alpha}\lambda^{\nu-\alpha}}, \quad \lambda \rightarrow 0$$

and

$$m_1^{(2)}(t) \sim \begin{cases} (1 - \alpha)\nu^2 t^2, & \alpha < \nu; \\ \frac{A(1 - \alpha)\nu^2}{A + B\nu^\alpha} t^2, & \alpha = \nu; \\ \frac{2A\Gamma(2 - \alpha)\nu^{2-\alpha}}{B\Gamma(3 - \alpha + \nu)\Gamma(1 - \nu)} t^{2-\alpha+\nu}, & \alpha > \nu. \end{cases}$$

Therefore, these cases reveal superdiffusion regime. Subdiffusion arises only if the mean free path is finite and $\nu < \delta - 1$ as it follows from Eq. (3.81).

3.6.7 Exact solution for one-dimensional walk

To obtain a numerical solution of the problem under consideration, we can use different ways. We can perform the inverse Fourier-Laplace transformation of Eq. (3.70), or reconstruct the pdf by using of spatial moments (Uchaikin and Yarovikova, 2003), or using one of the finite-difference schemes, described in the Sect. 5.3, or by means of Monte Carlo method. Here we describe some special method, convenient for solving of one-dimensional problems. It is based on the Neumann's series expansion for integral equation solutions.

At first, we consider the problem without trapping when $q(\tau) = \delta(\tau)$. Taking for the sake of simplicity $p_0 = 0$ ($p_1 = 1$) we rewrite Eqs. (3.60)–(3.62) as follows

$$p(x, t) = (2v)^{-1} \int_0^{vt} d\xi P(\xi) [F_1(x - \xi, t - \xi/v) + F_1(x + \xi, t - \xi/v)], \quad (3.82)$$

$$F_0(x, t) = (1/2) \int_0^{vt} d\xi p(\xi) [F_1(x - \xi, t - \xi/v) + F_1(x + \xi, t - \xi/v)],$$

$$F_1(x, t) = F_0(x, t) + \delta(x)\delta(t).$$

Eliminating $F_0(x, t)$ between the two last equations one obtains

$$F_1(x, t) = (1/2) \int_0^{vt} d\xi p(\xi) [F_1(x - \xi, t - \xi/v) + F_1(x + \xi, t - \xi/v)] + g_0(x, t), \quad (3.83)$$

where

$$g_0(x, t) = \delta(x)\delta(t).$$

According to the Neumann's method we represent the solution of this equation in the form of the series

$$F_1(x, t) = \sum_{n=0}^{\infty} g_n(x, t), \quad (3.84)$$

terms of which obey the recurrent relationship

$$g_{n+1}(x, t) = (1/2) \int_0^{vt} d\xi p(\xi) [g_n(x - \xi, t - \xi/v) + g_n(x + \xi, t - \xi/v)]. \quad (3.85)$$

Using mathematical induction one finds that

$$g_n(x, t) = v(1/2)^{n+1} \sum_{k=0}^n \binom{n}{k} p_k(\xi_+) p_{n-k}(\xi_-), \quad (3.86)$$

where $\xi_+ = (vt + x)/2$, $\xi_- = (vt - x)/2$, and $p_k(\xi)$ are multifold convolutions of $p(\xi)$:

$$p_0(\xi) \equiv \delta(\xi), \quad p_1(\xi) \equiv p(\xi), \quad p_{k+1}(\xi) = \int_0^{\xi} p_k(\xi - \xi') p(\xi') d\xi' \equiv p^{*(k+1)}(\xi).$$

Thus one can write for $x > 0$,

$$p(x, t) = \sum_{n=0}^{\infty} (1/2)^{n+2} \sum_{k=0}^n \binom{n}{k} \left[\int_0^{\xi_+} d\xi P(\xi) p_k(\xi_+ - \xi) p_{n-k}(\xi_-) \right. \\ \left. + \int_0^{\xi_-} d\xi P(\xi) p_k(\xi_+) p_{n-k}(\xi_- - \xi) \right]. \quad (3.87)$$

The restriction $x > 0$ is not essential here because the function $p(x, t)$ is even: $p(x, t) = p(-x, t)$. Remembering that

$$P(x) = \int_x^{\infty} p(\xi) d\xi$$

and expanding this notation on multifold convolutions

$$P_k(x) = \int_x^{\infty} p_k(\xi) d\xi,$$

it is easy to get the following useful formula

$$\int_0^x d\xi P_m(\xi) p_n(x - \xi) = P_{m+n}(x) - P_n(x).$$

Applying it to Eq. (3.87) we have got

$$p(x, t) = \sum_{n=0}^{\infty} (1/2)^{n+2} \sum_{k=0}^n \binom{n}{k} [P'_k(\xi_+) p_{n-k}(\xi_-) \\ + P'_{n-k}(\xi_-) p_k(\xi_+)], \quad (3.88)$$

where

$$P'_k(x) \equiv P_{k+1}(x) - P_k(x).$$

Because $p_0(\xi)$ is Dirac's δ -function, the solution (3.88) has δ -singularities on the lines $x = vt$ and $x = -vt$. These singularities arise through the terms with $(n, k) = (0, 0)$, $(n, 0)$, and (n, n) and relate to the case when the particle moves without changing its direction.

Now we return to the case with trappings, when the master equations system is

$$p(x, t) = \int_0^t d\tau Q(\tau) F_0(x, t - \tau) + (2v)^{-1} \int_0^{vt} d\xi P(\xi) [F_1(x - \xi, t - \xi/v) \\ + F_1(x + \xi, t - \xi/v)], \\ F_0(x, t) = (1/2) \int_0^{vt} d\xi p(\xi) [F_1(x - \xi, t - \xi/v) + F_1(x + \xi, t - \xi/v)], \\ F_1(x, t) = \int_0^t d\tau q(\tau) F_0(x, t - \tau) + \delta(x) \delta(t). \quad (3.89)$$

Instead of Eq. (3.83), we have

$$F_1(x, t) = (1/2) \int_0^t d\tau q(\tau) \int_0^{v(t-\tau)} d\xi p(\xi) [F_1(x - \xi, t - \tau - \xi/v) + F_1(x + \xi, t - \tau - \xi/v)] + g_0(x, t). \quad (3.90)$$

We will seek a solution of the equation in the form

$$F_1(x, t) = g_0(x, t) + \sum_{n=1}^{\infty} \int_0^t d\tau f_n(\tau) g_n(x, t - \tau), \quad (3.91)$$

where the functions $g_n(x, t)$ have been defined by Eq. (3.86) and $f_n(\tau)$ are unspecified yet. Substituting Eq. (3.91) into Eq. (3.90) and taking into account Eq. (3.83), one can see that the latter functions satisfy the recurrence relation

$$f_{n+1}(\tau) = \int_0^t d\tau q(\tau) f_n(t - \tau) \quad (3.92)$$

with $f_1(\tau) = q(\tau)$. This means that the functions are nothing but the multifold convolutions of $q(\tau)$:

$$f_n(\tau) = q_n(\tau). \quad (3.93)$$

According to Eq. (3.89), the distribution density $p(x, t)$ falls into three parts relating to the particle being at rest, moving to the right and moving to the left. The general expressions for these components can be written in terms of the functions $g_n(x, t)$ as follows

$$p^{(0)}(x, t) = \sum_{n=0}^{\infty} \int_0^t d\tau Q(\tau) \int_0^{t-\tau} d\tau' q_n(\tau') g_{n+1}(x, t - \tau - \tau'), \quad (3.94)$$

$$p^{(+)}(x, t) = (2v)^{-1} \sum_{n=0}^{\infty} \int_0^{vt} d\xi P(\xi) \int_0^{t-\xi/v} d\tau q_n(\tau) g_n(x - \xi, t - \tau), \quad (3.95)$$

$$p^{(-)}(x, t) = (2v)^{-1} \sum_{n=0}^{\infty} \int_0^{vt} d\xi P(\xi) \int_0^{t-\xi/v} d\tau q_n(\tau) g_n(x + \xi, t - \tau). \quad (3.96)$$

Observe that the moving particle density

$$p^{(*)}(x, t) = p^{(+)}(x, t) + p^{(-)}(x, t) \quad (3.97)$$

contains the singularity

$$\left\{ p^{(*)}(x, t) \right\}_{\text{sing}} = (1/2) P(vt) [\delta(x - vt) + \delta(x + vt)],$$

which describes the noninteracting particle.

Equations (3.94)–(3.97) give an exact solution of the one-dimensional walk problem in question. Notice that phenomenological analysis performed in (West et al., 1997) gave for this case the result (Eq. (43) therein)

$$p(x, t) = (1/2)[\delta(x - vt) + \delta(x + vt)],$$

so the continuous part of the distribution had been lost.

The main problem of using this approach lies in the computation of multifold convolutions $p_n(\xi)$ and $q_n(\tau)$. Fortunately, there exists a class of distributions having the power type asymptotics and being very suitable for performing this operation. I mean one-dimensional stable distributions

$$p^{(\alpha)}(x) = g_+(x; \alpha),$$

for which

$$p_n^{(\alpha)}(x) = n^{-1/\alpha} p^{(\alpha)}(xn^{-1/\alpha}), \quad n = 2, 3, \dots$$

Using this distribution for $p(\xi)$ and $q(\tau)$ we performed the calculation of the distribution density $\psi(x, t)$ according to formulae (3.84)–(3.87).

3.7 Diffusion on fractals

3.7.1 Diffusion on the Sierpinski gasket

Fractal objects are instructive models of a media nonhomogeneous at all scales. They often reveal unexpected even shocked properties. As an example, we consider walks on the Sierpinski gasket (Fig. 3.14). Let T be the mean first passage time taken to traverse the lattice from the vertex at the apex to one of the remaining two vertices at the bottom (a), and T' be the equivalent time in a lattice rescaled by a factor of two. Let A and B be the first passage times from the inner vertices of the rescaled lattices to the lower O vertices (b). Then,

$$T' = T + A,$$

$$A = T + A/4 + B/4 + T'/4,$$

and

$$B = T + A/2.$$

The solution of this system is

$$T' = 5T, \quad A = 4T, \quad B = 3T.$$

Relating time T to length L in the power way

$$T \sim L^{d_w},$$

and rescaling of space by a factor of two

$$L \mapsto L' = 2L,$$

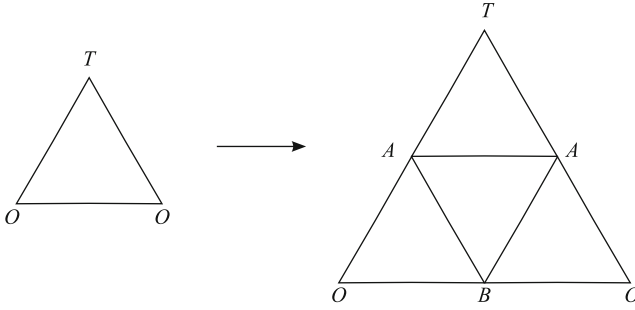


Fig. 3.14 Rescaling the first passage time. The triangle on the left denotes the mean first passage time T to travel from the top apex to the vertices. We then rescale the triangle by a factor of two, giving T' (Ben-Avraham and Havlin, 2000).

we see that

$$T \mapsto T' = 5T$$

and find

$$d_w = \frac{\log 5}{\log 2} \approx 2.32$$

for random walks on the Sierpinski gasket. The mean-square displacement is then given by

$$\langle R^2(t) \rangle \propto t^{2/d_w} \approx t^{0.86}.$$

This result looks unexpected because

$$\langle R^2(t) \rangle \propto t$$

for *any integer dimensionality*. Consequently, one should not interpret diffusion on a fractal as diffusion in a space of a fractional order. One should use also neither the normal propagator

$$p(\mathbf{x}, t) = \frac{1}{(4\pi Kt)^{d/2}} \exp\left(-\frac{|\mathbf{x}|^2}{4Kt}\right), \quad \mathbf{x} \in \mathbb{R}^d,$$

nor the d -dimensional diffusion equation

$$\frac{\partial p(\mathbf{x}, t)}{\partial t} = K \Delta p(\mathbf{x}, t) + \delta(\mathbf{x}) \delta(t).$$

3.7.2 Equation for diffusion on fractals

Evidently, diffusion on the Sierpinski gasket like on other fractal objects is anisotropic. In order to avoid additional complications, O'Shaughnessy and Procaccia (1985)

generalized the radial propagator

$$P(r, t) = \int_{U_d} p(r\mathbf{u}) d\mathbf{u}$$

being the solution of the equation

$$\frac{\partial P(r, t)}{\partial t} = \frac{1}{r^{d-1}} \frac{\partial}{\partial r} \left[K r^{d-1} \frac{\partial P(r, t)}{\partial r} \right], \quad r > 0, \quad K = \text{const} \quad (3.98)$$

under conditions

$$P(r, 0) = 0, \quad t > 0,$$

$$P(\infty, t) = 0, \quad t > 0,$$

and

$$d \int_0^\infty P(r, t) r^{d-1} dr = 1.$$

Substituting d_f and $K_1 r^{-\theta}$ for d and K respectively, they offered the result

$$\frac{\partial P(r, t)}{\partial t} = \frac{1}{r^{d_f-1}} \frac{\partial}{\partial r} \left[K_1 r^{-\theta} r^{d_f-1} \frac{\partial P(r, t)}{\partial r} \right], \quad r > 0, \quad K_1 = \text{const}, \quad (3.99)$$

as an equation for diffusion on fractals, because its solution

$$P(r, t) = \frac{2 + \theta}{d_f \Gamma[d_f/(2 + \theta)]} \left[\frac{1}{(2 + \theta)^2 K_1 t} \right]^{d_f/(2 + \theta)} \exp \left[-\frac{r^{2 + \theta}}{(2 + \theta)^2 K_1 t} \right] \quad (3.100)$$

is characterized by the mean square displacement growing with time according to the power law with a fractional exponent,

$$\langle R^2(t) \rangle = \frac{\Gamma[(d_f + 2)/(2 + \theta)]}{\Gamma[d_f/(2 + \theta)]} [(2 + \theta)^2 K_1 t]^{2/(2 + \theta)} = \langle R^2(1) \rangle t^{2/d_w}, \quad (3.101)$$

where $d_w \equiv 2 + \theta$ is called the *walk dimension*. The probability of return to the origin at time t is given by $P(0, t) \propto t^{-d_s}$, where the exponent

$$d_s = \frac{2d_f}{2 + \theta}$$

is called the *spectral (fracton) dimension*, because it relates to the vibrational density of states of the fractal regarded as a vibrating body (Alexander and Orbach, 1982).

Propagator (3.100) is an exact solution of Eq. (3.99), but the equation itself can not be considered as a strict interrelation: its validation does not look sufficient. Although propagator has a scaling form

$$P(r, t) = t^{-d_w/2} F(rt^{-1/d_w})$$

the form of $F(\xi)$ may vary from that given by Eq. (3.100). (Klafter et al., 1991) performed numerical calculations and found that the O'Shaughnessy and Procaccia form $F(\xi) \sim \exp(-c\xi)$ describes their numerical results well for $\xi \ll 1$, but for $\xi \gg 1$ approximation $F(\xi) \sim \xi^\lambda \exp(-c\xi^\alpha)$ fits the results better.

3.7.3 Diffusion on comb-structures

The concept *percolation* was introduced in (Broadbent and Hammersley, 1957) as opposing to the diffusion. While diffusion assumes a random walk of a particle in a regular environment, percolation consists in a regular motion (of a liquid or electricity) in a random environment. As a model of such an environment, they usually take a periodic lattice whose each site is “occupied” with probability p and empty with probability $1 - p$. The totality of the occupied sites together with their nearest neighbors forms a cluster. The standard problems of percolation theory are the study of distribution of sizes and other geometric characteristics of the clusters, the determination of the so-called percolation threshold, $p = p_c$, when the cluster becomes infinite, and so on. Along with the site percolation, the bond percolation is studied with clusters of connected conducting bonds. The bonds are conducting with the probability p and, correspondingly, blocked with the probability $1 - p$. The model is used for investigation of the problem of the electrical conductivity of a random resistor network. For this problem, another object is relevant—the “backbone” of an infinite percolation cluster defined as the network of unblocked connected bonds, through which one can go to infinity by at least two non-intersecting paths ².

In other words, the backbone is a set of bonds through which the electric current would flow if a voltage is applied to the cluster at infinitely remote electrodes. The rest of the cluster is referred to as a collection of “dead” or “dangling ends”. A dangling end can be disconnected from the cluster by cutting a single bond (see Fig. 3.15 (a)).

The effect of “dead ends” on the diffusion and the drift of particles along percolation cluster is studied in the frame of a simplified model called the “comb structure” (Fig. 3.15 (b)).

The “teeth” of this comb behave as traps in which x -coordinate of the particle stays during time while y -coordinate diffuses along the tooth.

If the length L of the teeth is bounded, the mean waiting time is finite and diffusion along the x -axis is regular, which is $X^2(t) = Kt$ and the Green function

$$p_1(x, t) = \frac{1}{\sqrt{4\pi Kt}} \exp[-x^2/(4Kt)]$$

as $t \rightarrow \infty$.

² The name “percolation” comes from the possible interpretation of bonds as channels open to the flow of a fluid in a porous medium (absent bonds represent blocked channels).

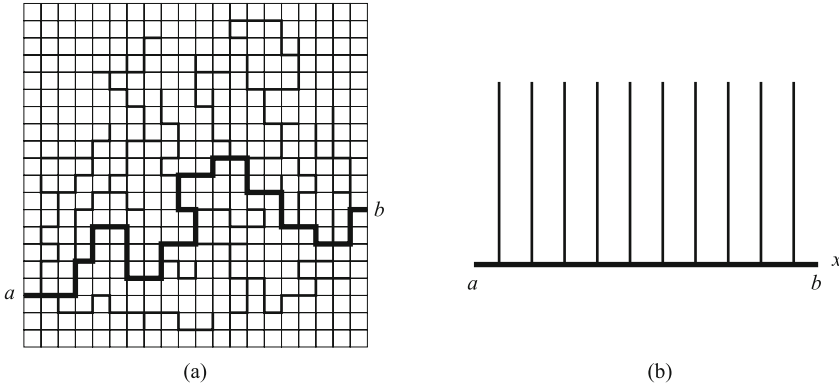


Fig. 3.15 Percolation on a two-dimensional lattice (the backbone is shown in heavy lines) **(a)**, A comb-like structure **(b)**.

For infinitely deep teeth, the waiting time distribution is simply the distribution of the first return time at the origin of a one-dimensional Brownian motion

$$P(T > t) \propto t^{-1/2}, \quad t \rightarrow \infty.$$

Thus the diffusion of the particle along x -axis asymptotically does not differ from the subdiffusion process described above.

This problem was considered in (Nigmatullin, 1986). The equation with the fractional time derivative of the order $1/2$ was obtained, and the correct asymptotic behavior of $\langle x^2(t) \rangle \propto t^\nu$, $\nu = 1/2$ was obtained by means of generating function technique, but the Green function was approximated by the Gauss law. The correct Green function is obtained in (Arkhipcheev and Baskin, 1991):

$$p_{1/2}(x, t) = \frac{D^{1/2}}{\pi \sqrt{Kt}} \int_0^\infty \exp\left(-\frac{x^2}{4K\tau} - \frac{D\tau^2}{t}\right) \frac{d\tau}{\sqrt{\tau}}.$$

Choosing the time scale in such a way that y -diffusivity $D = 1/4$ and using the notation

$$g_+(t; 1/2) = \frac{1}{2\sqrt{\pi}} \exp -\frac{1}{4t} t^{-3/2}, \quad t > 0,$$

we represent this result in the form

$$p_{1/2}(x, t) = \int_0^\infty p_1(x, (t/\tau)^{1/2}) g_+(\tau; 1/2) d\tau.$$

This form holds in case when the teeth have different independent random lengths with a common distribution $P(L > l) \propto l^{-\gamma}$, $0 < \gamma < 1$. Ben-Avraham and Havlin (2000) show that in this case $P(T > t) \propto t^{-\nu}$, $\nu = (1 + \gamma)/2$, therefore, one can expect that

$$p_v(x, t) = \int_0^\infty p_1(x, (t/\tau)^v) g_+(\tau; v) d\tau, \quad v \geq 1/2.$$

The importance of percolation lies in the fact that it models critical phase transitions of rich physical content, yet it may be formulated and understood in terms of very simple geometrical concepts. There exists a large variety of percolation models: the models defined on a triangular lattice, in different space dimensions, continuum percolation which is defined without resorting to a lattice, and so on (see for details (Ben-Avraham and Havlin, 2000)).

3.7.4 Some more on a one-dimensional fractal dust

Lévy flights/walks are often called *fractal flights/walks*, and some people understand them as *flights/walks on fractals*. However, this is not the case. The fact of the matter is that a fractal process loses its memory in each events, while a process on a fractal can preserve the memory about all preceding events related to its way through a particular environment. One can say that in the first case, the walker continues its motion after each event in a perfectly new environment than before.

This is why Eq. (3.98) should not be treated as an exact equation: it describes a Markovian process in a regular nonhomogeneous media, which looks regular at small scales, meanwhile a real walk on a fractal is a non-Markovian process. Perhaps, the most convenient way to satisfy ourselves of that is given by a one-dimensional flights: when the walker goes back it meets the same domain of the fractal as before (Fig. 3.16). Namely these correlations are simulated by introducing the spatial dependence of diffusivity on the distant,

$$K(r) = K_1 r^{-\theta}. \quad (3.102)$$

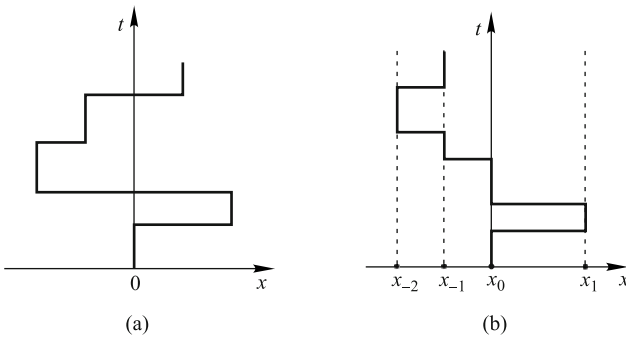


Fig. 3.16 Fractional diffusion (a) and diffusion on fractal (b).

In case of Eq. (3.99), the distance r is reading from a fixed point (the origin), but in case of a real fractal, this law must be valid relative to any point of the

fractal. This is especially clear in one-dimensional Lévy flights on the *Lorentz gas*. The latter concept terming a set of non-interacting point particles was introduced by Lorentz (1905) and modified by Barkai et al. (2000) into the *stochastic Lévy-Lorentz gas*. This is none other than the fractal dust model described in Sect. 3.5. According to this model, the substance is concentrated at points $\{X_j\} = \dots, X_{-2}, X_{-1}, X_0 = 0, X_1, X_2, \dots$ (atoms), randomly placed on x -axis. Distances $X_j - X_{j-1} = Y_j$ are independent identically distributed random variables with a common distribution function

$$F(x) = \mathbf{P}\{Y < x\} = \int_0^x p_Y(x) dx.$$

On this assumption, the random values X_j form a correlated random sequence, or one can say, two independent sequences $X_1, X_2, \dots$ and $X_{-1}, X_{-2}, \dots$ with a common initial point $X_0 = 0$.

Let $F_n(x)$ be the n -fold convolution of the distribution function $F(x)$:

$$F_{n+1}(x) = \int_0^x F_n(x-y) dF(y), \quad F_1(x) = F(x)$$

and let $N_+(x)$ be the random number of atoms belonging to the interval $(0, x]$. Then

$$W(n, x) \equiv \mathbf{P}\{N_+(x) = n\} = F_n(x) - F_{n+1}(x).$$

The similar relation takes place for the number $N_-(x)$ of atoms in the interval $[-x, 0)$. Note that the total number of atoms in the interval $[-x, x]$ is the sum

$$N(x) = N_+(x) + N_-(x) + 1.$$

Thus, choosing different functions $F(x)$ we obtain random sets of various kinds. Consider a few cases.

CASE 1. Choosing the distribution function in the form of the Heaviside function

$$F(x) = 1(x - a)$$

we obtain a one-dimensional lattice with parameter a

$$W(n, x) = 1(x - an) - 1(x - an - a) = \begin{cases} 1, & an \leq x < an + a, \\ 0, & x < an \text{ or } x \geq an + a. \end{cases}$$

CASE 2. Taking exponential distribution for $F(x)$

$$F(x) = 1 - \exp\{-\mu x\}$$

we have got

$$W(n, x) = \int_0^x \mu \frac{(\mu y)^n}{n!} e^{-\mu y} dy - \int_0^x \mu \frac{(\mu y)^{n+1}}{(n+1)!} e^{-\mu y} dy = \frac{(\mu x)^n}{n!} e^{-\mu x}.$$

This is nothing but a *homogeneous Poisson process* with the mean value

$$\langle N(x) \rangle = \mu x.$$

The relative deviation of the random variable $N(x)$

$$\frac{\Delta(x)}{\langle N(x) \rangle} \equiv \sqrt{\langle N^2(x) \rangle - \langle N(x) \rangle^2} / \langle N(x) \rangle = \langle N(x) \rangle^{-1/2} = (\mu x)^{-1/2}$$

disappear as $x \rightarrow \infty$.

Note that in this case the numbers of atoms belonging to differ disjoint domains are independent.

CASE 3. Now we don't use any concrete expression for $F(x)$. We suppose only that its variance σ^2 is finite. Naturally, on this assumption only asymptotical results are available. The larger x the larger values of n play the leading role in consideration. Application of the central limit theorem yields

$$F_n(x) \sim \Phi(\xi_n),$$

where

$$\Phi(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\xi} \exp\left(-\frac{x^2}{2}\right) dx$$

is normal distribution function, $\xi_n = (x - n/\mu)/(\sigma\sqrt{n})$ and $\mu = \langle Y \rangle^{-1}$. Asymptotically

$$F_n(x) - F_{n+1}(x) = \Phi(\xi_n) - \Phi(\xi_{n+1}) \sim \Phi'(\xi_n)(\xi_n - \xi_{n+1}),$$

and we arrive to normal distribution with mean value $\langle N(x) \rangle = \mu x$ and relative deviation

$$\frac{\Delta(x)}{\langle N(x) \rangle} \sim \frac{\mu\sigma}{\sqrt{\mu x}}, \quad x \rightarrow \infty.$$

CASE 4. Let the distribution function $F(x)$ obey the asymptotic relation

$$1 - F(x) \sim \frac{1}{\Gamma(1-\alpha)} (c_\alpha x)^{-\alpha}, \quad \alpha > 0, \quad x \rightarrow \infty.$$

For $\alpha > 2$, the variance of Y is finite and we arrive at the previous example. For $\alpha < 2$, the variance is infinite and we have a qualitatively another kind of medium called *Lévy-Lorentz gas*. Two cases arise: $1 < \alpha < 2$, when mathematical expectation exists, and $0 < \alpha < 1$, when mathematical expectation does not exist.

In the first case, according to the generalized limit theorem

$$F_n(x) \sim G(\xi_n; \alpha, \theta_\alpha), \quad \xi_n = c_\alpha(x - n/\mu)n^{-1/\alpha}, \quad n \rightarrow \infty,$$

where $G(\xi_n; \alpha, \theta_\alpha)$ is a totally skewed stable distribution function with characteristic exponent $\alpha \in (1, 2)$, and c_α is a positive constant. One can show that in this example $\langle N(x) \rangle \sim \mu x$ as before but the relative fluctuations fall at large distances slower:

$$\frac{\Delta(x)}{\langle N(x) \rangle} \propto (\mu x)^{-1/\alpha}, \quad x \rightarrow \infty.$$

Considering some characteristic of the medium as a smooth function $f(N(x), x)$ of the random variable $N(x)$ we get

$$f(N(x), x) \rightarrow f(\langle N(x) \rangle, x), \quad x \rightarrow \infty.$$

This means that the depth growth provides for averaging over whole statistical ensemble on the basis of a unique sample only. The property called *self-averaging* joins all above cases to a class of *asymptotically regular media*.

In the second case ($0 < \alpha < 1$), the generalized limit theorem yields

$$F_n(x) \sim G_+(\xi_n; \alpha), \quad \xi_n = c_\alpha x n^{-1/\alpha}$$

and therefore

$$W(n, x) \sim g_+(\xi_n, \alpha)(\xi_{n+1} - \xi_n) \sim \alpha^{-1} g_+(c_\alpha x n^{-1/\alpha})(c_\alpha x) n^{-1/\alpha-1}, \quad x \rightarrow \infty.$$

The distribution takes a scaling form

$$W(n, x) dn = w(z, \alpha) dz, \quad z = n / \langle N(x) \rangle = n \Gamma(1 + \alpha) (c_\alpha x)^{-\alpha},$$

$$w(z, \alpha) = \frac{1}{\alpha z} \left(\frac{\Gamma(1 + \alpha)}{z} \right)^{1/\alpha} g_+ \left(\left(\frac{\Gamma(1 + \alpha)}{z} \right)^{1/\alpha}; \alpha \right).$$

Thus, we obtain a random point structure with the following properties.

1. All atoms of the set are equal in rights by construction so that all processes $(X_j, X_j + x)$ are statistically equivalent: $N(X_j, X_j + x) \stackrel{d}{=} N(x)$ ($\stackrel{d}{=}$ means equality in distribution).
2. The mean number of atoms grows with depth x according to power law

$$\langle N(x) \rangle \propto x^\alpha, \quad 0 < \alpha < 1.$$

3. The relative deviations i.e., statistical fluctuations are the same at all depths and this structure is random at all scales.

This is a case of a *stochastic fractal Lorentz gas with fractal dimension α* embedded in a one-dimensional space. On the contrary to regular media stochastic fractals don't possess the property of self-averaging: while for a regular medium, we have

$$\langle f(N(x), x) \rangle \rightarrow f(\langle N(x) \rangle, x), \quad x \rightarrow \infty,$$

for a fractal Lorentz gas, we obtain

$$\langle f(N(x), x) \rangle \rightarrow \int_0^\infty f(N_1 x^\alpha z, x) w(z, \alpha) dz, \quad x \rightarrow \infty.$$

3.7.5 Flights on a single sample

Now we consider a random sequence of moments $\{T_j\} = T_1, T_2, T_3, \dots$ when the walker performs jumps. Let the sequence is built in the same manner as the sequence $X_1, X_2, X_3, \dots$ namely $0 < T_1 < T_2 < T_3 < \dots$, and $T_1, T_2 - T_1, T_3 - T_2, \dots$ are mutually independent and identically distributed random variables with a common probability distribution function $Q(t) = P\{T_{j+1} - T_j < t\}$.

The all four examples considered above can be extended to this case and we obtain discrete times in the first example, a homogeneous Poisson flow in the second one, an asymptotically (on large scales) homogeneous flow in the third example, and a non-homogeneous flow selfsimilar (in the stochastic sense) on all scales in the last example.

The random process $K(t)$ is defined in the same manner as the $N_+(x)$. Assuming in particular

$$1 - Q(t) \sim \frac{1}{\Gamma(1-\nu)} (c_\nu t)^{-\nu}, \quad t \rightarrow \infty, \quad 0 < \nu < 1,$$

we obtain

$$W(k, t) dk \sim w(z, \nu) dz,$$

where

$$z = k / \langle K(t) \rangle, \quad \langle K(t) \rangle = \frac{1}{\Gamma(1+\nu)} (c_\nu t)^\nu \equiv K_1 t^\nu.$$

Thus, the times of jumps $T_1, T_2, T_3, \dots$ form a fractal set on the time axis with fractal dimensionality β .

The next step is consideration of a test particle walking on the fixed set of atoms $\{x_j\}$ arranged irregularly on a line. The test particle appears at time $t = 0$ at the origin $x_0 = 0$ and stays there up to time $T_1 > 0$. At time $t = T_1$, it performs an instantaneous jump to one of the neighboring atoms: to x_{-1} with probability q_+ or to x_1 with probability q_- . It stays there up to time T_2 and then jumps again with the same probabilities to one of neighbors and so on.

This algorithm generates an ensemble of the particle trajectories $\{X_\theta(t; \{x_j\})\}$ on the set of fixed atoms $\{x_i\}$, where $\theta = q_+ - q_-$ is a skewness of the walking. Averaging over this ensemble will be denoted by the overbar:

$$F_\theta(x, t | \{x_j\}) \equiv P\{X_\theta(t; \{x_j\}) < x\} = \overline{H(x - X_\theta(t; \{x_j\}))}.$$

Let $J_\theta(t)$ be a random number of the atom where the particle stays at the moment t

$$X_\theta(t; \{x_j\}) = x_{J_\theta(t)}.$$

Taking into account monotonicity of the function x_j ($j = \dots, -1, 0, 1, \dots$) we obtain

$$F_\theta(x, t | \{x_j\}) = \overline{H(n(x) - J_\theta(t))},$$

where $n(x)$ obeys the equation

$$x_n \leq x \leq x_{n+1}, \quad n = \dots, -1, 0, 1, \dots$$

Further,

$$J_\theta(t) = \sum_{j=1}^{K(t)} U_j,$$

where the independent random variables $U_j = \pm 1$ with probabilities $q_\pm$ and $K(t)$ is the random number of jumps up to time t (U_j and K are independent of each other). According to the central limit theorem

$$\mathbf{P}\{J_\theta < n | K = k\} \sim G^{(2,0)}((n - k\theta)/\sqrt{(1 - \theta^2)k}), \quad k \rightarrow \infty,$$

where

$$G^{(2,0)}(x) \equiv (2\sqrt{\pi})^{-1} \int_{-\infty}^x \exp(-x^2/4) dx$$

is the stable probability function with the characteristic exponent 2, in other words the normal Gaussian with a doubled variance. Then

$$\mathbf{P}\{J_\theta < n\} \sim \overline{G^{(2,0)}((n - K\theta)/\sqrt{(1 - \theta^2)K})}$$

and as a result

$$F_\theta(x, t | \{x_j\}) \sim \sum_{k=1}^{\infty} G^{(2,0)}((n(x) - k\theta)/\sqrt{(1 - \theta^2)k}) W(k, t), \quad t \rightarrow \infty,$$

where

$$W(k, t) = \mathbf{P}\{K(t) = k\}.$$

We meet here with subdiffusion behavior which is described by the distribution function

$$F_\theta(x, t; \{x_j\}) \sim \int G^{(2,0)}((n(x) - z\theta K_1 t^\nu)/((1 - \theta^2)zK_1 t^\nu)^{-1/2}) w(z, \nu) dz.$$

In the symmetric case $\theta = 0$,

$$F_0(x, t; \{x_j\}) \sim \Psi^{(2,\nu)}(n(x)(Dt^\nu)^{-1/2}), \quad x \rightarrow \infty,$$

where

$$\Psi^{(2,\nu)}(x) = \int_{-\infty}^x \psi^{(2,\nu)}(x) dx$$

and $D = c_\nu^\nu/2$.

In the extreme case $\theta = 1$, the walker can not jump to the negative direction of the axis,

$$\lim_{\theta \rightarrow 1} G^{(2,0)} \left((n(x) - z\theta K_1 t^\nu) ((1 - \theta^2) z K_1 t^\nu)^{-1/2} \right) = H(n(x) - z K_1 t^\nu)$$

and the probability distribution function takes the form

$$F_1(x, t; \{x_j\}) = G_+([\Gamma(1 + \nu)]^{-1} [K_1/n(x)]^{1/\nu} t; \nu).$$

3.7.6 Averaging over the whole fractal ensemble

The final stage of the problem solution is averaging the conditional distribution function $F(x, t | \{x_j\})$ over all possible arrangements of atoms,

$$F(x, t) = \langle F(x, t; \{X_j\}) \rangle.$$

For an asymptotically regular medium, we obtain

$$\begin{aligned} F(x, t) &\sim \langle \Psi^{(2,\nu)}(N(x)(Dt^\nu)^{-1/2}) \rangle \\ &\sim \Psi^{(2,\nu)} \left(\langle N(x) \rangle (Dt^\nu)^{-1/2} \right) = \Psi^{(2,\nu)} \left(N_1 (Dt^\nu)^{-1/2} x \right). \end{aligned}$$

This result coincides with the corresponding FD distribution (formulas (35) and (36) in (Uchaikin, 1999)).

For a fractal Lévy-Lorentz gas, we have got for $x \geq 0$ (the distribution is symmetrical with respect to $x = 0$)

$$F(x, t) \sim \langle \Psi^{(2,\nu)} \left(N(x)(Dt^\nu)^{-1/2} \right) \rangle = \int_0^\infty \Psi^{(2,\nu)} \left(z N_1 (Dt^\nu)^{-1/2} x^\alpha \right) w(z, \alpha) dz.$$

Passing to the variate

$$y = \frac{z^{-1/\alpha}}{\Gamma(1 + \alpha)}$$

and taking into account that

$$w(z, \alpha) dz = g_+(y; \alpha) dy$$

we rewrite the above result in the form

$$F(x, t) \sim \Xi^{(\alpha,\nu)} \left((Dt^\nu)^{-\nu/2\alpha} x \right),$$

where

$$\Xi^{(\alpha, \nu)}(x) = \int_0^\infty \Psi^{(2, \nu)}((x/y)^\alpha) g_+(y; \alpha) dy$$

is the new distribution function describing the form of diffusion packet of the particle walking on the fractal with fractal dimensionality α , $D' = \text{const} > 0$.

Consider the case $\alpha = 1/2$ and $\nu = 1$. In this case

$$\Xi^{(1/2, 1)}(x) = \int_0^\infty \Psi^{(2, 1)}(\sqrt{x/y}) g_+(y; 1/2) dy$$

and the density function

$$\xi^{(1/2, 1)}(x) = d\Xi^{(1/2, 1)}(x)/dx = (\alpha/x^{1-\alpha}) \int_0^\infty \Psi^{(2, 1)}(\sqrt{x/y}) y^{-1/2} g_+(y; 1/2) dy.$$

Substituting here

$$\Psi^{(2, 1)}(x) = \frac{1}{2\sqrt{\pi}} \exp(-x^2/4)$$

and

$$g_+(y; 1/2) = \frac{1}{2\sqrt{\pi}} y^{-3/2} \exp(-1/4y)$$

we easily obtain:

$$\xi^{(1/2, 1)}(x) = \frac{1}{2\pi\sqrt{x}(x+1)}, \quad x > 0.$$

The DF distribution density $\xi^{(1/2, 1)}(x)$ with the corresponding DF distribution $\Psi^{(1/2, 1)}(x)$ are shown in Fig. 3.17.

Three important conclusions can be extracted from obtained results.

1. The fractal media does not possess of the self-averaging property:

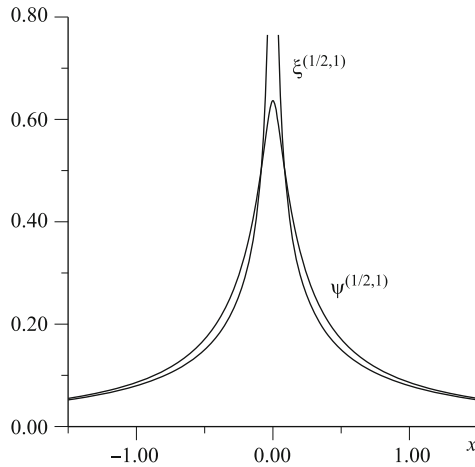


Fig. 3.17 DF- and FD- distribution densities with parameters $\alpha = 1/2$ and $\nu = 1$.

$$p(x, t | \{x_j\}) \neq \langle p(x, t | \{X_j\}) \rangle$$

for $t \rightarrow \infty$.

2. The DF-packet grows in width as $t^{\beta/2\alpha}$, i.e., much slower than the corresponding FD-packet the width of which is $\sim t^{\beta/\alpha}$. This is the effect of neighboring atoms playing the role of some kind of traps (see Fig. 3.17).
3. The DF- and FD- packet forms essentially differ from each other (see Figs. 3.17–3.19), but both of them are expressed through the stable distribution densities. The explicit expression are brought above.

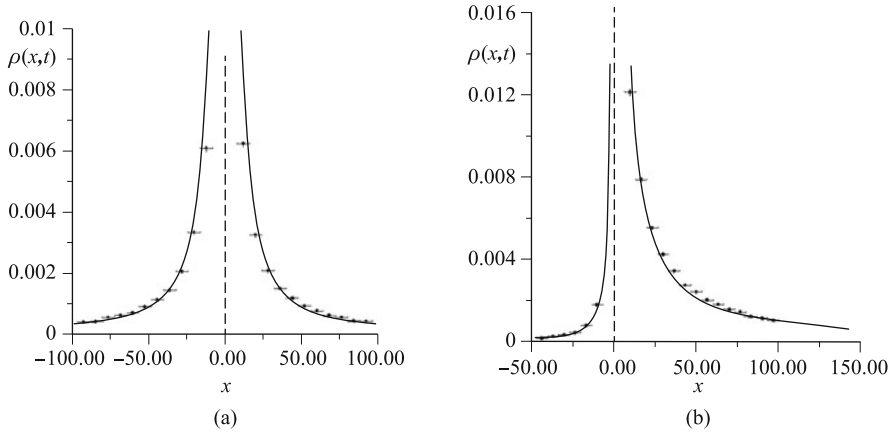


Fig. 3.18 Probability density function $p(x, t)$: **(a)** ($\alpha = 0.5$, $\nu = 0.5$, $q_+ = 0.5$, $t = 100$); **(b)** ($\alpha = 0.5$, $\nu = 0.5$, $q_+ = 0.7$, $t = 100$).

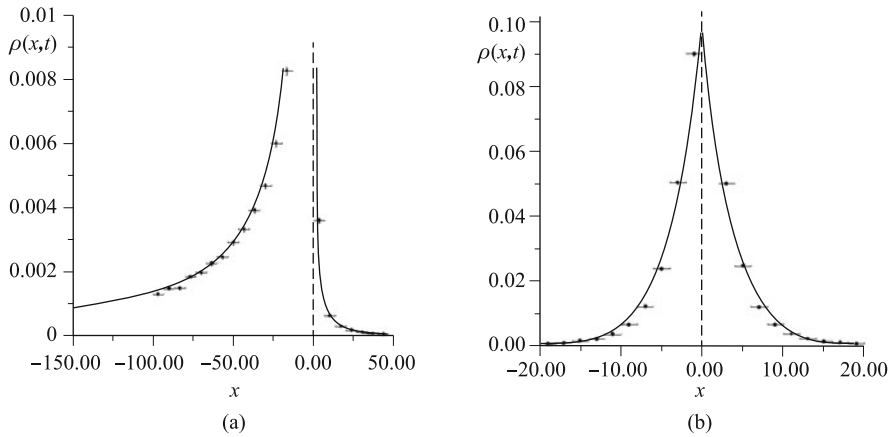


Fig. 3.19 Probability density function $p(x, t)$: **(a)** ($\alpha = 0.5$, $\nu = 0.5$, $q_+ = 0.2$, $t = 100$); **(b)** ($\alpha = 1$, $\nu = 0.5$, $q_+ = 0.5$, $t = 100$).

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Part II

Theory

Chapter 4

Fractional Differentiation

From mathematical point of view, fractional derivative ${}_a f^{(\nu)}(x)$ of order ν is a function of three variables: the lower limit a , the argument x and the order ν . Naming this functional the derivative, we believe that in case of integer ν , $\nu = n$, it coincides with the n -order derivative. Extending the interrelation $df^{(n)}(x)/dx = f^{(n+1)}(x)$ to negative values of order, we interpret $f^{(-m)}(x)$, $m > 0$, as integrals, and $f^{(0)}(x) = f(x)$. Now, the function ${}_a f^{(\nu)}(x)$ can be considered as an analytic continuation of $f^{(n)}(x)$, $n = \dots, -2, -1, 0, 1, 2, \dots$ saving basic properties of multiple derivatives.

4.1 Riemann-Liouville fractional derivatives

Reading the first part, we had possibility to make sure in ubiquitous of heredity, nonlocality and selfsimilarity in nature. Combining these properties of observed phenomena, we choose the power function Φ_μ , $0 < \mu < 1$ as an influence function in a hereditary integral:

$$g(x) = \int_0^x \Phi_\mu(x - \xi) f(\xi) d\xi = \Phi_\mu \star f(x).$$

Recall, that

$$\widehat{g}(\lambda) = \widehat{\Phi}_m(\lambda) \widehat{f}(\lambda) = \lambda^{-m} \widehat{f}(\lambda), \quad m = 1, 2, 3, \dots,$$

and consequently $\lambda^{-m} \widehat{f}(\lambda)$ is the Laplace image of m -fold integral of function $f(x)$,

$$\lambda^{-m} \widehat{f}(\lambda) = \mathcal{L} \{ {}_0 I_x^m f(x) \} (\lambda), \quad {}_0 I_x^m f(x) \equiv \int_0^x d\xi_m \int_0^{\xi_m} d\xi_{m-1} \cdots \int_0^{\xi_2} d\xi_1 f(\xi_1).$$

Thus, the product $\lambda^{-\mu} \widehat{f}(\lambda)$ can be considered as an interpolation of m -fold integral to fractional multiplicity μ :

$$\lambda^{-\mu} \widehat{f}(\lambda) = \mathcal{L} \{ {}_0I_x^\mu f(x) \} (\lambda),$$

$${}_0I_x^\mu f(x) = \frac{1}{\Gamma(\mu)} \int_0^x (x-\xi)^{\mu-1} f(\xi) d\xi = \Phi_\mu \star f(x), \quad \mu > 0.$$

Observe that change $\mu \mapsto \mu - 1$ in the integral is equivalent to action of the differentiation operator $D_x \equiv d/dx$. When $\mu \rightarrow 0$, the fractional integral $f_\mu(x)$ becomes the function $f(x)$ itself. Naively, one could expect that continuous passing μ from positive values to negative turns it into the derivative of a fractional order $\nu = -\mu$. However, the integral

$${}_0I_x^\mu f(x) = \frac{1}{\Gamma(\mu)} \int_0^x (x-\xi)^{\mu-1} f(\xi) d\xi$$

with $\mu < 0$ diverges. Thus, we have to choose another way: leaving $\mu = 1 - \nu$ inside the interval $(0, 1)$, to act on the integral by the operator D_x :

$${}_0D_x^\nu f(x) = D_x {}_0I_x^{1-\nu} f(x) = \frac{1}{\Gamma(1-\nu)} D_x \int_0^x (x-\xi)^{-\nu} f(\xi) d\xi = D_x \Phi_{1-\nu} \star f(x),$$

$$0 \leq \nu < 1.$$

This passage will look more natural, if we replace ${}_0I_x^\mu$, $\mu > 0$ by the symbol ${}_0D_x^{-\mu}$:

$${}_0D_x^\nu = D_x {}_0D_x^{\nu-1}, \quad 0 < \nu < 1.$$

We shall continue to interpret the fractional integral of order μ as a fractional derivative of the negative order $\nu = -\mu$:

$${}_0D_x^\nu f(x) = \frac{1}{\Gamma(-\nu)} \int_0^x (x-\xi)^{-\nu-1} f(\xi) d\xi = \Phi_{-\nu} \star f(x), \quad \nu < 0.$$

Using the notation

$${}_0f^{(\nu)}(x) = {}_0D_x^\nu f(x),$$

we can write the following consequence of equalities:

$$\begin{aligned} {}_0f^{(\nu)}(x) &= \Phi_{-\nu} \star f(x), \quad \nu < 0, \\ {}_0f^{(\nu)}(x) &= D_x {}_0f^{(\nu-1)}(x), \quad 0 \leq \nu < 1, \\ {}_0f^{(\nu)}(x) &= D_x^2 {}_0f^{(\nu-2)}(x), \quad 1 \leq \nu < 2, \\ {}_0f^{(\nu)}(x) &= D_x^3 {}_0f^{(\nu-3)}(x), \quad 2 \leq \nu < 3, \\ &\dots\dots\dots, \\ {}_0f^{(\nu)}(x) &= D_x^n {}_0f^{(\nu-n)}(x), \quad n-1 \leq \nu < n, \end{aligned}$$

and so on.

For the sake of evidence, we consider this procedure in detail. Suppose, we have to compute the derivative ${}_0f^{(3.4)}(x)$ (asterisk on ν -axis, Fig. 4.1). We are able to find derivatives of any (integer or fractional) negative order and integer positive orders.

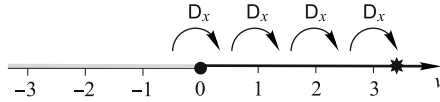


Fig. 4.1 Passage from fractional integral to fractional derivatives.

Let us make a few unit steps from point $v = 3.4$ to the left in order to reach the negative region. Let n be a number of such steps up to the first visit the region. It occurs in the interval $(-1, 0)$. In our case $n = 4$, and we arrive at a point with the negative coordinate $v - n = -0.6$. The corresponding derivative (of order -0.6) is computed by means of the fractional formula Cauchy:

$${}_0f^{(v-n)}(x) \equiv {}_0f^{0.6}(x) = \frac{1}{\Gamma(0.6)} \int_0^x (x - \xi)^{0.6-1} f(\xi) d\xi.$$

In order to return in the initial position, one has to make the same number of steps to the right. Each step involves the ordinary differential operator, and after n steps we obtain the n th power of the operator. As a result, we have

$${}_0f^{(3.4)}(x) = D_x^4 {}_0f^{(-0.6)}(x) = \frac{1}{\Gamma(0.6)} \frac{d^4}{dx^4} \int_0^x \frac{f(\xi) d\xi}{(x - \xi)^{0.4}}.$$

Observe that number n is $[v]$ (an integer part of v) plus 1, so in case of an arbitrary positive order v and any $a < x$,

$${}_af^{(v)}(x) = D^{[v]+1} {}_aI_x^{1-\{v\}} f(x) = \frac{1}{\Gamma(1 - \{v\})} D_x^{[v]+1} \int_a^x \frac{f(\xi) d\xi}{(x - \xi)^{\{v\}}},$$

where $\{v\}$ means a fractional part of v , $0 \leq \{v\} < 1$, $v = [v] + \{v\}$.

The explicit expressions for fractional derivatives, which extended to an arbitrary lower limit a , read:

$${}_af^{(v)}(x) \equiv {}_aD_x^v f(x) = \begin{cases} \frac{1}{\Gamma(-v)} \int_a^x (x - \xi)^{-v-1} f(\xi) d\xi, & v < 0, \\ \frac{1}{\Gamma(1-v)} D_x \int_a^x (x - \xi)^{-v} f(\xi) d\xi, & 0 \leq v < 1, \\ \frac{1}{\Gamma(2-v)} D_x^2 \int_a^x (x - \xi)^{-v+1} f(\xi) d\xi, & 1 \leq v < 2, \\ \dots\dots, & \dots\dots \end{cases}$$

In the fractional derivatives considered above, integration were performed from a constant lower limit to a variable upper one, which serves (in case of positive v) as a variable of differentiation. Such derivatives are called *left-sided fractional Riemann-Liouville derivatives*. Their *right-sided* counterparts are determined via relation

$$f_b^{(\nu)}(x) \equiv {}_x D_b^\nu f(x) = \begin{cases} \frac{1}{\Gamma(-\nu)} \int_x^b (\xi - x)^{-\nu-1} f(\xi) d\xi, & \nu < 0; \\ \frac{1}{\Gamma(1 - \{\nu\})} (-D_x)^{[\nu]+1} \int_x^b \frac{f(\xi) d\xi}{(\xi - x)^{\{\nu\}}}, & \nu \geq 0, \end{cases}$$

where $b > x$. Fractional derivatives ${}_{-\infty} f^{(\nu)}(x)$ are often called the *Liouville derivatives* (if $\nu > 0$) or *integrals* (if $\nu < 0$), while $f_\infty^{(\nu)}(x)$ are called the *Weyl derivatives* or *integrals* respectively.

For the reader who wish to delve more deeply into the theory of fractional operators, the books (Oldham and Spanier, 1974; Samko et al., 1993; Miller and Ross, 1993; Podlubny, 1999) can be recommended.

4.2 Properties of R-L fractional derivatives

Let us list basic properties of the Riemann-Liouville fractional derivatives.

4.2.1 Elementary properties

1. Linearity:

$${}_a D_x^\nu (bf(x) + cg(x)) = b {}_a D_x^\nu f(x) + c {}_a D_x^\nu g(x). \quad (4.1)$$

2. Linear transformation of limits:

$${}_{ab+c} D_x^\nu f(x) = b^{-\nu} {}_a D_z^\nu g(z), \quad g(z) = f(bz + c), \quad z = (x - c)/b, \quad b > 0. \quad (4.2)$$

3. Linear transformation of the argument:

$${}_a D_x^\nu f(bx + c) = b^\nu {}_{ab+c} D_z^\nu f(z), \quad z = bx + c, \quad b > 0.$$

4. Reflection formula: let

$$R_c f(x) \equiv f(c - x), \quad c \in \mathbb{R},$$

then for $\nu < 0$, the following relations take place

$${}_a D_x^\nu = R_{a+b} {}_x D_b^\nu R_{a+b} f(x),$$

and

$${}_{-\infty} D_x^\nu = R_0 {}_x D_\infty^\nu R_0 f(x).$$

4.2.2 The law of exponents

If μ and ν are positive real numbers, the following relations called the law of exponent take place:

$$\begin{aligned}
 {}_aD_x^{-\mu} {}_aD_x^{-\nu} f(x) &= {}_aD_x^{-\mu-\nu} f(x); \\
 {}_aD_x^{\mu} {}_aD_x^{-\nu} f(x) &= {}_aD_x^{\mu-\nu} f(x), \quad \nu \geq \mu; \\
 {}_aD_x^{-\mu} {}_aD_x^{\nu} f(x) &= {}_aD_x^{-\mu+\nu} f(x) - \sum_{j=1}^n a f^{(\nu-j)}(a+) \frac{(x-a)^{\mu-j}}{\Gamma(\mu-j+1)}; \\
 {}_aD_x^{\mu} {}_aD_x^{\nu} f(x) &= {}_aD_x^{\mu+\nu} f(x) - \sum_{j=1}^n a f^{(\nu-j)}(a+) \frac{(x-a)^{-\mu-j}}{\Gamma(1-\mu-j)}; \\
 m-1 < \mu \leq m, \quad n-1 < \nu \leq n, \quad \mu + \nu < n, m, n = 1, 2, 3, \dots
 \end{aligned}$$

Here

$${}_a f^{(\nu)}(a+) = \lim_{\varepsilon \downarrow 0} {}_a D_{a+\varepsilon}^{\nu} f(a+\varepsilon).$$

4.2.3 Inverse operators

Now, let us compose an operator from two differential operators with exponents of opposite signs. The equality

$${}_a D_x^{\nu} {}_a D_x^{-\nu} f(x) = f(x), \quad \nu > 0$$

holds for any summable function $f(x)$, meanwhile the reverse order of the operators yields a different result,

$${}_a D_x^{-\nu} {}_a D_x^{\nu} f(x) = f(x) - \sum_{j=1}^n a f^{(\nu-j)}(a+) \frac{(x-a)^{\nu-j}}{\Gamma(\nu-j)}, \quad n-1 < \nu \leq n = 1, 2, \dots,$$

generalizing the known Newton-Leibnitz formula. In particular, for $0 < \nu \leq 1$,

$${}_a D_x^{-\nu} {}_a D_x^{\nu} f(x) = f(x) - \frac{(x-a)^{\nu-1}}{\Gamma(\nu)} a f^{(\nu-1)}(a+).$$

Thereby, ${}_a D_x^{\nu}$ is the *left inverse operator* for ${}_a D_x^{-\nu}$.

4.2.4 Differentiation of a power function

Power functions play the central role in fractional calculus. Performing fractional differentiation of the power functions, we don't leave this family. Indeed, replacing

$f(x)$ by $(x-a)^m$, $m > -1$, in the R-L derivative of order ν , $n-1 \leq \nu < n$,

$${}_a[(x-a)^m]^{(\nu)} \equiv \frac{1}{\Gamma(n-\nu)} \left(\frac{d}{dx} \right)^n \int_a^x (x-\xi)^{n-\nu-1} (\xi-a)^m d\xi,$$

substituting $\theta = (\xi-a)/(x-a)$ and using the known formula

$$\int_0^1 (1-\theta)^{\alpha-1} \theta^{\beta-1} d\theta = B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)},$$

we find that

$${}_a[(x-a)^m]^{(\nu)} = \frac{\Gamma(m+1)}{\Gamma(m+1-\nu)} (x-a)^{m-\nu}.$$

In terms of standard power functions, $\Phi_\mu(x-a) = (x-a)_+^{\mu-1} / \Gamma(\mu)$, $\mu = m+1 > 0$, we have:

$${}_a D_x^\nu \Phi_\mu(x-a) = \Phi_{\mu-\nu}(x-a).$$

Firstly, let's draw attention to the case $\mu = 1$, when $\Phi_\mu = 1$:

$${}_a 1^{(\nu)}(x-a) = \frac{1}{\Gamma(1-\nu)} (x-a)_+^{-\nu}.$$

One fact can turn out to be unexpected for the reader: derivatives of all positive non-integer orders of the constant function differ from zero. Perhaps, it can easier be accepted if the reader looks at the graph of the derivative ${}_0 1^{(\nu)}(x)$ represented as a function of ν with a fixed x (Fig. 4.2). Plotting points ${}_0 1^{(1)}(1) = 0$, ${}_0 1^{(2)}(1) = 0$, ${}_0 1^{(3)}(1) = 0$, ... on the coordinate plane, we can see no grounds to expect the deviation of ${}_0 1^{(\nu)}(1)$ from zero between these integer values of ν . But let us put one more point corresponding to $\nu = 0$:

$${}_0 1^{(0)}(1) = 1.$$

Namely this value is determined by the axiom: the zero-order derivative of a function is the function itself. Another axiom requires that the fractional derivative must be an analytic function of ν , therefore, the ${}_0 1^{(\nu)}(1)$ should be plotted by a smooth curve, outgoing from the point $(0, 1)$ and intersecting ν -axis at points $1, 2, 3, \dots$. Each intersection changes the sign of the function, and the function oscillates. The oscillations relax with rising x and vanish in the limit $x \rightarrow \infty$.

The second observation yields: when the difference j between ν and μ is integer, $j = 0, 1, 2, \dots$, which is $\mu = \nu - j$ and $0 \leq j < \nu$, the ν th derivative of the function $\Phi_\mu(x-a)$ is identically zero,

$${}_a D_x^\nu \Phi_{\nu-j}(x-a) = 0.$$

Thus, each of the functions plays the role of a “constant function”. In other words, if $\phi_0(x)$ is a solution to the equation

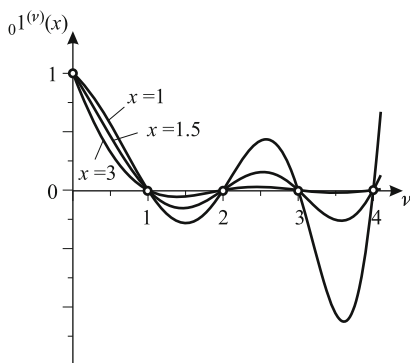


Fig. 4.2 Dependence of fractional derivative of the constant function on the order ν .

$${}_a D_x^\nu \phi_0(x) = f(x), \quad \nu > 0,$$

then the function

$$\phi(x) = \phi_0(x) + \sum_{j=0}^{[\nu]} c_j \Phi_{\nu-j}(x-a) = \phi_0(x) + \sum_{j=0}^{[\nu]} C_j (x-a)^{\nu-j-1}$$

will also be its solution (but the function $\phi_0(x) + C$ will not!). Note that as $\nu \uparrow 1$ the number j takes the unique value (0) and $\Phi_\nu(x-a) \uparrow 1$, that is the limit value as $\nu \uparrow 1$ agrees with the known fact: a solution to the first-order differential equation is determined up to an arbitrary constant.

4.2.5 Term-by-term differentiation

If the infinite series of functions $\sum f_j$ and $\sum {}_a D_x^\nu f_j$ converge uniformly in a circle $0 < |x-a| < R$, then the formula

$${}_a D_x^\nu \sum_{j=0}^{\infty} f_j(x) = \sum_{j=0}^{\infty} {}_a D_x^\nu f_j(x) = \sum_{j=0}^{\infty} {}_a f_j^{(\nu)}(x)$$

is valid in it for any ν . For example, the fractional derivative of the exponential function is represented as

$${}_0 D_x^\nu e^x = {}_0 D_x^\nu \sum_{k=0}^{\infty} \frac{x^k}{k!} = \sum_{k=0}^{\infty} \frac{1}{k!} {}_0 D_x^\nu x^k = \sum_{k=0}^{\infty} \frac{x^{k-\nu}}{\Gamma(k-\nu+1)}.$$

4.2.6 Differentiation of a product

There are a few forms generalizing the known Leibnitz formula to fractional order differentiation. One of them has the form of a series

$${}_aD_x^\nu [f(x)g(x)] = \sum_{k=-\infty}^{\infty} c \binom{\nu}{ck+\mu} {}_af^{(\nu-ck-\mu)}(x) {}_ag^{(ck+\mu)}(x),$$

where $0 < c \leq 1$, and μ is an arbitrary real number. For $c = 1$ and $\mu = 0$, this series becomes

$${}_aD_x^\nu [f(x)g(x)] = \sum_{k=0}^{\infty} \frac{\Gamma(\nu+1)}{k!\Gamma(\nu-k+1)} {}_af^{(\nu-k)}(x) g^{(k)}(x),$$

and in case of an integer $\nu = n$ reduces into the ordinary Leibnitz formula.

Another form is represented by means of integral with respect to the order of fractional derivatives:

$${}_aD_x^\nu [f(x)g(x)] = \int_{-\infty}^{\infty} \frac{\Gamma(\nu+1)}{\Gamma(\mu+1)\Gamma(\nu+1-\mu)} {}_af^{(\nu-\mu)}(x) {}_ag^{(\mu)}(x) d\mu.$$

As an example, we compute a fractional derivative of the product of two power type functions

$$f(x) = x^\alpha$$

and

$$g(x) = (bx+c)^\beta.$$

Using the known results

$${}_0f^{(\nu)}(x) = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1-\nu)} x^{\alpha-\nu}$$

and

$$g^{(k)}(x) = \frac{b^k \Gamma(\beta+1)}{\Gamma(\beta+1-k)} (bx+c)^{\beta-k},$$

we obtain

$$\begin{aligned} {}_0D_x^\nu [f(x)g(x)] &= \sum_{k=0}^{\infty} \frac{\Gamma(\nu+1)}{k!\Gamma(\nu-k+1)} {}_0f^{(\nu-k)}(x) g^{(k)}(x) \\ &= \sum_{k=0}^{\infty} \frac{\Gamma(\nu+1)}{k!\Gamma(\nu-k+1)} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1-\nu+k)} \\ &\quad \cdot \frac{b^k \Gamma(\beta+1)}{\Gamma(\beta+1-k)} x^{\alpha-\nu+k} (bx+c)^{\beta-k}. \end{aligned}$$

If $\alpha + \beta + 1 = \nu$, this series reduces to the one-term expression (Bologna, 2000)

$${}_0D_x^\nu [x^\alpha (bx+c)^\beta] = c^\nu \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1-\nu)} x^{\alpha-\nu} (bx+c)^{\beta-\nu}.$$

4.2.7 Differentiation of an integral

Following (Podlubny, 1999), we bring a formula generalizing the well-known rule of differentiating an integral with respect to its upper limit which serves also as a parameter of the integrand:

$$\frac{d}{dx} \int_a^x F(x, \xi) d\xi = \int_a^x \frac{\partial F(x, \xi)}{\partial x} d\xi + F(x, x-0).$$

In case of the fractional differentiation of order $\nu \in (0, 1)$ we perform the next transformations:

$$\begin{aligned} {}_a D_x^\nu \int_a^x F(x, \xi) d\xi &= \frac{1}{\Gamma(1-\nu)} D_x \int_a^x \frac{d\eta}{(x-\eta)^\nu} \int_a^\eta F(\eta, \xi) d\xi \\ &= \frac{1}{\Gamma(1-\nu)} \frac{d}{dx} \int_a^x \left[\int_\xi^x \frac{F(\eta, \xi) d\eta}{(x-\eta)^\nu} \right] d\xi \\ &= \frac{1}{\Gamma(1-\nu)} \left\{ \int_a^x \frac{\partial}{\partial x} \left[\int_\xi^x \frac{F(\eta, \xi) d\eta}{(x-\eta)^\nu} \right] d\xi + \lim_{\xi \uparrow x} \int_\xi^x \frac{F(\eta, \xi)}{(x-\eta)^\nu} d\eta \right\}. \end{aligned}$$

As a result, we obtain:

$${}_a D_x^\nu \int_a^x F(x, \xi) d\xi = \int_a^x {}_\xi D_x^\nu F(x, \xi) d\xi + \lim_{\xi \uparrow x} {}_\xi D_x^{\nu-1} F(x, \xi).$$

4.2.8 Generalized Taylor series

The ordinary Taylor series allows us to reconstruct an analytic function in its whole definitional domain knowing only a countable set of its integer-order derivatives at a point. Passing to fractional derivatives extends the sphere of such representation, and this is understandable: a nonlocal fractional derivative contains much more information about the function than a plain integer-order derivative.

The following formulas can be considered as the simplest generalizations of the Taylor integer-order series:

$$f(x) = \sum_{j=0}^{\infty} \frac{(x-a)^{\nu+j}}{\Gamma(\nu+j+1)} {}_a f^{(\nu+j)}(a), \quad \nu > 0$$

(see Eq. (2.63) in (Samko et al., 1993)) and

$$f(x) = \sum_{j=0}^{\infty} \frac{(x-a)^{\alpha j}}{\Gamma(1+\alpha j)} {}_a f^{(\alpha j)}(a), \quad 0 < \alpha \leq 1$$

(see (Oldham and Spanier, 1974; Trujillo et al., 1999)).

A more general representation has the form

$$f(x + \xi) = c \sum_{k=-\infty}^{\infty} \frac{1}{\Gamma(ck + \mu + 1)} {}_a f^{(ck+\mu)}(x) \xi^{ck+\mu}, \quad 0 < c \leq 1.$$

Inserting here $c = 1$, $\mu = 0$ and taking into account that $1/\Gamma(k+1) = 0$, as $k = -1, -2, -3, \dots$, we arrive at the Taylor series.

There exist versions using only fractional integrals. The formula

$$\int_0^{\infty} f(x + \xi) \xi^{\mu-1} d\xi = \Gamma(\mu) {}_x I_{\infty}^{\mu} f(x)$$

is nothing but the *Mellin transform* of the function $\varphi_x(\xi) \equiv f(x + \xi)$. Reversing this transform and substituting $x = 0$, we obtain the desired expansion in terms of the complex order fractional derivative:

$$f(\xi) = \frac{1}{2\pi i} \int_{\mu-i\infty}^{\mu+i\infty} \Gamma(z) {}_0 f^{(-z)}(\infty) \xi^{-z} dz, \quad \xi > 0, \quad \mu > 0.$$

One more generalization is of the form

$$f(x) = \int_{-\infty}^{\infty} \frac{{}_a f^{(\mu)}(x_0)}{\Gamma(\mu + 1)} (x - x_0)^{\mu} d\mu.$$

For more detailed information see also (Watanabe, 1931; Hardy, 1945; Osler, 1972; Munkhammar, 2005; Usero, 2007; Odibat and Shawagfeh, 2007).

4.2.9 Expression of fractional derivatives through the integers

In case of analytical function,

$$f(x) = \sum_{k=0}^{\infty} c_k (x - a)^k, \quad c_k = f^{(k)}(a)/k!,$$

fractional derivatives are expressed through the integer-order derivatives taken at the lower limit $f^{(k)}(a)$. Indeed,

$$\begin{aligned} {}_a f^{(\nu)}(x) &= {}_a D_x^{\nu} \left(\sum_{k=0}^{\infty} c_k (x - a)^k \right) = \sum_{k=0}^{\infty} c_k {}_a D_x^{\nu} (x - a)^k \\ &= \sum_{k=0}^{\infty} \frac{k!}{\Gamma(k - \nu + 1)} c_k (x - a)^{k-\nu} = \sum_{k=0}^{\infty} \frac{(x - a)^{k-\nu}}{\Gamma(1 + k - \nu)} f^{(k)}(a). \end{aligned}$$

This formula remains valid in case of negative orders as well (see Eq. (3.6) in (Samko et al., 1993)).

4.2.10 Indirect differentiation: the chain rule

Let $g(x)$ be a monotonic function having a continuous derivative. The integral

$${}_a I_{g(\cdot)}^\mu f(x) = \frac{1}{\Gamma(\mu)} \int_a^x \frac{f(\xi)g'(\xi)d\xi}{[g(x) - g(\xi)]^{1-\mu}}, \quad \mu > 0, \quad a \geq -\infty$$

is called the *fractional integral of the function $f(x)$ with respect to the function $g(x)$* . The *fractional derivative of the order $\alpha \in (0, 1)$ of the function $f(x)$ with respect to the function $g(x)$* is determined as

$${}_a D_{g(\cdot)}^\alpha f(x) = \frac{d}{g'(x)dx} {}_a I_{g(\cdot)}^{1-\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{g'(x)dx} \int_a^x \frac{f(\xi)g'(\xi)d\xi}{[g(x) - g(\xi)]^\alpha}.$$

It can also be represented in the Marchaud form (Sect. 4.2.13):

$${}_a D_{g(\cdot)}^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{f(x)}{[g(x) - g(a)]^\alpha} + \frac{\alpha}{\Gamma(1-\alpha)} \int_a^x \frac{[f(x) - f(\xi)]g'(\xi)d\xi}{[g(x) - g(\xi)]^{1+\alpha}}.$$

Similarly, derivatives with limits (x, ∞) are introduced. In particular, for $g(x) = x^\sigma$ and arbitrary ν , we have:

$${}_x^\sigma D_\infty^\nu f(x) = \begin{cases} \frac{\sigma}{\Gamma(-\nu)} \int_x^\infty \frac{f(\xi)\xi^{\sigma-1}d\xi}{(\xi^\sigma - x^\sigma)^{1+\nu}}, & \nu < 0, \\ f(x), & \nu = 0, \\ \frac{\sigma^{-n}}{\Gamma(1-\{\nu\})} \left(-\frac{d}{x^{\sigma-1}dx} \right)^{[\nu]+1} \int_x^\infty \frac{f(\xi)\xi^{\sigma-1}d\xi}{(\xi^\sigma - x^\sigma)^{\{\nu\}}}, & \nu > 0. \end{cases}$$

4.2.11 Asymptotic behavior as $x \rightarrow a$

When ν is a negative integer ($\nu = -m$, $m = 1, 2, \dots$), we are dealing with an ordinary integral for which

$${}_a f^{(-m)}(a) \equiv \lim_{x \downarrow a} {}_a I_x^m = \lim_{x \rightarrow a} \frac{1}{(m-1)!} \int_a^x f(\xi)(x-\xi)^{m-1} d\xi = 0.$$

In case ν is a natural number or zero ($\nu = n = 0, 1, 2, \dots$), the derivative ${}_a f^\nu(x)$ becomes the ordinary integer-order derivative or the function itself at point a :

$${}_a f^{(n)}(a) = f^{(n)}(a), \quad {}_a f^{(0)}(a) = f(a).$$

For all other values of ν , an analytic function

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k$$

has fractional derivatives with power type asymptotics at the lower limit:

$$\begin{aligned} {}_a f^{(\nu)}(x) &= \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{\Gamma(k-\nu+1)} (x-a)^{k-\nu} \\ &\sim \frac{f(a)}{\Gamma(1-\nu)} (x-a)^{-\nu} \rightarrow \begin{cases} 0, & \nu < 0, \\ \infty, & \nu > 0, \end{cases} x \downarrow a. \end{aligned}$$

If $f(x)$ is not an analytic function but belongs to the class of fractionally differentiable functions (Miller and Ross, 1993), which is

$$f(x) = (x-a)^p \varphi(x), \quad p > -1, \quad \varphi(x) = \sum_{k=0}^{\infty} \frac{\varphi^{(k)}(a)}{k!} (x-a)^k,$$

then

$${}_a f^{(\nu)}(x) \sim \frac{\varphi(a)\Gamma(p+1)}{\Gamma(p-\nu+1)} (x-a)^{p-\nu} \rightarrow \begin{cases} 0, & \nu < p, \\ \varphi(a)\Gamma(p+1), & \nu = p, \\ \infty, & \nu > p \end{cases}$$

as $x \downarrow a$.

4.2.12 Asymptotic behavior of ${}_a f^{(\nu)}(x)$ as $x \rightarrow \infty$

Now, we consider another limit $x \rightarrow \infty$, assuming that $f(x)$ is an analytic function. For $\nu < 0$,

$${}_a f^{(\nu)}(x) = \frac{1}{\Gamma(-\nu)} \int_a^x \frac{f(\xi) d\xi}{(x-\xi)^{\nu+1}} = \frac{1}{\Gamma(-\nu)} \int_0^{x-a} \frac{f(x-\xi) d\xi}{\xi^{\nu+1}}.$$

According to the Taylor expansion,

$$f(x-\xi) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} f^{(k)}(x) \xi^k.$$

Using it in the above formula for ${}_a f^{(\nu)}(x)$ and making term-by-term integration, we arrive at the expression

$${}_a f^{(\nu)}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k f^{(k)}(x) (x-a)^{k-\nu}}{k! \Gamma(-\nu) (k-\nu)}.$$

If $x \gg a$, then

$$(x-a)^{k-v} \sim x^{k-v} + (v-k)ax^{k-v-1}$$

and

$${}_af^{(v)}(x) \sim {}_0f^{(v)}(x) + (a/\pi)\Gamma(v+1)\sin(v\pi)f(0)x^{-v-1} \sim {}_0f^{(v)}(x), \quad x \rightarrow \infty.$$

Consequently, ${}_af^{(v)}(x)$ becomes independent of the lower limit which, particularly, can be replaced by 0.

Interpreting the variable x as time t and $f(x)$ as some dynamical process $f(t)$, one can say that the role of an initial state becomes negligible small in the process evolution as $t \rightarrow \infty$. This fact is referred to as the *limited memory principle* corresponding to the Volterra *fading memory principle*. It says, that it is not necessary to know the initial condition for prediction of behavior of such system at distant future t : it is quite enough to know its recent prehistory $(t - \tau, t)$ where τ is some characteristic time of the process (Podlubny, 1999):

$${}_af^{(v)}(t) \approx {}_{t-\tau}f^{(v)}(t), \quad t > a + \tau.$$

4.2.13 The Marchaud derivative

If the function under differentiation is sufficiently good, its R-L derivative may be reduced to a more convenient form. Let $v < 1$. Integrating by parts, first before differentiation with respect to x and then after it, yields:

$$\begin{aligned} {}_af^{(v)}(x) &= \frac{1}{\Gamma(1-v)} \frac{d}{dx} \int_a^x \frac{f(\xi)d\xi}{(x-\xi)^v} = \frac{1}{\Gamma(1-v)} \left\{ \frac{f(a)}{(x-a)^v} + \int_a^x \frac{df(\xi)}{(x-\xi)^v} \right\} \\ &= \frac{1}{\Gamma(1-v)} \left\{ \frac{f(a)}{(x-a)^v} + \int_a^x \frac{d[f(\xi) - f(x)]}{(x-\xi)^v} \right\} \\ &= \frac{1}{\Gamma(1-v)} \left\{ \frac{f(x)}{(x-a)^v} + v \int_a^x \frac{f(x) - f(\xi)}{(x-\xi)^{1+v}} d\xi \right\}, \quad 0 < v < 1. \end{aligned} \quad (4.3)$$

Assuming $a \rightarrow \infty$, we obtain:

$$\begin{aligned} {}_{-\infty}f^{(v)}(x) &= \frac{v}{\Gamma(1-v)} \int_{-\infty}^x \frac{f(x) - f(\xi)}{(x-\xi)^{1+v}} d\xi \\ &= \frac{v}{\Gamma(1-v)} \int_0^\infty \frac{f(x) - f(x-\xi)}{\xi^{1+v}} d\xi, \quad 0 < v < 1. \end{aligned}$$

The expression

$$\frac{v}{\Gamma(1-v)} \int_0^\infty \frac{f(x) - f(x-\xi)}{\xi^{1+v}} d\xi \equiv \mathbf{D}_+^v f(x)$$

is usually named after Marchaud, although, as noted in (Miller and Ross, 1993), H. Weyl considered it ten years before Marchaud, so one could call $D_+^\nu f(x)$ the right *Weyl-Marchaud fractional derivative*. Referring to Samko, Miller and Ross we take (4.3) as Marchaud's derivative for a finite interval ${}_a D_+^\nu f(x)$.

Acting in a similar way, one can obtain the left *Marchaud fractional derivative*.

$$\begin{aligned} f_\infty^{(\nu)}(x) &\equiv D_-^\nu f(x) = \frac{\nu}{\Gamma(1-\nu)} \int_x^\infty \frac{f(\xi) - f(x)}{(x-\xi)^{1+\nu}} d\xi \\ &= \frac{\nu}{\Gamma(1-\nu)} \int_0^\infty \frac{f(x) - f(x+\xi)}{\xi^{1+\nu}} d\xi, \quad 0 < \nu < 1. \end{aligned}$$

These operations can be extended to the case $\nu > 1$ in various ways. One of them is evident:

$$\begin{aligned} D_+^\nu f(x) &= \left(\frac{d}{dx} \right)^{[\nu]} \frac{\{\nu\}}{\Gamma(1-\{\nu\})} \int_0^\infty \frac{f(x) - f(x-\xi)}{\xi^{1+\{\nu\}}} d\xi \\ &= \frac{\{\nu\}}{\Gamma(1-\{\nu\})} \int_0^\infty \frac{f^{([\nu])}(x) - f^{([\nu])}(x-\xi)}{\xi^{1+\{\nu\}}} d\xi, \\ D_-^\nu f(x) &= \left(-\frac{d}{dx} \right)^{[\nu]} \frac{\{\nu\}}{\Gamma(1-\{\nu\})} \int_0^\infty \frac{f(x) - f(x+\xi)}{\xi^{1+\{\nu\}}} d\xi \\ &= \frac{(-1)^{[\nu]}\{\nu\}}{\Gamma(1-\{\nu\})} \int_0^\infty \frac{f^{([\nu])}(x) - f^{([\nu])}(x+\xi)}{\xi^{1+\{\nu\}}} d\xi. \end{aligned}$$

Another way is based on the observation that $f(x) - f(x-\xi)$ is merely the first-order finite difference $\Delta_\xi^1 f(x)$ of the function $f(x)$ so that

$$D_+^\nu f(x) = \frac{\nu}{\Gamma(1-\nu)} \int_0^\infty \frac{\Delta_\xi^1 f(x)}{\xi^{1+\nu}} d\xi, \quad 0 < \nu < 1.$$

Substituting the higher-order difference for the first-order

$$\Delta_\xi^1 f(x) \mapsto \Delta_\xi^m f(x) = \sum_{j=0}^m (-1)^j \binom{m}{j} f(x - j\xi),$$

we obtain

$$D_+^\nu f(x) = \frac{1}{\kappa(\nu, m)} \int_0^\infty \frac{\Delta_\xi^m f(x)}{\xi^{1+\nu}} d\xi.$$

Here m is any integer number exceeding ν and

$$\kappa(\nu, m) = \int_0^\infty \frac{(1 - e^{-\xi})^m}{\xi^{1+\nu}} d\xi = \Gamma(-\nu) \sum_{n=1}^m (-1)^n \binom{m}{n} n^\nu, \quad 0 < \nu < m$$

is a normalizing constant. For the right-side derivatives, we have

$$\mathbf{D}_{-}^{\nu} f(x) = \frac{(-1)^{[\nu]}}{\kappa(\nu, m)} \int_0^{\infty} \frac{\Delta_{-\xi}^m f(x)}{\xi^{1+\nu}} d\xi, \quad \nu < m = 1, 2, 3, \dots$$

Note that the right sides of equalities for $\mathbf{D}_{\pm}^{\nu} f(x)$ are independent of m whenever $m > \nu$. Let us show it confine ourselves to the case $0 < \nu < 1$. For any natural m ,

$$\begin{aligned} & \frac{1}{\kappa(\nu, m)} \int_{\varepsilon}^{\infty} \frac{\Delta_{\pm\xi}^m f(x)}{\xi^{1+\nu}} d\xi \\ &= \frac{1}{\kappa(\nu, m)} \frac{f(x)}{\nu \varepsilon^{\nu}} + \frac{1}{\kappa(\nu, m)} \sum_{n=1}^m (-1)^n \binom{m}{n} n^{\nu} \int_{n\varepsilon}^{\infty} \frac{f(x \mp \xi)}{\xi^{1+\nu}} d\xi \\ &= \frac{1}{\kappa(\nu, m)} \sum_{n=1}^m (-1)^n \binom{m}{n} n^{\nu} \int_{n\varepsilon}^{\infty} \frac{f(x \mp \xi) - f(x)}{\xi^{1+\nu}} d\xi \\ &\rightarrow \left\{ \frac{1}{\kappa(\nu, m)} \sum_{n=1}^m (-1)^n \binom{m}{n} n^{\nu} \right\} \int_0^{\infty} \frac{f(x \mp \xi) - f(x)}{\xi^{1+\nu}} d\xi \\ &= \frac{\nu}{\Gamma(1-\nu)} \int_0^{\infty} \frac{f(x) - f(x \mp \xi)}{\xi^{1+\nu}} d\xi, \quad \varepsilon \rightarrow 0+. \end{aligned}$$

4.3 Compositions and superpositions of fractional operators

4.3.1 Fractional operators

The known restrictions on the commutativity of fractional operators evoke us to consider *compound fractional operators*

$${}_a \mathbf{D}_x^{\langle \nu_1, \nu_2, \dots, \nu_k \rangle} \equiv {}_a \mathbf{D}_x^{\nu_1} {}_a \mathbf{D}_x^{\nu_2} \dots {}_a \mathbf{D}_x^{\nu_k}.$$

When the function differentiated is good enough (for example, a continuous function),

$${}_a \mathbf{D}_x^{\langle \nu_1, \nu_2, \dots, \nu_k \rangle} = {}_a \mathbf{D}_x^{\nu_1 + \nu_2 + \dots + \nu_k}.$$

A special kind of compound operators called the *sequential operator* has been introduced by Miller and Ross (1993):

$${}_a \mathcal{D}_x^{k\nu} = {}_a \mathbf{D}_x^{\langle \nu, 2\nu, \dots, k\nu \rangle}, \quad \nu \in (0, 1].$$

When $k = 2$, such a relation is given by

$${}_a \mathcal{D}_x^{2\nu} f(x) = {}_a \mathbf{D}_x^{2\nu} \left[f(x) - \frac{{}_a f^{(\nu-1)}(a+)}{\Gamma(\nu)} (x-a)^{\nu-1} \right].$$

If $\nu = n/k$, where n and k are integer and $f(x)$ is a continuous function, then

$${}_a\mathcal{D}_x^{k\nu} = {}_a\mathbf{D}_x^n.$$

A theoretical study of some aspects related to sequential differential equations and their applications, including existence and uniqueness theorems for the corresponding Cauchy-type problems, linear fractional differential equations and methods for solving them, can be found in (Kilbas et al., 2006).

4.3.2 The Gerasimov-Caputo derivative

The construction

$${}_a f^{(\nu)}(x) = \mathbf{D}_x^{[\nu]+1} {}_a \mathbf{D}_x^{\{\nu\}-1} f(x), \quad \nu > 0,$$

considered above is not a unique way providing the desired interpolation. Another way to define a fractional derivative is based on the reverse the operation sequence order:

$$\begin{aligned} {}_a^\nu f(x) &\equiv {}_a \mathbf{D}_x^{\{\nu\}-1} \mathbf{D}_x^{[\nu]+1} f(x) \\ &= \frac{1}{\Gamma(1-\{\nu\})} \int_a^x \frac{1}{(x-\xi)^{\{\nu\}}} \left(\frac{d}{d\xi} \right)^{[\nu]+1} f(\xi) d\xi, \quad \nu \neq 0, 1, 2, \dots \end{aligned}$$

This function is usually named after Italian mechanic M. Caputo referring to his work (Caputo, 1967), but Russian researcher A. Gerasimov used this derivative for solving the problem of motion of viscoelastic twenty years earlier (Gerasimov, 1948). This is why I will use for it the term *Gerasimov-Caputo (G-C) derivative*¹.

Under evident conditions imposed on a function $f(x)$, the derivative ${}_a^\nu f(x)$ coincides with conventional n -th derivative of the function $f(x)$ in the limit $\nu \uparrow n$. Indeed, integrating by parts yields

$$\begin{aligned} \lim_{\nu \uparrow n} {}_a^\nu f(x) &= \lim_{\nu \uparrow n} \left\{ \frac{f^{(n)}(a)(x-a)^{n-\nu}}{\Gamma(n-\nu+1)} + \frac{1}{\Gamma(n-\nu+1)} \int_a^x (x-\xi)^{n-\nu} f^{(n+1)}(\xi) d\xi \right\} \\ &= f^{(n)}(a) + \int_a^x f^{(n+1)}(\xi) d\xi = f^{(n)}(x). \end{aligned}$$

For *non-integer* $\nu > 0$, R-L and G-C derivatives of the same order take different values

$${}_a f^{(\nu)}(x) = {}_a^\nu f(x) + \sum_{k=0}^{[\nu]} \frac{f^{(k)}(a)}{\Gamma(1+k-\nu)} (x-a)^{k-\nu}, \quad \nu \neq 1, 2, 3, \dots,$$

¹ We give this name to indicate the scientists who began actively to use this derivatives in practice. The form itself was first written by Liouville (formula [A] on page 8 in (Liouville, 1832)).

and coincide with each other only when $f(a) = f^{(1)}(a) = f^{(2)}(a) = \dots = f^{[v]}(a) = 0$. In particular, for $0 < v < 1$.

$${}_a f^{(v)}(x) = {}^v_a f(x), \text{ when } f(a) = 0.$$

Observe that

$${}_a^v 1(x) \equiv 0, \quad v > 0,$$

while

$${}_a 1^{(v)}(x) = \frac{1}{\Gamma(1-v)}(x-a)^{-v}.$$

The backward form of interrelation (1) is

$$\begin{aligned} {}^v_a f(x) &= {}_a f^{(v)}(x) - \sum_{k=0}^{[v]} \frac{f^{(k)}(a)}{\Gamma(1+k-v)}(x-a)^{k-v} \\ &= {}_a D_x^v \left[f(x) - \sum_{k=0}^{[v]} \frac{f^{(k)}(a)}{k!}(x-a)^k \right], \quad 0 < v \neq 1, 2, 3, \dots, \end{aligned}$$

where $[v]$ is an integer number determined via equation $[v] < v \leq [v] + 1$.

Note that for integer orders $v = n = 1, 2, 3, \dots$

$${}_a^n f(x) \equiv \left(\frac{d}{dx} \right)^n f(x) = D_x^n \left[f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^k \right].$$

Both last formulas can be represented in a common form as follows:

$$\begin{aligned} {}^v_a f(x) &= {}_a D_x^v \left[f(x) - \sum_{k=0}^{[v]} \frac{f^{(k)}(a)}{k!}(x-a)^k \right] \\ &= {}_a f^{(v)}(x) - \sum_{k=0}^{[v]} \frac{f^{(k)}(a)}{\Gamma(1+k-v)}(x-a)^{k-v}, \quad 0 < v < \infty. \end{aligned}$$

In particular, when $0 < v < 1$, we have

$${}_a^v f_x^{(v)} = {}_a D_x^v [f(x) - f(a)] = {}_a f^{(v)}(x) - \frac{f(a)}{\Gamma(1-v)}(x-a)^{-v}, \quad 0 < v < 1.$$

The G-C derivatives are often preferable in problems of physical interest when the initial conditions for differential equations are expressed in terms of integer-order derivatives with known physical interpretations (velocities, accelerations, and so forth). Similarly, the following modifications of G-C derivative of “sufficiently good” functions can be introduced:

$$\begin{aligned}
{}_0^v f(x) &= \frac{1}{\Gamma(1-\{v\})} \int_0^x \frac{1}{(x-\xi)^{\{v\}}} \left(\frac{d}{d\xi} \right)^{[v]+1} f(\xi) d\xi, \quad x > 0, \\
{}_{-\infty}^v f(x) &= \frac{1}{\Gamma(1-\{v\})} \int_{-\infty}^x \frac{1}{(x-\xi)^{\{v\}}} \left(\frac{d}{d\xi} \right)^{[v]+1} f(\xi) d\xi, \quad -\infty < x < \infty, \\
{}_x^v f_b(x) &= \frac{1}{\Gamma(1-\{v\})} \int_x^b \frac{1}{(\xi-x)^{\{v\}}} \left(-\frac{d}{d\xi} \right)^{[v]+1} f(\xi) d\xi, \quad x < b, \\
{}_x^v f_{\infty}(x) &= \frac{1}{\Gamma(1-\{v\})} \int_x^{\infty} \frac{1}{(\xi-x)^{\{v\}}} \left(-\frac{d}{d\xi} \right)^{[v]+1} f(\xi) d\xi, \quad -\infty < x < \infty.
\end{aligned}$$

When $v \in (0, 1)$, the relations take the following forms:

$$\begin{aligned}
{}_0^v f(x) &= \frac{1}{\Gamma(1-v)} \int_0^x \frac{f'(\xi) d\xi}{(x-\xi)^v}, \quad x > 0, \\
{}_{-\infty}^v f(x) &= \frac{1}{\Gamma(1-v)} \int_{-\infty}^x \frac{f'(\xi) d\xi}{(x-\xi)^v}, \quad -\infty < x < \infty, \\
{}_x^v f_b(x) &= \frac{-1}{\Gamma(1-v)} \int_x^b \frac{f'(\xi) d\xi}{(\xi-x)^v}, \quad x < b, \\
{}_x^v f_{\infty}(x) &= \frac{-1}{\Gamma(1-v)} \int_x^{\infty} \frac{f'(\xi) d\xi}{(\xi-x)^v}, \quad -\infty < x < \infty.
\end{aligned}$$

The G-C derivatives produce a formula for fractional Taylor series (Li et al., 2010):

$$f(x) = \sum_{j=0}^{\infty} ({}_a^v D_x)^j f(x) \Big|_{x=a} \cdot \left[({}_a^v D_x^{-v})^j \cdot 1 \right] = \sum_{j=0}^{\infty} \frac{(x-a)^{\alpha_j}}{\Gamma(1+\alpha_j)} {}_a^{\alpha_j} f(a), \quad 0 < \alpha \leq 1.$$

We conclude the discussion of the G-C derivative with its two following properties. Let $\alpha \in [0, 1]$ and ${}_a^v f(x)$ exist and be continuous in $[a, x]$, then

$${}_a^v D_x^{-v} {}_a^v f(x) = \int_a^x f'(x') dx' = f(x) - f(a).$$

This is the first property. To obtain the second property, we rewrite the left-hand side of the last formula in an explicit form,

$${}_a^v D_x^{-v} {}_a^v f(x) = \frac{1}{\Gamma(v)} \int_a^x (x-\xi)^{v-1} {}_a^v f(\xi) d\xi,$$

and apply the Mean Value Theorem. As a result we have

$$f(x) - f(a) = {}_a^v f(\bar{\xi}) \frac{(x-a)^v}{\Gamma(v+1)},$$

where $\bar{\xi} \in [a, x]$.

Many authors prefer to use the G-C derivative because the corresponding initial conditions include integer-order derivatives having a simple conventional meaning. Moreover, they possess some properties that we are familiar with in classical calculus. For example, the G-C derivative of a constant function is zero. Nevertheless, some people don't share this opinion. Thus, Du and Wang (2011) write: "Actually, Caputo's derivative has also some defects in applications. For example, as shown in (Deng, 2007), one has

$$\lim_{v \uparrow n} {}^v_a D_t y(t) = y^{(n)}(t), \quad \lim_{v \downarrow n} {}^v_a D_t y(t) = y^{(n)}(t) - y^{(n)}(a).$$

In addition, when Caputo's fractional-order derivative is applied to describe the constitutive equations of viscoelastic material, say Nutting's law, $\sigma(t) = \eta {}_a D_t^k \varepsilon(t)$, a constant strain ε implies that $\sigma \equiv 0$, independent of the time t . This claim does not reflect the physical properties of real viscoelastic materials. No such defects are there in the theory of Riemann-Liouville derivative."

4.3.3 Hilfer's interpolation R-L and G-C fractional derivatives

A special kind of compound operators (*Hilfer's differential operators*) has been introduced in (Hilfer, 2000):

$${}_0 D_x^{\alpha, \beta} \equiv {}_0 D_x^{((\alpha-1)\beta, 1, (\alpha-1)(1-\beta))} = {}_0 I_x^{\beta(1-\alpha)} D_x {}_0 I_x^{(1-\beta)(1-\alpha)}, \quad \alpha \in (0, 1), \quad \beta \in [0, 1].$$

The type β allows us to pass continuously from the R-L derivative

$${}_a D_x^{\alpha, 0} f(x) = D_x {}_a I_x^{1-\alpha} f(x) = {}_a D_x^{\alpha} f(x)$$

to the G-C derivative

$${}_a D_x^{\alpha, 1} f(x) = {}_a I_x^{1-\alpha} D_x f(x) = {}_a D_x^{\alpha} f(x).$$

Three equations with this operator are considered there under the same initial condition

$${}_0 f^{-(1-\beta)(1-\alpha)}(0+) = b$$

with the use of the transformation formula

$$\mathcal{L} \left\{ {}_0 D_x^{\alpha, \beta} f(x) \right\} (\lambda) = \lambda^{\alpha} \widehat{f}(\lambda) - \lambda^{\beta(\alpha-1)} {}_0 f^{-(1-\beta)(1-\alpha)}(0+).$$

The first equation

$${}_0 D_x^{\alpha, \beta} f(x) = 0.$$

has the solution

$$f(x) = b \frac{x^{(1-\beta)(\alpha-1)}}{\Gamma((1-\beta)(\alpha-1)+1)}.$$

The solution of the second equation

$${}_0D_x^{\alpha,\beta} f(x) = a$$

is

$$f(x) = \frac{ax^\alpha}{\Gamma(\alpha+1)} + b \frac{x^{(1-\beta)(\alpha-1)}}{\Gamma((1-\beta)(\alpha-1)+1)}.$$

The third equation is

$${}_0D_x^{\alpha,\beta} f(x) + af(x) = 0.$$

The Laplace transform of its solution is of the form

$$\widehat{f}(\lambda) = b \frac{\lambda^{\beta(\alpha-1)}}{a + \lambda^\alpha} = b \sum_{j=0}^{\infty} (-a)^j \lambda^{-\alpha j - \gamma}, \quad \gamma = \alpha + \beta(1 - \alpha).$$

Its inverting yields the desired result

$$f(x) = bx^{\gamma-1} \sum_{j=0}^{\infty} \frac{(-ax^\alpha)^j}{\Gamma(\alpha j + \gamma)} = bx^{(1-\beta)(\alpha-1)} E_{\alpha, \alpha+\beta(1-\alpha)}(-ax^\alpha).$$

4.3.4 Weighted compositions of fractional operators

The R-L fractional integral ${}_0D_x^\alpha f(x)$ being represented in the form

$$\begin{aligned} {}_0D_x^\alpha f(x) &= \frac{(x-a)^{-\alpha-1}}{\Gamma(-\alpha)} \int_a^x \left(1 - \frac{\xi-a}{x-a}\right)^{-\alpha-1} f(\xi) d\xi \\ &= \frac{(x-a)^{-\alpha-1}}{\Gamma(-\alpha)} \int_a^x F\left(\alpha+1, \beta, \beta; \frac{\xi-a}{x-a}\right) f(\xi) d\xi, \end{aligned}$$

where $F(a, b, c; z)$ is the hypergeometric Gauss function and β is any number different from $0, -1, -2, \dots$, was generalized by Nakhushiev (1988) as follows:

$${}_0D_x^{\alpha,\beta,\gamma} f(x) = \frac{(x-a)^{-\alpha-1}}{\Gamma(-\alpha)} \int_a^x F\left(\alpha+1, \beta, \gamma; \frac{\xi-a}{x-a}\right) f(\xi) d\xi,$$

where $\alpha + \beta + 1 < \gamma \neq 0, -1, -2, \dots$. This fractional operator was called the *generalized fractional integral* of order $-\alpha$ with the origin at point a and the end at point $x > a$ from the function $f(x)$.

4.3.5 Fractional derivatives of distributed orders

If a linear differential equation contains integer-order derivatives

$$\left\{ \sum_{n=1}^N a_n \mathbf{D}_x^n \right\} f(x) + cf(x) = h(x),$$

the operator in the bracket can be treated as a differential operator of order distributed over values $1, 2, \dots, N$ with weights $a_1, a_2, \dots, a_n$ respectively. Existing fractional derivatives allows us to introduce differential operators of orders distributed over arbitrary discrete or even continuous set of values, called *distributed order operators*. In the latter case, one can introduce the weight distribution density $w(v)$ and represent the equation as

$$\left\{ \int_{-\infty}^{\infty} w(v) {}_a\mathbf{D}_x^v dv \right\} f(x) + cf(x) = h(x).$$

We will denote such operator by ${}_a\mathbf{D}_x^{\{w(\cdot)\}}$, so that the distributed-order differential equation is written in the form

$${}_a\mathbf{D}_x^{\{w(\cdot)\}} f(x) + cf(x) = h(x).$$

When the weights are positive and the total weight is normalized to 1, one can give the probabilistic interpretation to the operator, namely, consider the operator in curly brackets as the fractional operator averaged over all possible values of its *random order*:

$${}_a\mathbf{D}_x^{\{w(\cdot)\}} = \langle {}_a\mathbf{D}_x^v \rangle.$$

A word of caution has to be said: the solution of the equation with the averaged operator does not coincide with the averaged over v solution of the equation with a random v . From probabilistic point of view, the differential operator of distributed order can be treated as the operator ${}_a\mathbf{D}_x^v$ averaged over random order v because

$$\langle {}_a\mathbf{D}_x^v f(x; v) \rangle \neq \langle {}_a\mathbf{D}_x^v \rangle \langle f(x; v) \rangle.$$

4.4 Generalized functions approach

4.4.1 Generalized functions

As shown in Chapter 1, the memory function results from assumed linearity and continuity of the hereditary functional. The quantity observed at a moment t_1 is represented as

$$g \equiv g(t_1) = -{}_{\infty}F_{t_1}[f(\cdot)] = \int_{-\infty}^{\infty} \varphi_+(t_1 - t) f(t) dt, \quad (4.4)$$

where

$$\varphi_+(t) = 1(t)\varphi(t).$$

The Heavyside step-function $1(t)$ expresses the causality principle excluding any influence of the future on the past. The function $\varphi_+(t)$ has a physical sense only in expressions like (4.4) describing experimentally observed quantity. On its own account, $\varphi_+(t)$ can have points of singularity, as it takes place in case of the power function $t_+^{-\mu}$, $\mu > 0$, but any experiment gives us only finite measured values: a “good function” $f(t)$ spreads the singularities. Thus, the functions $f(t)$ can be identified with functionals, mapping a set of “good functions” $\{f(t)\}$ on the teal axis. Such mathematical objects are called the *distributions* (Schwartz, 1950; 1951) or the *generalized functions* (Gel’fand and Shilov, 1966–1968). We prefer to use the latter term to avoid the possible confusing with probability distributions.

The generalized functions represented by the integral (4.4), containing a locally integrated function $\varphi_+(t)$, are called *regular*, all other generalized functions are called *singular*. In the general case, the form of scalar product

$$F(f(\cdot)) = (\varphi, f)$$

is used instead of the integral form. The functions $f(t)$ undergoing the functional action should have continuous derivatives of all integer orders and be *finite* that is become zero outside some bounded domain. Such functions are called the *basic functions*.

The simplest example of the basic function is giving by the formula

$$f(x) = \begin{cases} \exp\left(-\frac{a^2}{a^2 - x^2}\right), & |x| < a, \\ 0, & |x| \geq a. \end{cases}$$

The most known, one can say, popular generalized function is the Dirac delta-function, $\delta(x)$, defined via relation

$$\int_{-\infty}^{\infty} f(x) \delta(x - x_1) dx = f(x_1).$$

From mechanical point of view, the delta-function can be interpreted as the point mass distribution density, from probabilistic point of view as the probability distribution density for a deterministic (non-random) variable.

4.4.2 Basic properties

Let us bring basic properties of the generalized functions.

- 1) All locally integrated functions enter into the class of generalized function.
- 2) Each singular generalized function is a weak limit of some regular function sequence. This property is realized in the delta-sequences $\delta_\mu(x)$, which are often used for physical interpretation of the delta function and their derivatives:

$$\frac{1}{\pi} \frac{\mu}{1 + (\mu x)^2} \Rightarrow \delta(x), \quad \mu \rightarrow \infty,$$

$$\frac{1}{2} \sqrt{\frac{\mu}{\pi}} \exp(-\mu x^2/4) \Rightarrow \delta(x), \quad \mu \rightarrow \infty,$$

$$\frac{\sin \mu x}{\pi x} \Rightarrow \delta(x), \quad \mu \rightarrow \infty,$$

$$\frac{1}{2\pi} \int_{-\mu}^{\mu} e^{ikx} dk \Rightarrow \delta(x), \quad \mu \rightarrow \infty,$$

$$\frac{1(x+\mu) - 1(x-\mu)}{2\mu} \Rightarrow \delta(x), \quad \mu \rightarrow \infty.$$

For the sake of better visualization, one usually imagines the delta function as a prelimit term of some foregoing sequence and then mentally traces its evolution under unlimited increasing μ . This is probably why the delta function is sometimes determined via the relation

$$\delta(x) = \begin{cases} \infty, & x = 0, \\ 0, & x \neq 0, \end{cases}$$

added by the normalizing condition

$$\int_{-\infty}^{\infty} \delta(x) dx = 1.$$

However, it is not difficult to find out a delta-sequence each term of which equals zero at the origin:

$$\delta_{\sigma}(x) = \frac{1}{\sqrt{2\pi}\sigma^3} x^2 \exp\left(-\frac{x^2}{2\sigma^2}\right).$$

Obviously, $\delta_{\sigma}(0) = 0$ for any σ and consequently $\lim \delta_{\sigma}(0) = 0$ for any $\sigma \rightarrow 0$. At any other point x , the sequence also tends to zero: $\delta_{\sigma}(x) \rightarrow 0$ as $x \neq 0$ and $\sigma \rightarrow 0$. Nevertheless, the sequence generates the delta function.

3) The characteristic property of generalized functions is their infinite differentiability: any generalized function has derivatives of all integer orders and all the derivatives are generalized functions. The converse is also valid: any generalized function is an integer-order derivative of some locally integrable function.

4) In accordance with the first property, all rules valid for “good functions” keep in force for the generalized. In particular, the rule of integration by parts takes place and has especially simple form:

$$(\varphi, f') = \int_{-\infty}^{\infty} \varphi(x) df(x) = - \int_{-\infty}^{\infty} f(x) d\varphi(x) = -(\varphi', f).$$

Reading this equality from right to left, we obtain the definition of a generalized function derivative as a new functional, that is a new generalized function:

$$(\varphi', f) = -(\varphi, f').$$

Applying this definition to the derivative $1'(x)$ of the step function $1(x) = x_+^0$, we obtain

$$(1'_+, f) = -(1_+, f') = -\int_0^\infty f'(x)dx = -f(\infty) + f(0) = f(0) = \int_{-\infty}^\infty \delta(x)f(x)dx,$$

accordingly,

$$1'(x) = \delta(x).$$

Continuing this operation yields

$$(\delta', f) = -(\delta, f') = -\int_{-\infty}^\infty \delta(x)f'(x)dx = -f'(0),$$

and on the whole

$$(\delta^{(k)}, f) = (-1)^k (\delta, f^{(k)}) = (-1)^k f^{(k)}(0).$$

5) There is developed a special operation for generalized functions called the *regularization* or p.f. (*partie finie – Hadamard finite part computing*). Consider the integral

$$(\varphi, f) = \int_{-\infty}^\infty \varphi(x)f(x)dx. \quad (4.5)$$

Obviously, it diverges when $\varphi(x) = |x|^{-1}$ and $f(x)$ is not zero at the origin and its vicinity. The regularization is the extension a definition in such a way that it would follow the result (4.5) for the basic functions integrals of which converge, and gives a finite values for all other basic functions being linear and continuous. The result of regularization of function $\varphi(x)$ is denoted by p.f. $\varphi(x)$. In the example under consideration is

$$\left(\text{p.f.} \frac{1}{t}, f\right) = \int_{-\infty}^{-a} \frac{f(x)}{x} dx + \int_{-a}^b \frac{f(x) - f(0)}{x} dx + \int_b^\infty \frac{f(x)}{x} dx,$$

appropriate $a, b > 0$.

4.4.3 Regularization of power functions

Let us come back to consideration of the standard power functions

$$\Phi_\mu(t) = \frac{1}{\Gamma(\mu)} t_+^{\mu-1}.$$

In case of a positive μ , they are locally integrable and no need in the regularization, but for $\mu < 0$, the integral

$$(\Phi_\mu, f) = \frac{1}{\Gamma(\mu)} \int_0^\infty t^{\mu-1} f(t) dt$$

requires regularization. Confining ourselves to interval $-1 < \mu < 0$, we apply the rule:

$$(t_+^{\mu-1}, f) = \mu^{-1} ((t_+^\mu)', f) = -\mu^{-1} \int_0^\infty t^\mu f'(t) dt.$$

On representing the improper integral as the limit of a proper integral sequence

$$\int_0^\infty t^\mu f'(t) dt = \lim_{\substack{a \rightarrow 0 \\ b \rightarrow \infty}} \int_a^b t^\mu f'(t) dt$$

and introducing new variables

$$t^\mu = U, \quad f'(t) dt = dV, \quad f(t) = V(t) + C,$$

we obtain:

$$\int_a^b t^\mu f'(t) dt = b^\mu [f(b) + C] - a^\mu [f(a) + C] - \int_a^b [f(t) + C] \mu t^{\mu-1} dt.$$

The basic function $f(t)$ is bounded and starting from some value of its argument identically equals zero. Taking b greater than this value and choosing the constant of integration $C = -f(0)$, we write

$$\int_a^b t^\mu f'(t) dt = -b^\mu f(0) - a^\mu [f(a) - f(0)] - \int_a^b [f(t) - f(0)] \mu t^{\mu-1} dt.$$

Because of differentiability of a basic function

$$\lim_{a \rightarrow 0} a^\mu [f(a) - f(0)] = 0, \quad \lim_{b \rightarrow \infty} b^\mu f(a) = 0$$

(recall that $-1 < \mu < 0$). In the double limit, we have

$$\int_0^\infty t^\mu f'(t) dt = - \int_0^\infty [f(t) - f(0)] \mu t^{\mu-1} dt,$$

so that

$$\text{p.f.} \mu \int_0^\infty t^{\mu-1} f(t) dt = \int_0^\infty [f(t) - f(0)] \mu t^{\mu-1} dt.$$

As one can see, the regularization replaces the integrand $f(t)$ by the difference $f(t) - f(0)$, what ensures the integral convergence on the lower limit and does not violate it on the upper. As a result, we have:

$$\text{p.f.} \int_0^\infty \Phi_\mu(t) f(t) dt = \frac{1}{\Gamma(\mu)} \int_0^\infty [f(t) - f(0)] t^{\mu-1} dt, \quad -1 < \mu < 0.$$

4.4.4 Marchaud derivative as a result of regularization

Let us replace μ by $-\nu$ and $f(t)$ by $f(x-t)$ in the last expression of the preceding subsection:

$$\text{p.f.} \int_0^\infty \Phi_{-\nu}(t) f(x-t) dt = \frac{1}{\Gamma(-\nu)} \int_0^\infty [f(x-t) - f(x)] t^{-\nu-1} dt.$$

The obtained expression is nothing other than a finite part of the fractional integral of negative order $-\nu$,

$$\text{p.f.} \int_0^\infty \Phi_{-\nu}(t) f(x-t) dt = \text{p.f.} \frac{1}{\Gamma(-\nu)} \int_0^\infty \frac{f(x-t)}{t^{\nu+1}} dt,$$

which can be considered as a fractional derivative of positive order ν

$$D_+^\nu f(x) = \frac{\nu}{\Gamma(1-\nu)} \int_0^\infty \frac{f(x) - f(x-t)}{t^{\nu+1}} dt = \frac{\nu}{\Gamma(1-\nu)} \int_{-\infty}^x \frac{f(x) - f(\xi)}{(x-\xi)^{\nu+1}} d\xi, \quad (4.6)$$

This is nothing else than the Marchaud derivative.

4.5 Integral transformations

4.5.1 The Laplace transformation

Integral transformations form a very powerful method of analysis of various mathematical problems. Each integral transformation is associated with some convolution relation and convolution theorem, which allows to reduce differential or integral operations to more simple algebraic ones. In case of fractional approach, the role of integral transforms increases: here they serve not only for solving fractional differential equations but for deriving them as well.

We begin with the one-sided *Laplace transform*

$$f(x) \xrightarrow{L} \widehat{f}(\lambda) = \int_0^\infty e^{-\lambda x} f(x) dx \equiv \mathcal{L}\{f(x)\}(\lambda). \quad (4.7)$$

The integral is supposed to be absolutely convergent in the complex half plane $\Re \lambda > 0$. The inverse transformation is given by the integral

$$f(x) = \mathcal{L}^{-1}\{\widehat{f}(\lambda)\}(x) \equiv \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} e^{\lambda x} \widehat{f}(\lambda) d\lambda, \quad x > 0,$$

where the integration contour is a straight line running parallel to the imaginary axis to the right of all singular points of the function $\widehat{f}(\lambda)$.

The Laplace convolution of two functions $f(x)$ and $g(x)$ is expressed by the integral

$$f(x) \star g(x) = \int_0^x f(x - \xi)g(\xi)d\xi,$$

and the Laplace transform of this convolution is

$$\mathcal{L}\{f(x) \star g(x)\}(\lambda) = \mathcal{L}\{f(x)\}(\lambda) \cdot \mathcal{L}\{g(x)\}(\lambda) = \widehat{f}(\lambda)\widehat{g}(\lambda).$$

For integer-order derivatives,

$$\widehat{f^{(n)}}(\lambda) = \mathcal{L}\{f^{(n)}\}(\lambda) = \lambda^n \widehat{f}(\lambda) - \sum_{k=0}^{n-1} \lambda^k f^{(n-k-1)}(0+), \quad n = 1, 2, \dots,$$

and for multiple integrals,

$$\widehat{f_m}(\lambda) = \lambda^{-m} \widehat{f}(\lambda), \quad m = 0, 1, 2, \dots$$

Both these formulas may be joined and rewritten for arbitrary (both positive and negative) integer orders

$$\widehat{{}_0f_x^{(n)}}(\lambda) = \lambda^n \widehat{f}(\lambda) - \sum_{k=0}^{n-1} \lambda^k {}_0f^{(n-k-1)}(0+), \quad n = 0, \pm 1, \pm 2, \dots,$$

where the sum vanishes when $n \leq 0$ because ${}_0f^{(m)}(0+)$ with negative m means a definite integral from 0 to 0.

The fractional order integral ${}_0f^{(-\mu)}(x)$ can be represented as a convolution of $f(x)$ with the power function

$$\Phi_\mu(x) = \frac{x_+^{\mu-1}}{\Gamma(\mu)} : \\ {}_0f^{(-\mu)}(x) = {}_0f(x) * \Phi_\mu(x) = f(x) \star \Phi_\mu(x).$$

According to the convolution theorem

$$\widehat{{}_0f^{(-\mu)}}(\lambda) = \widehat{f}(\lambda)\Phi_\mu(\lambda),$$

where

$$\Phi_\mu(\lambda) = \frac{1}{\Gamma(\mu)} \int_0^\infty e^{-\lambda x} x^{\mu-1} dx = \lambda^{-\mu}.$$

Therefore

$$\widehat{{}_0f^{(-\mu)}}(\lambda) = \Phi_\mu(\lambda)\widehat{f}(\lambda) = \lambda^{-\mu}\widehat{f}(\lambda), \quad \mu > 0.$$

In order to perform the Laplace transformation of the Riemann-Liouville fractional derivative ${}_0f^\nu(x)$, $\nu > 0$, we represent it in the form

$${}_0f^{(\nu)}(x) = \mathbf{D}_x^n {}_0f^{(-\mu)}(x), \quad \mu = n - \nu > 0,$$

and use the Laplace transform of an integer-order derivative obtained above. This yields

$$\begin{aligned}\widehat{{}_0f^{(\nu)}}(\lambda) &= \mathcal{L}\{{}_0\mathbf{D}_x^\nu f(x)\}(\lambda) = \mathcal{L}\{{}_0\mathbf{D}_x^n {}_0f^{(-\mu)}(x)\}(\lambda) \\ &= \lambda^\nu \widehat{f}(\lambda) - \sum_{k=0}^{[v]} \lambda^k {}_0f^{(\nu-k-1)}(0+).\end{aligned}\quad (4.8)$$

In particular, when $0 < \nu < 1$, equation (4.8) becomes

$$\widehat{{}_0f^{(\nu)}}(\lambda) = \lambda^\nu \widehat{f}(\lambda) - {}_0f^{(\nu-1)}(0+). \quad (4.9)$$

Observe that for sufficiently good functions

$${}_0f^{(\nu-1)}(0+) = \lim_{x \downarrow 0} \frac{1}{\Gamma(1-\nu)} \int_0^x \frac{f(\xi) d\xi}{(x-\xi)^\nu} = 0$$

and Eq. (4.9) takes a simpler form

$$\widehat{{}_0f^{(\nu)}}(\lambda) = \lambda^\nu \widehat{f}(\lambda), \quad 0 < \nu < 1. \quad (4.10)$$

The Laplace transform of the Gerasimov-Caputo derivative can be obtained in a similar way and looks as follows:

$$\widehat{{}_0^{\nu}f}(\lambda) = \lambda^\nu \widehat{f}(\lambda) - \sum_{k=0}^{n-1} \lambda^{\nu-k-1} f^{(k)}(0+), \quad n-1 < \nu \leq n.$$

In particular, if $0 < \nu \leq 1$, then

$$\widehat{{}_0^{\nu}f}(\lambda) = \lambda^\nu \widehat{f}(\lambda) - \lambda^{\nu-1} f(0+).$$

The essential difference between the R-L and G-C derivative transformations is that the former involves the fractional derivatives at the lower limit $x = 0$, whereas the latter involves only the integer-order derivatives. Note that both the derivatives have the same transform

$$\widehat{{}_0f^{(\nu)}}(\lambda) = \widehat{{}_0^{\nu}f}(\lambda) = \lambda^\nu \widehat{f}(\lambda)$$

if the function and corresponding number of its derivatives vanish at the lower limit.

4.5.2 The Mellin transform

Another integral transformation acting in the same domain $(0, \infty)$ is the Mellin transform defined by

$$f(x) \mapsto \bar{f}(s) = \mathcal{M}\{f(x)\}(s) = \int_0^\infty x^{s-1} f(x) dx.$$

Its inverse transform is given by

$$\bar{f}(s) \mapsto f(x) = \mathcal{M}^{-1}\{\bar{f}(s)\}(x) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} x^{-s} \bar{f}(s) ds, \quad \sigma = \Re s.$$

The Mellin convolution operation and convolution theorem read

$$f(x) \circ g(x) = \int_0^\infty f\left(\frac{x}{\xi}\right) g(\xi) \frac{d\xi}{\xi}$$

and

$$\mathcal{M}\{f(x) \circ g(x)\}(s) = \mathcal{M}\{f(x)\}(s) \cdot \mathcal{M}\{g(x)\}(s) = \bar{f}(s) \bar{g}(s).$$

The Mellin transform of the first derivative is calculated on integrating by parts:

$$\begin{aligned} \overline{f^{(1)}}(s) &= \int_0^\infty x^{s-1} f'(x) dx \\ &= [x^{s-1} f(x)]_0^\infty - (s-1) \int_0^\infty x^{s-2} f(x) dx = [x^{s-1} f(x)]_0^\infty - (s-1) \bar{f}(s-1). \end{aligned}$$

Continuing by induction yields

$$\overline{f^{(n)}}(s) = \sum_{k=0}^{n-1} \frac{\Gamma(1-s+k)}{\Gamma(1-s)} [x^{s-k-1} f^{(n-k-1)}(x)]_0^\infty + \frac{(-1)^n \Gamma(s)}{\Gamma(s-n)} \bar{f}(s-n).$$

If

$$\lim_{x \rightarrow 0} [x^{s-k-1} f^{(n-k-1)}(x)] = \lim_{x \rightarrow \infty} [x^{s-k-1} f^{(n-k-1)}(x)] = 0, \quad 0 \leq k < n,$$

the formula becomes simpler:

$$\overline{f^{(n)}}(s) = \frac{(-1)^n \Gamma(s)}{\Gamma(s-n)} \bar{f}(s-n) = \frac{\Gamma(1-s+n)}{\Gamma(1-s)} \bar{f}(s-n) = (-1)^n \frac{\Gamma(s)}{\Gamma(s-n)} \bar{f}(s).$$

The Mellin transformation of the Riemann-Liouville integral can be performed immediately for non-integer order, because the formula has the same form:

$${}_0f^{(-\mu)}(x) = \frac{1}{\Gamma(\mu)} \int_0^x (x-\xi)^{\mu-1} f(\xi) d\xi.$$

After multiplying it by $x^{s-1} dx$, integrating with respect to x from 0 to ∞ , and changing the order of integration by means of the Dirichlet rule, we obtain:

$$\overline{{}_0f^{(-\mu)}}(s) = \frac{1}{\Gamma(\mu)} \int_0^\infty d\xi f(\xi) \int_\xi^\infty (x-\xi)^{\mu-1} x^{s-1} dx.$$

The substitution $t = \xi/x$ reduces the inner integral to the beta-function

$$\int_{\xi}^{\infty} (x - \xi)^{\mu-1} x^{s-1} dx = \xi^{s+\mu-1} \int_0^1 (1-t)^{\mu-1} t^{-s-\mu} dt = \xi^{s+\mu-1} B(\mu, 1-s-\mu).$$

As a result, we have

$$\overline{{}_0f^{(-\mu)}}(s) = \frac{\Gamma(1-s-\mu)}{\Gamma(1-s)} \overline{f}(s+\mu), \quad \Re s < 1-\mu.$$

Let us evaluate the Mellin transform of the Riemann-Liouville fractional derivative. According to its definition

$${}_0f^{(\nu)}(x) = \mathbf{D}_x^n {}_0f^{(\nu-n)}(x) \equiv g^{(n)}(x),$$

$$g(x) = {}_0f^{(\nu-n)}(x), \quad n-1 \leq \nu < n, \quad n = [\nu] + 1.$$

Using the above formula for $\overline{g^{(n)}}(s)$ and inserting there

$$\overline{g}(s-n) = \overline{{}_0f^{(\nu-n)}}(s-n) = \frac{\Gamma(1-s+\nu)}{\Gamma(1-s+n)} \overline{f}(s-\nu),$$

we obtain

$$\overline{{}_0f^{(\nu)}}(s) = \sum_{k=0}^{n-1} \frac{\Gamma(1-s+k)}{\Gamma(1-s)} \left[{}_0f^{(\nu-k-1)}(x) x^{s-k-1} \right]_0^{\infty} + \frac{\Gamma(1-s+\nu)}{\Gamma(1-s)} \overline{f}(s-\nu).$$

If $f(x)$ is such a function that the first sum gives zero, then

$$\overline{{}_0f^{(\nu)}}(s) = \frac{\Gamma(1-s+\nu)}{\Gamma(1-s)} \overline{f}(s-\nu), \quad \Re s < 1-\mu.$$

For the Gerasimov-Caputo derivative,

$$\overline{{}_v f}(s) = \sum_{k=0}^{n-1} \frac{\Gamma(\nu-k-s)}{\Gamma(1-s)} \left[f^{(k)}(x) x^{s-\nu+k} \right]_0^{\infty} + \frac{\Gamma(1-s+\nu)}{\Gamma(1-s)} \overline{f}(s-\nu),$$

or in the reduced form

$$\overline{{}_v f}(s) = \frac{\Gamma(1-s+\nu)}{\Gamma(1-s)} \overline{f}(s-\nu).$$

Finally, we remark that the fractional integral

$$-_{\infty} f^{(-\mu)}(x) = \frac{1}{\Gamma(\mu)} \int_{-\infty}^x (x-\xi)^{\mu-1} f(\xi) d\xi$$

rewritten in the form

$${}_{-\infty}f^{(-\mu)}(x) = \frac{1}{\Gamma(\mu)} \int_0^\infty \xi^{\mu-1} f(x-\xi) d\xi$$

is (up to the factor $1/\Gamma(\mu)$) nothing other than the Mellin transform of the function $\phi_x(\xi) \equiv f(x-\xi)$:

$$\Gamma(\mu) {}_{-\infty}f^{(-\mu)}(x) = \overline{\phi_x(\mu)}.$$

Applying the inverse Mellin transformation

$$\phi_x(\xi) = f(x-\xi) = \frac{1}{2\pi i} \int_{\Re\mu-i\infty}^{\Re\mu+i\infty} \Gamma(\mu) {}_{-\infty}f^{-\mu}(x) \xi^{-\mu} d\mu, \quad \xi > 0.$$

and substituting $x = 0$, we obtain

$$\phi_0(\xi) = f(-\xi) = \frac{1}{2\pi i} \int_{\Re\mu-i\infty}^{\Re\mu+i\infty} \Gamma(\mu) {}_{-\infty}f^{-\mu}(0) \xi^{-\mu} d\mu, \quad \xi > 0.$$

This implies that $f(x)$ can be recovered from the values of its fractional integrals ${}_{-\infty}f^{(-\mu)}(0)$ taken as a function of all complex orders μ on the line $\Re\mu = \text{const} > 0$ (Butzer, 2000). So, we see here a version of the Taylor expansion.

4.5.3 The Fourier transform

When the function under consideration is given on a whole real axis $-\infty < x < \infty$, the third version of integral transformation is more suitable, the *Fourier transform*

$$f(x) \mapsto \tilde{f}(k) = \mathcal{F}\{f(x)\}(k) \equiv \int_{-\infty}^{\infty} e^{ikx} f(x) dx,$$

$$\tilde{f}(k) \mapsto f(x) = \mathcal{F}^{-1}\{\tilde{f}(k)\}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} \tilde{f}(k) dk.$$

Its convolution formulas look as follows:

$$f(x) * g(x) \equiv \int_{-\infty}^{\infty} f(x-\xi) g(\xi) d\xi,$$

$$\mathcal{F}\{f(x) * g(x)\}(k) = \mathcal{F}\{f(x)\}(k) \cdot \mathcal{F}\{g(x)\}(k) = \tilde{f}(k) \tilde{g}(k).$$

For integer-order operations, we have

$$\tilde{f}_n(k) = \{\mathcal{F} {}_{-\infty}\mathbf{D}_x^{-n} f(x)\}(k) = (-ik)^{-n} \tilde{f}(k), \quad n = 1, 2, \dots$$

and

$$\tilde{f}^{(n)}(k) = \mathcal{F}\{\mathbf{D}_x^n f(x)\}(k) = (-ik)^n \tilde{f}(k), \quad n = 1, 2, \dots$$

For fractional order Riemann-Liouville integrals, we use again the convolution representation

$$-_{\infty}f^{(-\mu)}(x) = \Phi_{\mu}(x) * f(x),$$

the convolution theorem

$$-_{\infty}\tilde{f}^{(-\mu)}(k) = \tilde{\Phi}_{\mu}(k)\tilde{f}(k),$$

and the explicit expression for the Fourier transform of the function $\Phi_{\mu}(x)$ (see Sect. 2.1):

$$\tilde{\Phi}_{\mu}(k) = |k|^{-\mu} \exp\{i(\mu\pi/2)\text{sign}(k)\},$$

and accordingly

$$-_{\infty}\tilde{f}^{(-\mu)}(k) = |k|^{-\mu} \exp\{i(\mu\pi/2)\text{sign}(k)\}\tilde{f}(k).$$

In a similar way,

$$\tilde{f}_{\infty}^{(-\mu)}(k) = (ik)^{-\mu}\tilde{f}(k) = |k|^{-\mu} \exp\{-i(\mu\pi/2)\text{sign}(k)\}\tilde{f}(k).$$

These results cannot be extended to orders $\nu > 1$ immediately, but if the function is sufficiently good, these expressions become valid for all $\nu \geq 0$ (Butzer and Westphal, 2000).

As to fractional derivatives of positive orders, we find

$$\begin{aligned} -_{\infty}\tilde{f}^{(\nu)}(k) &= \mathcal{F}\{\Phi_{\nu-n}(x) * f^{(n)}(x)\}(k) \\ &= \mathcal{F}\{\Phi_{\nu-n}(x)\}(k) \cdot \mathcal{F}\{f^{(n)}(x)\}(k) = (-ik)^{-(\nu-n)} \cdot (-ik)^n \tilde{f}(k) \\ &= (-ik)^{\nu} \tilde{f}(k) = |k|^{\nu} \exp\{-i(\nu\pi/2)\text{sign}(k)\}\tilde{f}(k), \end{aligned}$$

and similarly

$$\tilde{f}_{\infty}^{(\nu)}(k) = (ik)^{\nu} \tilde{f}(k) = |k|^{\nu} \exp\{i(\nu\pi/2)\text{sign}(k)\}\tilde{f}(k).$$

4.6 Potentials and fractional derivatives

4.6.1 The Riesz potentials on a straight line

The fractional integrals considered above belong to the Volterra type: one of their limits is variable. However, many physical problems need integration over some fixed interval or in general over the whole real axis from $-\infty$ to ∞ . Such situation takes place, for instance, when we calculate the scalar potential $\phi(x)$ created by a charge distributed on the x -axis with the density $f(x)$:

$$\phi(x) = \frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{f(\xi)d\xi}{|x-\xi|}.$$

When we are interested in the stress $E_x(x) = -d\phi(x)/dx$, we may deal with integral

$$E_x(x) = \frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{\text{sign}(x - \xi) f(\xi) d\xi}{|x - \xi|^2}.$$

Generalization of such integrals to fractional operators performed by M. Riesz (Riesz, 1949). One of the fractional integrals, having the form

$$\mathbb{D}^{-\nu} f(x) \equiv {}_{-\infty}\mathbb{D}_{\infty}^{-\nu} f(x) = \frac{1}{2\Gamma(\nu) \cos(\nu\pi/2)} \int_{-\infty}^{\infty} \frac{f(\xi) d\xi}{|x - \xi|^{1-\nu}}, \quad \nu \in (0, 1),$$

Which is called the *Riesz potential* of $f(x)$ of order ν , another integral

$${}^{\circ}\mathbb{D}^{\nu} f(x) = \frac{1}{2\Gamma(\nu) \sin(\nu\pi/2)} \int_{-\infty}^{\infty} \frac{\text{sign}(x - \xi) f(\xi) d\xi}{|x - \xi|^{1-\nu}}, \quad \nu \in (0, 1),$$

Which is called the *conjugate Riesz potential*. The potentials can be extended to all real positive or even complex values of ν (Samko et al., 1993).

It is easy to see, that Riesz potentials are linked to Riemann-Liouville integrals via relations:

$$\mathbb{D}^{\nu} = \frac{1}{2\cos(\nu\pi/2)} ({}_{-\infty}\mathbb{D}_x^{-\nu} + {}_x\mathbb{D}_{\infty}^{-\nu})$$

and

$${}^{\circ}\mathbb{D}^{\nu} = \frac{1}{2\sin(\nu\pi/2)} ({}_{-\infty}\mathbb{D}_x^{-\nu} - {}_x\mathbb{D}_{\infty}^{-\nu}).$$

Below, we will use also the inversion relations:

$$\begin{aligned} {}_{-\infty}\mathbb{D}_x^{-\nu} &= \cos(\nu\pi/2) \mathbb{D}^{-\nu} + \sin(\nu\pi/2) {}^{\circ}\mathbb{D}^{-\nu}, \\ {}_x\mathbb{D}_{\infty}^{-\nu} &= \cos(\nu\pi/2) \mathbb{D}^{-\nu} - \sin(\nu\pi/2) {}^{\circ}\mathbb{D}^{-\nu}. \end{aligned}$$

The potentials $\mathbb{D}^{\nu} f(x)$ and ${}^{\circ}\mathbb{D}^{\nu} f(x)$ are related to each other via the Hilbert transform $\mathcal{H}$, namely

$${}^{\circ}\mathbb{D}^{-\nu} \{f(\xi)\}(x) = \mathbb{D}^{-\nu} \mathcal{H} \{f(\xi)\}(x),$$

where the operator $\mathcal{H}$ is defined by the Cauchy principal value

$$\mathcal{H} \{f(\xi)\}(x) = \lim_{\varepsilon \downarrow 0} \frac{1}{\pi} \int_{|x-\xi| \geq \varepsilon} \frac{f(\xi) d\xi}{x - \xi}$$

at every point x where this limit exists. Moreover, the following additivity laws are satisfied: if $\mu > 0$, $\nu > 0$ such that $\mu + \nu < 1$, then (Butzer and Westphal, 2000)

$$\mathbb{D}^{-\mu} \mathbb{D}^{-\nu} f(x) = \mathbb{D}^{-\mu-\nu} f(x)$$

and

$${}^{\circ}\mathbb{D}^{-\mu} {}^{\circ}\mathbb{D}^{-\nu} f(x) = -\mathbb{D}^{-\mu-\nu} f(x).$$

4.6.2 The Fourier transforms of the Riesz potentials

The Riesz potential can be represented up to the factor $[2\Gamma(\nu)\cos(\nu\pi/2)]^{-1}$ as the convolution (in the Fourier sense) of functions $f(x)$ and $g_\nu(x) = |x|^{\nu-1}$. The Fourier transform of the latter function (for $0 < \nu < 1$)

$$\tilde{g}_\nu(k) = \int_{-\infty}^{\infty} |\xi|^{\nu-1} e^{ik\xi} d\xi = 2|k|^{-\nu} \int_0^{\infty} |\xi|^{\nu-1} \cos(\xi) d\xi = 2|k|^{-\nu} \Gamma(\nu) \cos(\nu\pi/2),$$

hence, the Fourier transform of $\mathbb{D}^{-\nu}f(x)$ takes the form

$$\begin{aligned} \mathcal{F}\{\mathbb{D}^{-\nu}f(x)\}(k) &= \frac{1}{2\Gamma(\nu)\cos(\nu\pi/2)} \mathcal{F}\{g_\nu(x) * f(x)\}(k) \\ &= \frac{1}{2\Gamma(\nu)\cos(\nu\pi/2)} \tilde{g}_\nu(k) \cdot \tilde{f}(k) = |k|^{-\nu} \tilde{f}(k). \end{aligned}$$

Acting in an analogous manner, one can find the Fourier transform of the conjugate Riesz potential:

$$\mathcal{F}\{\mathbb{D}^\nu f(x)\}(k) = i \operatorname{sign}(k) |k|^{-\nu} \tilde{f}(k).$$

It is easy to add these transforms by two other transforms of operators containing the simple differential operator:

$$\mathcal{F}\{\mathbf{D}_x \mathbb{D}^{-1+\nu} f(x)\}(k) = (-ik) |k|^{\nu-1} \tilde{f}(k) = -i \operatorname{sign}(k) |k|^\nu \tilde{f}(k),$$

and

$$\mathcal{F}\{\mathbf{D}_x \circ \mathbb{D}^{-1+\nu} f(x)\}(k) = (-ik)(i \operatorname{sign}(k)) |k|^{\nu-1} \tilde{f}(k) = |k|^\nu \tilde{f}(k).$$

Comparing the four Fourier transforms, obtained in this section, one is led to take $\mathbf{D}_x \circ \mathbb{D}^{-1+\nu} f(x)$ as a candidate for the ν -th Riesz derivative, while $\mathbf{D}_x \mathbb{D}^{-1+\nu} f(x)$ plays the corresponding role with respect to the conjugate Riesz potential (Butzer and Westphal, 2000).

4.6.3 The Riesz derivatives

The *Riesz derivative* can be defined as a regularized Riesz potential of negative order:

$$\begin{aligned} \mathbb{D}^\nu f(x) &= \text{p.f.} \frac{1}{K_{\nu,2}} \int_{-\infty}^{\infty} \frac{f(\xi) d\xi}{|x-\xi|^{1+\nu}} \\ &= \frac{1}{K_{\nu,2}} \int_{-\infty}^{\infty} \frac{f(x-\xi) - f(x)}{|\xi|^{1+\nu}} d\xi, \quad \nu \in (0,2), \nu \neq 1. \end{aligned}$$

Here

$$K_{\nu,2} = \begin{cases} 2\Gamma(-\nu)\cos(\nu\pi/2), & 0 < \nu < 2, \quad \nu \neq 1, \\ -\pi & \nu = 1, \end{cases}$$

and the improper integral has to be interpreted as the limit

$$\int_{-\infty}^{\infty} \frac{f(x-\xi) - f(x)}{|\xi|^{1+\nu}} d\xi = \lim_{\varepsilon \rightarrow 0+} \int_{|\xi| \geq \varepsilon} \frac{f(x-\xi) - f(x)}{|\xi|^{1+\nu}} d\xi.$$

The formula for the Riesz fractional derivative

$$\mathbb{D}^{\nu} f(x) = \frac{1}{K_{\nu,2}} \int_{-\infty}^{\infty} \frac{f(x-\xi) - f(x)}{|\xi|^{1+\nu}} d\xi$$

is an integral of the Marchaud type. It may be reduced to the form

$$\begin{aligned} \mathbb{D}^{\nu} f(x) &= \frac{1}{K_{\nu,2}} \left\{ \int_{-\infty}^0 \frac{f(x-\xi) - f(x)}{|\xi|^{1+\nu}} d\xi + \int_0^{\infty} \frac{f(x-\xi) - f(x)}{|\xi|^{1+\nu}} d\xi \right\} \\ &= \frac{1}{K_{\nu,2}} \int_0^{\infty} \frac{f(x+\xi) - 2f(x) + f(x-\xi)}{\xi^{1+\nu}} d\xi, \quad 0 < \nu < 2, \end{aligned} \quad (4.11)$$

with the second central difference of $f(x)$ in the numerator of the integrand. This suggests for an arbitrary $\nu > 0$ the following regularization

$$\mathbb{D}^{\nu} f(x) = \frac{1}{K_{\nu,2j}} \int_0^{\infty} \frac{\bar{\Delta}_{\xi}^{2j} f(x)}{\xi^{1+\nu}} d\xi, \quad 0 < \nu < 2j,$$

where the central difference of $f(x)$ of even order $2j$ is given by

$$\bar{\Delta}_{\xi}^{2j} f(x) = \sum_{k=0}^{2j} (-1)^k \binom{2j}{k} f(x + (j-k)\xi)$$

and

$$K_{\nu,2j} = (-1)^j 2^{2j-\nu} \int_0^{\infty} \frac{\sin^{2j} \xi}{\xi^{1+\nu}} d\xi.$$

Note that the Riesz derivative is expressed through the Marchaud derivatives via relation

$$\mathbb{D}^{\nu} f(x) = |2\cos(\nu\pi/2)|^{-1} (\mathbf{D}_{+}^{\nu} f(x) + \mathbf{D}_{-}^{\nu} f(x)). \quad (4.12)$$

Let us consider now the conjugate Riesz potential for $0 < \nu < 2$. Acting as above, we obtain:

$$\begin{aligned} {}^{\circ}\mathbf{D}^{\nu} f(x) &= \text{p.f.} \frac{-1}{2\Gamma(-\nu)\sin(\nu\pi/2)} \int_{-\infty}^{\infty} \frac{\text{sign}(x-\xi)}{|x-\xi|^{1+\nu}} f(\xi) d\xi \\ &= \frac{-1}{2\Gamma(-\nu)\sin(\nu\pi/2)} \int_{-\infty}^{\infty} \frac{\text{sign}(x-\xi)}{|x-\xi|^{1+\nu}} [f(\xi) - f(x)] d\xi \end{aligned}$$

$$\begin{aligned}
&= \frac{-1}{2\Gamma(-\nu)\sin(\nu\pi/2)} \left\{ \int_{-\infty}^x \frac{f(\xi) - f(x)}{(x - \xi)^{1+\nu}} d\xi - \int_x^{\infty} \frac{f(\xi) - f(x)}{(\xi - x)^{1+\nu}} d\xi \right\} \\
&= \frac{\nu}{2\Gamma(1-\nu)\sin(\nu\pi/2)} \int_0^{\infty} [f(x - \xi) - f(x + \xi)] \xi^{-1-\nu} d\xi, \quad 0 < \nu < 2.
\end{aligned}$$

4.6.4 The Fourier transforms of the Riesz derivatives

In order to find the Fourier transform of the Riesz derivative, we apply the operator $\mathcal{F}$ to both sides of Eq. (4.11)

$$\widetilde{f_x^\nu}(k) = \mathcal{F}\{\mathbb{D}^\nu f(x)\}(k) = \frac{1}{2\Gamma(-\nu)\cos(\nu\pi/2)} \int_0^{\infty} F(k, \xi) \xi^{-1-\nu} d\xi,$$

where

$$F(k, \xi) = \mathcal{F}\{f(x + \xi) - 2f(x) + f(x - \xi)\}(k) = [e^{-ik\xi} + e^{ik\xi} - 2]\widetilde{f}(k).$$

Changing the integration variable and using the formula

$$\int_0^{\infty} [1 - e^{-\xi}] \xi^{-1-\nu} d\xi = -\Gamma(-\nu), \quad 0 < \nu < 1,$$

we find the result

$$\mathcal{F}\{\mathbb{D}^\nu f(x)\}(k) = \frac{\Gamma(-\nu)[(-ik)^\nu] + (ik)^\nu}{2\Gamma(-\nu)\cos(\nu\pi/2)} \widetilde{f}(k) = |k|^\nu \widetilde{f}(k).$$

As to the conjugate counterpart to the derivative

$${}^\circ\mathbb{D}^\nu f(x) = \frac{\nu}{2\Gamma(1-\nu)\sin(\nu\pi/2)} \int_0^{\infty} [f(x - \xi) - f(x + \xi)] \xi^{-1-\nu} d\xi,$$

we have

$$\mathcal{F}\{{}^\circ\mathbb{D}^\nu f(x)\}(k) = \frac{\nu}{2\Gamma(1-\nu)\sin(\nu\pi/2)} \int_0^{\infty} {}^\circ G(k, \xi) \xi^{-1-\nu} d\xi,$$

where

$${}^\circ G(k, \xi) = \mathcal{F}\{f(x - \xi) - f(x + \xi)\}(k) = [e^{ik\xi} - e^{-ik\xi}]\widetilde{f}(k).$$

Changing the variable of integration,

$$\int_0^{\infty} \xi^{-1-\nu} \sin(k\xi) d\xi = \text{sign}(k)|k|^\nu \int_0^{\infty} t^{-1-\nu} \sin t dt$$

and using the formula

$$\int_0^\infty t^{-1-\nu} \sin t dt = -\Gamma(-\nu) \sin(\nu\pi/2), \quad 0 < \nu < 1,$$

we obtain

$$\mathcal{F}\{\circ\mathbb{D}^\nu f(x)\}(k) = i \operatorname{sign}(k) |k|^\nu \tilde{f}(k).$$

4.6.5 The Feller potential

As noticed above, Riesz potentials are linked to Riemann-Liouville integrals via relations:

$$\mathbb{D}^{-\nu} = \frac{1}{2 \cos(\nu\pi/2)} (-\infty \mathbb{D}_x^{-\nu} + {}_x \mathbb{D}_\infty^{-\nu})$$

and

$$\circ\mathbb{D}^{-\nu} = \frac{1}{2 \sin(\nu\pi/2)} (-\infty \mathbb{D}_x^{-\nu} - {}_x \mathbb{D}_\infty^{-\nu}).$$

A more general operator was introduced by Feller (1952). Called the *Feller potential*, it is represented by linear superposition of R-L integrals:

$$\mathbb{D}_{p,q}^{-\nu} f(x) = A^{-1/2} [p {}_{-\infty} \mathbb{D}_x^{-\nu} f(x) + q {}_x \mathbb{D}_\infty^{-\nu} f(x)], \quad A > 0.$$

Passing to Riesz potentials and then to explicit integral forms, we obtain:

$$\begin{aligned} \mathbb{D}_{p,q}^\nu f(x) &= A^{-1/2} [(p+q) \cos(\nu\pi/2) \mathbb{D}^{-\nu} f(x) + (p-q) \sin(\nu\pi/2) {}^\dagger \mathbb{D}^{-\nu}] \\ &= \frac{A^{-1/2}}{2\Gamma(\nu)} \int_{-\infty}^\infty \frac{p+q + (p-q) \operatorname{sign}(x-\xi)}{|x-\xi|^{1-\nu}} f(\xi) d\xi, \quad 0 < \nu < 1. \end{aligned}$$

Applying the Fourier transform operator and using corresponding expressions for R-L fractional integrals or of Riesz potential, one can easily find the Fourier transform of the Feller potential:

$$\begin{aligned} \mathcal{F}\{\mathbb{D}_{p,q}^{-\nu} f(x)\}(k) &= A^{-1/2} \cos(\nu\pi/2) [(p+q) + i(p-q) \tan(\nu\pi/2) \operatorname{sign}(k)] |k|^{-\nu} \tilde{f}(k), \\ &0 < \nu < 1. \end{aligned}$$

The inverse Feller potential is defined as Hadamard's finite part of the Feller potential of negative order :

$$\begin{aligned} \mathbb{D}_{p,q}^\nu f(x) &= \frac{A^{-1/2}}{2\Gamma(-\nu)} \int_{-\infty}^\infty \frac{p+q + (p-q) \operatorname{sign}(x-\xi)}{|x-\xi|^{1+\nu}} [f(x) - f(\xi)] d\xi, \\ &0 < \nu < 1. \end{aligned}$$

Performing evident mathematical transformations yields

$$\begin{aligned}
& \int_{-\infty}^{\infty} \frac{p+q+(p-q)\text{sign}(x-\xi)}{|x-\xi|^{1+\nu}} [f(x) - f(\xi)] d\xi \\
&= \int_{-\infty}^x \frac{p+q+(p-q)}{(x-\xi)^{1+\nu}} [f(x) - f(\xi)] d\xi + \int_x^{\infty} \frac{p+q-(p-q)}{(\xi-x)^{1+\nu}} [f(x) - f(\xi)] d\xi \\
&= -2 \int_0^{\infty} [(p+q)f(x) - pf(x-\xi) - qf(x+\xi)] \xi^{-1-\nu} d\xi.
\end{aligned}$$

As a result, we have

$$\mathbb{D}_{p,q}^{\nu} f(x) = \frac{\nu A^{-1/2}}{\Gamma(1-\nu)} \int_0^{\infty} [(p+q)f(x) - pf(x-\xi) - qf(x+\xi)] \xi^{-1-\nu} d\xi.$$

In conclusion, we find the Fourier transform of the Feller fractional derivative:

$$\begin{aligned}
\mathcal{F}\{\mathbb{D}_{p,q}^{\nu} f(x)\}(k) &= \frac{\nu A^{-1/2}}{\Gamma(1-\nu)} \int_0^{\infty} \mathbf{F}\{(p+q)f(x) - pf(x-\xi) - qf(x+\xi)\}(k) \xi^{-1-\nu} d\xi \\
&= \frac{\nu A^{-1/2}}{\Gamma(1-\nu)} \left\{ \int_0^{\infty} [(p+q) - pe^{ik\xi} - qe^{-ik\xi}] \xi^{-1-\nu} d\xi \right\} \tilde{f}(k) \\
&= A^{-1/2} [p(-ik)^{\nu} + q(ik)^{\nu}] \tilde{f}(k) \\
&= A^{-1/2} \cos(\nu\pi/2) [(p+q) - i(p-q) \tan(\nu\pi/2) \text{sign}(k)] |k|^{\nu} \tilde{f}(k).
\end{aligned}$$

In order to satisfy the condition

$$\mathcal{F}\{\mathbb{D}_{p,q}^{\nu} \mathbb{D}_{p,q}^{-\nu} f(x)\}(k) = \tilde{f}(k)$$

and therefore to be called the fractional derivative, this expression has to contain the constant A determined as

$$A = [(p+q) \cos(\nu\pi/2)]^2 + [(p-q) \sin(\nu\pi/2)]^2.$$

As one can see from above, the Feller fractional derivative is expressed through two R-L derivatives as follows:

$$\mathbb{D}_{p,q}^{\nu} f(x) = A^{-1/2} [{}_x^{-\infty} \mathbf{D}_x^{\nu} + {}_x^{\infty} \mathbf{D}_{\infty}^{\nu}].$$

When $p = q$, the Feller derivative becomes the Riesz derivative:

$$\mathbb{D}_{p,p}^{\nu} f(x) = |2 \cos(\nu\pi/2)|^{-1} [{}_x^{-\infty} \mathbf{D}_x^{\nu} + {}_x^{\infty} \mathbf{D}_{\infty}^{\nu}]$$

(compare with Eq. (4.12)).

4.7 Fractional operators in multidimensional spaces

4.7.1 The Riesz potentials and derivatives

We begin consideration of fractional operators in $\mathbb{R}^d$ with the Riesz operators as mostly used in applications. It is known that the scalar potential of the electrostatic field $\phi(\mathbf{x})$ created by the charge distribution density $f(\mathbf{x})$ is given by the integral:

$$\phi(\mathbf{x}) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{f(\mathbf{x}') d\mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|}.$$

The three-dimensional Fourier transformation of the relation yields

$$\tilde{\phi}(\mathbf{k}) = \int_{\mathbb{R}^3} e^{i\mathbf{k} \cdot \mathbf{x}} \phi(\mathbf{x}) d\mathbf{x} = |\mathbf{k}|^{-2} \tilde{f}(\mathbf{k}).$$

The Riesz potential generalizes the operation to arbitrary order ν and space dimension d :

$$\mathbb{D}_d^{-\nu} f(\mathbf{x}) \equiv f_d^{(-\nu)}(\mathbf{x}) = \frac{1}{\gamma_d(\nu)} \int_{\mathbb{R}^d} \frac{f(\mathbf{x}') d\mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^{d-\nu}}, \quad \mathbf{x} \in \mathbb{R}^d, \quad \nu > 0, \quad \nu \neq d, d+2, d+4, \dots$$

The normalized constants $\gamma_d(\nu)$ are defined so that the relation in Fourier space preserves the same form:

$$\tilde{f}_d^{(-\nu)}(\mathbf{k}) = |\mathbf{k}|^{-\nu} \tilde{f}(\mathbf{k}), \quad \mathbf{k} \in \mathbb{R}^d.$$

According to the convolution theorem

$$\begin{aligned} \tilde{f}_d^{(-\nu)}(\mathbf{k}) &= \frac{1}{\gamma_d(\nu)} \mathcal{F} \left\{ \int_{\mathbb{R}^d} \frac{f(\mathbf{x}') d\mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^{d-\nu}} \right\}(\mathbf{k}) = \frac{1}{\gamma_d(\nu)} \mathcal{F} \{ f(\mathbf{x}) * |\mathbf{x}|^{v-d} \}(\mathbf{k}) \\ &= \frac{1}{\gamma_d(\nu)} \mathcal{F} \{ f(\mathbf{x}) \}(\mathbf{k}) \cdot \mathcal{F} \{ |\mathbf{x}|^{v-d} \}(\mathbf{k}). \end{aligned}$$

Direct calculations show that

$$\begin{aligned} \mathcal{F} \{ |\mathbf{x}|^{v-d} \}(\mathbf{k}) &= \int_{\mathbb{R}^d} e^{i\mathbf{k} \cdot \mathbf{x}} |\mathbf{x}|^{v-d} d\mathbf{x} = |\mathbf{k}|^{-\nu} (2\pi)^{d/2} \int_0^\infty \xi^{v-d/2} J_{d/2-1}(\xi) d\xi \\ &= |\mathbf{k}|^{-\nu} 2^\nu \pi^{d/2} \Gamma(\nu/2) / \Gamma((d-\nu)/2). \end{aligned}$$

Thus,

$$\gamma_d(\nu) = 2^\nu \pi^{d/2} \Gamma(\nu/2) / \Gamma((d-\nu)/2).$$

Note that this consideration is valid under condition $(d+1)/2 < \nu < d$, but the found expression for $\gamma_d(\nu)$ is applicable to all cases if $\nu \neq d+2k$ and $\nu \neq -2k$, where $k = 0, 1, 2, \dots$ (Samko et al., 1993).

Let us bring here elementary properties of the Riesz potential:

$$\begin{aligned}
\mathcal{F}\{\mathbb{D}_d^{-\nu} f(\mathbf{x})\}(\mathbf{k}) &= |\mathbf{k}|^{-\nu} \tilde{f}(\mathbf{k}), \quad \nu > 0, \\
\Delta \mathbb{D}_d^{-\nu} f(\mathbf{x}) &= -\mathbb{D}_d^{-\nu+2} f(\mathbf{x}), \quad \nu > 2, \\
\mathbb{D}_d^{-\mu} \mathbb{D}_d^{-\nu} f(\mathbf{x}) &= \mathbb{D}_d^{-\mu-\nu} f(\mathbf{x}), \quad \mu > 0, \nu > 0, \\
\mathbb{D}_d^{-\nu} e^{i\mathbf{k}\cdot\mathbf{x}} &= |\mathbf{k}|^{-\nu} e^{i\mathbf{k}\cdot\mathbf{x}}, \quad \mathbf{k} \in \mathbb{R}^d.
\end{aligned}$$

The Riesz potential of $e^{i\mathbf{k}\cdot\mathbf{x}}$ is a conventionally convergent integral if $0 < \nu < (d+1)/2$. In the case $\nu \geq (d+1)/2$, it is to be interpreted as the analytic continuation in the parameter ν .

The Riesz differentiation generalizes another classical relation between the potential $\phi(\mathbf{x})$ and the source density $f(\mathbf{x})$:

$$-\Delta \phi(\mathbf{x}) = f(\mathbf{x}),$$

which in Fourier space looks as follows:

$$|\mathbf{k}|^2 \tilde{\phi}(\mathbf{k}) = \tilde{f}(\mathbf{k}).$$

Formally, it can be designated by

$$\phi_d^{(\nu)}(\mathbf{x}) \equiv [(-\Delta_d)^{1/2}]^\nu \phi(\mathbf{x}) = (-\Delta_d)^{\nu/2} \phi(\mathbf{x}) = f(\mathbf{x}).$$

The notation in the left-hand side of the formula means only that

$$\mathcal{F}(-\Delta_d)^{\nu/2} \phi(\mathbf{x}) = |\mathbf{k}|^\nu \tilde{\phi}(\mathbf{k}).$$

Comparing with corresponding formula for the transform of the potential

$$\mathcal{F}\mathbf{R}_d^\nu f(\mathbf{x}) = |\mathbf{k}|^{-\nu} \tilde{f}(\mathbf{k})$$

intimate to existence of interrelation

$$(-\Delta_d)^{\nu/2} = \mathbb{D}_d^\nu,$$

according to which the representation

$$(-\Delta_d)^{\nu/2} f(\mathbf{x}) = \frac{1}{\gamma_d(-\nu)} \int_{\mathbb{R}^d} \frac{f(\mathbf{x}') d\mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^{d+\nu}}, \quad \nu > 0$$

could be expected.

In other words, the action of the fractional differential operator on the function $f(\mathbf{x})$ is to be realized as a convolution of $f(\mathbf{x})$ with the generalized function $|\mathbf{x}|^{-d-\nu}$. Such a convolution, i.e., an integral with the kernel $|\mathbf{x} - \mathbf{x}'|^{-d-\nu}$, in contrast to the Riesz potential has an order of singularity higher than the dimension of the space d and so it is called hypersingular integral. Such an integral diverges and so our convolution needs to be properly defined.

Let $0 < \nu < 1$ at first. We can guarantee convergence of the convolution of the function $|\mathbf{x}|^{-d-\nu}$ (with sufficiently good functions) introducing it as

$$\int_{\mathbb{R}^d} \frac{f(\mathbf{x}') - f(\mathbf{x})}{|\mathbf{x} - \mathbf{x}'|^{d+\nu}} d\mathbf{x}' = - \int_{\mathbb{R}^d} \frac{f(\mathbf{x}) - f(\mathbf{x} - \mathbf{x}')}{|\mathbf{x}'|^{d+\nu}} d\mathbf{x}'.$$

This integral converges if $0 < \nu < 1$ for bounded differentiable functions and may be considered as a multidimensional analogue of the Marchaud derivative. An extension to the case $\nu \geq 1$ may be given either in terms of regularization, or by taking finite differences. In the last case, we obtain the representation of $(-\Delta)^{\nu/2}_d$, $\nu > 0$ in the form

$$(-\Delta_d)^{\nu/2} = \frac{1}{\delta_d(\nu, l)} \int_{\mathbb{R}^d} \frac{(\Delta_{\mathbf{h}}^l f)(\mathbf{x})}{|\mathbf{h}|^{d+\nu}} d\mathbf{h},$$

where $(\Delta_{\mathbf{h}}^l f)(\mathbf{x})$ is the finite difference of a function $f(\mathbf{x})$ of many variables with a vector step $\mathbf{h}$.

One can deal both with centered differences

$$(\Delta_{\mathbf{h}}^l f)(\mathbf{x}) = \sum_{k=0}^l (-1)^k \binom{l}{k} f[\mathbf{x} + (l/2 - k)\mathbf{h}]$$

and with non-centered differences

$$(\Delta_{\mathbf{h}}^l f)(\mathbf{x}) = \sum_{k=0}^l (-1)^k \binom{l}{k} f(\mathbf{x} - k\mathbf{h}).$$

The normalized constants guaranteeing the equality

$$\mathcal{F} \left\{ \frac{1}{\delta_d(\nu, l)} \int_{\mathbb{R}^d} \frac{(\Delta_{\mathbf{h}}^l f)(\mathbf{x})}{|\mathbf{h}|^{d+\nu}} d\mathbf{h} \right\} = |\mathbf{k}|^{\nu} \tilde{f}(\mathbf{k})$$

are defined by the integrals

$$\delta_d(\nu, l) = \begin{cases} \int_{\mathbb{R}^d} (1 - e^{ix_1})^l |\mathbf{x}|^{-d-\nu} d\mathbf{x} & \text{for non-centered differences,} \\ 2^{l-\nu} i^l \int_{\mathbb{R}^d} \sin^l x_1 |\mathbf{x}|^{-d-\nu} d\mathbf{x} & \text{for centered differences.} \end{cases}$$

Here, x_1 is the first (in principle, any) of coordinates of d -dimensional vector $\mathbf{x} = \{x_1, x_2, \dots, x_d\}$ (see for details (Samko et al., 1993)).

Stress that the Riesz operator $(-\Delta)^{\nu/2}$ being expressed through the difference $\Delta_{\mathbf{h}}^l$ does not depend on the choice of l , $l > \nu$.

Combining both operations we represent them as the Riesz differintegral:

$$(-\Delta_d)^{\nu/2} f(\mathbf{x}) = \begin{cases} \frac{1}{\gamma_d(-\nu)} \int_{\mathbb{R}^d} \frac{f(\mathbf{x}') d\mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^{d+\nu}}, & \nu < 0, \nu \neq -d - 2k, \\ \frac{1}{\delta_{d,n}(\nu)} \int_{\mathbb{R}^d} \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{f(\mathbf{x} - k\mathbf{x}')}{|\mathbf{x}'|^{d+\nu}} d\mathbf{x}', & \nu > 0, n = [\nu] + 1. \end{cases}$$

Here

$$\gamma_d(\nu) = 2^\nu \pi^{d/2} \Gamma(\nu/2) / \Gamma((d - \nu)/2),$$

and

$$\delta_{d,n}(\nu) = \frac{\pi^{1+d/2}}{2^\nu \Gamma(1 + \nu/2) \Gamma((d + \nu)/2) \sin(\nu\pi/2)} \sum_{k=1}^n (-1)^{(k-1)} \binom{n}{k} k^\nu.$$

4.7.2 Directional derivatives and gradients

Some authors define the fractional directional derivative of order ν of the function $f(\mathbf{x})$ as the Riemann-Liouville fractional derivative of the function $f(\mathbf{x} + \Omega\xi)$ with respect to ξ at $\xi = 0$:

$$\mathbf{D}_\Omega^\nu f(\mathbf{x}) \equiv -\rho(\mathbf{x}, \Omega) \mathbf{D}_\xi^\nu f(\mathbf{x} + \Omega\xi) \Big|_{\xi=0} = \mathbf{D}_\xi^n -\rho(\mathbf{x}, \Omega) I_\xi^{n-\nu} f(\mathbf{x} + \Omega\xi) \Big|_{\xi=0},$$

where $\rho(\mathbf{x}, \Omega)$ is the distance between the “observation point” $\mathbf{x}$ and the boundary point on the ray in the opposite direction. When the order takes negative values, $\nu = -\mu$, $\mu > 0$, the fractional derivative becomes the integral over the segment $(\mathbf{x} - \Omega\rho, \mathbf{x})$:

$$\mathbf{D}_\Omega^{-\mu} f(\mathbf{x}) \equiv \frac{1}{\Gamma(\mu)} \int_0^{\rho(\mathbf{x}, \Omega)} f(\mathbf{x} - \Omega\xi) \xi^{\mu-1} d\xi. \quad (4.13)$$

Choosing j th ort of coordinate system as the vector Ω , one can introduce the *partial fractional derivative*

$$a_j f_j^{(\nu_j)}(\mathbf{x}) = a_j \mathbf{D}_{x_j}^{\nu_j} f(x_1, \dots, x_j, \dots, x_d),$$

and the mixed fractional derivatives

$$a_j f_{\mathbf{x}}^{(\mathbf{v})}(\mathbf{x}) = \mathbf{D}_{x_1}^{\nu_1} \dots \mathbf{D}_{x_d}^{\nu_d} f(\mathbf{x}), \quad \mathbf{v} = (\nu_1, \dots, \nu_d)$$

(see for details (Kilbas et al., 2006)).

Considering $\mathbf{D}_{x_j}^\nu$, $j = 1, 2, \dots, d$, $0 < \nu \leq 1$ as components of the “vector” $\nabla_{\mathbf{x}}^\nu$, one can obtain the fractional generalization of the gradient, and define the the operator $\nabla_{\mathbf{x}}^{2\nu}$ as the scalar square of the gradient: $\nabla_{\mathbf{x}}^\nu \nabla_{\mathbf{x}}^\nu$. In a formal writing

$$\nabla_{\mathbf{x}}^{2\nu} \equiv \sum_{j=1}^d \mathbf{D}_{x_j}^{2\nu}.$$

As $\nu \rightarrow 1$, this operator becomes the ordinary Laplacian, however, in case $\nu < 1$, it loses the major property of this operator: the rotational invariance. Fractional “successor” of the Laplacian keeping this property is looks as follows:

$$(-\Delta)^\nu \equiv \left(-\sum_{j=1}^d \mathbf{D}_{x_j}^2 \right)^\nu.$$

Fourier transforms of these operators,

$$\mathcal{F} \{ \nabla_{\mathbf{x}}^{2\nu} \} (k) = \sum_{j=1}^d (-ik_j)^{2\nu}$$

and

$$\mathcal{F} \{ (-\Delta)^\nu \} (k) = \left(-\sum_{j=1}^d (-ik_j)^2 \right)^\nu = \left(\sum_{j=1}^d k_j^2 \right)^\nu = |\mathbf{k}|^{2\nu}$$

demonstrate their difference for $\nu < 1$. Particularly, in case $\nu = 1/2$,

$$\nabla_{\mathbf{x}}^1 f(\mathbf{x}) = \sum_{j=1}^d \frac{\partial f(\mathbf{x})}{\partial x_j},$$

while

$$\Delta^{1/2} = \sqrt{\sum_{j=1}^d \frac{\partial^2 f(\mathbf{x})}{\partial x_j^2}}.$$

In (Li et al., 2010) the Laplace equations with fractional radius derivative in 3-d and 2-d spaces are discussed. The authors generalize the gradient of a scalar function $f(r, \theta, \phi)$ as

$$\text{grad}^\alpha f = \left(\mathbf{e}_r {}^\alpha \mathbf{D}_r + \mathbf{e}_\theta \frac{\Gamma(\alpha+1)}{r^\alpha} \frac{\partial}{\partial \theta} + \mathbf{e}_\phi \frac{\Gamma(\alpha+1)}{r^\alpha \sin^\alpha \theta} \frac{\partial}{\partial \phi} \right) f(r, \theta, \phi).$$

By this definition, the real space metric is changed to an effective metric. This transformation is explained in (Li et al., 2010) by means of Fig. 1: the radius $|OA|$ becomes ${}_0l_r 1 = \frac{1}{\Gamma(1+\alpha)} r^\alpha$, the arc length $\widehat{NA} = \frac{1}{\Gamma(1+\alpha)} r^\alpha \theta$ and the arc length $\widehat{LA'} = \frac{1}{\Gamma(1+\alpha)} r^\alpha \sin^\alpha \theta \phi$. Due to the nonlocality of the fractional operations, this kind of metric is not additive ($|OB| \neq |OA| + |AB|$ even when A, B , and O are on the same straight line).

Following this way, the authors expressed the divergence of a vector function $\mathbf{A} = (A_r, A_\theta, A_\phi)$,

$$\begin{aligned} \text{div}^\alpha \mathbf{A}(r, \theta, \phi) &= \frac{1}{r^{2\alpha} \sin^\alpha \theta} {}^\alpha \mathbf{D}_r (r^{2\alpha} \sin^\alpha \theta A_r) \\ &+ \frac{\Gamma(1+\alpha)}{r^{2\alpha} \sin^\alpha \theta} \frac{\partial}{\partial \theta} (r^\alpha \sin^\alpha \theta A_\theta) + \frac{\Gamma(1+\alpha)}{r^{2\alpha} \sin^\alpha \theta} \frac{\partial}{\partial \phi} (r^\alpha A_\phi), \end{aligned}$$

and the Laplacian of a scalar function $f(r, \theta, \phi)$,

$$\Delta^\alpha f(r, \theta, \phi) = \frac{1}{r^{2\alpha}} {}_0^{\alpha}D_r (r^{2\alpha} {}_0^{\alpha}D_r f) + \frac{\Gamma^2(1+\alpha)}{r^{2\alpha} \sin^\alpha \theta} \frac{\partial}{\partial \theta} \left(\sin^\alpha \theta \frac{\partial f}{\partial \theta} \right) + \frac{\Gamma^2(1+\alpha)}{r^{2\alpha} \sin^{2\alpha} \theta} \frac{\partial^2 f}{\partial \phi^2}.$$

4.7.3 Various fractionalizing grad, div, and curl operators

It is unnecessary to say how the vector analysis is important for theoretical physics in general and the electrodynamics particularly. This is a strong impetus for combining the fractional differential ideas with the vector calculus and constructing on this basis fractional vector analysis.

From naive point of view, such a generalization seems to be almost evident:

$$\begin{aligned} \text{grad} f(\mathbf{x}) &= \sum_{j=1}^d \mathbf{e}_j D_{x_j} f(\mathbf{x}) \rightarrow \text{grad}^\alpha f(\mathbf{x}) = \sum_{j=1}^d \mathbf{e}_j D_{x_j}^\alpha f(\mathbf{x}), \\ \text{div} \mathbf{F}(\mathbf{x}) &= \sum_{j=1}^d D_{x_j} F_j(\mathbf{x}) \rightarrow \text{div}^\alpha \mathbf{F}(\mathbf{x}) = \sum_{j=1}^d D_{x_j}^\alpha F_j(\mathbf{x}), \\ \text{curl} \mathbf{F}(\mathbf{x}) &= \sum_{j=1}^d \mathbf{e}_j \varepsilon_{jkl} D_{x_k} F_l(\mathbf{x}) \rightarrow \text{curl}^\alpha \mathbf{F}(\mathbf{x}) = \sum_{j=1}^d \mathbf{e}_j \varepsilon_{jkl} D_{x_k}^\alpha F_l(\mathbf{x}). \end{aligned}$$

4.7.3.1 Engheta curl operator

Engheta (1998) has suggested that, in order to fractionalize the curl operator

$$\text{curl} \mathbf{F} = \nabla \times \mathbf{F},$$

one should first fractionalize the cross product operator $-i\mathbf{k} \times \mathbf{F}$ representing the curl operator in the Fourier space. In order to do this, he considers a class of linear operators where the domain and range of any linear operator are similar to each other and have the same dimensions. That is, any linear operator $\mathbf{L}$ of this class should map an element from the d -dimensional complex space $\mathbb{C}^d$ into generally another element in $\mathbb{C}^d$. Then the new operator $\mathbf{L}^\alpha$ can be considered as the fractionalized operator $\mathbf{L}$ if $\mathbf{L}^1 = \mathbf{L}$, $\mathbf{L}^0 = \mathbf{1}$, and $\mathbf{L}^\alpha \mathbf{L}^\beta = \mathbf{L}^\beta \mathbf{L}^\alpha = \mathbf{L}^{\alpha+\beta}$. Taking into account that operators $\mathbf{L}$ and $\mathbf{L}^\alpha$ have the same eigenvectors and different eigenvalues, λ and λ^α , and performing some manipulations, Engheta arrives at the representation of $\text{curl}^\alpha \mathbf{F}(\mathbf{x})$ which in case $\mathbf{F}(\mathbf{x}) = \mathbf{F}(z)$ has the form

$$\begin{aligned} \text{curl}^\alpha \mathbf{F} &= \mathbf{e}_x [\cos(\alpha\pi/2) {}_{-\infty}D_z^\alpha F_x(z) - \sin(\alpha\pi/2) {}_{-\infty}D_z^\alpha F_y(z)] \\ &+ \mathbf{e}_y [\sin(\alpha\pi/2) {}_{-\infty}D_z^\alpha F_x(z) + \cos(\alpha\pi/2) {}_{-\infty}D_z^\alpha F_y(z)] + \mathbf{e}_z {}_{0,\alpha}D_z^\alpha F_z(z). \end{aligned}$$

4.7.3.2 The Ben Adda approach

Ben Adda (2001) uses approach based on three definitions. Firstly, he defines the α -transform of $f(z)$ at $z \in \mathbb{C}$ under z_0 by

$$g_\alpha(z) = (T_\alpha f)(z, z_0) = \frac{1}{2\pi i} \int_{C(z_0)} \frac{f(\xi) d\xi}{(\xi - z_0) - (z - z_0)^\alpha (\xi - z_0)^{1-\alpha}},$$

where $\alpha \in (0, 1]$, $C(z_0) = \{C_-, C_+\}$, $C_-(z_0)$ is the curve along the cut joining two points z_0 and $-\infty + i\text{Im}(z_0)$, and $C_+(z_0) = \{C_-, C_+\}$, $C_+(z_0)$ is the curve along the cut joining two points z_0 and $+\infty + i\text{Im}(z_0)$. Secondly, the function regular in a strip including the real axis is α -differentiable at the point x_0 if there exist $L \in \mathbb{R}$ and a function $\varepsilon(x)$ such that $\lim_{x \rightarrow x_0} \varepsilon(x) = 0$, and

$$(T_\alpha f)(x, x_0) = f(x_0) + L(x - x_0)^\alpha + (x - x_0)^\alpha \varepsilon((x - x_0)^\alpha).$$

The constant L is given by $L = N^\alpha f(x_0)/\Gamma(1 + \alpha)$, and the operator N^α represents the Nishimoto fractional derivative. Third definition generalizes the two to d -dimensional space $\mathbb{R}^d$: the transformation of f of order α at the point $\mathbf{x} + \mathbf{h} \in \mathbb{R}^d$ is given by

$$(T_\alpha f)(\mathbf{x} + \mathbf{h}, \mathbf{x}) = ((T_\alpha f)_1(\mathbf{x} + \mathbf{h}, \mathbf{x}) \cdots (T_\alpha f)_p(\mathbf{x} + \mathbf{h}, \mathbf{x})),$$

with

$$(T_\alpha f)_j(\mathbf{x} + \mathbf{h}, \mathbf{x}) = \frac{1}{(2\pi i)^d} \int_{C(x_1)} \cdots \int_{C(x_n)} \frac{f_j(\xi_1, \dots, \xi_d) d\xi_1 \cdots d\xi_d}{\prod_{k=1}^d (((\xi_k - x_k) - h_k^\alpha)(\xi_k - x_k)^{1-\alpha})},$$

$$j = 1, \dots, p.$$

On the basis, Ben Adda (2001) suggests the following generalizations of the fractional operators:

$$\begin{aligned} \text{grad}^\alpha f(\mathbf{x}) &= \frac{1}{\Gamma(\alpha + 1)} \sum_{j=1}^d \mathbf{e}_j D_{x_j}^\alpha f(\mathbf{x}), \\ \text{div}^\alpha \mathbf{F}(\mathbf{x}) &= \frac{1}{\Gamma(\alpha + 1)} \sum_{j=1}^d D_{x_j}^\alpha F_j(\mathbf{x}), \\ \text{curl}^\alpha \mathbf{F}(\mathbf{x}) &= \frac{1}{\Gamma(\alpha + 1)} \sum_{j=1}^d \mathbf{e}_j \varepsilon_{jkl} D_{x_k}^\alpha F_l(\mathbf{x}). \end{aligned}$$

4.7.3.3 Meerschaert-Mortensen-Wheatcraft approach

Meerschaert et al. (2006) start from the fractional integration operator

$$I_M^{1-\nu} f(\mathbf{x}) = \int \Omega I_\Omega^{1-\nu} \Omega^T f(\mathbf{x}) M(d\Omega), \quad 0 < \beta < 1,$$

where Ω is a unit vector, $M(\cdot)$ is a distribution measure, $D_\Omega^\nu f(\mathbf{x})$ is the inverse Fourier transform of $(i\mathbf{k} \cdot \Omega) \tilde{f}(\mathbf{k})$. With this definition, the fractional operators are expressed as follows:

$$\text{grad}_M^\nu f(\mathbf{x}) = I_M^{1-\nu} \nabla f(\mathbf{x}) = \int \Omega D_\Omega^{\nu-1} \Omega^T \cdot \nabla f(\mathbf{x}) M(d\Omega) = \int \Omega D_\Omega^\nu f(\mathbf{x}) M(d\Omega),$$

$$\text{div}_M^\nu \mathbf{F}(\mathbf{x}) = \nabla \cdot I_M^{1-\nu} \mathbf{F}(\mathbf{x}) = \int \nabla \cdot \Omega D_\Omega^{\nu-1} \Omega^T \cdot \nabla \mathbf{F}(\mathbf{x}) M(d\Omega) = \int D_\Omega^\nu \mathbf{F}(\mathbf{x}) \cdot \Omega M(d\Omega),$$

and

$$\text{curl}_M^\nu \mathbf{F}(\mathbf{x}) = \nabla \times I_M^{1-\nu} \mathbf{F}(\mathbf{x}) = \int \nabla \times \Omega D_\Omega^{\nu-1} \Omega^T \cdot \mathbf{F}(\mathbf{x}) M(d\Omega).$$

4.7.3.4 Tarasov's approach

Tarasov (2008) defines the fractional differential operators such that fractional generalizations of integral theorems (Green's, Stokes', Gauss') can be realized. Using one-dimensional Gerasimov-Caputo fractional derivatives ${}^\alpha D_W[x_i] \equiv {}_{a_i}^\alpha D_{b_i}$ inside the parallelepiped $W = \{a_i \leq x_i \leq b_i\}'$, $i = 1, 2, \dots, d$, he introduces the operators as follows:

$$\begin{aligned} {}^\alpha \text{grad} f(\mathbf{x}) &= \sum_{j=1}^d \mathbf{e}_j {}^\alpha D_W[x_j] f(\mathbf{x}), \\ {}^\alpha \text{div} \mathbf{F}(\mathbf{x}) &= \sum_{j=1}^d {}^\alpha D_W[x_j] F_j(\mathbf{x}), \\ {}^\alpha \text{curl} \mathbf{F}(\mathbf{x}) &= \sum_{j=1}^d \mathbf{e}_j \varepsilon_{jkl} {}^\alpha D_W[x_k] F_l(\mathbf{x}). \end{aligned}$$

4.7.3.5 Baleanu et al. approach

The Baleanu et al. (2009) approach is based on generalization of the classical exterior differential operator

$$d = \sum_{j=1}^d x_j \frac{\partial}{\partial x_j}$$

by using the Gerasimov-Caputo fractional operators:

$$d^\nu = \sum_{j=1}^d x_j^\nu {}_0^\nu D_{x_j}.$$

Let

$$\sigma = \sum_{i=1}^d \sigma_i dx_i^v$$

and

$$d^v \sigma = \sum_{j=1}^d d^v (\sigma_j dx_j^v).$$

Using the product rule of exterior fractional derivative yields

$$d^v \sigma = \sum_{i=1}^d \sum_{j=1}^d {}^v D_{x_j} \sigma_i dx_j^v dx_i^v.$$

The fractional gradient, divergence, and curl are defined as follows:

$${}^v \text{grad} f(\mathbf{x}) = \mathbf{e}_x {}^v D_x f(\mathbf{x}) + \mathbf{e}_y {}^v D_y f(\mathbf{x}) + \mathbf{e}_z {}^v D_z f(\mathbf{x}),$$

$${}^v \text{div} \mathbf{F}(\mathbf{x}) = {}^v D_x F_x(\mathbf{x}) + {}^v D_y F_y(\mathbf{x}) + {}^v D_z F_z(\mathbf{x}),$$

$$\begin{aligned} {}^v \text{curl} \mathbf{F}(\mathbf{x}) &= \mathbf{e}_x ({}^v D_y F_z(\mathbf{x}) - {}^v D_z F_y(\mathbf{x})) + \mathbf{e}_y ({}^v D_z F_x(\mathbf{x}) - {}^v D_x F_z(\mathbf{x})) \\ &\quad + \mathbf{e}_z ({}^v D_x F_y(\mathbf{x}) - {}^v D_y F_x(\mathbf{x})). \end{aligned}$$

Some important results concerning development of fractional geometry can also be found in (Albu and Opris, 2009; Vacaru, 2010).

4.8 Concluding remarks

4.8.1 Leibniz's definition

It is customary to consider that the history of the fractional calculus began since the question raised in the year 1695 by Marquis de Lôpital to Gottfried Wilhelm Leibniz, which sought the meaning of Leibniz's (currently popular) notation $\frac{d^n y}{dx^n}$ for the derivative of the order $n = 0, 1, 2, \dots$, when $n = 1/2$. In his reply, Leibniz wrote: "This is an apparent paradox from which, one day, useful consequences will be drawn." He introduced (1710) fractional derivatives of exponential functions $e^{\lambda x}$ by generalizing the known formula

$$\frac{d^n e^{\lambda x}}{dx^n} = \lambda^n e^{\lambda x}, \quad n = 1, 2, 3, \dots$$

to fractional values of $n = v$,

$$\frac{d^v e^{\lambda x}}{dx^v} = \lambda^v e^{\lambda x}.$$

4.8.2 Euler-Lacroix's definition

The answer to Lôpital's question was also offered by Euler (1738) who generalized the formula

$$\frac{d^n x^\mu}{dx^n} = \mu(\mu-1)\cdots(\mu-n+1)x^{\mu-n}, \quad \mu \geq 0,$$

with integer n to non-integer orders

$$\frac{d^\nu x^\mu}{dx^\nu} = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\nu+1)} x^{\mu-\nu}, \quad \mu \geq \nu \geq 0,$$

by passing to the Gamma function,

$$\mu(\mu-1)\cdots(\mu-n+1) = \frac{\Gamma(\mu+1)}{\Gamma(\mu-n+1)},$$

and replacing integer n by an arbitrary ν :

$$\frac{\Gamma(\mu+1)}{\Gamma(\mu-n+1)} \Rightarrow \frac{\Gamma(\mu+1)}{\Gamma(\mu-\nu+1)}.$$

The same definition for fractional derivatives of power kind functions was also introduced by Lacroix (1819).

Following this way, one can extend the definition on every function representable in the form of polynomial or power series:

$$f(x) = \sum_{k=m}^{\infty} a_k x^k,$$

$$\frac{d^\nu f(x)}{dx^\nu} = \sum_{k=m}^{\infty} \frac{k!}{\Gamma(k-\nu+1)} a_k x^{k-\nu}, \quad \nu \leq m.$$

4.8.3 The Fourier definitions

Three years after Lacroix's definition, Fourier (1822) suggested the definition of fractional derivatives by generalizing the formula for trigonometric functions,

$$\frac{d^\nu}{dx^\nu} \cos(x) = \cos\left(x + \frac{\pi\nu}{2}\right).$$

Later, this approach was developed to the definition of fractional derivatives through their Fourier transformations:

$$\frac{d^\nu f(x)}{dx^\nu} \equiv f^{(\nu)}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (-ik)^\nu e^{-ikx} \tilde{f}(k) dk.$$

In the Fourier space, we have:

$$\widetilde{f^{(v)}}(k) = (-ik)^v \widetilde{f}(k).$$

4.8.4 The Liouville definitions

The Leibniz idea was developed by Liouville (1832) who considered functions $f(x)$ representable in the form

$$f(x) = \sum_{k=1}^{\infty} c_k e^{a_k x}$$

and obtained

$$\frac{d^v}{dx^v} f(x) = \sum_{k=1}^{\infty} c_k a_k e^{a_k x}$$

on assumption the series converges. This approach became the basis for introducing fractional derivatives by using their Laplace transforms.

The second Liouville definition is connected with the integral

$$J(x) = \int_0^{\infty} e^{-xt} t^{\mu-1} dt, \quad \mu > 0, x > 0.$$

By changing the variable $xt = \xi$, it can be represented in the form

$$x^{-\mu} = \frac{1}{\Gamma(\mu)} J(x).$$

Acting on the function by fractional differential operator of order v , one gets

$$\begin{aligned} \frac{d^v}{dx^v} x^{-\mu} &= \frac{1}{\Gamma(\mu)} \int_0^{\infty} \frac{d^v}{dx^v} e^{-xt} t^{\mu-1} dt = \frac{1}{\Gamma(\mu)} \int_0^{\infty} (-t)^v e^{-xt} t^{\mu-1} dt \\ &= \frac{(-1)^v}{\Gamma(\mu)} \int_0^{\infty} e^{-xt} t^{\mu+v-1} dt = (-1)^v \frac{\Gamma(\mu+v)}{\Gamma(\mu)} x^{-\mu-v}, \quad \mu > 0, \quad x > 0. \end{aligned}$$

4.8.5 Riemann's definition with complementary function

Although up to that time the definitions related to fractional derivatives, Riemann (1876) defined the fractional integral of $f(x)$ to be

$$f^{(v)}(x) = \frac{1}{\Gamma(-v)} \int_a^x (x-\xi)^{-v-1} f(\xi) d\xi + \Psi(x), \quad v < 0,$$

where $\Psi(x)$ called a *complementary function* was included on definition because of the ambiguity of the lower limit of integration. In one of his memoirs, he wrote: “The ordinary differential equation $dy^n/dx^n = 0$ has the complementary solution $y_c = c_0 + c_1x + c_2x^2 + \dots + c_{n-1}x^{n-1}$. Thus, $d^uy/dx^u = 0$ (u arbitrary) should have a corresponding complementary solution.” Further mention of it is made later, for it played a role in planting the seeds of distrust in the general theory of fractional operators (Miller and Ross, 1993). Peacock (1833) and Greatheed (1839) published papers which dealt with the complementary functions. Greatheed was the first to call attention to the indeterminate nature of the complementary function. New results in the development of this idea are contained in works of Lorenzo and Hartley (see Sect. 5.1).

4.8.6 From Sonin's to Nishimoto's fractional operators

As noted in the book (Miller and Ross, 1993), “the earliest work that ultimately led to what is now called the Riemann-Liouville definition appears to be the paper by N. Ya. Sonin (1869) entitled ‘On differentiation with arbitrary index’. His starting point was Cauchy's integral formula. A. V. Letnikov wrote four papers on this topic from 1868 to 1872. His paper ‘An explanation of the main concepts of the theory of differentiation of arbitrary index’ (1872) is an extension of Sonin's paper. The n th derivative of Cauchy's integral formula is given by

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{C_0} \frac{f(\zeta)d\zeta}{(\zeta - z)^{n+1}}.$$

There is no problem in generalizing $n!$ to arbitrariness values since $v! = \Gamma(v + 1)$. However, when n is not an integer, the integrand no longer contains a pole, but a branch point.” For this reason, H. Laurent (1884) introduced integration along an open circuit C on a Riemann surface, in contrast to the closed circuit C_0 of Sonin and Letnikov. The final form of this concept represented by K. Nishimoto (1984–1994) looks as follows.

Let

$$D = \{D^-, D^+\}, \quad C = \{C^-, C^+\},$$

C^- be a curve along the cut joining two points z and $-\infty + i\Im(z)$, C^+ be a curve along the cut joining points z and $\infty + i\Im(z)$, D^- be a domain surrounded by C^- , and D^+ a domain surrounded by C^+ . Moreover, let $f(z)$ be a regular function in D ,

$$f^{(v)}(z) = \frac{\Gamma(v+1)}{2\pi i} \int_C \frac{f(\zeta)d\zeta}{(\zeta - z)^{v+1}}, \quad z \in C, \quad v \in \mathbb{R}, \quad \mathbb{Z}^-, \quad \mathbb{Z}^- = \{-1, -2, \dots\},$$

and

$$f^{(-n)}(z) = \lim_{v \rightarrow -n} f^{(v)}(z), \quad n \in \mathbb{Z}^+,$$

where $\zeta \neq z$,

$$-\pi \leq \arg(\zeta - z) \leq \pi \text{ for } C^-$$

and

$$0 \leq \arg(\zeta - z) \leq 2\pi \text{ for } C^+,$$

then $f^{(\nu)}(z)$, $\nu > 0$ is said to be the Nishimoto fractional derivative of $f(z)$ of order ν , and $f^{(\nu)}(z)$, $\nu < 0$ is said to be the Nishimoto fractional integral of $f(z)$ of order $-\nu$, provided in each case that $|f^{(\nu)}(z)| < \infty$.

For a more in-depth view of fractional derivatives and their history (see (Post, 1930; Oldham and Spanier, 1974; Ross, 1975 and 1977; Samko et al., 1993; Miller and Ross, 1993; Vinagre, 2002; Loverro, 2004; Dalir and Bashour, 2010)) and other articles given in the References to this chapter. Especially, I'd like to draw the reader's attention to the dissertation of Marc Weilbeer (2005), containing remarkable historical chapter.

4.8.7 Local fractional derivatives

The limit behavior ${}_a f^\nu(x)$ at $x \rightarrow a$ is connected to a new kind of fractional differential operator, producing the local fractional derivative (LFD) (Kolwankar and Calgar, 1996 and 1998). To motivate the new definition, its authors note that the operator ${}_a I_x^\nu$ is non-local, and, further, the ${}_a f^\nu(x) \neq 0$ for $f(x) = \text{const}$ and the motivation were to correct for both of this features. These two features play an important role in defining local fractional differentiability. The authors have chosen the following way to modify the Riemann-Liouville concept of fractional derivative.

“(1) We subtract, from the function, the value of the function at the point where fractional differentiability is to be checked. This makes the value of the function zero at that point, washing out the effect of any constant term. (2) The natural choice of a lower limit will be that point, where we intend to examine the fractional differentiability, itself. This has an advantage in that it preserves the local nature of the differentiability property.”

As a result, the LFD has been defined as the limit

$${}_x f^{(\nu)}(x) = \lim_{\xi \rightarrow x} {}_x D_\xi^\nu [f(\xi) - f(x)],$$

if it exists. The other definition, introduced in the work (Kolwankar and Gangal, 1996) concerns to the function under differentiation: critical order $\alpha(x)$ of the function $f(x)$ at point x is defined as $\sup\{\nu \mid \text{all } {}_a f^{(\nu)}(x) \text{ of order less than } \nu \text{ exist at } x\}$. The local fractional derivatives appear to be suitable to deal with non-differentiable functions having fractal graphs. One can show, in particular, that the LFD of Weierstrass nowhere differentiable function exists up to (critical) order $1 - \gamma$, where $1 + \gamma$ ($0 < \gamma < 1$) is the box dimension of the graph of the function. The use of LFD to study pointwise behavior of multifractal functions was also demonstrated. Generalized to $\nu \in (n, n + 1]$, the LFD appear naturally in the fractional Taylor expansion as

the coefficient of the power with fractional exponent:

$$f(x + \xi) = \sum_{j=0}^n \frac{f^{(j)}(x)}{\Gamma(j+1)} \xi^j + \frac{x f^{(\nu)}(x)}{\Gamma(\nu+1)} \xi^\nu + R_\nu(x, \xi),$$

where $R_\nu(x, \xi)$ is the remainder.

4.8.8 The Jumarie nonstandard approach

A nonstandard way defining fractional derivatives, offered by Jumarie, is based on integration with respect to $(dx)^\alpha$. Assume that $f(x)$ is a constant C . Then its fractional J-R-L derivative of order α is determined as

$${}_0\bar{D}_x^\alpha C = \begin{cases} \frac{C}{\Gamma(1-\alpha)} x^{-\alpha}, & \alpha \leq 0; \\ 0, & \alpha > 0. \end{cases}$$

In case $f(x)$ is not a constant, it is represented as

$$f(x) = f(0) + (f(x) - f(0)),$$

and its J-R-L derivative is defined by the expression

$$\begin{aligned} {}_0\bar{f}^{(\alpha)}(x) &= {}_0\bar{D}_x^\alpha f(x) = {}_0\bar{D}_x^\alpha f(0) + {}_0\bar{D}_x^\alpha (f(x) - f(0)) \\ &= \begin{cases} \frac{f(0)}{\Gamma(1-\alpha)} x^{-\alpha} + \frac{1}{\Gamma(-\alpha)} \int_0^x (x-\xi)^{-\alpha-1} f(\xi) d\xi, & \alpha \leq 0; \\ \frac{1}{\Gamma(-\alpha)} D_x \int_0^x (x-\xi)^{-\alpha} (f(\xi) - f(0)) d\xi, & 0 < \alpha < 1. \end{cases} \end{aligned}$$

When $m < \alpha \leq m+1$,

$${}_0\bar{f}^{(\alpha)}(x) = D_x^m {}_0\bar{f}^{(\alpha-m)}(x), \quad m < \alpha \leq m+1, \quad m \geq 1.$$

The main idea in this definition is that a fractional differentiable function is considered as the sum of a constant with a selfsimilar function which, as such, takes on the value zero at $x = 0$.

The fractional Taylor's series written with these derivatives

$$f(x+h) = \sum_{k=0}^{\infty} \frac{h^{k\alpha}}{\Gamma(1+k\alpha)} ({}_0\bar{D}_x^\alpha)^k f(x), \quad 0 < \alpha < 1,$$

leads to relation

$$f(x+h) - f(x) \sim \frac{{}_0\bar{f}^{(\alpha)}(x)}{\Gamma(1+\alpha)} h^\alpha, \quad h \rightarrow 0.$$

Being rewritten in terms of differentials, it looks as

$$df(x) = \frac{{}_0\bar{f}^{(\alpha)}(x)}{\Gamma(1+\alpha)}(dx)^\alpha.$$

Introducing the fractional differential as

$$d^\alpha f = {}_0\bar{f}^{(\alpha)}(x)(dx)^\alpha,$$

one can reduce it in the useful form

$$d^\alpha f = \Gamma(1+\alpha)df, \quad 0 < \alpha \leq 1.$$

The integral with respect to $(dx)^\alpha$ is defined as the solution of the equation

$$\begin{aligned} df &= F(x)(dx)^\alpha, \quad x \geq 0, \quad f(0) = 0, \quad 0 < \alpha \leq 1, \\ f(x) &= \int_0^x F(\xi)(d\xi)^\alpha = \alpha \int_0^x (x-\xi)^{\alpha-1} F(\xi)d\xi. \end{aligned}$$

The following equalities hold:

$$\begin{aligned} {}_0\bar{D}_x^\alpha x^\mu &= \Gamma(\mu+1-\alpha)x^{\mu-\alpha}, \quad \mu > 0, \\ {}_0\bar{D}_x^\alpha (u(x)v(x)) &= \bar{u}^{(\alpha)}(x)v(x) + u(x)\bar{v}^{(\alpha)}(x), \\ \bar{f}_t^{(\alpha)}(x(t)) &= \Gamma(2-\alpha)x^{\alpha-1}\bar{f}_x^{(\alpha)}(x)\bar{x}^{(\alpha)}(t), \quad 0 < \alpha \leq 1. \end{aligned}$$

For the proof of these formulas and other details, see (Jumarie, 2009).

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Chapter 5

Equations and Solutions

As said in Introduction, the passage from integer orders of derivatives to real orders drastically enriches the family of differential equations: all gaps between integer-order equations become filled up by differential equations of intermediate noninteger orders.

Abel (1826) was the first who wrote a fractional equation for solving the tau-tochrone problem. The subsequent history of the theory can be retraced by reading the works (Ross, 1975; Oldham and Spanier, 1974; Samko et al., 1993; Miller and Ross, 1993). I add them by the remark that the first field of broad application was viscoelasticity (Gerasimov, 1948; Caputo and Mainardi, 1971; Bagley and Torvik, 1983, 1986). Nigmatullin (1986) pointed out that many of the universal electromagnetic, acoustic, and mechanical responses can be modeled accurately using the fractional diffusion-wave equations. Wyss (1986) used Mellin transform theory to obtain a closed form solution of the fractional diffusion equation in terms of Fox's H-function. Schneider and Wyss (1989) presented the diffusion and wave equations in terms of integro-differential equations, and obtained the associated Green's functions in closed form in terms of the Fox functions. During last decades, the scientific activity in this field kept steadily growing, so nowadays the authors are estimated at hundreds and published articles at thousands.

This chapter does not pretend to a systematic description of methods of solution, it is rather some sort of excursion to see the most important sights of this "town". Nevertheless, the reader will become familiar with basic analytical and numerical methods and will see some examples of their applications to relatively simple problems.

5.1 Ordinary equations

5.1.1 Initialization

We start with the Riemann-Liouville fractional equation of the simplest type

$${}_0\mathbf{D}_x^\nu f(x) = h(x), \quad (5.1)$$

where ν is an arbitrary positive number, $h(x)$ is a given function, and $f(x)$ is an unknown function which should be found from this equation. One is tempted to apply the operator ${}_0\mathbf{I}_x^\nu$ to both sides of the equation and perform the “inversion”

$$f(x) = {}_0\mathbf{D}_x^{-\nu} h(x),$$

but this will not be the most general solution.

From the R-L derivative definition, one can see that if two functions $f(x)$ and $g(x)$ have the same fractional derivative of order ν , with $n-1 < \nu \leq n$, $n = 1, 2, 3, \dots$,

$${}_a f^{(\nu)}(x) = {}_a g^{(\nu)}(x)$$

then they are related to each other via the relation

$$f(x) = g(x) + \sum_{k=1}^n c_k (x-a)^{\nu-k},$$

where the coefficients c_k are arbitrary constants. Thus, the general solution of the above equation is written in the form

$$f(x) = {}_a\mathbf{D}_x^{-\nu} h(x) + \sum_{k=1}^n c_k (x-a)^{\nu-k},$$

where constants $n = [\nu] + 1$ have to be determined from initial conditions. Acting on both sides of the equation by the operator ${}_0\mathbf{D}_x^\mu$, $\mu \leq \nu - 1$,

$${}_a f^{(\mu)}(x) = {}_a\mathbf{D}_x^{\mu-\nu} h(x) + \sum_{k=1}^n c_k {}_a\mathbf{D}_x^\mu (x-a)^{\nu-k},$$

and taking into account that

$${}_a\mathbf{D}_x^\mu (x-a)^\mu x^{\nu-k} = \frac{\Gamma(\nu-k+1)}{\Gamma(\nu-k+1-\mu)} (x-a)^{\nu-k-\mu},$$

we find the following interrelation between coefficients c_k and initial conditions

$$c_k = \frac{{}_a f^{(\nu-k)}(a+0)}{\Gamma(\nu-k+1)}.$$

Consequently, the general solution to Eq. (5.1) is

$$f(x) = {}_a\mathbf{D}_x^{-\nu} h(x) + \sum_{k=1}^n \frac{{}_a f^{(\nu-k)}(a+0)}{\Gamma(\nu-k+1)} (x-a)^{\nu-k}. \quad (5.2)$$

For example, the equation

$${}_0\mathbf{D}_x^{3/2}f(x) = x^5$$

has the general solution

$$f(x) = {}_0\mathbf{D}_x^{-3/2}x^5 + c_1x^{1/2} + c_2x^{-1/2} = [\Gamma(6)/\Gamma(15/2)]x^{13/2} + c_1x^{1/2} + c_2x^{-1/2},$$

containing two arbitrary constants.

In case of the G-C fractional equation

$${}_0^V\mathbf{D}_xf(x) = h(x) \quad (5.3)$$

with given $f^{(j)}(a+0)$, $j = 0, 1, \dots, n-1$, we obtain:

$$f(x) = {}_a\mathbf{D}_x^{-\nu}F(x) + \sum_{j=0}^{n-1} \frac{f^{(j)}(a+0)}{j!} (x-a)^j \quad (5.4)$$

(Gorenflo and Mainardi, 1997).

The process of initialization of fractional differential equations has systematically been studied by Lorenzo and Hartley (1998, 2000, 2008) (see also (Ortigueira, 2003; Fukunaga and Shimizu, 2004; Ortigueira and Coito, 2008; Sabatier et al., 2010; Heymans and Podlubny, 2006; Trigeassou and Maamri, 2011; Du and Wang, 2011)). Based on the evident property of a definite integral

$$\int_a^x (x-\xi)^{\mu-1} f(\xi) d\xi = \int_a^c (x-\xi)^{\mu-1} f(\xi) d\xi + \int_c^x (x-\xi)^{\mu-1} f(\xi) d\xi, \quad a < c < x,$$

the *initialization function* was defined via equation

$$\psi(f, \mu, a, c, x) = \frac{1}{\Gamma(\mu)} \int_a^c (x-\xi)^{\mu-1} f(\xi) d\xi,$$

So the following interpretation between two fractional integrals with different lower limits takes place:

$${}_af^{(-\mu)}(x) = {}_cf^{(-\mu)}(x) + \psi(f, -\mu, a, c, x), \quad \mu > 0.$$

The function ψ is interpreted as a prehistory of the process in the interval (a, c) . For $\mu = 1$, we see that $\psi = \int_a^c f(\xi) d\xi = \text{const}$, but for other values of μ , ψ depends on x .

The concept of initialization function is extended to fractional derivatives with positive orders,

$${}_0f^{(\nu)}(x) = {}_cf^{(\nu)}(x) + \psi(f, \nu, a, c, x), \quad \nu > 0.$$

In this case, expression for ψ has a more complicated form (see for details the articles cited above).

Practically this means that for solution of equation ${}_c\mathbf{D}f(x) = F(f(x), x)$ for a process starting at $x = a < c$, we should know not only $f(x)$ at point c , but its history

on (a, c) . It may seem to be less unexpected if we recall that in case of integer ν we have to know $\nu - 1$ derivatives at $x = c$. Together with $f(c)$, they can be used for reconstruction $f(x)$ in (a, c) .

5.1.2 Reduction to an integral equation

A rather universal method of solving fractional differential equations applicable among others to equations with variable coefficients is based on their transformation into integral equations. To get the latter from the solutions obtained in the previous section, it is enough to replace the known function $F(t)$ by a function containing the desired solution $F(t, f(t))$.

Thus, the Cauchy problem for the R-L fractional equation of order $\nu > 0$,

$${}_a D_x^\nu f(x) = F(x, f(x)), \quad {}_a f^{(\nu-k)}(a+0) = b_k, \quad k = 1, \dots, n, \quad n = [\nu] + 1$$

is reduced to the Volterra integral equation of the second kind:

$$f(x) = {}_a D_x^{-\nu} F(x, f(x)) + \sum_{k=1}^n \frac{b_k}{\Gamma(\nu - k + 1)} (x - a)^{\nu-k}.$$

The same problem for the G-C fractional equation

$${}_a^V D_x f(x) = F(x, f(x)), \quad f^{(k)}(a+0) = c_k, \quad k = 0, \dots, n-1$$

is expressed as

$$f(x) = {}_a D_x^{-\nu} F(x, f(x)) + \sum_{k=0}^{n-1} \frac{c_k}{k!} (x - a)^k.$$

Replacing x by t , $f(x)$ by $x(t)$, and $F(x, f(x))$ by $-\omega^\alpha x(t)$, we obtain the equation for the system called the *fractional oscillator*. Narahari Achar et al. (2005) started to investigate this system with generalization of the integral equation of harmonic oscillator

$$x(t) = x(0) + \dot{x}(0)t - \omega^2 \int_0^t dt_2 \int_0^{t_2} x(t_1) dt_1,$$

to the integral equation of a fractional order $\alpha \in (1, 2]$:

$$x(t) = x(0) + \dot{x}(0)t - \frac{\omega^\alpha}{\Gamma(\alpha)} \int_0^t (t - t')^{\alpha-1} x(t') dt'.$$

Taking the Laplace transform on both sides,

$$\hat{x}(\lambda) = \lambda^{-1} x(0) + \lambda^{-2} \dot{x}(0) - \omega^\alpha \lambda^{-\alpha} \hat{x}(\lambda),$$

solving the obtained equation,

$$\hat{x}(\lambda) = \frac{x(0)\lambda^{-1}}{1 + (\omega/\lambda)^\alpha} + \frac{\dot{x}(0)\lambda^{-2}}{1 + (\omega/\lambda)^\alpha}.$$

The solution is obtained by taking the inverse Laplace transform and using the Mittag-Leffler representation,

$$x(t) = x(0)E_{\alpha,1}[-(\omega t)^\alpha] + \dot{x}(0)tE_{\alpha,2}[-(\omega t)^\alpha].$$

5.1.3 Solution of inhomogeneous R-L fractional equation

For the sake of simplicity, we consider the Cauchy problem for R-L linear equation assuming $F(x, f(x)) = \lambda f(x) + h(x)$:

$${}_0D_x^\nu f(x) - \lambda f(x) = h(x), \quad {}_a f^{(\nu-k)}(a+0) = b_k, \quad k = 1, \dots, n, \quad n = [\nu] + 1.$$

Its integral counterpart looks as follows:

$$f(x) = f_0(x) + {}_aD_x^{-\nu}[\lambda f(x) + h(x)],$$

$$f_0(x) \equiv \sum_{k=1}^n \frac{b_k}{\Gamma(\nu - k + 1)}(x - a)^{\nu-k}.$$

According to the *method of successive approximations* known also as the *step-by-step method*, we write

$$f_m(x) = f_0(x) + {}_aD_x^\nu[\lambda f_{m-1}(x) + h(x)].$$

The successive calculation of these terms yields

$$\begin{aligned} f_1(x) &= \sum_{j=1}^2 \lambda^{j-1} \sum_{k=1}^n b_k \frac{(x-a)^{j\nu-k}}{\Gamma(j\nu-k+1)} + \int_a^x \left[\frac{(x-\xi)^{\nu-1}}{\Gamma(\alpha)} \right] h(\xi) d\xi, \\ f_2(x) &= \sum_{j=1}^3 \lambda^{j-1} \sum_{k=1}^n b_k \frac{(x-a)^{j\nu-k}}{\Gamma(j\nu-k+1)} + \int_a^x \left[\sum_{j=1}^2 \frac{\lambda^{j-1}(x-\xi)^{j\nu-1}}{\Gamma(j\nu)} \right] h(\xi) d\xi, \\ &\dots\dots\dots \\ f_m(x) &= \sum_{j=1}^{m+1} \lambda^{j-1} \sum_{k=1}^n b_k \frac{(x-a)^{j\nu-k}}{\Gamma(j\nu-k+1)} + \int_a^x \left[\sum_{j=1}^m \frac{\lambda^{j-1}(x-\xi)^{j\nu-1}}{\Gamma(j\nu)} \right] h(\xi) d\xi, \end{aligned}$$

and so on. As $m \rightarrow \infty$, $f_m(x) \rightarrow f(x)$, where

$$f(x) = \sum_{k=1}^n b_k \sum_{j=0}^{\infty} \frac{\lambda^j (x-a)^{j\nu+\nu-k}}{\Gamma(j\nu+\nu-k+1)} + \int_a^x \left[\sum_{j=0}^{\infty} \frac{\lambda^j (x-\xi)^{j\nu+\nu-1}}{\Gamma(j\nu+\nu)} \right] h(\xi) d\xi.$$

Consequently, the solution of the posted equation is expressed through the Mittag-Leffler function:

$$f(x) = \sum_{k=1}^n b_k (x-a)^{\nu-k} E_{\nu, \nu-k+1}[\lambda(x-a)^\nu] + \int_a^x (x-\xi)^{\nu-1} E_{\nu, \nu}[\lambda(x-\xi)^\nu] h(\xi) d\xi.$$

In particular, in case $\nu \in (0, 1)$, $n = 1$, and we have

$$f(x) = b_1 (x-a)^{\nu-1} E_{\nu, \nu}[\lambda(x-a)^\nu] + \int_a^x (x-\xi)^{\nu-1} E_{\nu, \nu}[\lambda(x-\xi)^\nu] h(\xi) d\xi,$$

while in case $\nu \in (1, 2)$, $n = 2$, and consequently

$$\begin{aligned} f(x) &= b_1 (x-a)^{\nu-1} E_{\nu, \nu}[\lambda(x-a)^\nu] + b_2 (x-a)^{\nu-2} E_{\nu, \nu-1}[\lambda(x-a)^\nu] \\ &\quad + \int_a^x (x-\xi)^{\nu-1} E_{\nu, \nu}[\lambda(x-\xi)^\nu] h(\xi) d\xi. \end{aligned}$$

5.1.4 Solution of the inhomogeneous G-C fractional equation

Let us consider the Cauchy problem for the G-C equation

$${}_a^{\nu} D_x f(x) - \lambda f(x) = h(x), \quad f^{(k)}(a+0) = c_k, \quad k = 0, \dots, n-1, \quad n = [\nu] + 1.$$

Its integral counterpart varies from the R-L equation only by the initial part of the free term

$$f(x) = f_0(x) + \lambda {}_a D_x^{\nu} f(x) + {}_a D_x^{\nu} h(x),$$

$$f_0(x) = \sum_{k=0}^{n-1} \frac{c_k}{k!} (x-a)^k,$$

including initial conditions in terms of integer-order polynomials.

Following the way described above, we obtain:

$$\begin{aligned} f_m(x) &= f_0(x) + {}_0 D_x^{\nu} [\lambda f(x) + h(x)] \\ &= \sum_{k=0}^{n-1} c_k \left[\sum_{j=0}^m \lambda^j \frac{(x-a)^{j\nu+k}}{\Gamma(j\nu+k+1)} \right] + \int_a^x \left[\sum_{j=1}^m \lambda^{j-1} \frac{(x-\xi)^{j\nu+1}}{\Gamma(j\nu)} \right] h(\xi) d\xi. \end{aligned}$$

At the limit $m \rightarrow \infty$,

$$\begin{aligned} f_m(x) \rightarrow f(x) &= \sum_{k=0}^{n-1} c_k (x-a)^k E_{\nu, k+1}[\lambda(x-a)^\nu] \\ &\quad + \int_a^x (x-t)^{\nu-1} E_{\nu, \nu}[\lambda(x-\xi)^\nu] h(\xi) d\xi. \end{aligned}$$

In particular, for $\nu \in (0, 1)$,

$$f(x) = c_0 E_\nu[\lambda(x-a)^\nu] + \int_a^x (x-\xi)^{\nu-1} E_{\nu,\nu}[\lambda(x-\xi)^\nu] h(\xi) d\xi,$$

while for $\nu \in (1, 2)$,

$$\begin{aligned} f(x) &= c_0 E_\nu[\lambda(x-a)^\nu] + c_1(x-a) E_{\nu,2}[\lambda(x-a)^\nu] \\ &\quad + \int_a^x (x-\xi)^{\nu-1} E_{\nu,\nu}[\lambda(x-a)^\nu] h(\xi) d\xi. \end{aligned}$$

The homogeneous version of this equation has been considered in (Gorenflo and Rutman, 1995). The authors found also that the solution $f(x) \equiv w_1(x)$ in $1 < \nu < 2$ is expressed through the solution $f(x) \equiv w_0(x)$ in $0 < \nu < 1$ via relation

$$w_1(x) = w_0(x) + [f'(0)/f(0)] \int_0^x w_0(\xi) d\xi.$$

5.1.5 Indicial polynomial method

Recall, that linearly independent solutions of the integer-order differential equation

$$P(D_x)y(x) \equiv D_x^n y(x) + a_1 D_x^{n-1} y(x) + \cdots + a_n D_x^0 y(x) = 0$$

with constant coefficients a_j are determined by zeros of the indicial polynomial

$$P(z) = z^n + a_1 z^{n-1} + \cdots + a_n.$$

Trying $y(x) = e^{\lambda x}$, we find that

$$P(D_x)e^{\lambda x} = P(\lambda)e^{\lambda x}.$$

Thus, if $\lambda = \alpha$ is a root of the indicial equation $P(z) = 0$, then $e^{\alpha x}$ is a solution of the differential equation.

Now, we come back to fractional equations. The indicial polynomial approach can not be generalized to the non-integer order equation

$${}_0 D_x^{\nu_n} f(x) + a_1 {}_0 D_x^{\nu_{n-1}} f(x) + \cdots + a_n {}_0 D_x^{\nu_0} f(x) = 0$$

without imposing additional requirements, namely that ν_j should be rational numbers (Miller and Ross, 1993). Thus, if q is the least common multiple of the denominators of the ν_j , one may take

$$\nu_n = n\nu, \quad \nu = 1/q,$$

and rewrite the fractional equation as

$${}_0 D_x^{n\nu} f(x) + a_1 {}_0 D_x^{(n-1)\nu} f(x) + \cdots + a_n {}_0 D_x^0 f(x) = 0.$$

This is a fractional differential equation of order (n, q) (Miller and Ross, 1993). If $q = 1$, then $\nu = 1$ and we deal simply with an ordinary differential equation. Introducing the “indicial” polynomial

$$P(z) = z^n + a_1 z^{n-1} + \cdots + a_n,$$

one may write the fractional differential equation compactly as

$$P({}_0\mathbf{D}_x^\nu)f(x) = 0.$$

The reason for using an indicial polynomial in the classical case is that the integer-order derivatives of $e^{\lambda x}$ are of the same form:

$$\mathbf{D}_x^n e^{\lambda x} = \lambda^n e^{\lambda x}, \quad n = 0, 1, 2, \dots$$

However, if we apply the fractional differential operator ${}_0\mathbf{D}_x^\nu$ to $e^{\lambda x}$, we get

$${}_0\mathbf{D}_x^\nu e^{\lambda x} = E_x(-\nu, \lambda).$$

Nevertheless, there exist a function meeting this requirement:

$${}_0\mathbf{D}_x^\nu E_x(\mu, \lambda) = E_x(\mu - \nu, \lambda).$$

This function is used in the following theorem (Miller and Ross, 1993).

Theorem Let

$$\left[{}_0\mathbf{D}_x^{n\nu} + a_1 {}_0\mathbf{D}_x^{(n-1)\nu} + \cdots + a_n {}_0\mathbf{D}_x^0 \right] f(x) = 0$$

be a fractional differential equation of order (n, q) , and let

$$P(z) = z^n + a_1 z^{n-1} + \cdots + a_n$$

be the corresponding indicial polynomial. Let $\alpha_1, \dots, \alpha_n$ with $\alpha_i \neq \alpha_j$ for $i \neq j$ be the zeros of $P(z)$ and let

$$A_m^{-1} = P'(\alpha_m), \quad m = 1, 2, \dots, n.$$

Then the fractional equation has N linearly independent solutions

$$y_j(x) = \sum_{m=1}^n A_m \sum_{k=0}^{q-1} \alpha_m^{qj-k-1} E_x(-k\nu, \alpha_m^q), \quad j = 1, 2, \dots, N,$$

where N is the smallest integer greater than or equal to $n\nu$. The general solution is the linear superposition of these functions.

According to this theorem, the simple fractional equation of order (4,3)

$${}_0\mathbf{D}_x^{4/3}f(x) = 0$$

has the general solution

$$f(x) = C_1x^{1/3} + C_2x^{-2/3},$$

where C_1 and C_2 are arbitrary constants ($N = 2$).

Another example: the equation

$$\left[{}_0\mathbf{D}_x^1 + a {}_0\mathbf{D}_x^{1/2} + b {}_0\mathbf{D}_x^0 \right] f(x) = 0$$

of the order $(2, 2)$, ($N = 1$) has the solution

$$\begin{aligned} f(x) &= \alpha_1 E_x(0, \alpha_1^2) - \alpha_2 E_x(0, \alpha_2^2) + E_x(-1/2, \alpha_1^2) - E_x(-1/2, \alpha_2^2) \\ &= \alpha_1 \exp(\alpha_1^2 x) \operatorname{erfc}(-\alpha_1 \sqrt{x}) - \alpha_2 \exp(\alpha_2^2 x) \operatorname{erfc}(-\alpha_2 \sqrt{x}) \end{aligned}$$

(if $\alpha_1 \neq \alpha_2$).

The solution of a nonhomogeneous fractional differential equation

$$[{}_0\mathbf{D}_x^{nv} + a_1 {}_0\mathbf{D}_x^{(n-1)v} + \cdots + a_n {}_0\mathbf{D}_x^0]f(x) = h(x), \quad f(0) = f'(0) = \cdots = f^{(N-1)}(0),$$

is given by

$$f(x) = \int_0^x G(x - \xi) h(\xi) d\xi,$$

where $G(x)$ is the fractional Green's function determined from the indicial polynomial $P(z)$ by using the inverse Laplace transform:

$$G(x) = \mathcal{L}^{-1}\{P^{-1}(\lambda^v)\}(x).$$

5.1.6 Power series method

Many classical methods including *power series method* remain effective tools for solving fractional equations as well. Consider one-term equation

$${}_0\mathbf{D}_x^{(v)}f(x) = h(x), \quad x > 0, \quad 0 < v < 1,$$

under initial condition $f(0) = 0$. Assuming that the right side of the equation $h(x)$ can be expanded in the Taylor series converging for $0 \leq x < R$, where R is a some positive constant, we seek the solution in the form

$$\begin{aligned} f(x) &= x^\alpha \sum_{n=0}^{\infty} a_n x^n, \\ {}_0\mathbf{D}_x^v \left(\sum_{n=0}^{\infty} a_n x^{n+\alpha} \right) &= \sum_{n=0}^{\infty} \frac{h^{(n)}(0)}{n!} x^n. \end{aligned}$$

Taking into account that

$${}_0\mathbf{D}_x^\nu x^\alpha = \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-\nu)} x^{\alpha-\nu},$$

we find from here

$$\alpha = \nu$$

and

$$a_n = \frac{h^{(n)}(0)}{\Gamma(1+n+\nu)}.$$

Finally,

$$f(x) = \sum_{n=0}^{\infty} \frac{h^{(n)}(0)}{n!} \frac{\Gamma(n+1)}{\Gamma(n+1+\nu)} x^{n+\nu}.$$

It is easy to prove that this solution is identical to that expressed through the Green function.

5.1.7 Series expansion of inverse differential operators

In this section, we consider power series expansion applying to an operator. Let $F(z)$ be such a function that $[F(z)]^{-1}$ may be expanded in a power series of z :

$$[F(z)]^{-1} = \sum_{j=0}^{\infty} a_j z^j.$$

Then, a special solution of non-homogeneous Volterra's integral equation

$$F({}_0\mathbf{D}_x^\nu)f(x) = h(x), \quad \nu < 0$$

is represented in the form

$$f(x) = [F({}_0\mathbf{D}_x^\nu)]^{-1}h(x) = \sum_{j=0}^{\infty} a_j {}_0\mathbf{D}_x^{j\nu}h(x) = \sum_{j=0}^{\infty} a_j {}_0h^{(j\nu)}(x).$$

Example. Let $F(z) = 1 + \lambda z$, $\lambda = \text{const}$. In this case,

$$[F(z)]^{-1} = \frac{1}{1+\lambda z} = \sum_{j=0}^{\infty} (-\lambda z)^j, \quad |\lambda z| < 1,$$

and the solution of equation

$$(1 + \lambda {}_0\mathbf{D}_x^\nu)f(x) = h(x)$$

can be written in one of the forms:

$$\begin{aligned}
 f(x) &= \sum_{j=0}^{\infty} (-1)^j \lambda^j {}_0h_x^{(j\nu)} \\
 &= \sum_{j=0}^{\infty} (-1)^j \frac{1}{\Gamma(-j\nu)} \int_0^x \frac{h(\xi) d\xi}{(x-\xi)^{j\nu+1}} = \frac{d}{dx} \int_0^x g_\nu(x-\xi) h(\xi) d\xi, \quad \nu < 0,
 \end{aligned}$$

where

$$g_\nu(x) = \sum_{j=0}^{\infty} (-1)^j \frac{x^{-j\nu}}{\Gamma(1-j\nu)}.$$

Particularly, for $\nu = -1/2$,

$$g_{-1/2}(x) = e^{\lambda^2 x} \operatorname{erfc}(\lambda \sqrt{x}), \quad x > 0.$$

5.1.8 Method of integral transformations

Like integer-order differential equations with constant coefficients, their fractional analogous can easily be solved by using the *integral transforms method*. Following (Podlubny, 1999), we demonstrate here, how the Laplace method does work in solving fractional equation

$${}_0f^{(\nu)}(x) + af(x) = h(x), \quad x > 0,$$

under initial conditions

$${}_0f^{(\nu-k)}(0+) = b_k, \quad k = 1, 2, \dots, n, \quad \text{for } \nu \in (n-1, n].$$

The Laplace transformation reduces it to the algebraic equation

$$\lambda^\nu \widehat{f}(\lambda) + a\widehat{f}(\lambda) = \widehat{h}(\lambda) + \sum_{k=1}^n b_k \lambda^{k-1},$$

the solution of which is

$$\widehat{f}(\lambda) = \frac{\widehat{h}(\lambda)}{\lambda^\nu + a} + \sum_{k=1}^n b_k \frac{\lambda^{k-1}}{\lambda^\nu + a}.$$

Usually, the most difficult phase of the process is inverting the Laplace transform, but our problem does not require a complicated technique. It is enough to know that

$$\mathcal{L}^{-1}\{\lambda^{-\gamma}\}(x) = \frac{x^{\gamma-1}}{\Gamma(\gamma)}.$$

Rewriting the fraction $\lambda^{k-1}/(\lambda^\nu + a)$ as

$$\frac{\lambda^{k-1}}{\lambda^v + a} = \lambda^{k-v-1} \frac{1}{1 + a\lambda^{-v}} = \sum_{j=0}^{\infty} (-a)^j \lambda^{-\gamma}$$

with $\gamma = vj + v - k + 1$ and inverting the series term by term, we obtain

$$\begin{aligned} \mathcal{L}^{-1} \left\{ \frac{\lambda^{k-1}}{\lambda^v + a} \right\} (x) &= \sum_{j=0}^{\infty} (-a)^j \mathcal{L}^{-1} \{ \lambda^{-\gamma} \} (x) \\ &= x^{v-k} \sum_{j=0}^{\infty} \frac{(-ax^v)^j}{\Gamma(vj + v - k + 1)} = x^{v-k} E_{v, v+1-k}(-ax^v), \end{aligned}$$

where

$$E_{\alpha, \beta}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\alpha j + \beta)}$$

is the two-parameter Mittag-Leffler function.

Finally, using the convolution theorem in the reversed form ($k = 1$)

$$\begin{aligned} \frac{1}{\lambda^v + a} \cdot \widehat{h}(\lambda) &= \mathcal{L} \{ x^{v-1} E_{v, v}(-ax^v) \} (\lambda) \cdot \mathcal{L} \{ h(x) \} (\lambda) \\ &= \mathcal{L} \{ (x^{v-1} E_{v, v}(-ax^v)) \star h(x) \} (\lambda), \end{aligned}$$

we arrive at the desired result

$$f(x) = \int_0^x (x - \xi)^{v-1} E_{v, v}(-a(x - \xi)^v) h(\xi) d\xi + \sum_{k=1}^n b_k x^{v-k} E_{v, v-k+1}(-ax^v).$$

A particular case of the equation, the semidifferential equation

$${}_0f^{(1/2)}(x) + f(x) = 0, \quad x > 0,$$

with the initial condition

$${}_0f^{(-1/2)}(0+) = b,$$

has been solved in (Oldham and Spanier, 1974):

$$f(x) = bx^{-1/2} E_{1/2, 1/2}(-\sqrt{x}) = b(1/\sqrt{\pi x} - e^x \operatorname{erfc}(\sqrt{x})).$$

This solution is in agreement with the more general result, obtained in (Podlubny, 1999) and described above.

Now we apply the same method to Solve the fractional Gerasimov-Caputo equation

$${}_0^v f(x) + af(x) = h(x), \quad x > 0, \quad v \in (n-1, n],$$

provided by corresponding initial conditions

$$f^{(k)}(0+) = c_k, \quad k = 0, 1, \dots, n-1.$$

Repeating all the operations,

$$\begin{aligned}\widehat{f}(\lambda) &= \frac{\widehat{h}(\lambda)}{\lambda^\nu + a} + \sum_{k=0}^{n-1} c_k \frac{\lambda^{\nu-k-1}}{\lambda^\nu + a}, \\ \frac{\lambda^{\nu-k-1}}{\lambda^\nu + a} &= \lambda^{-k-1} \frac{1}{1 + a\lambda^{-\nu}} = \sum_{j=0}^{\infty} (-a)^j \lambda^{-\gamma}, \quad \gamma = \nu j + k + 1, \\ \mathcal{L}^{-1} \left\{ \frac{\lambda^{\nu-k-1}}{\lambda^\nu + a} \right\} (x) &= \sum_{j=0}^{\infty} (-a)^j \mathcal{L}^{-1} \{ \lambda^{-\gamma} \} (x) \\ &= x^k \sum_{j=0}^{\infty} \frac{(-ax^\nu)^j}{\Gamma(\nu j + k + 1)} = x^k E_{\nu, k+1}(-ax^\nu),\end{aligned}$$

we arrive at the result

$$f(x) = \int_0^x (x - \xi)^{\nu-1} E_{\nu, \nu}(-a(x - \xi)^\nu) h(\xi) d\xi + \sum_{k=0}^{n-1} c_k x^k E_{\nu, k+1}(-ax^\nu).$$

In particular case,

$${}_0^{1/2} f(x) + f(x) = 0, \quad x > 0, \quad f(0+) = c,$$

we find

$$f(x) = c E_{1/2, 1}(-\sqrt{x}) = c e^x \operatorname{erfc}(\sqrt{x}).$$

The equation

$${}_0^\nu f(t) + \omega^\nu f(t) = 0, \quad 0 < \nu \leq 2,$$

in which the independent variable t has the sense of time and positive constant ω with the dimension t^{-1} can be regarded as a frequency, was considered by many authors, particularly, by Mainardi (1996). For $\nu = 2$, the solution $f(x)$ is interpreted as the coordinate of a mass-point (oscillation process), and for $\nu = 1$, it can be treated as its velocity (relaxation process). According to the cases $0 < \nu < 1$ and $1 < \nu < 2$, the equation can be referred to as the *fractional relaxation* or the *fractional oscillation* equation, respectively. In the former case, it must be equipped with a single initial condition, say $f(0+) = f_0$, and in the latter with the two initial conditions, say $f(0+) = f_0$ and $f'(0+) = f'_0$. We assume below $f'_0 = 0$ in order to ensure the continuous dependence of the solution on the parameter ν also in the transition from $\nu = 1 - 0$ to $\nu = 1 + 0$. On these assumptions,

$$\widehat{f}(\lambda) = \frac{f_0 \lambda^{\nu-1}}{\lambda^\nu + \omega^\nu}$$

and

$$f(t) = f_0 E_\nu(-(\omega t)^\nu).$$

The series representation

$$f(t) = f_0 \sum_{k=0}^{\infty} (-1)^k \frac{(\omega t)^{\nu k}}{\Gamma(\nu k + 1)}$$

is suitable only for short times since it exhibit a very slow numerical convergence. For large times, the asymptotic expansion is more appropriate:

$$f(t) \sim -f_0 \sum_{k=1}^{\infty} (-1)^k \frac{(\omega t)^{-\nu k}}{\Gamma(1 - \nu k)}, \quad t \rightarrow \infty.$$

The leading asymptotic behavior for $t \rightarrow 0+$ and $t \rightarrow \infty$ is given by

$$f(t) \sim \begin{cases} 1 - f_0(\omega t)^{\nu} / \Gamma(1 + \nu), & \text{as } t \rightarrow 0+, \\ (\omega t)^{-\nu} / \Gamma(1 - \nu), & \text{as } t \rightarrow \infty. \end{cases}$$

Compared to the solution $f(t) = f_0 \exp(-\omega t)$ of the ordinary relaxation equation ($\nu = 1$), the solution of the fractional relaxation equation ($0 < \nu < 1$) exhibits for small times a much faster decay (the derivative tends to $-\infty$ in comparison with $-\omega$), and for large times a much slower decay. This phenomenon of fractional relaxation is usually referred to as a *superslow process*.

Compared to the solution $f(t) = f_0 \cos(\omega t)$ of the ordinary oscillation equation ($\nu = 2$), the solution $f(t) = f_0 E_{\nu}(-(\omega t)^{\nu})$ of the fractional oscillation equation ($1 < \nu < 2$) does not exhibit permanent oscillations but an asymptotic algebraic decay. Only as a transient state, the fractional solution presents some attenuated oscillations, whose number and initial amplitude are increasing with ν . This phenomenon is usually referred to as an intermediate process.

5.1.9 Green's function method

The equations discussed above contain only one term with an unknown function. Now, we consider the equation with two terms both being fractional derivatives of different orders:

$${}_0f^{(\mu)}(x) + a {}_0f^{(\nu)}(x) = h(x), \quad 0 < \nu < \mu < 1.$$

Its Laplace transform is

$$(\lambda^{\mu} + a\lambda^{\nu})\hat{f}(\lambda) = c + \hat{h}(\lambda), \quad c = {}_0f_0^{(\mu-1)} + a {}_0f_0^{(\nu-1)},$$

whence

$$\hat{f}(\lambda) = \frac{c + \hat{h}(\lambda)}{\lambda^{\mu} + a\lambda^{\nu}}.$$

The right side of this equation can be represented as a power series,

$$\widehat{f}(\lambda) = [c + \widehat{h}(\lambda)] \sum_{j=0}^{\infty} (-a)^j \lambda^{(\nu-\mu)j-\mu},$$

which is nothing but the Laplace transform of production of $x^{\mu-1}$ and the two-parameter Mittag-Leffler function:

$$\mathcal{L}^{-1} \left\{ \sum_{j=0}^{\infty} (-a)^j \lambda^{(\nu-\mu)j-\mu} \right\} (x) = x^{\mu-1} E_{\mu-\nu, \mu}(-ax^{\mu-\nu}) \equiv G(x).$$

Using the convolution theorem, we arrive at the result

$$f(x) = cG(x) + \int_0^x G(x-\xi)h(\xi)d\xi,$$

where $G(x)$ has a sense of the *Green function* for this fractional equation.

The Green function method was successfully used by (Podlubny, 1999) for solution of fractional equations with constant coefficients and various order derivatives. Let us collect these results. One-term equation:

$$a {}_0D_x^{\nu} G(x) = \delta(x),$$

$$G(x) = \frac{x_+^{\nu-1}}{a\Gamma(\nu)}.$$

Two-term equation:

$$a {}_0D_x^{\nu} G(x) + bG(x) = \delta(x),$$

$$G(x) = \frac{1}{ax^{\nu-1}} E_{\nu, \nu}(-(b/a)x^{\nu}).$$

Three-term equation:

$$a {}_0D_x^{\alpha} G(x) + b {}_0D_x^{\beta} G(x) + cG(x) = \delta(x), \quad \alpha > \beta,$$

$$G(x) = \frac{1}{a} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left(\frac{c}{a}\right)^k x^{\alpha(k+1)-1} E_{\alpha-\beta, \alpha+\beta k}^{(k)}(-(b/a)x^{\alpha-\beta}). \quad (5.5)$$

5.1.10 The Adomian decomposition method

This is an effective non-numerical method applicable both to ordinary and partial differential equations including equations of fractional orders (Adomian, 1988, 1994).

To explain the idea of this method, we consider an equation in the operator form

$$\mathbf{L}f + \mathbf{M}f = F, \quad (5.6)$$

where $\mathbf{L}$ is a linear easily invertible operator. Acting on both sides of Eq. (5.6) by the inverse operator $\mathbf{L}^{-1}$, we obtain

$$\mathbf{L}^{-1}\mathbf{L}f = \mathbf{L}^{-1}F - \mathbf{L}^{-1}\mathbf{M}f. \quad (5.7)$$

For example, if $\mathbf{L} = \mathbf{D}_t^m$, $m = 1, 2, 3, \dots$, we have

$$f(x) = \sum_{k=0}^{m-1} \frac{1}{k!} f^{(k)}(0)x^k + \mathbf{L}^{-1}F - \mathbf{L}^{-1}\mathbf{M}f(x).$$

The general solution f is decomposed into the sum

$$f = \sum_{n=0}^{\infty} f_n, \quad (5.8)$$

where f_0 is the complete solution of the equation

$$\mathbf{L}f_0 = F.$$

Respectively, Equation (5.7) takes the form

$$f = f_0 - \mathbf{L}^{-1}\mathbf{M} \sum_{n=0}^{\infty} f_n$$

with

$$f_0 = \phi + \mathbf{L}^{-1}F,$$

where ϕ is the solution of the homogeneous equation

$$\mathbf{L}\phi = 0.$$

As a result, we arrive at the recurrent algorithm

$$f_{n+1} = -\mathbf{L}^{-1}\mathbf{M}f_n, \quad n = 0, 1, 2, \dots,$$

which with Eq. (5.8) solves the problem.

Let us consider some examples of application of this method.

5.1.10.1 Example 1

Following (Momani and Ibrahim, 2007), we apply the method to the simple fractional equation

$${}_0^{\alpha}\mathbf{D}_t x(t) + \omega^{\alpha}x(t) = F(t), \quad n-1 < \alpha \leq n = 1, 2, 3, \dots,$$

subject to the initial conditions

$$x^{(k)}(0) = a_k, \quad k = 0, 1, \dots, n-1,$$

where $F(t)$ is a given function. Rewriting the equation as

$${}_0^{\alpha}D_t x(t) = F(t) - \omega^{\alpha} x(t)$$

and operating with ${}_0I_t^{\alpha}$ on both its sides yield

$$x(t) = \phi(t) + {}_aI_t^{\alpha} F(t) - \omega^{\alpha} {}_0I_t^{\alpha} x(t), \quad (5.9)$$

where

$$\phi(t) = \sum_{k=0}^{n-1} \frac{x^{(k)}(0)}{k!} t^k.$$

The decomposition method assumes a series solution

$$x(t) = \sum_{m=0}^{\infty} x_m(t). \quad (5.10)$$

Substituting (5.10) and the initial condition into (5.9) and identifying the zeroth component x_0 by the term arising from the initial condition and from the force function, then we have the following recursive relations:

$$x_0 = \phi + {}_0I_t^{\alpha} F,$$

$$x_1 = -\omega^{\alpha} {}_0I_t^{\alpha} x_0 = -\omega^{\alpha} ({}_0I_t^{\alpha} \phi + {}_0I_t^{2\alpha} F),$$

$$x_2 = -\omega^{\alpha} {}_0I_t^{\alpha} x_1 = (-\omega^{\alpha})^2 ({}_0I_t^{2\alpha} \phi + {}_0I_t^{3\alpha} F),$$

and so on, where ${}_0I_t^{n\alpha}$ means ${}_0I_t^{\alpha} {}_0I_t^{\alpha} \dots {}_0I_t^{\alpha}$ (n times).

Consequently, the N -term approximation is

$$x^N(t) = \sum_{m=0}^{N-1} x_m(t)$$

and the exact solution is given by the limit

$$x(t) = \lim_{N \rightarrow \infty} x^N(t).$$

5.1.10.2 Example 2

Let us apply this algorithm to solution of the homogeneous one-term fractional equation

$${}_0^{\alpha}D_t x(t) + \omega^{\alpha} x(t) = 0,$$

of order $\alpha \in (1, 2]$ under initial conditions

$$x(0) = 1, \quad \dot{x}(0) = 0.$$

Using the above recursive relationship, the terms of the decomposition series are given by

$$\begin{aligned} x_1 &= -\frac{(\omega t)^\alpha}{\Gamma(\alpha+1)}, \\ x_2 &= \frac{(\omega t)^{2\alpha}}{\Gamma(2\alpha+1)}, \\ x_3 &= -\frac{(\omega t)^{3\alpha}}{\Gamma(3\alpha+1)}, \\ &\dots\dots \end{aligned}$$

so we obtain

$$x(t) = \sum_{m=0}^{\infty} \frac{(-(\omega t)^\alpha)^m}{\Gamma(m\alpha+1)}.$$

5.1.10.3 Example 3

The equation considered above can be interpreted as an equation for a free fractional oscillator. Now, we consider the fractional oscillator driven by the step-wise force,

$${}_0^{\alpha}\mathbf{D}_t x(t) + \omega^\alpha x(t) = A1(t), \quad 1 < \alpha \leq 2,$$

with initial conditions

$$x(0) = a, \quad \dot{x}(0) = 0.$$

In this case,

$$\begin{aligned} x_0 &= a + {}_0 I_t^\alpha A = a + \frac{At^\alpha}{\Gamma(\alpha+1)}, \\ x_1 &= -\omega^\alpha \left[\frac{at^\alpha}{\Gamma(\alpha+1)} + \frac{At^{2\alpha}}{\Gamma(2\alpha+1)} \right], \\ x_2 &= \omega^{2\alpha} \left[\frac{at^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{At^{3\alpha}}{\Gamma(3\alpha+1)} \right], \\ x_3 &= -\omega^{3\alpha} \left[\frac{at^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{At^{4\alpha}}{\Gamma(4\alpha+1)} \right], \end{aligned}$$

and so on. As a result, we have

$$x(t) = \sum_{m=0}^{\infty} (-1)^m \omega^{m\alpha} \left[\frac{at^{m\alpha}}{\Gamma(m\alpha+1)} + \frac{At^{(m+1)\alpha}}{\Gamma((m+1)\alpha+1)} \right].$$

5.1.10.4 Example 4

The problem is also easily solvable if the force function is not a constant but can be expressed in Taylor series at $x_0 = 0$. Let, for example, $F(t) = \sin(\omega t)$:

$${}_0^{\alpha}D_t x(t) + \omega^{\alpha} x(t) = \sin(\omega t), \quad 1 < \alpha \leq 2,$$

and

$$x(0) = 0, \quad \dot{x}(0) = 0.$$

In this case, the zeroth component is

$$x_0 = \frac{(\omega t)^{\alpha+1}}{\Gamma(\alpha+2)} - \frac{(\omega t)^{\alpha+3}}{\Gamma(\alpha+4)} + \frac{(\omega t)^{\alpha+5}}{\Gamma(\alpha+6)} - \cdots,$$

and the remaining part of the Adomian decomposition series is determined by the following sequence

$$\begin{aligned} x_1 &= -\omega^{\alpha} \left[\frac{(\omega t)^{2\alpha+1}}{\Gamma(2\alpha+2)} - \frac{(\omega t)^{2\alpha+3}}{\Gamma(2\alpha+4)} + \frac{(\omega t)^{2\alpha+5}}{\Gamma(2\alpha+6)} - \cdots \right], \\ x_2 &= \omega^{2\alpha} \left[\frac{(\omega t)^{3\alpha+1}}{\Gamma(3\alpha+2)} - \frac{(\omega t)^{3\alpha+3}}{\Gamma(3\alpha+4)} + \frac{(\omega t)^{3\alpha+5}}{\Gamma(3\alpha+6)} - \cdots \right], \\ x_3 &= -\omega^{3\alpha} \left[\frac{(\omega t)^{4\alpha+1}}{\Gamma(4\alpha+2)} - \frac{(\omega t)^{4\alpha+3}}{\Gamma(4\alpha+4)} + \frac{(\omega t)^{4\alpha+5}}{\Gamma(4\alpha+6)} - \cdots \right], \end{aligned}$$

and so on with a common term

$$x_m = (-1)^m \omega^{m\alpha} \sum_{j=0}^{\infty} \frac{(\omega t)^{2j+1+(m+1)\alpha}}{\Gamma(2j+2+(m+1)\alpha)}.$$

Using this terms gives the desired $x(t)$ in a series form solution by

$$x(t) = \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^{m+j} \omega^{m\alpha} (\omega t)^{2j+1+(m+1)\alpha}}{\Gamma(2j+2+(m+1)\alpha)}.$$

As shown by Momani and Ibrahim (2007), the results are in good agreement with the analytical solutions obtained by the Laplace transform method.

5.1.10.5 Example 5

In (Bazak et al., 2009), the R-L differential equation of

$${}_0f^{(\nu)}(x) + f(x) = xe^{-x}, \quad 1 < \nu \leq 2 \quad (5.11)$$

with zeroth initial conditions was solved. Under these conditions

$${}_0D_x^{2-\nu} {}_0D_x^\nu f(x) = D_x^2 f(x).$$

Applying ${}_0D_x^{2-\nu}$ to both sides of Eq. (5.11) yields

$$D_x^2 f(x) + {}_0D_x^{2-\nu} f(x) = {}_0D_x^{2-\nu} [xe^{-x}].$$

The authors choose

$$\begin{aligned} L &= D_x^2, \\ M &= {}_0D_x^{2-\nu}, \end{aligned}$$

and

$$F(t) = {}_0D_x^{2-\nu} [xe^{-x}].$$

The corresponding series consists of the terms

$$\begin{aligned} f_0(x) &= {}_0D_x^{-2} {}_0D_x^{2-\nu} [xe^{-x}] = {}_0D_x^{-\nu} [xe^{-x}] \\ &= {}_0D_x^{-\nu} \sum_{j=0}^{\infty} \frac{(-1)^j x^{j+1}}{j!} = \sum_{j=0}^{\infty} (-1)^j \frac{(j+1)x^{j+\nu+1}}{\Gamma(j+2+\nu)}, \\ f_n(x) &= -{}_0D_x^{-2} {}_0D_x^{2-\nu} f_{n-1}(x) = (-1)^n {}_0D_x^{-(n+1)\nu} [e^{-x}] \\ &= \frac{(-1)^n}{\Gamma((n+1)\nu)} \int_0^x \frac{\xi e^{-\xi} d\xi}{(x-\xi)^{1-(n+1)\nu}}, \end{aligned}$$

their summing exactly leads to the known result:

$$f(x) = \int_0^x G(x-\xi) \xi e^{-\xi} d\xi,$$

where

$$G(x) = x^{\nu-1} E_{\nu,\nu}(-x^\nu).$$

5.1.11 Equations with compositions of fractional operators

Generally, a compound fractional operator depends on the indices sequence order. Let us refer to the instructive example discussed in (Gorenflo and Mainardi, 1997). Three fractional differential equations of the total order $n\mu + (1-\nu) = 1$, $\nu \in (0, 1)$ are considered:

$$\begin{aligned} \text{a)} \quad D_x u(x) &\equiv {}_0D_x^{1-\nu+\nu} u(x) = h(x), \\ \text{b)} \quad {}_0D_x^{(1-\nu,\nu)} f(x) &\equiv {}_0D_x^{1-\nu} {}_0D_x^\nu f(x) = h(x), \end{aligned}$$

and

$$\text{c)} \quad {}_0\mathbf{D}_x^{(v, 1-v)} w(x) \equiv {}_0\mathbf{D}_x^v {}_0\mathbf{D}_x^{1-v} w(x) = h(x).$$

The general solution of the first equation is

$$\text{a')} \quad u(x) = \int_0^x h(\xi) d\xi + a,$$

where a is an arbitrary constant. In order to find the solution $f(x)$ of the second equation, we apply firstly the operator ${}_0\mathbf{D}_x^{v-1}$, and then ${}_0\mathbf{D}_x^{-v}$. As a result, we have

$$\text{b')} \quad f(x) = \int_0^x h(\xi) d\xi + b_1 + b_2 x^{v-1}.$$

Solving the last third equation, we use firstly ${}_0\mathbf{D}_x^{-v}$, and then ${}_0\mathbf{D}_x^{v-1}$:

$$\text{c')} \quad w(x) = \int_0^x h(\xi) d\xi + c_1 + c_2 x^{-v}.$$

The additional summands appear due to the relations

$${}_0\mathbf{D}_x^\gamma x^{\gamma-1} \equiv 0, \quad {}_0\mathbf{D}_x^{\gamma-1} x^{\gamma-1} = \Gamma(\gamma), \quad \gamma = v, 1-v.$$

Observe that as opposed to the first case when the solution contains only one arbitrary constant, the solution of each other equation contains two constants. If $v = 1/2$, the solutions b') and c') coincide, if not they differ due to the additional summands having an origin singularity of different orders. Gorenflo and Mainardi underline that introducing compound operators (or splitting initial operators) increases the number of free constants and required initial conditions.

Three equations with Hilfer's operator are considered in (Hilfer, 2000) under the same initial condition

$${}_0f^{-(1-\beta)(1-\alpha)}(0+) = b$$

with the use of the transformation formula

$$\mathcal{L} \left\{ {}_0\mathbf{D}_x^{\alpha, \beta} f(x) \right\} (\lambda) = \lambda^\alpha \widehat{f}(\lambda) - \lambda^{\beta(\alpha-1)} {}_0f^{-(1-\beta)(1-\alpha)}(0+).$$

The first equation

$${}_0\mathbf{D}_x^{\alpha, \beta} f(x) = 0$$

has the solution

$$f(x) = b \frac{x^{(1-\beta)(\alpha-1)}}{\Gamma((1-\beta)(\alpha-1)+1)}.$$

The solution of the second equation

$${}_0\mathbf{D}_x^{\alpha, \beta} f(x) = a$$

is

$$f(x) = \frac{ax^\alpha}{\Gamma(\alpha+1)} + b \frac{x^{(1-\beta)(\alpha-1)}}{\Gamma((1-\beta)(\alpha-1)+1)}.$$

The third equation is

$${}_0D_x^{\alpha,\beta} f(x) + af(x) = 0.$$

The Laplace transform of its solution is of the form

$$\widehat{f}(\lambda) = b \frac{\lambda^{\beta(\alpha-1)}}{a + \lambda^\alpha} = b \sum_{j=0}^{\infty} (-a)^j \lambda^{-\alpha j - \gamma}, \quad \gamma = \alpha + \beta(1 - \alpha).$$

Its inverting yields the desired result

$$f(x) = bx^{\gamma-1} \sum_{j=0}^{\infty} \frac{(-ax^\alpha)^j}{\Gamma(\alpha j + \gamma)} = bx^{(1-\beta)(\alpha-1)} E_{\alpha, \alpha+\beta(1-\alpha)}(-ax^\alpha).$$

5.1.12 Equations with superpositions of fractional operators

If a linear differential equation contains integer-order derivatives,

$$\left\{ \sum_{n=1}^N a_n D_x^n \right\} f(x) + cf(x) = h(x),$$

the operator in the bracket can be treated as a differential operator of order distributed over values $1, 2, \dots, N$ with weights $a_1, a_2, \dots, a_N$ respectively. Existing fractional derivatives allows us to introduce differential operators of orders distributed over arbitrary discrete or even continuous set of values, called *distributed order operators*. In the latter case, one can introduce the weight distribution density $w(v)$ and represent the equation as

$$\left\{ \int_{-\infty}^{\infty} w(v) {}_aD_x^v dv \right\} f(x) + cf(x) = h(x).$$

We will denote such operator by ${}_aD_x^{\{w(\cdot)\}}$, so that the distributed-order differential equation is written in the form

$${}_aD_x^{\{w(\cdot)\}} f(x) + cf(x) = h(x).$$

When the weights are positive and the total weight is normalized to 1, one can give the probabilistic interpretation to the operator, namely, consider the operator in curly brackets as the fractional operator averaged over all possible values of its *random order*:

$${}_aD_x^{\{w(\cdot)\}} = \langle {}_aD_x^v \rangle.$$

A word of caution has to be said: the solution of the equation with the averaged operator does not coincide with the averaged over v solution of the equation with a random v . From probabilistic point of view, the differential operator of distributed order can be treated as the operator ${}_a D_x^v$ averaged over random order v because

$$\langle {}_a D_x^v f(x; v) \rangle \neq \langle {}_a D_x^v \rangle \langle f(x; v) \rangle.$$

5.1.13 Equations with varying coefficients

All cases considered above relate to equations with constant coefficients. Here, we refer to the example of a fractional equation with a varying coefficient considered in (Nonnenmacher, 1990):

$${}_0 D_x^v f(x) = a^{-\mu} D_x^n (x^{2\mu} f(x)), \quad \mu = n - v > 0,$$

where $n = 1$ if $0 < \mu < 1$, $n = 2$ if $1 < \mu < 2$, etc. Under initial conditions $f^{(k)}(0+) = 0$, $k = 0, 1, 2, \dots, n-1$, it is equivalent to the fractional integral equation

$$x^{2\mu} f(x) = a^\mu {}_0 D_x^{-\mu} f(x).$$

For $a = \pi$ and $\mu = 1/2$, this equation leads to the Abel-type integral equation

$$xf(x) = \int_0^x (x - \xi)^{-1/2} f(\xi) d\xi,$$

which has been solved by Laplace transform techniques. The solution is

$$f(x) = x^{-3/2} \exp(-\pi/x).$$

In (Nonnenmacher, 1990), the solution for arbitrary non-integer positive $\mu = q$ is offered in the form

$$f(x) = \frac{a^q}{\Gamma(q)} x^{-q-1} \exp(-a/x), \quad a > 0, \quad x > 0.$$

It is easy to check this statement inserting the expression into the integral and substituting $\xi = ax/(xz + a)$. This leads to the result

$$\begin{aligned} a^\mu {}_0 I_x^\mu f(x) &= \frac{a^\mu}{\Gamma(\mu)} \frac{a^q}{\Gamma(q)} \int_0^x (x - \xi)^{\mu-1} \xi^{-q-1} e^{-a/\xi} d\xi \\ &= \frac{a^\mu}{\Gamma(\mu)} \frac{\exp(-a/x)}{x^{q+1-2\mu}} \frac{1}{\Gamma(q)} \int_0^\infty \frac{z^{\mu-1} e^{-z} dz}{(xz + a)^{q-\mu}}. \end{aligned}$$

For $q = \mu$, the remaining integral is just $\Gamma(\mu)$, $\mu > 0$, and getting

$$a^\mu {}_0I_x^\mu f(x) = \frac{a^\mu}{\Gamma(\mu)} \frac{\exp(-a/x)}{x^{1-\mu}} = x^{2\mu} f(x)$$

completes the proof.

Now, we consider solution of the equation

$${}_aD_x^\nu f(x) - (x-a)^\mu f(x) = 0, \quad x > 0, \nu > 0, \mu > -\{\nu\},$$

under initial conditions

$${}_af^{\nu-k}(a+0) = b_k, \quad k = 1, 2, \dots, n, \quad n = [\nu] + 1.$$

As we show in Sect. 5.1.2, this problem can be transformed into the form of an integral equation, namely

$$f(x) = f_0(x) + \frac{1}{\Gamma(\nu)} \int_a^x (\xi - a)^\mu (x - \xi)^{\nu-1} f(\xi) d\xi,$$

where

$$f_0(x) = \sum_{k=1}^n b_k \frac{(x-a)^{\nu-k}}{\Gamma(\nu-k+1)}.$$

Using again the method of successive approximations,

$$\begin{aligned} f_m(x) &= f_0(x) + \frac{1}{\Gamma(\nu)} \int_a^x (\xi - a)^\mu (x - \xi)^{\nu-1} f_{m-1}(\xi) d\xi \\ &= f_0(x) + {}_aD_x^{-\nu} [(x-a)^\mu f_{m-1}(x)], \end{aligned}$$

we obtain for $m = 1$:

$$\begin{aligned} f_1(x) &= f_0(x) + {}_aD_x^{-\nu} [(x-a)^\mu f_0(x)] \\ &= \sum_{k=1}^n b_k \frac{(x-a)^{\nu-k}}{\Gamma(\nu-k+1)} \left\{ 1 + \frac{\Gamma[(\mu+\nu)-k+1]}{\Gamma[(\mu+\nu)+\nu-k+1]} (x-a)^{\mu+\nu} \right\}. \end{aligned}$$

Continuing this procedure, we find

$$f_m(x) = \sum_{k=1}^n b_k \frac{(x-a)^{\nu-k}}{\Gamma(\nu-k+1)} \left\{ \sum_{j=0}^m A_j [(x-a)^{\mu+\nu}]^j \right\},$$

where

$$A_0 = 1, \quad A_j = \prod_{l=1}^j \frac{\Gamma[l(\mu+\nu)-k+1]}{\Gamma[l(\mu+\nu)+\nu-k+1]}.$$

Letting $m \rightarrow \infty$ and using the series representation of the Mittag-Leffler functions, we arrive at the desired solution:

$$f(x) = \sum_{k=1}^n b_k \frac{(x-a)^{v-k}}{\Gamma(v-k+1)} E_{v,1+\mu/v,1+(\mu-k)/v} [(x-a)^{\mu+v}].$$

In particular, for $v \in (0, 1)$,

$$f(x) = b_1 \frac{(x-a)^{v-1}}{\Gamma(v)} E_{v,1+\mu/v,1+(\mu-1)/v} [(x-a)^{\mu+v}],$$

and for $v \in (1, 2)$,

$$\begin{aligned} f(x) &= b_1 \frac{(x-a)^{v-1}}{\Gamma(v)} E_{v,1+\mu/v,1+(\mu-1)/v} [(x-a)^{\mu+v}] \\ &\quad + b_2 \frac{(x-a)^{v-2}}{\Gamma(v-1)} E_{v,1+\mu/v,1+(\mu-2)/v} [(x-a)^{\mu+v}]. \end{aligned}$$

We end this section with the solution of the G-C equation having the same structure:

$${}_0^v D_x f(x) - (x-a)^\mu f(x) = 0, \quad f^{(k)}(a+0) = c_k, \quad k = 0, 1, \dots, n-1.$$

Its general solution is

$$f(x) = \sum_{k=0}^{n-1} c_k \frac{(x-a)^k}{k!} E_{v,1+\mu/v,(\mu+k)/v} [(x-a)^{\mu+v}].$$

5.1.14 Nonlinear ordinary equations

5.1.14.1 Equation with the Liouville fractional operator

We end this section by considering two referring to the article (Campos, 1990) devoted to equations containing fractional operators of Liouville's type, ${}_{-\infty} D_z^\nu$, with the order ν allowed to take complex values.

We focus on the equation

$$z^\mu f^{(\nu)}(z) = a[f(z)]^b, \quad (5.12)$$

where $f^{(\nu)}(z) = {}_{-\infty} D_z^\nu$. First we consider the case $\mu = 0$, $b = 1$:

$$f^{(\nu)}(z) = a f(z). \quad (5.13)$$

Taking into account that

$$-_{\infty}D_z^{\nu}e^{cz} = c^{\nu}e^{cz}, \quad c = \text{const},$$

we observe that the function $f(z) = e^{cz}$ satisfies Eq. (5.13) under condition $c^{\nu} = a$. The latter is an algebraic equation with roots c_k given by the equation

$$\log c_k = \frac{\log a}{\nu} = \frac{\text{Log } a}{\nu} + \frac{2\pi ki}{\nu},$$

where Log denotes the principal branch of the logarithm and the integer k identifies all other branches, each leading to one value c_k of c . Thus, the functions

$$f_k(z) = \exp\left(a^{1/\nu} e^{2\pi i k/\nu} z\right)$$

are linearly independent particular integrals and the general integral of Eq. (5.13) is a linear combination of $f_k(z)$,

$$f(z) = \sum_{k=\alpha}^{\beta} C_k f_k(z),$$

where C_k are arbitrary constants, $\alpha = 0$ and $\beta = n$ numbers (see for details (Campos, 1990)).

The second case we will consider is $\mu = \nu$ and $b = 1$:

$$z^{\nu} f^{(\nu)}(z) = a f(z).$$

This equation can be satisfied by a power type function

$$f(z) = z^c$$

under condition that

$$\frac{\Gamma(1+c)}{\Gamma(1+c-\nu)} = a.$$

Denoting roots of the algebraic equation by c_k , we obtain the general integral of the equation under consideration in the form

$$f(z) = \sum_{k=\alpha}^{\beta} C_k z^{c_k}$$

with constant coefficients C_k .

Finishing his article by consideration of the nonlinear equation (5.12), Campos remarks: “we cannot expect the very simple methods used in the present paper to go very far towards solving a nonlinear differential equation, such as (5.12)” and restricts himself only by the question of existence of a power type solution of Eq. (5.12). Substitution

$$f(z) = z^b \quad (5.14)$$

into (5.12) yields

$$\frac{\Gamma(1+c)}{\Gamma(1+c-\nu)} z^{c+\mu-\nu} = az^{bc}.$$

This equation implies two identities

$$a = \frac{\Gamma(1+c)}{\Gamma(1+c-\nu)}$$

and

$$c = \frac{\mu - \nu}{b - 1}. \quad (5.15)$$

On eliminating c among them, we obtain

$$\Gamma(1 + (\mu - \nu)/(b - 1)) = a\Gamma(1 + (\mu - b\nu)/(b - 1)). \quad (5.16)$$

Consequently, the nonlinear fractional equation (5.12) has a power solution (5.14) if and only if the parameters μ, ν, a and b satisfy condition (5.16); in the latter case, the exponent c is given by Eq. (5.15).

For example, the equation

$$z^{1/2} f^{(-1/2)}(z) = a[f(z)]^2$$

has the power type solution

$$f(z) = z$$

under condition

$$\lambda = \Gamma(2)/\Gamma(5/2) = 4/(3\sqrt{\pi}).$$

5.1.14.2 Adomian decomposition method

Another instructive example can be found in (Mittal and Nigam, 2008). These authors have applied the decomposition method to solution of the nonlinear integro-differential equation with a fractional Gerasimov-Caputo differential operator:

$${}_0^V D_x f(x) = a(x)f(x) + \int_0^x K(x, \xi)F(f(\xi))d\xi.$$

Operating with ${}_0 I_x^\nu$ on both sides of the equation yields

$$f(x) = \sum_{j=0}^{J-1} \frac{1}{j!} f^{(j)}(0+) x^j + {}_0 D_x^\nu \left[a(x)f(x) + \int_0^x K(x, \xi)F(f(\xi))d\xi \right].$$

Along with the solution f , the nonlinear function F is decomposed as

$$F = \sum_{n=0}^{\infty} A_n,$$

where A_n are the Adomian polynomials given by

$$A_n = \frac{1}{n!} \left[\frac{d^n}{d\lambda^n} F \left(\sum_{i=0}^{\infty} \lambda^i f_i \right) \right]_{\lambda=0}, \quad n = 0, 1, 2, \dots$$

The values $f_j \equiv f(x_j)$, $j = 0, 1, 2, \dots$ are determined by the recurrent relation

$$f_{j+1} = {}_0\mathbf{D}_x^\nu \left[a(x)f_j + \int_0^x K(x, \xi) A_j d\xi \right].$$

5.1.14.3 Variational iteration method

Kurulay and Secer (2011) solve the fractional integro-differential equation of the type

$${}_0^\alpha \mathbf{D}_t x(t) = f \left(t, x(t), \int_0^t k(y, x) dy \right)$$

with the initial condition

$$x(0) = c$$

by using the variational method. They construct the correction functional as

$$x_{n+1}(t) = x_n(t) + {}_0\mathbf{I}_t^\alpha F(t),$$

with

$$F(t) = \lambda \left[{}_0^\alpha \mathbf{D}_t x_n(t) - f \left(t, x_n, \int_0^t k(s, x_n) ds \right) \right],$$

for $(n+1)st$ approximation $x_{n+1}(t)$. The Lagrange multiplier λ can be easily determined from the latter equality by the variational theory, and the following iteration procedure is obtained:

$$x_{n+1}(t) = x_n(t) + {}_0\mathbf{I}_t^\alpha \left[{}_0^\alpha \mathbf{D}_t x_n(t) - f \left(t, x_n(t), \int_0^t k(s, x_n) ds \right) \right], \quad n = 0, 1, 2, \dots$$

The authors apply this method to the system of nonlinear equations

$$\begin{aligned} {}_0^\alpha \mathbf{D}_t x(t) &= \left[K_1 - \gamma_1 y - \int_{t-T_0}^t k(t-s)y(s)ds \right] x(t), \\ {}_0^\alpha \mathbf{D}_t y(t) &= \left[K_2 - \gamma_1 x - \int_{t-T_0}^t k(t-s)x(s)ds \right] y(t), \end{aligned}$$

with parameters $K_1 > 0$, $K_2 > 0$, $0 < \alpha < 1$ and initial conditions

$$x(0) = M, y(0) = N.$$

This system is solved by introducing functionals

$$x_{n+1}(t) = x_n(t) + {}_0I_t^\alpha \{ \lambda [{}_0^{\alpha}D_t x_n(t) - g_1[x_n(t), y_n(t)]] \},$$

$$y_{n+1}(t) = y_n(t) + {}_0I_t^\alpha \{ \lambda [{}_0^{\alpha}D_t y_n(t) - g_2[x_n(t), y_n(t)]] \},$$

where

$$g_1[x(t), y(t)] = \left[K_1 - \gamma_1 y(t) - \int_{t-T_0}^t k(t-s)y(s)ds \right] x(t)$$

and

$$g_2[x(t), y(t)] = \left[K_2 - \gamma_2 x(t) - \int_{t-T_0}^t k(t-s)x(s)ds \right] y(t).$$

As a result, they have

$$x(t) = \{ 1 + [K_1 - \gamma_1 N - (1 - e^{-T_0})N] t^\alpha / \Gamma(\alpha + 1) + K_1^2 t^{2\alpha} / \Gamma(2\alpha + 1) \} M,$$

$$y(t) = \{ 1 + [K_2 - \gamma_2 M - (1 - e^{-T_0})M] t^\alpha / \Gamma(\alpha + 1) + K_2^2 t^{2\alpha} / \Gamma(2\alpha + 1) \} N.$$

When $\alpha = 1$, this solution coincides with the found by Biazar (2005) who used the Adomian decomposition method.

5.2 Partial fractional equations

5.2.1 Super-ballistic equation

In this section, we consider generalizations of integer-order partial differential equations

$$\frac{\partial^m f(x, t)}{\partial t^m} = \frac{\partial^n f(x, t)}{\partial x^n} + h(x, t), \quad m, n = 1, 2, \dots, \quad t > 0, \quad -\infty < x < \infty,$$

to fractional order equations, containing fractional time-derivatives, space-derivatives and space- and time-derivatives simultaneously. Special attention will be paid to orders $\nu \in [0, 2]$ containing three most important equations of classical mathematical physics: the ballistic equation ($m = 1, n = 1$), the diffusion equation ($m = 1, n = 2$), and the wave equation ($m = 2, n = 2$). The fractional calculus allows us to observe continuous transitions between these integer-order classical cases.

We begin with the Cauchy problem for the equation

$$\frac{\partial f(x, t)}{\partial t} + {}_0D_x^\alpha f(x, t) = 0, \quad x > 0, \quad t > 0, \quad \alpha \in (0, 1], \quad f(x, 0+) = \varphi(x).$$

Treating t as a coordinate and x as a time, Berens and Westphal (1968) called it the *generalized wave equation*. However, with the same or even greater right, it can be called the fractional *diffusion-ballistic equation*, because the limit case $\alpha = 1$ of the equation describes the *ballistic regime*, i.e., freely moving particles with the unit speed.

The solution of the equation was formulated in terms of semigroups and for $\alpha = 1/2$ was found in the explicit form (Butzer and Westphal, 2000):

$$f(x, t) = \frac{t}{\sqrt{4\pi}} \int_0^x \varphi(x - \xi) \xi^{-3/2} \exp(-t^2/4\xi) d\xi.$$

Applying the Laplace transformation with respect to x and taking into account that $\exp(-\lambda^\alpha)$ is the Laplace transform of the one-sided stable density $g_+(x; \alpha)$, one can easily express the solution of the problem in an explicit form for an arbitrary $\alpha \in (0, 1)$:

$$f(x, t) = t^{-1/\alpha} \int_0^x \varphi(x - \xi) g_+(t^{-1/\alpha} \xi; \alpha) d\xi.$$

Recall that

$$g_+(x; 1/2) = \frac{1}{4\pi} x^{-3/2} \exp(-1/4x),$$

consequently the solution is in agreement with the special case solved before.

It is not hard to see that if $\varphi(x) \geq 0$ and $\int_{-\infty}^{\infty} \varphi(x) dx = 1$, then

$$\int_{-\infty}^{\infty} f(x, t) dx = \int_{-\infty}^{\infty} \varphi(x) dx = 1,$$

and the function $f(x, t)$ can be interpreted as the probability density function for the random coordinate of a particle at time t . If the initial coordinate is not random and equal to 0, then $\varphi(x) = \delta(x)$ and

$$f(x, t) = t^{-1/\alpha} g_+(t^{-1/\alpha} x; \alpha). \quad (5.17)$$

In case of $\alpha = 1$,

$$f(x, t) = t^{-1} \delta(t^{-1} x - 1) = \delta(x - t),$$

and we have a deterministic process: the whole probability is concentrated at point $x = t$. This is the ballistic regime. If $\alpha < 1$, the probability is continuously distributed over positive semiaxis. With the course of time, this probability spreads more and more over the semiaxis, its mode $x_{\max}(t)$ moves away from the origin, but the mean value is infinite for all $t > 0$:

$$\int_0^{\infty} f(x, t) x dx = \infty.$$

Therefore, it is more suitable to follow the probability distribution mode moving according to the power law:

$$x_{\max} \propto t^{1/\alpha}, \quad \alpha < 1.$$

Because the exponent $1/\alpha$ is greater than 1, the mode moves with acceleration. Such kind of motion together with Eq. (5.17) is called *superballistic*.

5.2.2 Subballistic equation

Let us pass to another kind of motion governed by the equation

$${}_0^v D_t f(x, t) + \frac{\partial f(x, t)}{\partial x} = 0, \quad x > 0, t > 0, v \in (0, 1). \quad (5.18)$$

Observe that fractional differentiation is realized now by the Gerasimov-Caputo operator

$${}_0^v D_t f(x, t) \equiv {}_0 D_t^v f(x, t) - \frac{t^{-v}}{\Gamma(1-v)} f(x, 0+) = {}_0 D_t^v [f(x, t) - \varphi(x)]. \quad (5.19)$$

This change is necessitated to keep the normalization because

$$\int_{-\infty}^{\infty} \frac{\partial f(x, t)}{\partial x} dx = f(x, t) \Big|_{x=-\infty}^{x=\infty} = 0,$$

the condition

$$\int_{-\infty}^{\infty} {}_0^v D_t f(x, t) dx = {}_0^v D_t \int_{-\infty}^{\infty} f(x, t) dx = {}_0^v D_t 1 = 0$$

should be satisfied. Inserting (5.19) into (5.18), we can return to the R-L representation with appropriate normalization of its solution:

$${}_0 D_t^v f(x, t) + \frac{\partial f(x, t)}{\partial x} = \frac{t^{-v}}{\Gamma(1-v)} \varphi(x).$$

If initially the particle is localized at the origin, the equation becomes

$${}_0 D_t^v f(x, t) + \frac{\partial f(x, t)}{\partial x} = \frac{t^{-v}}{\Gamma(1-v)} \delta(x).$$

Its solution is again expressed through the stable density $g_+(x; v)$:

$$f(x, t) = \frac{1}{v} x^{-1/v-1} g_+(tx^{-1/v}; v). \quad (5.20)$$

The corresponding mean value is finite now and moves according to law t^v (decelerated motion, subballistic regime).

5.2.3 Subdiffusion equation

Now, we pass to the time-fractional equation with the second space-derivative

$${}_0D_t^\nu f(x, t) = \frac{\partial^2 f(x, t)}{\partial x^2}, \quad t > 0, \quad -\infty < x < \infty,$$

under conditions

$$\lim_{x \rightarrow \pm\infty} f(x, t) = 0, \quad \left[{}_0f_t^{(\nu-1)}(x, t) \right]_{t=0} = \varphi(x), \quad 0 < \nu < 1.$$

On $\nu = 1$, it coincides with an ordinary diffusion equation, the case $\nu < 1$ is treated as *retarding diffusion (subdiffusion)*. In case of $\nu = 1/2$, this equation was interpreted in terms of *percolation* (“*comb*”) model (Nigmatullin, 1986), its solution has been investigated by Mainardi (1994) and others.

The Fourier-Laplace transform with respect to both independent variables

$$f(x, t) \mapsto \widehat{f}(k, \lambda) \equiv \int_0^\infty dt e^{-\lambda t} \int_{-\infty}^\infty dx e^{ikx} f(x, t)$$

yields

$$\widehat{f}(k, \lambda) = \frac{\widetilde{\varphi}(k)}{\lambda^\nu + k^2}.$$

Inverting the Laplace transform leads to the Mittag-Leffler function

$$\widetilde{f}(k, t) = \widetilde{\varphi}(k) t^{\nu-1} E_{\nu, \nu}(-k^2 t^\nu),$$

and then the next inversion of the Fourier transform produces the solution of the initial-valued problem in terms of Green’s function:

$$f(x, t) = \int_{-\infty}^\infty G(x - \xi, t) \varphi(\xi) d\xi,$$

$$G(x, t) = \frac{1}{\pi} \int_0^\infty t^{\nu-1} E_{\nu, \nu}(-k^2 t^\nu) \cos(kx) dk.$$

Applying to this function the Laplace transform with respect to time, one can represent it as an integral along the Bromwich contour (Br),

$$\widehat{G}(x, \lambda) = \frac{1}{\pi} \int_0^\infty \frac{\cos(kx) dk}{\lambda^\nu + k^2} = (1/2) \lambda^{-\nu/2} \exp(-|x| \lambda^{\nu/2}),$$

$$G(x, t) = \frac{1}{4\pi i} \int_{\text{Br}} \exp(\lambda t - |x| \lambda^{\nu/2}) \lambda^{-\nu/2} d\lambda.$$

Substituting $\sigma = \lambda t$ and $z = |x| t^{-\nu/2}$ and transforming the Bromwich contour (Br) to the Hankel contour (Ha), as was done in a similar case by Mainardi (1994), we

obtain

$$G(x, t) = \frac{1}{4\pi i} t^{1-\nu/2} \int_{\text{Ha}} \exp(\sigma - z\sigma^{\nu/2}) \sigma^{-\nu/2} d\sigma = (1/2) t^{\nu/2-1} W(-z, -\nu/2, \nu/2),$$

where

$$W(z, \mu, \nu) = \sum_{k=0}^{\infty} \frac{z^k}{k! \Gamma(\mu k + \nu)}$$

is the Wright function.

Although for $\nu = 1$, we arrive at the known result of the traditional diffusion theory,

$$G(x, t) = \frac{1}{2\sqrt{\pi t}} \exp(-x^2/(4t)),$$

for $\nu < 1$, some deficiency arises, namely, the spatial integral of the $f(x, t)$ over whole space does not remain constant but vanishes with time:

$$\int_{-\infty}^{\infty} f(x, t) dx \equiv \tilde{f}(0, t) = \tilde{\varphi}(0) t^{\nu-1} E_{\nu, \nu}(0) = \tilde{\varphi}(0) \frac{t^{\nu-1}}{\Gamma(\nu)}.$$

This violation of normalizing does not allow us to interpret the solution as a distribution density of particle number, and the Green function as a probability distribution density.

5.2.4 The normalization problem

Evidently, the integral nonconservation is a property of the equation itself but not of the process. Although, in principle, there are exist kinetic processes in which particles disappear (due to absorption, for example) we have not built such property in the process under consideration. Nevertheless, the equation shows a slow (of the power type) law of decreasing total probability in the course of time. The point is the equation itself and its deriving. The careful consideration reveals that there exist at least two different generalizations following from three different representations of the same classical diffusion process. Here they are:

$$\frac{\partial f(x, t)}{\partial t} = \frac{\partial^2 f(x, t)}{\partial x^2}, \quad f(x, 0+) = \varphi(x), \quad t > 0, \quad (5.21)$$

$$\frac{\partial f(x, t)}{\partial t} = \frac{\partial^2 f(x, t)}{\partial x^2} + \varphi(x) \delta(t), \quad f(x, 0-) = 0, \quad t \geq 0, \quad (5.22)$$

and

$$f(x, t) - f(x, 0+) = \int_0^t \frac{\partial^2 f(x, \tau)}{\partial x^2} d\tau, \quad t > 0. \quad (5.23)$$

Taking into account that $f(x, 0+) = \varphi(x)$ and that the Dirac delta-function is the derivative of the Heavyside step-function $1(x)$ two last equations can be rewritten as

$$\frac{\partial}{\partial t} [f(x, t) - \varphi(x) 1(t)] = \frac{\partial^2 f(x, t)}{\partial x^2}, \quad f(x, 0-) = 0 \quad t \geq 0 \quad (5.24)$$

and

$$f(x, t) - \varphi(x) = \int_0^t d\tau \frac{\partial^2 f(x, \tau)}{\partial x^2}, \quad t > 0, \quad (5.25)$$

respectively. Substituting the fractional derivative for the first time-derivative in homogeneous Eq. (5.21) entails the normalization violation. The same operation applying to Eq. (5.15) leads to the non-homogeneous equation

$${}_0D_t^\nu f(x, t) = \frac{\partial^2 f(x, t)}{\partial x^2} + \frac{t^{-\nu}}{\Gamma(1-\nu)} \varphi(x). \quad (5.26)$$

In case (5.25), we handle a time-integral equation and its generalization should evidently be looked for by applying the fractional integration operator:

$$f(x, t) - \varphi(x) = {}_0D_t^{-\nu} \frac{\partial^2 f(x, t)}{\partial x^2}.$$

After fractional time differentiation,

$${}_0D_t^\nu f(x, t) = \frac{\partial^2 f(x, t)}{\partial x^2} + \frac{t^{-\nu}}{\Gamma(1-\nu)} \varphi(x),$$

we arrive at the same non-homogeneous equation as in case (5.26). The corresponding equation for the Green function has the form:

$${}_0D_t^\nu G(x, t) = \frac{\partial^2 G(x, t)}{\partial x^2} + \frac{t^{-\nu}}{\Gamma(1-\nu)} \delta(x),$$

The Fourier-Laplace transformation of this function

$$\widehat{\widetilde{G}}(k, \lambda) = \frac{\lambda^{\nu-1}}{\lambda^\nu + k^2}$$

shows that the principle of probability conservation having in terms of Fourier-Laplace image form

$$\widehat{\widetilde{G}}(0, \lambda) = \int_0^\infty \widetilde{G}(0, t) e^{-\lambda t} dt = 1/\lambda$$

keeps in force.

5.2.5 Subdiffusion on a half-axis

The solution of time-fractional one-dimensional diffusion equation

$${}_0D_t^\nu f(x, t) = \frac{\partial^2 f(x, t)}{\partial x^2}, \quad x, t \in \mathbb{R}_+, \quad 0 < \nu < 1,$$

under two boundary conditions

- 1) $f(x, 0) = 1(x), \quad f(0, t) = 0, \quad t > 0;$
- 2) $f(x, 0) = 1(x), \quad f(0, t) = -1, \quad t > 0$

was investigated in (Wyss, 1986).

For $\nu = 1$, the problems reduce to classical diffusion problems which have the solutions

- 1) $f(x, t) = \operatorname{erf}(t^{-1/2}x/2) \sim \pi^{-1/2}xt^{-1/2}, \quad t \rightarrow \infty,$
- 2) $f(x, t) = \operatorname{erf}(t^{-1/2}x/2) - 1 \sim \pi^{-1/2}xt^{-1/2}, \quad t \rightarrow \infty.$

For $\nu < 1$, the scaling substitution

$$f(x, t) = F(y), \quad y = t^{-\nu/2}x$$

leads to the integro-differential equation

$$\frac{d^2 F(y)}{dy^2} = y^{2/\nu} \int_0^\infty \eta^{-1-2/\nu} g(y/\eta) F(\eta) d\eta, \quad (5.27)$$

where

$$g(y) = \frac{2}{\nu \Gamma(-\nu)} (1 - y^{2/\nu})^{-\nu-1}.$$

The boundary conditions are as follows

- 1) $F(0) = 0, \quad F(\infty) = 1;$
- 2) $F(0) = -1, \quad F(\infty) = 0.$

The Mellin transform converts differential equation (5.27) into the difference equation

$$(s-1)(s-2)\overline{F}(s-2) = \overline{g}(s+2/\nu)\overline{F}(s).$$

Its solutions, compatible with the boundary conditions, are as follows:

- 1) $\overline{F}(s) = \frac{2^s \Gamma(-s) \Gamma(1/2 + s/2) \Gamma(1 + s/2)}{\sqrt{\pi} \Gamma(1-s) \Gamma(1 + s\nu/2)},$
- 2) $\overline{F}(s) = -\frac{2^s \Gamma(s) \Gamma(1/2 + s/2) \Gamma(1 + s/2)}{\sqrt{\pi} \Gamma(1+s) \Gamma(1 + s\nu/2)}.$

Computing the inverse Mellin transform, one gets

$$F(y) = 1/\sqrt{\pi}h(y/2),$$

where

$$\begin{aligned} 1) h(z) &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+\infty} \frac{\Gamma(s)\Gamma(1/2-s/2)\Gamma(1-s/2)}{\Gamma(1+s)\Gamma(1-s\nu/2)} z^s ds, \\ 2) h(z) &= -\frac{1}{2\pi i} \int_{c-i\infty}^{c+\infty} \frac{\Gamma(-s)\Gamma(1/2-s/2)\Gamma(1-s/2)}{\Gamma(1-s)\Gamma(1-s\nu/2)} z^s ds. \end{aligned}$$

These formulas allows us to represent the result in terms of the Fox functions

$$\begin{aligned} 1) f(x,t) &= \frac{1}{\sqrt{\pi}} H_{23}^{21} \left(\frac{x}{2t^{\nu/2}} \middle| \begin{matrix} (1,1); (1,\nu/2) \\ (1/2,1/2), (1,1/2); (0,1) \end{matrix} \right), \\ 2) f(x,t) &= -\frac{1}{\sqrt{\pi}} H_{23}^{30} \left(\frac{x}{2t^{\nu/2}} \middle| \begin{matrix} -; (1,1), (1,\nu/2) \\ (0,1), (1/2,1/2), (1,1/2); - \end{matrix} \right). \end{aligned}$$

According to the theory of the Fox functions, one can check that the obtained solutions obey the initial conditions and have the following long-time behavior:

$$\begin{aligned} 1) f(x,t) &\sim 1/\Gamma(1-\nu/2)xt^{-\nu/2}, \quad t \rightarrow \infty, \\ 2) f(x,t) &\sim 1/\Gamma(1-\nu/2)xt^{-\nu/2} - 1, \quad t \rightarrow \infty. \end{aligned}$$

At the end of the article (Wyss, 1986), it is noted that the results remain valid for $1 \leq \nu < 2$.

5.2.6 The signalling problem

Recall that the signalling problem in the ordinary diffusion theory is formulated as solving the equation

$$\frac{\partial f(x,t)}{\partial t} = \frac{\partial^2 f(x,t)}{\partial x^2}, \quad x \geq 0, \quad t \geq 0$$

under the condition

$$f(0,t) = \phi(t).$$

Its solution is provided by making use of the Laplace transform and the convolution theorem

$$f(x,t) = \int_0^t S(x,t-\tau)\phi(\tau)d\tau \mapsto \hat{f}(x,\lambda) = \hat{S}(x,\lambda)\hat{\phi}(\lambda),$$

where

$$S(x,t) = \frac{x}{2\sqrt{\pi}}t^{-3/2}\exp[-x^2/(4t)]$$

represents the fundamental solution (Green's function) of the signalling problem, since it corresponds to $\phi(t) = \delta(t)$. We observe that the function

$$S(x, t) = g_+(x/\sqrt{t}; 1/2)$$

is expressed through the one-sided stable density with characteristic exponent $1/2$.

The signalling problem for fractional equation

$${}_0\mathbf{D}_t^{2\nu} f(x, t) = \frac{\partial^2 f(x, t)}{\partial x^2}, \quad x \geq 0, \quad t \geq 0, \quad f(0, t) = \phi(t)$$

has been extensively treated by F. Mainardi (1994, 1995, 1996, 1997).

The Laplace transform of the signalling Green's function is found in the form

$$\widehat{S}^\nu(x, \lambda) = \exp\{-x\lambda^\nu\}, \quad x \geq 0.$$

Introducing the similarity variable $r = xt^{-\nu}$ and the auxiliary function

$$F(r; \nu) = \frac{1}{2\pi i} \int_{\text{Br}} \exp(\sigma - r\sigma^\nu) d\sigma,$$

one finds

$$S_\nu(x, t) = t^{-1} F(r; \nu).$$

The integral for $F(r; \nu)$ can be analytically continued from $r > 0$ to any $z \in \mathbb{C}$ by deforming the Bromwich path into the Hankel path. One obtains an entire function of order $1/(1-\nu)$, which can be identified with a special function, known as the Wright function.

Another representation of the signalling fractional Green's function is given by the one-sided stable time density with characteristic exponent ν ,

$$S_\nu(x, t) = x^{-1/\nu} g_+(x^{-1/\nu} t).$$

5.2.7 The telegraph equation

There exist a few fractional equations under the common title *fractional telegraph equations*. We consider here one of them solution of which is described in (Camargo et al., 2008). These authors solved the equation

$$\left[{}_0^{2\alpha}\mathbf{D}_t + b {}_0^\beta\mathbf{D}_t - {}_0^{2\gamma}\mathbf{D}_x \right] G_{\alpha, \beta}^\gamma(x, t) = b_1 \delta(t) \delta(x),$$

under initial and boundary conditions

$$G_{\alpha,\beta}^\gamma(x,0) = b_2\delta(x), \quad \left. \frac{\partial G_{\alpha,\beta}^\gamma(x,t)}{\partial t} \right|_{t=0} = 0,$$

$$\lim_{x \rightarrow \pm\infty} G_{\alpha,\beta}^\gamma(x,t) = 0, \quad \lim_{x \rightarrow \pm\infty} \left[\frac{\partial G_{\alpha,\beta}^\gamma(x,t)}{\partial x} \right] = 0.$$

When $\alpha = \beta = \gamma = 1$, it becomes the ordinary telegraph equation.

The space-time Fourier-Laplace transformation reduces this differential equation into algebraic one whose solution is of the form

$$\tilde{G}_{\alpha,\beta}^\gamma(k,\lambda) \equiv \int_0^\infty dt e^{-\lambda t} \int_{-\infty}^\infty dx e^{ikx} G_{\alpha,\beta}^\gamma(x,t) = \frac{b_1 + b_2(\lambda^{2\alpha-1} + b\lambda^{\beta-1})}{\lambda^{2\alpha} + b\lambda^\beta - |k|^{2\gamma}}.$$

To perform the inverse transformation we bring it to the form

$$\begin{aligned} \tilde{G}_{\alpha,\beta}^\gamma(k,\lambda) &= \frac{b_1 + b_2(\lambda^{2\alpha-1} + b\lambda^{\beta-1})}{\lambda^{2\alpha} + b\lambda^\beta - |k|^{2\gamma}} \\ &= \frac{\Omega(\lambda)}{|k|^{2\gamma}} \frac{|k|^{2\gamma}\lambda^{-\beta}}{\lambda^{2\alpha-\beta} + b} \frac{1}{1 - \frac{|k|^{2\gamma}\lambda^{-\beta}}{\lambda^{2\alpha-\beta} + b}} = \Omega(\lambda) \sum_{n=0}^{\infty} \frac{|k|^{2\gamma n} \lambda^{-\beta n}}{(\lambda^{2\alpha-\beta} + b)^{n+1}}, \end{aligned}$$

where $\Omega(\lambda) = b_1 + b_2(\lambda^{2\alpha-\beta} + b\lambda^{\beta-1})$ and is assumed that the condition $||k|^{2\gamma}\lambda^{-\beta} / (\lambda^{2\alpha-\beta} + b)| < 1$ is fulfilled. Using the properties of the generalized three-parameter Mittag-Leffler function, we compute the Laplace inversion,

$$\begin{aligned} &\tilde{G}_{\alpha,\beta}^\gamma(k,t) \\ &= \sum_{n=0}^{\infty} |k|^{2\gamma n} \left\{ b_1 t^{2\alpha n + 2\alpha - 1} E_{2\alpha-\beta, 2\alpha n + 2\alpha}^{n+1}(-bt^{2\alpha-\beta}) \right. \\ &\quad \left. + b_2 t^{2\alpha n} \left[E_{2\alpha-\beta, 2\alpha n + 1}(-bt^{2\alpha-\beta}) + bt^{2\alpha-\beta} E_{2\alpha-\beta, 2\alpha n + 2\alpha - \beta + 1}^{n+1}(-2bt^{2\alpha-\beta}) \right] \right\}, \end{aligned}$$

and then the Fourier inversion:

$$\begin{aligned} &G_{\alpha,\beta}^\gamma(x,t) \\ &= -\frac{1}{\pi} \sum_{n=1}^{\infty} \sin(n\gamma\pi) \Gamma(2n\gamma + 1) |x|^{-2n\gamma-1} \left\{ b_1 t^{2\alpha n + 2\alpha - 1} E_{2\alpha-\beta, 2\alpha n + 2\alpha}^{n+1}(-bt^{2\alpha-\beta}) \right. \\ &\quad \left. + b_2 t^{2\alpha n} \left[E_{2\alpha-\beta, 2\alpha n + 1}^{n+1}(-bt^{2\alpha-\beta}) + bt^{2\alpha-\beta} E_{2\alpha-\beta, 2\alpha n + 2\alpha - \beta + 1}^{n+1}(-bt^{2\alpha-\beta}) \right] \right\}, \end{aligned}$$

where $b \neq 0, -1 - m, m = 0, 1, 2, \dots$

Assuming that $\alpha = \beta$, we obtain the solution of the equation

$$\left[{}_0^{2\alpha}\mathbf{D}_t + b {}_0^\alpha\mathbf{D}_t - {}_0^{2\gamma}\mathbf{D}_x \right] G_{\alpha,\alpha}^\gamma(x,t) = b_1 \delta(t) \delta(x)$$

in the form

$$\begin{aligned}
 & G_{\alpha,\alpha}^{\gamma}(x,t) \\
 &= -\frac{1}{\pi} \sum_{n=1}^{\infty} \sin(n\gamma\pi) \Gamma(2n\gamma+1) |x|^{-2n\gamma-1} \left\{ b_1 t^{\alpha(n+2)-1} E_{\alpha,n\alpha+2\alpha}(-bt^{\alpha}) \right. \\
 & \quad \left. + b_2 t^{2n\alpha} [E_{\alpha,\alpha n+1}(-bt^{\alpha}) + bt^{\alpha} E_{\alpha,\alpha n+\alpha+1}(-bt^{\alpha})] \right\}, \quad 0 < \gamma < 1, 0 < \alpha < 1.
 \end{aligned}$$

5.2.8 Multidimensional subdiffusion: the Schneider-Wyss solution

One of the first solution of the multidimensional fractional wave-diffusion equation was obtained in (Schneider and Wyss, 1989). Both kinds of media were considered in this work: boundless and bounded (a half-space). For demonstration of this method, we restrict ourselves here by consideration only of the first case.

The initial wave-diffusion equation in the d -dimensional space

$${}_0D_t^{\nu} f(\mathbf{x}, t) = \Delta_d f(\mathbf{x}, t), \quad 0 < \nu \leq 2$$

with condition

$$f(\mathbf{x}, 0) = \phi_0(\mathbf{x})$$

if $0 < \nu \leq 1$, or

$$f(\mathbf{x}, 0) = \phi_0(\mathbf{x}), \quad \left. \frac{\partial f(\mathbf{x}, t)}{\partial t} \right|_{t=0} = \phi_1(\mathbf{x})$$

if $1 < \nu \leq 2$, was reformulated as integrodifferential equation

$$f_n(\mathbf{x}, t) = \frac{1}{\Gamma(\nu)} \int_0^t d\tau (t - \tau)^{\nu-1} \Delta_d f_n(\mathbf{x}, \tau) + \sum_{j=0}^{n-1} \phi_j(\mathbf{x}) t^j$$

with $n-1 < \nu \leq n$, where $n = 1$ for a fractional diffusion process, and $n = 2$ for a fractional wave process. The restricted range of n makes the factorial actually superfluous.

The solution of the equations are of the form

$$f_n(\mathbf{x}, t) = \sum_{j=0}^{n-1} \int_{\mathbb{R}^d} G_j^{\nu}(|\mathbf{x} - \mathbf{x}'|, t) \phi_j(\mathbf{x}') d\mathbf{x}'.$$

Applying the Laplace transform with respect to time yields

$$\widehat{f}_n(\mathbf{x}, \lambda) = \sum_{j=0}^{n-1} \int_{\mathbb{R}^d} \widehat{G}_j^{\nu}(|\mathbf{x} - \mathbf{x}'|, \lambda) \phi_j(\mathbf{x}') d\mathbf{x}'$$

with

$$\widehat{G}_j^v(r, \lambda) = g(r, \lambda^{v/2}) \lambda^{v-j-1}.$$

As shown in (Schneider and Wyss, 1989), the function $g(r, a)$ satisfies the equation

$$(\Delta_d - a^2)g(r, a) = -\delta(\mathbf{x})$$

and is expressed through the modified Bessel function of the second kind:

$$g(r, a) = (2\pi)^{-d/2} (r/a)^{1-d/2} K_{d/2-1}(ar).$$

Then the authors noticed that direct inverting the Laplace transform does not seem to be feasible. This difficulty is circumvented by passing through the intermediate step of the Mellin transform and using the connection between the transforms. Thus, they obtain

$$\begin{aligned} \overline{G}_j^v(r, s) &= \frac{1}{\Gamma(1-s)} \int_0^\infty g(r, \lambda^{v/2}) \lambda^{v-j-s-1} d\lambda \\ &= v^{-1} \pi^{-d/2} 2^{-2(j+s)/v} r^{-d+2(j+s)/v} \Gamma(1 - (j+s)/v) \Gamma(d/2 + (j+s)/v) / \Gamma(1-s). \end{aligned}$$

Inverting the Mellin transform in terms of Fox functions leads to

$$G_j^v(r, t) = \pi^{-d/2} 2^{-1-2j/v} r^{-d+2j/v} H_{12}^{20} \left(\frac{r}{2t^{v/2}} \middle| \begin{matrix} (1, v/2) \\ (d/2 - j/v, 1/2), (1 - j/v, 1/2) \end{matrix} \right).$$

As follows from the theory of Fox's functions,

$$\begin{aligned} G_j^v(r, t) &\sim \gamma_j^v r^{-[(1-v)d+2j]/(2-v)} t^{-[dv/2-2j]/(2-v)} \\ &\quad \times \exp(-(2-v)v^{v/(2-v)}(r/2)^{2/(2-v)} t^{-v/(2-v)}) \end{aligned}$$

with

$$\gamma_j^v = \pi^{-d/2} 2^{(2j-d)/(2-v)} (2-v)^{-1/2} v^{[v(d+1)/2-2j-1]/(2-v)}.$$

The case $v = 1$ and $j = 0$ leads to

$$G_0^1(r, t) = (4\pi t)^{-d/2} \exp(-r^2/4t).$$

As proved in (Schneider and Wyss, 1989), the Green function $G_0^v(r, t)$, $0 < v \leq 1$, is a probability density on $\mathbb{R}^d$ and equivalently, due to rotational invariance,

$$\rho^v(r, t) = [2\pi^{d/2}/\Gamma(d/2)] G_0^v(r, t) r^{d-2}$$

is a probability density on $\mathbb{R}_+$:

$$\rho^v(r, t) \geq 0, \quad \int_0^\infty \rho^v(r, t) dr = 1.$$

The following expression for moments of even orders can be derived using the Mellin transform of $\rho^\nu(r, t)$ with respect to r :

$$\langle r^{2n}(t) \rangle = \int_{\mathbb{R}^d} G_0^\nu(|\mathbf{x}|, t) |\mathbf{x}|^{2n} d\mathbf{x} = 2^{2n} \frac{\Gamma(n+d/2)\Gamma(n+1)}{\Gamma(d/2)\Gamma(n\nu+1)} t^{n\nu}, \quad n = 0, 1, 2, \dots$$

In particular,

$$\langle r^0(t) \rangle = 1, \quad \langle r^2(t) \rangle = \frac{2d}{\Gamma(\nu+1)} t^\nu, \quad \langle r^4(t) \rangle = \frac{8(2+d)d}{\Gamma(2\nu+1)} t^{2\nu},$$

and so on.

5.2.9 One-dimensional symmetric superdiffusion

In one-dimensional case, the simplest generalization of such kind looks as follows:

$$\frac{\partial f(x, t)}{\partial t} = -C \left(-\frac{\partial^2}{\partial x^2} \right)^{\alpha/2} f(x, t),$$

$$t > 0, \quad -\infty < x < \infty, \quad 0 < \alpha \leq 2.$$

Taking the initial condition in the form

$$f(x, 0) = \phi(x),$$

we represent the solution as integral

$$f(x, t) = \int_{-\infty}^{\infty} G(x - \xi, t) \phi(\xi) d\xi,$$

where the Green function $G(x, t)$ obeys the equation:

$$\frac{\partial G(x, t)}{\partial t} = -C \left(-\frac{\partial^2}{\partial x^2} \right)^{\alpha/2} G(x, t) = \delta(x) \delta(t), \quad G(x, t) = 0, \text{ if } t < 0.$$

Performing the Fourier transform of both sides of the equation with respect to space-coordinate

$$\frac{d\tilde{G}(k, t)}{dt} = -C|k|^\alpha \tilde{G}(k, t) + \delta(t)$$

and solving this equation, we get:

$$\tilde{G}(k, t) = \exp(-C|k|^\alpha t).$$

This is nothing but the characteristic function of the symmetric stable density

$$\tilde{G}(k, t) = \tilde{g}((Ct)^{1/\alpha} k; \alpha, 0).$$

Inverting this relation yields

$$G(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} \tilde{g}((Ct)^{1/\alpha} k; \alpha, 0) = (Ct)^{-1/\alpha} g((Ct)^{-1/\alpha} x; \alpha, 0).$$

The multidimensional generalization of this problem is evident:

$$\begin{aligned} \frac{\partial f(\mathbf{x}, t)}{\partial t} &= -C(-\Delta)^{\alpha/2} f(\mathbf{x}, t), \\ t > 0, \quad \mathbf{x} \in \mathbb{R}^d, \quad 0 < \alpha \leq 2, \quad f(\mathbf{x}, 0) &= \phi(\mathbf{x}), \\ f(\mathbf{x}, t) &= \int_{-\infty}^{\infty} G(\mathbf{x} - \mathbf{x}', t) \phi(\mathbf{x}') d\mathbf{x}', \\ G(\mathbf{x}, t) &= (Ct)^{-d/\alpha} g_d((Ct)^{-1/\alpha} \mathbf{x}; \alpha, 0) \end{aligned}$$

Here $g_d(\mathbf{x}; \alpha, 0)$ stands for the d -dimensional isotropic α -stable density.

5.2.10 Equations with Lévy-superposition of R-L operators

As we could see above, the characteristic function of one-dimensional CTLM in A-representation has the form

$$\tilde{p}^{(\alpha, \beta)}(k, t) = \tilde{g}^{(\alpha, \beta)}(kt^{1/\alpha}) = \exp\{tL^{(\alpha, \beta)}(k)\},$$

where

$$L^{(\alpha, \beta)}(k) = \begin{cases} -|k|^\alpha [1 - i\beta \tan(\alpha\pi/2) \operatorname{sign} k], & \alpha \neq 1, |\beta| \leq 1; \\ -|k|, & \alpha = 1, \beta = 0, \end{cases}$$

can be considered as the Fourier image of some linear operator $\mathbf{L}^{(\alpha, \beta)}$. We shall call it the *Lévy operator*.

Differentiating these expressions with respect to time yields

$$\frac{\partial \tilde{p}^{(\alpha, \beta)}(k, t)}{\partial t} = L^{(\alpha, \beta)}(k) \tilde{p}^{(\alpha, \beta)}(k, t)$$

with the initial condition

$$\tilde{p}^{(\alpha, \beta)}(k, 0) = 1.$$

For $\alpha = 2, \beta = 0$,

$$\tilde{\mathbf{L}}^{(2,0)} = -k^2,$$

and we recognize here the image of ordinary diffusion equation for Brownian motion

$$\frac{\partial p^{(\alpha,\beta)}(x,t)}{\partial t} = \Delta_1 p^{(\alpha,\beta)}(x,t)$$

with one-dimensional Laplacian $\Delta_1 \equiv \mathbf{D}_x^2$ and initial condition

$$p^{(\alpha,\beta)}(x,0) = \delta(x).$$

In the general case, the Lévy operator can be expressed through a linear superposition of left-sided and right-sided Riemann-Liouville differential operators. Indeed, writing its Fourier image in the form

$$-|k|^\alpha [1 - i\beta \tan(\alpha\pi/2) \text{sign } k] = -[A(-ik)^\alpha + B(ik)^\alpha],$$

after elementary algebra we obtain:

$$\begin{aligned} A(-ik)^\alpha + B(ik)^\alpha &= |k|^\alpha \left[A e^{-i(\alpha\pi/2) \text{sign } k} + B e^{i(\alpha\pi/2) \text{sign } k} \right] \\ &= |k|^\alpha \left[(A+B) \cos\left(\frac{\alpha\pi}{2}\right) - i(A-B) \sin\left(\frac{\alpha\pi}{2}\right) \text{sign } k \right] \\ &= |k|^\alpha (A+B) \cos\left(\frac{\alpha\pi}{2}\right) \left[1 - i \frac{A-B}{A+B} \tan\left(\frac{\alpha\pi}{2}\right) \text{sign } k \right]. \end{aligned}$$

Equating correspondent coefficients from both sides, we arrive at the system of equations

$$\begin{aligned} A+B &= \frac{1}{\cos(\alpha\pi/2)}, \\ (\beta-1)A + (\beta+1)B &= 0, \end{aligned}$$

the solution of which is

$$A = \frac{1+\beta}{2\cos(\alpha\pi/2)}, \quad B = \frac{1-\beta}{2\cos(\alpha\pi/2)}.$$

The power functions $(\mp ik)^\alpha$ are Fourier images of Riemann-Liouville operators $_{-\infty}\mathbf{D}_x^\alpha$ and $_x\mathbf{D}_\infty^\alpha$ respectively, therefore

$$\mathbf{L}^{(\alpha,\beta)} = -[A {}_{-\infty}\mathbf{D}_x^\alpha + B {}_x\mathbf{D}_\infty^\alpha], \quad (5.28)$$

and the original Lévy motion equation takes the form

$$\frac{\partial p^{(\alpha,\beta)}(x,t)}{\partial t} = -[A {}_{-\infty}\mathbf{D}_x^\alpha + B {}_x\mathbf{D}_\infty^\alpha] p^{(\alpha,\beta)}(x,t), \quad t > 0. \quad (5.29)$$

5.2.11 Equations with the Feller, Riesz, and Marchaud operators

Let us come back to the evolution equation for the characteristic function and represent it in the form

$$\frac{1 + i\beta \tan(\alpha\pi/2) \operatorname{sign} k}{|k|^\alpha [1 + \beta^2 \tan^2(\alpha\pi/2)]} \frac{\partial \tilde{p}^{(\alpha,\beta)}(k, t)}{\partial t} = -\tilde{p}^{(\alpha,\beta)}(k, t), \quad t > 0.$$

Assuming

$$\delta^2 = [1 + \beta^2 \tan^2(\alpha\pi/2)] \cos(\alpha\pi/2)$$

and writing $\mathcal{F}p$ for $\tilde{p}$, we get

$$\frac{\cos(\alpha\pi/2) + i\beta \sin(\alpha\pi/2) \operatorname{sign} k}{|k|^\alpha \delta^2} \mathcal{F} \left(\frac{\partial p^{(\alpha,\beta)}(x, t)}{\partial t} \right) = -\mathcal{F} p^{(\alpha,\beta)}(x, t).$$

Comparing the left hand side of this equality with the Fourier transform of Feller's potential (see Sect. 4.6.5) and inverting the transform, we arrive at the equation

$$\mathbb{D}_{u,v}^{-\alpha} \frac{\partial p^{(\alpha,\beta)}(x, t)}{\partial t} = -p^{(\alpha,\beta)}(x, t), \quad t > 0$$

or

$$\frac{\partial p^{(\alpha,\beta)}(x, t)}{\partial t} = -\left(\mathbb{D}_{u,v}^\alpha\right)^{-1} p^{(\alpha,\beta)}(x, t), \quad t > 0 \quad (5.30)$$

with

$$u = \frac{1 + \beta}{2\delta^2}$$

and

$$v = \frac{1 - \beta}{2\delta^2}.$$

As shown in Sect. 4.6.5, the evolution equation (5.30) can be written in the following equivalent forms:

$$\begin{aligned} \frac{\partial p^{(\alpha,\beta)}(x, t)}{\partial t} &= -\frac{\alpha}{CF(1 - \alpha)} \\ &\times \int_{-\infty}^{\infty} \frac{1 + \beta \operatorname{sign}(x - \xi)}{|x - \xi|^{1+\alpha}} \left[p^{(\alpha,\beta)}(x, t) - p^{(\alpha,\beta)}(\xi, t) \right] d\xi \end{aligned}$$

and

$$\begin{aligned} \frac{\partial p^{(\alpha, \beta)}(x, t)}{\partial t} &= -\frac{\alpha}{C\Gamma(1-\alpha)} \\ &\times \int_0^\infty \left[2p_A(x, t; \alpha, \beta) - (1+\beta)p^{(\alpha, \beta)}(x-\xi, t) \right. \\ &\quad \left. - (1-\beta)p^{(\alpha, \beta)}(x+\xi, t) \right] \xi^{-1-\alpha} d\xi, \end{aligned}$$

where

$$C = [1 + \beta^2 \tan(\alpha\pi/2)]^{-1}.$$

In case of a symmetrical process ($\beta = 0$), the operator in the right side of the latter equation coincides with the Riesz derivative

$$\frac{\partial p^{(\alpha, 0)}(x, t)}{\partial t} = -(-\Delta_1)^{\alpha/2} p^{(\alpha, 0)}(x, t).$$

When $\beta = 1$, we have a one-sided stable process with the evolution equation

$$\frac{\partial p^{(\alpha, 1)}(x, t)}{\partial t} = -[\cos(\alpha\pi/2)]^{-1} \mathbf{D}_+^\alpha p^{(\alpha, 1)}(x, t),$$

where $\mathbf{D}_+^\alpha p$ is the fractional Marchaud derivative.

Observe that the last equation is not applicable to the case with $\alpha = 1$ because $\cos(\pi/2) = 0$. This inconvenience can be avoided by using form C. The correspondent characteristic function $\tilde{p}(x, t; \alpha, \theta)$ obeys the equation

$$\frac{\partial \tilde{p}(k, t; \alpha, \theta)}{\partial t} = -|k|^\alpha \exp\{-i\alpha\theta(\pi/2)\text{sign } k\} \tilde{p}(k, t; \alpha, \theta) \quad (5.31)$$

with the initial condition

$$\tilde{p}(k, 0; \alpha, \theta) = 1.$$

In order to pass from Eq. (5.31) for the characteristic function to the correspondent equation for the probability density $p(x, t; \alpha, \theta)$ we rewrite (5.31) in the form

$$|k|^{-\alpha(1-\theta)} \mathcal{F} \frac{\partial p(x, t; \alpha, \theta)}{\partial t} = |k|^{\alpha\theta} \exp\{-i\alpha\theta(\pi/2)\text{sign } k\} \mathcal{F} p(x, t; \alpha, \theta)$$

and apply the inverse Fourier transformation to both sides of the equation. As a result we obtain the equation

$$\mathbb{D}^{-\alpha(1-\theta)} \frac{\partial p(x, t; \alpha, \theta)}{\partial t} = -\mathbf{D}_+^{\alpha\theta} p(x, t; \alpha, \theta),$$

which can also be represented in the form

$$\frac{\partial p(x, t; \alpha, \theta)}{\partial t} = \mathbf{L}(\alpha, \theta) p(x, t; \alpha, \theta) \quad (5.32)$$

with

$$\mathbf{L}(\alpha, \theta) = -\mathbf{D}^{\alpha(1-\theta)} \mathbf{D}_+^{\alpha\theta}.$$

In the symmetrical case ($\theta = 0$),

$$\frac{\partial p(x, t; \alpha, 0)}{\partial t} = -(-\Delta_1)^{\alpha/2} p(x, t, \alpha, 0).$$

In the extremely asymmetrical case ($\alpha < 1$, $\theta = 1$), Equation (5.32) takes the form

$$\frac{\partial p(x, t; \alpha, 1)}{\partial t} = -{}_0\mathbf{D}_x^\alpha p(x, t; \alpha, 1) \quad (5.33)$$

(see for details (Uchaikin, 1999)).

5.2.12 Lévy-Feldheim motion equation

Now, we consider the d -dimensional centrally symmetric Lévy motion, including the isotropic motion as a special case. Fourier transform of the corresponding equation has the form

$$\frac{\partial \tilde{p}(\mathbf{k}, t)}{\partial t} = -CL^{(\alpha)}(\mathbf{k}, \mu) \tilde{p}(\mathbf{k}, t) + \delta(t) \quad (5.34)$$

and can be immediately solved:

$$\tilde{p}(\mathbf{k}, t) = \exp\{-CL^{(\alpha)}(\mathbf{k}, \mu)t\}.$$

Remind that

$$g_d(\mathbf{k}; \alpha, \Gamma) = \exp\left\{-\int_{U_d} |\mathbf{k} \cdot \mathbf{u}|^\alpha \mu(du)\right\}, \quad 0 < \alpha \leq 2$$

is the characteristic function of d -variate symmetric stable pdf $g_d(x; \alpha, \Gamma(\cdot))$ with the spectral measure Γ . On collecting the equalities and reversing the Fourier transformation we arrive at the fractional diffusion equation

$$\frac{\partial p(\mathbf{x}, t)}{\partial t} = \mathbf{L}^{(\alpha)}(\Gamma) p(\mathbf{x}, t) + \delta(\mathbf{x}) \delta(t). \quad (5.35)$$

Its solution is expressed in terms of the d -dimensional Lévy-Feldheim distributions

$$p(\mathbf{x}, t) = t^{-d/\alpha} g_d\left(xt^{-1/\alpha}; \alpha, \Gamma\right). \quad (5.36)$$

The operator $\mathbf{L}^{(\alpha)}(\Gamma)$ defined through the Fourier inversion of $L^{(\alpha)}(\mathbf{k}, \Gamma)$

$$\mathbf{L}^{(\alpha)}(\Gamma) p(\mathbf{x}, \cdot) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i\mathbf{k} \cdot \mathbf{x}} L^{(\alpha)}(\mathbf{k}, \Gamma) p(\mathbf{k}, \cdot) d\mathbf{k} \quad (5.37)$$

contains fractional derivatives with respect to multivariate coordinates.

Equation (5.35) describes the process called α -stable Lévy-Feldheim motion (LFM). This is a Markovian process generalizing multidimensional Brownian motion ($\alpha = 2$) to arbitrary characteristic exponents $\alpha \in (0, 2]$ and spectral measures Γ (Samorodnitsky and Taqqu, 1994). When the spectral measure is uniformly distributed over the unit sphere U ($\Gamma = \Gamma_0$) we obtain an isotropic LFM pdf of which depends only on $|x|$.

Note that operator (5.37) acting on isotropic distributions are expressed as a fractional power of multivariate Laplace operator

$$\mathbf{L}^{(\alpha)}(\Gamma_0)\rho_d(\mathbf{x}; \alpha) = -(-\Delta_d)^{\alpha/2}\rho_d(\mathbf{x}; \cdot).$$

If $\alpha = 2$, different components of a_n isotropically distributed stable random vector are independent of each other. However this property takes place only in this case. When $\alpha < 2$, the components are not independent and vice versa if the components are independent then the distribution is not isotropic. In the last case, the spectral measure Γ is concentrated at points of intersection of the sphere U with coordinate axis. Denoting this measure by $\Gamma_d^{(0)}$ we write the pdf in the form

$$g_d(x_1, x_2, \dots, x_d; \alpha, \Gamma_d^{(0)}) = g_1(x_1; \alpha, \Gamma_1)g_1(x_2; \alpha, \Gamma_1) \cdots g_1(x_d; \alpha, \Gamma_1).$$

These distributions can describe diffusion process in such anisotropic structures as crystals and others.

5.2.13 Fractional Poisson motion

Now we consider another random variable: the number of events (pulses) $N(t)$ arriving during the period t . According to the theory of renewal processes, the probabilities

$$p_n(t) \equiv \mathbf{P}(N(t) = n), \quad n = 0, 1, 2, \dots$$

obey the following system of integral equations (see Sect. 3.5.1)

$$p_n(t) = \delta_{n0} \int_t^\infty \psi_v(\tau) d\tau + [1 - \delta_{n0}] \int_0^t \psi_v(t - \tau) p_{n-1}(\tau) d\tau, \quad n = 0, 1, 2, \dots$$

After the Laplace transform with respect to time, we obtain from here

$$\lambda^v \tilde{p}_n(\lambda) = -\mu \tilde{p}_n(\lambda) + \mu \tilde{p}_{n-1}(\lambda) + \lambda^{v-1} \delta_{n0}, \quad n = 0, 1, 2, \dots, \quad \tilde{p}_{-1} = 0.$$

The inversion yields:

$${}_0\mathbf{D}_t^v p_n(t) = \mu[p_{n-1}(t) - p_n(t)] + \frac{t^{-v}}{\Gamma(1-v)} \delta_{n0}, \quad 0 < v \leq 1. \quad (5.38)$$

This is a master equation system for the fractional Poisson processes. When $\nu \rightarrow 1$, it becomes the well known system for the standard Poisson process:

$$\frac{dp_n(t)}{dt} = \mu[p_{n-1}(t) - p_n(t)] + \delta(t)\delta_{n0}. \quad (5.39)$$

for the generating function

$$g(u, t) \equiv \sum_{n=0}^{\infty} u^n p_n(t), \quad (5.40)$$

system (5.38) produces the following equation:

$${}_0D_t^\nu g(u, t) = \mu(u-1)g(u, t) + \frac{t^{-\nu}}{\Gamma(1-\nu)}. \quad (5.41)$$

When $\nu \rightarrow 1$, it becomes the well known equation for the standard Poisson process:

$$\frac{dg(u, t)}{dt} = \mu(u-1)g(u, t) + \delta(t). \quad (5.42)$$

Comparing (5.38) with (5.39) and (5.41) with (5.42), one can observe that the equations for standard processes are generalized to the equations for correspondent fractional processes by means of replacement of the operator d/dt with ${}_0D_t^\nu$ and of the right-side term $\delta(t)$ with $t^{-\nu}/\Gamma(1-\nu)$.

The solution to Eq. (5.41) is of the form

$$g(u, t) = E_\nu(\mu(u-1)t^\nu) \equiv \sum_{n=0}^{\infty} \frac{a^n}{\Gamma(\nu n + 1)} (u-1)^n, \quad a = \mu t^\nu.$$

Applying the binomial formula to each term of the sum and interchanging the summations, one can rewrite it as the series

$$g(u, t) = \sum_{n=0}^{\infty} u^n \left[\frac{a^n}{n!} \sum_{m=0}^{\infty} \frac{(m+n)!(-a)^m}{m! \Gamma(\nu(mk+n)+1)} \right]. \quad (5.43)$$

Comparing (5.43) with (5.40) yields

$$p_n(t) = \frac{a^n}{n!} \sum_{m=0}^{\infty} \frac{(m+n)!}{m!} \frac{(-a)^m}{\Gamma((m+n)\nu+1)}, \quad a = \mu t^\nu, \quad 0 < \nu \leq 1.$$

This distribution becoming the Poisson one when $\nu = 1$ can be considered as its fractional generalization, called *fractional Poisson distribution*. The correspondent mean value and variance are given by

$$\langle N(t) \rangle = \frac{\mu t^\nu}{\Gamma(\nu+1)}$$

and

$$\sigma^2(t) = \langle N(t) \rangle \{1 + \langle N(t) \rangle [2^{1-2\nu} \nu B(\nu, 1/2) - 1]\}.$$

5.2.14 Lévy-Poisson motion

Let us call the Lévy-Poisson motion such a compound Poisson process which has a stable distribution of random jumps, that is

$$p(x) = g(x; \alpha, \theta), \quad \tilde{p}(k) = \exp \{-|k|^\alpha \exp\{-i(\theta \alpha \pi/2) \text{sign } k\}\}.$$

It is readily seen that

$$\widetilde{p^{*j}}(k) = [\tilde{p}(k)]^j = \tilde{p}(j^{1/\alpha} k),$$

and consequently,

$$p^{*j}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} \tilde{p}(j^{1/\alpha} k) dk = j^{-1/\alpha} g(j^{-1/\alpha} x; \alpha, \theta).$$

Therefore, the strict solution of the compound Poisson process equation

$$\frac{\partial f(x, t)}{\partial t} = \mathbf{K} f(x, t) + \delta(x) \delta(t)$$

with

$$\mathbf{K} f(x, t) = -\mu f(x, t) + \mu \int_{-\infty}^{\infty} p(x - x') f(x', t) dx' \quad (5.44)$$

is represented by the series

$$f(x, t; \alpha, \theta) = \sum_{j=0}^{\infty} \frac{(\mu t)^j}{j!} e^{-\mu t} j^{-1/\alpha} g(j^{-1/\alpha} x; \alpha, \theta).$$

Substituting here $\alpha = 1$, $\theta = 1$ and taking into account that

$$g(x; 1, 1) = \delta(x - 1),$$

we obtain

$$f(x, t; 1, 1) = \sum_{j=0}^{\infty} \frac{(\mu t)^j}{j!} e^{-\mu t} j^{-1} \delta(j^{-1} x - 1) = \sum_{j=0}^{\infty} \frac{(\mu t)^j}{j!} e^{-\mu t} \delta(x - j).$$

After integrating this expression over small interval $(n - \varepsilon, n + \varepsilon)$, $n = 0, 1, 2, \dots$, $0 < \varepsilon < 1$, we arrive at the ordinary Poisson distribution

$$p_n(t) = \int_{n-\varepsilon}^{n+\varepsilon} f(x, t) dx = \sum_{j=0}^{\infty} \frac{(\mu t)^j}{j!} e^{-\mu t} \delta_{jn} = \frac{(\mu t)^n}{n!} e^{-\mu t}.$$

From the other side, at the asymptotic of large μt

$$\sum_{j=0}^{\infty} \frac{(\mu t)^j}{j!} e^{-\mu t} \dots \sim \int_0^{\infty} d\tau \delta(\tau - \mu t) \dots$$

and we obtain the Lévy motion:

$$f(x, t; \alpha, \theta) \sim f^{as}(x, t; \alpha, \theta) \equiv p(x, t; \alpha, \theta) = (\mu t)^{-1/\alpha} g((\mu t)^{-1/\alpha} x; \alpha, \theta).$$

As shown above, this pdf obeys the equation

$$\frac{\partial p(x, t; \alpha, \theta)}{\partial t} = \mathbf{L}(\alpha, \theta) p(x, t; \alpha, \theta), \quad t > 0,$$

with the initial condition

$$p(x, 0+) = \delta(x),$$

or, equivalently, the equation

$$\frac{\partial p(x, t; \alpha, \theta)}{\partial t} = \mathbf{L}(\alpha, \theta) p(x, t; \alpha, \theta) + \delta(x) \delta(t), \quad t \geq 0,$$

with the initial condition

$$p(x, 0-) = 0.$$

5.2.15 Fractional compound Poisson motion

Let us come back to Eq. (5.44) describing evolution of a jump-like Markov process starting from the origin at $t = 0$. It can be rewritten in the form

$$\frac{\partial}{\partial t} [f(x, t) - 1_+(t) f(x, 0)] = \mathbf{K} f(x, t), \quad (5.45)$$

where

$$1_+(t) = \begin{cases} 0, & t \leq 0, \\ 1, & t > 0, \end{cases} \quad f(x, 0) = \delta(x).$$

Replacing the differential operator $\partial/\partial t$ in Eq. (5.45) by its fractional counterpart ${}_0\mathbf{D}_t^\nu$, we arrive at the correspondent generalization of the fractional Poisson equation

$${}_0\mathbf{D}_t^\nu f(x, t) = \mathbf{K} f(x, t) + f(x, 0) \Phi_\nu(t), \quad (5.46)$$

where

$$\Phi_\nu(t) = {}_0\mathbf{D}_t^\nu 1(t) = \frac{t^{-\nu}}{\Gamma(1-\nu)}.$$

Observe that the presence of function $\Phi_v(t)$ in the right-hand side of the equation guarantees against violating normalization. Indeed, because

$$\begin{aligned} \int_{-\infty}^{\infty} \mathbf{K}f(x, t)dx &= \int_{-\infty}^{\infty} \left[-\mu f(x, t) + \mu \int_{-\infty}^{\infty} p(x - x')f(x', t)dx' \right] dx \\ &= -\mu \int_{-\infty}^{\infty} f(x, t)dx + \mu \int_{-\infty}^{\infty} p(x)dx \int_{-\infty}^{\infty} f(x', t)dx' = 0, \end{aligned}$$

the condition

$$\int_{-\infty}^{\infty} {}_0\mathbf{D}_t^v f(x, t)dx = {}_0\mathbf{D}_t^v \int_{-\infty}^{\infty} f(x, t)dx = {}_0\mathbf{D}_t^v 1(t) = \Phi_v(t)$$

must be fulfilled.

Following the same way as above we can arrive at the time-space bi-fractional differential equations for a model of fractional Lévy-motion being alternative to that considered in Sect. 3.4.6. The correspondent master equation is of the form:

$${}_0\mathbf{D}_t^v p(x, t; \alpha, \theta, v) = \mathbf{L}(\alpha, \theta)p(x, t; \alpha, \theta, v) + \delta(x)\Phi_v(t), \quad t \geq 0, \quad (5.47)$$

with the initialization condition

$$p(x, t) = 0, \quad t < 0.$$

5.2.16 The link between solutions

Let us dwell on a link between solutions of first- and fractional-order equations which can be used instead of a special computational algorithms for solving a fractional (with $v \in (0, 1)$) equation in case the solution of the corresponding first-order equation is known. First applications of this approach were realized by Saichev and Zaslavsky (1997, Eq. (7.30)) and Uchaikin (1999, Eqs. (3.5)–(3.6)) (see also (Baeumer and Meerschaert, 2001; Kosztolowicz, 2004; Meerschaert and Scheffler, 2004)).

Consider equations

$${}_0\mathbf{D}_t^v f_{v, \omega}(t) = \mathbf{K}f_{v, \omega}(t) + \Phi_{\omega}(t)$$

and

$$\frac{df}{dt} = \mathbf{K}f(t) + \delta(t).$$

Recall that $\Phi_0(t) = 1(t)$ and $\Phi_1(t) = \delta(t)$. The space variable x doesn't participate in this transformations and will be temporarily omitted.

Performing the Laplace transform of both equations,

$$\lambda^\nu \widehat{f}_{\nu,\omega}(\lambda) = \mathbf{K} \widehat{f}_{\nu,\omega}(\lambda) + \lambda^{-\nu},$$

$$\lambda \widehat{f}(\lambda) = \mathbf{K} \widehat{f}(\lambda) + 1,$$

and solving them with respect to $\widehat{f}_{\nu,\omega}$ and $\widehat{f}$, we come to the interrelation:

$$\widehat{f}_{\nu,\omega}(\lambda) = \lambda^{-\omega} \widehat{f}(\lambda^\nu).$$

The backward transformation yields

$$f_{\nu,\omega}(t) = \frac{1}{2\pi i} \int_{\mathbb{C}} e^{\lambda t} \lambda^{-\omega} \widehat{f}(\lambda^\nu) d\lambda = \int_0^\infty h_{\nu,\omega}(t, \tau) f(\tau) d\tau,$$

where

$$h_{\nu,\omega}(t, \tau) = \frac{1}{2\pi i} \int_{\mathbb{C}} e^{\lambda t - \lambda^\nu \tau} \lambda^{-\omega} d\lambda.$$

Direct calculations allow to verify that the solution of equation

$${}_0\mathbf{D}_t^\nu f_{\nu,0}(t) = \mathbf{K} f_{\nu,0}(t) + \delta(t)$$

is expressed through the solution of equation

$$\frac{df}{dt} = \mathbf{K} f(t) + \delta(t)$$

according to

$$f_{\nu,0}(t) = \nu t^{\nu-1} \int_0^\infty f((t/\tau)^\nu) g_+(\tau; \nu) \tau^{-\nu} d\tau = \int_0^\infty f(\tau) g_+(t \tau^{-1/\nu}; \nu) \tau^{-1/\nu} d\tau, \quad (5.48)$$

whereas the solution of equations

$${}_0\mathbf{D}_t^\nu f_{\nu,1-\nu}(t) = \mathbf{K} f_{\nu,1-\nu}(t) + \frac{t^{-\nu}}{\Gamma(1-\nu)}$$

is connected with it by the interrelation

$$f_{\nu,1-\nu}(t) = \int_0^\infty f((t/\tau)^\nu) g_+(\tau; \nu) d\tau = (t/\nu) \int_0^\infty f(\tau) g_+(t \tau^{-1/\nu}; \nu) \tau^{-1/\nu-1} d\tau. \quad (5.49)$$

The latter is also applicable to homogeneous equations with G-C derivatives:

$${}_0^\nu \mathbf{D}_t f_{\nu,1-\nu}(t) = \mathbf{K} f_{\nu,1-\nu}(t).$$

Let us apply the obtained interrelations to solution of fractional generalizations of equations considered in Sect. 3.1.6. Introducing the function

$$V^{\mu,\nu}(x) = \int_0^\infty \tau^{\mu\nu} e^{-x\tau^\nu} g_+(\tau; \nu) d\tau, \quad x > 0$$

and assuming that $f(x, 0) = \delta(x)$, we arrive at the following results.

Equation

$${}_0^v D_t f(x, t) = \frac{\partial^2}{\partial x^2} \left[|x|^{-\theta} f(x, t) \right]$$

has the solution

$$f(x, t) = \frac{|x|^{\theta} t^{-\nu(1+\theta)/(2+\theta)}}{2(2+\theta)^{\theta/(2+\theta)} \Gamma[(1+\theta)/(2+\theta)]} V^{\frac{1+\theta}{2+\theta}, \nu} \left[\frac{|x|^{2+\theta} t^{-\nu}}{(2+\theta)^2} \right].$$

Equation

$${}_0^v D_t f(x, t) = \frac{\partial}{\partial x} \left\{ |x|^{-\theta/2} \frac{\partial}{\partial x} \left[|x|^{-\theta/2} f(x, t) \right] \right\}$$

has the solution

$$f(x, t) = \frac{|x|^{\theta/2} t^{-\nu/2}}{2\sqrt{\pi}} V^{1/2, \nu} \left[\frac{|x|^{2+\theta} t^{-\nu}}{(2+\theta)^2} \right].$$

Equation

$${}_0^v D_t f(x, t) = \frac{\partial}{\partial x} \left[|x|^{-\theta} \frac{\partial f(x, t)}{\partial x} \right]$$

has the solution

$$f(x, t) = \frac{(2+\theta)^{\theta/(2+\theta)} t^{-\nu/(2+\theta)}}{2\Gamma[1/(2+\theta)]} V^{1/(2+\theta), \nu} \left[\frac{|x|^{2+\theta} t^{-\nu}}{(2+\theta)^2} \right].$$

The functions $V^{\mu, \nu}(x)$ do not have any analytical representation in terms of elementary functions but can be expressed through the Fox H -functions.

5.2.17 Subordinated Lévy motion

Applying the above result to solving the fractional equation of subordinated Lévy motion

$${}_0 D_t^\nu p(x, t; \alpha, \nu, \theta) = L(\alpha, \theta) p(x, t; \alpha, \nu, \theta) + \delta(x) \frac{t^{-\nu}}{\Gamma(1-\nu)}, \quad t \geq 0 \quad (5.50)$$

with the initial condition

$$p(x, 0-) = 0$$

and taking into account that the equation

$$\frac{\partial p(x, t; \alpha, \theta)}{\partial t} = L(\alpha, \theta) p(x, t; \alpha, \theta) + \delta(x) \delta(t), \quad t \geq 0$$

with the same initial condition has the solution

$$p(x, t; \alpha, \theta) = t^{-1/\alpha} g(t^{-1/\alpha} x; \alpha, \theta),$$

we obtain

$$\begin{aligned} p(x, t; \alpha, \nu, \theta) &= \int_0^\infty p(x, (t/\tau)^\nu; \alpha, \theta) g_+(\tau; \nu) d\tau \\ &= \int_0^\infty (t/\tau)^{-\nu/\alpha} g((t/\tau)^{-\nu/\alpha} x; \alpha, \theta) g_+(\tau; \nu) d\tau. \end{aligned} \quad (5.51)$$

Samples of trajectories of subordinated Lévy motion are represented in Fig. 5.1.

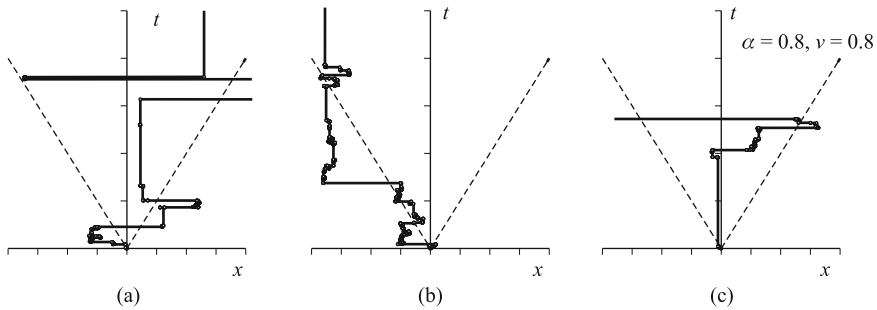


Fig. 5.1 Samples of trajectories of subordinated Lévy motion.

Introducing the notation determining the *fractional stable probability density*,

$$q(x; \alpha, \nu, \theta) = \int_0^\infty g(\tau^{\nu/\alpha} x; \alpha, \theta) g_+(\tau; \nu) \tau^{\nu/\alpha} d\tau,$$

we represent the result in the form:

$$p(x, t; \alpha, \nu, \theta) = t^{-\nu/\alpha} q(t^{-\nu/\alpha} x, \alpha, \nu, \theta).$$

The solution of the symmetrical fractional Lévy motion equation

$${}_0D_t^\nu p(x, t; \alpha, \nu, 0) = -(-\Delta_1)^{\alpha/2} p(x, t; \alpha, \nu, 0) + \delta(x) \frac{t^{-\alpha}}{\Gamma(1-\alpha)}$$

is expressed as follows:

$$p(x, t; \alpha, \nu, 0) = t^{-\nu/\alpha} q(t^{-\nu/\alpha} x, \alpha, \nu, 0).$$

Recall that

$$g(x; \alpha, 0) = \frac{1}{2\pi} \int_{-\infty}^\infty e^{-ikx - |k|^\alpha} dk.$$

A more general centrosymmetric process is described by the equation

$${}_0\mathbf{D}_t^\nu p(\mathbf{x}, t) = \mathbf{L}^{(\alpha)}(\Gamma)p(\mathbf{x}, t) + \delta(\mathbf{x}) \frac{t^{-\nu}}{\Gamma(1-\nu)}$$

whose solution is expressed through the corresponding fractional stable distribution

$$\Psi_d^{(\alpha, \nu)}(\mathbf{x}; \Gamma) = \int_0^\infty g_d(\mathbf{x}y^{\nu/\alpha}; \alpha, \Gamma) g_+(y; \nu) y^{\nu d/\alpha} dy$$

via relation

$$p(\mathbf{x}, t) = t^{-\nu d/\alpha} \Psi_d^{(\alpha, \nu)}(\mathbf{x}t^{-\nu/\alpha}; \Gamma).$$

5.2.18 Diffusion in a bounded domain

We consider now the time- and space-fractional equation describing a process of combined anomalous diffusion including subdiffusion ($\nu < 1$) and superdiffusion ($\alpha < 2$) features,

$${}_0^{\nu}\mathbf{D}_t f(x, t) = -(-\Delta_1)^{\alpha/2} f(x, t), \quad (5.52)$$

under homogeneous conditions at the ends of the finite section $[0, L]$,

$$f(0, t) = f(L, t) = 0$$

and the initial condition

$$f(x, 0) = f_0(x).$$

Following (Ilic et al., 2005), we apply the eigenfunction method to solve this problem. Trying the solution of Eq. (5.52) as a superposition of eigenfunctions of the operator $-\Delta_1 \equiv -\mathbf{D}_x^2$:

$$-\Delta_1 \phi_n(x) = \lambda_n \phi_n(x),$$

$$\phi_n(0) = \phi_n(L) = 0,$$

we obtain the eigenvalues

$$\lambda_n = (an)^2, \quad a = \pi/L, \quad n = 1, 2, 3, \dots$$

and the corresponding eigenfunctions

$$\phi_n(x) = c_n \sin(anx), \quad 0 \leq x \leq L,$$

where c_n are normalization constants.

The basis of this approach is the theorem stating that if λ_n and $\phi_n(x)$ are eigenvalues and eigenfunctions of a linear operator $\mathbf{A}$ and $\psi(z)$ is a sufficiently good function of z then eigenvalues of $\psi(\mathbf{A})$ are $\psi(\lambda_n)$ while the corresponding eigenfunctions coincide with $\phi_n(x)$. Inserting the expansion

$$f(x, t) = \sum_{n=1}^{\infty} f_n(t) \sin(anx)$$

into Eq. (5.52) and applying the above theorem,

$$(-\Delta_1)^{\alpha/2} \sin(anx) = (an)^{\alpha} \sin(anx),$$

we obtain

$$\sum_{n=1}^{\infty} [{}^{\nu}_0 D_t f_n(t) + (an)^{\alpha} f_n(t)] \sin(anx) = 0.$$

Multiplying the equation by $\sin(amx)$ and integrating over the section $[0, L]$ lead to the equation

$${}^{\nu}_0 D_t f_m(t) + (am)^{\alpha} f_m(t) = 0, \quad m = 1, 2, \dots$$

whose solution is

$$f_m(t) = f_m(0) E_{\nu} [-(am)^{\alpha} t^{\nu}],$$

where constants

$$f_m(0) = \int_0^L f(x, 0) \sin(amx) dx$$

are determined by initial conditions of the problem. As a result, we get

$$f(x, t) = \sum_{n=1}^{\infty} f_n(0) \sin(anx) E_{\nu} [-(an)^{\alpha} t^{\nu}].$$

In case $\alpha = 2$, this result coincides with the solution of the subdiffusion equation on this segment,

$$f(x, t) = \sum_{n=1}^{\infty} f_n(0) \sin(anx) E_{\nu} [-(an)^2 t^{\nu}],$$

is obtained in (Agrawal, 2002), while $\nu = 1$ corresponds to the superdiffusion equation, whose solution

$$f(x, t) = \sum_{n=1}^{\infty} f_n(0) \sin(anx) \exp [-(an)^{\alpha} t]$$

is given in (Ilic et al., 2005). Numerical approximation of the fractional-in-space diffusion equation with non-homogeneous boundary conditions is considered in the next work of the same authors (Ilic et al., 2006).

5.2.19 Equation for diffusion on fractals

Let us return to the diffusion process on fractal discussed in Sect. 3.7. Recall that Eq. (3.98)

$$\frac{\partial P(r,t)}{\partial t} = \frac{1}{r^{d-1}} \frac{\partial}{\partial r} \left[K r^{d-1} \frac{\partial P(r,t)}{\partial r} \right], \quad r > 0, K = \text{const} \quad (5.53)$$

with the solution

$$P(r,t) = \frac{2}{d\Gamma(d/2)} \left[\frac{1}{2Kt} \right]^{d/2} \exp\left(-\frac{r^2}{4Kt}\right)$$

has not been taken as a diffusion on fractals equation at fractional d because the mean square displacement

$$\langle R^2(t) \rangle = \langle R^2(1) \rangle t$$

turns out to be proportional to t for any integer d whereas in the fractal case

$$\langle R^2(t) \rangle = \langle R^2(1) \rangle t^{2/d_w}$$

the walk dimension d_w is fractional and depends on the fractal dimension d_f (Ren et al., 1999). To satisfy the latter requirement, the constant K has been replaced by the function $K(t) = K_1 t^{-\theta}$ (see Eqs. (3.100)–(3.101)). But the following trouble arose: what does r mean? A distance from what? From which point of the fractal? The origin of this trouble is quite clear: Eq. (3.100) describes diffusion in a non-homogeneous regular medium rather than in fractal.

A way to avoid this uncertainty lies through fractionalization of Eq. (5.53): we have to replace the time-differential operator by its fractional counterpart instead of replacing constant K by a power function $K_1 r^{-\theta}$. As a result, we obtain

$${}_0^v \mathbf{D}_t P_v(r,t) = \frac{1}{r^{d_f-1}} \frac{\partial}{\partial r} \left[K r^{d_f-1} \frac{\partial P_v(r,t)}{\partial r} \right].$$

According to Sect. 5.2.16, solution of these equations can be written in the form

$$P_v(r,t) = \int_0^\infty P_1(r, (t/\tau)^v) g_+(\tau; v) d\tau.$$

Because

$$P_1(r, (t/\tau)^v) = \frac{1}{\Gamma(d_f/2 + 1)(4Kt^v)^{d_f/2}} \exp\left(-\frac{r^2 \tau^v}{4Kt^v}\right),$$

one can represent it in terms of V -function (see Sect. 5.2.16):

$$\begin{aligned} P_v(r,t) &= \frac{1}{\Gamma(d_f/2 + 1)(4Kt^v)^{d_f/2}} \int_0^\infty \tau^{vd_f/2} \exp\left(-\frac{r^2 \tau^v}{4Kt^v}\right) g_+(\tau; v) d\tau \\ &= \frac{1}{\Gamma(d_f/2 + 1)(4Kt^v)^{d_f/2}} V^{d_f/2, v} \left(\frac{r^2}{4Kt^v} \right). \end{aligned}$$

In analogous way,

$$\langle R_v^2(t) \rangle = \int_0^\infty \langle R_1^2((t/\tau)^\nu) \rangle g_+(\tau; \nu) d\tau = \langle R_v^2(1) \rangle t^\nu, \quad \langle R_v^2(1) \rangle = \langle R_1^2(1) \rangle / \Gamma(1 + \nu).$$

Some authors prefer to keep the spatial dependence of the coefficient K , $K(r) = K_1 r^{-\theta}$, even after fractionalization of the equation. Such a system may be interpreted as a nonhomogeneous fractal.

5.2.20 Equation for flights on a fractal dust

Before deriving the requested equation, let us recall one of ways of derivation of equation for pdf

$$p_\theta(x, t) = \frac{\partial F_\theta(x, t)}{\partial x}$$

in case of a fractal walk. At first, we write down the equation for $W(m, t; \nu) = g_+(\tau_m; \nu) \partial \tau_m / \partial m$, considering m as a continuous variable:

$${}_0D_t^\nu W(m, t; \nu) = -c_\nu^\nu \frac{\partial W(m, t; \nu)}{\partial m} + \frac{t^{-\nu}}{\Gamma(1 - \nu)} \delta(m).$$

The conditional pdf

$$p_\theta(x|m) \equiv g(\xi_m; \alpha, \theta) \partial \xi_m / \partial x$$

obeys the equation

$$\frac{\partial p_\theta(x|m)}{\partial m} = c_\alpha^{-\alpha} L^{(\alpha, \theta)} p(x|m).$$

Using interrelations for conditional densities

$$p_\theta(x, t) = \int_0^\infty p_\theta(x|m) W(m, t; \nu) dm,$$

we obtain the equation

$${}_0D_t^\nu p_\theta(x, t) = -c_\nu^\nu \int_0^\infty p_\theta(x|m) \frac{\partial W(m, t; \nu)}{\partial m} dm + \frac{t^{-\nu}}{\Gamma(1 - \nu)} p(x, 0).$$

Integrating by parts,

$${}_0D_t^\nu p_\theta(x, t) = c_\nu^\nu \int_0^\infty \frac{\partial p_\theta(x|m)}{\partial m} W(m, t; \nu) dm + \frac{t^{-\nu}}{\Gamma(1 - \nu)} \delta(x),$$

we arrive at the fractal flights equation

$${}_0D_t^\nu p_\theta(x, t) = CL^{(\alpha, \theta)} p_\theta(x, t) + \frac{t^{-\nu}}{\Gamma(1 - \nu)} \delta(x), \quad 0 < \alpha \leq 2, \quad 0 < \nu \leq 1,$$

where $C = c_\nu^\nu / c_\alpha^\alpha$.

Observe that $W(m, t; \nu)$ obeys the equation

$${}_0D_t^\nu W(m, t; \nu) = -c_\nu^\nu \frac{\partial W(m, t; \nu)}{\partial m} + \frac{t^{-\nu}}{\Gamma(1-\nu)} \delta(m),$$

and conditional probability density

$$p_\theta(n|m) \simeq \frac{1}{\sqrt{2\pi m(1-\theta^2)}} \exp \left[-\frac{(n-m\theta)^2}{2m(1-\theta^2)} \right]$$

is a solution of the equation

$$\frac{\partial p_\theta(n|m)}{\partial m} = \frac{1-\theta^2}{2} \frac{\partial^2 p_\theta(n|m)}{n^2} - \theta \frac{\partial p_\theta(n|m)}{\partial n} \quad (5.54)$$

(m and n are here treated as continuous variables). Let us multiply equation for $W(m, t; \nu)$ by $p_\theta(n|m)$ and integrate it with respect to m . Taking into account that

$$p_\theta(n, t) = \int_0^\infty p_\theta(n|m) W(m, t; \nu) dm,$$

we obtain

$${}_0D_t^\nu p_\theta(n, t) = -c_\nu^\nu \int_0^\infty p_\theta(n|m) \frac{\partial p_\theta(m, t)}{\partial m} dm + \frac{t^{-\nu}}{\Gamma(1-\nu)} p(n, 0).$$

Integrating by parts and replacing $\partial p_\theta(n|m)/\partial m$ in this equation by the right hand side of Eq. (5.54), we obtain

$${}_0D_t^\nu p_\theta(n, t) = c_\nu^\nu \left[\frac{1-\theta^2}{2} \frac{\partial^2 p_\theta(n, t)}{\partial n^2} - \theta \frac{\partial p_\theta(n, t)}{\partial n} \right] + \frac{t^{-\nu}}{\Gamma(1-\nu)} \delta(n).$$

Multiplying both sides of the equation by $p_\theta(x|n)$, integrating result with respect to n and taken into account

$$p_\theta(x, t) = \int_0^\infty p_\theta(x|n) p_\theta(n, t) dn$$

and

$$\frac{\partial p_\theta(x|n)}{\partial n} = c_\alpha^{-\alpha} {}_0D_x^\alpha p_\theta(x, n),$$

we arrive at the equation for flights on fractal dust¹

$${}_0D_t^\nu p_\theta(x, t) = \theta C {}_0D_x^\alpha p_\theta(x, t) + c_\alpha^{-\alpha} (C/2)(1-\theta^2) {}_0D_x^{2\alpha} p_\theta(x, t) + \frac{t^{-\nu}}{\Gamma(1-\nu)} \delta(x),$$

¹ This equation is derived by R. T. Sibatov and published in our work (Uchaikin and Sibatov, 2004a).

with $C = c_v^\nu / c_\alpha^\alpha$. Note that in case of one-sided flights ($\theta = 1$) fractal flights and flights on fractal are equivalent processes:

$${}_0D_t^\nu p_1(x, t) = C {}_0D_x^\alpha p_1(x, t) + \frac{t^{-\nu}}{\Gamma(1-\nu)} \delta(x),$$

In case of symmetric flights

$${}_0D_t^\nu p_0(x, t) = c_\alpha^{-\alpha} (C/2) {}_0D_x^{2\alpha} p_0(x, t) + \frac{t^{-\nu}}{\Gamma(1-\nu)} \delta(x),$$

we observe a maximal difference between these processes.

5.2.21 Equation for percolation

Initially the fractional differential diffusion equation was associated with the comb model in the work (Nigmatullin, 1986). We are going to derive it, following the latter work (Arkhincheev and Baskin, 1991).

Assuming the set of comb's teeth, directed along the y axis, to be infinitely close and supposing an ordinary diffusion along them, and diffusion along x to proceed only through the axis itself, write an ordinary diffusion equation for the Green function

$$\frac{\partial G}{\partial t} - K \Delta G = \delta(\mathbf{x}) \delta(t)$$

in the form

$$\frac{\partial G}{\partial t} - K_x \delta(y) \frac{\partial^2 G}{\partial x^2} - K_y \frac{\partial^2 G}{\partial y^2} = \delta(x) \delta(y) \delta(t),$$

corresponding to diffusion in the considered heterogeneous anisotropic medium with diffusion coefficients K_x and K_y along the respective axes. Having performed the Laplace transform by time and the Fourier transform by x coordinate, we obtain

$$\left(\lambda + K_x k^2 \delta(y) - K_y \frac{\partial^2}{\partial y^2} \right) \tilde{G}(k, y, \lambda) = \delta(y).$$

The solution of the above equation has the form

$$\tilde{G}(k, y, \lambda) = \frac{\exp [-(\lambda / K_y)^{1/2} |y|]}{2(K_y \lambda)^{1/2} + K_x k^2}.$$

At the x axis, it takes the form

$$\left(\lambda^{1/2} + \frac{K_x}{2K_y^{1/2}} k^2 \right) \tilde{G}(k, 0, \lambda) = 1,$$

corresponding to fractional differential subdiffusion equation:

$${}_0D_t^{1/2}G(x,0,t) - \left(K_x/2K_y^{1/2}\right) \frac{\partial^2 G(x,0,t)}{\partial x^2} = \delta(x)\delta(t).$$

This equation describes the distribution of diffusing particles, situated at the x axis, and therefore

$$\int_{-\infty}^{\infty} G(x,0,t)dx < 1$$

decreasing with time within a power law: the increasing part of all particles leaves the x axis, having $y \neq 0$. If one analysis x coordinate irrespectively to another one, i.e., treating all particles including those, placed at the comb's teeth, he should use the following equation ²

$${}_0D_t^{1/2}G(x,t) - \left(K_x/2K_y^{1/2}\right) \frac{\partial^2 G(x,t)}{\partial x^2} = \delta(x) \frac{t^{-1/2}}{\Gamma(1/2)}. \quad (5.55)$$

We underline the fractional character of these equation to be caused by specific heterogeneous anisotropy of the media, which the diffusion proceeds in. Otherwise, if the particles had the probability of movement along x (while $y \neq 0$), then the diffusion equation would have the ordinary form

$$\frac{\partial G}{\partial t} - K_x \frac{\partial^2 G}{\partial x^2} - K_y \frac{\partial^2 G}{\partial y^2} = \delta(x)\delta(y)\delta(t).$$

Here the medium is also anisotropic (in different directions diffusion coefficients K differ), but this anisotropy is homogeneous: K_x and K_y don't depend on coordinates.

It is easy to imagine the further development of this model by introducing teeth of bounded length, randomizing this length, varying the teeth density along x , using randomly distributed (either accordingly to fractional Poisson law) teeth along the x axis, including external forces, drifting particles either x or y , or both x and y , introducing correlations on the base of fractional Brownian motion, say, and so on. Partially this program was done in next works of Arkhincheev (2000, 2002) and others.

5.2.22 Nonlinear equations

5.2.22.1 Nonlinear diffusion equation

Nonlinear equations play an important role in the theoretical physics. We met with them when discussed the Boltzmann equation containing bilinear integral opera-

² As one can see from Sect. 5.2.16, solution of Eq. (5.55) coincides with those given in Sect. 3.7.3.

tor. Following (Plastino, 1995; Tsallis and Bukman, 1996; Lenzi et al., 2006), we consider first the simplified version of the equation, namely, the nonlinear diffusion equation

$$\frac{\partial f(x,t)}{\partial t} = K \frac{\partial^2 [f(x,t)]^\gamma}{\partial x^2}, \quad -\infty < \gamma \leq 2, \quad (5.56)$$

treated also as *porous medium equation*. It is employed in the analysis of percolation of gases through porous media ($\gamma \geq 2$) (Muskat, 1937), thin saturated regions in porous media ($\gamma = 2$) (Polubarinova-Kochina, 1962), a standard solid-on-solid model for surface growth ($\gamma = 3$), thin liquid films spreading under gravity ($\gamma = 4$) and others (Kath, 1984).

Seeking the fundamental solution of Eq. (5.56) (on the assumption $K = 1$ and the initial condition $f(x,t) = \delta(x)$) in the self similar form

$$f(x,t) = \frac{1}{\phi(t)} F\left(\frac{x}{\phi(t)}\right),$$

the authors arrive at the equation

$$-\frac{\dot{\phi}(t)}{[\phi(t)]^2} \frac{d}{dz} [zF(z)] = \frac{1}{[\phi(t)]^{2+\gamma}} \frac{d^2}{dz^2} [F(z)]^\gamma,$$

admitting separation of variables:

$$\begin{aligned} [\phi(t)]^\gamma \dot{\phi}(t) &= k, \\ -k \frac{d[zF(z)]}{dz} &= \frac{d^2[F(z)]^\gamma}{dz^2}, \quad z = \frac{x}{\phi(t)}. \end{aligned}$$

On the assumption that the diffusion packet was initially concentrated at the origin ($x = 0$), the initial condition for the packet width is

$$\phi(0) = 0, \quad \gamma > -1.$$

The corresponding solution is

$$\phi(t) = [(1 + \gamma)kt]^{1/(1+\gamma)}.$$

The solution of the second equation under condition $dF^\gamma/dz|_{z=0} = 0$ is found as follows:

$$\begin{aligned} -k[zF(z)] &= \frac{d[F(z)]^\gamma}{dz} - C_1 = \frac{d[F(z)]^\gamma}{dz}, \\ -\frac{kz^2}{2} &= \gamma \frac{[F(z)]^{\gamma-1}}{\gamma-1} - C_2, \\ F(z) &= \frac{\gamma-1}{\gamma} \left[C_2 - \frac{kz^2}{2} \right]^{1/(\gamma-1)} = \left[C - \left(\frac{\gamma-1}{2\gamma} kz^2 \right) \right]^{1/(\gamma-1)}. \end{aligned}$$

As a result, we have the solution

$$f(x, t) = \frac{1}{\phi(t)} \left[C - \frac{\gamma-1}{2\gamma} k z^2 \right].$$

In case $C = 1$, it can be represented through the q -exponential function

$$\exp_q(x) = [1 + (1-q)x]^{1/(1-q)}$$

as

$$f(x, t) = \frac{1}{\phi(t)} \exp_{2-\gamma} \left[-\frac{k}{2\gamma} \left(\frac{x}{\phi(t)} \right)^2 \right].$$

The constant k provides the normalization condition

$$\int_{-\infty}^{\infty} f(x, t) dx = 1.$$

When $\gamma \rightarrow 1$, $q \rightarrow 1$, $\phi(t) \rightarrow \sqrt{2kt}$, $\exp_q(x) \rightarrow e^x$, $k \rightarrow 1$ and the solution takes the ordinary Gaussian form:

$$f(x, t) = \frac{1}{\sqrt{4\pi t}} \exp \left[-\left(\frac{x^2}{4t} \right) \right].$$

5.2.22.2 Fractional nonlinear equations

The generalized master equation, related to a fractional Fokker-Planck equation, has been used in the modeling of nonmarkovian dynamical processes in protein folding (Plotkin and Wolynes, 1998), describing relaxation to equilibrium in system (such as polymers chains and membranes) with long temporal memory (Douglas, 2000; Schiessel and Blumen, 1995), and in anomalous transport in disordered systems (Metzler et al., 1999). Another class of anomalous diffusion can be obtained using a spatial dependent diffusion coefficient $K = K(x)$. For instance, in (O'Shaughnessy and Procaccia, 1985) $K \propto |x|^{-\gamma}$ has been employed to analyze the diffusion in a fractal medium.

In (Bologna et al., 2000), the fractional counterpart of Eq. (5.49)

$$\frac{\partial f(x, t)}{\partial t} = {}_0D_x^\alpha [f(x, t)]^\gamma, \quad -\infty < \gamma \leq 2$$

was solved in a special case, when

$$\gamma = \frac{2-\alpha}{1+\alpha}.$$

Following the same way as above, the authors arrive at the result given by formulas (17)–(19) of the work cited,

$$f(x, t) = \frac{A}{t^{(\alpha+1)/(\alpha^2-\alpha+1)}} \left[\frac{z^{\alpha(\alpha+1)}}{(bz+1)^{1-\alpha^2}} \right]^{1/(1-2\alpha)},$$

where

$$z = \frac{x}{(kt)^{(\alpha+1)/(\alpha^2-\alpha+1)}},$$

and A and b are arbitrary constants. We refer the reader interested in various special cases of these solution and its applications to the original work (Bologna et al., 2000) and the Tsallis' book (2009).

5.2.22.3 Fractional nonlinear Fokker-Planck equation

In (Lenzi, 2002) the nonlinear version of fractional Fokker-Planck equation for particles in a nonhomogeneous medium

$$\frac{\partial f(x, t)}{\partial t} = \frac{\partial}{\partial x} \left\{ K(x) {}_0D_x^{\alpha-1} [f(x, t)]^\gamma - g(x)f(x) \right\}$$

is solved under conditions, that the (dimensionless) external drift force $g(x)$ and the diffusion coefficient $K(x)$ are of the form

$$g(x) = -k_1 x$$

and

$$K(x) = |x|^{-\theta}$$

respectively. The solution is again found for a special combination of parameters only,

$$\gamma = \frac{2 - \alpha}{1 + \alpha + \theta}.$$

It is represented in the form

$$f(x, t) = \frac{A}{\Phi(t)} \left[\frac{z^{(\alpha+\theta)(\alpha+\theta+1)}}{(bz+1)^{1/(1-2\alpha-\theta)}} \right]^{1/(1-2\alpha-\theta)},$$

where

$$\Phi(t) = \left[k/k_1 \left(1 - e^{-(\alpha+\gamma+\theta-1)k_1 t} \right) \right]^{1/(\alpha+\gamma+\theta-1)}$$

and $z = x/\Phi(t)$.

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Chapter 6

Numerical Methods

This chapter contains a short survey of basic numerical methods developed for solving various problems formulated in terms of fractional derivatives.

6.1 Grünwald-Letnikov derivatives

6.1.1 Fractional differences

We start with some facts from the *difference calculus*. The leading part in this approach belongs to the *shift operator*

$$\mathbf{E}^h f(x) = f(x+h), \quad h > 0.$$

It is a linear operator and its recurring is equivalently to multiplying the shift parameter h by a corresponding integer number:

$$\left(\mathbf{E}^h\right)^n f(x) = f(x+nh) = \mathbf{E}^{nh} f(x), \quad n = 0, \pm 1, \pm 2, \dots$$

Evidently,

$$\mathbf{E}^0 \equiv 1$$

is the identical operator and $\mathbf{E}^{-h}$ is the inverse with respect to $\mathbf{E}^h$.

If a function $f(x)$ admits the Taylor expansion, this series can be represented by means of the exponential operator function:

$$f(x+h) = \sum_{k=0}^{\infty} \frac{h^k}{k!} \mathbf{D}_x^k f(x) = \exp(h\mathbf{D}_x) f(x).$$

Therefore, one can write

$$\mathbf{E}^h = \exp(h\mathbf{D}_x)$$

and, as a particular case $h = 1$,

$$\mathbf{E} = \exp(\mathbf{D}_x).$$

We introduce further the *difference operator*

$$\Delta_h = \mathbf{1} - \mathbf{E}^{-h} = \mathbf{1} - e^{-h\mathbf{D}_x}.$$

It is not difficult to see, that for any differentiable at x function

$$\lim_{h \downarrow 0} \frac{\Delta_h}{h} = \mathbf{D}_x.$$

Successive application of the operator Δ_h to the function $f(x)$ produces its differences of higher orders:

$$\begin{aligned} \Delta_h \Delta_h \cdots \Delta_h f(x) &= \Delta_h^n f(x) = (\mathbf{1} - \mathbf{E}^{-h})^n f(x) \\ &= \sum_{k=0}^n (-1)^k \binom{n}{k} \mathbf{E}^{-hk} f(x) = \sum_{k=0}^n (-1)^k \binom{n}{k} f(x - kh). \end{aligned}$$

Replacing the integer n by an arbitrary real (or complex) v and continuing summation to infinity, we obtain the following expression for the *fractional difference*:

$$\Delta_h^v f(x) = \sum_{k=0}^{\infty} (-1)^k \binom{v}{k} f(x - kh).$$

Formally, one can rewrite it in the form

$$\Delta_h^v = (1 - e^{-h\mathbf{D}_x})^v = 1 - \frac{v}{1!} e^{-h\mathbf{D}_x} + \frac{v(v-1)}{2!} e^{-2h\mathbf{D}_x} - \frac{v(v-1)(v-2)}{3!} e^{-3h\mathbf{D}_x} + \dots$$

It makes evident the semigroup property of difference operators

$$\Delta_h^\mu \Delta_h^v = \Delta_h^{\mu+v},$$

which can be confirmed by direct computing as well:

$$\begin{aligned} \Delta_h^\mu \Delta_h^v f(x) &= \sum_{k=0}^{\infty} (-1)^k \binom{\mu}{k} \Delta_h^v f(x - kh) \\ &= \sum_{k=0}^{\infty} (-1)^k \binom{\mu}{k} \sum_{l=0}^{\infty} (-1)^l \binom{v}{l} f(x - (k+l)h) \\ &= \sum_{k=0}^{\infty} \binom{\mu}{k} \sum_{j=l}^{\infty} (-1)^j \binom{v}{j-k} f(x - jh) \\ &= \sum_{j=0}^{\infty} (-1)^j \left[\sum_{k=0}^j \binom{\mu}{k} \binom{v}{j-k} \right] f(x - jh) \\ &= \sum_{j=0}^{\infty} (-1)^j \binom{\mu+v}{j} f(x - jh) = \Delta_h^{\mu+v} f(x). \end{aligned}$$

Now, we are in a position to define fractional differentiation operators via the limit relation

$$\mathbf{d}_x^\nu \equiv \lim_{h \downarrow 0} \frac{\Delta_h}{h^\nu},$$

Using a formal representation, one can interpret this fractional operator as a fractional power of the ordinary differential operator:

$$\mathbf{d}_x^\nu = \lim_{h \downarrow 0} \frac{(1 - e^{-h\mathbf{D}_x})^\nu}{h^\nu} = (\mathbf{D}_x)^\nu.$$

The corresponding derivative

$$f^{(\nu)}(x) = \mathbf{d}_x^\nu f(x)$$

is called the *Grünwald-Letnikov* (G-L) *derivative*.

6.1.2 The G-L derivatives of integer orders

For deeper understanding of the G-L concept, we consider explicit expressions of foregoing operations starting from integer orders. The left-side fractional difference can be represented in the form

$$\Delta_h^\nu f(x) = f(x) - \frac{\nu}{1!} f(x-h) + \frac{\nu(\nu-1)}{2!} f(x-2h) - \frac{\nu(\nu-1)(\nu-2)}{3!} f(x-3h) + \dots.$$

In case of an integer ν , the series contains a finite number of terms, and we arrive at the usual formulas for finite differences

$$\Delta_h^0 f(x) = f(x),$$

$$\Delta_h^1 f(x) = f(x) - f(x-h),$$

$$\Delta_h^2 f(x) = f(x) - 2f(x-h) + f(x-2h) = [f(x) - f(x-h)] - [f(x-h) - f(x-2h)],$$

etc. The limit relations $\lim_{h \downarrow 0} \Delta_h^n f(x)/h^n$ coincide with ordinary derivatives of corresponding orders $f^{(n)}(x)$, $n = 1, 2, \dots$

In case of a negative integer $\nu = -m$, $m = 1, 2, 3, \dots$, the series does not break:

$$\Delta_h^{-1} f(x) = f(x) + f(x-h) + f(x-2h) + \dots,$$

$$\Delta_h^{-2} f(x) = f(x) + 2f(x-h) + 3f(x-2h) + \dots,$$

and so on. It is easy to recognize here the integral sums :

$$\Delta_h^{-1} f(x) = \sum_{k=0}^{\infty} f(x_k) = h^{-1} \sum_{k=0}^{\infty} f(x_k) \delta x_k, \quad x_k = x - kh, \quad \delta x_k = h,$$

$$\Delta_h^{-2} f(x) = \sum_{k=0}^{\infty} (k+1) f(x_k) = h^{-1} \sum_{k=0}^{\infty} (k+1) f(x_k) \delta x_k.$$

Application of the Dirichlet rule to a double sum bring this latter expression to the form

$$\Delta_h^{-2} f(x) = h^{-2} \sum_{l=0}^{\infty} \Delta x_l \sum_{k=l}^{\infty} f(x_k) \delta x_k.$$

Consequently,

$$\mathbf{f}^{(-1)}(x) = \int_{-\infty}^x d\xi f(\xi),$$

$$\mathbf{f}^{(-2)}(x) = \int_{-\infty}^x d\xi_2 \int_{-\infty}^{\xi_2} d\xi_1 f(\xi_1)$$

and on the whole

$$\mathbf{f}^{(-m)}(x) = {}_{-\infty}f^{(-m)}(x),$$

where

$${}_{-\infty}f^{(-m)}(x) = \int_{-\infty}^x d\xi_m \dots \int_{-\infty}^{\xi_2} d\xi_1 f(\xi_1).$$

6.1.3 The G-L derivatives of negative fractional orders

According the Cauchy formula, integrals ${}_{-\infty}f^{(-m)}(x)$ and ${}_x f^{(-m)}(\infty)$ can be reduced to one-fold forms

$${}_{-\infty}f^{(-m)}(x) = \frac{1}{\Gamma(m)} \int_{-\infty}^x f(\xi) (x - \xi)^{m-1} d\xi$$

and

$${}_x f^{(-m)}(\infty) = \frac{1}{\Gamma(m)} \int_x^{\infty} f(\xi) (\xi - x)^{m-1} d\xi, \quad m = 1, 2, 3, \dots$$

These expressions are easily extended to fractional positive orders $\mu > 0$:

$${}_{-\infty}f^{(-\mu)}(x) = \frac{1}{\Gamma(\mu)} \int_{-\infty}^x f(\xi) (x - \xi)^{\mu-1} d\xi$$

and

$${}_x f^{(-\mu)}(\infty) = \frac{1}{\Gamma(\mu)} \int_x^{\infty} f(\xi) (\xi - x)^{\mu-1} d\xi, \quad \mu > 0.$$

It is not hard to make sure that

$$f^{(-\mu)}(x) \equiv \lim_{\xi \rightarrow 0} \frac{\Delta_h^{-\mu}}{h^{-\mu}} = \frac{1}{\Gamma(\mu)} \int_{-\infty}^x f(h)(x-h)^{\mu-1} dh \equiv {}_{-\infty}f^{(-\mu)}(x)$$

and

$$f_-^{(-\mu)}(x) = {}_x f^{(-\mu)}(\infty).$$

6.1.4 The G-L derivatives on a semi-axis

Let a differentiable function $f(x)$ be identically zero on the semi-axis $x < a$. Then, the step h can be determined as

$$h = \frac{x-a}{N},$$

and the left derivative is written in the form

$${}_a f^{(\nu)}(x) = {}_a d_x^\nu f(x) = \lim_{h \downarrow 0} \frac{1}{h^\nu} \sum_{k=0}^{(x-a)/h-1} (-1)^k \frac{\Gamma(\nu+1)}{k! \Gamma(\nu-k+1)} f(x-kh), \quad (6.1)$$

or equivalently

$${}_a f^{(\nu)}(x) = \lim_{N \rightarrow \infty} \left(\frac{N}{x-a} \right)^\nu \sum_{k=0}^{N-1} (-1)^k \frac{\Gamma(\nu+1)}{k! \Gamma(\nu-k+1)} f\left(x - k \left(\frac{x-a}{N} \right)\right). \quad (6.2)$$

This representation of GL-derivatives being convenient for numerical calculations do not provide the limit transition to the negative integers $\nu = -1, -2, \dots$, but a simple transformation change the situation. Observing that

$$\binom{n}{k} = \frac{\Gamma(n+1)}{k! \Gamma(n-k+1)} = \frac{n!}{k! (n-k)!} = \frac{n(n-1) \cdots (n-k+1)}{k!},$$

and replacing here n by $-m$, we obtain:

$$\begin{aligned} \binom{-m}{k} &= \frac{-m(-m-1) \cdots (-m-k+1)}{k!} = (-1)^k \frac{(m+k-1)!}{k! (m-1)!} \\ &= (-1)^k \frac{\Gamma(m+k)}{k! \Gamma(m)}. \end{aligned}$$

Using this expression for analytic continuation of the binomial function $\binom{\nu}{k}$ to the range of negative orders $\nu = -\mu$, $\mu > 0$, we obtain

$$\binom{-\mu}{k} = (-1)^k \frac{\Gamma(\mu+k)}{k! \Gamma(\mu)},$$

and accordingly

$${}_a d_x^{-\mu} f(x) = \lim_{h \rightarrow 0} h^\mu \sum_{k=0}^{(x-a)/h-1} \frac{\Gamma(\mu+k)}{k! \Gamma(\mu)} f(x-kh).$$

Now, we are in a position to write the formula for G-L derivative for an arbitrary order $-\infty < \nu < \infty$:

$$\begin{aligned} {}_a d_x^\nu f(x) &= \lim_{h \rightarrow 0} h^{-\nu} \sum_{k=0}^{(x-a)/h-1} \frac{\Gamma(-\nu+k)}{k! \Gamma(-\nu)} f(x-kh) \\ &= \lim_{N \rightarrow \infty} \left(\frac{x-a}{N} \right)^{-\nu} \sum_{k=0}^{N-1} \frac{\Gamma(-\nu+k)}{k! \Gamma(-\nu)} f\left(x-k \left(\frac{x-a}{N} \right)\right). \end{aligned} \quad (6.3)$$

6.2 Finite-differences methods

6.2.1 Numerical approximation of R-L and G-C derivatives

Beginning with a negative order $\nu = -\mu < 0$, we divide the integration domain into N parts and represent the Riemann-Liouville fractional integral

$${}_0 f^{(-\mu)}(x) = \frac{1}{\Gamma(\mu)} \int_0^x (x-\xi)^{\mu-1} f(\xi) d\xi$$

as

$${}_0 f^{(-\mu)}(x) = \frac{1}{\Gamma(1+\mu)} \sum_{j=0}^{N-1} \int_{jh}^{(j+1)h} (x-\xi)^{\mu-1} f(\xi) d\xi,$$

where $h \equiv x/N$ is the integration step. The piecewise constant (“staircase”) approximation of the integrands leads to the *R1*-algorithm of the fractional integration

$${}_0 f^{(-\mu)}(x) \approx {}_0 \bar{f}^{(-\mu)}(x)_{R1} = \frac{h^\mu}{\Gamma(\mu)} \sum_{j=0}^{N-1} \frac{f_j + f_{j+1}}{2} [(j+1)^\mu - j^\mu] \quad (6.4)$$

while the piecewise linear representation yields a more precise approximation called the *R2* algorithm:

$$\begin{aligned} {}_0 \bar{f}^{(-\mu)}(x)_{R2} &= \frac{h^\mu}{\Gamma(\mu)} \sum_{j=0}^{N-1} \left\{ \frac{(j+1)f_j - jf_{j+1}}{\mu} [(j+1)^\mu - j^\mu] \right. \\ &\quad \left. + \frac{f_{j+1} - f_j}{1+\mu} [(j+1)^{1+\mu} - j^{1+\mu}] \right\}. \end{aligned}$$

Notice that $f_j \equiv f(x-jh)$, $j = 0, 1, \dots, N$, that is, $f_0 = f(x)$, $f_1 = f(x-h)$, $\dots$, $f_N = f(0)$.

Let us pass now to positive orders. In this region, it is more convenient to construct the numerical algorithm using the Gerasimov-Caputo version of the fractional derivative (the difference between R-L and G-C representations are computed analytically). For $\nu \in (0, 1)$, we obtain the $L1$ algorithm (by the Oldham-Spanier terminology):

$$\begin{aligned} {}^{\nu}_0 f(x) &= \frac{1}{\Gamma(1-\nu)} \int_0^x \frac{f'(\xi) d\xi}{(x-\xi)^{\nu}} = \frac{1}{\Gamma(1-\nu)} \sum_{j=0}^{N-1} \int_{jh}^{(j+1)h} \frac{f'(x-\xi) d\xi}{\xi^{\nu}} \\ &\approx \frac{1}{\Gamma(1-\nu)} \sum_{j=0}^{N-1} \frac{f(x-jh) - f(x-(j+1)h)}{h} \int_{jh}^{(j+1)h} \frac{d\xi}{\xi^{\nu}} \\ &= \frac{h^{-\nu}}{\Gamma(2-\nu)} \sum_{j=0}^{N-1} (f_j - f_{j+1}) [(j+1)^{1-\nu} - j^{1-\nu}] \equiv {}^{\nu}_0 \hat{f}(x)_{L1} \end{aligned}$$

and

$${}_0 \bar{f}^{\nu}(x)_{L1} = \frac{h^{-\nu}}{\Gamma(2-\nu)} \left\{ (1-\nu) f_N / N^{\nu} + \sum_{j=0}^{N-1} (f_j - f_{j+1}) [(j+1)^{1-\nu} - j^{1-\nu}] \right\}.$$

For the $[1, 2)$ range of orders, we obtain the $L2$ -algorithm

$${}^{\nu}_0 \bar{f}(x)_{L2} = \frac{h^{-\nu}}{\Gamma(3-\nu)} \sum_{j=0}^{N-1} (f_{j-1} - 2f_j + f_{j+1}) [(j+1)^{2-\nu} - j^{2-\nu}],$$

and

$${}_0 \bar{f}^{\nu}(x)_{L2} = \frac{h^{-\nu}}{\Gamma(3-\nu)} \left\{ \frac{(1-\nu)(2-\nu) f_N}{N^{\nu}} + \frac{(2-\nu)(f_{N-1} - f_N)}{N^{\nu-1}} \right\} + {}^{\nu}_0 \hat{f}(x)_{L2},$$

$$1 \leq \nu < 2$$

(Oldham and Spanier, 1974).

For $t_i = i/N$, $i = 0, 1, \dots, N$, it can also be represented in the Diethelm form

$${}^{\nu}_0 \bar{f}(x_i) = \sum_{j=0}^i c_{i,j} f(j/N), \quad i = 1, 2, \dots, N,$$

where

$$\begin{aligned} c_{i,j} &= \frac{N^{\nu}}{-2\Gamma(2-\nu)} \\ &\times \begin{cases} (\nu-1)i^{-\nu} - (i-1)^{1-\nu} + i^{1-\nu} + 1/\nu, & j=0; \\ 2(i-j)^{1-\nu} - (i-j-1)^{1-\nu} - (i-j+1)^{1-\nu}, & j=1, \dots, i-1; \\ -1, & j=i. \end{cases} \end{aligned}$$

(see (Diethelm, 1997, 2004)).

6.2.2 Numerical approximation of G-L derivatives

Another approach to the numerical differentiation is developed on the base of the Grünwald-Letnikov definition. Writing Eq. (6.3) in the prelimit form, we obtain the approximate representation of this derivative called G1-algorithm:

$${}_0f^{(\nu)}(x) \approx {}_0\hat{f}^{(\nu)}(x)_{G1} = h^{-\nu} \sum_{j=0}^{N-1} (-1)^j \binom{\nu}{j} f_j = h^{-\nu} \frac{1}{\Gamma(-\nu)} \sum_{j=0}^{N-1} \frac{\Gamma(j-\nu)}{j!} f_j.$$

It can be represented in the form

$${}_0\hat{f}^{(\nu)}(x) = \sum_{j=0}^{N-1} c_j f_j$$

with coefficients

$$c_j = h^{-\nu} w_j, \quad w_j = (-1)^j \binom{\nu}{j} = \frac{1}{\Gamma(-\nu)} \frac{\Gamma(j-\nu)}{j!},$$

which can be computed in a simple recurrence way:

$$c_0 = 1, \quad c_j = \left(1 - \frac{1+\nu}{j}\right) c_{j-1}, \quad j = 1, 2, 3, \dots$$

With the view of accuracy increase, one can use the modified G-L definition of fractional derivative

$${}_0f^{(\nu)}(x) = \lim_{N \rightarrow \infty} \left\{ \frac{h^{-\nu}}{\Gamma(-\nu)} \sum_{j=0}^{N-1} \frac{\Gamma(j-\nu)}{\Gamma(j+1)} f\left(x + \frac{\nu h}{2} - jh\right) \right\}.$$

Involving the approximation

$$\begin{aligned} f\left(x + \frac{\nu h}{2} - jh\right) &\approx \left(\frac{\nu}{4} + \frac{\nu^2}{8}\right) f(x+h-jh) \\ &\quad + \left(1 - \frac{\nu^2}{4}\right) f(x-jh) + \left(\frac{\nu^2}{8} - \frac{\nu}{4}\right) f(x-h-jh), \end{aligned}$$

based upon the Lagrange three-point approximation, one obtains a G2-algorithm

$$\begin{aligned} {}_0\hat{f}^{(\nu)}(x)_{G2} &= \left\{ h^{-\nu} \Gamma(-\nu) \sum_{j=0}^{N-1} \frac{\Gamma(j-\nu)}{j!} \right. \\ &\quad \times \left[f_j + \frac{\nu}{4}(f_{j-1} - f_{j+1}) + \frac{\nu^2}{8}(f_{j-1} - 2f_j + f_{j+1}) \right] \Big\}. \end{aligned}$$

There are used also the numerical representations of fractional G-C derivatives including the integer-order derivatives,

$${}_0^v \hat{f}(x) = \sum_{j=0}^{N-1} c_j [f_j - f(0) - f'(0)x]$$

(Gorenflo et al., 2002) and

$${}_0^v \hat{f}(x) = \sum_{j=0}^{N-1} c_j f_j - \frac{f(0)}{\Gamma(1-v)} t^{-v} - \frac{f'(0)}{\Gamma(2-v)} x^{1-v}$$

(Ciesielski and Leszczynski, 2003). If $v \in (0, 1)$, the last terms containing $f'(0)$ must be omitted in these equations. The comparison of the two approaches can be found in (Murillo and Yuste, 2009).

Important details of these algorithms, practical advices and error estimations can be found by the interested reader in books (Oldham and Spanier, 1974; Podlubny, 1997), and the others.

6.2.3 Estimation of accuracy

As an example of the accuracy estimation, we sketch the results of (Podlubny, 1997) concerning the case when the function $f(x)$ is smooth enough in order to be represented by a power series. In this case, one can confine oneself to investigation of one term of this series or simply of a power function x^m , $m = 1, 2, 3, \dots$. The fractional derivative of this function

$${}_0 f^{(v)}(x) = {}_0 D_x^v x^m = \frac{\Gamma(1+m)}{\Gamma(1+m-v)} x^{m-v}.$$

Our task is to compute

$${}_0 \bar{f}^{(v)}(x) \equiv \frac{1}{h^v} \sum_{j=0}^{[x/h]} (-1)^j \binom{v}{j} f(x-jh) = \frac{1}{h^v} \sum_{j=0}^{[x/h]} (-1)^j \binom{v}{j} (x-jh)^m$$

and to investigate the relative error

$$\varepsilon(h) \equiv \frac{{}_0 \bar{f}^{(v)}(x) - {}_0 f^{(v)}(x)}{{}_0 f^{(v)}(x)}.$$

Performing simple manipulations,

$${}_0 \bar{f}^{(v)}(x) = \frac{x^m}{h^v} \sum_{j=0}^n (-1)^j \binom{v}{j} \left(1 - j \frac{h}{x}\right)^m$$

$$\begin{aligned}
&= x^{m-\nu} n^{\nu} \sum_{j=0}^n \binom{j-\nu-1}{j} \left(1 - \frac{j}{n}\right)^m \\
&= x^{m-\nu} n^{\nu} \sum_{j=0}^n \binom{j-\nu-1}{j} \sum_{k=0}^m \binom{m}{k} \left(-\frac{j}{n}\right)^k \\
&= x^{m-\nu} \sum_{k=0}^m \binom{m}{k} n^{\nu-k} (-1)^k \Sigma_n^{(k)},
\end{aligned}$$

we express the derivative through coefficients

$$\Sigma_n^{(k)} = \sum_{j=0}^n \binom{j-\nu-1}{j} j^k$$

which are related to the Stirling numbers of the second kind $S_n^{(i)}$. Using the relation

$$j^k = \sum_{i=0}^k \frac{j!}{(j-i)!} S_k^{(i)}$$

and some properties of binomial coefficients, we obtain

$$\Sigma_n^{(k)} = \sum_{i=1}^k \frac{\Gamma(n-\nu+1)}{(i-\nu)\Gamma(-\nu)\Gamma(n-i+1)} S_k^{(i)}$$

and then

$${}_0\bar{f}^{\nu}(x) = \frac{x^{m-\nu}}{\Gamma(-\nu)} \sum_{k=0}^m (-1)^k \binom{m}{k} \sum_{i=1}^k \frac{n^{\nu-k} \Gamma(n-\nu+1)}{(i-\nu)\Gamma(n-i+1)} S_k^{(i)}.$$

Taking the asymptotic expression for the Gamma-function

$$z^{(b-a)} \frac{\Gamma(z+a)}{\Gamma(z+b)} = 1 + O(z^{-1}), \quad z \rightarrow \infty$$

into account, we can write

$$\frac{n^{\nu-k} \Gamma(n-\nu+1)}{\Gamma(n-i+1)} = n^{i-k} \left(n^{\nu-i} \frac{\Gamma(n-\nu+1)}{\Gamma(n-i+1)} \right) = n^{i-k} (1 + O(n^{-1})), \quad n \rightarrow \infty,$$

and arrive at the expression

$${}_0\bar{f}^{(\nu)}(x) = \frac{x^{m-\nu}}{\Gamma(-\nu)} \sum_{k=0}^m (-1)^k \binom{m}{k} \frac{S_k^{(k)}}{k-\nu} (1 + O(n^{-1})), \quad n \rightarrow \infty.$$

Finally, using the formula

$$\sum_{k=0}^m (-1)^k \binom{m}{k} \frac{S_k^{(k)}}{k-v} = \sum_{k=0}^m (-1)^k \binom{m}{k} \frac{1}{k-v} = B(-v, m+1),$$

we get

$${}_0\bar{f}^{(v)}(x) = \frac{1}{h^v} \sum_{j=0}^{[x/h]} (-1)^j \binom{v}{j} f(x-jh) = {}_0f^{(v)}(x) + O(h/x). \quad (6.5)$$

Consequently, for a fixed x we have

$$\delta(h) = O(h), \quad h \rightarrow 0.$$

Further improvement of formula (6.5) can be reached by passing to central differences,

$${}_0\bar{f}^{(v)}(x) = \frac{1}{h^v} \sum_{j=0}^{[x/h]} (-1)^j \binom{v}{j} f(x + (v/2 - j)h) = {}_0f^{(v)}(x) + O(h^2), \quad h \rightarrow \infty. \quad (6.6)$$

Equations (6.5) and (6.6) are applicable only if the transition of $f(x)$ to zero for $x < 0$ is sufficiently smooth at the origin $x = 0$ (Gorenflo, 1997). If one wants to numerically differentiate a function $f(x)$ given as a smooth function for $x \geq 0$, then the transition to its extension by zero for negative argument x may be non-smooth at the origin $x = 0$, even discontinuous, and the approximations (6.5) and (6.6) are of dubious value and may cause trouble. However, often this singularity, induced by the zero extension, has the form of a power function $p_1(x) = c_1 x_+^{\beta_1}$ or of a generalized polynomial $p_n(x) = c_1 x_+^{\beta_1} + \dots + c_n x_+^{\beta_n}$ which can be subtracted to obtain a function sufficiently smooth and vanishing for negative argument. The functions $p_n(x)$ can be differentiated exactly.

6.2.4 Approximation of the Riesz-Feller derivatives

Following the article (Ciesielski and Leszczynski, 2005), we rewrite the Riesz-Feller (R-F) derivative in the form

$$D_x^{\alpha, \theta} f(x) = -[c_-(\alpha, \theta) {}_{-\infty}D_x^\alpha + c_+(\alpha, \theta) {}_xD_\infty^\alpha] f(x),$$

where

$$c_\mp(\alpha, \theta) = \frac{\sin[(\alpha \mp \theta)\pi/2]}{\sin(\alpha\pi)}.$$

Using a uniform spatial grid $-\infty, \dots, < x_{i-1} < x_i < x_{i+1} < \dots, \infty$ with the step h , we denote $f(x_k)$ as f_k . Because of infinite limits of fractional derivatives involving in the R-F operator, their R-L and G-C versions coincide.

For $\alpha \in (0, 1)$,

$${}_x D_{\infty}^{\alpha} f(x) = -\frac{1}{\Gamma(1-\alpha)} \int_x^{\infty} \frac{f'(\xi) d\xi}{(\xi-x)^{\alpha}}$$

and one can write

$$\begin{aligned} D_{x_i}^{\alpha, \theta} f(x_i) &= - \left[\frac{c_-}{\Gamma(1-\alpha)} \int_{-\infty}^{x_i} \frac{f'(\xi) d\xi}{(x_i-\xi)^{\alpha}} - \frac{c_+}{\Gamma(1-\alpha)} \int_{x_i}^{\infty} \frac{f'(\xi) d\xi}{(\xi-x_i)^{\alpha}} \right] \\ &= \frac{-1}{\Gamma(1-\alpha)} \sum_{j=0}^{\infty} \left[c_- \int_{x_{i-j-1}}^{x_{i-j}} \frac{f'(\xi) d\xi}{(x_i-\xi)^{\alpha}} - c_+ \int_{x_{i+j}}^{x_{i+j+1}} \frac{f'(\xi) d\xi}{(\xi-x_i)^{\alpha}} \right] \\ &\approx \frac{-1}{\Gamma(1-\alpha)} \sum_{j=0}^{\infty} \left[c_- \hat{f}_{i-j}' \int_{x_{i-j-1}}^{x_{i-j}} \frac{d\xi}{(x_i-\xi)^{\alpha}} - c_+ \check{f}_{i+j}' \int_{x_{i+j}}^{x_{i+j+1}} \frac{d\xi}{(\xi-x_i)^{\alpha}} \right], \end{aligned}$$

where $\hat{f}_k'$ and $\check{f}_k'$ are approximations of $f'(x_k)$ with the use of intervals $[x_{k-1}, x_k]$ and $[x_k, x_{k+1}]$, respectively. To avoid the problem of singularity, the authors of the cited work have been used the following difference schemes for these approximations:

$$\begin{aligned} \hat{f}_k' &= (1/2h)[\lambda_1 f_{k+1} + 2(1-\lambda_1)f_k + (\lambda_1-2)f_{k-1}], \\ \check{f}_k' &= (1/2h)[(2-\lambda_1)f_{k+1} + 2(\lambda_1-1)f_k - \lambda_1 f_{k-1}], \end{aligned}$$

where $\lambda_1 = \lambda_1(\alpha, \theta) = \alpha - |\theta|$, $\lambda \in [0, 1]$.

After some algebra, the authors arrive at the algorithm

$$D_{x_i}^{\alpha, \theta} f(x_i) \approx \bar{f}^{(\alpha, \theta)}(x_i) = -\frac{1}{2\Gamma(2-\alpha)} \frac{1}{h^{\nu}} \sum_{j=-\infty}^{\infty} w_j f_{j+i}, \quad (6.7)$$

where

$$w_j = \begin{cases} ((|j|+2)^{1-\alpha}\lambda_1 + (|j|+1)^{1-\alpha}(2-3\lambda_1) \\ \quad + |j|^{1-\alpha}(3\lambda_1-4) + (|j|-1)^{1-\alpha}(2-\lambda_1))c_-, & \text{for } j \leq -2; \\ (3^{1-\alpha}\lambda_1 + 2^{1-\alpha}(2-3\lambda_1) + 3\lambda_1-4)c_- + \lambda_1 c_+, & \text{for } j = -1; \\ (2^{1-\alpha}\lambda_1 - 3\lambda_1 + 2)(c_- + c_+), & \text{for } j = 0; \\ (3^{1-\alpha}\lambda_1 + 2^{1-\alpha}(2-3\lambda_1) + 3\lambda_1-4)c_+ + \lambda_1 c_-, & \text{for } j = 1; \\ ((j+2)^{1-\alpha}\lambda_1 + (j+1)^{1-\alpha}(2-3\lambda_1) \\ \quad + j^{1-\alpha}(3\lambda_1-4) + (j-1)^{1-\alpha}(2-\lambda_1))c_+, & \text{for } j \geq 2. \end{cases}$$

For $\alpha \in (1, 2)$,

$${}_x D_{\infty}^{\alpha} f(x) = \frac{1}{\Gamma(2-\alpha)} \int_x^{\infty} \frac{f''(\xi) d\xi}{(\xi-x)^{\alpha}}$$

and consequently

$$\begin{aligned} \mathbf{D}_{x_i}^{\alpha, \theta} f(x_i) &= - \left[\frac{c_-}{\Gamma(2-\alpha)} \int_{-\infty}^{x_i} \frac{f''(\xi) d\xi}{(x_i - \xi)^{\alpha-1}} + \frac{c_+}{\Gamma(2-\alpha)} \int_{x_i}^{\infty} \frac{f''(\xi) d\xi}{(\xi - x_i)^{\alpha-1}} \right] \\ &\approx \frac{-1}{\Gamma(2-\alpha)} \sum_{j=0}^{\infty} \left[c_- \hat{f}_{i-j}'' \int_{x_{i-j-1}}^{x_{i-j}} \frac{d\xi}{(x_i - \xi)^{\alpha-1}} + c_+ \check{f}_{i+j}'' \int_{x_{i+j}}^{x_{i+j+1}} \frac{d\xi}{(\xi - x_i)^{\alpha-1}} \right], \end{aligned}$$

where $\hat{f}_k''$ and $\check{f}_k''$ are approximated as

$$\hat{f}_k'' = (1/2h^2)[(2 - \lambda_2)f_{k+1} + (3\lambda_2 - 4)f_k + (2 - 3\lambda_2)f_{k-1} + \lambda_2 f_{k-2}],$$

and

$$\check{f}_k'' = (1/2h^2)[\lambda_2 f_{k+2} + (2 - 3\lambda_2)f_{k+1} + (3\lambda_2 - 4)f_k + (2 - \lambda_2)f_{k-1}],$$

with $\lambda_2 = \lambda_2(\alpha, \theta) = 2 - (\alpha + |\theta|)$, $\lambda_2 \in [0, 1]$.

In this case,

$$\hat{\mathbf{D}}_{x_i}^{(\alpha, \theta)} f(x_i) = - \frac{1}{2\Gamma(3-\alpha)} \frac{1}{h^\nu} \sum_{j=-\infty}^{\infty} w_j f_{j+i}, \quad (6.8)$$

where

$$w_j = \begin{cases} ((|j|+2)^{2-\alpha}(2-\lambda_2) + (|j|+1)^{2-\alpha}(4\lambda_2-6) + |j|^{2-\alpha}(6-6\lambda_2) \\ \quad + (|j|-1)^{2-\alpha}(4\lambda_2-2) + (|j|-2)^{2-\alpha}(-\lambda_2))c_-, & \text{for } j \leq -2; \\ (3^{2-\alpha}(2-\lambda_2) + 2^{2-\alpha}(4\lambda_2-6) - 6\lambda_2+6)c_- + (2-\lambda_2)c_+, & \text{for } j = -1; \\ (2^{2-\alpha}(2-\lambda_2) + 4\lambda_2-6)(c_- + c_+), & \text{for } j = 0; \\ (3^{2-\alpha}(2-\lambda_2) + 2^{2-\alpha}(4\lambda_2-6) - 6\lambda_2+6)c_+ + (2-\lambda_2)c_-, & \text{for } j = 1; \\ ((j+2)^{2-\alpha}(2-\lambda_2) + (j+1)^{2-\alpha}(4\lambda_2-6) + j^{2-\alpha}(6-6\lambda_2) \\ \quad + (j-1)^{2-\alpha}(4\lambda_2-2) + (j-2)^{2-\alpha}(-\lambda_2))c_+, & \text{for } j \geq 2. \end{cases}$$

6.2.5 Predictor-corrector method

This is a well-known numerical method developed initially for solving first-order ordinary differential equations. It proceeds in two steps. Firstly, the *prediction step* calculates a rough approximation of the desired solution. Secondly, the *corrector step* refines the initial approximation using another means.

We demonstrate this method concerning the nonlinear fractional equation

$${}^v\mathbf{D}_x f(x) = F(x, f(x)), \quad v > 0, \quad x \in [0, l].$$

On given

$$f^{(k)}(0+) = c_k, \quad k = 0, 1, 2, \dots, n-1,$$

this equation is equivalent to the integral equation

$$f(x) = \frac{1}{\Gamma(v)} \int_0^x (x - \xi)^{v-1} F(\xi, f(\xi)) d\xi + \phi(x) \quad (6.9)$$

with the free term

$$\phi(x) = \sum_{k=0}^{n-1} c_k \frac{1}{k!} x^k.$$

Introducing the notation $g(\xi) = F(\xi, f(\xi))$ and the uniform grid $x_m = mh$, $m = 0, 1, 2, \dots, M$, we represent the integral term $J(x)$ at $x = x_{m+1}$ in the form of sum over the steps

$$J(t_{m+1}) \equiv J_{m+1} = \int_0^{x_{m+1}} (x_{m+1} - \xi)^{v-1} g(\xi) d\xi = \sum_{j=0}^m \int_{x_j}^{x_{j+1}} (x_{m+1} - \xi)^{v-1} g(\xi) d\xi.$$

Using the rough stepwise approximation of the integrand

$$g(\xi) \approx \hat{g}^P(\xi) = g_j \equiv g(x_j), \quad x_j < \xi < x_{j+1},$$

we obtain the predictor-approximation of the integral

$$\hat{J}_{m+1}^P = \frac{h^v}{v} \sum_{j=0}^{m+1} c_{m+1,j} g_j,$$

where

$$c_{m+1,j} = (m - j + 1)^{v-1} - (m - j)^{v-1}.$$

The corresponding solution obeys the equation

$$\hat{f}_{m+1}^P = \sum_{k=0}^{n-1} \frac{1}{k!} c_k + \frac{1}{\Gamma(v)} \sum_{j=0}^m c_{m+1,j} F(x_j, \hat{f}_j^P). \quad (6.10)$$

6.2.6 The linear scheme

For corrector-representation, we use the linear (trapezoidal) approximation of $g(\tau)$ on each segment:

$$\begin{aligned} \hat{J}_{m+1}^C &= \sum_{j=0}^m \int_{jh}^{(j+1)h} (x_{m+1} - \xi)^{v-1} \hat{g}(\xi) d\xi, \\ \hat{g}(\xi) &= g_j + \frac{\xi - x_j}{h} [g_{j+1} - g_j], \quad g_j \equiv g(x_j). \end{aligned}$$

Rearrangement of the terms yields

$$\hat{f}^C(x_{m+1}) = \sum_{j=0}^m [A_j g_j + B_j (g_{j+1} - g_j)], \quad (6.11)$$

where

$$A_j = \int_{x_j}^{x_{j+1}} (x_{m+1} - \xi)^{v-1} d\xi = \frac{h^v}{v} [(m-j+1)^v - (m-j)^v]$$

and

$$\begin{aligned} B_j &= \int_{x_j}^{x_{j+1}} (x_{m+1} - \xi)^{v-1} [(\xi - x_j)/h] d\xi \\ &= \frac{h^v}{v} (m-j+1) [(m-j+1)^v - (m-j)^v] - \frac{h^v}{v+1} [(m-j+1)^{v+1} - (m-j)^{v+1}]. \end{aligned}$$

It is convenient to separate the first and the last terms in Eq. (6.11) and rearrange this sum again:

$$\begin{aligned} \hat{f}_{m+1}^C &= \frac{h^v}{v(v+1)} \sum_{j=0}^{m+1} C_{m+1,j} g_j, \\ C_{m+1,j} &= \begin{cases} m^{v+1} - (m-v)(m+1)^v, & j=0; \\ (m+2-j)^{v+1} - 2(m+1-j)^{v+1} + (m-j)^{v+1}, & 1 \leq j \leq m-1; \\ 1, & j=m+1. \end{cases} \end{aligned}$$

So we obtain the formula

$$\hat{f}_{n+1} = \sum_{k=0}^{n-1} \frac{1}{k!} c_k x_{n+1}^k + \frac{h^v}{\Gamma(v+2)} \sum_{j=0}^m C_{m+1,j} F(x_j, \hat{f}_j) + \frac{h^v}{\Gamma(v+2)} F(x_{m+1}, \hat{f}_{m+1}).$$

Replacing $\hat{f}_{m+1}$ by $\hat{f}_{m+1}^P$ in the latter term, we arrive at the second equation of the predictor-corrector method:

$$\hat{f}_{m+1} = \sum_{k=0}^{n-1} \frac{1}{k!} c_k t_{m+1}^k + \frac{h^v}{\Gamma(v+2)} \sum_{j=0}^m C_{m+1,j} F(x_j, \hat{f}_j) + \frac{h^v}{\Gamma(v+2)} F(x_{m+1}, \hat{f}_{m+1}^P). \quad (6.12)$$

The improved system (6.11)–(6.12) completely describes the solving process of the fractional equation under consideration. As one can show,

$$\max |f_j - \hat{f}_j| = O(h^p), \quad p = \min\{2, 1+v\}.$$

The details of the algorithms can be found in (Diethelm et al., 2002; Diethelm, 2003).

6.2.7 The quadratic and cubic schemes

In the quadratic scheme, $m \equiv m'$ is taken as an even number $2m$, and $f(x)$ and $F(x, f(x))$ are approximated over two adjacent parts using quadratic polynomials. Assume that $f_j, j = 1, \dots, 2m$ have already been computed. Using Eq. (6.9), the expressions for f_{2m+1} and f_{2m+2} are given as

$$f_{2m+1} = \phi_{2m+1} + \frac{1}{\Gamma(v)} \int_0^{2mh} ((2m+1)h - t')^{v-1} F(\xi, f(\xi)) d\xi \quad (6.13)$$

$$+ \frac{1}{\Gamma(v)} \int_{2mh}^{(2m+1)h} ((2m+1)h - \xi)^{v-1} F(\xi, f(\xi)) d\xi$$

and

$$f_{2m+2} = \phi_{2m+2} + \frac{1}{\Gamma(v)} \int_0^{2mh} ((2m+2)h - \xi)^{v-1} F(\xi, f(\xi)) d\xi \quad (6.14)$$

$$+ \frac{1}{\Gamma(v)} \int_{2mh}^{(2m+2)h} ((2m+2)h - \xi)^{v-1} F(\xi, f(\xi)) d\xi.$$

Since $f_j, j = 0, \dots, 2m$ are known, the first integrals in both Eqs. (6.13) and (6.14) can be computed explicitly. To compute the second integral in Eq. (6.13), $F(x, f(x))$ is approximated over $[2mh, (2m+2)h]$ in terms of g_{2m}, g_{2m+1} , and g_{2m+2} , as

$$F(x, f(x)) = \sum_{j=0}^2 p_j(x) g_{2m+j} \quad (6.15)$$

where $p_j(x)$, $j = 0, 1$, and 2 are the quadratic interpolating polynomials, which is 1 at node $2m+j$ and 0 at two other nodes. Substituting Eq. (6.15) into Eq. (6.14), we obtain f_{2m+2} in terms of F_{2m+1} and F_{2m+2} . Note that F_{2m} is not included here as it can be computed directly from f_{2m} . To compute the second integral in Eq. (6.13), $F(t, f(t))$ is approximated over $[2mh, (2m+1)h]$ in terms of $F_{2m}, F_{2m+1/2}$ and F_{2m+1} using the procedure similar to given by Eq. (6.15). Then $f_{2m+1/2}$ is expressed in terms of F_{2m}, F_{2m+1} , and F_{2m+2} , and we obtain two equations in terms of two unknowns f_{2m+1} and f_{2m+2} , which can be solved using standard numerical methods (see details in (Agrawal, 2004; Agrawal and Kumar, 2007)).

In the cubic scheme is taken as a multiple of 3, and $F(x, f(x))$ are approximated over three adjacent parts using cubic polynomials, and expressions are generated for f_{3m+1}, f_{3m+2} and f_{3m+3} in terms of F_{3m+1}, F_{3m+2} and F_{3m+3} . Further details can be found in (Kumar and Agrawal, 2006).

6.2.8 The collocation spline method

A *spline* $S(x)$ is a piecewise-polynomial function on an interval $[a, b]$ composed of J ordered disjoint subintervals x_{j-1}, x_j , with $a = x_0 < x_1 < \dots < x_{J-1} < x_J = b$:

$$S(x) = \begin{cases} P_1(x), & x_0 \leq x < x_1, \\ P_2(x), & x_1 \leq x < x_2, \\ \dots & \dots \\ P_J(x), & x_{J-1} \leq x < x_J. \end{cases}$$

The highest order of the polynomials $P_j(x)$, $j = 1, \dots, J$ is called the *order of the spline*. The intrinsic property of the spline is its continuity and $N - 1$ times differentiability also at the inner knot dots x_j , $j = 1, \dots, J - 1$, that is, the spline polynomials should satisfy the conditions

$$P_j^{(i)}(x_j) = P_{j+1}^{(i)}(x_j),$$

for all j , $-1 \leq j \leq J - 1$ ($j = -1$ denotes the discontinuous splines) and all i , $0 \leq i \leq N - 1$.

The *collocation method* is a method for numerical solution of ordinary differential equation, partial differential equations and integral equations. It is based on using a finite-dimensional space of candidate solutions (usually, polynomials) and a number of points (*collocation points*) in the integration domain to select that solution which obeys the given equation at the collocation points. We will demonstrate it on the first step interval $[x_0, x_0 + h]$ of numerical solution of the equation $f'(x) = F(f(x))$:

$$f(x+h) - f(x) = \int_x^{x+h} F(f(x)) dx.$$

Let $0 \leq c_1 < \dots < c_k < \dots < c_K \leq 1$ be collocation points, then the collocation polynomial $u(x)$ is a polynomial of degree K satisfying conditions

$$u(x_0) = f(x_0), \quad u'(x_0 + c_k h) = F(u(x_0 + c_k h)), \quad k = 1, \dots, K,$$

and the numerical solution $f(x_1) = u(x_0 + h)$. Observe that the polynomial $u(x)$ passes through $f(x_0)$ and agrees with the initial equation at K nodes on $[x_0, x_1]$. It is not difficult to understand a continuation of the process.

6.2.9 The GMMP method

Considering the time-fractional subdiffusion equation, Gorenflo, Mainardi, Moretti and Paradisi (2002) discretize space and time by grid points and time-instant as follows

$$x_j = jh, \quad h > 0, \quad j = 0, \pm 1, \pm 2, \dots; \quad t_n = n\theta, \quad \theta > 0, \quad n = 0, 1, 2, \dots$$

The dependent variable is then also discretized,

$$\int_{x_j-h/2}^{x_j+h/2} f(x, t_n) dx \approx hf_j(t_n),$$

and the initial equation

$${}_0^v D_t f(x, t) = D_x^2 f(x, t)$$

becomes

$$\sum_{k=0}^{n+1} (-1)^k \binom{v}{k} \frac{f_j(t_{n+1-k}) - f_j(0)}{\theta^v} = \frac{f_{j+1}(t_n) - 2f_j(t_n) + f_{j-1}(t_n)}{h^2}, \quad 0 < v \leq 1.$$

Some rearrangement forms it as follows:

$$f_j(t_{n+1}) = b_n f_j(t_0) + \sum_{k=1}^n c_k f_j(t_{n+1-k}) + \mu [f_{j+1}(t_n) - 2f_j(t_n) + f_{j-1}(t_n)],$$

with the scale parameter

$$\mu = \frac{\theta^v}{h^2},$$

and coefficients

$$c_k = (-1)^{k+1} \binom{v}{k}, \quad \sum_{k=1}^{\infty} c_k = 1;$$

and

$$b_n = \sum_{k=n+1}^{\infty} c_k = \sum_{k=0}^n (-1)^k \binom{v}{k}.$$

This equation provides the transition rule from t_n to t_{n+1} for all $n = 0, 1, 2, \dots$ and $j = 0, \pm 1, \pm 2, \dots$:

$$\begin{aligned} f_j(t_1) &= (1 - 2\mu)f_j(t_0) + \mu[f_{j+1}(t_0) + f_{j-1}(t_0)], \\ f_j(t_2) &= b_1 f_1(t_0) + (v - 2\mu)f_j(t_1) + \mu[f_{j+1}(t_1) + f_{j-1}(t_1)], \\ f_j(t_{n+1}) &= b_n f_j(t_0) + \sum_{k=2}^n c_k f_j(t_{n+1-k}) + (v - 2\mu)f_j(t_n) + \mu[f_{j+1}(t_n) + f_{j-1}(t_n)]. \end{aligned}$$

Non-negativity and conservativity of this scheme proved in (Gorenflo et al., 2002) means that the scheme can be interpreted as a *redistribution scheme* of clumps $f_j(t_n)$. For closer knowledge of such aspects, the articles (Gorenflo, 1978) can be recommended.

6.2.10 The CL method

Ciesielski and Leszczynski (2003) have described a numerical algorithm on the base of the *finite element method*. Considering the same subdiffusion time-fractional equation, they semi-discretize (over space) the equation through the Galerkin method

and presented it in the matrix form. The main difference between GMMP and CL presentations is in the form of the discretized Gerasimov-Caputo derivative, namely

$${}_0^v D_t f_j(t_n) \approx \theta^{-v} \sum_{k=0}^n w_k [f_j(t_{n-k}) - f_j(0) - f'_j(0)t_n], \quad t_n = n\theta$$

for the GMMP method and

$${}_0^v D_t f_j(t_n) \approx \theta^{-v} \sum_{k=0}^n w_k f_j(t_{n-k}) - \frac{f_j(0)}{\Gamma(1-v)} t_n^{-v} - \frac{f'_j(0)}{\Gamma(2-v)} t_n^{1-v}$$

for the CL method (note that if $v < 1$, the last term with $f'_j(0)$ must be deleted in both formulas).

6.2.11 The YA method

Yuste and Acedo (2004) offer the Forward Time Centered Space (FTCS) difference scheme for solving the fractional subdiffusion equation

$$\frac{\partial f(x,t)}{\partial t} = K_v {}_0 D_t^{1-v} \frac{\partial^2 f(x,t)}{\partial t^2}.$$

Representing its finite-difference version in the form

$$\frac{f_j(t_{n+1}) - f_j(t_n)}{\theta} = K_v {}_0 D_t^{1-v} \frac{f_{j-1}(t_n) - 2f_j(t_n) + f_{j+1}(t_n)}{h^2},$$

inserting the G-L definition of the fractional derivative with step τ into this equation and rearranging its terms, the authors get the explicit FTCS difference scheme

$$f_j(t_{n+1}) = f_j(t_n) + S_v \sum_{k=0}^n w_k^{(1-v)} [f_{j-1}(t_{n-k}) - 2f_j(t_{n-k}) + f_{j+1}(t_{n-k})],$$

where $S_v = K_v \theta / [\tau^{1-v} h^2]$ and

$$w_k^{(\alpha)} = (-1)^k \binom{\alpha}{k}$$

as above. The authors underline that in this scheme, $f_j(t_{n+1})$, $t_{n+1} = n\theta$ is given *explicitly* in terms of all the previous states $f_j(t_k)$, $k = 0, 1, 2, \dots, n$. Because the estimates of $f_j(t_n)$ are made at the times $n\theta$ and because the evaluation of ${}_0 D_t^{1-v} f_j(t)$ requires knowing $f_j(t)$ at the times $m\tau$, $m = 0, 1, 2, \dots$, it is natural to choose $\tau = \theta$. Then

$$S_v = K_v \frac{\theta^v}{h^2}.$$

Comparison of these three numerical methods undertaken by Murillo and Yuste (2009) has shown that the three methods have the same stability and GMMP and YA schemes are equivalent, that is always lead to the same results for the fractional subdiffusion problem. Accuracy of the C-L method in considered numerical examples occurred a little be lower.

6.2.12 Galerkin's method

In works (Rozmej and Karczewska, 2005; Bandrowski et al., 2010), a numerical method using a finite subspace of an infinite basis in the time variable (Galerkin method) and discretization in space variables is developed. In the final step, a large-scale system of linear equations with a non-symmetric matrix is solved with the use of an iterative method. We confine ourselves to the first stage sketch only.

The fractional diffusion equation with time-derivative of G-C form

$${}_0^v D_t f(x, t) = \Delta f(x, t), \quad v \in (1, 2), \quad x \in \mathbb{R}^d \quad (6.16)$$

is represented in the time-integral form

$$f(x, t) = f(x, 0) + \int_0^t \Phi_v(t - \tau) \Delta f(x, \tau) d\tau. \quad (6.17)$$

Let $\{\phi_j : j = 1, 2, \dots\}$ be a set of real ortonormal functions on the interval $[0, t]$, spanning a Hilbert space H with an inner product

$$(f, g) = \int_0^t W(\tau) f(\tau) g(\tau) d\tau,$$

where W is a weight function. Approximating $f(x, t)$ by the superposition of the first n basis functions,

$$f_n(x, t) = \sum_{k=1}^n c_k(x) \phi_k(t), \quad (6.18)$$

and inserting this sum into Eq. (6.17), one obtains

$$f_n(x, t) = f_n(x, 0) + \int_0^t \Phi_v(t - \tau) \Delta f(x, \tau) d\tau + \varepsilon_n(x, t), \quad (6.19)$$

where the approximation error in case $d = 1$ is expressed as

$$\varepsilon_n(x, t) = \sum_{k=1}^n c_k(x) \phi_k(t) - f(x, 0) - \int_0^t \Phi_v(t - \tau) \sum_{k=1}^n D_x^2 [c_k(x) \phi_k(\tau)] d\tau. \quad (6.20)$$

The *Galerkin approximation* of the solution of Eq. (6.17) is a function $f_n \in H_n$ such that

$$(\varepsilon_n, \phi_k) = 0, \quad k = 1, 2, \dots, n.$$

From Eqs. (6.17) and (6.20), it follows that for any $j = 1, 2, \dots, n$

$$g_j(x) = c_j(x) - \sum_{k=1}^n a_{jk} \mathbf{D}_x^2 c_k(x), \quad j = 1, 2, \dots, n, \quad (6.21)$$

where

$$g_j(x) = f(x, 0) \int_0^t W(\tau) \phi_j(\tau) d\tau$$

and

$$a_{jk} = \int_0^t W(\tau) \phi_j(\tau) \left[\int_0^\tau \Phi_v(\tau-s) \phi_k(s) ds \right] d\tau.$$

The set of coupled equations (6.21) for the space-dependent coefficients $c_k(x)$ is solved by standard, centered three-point finite difference approximation to the second derivative (Laplacian) (Bandrowski et al., 2010):

$$g_j(x_i) = c_j(x_i) + \frac{1}{h^2} \sum_{k=1}^n a_{jk} [-c_k(x_i - h) + 2c_k(x_i) - c_k(x_i + h)],$$

where $j = 1, 2, \dots, n$; $i = 1, 2, \dots, m$.

Details of the next following computations and some numerical examples can be found in the cited works.

6.2.13 Equation with the Riesz fractional derivatives

In (Shen et al., 2008) a numerical solution of the one-dimensional *Riesz fractional advection-dispersive equation* (RFADE)

$$\begin{aligned} \frac{\partial f(x, t)}{\partial t} &= [A\Delta^{\alpha/2} + B\Delta^{\beta/2}] f(x, t), \quad x \in \mathbb{R}, \quad t \in \mathbb{R}^+, \\ f(x, 0) &= g(x), \quad x \in \mathbb{R}, \end{aligned}$$

with $\alpha \in (0, 1)$ and $\beta \in (1, 2)$ has been presented. It is based on the finite-difference approximation of READ problem by introducing a spatial grid $\dots < x_{i-1} < x_i < x_{i+1} < \dots$ with the step $h = x_k - x_{k-1}$. The key point of this approach is the discretization of the Riesz fractional derivative. For $\alpha \in (0, 1)$,

$$\begin{aligned} 2 \cos(\alpha\pi/2) \Delta^{\alpha/2} f(x_i) &= -[{}_{-\infty}\mathbf{D}_x^\alpha f(x_i) + {}_x\mathbf{D}_\infty^\alpha f(x_i)] \\ &\approx -\frac{1}{h^\alpha} \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} [f_{i+k} - f_{i-k}] \\ &= -\frac{1}{h^\alpha} \left[2f_i - \sum_{k=-\infty, k \neq 0}^{\infty} (-1)^k \binom{\alpha}{k} f_{i+k} \right], \end{aligned}$$

and finally we get

$$\Delta^{\alpha/2} f(x_i) \approx \frac{1}{h^\alpha} \sum_{k=-\infty}^{\infty} w_k^{(\alpha)} f_{i+k},$$

where $f_i \equiv f(x_i)$ and

$$\begin{cases} w_0^{(\alpha)} = -[\cos(\alpha\pi/2)]^{-1}, \\ w_{\pm k}^{(\alpha)} = (-1)^{(k+1)} [2\cos(\alpha\pi/2)]^{-1} \binom{\alpha}{k}, \quad k = 1, 2, \dots \end{cases}$$

For the next interval of orders, $\beta \in (1, 2]$,

$$\Delta^{\beta/2} f(x_i) \approx \frac{1}{h^\beta} \sum_{k=-\infty}^{\infty} w_k^{(\beta)} f_{i+k},$$

where

$$\begin{cases} w_0^{(\beta)} = [\cos(\beta\pi/2)]^{-1} \binom{\beta}{1}, \\ w_{\pm 1}^{(\beta)} = -[2\cos(\beta\pi/2)]^{-1} \left[\binom{\beta}{2} + 1 \right], \\ w_{\pm k}^{(\beta)} = (-1)^k [2\cos(\beta\pi/2)]^{-1} \binom{\beta}{k+1}, \quad k = 2, 3, \dots \end{cases}$$

Then, the authors of (Shen et al., 2008) introduce a temporal grid $0 = t_0 < t_1 < \dots < t_n < \dots$ with the grid step $\tau = t_{n+1} - t_n$ and arrive at the basic difference scheme for numerical calculating $f_{i,n} \equiv f(x_i, t_n)$:

$$\frac{f_{i,n+1} - f_{i,n}}{\tau} = \frac{A}{h^\alpha} \sum_{k=-\infty}^{\infty} w_k^{(\alpha)} f_{i+k,n} + \frac{B}{h^\beta} \sum_{k=-\infty}^{\infty} w_k^{(\beta)} f_{i+k,n}.$$

At the end of the cited work, its authors apply this method to the process in a finite spatial domain $[0, l]$. Dividing the domain into N subdomains with the step $h = l/N$ and introducing the Dirichlet boundary conditions assuming the same values of function $f(x, t)$ outside the segment limits as the values predicted on the boundary nodes x_0 and x_N ,

$$f(x_k, t) = \begin{cases} f(x_0, t) = 0, & k < 0; \\ f(x_N, t) = 0, & k > N, \end{cases}$$

leads to the following numerical scheme:

$$\frac{f_{i,n+1} - f_{i,n}}{\tau} = \frac{A}{h^\alpha} \sum_{k=-i}^{N-i} w_k^{(\alpha)} f_{i+k,n} + \frac{B}{h^\beta} \sum_{k=-i}^{N-i} w_k^{(\beta)} f_{i+k,n}.$$

The authors investigated a discrete random walk model based on the explicit finite-difference approximation and proved that the random walk model belongs to the domain of attraction of the corresponding stable distribution. Stability and convergence of these numerical methods are discussed. Some numerical examples are

given, to show that the numerical results are in good agreement with their theoretical analysis.

6.2.14 Equation with Riesz-Feller derivatives

Let us come back to the work (Ciesielski and Leszczynski, 2005), namely, to its second part devoted to numerical approximation of the equation on the segment $[L, R]$,

$$\frac{\partial f(x, t)}{\partial t} = K_\alpha {}_0D_x^{\alpha, \theta} f(x, t), \quad (6.22)$$

under the Dirichlet boundary conditions

$$f(x, t) = \begin{cases} g_L(t), & \text{for } x < L; \\ g_R(t), & \text{for } x > R. \end{cases}$$

The finite difference version of this problem for $L \leq x \leq R$, $h = (R - L)/N$ is

$$\frac{f(x_i, t_{j+1}) - f(x_i, t_j)}{\Delta t} = \frac{K_\alpha}{h^\alpha} \sum_{j=-\infty}^{\infty} f(x_{i+k}, t_j) w_k^{(\alpha)},$$

with

$$f_{x_k, t} = \begin{cases} g_L(t), & \text{for } x_k < L, \text{ that is } k < 0; \\ g_R(t), & \text{for } x_k > R, \text{ that is } k > N. \end{cases}$$

On the base of Sect. 6.2.4, containing description of the R-F derivative numerical approximation, we have:

$$D_x^{\alpha, \theta} f(x_i, t) \approx \frac{1}{h^\alpha} \left[\sum_{k=-i}^{N-i} f(x_{i+k}, t) w_k^\alpha + g_L(t) W_L i^\alpha + g_R(t) W_R i^\alpha \right], \quad i = 1, 2, \dots, N-1,$$

where

$$W_L i^\alpha = \sum_{k=-\infty}^{-i-1} w_k^\alpha = \frac{1}{\Gamma(4-\alpha)} [(i+2)^{3-\alpha} - 3(i+1)^{3-\alpha} + 3i^{3-\alpha} - (i-1)^{3-\alpha}] c_-,$$

$$\begin{aligned} W_R i^\alpha &= \sum_{k=N-i+1}^{\infty} w_k^\alpha \\ &= \frac{1}{\Gamma(4-\alpha)} [(N-i+2)^{3-\alpha} - 3(N-i+1)^{3-\alpha} \\ &\quad + 3(N-i)^{3-\alpha} - (N-i-1)^{3-\alpha}] c_+. \end{aligned}$$

After some simplifications, finite-difference representation of the time-derivative, the authors of the cited work have used this algorithm for numerical solution of the problem under consideration.

6.3 Monte Carlo technique

6.3.1 The inverse function method

Method of statistical modeling often called the *Monte Carlo method* is based on randomization of a problem under consideration that is on transformation of its solution, say a number x , into a random number X so that

$$\langle X \rangle = x,$$

and simulation of large number of its realizations Y_i , $m = 1, 2, \dots, n$. According to the law of large numbers,

$$\hat{x} \equiv \frac{1}{n} \sum_{i=1}^n X_i \stackrel{P}{=} x, \quad n \rightarrow \infty.$$

The general approach to generation of random variables with a given distribution law is the *method of an inverse cumulative probability function*. It is based on the following theorem.

Theorem on inverse function: Let U be a random variable distributed uniformly on the interval $(0, 1)$,

$$F_U(u) = \mathbf{P}(U < u) = u, \quad 0 < u < 1,$$

and let $F(x)$ be a monotonically increasing function on (a, b) possessing the derivative and the limits $F(x) \rightarrow 0$, $x \rightarrow a$ and $F(x) \rightarrow 1$, $x \rightarrow b$ (the cases $a = -\infty$ and/or $b = \infty$ are allowed). Then the inverse function $F^{-1}(u)$, $u \in (0, 1)$ exists, and the random variable

$$X = F^{-1}(U) \tag{6.23}$$

is distributed on the interval (a, b) with the distribution

$$\mathbf{P}(X < x) = F(x).$$

Proof: Since the function $F(x)$ is strictly increasing and $F_U(x) = x$, then

$$\mathbf{P}(X < x) = \mathbf{P}(F^{-1}(U) < x) = \mathbf{P}(U < F(x)) = F(x),$$

and we arrive at Eq. (6.23).

To illustrate some applications of the theorem, we consider a few simple examples.

(i) *Uniformly distributed angle Φ* . Its probability distribution function is

$$p_{\Phi}(\phi) = \frac{1}{\pi}, \quad \phi \in (-\pi/2, \pi/2),$$

consequently

$$F_{\Phi}(\phi) = \frac{1}{\pi}(\phi + \pi/2), \quad \phi = F^{-1}(u) = \pi(u - 1/2),$$

thus

$$\Phi = \pi(U - 1/2).$$

(ii) *Inverse power variable P* . Its distribution function is of the form

$$F_P(x) = \begin{cases} 0, & x < \varepsilon; \\ 1 - (x/\varepsilon)^{-\alpha}, & x \geq \varepsilon. \end{cases}$$

The theorem yields

$$P = \varepsilon U^{-1/\alpha},$$

because the random variables U and $1 - U$ are of the same distribution:

$$U \stackrel{d}{=} 1 - U.$$

(iii) *Exponential variable E* . This variable has the distribution function

$$F_E(x) = 1 - e^{-x},$$

hence

$$E = -\ln U.$$

The following algorithm corresponds to some simple cases of stable variables.

(iv) *Cauchy variable $C = S(1, 0)$* . Its distribution function is of the form

$$F_C(x) = \frac{1}{2} \int_{-\infty}^x \frac{d\xi}{\pi(1 + \xi^2)} = \frac{1}{\pi} \arctan \xi \Big|_{-\infty}^x.$$

By virtue of the theorem on inverse function, we obtain

$$C = \tan \Phi.$$

(v) *Gaussian variable (with the variance equal 2) $G = S(2, 0)$* . In this case, the distribution function is

$$F_G(x) = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^x e^{-\xi^2/4} d\xi$$

and its inversion function cannot be expressed in an elementary way. But the root of the sum of two squared independent Gaussian variables

$$R = \sqrt{G_1^2 + G_2^2}$$

is distributed with probability

$$F_R(r) = \mathbf{P}(R < r) = \frac{1}{4\pi} \iint_{x^2+y^2 < r^2} dx dy e^{-(x^2+y^2)/4} = \int_0^r e^{-r^2/4} d(r^2/4) = 1 - e^{-r^2/4},$$

hence $R = 2E^{1/2}$, and

$$G = 2E^{1/2} \cos \Phi \stackrel{d}{=} 2E^{1/2} \sin \Phi. \quad (6.24)$$

These two random variables are mutually independent.

(vi) *Smirnov-Lévy variable* $S = S_+(1/2)$. Let us consider the non-negative random variable G^{-2} . Its cumulative probability function is

$$F_{G^{-2}}(x) = \mathbf{P}(G^2 > 1/x) = 2\mathbf{P}(G > 1/\sqrt{x}) = \frac{1}{\sqrt{\pi}} \int_{1/\sqrt{x}}^{\infty} e^{-x^2/4} dx, \quad x > 0.$$

Differentiating this expression with respect to x , we obtain

$$p_{G^{-2}}(x) = \frac{1}{2\sqrt{\pi}} \exp\left\{-\frac{1}{4x}\right\} x^{-3/2} \equiv g_+(x; 1/2), \quad x > 0.$$

Thus, we obtain

$$S_+(1/2) = G^{-2} = [4E \cos^2 \Phi]^{-1}. \quad (6.25)$$

6.3.2 Density estimation

If the desired value is linearly connected with the solution $p(x)$ of some equation having a probability density sense,

$$J = \int_G h(x)p(x)dx, \quad p(x) \geq 0, \quad \int_G p(x)dx,$$

then its estimation is realized as

$$\hat{J} = \frac{1}{n} \sum_{i=1}^n h(X_i),$$

where $X_1, X_2, \dots, X_n$ are results of independent Monte Carlo simulations of random variables distributed according to solution $p(x)$ of the equation under consideration. When the problem is to find the function $p(x)$ itself, the histogram technique is usually used:

$$\hat{p}(x_k) = \frac{1}{n} \sum_{i=1}^n k(x_k, X_i),$$

where

$$k(x_k, y) = \begin{cases} (\Delta x_k)^{-1}, & x, y \in \Delta x_k, \\ 0, & \text{otherwise.} \end{cases}$$

Here, $y \in \Delta x_k$ means that point y belongs to k th bin of width Δx_k .

Being a popular tool for working with bounded distributions, histograms possess some unwanted properties: they are not smooth, depend on end points of bins, depend on width of bins. The histogram method meets some more troubles with unbounded distributions. Long tails of power type, $x^{-\alpha}$ aggravate the situation: for any n there exists a region with a poor statistics and correspondingly high level of statistical error, although we are often interested namely in this region. The increasing role of tails in case of small values of α becomes clear from Table 6.1.

Table 6.1 Probabilities $P(S(\alpha, 0) > x)$

α	$P(S(\alpha, 0) > 0.3)$	$P(S(\alpha, 0) > 3)$	$P(S(\alpha, 0) > 30)$
2.0	0.416	0.017	0.000
1.8	0.416	0.029	0.000
1.6	0.415	0.044	0.000
1.4	0.414	0.060	0.002
1.2	0.412	0.080	0.005
1.0	0.408	0.103	0.011
0.8	0.399	0.130	0.023
0.6	0.383	0.164	0.048
0.4	0.362	0.205	0.096
0.2	0.339	0.256	0.183
0.1	0.327	0.285	0.245
0.01	0.316	0.313	0.309

In connection with this peculiarity of long-tail distributions, some improvements of the estimation have been worked out. The basic idea of the improvement consists in replacing the rectangular response function $k(x, y)$ by a smooth function which allows to avoid discontinuity attributable to ordinary histograms and distribute the statistical error more uniformly on long tails. One kind of such estimations called the *Parzen-Rosenblatt window* or *kernel density estimator* is based on the representation

$$\hat{p}(x) = \frac{1}{n} \sum_{i=1}^n b_n^{-1} K((x - X_i)b_n^{-1}),$$

where the *kernel* $K(x)$ is a smooth absolutely integrable function satisfying the normalizing condition

$$\int_{-\infty}^{\infty} K(x)dx = 1.$$

The sequence of constants b_n is such that $b_n \rightarrow 0$ and $nb_n \rightarrow \infty$ as $n \rightarrow \infty$. (Rosenblatt, 1956; Parzen, 1962). It is known that estimator

$$\hat{p}(x) = \frac{1}{n} \sum_{j=1}^n b_n^{-1} K((x - X_j)b_n^{-1})$$

is asymptotically unbiased, consistent and asymptotically normal, and

$$\langle \hat{p}_n(x) \rangle - p(x) = p''(x)b_n^2/2 + o(b_n^2)$$

and

$$\text{Var} \hat{p}(x) = (nb_n)^{-1} p(x) \int_{-\infty}^{\infty} K^2(x)dx + o((nb_n)^{-1}).$$

In (Janicki and Weron, 1994), five various kernels have been investigated including rectangular, triangular, Gaussian, and others in application to stable random variables. The kernel

$$K(x) = \begin{cases} \frac{3}{4\sqrt{5}} \left(1 - \frac{x^2}{5}\right), & |x| \leq \sqrt{5}, \\ 0, & |x| > \sqrt{5}. \end{cases}$$

turned out to be optimal. Theoretical analysis of asymptotic behavior of $\hat{p}(x)$ as $n \rightarrow \infty$ has shown that the best choice of b_n is

$$b_n = b_1 n^{-1/5}.$$

The other class of estimations called *projection estimations* or *Chentsov's (1962) density estimator* is generated by kernels

$$K(x, y) = \sum_{k=1}^m w(y) \phi_k(x) \phi_k(y),$$

that is,

$$\hat{p}(x) = \sum_{k=1}^m \hat{a}_k \phi_k(x), \quad \hat{a}_k = \frac{1}{n} \sum_{i=1}^n w(X_i) \phi_k(X_i),$$

where $\{\phi_k(x)\}$ is an orthogonal with some weight $w(x)$ function system.

These kernel estimators are equally applicable both to simulated and observed data. But in the first case, such a favorable situation may occur when the sought for pdf $p(x)$ is representable as

$$p(x) = \int v(x, y) q(y) dy,$$

where $q(y)$ is a pdf again. Denoting this random variable by Y , we are in a position to estimate $p(x)$ avoiding any biases caused by involving smoothing or averaging over some small region kernels:

$$\hat{p}(x) = \frac{1}{n} \sum_{i=1}^n v(x, Y_i).$$

Following Monte Carlo terminology, we will call this way the *local estimation*. Using the same set of random variable Y_j for all values of arguments $x_1, x_2, \dots, x_k$, we arrive at the *method of correlated trials* which provides smoothed results without violence of unbiasedness.

6.3.3 Simulation of stable random variables

6.3.3.1 One-sided stable distributions

Three random variables, Gaussian, Cauchy, and Smirnov-Lévy random variables, described above, are stable variables with $\alpha = 2$, 2 and $1/2$. In this subsection, we consider algorithms for simulating other stable variables. As shown by Ibragimov and Chernin (1959), one-sided stable laws

$$g_+(x; \alpha) \equiv g(x; \alpha, 1) = (2\pi)^{-1} \int_{-\infty}^{\infty} \exp[-|k|^\alpha \exp(-i\alpha(\pi/2) \operatorname{sign} k)] dk, \quad 0 < \alpha \leq 1$$

can be transformed into a non-oscillating function integral being more convenient for numerical calculations:

$$G_+(x; \alpha) \equiv G(x; \alpha, 1) = \mathbf{P}(S_+(\alpha) < x) = \int_0^1 \exp[-x^{-\alpha/(1-\alpha)} A(u; \alpha)] du,$$

where

$$A(u; \alpha) = \frac{\sin((1-\alpha)\pi u) [\sin(\alpha\pi u)]^{\alpha/(1-\alpha)}}{(\sin \pi u)^{1/(1-\alpha)}}, \quad 0 < u < 1.$$

Evidently, this integral can be interpreted in terms of conditional probabilities and the total probability rule:

$$G_+(x; \alpha) = \int_0^1 G_+(x; \alpha|u) du,$$

$$G_+(x; \alpha|u) \equiv \mathbf{P}(S_+(\alpha) < x|U = u) = \exp[-x^{-\alpha/(1-\alpha)} A(u; \alpha)]$$

(observe that it is a nonnegative function monotonically increasing with x from 0 to 1). Replacing t by $x^{-\alpha/(1-\alpha)} A(u; \alpha)$ in the relation

$$e^{-t} = \mathbf{P}(E > t),$$

we see that

$$\exp\{-x^{-\alpha/(1-\alpha)}A(u;\alpha)\} = \mathbf{P}(E > x^{-\alpha/(1-\alpha)}A(u;\alpha)) = \mathbf{P}([A(u;\alpha)/E]^{(1-\alpha)/\alpha} < x).$$

Thereby,

$$\mathbf{P}(S_+(\alpha) < x|U = u) = \mathbf{P}([A(u;\alpha)/E]^{(1-\alpha)/\alpha} < x),$$

and we arrive at the formula

$$S_+(\alpha) \stackrel{d}{=} [A(U;\alpha)/E]^{(1-\alpha)/\alpha}$$

proofing the Kanter algorithm (1975):

$$S_+(\alpha) \stackrel{d}{=} [A(U_2;\alpha)/|\ln U_1|]^{(1-\alpha)/\alpha}, \quad (6.26)$$

where U_1 and U_2 are mutually independent random variables with uniform distribution on $(0,1)$.

6.3.3.2 Symmetrical stable distributions

Using the Kantor algorithm for one-dimensional stable random variables (subordinators), one can easily pass to simulation of s.r.v.'s with symmetrical distributions, i.e., with characteristic function

$$\tilde{g}(k; \alpha, 0) \equiv g(k; \alpha) = e^{-|k|^\alpha}.$$

To get the corresponding algorithm, let us compute a characteristic function of the product

$$\sqrt{S_+(\alpha/2)}G \equiv X$$

of mutually independent multipliers. Applying the total expectation rule yields

$$\begin{aligned} \tilde{p}_X(k) &= \left\langle \exp\left\{ik\sqrt{S_+(\alpha/2)}G\right\} \right\rangle = \left\langle \left\langle \exp\{ik\sqrt{s}G\} \mid S_+(\alpha/2) = s \right\rangle \right\rangle \\ &= \left\langle \exp\{-k^2 S_+(\alpha/2)\} \right\rangle = e^{-|k|^\alpha} \end{aligned}$$

as far as $\langle \exp(-\lambda S_+(\beta)) \rangle = \exp(-\lambda^\beta)$. Consequently, we obtain formula

$$X = \sqrt{S_+(\alpha/2)}G \stackrel{d}{=} S(\alpha, 0) \quad (6.27)$$

which represents the desired algorithm. Observe that when $\alpha = 2$, then $S_+(\alpha/2) = S_+(1) = 1$ and $S(2;0) = G$.

Evidently, this algorithm is easily extended to multidimensional case:

$$\begin{aligned} \mathbf{S}(\alpha, \Gamma_0) &\stackrel{d}{=} \sqrt{S_+(\alpha/2)}\mathbf{G} \\ &= \{[S(\alpha/2)]^{1/2}G_1, [S(\alpha/2)]^{1/2}G_2, \dots, [S(\alpha/2)G_d]^{1/2}\} \in \mathbb{R}^d. \end{aligned} \quad (6.28)$$

This is an isotropic stable random vector with the characteristic function

$$\tilde{g}_d(\mathbf{k}; \alpha, \Gamma_0(\cdot)) = \exp\{-|\mathbf{k}|^\alpha\}.$$

Its components $S_i(\alpha, 0) = [S_+(\alpha/2)]^{1/2} G_i$, $i = 1, 2, \dots, d$ are mutually independent only in case $\alpha = 2$. In order to obtain a family of symmetric stable vectors with independent component for all $\alpha \in (0, 2]$, one should take independent multipliers for different components, $S_i(\alpha; \Gamma_1) = [S_+(\alpha/2)]^{1/2} G_i$, $i = 1, 2, \dots, d$:

$$\mathbf{S}(\alpha, \Gamma_1) = \{[S_+(\alpha/2)]_1 G_1, [S_+(\alpha/2)]_2 G_2, \dots, [S_+(\alpha/2)]_d G_d\}. \quad (6.29)$$

Its characteristic function reads

$$\tilde{g}_d(\mathbf{k}; \alpha, \Gamma_1) = \exp\{-|k_1|^\alpha - \dots - |k_d|^\alpha\}.$$

Both the cases as well as other ones are considered in detail in the book (Samorodnitsky and Taqqu, 1994).

6.3.3.3 One-dimensional stable distributions

Generalizing the Kanter algorithm to the whole family of one-dimensional stable laws

$$\begin{aligned} \tilde{g}(k; \alpha, \theta) &= \exp[-|k|^\alpha \exp(-i\theta\alpha(\pi/2) \operatorname{sign} k)], \quad 0 < \alpha \leq 2, \\ |\theta| &\leq \theta_\alpha = \min\{1, 2/\alpha - 1\} \end{aligned}$$

has been realized by Chambers et al. (1976):

$$S(\alpha, \theta) \stackrel{d}{=} \begin{cases} \frac{\sin[\alpha(\Phi + \theta\pi/2)]}{(\cos \Phi)^{1/\alpha}} \left[\frac{\cos(\Phi - \alpha(\Phi + \theta\pi/2))}{E} \right]^{(1-\alpha)/\alpha}, & \alpha \neq 1; \\ \tan \Phi, & \alpha = 1, \theta = 0, \end{cases} \quad (6.30)$$

where $\Phi = \pi(U - 1/2)$.

For inspection, we reduce this formula to special cases considered above.

Gaussian random variable:

$$S(2, 0) = \frac{\sin(2\Phi)}{(\cos \Phi)^{1/2}} \left[\frac{\cos \Phi}{E} \right]^{-1/2} = 2E^{1/2} \sin \Phi.$$

Smirnov-Lévy random variable:

$$S(1/2, 1) = \frac{\sin(\Phi/2 + \pi/4)}{(\cos \Phi)^2} \left[\frac{\cos(\Phi/2 - \pi/4)}{E} \right] = \left(\frac{\sin(\Phi/2 + \pi/4)}{\cos \Phi} \right)^2 \frac{1}{E}.$$

Taking into account that

$$\cos^2 \Phi = \sin^2(\Phi + \pi/2) = [2 \sin(\Phi/2 + \pi/4) \cos(\Phi/2 + \pi/4)]^2$$

and

$$\sin^2 \Phi \stackrel{d}{=} \cos^2 \Phi \stackrel{d}{=} \sin^2(\Phi/2 \pm \pi/4) \stackrel{d}{=} \cos^2(\Phi/2 \pm \pi/4),$$

we obtain

$$S(1/2, 1) = \frac{\sin^2(\Phi + \pi/4)}{4 \cos^2(\Phi/2 + \pi/4) \sin^2(\Phi/2 + \pi/4) E} \stackrel{d}{=} \frac{1}{4E \cos^2 \Phi}.$$

Both these formulas are in accordance with Eqs. (6.24)–(6.25).

6.3.3.4 Fractional stable distributions

In Sect. 5.2.17, we derived the multidimensional fractional differential equation for subordinated Lévy-Feldheim motion (SLFM) when both time and space derivatives are of fractional orders. Its solution was expressed through the fractional stable distribution

$$\Psi_d^{(\alpha, \nu)}(\mathbf{x}; \Gamma) = \int_0^\infty g_d(\mathbf{x} \tau^{v/\alpha}; \alpha, \Gamma) g(\tau; \nu) \tau^{d\nu/\alpha} d\tau.$$

Rewriting this expression in the form

$$\Psi_d^{(\alpha, \nu)}(\mathbf{x}; \Gamma) = \int_0^\infty d\tau \int_{\mathbb{R}^d} d\mathbf{x} g(\tau; \nu) g_d(\mathbf{x}; \alpha, \Gamma_s) \delta(\mathbf{z} - \tau^{-v/\alpha} \mathbf{x}),$$

we recognize it as pdf of random vector

$$\mathbf{Z}(\alpha, \nu; \Gamma_s) = [S_+(\nu)]^{-v/\alpha} \mathbf{S}(\alpha, \Gamma_s), \quad s = 0, 1.$$

Substitution Eqs. (6.28)–(6.29) instead of $\mathbf{S}(\alpha, \Gamma_s)$ yields:

$$\mathbf{Z}(\alpha, \nu; \Gamma_0) = [S_+(\nu)]^{-v/\alpha} [S_+(\alpha/2)]^{1/2} \{G_1, \dots, G_n\}, \quad (6.31)$$

$$\mathbf{Z}(\alpha, \nu; \Gamma_1) = [S_+(\nu)]^{-v/\alpha} \{[S_+(\alpha/2)]_1^{1/2} G_1, \dots, [S_+(\alpha/2)]_d^{1/2} G_d\} \quad (6.32)$$

(all random variables with different symbols or/and subscripts are mutually independent here).

These formulas together with Eqs. (6.24) and (6.26) give a clear representation of algorithms for simulation of random vectors with fractional stable distributions of both types Γ_0 and Γ_1 . For calculation of the density itself, one can use the local estimation method according which

$$\hat{\Psi}_d^{(\alpha, \nu)}(\mathbf{x}; \Gamma_0) = \frac{1}{n} \sum_{j=1}^n \frac{1}{[2\sqrt{\pi} Q_j]^d} \exp \left\{ -\frac{|\mathbf{x}|^2}{4Q_j^2} \right\}, \quad Q_j = [S_+(\nu)]_j^{-v/\alpha} [S_+(\alpha/2)]_j^{1/2} \quad (6.33)$$

with independent for each j random variables $S_+(\nu)$ and $S_+(\alpha/2)$. This estimator is for isotropic distribution ($\Gamma = \Gamma_0$). In case of distribution with independent components, it is enough to give an algorithm for computing of one-dimensional density:

$$\Psi_d^{(\alpha, \nu)}(\mathbf{x}; \Gamma_1) = \Psi_1^{(\alpha, \nu)}(x_1; 0) \cdots \Psi_1^{(\alpha, \nu)}(x_d; 0), \quad (6.34)$$

$$\Psi_1^{(\alpha, \nu)}(x; 0) = \frac{1}{n} \sum_{j=1}^n \frac{1}{2\sqrt{\pi}Q_j} \exp\left\{-\frac{x^2}{4Q_j^2}\right\}, \quad Q_j = [S_+(\nu)]_j^{-\nu/\alpha} [S_+(\alpha/2)]_j^{1/2}.$$

Figure 1 presents the results of such calculations of two-dimensional densities for $\alpha = 1, \nu = 1$ (a, b) and $\alpha = 1, \nu = 1/2$ (c, d). The cases a, c relate to $\Gamma = \Gamma_0$ and b, d to $\Gamma = \Gamma_1$. The distributions with $\nu = 1/2$ (panels c and d) have singularities in the center of coordinates, dependent of β . When $\nu = 1$, they disappear (panels a and b).

6.3.4 Simulation of fractional exponential distribution

Let us come back to Sect. 3.5.3, where the pdf

$$\psi_\nu(t) = \mu t^{\nu-1} E_{\nu, \nu}(-\mu t^\nu)$$

for waiting time T in the fractional Poisson process is derived (Eq. (3.35)). In (Uchaikin et al., 2008) the following theorem is proved: *The random variable T has the same distribution as*

$$T' = \frac{|\ln U|^{1/\alpha}}{\mu^{1/\alpha}} S_+(\alpha),$$

where U and S_+ are mutually independent random variables. The proof is based on the theorem of total probability

$$\mathbf{P}(T > t) = \int_0^\infty \mathbf{P}(T > t | S_+(\alpha) = \tau) g_+(\tau; \alpha) d\tau.$$

Really,

$$\mathbf{P}(T > t | S_+(\alpha) = \tau) = e^{-\mu t^\alpha / \tau^\alpha} = \mathbf{P}(U < e^{-\mu t^\alpha / \tau^\alpha}) = \mathbf{P}\left(\frac{|\ln U|^{1/\alpha}}{\mu^{1/\alpha}} \tau > t\right),$$

or

$$T|_{S_+(\alpha)=\tau} \stackrel{d}{=} \frac{|\ln U|^{1/\alpha}}{\mu^{1/\alpha}} \tau.$$

Thus,

$$T \stackrel{d}{=} \frac{|\ln U_1|^{1/\alpha}}{\mu^{1/\alpha}} \frac{\sin(\alpha \pi U_2) [\sin((1-\alpha)\pi U_2)]^{1/\alpha-1}}{[\sin(\pi U_2)]^{1/\alpha} [\ln U_3]^{1/\alpha-1}}, \quad (6.35)$$

where U_1 , U_2 , and U_3 are independent uniformly distributed on $[0,1]$ random variables.

Note that if $\alpha \rightarrow 1$, the algorithm becomes the standard rule for simulation of random variables with the exponential distribution:

$$T \stackrel{d}{=} \frac{|\ln U|}{\mu}.$$

6.3.5 Fractional R-L integral

Continuing treatment of fractional Poisson process $N(t)$, we consider the following integral of a function $f(t)$:

$$\int_0^t f(t-t') dN(t') = \sum_{j=1}^{\infty} f(T(j)) 1(t - T(j)),$$

where

$$T(j) = T_1 + T_2 + \cdots + T_j,$$

and $T_1, T_2, \dots, T_j$ are independent identically distributed random variables with a common pdf

$$\psi_v(t) = \mu t^{v-1} E_{v,v}(-\mu t^v).$$

Averaging both parts of the equality, we arrive at the expression

$$\int_0^t f(t-t') \langle dN(t') \rangle = \left\langle \sum_{j=1}^{\infty} f(T(j)) 1(t - T(j)) \right\rangle = \left\langle \sum_{j=1}^{N(t)} f(T(j)) \right\rangle,$$

left hand side of which, according to Eq. (3.32), can be represented in the form

$$\int_0^t f(t-t') \langle dN(t') \rangle = \frac{\mu}{\Gamma(v)} \int_0^t (t-\tau)^{v-1} f(\tau) d\tau.$$

Consequently, we see that

$${}_0\mathbf{D}_t^{-v} f(t) \equiv \frac{1}{\Gamma(v)} \int_0^t (t-\tau)^{v-1} f(\tau) d\tau = \left\langle \frac{1}{\mu} \sum_{j=1}^{N(t)} f(T(j)) \right\rangle,$$

so that this R-L integral can be estimated by simulating n independent fractional Poisson trajectories and computing

$${}_0\hat{f}^{(-v)}(t) = \frac{1}{n} \sum_{i=1}^n \left\{ \frac{1}{\mu} \sum_{j=1}^{N_i(t)} f(T_i(j)) \right\}, \quad 0 < v < 1. \quad (6.36)$$

Statistical error is estimated in a usual way. Observe that the value of μ effects on the error only.

6.3.6 Simulation of a fractal dust in d -dimensional space

A spatial counterpart of Eq. (3.31)

$$f(\mathbf{x}) = \psi(\mathbf{x}) + \int f(\mathbf{x} - \mathbf{x}')\psi(\mathbf{x}')d\mathbf{x}' \quad (6.37)$$

describes the mean density of collision number of a non-vanishing particle which begins its motion at the origin of coordinates and performs rectilinear flights with transition probability density $p(\mathbf{x})$ ¹.

As a matter of fact, the motion itself does not interest us in case under consideration, we are going to study a trail of the particle as a set of all collision points (nods of the trajectory). As in one-dimensional case, we will call it the *dust*. The image of a walking particle helps us to understand and consequently to describe a general picture of correlations in this system of random points. Mathematical tools for solving this problem are developed in (Uchaikin and Gusarov, 1997).

In case of the isotropic exponential transition density

$$\psi(\mathbf{x}) = \frac{1}{4\pi r^2} e^{-r}, \quad r = |\mathbf{x}|$$

is well-known, its asymptotic behavior has the form

$$f^{as}(\mathbf{x}) \propto r^{-1}, \quad r \rightarrow \infty.$$

One can show that the same asymptotics is obtained with any other choice of transition probability but if only

$$\sigma^2 \equiv \int p(\mathbf{x})|\mathbf{x}|^2 d\mathbf{x} < \infty.$$

If not, but the condition

$$\psi(\mathbf{x}) \propto r^{-3-\alpha}, \quad r \rightarrow \infty, \quad 0 < \alpha < 2$$

holds true (such transitions are often called Lévy flights), we obtain also inverse power behavior but with different exponent depending on initial α

$$f^{as}(\mathbf{x}) \propto r^{-3+\alpha}, \quad r \rightarrow \infty.$$

¹ This is not a unique interpretation of Eq. (6.37): the assumption that the particle disappears in each collision with probability $1 - 1/n$ or fall to n secondary independent particles with probability $1/n$ leads to the same equation if the daughter particles move with the same transition probability.

Indeed, direct Fourier transformation of the integral equation yields

$$[1 - \tilde{\psi}(\mathbf{k})]\tilde{f}(\mathbf{k}) = \tilde{\psi}(k).$$

In the region of small $|\mathbf{k}|$,

$$1 - \tilde{\psi}(\mathbf{k}) \propto |k|^\alpha,$$

and consequently

$$|\mathbf{k}|^\alpha \tilde{f}^{as}(\mathbf{k}) = 1.$$

This expression is equivalent to equation with fractional Laplacian

$$(-\Delta)^{\alpha/2} f^{as}(\mathbf{x}) = \delta(\mathbf{x}).$$

As we know, its solution (under zero conditions at infinity) is

$$f^{as}(\mathbf{x}) = C(\alpha)r^{-3+\alpha}, \quad 0 < \alpha \leq 2,$$

where $C(\alpha)$ is a normalized constant. Like in one-dimensional case, the change of sign in front of α changes from $-$ to $+$ by transition from $\psi^{as}(\mathbf{x})$ to $f^{as}(\mathbf{x})$. Such behavior results from infiniteness of the trajectory length: if the trajectory has a non-zero probability p_b to break at each node, the sign does not change, the concentration $f(\mathbf{x})$ falls under the same law as $\psi(\mathbf{x})$. However, as shown in (Uchaikin and Gusarov, 1997), if p_b is small enough, then there exists some long distance region (r_1, r_2) where the law $f(\mathbf{x}) \propto r^{-3+\alpha}$ is observed and only behind this interval $f(\mathbf{x})$ reaches its final asymptotics $f(\mathbf{x}) \propto r^{-3-\alpha}$.

In case of infinite trajectory, the mean number of nodes in the sphere of radius R centered at the origin, grows as

$$\langle N(R) \rangle \propto R^\alpha, \quad \alpha \in (0, 2].$$

Let us define the problem: to find such transition pdf $\psi(\mathbf{x})$ (generator of the fractal dust) that the condition

$$\langle N(r) \rangle = Br^D, \quad r = |\mathbf{x}|$$

holds true at all distances $r > 0$. Consequently,

$$f(\mathbf{x}) = \frac{d\langle N(r) \rangle}{S_d r^{d-1} dr} = \frac{BD}{S_d} r^{D-d}, \quad 0 < r < \infty.$$

Evidently, it is the same problem as that was solved for one-dimensional case (Sect. 3.5.2). The only difference is that we suppose now the distribution of direction after each collision to be isotropic.

Applying the Fourier transform with respect to vector $\mathbf{x}$

$$\tilde{f}(\mathbf{k}) = \tilde{\psi}(\mathbf{k})\tilde{f}(\mathbf{k}) + \tilde{\psi}(\mathbf{k}),$$

we find

$$\tilde{\psi}(\mathbf{k}) = \frac{\tilde{f}(\mathbf{k})}{\tilde{f}(\mathbf{k}) + 1}.$$

One can show that

$$\tilde{f}(\mathbf{k}) \equiv \int_{\mathbb{R}^d} e^{i\mathbf{k}\mathbf{x}} f(\mathbf{x}) d\mathbf{x} = \frac{BD}{S_d} \int_{\mathbb{R}^d} e^{i\mathbf{k}\mathbf{x}} r^{D-d} d\mathbf{x} = C_d(D) |\mathbf{k}|^{-D},$$

where

$$C_d(D) = \frac{BD}{S_d} \frac{2^D \pi^{d/2} \Gamma(D/2)}{\Gamma((d-D)/2)}.$$

Thus, the Fourier image

$$\tilde{\psi}(\mathbf{k}) = \frac{1}{1 + C_d^{-1} |\mathbf{k}|^D}$$

obeys the equation

$$\tilde{\psi}(\mathbf{k}) = -C_d^{-1} |\mathbf{k}|^D \tilde{\psi}(\mathbf{k}) + 1,$$

and the original is a solution of the fractional equation

$$\psi(\mathbf{x}) = -(-\Delta_d)^{D/2} \psi(\mathbf{x}) + \delta(\mathbf{x}).$$

Using identity

$$\frac{1}{1 + C_d^{-1} |\mathbf{k}|^D} = \int_0^\infty e^{-[1 + C_d^{-1} |\mathbf{k}|^D]t} dt,$$

we rewrite the Fourier transform in the form

$$\tilde{\psi}(\mathbf{k}) = \int_0^\infty e^{-t} \tilde{g}_d((C_d^{-1}t)^{1/D} \mathbf{k}; D) dt, \quad D \in (0, 2].$$

The inverse transform yields

$$\begin{aligned} \psi(\mathbf{x}) &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} d\mathbf{k} e^{-i\mathbf{k}\mathbf{x}} \int_0^\infty e^{-t} \tilde{g}_d((C_d^{-1}t)^{1/D} \mathbf{k}; D) dt \\ &= \int_0^\infty dt e^{-t} (C_d/t)^{d/D} g_d((C_d/t)^{1/D} \mathbf{x}; D) \\ &= C_d D r^{D-d} \int_0^\infty e^{-C_d(r/\xi)^D} \xi^{d-D-1} g_d(\xi; D) d\xi. \end{aligned} \quad (6.38)$$

We have isotropy of distributions taken into account and written r for $|\mathbf{x}|$ and ξ for $(C_d/t)^{1/D} |\mathbf{x}|$. Observe that the transition density has the singularity $r^{-(d-D)}$ at zero:

$$\psi(\mathbf{x}) \sim DC_d r^{D-d} \int_0^\infty \xi^{d-D-1} g_d(\xi; D) d\xi = \frac{2^{-D} C_d \Gamma((d-D)/2)}{\Gamma(D/2)} r^{-(d-D)}, \quad r \rightarrow 0.$$

Asymptotic expansion of the fractal dust generator is

$$\begin{aligned}\psi(\mathbf{x}) &= \frac{D^2}{2\pi^{d/2}} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{2^{nD} n \Gamma((nD+d)/2)}{C_d^n \Gamma(1-nD/2)} r^{-nD-d} \\ &\sim \frac{D^2}{2\pi^{d/2}} \frac{2^D \Gamma((D+d)/2)}{C_d \Gamma(1-D/2)} r^{-D-d}, \quad r \rightarrow \infty.\end{aligned}$$

6.3.7 Multidimensional Riesz potential

Of course, the Monte Carlo approach is more effective in multidimensional problems. As an example, we consider calculation of the d -dimensional Riesz potential

$$f_d^{(-v)}(\mathbf{x}) = \frac{1}{\gamma_d(v)} \int_{\mathbb{R}^d} r^{v-d} f(\mathbf{x} - \mathbf{r}) d\mathbf{r}.$$

Writing $n(\mathbf{r})$ for the mean spatial density of nodes on the trajectory with transition probability $\psi(\mathbf{r})$ and using designation $A = S_d/[Bv\gamma_d(v)]$, we obtain

$$f_d^{(-v)}(\mathbf{x}) = A \int_{\mathbb{R}^d} n(\mathbf{r}) f(\mathbf{x} - \mathbf{r}) d\mathbf{r} = \left\langle \sum_{j=1}^{\infty} f(\mathbf{x} - \mathbf{X}_j) \right\rangle,$$

where $\mathbf{X}_j = \mathbf{R}_1 + \mathbf{R}_2 + \dots + \mathbf{R}_j$ are positions of nodes of a random trajectories with mutually independent and identically distributed summands $\mathbf{R}_i$. There remains one thing to do: to formulate an algorithm of simulation of random vectors $\mathbf{R}_i$.

As follows from Sect. 6.3.6, the cumulative probability function of $\mathbf{R}$ can be represented in the form

$$\mathbf{P}(|\mathbf{R}| < r) = \int_{|\mathbf{x}| < r} \psi(\mathbf{x}) d\mathbf{x} = \int_{\mathbb{R}^d} \mathbf{P}(|\mathbf{R}| < r|\mathbf{x}) g_d(\mathbf{x}; v, \Gamma_0) d\mathbf{x},$$

where $g_d(\mathbf{x}; v, \Gamma_0)$ is the isotropic Lévy-Feldheim density for d -dimensional vector $\mathbf{S}(v)$ and

$$\mathbf{P}(|\mathbf{R}| < r|\mathbf{x}) = \int_0^r \exp(-C_d(r/|\mathbf{x}|)^v) d(C_d(r/|\mathbf{x}|)^v) = 1 - \exp(-C_d(r/|\mathbf{x}|)^v).$$

Two these formulas lead to algorithm

$$\mathbf{R} = \mathbf{S}(v)(E/C_d)^{1/v},$$

where E is independent of $\mathbf{S}(v)$ exponentially distributed random variable. Finally, we arrive at the following estimation of d -dimensional Riesz potential:

$$\hat{f}_d^{(-v)}(\mathbf{x}) = \frac{1}{n} \sum_{i=1}^n \left\{ \sum_{j=1}^{\infty} \frac{1}{\gamma_d(v)} f\left(\mathbf{S}_{i,j}(v)(E_{i,j}/C_d)^{1/v}\right) \right\}. \quad (6.39)$$

The difficulty with the inner infinite summation is easily overcome by passing to branch finishing when a trajectory breaks with probability $1 - 1/k$ or falls to k daughter particles with probability $1/k$. These daughter particles move independently of each other according to the same law as the mother particle.

6.3.8 Bifractional diffusion equation

6.3.8.1 Stochastic interpretation

Let us consider the bifractional diffusion (or heat conduction) equation

$${}_0 D_t^\nu f(\mathbf{x}, t) = a \Delta_d f(\mathbf{x}, t) + b \frac{t^{-\nu}}{\Gamma(1-\nu)} \delta(\mathbf{x}). \quad (6.40)$$

This book is constructed in such a way, that the reader reading it without missing should understand the sense of the equation. Nevertheless, we ask him temporarily to forget it or make out like he doesn't know it and wants to know. To help him, we pass in Eq. (6.40) from space-time to Fourier-Laplace variables $t \mapsto \lambda$, $\mathbf{r} \mapsto \mathbf{k}$,

$$[b\lambda^\nu + a|\mathbf{k}|^\alpha] f(\mathbf{k}, \lambda) = b\lambda^{\nu-1}. \quad (6.41)$$

Then we rewrite Eq. (6.40) in the following asymptotically ($\lambda \rightarrow 0$, $|\mathbf{k}| \rightarrow 0$) equivalent form:

$$[1 - (1 - b\lambda^\nu)(1 - a|\mathbf{k}|^\alpha)] f(\mathbf{k}, \lambda) = \frac{1 - (1 - b\lambda^\nu)}{\lambda}. \quad (6.42)$$

The terms $1 - b\lambda^\nu$ and $1 - a|\mathbf{k}|^\alpha$ can be considered as main asymptotical terms in integral transforms of some time and space probability densities $q(t)$ and $p(\mathbf{r})$,

$$\hat{q}(\lambda) = \int_0^\infty e^{-\lambda t} q(t) dt \sim 1 - b\lambda^\nu, \quad \lambda \rightarrow 0 \quad (6.43)$$

and

$$\tilde{p}(\mathbf{k}) = \int_{\mathbb{R}^d} e^{i\mathbf{k}\mathbf{r}} p(\mathbf{r}) d\mathbf{r} \sim 1 - a|\mathbf{k}|^\alpha, \quad |\mathbf{k}| \rightarrow 0. \quad (6.44)$$

Thus, Equation (6.40) can be considered as the asymptotical representation of some “exact” equation

$$f(\mathbf{k}, \lambda) = \hat{q}(\lambda) \tilde{p}(\mathbf{k}) f(\mathbf{k}, \lambda) + \hat{Q}(\lambda), \quad (6.45)$$

looking in time-space variables as

$$f(\mathbf{r}, t) = \int_0^t dt' \int d\mathbf{r}' q(t') p(\mathbf{r}') f(\mathbf{r} - \mathbf{r}', t - t') + \delta(\mathbf{r}) Q(t), \quad Q(t) = \int_t^\infty q(t') dt'. \quad (6.46)$$

Equation (6.46) describes the following jump process. A particle placed at origin $\mathbf{r} = 0$ at time $t = 0$, stays there during a random time T , which is distributed with the density $q(t)$, and then performs the random jump $0 \mapsto \mathbf{r}' = 0 + \mathbf{R}$ with probability density $p(\mathbf{r}')$. Here, the particle has been waited again for a random time T and then jumps again, and so on. It is very important to stress that 1) waiting time and jump vector are independent of each other, and 2) the jumps are instantaneous, the particle disappears at point $\mathbf{r}$ at time t and appears at point $\mathbf{r} + \mathbf{R}'$ at the same time t independently of the displacement distance $\mathbf{R}'$. In case of normal diffusion ($\alpha = 2$, $\nu = 1$), random variables T and $\mathbf{R}$ become negligibly small in the asymptotical regime, but in case of anomalous diffusion ($\alpha < 2$ or/and $\nu < 1$) conditions (6.43)–(6.44) generate long tail distributions of inverse power type, and some of random summands T_j , $\mathbf{R}_j$ are observable at any scales.

Observe that non-Brownian trajectories differ from Brownian ones by presence of jumps along x -axis (when $\alpha < 2$) and along t -axis (when $\nu < 1$) (Fig. 3.3). The first of them are considered as a consequence of a fractal character of the structure and called Lévy flights, the second ones are interpreted as long-acting traps.

6.3.8.2 Estimator, bias, and variance

Two previous sections show an evident way to construct Monte Carlo estimators of function $f(x, t)$ under conditions (6.43)–(6.44). The simplest one looks as follows.

Every history of the walking particle begins with the state of a rest at the origin of coordinates. A random moment T_1 is chosen from the density

$$q(t) = \begin{cases} \beta t_0^\beta t^{-\beta-1}, & t > t_0, \\ 0, & t < t_0. \end{cases} \quad (6.47)$$

Then, the direction of its flight is chosen (to the right or to the left with equal probabilities), and the random path of the particle is chosen from the density

$$p(\xi) = \begin{cases} \alpha \xi_0^\alpha \xi^{-\alpha-1}, & \xi > \xi_0, \\ 0, & \xi < \xi_0. \end{cases} \quad (6.48)$$

Repeating this algorithm step by step up to fixed time t^* , we build a random trajectory and calculate its contribution into the quantity under calculation. Examples of such trajectory by comparison with normal Brownian trajectories are show in Fig. 6.1. The main difference between them consists in the absence of continuity in the anomalous case described by the fractional differential equation.

If we calculate the very function $f(x, t)$, we may use the ordinary histogram technique, or some modification like the local estimator of the flux. Writing h_j for the contribution of j th trajectory in the result we have the following estimator

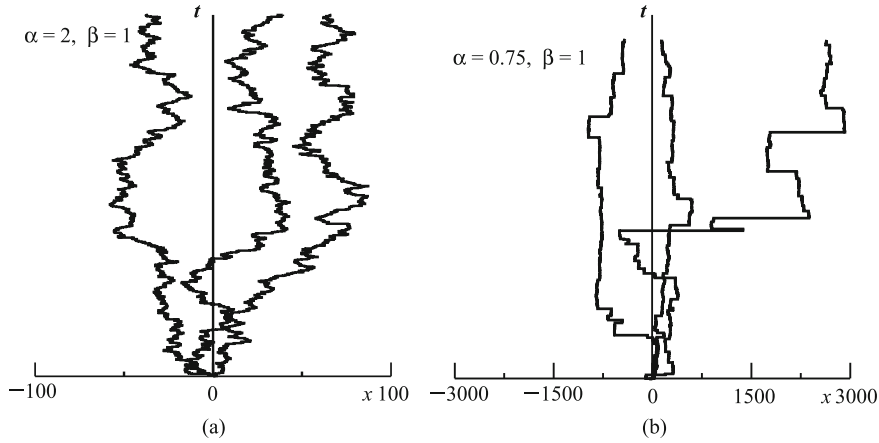


Fig. 6.1 Examples of Brownian trajectories **(a)** and anomalous trajectories **(b)**.

$$\hat{f}(x, t) = \frac{1}{N} \sum_{j=1}^N h_j \quad (6.49)$$

on the ensemble of N independent trajectories. For the histogram technique h_j have form (direct estimator, DE)

$$h_j = \frac{N(\Delta x)}{\Delta x}$$

and for local estimator (LE)

$$h_j = \sum_{k=1}^{K(t^*)} \rho(|x^* - x_k|, t^* - t_k),$$

where $\rho(\xi, t) = (1/2)p(\xi)Q(t)$ and $K(t^*)$ stands for random number of scattering in interval $(0, t^*)$.

Solving Eq. (6.46) for homogeneous boundless medium, we have possibility to compare results of two modifications with exact solution obtained by numerical methods (Fig. 6.1). The difference of results for $\alpha = 0.75$ shows that scale parameters a and b are not small enough and asymptotical regime is not recall yet. Numerical results obtained by simulation of $N = 10^4$ trajectories with parameters $a = 0.1$, $b = 0.01$ for time $t = 10$.

As follows from above estimator (6.49) is an asymptotically (with respect to a and b) unbiased one:

$$\hat{f}(x, t) = [1 + \delta + \hat{\delta}] f(x, t),$$

where $f(x, t)$ is the exact solution of equation (6.46), δ is the relative systematical error and $\hat{\delta}$ is the relative statistical error:

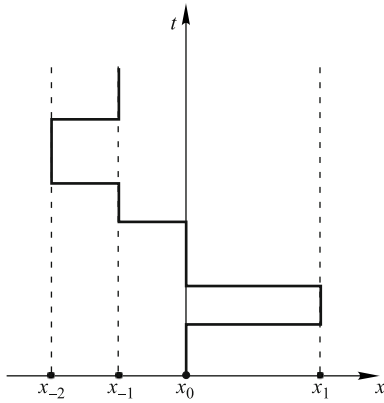


Fig. 6.2 Solution of FDE by direct imitation of anomalous transport process (solid curves present exact results for $t = 10$).

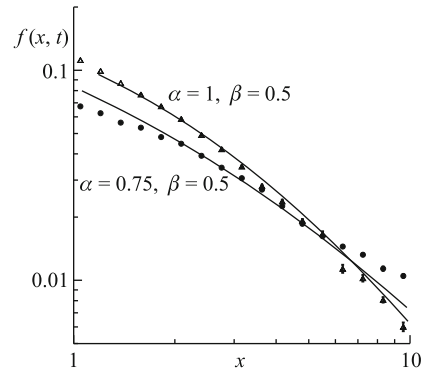


Fig. 6.3 Solution of FDE by LFE (solid curves present exact results for $t = 10$).

$$\delta \rightarrow 0, \quad a \rightarrow 0, \quad b \rightarrow 0, \\ \hat{\delta} \propto N^{-1/2}, \quad N \rightarrow \infty.$$

6.3.8.3 Modified method

Evidently, there exist an infinite family of distributions $q(t)$ and $p(\mathbf{r})$ satisfying the conditions (6.43)–(6.44). One can choose, for example, instead of (6.47)–(6.48) distributions corresponding to transforms

$$\hat{q}(\lambda) = \frac{\mu}{\mu + \lambda^\nu}, \quad \mu = 1/b, \\ \tilde{p}(\mathbf{k}) = e^{-a|\mathbf{k}|^\alpha}.$$

With this choice, time-intervals between jumps are simulated according to pdf $\psi_\nu(t)$ and flight lengths from d -dimensional isotropic stable distribution $g_d(\mathbf{x}; \alpha)$. Both algorithms are described above. Using them we can simulate a position of the walker by computing sum

$$\mathbf{X}(t) = \sum_j^{N_V(t)} a^{-1/\alpha} \mathbf{S}_j(\alpha).$$

If we are interested in a linear functional of solution $f(\mathbf{x}, t)$, say

$$J = \int_{\mathbb{R}^d} h(\mathbf{x}, t) f(\mathbf{x}, t) d\mathbf{x},$$

we should simulate a large number of independent trajectories and compute

$$J_{est} = \frac{1}{M} \sum_{m=1}^M h(\mathbf{X}_m(t)).$$

6.4 Variations, Homotopy and Differential Transforms

6.4.1 Variational iteration method

The variational iteration method (VIM) was first proposed by Chinese mathematician J.H. He (1998). The basic idea of VIM can be clarified as follows. Let us take the differential equation

$$[\mathbf{L} + \mathbf{N}]f(x, t) = h(x, t),$$

where $\mathbf{L}$ is a linear operator, $\mathbf{N}$ is an operator which can be nonlinear and $h(t)$ is a heterogeneous term. Using method of successive approximations, we can write the $(n+1)$ th approximation of the solution as the n th plus some corrections. According to VIM, this correction is represented by means of a *correction functional*

$$\mathbf{C}[f_n(x, \cdot); t] = \int_0^t \lambda(\tau) [\mathbf{L}f_n(x, \tau) + \mathbf{N}\tilde{f}_n(x, \tau) - h(x, \tau)] d\tau,$$

so that the resulting relation looks as follows:

$$f_{n+1}(x, t) = f_n(x, t) + \mathbf{C}[f_n(x, \cdot); t].$$

The correction functional contains a general Lagrange multiplier $\lambda(t)$ which can be identified in an optimal way by the variational theory, and $\tilde{f}_n$ is considered as a restricted variation ($\delta\tilde{f}_n(t) = 0$). The first step of the VIM requires the determination of lambda. Having determined the Lagrange multiplier, the successive approximation f_n of the solution, and the evaluating

$$f(x, t) = \lim_{n \rightarrow \infty} f_n(x, t).$$

As an example, we consider application of VIM to the one-dimensional linear inhomogeneous equation arising in fluid mechanics where the unknown function $u = u(x, t)$ is assumed to be a causal function of time, i.e. vanishing for $t < 0$ (Moman and Odibat, 2006):

$${}^{\alpha}\mathbf{D}_t u + a_0(x)u + a_1(x)\frac{\partial u}{\partial x} + a_2(x)\frac{\partial^2 u}{\partial x^2} = h(x, t), \quad t > 0, -\infty < x < \infty.$$

The solution subject to the boundary conditions

$$u(x, t) \rightarrow 0 \quad \text{as} \quad |x| \rightarrow \infty, \quad t > 0,$$

and to the initial conditions

$$u(x, 0) = f(x), \quad \text{if } 0 < \alpha \leq 1,$$

and

$$u(x, 0) = f(x), \quad \left. \frac{\partial u(x, t)}{\partial x} \right|_{t=0} = g(x) \quad \text{if } 1 < \alpha \leq 2.$$

The functions $a_i(x)$, $f(x)$, $g(x)$, and $h(x, t)$ all are continuous. The correction functional can be approximately expressed as

$$\begin{aligned} \mathbb{C}[u_n(x, \cdot); t] \approx & \int_0^t \lambda(\tau) \left[\frac{\partial^m u_n}{\partial \tau^m} + a_0(x) \tilde{u}_n \right. \\ & \left. + a_1(x) \frac{\partial \tilde{u}_n}{\partial x} + a_2(x) \frac{\partial^2 \tilde{u}_n}{\partial x^2} - h(x, \tau) \right] d\tau, \end{aligned}$$

where the natural number m is equal to 1 in case $\alpha \in (0, 1]$ and to 2 in case $\alpha \in (1, 2]$. Making the above functional stationary and taking into account that $\delta \tilde{u}_n = 0$,

$$\delta u_{n+1}(x, t) = \delta u_n(x, t) + \delta \int_0^t \lambda(\tau) \left[\frac{\partial^m u_n(x, \tau)}{\partial \tau^m} - h(x, \tau) \right] d\tau,$$

yields the following Lagrange multipliers

$$\begin{aligned} \lambda &= -1, \quad m = 1, \\ \lambda &= \tau - t, \quad m = 2. \end{aligned}$$

Therefore, the iteration process is

$$\begin{aligned} u_0(x, t) &= f(x), \\ u_{n+1}(x, t) &= u_n(x, t) - \int_0^t \left[{}_0^{\alpha} \mathbf{D}_t u_n + a_0(x) u_n \right. \\ &\quad \left. + a_1(x) \frac{\partial u_n}{\partial x} + a_2(x) \frac{\partial^2 u_n}{\partial x^2} - h(x, \tau) \right] d\tau, \quad n = 0, 1, 2, \dots \end{aligned}$$

in the first case, and

$$\begin{aligned} u_0(x, t) &= f(x) + tg(x), \\ u_{n+1}(x, t) &= u_n(x, t) - \int_0^t (\tau - t) \left[{}_0^{\alpha} \mathbf{D}_t u_n + a_0(x) u_n \right. \\ &\quad \left. + a_1(x) \frac{\partial u_n}{\partial x} + a_2(x) \frac{\partial^2 u_n}{\partial x^2} - h(x, \tau) \right] d\tau, \end{aligned}$$

in the second case.

Momani and Odibat(2006) demonstrate this method solving the equation

$${}_0^{\alpha} \mathbf{D}_t = \frac{\partial^2 u}{\partial x^2}, \quad t > 0, \quad -\infty < x < \infty, \quad 0 < \alpha \leq 1$$

subject to the initial condition $u(x, 0) = \sin x$. By the above VIM formulas, we obtain

$$u_{n+1}(x, t) = u_n(x, t) - \int_0^t \left[{}_0^\alpha \mathbf{D}_\tau u_n(\tau) - \frac{\partial^2 u_n(x, t)}{\partial x^2} \right] d\tau,$$

and then

$$\begin{aligned} u_0(x, t) &= \sin x, \\ u_1(x, t) &= (1 - t) \sin x, \\ u_2(x, t) &= \left[1 + \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} - 2t + \frac{t^2}{2} \right], \\ u_3(x, t) &= \left[1 - \frac{t^{3-2\alpha}}{\Gamma(4-2\alpha)} + \frac{3t^{2-\alpha}}{\Gamma(3-\alpha)} - \frac{2t^{3-\alpha}}{\Gamma(4-\alpha)} - 3t + \frac{3t^2}{2} - \frac{t^3}{6} \right] \sin x, \end{aligned}$$

and so on.

6.4.2 Homotopy analysis method

The *homotopy analysis method* (HAM) has been proposed by Chinese mathematician Liao (1992) for solving strongly nonlinear problems. It is based on *homotopy*, a fundamental concept in topology and differential geometry. By means of the HAM, a continuous mapping of an initial guess approximation to the exact solution is constructed. An auxiliary linear operator is chosen for continuous mapping, and an auxiliary parameter is introduced to ensure the convergence of solution series (Liao, 2003; 2009).

Let us demonstrate here the basic idea of HAM. Let $\mathbf{N}$ be a nonlinear operator, x denote independent variables and $u(x)$ be an unknown solution of the equation

$$\mathbf{N}[u(x)] = 0.$$

Generalizing the traditional homotopy method, Liao constructs the so-called *zero-order deformation equation*

$$(1 - q)\mathbf{L}[\phi(x, q) - u_0(x)] = qhH(x)\mathbf{N}[\phi(x, q)],$$

where $q \in [0, 1]$, $u_0(x)$ is an initial guess of $u(x)$, $\mathbf{L}$ is an auxiliary linear operator, $H(x) \neq 0$ and $\phi(x, t)$ are auxiliary functions and $q \in [0, 1]$ is the embedding parameter. Obviously, as q increases from 0 to 1, the solution $\phi(x, q)$ varies from the initial guesses $u_0(x)$ to the desired solution $u(x)$. Expanding it in Taylor series yields

$$\phi(x, q) = u_0(x) + \sum_{m=1}^{\infty} u_m(x)q^m,$$

where

$$u_m(x) = \frac{1}{m!} \frac{\partial^m \phi(x, q)}{\partial q^m} \bigg|_{q=0}.$$

Inserting it into zero-order deformation equation, we obtain

$$\mathbf{L}[u_m(x) - 1_m u_{m-1}(x)] = hH(x)R_m[u_0(x), u_1(x), \dots, u_{m-1}(x)] \quad m \geq 1, \quad (6.50)$$

where

$$R_m[u_0(x), u_1(x), \dots, u_{m-1}(x)] = \frac{1}{(m-1)!} \frac{\partial^{m-1} \mathbf{N}[\phi(x, q)]}{\partial q^{m-1}} \bigg|_{q=0}, \quad k = 1, 2, \dots, m-1,$$

and

$$1_m = \begin{cases} 0, & m \leq 1; \\ 1, & m > 1. \end{cases}$$

Applying $\mathbf{L}^{-1}$ on both side of Eq. (6.50), we get

$$u_m(x) = 1_m u_{m-1}(x) + h\mathbf{L}^{-1}[H(x)R_m(u_0(x), u_1(x), \dots, u_{m-1}(x))].$$

Finally, we get

$$u(x) = \sum_{m=0}^{\infty} u_m(x).$$

Example

To give an example of application of the HAM, we consider the following Fredholm integral equation (Hossein Zadeh et al., 2010)

$$u(x) = \cos x + \frac{1}{2} \int_0^{\pi/2} \sin x u(x) dx.$$

Choosing the linear operator

$$\mathbf{L}[\phi(x, q)] = \phi(x, q)$$

and the guesses solution

$$u_0(x) = \cos x,$$

we define a nonlinear operator as

$$\mathbf{N}[\phi(x, q)] = \phi(x, q) - \cos x - \frac{1}{2} \int_0^{\pi/2} \sin x \phi(x) dx.$$

Thus, we arrive at the m th-order deformation equation

$$\mathbb{L}[u_m(x) - 1_m u_{m-1}(x)] = h R_m = h \left[u_{m-1}(x) - (1 - 1_m) \cos x - \frac{1}{2} \int_0^{\pi/2} \sin x u_{m-1}(x) dx \right].$$

Sequentially, we obtain:

$$u_0(x) = \cos x,$$

$$u_1(x) = -\frac{h}{2} \sin x,$$

$$u_2(x) = -\frac{h}{2} \sin x + \frac{h}{4} \sin x,$$

and so on. When $h = -1$, we have

$$u(x) = \cos x + \frac{-h}{2} \sin x + \frac{-h}{2} \sin x + \frac{-h}{4} \sin x + \dots = \cos x + \sin x.$$

This is the exact solution of the equation under consideration.

6.4.3 Differential transform method

The *differential transform method* (DTM) is based on the Taylor series expansion. It constructs an analytic solution in the form of polynomial series by means of an iterative procedure. In case of a one-variable function $f(x)$ this transform is defined as follows

$$f(x) \mapsto F(k) = \frac{1}{k!} \left[\frac{d^k f(x)}{dx^k} \right]_{x=x_0}.$$

The inverse transform is given by the Taylor series

$$f(x) = \sum_{k=0}^{\infty} F(k)(x - x_0)^k.$$

To demonstrate how this method works, we show the following example given in (Biazar and Eslami, 2010). The problem is to find the solution of the equation

$$\frac{df}{dt} = -f^2(t) + 1,$$

subject to the homogeneous initial condition $y(0) = 0$. After the differential transform, it becomes

$$(k+1)F(k+1) = - \sum_{j=0}^k F(j)F(k-j) + \delta_{k,0}.$$

Starting from the initial condition, we get

$$F(0) = 0, \quad F(1) = 1, \quad F(2) = 0, \quad F(3) = -1/3, \dots$$

and finally

$$f(t) = \sum_{k=0}^{\infty} F(k)t^k = t - \frac{1}{3}t^3 + \frac{2}{15}t^5 - \frac{17}{315}t^7 + \frac{62}{2835}t^9 - \dots = \frac{e^{2t} - 1}{e^{2t} + 1}.$$

This is the known solution of this equation.

According to the two-dimensional version of the generalized DTM related to fractional differential equations (Momani et al, 2007), a function of two variables $f(x, y)$ is represented as

$$f(x, y) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} F_{\alpha, \beta}(j, k)(x - x_0)^{j\alpha}(y - y_0)^{k\beta}$$

where $0 < \alpha, \beta \leq 1$, and $F_{\alpha, \beta}(j, k)$ is called the spectrum of u . The latter is given by

$$F_{\alpha, \beta}(j, k) = \frac{1}{\Gamma(\alpha j + 1)\Gamma(\beta k + 1)} \left[({}_0^{\alpha}D_x)^j ({}_0^{\beta}D_y)^k f(x, y) \right]_{x_0, y_0}.$$

Some properties of the transformation are represented in the Table 6.2.

Table 6.2 Main properties of differential transforms

No.	$f(x, y)$	$F_{\alpha, \beta}(j, k)$
1	$u(x, y) + v(x, y)$	$U_{\alpha, \beta}(j, k) + V_{\alpha, \beta}(j, k)$
2	$au(x, y)$	$aU_{\alpha, \beta}(j, k)$
3	$u(x, y)v(x, y)$	$\sum_{r=0}^j \sum_{s=0}^k U_{\alpha, \beta}(r, k-s) V_{\alpha, \beta}(j-r, s)$
4	$(x - x_0)^{n\alpha}(y - y_0)^{m\beta}$	$\delta_{j, n} \delta_{k, m}$
5	${}_0^{\nu}D_x u(x, y)$	$[\Gamma(\alpha j + \nu + 1)/\Gamma(\alpha j + 1)] U_{\alpha, \beta}(j + \nu/\alpha, k)$

Let us take a look at example 3.1 from the article (Garg et al., 2011) where the following space-time fractional telegraph equation

$$\left[{}_0^{p\beta}D_t + {}_0^{r\beta}D_t \right] f(x, t) + f(x, t) = {}_0^{3/2}D_x f(x, t), \quad 0 < x < 1, \quad t > 0,$$

is considered. Here $\beta = 1/q$, p, q, r are integer, $p\beta \in (1, 2]$, $r\beta \in (0, 1]$, $p + r$ are odd and boundary conditions

$$f(0, t) = E_{\beta}(-t^{\beta}), \quad \left. \frac{\partial f(x, t)}{\partial x} \right|_{x=0} = E_{\beta}(-t^{\beta}).$$

Applying the generalized differential transform with $x_0 = y_0 = 0$, $\alpha = 1/2$ to this equation, one obtains

$$\begin{aligned}
F_{1/2,\beta}(0,k) &= \frac{(-1)^k}{\Gamma(\beta k + 1)}, \\
F_{1/2,\beta}(1,k) &= 0, \\
F_{1/2,\beta}(2,k) &= \frac{(-1)^k}{\Gamma(\beta k + 1)}, \\
&\dots\dots \\
F_{1/2,\beta}(j+3,k) &= \frac{\Gamma((j/2)+1)}{\Gamma((j/2+3/2)+1)} \\
&\quad \times \left[\frac{\Gamma(\beta(k+p)+1)}{\Gamma(\beta k + 1)} F_{1/2,\beta}(j,k+p) \right. \\
&\quad \left. + \frac{\Gamma(\beta(k+r)+1)}{\Gamma(\beta k + 1)} F_{1/2,\beta}(j,k+r) + F_{1/2,\beta}(j,k) \right].
\end{aligned}$$

Taking into account that $p+r$ is odd, one obtains for $l, k = 0, 1, 2, \dots$

$$\begin{aligned}
F_{1/2,\beta}(3l,k) &= \frac{(-1)^k}{\Gamma(3l/2+1)\Gamma(\beta k + 1)}, \\
F_{1/2,\beta}(3l+1,k) &= 0, \\
F_{1/2,\beta}(3l+2,k) &= \frac{(-1)^k}{\Gamma(3l/2+2)\Gamma(\beta k + 1)}.
\end{aligned}$$

All that has remained is to make sure that the expansion

$$f(x,t) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} F_{1/2,\beta}(j,k) x^{j/2} t^{\beta k}$$

coincides with the expression

$$f(x,t) = \left[E_{3/2} \left(x^{3/2} \right) + x E_{3/2,2} \left(x^{3/2} \right) \right],$$

obtained by Garg and Scharma (2011) using Adomian decomposition method.

The following equations have also been solved in (Garg et al, 2011) by means of the generalized DTM ($0 < x < 1$, $t > 0$, $2\alpha \in (1, 2]$):

$$\left[{}_0^{p\beta} \mathbf{D}_t + {}_0^{r\beta} \mathbf{D}_t \right] f(x,t) + f(x,t) = {}_0^{2\alpha} \mathbf{D}_x f(x,t)$$

with $p+r$ odd, and

$$\left[{}_0^{p\beta} \mathbf{D}_t + {}_0^{r\beta} \mathbf{D}_t \right] f(x,t) + f(x,t) = {}_0^{2\alpha} \mathbf{D}_x f(x,t) + 2E_\alpha(x^\alpha) E_\beta(-t^\beta),$$

with p and r even. Under identical boundary conditions

$$f(0,t) = E_\beta(-t^\beta), \quad \left. {}_0^\alpha \mathbf{D}_x f(x,t) \right|_{x=0} = E_\beta(-t^\beta);$$

they have the same solutions

$$f(x, t) = E_{\alpha}(x^{\alpha}) E_{\beta}(-t^{\beta}).$$

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