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Electroweak Processes in External Active Media

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Electroweak Processes in External Active Media

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Preface

In the late twentieth and the early twenty-first centuries, the most intensive progress was observed in the sciences, that develop at the junction of two sciences. The most interesting among them, seem to be those that combine the branches most distant from each other. If so, then it is easy to identify the leader of such sciences. It is one that connects the smallest objects available for research, elementary particles, and these giant objects like stars, and has the name of *Particle Astrophysics*. It is not difficult to identify the major milestones in the life of this relatively young, but very rapidly developing science. The birth is most likely to be dated to the beginning of the 1930s. Just then, after the discovery of a neutron by J. Chadwick in 1932, the concept of a neutron star was proposed by L. D. Landau, and independently by W. Baade and F. Zwicky. The start of the maturation of this science can be more or less confidently dated to 1987 when extragalactic neutrinos were registered for the first time from the supernova SN1987A explosion in the Large Magellanic Cloud, a satellite galaxy of our Milky Way. For the date of the endpoint of the maturation period for particle astrophysics, one can propose 2001 when the solar-neutrino puzzle was solved in a unique experiment at the heavy-water detector installed at the Sudbury Neutrino Observatory. This experiment confirmed B. Pontecorvo's key idea concerning neutrino oscillations and, along with experiments that studied atmospheric and reactor neutrinos, thereby proved the existence of a nonzero neutrino mass and the existence of mixing in the lepton sector. The Sun appeared in this case as a natural laboratory for investigations of neutrino properties.

There exist some books on the topic where the basics of this new science can be studied. However, new facts and ideas appear so fast that it is necessary for specialists to follow not only journal papers but also electronic preprints, in order to keep abreast of the latest developments.

A page of this new science, which on the one hand is rather difficult and on the other hand is not covered enough by books or reviews, deals with the particle processes under the extreme conditions of the stellar interior—hot dense plasma and strong electromagnetic fields. This discipline, which can be called *Quantum Field Theory in an External Active Media*, was founded in the 1970s, and now it continues in motion. As an attempt to set some milestone, the objective of our previous monograph [1] was to give a systematic description of the methods of calculation of the quantum processes, both at the tree and loop levels, in external electromagnetic fields. The aim of the present monograph is to consider the quantum processes under an influence of, along with a magnetic field, one more external active media which is hot dense plasma.

The review is based in part on the special lecture course given to the second-year master-course students studying at the Theoretical Physics Department of the Yaroslavl State University, Yaroslavl, Russia. It can be used by graduate and postgraduate students specializing in theoretical physics and being familiar with the basics of the Quantum Field Theory and the Standard Model of the Electroweak Interactions. The authors make a great effort to give all the details that will make this book a valuable text for students. The monograph can be also useful for specialists in the Quantum Field Theory and particle physics, who are interested in the problems of physics of quantum phenomena in external active media.

We have obtained a part of the results presented in this monograph in co-authorship with our colleagues and with our graduate and postgraduate students at the Department of Theoretical Physics of Yaroslavl State University. We thank L. A. Vassilevskaya, A. A. Gvozdev, A. Ya. Parkhomenko, M. V. Chistyakov, I. S. Ognev, E. N. Narynskaya, D. A. Rumyantsev, A. A. Okrugin, R. A. Anikin, A. M. Shitova, and M. S. Radchenko for collaboration and helpful discussions.

We are grateful to S.I. Blinnikov, V.A. Rubakov, V.B. Semikoz, and M.I. Vysotsky for many fruitful discussions and to G.G. Raffelt for collaboration in obtaining the results, concerning the self-energy neutrino operator in a magnetic field. We are thankful to H.-T. Janka and B. Müller for providing us with detailed data on radial distributions and time evolution of physical parameters in the supernova core, obtained in their model of the SN explosion. With a warm feeling we want to mention the many lively discussions with K.A. Ter-Martirosian, whose strong support in the 1990s proved to be crucial for our research group.

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Yaroslavl, March 2013 Alexander Kuznetsov
Nickolay Mikheev

Reference

1. A.V. Kuznetsov, N.V. Mikheev, *Electroweak Processes in External Electromagnetic Fields* (Springer, New York, 2003)

Contents

Chapter 1
Introduction

Astroparticle physics has manifested itself in recent decades as a vigorously growing and prospective line of investigation at the junction of particle physics, astrophysics, and cosmology; see e.g. [1–3]. An important stimulus of its development is an understanding of the essential role of quantum processes in the dynamics of astrophysical objects and of the early Universe. On the other hand, extreme physical conditions existing inside such objects, namely, the presence of hot dense plasma and strong electromagnetic fields, make an active influence on the run of quantum processes, thus allowing or enhancing the transitions that are forbidden or strongly suppressed in a vacuum. In this connection, there exists a stable interest in investigations of particle interactions in external active media.

This line of research is relevant to at least three from the list of the 30 top problems of physics and astrophysics for the beginning of the twenty-first century, formulated by Prof. V.L. Ginzburg in 1999 [4]. They are:

- the behavior of matter in superstrong magnetic fields;
- neutron stars and pulsars, supernova stars;
- neutrino physics and astronomy, neutrino oscillations.

It is known, that matter on the Earth in a natural form is rarely in the plasma state, or it exists during a very short time. In contrast, most of the baryonic matter in the Universe as a whole is a plasma in any form. Theoretical and experimental study of this state of matter has a long history and is still relevant. Nowadays, one of the priority areas of research is the study of plasma in extreme conditions. Such states, usually occur either at high temperature or density, or in ultrahigh external fields. In these conditions, plasma often has a completely new and unusual properties. The study of them is necessary to describe the behavior of the plasma as well as objects in which it is present. Appropriate conditions for the emergence of this plasma could occur in the early stages of the evolution of the Universe when it was very hot. A similar situation can also be realized in high-power stellar cataclysms and in compact astrophysical objects having very high density. Extreme values of physical parameters: temperature, density, magnetic field intensity, component composition, arising in Supernova explosions [5, 6], allow to characterize these objects

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as a unique natural laboratories to study the physical properties of the plasma under conditions that are currently (and may be ever) can not be implemented in terrestrial experiments [7–9].

The close relationship of the laws of microcosm and macrocosm, which is realized in the core collapse supernovae [10], where the laws are simultaneously valid both of the general relativity and nuclear and particle physics, allows to analyze the physical properties of the plasma in these unique environment, and also to investigate the effects of hot, dense plasma on the quantum processes, and to determine the fundamental characteristics of particles on the basis of astrophysical data, and finally, to study the impact of microphysics on astrodynamics [2, 3].

The study of plasma at extreme physical parameters which exist in a supernova explosion is one of the best examples of an interaction of branches of physical science which seem to be far from each other. The fact is that for a short time, such a plasma can be obtained in collisions of elementary particles and nuclei in accelerators.

In the recent years, the most significant progress has been made in the experimental study of the plasma. This is primarily due to the discovery at CERN of a new state of matter called quark-gluon plasma, which was obtained in collisions of heavy nuclei [11]. Today, the investigation continues actively at the accelerator of heavy ions RHIC [12], and the studies have been started at the Large Hadron Collider (LHC) [13, 14]. It is well-known that the quarks, by reason of the strong interaction, are associated into colorless objects, hadrons, and can not be observed in the free state. This phenomenon called quark confinement is sufficiently well studied. However, at high collision energies plasma can be formed, in which quarks and gluons are unconnected, then there is a deconfinement. The duration of the quark-gluon stage is only a small fraction of the evolution time of a system of colliding particles, though its influence is very essential and can be observed by an increased output of strange mesons, the decrease in the output of heavy J/ψ mesons, and by an increased output of photons and lepton-antilepton pairs with high energy [15]. It should be noted that, despite the fact that the properties of the quark-gluon plasma significantly different from all known states of matter, it has a lot of similarity with conventional electromagnetic plasmas [16].

A separate chapter in the physics of hot dense plasma is the research of its impact on quantum processes, which have a critical influence on the macroscopic characteristics of some astrophysical objects like supernovae and young neutron stars. The influence of the plasma on the quantum processes is twofold. On the one hand, under its influence the matrix elements may be modified, which means the change in the dynamics of the processes. On the other hand, the plasma influence changes the dispersion properties of particles, i.e., the process kinematics. As the result, the reactions can be opened or significantly enhanced which are kinematically forbidden or strongly suppressed in a vacuum. Among the best known processes, the photon decay into a pair of neutrino and antineutrino, $\gamma \to \nu\bar{\nu}$, can be indicated. This process, being forbidden in vacuum, is possible due to the plasma influence on the dispersion properties of a photon which acquires an effective mass. As a result, the decay $\gamma \to \nu\bar{\nu}$ is kinematically allowed and may occur in stars [17, 18]. In fact, this so-called plasma process is the primary mechanism of the neutrino emission by stars in a wide range of temperatures and densities, including, for example, the physical conditions inside the white dwarfs and red giants.

Along with the hot dense plasma, a significant effect on the quantum processes can be provided by another component of active astrophysical environments, which are strong magnetic fields. However, the magnetic field significantly influences the quantum processes only in the case when it is strong enough. There exists a natural scale for the field strength which is the so-called critical value $B_e = m_e^2/e \simeq 4.41 \times 10^{13}$ G (we use natural units in which $c = \hbar = 1$).

The fields of such strength are unattainable in a laboratory. However, the astrophysical objects and processes inside them give us unique possibilities for investigations of the particle physics, and of the neutrino physics especially under the extreme conditions of a strong magnetic field. The concept of the astrophysically strong magnetic field has changed over the years (see Fig. 1.1).

Whereas magnetic fields with strength 10^9–10^{11} G were considered as "very strong" nearly forty years ago [19], the fields observed at the surface of pulsars have appeared to be much stronger, of the order of 10^{12}–10^{13} G. The physics of pulsars, i.e. neutron stars, is described in detail in monographs, see e.g. [20–23]. Now the fields $\sim 10^{12}$–10^{13} G are treated as the so-called "old" magnetic fields [21]. There are grounds to expect that fields on even larger scale can arise in astrophysical objects. For example, there exist two classes of stars, the so-called soft gamma-ray repeaters (SGR) [24, 25] and anomalous X-ray pulsars (AXP) [26, 27] which are believed to be magnetars [28], neutron stars with magnetic field strength $\sim 10^{14}$–10^{15} G. To the date (March, 2013), the McGill SGR/AXP Online Catalog contains the current information available on 23 magnetars: 11 SGRs, and 12 AXPs [29].

The fields at the moment of a cataclysm like a supernova explosion, when a neutron star is born, or a coalescence of neutron stars, could be much greater, $\sim 10^{15}$–10^{17} G.

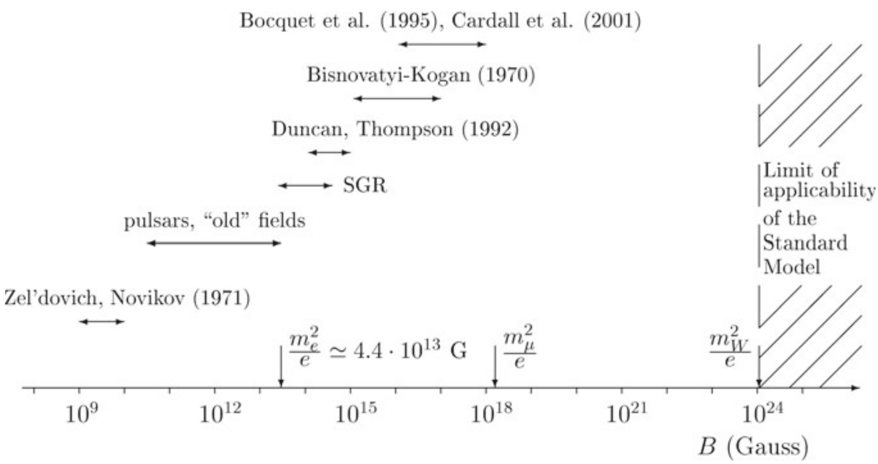

Fig. 1.1 Evolution of the notion "strong magnetic field" in astrophysics

The possible existence of such fields, both of toroidal and poloidal types, is the subject of wide discussions [30–38].

In the early Universe, in the interval between the stages of the QCD phase transition ($\sim 10^{-5}$ s) and nucleosynthesis ($\sim 10^{-2}$–10^{2} s), very strong magnetic fields, the so called "primary" fields, in principle, could exist, with an initial strength of the order 10^{23} G [39] and even more ($\sim 10^{33}$ G [40]). Their evolution during the expansion of the Universe could determine the existence at the present stage of coherent large-scale (~ 100 kpc) magnetic fields with an intensity $\sim 10^{-21}$ G. These fields, in turn, could be enhanced by the galactic dynamo mechanism to the observed values of galactic magnetic fields $\sim 10^{-6}$ G. Possible origins of primary strong magnetic fields and dynamics of their evolution in the expanding Universe are the subject of intense research (see, for example, the surveys [41, 42] and references cited therein).

Note that, in contrast to the magnetic field, the electric field corresponding to the critical value m_e^2/e is the maximal one, since the generation of an electric field of the order of the critical value in a macroscopic space region lead to intensive production of electron–positron pairs from the vacuum, which is equivalent to a short circuit of a "machine" generating the electric field. On the contrary, the magnetic field can exceed the critical value B_e due to the stability of a vacuum. Furthermore, the magnetic field plays a stabilizing role, if directed perpendicular to the electric one. In this configuration, the electric field $\mathcal{E}$ can exceed the critical value of B_e. The vacuum stability condition can be written in the invariant form as

$$F^{\mu\nu} F_{\mu\nu} = 2\left(B^2 - \mathcal{E}^2\right) \geqslant 0.$$

So far, essentially one-dimensional problems have been solved in astrophysical calculations of processes such as supernova explosions, and analyses of the influence of the active medium on quantum processes have only contained the plasma contribution. However, serious arguments have been put forward to suggest that the physics of supernovas is considerably more complex. In particular, we need to allow for rotation of the shell and also for the possible existence of a strong magnetic field, with these two phenomena being interrelated. In fact, the magnetic field generated during the collapse of a supernova nucleus may reach the critical value $\sim 10^{13}$ G. The presence of rotation can lead to generation of a toroidal magnetic field with increasing the field strength by an additional factor of 10^3–10^4 [30, 31].

In astrophysical phenomena such as stellar collapse, the absence of strong magnetic fields is an exotic rather than a typical case. It is appropriate to discuss the following set of questions.

(i) Which can be considered to be the more exotic object: a star possessing a magnetic field or a star without such a field? As far as we know from astrodynamics, a star without a magnetic field should be taken as an exotic rather than a typical case. In exactly the same way the presence of a primary magnetic field may be considered natural for a presupernova. As we know, a primary magnetic field of 100 G leads to the generation of a field on the scale of 10^{12}–10^{13} G during the collapse process as a result of the conservation of magnetic flux.

(ii) Which can be considered to be the more typical case: a star possessing rotation or a star without rotation? Evidently a star without rotation appears to be the more exotic object.
(iii) Which type of collapse looks more exotic: compression without or with an angular velocity gradient? Since the velocities at the edge of a compressible astrophysical object may reach relativistic scales, compression with differential rotation, i.e., with an angular velocity gradient, seems more probable.

All these factors are required to achieve the Bisnovatyi-Kogan scenario for the rotational explosion of a supernova [30, 31]. The main component of this scenario is that the initially poloidal magnetic field lines of a field of 10^{12}–10^{13} G are twisted and compacted as a result of the angular velocity gradient to form an almost toroidal field of $\sim 10^{15}$–10^{17} G. It should be emphasized that a field of the order of 10^{16} G is really a rather dense medium with mass density

$$\rho = \frac{B^2}{8\pi} \simeq 0.4 \times 10^{10} \left(\frac{B}{10^{16} \mathrm{G}}\right)^2 \frac{\mathrm{g}}{\mathrm{cm}^3}, \tag{1.1}$$

which is comparable with the plasma mass density 10^{10}–10^{12} g/cm^3, typical for the envelope of an exploding supernova. Thus, in detailed studies of astrophysical processes such as supernova collapse it is absolutely essential to take into account the influence of the complex active medium including the plasma and the magnetic field.

A dramatic possibility exists [43, 44], that the topic of an asymmetric supernova explosion or merger of neutron stars in our galaxy may appear vitally important for humankind, because of the possible production of a highly beamed gamma ray jet pointed in our direction, which could devastate life on Earth. The strong magnetic field is one typical characteristic of the asymmetry in such an astrophysical cataclysm.

Thus, the problem of particle interactions with external active media is of considerable interest for modern physics. At the same time, this problem is not covered comprehensively in the textbooks on Quantum Field Theory. There are a few classical books, e.g. [45–48], where the technique of calculations of quantum processes in external media is partially concerned. A more detailed presentation of this topic is made in the book [49], and in the reviews [50–53].

It is well known that processes forbidden in a vacuum become possible in intense external fields (such as the photon decay into an electron–positron pair $\gamma \to e^- e^+$ [54], the photon splitting into two photons $\gamma \to \gamma\gamma$ [55–65], the neutrino production of an electron–positron pair, $\nu \to \nu e^- e^+$ [66–74], the radiative transition of massless neutrinos, the so-called neutrino Cherenkov process $\nu \to \nu\gamma$ [75–78], the photon decay into neutrino pair $\gamma \to \nu\bar{\nu}$ [75, 76, 79], and the axion decay $a \to f\bar{f}$ [80]). Apart from this, intense external electromagnetic fields catalyze some processes allowed in a vacuum, for example, the radiative decay of a massive neutrino $\nu \to \nu'\gamma$ [81, 82], and the double-radiative decay of an axion, $a \to \gamma\gamma$ [83, 84].

The method in which the external field effect is taken into account on the basis of exact solutions of the field theory equations for a charged particle in an external electromagnetic field rather than on the basis of perturbation theory, has become an important tool for studying some fundamental problems of particle interactions with an electromagnetic field. The extent to which the motion of a particle is influenced by the field depends on its specific charge, i.e. the ratio of the particle charge to its mass. The hierarchy of masses of elementary particles existing in Nature leads to the inverse hierarchy of specific charges. Thus, particles that are the most sensitive to the external field influence are the lightest charged fermions: the electron is the first one, and then the muon and the u and d quarks follow. All these particles are described by the Dirac equation, and its solutions in the presence of an external electromagnetic field should be used.

In the Quantum Field Theory, the number of cases in which the Dirac equation can be solved analytically is relatively small. These are the problem of electron motion in a Coulomb field (hydrogen atom) and the problems of electron motion in a uniform magnetic field, in the field of a plane electromagnetic wave, and in some particular combinations of uniform electric and magnetic fields. Specific physical phenomena are usually calculated on the basis of a diagram technique (which is in fact the Feynman technique) where the initial and final states feature charged fermions in an external field, which are described by solutions of the Dirac equation in this field, and where internal lines for charged fermions represent their propagators constructed on the basis of the above solutions. This method is advantageous in that it enables us to analyze processes in high-strength fields—that is, in the case where it is impossible to treat field effects within perturbation theory. Since the vacuum is stable in superstrong magnetic fields, one can consider processes in magnetic fields with the strength significantly exceeding the critical value B_e. Thus, these problems form a separate line of investigation in the Quantum Field Theory having an independent conceptual interest. On the other hand, as was mentioned above, such fields can exist near young pulsars; they can also arise in mergers of neutron stars and in supernova explosions.

The above method has proved to be highly efficient in studying some processes in intense electromagnetic fields that are important for various applications (among others, we mean here beta decay in the field of intense laser radiation and quantum effects accompanying the propagation of ultrarelativistic particles through monocrystals).

The objective of the present review is to give a systematic description of the methods of calculation of the quantum processes, both at the tree and loop levels, in external electromagnetic fields. The consideration is accented on the two limiting cases: (i) the case of a very strong magnetic field when the charged fermions occupy the ground Landau level; (ii) the case of a crossed field when all the pure field invariants are equal to zero. These are the cases that allow us to make the analytical calculations in great detail. The review is based for the most part on the original results obtained by the authors with their collaborators [64, 65, 70, 71, 73, 74, 78, 82, 85–113].

The monograph is constructed as follows. In Chap. 2, the solutions of the Dirac equation for a fermion in an external electromagnetic field are presented for the cases of a pure magnetic field of arbitrary strength, of a strong magnetic field when fermions occupy the ground Landau level, and of a crossed field. Propagators of charged particles in an external electromagnetic field for the same cases are presented in Chap. 3. Chapter 4 is devoted to an analysis of the dispersion properties of photons and neutrinos in external active media: magnetic field, plasma, and magnetized plasma. In Chap. 5, electromagnetic interactions in external active media are analysed. They are the processes of the photon decay into an electron–positron pair, and of the photon emission by an electron in magnetic fields, and also the electromagnetic interactions of the Dirac neutrino with a magnetic moment. Chapters 6 and 7 are devoted to the analyses of neutrino–electron and neutrino–photon interactions in external active media. Astrophysical manifestations of the most physical processes are also analyzed.

Notations

The 4-metrics with the signature $(+---)$ and the natural units in which $\hbar = 1$, $c = 1$, are used.

$e = |e|$ is the elementary charge.

m_e is the electron mass, m_f is the fermion mass.

μ_ν is the neutrino magnetic moment, $\tilde{\mu}_\nu$ is the chemical potential of the neutrino gas.

$F_{\alpha\beta}$ is the tensor of the external constant uniform electromagnetic field, $\tilde{F}_{\alpha\beta} = \frac{1}{2}\varepsilon_{\alpha\beta\mu\nu}F^{\mu\nu}$ is the dual tensor ($\varepsilon^{0123} = -\varepsilon_{0123} = +1$).

$\varphi_{\alpha\beta} = F_{\alpha\beta}/B$ is the dimensionless tensor of the external magnetic field, $\tilde{\varphi}_{\alpha\beta} = \frac{1}{2}\varepsilon_{\alpha\beta\mu\nu}\varphi^{\mu\nu}$ is the dual dimensionless tensor.

The tensor indices of four-vectors and tensors standing inside the parentheses are contracted consecutively, for example:

$$(pFFp) = p^\alpha F_{\alpha\beta}F^{\beta\delta}p_\delta;$$
$$(FFp)_\alpha = F_{\alpha\beta}F^{\beta\delta}p_\delta;$$
$$(FF) = F_{\alpha\beta}F^{\beta\alpha}.$$

The dimensionless tensors $\Lambda_{\alpha\beta} = (\varphi\varphi)_{\alpha\beta}$, $\widetilde{\Lambda}_{\alpha\beta} = (\tilde{\varphi}\tilde{\varphi})_{\alpha\beta}$ are connected by the relation $\widetilde{\Lambda}_{\alpha\beta} - \Lambda_{\alpha\beta} = g_{\alpha\beta}$.

In the frame where the magnetic field $\mathbf{B}$ is only presented, directed along the 3d axis, the four-vectors with the indices $\perp$ and $\|$ belong to the Euclidean $\{1, 2\}$-subspace and the Minkowski $\{0, 3\}$-subspace, correspondingly. Then

$$\Lambda_{\alpha\beta} = \mathrm{diag}(0, 1, 1, 0), \qquad \widetilde{\Lambda}_{\alpha\beta} = \mathrm{diag}(1, 0, 0, -1). \tag{1.2}$$

For arbitrary four-vectors p_μ, q_μ one has

$$p_\perp^\mu = (0, p_1, p_2, 0), \qquad p_\parallel^\mu = (p_0, 0, 0, p_3), \tag{1.3}$$

$$(pq)_\perp = (p\Lambda q) = p_1 q_1 + p_2 q_2, \qquad (pq)_\parallel = (p\widetilde{\Lambda} q) = p_0 q_0 - p_3 q_3. \tag{1.4}$$

The Dirac gamma matrices are used in the standard representation [114]:

$$\gamma_0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \boldsymbol{\gamma} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix}, \gamma_5 = \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \tag{1.5}$$

$\boldsymbol{\sigma}$ are the Pauli matrices.

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Chapter 2
Solutions of the Dirac Equation in an External Electromagnetic Field

In this chapter, the solutions of the Dirac equation for a fermion in an external electromagnetic field are presented for the cases of a pure magnetic field of arbitrary strength, of a strong magnetic field when fermions occupy the ground Landau level, and of a crossed field. The density matrix of the plasma electron in a magnetic field with the fixed number of a Landau level is calculated. In this chapter, we use the notation for the 4-vectors and their components: $X^\mu = (t, x, y, z)$.

2.1 Magnetic Field

For calculation of the S matrix elements of quantum processes in external fields, the standard procedure is applied, which is based on the Feynman diagram technique using the field operators of charged fermions expanded over the solutions of the Dirac equation in an external magnetic field

$$\hat{\Psi}(X) = \sum_{\mathbf{p},s} \Big(\hat{a}_{\mathbf{p},s} \Psi^{(+)}_{\mathbf{p},s}(X) + \hat{b}^\dagger_{\mathbf{p},s} \Psi^{(-)}_{\mathbf{p},s}(X) \Big), \tag{2.1}$$

where $\hat{a}$ is the destruction operator for fermions, $\hat{b}^\dagger$ is the creation operator for antifermions, and $\Psi^{(+)}(X)$ and $\Psi^{(-)}(X)$ are the normalized solutions of the Dirac equation in a magnetic field with positive and negative energy, correspondingly.

There exist several methods of solving the Dirac equation in a magnetic field which are basically the similar but have some variations in details, see e.g. [1–5]. Here we present the basic points of the procedure which is the most simple and clear, in our opinion. The description is similar to the one of Ref. [5]. As a charged fermion, we consider an electron being the particle having the largest specific charge, i.e. being the most sensitive to the external field influence. More general case for an arbitrary charged fermion can be found e.g. in [1].

A. Kuznetsov and N. Mikheev, *Electroweak Processes in External Active Media*, Springer Tracts in Modern Physics 252, DOI: 10.1007/978-3-642-36226-2_2,

The Dirac equation for an electron with the mass m_e and the charge $(-e)$ in an external electromagnetic field with the four-potential $A_\mu = A_\mu(X)$ has the form

$$\Big(\mathrm{i}(\partial\gamma) + e(A\gamma) - m_e\Big)\Psi(X) = 0, \tag{2.2}$$

where $(\partial\gamma) = \partial_\mu\gamma^\mu$ and $(A\gamma) = A_\mu\gamma^\mu$. For solving the Eq. (2.2) in a pure magnetic field $\mathbf{B}$, we take the frame where the field is directed along the z axis, and the Landau gauge where the four-potential is: $A^\mu = (0, 0, xB, 0)$.

To solve the Eq. (2.2), let us rewrite it in the Schrödinger form:

$$\mathrm{i}\frac{\partial}{\partial t}\Psi(X) = \hat{H}\Psi(X), \tag{2.3}$$

with the Hamiltonian:

$$\hat{H} = \gamma_0\left[\boldsymbol{\gamma}\left(\hat{\mathbf{p}} + e\mathbf{A}\right)\right] + m_e\gamma_0. \tag{2.4}$$

Here, $\hat{\mathbf{p}} = -\mathrm{i}\boldsymbol{\nabla}$ is the momentum operator.

Since the Hamiltonian does not depend explicitly on time, the problem reduces to finding the eigenvalues and eigenfunctions of the Schrödinger stationary equation:

$$\Psi(X) = \mathrm{e}^{-\mathrm{i}p_0 t}\psi(x, y, z), \quad \hat{H}\psi(x, y, z) = p_0\psi(x, y, z). \tag{2.5}$$

Consider the auxiliary operator, called the longitudinal polarization operator:

$$\hat{T}^0 = \frac{1}{m_e}\left[\boldsymbol{\Sigma}\left(\hat{\mathbf{p}} + e\mathbf{A}\right)\right], \tag{2.6}$$

where $\boldsymbol{\Sigma}$ is the 3-dimensional double spin operator:

$$\boldsymbol{\Sigma} = \gamma_0\boldsymbol{\gamma}\gamma_5 = \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}, \tag{2.7}$$

and $\boldsymbol{\sigma}$ are the Pauli matrices. It is easy to verify by direct calculation that the operator $\hat{T}^0$ commutes with the Hamiltonian (2.4).

First, we find the eigenvalues and the eigenfunctions of the operator $\hat{T}^0$,

$$\hat{T}^0\psi_T(x, y, z) = T^0\psi_T(x, y, z). \tag{2.8}$$

The functions $\psi_T(x, y, z)$ are also the eigenfunctions of the Hamiltonian (2.4), due to commutativity of $\hat{H}$ and $\hat{T}^0$.

It is convenient to represent the operator $\hat{T}^0$ in the form

$$\hat{T}^0 = \begin{pmatrix} \hat{\tau}^0 & 0 \\ 0 & \hat{\tau}^0 \end{pmatrix}, \tag{2.9}$$

where

$$\hat{\tau}^0 = \frac{1}{m_e}\left[\boldsymbol{\sigma}\left(\hat{\mathbf{p}} + e\mathbf{A}\right)\right]. \tag{2.10}$$

By the structure of the operator $\hat{T}^0$, the system (2.8) of 4 equations splits into two exactly coinciding equations for the upper and lower spinors forming the bispinor $\psi_T(x, y, z)$.

In the chosen gauge, the operator $\hat{\tau}^0$ has the form:

$$\hat{\tau}^0 = \frac{1}{m_e}\left[\sigma_x\left(-\mathrm{i}\frac{\partial}{\partial x}\right) + \sigma_y\left(-\mathrm{i}\frac{\partial}{\partial y} + \beta x\right) + \sigma_z\left(-\mathrm{i}\frac{\partial}{\partial z}\right)\right], \tag{2.11}$$

where the notation is used: $\beta = eB$. Given the operator $\hat{T}^0$ not depending explicitly on the coordinates of y and z, one can write the bispinor $\psi_T(x, y, z)$ in the form:

$$\psi_T(x, y, z) = \mathrm{e}^{\mathrm{i}(p_y y + p_z z)}\begin{pmatrix} F(x) \\ \varkappa F(x) \end{pmatrix}, \quad F(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}, \tag{2.12}$$

where $\varkappa$ is an arbitrary number. Introducing a new variable

$$\xi = \sqrt{\beta}\left(x + \frac{p_y}{\beta}\right), \tag{2.13}$$

one can transform the equation for the spinor $F(x)$ to the form:

$$\frac{1}{m_e}\begin{pmatrix} p_z & -\mathrm{i}\sqrt{2\beta}a^- \\ \mathrm{i}\sqrt{2\beta}a^+ & -p_z \end{pmatrix}\begin{pmatrix} f_1(\xi) \\ f_2(\xi) \end{pmatrix} = T^0\begin{pmatrix} f_1(\xi) \\ f_2(\xi) \end{pmatrix}, \tag{2.14}$$

where the raising and lowering operators of the problem of the quantum harmonic oscillator arise:

$$a^+ = \frac{1}{\sqrt{2}}\left(\xi - \frac{\mathrm{d}}{\mathrm{d}\xi}\right), \quad a^- = \frac{1}{\sqrt{2}}\left(\xi + \frac{\mathrm{d}}{\mathrm{d}\xi}\right). \tag{2.15}$$

The expression (2.14) is a system of differential equations for the functions $f_1(\xi)$ and $f_2(\xi)$. We obtain:

$$\begin{gathered} f_1(\xi) = \frac{-\mathrm{i}\sqrt{2\beta}}{m_e T^0 - p_z}\, a^-\, f_2(\xi), \\ \left(a^+ a^- - \frac{m_e^2 (T^0)^2 - p_z^2}{2\beta}\right) f_2(\xi) = 0. \end{gathered} \tag{2.16}$$

Multiplying the operators (2.15), one can see that the equation for the function $f_2(\xi)$ is reduced to an equation for eigenfunctions of the quantum harmonic oscillator:

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} - \xi^2 + 1 + \frac{m_e^2(T^0)^2 - p_z^2}{\beta}\right) f_2(\xi) = 0. \tag{2.17}$$

Hence, we find the eigenvalues T^0 of the operator $\hat{T}^0$:

$$T^0 = \pm\frac{1}{m_e}\sqrt{p_z^2 + 2n\beta}. \tag{2.18}$$

Here, $n = 0, 1, 2, \ldots$. These numbers, as we shall see below, will determine the electron energy, i.e., will number the Landau levels. It should be noted that the eigenvalues T^0 are gauge invariant, being the eigenvalues of the Hermitian operator, i.e. the physically observable quantities.

The functions $f_1(\xi)$ and $f_2(\xi)$ are

$$f_1(\xi) = C\,\frac{-\mathrm{i}\sqrt{2n\beta}}{m_e T^0 - p_z}\,V_{n-1}(\xi), \qquad f_2(\xi) = C\,V_n(\xi), \tag{2.19}$$

where C is the normalization coefficient, and $V_n(\xi)$ ($n = 0, 1, 2, \ldots$) are the normalized harmonic oscillator functions, which are expressed in terms of Hermite polynomials $H_n(\xi)$:

$$V_n(\xi) = \frac{\beta^{1/4}}{\sqrt{2^n\, n!\,\sqrt{\pi}}}\,\mathrm{e}^{-\xi^2/2}\,H_n(\xi), \quad H_n(\xi) = (-1)^n\,\mathrm{e}^{\xi^2}\,\frac{\mathrm{d}^n}{\mathrm{d}\xi^n}\,\mathrm{e}^{-\xi^2},$$
$$\int\limits_{-\infty}^{+\infty} |V_n(\xi)|^2\,\mathrm{d}x = 1, \tag{2.20}$$

and for negative values of the index n the function $V_n(\xi)$ is assumed to be zero.

Returning to the stationary Schrödinger equation (2.5), let us substitute into it the found function $\psi_T(x, y, z)$ as an eigenfunction. The Hamiltonian (2.4) can be expressed in terms of the operator $\hat{\tau}^0$:

$$\hat{H} = m_e\begin{pmatrix} I & \hat{\tau}^0 \\ \hat{\tau}^0 & -I \end{pmatrix}. \tag{2.21}$$

In view of (2.14), we obtain the equation:

$$m_e\left[\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} + T^0\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\right]\begin{pmatrix} F(x) \\ \varkappa\,F(x) \end{pmatrix} = p_0\begin{pmatrix} F(x) \\ \varkappa\,F(x) \end{pmatrix}, \tag{2.22}$$

which is transformed to a system of algebraic equations for p_0 and $\varkappa$ having two solutions. Two eigenvalues of the stationary Schrödinger equation (2.5) are:

$$(p_0)_{1,2} = \pm E_n, \tag{2.23}$$

where

$$E_n = m_e\sqrt{(T^0)^2+1} = \sqrt{p_z^2+m_e^2+2n\beta}. \tag{2.24}$$

The values of $\varkappa$ corresponding to the two eigenvalues p_0 are:

$$\varkappa_1 = \mathrm{sign}(T^0)\sqrt{\frac{E_n-m_e}{E_n+m_e}}, \quad \varkappa_2 = -\mathrm{sign}(T^0)\sqrt{\frac{E_n+m_e}{E_n-m_e}}. \tag{2.25}$$

Thus, given the ambiguity of T^0 (2.18), there exist four independent solutions of the Eq. (2.2).

(i) The eigenvalue $p_0 = +E_n$.

The solutions corresponding to the positive eigenvalue $p_0 = +E_n$, called the solutions with positive energy, which differ in sign of T^0, can be written as:

$$\Psi^{(+\pm)}(X) = A^{(+\pm)}\,\mathrm{e}^{-\mathrm{i}(E_n t - p_y y - p_z z)}\,u^{(+\pm)}(\xi). \tag{2.26}$$

Here, the first of two signs in the superscript refers to p_0, while the second one refers to T^0. For the bispinors $u^{(+\pm)}(\xi)$ we obtain the expressions:

$$u^{(++)}(\xi) = \begin{pmatrix} \frac{-\mathrm{i}\sqrt{2n\beta}}{m_e|T^0|-p_z}\,V_{n-1}(\xi) \\ V_n(\xi) \\ \sqrt{\frac{E_n-m_e}{E_n+m_e}}\,\frac{-\mathrm{i}\sqrt{2n\beta}}{m_e|T^0|-p_z}\,V_{n-1}(\xi) \\ \sqrt{\frac{E_n-m_e}{E_n+m_e}}\,V_n(\xi) \end{pmatrix}, \tag{2.27}$$

$$u^{(+-)}(\xi) = \begin{pmatrix} \frac{-\mathrm{i}\sqrt{2n\beta}}{-m_e|T^0|-p_z}\,V_{n-1}(\xi) \\ V_n(\xi) \\ -\sqrt{\frac{E_n-m_e}{E_n+m_e}}\,\frac{-\mathrm{i}\sqrt{2n\beta}}{-m_e|T^0|-p_z}\,V_{n-1}(\xi) \\ -\sqrt{\frac{E_n-m_e}{E_n+m_e}}\,V_n(\xi) \end{pmatrix}. \tag{2.28}$$

The functions (2.26)–(2.28), as well as any of their linear combinations, are the solutions of the Dirac equation (2.2), corresponding to the eigenvalue $p_0 = +E_n$.

As in the analysis of solutions of the Dirac equation in vacuum, the solutions in a magnetic field are typically used in the form of linear combinations of the functions (2.26)–(2.28), in which the upper two components of the bispinor correspond to the states of the electron with the spin projections 1/2 and −1/2 on some direction, in this case, on the direction of the magnetic field.

Given the normalization

$$\int |\Psi(X)|^2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = 1, \tag{2.29}$$

we obtain the final form of the exact solutions of the Dirac equation for an electron in an external magnetic field on the n-th Landau level:

$$\Psi^{(+)}_{n,\,p_y,\,p_z,\,s}(X) = \frac{\mathrm{e}^{-\,\mathrm{i}\,(E_n\,t - p_y\,y - p_z\,z)}}{\sqrt{2\,E_n\,(E_n + m_e)\,L_y\,L_z}}\,U^{(+)}_{n,\,p_y,\,p_z,\,s}(\xi), \tag{2.30}$$

where L_y and L_z are the normalizing sizes along the axes of y and z; the number $s = \pm\,1$ is the eigenvalue of the double spin operator σ_z acting on the spinor composed of the upper two components of the bispinor.

The bispinor $U^{(+)}$ has different forms for the cases $s = +1$ and $s = -1$:

$$U^{(+)}_{n,\,p_y,\,p_z,\,s=+1}(\xi) = \begin{pmatrix} (E_n + m_e)\,V_{n-1}(\xi) \\ 0 \\ p_z\,V_{n-1}(\xi) \\ \mathrm{i}\,\sqrt{2n\beta}\,V_n(\xi) \end{pmatrix}, \tag{2.31}$$

$$U^{(+)}_{n,\,p_y,\,p_z,\,s=-1}(\xi) = \begin{pmatrix} 0 \\ (E_n + m_e)\,V_n(\xi) \\ -\,\mathrm{i}\,\sqrt{2n\beta}\,V_{n-1}(\xi) \\ -\,p_z\,V_n(\xi) \end{pmatrix}. \tag{2.32}$$

One can see that in each of the bispinors (2.31) and (2.32), the upper two components form a spinor being the eigenfunction of the operator σ_z. For the ground Landau level, $n = 0$, the solution exists only at $s = -1$.

Note that the value p_z in above expressions is a conserved component of the electron momentum along the z axis, i.e. along the field, while the value p_y is the generalized momentum, which determines the position of a center of the Gaussian packet along the x axis by the relation $x_0 = -p_y/\beta$ (see (2.13)).

(ii) The eigenvalue $p_0 = -E_n$.

The solutions $\Psi^{(-\pm)}(X)$ corresponding to this eigenvalue describe the states of an electron with negative energy in the Dirac sea. To obtain the functions corresponding to the states of a positron as a physical particle with the energy E_n and the momentum components p_y and p_z, one should construct the solutions $\Psi^{(-\pm)}(X)$ which are similar to the functions (2.26)–(2.28), in view of (2.25), and then change the signs of p_y and p_z. One should also remember that the projection of the spin of a positron, i.e. of a hole in the sea of negative energies, on any special direction is opposite to the spin projection of the electron, described by a bispinor.

There are two main variants of constructing the solutions with a negative energy, with using of different linear combinations of the functions $\Psi^{(-+)}(X)$ and $\Psi^{(--)}(X)$, which, of course, lead to identical results in calculations of observable quantities. In the first case, one can simply use the solutions (2.30)–(2.32) and change there the signs of E_n, p_y, and p_z. The second way is perhaps more physically justified. As in the analysis of the Dirac equation in vacuum, one can consider the solutions in which the upper two components of a bispinor are small, if the nonrelativistic limit,

$p_z^2 \ll m_e^2$, and the case of a weak field, $\beta \ll m_e^2$, are taken. In this case, the linear combinations of the functions $\Psi^{(-+)}(X)$ and $\Psi^{(--)}(X)$ should be used, in which the spinor composed of the two lower components of a bispinor, describes the states of the electron with the spin projections 1/2 and −1/2 on some direction, in this case, on the direction of the magnetic field.

The exact solutions of the Dirac equation corresponding to the positron states in an external magnetic field, on the nth Landau level have the form:

$$\Psi^{(-)}_{n,\,p_y,\,p_z,\,s}(X) = \frac{\mathrm{e}^{\mathrm{i}\,(E_n\,t - p_y\,y - p_z\,z)}}{\sqrt{2\,E_n\,(E_n + m_e)\,L_y\,L_z}}\,U^{(-)}_{n,\,p_y,\,p_z,\,s}(\xi^{(-)}), \tag{2.33}$$

where the number $s = \pm 1$ is the eigenvalue of the double electron spin operator σ_z acting on the spinor composed of the two lower components of the bispinor,

$$\xi^{(-)} = \sqrt{\beta}\left(x - \frac{p_y}{\beta}\right), \tag{2.34}$$

$$U^{(-)}_{n,\,p_y,\,p_z,\,s=+1}(\xi^{(-)}) = \begin{pmatrix} p_z\,V_{n-1}(\xi^{(-)}) \\ -\mathrm{i}\sqrt{2n\beta}\,V_n(\xi^{(-)}) \\ (E_n + m_e)\,V_{n-1}(\xi^{(-)}) \\ 0 \end{pmatrix}, \tag{2.35}$$

$$U^{(-)}_{n,\,p_y,\,p_z,\,s=-1}(\xi^{(-)}) = \begin{pmatrix} \mathrm{i}\sqrt{2n\beta}\,V_{n-1}(\xi^{(-)}) \\ -\,p_z\,V_n(\xi^{(-)}) \\ 0 \\ (E_n + m_e)\,V_n(\xi^{(-)}) \end{pmatrix}. \tag{2.36}$$

For the ground Landau level, $n = 0$, the solution exists only for the value of the double spin $s = -1$ of an electron with negative energy. This corresponds to the positron state with a value of the double spin $s = +1$.

2.2 The Ground Landau Level

If some physical process with electrons/positrons, with a typical energy E is realised in a strong magnetic field, where the field induction B determines the maximum energy scale of a problem, namely, $eB > E^2, m_e^2$, electrons/positrons can occupy only the states that correspond to the ground Landau level, $n = 0$. Contrary to other Landau levels with $n \geqslant 1$, which are doubly degenerate with respect to spin, the ground level is not degenerate, i.e. the electron/positron spin is fixed, $s = -1/+1$.

The solution of the Dirac equation for the electron with energy E and momentum components p_y and p_z can be presented in this case in the following form

$$\Psi^{(+)}_{0,\, p_y,\, p_z,\, s=-1}(X) = \frac{\beta^{1/4}\, \mathrm{e}^{-\mathrm{i}(Et - p_y y - p_z z)}}{(\sqrt{\pi}\, 2\, E\, (E + m_e) L_y L_z)^{1/2}}\, \mathrm{e}^{-\xi^2/2}\, u(p_\parallel)\,, \tag{2.37}$$

where $p_\parallel$ is the energy-momentum vector of an electron in the Minkowski $\{0{,}3\}$ plane. Here, $E = \sqrt{{p_z}^2 + m_e^2}$, and ξ is defined by (2.13) and describes the motion along the x axis.

The bispinor amplitude is given by

$$u(p_\parallel) = \begin{pmatrix} 0 \\ E + m_e \\ 0 \\ -p_z \end{pmatrix}. \tag{2.38}$$

It is interesting to note that the bispinor amplitude (2.38) is exactly the same as the solution of the free Dirac equation for an electron having a momentum directed along the z axis. This separation of a bispinor amplitude that does not depend on the spatial coordinate x is typical for the ground Landau level only.

The calculation technique of electroweak processes in a strong magnetic field, where electrons occupy the ground Landau level, the so-called two-dimensional electrodynamics, was developed by Loskutov and Skobelev [6, 7]; for details and a complete list of references see e.g. [8]. That technique was essentially improved, with a covariant extension, in our papers; see e.g. [9–14]. For example, the antisymmetric tensor $\varepsilon_{\alpha\beta}$ ($\varepsilon_{30} = -\varepsilon_{03} = 1$) in the subspace $\{0, 3\}$, used in that technique, appears to be not a mathematical abstraction, but has a clear physical meaning of the dimensionless dual magnetic field tensor, $\varepsilon_{\alpha\beta} = -\tilde{\varphi}_{\alpha\beta}$. Similarly, all the formulae can be written in a covariant form with obvious rules of transformation to any frame.

2.3 Crossed Field

There exists a special case of external electromagnetic field, in which the analysis of quantum processes is essentially simplified. It is the case of a crossed field, where the vectors of the electric field $\boldsymbol{\mathcal{E}}$ and the magnetic field $\mathbf{B}$ are orthogonal and their values are equal, $\boldsymbol{\mathcal{E}} \perp \mathbf{B}$, $\mathcal{E} = B$. The calculation technique of electromagnetic processes in the crossed field was developed by Nikishov and Ritus; for details and the list of references see e.g. [15, 16].

The particular case of a crossed field is in fact more general than it may seem at first glance. Really, the situation is possible when the so-called field dynamical parameter χ of the relativistic particle propagating in a relatively weak electromagnetic field, $F < B_e$ ($F = \mathcal{E}$ and/or B), could appear rather high. The definition of the dynamical parameter χ is

$$\chi = \frac{e(pFFp)^{1/2}}{m_e^3}, \tag{2.39}$$

where p^{α} is the particle four-momentum, and $F^{\alpha\beta}$ is the electromagnetic field tensor. In this case the field in the particle rest frame can exceed essentially the critical value and is very close to the crossed field. Even in a magnetic field whose strength is much greater than the critical value, the result obtained in a crossed field will correctly describe the leading contribution to the probability of a process in a pure magnetic field, provided that $\chi \gg B/B_e$. If, in addition, the invariant $|e^2(pFFp)|^{1/3}$ for a particle moving in an arbitrary electromagnetic field considerably exceeds the pure field invariants $|e^2(FF)|^{1/2}$ and $|e^2(\tilde{F}F)|^{1/2}$, the problem is reducible to a still simpler calculation, that in a crossed field for which one has $(FF) = 0$ and $(\tilde{F}F) = 0$. Thus, the calculation in a constant crossed field is the relativistic limit of the calculation in an arbitrary relatively weak smooth field. Consequently, the results obtained in a crossed field possess a great extent of generality, and acquire interest by itself.

The crossed field is described by the 4-vector potential $A^{\mu} = a^{\mu}\varphi$, where $\varphi = (kX)$, and a^{μ} and k^{μ} are the constant 4-vectors, $(kk) = 0,\ (ak) = 0$.

The field tensor in this case is $F^{\mu\nu} = k^{\mu}a^{\nu} - k^{\nu}a^{\mu}$, and the contraction of the two tensors over one index is $(FF)^{\mu\nu} = -k^{\mu}k^{\nu}(aa)$.

The solution of the Dirac equation for an electron in the crossed field can be found as a particular case of the Dirac equation solution in the field of a plane electromagnetic wave obtained by Volkov [17, 18], where the above-mentioned linear dependence of the field vector potential on the phase φ, $A^{\mu} = a^{\mu}\varphi$, should be taken. The solution has the form

$$\Psi_p(X) = \left(1 - \frac{e(k\gamma)(a\gamma)}{2(kp)}\,\varphi\right)\frac{u(p)}{\sqrt{2EV}} \times \exp\left[-\mathrm{i}\left((pX) - \frac{e(ap)}{2(kp)}\,\varphi^2 - \frac{e^2(aa)}{6(kp)}\,\varphi^3\right)\right]. \tag{2.40}$$

where $u(p)$ is the bispinor amplitude of a free electron with the 4-momentum $p^{\mu} = (E, \mathbf{p})$.

The solution with negative energy corresponding to an antiparticle can be obtained from (2.40) by the change of sign of all the components of the 4-momentum p^{μ}.

The directions of the coordinate frame axes can be taken as follows, without loss of generality:

$$k^{\mu} = (k_0, k_0, 0, 0),\ a^{\mu} = (0, 0, -a, 0). \tag{2.41}$$

In this case

$$\varphi = (kX) = k_0(t - x),\ \boldsymbol{\mathcal{E}} = (0, \mathcal{E}, 0),\ \mathbf{B} = (0, 0, B),\ \mathcal{E} = B = k_0 a.$$

It is worthwhile to introduce also the vector $b^{\mu} = (0, 0, 0, -a)$, which can be used for representing the dual tensor $\tilde{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$ by the following form $\tilde{F}^{\mu\nu} = k^{\mu}b^{\nu} - k^{\nu}b^{\mu}$.

2.4 Density Matrix of the Plasma Electron in a Magnetic Field with the Fixed Number of a Landau Level

When quantum processes in a magnetized plasma are investigated, it is occasionally necessary to calculate the plasma electron density matrix in the coordinate space, summed over all quantum states, except the Landau level number. In this Section, we present the calculations of this matrix which can be defined by the formula:

$$R_n(X, X') = \sum_s \int \frac{\mathrm{d}p_y \mathrm{d}p_z}{(2\pi)^2} L_y\, L_z\, f(E_n)\, \Psi^{(+)}_{n,\, p_y,\, p_z,\, s}(X)\, \bar{\Psi}^{(+)}_{n,\, p_y,\, p_z,\, s}(X'). \quad (2.42)$$

Here, $\Psi^{(+)}_{n,\, p_y,\, p_z,\, s}(X)$ are the solutions (2.30)–(2.32) of the Dirac equation for an electron in an external magnetic field, $E_n = \sqrt{p_z^2 + m_e^2 + 2n\beta}$ is the energy of the electron at the nth Landau level, $\beta = eB$, and $f(E_n)$ is the electron distribution function that allows for the presence of a plasma. In the plasma rest frame, it is

$$f(E) = [\mathrm{e}^{(E-\mu)/T} + 1]^{-1},$$

where μ is the chemical potential of plasma and T is its temperature.

Substituting the explicit form of the electron wave functions (2.30)–(2.32) into Eq. (2.42) for the density matrix, we can reduce it to the form

$$R_n(X, X') = \mathrm{e}^{\mathrm{i}\Phi(X,X')} \sum_s R_{n\parallel}((X - X')_\parallel)\, R_{ns\perp}((X - X')_\perp). \quad (2.43)$$

Here, the following functions are introduced:

$$\Phi(X, X') = -\frac{\beta}{2}\,(x + x')\,(y - y'), \quad (2.44)$$

$$R_{n\parallel}(X_\parallel) = \int\limits_{-\infty}^{+\infty} \frac{\mathrm{d}p_z}{E_n(E_n + m_e)}\, f(E_n)\, \mathrm{e}^{-\mathrm{i}(pX)_\parallel}, \quad (2.45)$$

$$R_{ns\perp}(X_\perp) = \frac{\sqrt{\beta}}{8\pi^2} \int\limits_{-\infty}^{+\infty} \mathrm{d}\xi\, U_s(\xi)\, \bar{U}_s(\xi - \sqrt{\beta}x)\, \mathrm{e}^{-\mathrm{i}\sqrt{\beta}(\sqrt{\beta}xy/2 - \xi y)}, \quad (2.46)$$

where we changed the integration variable from p_y to ξ, see (2.13). In Eq. (2.46), we have omitted all the indexes except s of the bispinors $U^{(+)}_{n,\, p_y,\, p_z,\, s}$. However, one should keep in mind that there is not a simple product of the functions $R_{n\parallel}$ and $R_{ns\perp}$ stands in Eq. (2.43), because $R_{ns\perp}$ depends on p_z.

The function $R_{ns\perp}(X_\perp)$ as a function of two variables x and y can be expanded into a Fourier integral:

$$R_{ns\perp}(X_\perp) = \int \frac{d^2 p_\perp}{(2\pi)^2}\, e^{i(pX)_\perp}\, R_{ns\perp}(p_\perp), \tag{2.47}$$

$$R_{ns\perp}(p_\perp) = \int d^2 X_\perp\, e^{-i(pX)_\perp}\, R_{ns\perp}(X_\perp)\,. \tag{2.48}$$

Integrating the function $R_{ns\perp}(p_\perp)$ over the coordinates x and y and substituting the result into (2.43) yields

$$\begin{aligned} R_n(X, X') = {} & \frac{e^{i\Phi(X,X')}}{(2\pi)^3\sqrt{\beta}} \int \frac{d^3 p\, f(E_n)}{E_n(E_n + m_e)}\, e^{-i(p(X-X'))}\, e^{2ip_x p_y/\beta} \\ & \times \int\limits_{-\infty}^{+\infty} d\xi\, e^{-2ip_x\xi/\sqrt{\beta}} \sum_s U_s(\xi)\, \bar{U}_s(\xi'), \end{aligned} \tag{2.49}$$

where $\xi' = 2p_y/\sqrt{\beta} - \xi$.

After simple but slightly cumbersome calculations, including the summation over the spin states of the initial and final electrons that occupy the same Landau level n, the product of the bispinor amplitudes can be reduced to

$$\begin{aligned} \sum_s U_s(\xi)\, \bar{U}_s(\xi') = {} & (E_n + m_e) \\ & \times \{((p\gamma)_\parallel + m_e)\ [\Pi_+ V_{n-1}(\xi) V_{n-1}(\xi') + \Pi_- V_n(\xi) V_n(\xi')] \\ & - \sqrt{2n\beta}\ [\Pi_+ \gamma^2 V_{n-1}(\xi) V_n(\xi') + \Pi_- \gamma^2 V_n(\xi) V_{n-1}(\xi')]\}. \end{aligned} \tag{2.50}$$

Here, the projection operators are introduced:

$$\Pi_\pm = \frac{1}{2}\,(I \pm i\gamma^1\gamma^2)\,, \quad \Pi_\pm \Pi_\pm = \Pi_\pm\,, \quad \Pi_\pm \Pi_\mp = 0. \tag{2.51}$$

The integral over the variable ξ in Eq. (2.49) can be calculated using the formula

$$\begin{aligned} J_{n,n'} = {} & \frac{e^{iab/2}}{\sqrt{\beta}} \int\limits_{-\infty}^{+\infty} d\xi\, e^{-ia\,\xi}\, V_n(\xi) V_{n'}(b - \xi) \\ = {} & (-1)^{n'}\, e^{-i(n-n')\varphi}\, F_{n',n}(u), \qquad n \geqslant n'\,, \end{aligned} \tag{2.52}$$

where

$$\tan\varphi = \frac{a}{b}\,, \quad u = \frac{a^2 + b^2}{4},$$

$$F_{n',n}(u) = \sqrt{\frac{n'!}{n!}}\ (2u)^{(n-n')/2}\ e^{-u}\ L_{n'}^{n-n'}(2u),$$

and the associated Laguerre polynomials $L_k^s(x)$ are defined as

$$L_k^s(x) = \frac{1}{k!}\,\mathrm{e}^x\,x^{-s}\,\frac{\mathrm{d}^k}{\mathrm{d}x^k}\,(\mathrm{e}^{-x}\,x^{k+s})\,. \tag{2.53}$$

Finally, the electron density matrix can be reduced to a triple integral convenient for the subsequent use:

$$\begin{aligned} R_n(X, X') = {} & \mathrm{e}^{\mathrm{i}\Phi(X,X')}\,(-1)^n \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\,\frac{f(E_n)}{E_n}\,\mathrm{e}^{-u}\,\mathrm{e}^{-\mathrm{i}p(X-X')} \\ & \times \{((p\gamma)_\parallel + m_e)[L_n(2u)\Pi_- - L_{n-1}(2u)\Pi_+] + 2(p\gamma)_\perp L^1_{n-1}(2u)\}, \end{aligned} \tag{2.54}$$

where $u = p_\perp^2/\beta$. Equation (2.54) can be used to investigate quantum processes in a plasma in the presence of a magnetic field with an arbitrary strength.

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Chapter 3
Propagators of Charged Particles in External Active Media

In this chapter, we give different representations of the charged particle propagators in an external active environment that will be needed for the analysis of quantum processes. The transformation from one representation to another are provided which can be useful from a methodological point of view. The exact propagator for an electron in a constant uniform magnetic field as the sum over Landau levels is obtained by the direct derivation by standard methods of quantum field theory from exact solutions of the Dirac equation in the magnetic field. In this chapter, we use the notation for the 4-vectors and their components: $X^\mu = (t, x, y, z)$; digital indices are used to enumerate the various 4-vectors. Throughout the chapter, all the masses squared are assumed to have small negative imaginary parts, $m^2 \to m^2 - \mathrm{i}\epsilon$.

3.1 Propagators of Charged Particles in a Magnetic Field

The magnetic field influence on the particle properties is determined by the specific charge, i. e. by the particle charge and mass ratio. Hence, the charged fermion which is the most sensitive to the external field influence is the electron. The calculations of specific physical phenomena in strong external field are based on the application of Feynman diagram technique generalization. It consists in the following procedure: in initial and final states the electron is described by the exact solution of the Dirac equation in the external field, and internal electron lines in quantum processes correspond to exact propagators that are constructed on the basis of these solutions.

The expression for the exact electron propagator in the constant uniform magnetic field was obtained by J. Schwinger [1] in the Fock proper-time formalism [2]; see e.g. [3]. There are another propagator representations given in a number of works. Thus, in Refs. [4, 5] the case was considered of superstrong field and the contribution of the ground Landau level to the electron propagator was obtained. In Ref. [6], see also Ref. [7], the propagator was transformed from the form of Ref. [1] into the sum over Landau levels. Also in Ref. [7] the electron propagator decomposition over the power series of the magnetic field strength was given.

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In our opinion, it is quite important to know different representations of the electron propagator in the external magnetic field and the conditions of their applicability. There were some examples where misunderstanding of such conditions has led to erroneous papers. Thus, in Refs. [8, 9] the self-energy operator of a neutrino in the magnetic field was calculated by the analysis of the one-loop diagram $\nu \to e^- W^+ \to \nu$. The authors of the paper restricted themselves by consideration of the ground Landau level contribution to the electron propagator. As was shown in Ref. [10], because of large virtuality of an electron, $q^2 \sim m_W^2$, the ground Landau level contribution was not dominant and the next levels gave the contributions of the same order of magnitude. Ignoring that fact led the authors [8, 9] to erroneous results. Another example of this kind was an attempt to re-analyse the probability of the ultrahigh-energy neutrino decay $\nu \to e^- W^+$ in the external magnetic field, which was calculated through the imaginary part of the one-loop amplitude of the above-mentioned transition $\nu \to e^- W^+ \to \nu$. Initially, this probability was calculated in Ref. [11]. Another authors [12] performed a new calculation, insisting on a different result. The third calculation, we carried out [13], confirmed the result of Ref. [11]. The most likely reason for the error in the calculation [12] was that the authors used the propagator of the W-boson in an external field in the decomposition over the tensor $F^{\mu\nu}$, and they limited themselves with only linear terms, while the quadratic terms were also significant in that case.

Among papers devoted to the study of the particle propagators in an external field, an article [14] should be highlighted, where the computation carried out of the neutrino self-energy operator in a magnetic field in an arbitrary ξ-gauge. It was demonstrated that, although the self-energy operator depended on the gauge parameter ξ, the neutrino observable characteristics arising from its dispersion law, as expected, were gauge-invariant.

3.1.1 Propagators in the Fock Proper-Time Presentation

The electron propagator in the constant uniform magnetic field in the Fock proper-time formalism can be presented in the form

$$S^{(e)}(X_1, X_2) = \mathrm{e}^{\mathrm{i}\Phi(X_1,X_2)} S(X_1 - X_2). \tag{3.1}$$

Here, $S(X)$ is the translational and gauge invariant part of the propagator

$$\begin{aligned} S(X) = -\frac{\mathrm{i}\beta}{2(4\pi)^2} \int\limits_0^\infty \frac{\mathrm{d}s}{s\sin(\beta s)} \Bigg\{ \frac{1}{s} \Big[\cos(\beta s)(X\tilde{\varphi}\tilde{\varphi}\gamma) - \mathrm{i}\sin(\beta s)(X\tilde{\varphi}\gamma)\gamma_5 \Big] \\ - \frac{\beta}{\sin(\beta s)}(X\varphi\varphi\gamma) + m_e\left[2\cos(\beta s) - \sin(\beta s)(\gamma\varphi\gamma)\right] \Bigg\} \\ \times \exp\left\{ -\mathrm{i}\left[m_e^2 s + \frac{(X\tilde{\varphi}\tilde{\varphi}X)}{4s} - \frac{\beta}{4\tan(\beta s)}(X\varphi\varphi X) \right] \right\}, \end{aligned} \tag{3.2}$$

where $\beta = eB$, e being the elementary charge, m_e is the electron mass, $X_\mu = (X_1 - X_2)_\mu$, the s variable is the Fock proper time, $\varphi_{\alpha\beta}$ being the dimensionless strength tensor of the external field, $\varphi_{\alpha\beta} = F_{\alpha\beta}/B$; and $\tilde{\varphi}_{\alpha\beta} = \frac{1}{2}\varepsilon_{\alpha\beta\rho\sigma}\varphi^{\rho\sigma}$ is the dual tensor.

Integration over the variable s in (3.2) should be correctly defined because the integrand has the poles in the points $s = \pi k/\beta$, where $k = 0, 1, 2\ldots$. The integration is supposed to be performed in the complex plane s along a contour starting from the point $s = 0$ and underlying the real axis. The contour can be also turned down to the negative imaginary axis; see Sect. 3.1.5 below.

The phase $\Phi(X_1, X_2)$ is the translational and gauge noninvariant value, and can be defined in terms of an integral along an arbitrary contour as

$$\Phi(X_1, X_2) = -e \int\limits_{X_1}^{X_2} \mathrm{d}X_\mu \, K^\mu(X), \tag{3.3}$$

$$K^\mu(X) = A^\mu(X) + \frac{1}{2} F^{\mu\nu}(X - X_2)_\nu. \tag{3.4}$$

The integration path from X_1 to X_2 in (3.3) is arbitrary due to the relation $\partial^\mu K^\nu - \partial^\nu K^\mu = 0$. For more details on the noninvariant phase see below, Sect. 3.1.2.

Similarly to Eq. (3.1), one can define the propagators of the W boson and the charged scalar Φ boson in a magnetic field (we consider negative charged W^- and Φ^- bosons as particles):

$$G^{(W)}_{\mu\nu}(X_1, X_2) = \mathrm{e}^{\mathrm{i}\Phi(X_1,X_2)} G_{\mu\nu}(X_1 - X_2), \tag{3.5}$$

$$D^{(\Phi)}(X_1, X_2) = \mathrm{e}^{\mathrm{i}\Phi(X_1,X_2)} D(X_1 - X_2), \tag{3.6}$$

where the phase $\Phi(X_1, X_2)$ is defined by the same Eqs. (3.3), (3.4).

It can be convenient to use the Fourier transforms of the translational invariant parts of the propagators:

$$S(X) = \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \, S(q) \, \mathrm{e}^{-\mathrm{i}qX}, \tag{3.7}$$

$$G_{\mu\nu}(X) = \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \, G_{\mu\nu}(q) \, \mathrm{e}^{-\mathrm{i}qX}, \tag{3.8}$$

$$D(X) = \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \, D(q) \, \mathrm{e}^{-\mathrm{i}qX}. \tag{3.9}$$

From Eqs. (3.2) and (3.7), one can obtain the Fourier transform of the electron propagator in the form

$$S(q) = \int_0^\infty \frac{\mathrm{d}s}{\cos(\beta s)} \exp\left[-\mathrm{i}s\left(m_e^2 - q_\parallel^2 + q_\perp^2 \frac{\tan(\beta s)}{\beta s}\right)\right]$$
$$\times \left\{\left[(q\gamma)_\parallel + m_e\right]\left[\cos(\beta s) - \frac{(\gamma\varphi\gamma)}{2}\sin(\beta s)\right] - \frac{(q\gamma)_\perp}{\cos(\beta s)}\right\}. \tag{3.10}$$

The Fourier transforms of the W boson propagator (3.5), (3.8) and of the charged scalar Φ boson propagator (3.6), (3.9) are gauge dependent. In an arbitrary ξ-gauge they have the forms [14]:

$$G_{\mu\nu}(q) = -\int_0^\infty \frac{\mathrm{d}s}{\cos(\beta s)} \exp\left[\mathrm{i}s\left(q_\parallel^2 - q_\perp^2 \frac{\tan(\beta s)}{\beta s}\right)\right]$$
$$\times \left\{\mathrm{e}^{-\mathrm{i}sm_W^2}\left[g_{\mu\nu} + (\varphi\varphi)_{\mu\nu}\,(1 - \cos(2\beta s)) - \varphi_{\mu\nu}\,\sin(2\beta s)\right]\right.$$
$$- \left[\left(q_\mu + (\varphi q)_\mu \tan(\beta s)\right)\left(q_\nu + (q\varphi)_\nu \tan(\beta s)\right)\right.$$
$$\left. + \mathrm{i}\,\frac{\beta}{2}\left(\varphi_{\mu\nu} - (\varphi\varphi)_{\mu\nu}\tan(\beta s)\right)\right]$$
$$\left. \times \frac{1}{m_W^2}\left(\mathrm{e}^{-\mathrm{i}sm_W^2} - \mathrm{e}^{-\mathrm{i}s\,\xi\,m_W^2}\right)\right\}, \tag{3.11}$$

$$D(q) = \int_0^\infty \frac{\mathrm{d}s}{\cos(\beta s)} \exp\left[-\mathrm{i}s\left(\xi\,m_W^2 - q_\parallel^2 + q_\perp^2 \frac{\tan(\beta s)}{\beta s}\right)\right]. \tag{3.12}$$

In the Feynman gauge, when $\xi = 1$, the Fourier transform of the W boson propagator is essentially simplified [15]:

$$G_{\mu\nu}(q) = -\int_0^\infty \frac{\mathrm{d}s}{\cos(\beta s)} \exp\left[-\mathrm{i}s\left(m_W^2 - q_\parallel^2 + q_\perp^2 \frac{\tan(\beta s)}{\beta s}\right)\right]$$
$$\times \left[g_{\mu\nu} + (\varphi\varphi)_{\mu\nu}\,(1 - \cos(2\beta s)) - \varphi_{\mu\nu}\,\sin(2\beta s)\right]. \tag{3.13}$$

Finally, the Fourier transform of the charged scalar Φ boson in the Feynman gauge has the form

$$D(q) = \int_0^\infty \frac{\mathrm{d}s}{\cos(\beta s)} \exp\left[-\mathrm{i}s\left(m_W^2 - q_\parallel^2 + q_\perp^2 \frac{\tan(\beta s)}{\beta s}\right)\right]. \tag{3.14}$$

3.1.2 A Note on the Noninvariant Phase

At first glance, the expression for the translational and gauge noninvariant phase $\Phi(X_1, X_2)$ written in the covariant form (3.3), (3.4) is rather cumbersome. Some authors prefer to fix the gauge by the choice of the 4-potential of an external field as $A^\mu(X) = (0, 0, x B, 0)$, to write the phase in a more compact form:

$$\Phi(X, X') = -\frac{eB}{2}(x + x')(y - y'). \tag{3.15}$$

However, just the covariant form (3.3), (3.4) of a phase is much more convenient in the analysis of closed loops containing multiple propagators of charged particles.

In a case of the two-vertex loop, the sum of the phases, arising in the amplitude, is zero:

$$\Phi(X_1, X_2) + \Phi(X_2, X_1) = 0. \tag{3.16}$$

In the case of three or more vertices in the loop, the total phase of all propagators is translational- and gauge-invariant. It can be easily shown by presenting the 4-potential of the constant uniform external field in an arbitrary gauge in the form:

$$A^\mu(X) = \frac{1}{2} X_\nu F^{\nu\mu} + \partial^\mu \chi(X), \tag{3.17}$$

where $\chi(X)$ is an arbitrary function. With (3.17), one automatically has $\partial^\mu A^\nu - \partial^\nu A^\mu = F^{\mu\nu}$. Integrating (3.3) with (3.17) one obtains:

$$\Phi(X_1, X_2) = -\frac{e}{2}(X_1 F X_2) - e\,[\chi(X_2) - \chi(X_1)]. \tag{3.18}$$

It is seen from Eq. (3.18) that the terms with the function $\chi(X)$ totally cancel each other in the sum of phases inside a closed loop, providing the gauge invariance. It is easy to check that the sum of phases (3.18) inside a closed loop is translational invariant also. For example, the total phases of three and four propagators of charged particles in the loop are the following:

$$\Phi(X_1, X_2) + \Phi(X_2, X_3) + \Phi(X_3, X_1) = -\frac{e}{2}(X_1 - X_2)_\mu F^{\mu\nu}(X_2 - X_3)_\nu, \tag{3.19}$$

$$\Phi(X_1, X_2) + \Phi(X_2, X_3) + \Phi(X_3, X_4) + \Phi(X_4, X_1) = -\frac{e}{2}(X_1 - X_3)_\mu F^{\mu\nu}(X_2 - X_4)_\nu. \tag{3.20}$$

In a general case of the sum of n phases one has:

$$\Phi_{tot} = -\frac{e}{2}\sum_{i=1}^{n}(X_i F X_{i+1})\bigg|_{X_{n+1}\equiv X_1} = -\frac{e}{2}\sum_{l=2}^{n-1}\sum_{k=1}^{l-1}(Z_k F Z_l), \tag{3.21}$$

where

$$Z_i = X_i - X_{i+1}.$$

3.1.3 Propagators in the Weak-Field Expansion

Manipulations with the exact expressions (3.10) and (3.13) are extremely cumbersome. On the other hand, magnetic fields existing in Nature, except the early Universe, are always weak compared with the critical field for the W boson, $m_W^2/e \simeq 10^{24}$ G. Therefore, the propagators of the W boson and the charged scalar Φ boson can be expanded in powers of $\beta = eB$ as a small parameter. We find up to second order, using the Feynman gauge:

$$\begin{aligned} G_{\mu\nu}(q) = -\mathrm{i}\,\frac{g_{\mu\nu}}{q^2 - m_W^2} \; - \; \beta\,\frac{2\,\varphi_{\mu\nu}}{(q^2 - m_W^2)^2} \\ + \mathrm{i}\,\beta^2 \Bigg[g_{\mu\nu} \left(\frac{1}{(q^2 - m_W^2)^3} + \frac{2\,q_\perp^2}{(q^2 - m_W^2)^4} \right) \\ + 4\,(\varphi\varphi)_{\mu\nu}\,\frac{1}{(q^2 - m_W^2)^3} \Bigg] + \mathcal{O}(\beta^3). \end{aligned} \tag{3.22}$$

It is not difficult to find the similar expansion for the propagator of the W boson in an arbitrary ξ-gauge, however the resulting expression appear to be rather cumbersome, and we do not present it here.

Comparing Eqs. (3.13) and (3.14), one can easily see that the Φ boson propagator $D(q)$ differs only in sign from the coefficient at the term $g_{\mu\nu}$ in the expansion of the propagator $G_{\mu\nu}(q)$ over the three independent tensor structures. One obtains

$$D(q) = \frac{\mathrm{i}}{q^2 - m_W^2} - \mathrm{i}\,\beta^2 \left(\frac{1}{(q^2 - m_W^2)^3} + \frac{2\,q_\perp^2}{(q^2 - m_W^2)^4} \right) + \mathcal{O}(\beta^3). \tag{3.23}$$

Likewise, the asymptotic expression for the electron propagator $S(q)$ is realised when the field strength is the smallest dimensional parameter, $\beta \ll m_e^2 \ll m_W^2$. In this "weak field approximation" the charged-lepton propagator can be expanded as [7]

$$\begin{aligned} S(q) = \mathrm{i}\,\frac{(q\gamma) + m_e}{q^2 - m_e^2} + \beta\,\frac{(q\gamma)_\parallel + m_e}{2(q^2 - m_e^2)^2}(\gamma\varphi\gamma) \\ + \beta^2\,\frac{2\mathrm{i}\left[(q_\parallel^2 - m_e^2)(q\gamma)_\perp - q_\perp^2((q\gamma)_\parallel + m_e)\right]}{(q^2 - m_e^2)^4} + \mathcal{O}(\beta^3). \end{aligned} \tag{3.24}$$

One can see from this expansion that the contribution of the region of small virtual momenta $q^2 \sim m_e^2 \ll m_W^2$ is enhanced in each succeeding term. If the propagator is used for a moderate field, $m_e^2 \ll \beta \ll m_W^2$, the expansion (3.24) is not applicable and the exact propagator Eq. (3.10) must be used.

3.1.4 Propagators in an Expansion over Landau Levels

3.1.4.1 Electron Propagator

If a magnetic field is sufficiently large, $B \gtrsim B_e = m_e^2/e$, it is convenient to use the expression for the electron propagator in an expansion over the Landau levels. We present here the procedure for obtaining such an expression, following to Ref. [6] (see also [7]). It should be noted, that there was an error in expression for the propagator in Ref. [6], namely, the term in the second line of Eq. (4.33) should contain the extra factor $(-\mathrm{i})$. This error was corrected in Ref. [7], Eqs. (39) and (40), and also in Ref. [16], Eqs. (13) and (14), but without any comments.

Let us rewrite the Fourier transform of the translationally and gauge invariant part of the electron propagator (3.10) by introducing a new integration variable $v = \beta s$, to obtain:

$$S(q) = \frac{1}{\beta} \int\limits_0^\infty \mathrm{d}v \exp(-\mathrm{i}\rho v) \bigg\{ \big[(q\gamma)_\parallel + m_e\big] f_1(v) - \big[(q\gamma)_\parallel + m_e\big] \frac{(\gamma\varphi\gamma)}{2} f_2(v) - (q\gamma)_\perp f_3(v) \bigg\}, \tag{3.25}$$

where the following notations are used:

$$\begin{aligned} f_1(v) &= \exp(-\mathrm{i}\alpha \tan v), \\ f_2(v) &= \tan v \exp(-\mathrm{i}\alpha \tan v), \\ f_3(v) &= \frac{1}{\cos^2 v} \exp(-\mathrm{i}\alpha \tan v), \end{aligned} \tag{3.26}$$

and $\rho = (m_e^2 - q_\parallel^2)/\beta$, $\alpha = q_\perp^2/\beta$. Since the functions $f_j(v)$ $(j = 1, 2, 3)$ are periodic, $f_j(v) = f_j(v + n\pi)$, let us divide the integration domain $(0, \infty)$ into intervals $(0, \pi), (\pi, 2\pi), \ldots (n\pi, (n+1)\pi) \ldots$. Making in each segment the change of variable, $v \to v + n\pi$, we can write:

$$\int\limits_0^\infty \mathrm{d}v \exp(-\mathrm{i}\rho v) f_j(v) = \sum_{n=0}^\infty \exp(-\mathrm{i}\rho n\pi) \int\limits_0^\pi \mathrm{d}v \exp(-\mathrm{i}\rho v) f_j(v)$$

$$= \frac{1}{1 - \exp(-\mathrm{i}\rho\pi)} A_j, \tag{3.27}$$

where

$$A_j = \int_0^{\pi} \mathrm{d}v \exp(-\mathrm{i}\rho v) f_j(v). \tag{3.28}$$

It suffices to compute the integral A_1, because the other two integrals can then be found by the formulas:

$$\begin{aligned} A_2 &= \mathrm{i}\frac{\partial}{\partial\alpha} A_1, \\ A_3 &= -\frac{\mathrm{i}}{\alpha}\left(1 - \mathrm{e}^{-\mathrm{i}\rho\pi}\right) - \frac{\rho}{\alpha} A_1. \end{aligned} \tag{3.29}$$

The validity of the last relation is easily seen by representing the integral A_3 in the form:

$$A_3 = \frac{\mathrm{i}}{\alpha} \int_0^{\pi} \mathrm{d}v \exp(-\mathrm{i}\,\rho v) \frac{\mathrm{d}}{\mathrm{d}v}\Big(\exp(-\mathrm{i}\alpha \tan v)\Big) \tag{3.30}$$

and integrating by parts.

To calculate A_1, we write $f_1(v)$ in the form

$$f_1(v) = \exp(-\mathrm{i}\alpha \tan v) = \exp\left(\alpha \frac{-\mathrm{e}^{-2\mathrm{i}v} + 1}{-\mathrm{e}^{-2\mathrm{i}v} - 1}\right). \tag{3.31}$$

The right-hand side of this equation can be expressed through the Laguerre polynomials:

$$L_n(x) = \frac{1}{n!} \mathrm{e}^x \frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(x^n \mathrm{e}^{-x}\right). \tag{3.32}$$

The generating function for Laguerre polynomials is determined by:

$$\frac{1}{1-t} \exp\left(-\frac{xt}{1-t}\right) = \sum_{n=0}^{\infty} L_n(x) t^n \tag{3.33}$$

for $|t| < 1$, where you can get

$$\exp\left(-\frac{xt}{1-t}\right) = \sum_{n=0}^{\infty} \left[L_n(x) - L_{n-1}(x)\right] t^n. \tag{3.34}$$

with a completion of $L_{-1}(x) \equiv 0$. Denoting $-\mathrm{e}^{-2\mathrm{i}v} = t$ and $2\alpha = x$ in the right-hand side of Eq. (3.31) and using the identity

$$\exp\left(\frac{x}{2}\frac{t+1}{t-1}\right) = \exp\left(-\frac{xt}{1-t}\right)\exp\left(-\frac{x}{2}\right), \tag{3.35}$$

we transform the expression for A_1 to the form:

$$\begin{aligned} A_1 &= \int_0^\pi \mathrm{d}v\, \mathrm{e}^{-\alpha} \sum_{n=0}^{\infty} \left[L_n(2\alpha) - L_{n-1}(2\alpha)\right] (-1)^n \exp(-2\mathrm{i}nv)\exp(-\mathrm{i}\rho v) \\ &= \mathrm{e}^{-\alpha} \sum_{n=0}^{\infty} (-1)^n \left[L_n(2\alpha) - L_{n-1}(2\alpha)\right] \int_0^\pi \mathrm{d}v \exp[-\mathrm{i}(\rho + 2n)v] \\ &= -\mathrm{i}\mathrm{e}^{-\alpha}\left(1 - \mathrm{e}^{-\mathrm{i}\rho\pi}\right) \sum_{n=0}^{\infty} \frac{(-1)^n}{\rho + 2n}\left[L_n(2\alpha) - L_{n-1}(2\alpha)\right]. \end{aligned} \tag{3.36}$$

Finally, using Eqs. (3.25), (3.26), (3.28), (3.29) and (3.36), we write the Fourier transform of the translationally and gauge invariant part of the electron propagator in the form:

$$\begin{aligned} S(q) = \sum_{n=0}^{\infty} \frac{\mathrm{i}}{q_\parallel^2 - m_e^2 - 2n\beta} \Bigg\{ &\left[(q\gamma)_\parallel + m_e\right]\left[d_n(\alpha) - \frac{\mathrm{i}}{2}(\gamma\varphi\gamma)\, d_n'(\alpha)\right] \\ &- (q\gamma)_\perp 2n\, \frac{d_n(\alpha)}{\alpha} \Bigg\}, \end{aligned} \tag{3.37}$$

where $\alpha = q_\perp^2/\beta$, and the functions are introduced:

$$d_n(\alpha) = (-1)^n \mathrm{e}^{-\alpha}[L_n(2\alpha) - L_{n-1}(2\alpha)]. \tag{3.38}$$

3.1.4.2 Propagators of the W and Φ Bosons

Similarly to the electron propagator, the propagators of the W and Φ bosons can also be represented as an expansions over the Landau levels. As it was noted in the Introduction, magnetic fields could exist in the early Universe of the scale of the critical field value for a W-boson, $B_W = m_W^2/e \simeq 10^{24}$ gauss. In this case, a knowledge of the vector-boson propagator expanded over the Landau levels can be helpful for investigations of processes in the early Universe.

The Fourier transform of the translationally invariant part of the W boson propagator (3.8) in the ξ-gauge is presented in Eq. (3.11). Similarly to the transformations of the electron propagator, let us rewrite (3.11) in a more convenient form:

$$G_{\mu\nu}(q) = -\frac{1}{\beta}\int_0^\infty \mathrm{d}v\, \mathrm{e}^{-\mathrm{i}\rho v}\Big[(\tilde{\varphi}\tilde{\varphi})_{\mu\nu} f_4(v) - (\varphi\varphi)_{\mu\nu} f_5(v) - \varphi_{\mu\nu} f_6(v)\Big]$$
$$+ \frac{1}{\beta m_W^2}\int_0^\infty \mathrm{d}v\left(\mathrm{e}^{-\mathrm{i}\,\rho v} - \mathrm{e}^{-\mathrm{i}\,\rho_\xi v}\right)\Bigg[\left(q_\mu q_\nu + \mathrm{i}\frac{\beta}{2}\varphi_{\mu\nu}\right) f_4(v) \tag{3.39}$$
$$+ \left((\varphi q)_\mu q_\nu + q_\mu (q\varphi)_\nu - \mathrm{i}\frac{\beta}{2}(\varphi\varphi)_{\mu\nu}\right) f_7(v) + (\varphi q)_\mu (q\varphi)_\nu f_8(v)\Bigg],$$

where the functions are introduced:

$$f_4(v) = \frac{1}{\cos v}\exp(-\mathrm{i}\alpha\tan v),$$
$$f_5(v) = \frac{\cos(2v)}{\cos v}\exp(-\mathrm{i}\alpha\tan v),$$
$$f_6(v) = \frac{\sin(2v)}{\cos v}\exp(-\mathrm{i}\alpha\tan v),$$
$$f_7(v) = \frac{\tan v}{\cos v}\exp(-\mathrm{i}\alpha\tan v) = \mathrm{i}\frac{\partial}{\partial\alpha} f_4(v),$$
$$f_8(v) = \frac{\tan^2 v}{\cos v}\exp(-\mathrm{i}\alpha\tan v) = -\frac{\partial^2}{\partial\alpha^2} f_4(v), \tag{3.40}$$

and $\rho = (m_W^2 - q_\parallel^2)/\beta,\ \rho_\xi = (\xi\, m_W^2 - q_\parallel^2)/\beta,\ \alpha = q_\perp^2/\beta.$

Through the same procedure as in the case of the fermion propagator and noting that $f_j(v + \pi n) = (-1)^n f_j(v)$ $(j = 4, 5, 6, 7, 8)$, we can write

$$\int_0^\infty \mathrm{d}v \exp(-\mathrm{i}\rho v) f_j(v) = \frac{1}{1 + \exp(-\mathrm{i}\rho\pi)} A_j, \tag{3.41}$$

where the integrals similar to Eq. (3.28) are introduced:

$$A_j = \int_0^\pi \mathrm{d}v \exp(-\mathrm{i}\rho v) f_j(v) \quad (j = 4, 5, 6, 7, 8). \tag{3.42}$$

It is worthwhile to introduce the auxiliary integrals:

$$\mathrm{C}(\alpha) = \int_0^\pi \mathrm{d}v \exp(-\mathrm{i}\rho v)\exp(-\mathrm{i}\alpha\tan v)\cos v, \tag{3.43}$$

$$\mathrm{S}(\alpha) = \int_0^\pi \mathrm{d}v \exp(-\mathrm{i}\rho v)\exp(-\mathrm{i}\alpha\tan v)\sin v, \tag{3.44}$$

$$\begin{aligned} \mathrm{E}^{(\pm)}(\alpha) &= \mathrm{C}(\alpha) \pm \mathrm{i}\mathrm{S}(\alpha) \\ &= \int\limits_0^\pi \mathrm{d}v \exp\big[-\mathrm{i}(\rho \mp 1)v\big] \exp(-\mathrm{i}\alpha \tan v). \end{aligned} \tag{3.45}$$

The integral A_4 can be represented in a form

$$A_4 = \frac{\mathrm{i}}{\alpha} \int\limits_0^\pi \mathrm{d}v \exp(-\mathrm{i}\rho v) \cos v \frac{\mathrm{d}}{\mathrm{d}v}\Big(\exp(-\mathrm{i}\alpha \tan v)\Big) \tag{3.46}$$

and further, integrating by parts, we write:

$$A_4 = \frac{\mathrm{i}}{\alpha}\Big[-1 - \exp(-\mathrm{i}\rho\pi) + \mathrm{i}\rho\,\mathrm{C}(\alpha) + \mathrm{S}(\alpha)\Big]. \tag{3.47}$$

The integrals A_5 and A_6 are expressed in terms of A_4, $\mathrm{C}(\alpha)$ and $\mathrm{S}(\alpha)$:

$$A_5 = 2\,\mathrm{C}(\alpha) - A_4, \tag{3.48}$$

$$A_6 = 2\,\mathrm{S}(\alpha). \tag{3.49}$$

To find the integrals $\mathrm{C}(\alpha)$ and $\mathrm{S}(\alpha)$, let us compute $\mathrm{E}^{(\pm)}(\alpha)$ and apply the relations:

$$\mathrm{C}(\alpha) = \frac{1}{2}\Big[E^{(+)}(\alpha) + E^{(-)}(\alpha)\Big], \tag{3.50}$$

$$\mathrm{S}(\alpha) = \frac{1}{2\mathrm{i}}\Big[E^{(+)}(\alpha) - E^{(-)}(\alpha)\Big]. \tag{3.51}$$

The integral $\mathrm{E}^{(\pm)}(\alpha)$ is computed similarly to the integral A_1 for the fermion propagator and is

$$\mathrm{E}^{(\pm)}(\alpha) = -\mathrm{i}\big[1 + \exp(-\mathrm{i}\rho\pi)\big]\sum_{n=0}^{\infty} \frac{d_n(\alpha)}{\rho + 2n \mp 1}. \tag{3.52}$$

Here, as before, the functions $d_n(v)$ are defined by the expression (3.38). We obtain the integrals $\mathrm{C}(\alpha)$ and $\mathrm{S}(\alpha)$ as

$$\mathrm{C}(\alpha) = -\frac{\mathrm{i}}{2}\big[1 + \exp(-\mathrm{i}\rho\pi)\big]\sum_{n=0}^{\infty} \frac{d_n(\alpha) + d_{n-1}(\alpha)}{\rho + 2n - 1}, \tag{3.53}$$

$$\mathrm{S}(\alpha) = -\frac{1}{2}\big[1 + \exp(-\mathrm{i}\rho\pi)\big]\sum_{n=0}^{\infty} \frac{d_n(\alpha) - d_{n-1}(\alpha)}{\rho + 2n - 1}. \tag{3.54}$$

To obtain the final expressions for A_4, A_5, A_6, the relation should be used $\sum_{n=0}^{\infty} d_n(\alpha) = 1$. The result can be written in a more compact form, being expressed in terms of the functions:

$$\ell_n(\alpha) = \frac{(n+1)\, d_{n+1}(\alpha) + n\, d_n(\alpha)}{2\alpha} = (-1)^n \mathrm{e}^{-\alpha} L_n(2\alpha). \tag{3.55}$$

One obtains:

$$A_4 = -2\mathrm{i}\left(1 + \mathrm{e}^{-\mathrm{i}\rho\pi}\right) \sum_{n=0}^{\infty} \frac{\ell_{n-1}(\alpha)}{\rho + 2n - 1}, \tag{3.56}$$

$$A_5 = -\mathrm{i}\left(1 + \mathrm{e}^{-\mathrm{i}\rho\pi}\right) \sum_{n=0}^{\infty} \frac{\ell_n(\alpha) + \ell_{n-2}(\alpha)}{\rho + 2n - 1}, \tag{3.57}$$

$$A_6 = -\left(1 + \mathrm{e}^{-\mathrm{i}\rho\pi}\right) \sum_{n=0}^{\infty} \frac{\ell_n(\alpha) - \ell_{n-2}(\alpha)}{\rho + 2n - 1}, \tag{3.58}$$

$$A_7 = 2\left(1 + \mathrm{e}^{-\mathrm{i}\rho\pi}\right) \sum_{n=0}^{\infty} \frac{\ell'_{n-1}(\alpha)}{\rho + 2n - 1}, \tag{3.59}$$

$$A_8 = 2\mathrm{i}\left(1 + \mathrm{e}^{-\mathrm{i}\rho\pi}\right) \sum_{n=0}^{\infty} \frac{\ell''_{n-1}(\alpha)}{\rho + 2n - 1}. \tag{3.60}$$

Substituting the integrals (3.56)–(3.60) into the expression for the propagator (3.39), we find:

$$\begin{aligned} G_{\mu\nu}(q) = \sum_{n=0}^{\infty} \frac{-\mathrm{i}}{q_\parallel^2 - m_W^2 - \beta(2n-1)} \Big\{ & 2(\tilde{\varphi}\tilde{\varphi})_{\mu\nu}\, \ell_{n-1}(\alpha) \\ & - (\varphi\varphi)_{\mu\nu}\Big(\ell_n(\alpha) + \ell_{n-2}(\alpha)\Big) + \mathrm{i}\varphi_{\mu\nu}\Big(\ell_n(\alpha) - \ell_{n-2}(\alpha)\Big) \\ & + \frac{\xi - 1}{q_\parallel^2 - \xi\, m_W^2 - \beta(2n-1)} \Big[\Big(2q_\mu q_\nu + \mathrm{i}\beta\, \varphi_{\mu\nu}\Big)\ell_{n-1}(\alpha) \\ & + \mathrm{i}\Big(2(\varphi q)_\mu q_\nu + 2q_\mu (q\varphi)_\nu - \mathrm{i}\beta(\varphi\varphi)_{\mu\nu}\Big)\ell'_{n-1}(\alpha) \\ & - 2(\varphi q)_\mu (q\varphi)_\nu\, \ell''_{n-1}(\alpha)\Big]\Big\}. \end{aligned} \tag{3.61}$$

It is worth noting a singularity that the contribution of the ground level, $n = 0$, into the W boson propagator contains, in contrast to the contribution of the ground Landau level into the electron propagator (3.37). For the W boson, this contribution has the gauge independent form

$$G^{(0)}_{\mu\nu}(q) = \frac{-\mathrm{i}}{q_\parallel^2 - m_W^2 + \beta} \mathrm{e}^{-q_\perp^2/\beta} \left[-(\varphi\varphi)_{\mu\nu} + \mathrm{i}\varphi_{\mu\nu} \right], \tag{3.62}$$

that is, it contains a pole at $q_\parallel^2 = m_W^2 - \beta$. Thus, if the magnetic field approaches the critical value of the field for the W boson, $B_W = m_W^2/e \simeq 10^{24}$ G, the so-called instability arises of the perturbation theory for a W boson vacuum (see, e.g., [17]).

The propagator of the Φ boson in the ξ gauge, $D(q)$, as in the case of a weak field, is reconstructed from (3.61) in the form

$$D(q) = \sum_{n=0}^{\infty} \frac{2\mathrm{i}\, \ell_{n-1}(\alpha)}{q_\parallel^2 - \xi\, m_W^2 - \beta(2n-1)}. \tag{3.63}$$

It should be noted, that the summation over n in Eq. (3.61) formally starts from $n = 0$, but in fact it starts from $n = 1$, because $\ell_{-1}(\alpha) = 0$ by definition. This means that the propagator of the Φ boson, as one could expect, does not contain a pole at $q_\parallel^2 = \xi\, m_W^2 - \beta$.

3.1.5 Electron Propagator in a Strong Magnetic Field

Translationally invariant part of the electron propagator $S(X)$ has also other representations. For example, to analyze the processes in a strong magnetic field, it is worthwhile to use the asymptotic expression for the propagator. To obtain this, let us perform the rotation of the contour of integration in the complex plane of the variable s in the integral (3.2) onto the negative imaginary axis, $s = -\mathrm{i}\tau$, and perform a partial decomposition into the Fourier integral over the coordinates $t = X^0$ and $z = X^3$ (the magnetic field is directed along the third axis):

$$\begin{aligned} S(X) = -\frac{\mathrm{i}}{4\pi} \int_0^\infty \frac{\mathrm{d}\tau}{\tanh\tau} \int \frac{\mathrm{d}^2 q_\parallel}{(2\pi)^2} \Big\{ &[(q\gamma)_\parallel + m_e]\Pi_-(1+\tanh\tau) \\ &+ [(q\gamma)_\parallel + m_e]\Pi_+(1-\tanh\tau) - (X\gamma)_\perp \frac{\mathrm{i}\beta}{2\tanh\tau}(1-\tanh^2\tau) \Big\} \\ &\times \exp\left(-\frac{\beta X_\perp^2}{4\tanh\tau} - \frac{\tau(m_e^2 - q_\parallel^2)}{\beta} - \mathrm{i}(qX)_\parallel \right), \end{aligned} \tag{3.64}$$

Here, γ_α are the Dirac matrices in the standard representation, $\Pi_\pm$ are the projection operators (2.51),

$$\mathrm{d}^2 q_\parallel = \mathrm{d}q_0 \mathrm{d}q_3, \qquad [\Pi_\pm, (a\gamma)_\parallel] = 0.$$

The asymptotic expression for the propagator in a strong magnetic field can be obtained from Eq. (3.64) by an approximate estimate of the integral over τ in the limit $\beta/|m_e^2 - q_\parallel^2| \gg 1$. In this case, the main contribution into the integral over τ comes from the region $\tau \sim \beta/|m_e^2 - q_\parallel^2|$. Considering that $\tanh \tau \simeq 1 - 2e^{-2\tau}$ for $\tau \gg 1$, we obtain the following asymptotic expressions for the translationally invariant part of the electron propagator in a strong magnetic field:

$$S(X) \simeq \frac{\mathrm{i}\beta}{2\pi} \exp\left(-\frac{\beta X_\perp^2}{4}\right) \int \frac{\mathrm{d}^2 q_\parallel}{(2\pi)^2} \frac{(q\gamma)_\parallel + m_e}{q_\parallel^2 - m_e^2} \Pi_- \, \mathrm{e}^{-\mathrm{i}(qX)_\parallel}, \tag{3.65}$$

which was first obtained in Refs. [4, 5]. It is easy to see that the expression (3.65) coincides with the contribution of the ground Landau level. Indeed, substituting the term with $n = 0$ from (3.37) into (3.7) and integrating over $\mathrm{d}^2 q_\perp = \mathrm{d}q_1 \mathrm{d}q_2$, we reproduce the formula (3.65).

3.2 Propagators of Charged Particles in a Crossed Field

In the case of a crossed field, the electron propagator in the Fock proper-time formalism has the same form of (3.1), where the translational and gauge invariant part $S(X)$ can be obtained from (3.2) by the limiting transition when the field invariant $\beta \sim [-(FF)]^{1/2}$ is made to tend to zero in such a way that the field tensor $F_{\alpha\beta} \sim \beta\varphi_{\alpha\beta}$ remains finite. Thus one obtains

$$\begin{aligned} S(X) = -\frac{\mathrm{i}}{16\pi^2} \int\limits_0^\infty \frac{\mathrm{d}s}{s^2} \bigg[\frac{1}{2s}(X\gamma) - \frac{\mathrm{i}e}{2}(X\tilde{F}\gamma)\gamma_5 - \frac{se^2}{3}(XFF\gamma) + m_e \\ - \frac{sm_e e}{2}(\gamma F\gamma) \bigg] \exp\left\{ -\mathrm{i}\left[m_e^2 s + \frac{X^2}{4s} + \frac{se^2}{12}(XFFX) \right] \right\}, \end{aligned} \tag{3.66}$$

where $F_{\mu\nu}$ and $\tilde{F}_{\mu\nu}$ are the strength tensor and the dual strength tensor for the external crossed field.

The Fourier transforms of the translational invariant parts of the propagators has the form:

$$\begin{aligned} S(q) = \int\limits_0^\infty \mathrm{d}s \, \mathrm{e}^{-\mathrm{i}\Omega_e} \bigg[(q\gamma) + \mathrm{i}se(q\tilde{F}\gamma)\gamma_5 - s^2 e^2 (qFF\gamma) \\ + m_e - \frac{1}{2} s m_e e(\gamma F\gamma) \bigg], \end{aligned} \tag{3.67}$$

$$G_{\mu\nu}^{(W)}(q) = -\int\limits_0^\infty \mathrm{d}s\, \mathrm{e}^{-\mathrm{i}\Omega_W} \Big[g_{\mu\nu} + 2s^2 e^2 (FF)_{\mu\nu} - 2seF_{\mu\nu} \Big], \tag{3.68}$$

$$D^{(\Phi)}(q) = \int\limits_0^\infty \mathrm{d}s\, \mathrm{e}^{-\mathrm{i}\Omega_W}, \tag{3.69}$$

where the Feynman gauge is taken for the W and Φ-bosons, and the notation is used ($j = e, W$):

$$\Omega_j = s(m_j^2 - q^2) + \frac{s^3}{3} e^2 (qFFq). \tag{3.70}$$

3.3 Direct Derivation of the Electron Propagator in a Magnetic Field as the Sum over Landau Levels on a Basis of the Dirac Equation Exact Solutions

In this section, we explore such a methodologically important issue as a direct derivation by the standard quantum field theory methods of the exact electron propagator in the external magnetic field in the form of the sum over Landau levels from the exact solutions of the Dirac equation in a magnetic field. The presentation is based on the paper [18].

To calculate the electron propagator, the standard method is applied based on using the field operators which include the Dirac equation solutions in a magnetic field:

$$\hat{\Psi}(X) = \sum_{n,\, p_y,\, p_z,\, s} \Big(a_{n,\, p_y,\, p_z,\, s} \Psi^{(+)}_{n,\, p_y,\, p_z,\, s}(X) + b^{\dagger}_{n,\, p_y,\, p_z,\, s} \Psi^{(-)}_{n,\, p_y,\, p_z,\, s}(X) \Big). \tag{3.71}$$

Here, a is the destruction operator of the electron, $b^{\dagger}$ is the creation operator of the positron, $\Psi^{(+)}$ and $\Psi^{(-)}$ are the normalized solutions of the Dirac equation (2.2) in a magnetic field with positive and negative energy correspondingly, presented in Sect. 2.1.

The propagator is defined as the difference of time-ordered and normal-ordered productions of the field operators (3.71):

$$S^{(e)}(X, X') = T\left(\hat{\Psi}(X)\overline{\hat{\Psi}}(X')\right) - \mathcal{N}\left(\hat{\Psi}(X)\overline{\hat{\Psi}}(X')\right). \tag{3.72}$$

Using anticommutation relations for the creation and destruction operators, we obtain, that the propagator at $t > t'$ and at $t < t'$ is expressed in terms of the solutions with positive energy (2.30)–(2.32) and negative energy correspondingly:

$$S^{(e)}(X, X')\Big|_{t \gtrless t'} = \pm \sum_{n,\, p_y,\, p_z,\, s} \Psi^{(\pm)}_{n,\, p_y,\, p_z,\, s}(X) \overline{\Psi}^{(\pm)}_{n,\, p_y,\, p_z,\, s}(X'). \tag{3.73}$$

Thus, the propagator is divided into the sum over Landau levels:

$$S^{(e)}(X, X') = \sum_{n=0}^{\infty} S^{(e)}_n(X, X'). \tag{3.74}$$

Further we will find the nth Landau level contribution into the propagator (3.73). It is convenient to come from the summation over the momenta p_y and p_z to the integration, by the substitution

$$\frac{1}{L_y L_z} \sum_{p_y,\, p_z} \to \int \frac{\mathrm{d}p_y \mathrm{d}p_z}{(2\pi)^2}. \tag{3.75}$$

For the nth level contribution we found:

$$\begin{aligned} S^{(e)}_n(X, X')\Big|_{t \gtrless t'} = &\int \frac{\mathrm{d}p_y\, \mathrm{d}p_z}{(2\pi)^2 2\, E_n(\pm E_n + m)} \\ &\times \exp\left\{\mathrm{i}\left[\mp E_n(t - t') \pm p_y(y - y') \pm p_z(z - z')\right]\right\} \\ &\times \sum_{s=\pm 1} U^{(\pm)}_{n,\, p_y,\, p_z,\, s}(\xi^{(\pm)})\, \overline{U}^{(\pm)}_{n,\, p_y,\, p_z,\, s}(\xi^{(\pm)\prime}). \end{aligned} \tag{3.76}$$

After simple but quite cumbersome transformations one can reduce the matrices in Eq. (3.76), which are constructed from the bispinors (2.31), (2.32) and the corresponding bispinors of the solution with negative energy, to:

$$\begin{aligned} &\frac{1}{\pm E_n + m} \sum_{s=\pm 1} U^{(\pm)}_{n,\, p_y,\, p_z,\, s}(\xi^{(\pm)})\, \overline{U}^{(\pm)}_{n,\, p_y,\, p_z,\, s}(\xi^{(\pm)\prime}) \\ &= \frac{1}{2^n\, n!}\sqrt{\frac{\beta}{\pi}} \exp\left[-\frac{1}{2}(\xi^{(\pm)})^2 - \frac{1}{2}(\xi^{(\pm)\prime})^2\right] \Big\{ \left(\pm E_n \gamma_0 \mp p_z \gamma^3 + m\right) \\ &\times \left[\Pi_-\, H_n(\xi^{(\pm)})\, H_n(\xi^{(\pm)\prime}) + \Pi_+\, 2n\, H_{n-1}(\xi^{(\pm)})\, H_{n-1}(\xi^{(\pm)\prime})\right] \\ &+ \mathrm{i}2n\sqrt{\beta}\gamma^1 \left[\Pi_-\, H_{n-1}(\xi^{(\pm)})\, H_n(\xi^{(\pm)\prime}) - \Pi_+\, H_n(\xi^{(\pm)})\, H_{n-1}(\xi^{(\pm)\prime})\right] \Big\}, \end{aligned} \tag{3.77}$$

where $\Pi_\pm$ are the projection operators (2.51). One can see, that after changing the signs of integration variables $p_y \to -p_y$ and $p_z \to -p_z$ in the expression (3.76) at $t < t'$, the $\pm$ sign at $t > t'$ and $t < t'$ still remains just in the sign at E_n. It is appropriate to use the following relation, where the expression for energy (2.24) is taken into account:

$$\frac{f(\pm E_n)}{2\,E_n}\mathrm{e}^{\mp\mathrm{i}\,E_n(t-t')}\Big|_{t\gtrless t'} = \frac{\mathrm{i}}{2\pi}\int_{-\infty}^{+\infty}\frac{\mathrm{d}p_0\,f(p_0)\,\mathrm{e}^{-\mathrm{i}p_0(t-t')}}{p_\parallel^2 - m^2 - 2\beta n + \mathrm{i}\epsilon}, \tag{3.78}$$

where $p_\parallel^2 = p_0^2 - p_z^2$.

Using the relation (3.78) we add to the expression (3.76) an integration over the zero momentum component. As a result the propagator can be written at $t > t'$ and at $t < t'$ identically. Renaming the variables $\xi^{(+)} = \xi,\ \xi^{(+)\prime} = \xi'$, we reduce (3.76) with taking into account (3.77) and (3.78) to the form

$$\begin{aligned} S_n^{(e)}(X, X') = {} & \frac{\mathrm{i}}{2^n\,n!}\sqrt{\frac{\beta}{\pi}}\exp\left(-\beta\frac{x^2 + x'^2}{2}\right)\int\frac{\mathrm{d}p_0\,\mathrm{d}p_y\,\mathrm{d}p_z}{(2\pi)^3} \\ & \times \frac{\mathrm{e}^{-\mathrm{i}(p(X-X'))_\parallel}}{p_\parallel^2 - m^2 - 2\,\beta n + \mathrm{i}\,\epsilon}\exp\left\{-\frac{p_y^2}{\beta} - p_y\left[x + x' - \mathrm{i}(y - y')\right]\right\} \\ & \times \Big\{\left[(p\gamma)_\parallel + m\right]\left[\Pi_- H_n(\xi)\,H_n(\xi') + \Pi_+ 2n H_{n-1}(\xi)\,H_{n-1}(\xi')\right] \\ & + \mathrm{i}\,2n\sqrt{\beta}\,\gamma^1\left[\Pi_- H_{n-1}(\xi)\,H_n(\xi') - \Pi_+ H_n(\xi) H_{n-1}(\xi')\right]\Big\}. \end{aligned} \tag{3.79}$$

It is worthwhile to note that the expression (3.74) with (3.79) for the electron propagator in a constant uniform magnetic field as the sum over Landau levels in the x-space has its own significance. In some cases, this form of the propagator can be more convenient than other representations.

One can make an integration over p_y in the propagator (3.79) by introducing a new variable

$$u = \frac{p_y}{\sqrt{\beta}} + \frac{\sqrt{\beta}}{2}\left[x + x' - \mathrm{i}(y - y')\right],$$

and using the well-known integrals being expressed via the Laguerre polynomials [19]:

$$\begin{aligned} & \int_{-\infty}^{\infty}\mathrm{e}^{-u^2}\,H_n(u + a)\,H_n(u + b)\,\mathrm{d}u = 2^n\,n!\,\sqrt{\pi}\,L_n(-2\,a\,b), \\ & \int_{-\infty}^{\infty}\mathrm{e}^{-u^2}\,H_n(u + a)\,H_{n-1}(u + b)\,\mathrm{d}u \\ & = 2^{n-1}\,n!\,\sqrt{\pi}\,\frac{1}{b}\left[L_n(-2\,a\,b) - L_{n-1}(-2\,a\,b)\right]. \end{aligned} \tag{3.80}$$

As a result, the nth Landau level contribution into the electron propagator in a magnetic field can be presented in the form:

$$S_n^{(e)}(X, X') = \mathrm{e}^{\mathrm{i}\Phi(X,\,X')}S_n(X - X'), \tag{3.81}$$

where $\Phi(X, X')$ is the translational and gauge non-invariant phase, which is equal for all Landau levels:

$$\Phi(X, X') = -\frac{\beta}{2}(x + x')(y - y').$$

For more details about properties of the phase, see, e.g., Sect. 3.1.2. $S_n(Z)$ is the gauge and translational invariant part of the propagator ($Z = X - X'$), represented in the form of the double integral over $p_\parallel$:

$$S_n(Z) = \frac{\mathrm{i}\beta}{2\pi}\exp\left(-\frac{\beta}{4}Z_\perp^2\right)\int\frac{\mathrm{d}^2 p_\parallel}{(2\pi)^2}\frac{\mathrm{e}^{-\mathrm{i}(p\,Z)_\parallel}}{p_\parallel^2 - m^2 - 2\beta n + \mathrm{i}\epsilon}$$
$$\times\left\{\left[(p\gamma)_\parallel + m\right]\left[\Pi_- L_n\left(\frac{\beta}{2}Z_\perp^2\right) + \Pi_+ L_{n-1}\left(\frac{\beta}{2}Z_\perp^2\right)\right]\right.$$
$$\left. + 2\mathrm{i}n\frac{(Z\,\gamma)_\perp}{Z_\perp^2}\left[L_n\left(\frac{\beta}{2}Z_\perp^2\right) - L_{n-1}\left(\frac{\beta}{2}Z_\perp^2\right)\right]\right\}. \tag{3.82}$$

Let us compare the obtained expression (3.82) with the electron propagator (3.7) expanded over Landau levels (3.37). To ensure that the expressions for the propagator are consistent, it is enough to perform in Eqs. (3.7), (3.37) the integration over the momentum components $p_x,\ p_y$, which are transverse to the field. Thus, the nth Landau level contribution to the propagator is expressed via three different integrals $\mathrm{I}_{1,2,3}(Z_\perp)$ in the Euclidean plane $(p_x,\ p_y)$:

$$S_n(z) = \int\frac{\mathrm{d}^2 p_\parallel}{(2\pi)^2}\frac{\mathrm{i}\mathrm{e}^{-\mathrm{i}(pZ)_\parallel}}{p_\parallel^2 - m^2 - 2\beta n + \mathrm{i}\epsilon}$$
$$\times\left\{\left[(p\gamma)_\parallel + m\right]\left[\mathrm{I}_1(Z_\perp) - \frac{\mathrm{i}}{2}(\gamma\varphi\gamma)\mathrm{I}_2(Z_\perp)\right] - 2\,n\mathrm{I}_3(Z_\perp)\right\}. \tag{3.83}$$

An integration over the polar angle leads to the Bessel integral:

$$\int\limits_0^{2\pi}\mathrm{e}^{\mathrm{i}\,(\xi\,\cos\varphi - n\,\varphi)}\,\mathrm{d}\varphi = 2\,\pi\,\mathrm{i}^n\,J_n(\xi), \tag{3.84}$$

where $J_n(\xi)$ is the Bessel function. As a result, the integrals $\mathrm{I}_{1,2,3}(Z_\perp)$ take the form:

$$\mathrm{I}_1(Z_\perp) = \int\frac{\mathrm{d}^2 p_\perp}{(2\pi)^2}\,d_n(v)\,\mathrm{e}^{\mathrm{i}\,(p\,Z)_\perp} = \frac{\beta}{4\pi}\int\limits_0^\infty \mathrm{d}v\,J_0\left(\sqrt{\beta}\,Z_\perp\sqrt{v}\right)\,d_n(v),$$

$$\mathrm{I}_2(Z_\perp) = \int \frac{\mathrm{d}^2 p_\perp}{(2\pi)^2}\, d_n'(v)\, \mathrm{e}^{\mathrm{i}\,(p\,Z)_\perp} = \frac{\beta}{4\pi} \int\limits_0^\infty \mathrm{d}v\, J_0\left(\sqrt{\beta}\, Z_\perp \sqrt{v}\right) d_n'(v),$$

$$\mathrm{I}_3(Z_\perp) = \int \frac{\mathrm{d}^2 p_\perp}{(2\pi)^2}\, \frac{d_n(v)}{v}\, \mathrm{e}^{\mathrm{i}\,(p\,Z)_\perp}\, (p\,\gamma)_\perp$$
$$= \mathrm{i}\, \frac{\beta^{3/2}}{4\pi}\, \frac{(Z\,\gamma)_\perp}{Z_\perp} \int\limits_0^\infty \mathrm{d}v\, J_1\left(\sqrt{\beta}\, Z_\perp \sqrt{v}\right) \frac{d_n(v)}{\sqrt{v}},$$

where $Z_\perp = \sqrt{Z_\perp^2} = \sqrt{(x - x')^2 + (y - y')^2}$. Calculating the integrals [19]:

$$\mathrm{I}_1(Z_\perp) = \frac{\beta}{4\pi} \exp\left(-\frac{\beta}{4}\, Z_\perp^2\right) \left[L_n\left(\frac{\beta}{2}\, Z_\perp^2\right) + L_{n-1}\left(\frac{\beta}{2}\, Z_\perp^2\right)\right],$$
$$\mathrm{I}_2(Z_\perp) = -\frac{\beta}{4\pi} \exp\left(-\frac{\beta}{4}\, Z_\perp^2\right) \left[L_n\left(\frac{\beta}{2}\, Z_\perp^2\right) - L_{n-1}\left(\frac{\beta}{2}\, Z_\perp^2\right)\right],$$
$$\mathrm{I}_3(Z_\perp) = -\,\mathrm{i}\, \frac{\beta}{2\pi}\, \frac{(Z\,\gamma)_\perp}{Z_\perp^2} \exp\left(-\frac{\beta}{4}\, Z_\perp^2\right) \left[L_n\left(\frac{\beta}{2}\, Z_\perp^2\right) - L_{n-1}\left(\frac{\beta}{2}\, Z_\perp^2\right)\right],$$

and substituting them into (3.83), one finally obtains the expression, which coincides with (3.82).

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Chapter 4
Particle Dispersion in External Active Media

This chapter is devoted to an analysis of the dispersion properties of photons and neutrinos in external active media: magnetic field, plasma, and magnetized plasma. Possible astrophysical manifestations of particle processes influenced by external active media are also considered.

4.1 Dispersion in Media: Main Definitions

Dispersion effects in the medium significantly affect the propagation of particles with small masses (photons, neutrinos), while other particles remain almost insensitive to the influence of the environment (e.g., axions and other Nambu–Goldstone bosons). The direct way to investigate the dispersion relations of photons and neutrinos is to analyze the link between forward scattering and refractive index.

In accordance with the general concepts of quantum field theory, particles are quantized excitations of the corresponding fields: the electromagnetic field produces photons, the electron-positron field produces the electrons, and so on. Usually, it is convenient to describe these fields by means of plane waves, characterized by a frequency ω and wavevector $\mathbf{k}$. Then the excitations of these modes have the time and spatial dependence, which is described by a factor $\exp\left[-\mathrm{i}\left(\omega t - \mathbf{kx}\right)\right]$. With a wave vector given, the frequency is determined by the dispersion relation. Since $(\omega,\ \mathbf{k})$ is a 4-vector, basing on Lorentz invariance we find that in vacuum the value $\omega^2 - \mathbf{k}^2 = m^2$ is the same for all frequencies and m is the particle mass. One consequence of the covariant dispersion relation is that the decay of the form $1 \to 2 + 3$ is possible only if $m_1 > m_2 + m_3$, so that the particle 1 in its rest frame had enough energy for production of the final state.

In a medium, dispersion relations are changed, as a rule, by the coherent interaction with the background. In the simplest case, a particle acquires an effective mass caused by the presence of a medium. For example, dispersion relation for photons in a nonrelativistic plasma is of the form $\omega^2 = \omega_P^2 + \mathbf{k}^2$, where ω_P is the so-called plasma frequency, defined by the expression

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$$\omega_P^2 = \frac{4\pi\,\alpha\,N_e}{m_e}\,, \tag{4.1}$$

where N_e is the electron density. For a process in an environment which induces effective masses of particles, the kinematic condition for the process realization should be considered more carefully. For example, the kinematic condition for the decay $1 \to 2 + 3$, instead of the simplified vacuum relation $m_1 > m_2 + m_3$, is expressed in its original form through the squares of masses:

$$m_1^4 - 2\,m_1^2\left(m_2^2 + m_3^2\right) + \left(m_2^2 - m_3^2\right)^2 > 0\,. \tag{4.2}$$

In this form, the kinematic condition for the possibility of the process is applicable for the case of the negative effective mass squared. The appearance of the photon effective mass in a medium leads to the fact that if $\omega_P^2 > 4\,m_\nu^2$, the decay $\gamma \to \nu\bar{\nu}$ becomes kinematically allowed, which can occur in stars. In fact, this so-called plasma process is the main mechanism of the neutrino emission in a wide range of temperatures and densities, including, for example, the physical conditions inside the white dwarfs and red giants.

Note that the dispersion relation can be such that the 4-momentum $P^\mu = (E, \mathbf{p})$ be a space-like, $P^2 = E^2 - \mathbf{p}^2 < 0$. This means an appearance of the negative effective mass squared, $P^2 = m_{\text{eff}}^2 < 0$. No physical problems with such a “tachyon” would arise, because the speed of propagation is determined by the group velocity, which is always less than the speed of light. The dispersion relation in a homogeneous medium is often written in terms of the refractive index n as $k = |\mathbf{k}| = n\,\omega$. Space-like excitations correspond to the condition $n > 1$; an example of such kind is a photon in water or in air. In this case, the well-known process of the Cherenkov radiation which can be treated as the “decay” $e \to e\gamma$, is kinematically allowed for a sufficiently fast moving electrons. Similarly, the neutrino Cherenkov process $\nu \to \nu\gamma$ is possible for a massless neutrino in an external magnetic field, where the photon 4-momentum can be space-like.

Neutrinos can participate in non-standard electromagnetic processes, for example, due to the intrinsic magnetic moments. This can lead to plasma processes of the creation of sterile neutrinos, and thus, to the cooling of stars. Limits on an anomalous cooling rate derived from the observations of white dwarfs and stars of the globular star clusters, have allowed to establish the most stringent limits on the electromagnetic interactions of neutrinos.

In the standard model, all fermions are initially massless. They acquire effective masses due to interaction with the Higgs scalar field. Its vacuum expectation value Φ_0 is the main factor determining the values of the masses. Therefore, even the vacuum masses can be interpreted as a phenomenon of refraction. Since the scalar Φ_0 is a Lorentz-invariant, the dispersion relation is thus derived from the standard formula $E^2 - \mathbf{p}^2 = m^2$. In general, the active medium changes this relation, and the dependence $E(\mathbf{p})$ is usually more complicated function than $(m^2 + \mathbf{p}^2)^{1/2}$.

The dispersion relation may also depend on the polarization of the radiation. In the optically active medium, the left- and right-polarized photons have different refractive indices. In this sense, the entire medium is optically active for neutrinos, since only left-handed neutrinos are involved in interactions, and the right-handed neutrinos are sterile.

The interaction of the muon and tau neutrinos, ν_μ and ν_τ, with an ordinary astrophysical environments, that is, containing no thermal muons and tau leptons is different from the interaction of the electron neutrinos ν_e due to the contribution of the charged current $(\nu_e e^-)$ to the scattering amplitude. Therefore, this environment is a birefringent medium with respect to the flavor of neutrino, in the sense that the environment induces a variety of dispersion relations for neutrinos of different flavors. The importance of this effect for neutrino oscillations, which are actually determined by the relation of phases in the propagation of neutrinos of different flavors, is extremely high.

Considering different quantum processes in active media, one should take into account that all the particles have non-trivial dispersion properties, while it depends on the circumstances, whether an effect of refraction is significant or not. For example, the statement appeared in the literature that in a sufficiently dense plasma, where $\omega_P > 2m_e$, photons decay with a pair creation, $\gamma \to e^+e^-$. However, this is not true, because the effective masses induced by plasma, which the charged leptons also acquire, are so large that such decays do not occur [1], see also the discussion below in Sect. 4.5.3.

In addition to the modification of the particle dispersion relations, the presence of medium can lead to an appearance of entirely new excitations. The well-known example is the longitudinally polarized state of the electromagnetic field that exists in plasma in addition to the normal state with transverse polarization. These objects, usually called the longitudinal plasmons, were first discussed in 1926 by Langmuir. In many cases these quantized collective excitations play a role similar to that of ordinary particles. For example, both the usual states with the transverse polarization, called the transverse plasmons or simply photons, and longitudinal plasmons can decay into neutrino pairs and thus contribute to the plasma neutrino emission processes.

While the dispersion relations and particle interactions in a plasma are formally best described in terms of field theory at finite temperatures and densities, most of the important results of elementary particle physics in stars have been obtained before the development of this formalism by using simpler tools of kinetic theory. Indeed, for many problems in describing the dispersion properties of particles and collective effects, the kinetic approach often seems more physically transparent, leading to identical results. Further discussion is entirely based on the kinetic theory.

To obtain the dispersion relation in a plasma for a given particle with known properties, it is usually sufficient to use the simplest approximation, calculating the forward scattering amplitude off the corresponding field excitations, being the components of the plasma.

Along with the hot dense plasma, another component of active astrophysical environment, a strong magnetic field could have a significant influence on the dispersion properties of particles. However, this effect of the field is significant only in a case of the sufficiently high field intensity. There exists a natural scale of the magnetic field, the so-called critical value $B_e = m_e^2/e \simeq 4.41 \times 10^{13}$ gauss. A detailed analysis of the magnetic field influence on the photon and neutrino dispersion properties is presented below in Sects. 4.2 and 4.6.

4.2 Photon Polarization Operator in an External Magnetic Field

The dispersion properties of photons in a magnetic field are determined by the polarization operator, which can be obtained from the amplitude of the photon to photon transition, $\mathcal{M}_{\gamma\to\gamma}$:

$$\mathcal{M}_{\gamma\to\gamma} = -\varepsilon^*_\alpha \, \Pi_{\alpha\beta} \, \varepsilon_\beta \,, \tag{4.3}$$

described by the Feynman diagram shown in Fig. 4.1. In this case, the dominant role is played by the electron as a particle with a maximal specific charge, e/m_e, which is the most sensitive to the influence of an external field. The photon polarization operator in an external field was studied in a number of papers, see, e.g., [2–6]. It is convenient to represent the polarization operator in the form:

$$\Pi_{\alpha\beta} = \sum_{\lambda=1}^{3} \frac{b_\alpha^{(\lambda)} b_\beta^{(\lambda)}}{(b^{(\lambda)})^2} \, \Pi^{(\lambda)}(q), \tag{4.4}$$

where $\Pi^{(\lambda)}$ are the eigenvalues of the polarization operator, $b_\alpha^{(\lambda)}$ are the eigenvectors of the orthogonal basis:

$$\begin{aligned} b_\alpha^{(1)} &= (q\varphi)_\alpha, \qquad b_\alpha^{(2)} = (q\tilde{\varphi})_\alpha, \\ b_\alpha^{(3)} &= q^2 (q\varphi\varphi)_\alpha - q_\alpha \, (q\varphi\varphi q), \qquad b_\alpha^{(4)} = q_\alpha \,. \end{aligned} \tag{4.5}$$

The functions $\Pi^{(\lambda)}(q)$ obtained in Ref. [6] can be written as

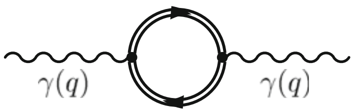

Fig. 4.1 Photon polarization operator in a strong magnetic field: the *double line* in the loop corresponds to the exact propagator of a charged fermion in a magnetic field

$$\Pi^{(\lambda)}(q) = -\frac{\alpha}{\pi}\int_0^1 \mathrm{d}u \int_0^\infty \frac{\mathrm{d}t}{t}\left[\frac{\beta t}{\sin\beta t}\,\varrho^{(\lambda)}\,\mathrm{e}^{-\mathrm{i}\Omega} - q^2\frac{1-u^2}{2}\,\mathrm{e}^{-\mathrm{i}\Omega_0}\right] + \Pi^{(\mathrm{vac})}(q^2),$$

$$\varrho^{(1)} = \frac{q_\parallel^2}{2}\left(\cos\beta tu - \frac{u\sin\beta tu}{\tan\beta t}\right) - q_\perp^2\frac{\cos\beta tu - \cos\beta t}{\sin^2\beta t},$$

$$\varrho^{(2)} = q_\parallel^2\frac{1-u^2}{2}\cos\beta t - \frac{q_\perp^2}{2}\left(\cos\beta tu - \frac{u\sin\beta tu}{\tan\beta t}\right),$$

$$\varrho^{(3)} = \frac{q^2}{2}\left(\cos\beta tu - \frac{u\sin\beta tu}{\tan\beta t}\right), \tag{4.6}$$

where

$$\Omega = \Omega_0 + \frac{q_\perp^2}{2}\left(\frac{\cos\beta tu - \cos\beta t}{\beta\sin\beta t} - \frac{1-u^2}{2}\,t\right),$$

$$\Omega_0 = t\left(m_e^2 - q^2\frac{1-u^2}{4}\right).$$

In Eq. (4.6), the subtraction is made of the vacuum polarization operator, resulting in a convergence of the integral over t, and then the renormalized vacuum polarization operator was added. The function $\Pi^{(\mathrm{vac})}(q^2)$ describes the vacuum polarization in the absence of a field and has the form, see, e.g., [7]:

$$\Pi^{(\mathrm{vac})}(q^2) = \frac{\alpha}{2\pi}\,q^2\,v(q^2)\,, \tag{4.7}$$

$$v(q^2) = \int_0^1 \mathrm{d}u\,(1-u^2)\,\ln\left(1 - \frac{q^2}{4m_e^2}\,(1-u^2)\right). \tag{4.8}$$

The dispersion equations for a real photon in a magnetic field has the form:

$$q^2 - \Pi^{(\lambda)}(q) = 0 \quad (\lambda = 1, 2, 3)\,. \tag{4.9}$$

An analysis of Eq. (4.9) shows that only two transverse polarizations, $\lambda = 1, 2$, are physical, while the third photon polarization, $\lambda = 3$, is unphysical. Indeed, substituting the expression for $\Pi^{(3)}(q)$ into Eq. (4.9), we see that it has a unique solution $q^2 = 0$. As it follows from (4.5), in this case the basis vector $b_\alpha^{(3)}$ is proportional to the photon four-momentum q_α, i.e., the corresponding operator of the electromagnetic field is proportional to the total divergence and can be removed by a gauge transformation.

The polarization vectors of photons with the certain dispersion laws are proportional to the eigenvectors $b_\alpha^{(1,2)}$:

$$\varepsilon_\alpha^{(1)} = \sqrt{\mathcal{Z}_1}\,\frac{(q\varphi)_\alpha}{\sqrt{q_\perp^2}}, \qquad \varepsilon_\alpha^{(2)} = \sqrt{\mathcal{Z}_2}\,\frac{(q\tilde{\varphi})_\alpha}{\sqrt{q_\parallel^2}}. \tag{4.10}$$

The factors $\sqrt{\mathcal{Z}_\lambda}$ are caused by *renormalization of the photon wave function*

$$\mathcal{Z}_\lambda^{-1} = 1 - \frac{\partial \Pi^{(\lambda)}}{\partial q_\parallel^2}. \tag{4.11}$$

These renormalizations are especially significant near the values of $q_\parallel^2$ corresponding to the so-called cyclotron resonances:

$$q_\parallel^2 = \left(\sqrt{m_e^2 + 2neB} + \sqrt{m_e^2 + 2n'eB}\right)^2, \tag{4.12}$$

where the functions $\Pi^{(\lambda)}(q)$ have the square-root singularity.

There exists some discordance of terms for these polarization vectors (4.10). In the classical paper by S. Adler [8] they were called as "longitudinal" $\parallel$ and the "transversal" $\perp$ photon modes, $\varepsilon_\alpha^{(1)} = \varepsilon_\alpha^{(\parallel)}$, $\varepsilon_\alpha^{(2)} = \varepsilon_\alpha^{(\perp)}$. These notations were based on the position of the *magnetic field* vector of the photon electromagnetic wave with respect to the plane formed by the vectors of external magnetic field, **B**, and of the photon momentum, **q**. Later on, some authors decided that it was more natural to consider the position of the *electric field* vector of the photon wave with respect to that plane, and they used the opposite notations; see e.g. [9], and [10]. As a result, some authors–see e.g. [11]–confused these notations, using the ones of [9] while referring to [8]. Sometimes attempts were also made to introduce another notations for these two photon polarizations, B and C, I and II–see e.g. [12]–or σ and π polarizations (to the gauge transformation); see e.g. [13]. In our previous book [14] we used the terms "ordinary" and "extraordinary" for the photon 1 and 2 polarizations in a magnetic field (4.10): $\varepsilon_\alpha^{(1)} = \varepsilon_\alpha^{(O)}$ and $\varepsilon_\alpha^{(2)} = \varepsilon_\alpha^{(E)}$. Introducing such notations, we based on the properties of these modes with respect to the CP transformation. Here, we use the notation $\varepsilon_\alpha^{(1,2)}$ (see (4.10)).

In the limit of strong fields, in the kinematic region $q_\parallel^2 \ll eB$, the expressions for the functions $\Pi^{(\lambda)}(q)$ are simplified and can be written as

$$\begin{aligned} \Pi^{(1)}(q) &= -\frac{\alpha}{3\pi}\, q_\perp^2 + \frac{\alpha}{3\pi}\, q^2 \left(\ln\frac{B}{B_e} - C - \gamma_{\rm E} + \frac{3}{2}\, v(q^2)\right) \\ &\quad + O\left(\frac{1}{eB}\right), \qquad (4.13) \\ \Pi^{(2)}(q) &= -\frac{2\alpha}{\pi}\, eB\, H\!\left(\frac{q_\parallel^2}{4m_e^2}\right) + \frac{\alpha}{3\pi}\, q^2 \left(\ln\frac{B}{B_e} - C - \gamma_{\rm E} + \frac{3}{2}\, v(q^2)\right) \end{aligned}$$

$$+ O\left(\frac{1}{eB}\right), \tag{4.14}$$

$$\Pi^{(3)}(q^2) = \frac{\alpha}{3\pi} q^2 \left(\ln\frac{B}{B_e} - C - \gamma_{\mathrm{E}} + \frac{3}{2} v(q^2)\right) + O\left(\frac{1}{eB}\right), \tag{4.15}$$

where $\gamma_{\mathrm{E}} = 0.577\ldots$ is the Euler constant, $C \simeq 1.2147$ is the numerical value of the integral

$$C = \frac{1}{2} \int\limits_0^\infty \frac{\mathrm{d}z}{z} \left(\frac{1+3z}{1+z} + \frac{3}{z\tanh z} - \frac{3}{\tanh^2 z}\right). \tag{4.16}$$

The function $H(z)$ introduced in Eq. (4.14) is

$$H(z) = \int\limits_0^1 \frac{\mathrm{d}u}{1 - z(1-u^2) - \mathrm{i}0} - 1\,. \tag{4.17}$$

In different areas of the argument the function takes the form:

$$\begin{aligned} H(z) &= \frac{1}{2\sqrt{-z(1-z)}} \ln\frac{\sqrt{1-z}+\sqrt{-z}}{\sqrt{1-z}-\sqrt{-z}} - 1\,, \quad z < 0\,, \\ H(z) &= \frac{1}{\sqrt{z(1-z)}} \arctan\sqrt{\frac{z}{1-z}} - 1\,, \quad 0 < z < 1\,, \\ H(z) &= -\frac{1}{2\sqrt{z(z-1)}} \ln\frac{\sqrt{z}+\sqrt{z-1}}{\sqrt{z}-\sqrt{z-1}} - 1 + \frac{\mathrm{i}\pi}{2\sqrt{z(z-1)}}\,, \quad z > 1\,. \end{aligned} \tag{4.18}$$

The function has the asymptotics:

$$H(z) \simeq \frac{2}{3} z + \frac{8}{15} z^2 + \frac{16}{35} z^3\,, \quad |z| \ll 1\,, \tag{4.19}$$

$$H(z) \simeq -1 - \frac{1}{2z} \ln 4|z| + \frac{\mathrm{i}\pi}{2z} \Theta(z)\,, \quad |z| \gg 1\,, \tag{4.20}$$

where $\Theta(z)$ is the step function.

It should be noted that in real calculations, the terms with q^2 contained in Eqs. (4.13) and (4.14) are inessential, because they determine the corrections of the α order, in accordance with the dispersion Eq. (4.9).

The solutions of the dispersion Eq. (4.9) for photons of the 1st and 2nd modes defined by Eqs. (4.13) and (4.14) are shown in Fig. 4.2. The dotted line corresponds to the vacuum dispersion $q^2 = 0$. In the region above this line, the square of the "photon mass" Re $\Pi^{(2)}$ has the positive sign, while below the line the sign is negative. The vertical distance from the given point of the dispersion curve to the line $q^2 = 0$ is

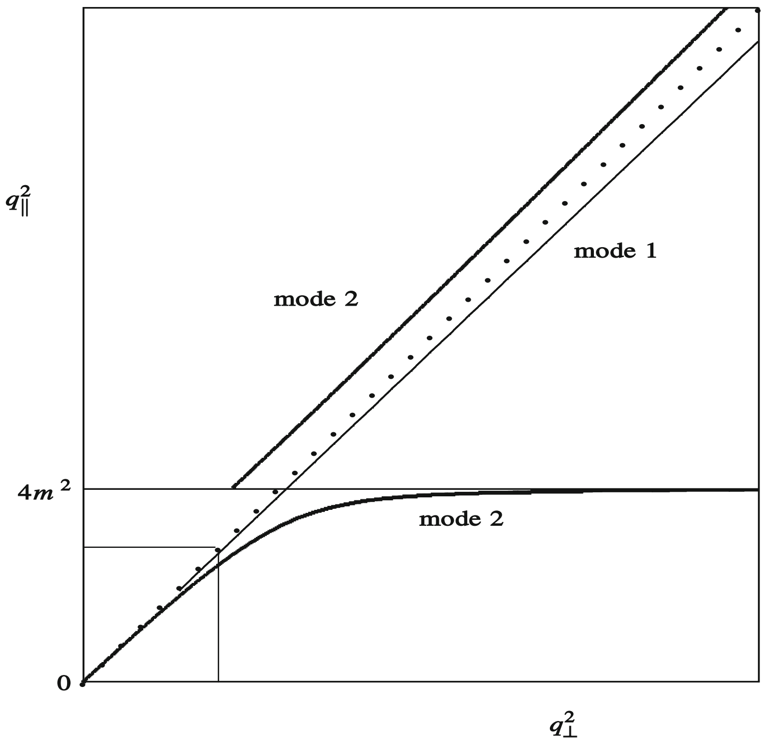

Fig. 4.2 The dispersion in a strong magnetic field of the first and second photon modes; the dispersion curve for the mode 2 photon above the *line* $q_{\parallel}^2 = 4m_e^2$ is a real part of the function $\Pi^{(2)}$ (see Eq. (4.14)); the dotted line corresponds to the vacuum dispersion at $q^2 = 0$

$|q^2|$. The line $q^2 = 0$ and the horizontal line $q_{\parallel}^2 = 4m_e^2$ divide the plane into regions corresponding to the physical processes with essentially different kinematics.

The solution of Eq. (4.9) for a photon of the 1st mode, as seen from the expression for the function (4.13), in the considered kinematic area is a straight line, slightly deviating from the vacuum line $q^2 = 0$ into the region of negative q^2.

4.3 Generalized Two-Point Loop Amplitude $j \to f\bar{f} \to j'$ in an External Electromagnetic Field

The result obtained for the photon polarization operator in external magnetic field, can be easily generalized by performing the one-loop calculation of the two-point amplitude of the transition $j \to f\bar{f} \to j'$ in a constant uniform magnetic field for various combinations of scalar, pseudoscalar, vector and pseudovector currents j and j' interacting with charged fermions. By the currents j and j', we mean generalized local quantum-field objects that can be currents, as such, or the wave functions of the corresponding particles. In this section, we present the basic points of such calculation in a magnetic field and in a crossed field, and give the results in the cases of the vector and pseudovector currents j and j'. As a charged fermion, we consider an electron as a particle with a maximal specific charge, e/m_e, which is the most sensitive to the influence of an external field. Both a more detailed calculation procedure and the results for the other combinations of currents are presented in Refs. [14, 15].

The field-induced one-loop contributions to the amplitude for the transition $j \to f\bar{f} \to j'$, presented here, can be used in the investigations of both tree-level and loop-level quantum processes in external electromagnetic fields. The field effects are taken into account exactly, because exact solutions of the Dirac equation are used. Owing to this, the expression obtained here for the amplitude is quite general. The amplitude $\Delta\mathcal{M}_{VV}$ defines, for example, the field-induced part of the photon polarization operator. Upon the substitutions

$$j_{V\alpha} \to \frac{G_{\mathrm{F}}}{\sqrt{2}} C_V j_\alpha^{(\nu)}, \quad j_{A\alpha} \to \frac{G_{\mathrm{F}}}{\sqrt{2}} C_A j_\alpha^{(\nu)}, \quad j'_{V\alpha} \to e\varepsilon_\alpha, \tag{4.21}$$

the sum of $\Delta\mathcal{M}_{VV}$ and $\Delta\mathcal{M}_{VA}$ describes the process amplitude for the radiative transition of massless neutrino $\nu \to \nu\gamma$. In (4.21), C_V and C_A are, respectively, the vector and axial-vector coupling constants in the effective Lagrangian for neutrino interaction with electrons in the Standard Model; $j_\alpha^{(\nu)}$ is the neutrino current; and ε_α is the photon polarization vector. Similarly, combining the amplitudes $\Delta\mathcal{M}_{VV}$, $\Delta\mathcal{M}_{AA}$ and $\Delta\mathcal{M}_{VA}$ where the neutrino currents (4.21) are substituted, one can also analyze the process $\nu\bar{\nu} \to e^- e^+$ by using the imaginary parts of the amplitudes.

4.3.1 Magnetic Field

The generalized amplitude of the transition $j \to f\bar{f} \to j'$ will be analyzed by using the effective Lagrangian for the interaction of the current j with electrons in the form

$$\mathcal{L}(X) = \sum_n j_n(X) \left(\overline{\hat{\Psi}}(X) \Gamma_n \hat{\Psi}(X) \right), \tag{4.22}$$

where $\hat{\Psi}(X)$ is the field operator (2.1), the generic index $n = S, P, V, A$ numbers the matrices

$$\Gamma_n = 1, \gamma_5, \gamma_\alpha, \gamma_5\gamma_\alpha, \tag{4.23}$$

while $j_n(X)$ is the generalized current including the coupling constant.

The one-loop amplitude for the transition $j \to j'$ is described by the Feynman diagram in the Fig. 4.3, and has the form

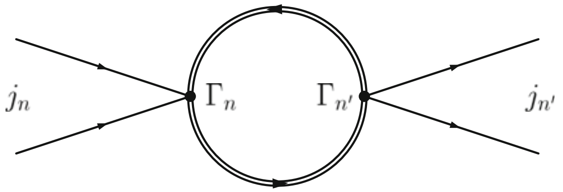

Fig. 4.3 Feynman diagram for the transition $j \to j'$. *Double lines* indicate that the effect of an external electromagnetic field is taken exactly into account in the propagators of virtual fermions

$$\mathcal{M}_{nn'} = -\,\mathrm{i}\, j_n j_{n'} \int \mathrm{d}^4 Z\ \mathrm{Tr}\,[S(-Z)\varGamma_n S(Z)\varGamma_{n'}]\,\mathrm{e}^{-\mathrm{i}qZ}. \tag{4.24}$$

Here, $S(Z)$ is the translational invariant part of the fermion propagator in a magnetic field (3.2), $Z = X - X'$, j_n and $j_{n'}$ are the Fourier transforms of the corresponding currents, while q is the momentum transfer. From expression (3.2) for the propagator, it can be seen that the amplitude in (4.24) diverges at the lower limit of integration with respect to the proper time. This divergence, an ultraviolet one, as a matter of fact, is due the use of a local limit in the Lagrangian (4.22). Below, only the field-induced part of the amplitude will be analyzed,

$$\Delta\mathcal{M}_{nn'} = \mathcal{M}_{nn'} - \mathcal{M}_{nn'}\Big|_{B=0}. \tag{4.25}$$

As can be deduced from the corresponding analysis, the difference in (4.25) is free from ultraviolet divergences.

Given the bilinear dependence of the phase of the translational invariant part $S(Z)$ (3.2) of the fermion propagator on the Z variable, the integration with respect to Z in the expression for the amplitude (4.24) is reduced to the calculation of the generalized Gaussian integrals of the scalar, vector, and tensor types. The scalar integral has the form

$$\varPhi = \int \mathrm{d}^4 Z\ \exp\left[-\mathrm{i}\left((Zp) + \frac{1}{4}(ZGZ)\right)\right], \tag{4.26}$$

where

$$G^{\mu\nu} = \frac{v+s}{vs}\,\tilde{\varLambda}^{\mu\nu} - \beta\,\frac{\sin(\beta(v+s))}{\sin(\beta v)\,\sin(\beta s)}\,\varLambda^{\mu\nu}.$$

Here, $\beta = eB$, the variables v and s are the Fock proper-times in electron propagators. The matrices $\varLambda_{\mu\nu}$ and $\tilde{\varLambda}_{\mu\nu}$ are defined in (1.2). The vector and tensor integrals can be defined from the scalar one by taking the derivatives of $\varPhi$ with respect to the momentum p:

$$\Phi_\mu = \int \mathrm{d}^4Z\, Z_\mu \exp\left[-\mathrm{i}\left((Zp) + \frac{1}{4}(ZGZ)\right)\right] = \mathrm{i}\frac{\partial\Phi}{\partial p_\mu}, \tag{4.27}$$

$$\Phi_{\mu\nu} = \int \mathrm{d}^4Z\, Z_\mu Z_\nu \exp\left[-\mathrm{i}\left((Zp) + \frac{1}{4}(ZGZ)\right)\right] = -\frac{\partial^2\Phi}{\partial p_\mu \partial p_\nu}. \tag{4.28}$$

Performing integrations over the spacetime variable Z one obtains

$$\begin{aligned} \Phi &= -(4\pi)^2(\det G)^{-1/2} \exp\left(\mathrm{i}(pG^{-1}p)\right), \\ \Phi_\mu &= -2(pG^{-1})_\mu \Phi\,, \\ \Phi_{\mu\nu} &= 2\left[2(pG^{-1})_\mu (pG^{-1})_\nu - \mathrm{i}G^{-1}_{\mu\nu}\right]\Phi\,, \end{aligned} \tag{4.29}$$

where the inverse matrix G^{-1} is

$$G^{-1}_{\mu\nu} = \frac{vs}{v+s}\,\tilde{\Lambda}_{\mu\nu} - \frac{\sin(\beta v)\,\sin(\beta s)}{\beta\sin(\beta(v+s))}\,\Lambda_{\mu\nu}\,,$$

and the determinant of the G matrix is

$$\det\ G = -\left\{\frac{(v+s)\,\beta\,\sin(\beta(v+s))}{sv\,\sin(\beta v)\,\sin(\beta s)}\right\}^2. \tag{4.30}$$

After performing integrations over Z, the generalized amplitude can be expressed in the form of a double integral.

Here, we present a complete set of expressions for the amplitudes $\Delta\mathcal{M}_{nn'}$ in the magnetic field ($n, n' = V, A$).

If one of the currents is a vector one ($j_n \equiv j_{V\alpha}$, $\Gamma_n \equiv \gamma_\alpha$), it can be shown by a direct calculation that this currents appears in the amplitude only through the combination $f_{\alpha\beta} = q_\alpha j_{V\beta} - q_\beta j_{V\alpha}$. If, in addition, the current j_V appears to be the photon polarization vector, the tensor $f_{\alpha\beta}$ has the meaning of the strength tensor of the photon electromagnetic field. This corresponds to the gauge invariance of the amplitude for the processes being considered.

Thus, the vector–vector amplitude ($n, n' = V$) is described in terms of the tensors $f_{\alpha\beta}$ and $f'_{\alpha\beta}$; that is,

$$\begin{aligned} \Delta\mathcal{M}_{VV} = \frac{1}{4\pi^2}\Bigg[&\frac{(f\varphi)(f'^*\varphi)}{4q_\perp^2} Y^{(1)}_{VV} + \frac{(f\tilde{\varphi})(f'^*\tilde{\varphi})}{4q_\parallel^2} Y^{(2)}_{VV} \\ &+ \frac{(q\varphi\varphi f q)(q\varphi\varphi f'^* q)}{q^2 q_\parallel^2 q_\perp^2} Y^{(3)}_{VV}\Bigg], \end{aligned} \tag{4.31}$$

where

$$
\begin{aligned}
Y_{VV}^{(i)} &= \int\limits_0^1 \mathrm{d}u \int\limits_0^\infty \frac{\mathrm{d}t}{t}\left[\frac{\beta t}{\sin\beta t}\, y_{VV}^{(i)}\,\mathrm{e}^{-\mathrm{i}\Omega} - q^2\frac{1-u^2}{2}\,\mathrm{e}^{-\mathrm{i}\Omega_0}\right],\\
y_{VV}^{(1)} &= \frac{q_\parallel^2}{2}\left(\cos\beta tu - \frac{u\sin\beta tu}{\tan\beta t}\right) - q_\perp^2\,\frac{\cos\beta tu - \cos\beta t}{\sin^2\beta t},\\
y_{VV}^{(2)} &= q_\parallel^2\frac{1-u^2}{2}\cos\beta t - \frac{q_\perp^2}{2}\left(\cos\beta tu - \frac{u\sin\beta tu}{\tan\beta t}\right),\\
y_{VV}^{(3)} &= \frac{q^2}{2}\left(\cos\beta tu - \frac{u\sin\beta tu}{\tan\beta t}\right),\\
f_{\alpha\beta} &= q_\alpha j_{V\beta} - q_\beta j_{V\alpha},\;\; f'_{\alpha\beta} = q_\alpha j'_{V\beta} - q_\beta j'_{V\alpha}.
\end{aligned}
$$

In the above expressions, the as-yet-undefined quantities are given by

$$
\begin{aligned}
\Omega_0 &= t\left(m_e^2 - q^2\frac{1-u^2}{4}\right),\\
\Omega &= \Omega_0 + \frac{q_\perp^2}{2}\left(\frac{\cos\beta tu - \cos\beta t}{\beta\sin\beta t} - \frac{1-u^2}{2}t\right),\\
q_\perp^2 &= (q\varphi\varphi q) = q_\mu\varphi^{\mu\nu}\varphi_{\nu\rho}q^\rho,\\
q_\parallel^2 &= (q\tilde\varphi\tilde\varphi q),\;\; q_\parallel^2 - q_\perp^2 = q^2.
\end{aligned}
$$

The amplitude for transitions between axial-vector currents ($n, n' = A$) has the form

$$
\begin{aligned}
\Delta\mathcal{M}_{AA} = \frac{1}{4\pi^2}\Bigg[&\frac{(j_A\varphi q)(j_A'^{*}\varphi q)}{q_\perp^2}Y_{AA}^{(1)} + \frac{(j_A\tilde\varphi q)(j_A'^{*}\tilde\varphi q)}{q_\parallel^2}Y_{AA}^{(2)}\\
&+ \frac{q^2}{q_\parallel^2 q_\perp^2}(j_A\varphi\varphi q)(j_A'^{*}\varphi\varphi q)Y_{AA}^{(3)}\\
&- \frac{(j_A\varphi\varphi q)(j_A'^{*}q) + (j_A'^{*}\varphi\varphi q)(j_A q)}{q_\parallel^2}Y_{AA}^{(4)} + \frac{(j_A q)(j_A'^{*}q)}{q_\parallel^2}Y_{AA}^{(5)}\Bigg],
\end{aligned} \tag{4.32}
$$

where

$$
\begin{aligned}
Y_{AA}^{(i)} &= \int\limits_0^1 \mathrm{d}u \int\limits_0^\infty \frac{\mathrm{d}t}{t}\left[\frac{\beta t}{\sin\beta t}\, y_{AA}^{(i)}\,\mathrm{e}^{-\mathrm{i}\Omega} + \left(2m_e^2 - q^2\frac{1-u^2}{2}\right)\mathrm{e}^{-\mathrm{i}\Omega_0}\right],\\
i &= 1, 2, 3, 4,
\end{aligned}
$$

$$Y_{AA}^{(5)} = \int_0^1 \mathrm{d}u \int_0^\infty \frac{\mathrm{d}t}{t}\left[\frac{\beta t}{\sin\beta t}\, y_{AA}^{(5)}\, \mathrm{e}^{-\mathrm{i}\Omega} - \left(2m_e^2 + q_\perp^2 \frac{1-u^2}{2}\right)\mathrm{e}^{-\mathrm{i}\Omega_0}\right].$$

In the above expressions, the following notations are used:

$$y_{AA}^{(1)} = \frac{q_\parallel^2}{2}\left(\cos\beta tu - \frac{u\sin\beta tu}{\tan\beta t}\right) - q_\perp^2 \frac{\cos\beta tu - \cos\beta t}{\sin^2\beta t} - 2m_e^2\cos\beta tu,$$

$$y_{AA}^{(2)} = q_\parallel^2 \frac{1-u^2}{2}\cos\beta t - \frac{q_\perp^2}{2}\left(\cos\beta tu - \frac{u\sin\beta tu}{\tan\beta t}\right) - 2m_e^2\cos\beta t,$$

$$y_{AA}^{(3)} = \frac{q^2}{2}\left(\cos\beta tu - \frac{u\sin\beta tu}{\tan\beta t}\right) + \frac{2m_e^2}{q^2}\left(q_\perp^2\cos\beta t - q_\parallel^2\cos\beta tu\right),$$

$$y_{AA}^{(4)} = \frac{q^2}{2}\left(\cos\beta tu - \frac{u\sin\beta tu}{\tan\beta t}\right) - 2m_e^2\cos\beta t,$$

$$y_{AA}^{(5)} = \frac{q_\perp^2}{2}\left(\cos\beta tu - \frac{u\sin\beta tu}{\tan\beta t}\right) + 2m_e^2\cos\beta t.$$

In the case of the vector and axial-vector vertices ($\Gamma_n \equiv \gamma_\alpha,\ \Gamma_{n'} \equiv \gamma_5\gamma_\beta$), the field-induced part of the amplitude is given by

$$\Delta\mathcal{M}_{VA} = -\frac{1}{4\pi^2}\beta\left[\frac{(f\tilde{\varphi})(j_A^*\varphi\varphi q)}{2q_\parallel^2} Y_{VA}^{(1)} + \frac{(j_A^*\tilde{\varphi}q)(qf\varphi\varphi q)}{q_\parallel^2 q_\perp^2} Y_{VA}^{(2)} + \frac{(f\tilde{\varphi})(j_A^* q)}{2q_\parallel^2} Y_{VA}^{(3)}\right], \tag{4.33}$$

where

$$Y_{VA}^{(1)} = \mathrm{i}\int_0^1 \mathrm{d}u \int_0^\infty \mathrm{d}t \left(\frac{q_\perp^2 + q_\parallel^2}{q_\perp^2} m_e^2 - q^2 \frac{q_\parallel^2}{q_\perp^2}\frac{1-u^2}{4}\right)\mathrm{e}^{-\mathrm{i}\Omega} - \frac{q_\parallel^2}{q_\perp^2},$$

$$Y_{VA}^{(2)} = \mathrm{i}\int_0^1 \mathrm{d}u \int_0^\infty \mathrm{d}t \left(m_e^2 - q_\parallel^2 \frac{1-u^2}{4}\right)\mathrm{e}^{-\mathrm{i}\Omega} - \frac{q_\parallel^2}{q^2},$$

$$Y_{VA}^{(3)} = \mathrm{i}\int_0^1 \mathrm{d}u \int_0^\infty \mathrm{d}t \left(m_e^2 + q_\parallel^2 \frac{1-u^2}{4}\right)\mathrm{e}^{-\mathrm{i}\Omega}.$$

It should be emphasized that, in using our results to calculate the amplitudes of processes featuring axial-vector currents, care should be taken in dealing with terms linear in an external field in diagrams of the type shown in Fig. 4.3. The point is that such terms may prove incorrect because of the Adler triangle anomaly. Strictly speaking, it is therefore necessary to specify a procedure for subtracting terms linear in the field, which must then be recovered. Thus, the correct expression for the field-induced part of the amplitude must have the form

$$\Delta\widetilde{\mathcal{M}} = \left(\mathcal{M} - \mathcal{M}\Big|_{B=0} - B\frac{\partial \mathcal{M}}{\partial B}\Big|_{B=0}\right) + \widetilde{\mathcal{M}}^{(1)}, \tag{4.34}$$

where the expression in parentheses is free from the Adler anomaly. A scheme for recovering the correct form of the term $\widetilde{\mathcal{M}}^{(1)}$ linear in the field is determined by a specific type of process and by the origin of the triangle anomaly. An example of recovering the linear term for the vector—axial-vector part of the amplitude of the neutrino Cherenkov process in a strong magnetic field, $\nu \to \nu + \gamma$ [10, 16], is presented below in Sect. 7.1.1. In this case, the origin of the triangle anomaly is connected with the transition to the local limit of weak interaction.

4.3.2 Crossed Field

The amplitude for the transition $j \to j'$ in a crossed field can be derived by performing once again the calculations outlined in the previous section, but the fermion propagator in a crossed field (3.66) should be used now.

The field-induced parts of the amplitudes $\Delta\mathcal{M}_{nn'}$ $(n, n' = V, A)$ can be written as follows.

The vector—vector amplitude is:

$$\Delta\mathcal{M}_{VV} = \frac{1}{4\pi^2}\left[\frac{(fF)(f'^*F)}{4(qFFq)} Y_{VV}^{(1)} + \frac{(f\tilde{F})(f'^*\tilde{F})}{4(qFFq)} Y_{VV}^{(2)} \right.$$
$$\left. + \frac{(qFFfq)(qFFf'^*q)}{q^2(qFFq)^2} Y_{VV}^{(3)}\right], \tag{4.35}$$

where

$$Y_{VV}^{(1)} = -\int_0^1 \mathrm{d}u\left[\frac{1}{6}m_e^2\chi_q^{2/3}\left(\frac{4}{1-u^2}\right)^{1/3}(3+u^2)\frac{\mathrm{d}f(x)}{\mathrm{d}x} - q^2\frac{1-u^2}{2}f_1(x)\right],$$

$$Y_{VV}^{(2)} = -\int_0^1 \mathrm{d}u\left[\frac{1}{3}m_e^2\chi_q^{2/3}\left(\frac{4}{1-u^2}\right)^{1/3}(3-u^2)\frac{\mathrm{d}f(x)}{\mathrm{d}x} - q^2\frac{1-u^2}{2}f_1(x)\right],$$

$$Y_{VV}^{(3)} = \frac{q^2}{2}\int_0^1 \mathrm{d}u(1-u^2)f_1(x).$$

Here, the notations are used:

$$\chi_q^2 = \frac{e^2(qFFq)}{m_e^6},$$
$$x = \left(\frac{4}{\chi_q(1-u^2)}\right)^{2/3}\left(1 - \frac{q^2}{4m_e^2}(1-u^2)\right),$$
$$f(x) = \mathrm{i}\int_0^\infty \mathrm{d}t\ \mathrm{e}^{-\mathrm{i}(tx+\frac{t^3}{3})}, \tag{4.36}$$
$$f_1(x) = \int_0^\infty \frac{\mathrm{d}t}{t}\left(\mathrm{e}^{-\mathrm{i}(tx+\frac{t^3}{3})} - \mathrm{e}^{-\mathrm{i}tx}\right)$$
$$= -\int_0^x f(z)\mathrm{d}z + \ln x + \frac{1}{3}\ln 3 + \frac{2}{3}\gamma_{\mathrm{E}} + \frac{\mathrm{i}\pi}{3}, \tag{4.37}$$

$f(x)$ being the Hardy—Stokes function, $\gamma_{\mathrm{E}} = 0.577\ldots$ being the Euler constant.
The axial vector—axial vector amplitude is:

$$\begin{aligned}\Delta\mathcal{M}_{AA} = \frac{1}{4\pi^2}\Bigg[&\frac{(j_A Fq)(j_A'^* Fq)}{(qFFq)}Y_{AA}^{(1)} + \frac{(j_A\tilde{F}q)(j_A'^*\tilde{F}q)}{(qFFq)}Y_{AA}^{(2)}\\ &+ q^2\frac{(j_A FF j_A'^*)}{(qFFq)}Y_{AA}^{(3)}\\ &- \frac{(j_A FFq)(j_A'^* q) + (j_A'^* FFq)(j_A q)}{(qFFq)}Y_{AA}^{(4)} + (j_A q)(j_A'^* q)Y_{AA}^{(5)}\Bigg],\end{aligned} \tag{4.38}$$

where

$$Y_{AA}^{(1,2)} = Y_{VV}^{(1,2)} - 2m_e^2\int_0^1 \mathrm{d}u f_1(x),$$
$$\begin{aligned}Y_{AA}^{(3)} = -\int_0^1 \mathrm{d}u\Bigg[&4\frac{m_e^2}{q^2}m_e^2\chi_q^{2/3}\left(\frac{4}{1-u^2}\right)^{1/3}\frac{\mathrm{d}f(x)}{\mathrm{d}x}\\ &+\left(2m_e^2 - q^2\frac{1-u^2}{2}\right)f_1(x)\Bigg],\end{aligned}$$
$$Y_{AA}^{(4)} = -\int_0^1 \mathrm{d}u\left(2m_e^2 - q^2\frac{1-u^2}{2}\right)f_1(x),$$

$$Y^{(5)}_{AA} = \int\limits_0^1 \mathrm{d}u \frac{1-u^2}{2} f_1(x).$$

The vector—axial vector amplitude is:

$$\Delta\mathcal{M}_{VA} = -\frac{e}{4\pi^2}\left[\frac{(f\tilde{F})(j_A^* FFq)}{2(qFFq)} Y^{(1)}_{VA} + \frac{(j_A^*\tilde{F}q)(qfFFq)}{q^2(qFFq)} Y^{(2)}_{VA} + \frac{(f\tilde{F})(j_A^* q)}{2q^2} Y^{(3)}_{VA}\right], \tag{4.39}$$

where

$$\begin{aligned}
Y^{(1)}_{VA} &= -\frac{1}{m_e^2 \chi_q^{2/3}} \int\limits_0^1 \mathrm{d}u \left(\frac{4}{1-u^2}\right)^{2/3} \left(2m_e^2 - q^2 \frac{1-u^2}{4}\right) f(x) + 1,\\
Y^{(2)}_{VA} &= -Y^{(3)}_{VA} + 1,\\
Y^{(3)}_{VA} &= -\frac{1}{m_e^2 \chi_q^{2/3}} q^2 \int\limits_0^1 \mathrm{d}u \left(\frac{1-u^2}{4}\right)^{1/3} f(x).
\end{aligned}$$

It should be noted that, in general, the expression for the amplitude $\Delta\mathcal{M}_{VA}$ involves indefinite forms associated with the Adler anomaly. The procedure for removing them is described above in (4.34).

The expressions obtained for the amplitudes in a crossed field can be used to test the correctness of a more cumbersome calculation in the presence of a magnetic field. If, in the amplitudes calculated in the previous section, the field invariant $\beta \sim [-(FF)]^{1/2}$ is made to tend to zero in such a way that the field tensor $eF_{\alpha\beta} = \beta\varphi_{\alpha\beta}$ remains finite, the required amplitudes in a crossed field can be obtained from the resulting expressions.

4.4 Photon Polarization Operator in Plasma

In describing the electromagnetic processes with virtual photons in plasma, the principal point is to use the photon propagator $G_{\alpha\beta}(Q)$ with the plasma polarization effects taken into account. We use the straightforward way of taking account of these effects by summation of the Feynman diagrams of the forward photon scattering off plasma particles. Similarly to the vacuum case, this summation leads to the Dyson equation which provides a correct result for the photon propagator in plasma in the region where the photon polarization operator is real, in the form:

$$G_{\alpha\beta}(Q) = \frac{\mathrm{i}\,\rho^{(t)}_{\alpha\beta}}{Q^2 - \Pi_t} + \frac{\mathrm{i}\,\rho^{(\ell)}_{\alpha\beta}}{Q^2 - \Pi_\ell}\,, \tag{4.40}$$

where $\Pi_{t,\ell}$ are the eigenvalues of the photon polarization tensor $\Pi_{\alpha\beta}$ for the transverse and longitudinal plasmon,

$$\Pi_{\alpha\beta} = -\Pi_t\,\rho^{(t)}_{\alpha\beta} - \Pi_\ell\,\rho^{(\ell)}_{\alpha\beta}\,, \tag{4.41}$$

and $\rho^{(t,\ell)}_{\alpha\beta}$ are the corresponding density matrices

$$\rho^{(t)}_{\alpha\beta} = -\left(g_{\alpha\beta} - \frac{Q_\alpha Q_\beta}{Q^2} - \frac{L_\alpha L_\beta}{L^2}\right), \tag{4.42}$$

$$\rho^{(\ell)}_{\alpha\beta} = -\frac{L_\alpha L_\beta}{L^2}\,, \tag{4.43}$$

$$L_\alpha = Q_\alpha\,(u\,Q) - u_\alpha\,Q^2\,, \tag{4.44}$$

u_α is the four-vector of the plasma velocity. The density matrices $\rho^{(\lambda)}_{\alpha\beta}$ with $\lambda = t, \ell$ have properties of the projection operators:

$$\rho^{(\lambda)}_{\alpha\mu}\,\rho^{\mu(\lambda')}_{\beta} = -\delta_{\lambda\lambda'}\,\rho^{(\lambda)}_{\alpha\beta}\,. \tag{4.45}$$

In the region where the eigenvalues $\Pi_{t,\,\ell}$ of the photon polarization tensor develop imaginary parts, they can be written as:

$$\Pi_\lambda = R_\lambda + \mathrm{i}\,I_\lambda\,, \tag{4.46}$$

where R_λ and I_λ are the real and imaginary parts, containing the contributions of all components of the active medium. For extracting the imaginary parts $I_{t,\,\ell}$, it will suffice to make an analytical extension $q_0 \to q_0 + \mathrm{i}\,\epsilon$ corresponding to the retarded polarization operator.

The eigenvalues $\Pi_{t,\,\ell}$ of the photon polarization tensor are presented below both in the general form and in some particular cases.

The expressions for the contributions of a charged fermion into the polarization functions $\Pi_{t,\,\ell}$ in the hard thermal loop approximation can be found e.g. in [17] and have the form

$$\Pi_t = \frac{4\alpha}{\pi}\int\limits_0^\infty \frac{\mathrm{d}\mathcal{P}\mathcal{P}^2}{\mathcal{E}}\,\left[f_F(\mathcal{E}) + \bar{f}_F(\mathcal{E})\right]$$
$$\times\left(\frac{q_0^2}{q^2} - \frac{q_0^2 - q^2}{q^2}\,\frac{q_0}{2vq}\,\ln\frac{q_0 + vq}{q_0 - vq}\right), \tag{4.47}$$

$$\Pi_\ell = \frac{4\alpha}{\pi}\,\frac{q_0^2 - q^2}{q^2} \int\limits_0^\infty \frac{\mathrm{d}\mathcal{P}\,\mathcal{P}^2}{\mathcal{E}}\,\left[f_F(\mathcal{E}) + \bar{f}_F(\mathcal{E})\right] \times \left(\frac{q_0}{vq}\ln\frac{q_0 + vq}{q_0 - vq} - \frac{q_0^2 - q^2}{q_0^2 - v^2 q^2} - 1\right), \tag{4.48}$$

where $\mathcal{E} = \sqrt{\mathcal{P}^2 + m_f^2}$, $v = \mathcal{P}/\mathcal{E}$, m_f is the effective fermion mass in plasma, and the Fermi–Dirac distribution functions for the fermions and anti-fermions are

$$f_F(\mathcal{E}) = \frac{1}{\mathrm{e}^{(\mathcal{E}-\mu)/T} + 1}\,, \qquad \bar{f}_F(\mathcal{E}) = \frac{1}{\mathrm{e}^{(\mathcal{E}+\mu)/T} + 1}\,, \tag{4.49}$$

μ is the fermion chemical potential.

For the supernova core conditions, the main contribution comes from the plasma electrons and protons:

$$R_{t,\,\ell} \simeq R^{(e)}_{t,\,\ell} + R^{(p)}_{t,\,\ell}\,, \qquad I_{t,\,\ell} \simeq I^{(e)}_{t,\,\ell} + I^{(p)}_{t,\,\ell}\,. \tag{4.50}$$

In these conditions, there is a good approximation to consider the electron fraction as the relativistic plasma ($\mu_e, T \gg m_e$).

The real and imaginary parts (4.50) of the electron contributions into the photon polarization functions take the following form:

$$R_t^{(e)} = m_\gamma^2 \left(x^2 + \frac{x\left(1 - x^2\right)}{2}\,\ln\left|\frac{1+x}{1-x}\right|\right), \tag{4.51}$$

$$I_t^{(e)} = -\frac{\pi}{2}\,m_\gamma^2\,x\left(1 - x^2\right), \tag{4.52}$$

$$R_\ell^{(e)} = 2\,m_\gamma^2\left(1 - x^2\right)\left(1 - \frac{x}{2}\,\ln\left|\frac{1+x}{1-x}\right|\right), \tag{4.53}$$

$$I_\ell^{(e)} = \pi\,m_\gamma^2\,x\left(1 - x^2\right), \tag{4.54}$$

where $x = q_0/q$, $|x| < 1$, m_γ is the so-called photon thermal mass,

$$m_\gamma^2 = \frac{2\,\alpha}{\pi}\left(\mu_e^2 + \frac{\pi^2 T^2}{3}\right). \tag{4.55}$$

For the proton contributions, the situation appears to be more complicated. For the real and imaginary parts of the proton contribution into the polarization functions (4.47), (4.48), for the conditions $\mu_p \gg T$, where μ_p is the proton chemical potential, one obtains:

$$R_t^{(p)} = \frac{4\alpha}{\pi} \int\limits_0^\infty \frac{\mathrm{d}\mathcal{P}\,\mathcal{P}^2}{\mathcal{E}\left(\mathrm{e}^{(\mathcal{E}-\mu_p)/T}+1\right)} \left(x^2 + \frac{x\left(1-x^2\right)}{2v} \ln\left|\frac{x+v}{x-v}\right| \right), \tag{4.56}$$

$$I_t^{(p)} = -2\alpha\, x\left(1-x^2\right) \int\limits_{\mathcal{P}_{\min}}^\infty \frac{\mathrm{d}\mathcal{P}\,\mathcal{P}}{\mathrm{e}^{(\mathcal{E}-\mu_p)/T}+1}\,, \quad \mathcal{P}_{\min} = \frac{m_p|x|}{\sqrt{1-x^2}}\,, \tag{4.57}$$

$$R_\ell^{(p)} = \frac{4\alpha}{\pi}\left(1-x^2\right) \int\limits_0^\infty \frac{\mathrm{d}\mathcal{P}\,\mathcal{P}^2}{\mathcal{E}\left(\mathrm{e}^{(\mathcal{E}-\mu_p)/T}+1\right)}$$
$$\times \left(1 + \frac{1-x^2}{v^2-x^2} - \frac{x}{v}\ln\left|\frac{x+v}{x-v}\right| \right), \tag{4.58}$$

$$I_\ell^{(p)} = -2\, I_t^{(p)} + 2\alpha\, m_p^2\, x \left[\exp\left(\frac{m_p}{T\sqrt{1-x^2}} - \frac{\mu_p}{T} \right) + 1 \right]^{-1}, \tag{4.59}$$

where m_p is the effective proton mass in plasma [18]. For example, at the nuclear density $3 \times 10^{14}\,\mathrm{g/cm^3}$, one has $m_p \simeq 700\,\mathrm{MeV}$.

The proton chemical potential μ_p is defined from the equation

$$N_p \simeq N_e \simeq \frac{\mu_e^3}{3\,\pi^2} = \frac{1}{\pi^2} \int\limits_0^\infty \frac{\mathrm{d}\mathcal{P}\,\mathcal{P}^2}{\mathrm{e}^{(\mathcal{E}-\mu_p)/T}+1}\,. \tag{4.60}$$

As the analysis of Eq. (4.60) shows, the difference $\mu_p - m_p$ (the so-called non-relativistic proton chemical potential) appears to be of the positive sign at the temperatures $T \simeq 30-60\,\mathrm{MeV}$, and of the same order of magnitude, as the temperature. Thus, in the supernova core conditions both the approximations of the degenerate Fermi gas and of the classical Boltzmann gas should be, in general, hardly applicable for protons. However, as it will be shown later in Sect. 5.4, the observables such as the neutrino luminosity appear to be rather stable with respect to the choice of the approximation for the proton distribution function.

In the Figs. 4.4, 4.5, 4.6, and 4.7, we present for the sake of illustration the electron and proton contributions into the eigenvalues $\Pi_{\ell,t}$ for the longitudinal and transverse plasmon. It is seen that the electron and proton contributions are of the same order of magnitude.

Together with electrons and protons, in general, a small fraction Y_i of the free ions could also present in plasma, $Y_i = N_i/N_B$, N_B is the barion density. This fraction can be considered with a good accuracy as the classical Boltzmann gas. The real and imaginary parts of the corresponding polarization functions have the form:

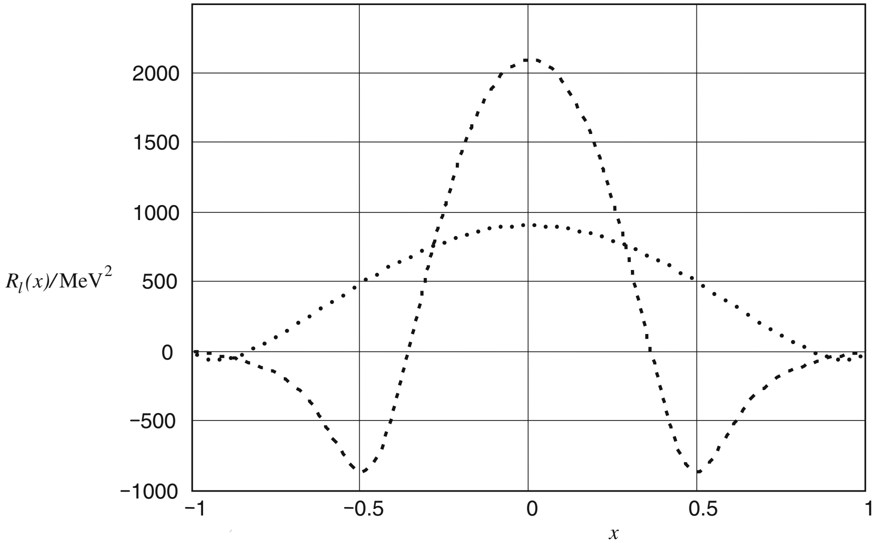

Fig. 4.4 Electron contribution (*dotted line*) and proton contribution (*dashed line*) at $T = 30\,\mathrm{MeV}$ to the real part of Π_ℓ

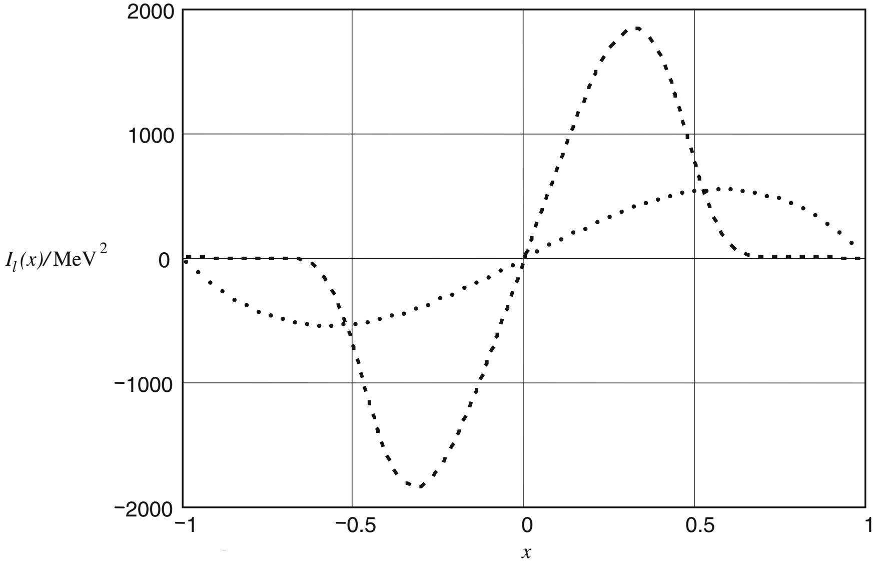

Fig. 4.5 Electron contribution (*dotted line*) and proton contribution (*dashed line*) at $T = 30\,\mathrm{MeV}$ to the imaginary part of Π_ℓ

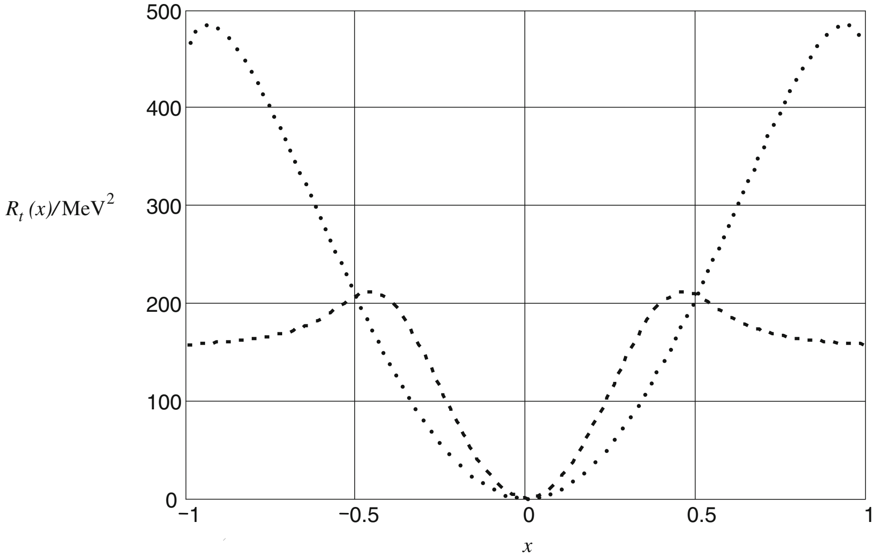

Fig. 4.6 Electron contribution (*dotted line*) and proton contribution (*dashed line*) at $T = 30\,\mathrm{MeV}$ to the real part of Π_t

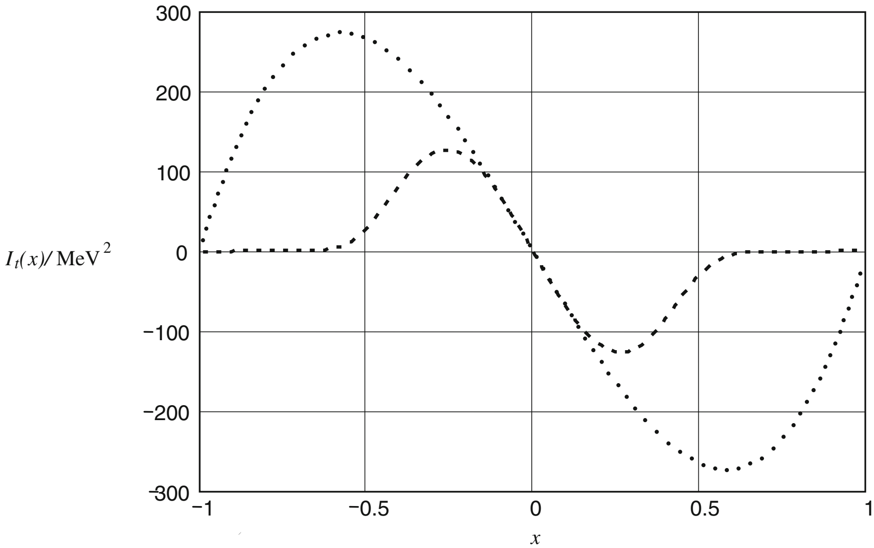

Fig. 4.7 Electron contribution (*dotted line*) and proton contribution (*dashed line*) at $T = 30\,\mathrm{MeV}$ to the imaginary part of Π_t

$$R_\ell^{(i)} = 4\,\pi\,\alpha\,\frac{Z_i^2\,N_i}{T}\left[1-\phi\left(\frac{x}{x_0}\right)\right],$$
$$I_\ell^{(i)} = 8\,\pi^{3/2}\,\alpha\,Z_i^2\,N_i\,\frac{1}{x_0\,q}\,\sinh\frac{q_0}{2\,T}\,\exp\left(\frac{q^2}{8\,m_i\,T}\right)\exp\left(-\frac{x^2}{x_0^2}\right), \tag{4.61}$$

where $x_0 = \sqrt{2\,T/m_i}$, and the function is introduced:

$$\phi(y) = \frac{2}{\sqrt{\pi}}\,|y|^3\int\limits_0^\infty u\,\ln\left|\frac{1+u}{1-u}\right|\,\mathrm{e}^{-y^2u^2}\,\mathrm{d}u\,. \tag{4.62}$$

As is seen from Eq. (4.61), the function $I_\ell^{(i)}$ differs from zero only in the narrow area of the variable $x = q_0/q$, namely, $x \lesssim x_0 \sim \sqrt{T/m_i} \ll 1$.

The functions $R_t^{(i)}$ and $I_t^{(i)}$ for the transversal plasmon are of the order $\alpha\,Z_i^2\,N_i/m_i$ and thus are suppressed by the large mass of the ion in the denominator. Thus, the contribution of the neutrino scattering off free ions via the longitudinal plasmon ($\lambda = \ell$) is only essential.

The ion contribution (4.61) comes with the factor $Z_i^2\,Y_i$, and it is negligibly small in the supernova core conditions, because of the smallness of Y_i. However, it could be essential in the upper layers of the supernova envelope, which are believed to be rich in elements of the iron group.

4.5 Neutrino Self-energy Operator in Plasma

The most important event in neutrino physics of the last decades was the solving of the Solar neutrino problem, made in the unique experiment on the heavy-water detector at the Sudbury Neutrino Observatory [19–21]. This experiment, together with the atmospheric and the reactor neutrino experiments [22–25] has confirmed the key idea by B. Pontecorvo on neutrino oscillations [26, 27]. The existence of non-zero neutrino mass and lepton mixing is thereby established. On the one hand, the Sun appeared in this case as a natural laboratory for investigations of neutrino properties. On the other hand, the process of solving of the Solar neutrino problem significantly stimulated the progress of the Solar physics in different aspects [28] and of several sciences investigating microscopic matter properties: physics of nuclear reactions, radiochemistry, etc.

Another direction of neutrino astrophysics, which also interact with several branches of physical science, is the registration of neutrinos from a supernova explosion. At the moment, there is only one registered neutrino signal from the supernova *SN1987A* in the Large Magellanic Cloud, where four underground neutrino detectors, Kamiokande 2, IMB, LSD and Baksan scintillation telescope, for the first time registered electron antineutrinos in the reaction $\bar{\nu}_e + p \to n + e^+$.

Supernova explosions can be called unique natural laboratories for studying the fundamental properties of matter under extreme physical conditions. At the same time, one of the most important factors almost completely determining the energetics of the process is the presence of giant neutrino fluxes. This means that the presence of microscopic characteristics of the neutrino, determined by its dispersion in the active medium, could have a critical impact on macroscopic properties of these astrophysical events.

In real astrophysical conditions, the external active medium is usually represented by two components: a strong magnetic field and the hot dense plasma. Therefore, the investigation of the neutrino dispersion properties in a medium containing both plasma and field is of the most interest. However, due to the large computational complexity of such studies, the analyses were initially carried out, where the dominance of one of the two indicated components of the active medium, or strong magnetic field, or the hot dense plasma was supposed.

The calculation of the neutrino self-energy operator in a hot dense plasma without a magnetic field was carried out in Refs. [29–31]. The contribution of the external magnetic field into the neutrino self-energy operator, without taking into account the plasma has been studied in a series of papers [32–37]. The series of papers [38–41] has been devoted to the analysis of the operator $\Sigma(p)$ with taking into account both components of the environment, both field and plasma, with the dominance of the influence of the latter, that is, the contribution of the field has been taken into account in the form of small corrections. Finally, in the papers [42, 43] the calculation of the operator $\Sigma(p)$ in a magnetized plasma is carried out over a wide range of magnetic field intensity.

The early Universe can be treated as another natural laboratory for fundamental physics, where the role of neutrinos is also high. Thus, there has been a steady growth of interest in neutrino physics in the external active media.

Investigation of the active media influence on the neutrino dispersion is based on the analysis of the neutrino self-energy operator $\Sigma(p)$. Knowing of the operator $\Sigma(p)$ can solve at least three important tasks:

(i) From the neutrino self-energy operator, an additional energy can be easily determined acquired by neutrinos in a medium. The astrophysical medium is asymmetric with respect to lepton flavors: it contains electrons and positrons, but no muons and tau leptons. Due to this, neutrinos of different flavors acquire a variety of additional energy, which is the determining factor in the influence of environment on the neutrino flavor oscillation.

(ii) The importance of calculating the self-energy operator is supported by the fact that you can extract from it the neutrino anomalous magnetic moment.

(iii) The imaginary part of the neutrino self-energy in the medium determines the probability of the neutrino decay into the W^+-boson and the charged lepton, $\nu_\ell \to \ell^- W^+$.

Further we discuss each of these tasks.

4.5.1 Definition of the Operator $\Sigma(p)$ in Plasma

The neutrino self-energy operator $\Sigma(p)$ can be defined from the invariant amplitude of the transition $\nu \to \nu$ by the relation

$$\mathcal{M}(\nu \to \nu) = -\left[\bar{\nu}(p)\,\Sigma(p)\,\nu(p)\right] = -\mathrm{Tr}\left[\Sigma(p)\,\rho(p)\right], \tag{4.63}$$

where $p^\alpha = (E,\, \mathbf{p})$ is the neutrino 4-momentum, $\rho(p) = \nu(p)\bar{\nu}(p)$ is the density matrix of neutrinos. An additional energy ΔE, acquired by neutrinos in the external active medium is determined by the invariant amplitude (4.63) as follows:

$$\Delta E = -\frac{1}{2\,E}\,\mathcal{M}(\nu \to \nu) = \frac{1}{2\,E}\,\mathrm{Tr}\left[\Sigma(p)\,\rho(p)\right]. \tag{4.64}$$

It is convenient to represent the operator $\Sigma(p)$ in plasma in a general form of an expansion over the linearly independent covariant structures:

$$\begin{aligned}\Sigma(p) = &\left[\mathcal{A}_L\,(p\gamma) + \mathcal{B}_L\,(u\gamma)\right]\,\gamma_L \\ &+ \left[\mathcal{A}_R\,(p\gamma) + \mathcal{B}_R\,(u\gamma)\right]\,\gamma_R + \mathcal{K}_1\,m_\nu\,.\end{aligned} \tag{4.65}$$

Here, $\gamma_L = (1-\gamma_5)/2$ and $\gamma_R = (1+\gamma_5)/2$ are, respectively, the left-handed and the right-handed chiral projection operators, u^α is the 4-velocity vector of the medium.

Note that the coefficients $\mathcal{A}_L$, $\mathcal{A}_R$ and $\mathcal{K}_1$ in Eq. (4.65) contain an ultraviolet divergence. But it does not have an independent meaning, since it does not contribute into the real energy of neutrinos in the external media at the one-loop level, taking into account the renormalization of the vacuum wave function and the mass of a neutrino.

4.5.2 Neutrino Additional Energy in Hot Dense Plasma

As was first shown by L. Wolfenstein [44], studying the propagation of neutrinos in a medium one must take into account the effect of coherent forward scattering. In astrophysical conditions, the influence of a medium on the neutrino properties is primarily due to the additional energy, which only a left-handed neutrino (with the spin oriented opposite to the direction of motion) acquires. For illustration, we present here a detailed calculation of the contribution into the neutrino additional energy from the electron-positron plasma component in accordance with Eq. (4.64). Note that in the approximation of the massless left-handed neutrino, there are only two linearly independent covariant structures present in the expression (4.65) for the operator $\Sigma(p)$ with the coefficients $\mathcal{A}_L$ and $\mathcal{B}_L$.

Let us consider the process of a coherent neutrino forward scattering on electrons and positrons of the plasma. To begin with, we consider the local limit of the weak

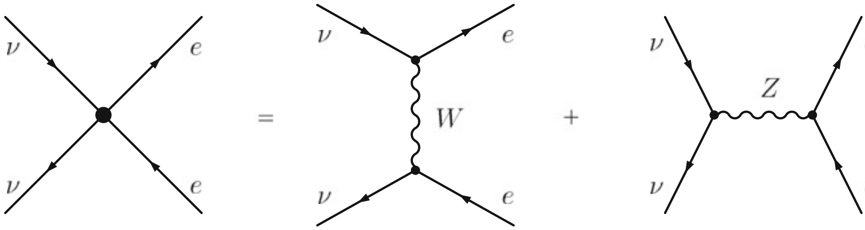

Fig. 4.8 Feynman diagrams illustrating the transition to an effective $\nu - e$ interaction in the local limit

interaction of the left-handed neutrinos with electrons, when the propagators of the intermediate W and Z bosons "shrink" to the point, as shown in Fig. 4.8. The effects of non-locality of the weak interaction which must be taken into account in a case of high neutrino energies, will be considered below in Sect. 4.5.4.

The effective local Lagrangian of the neutrino—electron interaction can be written in the form[1]

$$\mathcal{L} = -\frac{G_F}{\sqrt{2}}\left[\bar{e}\gamma_\alpha(C_V^{(\ell)} - C_A^{(\ell)}\gamma_5)e\right]\left[\bar{\nu}\gamma^\alpha(1-\gamma_5)\nu\right], \tag{4.66}$$

where the constants $C_V^{(\ell)}$ and $C_A^{(\ell)}$ are different in two cases:

- if the neutrinos in the Lagrangian (4.66) are of the electron type, $\nu = \nu_e$, a contribution from the exchange of Z and W boson appears, and we have:

$$C_V^{(e)} = +\frac{1}{2} + 2\sin^2\theta_W\,, \qquad C_A^{(e)} = +\frac{1}{2}\,, \tag{4.67}$$

  where θ_W is the Weinberg angle, $\sin^2\theta_W \simeq 0.231$;
- if we consider the muon and tau neutrinos, $\nu = \nu_\mu, \nu_\tau$, only the Z boson contributes in this case, and we have:

$$C_V^{(\mu)} = -\frac{1}{2} + 2\sin^2\theta_W\,, \qquad C_A^{(\mu)} = -\frac{1}{2}\,. \tag{4.68}$$

Let us start with the scattering by electrons. We write the $\mathcal{S}$ matrix element of the process in the standard form:

$$\mathcal{S}^{(e-)} = \frac{\mathrm{i}(2\pi)^4\delta^{(4)}(p'+k'-p-k)}{\sqrt{2EV\,2\varepsilon V\,2E'V\,2\varepsilon' V}}\,\mathcal{M}\{\nu(p)+e^-(k)\to\nu(p')+e^-(k')\}\,. \tag{4.69}$$

[1] Note that the sign of the effective Lagrangian is significant in this case, since the additional neutrino energy is the linear in G_F effect. In the calculation of probabilities and cross sections of weak processes, which are proportional to G_F^2, the sign of the effective Lagrangian does not appear.

Here, $p^\alpha = (E, \mathbf{p})$ and $k^\alpha = (\varepsilon, \mathbf{k})$ are, respectively, the 4-momenta of the initial neutrino and electron, p' and k' are the 4-momenta of the final neutrino and electron, $\mathcal{M}$ is the invariant amplitude:

$$\mathcal{M}\{\nu(p) + e^-(k) \to \nu(p') + e^-(k')\} = -\frac{G_\mathrm{F}}{\sqrt{2}}\big[\bar{e}(k')\gamma_\alpha(C_V^{(e)} - C_A^{(e)}\gamma_5)e(k)\big] \times \big[\bar{\nu}(p')\gamma^\alpha(1-\gamma_5)\nu(p)\big]. \tag{4.70}$$

Given the process is a forward scattering, we need to put $p' = p$ and $k' = k$ in the $\mathcal{S}$ matrix element (4.69). At the same time

$$(2\pi)^4\delta^{(4)}(0) = \int \mathrm{d}^4x\, \mathrm{e}^{\mathrm{i}0} = V\mathcal{T}, \tag{4.71}$$

where V is the total volume of the interaction region, $\mathcal{T}$ is the total time of interaction. The $\mathcal{S}$ matrix element of the forward neutrino scattering off the electrons takes the form

$$\mathcal{S}_{forw}^{(e-)} = \frac{\mathrm{i}\,V\mathcal{T}}{2EV\,2\varepsilon V}\,\mathcal{M}\{\nu(p) + e^-(k) \to \nu(p) + e^-(k)\}. \tag{4.72}$$

Since this is a coherent process, the total scattering amplitude is obtained by summing the scattering amplitudes for all electrons of the medium:

$$\mathcal{S}_{tot}^{(e-)} = \sum_{\mathbf{k},s}\mathcal{S}_{forw}^{(e-)} = 2\int\frac{\mathrm{d}^3\mathbf{k}\,V}{(2\pi)^3} f_e(k)\,\mathcal{S}_{forw}^{(e-)}, \tag{4.73}$$

where the coefficient 2 takes into account two electronic spin states s, $f_e(k)$ is the distribution function of the electrons of medium. We assume this distribution to be in an equilibrium and consider the reference frame where the medium moves as a whole with the 4-velocity vector u. The Fermi—Dirac distribution function is written as

$$f_e(k) = \left(\exp\frac{(ku) - \mu_e}{T} + 1\right)^{-1}, \tag{4.74}$$

where μ_e is the chemical potential of the electron-positron plasma, T is the plasma temperature.

We can now determine the contribution to the invariant transition amplitude $\mathcal{M}^{(e-)}(\nu \to \nu)$ caused by the coherent forward scattering on the electron fraction of plasma, from the expression

$$\mathcal{S}_{tot}^{(e-)} = \mathcal{S}^{(e-)}(\nu \to \nu) = \frac{\mathrm{i}\,V\mathcal{T}}{2EV}\,\mathcal{M}^{(e-)}(\nu \to \nu). \tag{4.75}$$

From Eqs. (4.70)–(4.75), we obtain

$$\mathcal{M}^{(e-)}(\nu \to \nu) = -\sqrt{2}\, G_{\mathrm{F}}\, C_V^{(e)} \big[\bar{\nu}(p)\gamma^{\alpha}\gamma_L \nu(p)\big] 2 \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} f_e(k)\, \frac{k_\alpha}{\varepsilon} \,. \tag{4.76}$$

The arising integral is a relativistic covariant and can be easily calculated:

$$2 \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} f_e(k)\, \frac{k_\alpha}{\varepsilon} = u_\alpha N_e \,, \tag{4.77}$$

where N_e is the electron density.

The contribution from the coherent forward scattering on the positron fraction of plasma can be calculated quite similarly. As a result, taking into account Eq. (4.63), we finally obtain the neutrino self-energy operator in the electron-positron plasma in the form

$$\Sigma(p) = \sqrt{2}\, G_{\mathrm{F}}\, C_V^{(e)}\, (u\gamma)\gamma_L \left(N_e - \bar{N}_e\right) \,, \tag{4.78}$$

where N_e and $\bar{N}_e$ are the densities of electrons and positrons. Comparing (4.78) with (4.65), one can see that only one structure with a coefficient $\mathcal{B}_L = \sqrt{2}\, G_{\mathrm{F}}\, C_V^{(e)} \left(N_e - \bar{N}_e\right)$ presents in the operator $\Sigma(p)$ in this case.

According to Eq. (4.64), for the additional neutrino energy in electron-positron plasma we obtain

$$\Delta E = \sqrt{2}\, G_{\mathrm{F}}\, C_V^{(e)}\, \frac{(pu)}{E} \left(N_e - \bar{N}_e\right) \,. \tag{4.79}$$

In the transition from an arbitrary reference frame to the plasma rest frame one should put $(pu) = E$.

In the analysis of the neutrino dispersion properties in the active astrophysical media one should generally take into account, along with the electron-positron plasma, the presence of other components. The contribution of protons and neutrons can be found similarly to the previous analysis, with the effective Lagrangian caused only by the exchange of Z boson (see Fig. 4.8). In a dense plasma of the supernova core, the contribution of the neutrino gas which can be regarded to be in approximate equilibrium, could also be significant. The general expression for the additional energy of the electron, muon and tau neutrinos, $i = e, \mu, \tau$, is given by

$$\begin{aligned} \Delta E_i = \sqrt{2}\, G_{\mathrm{F}} \Bigg[& \left(\delta_{ie} - \frac{1}{2} + 2\, \sin^2\theta_{\mathrm{W}}\right) \left(N_e - \bar{N}_e\right) \\ & + \left(\frac{1}{2} - 2\, \sin^2\theta_{\mathrm{W}}\right) \left(N_p - \bar{N}_p\right) \\ & - \frac{1}{2} \left(N_n - \bar{N}_n\right) + \sum_{\ell = e, \mu, \tau} (1 + \delta_{i\ell}) \left(N_{\nu_\ell} - \bar{N}_{\nu_\ell}\right) \Bigg] \,, \end{aligned} \tag{4.80}$$

where N_e, N_p, N_n, and N_{ν_ℓ} are the densitis of electrons, protons, neutrons and neutrinos, $\bar{N}_e, \bar{N}_p, \bar{N}_n$, and $\bar{N}_{\nu_\ell}$ are the densitis of the corresponding antiparticles.

To find the additional energy of an antineutrino in plasma, one should change the overall sign in the right-hand side of Eq. (4.80).

The formula (4.80) obtained in the local limit of weak interaction, is not sufficient for the case when the plasma is nearly charge-symmetric, for example, in the early Universe. In this case the value of ΔE_i in Eq. (4.80) tends to zero, and the contribution into the neutrino energy becomes significant caused by the nonlocality of the weak interaction. This non-local contribution was investigated in Refs. [29, 39, 45] in the form of the next terms in the expansion of the $W-$ and Z–boson propagators by the inverse powers of their masses $m_{W,Z}^{-2}$. The result can be presented as follows:

$$\Delta^{(\mathrm{nloc})} E^{\nu_\ell} = -\frac{16 G_{\mathrm{F}} E}{3\sqrt{2}} \left(\frac{\langle E_{\nu_\ell} \rangle N_{\nu_\ell} + \langle E_{\bar{\nu}_\ell} \rangle \bar{N}_{\nu_\ell}}{m_Z^2} + \delta_{\ell e} \frac{\langle E_e \rangle N_e + \langle E_{\bar{e}} \rangle \bar{N}_e}{m_W^2} \right). \tag{4.81}$$

Here, $\langle E_{\nu_\ell} \rangle$, $\langle E_{\bar{\nu}_l} \rangle$, $\langle E_e \rangle$, $\langle E_{\bar{e}} \rangle$ are the average energies of plasma neutrinos, antineutrinos, electrons and positrons respectively. In a particular case of a charge symmetric hot plasma, the expression (4.81) reproduces the result of Refs. [29, 39]:

$$\Delta^{(\mathrm{nloc})} E^{\nu_\ell} = -\frac{7\sqrt{2}\,\pi^2\, G_{\mathrm{F}}\, T^4}{45} \left(\frac{1}{m_Z^2} + \frac{2\,\delta_{\ell e}}{m_W^2} \right) E\,. \tag{4.82}$$

However, the correction of the type of Eq. (4.81) can be insufficient in the case of ultra-high neutrino or antineutrino energies. The neutrino self-energy operator with using the exact dependence of the propagators of gauge bosons on the momentum transferred was investigated in Ref. [46], see Sect. 4.5.4 below.

For a typical astrophysical plasma, with the exception of the early Universe and supernova core, we have $\bar{N}_e \simeq \bar{N}_p \simeq \bar{N}_n \simeq N_{\nu_\ell} \simeq \bar{N}_{\nu_\ell} \simeq 0$ and $N_p \simeq N_e = Y_e\, N_B$, $N_n \simeq (1 - Y_e)\, N_B$, where N_B is the density of baryons. If the neutrino energy is not extremely high, for the additional energy of neutrinos of different flavors, we obtain

$$\Delta E_e = \frac{G_{\mathrm{F}}\, N_B}{\sqrt{2}}\, (3\, Y_e - 1)\,, \tag{4.83}$$

$$\Delta E_{\mu,\tau} = -\frac{G_{\mathrm{F}}\, N_B}{\sqrt{2}}\, (1 - Y_e)\,. \tag{4.84}$$

Since $Y_e < 1$, the additional energy of the left-handed muon and tau neutrinos is always negative. At the same time, the additional energy of the left-handed electron neutrinos is positive for $Y_e > 1/3$. Conversely, the additional energy of the electron antineutrinos is positive for $Y_e < 1/3$, while it is always positive for the muon and tau antineutrinos. In turn, the right-handed neutrino, with the spin oriented in the direction of movement, and its antiparticle, the left-handed antineutrino, being sterile with respect to weak interaction, do not acquire the additional energy.

4.5.3 On the Neutrino Radiative Decay in Plasma

It should be noted that the history of studies of the neutrino dispersion modifications by plasma has not been without its oddities. In this section, we illustrate how a consideration of the plasma influence on the neutrino dispersion, with ignoring the photon dispersion in plasma, has led the authors [47], for a comprehensive list of references see [48], to a detailed discussion of an effect, which is physically impossible, strictly speaking.

It is known that the effect of plasma on the particle properties may open new possibilities for the realization of processes, forbidden in vacuum by conservation laws. However, it is necessary to consider the plasma impact on all components of the process, and it can complicate the kinematics essentially.

The additional energy ΔE, defined by the expression (4.83), results in the appearance of the effective mass square m_L^2 for the left-handed electron neutrinos:

$$m_L^2 = \mathcal{P}^2 = (E + \Delta E)^2 - \mathbf{p}^2 \,, \tag{4.85}$$

where $\mathcal{P}$ is the 4-momentum of the neutrino in a plasma in its rest frame, while the 4-vector $(E,\ \mathbf{p})$ would be a 4-momentum of the neutrino in vacuum, $E = \sqrt{\mathbf{p}^2 + m_\nu^2}$.

Given the neutrino magnetic moment interaction with a photon, which leads to the neutrino helicity-flip, the appearance of an additional energy for left-handed neutrinos in plasma would open new kinematic possibilities for the neutrino radiative transition:

$$\nu_L \to \nu_R + \gamma \,. \tag{4.86}$$

It can be considered as the radiative decay of the left-handed neutrino which becomes heavier in plasma, into lighter right-handed neutrino.

At the same time it should be obvious that it is necessary to take into account the influence of plasma on the dispersion of the photon $\omega = |\mathbf{k}|/n$, where $n \neq 1$ is the index of refraction.

First of all, the plasma influence can provide the condition $n > 1$ to be satisfied (the square of the effective photon mass is negative, $m_\gamma^2 \equiv q^2 = \omega^2 - \mathbf{k}^2 < 0$), which corresponds to a well-known effect of the neutrino Cherenkov radiation [10, 49, 50]. In this situation, a change of the neutrino dispersion properties under the plasma influence could be neglected at all. Really, while the neutrino dispersion is defined by a weak interaction, the change of the photon dispersion depends on its much more intense electromagnetic interaction with plasma.

Theoretically, one can consider another situation: if the photon dispersion in plasma was the same as in vacuum,[2] the process would occur of the neutrino radiative transition $\nu \to \nu\gamma$, caused only by the neutrino dispersion. Since the effect of the plasma changes the dispersion properties of the only left-handed neutrino, the transitions (4.86) would be possible due to the photon interaction with the neutrino magnetic moment. Such an imaginary effect, called "neutrino spin light" ($SL\nu$), has been proposed and studied in detail in an extensive series of papers, see [48]. However, in analyzing this effect the authors have not considered such an important phenomenon as the above-mentioned plasma influence on the photon dispersion. As was shown in [51, 52], this phenomenon makes the $SL\nu$ effect forbidden for all real astrophysical situations.

Following the papers [51, 52], we analyze here the process $\nu_L \to \nu_R\gamma$, taking into account the dispersion properties of both the neutrino and photon in astrophysical plasmas.

To analyze the kinematics of the process, it is worthwhile to estimate the scales of the values of additional neutrino energy $\varDelta E$ and the effective mass of the photon (plasmon) m_γ.

From the expression (4.83) for the electron antineutrino, we obtain

$$\varDelta E \simeq 6\ \text{eV} \left(\frac{N_B}{10^{38}\ \text{cm}^{-3}} \right) (1 - 3\, Y_e)\,, \tag{4.87}$$

where the scale of the baryon density is taken, which is typical e.g. for the interior of a neutron star.

In turn, a plasmon acquires in medium an effective mass m_γ, which is approximately constant at high energies. For the transverse plasmon, the value of m_γ^2 is always positive and is determined by the so-called plasma frequency ω_P. For a non-relativistic classical plasma (e.g. in the Sun), we obtain

$$m_\gamma \equiv \omega_\text{P} = \sqrt{\frac{4\pi\,\alpha\,N_e}{m_e}} \simeq 4 \times 10^2\ \text{eV} \left(\frac{N_e}{10^{26}\ \text{cm}^{-3}} \right)^{1/2} . \tag{4.88}$$

For the ultra-relativistic dense matter one has:

$$m_\gamma^2 = \frac{2\,\alpha}{\pi} \left(\mu_e^2 + \frac{\pi^2}{3}\, T^2 \right), \tag{4.89}$$

where μ_e is the chemical potential of plasma electrons. For the case of cold degenerate plasma one obtains from Eq. (4.89):

[2] Strictly speaking, a particle that interacts with the magnetic moment of neutrinos, and at the same time, is sterile with respect to the interactions with electrically charged plasma particles, should not be called a photon.

$$m_\gamma = \sqrt{\frac{3}{2}}\,\omega_{\text{pl}} = \left(\frac{2\,\alpha}{\pi}\right)^{1/2}\left(3\,\pi^2\,N_e\right)^{1/3} \simeq 10^7\,\text{eV}\left(\frac{N_e}{10^{37}\,\text{cm}^{-3}}\right)^{1/3}. \qquad (4.90)$$

In the case of hot plasma, where its temperature is the largest physical parameter, the effective mass of the plasmon is

$$m_\gamma = \sqrt{\frac{2\,\pi\,\alpha}{3}}\,T \simeq 1.2\times 10^7\,\text{eV}\left(\frac{T}{100\,\text{MeV}}\right). \qquad (4.91)$$

Comparison of the scales of m_γ (4.88)–(4.91) with the scale of ΔE (4.87) should indicate that neglecting the mass of the plasmon, made in the consideration of the $SL\nu$ effect [48], was obviously incorrect. At the same time another physical parameter, a great attention was paid to in the $SL\nu$ analysis, was the neutrino vacuum mass m_ν. As the scale of neutrino vacuum mass could not exceed essentially a few electron-volts, which is much less than typical plasmon mass scales for real astrophysical situations, see Eqs. (4.88)–(4.91), it is reasonable to neglect m_ν in our analysis.

Thus, in accordance with (4.85), a simple condition for the kinematic opening of the process $\nu_L \to \nu_R\gamma$ is:

$$m_L^2 \simeq 2\,E\,\Delta E > m_\gamma^2\,. \qquad (4.92)$$

This means that the process becomes kinematically opened when the neutrino energy exceeds the threshold value,

$$E > E_0 = \frac{m_\gamma^2}{2\,\Delta E}\,. \qquad (4.93)$$

The appearance of the threshold energy of neutrinos can be demonstrated by considering the range of integration over the energies and momenta of the photon (plasmon) in the $\nu_L \to \nu_R\gamma$ process, taking into account the dispersion properties of both neutrinos and photons in astrophysical plasmas. In Fig. 4.9, the line of the photon dispersion in vacuum, $q_0 = k$, lies inside the allowed kinematical region (left panel), whereas the line of the photon dispersion, modified by plasma, may be outside this area if the neutrino energy is not large enough (right panel). In this case the phase volume, and hence the process probability is zero.

For fixed plasma parameters, the value ω_P remains constant. The value ΔE remains constant also, if we disregard the contribution to the neutrino energy from the non-locality of the weak interaction. Therefore, in order to obtain the non-zero phase volume and the process probability or, in other words, in order to put a part of the plasmon dispersion curve into the integration region, it is necessary to increase the neutrino energy E, i.e., the width of the oblique rectangle in Fig. 4.9. It should be clear, that there is the minimum energy E_0 for the integration region to exist. This is just the threshold energy (4.93).

Let us estimate these threshold energies for various astrophysical situations.

In the approximation of nonrelativistic classical plasma, one obtains from Eqs. (4.87) and (4.88) that the threshold neutrino energy does not depend on density, and do depend on the chemical composition only:

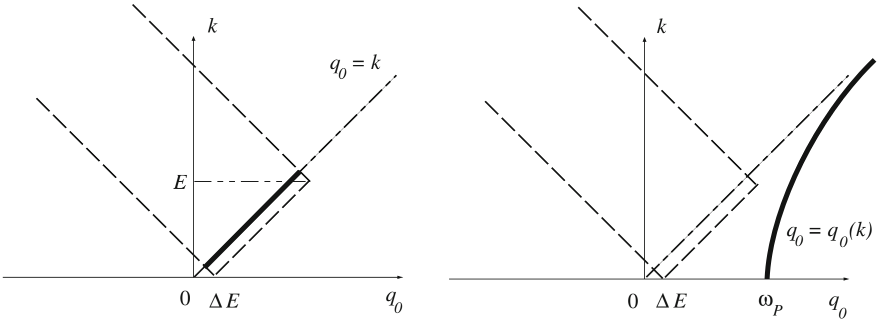

Fig. 4.9 The integration region for the calculation of the probability of the process $\nu_L \to \nu_R\gamma$ at fixed energy E of the initial neutrino (inside the oblique rectangle drawn by *dashed lines*) and the *line* of the photon dispersion (*thick line*) in a vacuum (*left panel*) and in plasma (*right*)

$$E_0 \simeq \frac{Y_e}{3Y_e - 1}\, 4\sin^2\theta_{\mathrm{W}}\, \frac{m_{\mathrm{W}}^2}{m_e}\,. \tag{4.94}$$

For the solar interior $Y_e \simeq 0.6$, and the threshold neutrino energy is

$$E_0 \simeq 10^{10}\,\mathrm{MeV}\,, \tag{4.95}$$

to be compared with the upper bound $\sim$20 MeV for the solar neutrino energies.

For the interior of a neutron star, where $Y_e \ll 1$, the additional energy for neutrinos (4.83), (4.84) is negative, and the process $\nu_L \to \nu_R + \gamma$ is closed. On the other hand, there exists a possibility for opening the antineutrino decay. Taking for the estimation $Y_e \simeq 0.1$, one obtains from (4.87) and (4.89) the threshold value

$$E_0 \simeq 10^{7}\,\mathrm{MeV}\,, \tag{4.96}$$

to be compared with the typical energy $\sim$1—0.1 MeV of neutrinos emitted via the direct or modified Urca processes [53].

For the conditions of a supernova core, the additional energy of left-handed electron neutrinos can be obtained from Eq. (4.80) in the form:

$$\Delta E_e = \frac{G_{\mathrm{F}}\, N_B}{\sqrt{2}}\,\left(3\, Y_e + 4\, Y_{\nu_e} - 1\right)\,, \tag{4.97}$$

where Y_{ν_e} describes the fraction of the trapped electron neutrinos in the supernova core, $N_{\nu_e} = Y_{\nu_e} N_B$. Using the typical parameters of a supernova core, we obtain

$$E_0 \simeq 10^{7}\,\mathrm{MeV}\,, \tag{4.98}$$

to be compared with the averaged energy $\sim 10^2$ MeV of trapped neutrinos.

In the early Universe, when the plasma was almost charge symmetric, the formula (4.80), which gives a null result must be supplemented by the non-local contribution (4.82), which is the same for neutrinos and antineutrinos. The minus sign in (4.82) unambiguously shows that in the early Universe, in contrast to the neutron star interior, the process of the radiative spin-flip transition is forbidden both for neutrinos and antineutrinos regardless of their energy.

An analysis of the sum of the local and non-local weak contributions (4.80) and (4.81) in a case if the neutrino energy is not ultra-high, shows that adding of the non-local term leads in general to the decreasing of the additional neutrino energy in plasma, i.e. to the increasing of the threshold energy (4.93). Strictly speaking, one has to perform an analysis of the kinematical inequality (4.92), which leads to the solving of the quadratic equation. As a result, there arises the window in the neutrino energies for the process to be kinematically opened, $E_0 < E < E_{\max}$, where E_0 and $E_{\max}$ are the lower and the upper limits connected with the roots of the above-mentioned quadratic equation, if they exist. For example, in the solar interior there is no window for the process with electron neutrinos at all, i.e. the transition $\nu_{eL} \to \nu_{eR} + \gamma$ is forbidden kinematically.

Thus, the above analysis shows that the nice effect of the "spin light of neutrino", unfortunately, has no place in real astrophysical conditions if the dispersion properties of neutrinos and photons are properly taken into account. The sole possibility for the discussed process $\nu_L \to \nu_R + \gamma$ to be theoretically possible, could be connected only with the situation when an ultra-high energy neutrino threads a star. Obviously, this task can only have a purely methodological sense. In the papers [51, 52], the mean free path L of the ultra-high energy neutrino with respect to the radiative decay process was correctly calculated in the situation where a neutrino arrived from outside penetrates a neutron star.

Based on the typical neutron star parameters $N_B \simeq 10^{38}\,\text{cm}^{-3}$, $Y_e \simeq 0.05$, the mean free path was obtained:

$$L \gtrsim 10^{19}\,\text{cm} \times \left(\frac{10^{-12}\,\mu_{\text{B}}}{\mu_\nu} \right)^2 , \tag{4.99}$$

where μ_ν is the neutrino magnetic moment, μ_{B} is the Bohr magneton. This mean free path should be compared with the radius of the neutron star $\sim 10^6$ cm, to illustrate the extremely low probability of the process.

It is interesting to note that it was not the first case when the plasma influence was taken into account for one participant of the physical process while it was not taken for other participant. As E. Braaten wrote in Ref. [1]:

"In Ref. [54], it was argued that their calculation for the emissivities from photon and plasmon decay would break down at temperatures large enough that $m_\gamma > 2\,m_e$, since the decay $\gamma \to e^+e^-$ is then kinematically allowed. This statement, which has been repeated in subsequent papers, [55–58] is simply untrue. The plasma effects which generate the photon mass m_γ also generate corrections to the electron mass such that the decay $\gamma \to e^+e^-$ is always kinematically forbidden."

Thus, a history repeated itself. The authors [48] made the same mistake when they considered the plasma-induced additional neutrino energy and ignored the effective photon mass m_γ arising by the same reason.

The only question remained open whether this effect was possible in the case of ultra-high neutrino energies [59]. This gap was eliminated in Ref. [46]. In the next section, we reproduce that analysis.

4.5.4 Ultra-High Energy Neutrino Dispersion in Plazma

As it was already mentioned, the accounting of the non-local contribution to the neutrino additional energy made by the retention of the next term in the expansion of the W– and Z–boson propagators in the inverse powers of their masses [29, 39, 45, 52] would be irrelevant in the limit of the ultra-high neutrino energies. Therefore, it is necessary to use the exact expressions for the W– and Z–boson propagators. Analysis of the neutrino additional energy in a plasma in the limit of ultra-high energies, with taking account of the nonlocality of the weak interaction was made in a series of papers, Refs. [60–62], with respect to the neutrino oscillations. In this section, we consider the neutrino self-energy operator in medium similarly to the procedure described in Sect. 4.5.2 but with taking into account the dependence of the W and Z–boson propagators on the momentum transferred, and we reanalyse its effects on the neutrino radiative conversion (4.86). The presentation is based mainly on Ref. [46].

We first consider the electron neutrino scattering on the electron-positron component of plasma. For the channel of the $\nu_e e$ scattering through the W–boson exchange, the Lagrangian of the interaction is:

$$\mathcal{L} = \frac{g}{2\sqrt{2}} \left(\bar{e}\, \gamma_\alpha \,(1-\gamma_5)\, \nu_e\, \right)\, W_\alpha + \frac{g}{2\sqrt{2}} \left(\bar{\nu}_e\, \gamma_\alpha \,(1-\gamma_5)\, e\, \right)\, W_\alpha^\dagger\,. \tag{4.100}$$

It leads to the invariant amplitude of the process:

$$\begin{aligned} M_{\nu_e e^- \to \nu_e e^-} = &-\frac{G_\mathrm{F}}{\sqrt{2}} \left[\bar{e}(k')\gamma_\alpha(1-\gamma_5)e(k)\right] \\ &\times \left[\bar{\nu}_e(p')\gamma^\alpha(1-\gamma_5)\nu_e(p)\right] \frac{m_W^2}{m_W^2 - q_1^2}\,, \end{aligned} \tag{4.101}$$

where we use the notation $q_1 = k' - p$ for the W^-–boson momentum (see Fig. 4.10a). Here, the Fiertz transformation is performed, and the term in the W–boson propagator leading to the small term of the order of $(m_e/m_W)^2$ is neglected.

The amplitude of the neutrino-positron scattering process can be written in the similar form (see Fig. 4.10b):

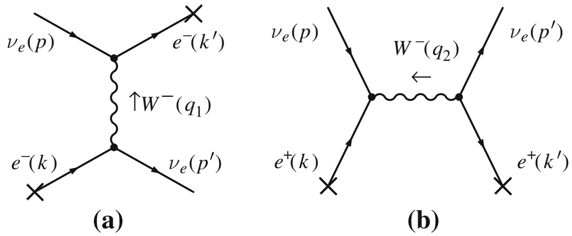

Fig. 4.10 The Feynman diagrams for the neutrino scattering through W–boson: **a** on plasma electrons; **b** on plasma positrons

$$M_{\nu_e e^+ \to \nu_e e^+} = \frac{G_F}{\sqrt{2}} \left[\bar{e}(-k) \gamma_\alpha (1-\gamma_5) e(-k') \right] \times \left[\bar{\nu}_e(p') \gamma^\alpha (1-\gamma_5) \nu_e(p) \right] \frac{m_W^2}{{m_W^*}^2 - q_2^2}\,, \tag{4.102}$$

where W^-–boson momentum is $q_2 = -p - k$. Note that the amplitudes (4.101) and (4.102), described by the diagrams in Figs. 4.10a,b differ essentially. Namely, in the s-channel process of the neutrino scattering off positrons, Fig. 4.10b, we have $q_2^2 > 0$, i.e. a resonance behavior of the W–boson propagator manifests itself. On the contrary, in the u-channel process of the neutrino scattering off electrons, Fig. 4.10a, we have $q_1^2 < 0$, and no resonance arises. Taking account of this type of resonance is made by introducing a complex mass of W–boson, $m_W^* = m_W - \frac{1}{2}\,\mathrm{i}\,\Gamma_W$, where Γ_W is the total decay width of W–boson, $\Gamma_W \simeq 2.1$ GeV, see e.g.[63].

Because of the t–channel behavior of the neutrino-electron and neutrino-positron scattering diagrams for neutrinos of all flavors through Z–boson, see Fig. 4.11, and keeping in mind that the forward scattering is considered, i.e. the scattering with zero-momentum transfer, one concludes that the contribution to the energy from these subprocesses is described by the local limit of the weak interaction.

The total contribution to the ℓ–flavor neutrino self-energy operator from the scattering processes on plasma electrons and positrons can be found by the same way as Eq. (4.78) and be represented in the form:

$$\Sigma^{\nu_\ell}_{(e^- e^+)}(p) = \sqrt{2} G_F \Big[C_V^{(\mu)} (u\gamma) \gamma_L (N_e - \bar{N}_e) + \delta_{\ell e} \gamma^\alpha \gamma_L \, m_W^2 \left(j_\alpha^- - j_\alpha^+ \right) \Big], \tag{4.103}$$

where $N_e, \bar{N}_e = 2(2\pi)^{-3} \int \mathrm{d}^3 k \,(\exp((\varepsilon \mp \mu_e)/T) + 1)^{-1}$ are the electron and positron densities respectively, and we use the notation

$$j_\alpha^\mp = 2 \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{k_\alpha}{\varepsilon} \left(\exp \frac{\varepsilon \mp \mu_e}{T} + 1 \right)^{-1} \left(m_W^2 \pm 2(kp) \right)^{-1}. \tag{4.104}$$

The constant $C_V^{(\mu)}$ in Eq. (4.103) comes from the electron Z–current and is the same for $\ell = e, \mu, \tau$, see Eq. (4.68). For taking account of the resonance behavior in the denominator of the integral j_α^+, m_W should be replaced by m_W^*.

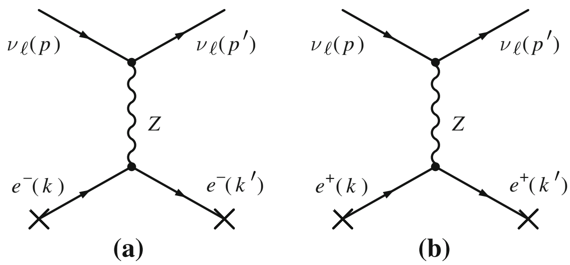

Fig. 4.11 The Feynman diagrams for the neutrino scattering on plasma electrons and positrons through Z–boson

In accordance with Eq. (4.64), the neutrino ν_ℓ additional energy in the electron and positron medium takes the form:

$$\Delta E^{\nu_\ell}_{(e^-e^+)} = \sqrt{2} G_{\mathrm{F}} \Big[C_V^{(\mu)} (N_e - \bar{N}_e) + \delta_{\ell e} \left(F_1(\mu_e, m_W) - F_2(-\mu_e, m_W^*) \right) \Big] , \tag{4.105}$$

where we introduce the functions

$$F_{1,2}(\mu, m) = \frac{2\, m^2}{(2\pi)^3 E} \int \frac{\mathrm{d}^3 k}{\varepsilon} \left(\exp \frac{\varepsilon - \mu}{T} + 1 \right)^{-1} \frac{(pk)}{m^2 \pm 2(pk)} . \tag{4.106}$$

In order to obtain the antineutrino additional energy in the same medium, one has to make the replacement $\mu_e \to -\mu_e$ in the right-hand side of Eq. (4.105). In the first term with the difference of the electron and positron densities it simply means a change of sign.

In the analysis of the neutrino dispersion in active astrophysical medium in a general case, the presence of the other plasma components, protons and neutrons, must be considered. In a dense plasma of the supernova core the donation from thermal neutrinos that can be considered to be approximately in equilibrium, can also be significant. The corresponding Feynman diagrams are shown in Fig. 4.12. The two Feynman diagrams, Fig. 4.12c,d contain a contribution from the non-locality of weak interaction.

A complete formula for the ν_ℓ neutrino and $\bar{\nu}_\ell$ antineutrino additional energy can be written in the following way:

$$\begin{aligned} \Delta E^{\nu_\ell, \bar{\nu}_\ell} = \sqrt{2} G_{\mathrm{F}} \Big\{ & \mp \frac{1}{2} (N_n - \bar{N}_n) \pm (N_{\nu_e} - \bar{N}_{\nu_e}) \\ & \pm (N_{\nu_\mu} - \bar{N}_{\nu_\mu}) \pm (N_{\nu_\tau} - \bar{N}_{\nu_\tau}) \\ & + \delta_{\ell e} \left[F_1(\pm\mu_e, m_W) - F_2(\mp\mu_e, m_W^*) \right] \\ & + \frac{1}{2} \left[F_1(\pm\tilde{\mu}_{\nu_\ell}, m_Z) - F_2(\mp\tilde{\mu}_{\nu_\ell}, m_Z^*) \right] \Big\} . \end{aligned} \tag{4.107}$$

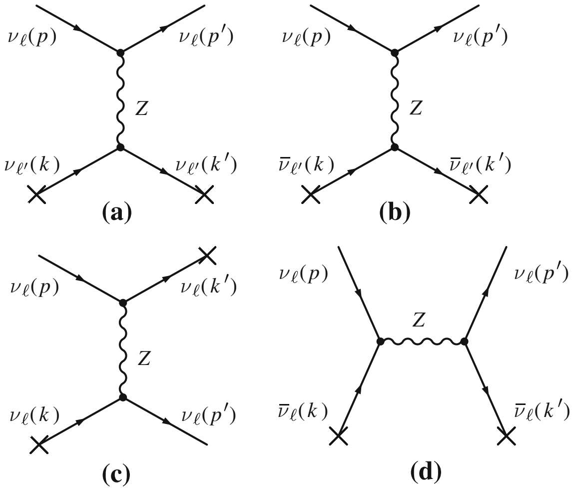

Fig. 4.12 The Feynman diagrams for the neutrino scattering on neutrinos and antineutrinos of the supernova core dense plasma

In this expression, N_n, N_{ν_ℓ} are the neutron and neutrino densities and $\bar{N}_n, \bar{N}_{\nu_\ell}$ are the densities of the corresponding antiparticles. The proton contribution in Eq. (4.107) is cancelled by the electron contribution, because of plasma electroneutrality. Note that in both functions F_2 there exists the mentioned above resonance behavior, which is accounted by the introduction of complex masses of W and Z bosons, $m^*_{W,Z} = m_{W,Z} - \frac{1}{2}\,\mathrm{i}\,\Gamma_{W,Z}$, where the total decay width of the Z– boson is $\Gamma_Z \simeq 2.5\,\mathrm{GeV}$.

Tending formally m_W and m_Z in Eq. (4.107) to infinity, one obtains the neutrino additional energy in the local limit of weak interaction, the so-called Wolfenstein energy [44]. Taking the next terms in the expansion of the W– and Z–boson propagators by the inverse powers of their masses, $m^{-2}_{W,Z}$, i.e. retaining the first term in the expansion of the functions $F_{1,2}$ by m^{-2}, one obtains the first non-local correction (4.81) to the Wolfenstein energy.

Further we consider the kinematical possibilities of the ultra-high-energy neutrino radiative conversion (4.86) for different astrophysical situations.

4.5.4.1 Nonrelativistic Cold Plasma: The Sun and Red Giants

Let us consider first the limit of "cold" plasma, $T \to 0$. In this case, the electron gas is completely degenerate, and there are no positrons in a medium. Calculation of the additional neutrino energy reduces in this case to a simplified calculation of the function $F_1(\mu_e, m_W)$, taking the form in the limit $T \to 0$:

$$F_1(\mu_e, m_W) = \frac{m_W^2}{2\pi^2 E} \int\limits_{m_e}^{\mu_e} v\,\varepsilon\,\mathrm{d}\varepsilon \left(1 - \frac{m_W^2}{4Ev\,\varepsilon} \ln \frac{m_W^2 + 2E\,\varepsilon(1+v)}{m_W^2 + 2E\,\varepsilon(1-v)}\right), \tag{4.108}$$

where ε is the energy of a plasma electron, and $v = \sqrt{1 - m_e^2/\varepsilon^2}$ is its velocity.

Similarly, calculation of the additional antineutrino energy reduces to the calculation of the integral:

$$\begin{aligned} F_2(\mu_e, m_W^*) = &-\frac{m_W^2}{2\pi^2 E} \int\limits_{m_e}^{\mu_e} v\varepsilon\mathrm{d}\varepsilon \\ &\times\left\{1 - \frac{m_W^2}{8Ev\varepsilon} \ln \frac{\left[m_W^2 - 2E\varepsilon(1-v)\right]^2 + m_W^2\Gamma_W^2}{\left[m_W^2 - 2E\varepsilon(1+v)\right]^2 + m_W^2\Gamma_W^2}\right. \\ &\left.-\frac{\mathrm{i}\,m_W^2}{4Ev\varepsilon}\left(\arctan\frac{m_W^2 - 2E\,\varepsilon(1-v)}{m_W\Gamma_W} - \arctan\frac{m_W^2 - 2E\,\varepsilon(1+v)}{m_W\Gamma_W}\right)\right\}. \end{aligned} \tag{4.109}$$

Here, we have neglected the terms of order Γ_W/m_W compared to unity wherever it does not cause problems. Thus, the imaginary part of the additional antineutrino energy, in general, differs from zero. The presence of the imaginary part in the self-energy of a particle indicates its instability, that is, an electron antineutrino is unstable with respect to the process $\bar{\nu}_e + e^- \to W^-$ on the plasma electrons. The width of this process can be found, using the formula:

$$w = -2\,\mathrm{Im}\,\Delta E\,. \tag{4.110}$$

In the case of non-relativistic cold plasma, the integral (4.108), with taking account of the smallness of the Fermi momentum, $p_\mathrm{F} = \sqrt{\mu_e^2 - m_e^2} \ll m_e$, can be obtained in the form:

$$F_1^{(nr)}(\mu_e, m_W) = \frac{p_\mathrm{F}^3}{3\pi^2\left(1 + 2m_e E(m_W)^{-2}\right)} = \frac{Y_e N_B}{1 + 2m_e E(m_W)^{-2}}\,, \tag{4.111}$$

where N_B is the baryon density, $Y_e = N_e/N_B$ is the fraction of electrons.

Let us consider the high-energy neutrino propagation through the "cold" plasma of the Sun or of red giants, where the temperature is $T \sim (10^7 - 10^8)\,\mathrm{K} \sim (10^{-3} - 10^{-2})\,m_e$, and the electron density is $N_e \sim 10^{26}\,\mathrm{cm}^{-3}$. The effective plasmon mass in these conditions takes the form: $m_\gamma = \sqrt{4\pi\alpha N_e/m_e}$. The stellar substance is transparent for the neutrino radiation, thus the contribution into the neutrino additional energy from thermal neutrinos can be neglected.

As a result, the additional energy of a neutrino ν_ℓ in the nonrelativistic cold plasma becomes:

$$\Delta E^{\nu_\ell} = \sqrt{2} G_{\rm F} N_B \left(\frac{\delta_{\ell e} Y_e}{1 + 2 m_e E (m_W)^{-2}} - \frac{1}{2} (1 - Y_e) \right). \tag{4.112}$$

Accordingly, the additional energy of an antineutrino $\bar{\nu}_\ell$ in the same conditions can be written as:

$$\Delta E^{\bar{\nu}_\ell} = \sqrt{2} G_{\rm F} N_B \left(\frac{-\delta_{\ell e} Y_e}{1 - 2 m_e E (m_W)^{-2} - {\rm i} \Gamma_W (m_W)^{-1}} + \frac{1}{2} (1 - Y_e) \right). \tag{4.113}$$

The analysis of the threshold inequality (4.92) for the electron neutrino reduces, in view of (4.112), to the investigation of the positiveness of the square trinomial with respect to the energy E. Assuming that $Y_e \simeq 0.6$ inside the Sun, we conclude that the inequality (4.92) is not satisfied for any neutrino energies. One can see that taking account of the non-locality of the weak interaction dramatically changes the conclusion on a possibility of the electron neutrino radiative conversion in the nonrelativistic cold plasma. Really, in the earlier papers [51, 52] where the local limit of the weak interaction was used, it was concluded that the neutrino radiative conversion in the considered conditions was possible for neutrino energies E greater than threshold energy $E_0 \simeq 10^7$ GeV. However, in reality the effect for ν_e is totally closed.

Consider now the possibilities for a trueness of the inequality (4.92) in the same conditions for other neutrino flavors. Note that the question on any observational realization of this process remains open.

The analysis of the inequality (4.92) for the electron antineutrino, in view of (4.113), where a real part of ΔE should be taken, shows that the radiative neutrino conversion is possible for antineutrino energies greater than the threshold energy value, $E > E_0 \simeq 0.6 \times 10^7$ GeV.

As it was already mentioned, the imaginary part of $\Delta E^{\bar{\nu}_e}$ causes the instability of the electron antineutrino with respect to the process $\bar{\nu}_e + e^- \to W^-$ on plasma electrons. Using the formula (4.110) one obtains from Eq. (4.113) the width of the process:

$$w(\bar{\nu}_e + e^- \to W^-) = 2\sqrt{2}\, G_{\rm F} N_e E_0 \, \frac{\Gamma_W E_0 / m_W}{(E - E_0)^2 + (\Gamma_W E_0 / m_W)^2} \,, \tag{4.114}$$

where $E_0 = m_W^2/(2 m_e)$. Evaluation of a mean free path with respect to this process, $\lambda = 1/w$, for $N_e \sim 10^{26}\,\text{cm}^{-3}$, $E \sim 10^7$ GeV provides $\lambda \sim 100$ km, while the minimum value is reached at $E = E_0$, to be: $\lambda \sim 200$ m. It is obvious, that the process $\bar{\nu}_e + e^- \to W^-$ dominates the radiative neutrino conversion, see e.g. Eq. (4.99). If one formally takes the limit $\Gamma_W \to 0$ in Eq. (4.114) to obtain:

$$w(\bar{\nu}_e + e^- \to W^-) = 2\sqrt{2}\, \pi\, G_{\rm F} N_e E_0 \; \delta(E - E_0) \,. \tag{4.115}$$

It coinsides with the result of a direct calculation of the W–boson production by $\bar{\nu}_e$ scattered off nonrelativistic electron gas, without taking account of the instability of the W–boson.

The interaction of the μ- and τ-neutrinos with medium occurs only through the Z–boson exchange with the zero momentum transfer and, as it was pointed above, it is completely described by the local limit of the weak interaction. As it can be seen from Eq. (4.112), the additional energy of ν_μ and ν_τ is negative, consequently, the neutrino radiative conversion process is closed for these neutrino flavors.

In turn, the additional energy (4.113) of antineutrinos $\bar{\nu}_\mu$ and $\bar{\nu}_\tau$ is positive. To estimate the border of the kinematically possible region for the $SL\nu$ process in this case one can use a simple inequality:

$$E > E_0 = 4\,\sin^2\,\theta_W\,\frac{Y_e}{1 - Y_e}\,\frac{m_W^2}{m_e}. \tag{4.116}$$

For $Y_e \simeq 0.6$, the process is kinematically opened for μ – and τ –antineutrino energies greater than $E_0 \simeq 2 \times 10^7$ GeV.

4.5.4.2 Relativistic Cold Plasma: Neutron Stars

The substance of a neutron star is transparent for the neutrino radiation, as in the previous case. Electrons in extremely dense neutron stars are ultra-relativistic, therefore $\mu_e \simeq p_{\mathrm{F}} \simeq 120\,(N_e/(0.05\,N_0))^{1/3}$ MeV, where p_{F} is the electron Fermi momentum, and $N_0 = 0.16$ Fm^{-3} is the typical nuclear density [64]. Due to the modern estimations, the temperature inside neutron stars does not exceed a part of MeV, so the electron gas can be considered to be degenerate and the approximation of the zero temperature can be used. In this case the electron density is $N_e = \mu_e^3/(3\pi^2)$ and the square effective plasmon mass is $m_\gamma^2 = 2\alpha\mu_e^2/\pi$.

The functions $F_{1,2}(\mu, m)$ for ultrarelativistic electrons can be obtained from Eqs. (4.108) and (4.109) by taking the limit $m_e \to 0$.

The additional energy for a neutrino ν_ℓ under conditions being considered takes the following form:

$$\Delta E^{\nu_\ell} = \sqrt{2}G_{\mathrm{F}}\left(-\frac{1}{2}\,(1 - Y_e)\,N_B + \frac{\delta_{\ell e}}{2\pi^2}A(E, \mu_e)\right), \tag{4.117}$$

$$A(E, \mu_e) = \frac{1}{16E^3}\left[4Em_W^2\mu_e(m_W^2 + 2E\mu_e) - (m_W^6 + 4E\mu_e m_W^4)\ln\left(1 + \frac{4E\mu_e}{m_W^2}\right)\right]. \tag{4.118}$$

The analysis of the threshold inequality (4.92) with taking account of Eqs. (4.117), (4.118) indicates that the $SL\nu$ process for the electron neutrino is forbidden in the conditions of a neutron star.

The similar analysis can be held for an antineutrino $\bar{\nu}_\ell$. The additional energy in this case is

$$\Delta E^{\bar{\nu}_\ell} = \sqrt{2} G_{\mathrm{F}} \left(\frac{1}{2} (1 - Y_e) N_B - \frac{\delta_{\ell e}}{2\pi^2} \bar{A}(E, \mu_e) \right), \tag{4.119}$$

$$\bar{A}(E, \mu_e) = \int\limits_0^{\mu_e} k^2 \mathrm{d}k \int\limits_{-1}^{1} \frac{(1 - x)\mathrm{d}x}{1 - 2E(1 - x)k(m_W)^{-2} - \mathrm{i}\Gamma_W (m_W)^{-1}} . \tag{4.120}$$

This integral can be easily calculated analytically but the final expression is too cumbersome. From the analysis of the kinematically possible region (4.92), where a real part of ΔE should be taken, we can conclude that the radiative conversion process (4.86) is permitted for the electron antineutrino for energies greater than the threshold value $E_0 \simeq 8 \times 10^4$ GeV, for $Y_e \simeq 0.1$ and $N_B \simeq 10^{37}$ cm^{-3}.

A comparison of these conclusions with the results of Refs. [51, 52] shows that taking account of the non-locality of the weak interaction does not lead to any qualitative changes of the conclusions on kinematical possibilities of the radiative conversion for the electron neutrino and antineutrino in the conditions of a neutron star.

Again, as in the considered case of nonrelativistic cold plasma, the imaginary part of $\Delta E^{\bar{\nu}_e}$ means an instability of the electron antineutrino with respect to the process $\bar{\nu}_e + e^- \to W^-$ on plasma electrons. A width of the process can be obtained from Eqs. (4.110), (4.119), and (4.120), but in a general case the expression is rather cumbersome. It is esssentially simplified for high neutrino energies, $E \gg m_W \Gamma_W / \mu_e$, taking the form:

$$w(\bar{\nu}_e + e^- \to W^-) = \frac{G_{\mathrm{F}} m_W^4 \mu_e}{2\sqrt{2}\, \pi\, E^2} \left(1 - \frac{m_W^2}{4\mu_e E} \right) \Theta \left(E - \frac{m_W^2}{4\mu_e} \right) . \tag{4.121}$$

Evaluation of a mean free path with respect to this process for $\mu_e \simeq 120$ MeV, $E \simeq 5 \times 10^4$ GeV provides $\lambda \sim 10^{-5}$ cm. Domination of the process $\bar{\nu}_e + e^- \to W^-$ over the radiative neutrino conversion in the neutron star conditions is undoubted, see Eq. (4.99).

For $\mu-$, $\tau-$neutrino and antineutrino, as well as in the case of nonrelativistic cold plasma, it is correct to use the local limit of the weak interaction. Substituting the additional energy for $\ell = \mu, \tau$

$$\Delta E^{\nu_\ell, \bar{\nu}_\ell} = \mp \frac{G_{\mathrm{F}}}{\sqrt{2}} (1 - Y_e) N_B , \tag{4.122}$$

and the plasmon mass in the case of a cold degenerate plasma

$$m_\gamma = \left(\frac{2\,\alpha}{\pi}\right)^{1/2} \left(3\,\pi^2\,Y_e\,N_B\right)^{1/3} \tag{4.123}$$

into the threshold inequality (4.92), we come to the conclusion that for $\nu_\mu,\ \nu_\tau$ the radiative conversion process (4.86) is forbidden. For $\bar{\nu}_\mu,\ \bar{\nu}_\tau$ the process is kinematically permitted for the energies greater than

$$E > E_0 = \frac{2\,\sin^2\,\theta_W}{1 - Y_e} \left(\frac{3\,Y_e}{\pi}\right)^{2/3} \frac{m_W^2}{N_B^{1/3}}. \tag{4.124}$$

Using for estimation the values $Y_e \simeq 0.1, N_B \simeq 10^{37}\ \text{cm}^{-3}$, we obtain $E_0 \simeq 2 \times 10^4$ GeV.

4.5.4.3 Hot Plasma of a Supernova Core

In this case one needs to use the general expression for the additional energy (4.107) of the neutrino ν_ℓ and antineutrino $\bar{\nu}_\ell$ with taking account of the scattering on all plasma components. The additional energy can be written as:

$$\begin{aligned}
\Delta E^{\nu_\ell,\bar{\nu}_\ell} &= \sqrt{2} G_{\text{F}} \Bigg\{ \mp \frac{1}{2}(N_n - \bar{N}_n) \pm (N_{\nu_e} - \bar{N}_{\nu_e}) \\
&\pm (N_{\nu_\mu} - \bar{N}_{\nu_\mu}) \pm (N_{\nu_\tau} - \bar{N}_{\nu_\tau}) \\
&+ \frac{T^3}{2\pi^2} \Bigg[\delta_{\ell e} \Big(B(\pm\mu_e, m_W, T) - B(\pm\mu_e, m_W^*, -T) \Big) \\
&+ \frac{1}{2} \Big(B(\pm\tilde{\mu}_{\nu_\ell}, m_Z, T) - B(\pm\tilde{\mu}_{\nu_\ell}, m_Z^*, -T) \Big) \Bigg] \Bigg\},
\end{aligned} \tag{4.125}$$

where we use the notation

$$B(\mu, m, T) = -\frac{m^2}{ET} \left[\text{Li}_2 \left(-\text{e}^{\mu/T} \right) + b \int\limits_0^\infty \frac{\text{d}y}{\exp\,(y - \mu/T) + 1} \ln\left(1 + \frac{y}{b}\right) \right]. \tag{4.126}$$

Here, $\text{Li}_2(z)$ is the Euler dilogarithm, and b is the dimensionless parameter, $b = m^2/4ET$. The complex masses are introduced in the functions $B(\pm\mu_e, m_W^*, -T)$ and $B(\pm\tilde{\mu}_{\nu_\ell}, m_Z^*, -T)$ of Eq. (4.125) for proper taking account of imaginary parts, similarly to Eqs. (4.109) and (4.120).

In the limit $m_W^2 \gg 4ET$, that is $b \gg 1$, assuming that plasma is not degenerate ($\mu_e \sim T$), the integral in Eq. (4.126) can be represented as the series expansion that can be calculated analytically:

$$\int_0^\infty \frac{\mathrm{d}y}{\mathrm{e}^{-\mu/T}\mathrm{e}^y + 1} \ln\left(1 + \frac{y}{b}\right)$$
$$= \frac{1}{b} \int_0^\infty \frac{y\mathrm{d}y}{\mathrm{e}^{-\mu/T}\mathrm{e}^y + 1} - \frac{1}{2\,b^2} \int_0^\infty \frac{y^2\mathrm{d}y}{\mathrm{e}^{-\mu/T}\mathrm{e}^y + 1}$$
$$+ \frac{1}{3\,b^3} \int_0^\infty \frac{y^3\mathrm{d}y}{\mathrm{e}^{-\mu/T}\mathrm{e}^y + 1} - \cdots \tag{4.127}$$

Taking into account that the arising Fermi integrals are expressed in terms of polylogarithms:

$$\int_0^\infty \frac{y^n\mathrm{d}y}{\mathrm{e}^{-\mu/T}\mathrm{e}^y + 1} = -n!\,\mathrm{Li}_{n+1}\left(-\mathrm{e}^{\mu/T}\right), \tag{4.128}$$

and using the recurrent connections between the polylogarithms $\mathrm{Li}_n(x)$ and $\mathrm{Li}_n(x^{-1})$ [65], one obtains the following expression:

$$\Delta E^{\nu_e} = \sqrt{2}G_\mathrm{F}\Bigg[C_V^{(e)}\,\frac{\mu_e}{3\pi^2}\left(\mu_e^2 + \pi^2T^2\right)$$
$$-\frac{2}{3\pi^2}\frac{E}{m_W^2}\left(\mu_e^4 + 2\pi^2\mu_e^2T^2 + \frac{7\pi^4}{15}T^4\right)$$
$$+\frac{8}{5\pi^2}\frac{E^2\mu_e}{m_W^4}\left(\mu_e^4 + \frac{10\pi^2}{3}\mu_e^2T^2 + \frac{7\pi^4}{3}T^4\right)$$
$$-\frac{64}{15\pi^2}\frac{E^3}{m_W^6}\left(\mu_e^6 + 5\pi^2\mu_e^4T^2 + 7\mu_e^2\pi^4T^4 + \frac{31}{21}\pi^6T^6\right) + \cdots\Bigg]. \tag{4.129}$$

As it is illustrated in Fig. 4.13, taking account of only few terms in the series (4.129) for the additional electron neutrino energy ΔE as a function of the initial neutrino energy E, leads to an overestimation or understatement of the additional energy.

For a numerical estimation of the borders of the kinematically possible region for the $SL\nu$ process in a general case with using of Eq. (4.125), let us take $\mu_e \simeq 160$ MeV, $\tilde{\mu}_\nu \simeq \mu_e/4 \simeq 40$ MeV, and $T \simeq 30$ MeV, see e.g. Refs. [66] and [67]. The analysis displays that the process is forbidden for neutrinos of all flavors. For all types of antineutrinos the effect becomes possible for energies greater than 2×10^4 GeV.

As in the considered cases of nonrelativistic cold plasma and of the neutron star interior, for electron neutrinos and antineutrinos the processes of the W–boson production on plasma electrons and positrons, $\nu_e + e^+ \to W^+$ and $\bar{\nu}_e + e^- \to W^-$, are dominating. Using Eqs. (4.107), (4.110), one obtains the width of the process in the conditions of a hot dense plasma, $\mu_e \sim T \gg m_e$, for high neutrino energies, $E \gg m_W\,\Gamma_W/\mu_e$:

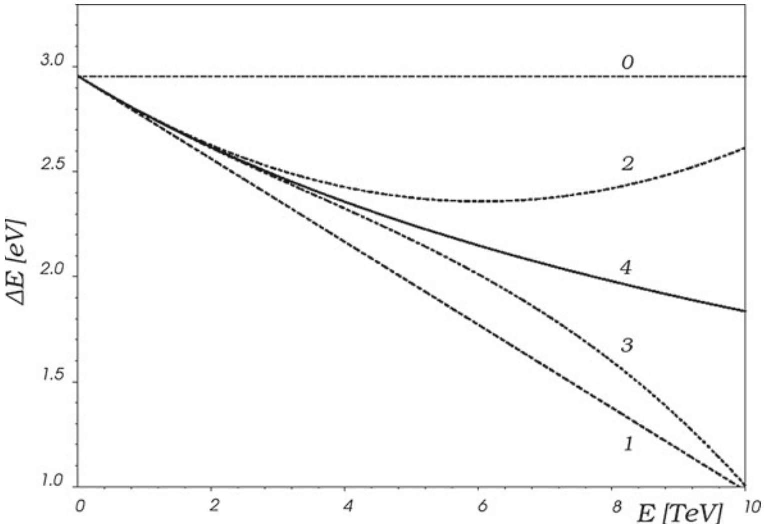

Fig. 4.13 Additional electron neutrino energy in the electron-positron medium ($\mu_e \simeq 160\,\text{MeV}$, $T \simeq 30\,\text{MeV}$) as an expansion into the series by initial neutrino energy: *0* is the local contribution; *1*, *2* and *3*—with consecutive adding of non-local terms $\sim E$, $\sim E^2$ and $\sim E^3$; *4* is the exact function (Figure reprinted from [46] with the World Scientific Publishing Company's permission.)

$$w(\bar{\nu}_e + e^- \to W^-) = \frac{G_F m_W^4 T}{2\sqrt{2}\,\pi\,E^2} \ln\left[1 + \exp\left(\frac{4\mu_e E - m_W^2}{4ET}\right)\right]. \tag{4.130}$$

Taking here the limit of cold plasma, $T \to 0$, one readily comes to Eq. (4.121). The width of the W^+ production by ν_e on positrons can be obtained from Eq. (4.130) by the replacement $\mu_e \to -\mu_e$.

Since in a dense plasma of the supernova core thermal neutrinos and antineutrinos of all flavors present, the processes of the Z–boson production should be also considered for the sake of completeness. Using Eqs. (4.107), (4.110), one obtains the width of the process where a high-energy antineutrino of the flavor ℓ scatters off a thermal ν_ℓ:

$$w(\bar{\nu}_\ell + \nu_\ell \to Z) = \frac{G_F m_Z^4 T}{4\sqrt{2}\,\pi\,E^2} \ln\left[1 + \exp\left(\frac{4\tilde{\mu}_{\nu_\ell} E - m_Z^2}{4ET}\right)\right]. \tag{4.131}$$

The width of the process with a high-energy neutrino and a thermal antineutrino can be obtained from Eq. (4.131) by the replacement $\tilde{\mu}_{\nu_\ell} \to -\tilde{\mu}_{\nu_\ell}$. It should be noted that in the supernova core conditions, $\tilde{\mu}_{\nu_\ell} \simeq 0$ for $\ell = \mu,\ \tau$.

Thus, the analysis of a possibility of the neutrino radiative conversion effect $\nu_L \to \nu_R + \gamma$ ("spin light of neutrino", $SL\nu$) based on the additional neutrino energy in plasma in the case of ultra-high neutrino energies [59] should be performed only with taking into account the dependence of the W and Z–boson propagators on the momentum transferred. It should be noted that the question about any observational realization of the studied process requires a separate consideration. For high energy neutrinos and antineutrinos, the processes of the $W-$ and Z–boson production on plasma, $\nu_e + e^+ \to W^+$, $\bar{\nu}_e + e^- \to W^-$ and $\bar{\nu}_\ell + \nu_\ell \to Z$, are dominating.

4.6 Neutrino Self-energy Operator in an External Magnetic Field

4.6.1 Definition of the Operator $\Sigma(p)$ in a Magnetic Field

As it was noted above, the analysis of the influence, along with plasma, of another component of external active astrophysical environment, which is strong magnetic field, onto the properties of neutrinos, in particular onto the neutrino oscillation mechanism, is of considerable interest. However, this effect of the field could be significant only in a case of its sufficiently high intensity. As already noted, there is a natural scale of the magnetic field, called the critical value, $B_e = m_e^2/e \simeq 4.41 \times 10^{13}$ gauss. There are arguments in favor of the field of such and larger scales to be generated in astrophysical processes, such as supernova explosions and mergings of neutron stars, which are characterized also by giant neutrino fluxes.

It should be noted that the study of the self-energy operator of a neutrino in a magnetic field has a 30-year-old history [32–34, 36, 37, 68].

A general Lorentz structure of the self-energy neutrino operator $\Sigma(p)$ in a magnetic field can be presented in a form similar to the expression (4.65), in terms of linearly independent covariant structures:

$$\begin{aligned}\Sigma(p) = &\left[\mathcal{A}_L\,(p\gamma) + \bar{\mathcal{B}}_L\,e^2\left(p\tilde{F}\tilde{F}\gamma\right) + \bar{\mathcal{C}}_L\,e\left(p\tilde{F}\gamma\right)\right]\gamma_L \\ &+ \left[\mathcal{A}_R\,(p\gamma) + \bar{\mathcal{B}}_R\,e^2\left(p\tilde{F}\tilde{F}\gamma\right) + \bar{\mathcal{C}}_R\,e\left(p\tilde{F}\gamma\right)\right]\gamma_R \\ &+ m_\nu\,\left[\mathcal{K}_1 + \mathrm{i}\,\mathcal{K}_2\,e\,(\gamma F\gamma)\right]\,. \end{aligned} \tag{4.132}$$

Similarly to Eq. (4.65), if the approximation is used of the massless left-handed neutrino, only three terms with the coefficients $\mathcal{A}_L$, $\bar{\mathcal{B}}_L$, and $\bar{\mathcal{C}}_L$ present in the operator $\Sigma(p)$.

The analysis shows that the results of calculations of the invariant coefficients $\bar{\mathcal{B}}_L$, and $\bar{\mathcal{C}}_L$ in Eq. (4.132), obtained by different authors, are not consistent. In Table 4.1, we give the values of these coefficients obtained in previous studies, and the results of our calculations, which are discussed in detail below. The field B is called "weak" at $eB \ll m_\ell^2$ and "moderate" at $m_\ell^2 \ll eB \ll m_W^2$.

Below, it will be demonstrated in detail that for massive neutrino the coefficient $\bar{\mathcal{C}}_L$ defines its anomalous magnetic moment. As a validation of the calculations of the coefficient $\bar{\mathcal{C}}_L$, it should be its agreement with the known result for the neutrino anomalous magnetic moment [69, 70]:

Table 4.1 The coefficients in the formula (4.132) for the self-energy neutrino operator $\Sigma(p)$ in an external magnetic field. *)

Authors	Field	$\bar{\mathcal{B}}_L \times \dfrac{\sqrt{2}\,\pi^2}{G_F}$	$\bar{\mathcal{C}}_L \times \dfrac{\sqrt{2}\,\pi^2}{G_F}$
McKeon [32]	–	0	$+3$
Erdas et al. [34]	Mod.	$-\dfrac{1}{3m_W^2}\left(\ln\dfrac{m_W^2}{m_\ell^2}+\dfrac{3}{4}\right)$	0
Elizalde et al. [35]	Mod.	$+\dfrac{1}{2eB}$	$-\dfrac{1}{2}$
Elizalde et al. [41]	Mod.	$+\dfrac{1}{4eB}\,\mathrm{e}^{-p_\perp^2/(2eB)}$	$-\dfrac{1}{4}\,\mathrm{e}^{-p_\perp^2/(2eB)}$
Our result [36]	Weak	$-\dfrac{1}{3m_W^2}\left(\ln\dfrac{m_W^2}{m_\ell^2}+\dfrac{3}{4}\right)$	$+\dfrac{3}{4}$
Our result [36]	Mod.	$-\dfrac{1}{3m_W^2}\left(\ln\dfrac{m_W^2}{eB}+2.542\right)$	$+\dfrac{3}{4}$

*) It is indicated in Ref. [41] that their result is valid in the region of the neutrino momenta $0 < p_\perp^2 \ll eB$. Our result is valid in the region $0 < p_\perp^2 \ll m_W^4/\beta$.

$$\mu_\nu \simeq \frac{e\,m_\nu\,\bar{\mathcal{C}}_L}{2} = \frac{3e\,G_F m_\nu}{8\pi^2\sqrt{2}}\,. \tag{4.133}$$

The comparison shows that in Ref. [32] the coefficient $\bar{\mathcal{C}}_L$ was overstated by 4 times, while in Refs. [35] and [41] it contains the extra factors: $-2/3$ and $-1/3$, respectively. In addition, in Ref. [32], a non-zero value is declared for the coefficient at the structure of the form $(pF\gamma)$, defining the electric dipole moment of the neutrino. However, this contribution to the neutrino self-energy operator can differ from zero only in the presence of the electromagnetic field with a nonzero CP-odd field invariant $(F\tilde{F}) = 4\,(\mathcal{E}\mathbf{B})$. But even in this case, it is strongly suppressed (see [33]). One should conclude that the result for the neutrino electric dipole moment obtained in Ref. [32], where a purely magnetic field was considered, was erroneous.

The differences in the results for the coefficient of $\bar{\mathcal{B}}_L$ are the most significant. In Ref. [32], it was not calculated as negligible. Computation of the $\bar{\mathcal{B}}_L$, carried out in Ref. [34], led to the magnitude scale G_F/m_W^2. If compared with this value, the result for $\bar{\mathcal{B}}_L$ obtained in Refs. [35, 41], has a huge amplification factor of m_W^2/eB. Being correct, that result would lead to important consequences for the physics of neutrinos in medium (see [71]) because the field contribution to the additional neutrino energy would exceed the plasma contribution.

Earlier calculations of the plasma contribution to the operator $\Sigma(p)$ both excluding and including the magnetic field, were performed in a number of papers (see, e.g., [30, 38, 39]).

In Ref. [39], the neutrino dispersion properties were studied in the approximations $m_e \ll T \ll m_W$ and $B \lesssim T^2$ for the sake of applying the results to the early Universe. In particular, for a charge-symmetric plasma it is possible to extract from Ref. [39]

the difference of the neutrino self-energies which is the same for neutrinos and antineutrinos,[3] $\Delta E = E_{\nu_e} - E_{\nu_i}$ $(i = \mu, \tau)$, in the form:

$$\Delta E^{(T,\,B)} \simeq -6.0\,\frac{G_F\,T^4}{m_W^2}\,|\mathbf{p}\,| + 0.47\,\frac{G_F\,T^2}{m_W^2}\,e\,(\mathbf{B}\,\mathbf{p}\,)\,, \tag{4.134}$$

where $\mathbf{p}$ is the momentum of a neutrino or antineutrino. The first term is the dominating contribution of pure plasma, and the second term is caused by the collective influence of the plasma and magnetic field.

The pure field contribution to the neutrino self-energy was not considered by the authors [39] as insignificant. In contrast, the authors [35, 41] argue that just the field contribution is dominant. The result of Ref. [41] for the pure field contribution to the difference of the neutrino self-energies, which is the same for neutrinos and antineutrinos, can be written as

$$\Delta E^{(B)} \simeq \frac{G_F\,eB}{4\sqrt{2}\pi^2}\,|\mathbf{p}\,|\,\sin^2\phi\,, \tag{4.135}$$

where ϕ is the angle between $\mathbf{B}$ and $\mathbf{p}$. A comparison of the formulas (4.134) and (4.135) shows that the pure field contribution obtained by the authors [35, 41] may significantly exceed the plasma contribution (4.134). Indeed, for the ratio of the contributions one obtains

$$R = \left|\frac{\Delta E^{(B)}}{\Delta E^{(T,\,B)}}\right| \simeq 1.5\times 10^{-3}\,\frac{eB}{T^2}\,\frac{m_W^2}{T^2}\,, \tag{4.136}$$

where an averaging over the angle ϕ is performed in the value $\Delta E^{(B)}$, and only the leading term is taken in the value $\Delta E^{(T,\,B)}$. Since the temperature during the considered stage of the evolution of the Universe $T \ll m_W$, the ratio R can appear significantly greater than one due to a large factor $(m_W/T)^2$.

Thus, since the question was of fundamental importance, whether the contribution of the external magnetic field into the neutrino energy was negligible or dominant, the necessity of its independent calculation was obvious. This calculation was performed in Ref. [36]. Here we reproduce the calculation of the neutrino self-energy operator in a constant uniform magnetic field which is weaker than the critical field for a W boson, $eB \ll m_W^2$.

The $\mathcal{S}$-matrix element for the transition $\nu \to \nu$ corresponds to the Feynman diagrams in Fig. 4.14.

Similarly to the procedure described in Sect. 4.5.2, the self-energy operator of a neutrino in a magnetic field can be found to be

[3] The sign $\pm$ at the linear in the field term in Eq. (13) of Ref. [39] came from poorly chosen notations: the neutrino momentum in this article was $\mathbf{k}$, while the antineutrino momentum was $-\mathbf{k}$ (G. Raffelt, private communication).

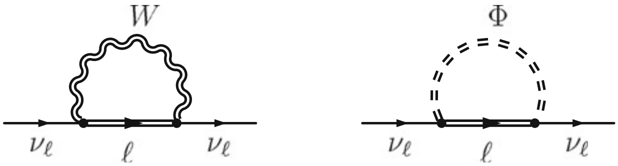

Fig. 4.14 The Feynman diagrams describing the magnetic field-induced contribution to the neutrino self-energy operator in the Feynman gauge: the double lines correspond to the exact propagators of the charged lepton, W-boson and the unphysical charged scalar Φ-boson in an external magnetic field

$$\Sigma(p) = -\frac{\mathrm{i}\, g^2}{2}\Big[\gamma^\alpha\, \gamma_L\, J^{(W)}_{\alpha\beta}(p)\, \gamma^\beta\, \gamma_L \qquad (4.137)$$
$$+\frac{1}{m_W^2}\,(m_\ell\gamma_R - m_\nu\gamma_L)\, J^{(\Phi)}(p)\, (m_\ell\gamma_L - m_\nu\gamma_R)\Big],$$

where g is the constant of the electroweak standard model. The integrals introduced in Eq. (4.137), have the form

$$J^{(W)}_{\alpha\beta}(p) = \int \frac{\mathrm{d}^4 q}{(2\pi)^4}\, S(q)\, G^{(W)}_{\beta\alpha}(q-p)\,, \qquad (4.138)$$

$$J^{(\Phi)}(p) = \int \frac{\mathrm{d}^4 q}{(2\pi)^4}\, S(q)\, D^{(\Phi)}(q-p)\,, \qquad (4.139)$$

where $S(q)$, $G^{(W)}_{\beta\alpha}(q-p)$ and $D^{(\Phi)}(q-p)$ are the Fourier transforms of the translationally invariant parts of the propagators for a charged lepton, W^--boson and charged scalar Φ-boson respectively, see Eqs. (3.10), (3.13) and (3.14). We emphasize that m_ν in Eq. (4.137) is generally a non-diagonal Dirac neutrino mass matrix caused by the mixing in the lepton sector.

It should be noted that the coefficients $\mathcal{A}_R$, $\bar{\mathcal{B}}_R$, $\bar{\mathcal{C}}_R$, and $\mathcal{K}_{1,2}$ in Eq. (4.132) originated from the Feynman diagram with a scalar Φ-boson and are suppressed by the square of the ratio of the lepton mass to the mass of the W-boson, while the coefficients $\mathcal{A}_L$, $\bar{\mathcal{B}}_L$, and $\bar{\mathcal{C}}_L$ contain the contributions from both diagrams.

Further, we calculate the contribution to the neutrino self-energy operator from the nth Landau level in the propagator of the charged lepton in combination with the exact W-propagator. It is shown that the contribution of the ground Landau level is not dominant and the higher levels give contributions of the same order, contrary to the assumption used in Refs. [35, 41]. Then we present a detailed calculation of the neutrino self-energy operator in a magnetic field in two limiting cases, of a relatively weak field, $eB \ll m_\ell^2$, and moderately strong field, $m_\ell^2 \ll eB \ll m_W^2$. The additional energy acquired by a neutrino in an external magnetic field is calculated, and possible cosmological and astrophysical implications are analyzed.

4.6.2 Low Landau Level Contribution into the Operator $\Sigma(p)$

As we have already mentioned, our results for the coefficients of the neutrino self-energy operator (4.132) strongly disagree with that of Refs. [35, 41]. We think that the disagreement arises because these authors used only one lowest Landau level contribution in the charged-lepton propagator in the case of moderate field strengths which they call "strong fields." However, the contributions of the next Landau levels can be of the same order as the ground-level contribution because in the integration over the virtual lepton four-momentum in the loop the region $q_\parallel^2 \sim m_W^2 \gg \beta$ appears to be essential.

To substantiate this point we calculate the contribution to the neutrino self-energy operator from the nth charged-lepton Landau level in conjunction with the exact W-propagator in the limit $p_\perp^2/m_W^2 \ll m_W^2/\beta$. Substituting the exact W-propagator (3.13) and the nth Landau level contribution to the charged-lepton propagator from Eq. (3.37) into Eq. (4.138) we find

$$\begin{aligned}
J_{\sigma\rho}^{(n)}(p) = &-\int \frac{\mathrm{d}^4 q}{(2\pi)^4}\, \frac{\mathrm{i}}{q_\parallel^2 - m_\ell^2 - 2n\beta} \Bigg\{ (q\gamma)_\parallel \left[d_n(v) - \frac{\mathrm{i}}{2}\,(\gamma\varphi\gamma)\; d_n'(v) \right] \\
&- (q\gamma)_\perp\, 2n\, \frac{d_n(v)}{v} \Bigg\} \\
&\times \int_0^\infty \frac{\mathrm{d}s}{\cos(\beta s)}\, \exp\!\left[-\mathrm{i}s \left(m_W^2 - (q-p)_\parallel^2 + \frac{\tan(\beta s)}{\beta s}\,(q-p)_\perp^2 \right) \right] \\
&\times \left[(\tilde\varphi\tilde\varphi)_{\rho\sigma} - (\varphi\varphi)_{\rho\sigma}\, \cos(2\beta s) - \varphi_{\rho\sigma}\, \sin(2\beta s) \right].
\end{aligned} \tag{4.140}$$

The terms with even numbers of γ matrices were omitted because they are removed by the chiral structure of the operator Eq. (4.132). Next we perform a clockwise rotation in the complex plane $s = -\mathrm{i}\tau$ and use the identity

$$\frac{1}{q_\parallel^2 - m_\ell^2 - 2n\beta} = -\int_0^\infty \mathrm{d}\tau'\, \exp\left[-\tau' \left(m_\ell^2 + 2n\beta - q_\parallel^2 \right) \right]. \tag{4.141}$$

These manipulations allow us to rewrite the integral Eq. (4.140) as

$$\begin{aligned}
J_{\sigma\rho}^{(n)}(p) = &\int \frac{\mathrm{d}^4 q}{(2\pi)^4} \left\{ (q\gamma)_\parallel \left[d_n(v) - \frac{\mathrm{i}}{2}\,(\gamma\varphi\gamma)\, d_n'(v) \right] - (q\gamma)_\perp\, 2n\, \frac{d_n(v)}{v} \right\} \\
&\times \int_0^\infty \frac{\mathrm{d}\tau\,\mathrm{d}\tau'}{\cosh(\beta\tau)} \left[(\tilde\varphi\tilde\varphi)_{\rho\sigma} - (\varphi\varphi)_{\rho\sigma}\, \cosh(2\beta\tau) + \mathrm{i}\,\varphi_{\rho\sigma}\, \sinh(2\beta\tau) \right] \\
&\times \exp\Bigg[-\tau' \left(m_\ell^2 + 2n\beta - q_\parallel^2 \right) - \tau \left(m_W^2 - (q-p)_\parallel^2 \right) \\
&- \frac{\tanh(\beta\tau)}{\beta}\,(q-p)_\perp^2 \Bigg].
\end{aligned} \tag{4.142}$$

In the integration over $d^4q = d^2q_\parallel\, d^2q_\perp$, the integrals over $d^2q_\parallel$ can be easily calculated because they are of Gaussian form. As a result we find

$$
\begin{aligned}
J^{(n)}_{\sigma\rho}(p) &= \frac{\mathrm{i}}{16\pi^3 m_W^2}\int_0^\infty \frac{\mathrm{d}x\,\mathrm{d}y}{(x+y)\cosh(\eta x)}\,\exp\left[-x+\xi\frac{xy}{x+y}-y\,(2n\eta+\lambda)\right]\\
&\times\left[(\tilde\varphi\tilde\varphi)_{\rho\sigma}-(\varphi\varphi)_{\rho\sigma}\,\cosh(2\eta x)+\mathrm{i}\,\varphi_{\rho\sigma}\,\sinh(2\eta x)\right]\\
&\times\int d^2q_\perp\,\exp\left[-\frac{\tanh(\eta x)}{\beta}\,(q-p)^2_\perp\right]\\
&\times\left\{(p\gamma)_\parallel\,\frac{x}{x+y}\left[d_n(v)-\frac{\mathrm{i}}{2}\,(\gamma\varphi\gamma)\,d_n'(v)\right]-(q\gamma)_\perp\,2n\,\frac{d_n(v)}{v}\right\},
\end{aligned}
\tag{4.143}
$$

where the dimensionless variables $x = m_W^2\tau$ and $y = m_W^2\tau'$ have been introduced as well as the parameters $\eta = \beta/m_W^2$, $\xi = p_\parallel^2/m_W^2 \simeq p_\perp^2/m_W^2$ and $\lambda = m_\ell^2/m_W^2$. It follows from Eq. (4.143) that the essential region of the x variable is $x \sim 1$ due to the exponential e^{-x}. Given the condition $\eta \ll 1$, the argument of the hyperbolic functions is small, $\eta x \ll 1$, leading to an obvious simplification. One should also take into account the condition $q_\perp^2 \sim \beta$ caused by the functions $d_n(v)$, see Eq. (3.38), containing the exponential e^{-v}. For a wide range of the numbers n the exponential in the integral over $d^2q_\perp$ is simplified, with the only restriction $n \ll 1/\eta = m_W^2/\beta$:

$$
\begin{aligned}
\exp\left[-\frac{\tanh(\eta x)}{\beta}\,(q-p)^2_\perp\right] &\simeq \exp\left(-x\,\frac{p_\perp^2}{m_W^2}\right)\\
&\times\exp\left(-x\,\frac{q_\perp^2-2(qp)_\perp}{m_W^2}\right).
\end{aligned}
\tag{4.144}
$$

Here, the first exponential is equal to $e^{-\xi x}$. We consider the value $p_\perp^2$ to vary in a very wide range, $0 < p_\perp^2 \ll m_W^4/\beta$. The second exponential is equal to unity with a good accuracy, because $q_\perp^2 \sim \beta \ll m_W^2$ and $(qp)_\perp \ll m_W^2$. With these approximations, the integration over $d^2q_\perp$ can be easily performed,

$$
\int d^2q_\perp\,d_n(v) = \pi\,\beta\,(2-\delta_{n0})\,,\qquad \int d^2q_\perp\,d_n{}'(v) = -\pi\,\beta\,\delta_{n0}\,,\tag{4.145}
$$

$$
\int d^2q_\perp\,(q\gamma)_\perp\,\frac{d_n(v)}{v} = 0\,.\tag{4.146}
$$

Let us return to the expression (4.144) and make additional comments, to prevent possible misunderstanding. At first glance it might seem that the replacement of the value $(p-q)_\perp^2$ by $p_\perp^2$, which actually occurred in the expression (4.144), meant that an additional condition had been taken: $p_\perp^2 \gg q_\perp^2 \sim eB$, which significantly narrowed the area of the values of $p_\perp$ under consideration. We show by direct calculation that this is not so and that the result is valid in the entire range $0 < p_\perp^2 \ll m_W^4/\beta$.

Let us save the second exponential term in Eq. (4.144) and substitute it into the first integral (4.146) denoting it as $j^{(n)}$:

$$j^{(n)} = \int \mathrm{d}^2 q_\perp \, d_n \left(\frac{q_\perp^2}{\beta} \right) \exp \left(-x \, \frac{q_\perp^2 - 2(qp)_\perp}{m_W^2} \right). \tag{4.147}$$

Consider first the case $n = 0$, with $d_0(v) = \exp(-v)$. Making a transition in the plane $q_\perp$ to polar coordinates $\{q_\perp, \phi\}$, while $(qp)_\perp = q_\perp p_\perp \cos\phi$, and using the known integral

$$\int\limits_0^{2\pi} \mathrm{d}\phi \, \mathrm{e}^{b \cos\phi} = 2\pi \, I_0(b) \,, \tag{4.148}$$

where $I_0(b)$ is the modified Bessel function of zero order, we obtain

$$j^{(0)} = 2\pi \int\limits_0^\infty q_\perp \, \mathrm{d}q_\perp \, \exp \left[-q_\perp^2 \left(\frac{1}{\beta} + \frac{x}{m_W^2} \right) \right] I_0 \left(\frac{2x \, q_\perp p_\perp}{m_W^2} \right). \tag{4.149}$$

We emphasize that no approximation has not been done yet. Using another well-known integral

$$\int\limits_0^\infty \mathrm{d}y \, \mathrm{e}^{-y} \, I_0(2z\sqrt{y}) = \mathrm{e}^{z^2} \,, \tag{4.150}$$

we finally obtain

$$j^{(0)} = \frac{\pi\beta}{1 + x\beta/m_W^2} \, \exp \left(\frac{x^2 \, p_\perp^2 \, \beta}{m_W^4 (1 + x\beta/m_W^2)} \right). \tag{4.151}$$

Recalling that $x \lesssim 1$, $\beta \ll m_W^2$ and $p_\perp^2 \ll m_W^4/\beta$, we exactly reproduce from (4.151) the first integral of Eq. (4.145) with $n = 0$:

$$j^{(0)} \simeq \pi\beta \,. \tag{4.152}$$

The similar calculation for $n = 1$, when $d_1(v) = 2v \exp(-v)$, leads to one more well-known integral:

$$\int\limits_0^\infty y \, \mathrm{d}y \, \mathrm{e}^{-y} \, I_0(2z\sqrt{y}) = (1 + z^2) \, \mathrm{e}^{z^2} \,. \tag{4.153}$$

In the approximation used, this gives

$$j^{(1)} \simeq 2\pi\beta\,. \tag{4.154}$$

The similar analysis can be performed for any n. Thus, the approximation used in the calculation of the integrals (4.145)–(4.146), is justified.

Given the relations (4.145)–(4.146), the integral (4.143) acquires the form

$$\begin{aligned} J^{(n)}_{\sigma\rho}(p) &= \frac{\mathrm{i}\,\beta}{16\pi^2 m_W^2}\,(p\gamma)_{\parallel}\, g_{\rho\sigma}\left\{2 - \left[1 - \frac{\mathrm{i}}{2}\,(\gamma\varphi\gamma)\right]\delta_{n0}\right\} \\ &\times \int_0^{\infty} \frac{x\,\mathrm{d}x\,\mathrm{d}y}{(x+y)^2}\,\exp\left[-x - \xi\frac{x^2}{x+y} - y\,(2n\eta + \lambda)\right]. \end{aligned} \tag{4.155}$$

Taking into account the smallness of the parameters η and λ, one finally obtains for $n \ll m_W^2/\beta$

$$J^{(n)}_{\sigma\rho}(p) = \frac{\mathrm{i}\,\beta}{16\pi^2\,p_{\perp}^2}\,\ln\left(1 + \frac{p_{\perp}^2}{m_W^2}\right)(p\gamma)_{\parallel}\, g_{\rho\sigma}\left\{2 - \left[1 - \frac{\mathrm{i}}{2}\,(\gamma\varphi\gamma)\right]\delta_{n0}\right\}. \tag{4.156}$$

Substituting Eq. (4.156) into Eq. (4.137) we finally find the contribution of the nth Landau level of the lepton propagator to the neutrino self-energy operator

$$\Sigma^{(n)}(p) = -\frac{G_{\mathrm{F}}\,eB}{\sqrt{2}\,2\pi^2}\,\frac{m_W^2}{p_{\perp}^2}\,\ln\left(1 + \frac{p_{\perp}^2}{m_W^2}\right)\left[(2-\delta_{n0})\,(p\gamma)_{\parallel} - \delta_{n0}\,(p\tilde{\varphi}\gamma)\right]\gamma_L\,. \tag{4.157}$$

We conclude from Eq. (4.157) that, contrary to the treatment of Refs. [35, 41], the lowest Landau level does not dominate.

For higher Landau levels, $n \gtrsim m_W^2/\beta$, the calculation is more cumbersome. Therefore, using the lepton propagator expanded in terms of the Landau levels, with a further summation, is extremely inconvenient. It is much simpler to take the exact lepton propagator in the form of Eq. (3.10). This approach is used in Sect. 4.6.4 below.

4.6.3 Calculation of the Operator $\Sigma(p)$ in a "Weak" Field

Because of the discrepancy of our results with the results of Refs. [35, 41], we present here our calculations of the operator $\Sigma(p)$ in detail. We start the analysis with the simpler case of a relatively weak field, when the field strength is the smallest dimensional parameter of the problem, $eB \ll m_\ell^2 \ll m_W^2$. In this case, for the Fourier transforms of the propagators both the W-boson and lepton one can use the field decompositions (3.22) and (3.24) and evaluate the integral (4.138) as a series in powers of β/m_W^2:

$$
\begin{aligned}
J_{\alpha\beta}(p) = &\int \frac{d^4q}{(2\pi)^4} \left[S^{(0)}(q) + S^{(1)}(q) + S^{(2)}(q) + \cdots\right] \\
&\times \left[G^{(0)}_{\beta\alpha}(q-p) + G^{(1)}_{\beta\alpha}(q-p) + G^{(2)}_{\beta\alpha}(q-p) + \cdots\right] \\
= &\,\Delta_0 J_{\alpha\beta}(p) + \Delta_1 J_{\alpha\beta}(p) + \Delta_2 J_{\alpha\beta}(p) + \cdots
\end{aligned}
\tag{4.158}
$$

It is easy to show that the fieldless term $\Delta_0 J_{\alpha\beta}(p)$ containing an ultraviolet divergence, has the structure of $g_{\alpha\beta}\,(p\gamma)$ and contributes only to the coefficient $\mathcal{A}_L$ of the operator $\Sigma(p)$, see (4.132), which is absorbed to the renormalization of the neutrino wave function.

The first-order term consists of two parts:

$$
\begin{aligned}
\Delta_1 J_{\alpha\beta}(p) &= J^{(\ell_0 W_1)}_{\alpha\beta}(p) + J^{(\ell_1 W_0)}_{\alpha\beta}(p) \\
&= \int \frac{d^4q}{(2\pi)^4}\, S^{(0)}(q)\, G^{(1)}_{\beta\alpha}(q-p) + \int \frac{d^4q}{(2\pi)^4}\, S^{(1)}(q)\, G^{(0)}_{\beta\alpha}(q-p)\,.
\end{aligned}
\tag{4.159}
$$

The part containing the zero-order term of the lepton propagator and the first-order term of the W-propagator has the form

$$
J^{(\ell_0 W_1)}_{\alpha\beta}(p) = 2\mathrm{i}\,\beta\,\varphi_{\alpha\beta} \int \frac{d^4q}{(2\pi)^4}\, \frac{(q\gamma) + m_\ell}{q^2 - m_\ell^2}\, \frac{1}{[(q-p)^2 - m_W^2]^2}\,. \tag{4.160}
$$

Due to the chiral structure of the operator (4.137), only the terms with an odd number of γ-matrices should be taken. Using the expansion

$$
\frac{1}{[(q-p)^2 - m_W^2]^n} \simeq \frac{1}{(q^2 - m_W^2)^n} + \frac{2n\,(qp)}{(q^2 - m_W^2)^{n+1}}\,, \tag{4.161}
$$

where we have neglected the neutrino mass $p^2 = m_\nu^2$, we obtain in the approximation $m_\ell^2 \ll m_W^2$:

$$
J^{(\ell_0 W_1)}_{\alpha\beta}(p) \simeq \frac{1}{16\pi^2}\, \frac{\beta}{m_W^2}\, \varphi_{\alpha\beta}\,(p\gamma)\,. \tag{4.162}
$$

The part containing the first-order term of the lepton propagator and the zero-order term of the W-propagator is calculated similarly:

$$
\begin{aligned}
J^{(\ell_1 W_0)}_{\alpha\beta}(p) &= -\frac{\mathrm{i}}{2}\,\beta\, g_{\alpha\beta}\,(\gamma\varphi\gamma) \int \frac{d^4q}{(2\pi)^4}\, \frac{(q\gamma)_\parallel + m_\ell}{(q^2 - m_\ell^2)^2}\, \frac{1}{(q-p)^2 - m_W^2} \\
&\simeq -\frac{\mathrm{i}}{32\pi^2}\, \frac{\beta}{m_W^2}\, g_{\alpha\beta}\,(p\tilde{\varphi}\gamma)\,\gamma_5\,,
\end{aligned}
\tag{4.163}
$$

where we use the identity

$$(p\gamma)_{\|}\,(\gamma\varphi\gamma) = 2\mathrm{i}\,(p\tilde{\varphi}\gamma)\,\gamma_5\,. \tag{4.164}$$

Contribution of the second order into the integral $J_{\alpha\beta}(p)$ consists of three parts:

$$\begin{aligned}\Delta_2 J_{\alpha\beta}(p) &= J_{\alpha\beta}^{(\ell_0 W_2)}(p) + J_{\alpha\beta}^{(\ell_1 W_1)}(p) + J_{\alpha\beta}^{(\ell_2 W_0)}(p)\\ &= \int \frac{\mathrm{d}^4 q}{(2\pi)^4}\, S^{(0)}(q)\, G^{(2)}_{\beta\alpha}(q-p) + \int \frac{\mathrm{d}^4 q}{(2\pi)^4}\, S^{(1)}(q)\, G^{(1)}_{\beta\alpha}(q-p)\\ &\quad + \int \frac{\mathrm{d}^4 q}{(2\pi)^4}\, S^{(2)}(q)\, G^{(0)}_{\beta\alpha}(q-p)\,.\end{aligned} \tag{4.165}$$

In the approximation considered, as well as for $p_\perp^2 \simeq p_\|^2 \ll m_W^2$, using the expansion (4.161), one can obtain the part containing the zero-order term of the lepton propagator and the second-order term of the W-propagator in the form

$$\begin{aligned}J_{\alpha\beta}^{(\ell_0 W_2)}(p) &= -\beta^2 \int \frac{\mathrm{d}^4 q}{(2\pi)^4}\, \frac{(q\gamma) + m_\ell}{q^2 - m_\ell^2} \Bigg[g_{\alpha\beta} \Bigg(\frac{1}{[(q-p)^2 - m_W^2]^3}\\ &\quad + \frac{2\,(q-p)_\perp^2}{[(q-p)^2 - m_W^2]^4} \Bigg) + 4\,(\varphi\varphi)_{\alpha\beta}\, \frac{1}{[(q-p)^2 - m_W^2]^3} \Bigg]\\ &\simeq \frac{\mathrm{i}}{16\pi^2} \left(\frac{\beta}{m_W^2} \right)^2 \left[\frac{1}{18}\, g_{\alpha\beta}\,(p\gamma)_\| - (\varphi\varphi)_{\alpha\beta}\,(p\gamma) \right] + \cdots\,,\end{aligned} \tag{4.166}$$

where dots mean the term of the form $g_{\alpha\beta}\,(p\gamma)$, which contributes only to the coefficient $\mathcal{A}_L$.

The part containing the first-order term of the lepton propagator and the first-order term of the W-propagator is found to be

$$\begin{aligned}J_{\alpha\beta}^{(\ell_1 W_1)}(p) &= \beta^2\, \varphi_{\alpha\beta}\,(\gamma\varphi\gamma) \int \frac{\mathrm{d}^4 q}{(2\pi)^4}\, \frac{(q\gamma)_\| + m_\ell}{(q^2 - m_\ell^2)^2}\, \frac{1}{[(q-p)^2 - m_W^2]^2}\\ &\simeq -\frac{1}{16\pi^2} \left(\frac{\beta}{m_W^2} \right)^2 \varphi_{\alpha\beta}\,(p\tilde{\varphi}\gamma)\,\gamma_5\,.\end{aligned} \tag{4.167}$$

The combination of the second-order term of the lepton propagator and the zero-order term of the W-propagator is

$$\begin{aligned}J_{\alpha\beta}^{(\ell_2 W_0)}(p) &= 2\,\beta^2\, g_{\alpha\beta} \int \frac{\mathrm{d}^4 q}{(2\pi)^4}\, \frac{(q_\|^2 - m_\ell^2)\,(q\gamma)_\perp - q_\perp^2\,[(q\gamma)_\| + m_\ell]}{(q^2 - m_\ell^2)^4}\\ &\quad \times \frac{1}{(q-p)^2 - m_W^2}\,.\end{aligned} \tag{4.168}$$

As it was already noted, this part contains the increased contribution of the region of the relatively small virtual momenta, $q^2 \sim m_\ell^2 \ll m_W^2$. Making the Wick rotation in the complex plane q_0, $q_0 = \mathrm{i}q_4$, $q_\parallel^2 = q_0^2 - q_3^2 = -(q_3^2 + q_4^2)$, and after the standard transformations we rewrite the integral (4.168) as

$$J_{\alpha\beta}^{(\ell_2 W_0)}(p) = \frac{\mathrm{i}}{8\pi^2}\left(\frac{\beta}{m_W^2}\right)^2 g_{\alpha\beta}\,(p\gamma)_\parallel\,(2\,I_1 + I_2)\,. \tag{4.169}$$

Here, the following integrals are introduced, using the notations $x = -q_\parallel^2/m_W^2$, $y = q_\perp^2/m_W^2$, $\lambda = m_\ell^2/m_W^2 \ll 1$:

$$I_1 = \int\limits_0^\infty \frac{x\mathrm{d}x\, y\mathrm{d}y}{(x+y+\lambda)^4(x+y+1)^2}\,,$$
$$I_2 = \lambda\int\limits_0^\infty \frac{\mathrm{d}x\, y\mathrm{d}y}{(x+y+\lambda)^4(x+y+1)^2}\,, \tag{4.170}$$

which, due to the smallness of the parameter λ, can be easily calculated:

$$I_1 \simeq -\frac{1}{6}\ln\lambda - \frac{17}{36}\,, \qquad I_2 \simeq \frac{1}{6}\,. \tag{4.171}$$

Finally, for the contribution (4.168), we obtain

$$J_{\alpha\beta}^{(\ell_2 W_0)}(p) \simeq \frac{\mathrm{i}}{24\pi^2}\left(\frac{\beta}{m_W^2}\right)^2 g_{\alpha\beta}\,(p\gamma)_\parallel\left(\ln\frac{m_W^2}{m_\ell^2} - \frac{7}{3}\right). \tag{4.172}$$

Collecting the calculated contributions to the expression (4.158), we find the integral $J_{\alpha\beta}(p)$ as the following expansion in powers of the field strength:

$$\begin{aligned} J_{\alpha\beta}(p) \simeq \frac{1}{16\pi^2}&\left\{\frac{eB}{m_W^2}\left[-\frac{\mathrm{i}}{2}\,g_{\alpha\beta}\,(p\tilde{\varphi}\gamma)\gamma_5 + \varphi_{\alpha\beta}\,(p\gamma)\right]\right. \\ &\left.+\mathrm{i}\left(\frac{eB}{m_W^2}\right)^2\left[g_{\alpha\beta}(p\gamma)_\parallel\left(\frac{2}{3}\ln\frac{m_W^2}{m_\ell^2} - \frac{3}{2}\right) + \mathrm{i}\varphi_{\alpha\beta}(p\tilde{\varphi}\gamma)\gamma_5 - (\varphi\varphi)_{\alpha\beta}(p\gamma)\right]\right\}. \end{aligned} \tag{4.173}$$

Here, the terms are omitted which have the structure of $g_{\alpha\beta}\,(p\gamma)$ and are totally absorbed by the renormalization of the neutrino wave function, as well as the terms of the even- number of γ matrices, which are removed due to the chiral structure of the operator (4.137).

Substituting the result (4.173) to Eq. (4.137), we finally obtain the neutrino self-energy operator in a relatively weak field in the form:

$$\Sigma(p) = \frac{G_{\rm F}\, eB}{\sqrt{2}\, 4\pi^2} \left[3(p\tilde{\varphi}\gamma) - \frac{eB}{m_W^2} \left(\frac{4}{3} \ln \frac{m_W^2}{m_\ell^2} + 1 \right) (p\gamma)_\parallel \right] \gamma_L \,. \tag{4.174}$$

Comparing (4.174) with Eqs. (4.132) and (4.133), one can conclude that the coefficient $\bar{\mathcal{C}}_L$ is in agreement with the well-known results for the neutrino anomalous magnetic moment [69, 70], and with the results of Refs. [72, 73], where the non-diagonal transitions $\nu_i \leftrightarrow \nu_j$ $(i \neq j)$ in an external electromagnetic field were investigated.

In turn, the coefficient $\bar{\mathcal{B}}_L$ coincides with the result of Ref. [34], but not in the case of a moderately strong field $eB \ll m_W^2$, as stated in that paper, but only in the considered weak field limit.

There is another criterion for the correctness of the presented calculation of the coefficient $\bar{\mathcal{B}}_L$. Really, an effective Lagrangian of the $\nu\nu\gamma\gamma$ interaction constructed on the basis of the corresponding term of the amplitude (4.63) with (4.132) and (4.174), and with the replacement of the external field by the photon field operators, is in agreement, to the definitions, with the result of Ref. [74].

4.6.4 The Case of a Moderately Strong Field

In the case of a moderately strong field, $m_\ell^2 \ll eB \ll m_W^2$, as was noted above, the expansion of the lepton propagator in powers of the field (3.24) is inapplicable. Using the exact expression (3.10) and the expansion (3.22) for the propagator of the W-boson, we represent the integral $J_{\alpha\beta}(p)$ as

$$\begin{aligned} J_{\alpha\beta}(p) &= \int \frac{d^4 q}{(2\pi)^4}\, S(q) \left[G^{(0)}_{\beta\alpha}(q-p) + G^{(1)}_{\beta\alpha}(q-p) + G^{(2)}_{\beta\alpha}(q-p) + \cdots \right] \\ &= J^{(\ell_E W_0)}_{\alpha\beta}(p) + J^{(\ell_E W_1)}_{\alpha\beta}(p) + J^{(\ell_E W_2)}_{\alpha\beta}(p) + \cdots \end{aligned} \tag{4.175}$$

For the first of the integrals (4.175) we get

$$\begin{aligned} J^{(\ell_E W_0)}_{\alpha\beta}(p) = -{\rm i}\, g_{\alpha\beta} &\int \frac{d^4 q}{(2\pi)^4} \frac{1}{(q-p)^2 - m_W^2} \\ &\times \int\limits_0^\infty \frac{ds}{\cos(\beta s)} \exp\left[-{\rm i}s \left(m_\ell^2 - q_\parallel^2 + \frac{\tan(\beta s)}{\beta s}\, q_\perp^2 \right) \right] \\ &\times \left\{ [(q\gamma)_\parallel + m_\ell] \left[\cos(\beta s) - \frac{1}{2}\,(\gamma\varphi\gamma)\, \sin(\beta s) \right] - \frac{(q\gamma)_\perp}{\cos(\beta s)} \right\} . \end{aligned} \tag{4.176}$$

Given the decomposition of the W-propagator (4.161), making a rotation in the complex plane s, $s = -\mathrm{i}\tau$, and omitting the terms with an even number of the γ matrices, we obtain

$$J_{\alpha\beta}^{(\ell_E W_0)}(p) = -2 g_{\alpha\beta} \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \int\limits_0^\infty \mathrm{d}\tau \, \exp\left[-\tau(m_\ell^2 - q_\parallel^2) - \frac{q_\perp^2}{\beta} \tanh(\beta\tau)\right]$$
$$\times \left\{(q\gamma)_\parallel \left[1 + \frac{\mathrm{i}}{2}(\gamma\varphi\gamma)\tanh(\beta\tau)\right] - \frac{(q\gamma)_\perp}{\cosh^2(\beta\tau)}\right\} \frac{(qp)_\parallel - (qp)_\perp}{(q^2 - m_W^2)^2} \,. \quad (4.177)$$

Making the Wick rotation in the complex plane q_0, $q_0 = \mathrm{i}q_4$, integrating over the angles in the Euclidean planes $\{q_1, q_2\}$ and $\{q_3, q_4\}$, passing to the dimensionless variables $u = -q_\parallel^2/m_W^2$, $v = q_\perp^2/\beta$, $x = m_W^2\tau$ and introducing the dimensionless small parameter $\eta = \beta/m_W^2 \ll 1$, we can rewrite the integral (4.177) in the form

$$J_{\alpha\beta}^{(\ell_E W_0)}(p) = \frac{\mathrm{i}\, g_{\alpha\beta}}{16\pi^2}\, \eta \int\limits_0^\infty \mathrm{d}x \int\limits_0^\infty \mathrm{d}u \int\limits_0^\infty \mathrm{d}v\, \frac{\mathrm{e}^{-ux - v\tanh(\eta x)}}{(1 + u + \eta v)^2} \quad (4.178)$$
$$\times \left\{(p\gamma)_\parallel u\left[1 + \frac{\mathrm{i}}{2}(\gamma\varphi\gamma)\tanh(\eta x)\right] - (p\gamma)_\perp \frac{\eta v}{\cosh^2(\eta x)}\right\} .$$

The integral over x requires a careful handling both at the lower and upper limits. Using the smallness of the parameter η, it is advisable to choose the intermediate scale A for the x variable, such that $A \gg 1$, but $\eta A \ll 1$. The region of integration over x is then divided into two parts, $0 < x < A$ and $A < x < \infty$:

$$J_{\alpha\beta}^{(\ell_E W_0)}(p) = J_{\alpha\beta}^{(0A)}(p) + J_{\alpha\beta}^{(A\infty)}(p) \,. \quad (4.179)$$

In the region $0 < x < A$, the argument of the hyperbolic functions is small, $\eta x \ll 1$, and the first of the integrals (4.179) is essentially simplified with the change of the variable $\eta v = w$:

$$J_{\alpha\beta}^{(0A)}(p) = \frac{\mathrm{i}\, g_{\alpha\beta}}{16\pi^2} \int\limits_0^A \mathrm{d}x \int\limits_0^\infty \frac{\mathrm{d}u\,\mathrm{d}w}{(1 + u + w)^2}\left(1 + \frac{1}{3}\eta^2 x^3 w\right)\mathrm{e}^{-x(u+w)}$$
$$\times \left\{(p\gamma)_\parallel\, u\left[1 + \frac{\mathrm{i}}{2}(\gamma\varphi\gamma)\,\eta x\right] - (p\gamma)_\perp\, w\left(1 - \eta^2 x^2\right)\right\} . \quad (4.180)$$

Passing from the variables $\{u, w\}$ to the new variables $\{z, \xi\}$:

$$u = z\,\frac{1+\xi}{2}\,, \quad w = z\,\frac{1-\xi}{2}\,, \quad \int\limits_0^\infty \mathrm{d}u\,\mathrm{d}w = \frac{1}{2}\int\limits_0^\infty z\,\mathrm{d}z \int\limits_{-1}^1 \mathrm{d}\xi\,, \quad (4.181)$$

we can integrate over ξ. Omitting as before the terms of the form $g_{\alpha\beta}\,(p\gamma)$, let us rewrite the integral (4.180) with the identity (4.164) in account, as

$$J^{(0A)}_{\alpha\beta}(p) = \frac{\mathrm{i}\, g_{\alpha\beta}}{32\pi^2}\left[-\,(p\tilde{\varphi}\gamma)\,\gamma_5\,\eta\, I_3 + (p\gamma)_{\parallel}\,\eta^2\left(I_4 - \frac{1}{9}\, I_5\right)\right]. \tag{4.182}$$

Here, the following integrals are introduced:

$$I_3 = \int\limits_0^A x\,\mathrm{d}x \int\limits_0^\infty \frac{z^2\,\mathrm{d}z}{(1+z)^2}\,\mathrm{e}^{-xz}\,, \quad I_4 = \int\limits_0^A x^2\,\mathrm{d}x \int\limits_0^\infty \frac{z^2\,\mathrm{d}z}{(1+z)^2}\,\mathrm{e}^{-xz}\,,$$
$$I_5 = \int\limits_0^A x^3\,\mathrm{d}x \int\limits_0^\infty \frac{z^3\,\mathrm{d}z}{(1+z)^2}\,\mathrm{e}^{-xz}\,. \tag{4.183}$$

Given that A is the large parameter, we obtain up to the terms of $O(1/A)$:

$$I_3 = 1\,, \quad I_4 = 2\,\ln A - 5 + 2\,\gamma_E\,, \quad I_5 = 6\,\ln A - 17 + 6\,\gamma_E\,, \tag{4.184}$$

where $\gamma_E = 0.577\ldots$ is the Euler constant. As a result, we have for the integral $J^{(0A)}_{\alpha\beta}(p)$:

$$J^{(0A)}_{\alpha\beta}(p) = \frac{\mathrm{i}\, g_{\alpha\beta}}{32\pi^2}\left[-\,(p\tilde{\varphi}\gamma)\,\gamma_5\,\eta + \frac{4}{3}\,(p\gamma)_{\parallel}\,\eta^2\left(\ln A - \frac{7}{3} + \gamma_E\right)\right]. \tag{4.185}$$

The second of the integrals (4.179) can also be simplified. As one can see from Eq. (4.178), the exponential in the numerator provides for $A < x < \infty$ that the region of integration is only significant where the terms u and ηv in the denominator are small if compared with unity. It is worthwhile to move to the new variables $z = \eta x$, $y = u/\eta$ to obtain

$$\begin{aligned} J^{(A\infty)}_{\alpha\beta}(p) = {} & \frac{\mathrm{i}\, g_{\alpha\beta}}{16\pi^2}\,\eta^2 \int\limits_{\eta A}^\infty \mathrm{d}z \int\limits_0^\infty \mathrm{d}y \int\limits_0^\infty \mathrm{d}v\, \mathrm{e}^{-yz}\,\mathrm{e}^{-v\tanh z} \\ & \times \left\{(p\gamma)_{\parallel} y\left[1 + \frac{\mathrm{i}}{2}\,(\gamma\varphi\gamma)\,\tanh z\right] - (p\gamma)_{\perp}\,\frac{v}{\cosh^2 z}\right\} \\ & \times \frac{1}{(1+\eta y+\eta v)^2}\,. \end{aligned} \tag{4.186}$$

Replacing the last fraction by 1, we see that the integrals over y and v are easily calculated. Neglecting the term $O(1/A)$, we get

$$J_{\alpha\beta}^{(A\infty)}(p) = \frac{\mathrm{i}\, g_{\alpha\beta}}{16\pi^2}\,\eta^2 \left\{ (p\gamma)_{\parallel} \int\limits_{\eta A}^{\infty} \frac{\mathrm{d}z}{z^2\,\tanh z} - (p\gamma)_{\perp} \int\limits_{\eta A}^{\infty} \frac{\mathrm{d}z}{z\,\sinh^2 z} \right\}. \tag{4.187}$$

Here, the first integral can be converted to the second one using the integration by parts:

$$\int\limits_{\eta A}^{\infty} \frac{\mathrm{d}z}{z^2\,\tanh z} = -\int\limits_{\eta A}^{\infty} \frac{\mathrm{d}z}{z\,\sinh^2 z} + \frac{1}{(\eta A)^2} + \frac{1}{3} + O\left((\eta A)^2\right). \tag{4.188}$$

Given that $\eta A \ll 1$, the remaining integral can be rewritten as

$$\int\limits_{\eta A}^{\infty} \frac{\mathrm{d}z}{z\,\sinh^2 z} = \int\limits_{\eta A}^{\infty} \frac{\mathrm{d}z}{z^3}\left(\frac{z^2}{\sinh^2 z} - \frac{3}{3+z^2}\right) + 3\int\limits_{\eta A}^{\infty} \frac{\mathrm{d}z}{z^3(3+z^2)}\,. \tag{4.189}$$

Here, the added and subtracted term is chosen in such a way that, on the one hand, it provided a convergence of the first integral at both the lower and upper limits, and on the other hand, it was easily calculable. So, the first of the integrals (4.189) is finite, if we tend the lower limit to zero. Its numerical value is

$$C = \int\limits_{0}^{\infty} \frac{\mathrm{d}z}{z^3}\left(\frac{z^2}{\sinh^2 z} - \frac{3}{3+z^2}\right) \simeq -0.055\,. \tag{4.190}$$

For the integral (4.189) we obtain

$$\int\limits_{\eta A}^{\infty} \frac{\mathrm{d}z}{z\,\sinh^2 z} = \frac{1}{2(\eta A)^2} + \frac{1}{3}\,\ln A + \frac{1}{3}\,\ln\eta - \frac{1}{6}\,\ln 3 + C \tag{4.191}$$

up to terms of higher order. Collecting the calculated contributions and omitting, as before, the terms of the form $g_{\alpha\beta}\,(p\gamma)$, for the integral in the region $A < x < \infty$ we have

$$J_{\alpha\beta}^{(A\infty)}(p) = \frac{\mathrm{i}\, g_{\alpha\beta}}{24\pi^2}\,(p\gamma)_{\parallel}\,\eta^2 \left(-\ln A + \ln\frac{1}{\eta} + \frac{1}{2} + \frac{1}{2}\,\ln 3 - 3C\right). \tag{4.192}$$

The final expression for the integral (4.179), as expected, does not contain the intermediate scale A:

$$
\begin{aligned}
J^{(\ell_E W_0)}_{\alpha\beta}(p) = \frac{\mathrm{i}\, g_{\alpha\beta}}{32\pi^2} \Bigg[&- (p\tilde{\varphi}\gamma)\,\gamma_5\,\eta \\
&+ \frac{4}{3}\,(p\gamma)_{\parallel}\,\eta^2 \left(\ln\frac{1}{\eta} - \frac{11}{6} + \frac{1}{2}\,\ln 3 + \gamma_E - 3C \right) \Bigg] . \qquad (4.193)
\end{aligned}
$$

The presence of the term $\eta^2 \ln\eta$ ($\eta = \beta/m_W^2$) shows again that the expansion of the lepton propagator over β as the small parameter is impossible in this case.

The similar analysis of the second and the third terms of the expansion (4.175) shows that the "dangerous" contribution with the logarithm of β appears here in the next orders of the η parameter, so it is possible to use the field expansion of the lepton propagator, writing the integrals as

$$
\begin{aligned}
J^{(\ell_E W_1)}_{\alpha\beta}(p) &= J^{(\ell_0 W_1)}_{\alpha\beta}(p) + J^{(\ell_1 W_1)}_{\alpha\beta}(p)\,, \\
J^{(\ell_E W_2)}_{\alpha\beta}(p) &= J^{(\ell_0 W_2)}_{\alpha\beta}(p)\,. \qquad (4.194)
\end{aligned}
$$

Summing (4.193), (4.162), (4.166), and (4.167) we find

$$
\begin{aligned}
J_{\alpha\beta}(p) = \frac{1}{16\pi^2} \Bigg\{ &\frac{eB}{m_W^2} \left[-\frac{\mathrm{i}}{2}\, g_{\alpha\beta}\,(p\tilde{\varphi}\gamma)\gamma_5 + \varphi_{\alpha\beta}\,(p\gamma)_{\parallel} \right] \\
&+ \mathrm{i} \left(\frac{eB}{m_W^2} \right)^2 \Bigg[g_{\alpha\beta}\,(p\gamma)_{\parallel} \left(\frac{2}{3}\,\ln\frac{m_W^2}{eB} - \frac{7}{6} + \frac{1}{3}\,\ln 3 + \frac{2}{3}\,\gamma_E - 2C \right) \\
&+ \mathrm{i}\,\varphi_{\alpha\beta}\,(p\tilde{\varphi}\gamma)\gamma_5 - (\varphi\varphi)_{\alpha\beta}\,(p\gamma) \Bigg] \Bigg\} . \qquad (4.195)
\end{aligned}
$$

For the neutrino self-energy operator in the case of a moderately strong field, $m_\ell^2 \ll eB \ll m_W^2$, we finally obtain

$$
\Sigma(p) = \frac{G_{\mathrm{F}}\, eB}{\sqrt{2}\, 4\pi^2} \left[3(p\tilde{\varphi}\gamma) - \frac{eB}{m_W^2} \left(\frac{4}{3}\,\ln\frac{m_W^2}{eB} + 3.389 \right) (p\gamma)_{\parallel} \right] \gamma_L\,. \qquad (4.196)
$$

At first glance, the second terms in Eqs. (4.174) and (4.196), which contain the small extra factor eB/m_W^2, can be neglected. However, as we will show below, just these terms give the dominant contribution to the neutrino additional energy in an external magnetic field.

In Ref. [35], the authors made an attempt to test the correctness of their analytical calculations by producing a numerical evaluation of the coefficients $\bar{\mathcal{B}}_L$ and $\bar{\mathcal{C}}_L$ of the operator $\Sigma(p)$, being written in the form of a double integrals (see Eqs. (89) and (90) of Ref. [35]). As we show below, this numerical calculation is also incorrect. The main reason for the error is, probably, in the attempt of the authors to calculate numerically the integral of the difference between the two, in fact, infinite quantities. The analysis shows that the integral is finite and has an order of the value $eB/m_W^2 \lesssim 10^{-6}$ for the corresponding field strength, but not of the order of unity, as the authors [35] claim.

Similarly to Ref. [35], let us represent the expressions for the coefficients of the operator $\Sigma(p)$ in the form of double integrals. We substitute the exact expressions for the propagators (3.10) and (3.13), where it is convenient to make a turn in the complex plane s, $s = -\mathrm{i}\tau$, in the integral (4.138). In this case, the integrals over the 4-momentum $\mathrm{d}^4 q = \mathrm{d}^2 q_\parallel \, \mathrm{d}^2 q_\perp$ can be rather easily calculated. Substituting the result into Eq. (4.137) and comparing it with the definition of the self-energy operator, Eq. (4.132), one can present the coefficients $\mathcal{A}_L$, $\bar{\mathcal{B}}_L$, and $\bar{\mathcal{C}}_L$ as follows:

$$\mathcal{A}_L = -\frac{g^2 \eta}{16\pi^2} \int\limits_0^\infty \frac{\mathrm{d}x \, \mathrm{d}y \, \sinh(\eta x)}{(x+y) \, \sinh^2[\eta(x+y)]} \exp[-\Phi(x, y, \lambda, p, m_W)] \,, \tag{4.197}$$

$$\bar{\mathcal{B}}_L = \frac{g^2 \eta}{16\pi^2} \int\limits_0^\infty \frac{\mathrm{d}x \, \mathrm{d}y}{(x+y) \, \sinh[\eta(x+y)]} \left[\frac{\sinh(\eta x)}{\sinh[\eta(x+y)]} - \frac{x \cosh[\eta(2x+y)]}{x+y} \right]$$
$$\times \exp[-\Phi(x, y, \lambda, p, m_W)] \,, \tag{4.198}$$

$$\bar{\mathcal{C}}_L = \frac{g^2 \eta}{16\pi^2} \int\limits_0^\infty \frac{x \, \mathrm{d}x \, \mathrm{d}y}{(x+y)^2} \frac{\sinh[\eta(2x+y)]}{\sinh[\eta(x+y)]} \exp[-\Phi(x, y, \lambda, p, m_W)] \,, \tag{4.199}$$

where

$$\Phi(x, y, \lambda, p, m_W) = x + \lambda y$$
$$- \frac{x y}{x+y} \frac{p^2}{m_W^2} - \left(\frac{x y}{x+y} - \frac{\sinh(\eta x) \, \sinh(\eta y)}{\eta \, \sinh[\eta(x+y)]} \right) \frac{p_\perp^2}{m_W^2} \,, \tag{4.200}$$

and the notation are also introduced: $\eta = \beta/m_W^2$, $\lambda = m_\ell^2/m_W^2$.

It is easy to see that the integral for the coefficient $\mathcal{A}_L$ is divergent. As has been noted, this coefficient is absorbed by the renormalization of the neutrino wave function.

We note that the expressions for the coefficients of $\bar{\mathcal{B}}_L$ and $\bar{\mathcal{C}}_L$ are in agreement with Eqs. (89) and (90) of Ref. [35] up to an obvious error in the sign of Eq. (90). However, as one can see from Eq. (4.198), the coefficient $\bar{\mathcal{B}}_L$ is an even function of the η parameter, therefore, the linear dependence of $\bar{\mathcal{B}}_L$ on η declared in Ref. [35] is an obvious error.

To verify the correctness of our analytical calculations let us consider the limiting case: $m_\ell^2 \ll m_W^2$ and $p_\parallel^2 \simeq p_\perp^2 \ll m_W^2$. Moving to a new variable $z = x + y$, we can simplify the integrals (4.198) and (4.199) as:

$$\bar{\mathcal{B}}_L = \frac{g^2\,\eta}{16\pi^2} \int\limits_0^\infty \frac{\mathrm{d}z}{z\,\sinh(\eta z)} \int\limits_0^z \mathrm{d}x\,\mathrm{e}^{-x} \left[\frac{\sinh(\eta x)}{\sinh(\eta z)} - \frac{x\,\cosh[\eta(z+x)]}{z} \right], \tag{4.201}$$

$$\bar{\mathcal{C}}_L = \frac{g^2\,\eta}{16\pi^2} \int\limits_0^\infty \frac{\mathrm{d}z}{z^2\,\sinh(\eta z)} \int\limits_0^z x\mathrm{d}x\,\mathrm{e}^{-x}\,\sinh[\eta(z+x)]\,. \tag{4.202}$$

The results of numerical calculation of $\bar{\mathcal{B}}_L$ and $\bar{\mathcal{C}}_L$ as the functions of the η parameter demonstrate a good agreement with the previous approximate formulas, especially for small values of η.

4.6.5 The Neutrino Operator $\Sigma(p)$ in a Crossed Field

In addition to the limiting cases of a weak ($eB \ll m_\ell^2$) and a moderately strong ($m_\ell^2 \ll eB \ll m_W^2$) field, which were considered in Ref. [36], there is yet another region of values of the physical parameters that requires a dedicated analysis. We mean here the situation where the neutrino transverse momentum $p_\perp$ with respect to the magnetic field is rather high — for example, $p_\perp \gtrsim m_W$ or $p_\perp \gg m_W$. This region of parameter values is of importance in connection with problems of the physics of ultrahigh-energy cosmic rays. In particular, the possibility of detecting cosmic neutrinos of ultrahigh energy ($E_\nu \sim 10^{7\div 17}$GeV) is widely discussed (see, for example, Ref. [75] and references therein). Apparently, the propagation of neutrinos having such energies cannot be described adequately without taking into account their interaction with magnetic fields of astrophysical nature.

The above region of parameter values corresponds to the crossed-field approximation, where the Fourier transforms of the translation-invariant parts of the propagators from expressions (4.138) and (4.139), are presented in Eqs. (3.67), (3.68), and (3.69).

The general Lorentz structure of the operator $\Sigma(p)$ in the presence of a magnetic field is represented in Eq. (4.132).

In the crossed field approximation, the coefficients in Eq. (4.132) we are interested in, are expressed in terms of the integrals containing the Hardy—Stokes function $f(u)$, see Eq. (4.36), and its derivative:

$$\bar{\mathcal{B}}_L = \frac{G_\mathrm{F}}{12\sqrt{2}\,\pi^2\,m_W^2} \int\limits_0^1 \frac{\mathrm{d}v\,v\,[2(1+v)(2+v) + \lambda\,(1-v)(2-v)]}{[v + \lambda\,(1-v)]^2} \times u^2\,\frac{\mathrm{d}f(u)}{\mathrm{d}u}\,, \tag{4.203}$$

$$\bar{\mathcal{C}}_L = \frac{G_F}{4\sqrt{2}\,\pi^2} \int_0^1 \frac{dv\, v\,[2(1+v) - \lambda\,(1-v)]}{v + \lambda\,(1-v)}\, u f(u)\,, \tag{4.204}$$

$$\bar{\mathcal{B}}_R = \frac{G_F}{12\sqrt{2}\,\pi^2\, m_W^2} \frac{m_\nu^2}{m_W^2} \int_0^1 \frac{dv\, v\,(1-v)(2-v)}{[v + \lambda\,(1-v)]^2}\, u^2\, \frac{df(u)}{du}\,, \tag{4.205}$$

$$\bar{\mathcal{C}}_R = \frac{G_F}{4\sqrt{2}\,\pi^2} \frac{m_\nu^2}{m_W^2} \int_0^1 \frac{dv\, v\,(1-v)}{v + \lambda\,(1-v)}\, u f(u)\,, \tag{4.206}$$

$$\mathcal{K}_2 = \frac{G_F\,\lambda}{8\sqrt{2}\,\pi^2} \int_0^1 \frac{dv\,(1-v)}{v + \lambda\,(1-v)}\, u f(u)\,, \tag{4.207}$$

where $\lambda = m_\ell^2/m_W^2$. The argument of the function $f(u)$ in Eqs. (4.203)–(4.207) has the form

$$u = \frac{v + \lambda\,(1-v)}{[\chi\, v(1-v)]^{2/3}}\,, \tag{4.208}$$

where χ is the dynamical field parameter, $\chi^2 = e^2\,(pFFp)/m_W^6$.

For the dynamical parameter χ, there are three regions of values where one can obtain simple approximate analytic expressions for the integrals in Eqs. (4.203)–(4.207).

(i) Region where χ is the smallest parameter in the problem, $\chi^2 \ll \lambda$, or $eB\,p_\perp \ll m_\ell\, m_W^2$. In this region, we have

$$\bar{\mathcal{B}}_L \simeq -\frac{G_F}{3\sqrt{2}\,\pi^2\, m_W^2} \left[\ln\frac{1}{\lambda} + \frac{3}{4} + i\pi\, \frac{\sqrt{3\lambda}}{\chi}\, \exp\left(-\frac{\sqrt{3\lambda}}{\chi}\right)\right], \tag{4.209}$$

$$\begin{aligned}\bar{\mathcal{C}}_L \simeq \frac{3G_F}{4\sqrt{2}\,\pi^2} \Bigg[1 - \frac{2}{3}\lambda\left(\ln\frac{1}{\lambda} - \frac{1}{4}\right) + \frac{4}{3}\chi^2\left(\ln\frac{1}{\lambda} - 3\right) \\ + i\,\frac{4\pi\lambda}{3}\, \exp\left(-\frac{\sqrt{3\lambda}}{\chi}\right)\Bigg],\end{aligned} \tag{4.210}$$

$$\bar{\mathcal{B}}_R \simeq -\frac{G_F}{6\sqrt{2}\,\pi^2\, m_W^2} \frac{m_\nu^2}{m_W^2} \left(\ln\frac{1}{\lambda} - \frac{9}{4}\right), \tag{4.211}$$

$$\bar{\mathcal{C}}_R \simeq \frac{G_F}{8\sqrt{2}\,\pi^2} \frac{m_\nu^2}{m_W^2}\,, \tag{4.212}$$

$$\mathcal{K}_2 \simeq \frac{G_F}{8\sqrt{2}\,\pi^2} \left[\lambda\left(\ln\frac{1}{\lambda} - 1\right) + \frac{2}{3}\chi^2\right]. \tag{4.213}$$

(ii) Region of intermediate values of the dynamical parameter, $\lambda \ll \chi^2 \ll 1$. This region is likely to be of greatest interest. We recall that $\lambda = m_e^2/m_W^2 \simeq 4 \times 10^{-11}$. Representing the parameter χ in the form

$$\chi^2 \simeq 3 \times 10^{-3} \left(\frac{B}{B_e}\right)^2 \left(\frac{E}{10^{20}\,\mathrm{e}V}\right)^2 , \tag{4.214}$$

we can see that, over very broad ranges of magnetic-field strengths and neutrino energies, the parameter χ falls within this intermediate region. Here, we have

$$\bar{\mathcal{B}}_L \simeq -\frac{G_\mathrm{F}}{3\sqrt{2}\,\pi^2\, m_W^2} \left(2\ln\frac{1}{\chi} - \frac{5}{4} + \ln 3 + 2\,\gamma_\mathrm{E} + \mathrm{i}\pi\right), \tag{4.215}$$

$$\bar{\mathcal{C}}_L \simeq \frac{3G_\mathrm{F}}{4\sqrt{2}\,\pi^2} \left[1 + \frac{4}{3}\chi^2 \left(2\ln\frac{1}{\chi} - \frac{17}{3} + \ln 3 + 2\,\gamma_\mathrm{E} + \mathrm{i}\pi\right)\right], \tag{4.216}$$

$$\bar{\mathcal{B}}_R \simeq -\frac{G_\mathrm{F}}{6\sqrt{2}\,\pi^2\, m_W^2} \frac{m_\nu^2}{m_W^2} \left(2\ln\frac{1}{\chi} - \frac{17}{4} + \ln 3 + 2\,\gamma_\mathrm{E} + \mathrm{i}\pi\right), \tag{4.217}$$

$$\bar{\mathcal{C}}_R \simeq \frac{G_\mathrm{F}}{12\sqrt{2}\,\pi^2} \frac{m_\nu^2}{m_W^2} , \tag{4.218}$$

$$\mathcal{K}_2 \simeq \frac{G_\mathrm{F}}{8\sqrt{2}\,\pi^2} \frac{m_\ell^2}{m_W^2} \left(2\ln\frac{1}{\chi} - 1 + \ln 3 + 2\,\gamma_\mathrm{E} + \mathrm{i}\pi\right). \tag{4.219}$$

(iii) Region of large values of the dynamical parameter $\chi \gg 1$. Our results in this region are:

$$\bar{\mathcal{B}}_L \simeq -\mathrm{i}\,\frac{\sqrt{3}}{2\sqrt{2}\,\pi} \frac{G_\mathrm{F}}{m_W^2\,\chi} , \tag{4.220}$$

$$\bar{\mathcal{C}}_L \simeq \frac{\pi\, G_\mathrm{F} \left(1 + \mathrm{i}\sqrt{3}\right)}{5\sqrt{2}\, 3^{1/6}\, \Gamma^4(2/3)\, \chi^{2/3}} , \tag{4.221}$$

$$\bar{\mathcal{B}}_R \simeq \frac{3^{7/6}\, \Gamma^4(2/3)\, G_\mathrm{F} \left(1 - \mathrm{i}\sqrt{3}\right)}{32\sqrt{2}\,\pi^3\, \chi^{4/3}} \frac{m_\nu^2}{m_W^4} , \tag{4.222}$$

$$\bar{\mathcal{C}}_R \simeq \frac{\pi\, G_\mathrm{F} \left(1 + \mathrm{i}\sqrt{3}\right)}{90\sqrt{2}\, 3^{1/6}\, \Gamma^4(2/3)\, \chi^{2/3}} \frac{m_\nu^2}{m_W^2} , \tag{4.223}$$

$$\mathcal{K}_2 \simeq \frac{\pi\, G_\mathrm{F} \left(1 + \mathrm{i}\sqrt{3}\right)}{36\sqrt{2}\, 3^{1/6}\, \Gamma^4(2/3)\, \chi^{2/3}} \frac{m_\ell^2}{m_W^2} , \tag{4.224}$$

where $\Gamma(x)$ is a gamma function, $\Gamma(2/3) = 1.354...$

4.6.6 Field-Induced Neutrino Magnetic Moment

The field-induced correction to the magnetic moment μ_{ν_ℓ} of the neutrino ν_ℓ is yet another quantity that can be extracted from the neutrino self-energy operator. The anomalous neutrino magnetic moment is expressed in terms of the coefficients in the self-energy operator (4.132), for the details see Sect. 4.7.4 below, as

$$\mu_{\nu_\ell} = \frac{e\, m_\nu}{2} \left[\bar{\mathcal{C}}_L - \bar{\mathcal{C}}_R - \left(\bar{\mathcal{B}}_L - \bar{\mathcal{B}}_R \right) e(\mathbf{B}\mathbf{v}) + 4\, \mathcal{K}_2 \right] , \tag{4.225}$$

where $\mathbf{v} = \mathbf{p}/E$ is the neutrino velocity.

In the limiting case of $\chi^2 \ll \lambda = m_\ell^2/m_W^2$, the neutrino magnetic moment becomes

$$\mu_{\nu_\ell} \simeq \mu_{\nu_\ell}^{(0)} \left[1 - \frac{1}{2}\lambda + \frac{4}{3}\chi^2 \left(\ln\frac{1}{\lambda} - 3 + \frac{1}{3} \right) \right] , \tag{4.226}$$

$$\mu_{\nu_\ell}^{(0)} = \frac{3e\, G_{\mathrm{F}} m_\nu}{8\pi^2 \sqrt{2}} ,$$

where $\mu_{\nu_\ell}^{(0)}$ is the neutrino magnetic moment in a vacuum [69, 70]. In the field-induced corrections in Eq. (4.226), the leading term of order $\sim\chi^2$, which involves a large logarithm, coincides with the result presented in Ref. [33], where the postlogarithmic terms were disregarded. The last term in the field-induced correction in Eq. (4.226) originates from the Φ–boson contribution. One can see that it is relatively small but does not involve a parametric suppression.

4.7 Neutrino Self-energy Operator in Magnetized Plasma

One of the topical questions of the physics of elementary particles in an external medium is the question of how the external active medium influences the neutrino dispersion properties.

An analysis of the neutrino self-energy operator $\Sigma(p)$ in a magnetized plasma, defining, in particular, the neutrino dispersion law, was performed in a number of papers (see, e.g., [29, 39–41]).

In Refs. [29, 39, 40], the neutrino dispersion was investigated in a charge-symmetric, weakly magnetized plasma under physical conditions

$$m_W^2 \gg T^2 \gg eB \gg m_e^2 . \tag{4.227}$$

The additional energy of electron neutrinos was written in the form

$$\frac{\Delta E}{|\mathbf{p}|} = -\frac{7\sqrt{2}\, G_F\, \pi^2\, T^4}{45}\left(\frac{1}{m_Z^2} + \frac{2}{m_W^2}\right)$$
$$+ \frac{\sqrt{2}\, G_F\, T^2\, eB}{3m_W^2}\cos\phi + \frac{\sqrt{2}\, G_F\, (eB)^2}{3\,\pi^2\, m_W^2}\ln\left(\frac{T}{m_e}\right)\sin^2\phi. \qquad (4.228)$$

Here, E and $\mathbf{p}$ are the neutrino energy and momentum, respectively; T is the plasma temperature; and ϕ is the angle between the direction of the magnetic field $\mathbf{B}$ and the momentum vector $\mathbf{p}$. The first term in Eq. (4.228) describes the additional neutrino energy in a plasma without a magnetic field [29], while the second [39] and the third [40] terms are attributable to the simultaneous presence of a plasma and a magnetic field. As we see from Eq. (4.228), the term proportional to the square of the magnetic field strength contains amplification by the logarithmic factor $\ln(T/m_e)$, which, in general, raises doubts under the indicated physical conditions (4.227). Indeed, under such conditions the contribution to the neutrino energy is determined by the plasma electrons and positrons that populate the highest Landau levels. The energy of these electrons and positrons at the nth Landau level is given by the formula

$$\omega_n = \sqrt{m_e^2 + k_3^2 + 2eBn}, \qquad n \gg 1. \qquad (4.229)$$

Since the electron mass under the presumed conditions is the smallest parameter of the problem, it can be neglected in Eq. (4.229) for the energy. Therefore, it is unlikely that the electron mass could present in the final result in the principal approximation. Thus, an independent calculation of the neutrino dispersion in a magnetized plasma was of considerable interest.

In this section, we present, following the papers [76, 77], the results of our analysis of the charge-symmetric magnetized plasma influence on the neutrino dispersion in the presence of an external magnetic field. A general expression for the neutrino self-energy operator $\Sigma(p)$ is derived. The neutrino dispersion under the physical conditions of weakly, moderately, and strongly magnetized plasmas is analyzed in detail.

4.7.1 Neutrino Scattering on Magnetized Plasma

Similarly to consideration performed in Sect. 4.5.1, an expression for the neutrino self-energy operator $\Sigma(p)$ is defined via the amplitude of the neutrino forward scattering (4.63). The additional neutrino energy due to the neutrino forward scattering in the medium is expressed in terms of the amplitude of this process,

$$\Delta E = -\frac{1}{2E}\, M(\nu \to \nu) = \frac{1}{4E}\,\mathrm{Tr}\left\{\left((p\gamma) + m_\nu\right)\left(1 - (s\gamma)\,\gamma_5\right)\Sigma(p)\right\}, \qquad (4.230)$$

where $E = \sqrt{\mathbf{p}^2 + m_\nu^2}$ is the neutrino energy in a vacuum, m_ν is the neutrino mass, and s^μ is the neutrino doubled spin 4-vector.

A detailed description of calculation of the neutrino self-energy operator $\Sigma(p)$ in a magnetized plasma can be found, for example, in Ref. [77]. The operator $\Sigma(p)$ is represented there in a general form of an expansion over the linearly independent covariant structures:

$$\begin{aligned}\Sigma(p) = &\left[\mathcal{A}_L\,(p\gamma) + \mathcal{B}_L\,(u\gamma) + \mathcal{C}_L\,e\,(p\tilde{F}\gamma)\right]\gamma_L \\ &+ \left[\mathcal{A}_R\,(p\gamma) + \mathcal{B}_R\,(u\gamma) + \mathcal{C}_R\,e\,(p\tilde{F}\gamma)\right]\gamma_R \\ &+ m_\nu\,\left[\mathcal{K}_1 + \mathrm{i}\,\mathcal{K}_2\,e\,(\gamma F\gamma)\right]\,, \end{aligned} \tag{4.231}$$

where u_α is the four-vector of the plasma velocity. Comparing this formula with Eq. (4.132) for $\Sigma(p)$ in a magnetic field, one can see that a replacement is made here of the structure $(p\tilde{F}\tilde{F}\gamma)$ to the structure $(u\gamma)$. Such a replacement is possible, due to the relation

$$(pu)(p\tilde{F}\tilde{F}\gamma) = (p\tilde{F}\tilde{F}p)(u\gamma) + (p\tilde{F}u)(p\tilde{F}\gamma). \tag{4.232}$$

This equality holds if the spatial part of the plasma velocity four-vector u_α is directed along the magnetic field. It should be kept in mind that under the term "magnetized plasma" we mean a situation where in the plasma rest frame, $u^\alpha = (1, \mathbf{0})$, the electromagnetic field is reduced to a purely magnetic. The covariance of the operator $\Sigma(p)$ means in this case that there are many reference frames moving parallel to the magnetic field in which the operator (4.231) retains its shape.

Given the relation (4.232), one can connect the coefficients of Eqs. (4.132) and (4.231) as follows:

$$\mathcal{B}_L = \bar{\mathcal{B}}_L\,\frac{e^2(p\tilde{F}\tilde{F}p)}{(pu)}\,, \qquad \mathcal{C}_L = \bar{\mathcal{C}}_L + \bar{\mathcal{B}}_L\,\frac{e(p\tilde{F}u)}{(pu)}\,, \tag{4.233}$$

and similarly for $\mathcal{B}_R, \mathcal{C}_R$ coefficients.

Using Eqs. (4.231) and (4.230), one can represent the additional neutrino energy ΔE in a magnetized plasma in the form

$$\begin{aligned}\Delta E = &\frac{1}{2}\,\mathcal{B}_L\,[1 - (\mathbf{sv})] + \frac{1}{2}\,\mathcal{B}_R\,[1 + (\mathbf{sv})] \\ &- \frac{e\,m_\nu}{2}\,(\mathcal{C}_L - \mathcal{C}_R + 4\mathcal{K}_2)\left[(\mathbf{sB}_t) + \frac{m_\nu}{E}\,(\mathbf{sB}_\ell)\right] \\ &+ \frac{m_\nu^2}{2E}\,(\mathcal{A}_L + \mathcal{A}_R + 2\mathcal{K}_1)\,, \end{aligned} \tag{4.234}$$

where $\mathbf{v} = \mathbf{p}/E$ is the neutrino velocity vector, $\mathbf{s}$ is the average neutrino doubled spin vector, $\mathbf{B}_{t,\ell}$ are the transversal and longitudinal components of a magnetic field $\mathbf{B}$ with respect to the neutrino momentum, $\mathbf{B} = \mathbf{B}_t + \mathbf{B}_\ell$.

As it was mentioned above, the coefficients $\mathcal{A}_R, \mathcal{B}_R, \mathcal{C}_R, \mathcal{K}_{1,2}$ are supressed by the square of the lepton and W boson mass ratio, and can be neglected. For ultrarelativistic neutrinos, we obtain from (4.234):

$$\Delta E \simeq \mathcal{B}_L \, \frac{(1-(\mathbf{ns}))}{2} - \mathcal{C}_L \, \frac{e m_\nu}{2} \, (\mathbf{B}[\, \mathbf{n} \times [\mathbf{s} \times \mathbf{n}]]), \tag{4.235}$$

where $\mathbf{n}$ is a unit vector in the direction of the neutrino momentum. The terms proportional to the square of the neutrino mass were omitted in Eq. (4.235).

Thus, finding the additional energy acquired by the neutrinos during their forward scattering in a magnetized plasma is reduced to calculating the parameters $\mathcal{B}_L$ and $\mathcal{C}_L$.

The term in Eq. (4.235) proportional to the first power of the neutrino mass corresponds to the additional neutrino energy attributable to the neutrino magnetic moment and will be further analysed in detail. The additional neutrino energy in the medium for left-handed massless neutrinos is defined only by the parameter $\mathcal{B}_L$:

$$\Delta E = \mathcal{B}_L .$$

Since the additional neutrino energy, being a physical quantity, is gauge-invariant, we will perform our calculations in a unitary gauge, which is convenient in that it contains no contribution from scalar bosons. In this gauge, the amplitude of the $\nu \to \nu$ scattering in a magnetized plasma can be represented as the sum of two terms:

$$M_{(\nu\to\nu)} = M^{W}_{(\nu\to\nu)} + M^{Z}_{(\nu\to\nu)}, \tag{4.236}$$

where the first term corresponds to the amplitude of the neutrino forward scattering by plasma electrons and positrons of the medium via a W boson (see Fig. 4.15) and the second term is attributable to the $\nu \to \nu$ transition via a Z boson (see Fig. 4.12 (c,d) where the forward scattering case $p' = p$ and $k' = k$ should be taken). The neutrino scattering by plasma neutrinos shown in Fig. 4.12 is insensitive to the presence of an external magnetic field; its contribution to the additional neutrino energy was investigated previously and was calculated in Ref. [29], see Eq. (4.82):

$$\frac{\Delta E^Z}{|\mathbf{p}|} = -\frac{7\sqrt{2}\, G_{\mathrm{F}}\, \pi^2\, T^4}{45\, m_Z^2}. \tag{4.237}$$

Note that we do not consider the diagrams of Figs. 4.11 and 4.12a,b where the 4-momentum of the intermediate Z boson is zero in the forward scattering. This is because such diagrams give only a local contribution, which is zero in a charge-symmetric plasma. Thus, our problem is reduced to calculating the contribution of a magnetized plasma to the additional neutrino energy from the W boson exchange.

The scattering process that corresponds to the diagrams in Fig. 4.15 is described by the Lagrangian (4.100). The corresponding S-matrix element of the neutrino forward scattering by plasma electrons is:

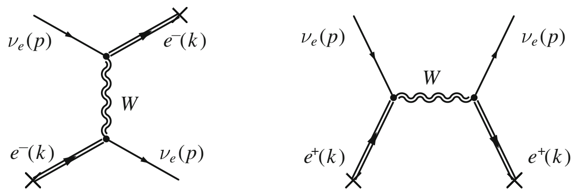

Fig. 4.15 The Feynman diagrams for the neutrino forward scattering on plasma electrons and positrons through W–boson. Double lines correspond to charged particles influenced by an external magnetic field

$$S^W_{\nu e^- \to \nu e^-} = \frac{g^2}{2} \sum_n \int \frac{\mathrm{d}^4 x \, \mathrm{d}^4 x'}{\sqrt{2EV}\sqrt{2E'V}} \, \mathrm{e}^{\mathrm{i}(px' - p'x)} \times \bar{\nu}(p')\gamma_\alpha \gamma_L R_n(x, x')\gamma_\beta \gamma_L \nu(p) G^{(W)}_{\beta\alpha}(x', x), \tag{4.238}$$

where $V = L_1 L_2 L_3$ is the normalization volume, $p^\mu = (E, \mathbf{p})$ and $p'^\mu = (E', \mathbf{p}')$ are the 4-momenta of the initial and final neutrinos, $\nu(p)$ is the bispinor neutrino amplitude, $\gamma_L = (1 - \gamma_5)/2$, $G^{(W)}_{\beta\alpha}(x', x)$ is the W boson propagator in a magnetic field (3.5), $R_n(x, x')$ is the density matrix of the plasma electron with a fixed Landau level number n:

$$R_n(x, x') = \sum_s \int \frac{\mathrm{d}k_2 \mathrm{d}k_3}{(2\pi)^2} L_2 \, L_3 f(\omega_n) \, \psi_e(x) \, \bar{\psi}_e(x'). \tag{4.239}$$

Here, $\psi_e(x)$ are the solutions (2.30) of the Dirac equation in the external magnetic field,[4] $\omega_n = \sqrt{k_3^2 + 2eBn + m_e^2}$ is the energy of the electron at the nth Landau level, k_3 is the kinetic momentum along the third axis, k_2 is a generalized momentum that defines the position $x_0 = -k_2/eB$ of the Gaussian packet center on the first axis, and $f(\omega_n)$ is the electron distribution function, which describes the presence of plasma. In the plasma rest frame, it is

$$f(\omega) = [\mathrm{e}^{(\omega_n - \mu_e)/T} + 1]^{-1},$$

where μ_e is the chemical potential of the plasma and T is its temperature.

A detailed calculation of the function $R_n(x, x')$ is presented in Sect. 2.4. In a constant uniform magnetic field, it can be reduced to the form

$$R_n(x, x') = \mathrm{e}^{-\mathrm{i}\Phi(x', x)} \, (-1)^n \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{f(\omega_n)}{\omega_n} \, \mathrm{e}^{-u} \, \mathrm{e}^{-\mathrm{i}k(x - x')} \times \{[(k\gamma)_\parallel + m_e][L_n(2u)\Pi_- - L_{n-1}(2u)\Pi_+] + 2(k\gamma)_\perp L^1_{n-1}(2u)\}, \tag{4.240}$$

[4] We perform our calculations in the gauge $A^\mu = (0, 0, Bx, 0)$; the magnetic field is directed along the third axis $\mathbf{B} = (0, 0, B)$.

$$\Phi(x', x) = \frac{eB}{2}\,(x_1 + x_1')\,(x_2 - x_2'), \tag{4.241}$$

where $u = k_\perp^2/eB$. The associated Laguerre polynomials are defined as follows:

$$L_n^s(x) = \frac{1}{n!}\,\mathrm{e}^x\, x^{-s}\,\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(x^{n+s}\,\mathrm{e}^{-x}\right). \tag{4.242}$$

Equation (4.240) can be used to investigate quantum processes in a plasma in the presence of a magnetic field with an arbitrary strength.

4.7.2 Neutrino Additional Energy in Magnetized Plasma

After the substitution of the function $R_n(x, x')$ in the form (4.240) and the W boson propagator (3.5) into Eq. (4.238) and the integration over the 4-coordinates, the S-matrix element of the $\nu e^- \to \nu e^-$ process can be reduced to the form

$$\begin{aligned} S^W_{\nu e^- \to \nu e^-} = {} & \frac{g^2 (2\pi)^4 \delta^4(p - p')}{2\sqrt{2EV}\sqrt{2E'V}} \sum_n (-1)^n \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{f(\omega_n)}{\omega_n}\,\mathrm{e}^{-u}\, G^W_{\beta\alpha}(k - p) \\ & \times \bar{\nu}(p)\,\gamma_\alpha \left\{(k\gamma)_\parallel [L_n(2u)\Pi_- - L_{n-1}(2u)\Pi_+] \right. \\ & \left. + \,2\,(k\gamma)_\perp\, L^1_{n-1}(2u)\right\} \gamma_\beta\gamma_L\,\nu(p)\,. \end{aligned} \tag{4.243}$$

The four-dimensional δ function corresponding to the energy and momentum conservation law has been separated out in the S-matrix element. Therefore, we can use the standard relation between the S-matrix element and the invariant transition amplitude,

$$S_{if} = \frac{\mathrm{i}(2\pi)^4 \delta^{(4)}(q - q')}{2\omega V}\, M_{if} \tag{4.244}$$

and separate out the invariant amplitude of the neutrino scattering by plasma electrons in the form

$$\begin{aligned} M^W_{\nu e^- \to \nu e^-} = {} & -\frac{\mathrm{i}\, g^2}{2} \sum_n (-1)^n \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{f(\omega_n)}{\omega_n}\,\mathrm{e}^{-u}\, G^W_{\beta\alpha}(k - p) \\ & \times \bar{\nu}(p)\,\gamma_\alpha \left\{(k\gamma)_\parallel\, [L_n(2u)\Pi_- - L_{n-1}(2u)\Pi_+] \right. \\ & \left. + \,2\,(k\gamma)_\perp\, L^1_{n-1}(2u)\right\} \gamma_\beta\gamma_L\,\nu(p). \end{aligned} \tag{4.245}$$

Calculating the amplitude $M^W_{\nu e^+ \to \nu e^+}$ of the neutrino scattering by plasma positrons is identical to calculating the amplitude $M^W_{\nu e^- \to \nu e^-}$. The result for the transition amplitude $M^W_{\nu e^+ \to \nu e^+}$ in a charge-symmetric plasma turned out to differ from (4.245) by the general sign and by the substitution $k_\mu \to -k_\mu$ in the argument of the W boson propagator. The amplitude of the coherent $\nu e \to \nu e$ scattering by all plasma electrons and positrons is

$$M^W_{\nu e \to \nu e} = M^W_{\nu e^- \to \nu e^-} + M^W_{\nu e^+ \to \nu e^+} = -\frac{\mathrm{i} g^2}{2} \sum_n (-1)^n \tag{4.246}$$
$$\times \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{f(\omega_n)}{\omega_n} \, \mathrm{e}^{-u} \left(G^W_{\beta\alpha}(k-p) - G^W_{\beta\alpha}(-k-p)\right)$$
$$\times \bar{\nu}(p) \, \gamma_\alpha \left\{ (k\gamma)_\parallel \left[L_n(2u) \, \Pi_- - L_{n-1}(2u) \, \Pi_+ \right] \right.$$
$$\left. + \, 2 \, (k\gamma)_\perp \, L^1_{n-1}(2u) \right\} \gamma_\beta \gamma_L \, \nu(p) \, .$$

Using Eq. (4.63) for the amplitude, we find the contribution of plasma electrons and positrons to the neutrino self-energy operator

$$\Sigma^W(p) = \frac{\mathrm{i} \, g^2}{2} \sum_n (-1)^n \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{f(\omega_n)}{\omega_n} \, \mathrm{e}^{-u} \left[G^W_{\beta\alpha}(k-p) - G^W_{\beta\alpha}(-k-p) \right] \tag{4.247}$$
$$\times \, \gamma_\alpha \left\{ (k\gamma)_\parallel \left[L_n(2u) \Pi_- - L_{n-1}(2u) \Pi_+ \right] + 2 (k\gamma)_\perp L^1_{n-1}(2u) \right\} \gamma_\beta \gamma_L \, .$$

The Fourier transform of the W boson propagator in a unitary gauge is fairly cumbersome. For the physical conditions under consideration, where the W boson mass is the largest physical parameter of the problem, the Fourier transform of the propagator can be represented as an expansion in terms of inverse powers of the W boson mass:

$$G^W_{\beta\alpha}(q) \simeq \frac{\mathrm{i} \, g_{\beta\alpha}}{m_W^2} - \frac{3 e \, F_{\beta\alpha}}{2 m_W^4} - \frac{\mathrm{i} \, q_\beta \, q_\alpha}{m_W^4} + \frac{\mathrm{i} \, q^2 \, g_{\beta\alpha}}{m_W^4} + O\left(\frac{1}{m_W^6} \right). \tag{4.248}$$

Here, the first and the second momentum-independent terms give a contribution in the local limit that is zero in a charge-symmetric plasma [29], as is clearly seen from Eq. (4.246). The third and the fourth terms allow for the nonlocality of the interaction. As our analysis shows, the third term in (4.248) contributes only to the parameter $\mathcal{A}_{L,R}$ and, hence, does not contribute to the additional neutrino energy.

Substituting the W boson propagator in the form (4.248) into Eq. (4.247) and discarding the terms that do not contribute to the additional neutrino energy, we obtain

$$\Sigma^W(p) = \frac{2 g^2 \, g_{\alpha\beta}}{m_W^4} \sum_n (-1)^n \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{f(\omega_n)}{\omega_n} \, (pk) \, \mathrm{e}^{-u}$$
$$\times \, \gamma_\alpha \left\{ (k\gamma)_\parallel \left[L_n(2u) \Pi_- - L_{n-1}(2u) \Pi_+ \right] + 2 (k\gamma)_\perp L^1_{n-1}(2u) \right\} \gamma_\beta \gamma_L .$$

Passing to integration over the variable $u = k_\perp^2/eB$ in this expression,

$$\int\limits_{-\infty}^{+\infty} \mathrm{d}^3 k = \pi e B \int\limits_{-\infty}^{+\infty} \mathrm{d}k_3 \int\limits_0^{\infty} \mathrm{d}u$$

and performing the integration over the variable u using the relations

$$\int\limits_0^{\infty} \mathrm{d}u\, \mathrm{e}^{-u} L_n(2u) = (-1)^n,$$

$$2u\, L_{n-1}^1(2u) = n\,[L_{n-1}(2u) - L_n(2u)],$$

we finally obtain

$$\begin{aligned}\Sigma^W(p) = {} & \frac{g^2\, eB}{2\,\pi^2\, m_W^4} \sum_{n=0}^{\prime} \int\limits_{-\infty}^{+\infty} \frac{\mathrm{d}k_3 f(\omega_n)}{\omega_n} \\ & \times \left\{ (p\gamma)eBn - (p\tilde{\varphi}\gamma) \left(\frac{(k\tilde{\varphi}p)}{\omega_n E}(k_3^2 - eBn) - \delta_{n0}\omega_n^2 \right) \right. \\ & \left. - (u\gamma) \left(E\,(\omega_n^2 + eBn + \frac{p_3^2}{E^2}(k_3^2 - eBn)) - \delta_{no}\, p_3\,(k_3^2 + \omega_n^2) \right) \right\} \gamma_L .\end{aligned} \tag{4.249}$$

Here, δ_{n0} is the Kronecker symbol, which is nonzero only for the ground Landau level; the sum over the Landau levels (with a prime) is defined as

$$\sum_{n=0}^{\prime} F(n) = \frac{1}{2}\, F(n=0) + \sum_{n=1}^{\infty} F(n).$$

Finally we find the contribution to the additional neutrino energy from the neutrino forward scattering by electrons and positrons of a magnetized plasma:

$$\begin{aligned}\mathcal{B}^W = {} & -\frac{2\sqrt{2}\, G_\mathrm{F}\, eBE}{\pi^2\, m_W^2} \sum_{n=0}^{\prime} \int\limits_{-\infty}^{+\infty} \frac{\mathrm{d}k_3 f(\omega_n)}{\omega_n} \\ & \times [\omega_n^2 + eBn + \cos^2\phi\,(k_3^2 - eBn) - \delta_{n0}\, \cos\phi\,(k_3^2 + \omega_n^2)],\end{aligned} \tag{4.250}$$

where ϕ is the angle between the magnetic field direction and the neutrino momentum vector.

Below, we will consider some limiting cases that can be of interest from the standpoint of possible astrophysical applications.

4.7.3 Asymptotic Expressions for the Neutrino Additional Energy in Magnetized Plasma

A general expression for the additional neutrino energy in a charge-symmetric magnetized plasma can be obtained by summing the contributions from the processes attributable to the Z boson and W boson exchange and can be represented in the form:

$$\frac{\Delta E}{|\mathbf{p}|} = -\frac{7\sqrt{2}\,G_{\mathrm{F}}\,\pi^2\,T^4}{45\,m_Z^2} - \frac{2\sqrt{2}\,G_{\mathrm{F}}\,eB}{\pi^2\,m_W^2}\sum_{n=0}^{\infty}{}' \int\limits_{-\infty}^{+\infty} \frac{\mathrm{d}k_3\,f(\omega_n)}{\omega_n} \times \left[\omega_n^2 + eBn + \cos^2\phi\,(k_3^2 - eBn) - \delta_{n0}\,\cos\phi\,(k_3^2+\omega_n^2)\right]. \tag{4.251}$$

It should be noted that Eq. (4.251) describes the partial contribution of a magnetized plasma to the additional neutrino energy. To obtain a complete expression for the neutrino energy in a magnetized plasma, the purely field contribution calculated in Ref. [36] must be added to the result (4.251).

The integral in Eq. (4.251) can be calculated in some limiting cases considered below.

(i) The limit of a weak magnetic field, when the magnetic field strength is the smallest physical parameter of the problem,

$$T^2 \gg m_e^2 \gg eB. \tag{4.252}$$

The additional neutrino energy in such a weakly magnetized plasma can be reduced to the form

$$\frac{\Delta E}{|\mathbf{p}\,|} = \frac{\sqrt{2}\,G_{\mathrm{F}}}{3m_W^2}\left\{-\frac{7\,\pi^2\,T^4}{15}\left(2+\frac{m_W^2}{m_Z^2}\right) + T^2\,eB\,\cos\phi + + \frac{(eB)^2}{2\,\pi^2}\left[\sin^2\phi\left(\ln\frac{T^2}{m_e^2}+0.635\right)-1\right]\right\}. \tag{4.253}$$

Equation (4.253) contains a logarithmic factor with the electron mass m_e. However, the electron mass is not the smallest parameter for the physical conditions (4.252) under consideration and, hence, the additional neutrino energy (4.253) cannot be investigated in the limit $m_e \to 0$.

(ii) The limit of a moderate magnetic field, when the field strength is small on the scale of physical parameters of the medium, but, at the same time, it is much larger than the critical field strength for the electron:

$$T^2 \gg eB \gg m_e^2. \tag{4.254}$$

Such a physical situation could take place, for example, in a supernova core after its collapse, where the plasma temperature $T \sim 70\, m_e$. Substituting this value into the conditions (4.254) yields

$$\frac{T^2}{m_e^2} \sim 5 \times 10^3 \gg \frac{B}{B_e} \gg 1. \tag{4.255}$$

Thus, we see that even the magnetic fields with strengths up to $B \sim 10^{15} - 10^{16}$ G satisfy the conditions (4.254) and, hence, may be considered as 'relatively weak'.

A large number of Landau levels are excited under the physical conditions (4.254). In this limit, we find the additional neutrino energy to be

$$\frac{\Delta E}{|\mathbf{p}\,|} = \frac{\sqrt{2}\, G_{\mathrm{F}}}{3m_W^2} \left\{ -\frac{7\,\pi^2\, T^4}{15} \left(2 + \frac{m_W^2}{m_Z^2} \right) + T^2\, eB\, \cos\phi \right.$$
$$\left. + \frac{(eB)^2}{2\,\pi^2} \left[\sin^2\phi \left(\ln\frac{T^2}{eB} + 2.93 \right) - 1 \right] \right\}. \tag{4.256}$$

As one can see from Eq. (4.256), in contrast to the result of Ref. [40], the additional neutrino energy under the physical conditions (4.254) contains no infrared divergence in the limit $m_e \to 0$.

(iii) The limit of a strong magnetic field, which corresponds to a physical situation where the magnetic field strength is the largest of all the physical parameters that characterize a magnetized plasma:

$$eB \gg T^2, m_e^2. \tag{4.257}$$

Under the conditions (4.257), the plasma electrons and positrons occupy mostly the ground Landau level.

In the limit of a strongly magnetized plasma, the additional neutrino energy is

$$\frac{\Delta E}{|\mathbf{p}\,|} = -\frac{\sqrt{2}\, G_{\mathrm{F}}}{3m_W^2} \left[\frac{7\,\pi^2\, T^4\, m_W^2}{15\, m_Z^2} + \frac{T^2\, eB}{2}\,(1 - \cos\phi)^2 \right. \tag{4.258}$$
$$\left. + 3\,(eB)^2 \left(\frac{2}{\pi} \right)^{3/2} \left(\frac{T^2}{2eB} \right)^{1/4} (3 - \cos^2\phi)\, \mathrm{e}^{-\sqrt{2eB}/T} \right].$$

Here, the second term is attributable to the contribution from the ground Landau level, and the third term containing the exponential suppression is caused by the first Landau level.

4.7.4 Induced Neutrino Magnetic Moment in Magnetized Plasma

As already noted, the additional interest in calculating the neutrino self-energy operator stems from the possibility of obtaining the data on the anomalous magnetic moment of a neutrino. However, there is some doubt concerning the validity of the data presented in the literature so far on the magnetic moment of the neutrino in a magnetized plasma, since the results imply that the magnetic moment of the neutrino is either independent of its mass m_ν [78] or exhibits a giant enhancement by a factor of $1/m_\nu$. As was reasonably pointed out [39], these results confuse the situation with the magnetic moment of the neutrino, instead of elucidating it. An independent calculation of the neutrino magnetic moment in a magnetized plasma was carried out in Ref. [77]. Here we reproduce a general scheme of the analysis.

Let us find a contribution to the energy of a neutrino, which is related to the presence of its magnetic moment μ_ν. This energy correction can be determined using the Lagrangian expressed as

$$\Delta\mathcal{L}_{int} = -\frac{\mathrm{i}\mu_\nu}{2}\,(\,\bar{\Psi}\,\sigma_{\mu\nu}\,\Psi\,)\,F^{\mu\nu}, \tag{4.259}$$

where Ψ is the fermion field and $\sigma_{\mu\nu} = (\gamma_\mu\,\gamma_\nu - \gamma_\nu\,\gamma_\mu)/2$.

Substituting this formula into an expression for the additional energy defined as

$$\Delta E^{(\mu)} = -\int \mathrm{d}V\,\langle\Delta\mathcal{L}_{int}\rangle\,, \tag{4.260}$$

we eventually obtain the following formula:

$$\Delta E^{(\mu)} = -\mu\left[(\mathbf{s}\,\mathbf{B}_t) + \frac{m_\nu}{E}\,(\mathbf{s}\,\mathbf{B}_\ell)\right], \tag{4.261}$$

where $\mathbf{s}$ is the average twice spin vector of a fermion, $\mathbf{B}_{t,\ell}$ are the transverse and longitudinal components of the magnetic field $\mathbf{B}$ with respect to the momentum of a fermion, introduced in Eq. (4.234).

Thus, in expression (4.234) for the additional energy of the neutrino in a magnetized plasma, the magnetic moment only enters into the structure, which is proportional to the following sum:

$$(\mathbf{s}\,\mathbf{B}_t) + \frac{m_\nu}{E}\,(\mathbf{s}\,\mathbf{B}_\ell)\,.$$

We should conclude that determination of the magnetic moment of the neutrino in Ref. [78] was incorrect because it was assumed that the entire additional energy of the neutrino (related to its dependence on the spin and magnetic field) in a magnetized plasma contributes to the induced magnetic moment. However, as was shown above,

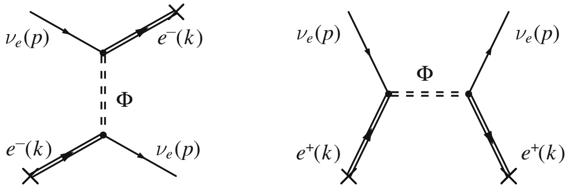

Fig. 4.16 The Feynman diagrams for the neutrino scattering on plasma electrons and positrons through the charged scalar Φ–boson

only one term in the expression for the additional energy of the neutrino refers to its magnetic moment.

A comparison of expression (4.234) to formula (4.261) for the additional energy of a neutrino shows that, in order to determine the magnetic moment of the neutrino in a magnetized plasma, it is sufficient to find the coefficients $\mathcal{C}_L, \mathcal{C}_R$ and $\mathcal{K}_2$, see Eq. (4.231), by which the magnetic moment is expressed as follows:

$$\mu_{\nu_\ell} = \frac{e\, m_\nu}{2}\, (\mathcal{C}_L - \mathcal{C}_R + 4\,\mathcal{K}_2)\ . \tag{4.262}$$

Further we calculate the terms of the neutrino self-energy operator $\Sigma(p)$, which contribute to the magnetic moment of a neutrino. For variety, the calculation will be given in the Feynman gauge.

In a magnetized plasma, this magnetic moment consists of two parts: the purely field contribution and the plasma contribution. The field contribution to the magnetic moment of the neutrino was calculated in a number of papers (see, e.g., Refs. [33, 37, 43]). An expression for the magnetic moment of the neutrino in a broad range of its energies and of magnetic fields strengths, such that

$$m_\ell^2/m_W^2 \ll (eB)^2 p_\perp^2/m_W^6 \ll 1\,,$$

can be written as follows [37]:

$$\mu_{\nu_\ell} \simeq \mu^0_{\nu_\ell} \left\{ 1 + \frac{4\,\chi^2}{3} \left(\ln\frac{1}{\chi} - \frac{17}{3} + \ln 3 + 2\,\gamma_E + \mathrm{i}\pi \right) \right\}. \tag{4.263}$$

Here, $\mu^0_{\nu_\ell}$ is the neutrino magnetic moment in vacuum [69, 70]:

$$\mu^0_{\nu_\ell} = \frac{3\, e\, G_{\mathrm{F}} m_{\nu_\ell}}{8\sqrt{2}\,\pi^2}, \tag{4.264}$$

m_{ν_ℓ} is the neutrino mass, $p_\perp$ is the neutrino transverse momentum with respect to the magnetic field direction, $\chi^2 = (eB)^2 p_\perp^2/m_W^6$, $\lambda = m_\ell^2/m_W^2$, $\gamma_E = 0,577\ldots$ is the Euler constant. The imaginary part of the magnetic moment (4.263) corresponds to the neutrino instability in the external electromagnetic field with respect to the decay $\nu_\ell \to \ell W$.

Then, in order to calculate the self-energy operator $\Sigma(p)$, it is sufficient to determine the amplitude (4.63) of the forward scattering of the neutrino in a magnetized plasma. Under real astrophysical conditions, the main contribution to the amplitude that accounts for the magnetic moment of the neutrino is due to scattering on plasma electrons and positrons. The amplitude of the $\nu \to \nu$ scattering process in a magnetized plasma (and, hence, the neutrino self-energy operator) can be represented as the sum of three terms that correspond to the diagrams depicted in Figs. 4.11 and 4.15 and also the diagrams with the charged scalar Φ–boson, Fig. 4.16, which appear in the Feynman gauge:

$$\Sigma(p) = \Sigma^W(p) + \Sigma^\Phi(p) + \Sigma^Z(p). \tag{4.265}$$

The calculation of $\Sigma(p)$ is similar to the one performed in Sect. 4.7.1. The contribution to $\Sigma(p)$ due to the scattering with W-boson exchange is:

$$\begin{aligned} \Sigma^W(p) = \frac{\mathrm{i}\, g^2}{2} \sum_{n=0}^{\infty} (-1)^n \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{\mathrm{e}^{-u}}{\omega_n} \\ \times \Big(f(\omega_n)\, G^W_{\beta\alpha}(p-k) - \bar{f}(\omega_n)\, G^W_{\beta\alpha}(p+k) \Big) \\ \times \gamma_\alpha \, [\, (k\gamma)_\parallel (L_n(2u)\, \Pi_- - L_{n-1}(2u)\, \Pi_+) + 2(k\gamma)_\perp L^1_{n-1}(2u) \,]\, \gamma_\beta \, \gamma_L, \end{aligned} \tag{4.266}$$

where g is the electroweak interaction constant in the standard model, $\Pi_\pm$ are the projection operators (2.51), $G_{\beta\alpha}(q)$ is the Fourier transform of the translationally invariant part of the W-boson propagator (3.13), $f(\omega_n)$ and $\bar{f}(\omega_n)$ are the distribution functions of electrons and positrons, respectively. In the plasma rest frame, the latter functions have the following form:

$$f(\omega_n) = [\mathrm{e}^{(\omega_n - \mu_e)/T} + 1]^{-1}, \quad \bar{f}(\omega_n) = [\mathrm{e}^{(\omega_n + \mu_e)/T} + 1]^{-1},$$

where μ_e and T are the chemical potential and temperature of the plasma, respectively, and ω_n is the electron (positron) energy on the nth Landau level.

Similarly, the contribution from the process of neutrino scattering with scalar Φ-boson exchange is as follows:

$$\begin{aligned} \Sigma^\Phi(p) = -\frac{\mathrm{i}\, g^2}{2} \sum_{n=0}^{\infty} (-1)^n \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{\mathrm{e}^{-u}}{\omega_n} \big(f(\omega_n) D(p-k) - \bar{f}(\omega_n) D(p+k) \big) \\ \times \Bigg\{ \frac{m_e^2\, m_\nu}{m_W^2} (L_n(2u)\, \Pi_- - L_{n-1}(2u)\, \Pi_+)\; [(k\gamma)_\parallel (L_n(2u)\, \Pi_- - L_{n-1}(2u)\, \Pi_+) \\ + 2(k\gamma)_\perp L^1_{n-1}(2u)\,] \left(\frac{m_e^2}{m_W^2} \gamma_L - \frac{m_\nu^2}{m_W^2} \gamma_R \right) \Bigg\}. \end{aligned} \tag{4.267}$$

Here, $D(q)$ is the Fourier transform of the translationally invariant part of the Φ-boson propagator (3.14).

Note that the contributions considered above refer only to the electron neutrino, since contributions with exchange by charged bosons for neutrinos of other types (ν_μ, ν_τ) vanish ($\Sigma^W(p) = \Sigma^\Phi(p) = 0$).

Below, we consider a realistic physical situation where the W-boson mass (m_W) is the largest parameter of the problem. This implies that parameters characterizing a magnetized plasma obey the following condition:

$$m_e^2,\, \mu^2,\, T^2,\, eB \ll m_W^2. \tag{4.268}$$

If the plasma is charge-asymmetric, it suffices to retain only the main contributions in the W and Φ boson propagators expanded in inverse powers of m_W^2, that is,

$$G_{\beta\alpha}(q) \simeq \frac{\mathrm{i}\, g_{\beta\alpha}}{m_W^2}, \qquad D(q) \simeq -\frac{\mathrm{i}}{m_W^2}. \tag{4.269}$$

Having accomplished simple calculations, we can write the two contributions to the neutrino self-energy as follows:

$$\Sigma^W(p) \simeq \sqrt{2}\, G_{\mathrm{F}} \left(-\frac{n_e^0 - \bar{n}_e^0}{EB}\, (p\tilde{F}\gamma) + \cdots \right) \gamma_L, \tag{4.270}$$

$$\begin{aligned} \Sigma^\Phi(p) \simeq \frac{G_{\mathrm{F}}}{\sqrt{2}} \Bigg\{ & \frac{n_e^0 - \bar{n}_e^0}{EB}\, (p\tilde{F}\gamma) \left(\frac{m_e^2}{m_W^2}\gamma_L + \frac{m_\nu^2}{m_W^2}\gamma_R \right) \\ & - \frac{\mathrm{i}e}{4\pi^2}\, m_\nu \frac{m_e^2}{m_W^2}\, (\gamma F\gamma) \int\limits_0^\infty \frac{\mathrm{d}k}{\omega_0}\, (f(\omega_0) - \bar{f}(\omega_0)) + \cdots \Bigg\}, \end{aligned} \tag{4.271}$$

where the dots corresponds to the terms not contributing to the magnetic moment of the neutrino; ω_0 is the electron (positron) energy on the ground $(n = 0)$ Landau level; and n_e^0 and $\bar{n}_e^0$ are the electron and positron densities, respectively, on this level. The difference of these densities is given by the following integral:

$$n_e^0 - \bar{n}_e^0 = \frac{eB}{2\pi^2} \int\limits_0^\infty \mathrm{d}k\, (f(\omega_0) - \bar{f}(\omega_0)). \tag{4.272}$$

Comparing expressions (4.270) and (4.271) to the parametrization in Eq. (4.231), we obtain the following formulas for the coefficients $\mathcal{C}_L$, $\mathcal{C}_R$, and $\mathcal{K}_2$:

$$\mathcal{C}_L^W = -\frac{e\,G_\mathrm{F}}{\sqrt{2}\,\pi^2\,E}\int\limits_0^\infty \mathrm{d}k\,(f(\omega_0) - \bar{f}(\omega_0)), \tag{4.273}$$

$$\mathcal{C}_R^W = \mathcal{K}_2^W = 0, \tag{4.274}$$

$$\mathcal{C}_L^\Phi = -\frac{m_e^2}{2m_W^2}\,\mathcal{C}_L^W, \quad \mathcal{C}_R^\Phi = -\frac{m_\nu^2}{2m_W^2}\,\mathcal{C}_L^W, \tag{4.275}$$

$$\mathcal{K}_2^\Phi = -\frac{e\,G_\mathrm{F}}{4\sqrt{2}\pi^2}\,\frac{m_e^2}{m_W^2}\int\limits_0^\infty \frac{\mathrm{d}k}{\omega_0}\,(f(\omega_0) - \bar{f}(\omega_0)). \tag{4.276}$$

Note that, as could be expected, the contributions from charged scalar exchange are suppressed by the small factors m_ν^2/m_W^2 and m_e^2/m_W^2.

The third term, Σ^Z, in the neutrino self-energy operator (4.265), which accounts for the contribution from neutrino scattering on charged fermions with Z-boson exchange, is readily calculated as follows:

$$\Sigma_f^Z = \sqrt{2}\,G_\mathrm{F}\left(-\frac{T_3^f}{BE}\,(n_f^0 - \bar{n}_f^0)\,(p\,\tilde{F}\,\gamma) + \cdots\right)\gamma_L\,. \tag{4.277}$$

where n_f^0, and $\bar{n}_f^0$ are the densities of charged fermions and antifermions, respectively, on the ground Landau level; T_3^f is the third component of the weak isospin of a charged fermion; and the dots correspond to terms not contributing to the magnetic moment of neutrino. Taking into account that the maximum density of particles on the ground Landau level corresponds to electron and positrons, we obtain from expression (4.277) the following formulas for the coefficients $\mathcal{C}_L$, $\mathcal{C}_R$ and $\mathcal{K}_2$:

$$\mathcal{C}_L^Z = -\frac{e\,G_\mathrm{F}}{2\sqrt{2}\,\pi^2\,E}\int\limits_0^\infty \mathrm{d}k\,(f(\omega_0) - \bar{f}(\omega_0)), \tag{4.278}$$

$$\mathcal{C}_R^Z = \mathcal{K}_2^Z = 0. \tag{4.279}$$

Thus, the neutrino magnetic moment induced by a charge-asymmetric plasma is expressed in terms of the coefficients $\mathcal{C}_L^Z$ and $\mathcal{C}_L^W$, with an addition of the purely field contribution (4.263). The final formula for $\mathcal{C}_L$ is:

$$\mathcal{C}_L \simeq \mathcal{C}_L^W + \mathcal{C}_L^Z \simeq \frac{3eG_\mathrm{F}}{4\sqrt{2}\,\pi^2}\left(1 \mp \frac{2}{3\,E}\int\limits_0^\infty \mathrm{d}k\,(f(\omega_0) - \bar{f}(\omega_0))\right), \tag{4.280}$$

where the upper sign refers to the electron neutrino (ν_e) and the lower sign, to the muon and tau neutrinos (ν_μ, ν_τ). Contributions proportional to $1/m_W^4$, $1/m_W^6$, etc., were neglected.

The integral in expression (4.280) is easily calculated for an ultrarelativistic plasma. In this case, the magnetic moment of the neutrino is given by the following simple formula:

$$\mu_\nu = \frac{\mathcal{C}_L\, m_\nu}{2} \simeq \frac{3e\, G_{\mathrm{F}}\, m_\nu}{8\sqrt{2}\,\pi^2} \left(1 \mp \frac{2}{3}\,\frac{\mu_e}{E}\right). \tag{4.281}$$

where μ_e is the chemical potential of electrons in the ultrarelativistic plasma. For weakly magnetized plasma, it reduces to

$$\mu_e \simeq (3\pi^2\,(n_e - \bar{n}_e))^{1/3}, \tag{4.282}$$

For strongly magnetized plasma, in which case magnetic field rather than the plasma is the dominant component of the active medium and plasma electrons occupy the ground Landau level, the chemical potential of electrons is

$$\mu_e \simeq 2\pi^2\,\frac{(n_e - \bar{n}_e)}{eB}, \tag{4.283}$$

where n_e and $\bar{n}_e$ are the total electron and positron densities, respectively.

Another situation for which analytical calculation of the neutrino magnetic moment can be performed refers to physical conditions of a charge-symmetric electron-positron plasma. In this case, the contribution from the diagram of neutrino scattering with Z-boson exchange vanishes and, hence, the ν_μ and ν_τ-type neutrinos possess no additional magnetic moment induced by a magnetized plasma.

For the electron neutrino in a charge-symmetric $e^- e^+$ plasma, the magnetic moment is determined by the following expression:

$$\mu_{\nu_e} \simeq \frac{3e\, G_{\mathrm{F}}\, m_\nu}{8\sqrt{2}\,\pi^2} \left(1 + \frac{4\pi^2}{9}\,\frac{T^2}{m_W^2}\right). \tag{4.284}$$

As one can see, under real astrophysical conditions, where $T \ll m_W$, the plasma contribution to the neutrino magnetic moment is suppressed.

Thus, we have shown that the presence of a plasma does not lead to an enhancement of the neutrino magnetic moment, in contrast to the statement of Ref. [78]. The plasma-induced part of the magnetic moment is suppressed by the neutrino mass m_ν; in a charge-symmetric plasma, it is also suppressed by a factor of $T^2/m_W^2 \ll 1$.

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Chapter 5
Electromagnetic Interactions in External Active Media

In this chapter, we present in great detail the technique of calculations of the electromagnetic processes in external active media. We consider mainly the two processes. The first one, which is forbidden in a vacuum but is possible in an intense external electromagnetic field, is the photon decay into the electron–positron pair $\gamma \to e^- e^+$. We calculate the probability of this process in an external field, in the two limiting cases where the detailed analytical calculations are possible. These are: (i) the case of a very strong magnetic field when electrons and positrons occupy the ground Landau level; (ii) the case of a relatively weak external field when the energy of the initial particle is the main physical parameter of a problem; this case can be analyzed in the crossed field approximation. Calculations are performed by the two methods: (i) using the exact solutions of the Dirac equation; (ii) via the imaginary part of the loop amplitude. We analyse also the process of the photon emission by an electron in magnetic fields which is the crossed process to the $\gamma \to e^- e^+$ decay. The second process considered is the electromagnetic interaction of the Dirac neutrino having a magnetic moment, with plasma. The plasma influence on the virtual photon, and contributions of plasma components into the neutrino scattering process are taken into account. The upper bound on the neutrino magnetic moment using the data on supernova $SN1987A$ is established. Possible effects of the neutrino magnetic moment: shock-wave revival in a supernova explosion and the time evolution of the neutrino signal (neutrino pulsar) are analysed.

5.1 Photon Decay into an Electron–Positron Pair in a Strong Magnetic Field

The $\gamma \to e^+ e^-$ process is kinematically forbidden in a vacuum. The magnetic field changes the kinematics of charged particles, electrons, and positrons, allowing the production of an electron–positron pair in the kinematic region $q_\|^2 = q_0^2 - q_z^2 \geqslant 4m_e^2$, where q_0 is the photon energy (the z axis is directed along the magnetic field).

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In 1954, Klepikov [1] examined the production of an electron–positron pair by a photon in a magnetic field and obtained the amplitude and width of the $\gamma \to e^+e^-$ decay in the semiclassical approximation. Later, the authors of Refs. [2–7] considered this process in the context of its astrophysical applications. It was pointed out in Ref. [6, 7] that the use of the expression derived in Ref. [1] for the width considerably overestimates the result in the strong magnetic field limit. In this case, one should use an exact expression for the width of one-photon production of a pair when electrons and positrons occupy only the ground Landau level. This calculation is demonstrated in the following section.

5.1.1 Direct Calculation Based on the Solutions of the Dirac Equation

Photon decay into the electron–positron pair $\gamma(q) \to e^-(p') + e^+(p)$ in a magnetic field is described by the Lagrangian of the electromagnetic interaction

$$\mathcal{L}_{\text{em}} = e\left(\overline{\Psi}(x)\hat{A}(x)\Psi(x)\right) \tag{5.1}$$

and is depicted by the Feynman diagram presented in Fig. 5.1.

In the first order of the perturbation theory with the interaction (5.1), one obtains the following expression for the matrix element S_{if}

$$S_{if} = \mathrm{i}\, e\, \langle f \mid N \int (\overline{\Psi}\hat{A}\Psi)\mathrm{d}^4x \mid i\rangle, \tag{5.2}$$

where A_α is the electromagnetic field operator,

$$A_\alpha = \sum_{\mathbf{q},\lambda} \frac{1}{\sqrt{2\omega V}} \left(c_\lambda \varepsilon_\alpha^{(\lambda)} \mathrm{e}^{-\mathrm{i}qx} + c_\lambda^+ \varepsilon_\alpha^{(\lambda)*} \mathrm{e}^{\mathrm{i}qx}\right),$$

Ψ is the operator of the electron–positron field,

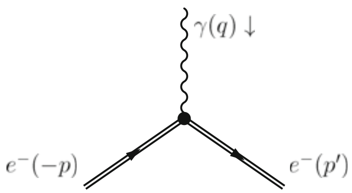

Fig. 5.1 The Feynman diagram for the process $\gamma \to e^-e^+$ in a magnetic field. Double lines indicate that the effect of an external field is taken exactly into account in the wave functions of the electron and the positron

$$\Psi = \sum_{\mathbf{p},s,n} \left(a_{p,s} \Psi^{(+)} + b^{+}_{p,s} \Psi^{(-)} \right),$$

$\Psi^{(+)}$ is the normalized solution of the Dirac equation in a magnetic field, with positive energy (2.30)–(2.32), and $\Psi^{(-)}$ is the corresponding solution with negative energy (2.33)–(2.36). In a strong magnetic field, the electron and the positron can be produced only in the states that correspond to the ground Landau level (2.37), (2.38), which are described by the wave functions:

$$\Psi^{(+)} = \frac{(eB)^{1/4}}{(\sqrt{\pi} 2E' L_y L_z)^{1/2}} \, \mathrm{e}^{-\mathrm{i}(E't - p'_y y - p'_z z)} \mathrm{e}^{-\xi'^2/2} \, u_{p'}, \tag{5.3}$$

$$\Psi^{(-)} = \frac{(eB)^{1/4}}{(\sqrt{\pi} 2E L_y L_z)^{1/2}} \, \mathrm{e}^{\mathrm{i}(Et - p_y y - p_z z)} \mathrm{e}^{-\xi^2/2} \, u_{-p}, \tag{5.4}$$

where

$$E = \sqrt{p_z^2 + m_e^2}, \quad E' = \sqrt{p_z'^2 + m_e^2},$$
$$\xi = \sqrt{eB} \left(x - \frac{p_y}{eB} \right), \quad \xi' = \sqrt{eB} \left(x + \frac{p'_y}{eB} \right),$$

$$u_{p'} = \frac{1}{\sqrt{E' + m_e}} \begin{pmatrix} 0 \\ E' + m_e \\ 0 \\ -p'_z \end{pmatrix}, \qquad u_{-p} = \frac{1}{\sqrt{E - m_e}} \begin{pmatrix} 0 \\ E - m_e \\ 0 \\ -p_z \end{pmatrix}. \tag{5.5}$$

Substituting the wave functions of the final state (5.3) and (5.4) into the expression (5.2) and integrating over $\mathrm{d}t \, \mathrm{d}y \, \mathrm{d}z$, one obtains

$$S_{if} = \frac{\mathrm{i}e(2\pi)^3 \delta^3(p + p' - q)}{2L_y L_z \sqrt{2\omega V E E'}} (\bar{u}_{p'} \hat{\varepsilon}^{(\lambda)} u_{-p}) \int \mathrm{e}^{\mathrm{i}q_x x} \mathrm{e}^{-\xi^2/2} \mathrm{e}^{-\xi'^2/2} \mathrm{d}x,$$

where $\delta^3(p + p' - q) = \delta(E + E' - \omega) \, \delta(p_y + p'_y - q_y) \, \delta(p_z + p'_z - q_z)$. It is convenient to perform the integration over x in the frame where the initial photon momentum has a form $\mathbf{q} = (q_x, 0, q_z)$, and therefore $p_y = -p'_y$, $\xi = \xi'$. As $u(p')$ and $u(-p)$ do not depend on x, the integral of the Gaussian type with respect to x can be easily calculated. Taking the strong field limit into account, we assume that $\exp(-q_x^2/2eB) \simeq 1$. The S matrix element squared can be written as follows

$$|S_{if}|^2 = \frac{e^2 (2\pi)^3 T}{8 L_y L_z V \omega E E'} |\bar{u}_{p'} \hat{\varepsilon}^{(\lambda)} u_{-p}|^2 \delta^3(p + p' - q), \tag{5.6}$$

where T is the total interaction time. The expression $|\bar{u}_{p'}\hat{\varepsilon}^{(\lambda)}u_{-p}|^2$ can be rewritten in terms of the trace calculation

$$|\bar{u}_{p'}\hat{\varepsilon}^{(\lambda)}u_{-p}|^2 = \mathrm{Tr}[\rho(p')\hat{\varepsilon}^{(\lambda)}\rho(-p)\hat{\varepsilon}^{(\lambda)}],$$

where the density matrix of polarized electrons is

$$\rho(p') = (\hat{p}'_{\|} + m_e)\Pi_-, \tag{5.7}$$

and for polarized positrons

$$\rho(-p) = (\hat{p}_{\|} - m_e)\Pi_-. \tag{5.8}$$

Π_- is the projecting operator (2.51) corresponding to the electron or positron state on the ground Landau level, where the electron spin direction is opposite to the external field direction, while the positron spin is directed along the field.

To simplify further calculations, we make the Lorentz transformation along the field direction, as $q_{\|}$ is the timelike vector,

$$q_{\|}^2 = \omega^2 - q_z^2 = (E + E')^2 - (p_z + p'_z)^2 > 0,$$

to the frame where $q_z = 0$. In this frame $p_z = -p'_z,\ \ E = E'$. Further we perform separate calculations for the definite photon polarizations, using the explicit form of the polarization vectors $\varepsilon^{(\lambda)}$ $(\lambda = 1, 2)$. The vectors describing the physical states of a photon in a magnetic field (for details see Sect. 4.2) are

$$\varepsilon_\alpha^{(1)} = \frac{(q\varphi)_\alpha}{\sqrt{q_\perp^2}}, \qquad \varepsilon_\alpha^{(2)} = \frac{(q\tilde{\varphi})_\alpha}{\sqrt{q_\|^2}}. \tag{5.9}$$

Substituting the polarization vector of the 1st mode photon, one obtains

$$\mathrm{Tr}[\rho(p')\hat{\varepsilon}^{(1)}\rho(-p)\hat{\varepsilon}^{(1)}] = 0.$$

By this means the 1st mode photon cannot decay into the electron–positron pair with both electron and positron being produced in the ground Landau level. Performing the similar calculation for the 2nd mode photon one obtains

$$\mathrm{Tr}[\rho(p')\hat{\varepsilon}^{(2)}\rho(-p)\hat{\varepsilon}^{(2)}] = 4m_e^2.$$

The resulting S matrix element squared for the decay of the 2-mode photon takes the form

$$|S_{if}|^2 = \frac{e^2(2\pi)^3 m_e^2 T}{2L_y L_z V\omega E E'}\delta^3(p + p' - q), \tag{5.10}$$

where $\delta^3(p + p' - q) = \delta(2E - \omega)\,\delta(p_y + p'_y - q_y)\,\delta(p_z + p'_z)$.

To find the decay probability, one should perform the integration over the phase space of final electrons and positrons:

$$\mathrm{d}W = \frac{|S_{if}|^2}{T}\,\mathrm{d}n_f, \tag{5.11}$$

where

$$\mathrm{d}n_f = \frac{\mathrm{d}p_y \mathrm{d}p_z \mathrm{d}p'_y \mathrm{d}p'_z}{(2\pi)^4}\,L_y^2 L_z^2. \tag{5.12}$$

The δ function for energies can be transformed into the following form

$$\frac{\delta(2E - \omega)}{\omega} = \frac{1}{4|p^*|}[\delta(p_z - p^*) + \delta(p_z + p^*)]\,\Theta(\omega^2 - 4m_e^2),$$

where $p^* = \pm\frac{1}{2}\sqrt{\omega^2 - 4m_e^2}$, $\Theta(x)$ is the step function.

The integration over $\mathrm{d}p'_y\,\mathrm{d}p'_z\,\mathrm{d}p_z$ removes the δ functions. It can easily be seen that the integrand is independent on p_y; hence, integration with respect to p_y actually determines the degeneracy multiplicity of the electron state at a given energy:

$$N_E = \frac{L_y}{2\pi}\int \mathrm{d}p_y = \frac{eBL_y}{2\pi}\int\limits_{-L_x/2}^{L_x/2} \mathrm{d}x_0 = \frac{eBL_xL_y}{2\pi}, \tag{5.13}$$

where $x_0 = p_y/eB$ determines the center of the Gaussian packet on the x axis; see (5.5). As a result, for the decay probability of the 2-mode photon one obtains

$$W^{(2)} = \frac{4\alpha eBm_e^2}{\omega^2\sqrt{\omega^2 - 4m_e^2}}\,\Theta(\omega^2 - 4m_e^2). \tag{5.14}$$

The Θ function is seen to define the threshold of the photon decay into the e^-e^+ pair. Making the inverse Lorentz transformation, in view of the invariance of the product ωW, one can rewrite the probability (5.14) in an arbitrary frame

$$W^{(2)} = \frac{4\alpha eBm_e^2}{\omega^2 \sin\theta\sqrt{\omega^2\sin^2\theta - 4m_e^2}}\,\Theta(\omega^2\sin^2\theta - 4m_e^2), \tag{5.15}$$

where θ is the angle between the photon momentum and the magnetic field direction.

The formula obtained shows that the photon decay process has a resonant character. It is enhanced essentially when the angle θ is close to $\theta_{\mathrm{res}} = \arcsin(2m_e/\omega)$.

5.1.2 Calculation Based on the Imaginary Part of the Loop Amplitude

There exists another way to calculate the probability of the photon decay in a magneic field, which is based on an application of the unitarity relation (see e.g. [8])

$$W(\gamma \to e^- e^+) = \frac{1}{\omega}\,\mathrm{Im}\,\mathcal{M}(\gamma \to \gamma), \tag{5.16}$$

where ω is the photon energy. The amplitude of the transition $\mathcal{M}(\gamma \to \gamma)$ in a magnetic field can be obtained from (4.31), where the vector currents should be replaced as follows, $j_{V\alpha} \to e\varepsilon_\alpha^{(\lambda)}$, here $\varepsilon_\alpha^{(\lambda)}$ $(\lambda = 1, 2)$ are the polarization vectors (4.10); the condition $q^2 = 0$ should be set also. By this means, we obtain from (4.31)

$$\Delta\mathcal{M}^{(\lambda)} \equiv \Delta\mathcal{M}\left(\gamma^{(\lambda)} \to \gamma^{(\lambda)}\right) = \frac{\alpha}{\pi}\,Y_{VV}^{(\lambda)}, \quad \lambda = 1, 2. \tag{5.17}$$

To take the strong field limit in the functions $Y_{VV}^{(\lambda)}$, it is worthwhile to make the Wick rotation of the integration contour in the complex plane t (see Chap. 3), replacing it on the negative imaginary axis, $t = -\mathrm{i}\tau$, where τ is a real variable. In this case $\sin\beta t = -\mathrm{i}\,\sinh\beta\tau$ and $\cos\beta t = \cosh\beta\tau$. Let us analyze first the amplitude (5.17) for the 2nd mode photon. We obtain:

$$\begin{aligned}\Delta\mathcal{M}^{(2)} = \frac{\alpha}{\pi}\int\limits_0^1 \mathrm{d}u \int\limits_0^\infty \frac{\mathrm{d}\tau}{\tau}\left\{\frac{\beta\tau}{\sinh\beta\tau}\,\mathrm{e}^{-\tau[m_e^2 - q_\parallel^2(1-u^2)/4]}\left[q_\parallel^2\,\frac{1-u^2}{2}\cosh\beta\tau\right.\right.\\ \left.\left. - \frac{q_\perp^2}{2}\left(\cosh\beta\tau u - \frac{u\,\sinh\beta\tau u}{\tanh\beta\tau}\right)\right] - q^2\,\frac{1-u^2}{2}\,\mathrm{e}^{-\tau[m_e^2 - q^2(1-u^2)/4]}\right\}.\end{aligned} \tag{5.18}$$

Taking the strong field limit we assume that the field parameter $\beta = eB$ is the maximal dimensional parameter of our problem, $\beta \gg q_\parallel^2, q_\perp^2, m_e^2$. It is seen from the integrand in (5.18) that the region $\tau \sim 1/m_e^2,\, 1/q_\parallel^2 \gg 1/\beta$ gives the main contribution. In this region one can assume

$$\cosh\beta\tau \simeq \sinh\beta\tau \simeq \frac{1}{2}\mathrm{e}^{\beta\tau}.$$

In the strong field limit, the field-induced part of the amplitude dominates and actually it defines the total amplitude of the transition $\gamma^{(2)} \to \gamma^{(2)}$, $\mathcal{M}^{(2)} \simeq \Delta\mathcal{M}^{(2)}$. The integral with respect to τ in (5.18) can be easily calculated to give

$$\mathcal{M}^{(2)} \simeq \frac{2\alpha\beta}{\pi}\,H\left(\frac{q_\parallel^2}{4m_e^2}\right), \tag{5.19}$$

where the function $H(z)$ is defined in (4.17). Using (4.18), one obtains the following expression for the imaginary part of the amplitude,

$$\mathrm{Im}\,\mathcal{M}^{(2)} = \frac{4\alpha\beta m_e^2}{\sqrt{q_\parallel^2(q_\parallel^2 - 4m_e^2)}}\,\Theta(q_\parallel^2 - 4m_e^2). \tag{5.20}$$

Substituting (5.20) into (5.16), in view of $\beta = eB$ and $q_\parallel^2 = \omega^2 - q_3^2 = \omega^2 \sin^2\theta$, we obtain the result for the probability that coincides with (5.15).

The similar analysis of the amplitude for the transition $\gamma^{(1)} \to \gamma^{(1)}$ shows that in the strong field limit, the integral $Y_{VV}^{(1)}$ does not have an interval where the enhancing factor β could arise, as it was for $Y_{VV}^{(2)}$. But it is more essential that the amplitude $\mathcal{M}^{(1)}$ does not have an imaginary part in the strong magnetic field limit $\beta \gg q_\parallel^2$. Thus, only the 2nd mode photon can decay into the e^-e^+ pair in the strong field limit.

5.2 The $\gamma \to e^-e^+$ Decay in a Crossed Field

As was already mentioned, the case of a relatively weak external field when the photon energy is the largest physical parameter, corresponds to the crossed field approximation. We perform the calculations by the two ways, first by using the exact solution of the Dirac equation (2.40), and second via the imaginary part of the loop amplitude $\mathcal{M}_{VV}$ (4.35) for the transition $\gamma \to e^-e^+ \to \gamma$.

5.2.1 *Direct Calculation Based on the Solutions of the Dirac Equation*

Substituting the solutions in a crossed field (2.40) for the electron and the positron into the S matrix element (5.2), one obtains

$$\begin{aligned} S_{if} = \frac{\mathrm{i}e}{\sqrt{2\omega V 2EV 2E'V}} &\int \mathrm{d}^4x \exp\left[-\mathrm{i}\left((Qx) - r^3\varkappa^3(\varphi_0\varphi^2 + \frac{1}{3}\varphi^3)\right)\right] \\ &\times \left[\bar{u}(p)\left(1 - \frac{e\hat{a}\hat{k}}{2(kp)}\varphi\right)\hat{\varepsilon}\left(1 + \frac{e\hat{k}\hat{a}}{2(kp')}\varphi\right)u(-p')\right], \end{aligned} \tag{5.21}$$

where the following notations are used: $Q = q - p - p'$, and

$$r = \left(\frac{\chi}{2\chi_1\chi_2}\right)^{1/3}, \quad \varkappa^2 = -\frac{e^2(aa)}{m_e^2}, \quad \varphi_0 = -\frac{e(qFp)}{m_e^4\varkappa\chi},$$
$$\chi = \left(\frac{e^2(qFFq)}{m_e^6}\right)^{1/2} = \frac{\varkappa(qk)}{m_e^2},$$
$$\chi_1 = \left(\frac{e^2(pFFp)}{m_e^6}\right)^{1/2} = \frac{\varkappa(pk)}{m_e^2},$$
$$\chi_2 = \left(\frac{e^2(p'FFp')}{m_e^6}\right)^{1/2} = \frac{\varkappa(p'k)}{m_e^2}. \tag{5.22}$$

Taking the frame (2.41) and keeping in mind that $\varphi = (kx) = k_0(t - x)$, we can write

$$(Qx) = (Q_0 - Q_x)t - Q_y y - Q_z z + s\varphi, \quad s = \frac{Q_x}{k_0}.$$

The integrals with respect to y and z give the two-dimensional δ function:

$$\int \mathrm{d}y\ \mathrm{d}z\ \mathrm{e}^{\mathrm{i}(Q_y y + Q_z z)} = (2\pi)^2\delta^2(\mathbf{Q}_\perp).$$

Changing the variables t, x to t, φ

$$\int \mathrm{d}t\ \mathrm{d}x = \frac{1}{k_0}\int \mathrm{d}t\ \mathrm{d}\varphi$$

and integrating with respect to t:

$$\int \mathrm{d}t\ \mathrm{e}^{-\mathrm{i}(Q_0 - Q_x)t} = 2\pi\delta(Q_0 - Q_x) = 2\pi k_0\delta(kQ) = 2\pi k_0\frac{\varkappa}{m_e^2}\delta(\chi - \chi_1 - \chi_2),$$

we transform the S matrix element (5.21) to the form

$$S_{if} = \frac{\mathrm{i}e(2\pi)^3\delta^2(\mathbf{Q}_\perp)\delta(kQ)}{\sqrt{2\omega V 2E V 2E' V}}\int_{-\infty}^{\infty}\mathrm{d}\varphi\left[\bar{u}(p)\gamma_\mu L^{\mu\nu}\varepsilon_\nu u(-p')\right]$$
$$\times \exp\left[-\mathrm{i}\left(s\varphi - r^3\varkappa^3(\varphi_0\varphi^2 + \frac{1}{3}\varphi^3)\right)\right], \tag{5.23}$$

where

$$L^{\mu\nu} = g^{\mu\nu} + \kappa_- F^{\mu\nu}\varphi - \mathrm{i}\,\kappa_+\gamma_5\tilde{F}^{\mu\nu}\varphi - \frac{e^2\varkappa^2}{2m_e^4\chi_1\chi_2}(FF)^{\mu\nu}\varphi^2, \tag{5.24}$$
$$\kappa_\pm = \frac{e\varkappa}{2m_e^2}\left(\frac{1}{\chi_1} \pm \frac{1}{\chi_2}\right). \tag{5.25}$$

It is worthwhile to perform further calculations for a photon of the definite polarization. The polarization vectors (4.10) in a crossed field can be presented in the form

$$\varepsilon_\alpha^{(1)} = \frac{(Fq)_\alpha}{\sqrt{(qFFq)}}, \qquad \varepsilon_\alpha^{(2)} = \frac{(\tilde{F}q)_\alpha}{\sqrt{(qFFq)}}. \tag{5.26}$$

We obtain

$$\begin{aligned} \left(L\varepsilon^{(1)}\right)_\mu &= \frac{1}{\sqrt{(qFFq)}}\Big[(Fq)_\mu + \kappa_-(FFq)_\mu(\varphi - \varphi_0)\Big], \\ \left(L\varepsilon^{(2)}\right)_\mu &= \frac{1}{\sqrt{(qFFq)}}\Big[(\tilde{F}q)_\mu - \mathrm{i}\kappa_+\gamma_5(FFq)_\mu(\varphi - \varphi_0)\Big]. \end{aligned} \tag{5.27}$$

Making a shift in the integral with respect to the variable φ, $\varphi \to \varphi - \varphi_0$, we can remove the terms in the exponent which are proportional to φ^2, to obtain

$$s\,\varphi - r^3\varkappa^3\left(\varphi_0\,\varphi^2 + \frac{1}{3}\,\varphi^3\right) \to \bar{s}\varphi - \frac{1}{3}r^3\varkappa^3\varphi^3 \ + \ A,$$

where

$$\bar{s} = s + r^3\varkappa^3\varphi_0^2.$$

The value A not depending on φ is inessential; it leads to the appearance of a constant phase factor in the S matrix element. Given the symmetry of the integral limits, this shift on φ allows to express the result in terms of the Airy function

$$\mathrm{Ai}(y) = \frac{1}{\pi}\int_0^\infty \mathrm{d}z\,\cos\left(yz + \frac{z^3}{3}\right), \tag{5.28}$$

satisfying the equation

$$\mathrm{Ai}''(y) - y\mathrm{Ai}(y) = 0\,. \tag{5.29}$$

Thus, the integrals over φ can be rewritten as follows:

$$\int_{-\infty}^{+\infty} \mathrm{d}\varphi\,\exp\left(-\mathrm{i}(\bar{s}\varphi - \frac{1}{3}r^3\varkappa^3\varphi^3)\right) = \frac{2\pi}{r\varkappa}\,\mathrm{Ai}(y)\,, \tag{5.30}$$

$$\int_{-\infty}^{+\infty} \mathrm{d}\varphi\,\varphi\,\exp\left(-\mathrm{i}(\bar{s}\varphi - \frac{1}{3}r^3\varkappa^3\varphi^3)\right) = -\frac{2\pi\mathrm{i}}{r^2\varkappa^2}\,\mathrm{Ai}'(y)\,, \tag{5.31}$$

$$\int_{-\infty}^{+\infty} \mathrm{d}\varphi\,\varphi^2\,\exp\left(-\mathrm{i}(\bar{s}\varphi - \frac{1}{3}r^3\varkappa^3\varphi^3)\right) = -\frac{2\pi}{r^3\varkappa^3}\,\mathrm{Ai}''(y)\,, \tag{5.32}$$

where

$$y = -\frac{\bar{s}}{r\varkappa}. \tag{5.33}$$

The S matrix elements for the decays of photons with definite polarizations have the form

$$\begin{aligned} S_{if}^{(1)} &= \mathrm{i}\,\mathrm{e}^{-\mathrm{i}A}\,\frac{e(2\pi)^4\delta^2(\mathbf{Q}_\perp)\delta(kQ)}{\sqrt{2\omega V 2EV 2E'V}}\,\frac{\left[\bar{u}(p)\gamma_\mu u(-p')\right]}{r\varkappa\sqrt{(qFFq)}} \\ &\quad\times\left[(Fq)_\mu\,\mathrm{Ai}(y) + \kappa_-(FFq)_\mu\left(-\frac{\mathrm{i}}{r\varkappa}\,\mathrm{Ai}'(y) - \varphi_0\,\mathrm{Ai}(y)\right)\right], \end{aligned} \tag{5.34}$$

$$\begin{aligned} S_{if}^{(2)} &= \mathrm{i}\,\mathrm{e}^{-\mathrm{i}A}\,\frac{e(2\pi)^4\delta^2(\mathbf{Q}_\perp)\delta(kQ)}{\sqrt{2\omega V 2EV 2E'V}}\,\frac{1}{r\varkappa\sqrt{(qFFq)}}\Big\{\left[\bar{u}(p)\gamma_\mu u(-p')\right](\tilde{F}q)_\mu\,\mathrm{Ai}(y) \\ &\quad+\mathrm{i}\left[\bar{u}(p)\gamma_5\gamma_\mu u(-p')\right]\kappa_+(FFq)_\mu\left(-\frac{\mathrm{i}}{r\varkappa}\,\mathrm{Ai}'(y) - \varphi_0\,\mathrm{Ai}(y)\right)\Big\}. \end{aligned} \tag{5.35}$$

The photon decay probability is defined as

$$W = \frac{1}{T}\int |S_{if}|^2\frac{\mathrm{d}^3p\,V}{(2\pi)^3}\,\frac{\mathrm{d}^3p'V}{(2\pi)^3}. \tag{5.36}$$

Substituting the matrix element, one should take into account, that, as usual,

$$\delta^2(\mathbf{Q}_\perp = 0) = \frac{L_yL_z}{(2\pi)^2}, \quad \delta(kQ = 0) = \frac{T}{2\pi k_0},$$

where L_x, L_y, L_z are the typical scales along the axes OX, OY, and OZ, and T is the total interaction time.

Integration over the positron momenta with the δ functions yields

$$\int\frac{\mathrm{d}^3p'}{E'}\delta^2(\mathbf{Q}_\perp)\delta(kQ)\{\dots\} = \frac{\varkappa}{m_e^2\chi_2}\,\{p' \to q - p - sk;\ \chi_2 \to \chi - \chi_1\}.$$

For the integration over the electron momenta it is convenient to insert the variables τ and u as follows

$$\tau = \frac{e(q\tilde{F}p)}{m_e^4\chi}, \quad u = 1 - 2\frac{\chi_1}{\chi}. \tag{5.37}$$

In this case

$$\chi_1 = \frac{1-u}{2}\chi, \quad \chi_2 = \frac{1+u}{2}\chi, \tag{5.38}$$

and we can write

$$\int \frac{d^3 p}{E} \frac{1}{\chi_2} = \frac{2m_e^2 \varkappa}{\chi} \int_{-1}^{1} \frac{du}{1-u^2} \int_{-\infty}^{\infty} d\tau \int d\varphi_0.$$

However, as the calculation shows, the integrand does not depend on φ_0. If the connection between φ and x is recalled, we can conclude that the integral with respect to φ_0 represents an arbitrariness of the choice of the zero point for the x coordinate. Analyzing a problem within the finite quantization volume $V = L_x L_y L_z$, we should obviously take the integration region over φ_0 to be finite and equal to $k_0 L_x$, i.e.

$$\int d\varphi_0 = k_0 \int dx_0 = k_0 L_x.$$

The argument (5.33) of the Airy function in the notations (5.37) has the form

$$y = r^2(\tau^2 + 1), \qquad r = \left(\frac{2}{\chi(1-u^2)}\right)^{1/3}. \tag{5.39}$$

The result of calculation of the decay probabilities for the photons of both polarizations (5.26) can be represented in the form

$$\begin{aligned} W^{(1,2)} = \frac{e^2 m_e^2 \chi^{1/3}}{2^{1/3}\pi\omega} \int_0^1 \frac{du}{(1-u^2)^{2/3}} \int_{-\infty}^{\infty} d\tau \Bigg\{ \left(\frac{1+u^2}{2} \mp \frac{1-u^2}{2}\right) \left[\mathrm{Ai}'(y)\right]^2 \\ + \left(\frac{2}{\chi(1-u^2)}\right)^{2/3} \left[1 + \tau^2 \left(\frac{1+u^2}{2} \pm \frac{1-u^2}{2}\right)\right] [\mathrm{Ai}(y)]^2 \Bigg\}. \end{aligned} \tag{5.40}$$

This result coincides, to the notations, with the result of [9], where the polarizations $\parallel$ and $\perp$ correspond to our 1 and 2.

To calculate the integrals with respect to the τ variable, which are involved in (5.40),

$$I_1 = \int_{-\infty}^{\infty} d\tau \, [\mathrm{Ai}(y)]^2, \quad I_2 = \int_{-\infty}^{\infty} d\tau \tau^2 \, [\mathrm{Ai}(y)]^2, \quad I_3 = \int_{-\infty}^{\infty} d\tau \left[\mathrm{Ai}'(y)\right]^2, \tag{5.41}$$

we use the known relations for the Airy function; see [9]:

$$y\,[\mathrm{Ai}(y)]^2 + \left[\mathrm{Ai}'(y)\right]^2 = \frac{1}{2} \frac{d^2}{dy^2} [\mathrm{Ai}(y)]^2, \tag{5.42}$$

$$\int_0^\infty \frac{\mathrm{d}t}{\sqrt{t}}\left[\mathrm{Ai}(t+a)\right]^2 = \frac{1}{2}\int_{2^{2/3}a}^\infty \mathrm{d}y\mathrm{Ai}(y), \tag{5.43}$$

$$\int_0^\infty \mathrm{d}t t^\sigma \left[\mathrm{Ai}(t+a)\right]^2 = \frac{\sigma}{2(2\sigma+1)}\left(\frac{\mathrm{d}^2}{\mathrm{d}a^2} - 4a\right)\int_0^\infty \mathrm{d}t t^{\sigma-1}\left[\mathrm{Ai}(t+a)\right]^2, \ \sigma > 0. \tag{5.44}$$

For the integrals (5.41) we obtain

$$\begin{aligned} I_1 &= \frac{1}{2r}\mathrm{Bi}(z), \quad I_2 = \frac{2^{1/3}}{8r^3}\left[-\mathrm{Ai}'(z) - z\mathrm{Bi}(z)\right], \\ I_3 &= \frac{2^{1/3}}{8r}\left[-3\mathrm{Ai}'(z) - z\mathrm{Bi}(z)\right], \end{aligned} \tag{5.45}$$

where

$$\mathrm{Bi}(z) = \int_z^\infty \mathrm{d}y\mathrm{Ai}(y), \qquad z = 2^{2/3}r^2 = \left(\frac{4}{\chi(1-u^2)}\right)^{2/3}. \tag{5.46}$$

Inserting the integrals (5.45) and turning to a new variable $v = 1/(1-u^2)$, we present the probability (5.40) in the form

$$W^{(1,2)} = \frac{\alpha m_e^2}{2\omega}\int_1^\infty \frac{\mathrm{d}v}{v\sqrt{v(v-1)}}\left\{\mathrm{Bi}(z) - \frac{4v-2\mp 1}{z}\mathrm{Ai}'(z)\right\}, \tag{5.47}$$

where $z = (4v/\chi)^{2/3}$. The expression (5.47) can be further simplified by using the Eq. (5.29) for the Airy function. We obtain

$$W^{(1,2)} = -\frac{\alpha m_e^2\chi}{16\,\omega}\int_{(4/\chi)^{2/3}}^\infty \frac{\mathrm{d}z}{\sqrt{z}}\frac{8v+1\mp 3}{v\sqrt{v(v-1)}}\mathrm{Ai}'(z), \qquad v = \frac{\chi z^{3/2}}{4}. \tag{5.48}$$

The formulae for the probability are simplified significantly in the two limiting cases: for small values of the dynamical parameter χ

$$W^{(1,2)}(\chi) = \sqrt{\frac{3}{2}}\frac{(3\mp 1)\alpha m_e^2}{16\omega}\chi\,\mathrm{e}^{-8/3\chi}, \qquad \chi \ll 1, \tag{5.49}$$

and for large dynamical parameter

$$W^{(1,2)}(\chi) = \frac{3(5\mp 1)\Gamma^4(2/3)\alpha m_e^2}{28\pi^2\omega}(3\chi)^{2/3}, \qquad \chi \gg 1. \tag{5.50}$$

Here $\Gamma(z)$ is the gamma function, $\Gamma(2/3) = 1.354\ldots$. The presented expressions for the probabilities coincide, to the notations, with corresponding formulas of [9].

5.2.2 Calculation Based on the Imaginary Part of the Loop Amplitude

Similarly to Sect. 5.1.2, the decay probability in a crossed field can be calculated via the unitarity relation. For this purpose the expression (4.35) should be substituted as the amplitude $\mathcal{M}(\gamma \to \gamma)$ into (5.16), replacing the vector current by the photon polarization vectors (5.26), $j_{V\alpha} \to e\varepsilon_\alpha^{(1,2)}$, and setting $q^2 = 0$. We obtain

$$\begin{aligned}
\mathcal{M}(\gamma^{(1)} \to \gamma^{(1)}) &= \frac{\alpha}{\pi}\, Y_{VV}^{(1)} \\
&= -\frac{\alpha m_e^2 \chi^{2/3}}{6\pi} \int\limits_0^1 \mathrm{d}u \left(\frac{4}{1-u^2}\right)^{1/3} (3+u^2) \frac{\mathrm{d}f(z)}{\mathrm{d}z}, \\
\mathcal{M}(\gamma^{(2)} \to \gamma^{(2)}) &= \frac{\alpha}{\pi}\, Y_{VV}^{(2)} \\
&= -\frac{\alpha m_e^2 \chi^{2/3}}{3\pi} \int\limits_0^1 \mathrm{d}u \left(\frac{4}{1-u^2}\right)^{1/3} (3-u^2) \frac{\mathrm{d}f(z)}{\mathrm{d}z},
\end{aligned} \tag{5.51}$$

$$z = \left(\frac{4}{\chi(1-u^2)}\right)^{2/3}. \tag{5.52}$$

Keeping in mind that the imaginary part of the Hardy–Stokes function is expressed via the Airy function,

$$\mathrm{Im}\, f(z) = \pi\, \mathrm{Ai}(z), \tag{5.53}$$

and changing the variable u to z in the integral

$$\int\limits_0^1 \mathrm{d}u = \frac{3}{4} \int\limits_{(4/\chi)^{2/3}}^{\infty} \frac{\mathrm{d}z}{z} \frac{1}{\sqrt{v(v-1)}},$$

where $v = \chi z^{3/2}/4$, we readily obtain the formula (5.48).

5.3 Photon Emission by Electron in a Strong Magnetic Field

Photon emission by an electron in an external electromagnetic field, $e \to e + \gamma$, is the crossed process to the photon decay into the pair $e^- e^+$. Therefore, it is described by the same diagram, Fig. 5.1, with the replacement $p \to -p, q \to -q$.

The S matrix element (5.2) for this process can be written in the form

$$S_{if} = \frac{\mathrm{i}e}{\sqrt{2\omega V}} \int (\overline{\Psi} \hat{\varepsilon}^{(\lambda)} \Psi) \mathrm{e}^{\mathrm{i}qx} \mathrm{d}^4 x, \tag{5.54}$$

where Ψ and $\bar{\Psi}$ correspond to the solutions of the Dirac equation in a magnetic field with positive energy (2.30)–(2.32), ω is the photon energy.

It should be noted that the photon emission process is impossible when the initial electron occupy the ground Landau level. To see this, it is enough to make the Lorentz transformation to the rest frame of the initial electron ($p_z = 0$) where its energy is equal to its mass. In another case, when both initial and final electrons occupy the first Landau level, and in the same frame, where $p_z = 0$, the energy conservation law taking the form $\sqrt{2eB + m_e^2} = \sqrt{2eB + m_e^2 + p_z'^2} + \omega$, obviously cannot be valid for the nonzero energy of the photon. Only the process is possible where the electron emitting the photon, passes from the first Landau level into the ground one. In a general case, only the processes could be realized where the electron passes into a lower Landau level.

Let us consider the case when the field is strong enough and the electrons, which are relativistic, can occupy only the ground and the first Landau levels. It is just the case when the electron emitting the photon passes from the first Landau level into the ground one. The energy of the relativistic electron in a magnetic field is (see (2.24)):

$$E_n \simeq \sqrt{p_z^2 + 2n\beta}.$$

The first Landau level ($n = 1$) is doubly degenerate because two spin states exist, $s = -1$ and $s = +1$.

It is convenient for further calculations to take the frame where the p_z component of the initial electron momentum is equal to zero. In this frame $p_z = 0,\ E \simeq \sqrt{2eB}$, and the wave functions describing the state of relativistic electrons that occupy the first Landau level, takes the following form, according to (2.30)–(2.32):

$$\Psi_{s=+1}^{(+)} = \left(\frac{eB}{\pi}\right)^{1/4} \frac{u_{p,s=+1}}{\sqrt{2L_y L_z}}\, \mathrm{e}^{-\xi^2/2} \mathrm{e}^{-\mathrm{i}(Et - p_y y)}. \tag{5.55}$$

$$\Psi_{s=-1}^{(+)} = \left(\frac{eB}{\pi}\right)^{1/4} \frac{u_{p,s=-1}}{\sqrt{2L_y L_z}}\, \mathrm{e}^{-\xi^2/2} \mathrm{e}^{-\mathrm{i}(Et - p_y y)}. \tag{5.56}$$

$$u_{p,s=+1} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \mathrm{i}\sqrt{2}\xi \end{pmatrix}, \qquad u_{p,s=-1} = \begin{pmatrix} 0 \\ \sqrt{2}\xi \\ -\mathrm{i} \\ 0 \end{pmatrix}.$$

Substituting the wave functions of the initial state (5.55) and (5.56) and of the final state (5.53) into the expression (5.54), we obtain the matrix elements S_{if} corresponding to the two projections of the initial electron spin on the field direction,

$$S_{if,s=\pm 1} = \frac{\mathrm{i}e(eB/\pi)^{1/2}}{2L_y L_z\sqrt{2\omega V}} \int (\bar{u}_{p'}\hat{\varepsilon}^{(\lambda)} u_{p,s=\pm 1})\mathrm{e}^{-\xi'^2/2}\mathrm{e}^{-\xi^2/2} \times \mathrm{e}^{\mathrm{i}qx}\mathrm{e}^{-\mathrm{i}(Et-p_y y)}\mathrm{e}^{\mathrm{i}(E't-p'_y y-p'_z z)}\mathrm{d}^4x, \tag{5.57}$$

where

$$\xi = \sqrt{eB}\left(x + \frac{p_y}{eB}\right), \; \xi' = \sqrt{eB}\left(x + \frac{p'_y}{eB}\right).$$

By choosing the coordinate axes in such a manner that the vector of the photon momentum would have the form $\mathbf{q} = (q_x, 0, q_z)$, the integration with respect to x in the expression S_{if} can be easily performed. In this frame we have $p_y = p'_y$ and $\xi = \xi'$, and the matrix element S_{if} is transformed to the form

$$S_{if,s=\pm 1} = \frac{\mathrm{i}e(eB/\pi)^{1/2}}{2L_y L_z\sqrt{2\omega V}}(2\pi)^3\delta^3(q + p' - p) \times \int (\bar{u}_{p'}\hat{\varepsilon}^{(\lambda)} u_{p,s=\pm 1})\mathrm{e}^{-\mathrm{i}q_x x}\mathrm{e}^{-\xi^2}\mathrm{d}x, \tag{5.58}$$

where $\delta^3(q + p' - p) = \delta(\omega + E' - E)\,\delta(p'_y - p_y)\,\delta(q_z + p'_z)$ and the integration over $\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}z$ is taken.

Calculating the values $(\bar{u}_{p'}\hat{\varepsilon}^{(\lambda)} u_p)$ for the initial electron with $s = +1$ and for the photon 1- and 2-modes (4.10), we obtain

$$\bar{u}_{p'}\hat{\varepsilon}^{(1)} u_{p,s=+1} = \frac{\mathrm{i}\eta q_x}{\sqrt{q_\perp^2}}, \qquad \bar{u}_{p'}\hat{\varepsilon}^{(2)} u_{p,s=+1} = \frac{\mathrm{i}\sqrt{2}\xi}{\sqrt{q_\parallel^2}}(\eta q_z - \omega), \tag{5.59}$$

where $\eta = p'_z/|p'_z|$. Note that in this frame $q_\perp^2 = q_x^2$. For the initial electron with the spin projection $s = -1$, we obtain

$$\bar{u}_{p'}\hat{\varepsilon}^{(1)} u_{p,s=-1} = -\frac{q_x}{\sqrt{q_\perp^2}}, \qquad \bar{u}_{p'}\hat{\varepsilon}^{(2)} u_{p,s=-1} = -\frac{\sqrt{2}\xi}{\sqrt{q_\parallel^2}}(q_z - \eta\omega). \tag{5.60}$$

The remaining integration with respect to x in (5.58) is reduced to the Gaussian integral, and the calculation of the S matrix elements yields

$$S^{(1)}_{if,s=+1} = \frac{\mathrm{i}e(2\pi)^3\delta^3(q+p'-p)}{2L_yL_z\sqrt{2\omega V}}\frac{\mathrm{i}\eta q_x}{\sqrt{q_\perp^2}}\mathrm{e}^{-q_x^2/4eB}\mathrm{e}^{\mathrm{i}q_xp_y/eB},$$
$$S^{(2)}_{if,s=+1} = \frac{\mathrm{i}e(2\pi)^3\delta^3(q+p'-p)}{2L_yL_z\sqrt{2\omega V}}\frac{(\eta q_z-\omega)}{\sqrt{q_\parallel^2}}\frac{(-\mathrm{i}q_x)}{\sqrt{2eB}}\mathrm{e}^{-q_x^2/4eB}\mathrm{e}^{iq_xp_y/eB},$$
$$S^{(1,2)}_{if,s=-1} = \mathrm{i}\eta S^{(1,2)}_{if,s=+1}. \tag{5.61}$$

Returning into a more general frame where $\mathbf{q} = (q_x, q_y, q_z)$, let us write the matrix elements squared

$$\left|S^{(1)}_{if,s=+1}\right|^2 = \left|S^{(1)}_{if,s=-1}\right|^2 = \frac{e^2(2\pi)^3T}{8L_yL_z\omega V}\delta^3(q+p'-p)\mathrm{e}^{-q_\perp^2/2eB}, \tag{5.62}$$
$$\left|S^{(2)}_{if,s=+1}\right|^2 = \left|S^{(2)}_{if,s=-1}\right|^2 = \frac{e^2(2\pi)^3T(\eta q_z-\omega)^2}{8L_yL_z\omega V\, 2eB}\frac{q_\perp^2}{q_\parallel^2} \times \delta^3(q+p'-p)\mathrm{e}^{-q_\perp^2/2eB}, \tag{5.63}$$

where $\delta^3(q+p'-p) = \delta(E-E'-\omega)\delta(p'_z+q_z)\delta(q_y+p'_y-p_y)$. Thus, the probability of the photon emission is seen to be independent on the polarization of the initial electron. To find the total probability of the photon emission, the integration over the phase space of final particles should be performed:

$$W = \int \frac{|S_{if}|^2}{T}\frac{\mathrm{d}^3qV}{(2\pi)^3}\frac{\mathrm{d}p'_y\mathrm{d}p'_zL_yL_z}{(2\pi)^2}. \tag{5.64}$$

Upon integrating (5.64) over $\mathrm{d}p'_y\,\mathrm{d}p'_z$ with (5.62) and (5.63) taken into account we obtain that the emission probabilities of the photons of the two modes, $\lambda = 1, 2$, coincide at $q^2 = 0$:

$$W^{(1)} = W^{(2)} \equiv W = \frac{\alpha}{8\pi}\int\frac{\mathrm{d}^2q_\perp\mathrm{d}q_z}{\omega}\mathrm{e}^{-q_\perp^2/2eB}\delta(E-|q_z|-\omega). \tag{5.65}$$

The δ function for energies can be presented as follows

$$\frac{\delta(E-|q_z|-\omega)}{\omega} = \frac{\delta(q_z+q^*)+\delta(q_z-q^*)}{\sqrt{2eB}},$$

where $q^* = (2eB - q_\perp^2)/(2\sqrt{2eB})$ defines the absolute value of q_z. From the condition $q^* > 0$ we find the integration limits over the $q_\perp^2$ variable,

$$0 < q_\perp^2 < 2eB.$$

Finally, we obtain

$$W = \frac{\alpha}{4\sqrt{2eB}} \int_0^{2eB} \mathrm{e}^{-q_\perp^2/2eB} \mathrm{d}q_\perp^2 = \frac{\alpha}{4}\sqrt{2eB}(1 - \mathrm{e}^{-1}). \tag{5.66}$$

The total probability of the process

$$e^- \to e^- + \gamma$$

averaged over the polarizations of the initial electron and summarized over polarizations of the final photon, in the frame where $p_z = 0$, is

$$\overline{W} = \frac{\alpha}{2}\sqrt{2eB}(1 - \mathrm{e}^{-1}). \tag{5.67}$$

Taking account of the Lorentz invariance of the product of the probability by the initial electron energy, we can rewrite the expression (5.67) to the arbitrary frame, to obtain

$$\overline{W} = \frac{\alpha eB}{\sqrt{p_z^2 + 2eB}}(1 - \mathrm{e}^{-1}). \tag{5.68}$$

5.4 Electromagnetic Interactions of the Dirac Neutrino with a Magnetic Moment

Throughout this section, we use the notation μ_ν for the magnetic moment of a neutrino, and the notation $\tilde{\mu}_\nu$ for a chemical potential of the neutrino gas.

5.4.1 Magnetic Moment of the Dirac Neutrino and its Astrophysical Manifestations

Nonvanishing neutrino magnetic moment leads to various chirality-flipping processes where the left-handed neutrinos produced in the stellar interior become the right-handed ones, i.e. sterile with respect to the weak interaction, and this can be important e.g. for the stellar energy-loss. In the standard model extended to include the neutrino mass m_ν, the well-known result for the neutrino magnetic moment is [10, 11]:

$$\mu_\nu^{(SM)} = \frac{3e\, G_\mathrm{F}\, m_\nu}{8\pi^2\sqrt{2}} = 3.20 \times 10^{-19} \left(\frac{m_\nu}{1\,\mathrm{eV}}\right) \mu_\mathrm{B}\,, \tag{5.69}$$

where $\mu_B = e/2m_e$ is the Bohr magneton. Thus, it is unobservably small given the known limits on neutrino masses. On the other hand, nontrivial extensions of the standard model such as left-right symmetry [12–19] can lead to more significant values for the neutrino magnetic moment [20–22].

First attempts of exploiting the mechanism of the neutrino chirality flipping were connected with the solar neutrino problem, and two different scenarios were analysed. The first one, based on the neutrino magnetic moment rotation in a stellar magnetic field, was investigated in the papers [23–25]. In the second scenario, a neutrino changed the chirality due to the electromagnetic interaction of its magnetic moment with plasma [26, 27]. For a more extended list of references see, e.g., [28]. In all these cases the effect appeared to be small to have an essential impact on the solar neutrino problem, if $\mu_\nu < 10^{-10}\,\mu_B$.

More stringent constraints on μ_ν are provided by other stars. For example, the cores of low-mass red giants are about 10^4 times denser than the Sun, and nonstandard neutrino losses would have a more essential effect there, delaying the ignition of heluim. Thus, the limit was obtained [29, 30]:

$$\mu_\nu < 0.3 \times 10^{-11}\,\mu_B\,. \tag{5.70}$$

An independent constraint on the magnetic moment of a neutrino was also obtained from the Early Universe [31, 32]:

$$\mu_\nu < 6.2 \times 10^{-11}\,\mu_B\,, \tag{5.71}$$

where spin-flip collisions would populate the sterile Dirac components in the era before the decoupling of the neutrinos. Thus, it doubles the effective number of thermally excited neutrino degrees of freedom and increases the expansion rate of the Universe, causing the overabundance of helium.

Interest in possible astrophysical and cosmological manifestations of the neutrino magnetic moment stimulated experiments on its measurement in laboratory conditions. The best constraint was obtained in the GEMMA experiment to study the scattering of antineutrinos by electrons carried out at the Kalinin nuclear power station by the collaboration of the Institute of Theoretical and Experimental Physics (Moscow) and the Joint Institute for Nuclear Research (Dubna). The upper bound for the neutrino magnetic moment was [33]:

$$\mu_\nu < 3.2 \times 10^{-11}\,\mu_B\,. \tag{5.72}$$

A considerable interest to the neutrino magnetic moment arised after the great event of SN1987A, in connection with the modelling of a supernova explosion, where gigantic neutrino fluxes define in fact the process energetics. It means that such a microscopic neutrino characteristic, as the neutrino magnetic moment, would have a critical influence on macroscopic properties of these astrophysical events. Namely, the left-handed neutrinos produced inside the supernova core during the collapse, could convert into the right-handed neutrinos due to the magnetic moment interaction with a virtual plasmon γ^* that can be both produced and absorbed:

$$\nu_L \to \nu_R + \gamma^*, \quad \nu_L + \gamma^* \to \nu_R\,. \tag{5.73}$$

These sterile neutrinos would escape from the core leaving no energy to explain the observed neutrino luminosity of the supernova. Thus, the upper bound on the neutrino magnetic moment can be established.

This matter was investigated by many authors in different aspects [34–38]. The authors [36] considered the neutrino spin-flip via both $\nu_L e^- \to \nu_R e^-$ and $\nu_L p \to \nu_R p$ scattering processes in the inner core of a supernova immediately after the collapse. Imposing for the ν_R luminosity Q_{ν_R} the upper limit of 10^{53} ergs/s, the authors obtained the upper bound on the neutrino magnetic moment:

$$\mu_\nu < (0.2 - 0.8) \times 10^{-11}\, \mu_{\rm B}\,. \tag{5.74}$$

However, the essential plasma polarization effects in the photon propagator were not considered in Ref. [36], and the photon dispersion was taken in a phenomenolical way, by inserting an *ad hoc* thermal mass into the vacuum photon propagator. A detailed investigation of this question was performed in Refs. [39, 40], where the formalism was used of the thermal field theory to take into account the influence of hot dense astrophysical plasma on the photon propagator. The upper bound on the neutrino magnetic moment compared with the result of the paper [36] was improved in Refs. [39, 40] by the factor of 2:

$$\mu_\nu < (0.1 - 0.4) \times 10^{-11}\, \mu_{\rm B}\,. \tag{5.75}$$

However, looking at the intermediate analytical results of the authors [39, 40], one can see that only the contribution of plasma electrons was taken into account there, while the proton fraction was omitted. This is despite the fact that the electron and proton contributions to the neutrino spin flip process were evaluated in Ref. [36] to be of the same order. It should be mentioned also that the improvement of the bound (5.75) with respect to the bound (5.74) was based in part on the enhancement by the factor of 2 of the supernova core volume made in Refs. [39, 40] if compared with Ref. [36], while the density was taken to be the same, $\rho_c \simeq 8 \times 10^{14}$ g cm^{-3}. This means that the core mass appeared to be in Ref. [39, 40] of the order of 3 $M_\odot$, which is nearly twice the mass of the supernova remnant believed to be typical.

The neutrino spin flip processes in the supernova core was reconsider more attentively in Refs. [41–43]. It was shown in part, that the proton contribution into the photon propagator was not less essential, than the electron contribution. In this section, we reproduce that analysis. We consider the Dirac neutrinos only, because in this case the neutrino magnetic moment interaction (both diagonal and non-diagonal) with a photon transforms the active left-handed neutrinos into the right-handed neutrinos which are sterile with respect to the weak interaction. We do not consider the Majorana neutrinos, because the produced right-handed antineutrino states are not sterile in this case.

The amplitude of the helicity flip through the scattering by plasma components is calculated. A general expression for the creation probability of right-handed neutrinos with a fixed energy is derived. We estimate the core luminosity with respect to the emission of neutrinos ν_R and obtain an upper limit on the neutrino magnetic moment by taking into account the radial distributions and time evolution of physical parameters.

5.4.2 Neutrino Interaction with Background

The neutrino chirality flip is caused by the scattering via the intermediate photon (plasmon) off the plasma electromagnetic current presented by electrons, $\nu_L e^- \to \nu_R e^-$, protons, $\nu_L p \to \nu_R p$, etc. The total process Lagrangian consists of two parts, the first one is the interaction of a neutrino having a magnetic moment μ_ν^{ij} (both diagonal and transition) with photons, while the second part describes the plasma interaction with photons:

$$\mathcal{L} = -\frac{\mathrm{i}}{2}\sum_{i,j}\mu_\nu^{ij}\left(\bar{\nu}_j\sigma_{\alpha\beta}\nu_i\right)F^{\alpha\beta} - e\,J_\alpha\,A^\alpha\,, \tag{5.76}$$

where $\sigma_{\alpha\beta} = (1/2)\,(\gamma_\alpha\gamma_\beta - \gamma_\beta\gamma_\alpha)$, $F^{\alpha\beta}$ is the tensor of the photon electromagnetic field, $J_\alpha = -\,(\bar{e}\gamma_\alpha e) + (\bar{p}\gamma_\alpha p) + \cdots$ is an electromagnetic current in the general sense, formed by different components of the medium, i.e. free electrons and positrons, protons, free ions, etc.

The neutrino magnetic moment is generally a matrix $\mu_{\nu_i\nu_j} \equiv \mu_\nu^{ij}$ that contains both diagonal and transition magnetic moments, where ν_i and ν_j are the states of a neutrino with a specific mass. The neutrino states ν_ℓ with specific flavors $\ell = e, \mu, \tau$ being produced in weak processes are superpositions of states ν_i:

$$\nu_\ell = \sum_i U_{\ell i}^*\nu_i\,, \tag{5.77}$$

where $U_{\ell i}$ is the unitary Pontecorvo–Maki–Nakagawa–Sakata leptonic mixing matrix [44–47]. Below, for simplicity, we will consider the diagonal neutrino magnetic moment μ_ν. The extension to the general case of the matrix of magnetic moments μ_ν^{ij} presents no difficulty and consists in the following: the magnetic moment in all of the succeeding expressions should be considered as an effective value. For example, for the processes with initial electron neutrinos, by μ_ν we should mean

$$\mu_\nu \to \mu_{\nu_e} \equiv \left(\sum_i\left|\sum_j \mu_\nu^{ij}U_{ej}\right|^2\right)^{1/2}. \tag{5.78}$$

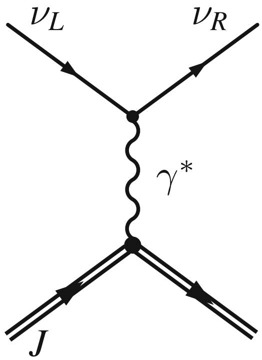

Fig. 5.2 The Feynman diagram for the neutrino spin-flip scattering via the intermediate plasmon γ^* on the plasma electromagnetic current J

and a similar quantity for initial muon and tau neutrinos.

With the Lagrangian (5.76), the process is described by the Feynman diagram shown in Fig. 5.2.

The technics of calculations of the neutrino spin-flip rate is rather standard. The invariant amplitude for the process of the neutrino scattering off the k-th plasma component can be written in the form

$$\mathcal{M}^{(k)} = -\mathrm{i}\, e\, \mu_\nu\, j^\alpha_{(\nu)}\, G_{\alpha\beta}(Q)\, J^\beta_{(k)}\,, \tag{5.79}$$

where $j^\alpha_{(\nu)}$ is the Fourier transform of the neutrino magnetic moment current,

$$j^\alpha_{(\nu)} = \left[\bar{\nu}_R(p')\, \sigma^{\mu\alpha}\, \nu_L(p)\right] Q_\mu\,,$$

$J^\beta_{(k)}$ is the Fourier transform of the k-th plasma component electromagnetic current, and $Q = (q_0, \mathbf{q})$ is the four-momentum transferred. The only principal point is to use the photon propagator $G_{\alpha\beta}(Q)$ with the plasma polarization effects taken into account, see Sect. 4.4.

5.4.3 The Rate of Creation of the Right-Handed Neutrino

The value of physical interest is the rate of creation of the right-handed neutrino ν_R, $\Gamma_{\nu_R}(E')$, with the fixed energy E' by all the left-handed neutrinos. This function can be obtained by integration of the amplitude (5.79) squared over the states of the initial left-handed neutrinos and over the states of the initial and final plasma particles forming the electromagnetic current $J^\beta_{(k)}$

$$\varGamma_{\nu_R}(E') = \sum_k \varGamma_{\nu_R}^{(k)}(E') \,, \tag{5.80}$$

$$\varGamma_{\nu_R}^{(k)}(E') = \frac{1}{16\,(2\pi)^5\,E'} \int \sum_{s,s'} |\mathcal{M}^{(k)}|^2\,\delta^{(4)}(p' + \mathcal{P}' - p - \mathcal{P}) \\ \times \frac{\mathrm{d}^3\mathbf{P}}{\mathcal{E}}\,f_k(\mathcal{E})\,\frac{\mathrm{d}^3\mathbf{P}'}{\mathcal{E}'}\,\big[1 \mp f_k(\mathcal{E}')\big]\,\frac{\mathrm{d}^3\mathbf{p}}{E}\,f_\nu(E) \,. \tag{5.81}$$

Here, $p^\alpha = (E, \mathbf{p})$ and $p'^\alpha = (E', \mathbf{p}')$ are the four-momenta of the initial and final neutrinos, $\mathcal{P}^\alpha = (\mathcal{E}, \mathbf{P})$ and $\mathcal{P}'^\alpha = (\mathcal{E}', \mathbf{P}')$ are the four-momenta of the initial and final plasma particles; $\sum_{s,s'}$ means the summation over the spins of these particles, the index $k = e, p, i, \ldots$ corresponds to the type of the plasma particles (electrons, protons, free ions, etc.) with the distribution function $f_k(\mathcal{E})$, which can be both fermions (the upper sign in $\big[1 \mp f_k(\mathcal{E}')\big]$) and bosons (the lower sign); $f_\nu(E) = \big(\mathrm{e}^{(E-\tilde{\mu}_\nu)/T} + 1\big)^{-1}$ is the Fermi—Dirac distribution function for the initial left-handed neutrinos in the plasma restframe.

It is convenient to pass in Eq. (5.81) from integration over the initial neutrino momentum $\mathbf{p}$ to the integration over the virtual plasmon momentum $p - p' = Q = (q_0, \mathbf{q})$, $|\mathbf{q}| \equiv q$, using the relation:

$$\frac{\mathrm{d}^3\mathbf{p}}{E}\,f_\nu(E) = \frac{2\,\pi}{E'}\,q\,\mathrm{d}q\,\mathrm{d}q_0\,\theta(-Q^2)\,\theta(2E' + q_0 - q)\,f_\nu(E' + q_0) \,.$$

Substituting the amplitude (5.79) squared into Eq. (5.81), one obtains

$$\varGamma_{\nu_R}(E') = \frac{\mu_\nu^2}{8\,\pi^2\,E'^2} \int\limits_{-E'}^{\infty} \mathrm{d}q_0 \int\limits_{|q_0|}^{2E'+q_0} q\,\mathrm{d}q\,f_\nu(E' + q_0)\,j^\alpha_{(\nu)}\,j^{\alpha'*}_{(\nu)} \\ \times \sum_{\lambda,\lambda'} \frac{\rho_{\alpha\beta}{}^{(\lambda)}\,\rho_{\alpha'\beta'}{}^{(\lambda')}}{(Q^2 - \Pi_\lambda)\,(Q^2 - \Pi^*_{\lambda'})}\,T^{\beta\beta'} \,, \tag{5.82}$$

where the following tensor integral is introduced:

$$T^{\alpha\beta} = \frac{e^2}{32\,\pi^2} \sum_k \sum_{s,s'} \int J^\alpha_{(k)}\,J^{\beta*}_{(k)}\,\mathrm{d}\,\varPhi \,, \tag{5.83}$$

$$\mathrm{d}\,\varPhi = \frac{\mathrm{d}^3\mathbf{P}\,\mathrm{d}^3\mathbf{P}'}{\mathcal{E}\,\mathcal{E}'}\,f_k(\mathcal{P})\,\big[1 \mp f_k(\mathcal{P}')\big]\,\delta^{(4)}(\mathcal{P}' - \mathcal{P} - Q) \,.$$

Further, we present the detailed calculation of the tensor $T^{\alpha\beta}$. To use the covariant properties of this tensor, one should write the distribution functions $f_k(\mathcal{P})$ in the arbitrary frame

$$f_k(\mathcal{P}) = \left[\exp\frac{(\mathcal{P}u) - \tilde{\mu}}{T} \pm 1\right]^{-1}, \tag{5.84}$$

where u_α is the four-vector of the plasma velocity. This vector and the four-vector Q_α are the building bricks for constructing the tensor $T^{\alpha\beta}$. This tensor is symmetric because the electromagnetic current $J^\alpha_{(k)}$ is real. The tensor is also orthogonal to the four-vector Q_α because of the electromagnetic current conservation. There exist only two independent structures having these properties, which are the density matrices (4.42) and (4.43), and thus one can write:

$$T^{\alpha\beta} = \mathcal{A}^{(t)}\,{\rho_{\alpha\beta}}^{(t)} + \mathcal{A}^{(\ell)}\,{\rho_{\alpha\beta}}^{(\ell)}\,. \tag{5.85}$$

Because of orthogonality of the tensors ${\rho_{\alpha\beta}}^{(t)}$ and ${\rho_{\alpha\beta}}^{(\ell)}$, see Eq. (4.45), one obtains

$$\mathcal{A}^{(t)} = \frac{1}{2}\,T^{\alpha\beta}\,{\rho_{\alpha\beta}}^{(t)} = \frac{e^2}{64\,\pi^2}\,{\rho_{\alpha\beta}}^{(t)}\sum_k\sum_{s,s'}\int J^\alpha_{(k)}J^{\beta *}_{(k)}\,\mathrm{d}\,\Phi\,, \tag{5.86}$$

$$\mathcal{A}^{(\ell)} = T^{\alpha\beta}\,{\rho_{\alpha\beta}}^{(\ell)} = \frac{e^2}{32\,\pi^2}\,{\rho_{\alpha\beta}}^{(\ell)}\sum_k\sum_{s,s'}\int J^\alpha_{(k)}J^{\beta *}_{(k)}\,\mathrm{d}\,\Phi\,. \tag{5.87}$$

As we show below, just these integrals (5.86) and (5.87) define the widths of absorption (at $q_0 > 0$) and creation (at $q_0 < 0$) of a plasmon by the plasma particles. Really, let us consider for definiteness the width of absorption of the transversal plasmon by plasma particles forming the electromagnetic current $J^\beta_{(k)}$. The amplitude of the process has the form

$$\mathcal{M}^{(k)(t)} = -e\,{\varepsilon_\alpha}^{(t)}\,J^\alpha_{(k)}\,. \tag{5.88}$$

where ${\varepsilon_\alpha}^{(t)}$ is the unit polarization four-vector. Performing standard calculations, one obtains for the width of the plasmon absorption by all the components of plasma:

$$\Gamma^{abs}_{(t)} = \frac{1}{32\,\pi^2\,q_0}\,\frac{1}{2}\sum_\tau\sum_k\sum_{s,s'}\int|\mathcal{M}^{(k)(t)}|^2\,\mathrm{d}\,\Phi\,, \tag{5.89}$$

where the summation is made both over the kth types of the plasma particles and over the polarizations of all particles participating in the process, τ for a plasmon and s, s' for plasma particles.

Substituting the amplitude (5.88) into (5.89),

$$\Gamma^{abs}_{(t)} = \frac{e^2}{64\,\pi^2\,q_0}\,{\rho_{\alpha\beta}}^{(t)}\sum_k\sum_{s,s'}\int J^\alpha_{(k)}J^{\beta *}_{(k)}\,\mathrm{d}\,\Phi\,, \tag{5.90}$$

where ${\rho_{\alpha\beta}}^{(t)} = \sum\limits_{\tau=1}^{2} \varepsilon_\alpha^{\tau(t)}\, \varepsilon_\beta^{\tau(t)}$, and comparing it with Eq. (5.86), one can find the value

$$\mathcal{A}^{(t)} = q_0\, \Gamma^{abs}_{(t)} \,. \tag{5.91}$$

Using the known relation [48] between the width of absorption of the transversal plasmon and the imaginary part I_t of the eigenvalue Π_t of the photon polarization tensor $\Pi_{\alpha\beta}$,

$$I_t(q_0) = -q_0 \left(1 - \mathrm{e}^{-q_0/T}\right) \Gamma^{abs}_{(t)} \,, \tag{5.92}$$

we express the value $\mathcal{A}^{(t)}$ in terms of I_t:

$$\mathcal{A}^{(t)} = -\frac{I_t}{1 - \mathrm{e}^{-q_0/T}} = -I_t \left[1 + f_\gamma(q_0)\right], \tag{5.93}$$

where $f_\gamma(q_0) = \left(\mathrm{e}^{q_0/T} - 1\right)^{-1}$ is the Bose–Einstein distribution function for a photon. This relation obtained in the case $q_0 > 0$ is also correct for the case $q_0 < 0$, which corresponds to the transversal plasmon creation with the energy $\omega = -q_0 > 0$. The connection should be used here between the imaginary part I_t and the width of creation of the transversal plasmon:

$$I_t(\omega) = -\omega \left(\mathrm{e}^{\omega/T} - 1\right) \Gamma^{cr}_{(t)} \,. \tag{5.94}$$

It is essential also that the function I_t is odd:

$$I_t(-q_0) = -I_t(q_0) \,, \tag{5.95}$$

and this is the feature of the retarded polarization operator.

Performing the similar calculations, one can see that the relation of the form (5.93) is valid for the longitudinal plasmon also. It is necessary to remember that ${\rho_{\alpha\beta}}^{(\ell)} = -{\varepsilon_\alpha}^{(\ell)}\, {\varepsilon_\beta}^{(\ell)}$, and

$$I_\ell(q_0) = q_0 \left(1 - \mathrm{e}^{-q_0/T}\right) \Gamma^{abs}_{(\ell)} \,. \tag{5.96}$$

Finally, we obtain the tensor $T^{\alpha\beta}$ in the form of decomposition over the density matrices (4.42), (4.43):

$$T^{\alpha\beta} = \left[-I_t\, \rho^{\alpha\beta(t)} - I_\ell\, \rho^{\alpha\beta(\ell)}\right] \left[1 + f_\gamma(q_0)\right], \tag{5.97}$$

where $I_{t,\,\ell}$ are the imaginary parts of the eigenvalues $\Pi_{t,\,\ell}$ of the photon polarization tensor; $f_\gamma(q_0)$ is the Bose–Einstein distribution function for a photon.

Substituting (5.97) into (5.82), using the orthogonality of the tensors $\rho^{\alpha\beta(t)}$ and $\rho^{\alpha\beta(\ell)}$, see Eq. (4.45), and taking into account the expressions for the contractions of the neutrino current with these tensors:

$$j^{\alpha}_{(\nu)}\, j^{\beta *}_{(\nu)}\, \rho_{\alpha\beta}{}^{(t)} = Q^4 \left[\frac{(2E' + q_0)^2}{q^2} - 1\right],$$

$$j^{\alpha}_{(\nu)}\, j^{\beta *}_{(\nu)}\, \rho_{\alpha\beta}{}^{(\ell)} = -Q^4\, \frac{(2E' + q_0)^2}{q^2}\,,$$

one finally obtains for the rate of creation of the right-handed neutrino:

$$\Gamma_{\nu_R}(E') = \frac{\mu_\nu^2}{16\,\pi^2\, E'^2} \int\limits_{-E'}^{\infty} \mathrm{d}q_0 \int\limits_{|q_0|}^{2E'+q_0} q^3\, \mathrm{d}q\; f_\nu(E' + q_0)\, (2E' + q_0)^2$$
$$\times \left(1 - \frac{q_0^2}{q^2}\right)^2 \left[1 + f_\gamma(q_0)\right] \left[\left(1 - \frac{q^2}{(2E' + q_0)^2}\right) \varrho_t - \varrho_\ell\right]. \quad (5.98)$$

Here, the plasmon spectral densities are introduced:

$$\varrho_\lambda = \frac{-2\, I_\lambda}{(Q^2 - R_\lambda)^2 + I_\lambda^2}\,, \quad (5.99)$$

which are defined by the eigenvalues (4.46) of the photon polarization tensor (4.41).

The formula (5.98) is in agreement, to the notations, with the rate obtained in Ref. [32] from the retarded self-energy operator of the right-handed neutrino. However, extracting from our general expression the electron contribution only, we obtain the result which is larger by the factor of 2 than the corresponding formula in the papers [39, 40]. It can be seen that an error was made there just in the first formula defining the production rate Γ of a right-handed neutrino.

The formula (5.98) being obtained for the process of the neutrino interaction with virtual photons, has in fact a more general sense, and can be used for neutrino-photon processes in any optically active medium. We only need to identify the photon spectral density functions ϱ_λ. For example, in the medium where $I_t \to 0$ in the space-like region $Q^2 < 0$ corresponding to the refractive index values $n > 1$, the spectral density function is transformed to δ-function, and we can reproduce the result of the paper [49] devoted to the study of the Cherenkov radiation of transversal photons by neutrinos.

If one formally takes the limit $I_\ell \to 0$, the result obtained in Ref. [50] can be reproduced, namely, as the authors believed, it would be the width of the Cherenkov radiation and absorption of longitudinal photons by neutrinos in the space-like region $Q^2 < 0$. However, the limit $I_\ell \to 0$ itself is irrelevant for $Q^2 < 0$ in the real astrophysical plasma conditions considered by those authors. As it was mentioned in Refs. [39, 40], see also Fig. 4.5, the space-like branch of the longitudinal photon mode developped a large imaginary part in the supernova core conditions. Thus, taking the limit $I_\ell \to 0$ leads to the strong overestimation of a result.

5.4.4 Contributions of Plasma Components into the Neutrino Scattering Process

As it was mentioned above, an analysis of the neutrino chirality flip process has to be performed with taking account of the neutrino scattering off various plasma components: electrons, protons, free ions, etc. For the first step we consider the contribution of the neutrino scattering off electrons into the right-handed neutrino production rate. This means that we take into account the electron contribution only into the function I_λ in the numerator of Eq. (5.99). It should be stressed however, that the functions R_λ and I_λ in the denominator of Eq. (5.99) contain the contributions of all plasma components. At this point our result for the neutrino scattering off electrons differs from the result of Ref. [39, 40], where the electron contribution only was taken both in the numerator and in the denominator of the plasmon spectral densities.

As the analysis shows, see Sect. 4.4, the electron and proton contributions into the imaginary parts I_λ of the eigenvalues Π_λ of the photon polarization tensor are of the same order of magnitude and have the same sign both for $\lambda = t$ and for $\lambda = \ell$, see Figs. 4.5 and 4.7. This fact itself should lead to a decreasing of the electron contribution into the function $\Gamma_{\nu_R}(E')$. On the other hand, it is seen from Fig. 4.4, that the electron and proton contributions into the real part R_ℓ of the eigenvalue Π_ℓ are of the same order of magnitude but have the opposite signs in the region where the imaginary part of the electron contribution into the numerator of Eq. (5.99) is relatively large. As a result, the contribution of the neutrino scattering off electrons into the right-handed neutrino production rate, obtained by us, appears to be close to the result of Ref. [39, 40], besides the above-mentioned factor of 2.

It is possible to consider similarly the contribution of the neutrino scattering off protons into the right-handed neutrino production rate. In this case, we take the proton contribution into the functions I_λ (4.57), (4.59) in the numerator of Eq. (5.99).

The results of our numerical analysis of the separate contributions of the neutrino scattering off electrons and protons, as well as the total ν_R production rate in the typical conditions of the supernova core are presented in Fig. 5.3.

The plotted dimensionless creation width $\mathcal{R}(E')$ is defined by the expression

$$\Gamma_{\nu_R}(E') = \mu_\nu^2 \, \mu_e^3 \, T^3 \, \mathcal{R}(E') \,. \tag{5.100}$$

For comparison, the result of Ref. [40] is also shown in Fig. 5.3, illustrating a strong underestimation of the neutrino chirality flip rate made by those authors.

We consider also the contribution of the neutrino scattering off free ions into the ν_R production rate. While the ions are believed to be absent in the supernova

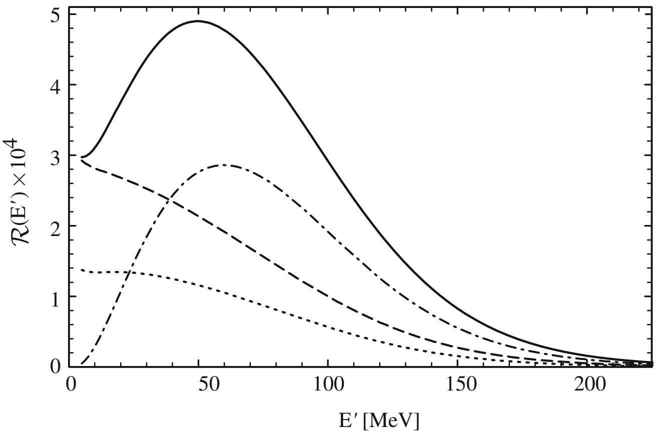

Fig. 5.3 Contributions from electrons (*dashed line*) and protons (*dashdotted line*) to the dimensionless creation width $\mathcal{R}(E')$ of a right-handed neutrino and total width (*solid line*) for plasma temperature $T = 25$ MeV and chemical potentials of electrons $\mu_e = 250$ MeV and neutrinos $\tilde{\mu}_{\nu_e} = 100$ MeV. The dotted line indicates the result of Ref. [40]

core, a significant fraction of them could be presented e.g. in the upper layers of the supernova envelope. It should be mentioned that longitudinal virtual plasmons give the main contribution into the ν_R production rate in this case. As is seen from Eq. (4.61), the function $I_\ell^{(i)}$ differs from zero only in the narrow area Δx of the variable $x = q_0/q$, namely, $\Delta x \sim \sqrt{T/m_i} \ll 1$, where m_i is the ion mass. This allows to perform calculations of the ion contribution into the ν_R production rate analitically, to obtain:

$$\Delta\,\Gamma^{(i)}_{\nu_R}(E') = \mu_\nu^2\,\alpha\,Z_i^2\,n_i\,f_\nu(E')\left(\ln\,\frac{4E'^2 + m_{\rm D}^2}{m_{\rm D}^2} - \frac{4E'^2}{4E'^2 + m_{\rm D}^2}\right), \qquad (5.101)$$

where α is the fine structure constant, eZ_i and n_i are the charge and the density of ions, $m_{\rm D}$ has a meaning of the Debye screening radius inversed, $m_{\rm D}^2 = \sum_k R_\ell^{(k)}(q_0 = 0)$. We remind that the summation is performed over all plasma components.

It is interesting to note that Eq. (5.101) obtained in the approximation of heavy ions, describes rather satisfactory the proton contribution.

Given the function $\Gamma_{\nu_R}(E')$, one can calculate the total number of right-handed neutrinos emitted per 1 MeV per unit time from the unit volume, i.e. the right-handed neutrino energy spectrum:

$$\frac{{\rm d}n_{\nu_R}}{{\rm d}E'} = \frac{E'^2}{2\,\pi^2}\,\Gamma_{\nu_R}(E')\,. \qquad (5.102)$$

One can see from Eq. (5.102), that very narrow peak of the function $\Gamma_{\nu_R}(E')$ at small neutrino energy, which was analysed in detail in Ref. [40], does not provide a huge number of soft right-handed neutrino production, as it was declared in [40], because of the factor E'^2.

The right-handed neutrino energy spectrum (5.102) can be useful for investigations of possible mechanisms of the energy transfer from these neutrinos to the outer layers of the supernova envelope. For example, a process is possible of the inverse conversion of a part of right-handed neutrinos into left-handed ones, with their subsequent absorption. Just these processes were proposed [51] and then investigated [52–54] as a possible mechanism for the stalled shock wave revival in the supernova explosion. A consistent analysis of such scenario would be doubtful without knowing the ν_R energy spectrum (5.102). We discuss this question below in Sect. 5.4.8.

The function $\Gamma_{\nu_R}(E')$ provides also the calculation of the spectral density of the supernova core luminosity via right-handed neutrinos as follows:

$$\frac{\mathrm{d}L_{\nu_R}}{\mathrm{d}E'} = V \, \frac{\mathrm{d}n_{\nu_R}}{\mathrm{d}E'} \, E' = V \, \frac{E'^3}{2\,\pi^2} \, \Gamma_{\nu_R}(E') \,. \tag{5.103}$$

Here, V is the volume of the neutrino-emitting region, $V \simeq 4 \times 10^{18}\,\mathrm{cm}^3$ [55]. The value $\mathrm{d}L_{\nu_R}/\mathrm{d}E'$ is presented in Fig. 5.4 for several values of the plasma temperature.

5.4.5 Illustration: Completely Degenerate Plasma at $T = 0$

In this section, we give a clear illustration of the fact that neutrino scattering by protons dominates over their scattering by plasma electrons, basing on an analysis of a simplified case of the completely degenerate plasma, $T = 0$.

The comparison of the typical parameters of the supernova core, where the temperature is believed to be of order $T \simeq$ 15–30 MeV, while the electron and neutrino chemical potentials are $\mu_e \simeq$ 200–250 MeV and $\tilde{\mu}_{\nu_e} \simeq 100$ MeV, respectively, shows that the temperature is the smallest physical parameter.[1] Thus, the limiting case of the completely degenerate plasma, $T = 0$, seems to give a reasonable estimate. It is remarkable that for the zero temperature limit the contributions from neutrino scattering by protons and electrons to the neutrino creation probability can be evaluated analytically using Eqs. (5.98) and (5.99) and the corresponding formulas of Sect. 4.4.

It is appropriate to analyse the function $\Gamma_{\nu_R}(E)$ defining the energy spectrum of right-handed neutrinos (5.102).

The contribution of ultrarelativistic electrons to the function $\Gamma_{\nu_R}(E)$ in the case $T = 0$ can be obtained from Eqs. (5.98) and (5.99) in the simple form:

[1] Hereafter we consider neutrinos as a quasiequilibrium gas described by the distribution functions: $f_\nu(T, \tilde{\mu}_{\nu_e})$ for the electron neutrinos, and $f_\nu(T, 0)$ for the muon and tau neutrinos. This is believed to be a rather good approximation inside the SN core during a few seconds after the collapse.

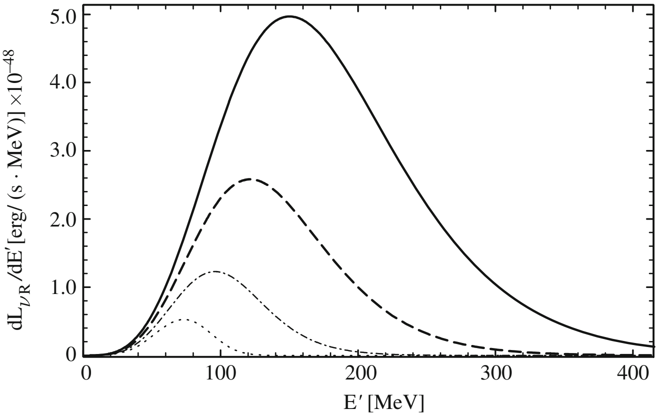

Fig. 5.4 Energy distributions of the right-handed neutrino luminosity for plasma temperatures $T = 35\,\mathrm{MeV}$ (*solid line*), $T = 25\,\mathrm{MeV}$ (*dashed line*), $T = 15\,\mathrm{MeV}$ (*dashdotted line*), $T = 5\,\mathrm{MeV}$ (*dotted line*) and for neutrino magnetic moment $\mu_\nu = 3 \times 10^{-13}\,\mu_\mathrm{B}$

$$\Gamma^{(e)}_{\nu_R}(E) = \frac{\mu^2_{\nu_e}\, m^2_\gamma}{2\,\pi}\,(\tilde{\mu}_{\nu_e} - E)\,\Theta(\tilde{\mu}_{\nu_e} - E)\,, \tag{5.104}$$

where E is the right-handed neutrino energy, μ_{ν_e} is the effective electron neutrino magnetic moment (5.78), $\tilde{\mu}_{\nu_e}$ is the electron neutrino chemical potential, $m^2_\gamma = 2\,\alpha\,\mu^2_e/\pi$ is the squared mass of a transverse plasmon at $T = 0$, and $\Theta(x)$ is the step function.

The analytical expression describing the proton contribution turns out to be more complicated since it depends also on the proton mass. The plasma charge neutrality condition for $T = 0$ takes the form $n_p = n_{e^-}$ and ensures that the electron and proton Fermi momenta are equal: $k^{(e)}_\mathrm{F} = k^{(p)}_\mathrm{F}$. Then, the proton chemical potential coinciding with the Fermi energy is $\mu_p = E^{(p)}_\mathrm{F} = \sqrt{m^2_p + \mu^2_e}$ and the proton contribution is expressed in terms of the proton Fermi velocity $v_\mathrm{F} = k^{(p)}_\mathrm{F}/E^{(p)}_\mathrm{F} = \mu_e/\mu_p = \mu_e/\sqrt{m^2_p + \mu^2_e}$. As a result, the proton contribution can be expressed in the form:

$$\Gamma^{(p)}_{\nu_R}(E) = \frac{\mu^2_{\nu_e}\, m^2_\gamma\, \tilde{\mu}_{\nu_e}}{2\,\pi}\,\varphi_p(y)\,, \quad y = \frac{E}{\tilde{\mu}_{\nu_e}}\,, \quad 0 \leqslant y \leqslant 1\,. \tag{5.105}$$

Here, the function $\varphi_p(y)$ has different forms in two intervals: it is

$$\varphi_p(y) = \frac{1 + v_\mathrm{F}/3}{1 - v_\mathrm{F}}\,y\,, \tag{5.106}$$

for $0 \leqslant y \leqslant (1 - v_\mathrm{F})/(1 + v_\mathrm{F})$, and

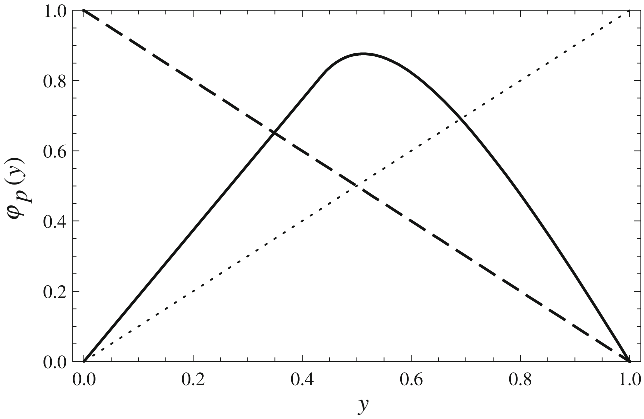

Fig. 5.5 Plots of the function $\varphi_p(y)$ for various $v_{\rm F}$ values. The dependence $\varphi_e(y) = (1 - y)$ for the electron contribution is reproduced for $v_{\rm F} = 1$ (*dashed line*). The value $v_{\rm F} = 0.394$ (*solid curve*) corresponds to the effective proton mass $m_p \simeq 700$ MeV. The case $v_{\rm F} = 0$ (*dotted line*) corresponds to the limit of infinitely large proton mass (Figs. 5.5–5.12 reprinted from [42] with the World Scientific Publishing Company's permission.)

$$\varphi_p(y) = \frac{1-y}{v_{\rm F}} \left[1 - \frac{(1-v_{\rm F})^2}{12\, y^2\, v_{\rm F}} \,(1-y)\,(1+2\,y) \right] , \qquad (5.107)$$

for $(1 - v_{\rm F})/(1 + v_{\rm F}) \leqslant y \leqslant 1$.

Note that the formal turn to the limit $m_p \to 0$, i.e. $v_{\rm F} \to 1$, in Eqs. (5.105)–(5.107) yields $\varphi_p(y) \to \varphi_e(y) = (1 - y)\, \theta(1 - y)$, where the function $\varphi_e(y)$ can be introduced in Eq. (5.104) in complete analogy with Eq. (5.105). Thus, as expected, Eq. (5.104) for the electron contribution is reproduced.

In Fig. 5.5, the plots are shown of the function $\varphi_p(y)$ for $v_{\rm F} = 1$, $v_{\rm F} = 0.394$, and $v_{\rm F} = 0$. The value $v_{\rm F} = 0.394$ corresponds to the effective proton mass $m_p \simeq 700$ MeV in a plasma with a nuclear density 3×10^{14} g cm^{-3} (see Ref. [55], p. 152). The value $v_{\rm F} = 0$ corresponds to the formal limit $m_p \to \infty$ for which this function is also significantly simplified: $\varphi_p(y) \to \varphi_\infty(y) = y\, \theta(1 - y)$.

The function $\Gamma_{\nu_R}(E)$ defined in Eq. (5.102) determines as well the right-handed neutrino emissivity of a supernova core, i.e. the energy passed away by right-handed neutrinos per 1 MeV of the neutrino energy spectrum per unit time from unit volume:

$$Q_{\nu_R} = E\, \frac{{\rm d}n_{\nu_R}}{{\rm d}E} = \frac{E^3}{2\,\pi^2}\, \Gamma_{\nu_R}(E)\,. \qquad (5.108)$$

According to Eqs. (5.102) and (5.108), the right-handed neutrino emissivity is given by the formula

$$Q_{\nu_R} = \frac{\mu_{\nu_e}^2\, m_\gamma^2\, \tilde{\mu}_{\nu_e}^4}{4\,\pi^3}\, y^3 \left[\varphi_e(y) + \varphi_p(y) \right] . \qquad (5.109)$$

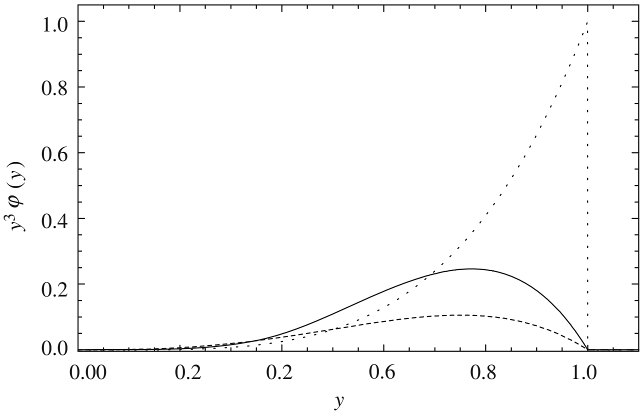

Fig. 5.6 The function $y^3\,\varphi(y)$ defining the contributions from electrons (*dashed line*) and protons with $m_p \simeq 700$ MeV (*solid line*) and $m_p \to \infty$ (*dotted line*) to the right-handed neutrino emissivity at $T = 0$

The difference between the electron and proton contributions to the quantity given by Eq. (5.109) is illustrated in Fig. 5.6. It is clearly seen that the factor y^3 causes the increasing of the proton contribution to the emissivity.

5.4.6 Uniform Ball Model for the Supernova Core

The spectral density of the supernova core luminosity via right-handed neutrinos is defined as follows, see Eq. (5.103):

$$\frac{\mathrm{d}L_{\nu_R}}{\mathrm{d}E} = V\,\frac{\mathrm{d}n_{\nu_R}}{\mathrm{d}E}\,E = V\,\frac{\mu_{\nu_e}^2\,m_\gamma^2\,\tilde{\mu}_{\nu_e}^4}{4\,\pi^3}\,y^3\,\varphi^{(\mathrm{num})}(y, T)\,. \tag{5.110}$$

Here, m_γ is the mass of a transverse plasmon,

$$m_\gamma^2 = \frac{2\,\alpha}{\pi}\left({\mu_e}^2 + \frac{\pi^2 T^2}{3}\right). \tag{5.111}$$

The function $\varphi^{(\mathrm{num})}(y, T)$ introduced in Eq. (5.110) similarly to Eqs. (5.105) and (5.109) can be extracted from Ref. [41]. The function $y^3\,\varphi^{(\mathrm{num})}(y, T)$ is plotted in Fig. 5.7 for two values of the averaged temperature and for the electron and electron-neutrino chemical potentials $\mu_e \simeq 300$ MeV and $\tilde{\mu}_{\nu_e} \simeq 160$ MeV. We neglected in our analysis [41] the contributions of the processes with the initial muon and tau neutrinos. However, as will be shown below, these contributions appear to be essential.

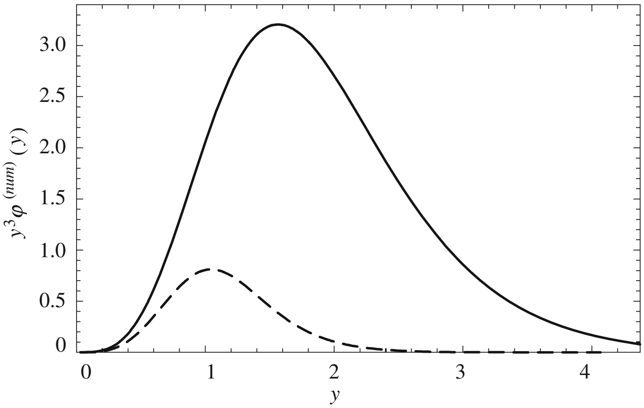

Fig. 5.7 The function $y^3\,\varphi^{(\mathrm{num})}(y, T)$ representing the result of the numerical calculation of the right-handed neutrino emissivity at $T = 30$ MeV (*dashed line*) and $T = 60$ MeV (*solid line*)

A comparison of Figs. 5.6 and 5.7 shows that taking of a nonzero temperature leads to a shift of the maximum of the energy distribution of the luminosity towards higher energies of right-handed neutrinos. This additionally enhances the proton contribution.

As a result, using the data on supernova $SN1987A$, a new astrophysical limit was imposed [41] on the electron-neutrino magnetic moment:

$$\mu_\nu < (0.7 - 1.5) \times 10^{-12}\,\mu_{\mathrm{B}}\,. \tag{5.112}$$

This is a factor of two better than the previous constraint [39, 40]. We have to remind, however, that both the previous and this improved bound on the electron-neutrino magnetic moment were based on a very simplified model of the supernova core as the uniform ball with some averaged values of physical parameters. In addition, the parameter values were set too high. For example, the upper limit $1.5 \times 10^{-12}\,\mu_{\mathrm{B}}$ in Eq. (5.112) corresponds to the SN core temperature 30 MeV, while the limit $0.7 \times 10^{-12}\,\mu_{\mathrm{B}}$ corresponds to the temperature 60 MeV. As is seen from Fig. 5.7, the right-handed neutrino emissivity grows with temperature very rapidly. However, according to recent simulations of the SN explosion, the temperature values inside the SN core are believed not to exceed 40 MeV, see e.g. Fig. 5.8. Anyway, taking account of the radial distribution of physical parameters inside the SN core would give more solid results.

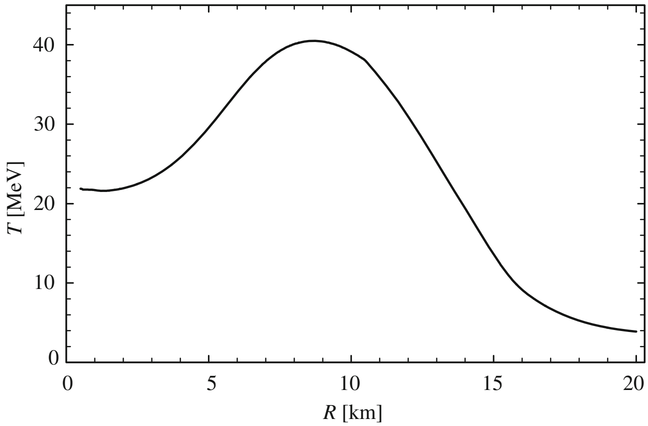

Fig. 5.8 The radial distribution for the temperature within the SN core at the moment $t = 1.0\,\mathrm{s}$ after the bounce, Ref. [56]

5.4.7 Models of the Supernova Core with Radial Distributions of Physical Parameters: Limits on the Neutrino Magnetic Moment

In this section we make the estimation of the upper bound on the Dirac neutrino magnetic moment by a more reliable way, with taking account of radial distributions and time dependences of physical parameters from realistic models of the SN core. Here we consider the models in the inverse chronology.

5.4.7.1 The Model of the O-Ne-Mg Core Collapse SN

This model was developed by H.-Th. Janka with collaborators who presented us the results of their simulations [56] of the O-Ne-Mg core collapse supernovae which were a continuation of their model simulations [57, 58]. The successful explosion results for this case were independently confirmed by the Arizona/Princeton SN modelling group [59, 60], which found very similar results. So we were provided with a model whose explosion behavior was comparatively well understood and generally accepted.

We should stress that this O-Ne-Mg core collapse model (for the initial stellar mass of $8.8\,M_{\odot}$) is not applicable directly to $SN1987A$ which was $15 - 20\,M_{\odot}$ prior to collapse and according to the evolution theory it had a collapsing core which consisted of iron-peak elements.

We redefine Eq. (5.103), where, instead of multiplying by the volume of the neutrino-emitting region V, we integrate over this volume to obtain the spectral density of the energy luminosity of a supernova core via right-handed neutrinos:

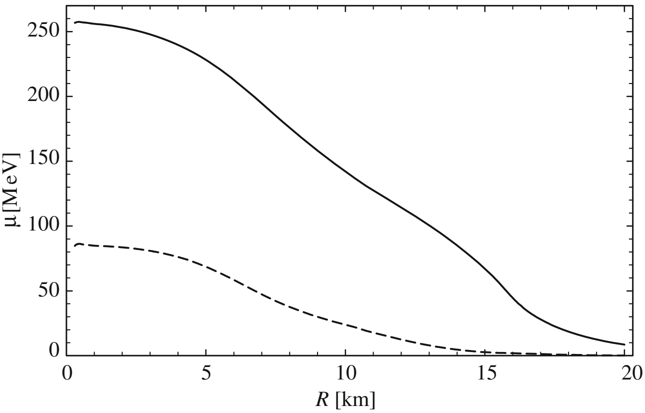

Fig. 5.9 The radial distributions for the chemical potentials of electrons (*solid line*) and electron neutrinos (*dashed line*) within the SN core at the moment $t = 1.0\,\mathrm{s}$ after the bounce

$$\frac{\mathrm{d}L_{\nu_R}}{\mathrm{d}E} = \int \mathrm{d}V\, \frac{E^3}{2\,\pi^2}\, \Gamma_{\nu_R}(E)\,. \tag{5.113}$$

Here, taking the values defined in Eqs. (5.98) and (5.99) and the corresponding formulas of Sect. 4.4, we take account of their dependence on the radius R and time t. A comprehensive set of parameter distributions used in our estimation includes the profiles [56] of the density ρ, the temperature T, the electron fraction Y_e, the fractions of electron neutrinos Y_{ν_e}, electron anti-neutrinos $Y_{\bar{\nu}_e}$, and the fractions Y_{ν_x} for one kind of heavy-lepton neutrino or antineutrino ($\nu_x = \nu_{\mu,\tau}, \bar{\nu}_{\mu,\tau}$), which are treated identically. The time evolution of the parameter distributions is calculated [56] within the interval until $\sim 2\,\mathrm{s}$ after the bounce. For the sake of illustration, we present in Figs. 5.8, 5.9 and 5.10 the radial distributions within the SN core, from 0 to 20 km, at the moment $t = 1.0\,\mathrm{s}$ after the bounce. The plots are presented for the temperature [56], for the chemical potentials of electrons μ_e and electron neutrinos $\tilde{\mu}_{\nu_e}$ (calculated on the base of the data of Ref. [56]), and for the proton nonrelativistic chemical potential $\mu_p^* = \mu_p - m_N^*$ defining the degeneracy of protons (calculated on the base of the data of Ref. [56] and the effective nucleon mass m_N^* in plasma, see Ref. [55], p. 152).

To analyse the influence of the right-handed neutrino emission on the SN energy loss, we also used the time evolution of the total luminosity of all species of left-handed neutrinos [56], presented in Fig. 5.11.

Integrating Eq. (5.113) over the neutrino energy, one obtains the time evolution of the right-handed neutrino luminosity:

$$L_{\nu_R}(t) = \frac{1}{2\,\pi^2} \int \mathrm{d}V \int\limits_0^\infty \mathrm{d}E\, E^3\, \Gamma_{\nu_R}(E)\,. \tag{5.114}$$

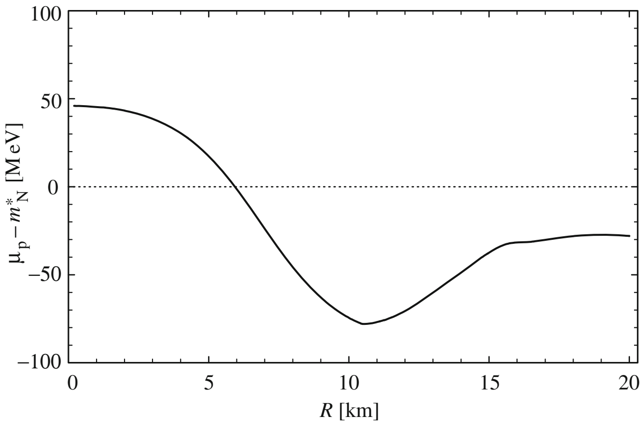

Fig. 5.10 The radial distribution for the proton nonrelativistic chemical potential $\mu_p^* = \mu_p - m_N^*$ within the SN core at the moment $t = 1.0$ s after the bounce

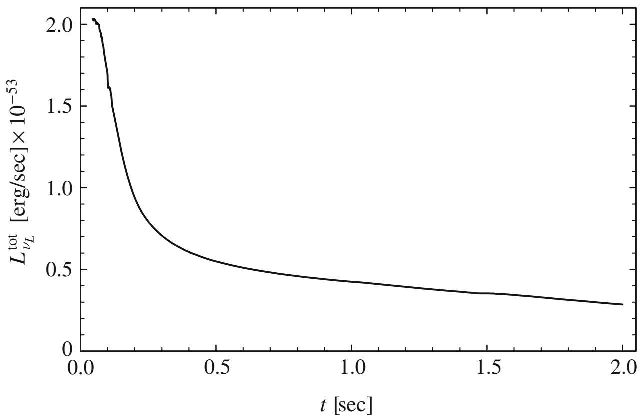

Fig. 5.11 The time evolution of the total luminosity of all active neutrino species, Ref. [56]

The right-handed neutrino is a novel cooling agent which would have to compete with the energy-loss via active neutrino species in order to affect the total cooling time scale significantly. Therefore, the observed $SN1987A$ signal duration indicates that a novel energy-loss via right-handed neutrinos is bounded by

$$L_{\nu_R} < L_{\nu_L}, \tag{5.115}$$

and we believe this estimation to be applicable also to the considered O-Ne-Mg core collapse model. Within the considered time interval until 2 s after the bounce, one obtains from Eqs. (5.114), (5.115) the time-dependent upper bound on the combination of the effective magnetic moments of the electron, muon and tau neutrinos. Assuming for simplicity that these effective magnetic moments are equal, one obtains

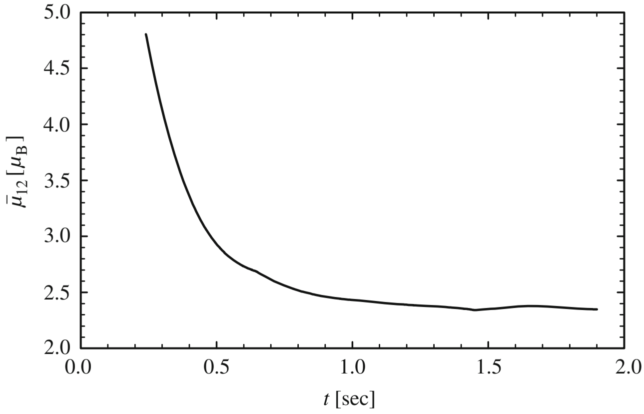

Fig. 5.12 The time evolution of the upper bound on the neutrino magnetic moment within the time interval until 2 s after the bounce (in assumption that the effective magnetic moments of electron, muon and tau neutrinos are equal)

the time evolution of the upper bound on some flavor-averaged neutrino magnetic moment $\bar{\mu}_\nu$ shown in Fig. 5.12, where $\bar{\mu}_{12} = \bar{\mu}_\nu/(10^{-12}\,\mu_B)$.

As is seen from Fig. 5.12, the averaged upper bound tends to some value, providing the limit

$$\bar{\mu}_\nu < 2.4\,\times\,10^{-12}\,\mu_B\,. \tag{5.116}$$

In a general case the combined limit on the effective magnetic moments of the electron, muon and tau neutrinos is

$$\left[\mu^2_{\nu_e} + 0.71\left(\mu^2_{\nu_\mu} + \mu^2_{\nu_\tau}\right)\right]^{1/2} < 3.7\,\times\,10^{-12}\,\mu_B\,, \tag{5.117}$$

where the effective magnetic moments are defined according to Eq. (5.78). This limit is less stringent than the bound (5.112) obtained in the frame of the uniform ball model for the SN core, but it is surely more reliable. Additionally, the upper bound on the effective magnetic moments of muon and tau neutrinos is established.

5.4.7.2 Earlier Models of the SN Explosion

The similar procedure of evaluation was performed with using of the data of the model [61] of the two-dimensional hydrodynamic core-collapse supernova simulation for a 15 $M_\odot$ star. Namely, the radial distributions of parameters at the moments $t = 0.2, 0.4, 0.6, 0.8$ s after the bounce in the model *s15Gio_32.a* were taken from Fig. 40 of Ref. [61]. Additionally, the fraction of electron neutrinos was evaluated as $Y_{\nu_e} \simeq (1/5)\,Y_e$. Calculating the right-handed neutrino luminosity with those parameters and putting the limit (5.115), where the total luminosity via active neutrino species L_{ν_L} in that model can be taken from Fig. 42 of Ref. [61], one obtains that the upper bound on the flavor-averaged neutrino magnetic moment $\bar{\mu}_\nu$ also varies in time as in the previous case. The time-averaged upper bound on $\bar{\mu}_\nu$ corresponding to the interval 0.4–0.8 s, is:

$$\bar{\mu}_\nu < 2.7 \times 10^{-12}\,\mu_{\rm B}\,, \tag{5.118}$$

to be compared with the limit (5.116).

Using the results of Ref. [62] where the thermal and chemical evolution during the Kelvin-Helmholtz phase of the birth of a neutron star was studied, taking the data from Figs. 9 and 14, we have obtained the time-averaged upper bound on $\bar{\mu}_\nu$ for the time interval 1–10 s of the post-bounce evolution in the form:

$$\bar{\mu}_\nu < 1.2 \times 10^{-12}\,\mu_{\rm B}\,. \tag{5.119}$$

We also used the results of Ref. [63] where the numerical simulations were performed of the neutrino-driven deleptonization and cooling of newly formed, hot, lepton-rich neutron star. Using the data presented in Figs. 3–9 on the SBH model (of the hot star with a "small" baryonic mass), we have evaluated the time-averaged upper bound on $\bar{\mu}_\nu$ for the time interval 0.5–5 s after the bounce in the form:

$$\bar{\mu}_\nu < 1.1 \times 10^{-12}\,\mu_{\rm B}\,. \tag{5.120}$$

One can summarize that the upper bound on the flavor- and time-averaged neutrino magnetic moment at the Kelvin-Helmholtz phase of the supernova explosion occurs to be

$$\bar{\mu}_\nu < (1.1 - 2.7) \times 10^{-12}\,\mu_{\rm B}\,, \tag{5.121}$$

depending on the explosion model.

5.4.8 Possible Effect of the Neutrino Magnetic Moment: Shock-Wave Revival in a Supernova Explosion

Two basic problems arise in numerical simulation of a supernova explosion [55, 61, 64–66]. First, the mechanism of damped-shock-wave stimulation has not yet been developed conclusively, but, without this mechanism, an explosion can hardly occur. We recall that the damping of a shock wave is due largely to the loss of energy spent on the dissociation of nuclei. Second, the energy deposition even in the case of a "successful" theoretical supernova explosion proves to be substantially less than the observed envelope kinetic energy of about $\sim 10^{51}$ erg [so-called FOE (ten to the Fifty-One Ergs) problem]. In order to describe self-consistently explosion dynamics, it is therefore necessary that, via some mechanism, the neutrino flux going from a supernova central part transfer an energy on the order $\sim 10^{51}$ erg to the envelope.

The mechanism proposed by Dar [51] and based on the assumption that the neutrino magnetic moment is not overly small is one of the possible means for solving the above problems. We note that this mechanism is operative only for Dirac but not for Majorana neutrinos. Left-handed electron neutrinos produced abundantly in the collapsing supernova core form a degenerate neutrino gas such that typical values of its chemical potential fall within the range $\tilde{\mu}_{\nu_e} \sim 150$–$200\,\mathrm{MeV}$ [55]. Since the values of $\tilde{\mu}_{\nu_e}$ are much higher than typical temperature values of $T \sim 30\,\mathrm{MeV}$, the density of electron neutrinos in the supernova core exceeds substantially the densities of neutrinos belonging to any other flavor. Part of left-handed electron neutrinos are converted into right-handed neutrinos via magnetic-moment interaction with plasma electrons and protons. In turn, right-handed neutrinos, which are sterile with respect to weak interaction, escape freely from the supernova central part if the neutrino magnetic moment lies in the range $\mu_\nu < 10^{-11}\,\mu_{\mathrm{B}}$. Some of these neutrinos may again transform into left-handed neutrinos owing to magnetic-moment interaction with a magnetic field in the envelope of the supernova core. According to currently prevalent ideas, the magnetic-field strength there may reach high values, on the scale of the critical value of $B_e = m_e^2/e \simeq 4.41 \times 10^{13}$ G, or even higher [67–69]. Newly produced left-handed neutrinos can transfer additional energy to the supernova envelope upon undergoing absorption in the course of $\nu_e n \to e^- p$ beta processes.

A sufficient motivation for reconsidering the Dar mechanism has appeared after publication of the papers [41, 42] where it was shown that the flux and luminosity of right-handed neutrinos from the supernova central part were strongly underestimated in previous studies. In this section, we analyze the $\nu_L \to \nu_R \to \nu_L$ double conversion of the neutrino helicity under supernova conditions and consider the possibility of stimulating a damped shock wave via this process.

At typical values of the supernova-core parameters (a temperature of $T \simeq 30\,\mathrm{MeV}$; electron and electron- neutrino chemical potentials of $\mu_e \simeq 300$ MeV and $\tilde{\mu}_{\nu_e} \simeq 160$ MeV, respectively; and a volume of $V \simeq 4 \times 10^{18}\,\mathrm{cm}^3$ [55]), the integrated luminosity of right-handed neutrinos is

$$L_{\nu_R} = 4 \times 10^{51}\,\frac{\mathrm{erg}}{\mathrm{s}} \left(\frac{\mu_\nu}{3 \times 10^{-13}\,\mu_{\mathrm{B}}}\right)^2 . \tag{5.122}$$

The energies of right-handed neutrinos that escaped from the core are on the same order of magnitude as the chemical potential of left-handed neutrinos captured in the core, $E_\nu \sim 100$–$200\,\mathrm{MeV}$.

For the sake of definiteness, we henceforth set the neutrino magnetic moment to $\mu_\nu = 3 \times 10^{-13}\,\mu_{\mathrm{B}}$. On one hand, this value is sufficiently small for the dynamics of the supernova core to remain undistorted; on the other hand, it ensures the required level of luminosity in (5.122).

If it were possible to convert the energy of right- handed neutrinos into the energy of left-handed neutrinos, for example, via the well-known mechanism of spin oscillations, then, within the typical shock-wave-stagnation time of about a few tenths of a second, an additional energy of about 10^{51} erg would be injected into the supernova envelope. We recall that we deal here with electron neutrinos, whose absorption in the envelope is due to beta processes.

We consider the part of the supernova envelope between the neutrinosphere (of radius R_ν) and the shock-wave-stagnation region (of radius R_s). According to currently prevalent ideas, typical values of R_ν and R_s change only slightly within the stagnation time, amounting to $R_\nu \sim 20$–50 km and $R_s \sim 100$–200 km (see, for example, [61]). If a rather strong magnetic field is present in this region, neutrino spin oscillations, which, under certain conditions, may have a resonance character, occur.

The effect of a magnetic field on a neutrino that has a magnetic moment can be the most conveniently illustrated in terms of the equation that describes neutrino-spin evolution in a uniform external magnetic field. With allowance for the additional energy $\Delta E_L^{(e)}$ that a left-handed electron neutrino acquires in a medium, see Eqs. (4.80) and (4.83), the spin-evolution equation can be represented in the form [24, 25, 53, 70, 71]

$$\mathrm{i}\,\frac{\partial}{\partial t}\begin{pmatrix}\nu_R\\ \nu_L\end{pmatrix} = \left[\hat{E}_0 + \begin{pmatrix}0 & \mu_\nu B_\perp\\ \mu_\nu B_\perp & \Delta E_L^{(e)}\end{pmatrix}\right]\begin{pmatrix}\nu_R\\ \nu_L\end{pmatrix}, \qquad (5.123)$$

where

$$\Delta E_L^{(e)} = \frac{3\,G_\mathrm{F}}{\sqrt{2}}\,\frac{\rho}{m_N}\left(Y_e + \frac{4}{3}\,Y_{\nu_e} - \frac{1}{3}\right), \qquad (5.124)$$

Here, $\rho/m_N = n_B$ is the nucleon density; $Y_e = n_e/n_B = n_p/n_B$; $Y_{\nu_e} = n_{\nu_e}/n_B$, with n_{e,p,ν_e} standing for the electron, proton, and neutrino densities; and $B_\perp$ is the transverse component of the magnetic field with respect to the direction of neutrino motion.

It should be explained why use is made here of expression (5.124) for the additional energy of left-handed electron neutrinos in an unpolarized medium, even though, in general, electrons must at least be partly polarized in a field on the scale of B_e. The following considerations prove the validity of the unpolarized-medium approximation in this case. As is well known, electron states in a magnetic field that correspond to all Landau levels, with the exception of the ground one, are doubly degenerate in the spin projection onto the magnetic-field direction and, hence, do not contribute to medium polarization. In order to assess the degree of polarization, it is therefore necessary to estimate the fraction of electrons that populate the ground Landau level and whose spins are not compensated. Under conditions typical of the supernova-envelope region being considered, we have $\mu_e \simeq 5$–10 MeV (see, for example, [61]); taking the ratio of the concentration of electrons populating the ground Landau level, $n_0 \simeq eB\mu_e/(2\pi^2)$ (see, for example, [72]), to the total electron concentration, $n \simeq \mu_e^3/(3\pi^2)$, we estimate the degree of medium polarization at

$$P \sim \frac{n_0}{n} \lesssim \frac{eB}{\mu_e^2} \sim 10^{-2}\,\frac{B}{B_e}\,. \tag{5.125}$$

Thus, the use of the unpolarized-medium approximation is legitimate at magnetic-field strengths around $B \sim B_e$, which are used here. A more rigorous condition of weak plasma magnetization under which the influence of the magnetic field on the polarization of the medium can be neglected is formulated as

$$B \ll \frac{(3\pi^2\, n_e)^{2/3}}{e} \simeq 0.6 \times 10^{16}\,\mathrm{G}\left(\frac{n_e}{10^{33}\,\mathrm{cm}^{-3}}\right)^{2/3}. \tag{5.126}$$

Expression (5.124) for the additional energy $\Delta E_L^{(e)}$ of left-handed electron neutrinos deserves a more detailed analysis. It is noteworthy that the discussed energy can appear to be exactly zero in the supernova-envelope region of our interest, and this is in turn the condition of the $\nu_R \to \nu_L$ resonance transition. Since the neutrino density is rather low in the supernova envelope, the quantity Y_{ν_e} in expression (5.124) can be disregarded, in which case the resonance condition is written as $Y_e = 1/3$. We note that, in the supernova envelope, Y_e takes values characteristic of collapsing matter, $Y_e \sim 0.4$–0.5. However, a shock wave causing the dissociation of heavy nuclei renders matter more transparent to neutrinos, thus leading to a so-called short neutrino burst and, hence, to a considerable deleptonization of matter in this region. According to existing estimates, the radial distribution of Y_e develops a characteristic dip, where Y_e may decrease to values of about 0.1 (see, for example, [61, 65]). Thus, a point where Y_e acquires a value of 1/3 does inevitably exist. It is noteworthy that there is only one such point where $\mathrm{d}Y_e/\mathrm{d}r > 0$ (see [61, 65]).

We emphasize that expression (5.124) refers only to the electron neutrino, in which case the amplitude for its scattering on medium electrons features channels of exchange of both a neutral Z boson and a charged W boson. For the muon neutrino and for the tau neutrino, which are scattered on electrons only via the exchange of a neutral Z boson, the additional energy has the form (4.84), or:

$$\Delta E_L^{(\mu,\tau)} = -\frac{G_\mathrm{F}}{\sqrt{2}}\,\frac{\rho}{m_N}\,(1 - Y_e)\,, \tag{5.127}$$

that is, it does not vanish anywhere, so that the above resonance transition is impossible.

A qualitative character of the dependence $Y_e(r)$ according to [61] is depicted in Fig. 5.13.

We note that the condition $Y_e = 1/3$ is necessary for the resonance conversion of right-handed neutrinos into left-handed ones, but it is not sufficient. In addition, fulfillment of the so-called adiabaticity condition is required. Its meaning is the following: upon moving off the resonance point by a distance of about one oscillation wavelength, the diagonal element $\Delta E_L^{(e)}$ in Eq. (5.123) at least must not exceed the off-diagonal element $\mu_\nu B_\perp$. This leads to the condition [52]

$$\mu_\nu B_\perp \gtrsim \left(\frac{\mathrm{d}\Delta E_L^{(e)}}{\mathrm{d}r}\right)^{1/2} \simeq \left(\frac{3\,G_\mathrm{F}}{\sqrt{2}}\,\frac{\rho}{m_N}\,\frac{\mathrm{d}Y_e}{\mathrm{d}r}\right)^{1/2}. \tag{5.128}$$

In the region being considered, typical parameter values are the following (see [61, 65]):

$$\frac{\mathrm{d}Y_e}{\mathrm{d}r} \sim 10^{-8}\,\mathrm{cm}^{-1}\,, \quad \rho \sim 10^{10}\,\mathrm{g\,cm}^{-3}\,. \tag{5.129}$$

For the magnetic-field strength ensuring fulfillment of the resonance condition, we obtain

$$B_\perp \gtrsim 2.6\times10^{13}\,\mathrm{G}\left(\frac{10^{-13}\,\mu_\mathrm{B}}{\mu_\nu}\right)\left(\frac{\rho}{10^{10}\,\mathrm{g\ cm}^{-3}}\right)^{1/2} \times\left(\frac{\mathrm{d}Y_e}{\mathrm{d}r}\times10^{8}\,\mathrm{cm}\right)^{1/2}. \tag{5.130}$$

Thus, our analysis has revealed that, if the neutrino has a magnetic moment in the range

$$10^{-13}\,\mu_\mathrm{B} < \mu_\nu < 10^{-12}\,\mu_\mathrm{B} \tag{5.131}$$

and if a magnetic field of strength about 10^{13} G exists in the region $R_\nu < R < R_s$, the mechanism of the double conversion of the neutrino helicity, $\nu_L \to \nu_R \to \nu_L$, according to Dar's scenario is operative. At energies estimated at $E_\nu \sim 100$–200 MeV, the neutrino mean free path with respect to beta processes is

$$\lambda \simeq 800\,\mathrm{m}\,\frac{1}{1-Y_e}\left(\frac{150\,\mathrm{MeV}}{E_\nu}\right)^2. \tag{5.132}$$

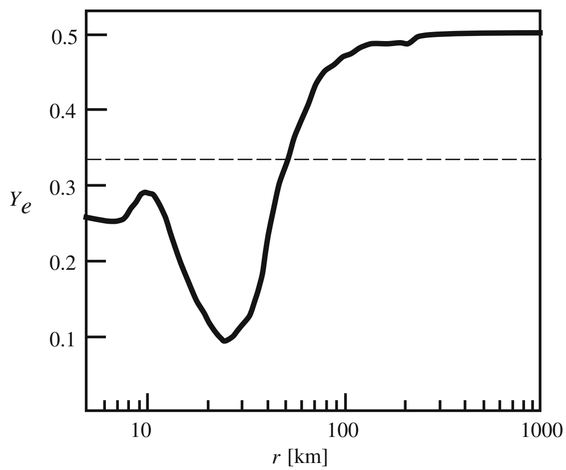

Fig. 5.13 Qualitative character of the radial distribution of $Y_e(r)$ approximately after 0.1–0.2 s from the generation of a shock wave featuring a dip caused by a "short" neutrino burst (see, for example [61]). The dashed line corresponds to the value of $Y_e = 1/3$

Therefore, the additional energy

$$\Delta E \simeq L_{\nu_R}\, \Delta t \sim 10^{51}\,\text{erg} \tag{5.133}$$

is injected into the supernova-envelope region within the time of shock-wave stagnation, $\Delta t \sim 0.2$–0.4 s. This solves the FOE problem.

5.4.9 *Possible Effect of the Neutrino Magnetic Moment: Neutrino Pulsar*

If the neutrino magnetic moment is less than the values of the range (5.131), the conversion of sterile neutrinos produced in the supernova core into active ones through the scattering mechanism of the Dirac neutrino magnetic moment with the microscopic electromagnetic field of a virtual plasmon (5.73), did not influence the supernova explosion dynamics. In this case, the process of the neutrino helicity flip in a strong magnetic field of the supernova envelope can lead to interesting observational consequences when the expected neutrino signal from an imminent supernova explosion is studied in detail [73, 74].

According to existing views, during the explosion of a Galactic supernova at a distance up to 10 kpc, the expected number of neutrino events in the Super-Kamiokande detector will be $\sim 10^4$. This will allow the time evolution of the neutrino flux to be recorded with a good accuracy.

In the presence of a sufficiently strong magnetic field in the supernova envelope, not only the above-mentioned conversion of right-handed neutrinos into left-handed ones, $\nu_R \to \nu_L$ [51, 52], but also the conversion of active electron neutrinos and antineutrinos of the main neutrino flux into a form sterile with respect to weak interactions, $\nu_L \to \nu_R$, $\bar{\nu}_R \to \bar{\nu}_L$, is possible.

Numerical analysis of Eq. (5.123) shows that after its passage through the resonance region ($Y_e = 1/3$), the flux of left-handed neutrinos is attenuated as a result of the above conversion by the factor W_{LL}, which has the meaning of the survival probability of left-handed neutrinos, $\nu_{eL} \to \nu_{eL}$, or, in other words, the transparency. Figure 5.14 shows the characteristic variation in W_{LL} when passing through the resonance point (placed here at the coordinate origin) for various magnetic field strengths. We see that the supernova envelope in the presence of a sufficiently strong magnetic field is virtually opaque to active electron neutrinos and antineutrinos, which can cause the expected neutrino signal from the supernova to be attenuated.

A more detailed analysis of the numerical solution of Eq. (5.123) allows us to establish a relationship between the magnetic field strength and parameters of the medium in the supernova envelope, on the one hand, and the survival probability of active neutrinos W_{LL}, on the other hand. Using typical scales of parameters in the region under consideration [61, 65]

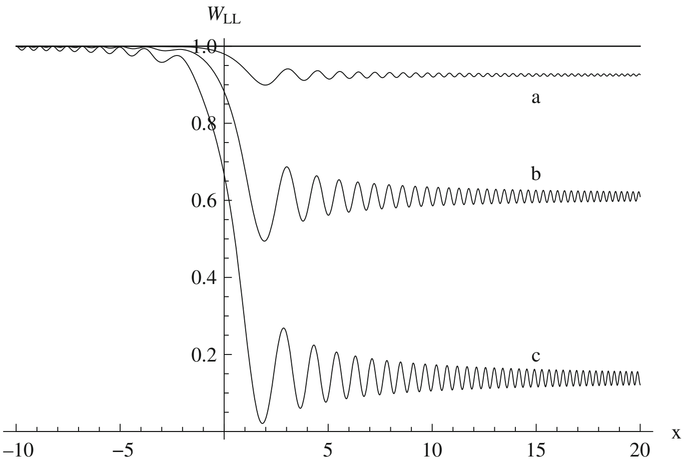

Fig. 5.14 Pattern of variations in W_{LL}, the survival probability of left-handed neutrinos, $\nu_{eL} \rightarrow \nu_{eL}$, (transparency), with distance x (in arbitrary units) when passing through the resonance point placed at the coordinate origin for several magnetic field strengths: $B = 0.2\,B_e$ (**a**), $B = 0.5\,B_e$ (**b**), $B = B_e$ (**c**). To be specific, the neutrino magnetic moment is assumed to be $10^{-13}\,\mu_{\mathrm{B}}$, the density is 10^{10} g cm^{-3}, and the gradient of the electron fraction is $\mathrm{d}Y_e/\mathrm{d}r \simeq 10^{-7}$ cm^{-1}

$$\frac{\mathrm{d}Y_e}{\mathrm{d}r} \sim 10^{-7}\,\mathrm{cm}^{-1}\,, \quad \rho \sim 10^{10}\,\mathrm{g\ cm}^{-3}\,, \tag{5.134}$$

we find an approximation formula,

$$\frac{B_\perp(t)}{B_e} = f(W_{LL})\left(\frac{10^{-13}\mu_{\mathrm{B}}}{\mu_\nu}\right) \times \left(\frac{\rho(t)}{10^{10}\,\mathrm{g\ cm}^{-3}}\right)^{1/2}\left(\frac{\mathrm{d}Y_e}{\mathrm{d}r}(t) \times 10^7\,\mathrm{cm}\right)^{1/2}. \tag{5.135}$$

Here, the factor

$$f(W_{LL}) = 0.88\,\frac{(1 - W_{LL})^{0.62}}{(W_{LL})^{0.13}} \tag{5.136}$$

characterizes the degree of adiabaticity of the conversion process. The literal adiabaticity corresponds to the limit $f \rightarrow \infty$ when $W_{LL} \rightarrow 0$; in this case, the left-handed neutrinos are completely converted into right-handed ones, $W_{LR} = (1 - W_{LL}) \rightarrow 1$.

The conservative value of $10^{-13}\mu_{\mathrm{B}}$ introduced in Eq. (5.135) as the scale for the neutrino magnetic moment was chosen in order not to distort the supernova explosion dynamics. Thus, we can use the parameters of the explosion model without allowance for the influence of the neutrino magnetic moment.

Our analysis based on detailed data on the radial distributions and time evolution of physical properties in a supernova core obtained in the specific model of a successful explosion [56] showed that the gradient of the electron fraction $\mathrm{d}Y_e/\mathrm{d}r$ in Eq. (5.135) grows fairly rapidly with time at point $Y_e = 1/3$ and, thus, the envelope becomes more transparent to active neutrinos at a fixed magnetic field strength. This means that the neutrino signal from the supernova can be attenuated within some limited time interval after its explosion.

Thus, if the Dirac neutrino had a magnetic moment and if the magnetic field in the supernova envelope were sufficiently strong, then the characteristic effect of a significant attenuation of the initial neutrino signal intensity peak predicted by supernova models could take place. For example, there would be a tenfold reduction in the neutrino signal ($W_{LL} = 0.1$) for typical parameters of the medium at a magnetic field strength

$$B_\perp = 4.9 \times 10^{13}\,\mathrm{G} \left(\frac{10^{-13}\mu_\mathrm{B}}{\mu_\nu}\right) \times \left(\frac{\rho}{10^{10}\,\mathrm{g\,cm^{-3}}}\right)^{1/2} \left(\frac{\mathrm{d}Y_e}{\mathrm{d}r} \times 10^7\,\mathrm{cm}\right)^{1/2}. \tag{5.137}$$

Note that the possible strengths of a magnetic field generated in a supernova envelope are believed to reach 10^{16} G [67, 68, 75–78].

Note another possible interesting manifestation of the neutrino magnetic moment. If a magnetar with a poloidal magnetic field of $10^{14} - 10^{15}$ G is formed during a supernova explosion, then, given that Eqs. (5.123) and (5.135) contain the transverse magnetic field component $B_\perp$, the neutrinos can avoid the conversion of their helicity only in a narrow region near the poles. When the nascent magnetar rotates around

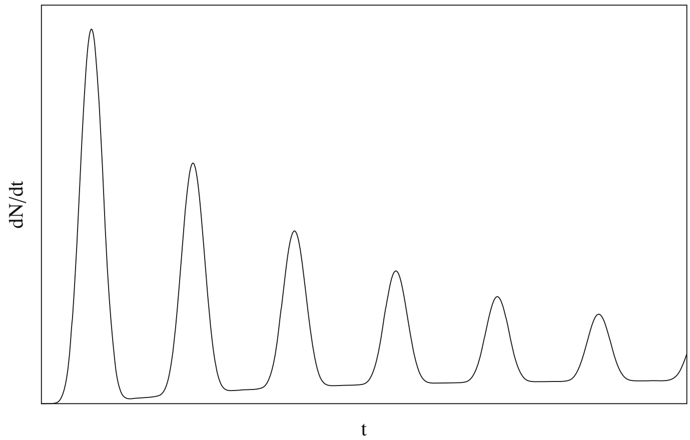

Fig. 5.15 Illustration of the pulsating behavior of the neutrino signal from a nascent magnetar rotating around an axis that does not coincide with its magnetic moment, a neutrino pulsar

an axis that does not coincide with its magnetic moment and if we are lucky with the orientation of the rotation axis, the neutrino signal will have a pulsating behavior, as is illustrated in Fig. 5.15, i.e., a kind of a neutrino pulsar can be observed.

It should be noted that, strictly speaking, the described influence of a strong magnetic field when the neutrino has a magnetic moment on the time evolution of the neutrino signal is incomplete without allowance for the effects of neutrino flavor oscillations (see, e.g., [79]). The combined action of these effects on the neutrino flux requires a special study.

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Chapter 6
Neutrino-Electron Interactions in External Active Media

As it was mentioned above, an intense electromagnetic field makes possible the processes which are forbidden in a vacuum such as the neutrino decay into the W^+ boson and a charged lepton, $\nu \to \ell^- W^+$ ($\ell = e, \mu, \tau$) and the production of a lepton pair by neutrino, $\nu \to \nu \ell^- \ell^+$. In this chapter, we present in details the technique of calculations of the neutrino-electron processes in external active media. We consider mainly the two processes. The first one, which is possible in an intense external electromagnetic field and in the case of sufficiently high neutrino energy, is the decay $\nu \to e^- W^+$. The second process is the electron–positron pair production by a neutrino $\nu \to \nu + e^- + e^+$. We present the procedure of calculation of the process probability in the case of a strong magnetic field, when an electron and a positron are created in the ground Landau level, and in the crossed field limit. We calculate also the four-vector of the mean values of the neutrino energy and momentum losses due to the process $\nu \to \nu + e^- + e^+$, which could be essential in astrophysical applications. The process of the electron–positron pair production by neutrino in a strong magnetic field, if one more component of the external active medium which is dense plasma is taken into account, should be suppressed by the Fermi—Dirac statistical factors. In this chapter, we also consider the electron–positron plasma influence on the process $\nu \to \nu + e^- + e^+$, and take into consideration the crossed neutrino-electron processes. We also try to apply the results obtained to the well-known problem of large kick velocities of pulsars born in supernova explosions.

6.1 The $\nu \to e^- W^+$ Process in a Strong Magnetic Field

The probability of the neutrino decay $\nu \to e^- W^+$ in an external electromagnetic field is one of the most interesting results that can be extracted from the neutrino self-energy operator. This probability can be expressed in terms of the imaginary part of the amplitude (4.63) with the neutrino self-energy operator (4.132).

A. Kuznetsov and N. Mikheev, *Electroweak Processes in External Active Media*, Springer Tracts in Modern Physics 252, DOI: 10.1007/978-3-642-36226-2_6,

For simplicity, hereafter we neglect the neutrino mass m_ν, taking the density matrix of the left-handed neutrino as $\rho(p) = \gamma_L\,(p\gamma)$. One obtains:

$$w(\nu \to e^- W^+) = \frac{1}{E}\,\mathrm{Im}\,\mathcal{M}(\nu_e \to \nu_e)$$
$$= -\frac{1}{E}\,\mathrm{Im}\,\mathrm{Tr}\,[\Sigma(p)\,\gamma_L\,(p\gamma)] = -2\,\frac{p_\perp^2}{E}\,\mathrm{Im}\,\bar{\mathcal{B}}_L\,. \tag{6.1}$$

An analysis of the neutrino decay $\nu \to e^- W^+$ in an external field is of interest only at ultrahigh neutrino energies.

In the papers [1–4], the neutrino decay width in an external electromagnetic field was calculated in the crossed field approximation, in which case the width is expressed in terms of the dynamical field parameter χ and the lepton mass parameter λ:

$$\chi = \frac{e(pFFp)^{1/2}}{m_W^3}\,, \qquad \lambda = \frac{m_e^2}{m_W^2}\,. \tag{6.2}$$

In the frame where the field is pure magnetic one, the dynamical field parameter takes the form:

$$\chi = \frac{eB\;p_\perp}{m_W^3}\,. \tag{6.3}$$

The decay width is expressed via the parameters (6.2) as follows, see Eqs. (4.203) and (5.53):

$$w(\nu \to e^- W^+) = \frac{\sqrt{2}\,G_\mathrm{F}\,m_W^4\,\chi^{2/3}}{12\pi\,E}$$
$$\times \int\limits_0^1 \frac{\mathrm{d}v\,v\,[2(1+v)(2+v) + \lambda\,(1-v)(2-v)]}{[v(1-v)]^{4/3}} \left(-\frac{\mathrm{dAi}(u)}{\mathrm{d}u}\right), \tag{6.4}$$

where $\mathrm{Ai}(u)$ is the Airy function (5.28) with the argument:

$$u = \frac{v + \lambda\,(1-v)}{[\chi\,v(1-v)]^{2/3}}\,. \tag{6.5}$$

The derivative of the Airy function is expressed via the modified Bessel function $K_\nu(x)$

$$-\frac{\mathrm{dAi}(u)}{\mathrm{d}u} = \frac{u}{\sqrt{3}\,\pi}\,K_{2/3}\left(\frac{2}{3}\,u^{3/2}\right). \tag{6.6}$$

Taking in Eq. (6.4) the limit $\chi,\,\lambda \ll 1$, one obtains the result which can be written in terms of the only modified dynamical field parameter

$$\xi = \frac{\chi}{\sqrt{\lambda}} = \frac{eB\, p_\perp}{m_e\, m_W^2}\,. \tag{6.7}$$

The range for the ξ parameter appears to be rather large, $0 < \xi \ll 1/\sqrt{\lambda}$, while $1/\sqrt{\lambda} \gg 1$. Taking account of the exponential decrease of the modified Bessel function $K_\nu(x)$ at large argument value, one can see that the region of small v gives the main contribution into the integral Eq. (6.4) at small χ. Changing the variable $v = \lambda x$, one can transform the decay width to the form

$$w(\nu \to e^- W^+) = \frac{\sqrt{2}\, G_{\mathrm{F}}}{3\pi}\, \frac{(eB\, p_\perp)^2}{m_W^2\, E}\, F(\xi)\,, \tag{6.8}$$

where

$$F(\xi) = \frac{1}{\sqrt{3}\,\pi\,\xi^2} \int\limits_0^\infty \mathrm{d}x\, \frac{1+x}{x} K_{2/3}\left(\frac{2}{3}\, \frac{(1+x)^{3/2}}{\xi\, x}\right). \tag{6.9}$$

We remind that these formulas are valid in the approximation $\xi \ll m_W/m_e$. The range being very wide for the electron, $\xi \ll 1.6 \times 10^5$, is not too wide for the τ lepton, $\xi \ll 45$.

The integration in Eq. (6.9) can be performed exactly to give

$$F(\xi) = \left(1 + \frac{\sqrt{3}}{\xi}\right) \exp\left(-\frac{\sqrt{3}}{\xi}\right). \tag{6.10}$$

The formulas (6.8)–(6.10) should be compared with the results of Refs. [1, 2, 4]. It should be mentioned that the decay width w defined in Refs. [1, 3] is the same, in the natural system of units, than the absorption coefficient α [2] and the damping rate of the neutrino γ [4]. One can see that the absorption coefficient α presented in Eq. (25) of Ref. [2] looks very similar to our Eqs. (6.8) and (6.10). However, the angular dependence in our formulas is quite different: instead of the factor $p_\perp^2/E = E \sin^2\theta$ standing in our Eq. (6.8), there is the factor $p_\perp = E \sin\theta$ in Eq. (25) of Ref. [2].

On the other hand, one can see that our result (6.8)–(6.10) surely contradicts the Eq. (58) of Ref. [4], where an attempt was made of reinvestigation of the process $\nu \to e^- W^+$ in the crossed field approximation. The difference is the most essential at small values of ξ, where the result of Ref. [4] appears to be strongly underestimated.

In the earlier paper by Borisov et al. [1] the calculations of the process $\nu \to e^- W^+$ width were performed in the two limiting cases of the small and large values of the parameter χ. In the limit $\chi^2 \ll \lambda$ their result can be presented in the form

$$w = \frac{\sqrt{2}\, G_{\mathrm{F}}}{\sqrt{3}\,\pi}\, m_e\, eB \sin\theta \exp\left(-\sqrt{3}\, \frac{m_e m_W^2}{eB\, p_\perp}\right), \tag{6.11}$$

and can be reproduced from the general formulas (6.8)–(6.10).

On the other hand, in the limit $\chi \gg 1$ ($\xi \gg 1/\sqrt{\lambda}$) the result of [1] can be written as

$$w = \frac{\sqrt{3}\, G_{\mathrm{F}}}{\sqrt{2}\, \pi}\, m_W\, eB \sin\theta\,, \tag{6.12}$$

and can be reproduced from our more general formula (6.4), or by an easier way from Eqs. (6.1) and (4.220).

A problem of the decay $\nu \to e^- W^+$ has a physical meaning only in the fields of the pulsar type, where the field strength is of order of the critical value $\sim 10^{13}$ G. The above formulas for the probability except for Eq. (6.12) are applicable for relatively weak fields only, $B \ll 10^{13}$ G. It is interesting to consider the process $\nu \to e^- W^+$ in strong magnetic fields of magnetars, of the order of $\sim 10^{14} - 10^{15}$ G, where the crossed-field approximation is inapplicable.

Thus, we will use the following hierarchy of the physical parameters: $p_\perp^2 \gg m_W^2 \gg eB \gg m_e^2$. A general expression for the process $\nu \to e^- W^+$ probability can be obtained by the substitution of Eq. (4.137) into Eq. (6.1) with taking account of Eqs. (3.10)–(3.14). After calculations which are not difficult but rather cumbersome, the process width can be presented in the form

$$w(\nu \to e^- W^+) = \frac{G_{\mathrm{F}}\, (eB)^{3/2}\, p_\perp}{\pi\sqrt{2\pi}\, E}\, \Phi(\eta)\,, \tag{6.13}$$

where $\Phi(\eta)$ is the function depending on the one parameter η only:

$$\eta = \frac{4\, eBp_\perp^2}{m_W^4}\,, \tag{6.14}$$

$$\Phi(\eta) = \frac{1}{\eta} \int\limits_0^\infty \frac{\mathrm{d}y}{y^{1/2}}\, \frac{(\tanh y)^{1/2}}{(\sinh y)^2}\, \frac{(\sinh y)^2 - y\, \tanh y}{(y - \tanh y)^{3/2}} \times \exp\left[-\frac{y\, \tanh y}{\eta(y - \tanh y)}\right]. \tag{6.15}$$

We stress that we have obtained this formula neglecting the electron mass as the smallest parameter in the hierarchy used.

The formulas (6.13), (6.15) are valid in a wide region of the parameter η values, $0 < \eta \ll m_W^2/(eB)$. The function $\Phi(\eta)$ is essentially simplified at large and small values of the argument.

In the limit $\eta \gg 1$, one obtains:

$$\Phi(\eta \gg 1) \simeq \frac{1}{3}\sqrt{\pi(\eta - 0.3)}\,, \tag{6.16}$$

and the error is less than 1 % for $\eta > 10$.

The formulas (6.13), (6.16) reproduce the probability (6.8), (6.10), where the limit $\xi \gg 1$ should be taken, and $F(\xi \gg 1) \simeq 1$.

In the other limit $\eta \ll 1$ one obtains

$$\Phi(\eta \ll 1) \simeq \exp\left(-\frac{1}{\eta}\right)\left(1 - \frac{1}{2}\eta + \frac{3}{4}\eta^2\right) \tag{6.17}$$

and the error is less than 1 % for $\eta < 0.5$.

The formulas obtained allow to establish an upper limit on the energy spectrum of neutrinos propagating in a strong magnetic field. Let us take the typical size R of the region with the strong magnetic field as $R \sim 10\,\text{km}$. If the neutrino mean free path $\lambda = 1/w$ is much less than the field size, $\lambda \ll R$, all the neutrinos are decaying inside such the field. For $\lambda = 1\,\text{km} \ll R$, we can find the cutoff energies E_c for the neutrino spectrum, depending on the magnetic field strength, as follows:

(i) for relatively weak field, $B \simeq 0.1 B_e \simeq 4 \times 10^{12}$ G, the neutrino mean free path can be obtained from Eq. (6.11):

$$\lambda \simeq \frac{4.9\,\text{m}}{B_{0.1}\,\sin\theta} \exp\left(\frac{219}{B_{0.1}\,E_{15}\,\sin\theta}\right), \tag{6.18}$$

where $B_{0.1} = B/(0.1 B_e)$, $E_{15} = E/(10^{15}\text{eV})$, and the cutoff energy corresponding to $\lambda = 1\,\text{km}$, at $B_{0.1} = 1$, $\theta = \pi/2$, is

$$E_c \simeq 0.4 \times 10^{17}\text{eV}; \tag{6.19}$$

(ii) for relatively strong field, $B \simeq 10 B_e \simeq 4 \times 10^{14}$ G, the neutrino mean free path can be obtained from Eqs. (6.13), (6.17):

$$\lambda \simeq \frac{3.2\,\text{cm}}{B_{10}^{3/2}\,\sin\theta} \exp\left(\frac{4.0}{B_{10}\,E_{15}^2\,\sin^2\theta}\right), \tag{6.20}$$

where $B_{10} = B/(10 B_e)$, and the cutoff energy corresponding to $\lambda = 1\,\text{km}$, at $B_{10} = 1$, $\theta = \pi/2$, is

$$E_c \simeq 0.6 \times 10^{15}\text{eV}. \tag{6.21}$$

The results obtained show an essential influence of the intense magnetic field on the process $\nu \to e^- W^+$ width. Despite the exponential character of suppression of the width in a strong field, Eqs. (6.13), (6.17), as well as in a weak field, Eq. (6.11), the decay width in a strong field is greater in orders of magnitude than the one in a weak field, for the same neutrino energy.

6.2 The $\nu \to \nu e^- e^+$ Process in a Strong Magnetic Field

The process of the electron–positron pair production by a neutrino

$$\nu(P) \to \nu(P') + e^-(p) + e^+(p')$$

for the relatively small momentum transfers $|q^2| \ll m_W^2$, where m_W is the W boson mass, is described by the effective local Lagrangian of the neutrino-electron interaction (4.66), when the propagators of intermediate W and Z bosons are shrunk into a point, as is shown in Fig. 4.8.

6.2.1 Calculation of the Differential Probability Based on the Solutions of the Dirac Equation

The total amplitude for neutrino-electron processes is obtained directly from the Lagrangian (4.66) where known solutions of the Dirac equation in a magnetic field (2.1) must be used. As was already mentioned in Chap. 2, in a strong magnetic field, $eB \gg E^2$, the electron and the positron can be produced only in the states that correspond to the ground Landau level (2.37).

Using the Lagrangian (4.66) and the wave functions (2.37), we write the S matrix element of the process $\nu \to \nu e^- e^+$ in the following form

$$S = \mathrm{i}\,\frac{G_{\mathrm{F}}}{\sqrt{2}}\,\frac{(2\pi)^3\,\delta(\varepsilon + \varepsilon' - q_0)\,\delta(p_y + p'_y - q_y)\,\delta(p_z + p'_z - q_z)}{\sqrt{2EV\,2E'V\,2\varepsilon(\varepsilon + m_e)L_yL_z\,2\varepsilon'(\varepsilon' - m_e)L_yL_z}}$$
$$\times\, \mathrm{e}^{-q_\perp^2/4eB - \mathrm{i}q_x(p_y - p'_y)/2eB}[\bar{u}(p_\parallel)\,\hat{j}(C_V - C_A\gamma_5)\,u(-p'_\parallel)], \qquad (6.22)$$

where $q = P - P' = p + p'$ is the change of the four-vector of the neutrino momentum equal to the four-momentum of the e^-e^+ pair, ε and ε' are the electron and positron energies, $q_\perp$ is the projection of the vector $\mathbf{q}$ on the plane perpendicular to the vector $\mathbf{B} = (0, 0, B)$, $q_\perp^2 = q_x^2 + q_y^2$, and $j_\alpha = \bar{\nu}(P')\gamma_\alpha(1 - \gamma_5)\nu(P)$ is the Fourier transform of the current of the left-handed neutrinos. Note that in this approximation where the field strength is the largest physical parameter of the problem, the exponential factor $\mathrm{e}^{-q_\perp^2/4eB}$ in the amplitude (6.22) slightly differs from unity and may be omitted. Direct calculations taking into account the conservation laws in (6.22) give

$$[\bar{u}(p_\parallel)\,\hat{j}(C_V - C_A\gamma_5)\,u(-p'_\parallel)] = \frac{m_e\sqrt{q_\parallel^2 - 4m_e^2}}{\sqrt{q_\parallel^2}}\,\frac{q_z}{|q_z|}\,[C_V(j\tilde{\varphi}q) + C_A(j\tilde{\varphi}\tilde{\varphi}q)]. \qquad (6.23)$$

The further calculations will be performed for the case when the electron mass is the smallest parameter of the problem, i.e. for the following hierarchy: $eB \gg E^2 \gg m_e^2$. In this case the expression (6.23) and thus the total amplitude (6.22) contain the suppression associated with the relative smallness of the electron mass. This suppression is not random and reflects the angular momentum conservation law. For example, in the crossed process $\nu\bar{\nu} \to e^- e^+$ being described by the same amplitude (6.22), the total spin of a neutrino–antineutrino pair in the center-of-inertia system is one, whereas the total spin of an electron–positron pair in the ground Landau level is zero. Consequently, the amplitude of the process would be zero for massless particles and contain the suppression in the relativistic limit under study. However, an analysis shows that when integration is performed over the phase volume, the main contribution arises from the kinematic region where $\sqrt{q_\parallel^2} \sim m_e$, and this suppression disappears.

For the probability of the process per unit time we obtain

$$W = \frac{1}{\mathcal{T}} \int \frac{d^3 P' \, V}{(2\pi)^3} \, |S|^2 \, dn_{e^-} \, dn_{e^+}, \tag{6.24}$$

where $\mathcal{T}$ is the total interaction time, and the elements of the phase volume are introduced for the electron and the positron occupying the ground Landau level:

$$dn_{e^-} = \frac{d^2 p \, L_y L_z}{(2\pi)^2}, \qquad dn_{e^+} = \frac{d^2 p' \, L_y L_z}{(2\pi)^2}. \tag{6.25}$$

Substituting (6.22) into (6.24) and integrating using δ functions with respect to $d^2 p'$ [where, as is usually the case $\delta^3(0) = T L_y L_z/(2\pi)^3$], we obtain for the total probability per unit time

$$\begin{aligned} W = {} & \frac{G_F^2}{32(2\pi)^4 E} \, \frac{1}{L_x} \int \frac{d^3 P'}{E'} \, \frac{dp_y dp_z}{\varepsilon(\varepsilon + m_e)\varepsilon'(\varepsilon' - m_e)} \, \delta(\varepsilon + \varepsilon' - q_0) \\ & \times |\bar{u}(p_\parallel) \, \hat{j} \, (C_V - C_A \gamma_5) \, u(-p'_\parallel)|^2, \end{aligned} \tag{6.26}$$

where we need to substitute $\varepsilon' = \sqrt{m_e^2 + (q_z - p_z)^2}$ and $p'_z = q_z - p_z$. The integrand in (6.26) does not depend on p_y, and the integration should be performed in accordance with (5.13). Upon integrating in (6.26) with respect to the electron momentum we obtain the probability of the $\nu \to \nu e^- e^+$ process in the form of the following integral over the final neutrino momentum:

$$W = \frac{G_F^2 eB m_e^2}{32\pi^4 E} \int \frac{d^3 P'}{E'} \, \Theta(q_0 - \sqrt{q_z^2 + 4m_e^2}) \, \frac{|C_V (j\tilde{\varphi} q) + C_A (j\tilde{\varphi}\tilde{\varphi} q)|^2}{(q_\parallel^2)^{3/2} (q_\parallel^2 - 4m_e^2)^{1/2}} \, . \tag{6.27}$$

6.2.2 Calculation Based on the Imaginary Part of the Loop Amplitude

As for the photon decay, we present here another method of calculation of the probability (6.27) based on the unitarity relation. The crossed process for the pair production by a neutrino $\nu \to \nu e^- e^+$ is the reaction of the conversion of the neutrino–antineutrino pair into the electron–positron pair $\nu\bar{\nu} \to e^- e^+$. It is well known that the cross section for this reaction is related to the imaginary part of the transition $\nu\bar{\nu} \to \nu\bar{\nu}$ via the electron loop (see Fig. 6.1) by the unitarity condition

$$\sigma(\nu\bar{\nu} \to e^- e^+) = \frac{1}{q^2}\, \mathrm{Im}\, \mathcal{M}(\nu\bar{\nu} \to \nu\bar{\nu}), \tag{6.28}$$

where q^α is the four-momentum of the neutrino–antineutrino pair. It can easily be seen that the relation (6.28) makes it possible to find the probability of the process $\nu \to \nu e^- e^+$, if we integrate this relation over the phase volume of the final neutrino. We have

$$w(\nu \to \nu e^- e^+)E \;=\; \frac{1}{16\pi^3} \int \frac{\mathrm{d}^3 P'}{E'}\, \mathrm{Im}\, \mathcal{M}(\nu\bar{\nu} \to \nu\bar{\nu}). \tag{6.29}$$

Remember that $P^\alpha = (E, \mathbf{P})$ and $P'^\alpha = (E', \mathbf{P}')$ are the four-momenta of the initial and final neutrinos, and $q = P - P'$.

The magnetic-field-induced part of the process amplitude, Fig. 6.1, can be easily constructed from the generalized amplitude (4.24) of the vector—vector type (4.31), the axial-vector—axial-vector type (4.32), and the vector—axial-vector type (4.33), with the corresponding substitutions of the generalized currents:

$$j_{V\alpha} \to \frac{G_\mathrm{F}}{\sqrt{2}} C_V\, j_\alpha, \quad j_{A\alpha} \to \frac{G_\mathrm{F}}{\sqrt{2}} C_A\, j_\alpha, \tag{6.30}$$

where j_α is the neutrino current. It should be noted also that $(jq) = 0$ and $\beta = eB$. We obtain

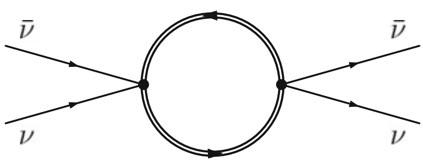

Fig. 6.1 The Feynman diagram for the process $\nu\bar{\nu} \to \nu\bar{\nu}$. The *double line* corresponds to the exact propagator of an electron in a magnetic field

$$\Delta\mathcal{M}_{j\to j} = \frac{G_F^2}{8\pi^2}\left\{ \left(C_V^2 Y_{VV}^{(1)} + C_A^2 Y_{AA}^{(1)}\right) \frac{|q\varphi j|^2}{q_\perp^2} \right.$$
$$+ \left(C_V^2 Y_{VV}^{(2)} + C_A^2 Y_{AA}^{(2)}\right) \frac{|q\tilde{\varphi} j|^2}{q_\parallel^2} + \left(C_V^2 Y_{VV}^{(3)} + C_A^2 Y_{AA}^{(3)}\right) \frac{q^2|q\varphi\varphi j|^2}{q_\perp^2 q_\parallel^2}$$
$$\left. + 2C_V C_A\, eB \left(Y_{VA}^{(1)} + \frac{q^2}{q_\perp^2}\, Y_{VA}^{(2)} \right) \frac{\mathrm{Re}[(q\tilde{\varphi} j)(q\varphi\varphi j^*)]}{q_\parallel^2} \right\}. \tag{6.31}$$

Turning to the strong field limit, as was done in Sect. 5.1.2, one can show that only the following functions have the imaginary parts, of all the functions Y included in (6.31)

$$\mathrm{Im} Y_{VV}^{(2)} = \frac{q^2}{q_\perp^2}\, \mathrm{Im} Y_{AA}^{(3)} = eB\ \mathrm{Im}\left(Y_{VA}^{(1)} + \frac{q^2}{q_\perp^2}\, Y_{VA}^{(2)} \right)$$
$$= \frac{4\pi e B m_e^2}{\sqrt{q_\parallel^2 (q_\parallel^2 - 4m_e^2)}}\, \Theta(q_\parallel^2 - 4m_e^2). \tag{6.32}$$

Substituting (6.31) into (6.29) and taking account of (6.32), we immediately obtain the expression (6.27) for the probability of the process $\nu \to \nu e^- e^+$.

6.2.3 The Total Process Probability

It is convenient to perform the further integration over the final neutrino momentum, without loss of generality, not in the arbitrary frame (referred to as K), but in the special frame K_0, where the initial neutrino momentum is perpendicular to the magnetic field direction, $P_z = 0$. In the case of a pure magnetic field we can then return from the frame K_0 to K by the Lorentz transformation along the field direction (we recall that the field is invariant with respect to this transformation). Really, the value EW defined by (6.27) is seen to contain the invariants only, including the sign of the Θ function argument.

It is worthwhile to introduce in (6.27) the dimensionless cylindrical coordinates in the space of the final neutrino momentum vector $\mathbf{P}'$,

$$\rho = \sqrt{P_x'^2 + P_y'^2}/E_\perp,\ \tan\phi = P_y'/P_x',\ \zeta = P_z'/E_\perp.$$

Here, $E_\perp$ is the initial neutrino energy in the K_0 frame, connected with its energy E in the arbitrary frame K by the relation $E_\perp = E\sin\theta$, where θ is the angle between the initial neutrino momentum and the field direction in the K frame.

Representing the expression (6.27) in the form of the integral over the ρ, ϕ and ζ variables one obtains:

$$
\begin{aligned}
EW = {} & \frac{G_F^2 m_e^2 eBE_\perp^2}{2\pi^3} \int\limits_0^{2\pi} \frac{\mathrm{d}\phi}{2\pi} \int\limits_0^{1-\lambda} \mathrm{d}\rho\, \rho\, \mathrm{e}^{-\varepsilon(1-2\rho\cos\phi+\rho^2)/2} \\
& \times \int\limits_{-\zeta_m}^{\zeta_m} \frac{\mathrm{d}\zeta}{\gamma\sqrt{\rho^2+\zeta^2}\,(1-2\sqrt{\rho^2+\zeta^2}+\rho^2)^2} \\
& \times \Big\{ (C_V^2+C_A^2)\Big[(1+\rho^2)\sqrt{\rho^2+\zeta^2}-2\rho^2\Big] - 2C_V C_A(1-\rho^2)\zeta \\
& - (C_V^2-C_A^2)\,\rho\,(1-2\sqrt{\rho^2+\zeta^2}+\rho^2)\cos\phi \Big\},
\end{aligned}
\tag{6.33}
$$

where

$$
\gamma = \sqrt{1-\frac{4m_e^2}{q_\parallel^2}} = \sqrt{1-\frac{\lambda^2}{1-2\sqrt{\rho^2+\zeta^2}+\rho^2}},
$$
$$
\lambda = \frac{2m_e}{E_\perp}, \quad \zeta_m = \frac{1}{2}\sqrt{\left(1+\rho^2-\lambda^2\right)^2 - 4\rho^2}\,.
$$

Note that the integrand in (6.33) has an enhancement that completely compensates for the suppression by the smallness of the electron mass. The main contribution then comes from the region near the upper limit of the integral over ρ corresponding to the relation $\sqrt{q_\parallel^2} \sim m_e$.

The term in (6.33) with the $C_V C_A$ product is caused by the interference of the vector and axial-vector electron currents. It determines the asymmetry of the electron emission with respect to the magnetic field, and obviously this term does not contribute to the probability. However, it could be important in calculating the asymmetry of the averaged neutrino momentum loss, see Sect. 6.4.1.

Neutrino energies in the region $E \gg m_e$ are typical for the above-mentioned astrophysical processes. It should be noted that expressions (6.27) and (6.33), which were obtained for the ground Landau level, have the physical meaning of the total probability of the process only for $eB > E^2/2$, in which case the contribution of other Landau levels is completely suppressed. For the sake of completeness, we nevertheless present here the asymptotic expressions for both strong ($eB \gg m_e^2$) and relatively weak ($eB \ll E^2$) fields in order to estimate below the relative contribution of the ground Landau level to the probability of the process. The cumbersome expression (6.33) is then replaced by simple formulas whose applicability ranges partially overlap.

(i) For $eB \gg m_e^2$, we have

$$
W = \frac{G_F^2(C_V^2+C_A^2)}{16\pi^3}\, eBE^3 \sin^4\theta\; f_1(\varepsilon)\,,
\tag{6.34}
$$

where

$$f_1(\varepsilon) = 4 \int_0^1 \mathrm{d}\rho\, \rho (1-\rho^2)\, \mathrm{e}^{-\varepsilon(1+\rho^2)/2}\, I_0(\varepsilon\rho)$$
$$= 1 - \frac{2}{3}\varepsilon + \frac{5}{16}\varepsilon^2 - \frac{7}{60}\varepsilon^3 + \cdots, \tag{6.35}$$

$\varepsilon = E_\perp^2/eB$, and $I_0(x)$ is the modified Bessel function of the zeroth order. For $eB \gg E_\perp^2$, the formula (6.34) takes the simple form

$$W = \frac{G_F^2(C_V^2 + C_A^2)}{16\pi^3}\, eBE^3 \sin^4\theta. \tag{6.36}$$

In this region, the result determines precisely the total probability of the process. It can be seen that the probability grows with neutrino energy in proportion to E^3, but it will be shown below that, at higher neutrino energies, higher Landau levels come into play. As a result, this type of behavior changes to a linear growth, which persists up to energies corresponding to the boundary of the applicability range of the effective local Lagrangian (4.66).

In the case of relatively weak fields ($m_e^2 \ll eB \ll E_\perp^2$), it follows from (6.34) that the contribution of the ground Landau level is given by

$$W = \frac{G_F^2(C_V^2 + C_A^2)}{2^{3/2}\pi^{7/2}}\, (eB)^{5/2}\, \sin\theta. \tag{6.37}$$

(ii) For $eB \ll E_\perp^2$, the general expression (6.33) yields

$$W = \frac{2^{1/2} G_F^2 m_e^2 (eB)^{3/2}}{\pi^{7/2}} \left[(C_V^2 + C_A^2) \int_1^\infty \mathrm{d}u\, u\, \mathrm{e}^{-2u^2/\eta}\, \mathbf{E}\left(\frac{\sqrt{u^2-1}}{u}\right) \right.$$
$$\left. - C_V^2 \int_1^\infty \frac{\mathrm{d}u}{u}\, \mathrm{e}^{-2u^2/\eta}\, \mathbf{K}\left(\frac{\sqrt{u^2-1}}{u}\right)\right] \sin\theta\,, \tag{6.38}$$

where $\eta = eB/m_e^2 = B/B_e$ is the field intensity parameter and $\mathbf{K}(k)$ and $\mathbf{E}(k)$ are the complete elliptic integrals of the first and second type, respectively [5]. It should be noted that the applicability ranges of formulas (6.34) and (6.38) partially overlap, and the region of overlap is $m_e^2 \ll eB \ll E_\perp^2$. Indeed, if we go over to the extreme case of $\eta \gg 1$ in (6.38), formula (6.37) is recovered, as might have been expected. In weak fields, $eB \ll m_e^2$, the result is exponentially small, as is usually the case; specifically, we have

$$W = \frac{G_F^2 C_A^2}{(2\pi)^{5/2}}\, (eB)^{5/2}\, \mathrm{e}^{-2/\eta}\, \sin\theta\,. \tag{6.39}$$

6.3 The $\nu \to \nu e^- e^+$ Process in a Crossed Field

6.3.1 A Historical Overview

Theoretical study of the process of the electron–positron pair production by a neutrino in the crossed field limit has a rather long history [6–14]. The correct type of dependence of the probability on the dynamical parameter χ:

$$\chi^2 = \frac{e^2 (P F F P)}{m_e^6}$$

in the leading log approximation, namely, $\sim\chi^2 \ln \chi$, was found in the paper [6], where the numerical coefficient was incorrect, however. In succeeding papers, attempts were made to adjust this coefficient and to find the next postlogarithm terms, which could appear quite essential when $\ln \chi$ is not very large.

According to the definition of the problem in the crossed field approximation, one should consider the ultrarelativistic neutrino only, which exists as the left-handed one due to the chiral type of its interaction in the frame of the Standard Model, even if the neutrino mass is nonzero. This remains true if we admit the existence of exotic properties of the neutrino, which could lead in certain physical conditions to the depolarizing effects, which were not observed yet. Lack of understanding that unpolarized ultrarelativistic neutrino fluxes do not exist in Nature, often caused erroneous extra factors of 1/2 in formulas for the process probabilities with a neutrino in the initial state because of the non-physical averaging on its polarizations (see, e.g., [12, 15]).

There are significant differences in the results for the probability of the process $\nu \to \nu e^- e^+$ in the crossed field, obtained in the listed papers. In Ref. [12], dedicated to the study of the decay of a massive neutrino $\nu_i \to \nu_j e^- e^+$ $(m_i > m_j + 2m_e)$ in an external field, the different formulas for the probability of the process were also compared, and a conclusion on the mutual agreement of the results was made. In our opinion, such an agreement is absent.

Indeed, the probability of the process in the limit $\chi \gg 1$ can be presented as follows:

$$W(\nu \to \nu e^- e^+) = K\, W_0\, \chi^2 \left(\ln \chi - \frac{1}{2} \ln 3 - \gamma_E + \Delta \right), \tag{6.40}$$

where

$$W_0 = \frac{G_F^2\, (C_V^2 + C_A^2)\, m_e^6}{27 \pi^3 E}, \tag{6.41}$$

$\gamma_E = 0.577\ldots$ is the Euler constant, E is the energy of the initial neutrino. The constants K and Δ entering the expression (6.40), which were obtained by different authors are given in Table 6.1. It should be noted that in Refs. [6, 12], calculations

Table 6.1 The constants K and Δ of the expression (6.40), obtained in different studies

Authors	K	Δ
Choban and Ivanov (1969) [6]	$\dfrac{29}{1024\pi}$	–
Borisov e a. (1983) [7]	1	$-2\ln 2 - \dfrac{389}{384} + \dfrac{9}{128}\,\dfrac{C_V^2 - C_A^2}{C_V^2 + C_A^2}$
Knizhnikov e a. (1984) [8]	$\dfrac{9}{16}\,\dfrac{E}{m_e}$	–
Borisov e a. (1993) [9]	$\dfrac{1}{2}$	$+\dfrac{5}{4}$
Our result (1997) [10]	1	$-\dfrac{29}{24}$
Borisov and Zamorin (1999) [12]	$\dfrac{1}{2}$	$-\dfrac{29}{24}$

were performed with taking account of the neutrino-electron interaction through the W boson only. For comparison of the result (6.40) with these studies, one should put in Eq. (6.40), respectively, $C_V = C_A = 1$ [6] and $C_V = C_A = |U_{ei} U_{e3}|$ [12]. The loss of the factor m_e/E in formulas of [8] for the probability is not a numerical but a physical errors, since it leads to a loss of relativistic invariance of the value $E\,W$.

As it was already noted, the formula (6.40) for the probability describes rather special case of $\ln\chi \gg 1$. There exist a number of physical tasks where the situation is realized when the dynamical parameter takes moderately high values, so that $\chi \gg 1$, but $\ln\chi \sim 1$. In this case, the crossed field approximation is applicable, but the above condition $\ln\chi \gg 1$ is not satisfied, so that the formula (6.40) is not enough. For example, one would wish to consider the next terms in the expansion in inverse powers of the large parameter χ. On the other hand, the formulas for the probability for arbitrary values of the χ parameter presented in some of the listed papers, have a very cumbersome form of multiple integrals, and are inconvenient for the analysis.

The final point in the analysis of the process $\nu \to \nu e^- e^+$ in the crossed field approximation was put, as we believe, in our papers [13, 14]. Here we present the calculation in some detail.

6.3.2 Calculation of the Differential Probability Based on the Imaginary Part of the Loop Amplitude

Because of differences in the results for the probability of the process $\nu \to \nu e^- e^+$ in the crossed field, see Table 6.1, a reliable analysis was necessary. For this sake, we performed the calculation of the differential probability of the process using the two different methods. The first one was based on the exact solutions of the Dirac equation in the crossed field (2.40). In the second method, the imaginary part of the loop amplitude of the transition $\nu\bar{\nu} \to e^- e^+ \to \nu\bar{\nu}$, see Fig. 6.1, was used. The calculation based on the solutions of the Dirac equation is similar in many details to the one performed in Sect. 5.2.1 for the photon decay $\gamma \to e^- e^+$ in a crossed field. We do not present here this analysis and refer the reader to Sect. 5.6.1 of our previous book [16]. In this section, we focus on exploiting the loop amplitude which allows to find the probability via the unitarity relation.

Note that the results for the process $\nu \to \nu e^- e^+$ are trivially generalized to other neutrino-lepton processes. For example, the probability of the process $\nu_e \to \nu_e e^- e^+$ with replacing $m_e \to m_\mu$ and the corresponding change of the constants C_V, C_A gives the probability of the process $\nu_\mu \to \nu_\mu \mu^- \mu^+$, etc.

As in Sect. 6.2.2, we use the relation (6.29) where the field-induced amplitude is constructed from the amplitudes of the vector—vector type (4.35), the axial-vector—axial-vector type (4.38), and the vector—axial-vector type (4.39), with the corresponding substitutions of the generalized currents (6.30). Similarly to Eq. (6.31) we obtain:

$$\begin{aligned}\Delta\mathcal{M}_{j\to j} = \frac{G_{\mathrm{F}}^2}{8\pi^2}\Bigg\{ &\left(C_V^2 Y_{VV}^{(1)} + C_A^2 Y_{AA}^{(1)}\right) \frac{|qFj|^2}{(qFFq)} \\ &+ \left(C_V^2 Y_{VV}^{(2)} + C_A^2 Y_{AA}^{(2)}\right) \frac{|q\tilde{F}j|^2}{(qFFq)} + \left(C_V^2 Y_{VV}^{(3)} + C_A^2 Y_{AA}^{(3)}\right) \frac{q^2 (jFFj^*)}{(qFFq)} \\ &+ 2eC_V C_A \left(Y_{VA}^{(1)} + Y_{VA}^{(2)}\right) \frac{\mathrm{Re}[(q\tilde{F}j)(qFFj^*)]}{(qFFq)}\Bigg\},\end{aligned} \tag{6.42}$$

where the functions Y for the crossed field from Eqs. (4.35), (4.38), and (4.39) should be substituted. We remind that the imaginary part of the Hardy—Stokes function is expressed via the Airy function, $\mathrm{Im} f(z) = \pi\,\mathrm{Ai}(z)$.

The resulting probability of the process takes the form of the following integral over the final neutrino momentum

$$\begin{aligned} W = \frac{G_{\mathrm{F}}^2 (C_V^2 + C_A^2) m_e^2}{2^7 \pi^4 E} \int \frac{\mathrm{d}^3 P'}{E'} \int\limits_0^1 \mathrm{d}u \Bigg\{ &i_0 \left[\frac{2}{3}\,\frac{3+u^2}{(1-u^2)^{1/3}} \left(\frac{\chi_q}{4}\right)^{2/3} \mathrm{Ai}'(U) \right. \\ &\left. - \frac{q^2(1-u^2)}{2m_e^2}\,\mathrm{Bi}(U)\right] + i_1\, \frac{q^2}{24 m_e^2}\,\frac{3+u^2}{(1-u^2)^{1/3}} \left(\frac{4}{\chi_q}\right)^{4/3} \mathrm{Ai}'(U) \\ &- i_2\, \frac{1}{8}\,(1-u^2)^{2/3} \left(\frac{4}{\chi_q}\right)^{4/3} \mathrm{Ai}'(U) \\ &+ \frac{C_A^2}{C_V^2 + C_A^2}\left[i_0\, 2\mathrm{Bi}(U) - i_1\, \frac{1}{(1-u^2)^{1/3}} \left(\frac{4}{\chi_q}\right)^{4/3} \mathrm{Ai}'(U)\right] \\ &- \frac{C_V C_A}{4(C_V^2 + C_A^2)}\, i_3 \left(\frac{4}{\chi_q}\right)^2 U\,\mathrm{Ai}(U)\Bigg\}. \end{aligned} \tag{6.43}$$

where the invariants are introduced that are constructed of the neutrino current and the field tensor:

$$i_0 = (jj^*), \quad i_1 = \frac{e^2 (jFFj^*)}{m_e^4}, \quad i_2 = \frac{e^2 (q\tilde{F}j)(q\tilde{F}j^*)}{m_e^6},$$
$$i_3 = \frac{e^3 \mathrm{Re}[(q\tilde{F}j)(qFFj^*)]}{m_e^8}, \tag{6.44}$$

$\mathrm{Ai}(U)$ is the Airy function (5.28) while $\mathrm{Bi}(U)$ is the integral:

$$\mathrm{Bi}(U) = \int_U^\infty \mathrm{d}y \, \mathrm{Ai}(y), \tag{6.45}$$

and the argument of the Airy function is

$$U = \left(\frac{4}{\chi_q (1-u^2)}\right)^{2/3} \left(1 - \frac{q^2 (1-u^2)}{4m_e^2}\right). \tag{6.46}$$

$q = P - P'$ is the four-momentum lost by a neutrino. Hereafter, we denote the dynamical parameter constructed of the initial neutrino momentum P as χ, and the dynamical parameter constructed of the q momentum as χ_q:

$$\chi = \left(\frac{e^2 (PFFP)}{m_e^6}\right)^{1/2}, \quad \chi_q = \left(\frac{e^2 (qFFq)}{m_e^6}\right)^{1/2}. \tag{6.47}$$

6.3.3 The Total Process Probability

To integrate the expression (6.43) with respect to the final neutrino momentum, let us introduce new variables κ, ξ, and ϕ, which are the relativistic invariants, as follows

$$\kappa = -\frac{q^2}{[4e^2 (PFFP)]^{1/3}}, \qquad \xi = \sqrt{\frac{(qFFq)}{(PFFP)}},$$
$$\cos\phi = \frac{(P\tilde{F}P')}{\sqrt{(P\tilde{F}P')^2 + (PFP')^2}}. \tag{6.48}$$

In the frame where the initial neutrino momentum $\mathbf{P}$ is perpendicular to the magnetic field vector $\mathbf{B}$, the angle ϕ has a meaning of the azimuthal angle in the plane perpendicular to the vector $\mathbf{P}$, between the magnetic field and the projection of the vector $\mathbf{P}'$ on this plane. With these variables, the invariants (6.44) take the form

$$
\begin{aligned}
i_0 &= -8\, m_e^2\, (2\chi)^{2/3}\ \kappa\,, \\
i_1 &= 16\, m_e^2\, \chi^2\, (1-\xi)\,, \\
i_2 &= m_e^2\, (2\chi)^{8/3}\ \kappa \Big[\xi^2 + 4(1-\xi)\sin^2\phi\Big]\,, \\
i_3 &= m_e^2\, (2\chi)^{10/3}\ \xi\,(2-\xi)\sqrt{\kappa(1-\xi)}\ \cos\phi\,.
\end{aligned} \tag{6.49}
$$

The integral over the final neutrino momentum can be written as

$$
\int \frac{\mathrm{d}^3 P'}{E'}\, \Theta(\chi_q) = 4\pi m_e^2 \left(\frac{\chi}{4}\right)^{2/3} \int\limits_0^1 \mathrm{d}\xi \int\limits_0^\infty \mathrm{d}\kappa \int\limits_0^{2\pi} \frac{\mathrm{d}\phi}{2\pi}\,. \tag{6.50}
$$

As was already mentioned, the interference term in (6.43), which is proportional to the product $C_V C_A$, does not contribute to the probability, but it could be important in calculating the averaged neutrino momentum loss.

Upon integrating over ϕ the expression (6.43) takes the form

$$
\begin{aligned}
W = \frac{G_{\mathrm{F}}^2 (C_V^2 + C_A^2) m_e^6 \chi^2}{16\pi^3 E} \int\limits_0^1 \mathrm{d}u \int\limits_0^1 \mathrm{d}\xi \int\limits_0^\infty \mathrm{d}\kappa \Bigg\{ &-2\kappa^2 (1-u^2)\mathrm{Bi}(U) \\
&- \kappa\, \frac{2-2\xi+\xi^2}{3\xi^{4/3}} \frac{9-u^2}{(1-u^2)^{1/3}} \mathrm{Ai}'(U) \\
&- \frac{2C_A^2}{C_V^2+C_A^2} \left(\frac{4}{\chi}\right)^{2/3} \left[\kappa\, \mathrm{Bi}(U) + 4\, \frac{1-\xi}{\xi^{4/3}} \frac{1}{(1-u^2)^{1/3}} \mathrm{Ai}'(U)\right]\Bigg\},
\end{aligned} \tag{6.51}
$$

where

$$
U = \kappa\, \frac{(1-u^2)^{1/3}}{\xi^{2/3}} + \left(\frac{4}{\chi(1-u^2)\xi}\right)^{2/3}.
$$

Performing integration over the variable κ, one obtains

$$
\begin{aligned}
W = \frac{G_{\mathrm{F}}^2\, (C_V^2 + C_A^2)\, m_e^6\, \chi^2}{27\pi^3 E} &\int\limits_0^1 \mathrm{d}u \int\limits_0^1 x \mathrm{d}x\, z\, \mathrm{Ai}(z) \\
\times &\left\{ \frac{3+x^2}{(1-u^2)(1-x)} + \frac{3}{8}(1-3x) + \frac{9}{4} \frac{C_A^2}{C_V^2+C_A^2}\ (5+x)\right\},
\end{aligned} \tag{6.52}
$$

where

$$
z = \left(\frac{4}{\chi(1-u^2)(1-x)}\right)^{2/3}.
$$

Finally, performing one more cumbersome integration, we present the result for the probability in a form of the single integral containing the Airy function:

$$
\begin{aligned}
W = {} & \frac{G_F^2\,(C_V^2 + C_A^2)\,m_e^6\,\chi^2}{27\pi^3 E} \int\limits_0^1 u^2 \mathrm{d}u\, t\, \mathrm{Ai}(t) \\
& \times \left\{ \frac{4}{1-u^2} \left(2L(u) - \frac{29}{24} \right) - \frac{15}{2}\, L(u) - \frac{47}{48} \right. \\
& + \frac{1}{8} \Big(1 + (1-u^2) L(u) \Big) \left(33 - \frac{47}{4}(1-u^2) \right) \\
& \left. + \frac{9}{16} \frac{C_A^2}{C_V^2 + C_A^2} \left[48 L(u) + 2 - \Big(1 + (1-u^2) L(u) \Big) \Big(28 - 3(1-u^2) \Big) \right] \right\}.
\end{aligned}
\tag{6.53}
$$

Here

$$
t = \left(\frac{4}{\chi(1-u^2)} \right)^{2/3}, \quad L(u) = \frac{1}{2u} \ln \frac{1+u}{1-u}\,. \tag{6.54}
$$

In the case $\chi \ll 1$, one obtains from Eq. (6.53) the formula for the probability which demonstrates the well-known exponential suppression, in agreement with Ref. [9]:

$$
W(\chi \ll 1) \simeq \frac{3\sqrt{6}\, G_F^2\, m_e^6}{(16\pi)^3 E}\, (3\, C_V^2 + 13\, C_A^2)\, \chi^4 \exp\left(-\frac{8}{3\chi} \right). \tag{6.55}
$$

In the case $\chi \gg 1$ (more exactly, in the case $\ln \chi \gg 1$) we obtain from Eq. (6.53) the formula (6.40) where $K = 1$ and $\Delta = -29/24$, in agreement with [10, 11]:

$$
W(\chi \gg 1) \simeq \frac{G_F^2\,(C_V^2 + C_A^2)\,m_e^6\,\chi^2}{27\pi^3 E} \left(\ln\chi - \frac{1}{2}\ln 3 - \gamma_E - \frac{29}{24} \right). \tag{6.56}
$$

As the dynamical parameter χ is proportional to the neutrino energy, the probability (6.56) is seen to grow with energy as $E \ln E$ instead of the growth $W \sim E^3$ in the strong field limit, cf. (6.36). Comparing also (6.56) with (6.37), one can see that the contribution of the ground Landau level into the probability is relatively small in the limit $E^2 \gg eB$ ($\sim \sqrt{eB}/E \ll 1$).

It is not difficult to find from (6.53) the next term of expansion over the inversed powers of the parameter χ, to obtain

$$
\begin{aligned}
W(\chi \gg 1) \simeq {} & \frac{G_F^2\,(C_V^2 + C_A^2)\,m_e^6\,\chi^2}{27\pi^3 E} \left\{ \ln\chi - \frac{1}{2}\ln 3 - \gamma_E - \frac{29}{24} \right. \\
& \left. - \frac{1}{\chi^{2/3}}\, \frac{9}{56}\, \frac{3^{1/3}\pi^2}{\left[\Gamma\left(\frac{2}{3}\right)\right]^4}\, \frac{19\, C_V^2 - 63\, C_A^2}{C_V^2 + C_A^2} \right\},
\end{aligned}
\tag{6.57}
$$

where $\Gamma(x)$ is the gamma function, $\Gamma(2/3) = 1.354\ldots$.

As is seen from (6.57), the correcting term $\sim \chi^{-2/3}$ is not universal with respect to the neutrino flavor. It is relatively small and negative for the process $\nu_e \to \nu_e e^- e^+$, while for the process $\nu_\mu \to \nu_\mu e^- e^+$ the correction term is positive and rather large.

The dependence of the probability of the process $\nu_e \to \nu_e e^- e^+$ on the dynamical parameter χ in the region where its value is moderately large, was analysed numerically in [13], see also [16]. It appears that the correction term $\sim \chi^{-2/3}$ is more likely to worsen than to improve the presentation of the probability in this region. A possible explanation of this could be that the next term of expansion over the parameter χ inversed has the form $\sim \chi^{-4/3} \ln \chi$ to be rather large. However, it appears to be a difficult problem to extract this term. On the other hand, it is unnecessary because the exact formula (6.53) can be used in a detailed analysis of the probability of the $e^- e^+$ pair production by a neutrino propagating in an external electromagnetic field, when the value of the dynamical parameter χ is moderately large.

6.4 Possible Astrophysical Manifestations of the $\nu \to \nu e^- e^+$ Process in an External Magnetic Field

6.4.1 Mean Losses of the Neutrino Energy and Momentum

The probability of the $\nu \to \nu e^- e^+$ process defines its partial contribution into the neutrino opacity of the medium. The estimation, e.g. of the electron neutrino mean free path with respect to this process, obtained from the probability (6.36) yields:

$$\lambda(\nu \to \nu e^- e^+) = \frac{1}{W} \sim 4400\,\mathrm{km} \left(\frac{10^3 B_e}{B}\right) \left(\frac{10\,\mathrm{MeV}}{E}\right)^3 . \tag{6.58}$$

It is too large compared with the typical size of a compact astrophysical object, e.g. the supernova remnant, where a strong magnetic field could exist. However, a mean free path does not exhaust the neutrino physics in a medium. In astrophysical applications, we could consider the values that probably are more essential, namely, the mean values of the neutrino energy and momentum loss and especially the asymmetry of the momentum loss, caused by the influence of an external magnetic field. These values can be described by the four-vector of losses Q^α,

$$Q^\alpha = E \int q^\alpha \, \mathrm{d}W = -E\,(\mathcal{I}, \mathbf{F})\,. \tag{6.59}$$

where q is the difference of the momenta of the initial and final neutrinos, $q = P - P'$, $\mathrm{d}W$ is the total differential probability of the process. The zeroth component of Q^α is connected with the mean energy lost by a neutrino per unit time due to the process considered, $\mathcal{I} = \mathrm{d}E/\mathrm{d}t$. The space components of the four-vector (6.59) are similarly connected with the mean neutrino momentum loss per unit time, $\mathbf{F} = \mathrm{d}\mathbf{P}/\mathrm{d}t$. We present here the results of our calculation of the four-vector Q^α in the two limiting cases considered above.

(i) In the case $eB \gtrsim E_\perp^2$ one obtains:

$$\begin{aligned} Q^\alpha &= \frac{G_\mathrm{F}^2 eB(P\varphi\varphi P)^2(C_V^2 + C_A^2)}{48\pi^3}[P^\alpha f_2(\varepsilon) - 2(\varphi\varphi P)^\alpha f_3(\varepsilon) \\ &\quad + \frac{2C_V C_A}{C_V^2 + C_A^2}(\tilde{\varphi}P)^\alpha f_2(\varepsilon)], \end{aligned} \tag{6.60}$$

$$\begin{aligned} f_2(\varepsilon) &= 6\int_0^1 \mathrm{d}\rho\, \rho(1-\rho^2)^2\, \mathrm{e}^{-\varepsilon(1+\rho^2)/2}\, I_0(\varepsilon\rho) \\ &= 1 - \frac{5}{8}\varepsilon + \frac{21}{80}\varepsilon^2 - \frac{7}{80}\varepsilon^3 + \cdots, \\ f_3(\varepsilon) &= 3\int_0^1 \mathrm{d}\rho\, \rho(1-\rho^2)\, \mathrm{e}^{-\varepsilon(1+\rho^2)/2}\, [(1+\rho^2)I_0(\varepsilon\rho) - 2\rho I_1(\varepsilon\rho)] \\ &= 1 - \frac{15}{16}\varepsilon + \frac{21}{40}\varepsilon^2 - \frac{7}{32}\varepsilon^3 + \cdots, \end{aligned}$$

where $\varepsilon = E_\perp^2/eB$, $I_0(x)$ and $I_1(x)$ are the modified Bessel functions. In the strong field limit, $eB \gg E_\perp^2$, one obtains for the neutrino energy and momentum loss,

$$\dot{\mathcal{E}} = \frac{1}{3}EW\left(1 + \frac{2C_V C_A}{C_V^2 + C_A^2}\cos\theta\right), \tag{6.61}$$

$$\mathcal{F}_z = \frac{1}{3}EW\left(\cos\theta + \frac{2C_V C_A}{C_V^2 + C_A^2}\right), \quad \mathcal{F}_\perp = EW\,\sin\theta, \tag{6.62}$$

where the OZ axis is directed along the field, the vector $\mathbf{F}_\perp$ orthogonal to the field direction belongs to the plane of the vectors $\mathbf{B}$ and $\mathbf{p}$. The probability W should be taken from (6.36).

(ii) In the limiting case $eB \ll E^2 \sin^2\theta$ corresponding to the crossed field limit we have obtained the following result for the four-vector Q^α of the neutrino energy and momentum losses due to the process $\nu \to \nu e^- e^+$:

$$\begin{aligned} Q^\alpha &= \frac{7G_\mathrm{F}^2(C_V^2 + C_A^2)m_e^6\chi^2}{432\pi^3}\Bigg[P^\alpha(\ln\chi - 1.888) - \sqrt{3}\,\frac{\eta^2}{\chi}\,(\varphi\varphi P)^\alpha \\ &\quad + 7.465\,\frac{C_V C_A}{C_V^2 + C_A^2}\,\frac{\eta}{\chi^{2/3}}\,(\tilde{\varphi}P)^\alpha\Bigg]. \end{aligned} \tag{6.63}$$

We recall that $\eta = eB/m_e^2 = B/B_e$ is the field intensity parameter.
In the limiting case of very large dynamical parameter $\ln \chi \gg 1$, the expression for the four-vector is simplified significantly:

$$Q^\alpha \simeq \frac{7}{16} E W P^\alpha, \tag{6.64}$$

where the probability W should be taken from (6.56).

6.4.2 Applicability of the Results Obtained in a Pure Magnetic Field for the Plasma Environment

Note that the formulas obtained are valid also in the presence of dense plasma with an electron density of about $10^{33}\,\text{cm}^{-3}$. This is due to the specificity of the ultrarelativistic electron gas statistics in a magnetic field, see Ref. [17]. Given the degeneracy with respect to the transverse momentum, see Eq. (5.13), the connection of the density of the ultrarelativistic electron–positron gas with the chemical potential μ_e and the temperature T is described by the sum over the Landau levels:

$$\begin{aligned} n_e = n_{e^-} - n_{e^+} \; &= \frac{eB}{2\pi^2} \int\limits_0^\infty \mathrm{d}p \left\{ \left(\exp\left(\frac{p - \mu_e}{T} \right) + 1 \right)^{-1} \right. \\ &\left. + 2 \sum_{k=1}^{\infty} \left(\exp\left(\frac{\sqrt{p^2 + 2keB} - \mu_e}{T} \right) + 1 \right)^{-1} - (\mu_e \to -\mu_e) \right\}. \end{aligned} \tag{6.65}$$

In a strong field, under the condition $\sqrt{eB} - \mu_e \gg T$, when practically the main Landau level is only occupied, the temperature dependence in Eq. (6.65) disappears and the chemical potential depends only on the plasma density and the field intensity:

$$\mu_e = \frac{2\pi^2 n_e}{eB} \simeq 2.6\,\text{MeV} \left(\frac{n_e}{10^{33}\,\text{cm}^{-3}} \right) \left(\frac{10^{16}\,\text{G}}{B} \right). \tag{6.66}$$

Thus, the chemical potential can be significantly less than in the absence of the field, $\mu_e \simeq (3\pi^2 n_e)^{1/3}$ for the same values of the density. However, it is clear that the chemical potential increases with the density much faster than in the absence of the field. For the density values

$$n > 3.5 \times 10^{33}\,\text{cm}^{-3} \left(\frac{B}{10^{16}\,\text{G}} \right)^{3/2}$$

the next Landau levels become to be occupied and the connection between the chemical potential and the density is given by

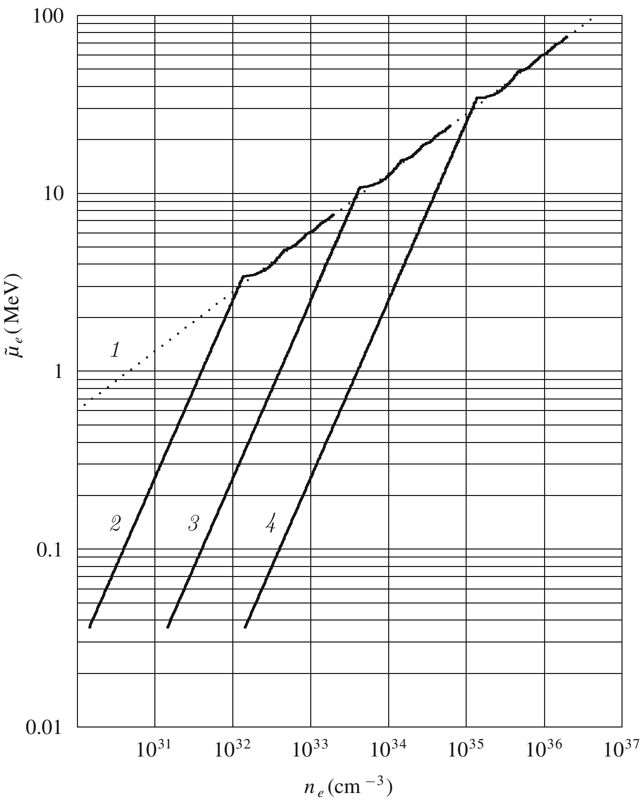

Fig. 6.2 The dependence of the chemical potential on the density of the relativistic electron gas: *1* in the absence of the field; *2,3,4* in a strong magnetic field (see Eq. (6.67)) for the values of the field strength 10^{15}, 10^{16}, 10^{17} G, respectively; *breaks of the curves* correspond to successive opening of the Landau levels

$$n_e \simeq \frac{eB\mu_e}{2\pi^2}\left(1 + 2\sum_{k=1}^{k_{max}}\sqrt{1 - k\frac{2eB}{\mu_e^2}}\right), \quad k_{max} = \left[\frac{\mu_e^2}{2eB}\right], \tag{6.67}$$

where $[x]$ is the integer part of x. The dependence of the chemical potential on the density of the relativistic electron gas is shown in Fig. 6.2.

It can be seen that the chemical potential almost coincides with its value in the absence of the field, when several lower Landau levels are excited. Thus, for the typical energies of the electrons and positrons produced by neutrinos with energies $\sim$20 MeV, when the plasma parameters correspond to the supernova envelope conditions, $n \sim 10^{33}$ cm^{-3} and $T \sim 1$ MeV, the suppressing statistical factors are unimportant.

In the other limiting case of very high temperatures $T \gg \sqrt{eB}$, μ_e, taking account of the medium influence leads to the appearance of the constant statistical factors equal to 1/2 both for electrons and positrons. It reduces the process probability in 4 times.

6.4.3 Possible Astrophysical Manifestations

To illustrate the formulae obtained we consider the astrophysical process of the birth of a magnetized neutron star (i.e. pulsar) in a supernova explosion. Let us suppose that a very strong magnetic field of the order of 10^{16}–10^{18} G [18–22] arises in the cataclysm in the vicinity of a neutrinosphere. The electron density in this region will be considered to be not too high, so a creation of the $e^- e^+$ pairs is not suppressed by statistical factors. In this case the neutrino propagating through the magnetic field would lose energy and momentum in accordance with the above formulae. A part of the total energy lost by neutrinos in the strong magnetic field due to the process of the $e^- e^+$ pair creation could be estimated from Eq. (6.61):

$$\frac{\Delta\mathcal{E}}{\mathcal{E}_{\text{tot}}} \sim 0.6 \times 10^{-2} \left(\frac{B}{10^{17}\,\text{G}}\right) \left(\frac{\bar{E}}{10\,\text{MeV}}\right)^3 \left(\frac{\Delta\ell}{10\,\text{km}}\right). \tag{6.68}$$

Here, $\Delta\ell$ is the characteristic size of the region where the field strength varies insignificantly, $\mathcal{E}_{\text{tot}}$ is the total energy carried off by neutrinos in a supernova explosion, and $\bar{E}$ is the neutrino energy averaged over the neutrino spectrum. Here we take the energy scales that are believed to be typical for supernova explosions [23, 24]. One can see from (6.68) that the effect could manifest itself at a level of about one percent. In principle, it could be essential in a detailed theoretical description of the process of a supernova explosion. Namely, if the magnetic field is strong enough, the well-known FOE problem could be solved due to the process of the production of electron–positron pairs by neutrinos, $\nu \to \nu e^- e^+$. The meaning of the FOE problem is the following: for the self-consisted description of a supernova explosion, it is necessary to find any mechanism of transferring the energy $\sim 10^{51}$ erg (ten to the Fifty One Erg) from the neutrino outflow to the supernova envelope i.e. near 1 % of the total energy $\sim 10^{53}$ erg produced in the explosion.

One more interesting effect is an asymmetry of outgoing neutrinos:

$$A = \frac{\left|\sum\limits_i \mathbf{p}_i\right|}{\sum\limits_i |\mathbf{p}_i|}. \tag{6.69}$$

In the same limit of the strong field we obtain

$$A \sim 3 \times 10^{-3} \left(\frac{B}{10^{16}\,\text{G}}\right) \left(\frac{\bar{E}}{20\,\text{MeV}}\right)^3 \left(\frac{\Delta\ell}{20\,\text{km}}\right). \tag{6.70}$$

Let us note that an origin of the asymmetry of the neutrino momentum loss with respect to the magnetic field direction is a manifestation of the parity violation in weak interaction, because the $\mathcal{F}_z$ value contains the term proportional to the product of the constants C_V and C_A. This asymmetry could result in the recoil "kick" velocity of the rest of the cataclysm. The long-standing problem of the observed high space velocities of pulsars is discussed in more detail below in Sect. 6.5.8. For the parameters used, the asymmetry due to the process $\nu \to \nu e^- e^+$ (6.70) would provide a "kick" velocity on the order of 150 km/s for a pulsar with a mass on the order of the solar mass.

It is important for astrophysical manifestations that all expressions obtained for the process $\nu \to \nu e^- e^+$ are also applicable for the process with antineutrino $\bar{\nu} \to \bar{\nu} e^- e^+$ due to the CP-invariance of the weak interaction.

In the limiting case $eB \ll E^2 \sin^2\theta$ corresponding to the crossed field limit, for the total energy loss via the production of electron–positron pairs by neutrinos $\nu \to \nu e^- e^+$ one obtains from (6.64):

$$
\begin{aligned}
\frac{\Delta\mathcal{E}}{\mathcal{E}_{\text{tot}}} \sim 10^{-6} \left(\frac{B}{10^{15}\,\text{G}}\right)^2 \left(\frac{\bar{E}}{20\,\text{MeV}}\right) \left(\frac{\Delta\ell}{10\,\text{km}}\right) \\
\times \left[4.7 + \ln\left(\frac{B}{10^{15}\,\text{G}}\,\frac{\bar{E}}{20\,\text{MeV}}\right)\right],
\end{aligned}
\tag{6.71}
$$

which is much less than (6.68). The asymmetry is suppressed in this case and has no practical interest.

6.5 Neutrino in Strongly Magnetized Electron–Positron Plasma

The process of the electron–positron pair production by neutrino in a strong magnetic field, if one more component of the external active medium which is dense plasma is taken into account, should be suppressed by the Fermi—Dirac statistical factors. In Sect. 6.4.2, the conditions are defined when such a suppression is inessential. These conditions could be realized, for example, in the process of the neutron star merging. For higher plasma densities corresponding to the conditions of a supernova explosion, the effect of plasma must be considered. At the same time, along with the above-mentioned suppression of the $e^- e^+$ pair birth, new channels of the neutrino-electron interaction arise.

In this section, the full set of the neutrino-electron processes in a magnetized plasma is considered according to Refs. [25, 26]. Besides the canonical scattering and annihilation reactions $\nu e^{\mp} \to \nu e^{\mp}$ and $\nu\bar{\nu} \to e^- e^+$, which are possible in the absence of the field, the processes are also analysed of "synchrotron" emission and absorption of a neutrino pair $e \leftrightarrow e\nu\bar{\nu}$ and of the electron–positron pair production by neutrino $\nu \to \nu e^- e^+$, which are possible only in a magnetic field. Finally, an "exotic" process of the plasma electron–positron pair capture by a neutrino, $\nu e^- e^+ \to \nu$, is also considered. This process is allowed only in the presence of both a magnetic field and hot plasma.

6.5.1 What Do We Mean Under Strongly Magnetized e^-e^+ Plasma

Here we discuss the conditions where, among all the physical parameters characterizing an electron–positron plasma, the field parameter is the dominant one. These conditions can be characterized simply by the relationship: $eB \gg \mu_e^2, T^2$, where μ_e is the chemical potential and T is the temperature of the electron–positron plasma. In order to find a better substantiated relationship we compare the energy densities of the magnetic field $B^2/8\pi$ and the electron–positron plasma.

As we know, a magnetic field changes the statistical properties of an electron–positron gas [17]. Taking into account degeneracy of the transverse momentum, the dependencies of the concentration and energy density of an electron–positron gas on the chemical potential and temperature are described by the following sums over Landau levels:

$$n = n_{e^-} - n_{e^+} = \frac{eB}{2\pi^2} \int_0^\infty \mathrm{d}p \; [\Phi(p, \mu_e, T) - \Phi(p, -\mu_e, T)] \, , \tag{6.72}$$

$$\mathcal{E} = \mathcal{E}_{e^-} + \mathcal{E}_{e^+} = \frac{eB}{2\pi^2} \int_0^\infty p \, \mathrm{d}p \; [\Phi(p, \mu_e, T) + \Phi(p, -\mu_e, T)] \, , \tag{6.73}$$

$$\begin{aligned} \Phi(p, \mu, T) = {} & \left(\exp\left(\frac{p - \mu}{T} \right) + 1 \right)^{-1} \\ & + 2 \sum_{k=1}^{\infty} \left(\exp\left(\frac{\sqrt{p^2 + 2keB} - \mu}{T} \right) + 1 \right)^{-1} . \end{aligned} \tag{6.74}$$

Here we used the approximation of an ultrarelativistic electron–positron gas since astrophysical processes are characterized by fairly high neutrino and plasma electron energies $E \gg m_e$. Thus, we shall neglect the electron mass wherever this causes no misunderstandings.

In a strong field and specifically, when the condition $\sqrt{eB} - \mu_e \gg T$ is satisfied, in practice only the ground Landau level is occupied. From (6.72) and (6.73) we then obtain

$$n = \frac{eB\mu_e}{2\pi^2} , \tag{6.75}$$

$$\mathcal{E} = \frac{eB\mu_e^2}{4\pi^2} + \frac{eBT^2}{12} . \tag{6.76}$$

Thus, a more exact condition that the electron–positron plasma is strongly magnetized may be written in the form

$$\frac{B^2}{8\pi} \gg \frac{\pi^2 n_e^2}{eB} + \frac{eBT^2}{12} \, . \tag{6.77}$$

Selecting values of the physical parameters typical for a supernova envelope as scales in the relationship (6.77), we rewrite this in the form

$$0.8 \times 10^{32} B_3^2 \gg 1.7 \times 10^{30} \, \frac{\rho_{12}^2 Y_{0.1}^2}{B_3} + 1.1 \times 10^{27} \, B_3 \, T_5^2 \qquad \left(\frac{\text{erg}}{\text{cm}^3}\right), \tag{6.78}$$

where

$$B_3 = \frac{B}{10^3 B_e}, \quad \rho_{12} = \frac{\rho}{10^{12}\ \text{g cm}^{-3}}, \quad Y_{0.1} = \frac{Y_e}{0.1}, \quad T_5 = \frac{T}{5\,\text{MeV}}, \tag{6.79}$$

ρ is the total plasma mass density in the envelope, and Y_e is the ratio of the number of electrons to the number of baryons. It can be seen that the plasma magnetization condition is definitely satisfied.

6.5.2 *Neutrino-Electron Processes in Strongly Magnetized Plasma: A Kinematic Analysis*

In this section, calculations are similar to the ones performed in Sect. 6.2. When the processes in strongly magnetized plasma are studied, the additional conditions of applicability of the Lagrangian (4.66) should be taken: eBT, $eB\mu_e \ll m_W^3$.

All neutrino-electron processes determined by the Lagrangian (4.66) can be divided into two groups.

(i) Processes in which a neutrino presents in both the initial and final states: $\nu e^{\mp} \to \nu e^{\mp}$, $\nu \to \nu e^- e^+$, $\nu e^- e^+ \to \nu$, and the similar antineutrino processes.
(ii) Processes involving creation or absorption of a neutrino-antineutrino pair: $e^- e^+ \to \nu\bar{\nu}$, $\nu\bar{\nu} \to e^- e^+$, $e \to e\nu\bar{\nu}$, $e\nu\bar{\nu} \to e$.

It can be seen from Eq. (6.23) that the square of the amplitude of each neutrino-electron process contains the factor $m_e^2/q_\parallel^2$. However, the value of $q_\parallel^2 = q_0^2 - q_z^2$ differs fundamentally for processes of the first and second types. For processes with a neutrino-antineutrino pair we have $q = P + P'$ (P and P' are the four-momenta of a neutrino and an antineutrino, respectively), and consequently $q^2 > 0$. Since $q_\parallel^2 = q^2 + q_\perp^2$, where both terms are positive, the value of $q_\parallel^2$ can only be small when both q^2, and $q_\perp^2$ are small which is only possible in a small region of a phase space. This implies that almost everywhere in the phase space one has $\sqrt{q_\parallel^2} \sim E \sim T \gg m_e$. This leads to reduction of the probability by a factor $m_e^2/T^2 \ll 1$.

At the same time, we have $q = P - P'$ for processes involving neutrinos in the initial and final states and consequently $q^2 < 0$ and the value of $q_\parallel^2$ may be small over a fairly wide region of phase space. Calculations confirm that kinematic amplification is achieved for these processes, leading to the disappearance of the factor m_e^2/T^2 in the probabilities.

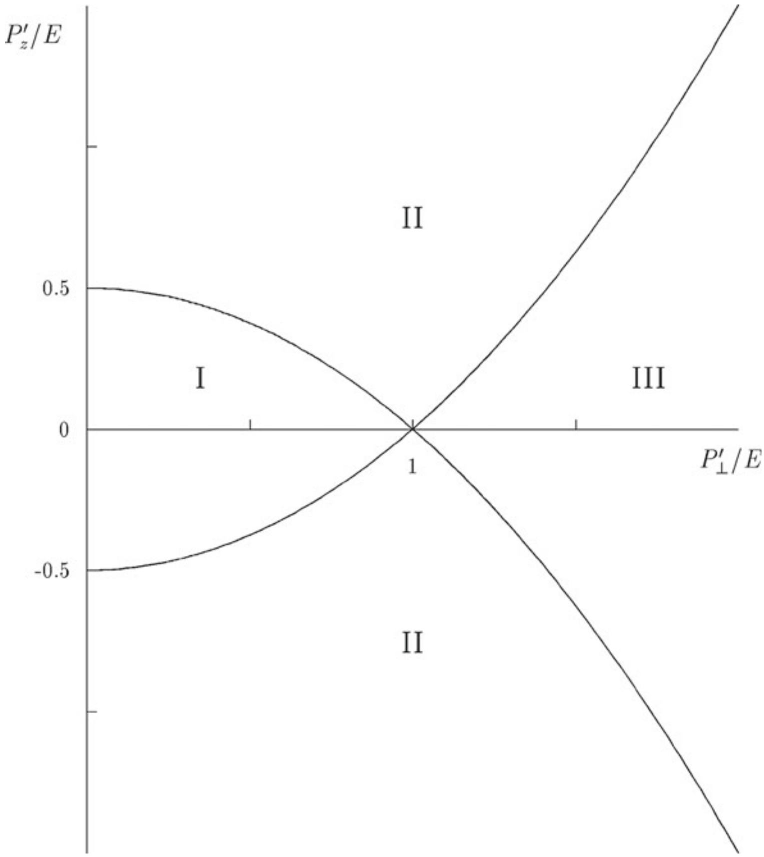

Fig. 6.3 Kinematic regions in the momentum space of a final neutrino: *I* for the pair creation process $\nu \to \nu e^- e^+$; *II* for the scattering channels $\nu e^- \to \nu e^-$, $\nu e^+ \to \nu e^+$; *III* for the pair capture process $\nu e^- e^+ \to \nu$; the lines correspond to the condition $q_\parallel^2 = 0$

Hence, neutrino interaction with a strongly magnetized electron–positron plasma is determined by the processes $\nu e^- \to \nu e^-, \nu e^+ \to \nu e^+, \nu \to \nu e^- e^+, \nu e^- e^+ \to \nu$. In Fig. 6.3, kinematic region in the momentum space of a finite neutrino is shown for the processes listed above in a convenient reference frame where the momentum of the initial neutrino is perpendicular to the magnetic field. The main contribution to the probability comes from regions near the parabola $q_\parallel^2 = 0$ where this kinematic amplification takes place.

It is interesting to analyze the kinematics of the processes $\nu \to \nu e^- e^+, \nu e^- e^+ \to \nu$ in the Minkowski plane $\{0,\ 3\}$. The energies of the electron and positron (see Eq. (2.24)) on the ground Landau level are $\varepsilon = \sqrt{p_z^2 + m_e^2}$ and $\varepsilon' = \sqrt{p_z'^2 + m_e^2}$. Under the condition $\varepsilon \sim \varepsilon' \sim T \gg m_e$, electrons and positrons can be treated as almost massless particles. In turn, in the energies of the initial and final neutrinos,

$E = \sqrt{P_z^2 + P_\perp^2}$ and $E' = \sqrt{P_z'^2 + P_\perp'^2}$, the transverse momentum components can play the role of effective masses in the plane $\{0, 3\}$, namely, $P_\perp^2 = (m_\nu)^2_{\rm eff}$ and $P_\perp'^2 = (m'_\nu)^2_{\rm eff}$. Thus, the process $\nu \to \nu e^- e^+$ is open at the condition $(m_\nu)_{\rm eff} > (m'_\nu)_{\rm eff}$. In the plane $\{0, 3\}$, it looks as the decay of "heavier" neutrino into the "lighter" one plus the "massless" electron and positron. Accordingly, the process of the capture of a pair, $\nu e^- e^+ \to \nu$, is open at $(m_\nu)_{\rm eff} < (m'_\nu)_{\rm eff}$.

6.5.3 The Probability of the Process $\nu \to \nu e^- e^+$

The $\mathcal{S}$ matrix element of the process

$$\nu(P) \to \nu(P') + e^-(p) + e^+(p')$$

has the form (6.22), and the amplitudes of the other neutrino-electron processes are then obtained by crossing transformations.

We express the probability of the creation of an $e^- e^+$ pair by neutrino per unit time in the following form:

$$W(\nu \to \nu e^- e^+) = \frac{1}{T} \int |\mathcal{S}|^2 \, \mathrm{d}\Gamma_{e^-} \, (1 - f_{e^-}) \, \mathrm{d}\Gamma_{e^+} \, (1 - f_{e^+}) \, \mathrm{d}\Gamma'_\nu \, (1 - f'_\nu), \quad (6.80)$$

where T is the total interaction time, and $\mathrm{d}\Gamma$ is an element of the particle phase volume,

$$\mathrm{d}\Gamma_{e^-} = \frac{\mathrm{d}^2 p \, L_y L_z}{(2\pi)^2}, \quad \mathrm{d}\Gamma_{e^+} = \frac{\mathrm{d}^2 p' \, L_y L_z}{(2\pi)^2}, \quad \mathrm{d}\Gamma'_\nu = \frac{\mathrm{d}^3 P' \, V}{(2\pi)^3}. \quad (6.81)$$

The electron and positron distribution functions

$$f_{e^-} = \frac{1}{\mathrm{e}^{(\varepsilon - \mu_e)/T} + 1}, \quad f_{e^+} = \frac{1}{\mathrm{e}^{(\varepsilon' + \mu_e)/T} + 1} \quad (6.82)$$

allow for the presence of a plasma; here μ_e and T are the chemical potential and temperature of the electron–positron gas. To be general, we also allowed for the possible presence of a quasiequilibrium neutrino gas described by the distribution function f'_ν. In general, the question of the accuracy of the description of the state of a neutrino gas under conditions of stellar collapse or another astrophysical process using an equilibrium distribution function and also the determination of this function is a complex astrophysical problem (see, e.g., [27]). Quite clearly, the approximation of an equilibrium neutrino Fermi gas using the distribution function

$$f_\nu^{(eq)} = \frac{1}{\mathrm{e}^{(E - \tilde{\mu}_\nu)/T_\nu} + 1}, \quad (6.83)$$

where $\tilde{\mu}_\nu$ and T_ν are the chemical potential and the temperature of the neutrino gas, should give satisfactory results inside the neutrinosphere. Outside the neutrinosphere, where an outgoing neutrino flux is formed and the neutrino momenta become asymmetric, a factorization of the local distribution is usually assumed

$$f_\nu = \frac{\Phi(\vartheta, R)}{\mathrm{e}^{(E-\tilde{\mu}_\nu)/T_\nu} + 1}\,, \tag{6.84}$$

where the energy distribution is assumed to be approximately equilibrium, the function $\Phi(\vartheta, R)$ determines the neutrino angular distribution, $\vartheta = \cos\alpha$, α is the angle between the neutrino momentum and the radial direction in the star, and R is the distance from the center of the star. An analysis shows [27], that in the vicinity of the neutrinosphere the function $\Phi(\vartheta, R)$ differs negligibly from unity. In order to calculate the probability we shall use the neutrino distribution function in the form (6.83), neglecting the asymmetry. Later in Sect. 6.5.6 when analyzing possible astrophysical manifestations of these neutrino-electron processes, we shall also allow for asymmetry in the distribution function (6.84) for the initial and final neutrinos.

Substituting the $\mathcal{S}$ matrix element (6.22) into (6.80) and integrating using δ-functions over $\mathrm{d}^2 p'$ [where, as is usually the case $\delta^3(0) = T L_y L_z/(2\pi)^3$], we obtain

$$W = \frac{G_{\mathrm{F}}^2}{32(2\pi)^4 E}\,\frac{1}{L_x}\int \frac{\mathrm{d}^3 P'}{E'}\,(1 - f'_\nu)\,\frac{\mathrm{d}p_y \mathrm{d}p_z}{\varepsilon(\varepsilon + m_e)\varepsilon'(\varepsilon' - m_e)}\,\delta(\varepsilon + \varepsilon' - q_0)$$
$$\times\,(1 - f_{e^-})\,(1 - f_{e^+})\,|\bar{u}(p_\parallel)\,\hat{j}(C_V - C_A\gamma_5)\,u(-p'_\parallel)|^2, \tag{6.85}$$

where we need to substitute $\varepsilon' = \sqrt{m_e^2 + (q_z - p_z)^2}$, $p'_z = q_z - p_z$. It is easy to see that the expression in the integrand in Eq. (6.85) does not depend on k_y and consequently integration over k_y essentially determines the degree of degeneracy of an electron having a given energy.

Integrating over the electron momentum in Eq. (6.85) with taking account of Eq. (5.13) we obtain the probability of the $\nu \to \nu e^- e^+$ process in the form of the following integral over the final neutrino momentum:

$$W = \frac{G_{\mathrm{F}}^2 eBm_e^2}{64\pi^4 E}\int \frac{\mathrm{d}^3 P'}{E'}\,\Theta(q_0 - \sqrt{q_z^2 + 4m_e^2})\,\frac{1}{(q_\parallel^2)^{3/2}(q_\parallel^2 - 4m_e^2)^{1/2}}$$
$$\times\,|C_V(j\tilde{\varphi}q) + C_A(j\tilde{\varphi}\tilde{\varphi}q)|^2\;(1 - f'_\nu)\,[(1 - f_{e^-})(1 - f_{e^+}) + (q_z \to -q_z)]. \tag{6.86}$$

In this expression the electron and positron energies ε and ε' appearing in the distribution functions $f_{e^\mp}$ are determined by the conservation law $\varepsilon + \varepsilon' - q_0 = 0$ and are given by

$$\varepsilon = \frac{1}{2}\left(q_0 + q_z\sqrt{1 - \frac{4m_e^2}{q_\parallel^2}}\right), \quad \varepsilon' = \frac{1}{2}\left(q_0 - q_z\sqrt{1 - \frac{4m_e^2}{q_\parallel^2}}\right). \tag{6.87}$$

Expression (6.86) is a generalization of the formula (6.27), where we investigated the neutrino-electron process $\nu \to \nu e^- e^+$ in a high-intensity purely magnetic field, to the case where electron–positron and neutrino gases are present.

Further integration over the final neutrino momentum can be conveniently performed as in Sect. 6.2 in a reference frame where the initial neutrino momentum is perpendicular to the magnetic field, $P_z = 0$. For the case of a purely magnetic field we could convert to this frame without any loss of generality by performing a Lorentz transformation parallel to the field. In fact, we can see that in addition to statistical Fermi factors the value of EW determined from Eq. (6.86) only contains invariants with respect to this transformation (including the sign of the argument of the Θ function). However, we now have a special reference frame, namely, the plasma rest frame, in which the distribution functions (6.82) and (6.83) are formulated. In order to convert to a frame where $P_z = 0$ we express these functions in the invariant form:

$$\begin{aligned} f_{e^-} &= \frac{1}{\mathrm{e}^{((pv)-\mu_e)/T} + 1}\,, \qquad f_{e^+} = \frac{1}{\mathrm{e}^{((p'v)+\mu_e)/T} + 1}\,, \\ f'_\nu &= \frac{1}{\mathrm{e}^{((P'v)-\tilde{\mu}_\nu)/T_\nu} + 1}\,. \end{aligned} \tag{6.88}$$

Here we introduce the four-vector of the plasma velocity v^α, $(v^2 = 1)$ which in its rest frame is $v^\alpha = (1, \mathbf{0})$ and the distribution functions (6.88) are exactly the same as the functions (6.82) and (6.83). In the frame $P_z = 0$ we have

$$v^\alpha = (v_0, 0, 0, v_z)\,, \quad v_0 = 1/\sin\theta\,, \quad v_z = -\cos\theta/\sin\theta\,,$$

where θ is the angle between the vectors of the initial neutrino momentum and the magnetic field in the plasma rest frame.

In formula (6.86) it is convenient to use the dimensionless cylindrical coordinates in the space of the final neutrino momentum vector $\mathbf{P}'$:

$$\rho = \sqrt{P_x'^2 + P_y'^2}/E_\perp,\ \tan\phi = P'_y/P'_x,\ \zeta = P'_z/E_\perp.$$

Here $E_\perp$ is the energy of the initial neutrino in the frame $P_z = 0$ which is related to its energy E in the plasma rest frame by $E_\perp = E\sin\theta$. In terms of the variables ρ, ζ, Eq. (6.86) is rewritten in the form

$$EW = \frac{G_{\mathrm{F}}^2 m_e^2 eBE_\perp^2}{4\pi^3}\int\limits_0^{1-\lambda}\mathrm{d}\rho\,\rho\int\limits_{-\zeta_m}^{\zeta_m}\frac{\mathrm{d}\zeta}{\beta\sqrt{\rho^2+\zeta^2}\,(1-2\sqrt{\rho^2+\zeta^2}+\rho^2)^2}$$

$$\times \left\{ (C_V^2 + C_A^2) \left[(1+\rho^2)\sqrt{\rho^2+\zeta^2} - 2\rho^2 \right] - 2C_V C_A (1-\rho^2)\zeta \right\}$$
$$\times \frac{1}{1+\mathrm{e}^{-(P'v)/T_\nu - \eta_\nu}} \left(\frac{1}{1+\mathrm{e}^{-(pv)/T+\eta}} \frac{1}{1+\mathrm{e}^{-(p'v)/T-\eta}} \bigg|_{\sigma=+1} \right.$$
$$\left. + \frac{1}{1+\mathrm{e}^{-(pv)/T+\eta}} \frac{1}{1+\mathrm{e}^{-(p'v)/T-\eta}} \bigg|_{\sigma=-1} \right), \tag{6.89}$$

where we need to substitute in the distribution functions (6.88)

$$(pv) = \frac{E_\perp}{2\sin\theta} \left[\left(1 - \sqrt{\rho^2+\zeta^2}\right)(1+\sigma\beta\cos\theta) - \zeta(\cos\theta + \sigma\beta) \right],$$
$$(p'v) = \frac{E_\perp}{2\sin\theta} \left[\left(1 - \sqrt{\rho^2+\zeta^2}\right)(1-\sigma\beta\cos\theta) - \zeta(\cos\theta - \sigma\beta) \right],$$
$$(P'v) = \frac{E_\perp}{\sin\theta} \left(\sqrt{\rho^2+\zeta^2} + \zeta\cos\theta \right),$$

and also introduce the notations $\eta = \mu_e/T$, $\eta_\nu = \tilde{\mu}_\nu/T_\nu$,

$$\beta = \sqrt{1 - \frac{4m_e^2}{q_\parallel^2}} = \sqrt{1 - \frac{\lambda^2}{1 - 2\sqrt{\rho^2+\zeta^2} + \rho^2}},$$
$$\lambda = \frac{2m_e}{E_\perp}, \quad \zeta_m = \frac{1}{2}\sqrt{\left(1+\rho^2-\lambda^2\right)^2 - 4\rho^2}.$$

Note that the expression in the integrand in (6.89) exhibits an enhancement which completely compensates for the suppression by the smallness of the electron mass. The main contribution then comes from the region near the upper limits of the integrals over ρ, ζ corresponding to the values $\sqrt{q^2} \sim m_e$. Converting to the new integration variables β and $x = E_\perp(1-\rho^2)/4T\sin\theta$ in Eq. (6.89) and extracting the leading contribution $\sim E_\perp^2/m_e^2$, we transform the expression for the probability to the form

$$EW = \frac{G_{\mathrm{F}}^2 eBE_\perp^2 T^2 \sin^2\theta}{2\pi^3} \int\limits_0^{\epsilon\tau/4} x\,\mathrm{d}x \int\limits_0^1 \mathrm{d}\beta \left\{ \frac{(C_V+C_A)^2}{1+\mathrm{e}^{-\epsilon+2x(1+u)/\tau+\eta_\nu}} \right.$$
$$\times \left[\mathrm{f}(\beta,u,\eta)\ \mathrm{f}(-\beta,u,-\eta) + \mathrm{f}(\beta,u,-\eta)\ \mathrm{f}(-\beta,u,\eta) \right]$$
$$+ \frac{(C_V-C_A)^2}{1+\mathrm{e}^{-\epsilon+2x(1-u)/\tau+\eta_\nu}} \left[\mathrm{f}(\beta,-u,\eta)\ \mathrm{f}(-\beta,-u,-\eta) \right.$$
$$\left.\left. + \mathrm{f}(\beta,-u,-\eta)\ \mathrm{f}(-\beta,-u,\eta) \right] \right\}, \tag{6.90}$$

where $\epsilon = E_\perp/(T_\nu\sin\theta)$, $u = \cos\theta$, $\tau = T_\nu/T$,

$$\mathrm{f}(\beta, u, \eta) = \frac{1}{1 + \mathrm{e}^{-x(1+\beta)(1+u)+\eta}}.$$

Integrating (6.90) over the variable β with using the relation

$$\int_0^1 \mathrm{d}\beta\, \mathrm{f}(\beta, u, \eta)\, \mathrm{f}(-\beta, u, -\eta) = \frac{1}{a(1 - \mathrm{e}^{-2a})} \ln\left(\frac{1 + \mathrm{e}^{-2a+\eta}}{1 + e^{\eta}} \frac{1 + \mathrm{e}^{a+\eta}}{1 + \mathrm{e}^{-a+\eta}}\right), \tag{6.91}$$

where $a = x(1 + u)$ and converting to the plasma rest frame, we finally obtain

$$W(\nu \to \nu e^- e^+) = \frac{G_{\mathrm{F}}^2 eBT^2 E}{4\pi^3} \Bigg\{ (C_V + C_A)^2 (1 - u)^2 \times \int_0^{\epsilon\tau \frac{1+u}{2}} \frac{\mathrm{d}\xi}{(1 - \mathrm{e}^{-\xi})(1 + \mathrm{e}^{-\epsilon+\xi/\tau+\eta_\nu})} \ln \frac{\cosh\xi + \cosh\eta}{1 + \cosh\eta} + + (C_A \to -C_A;\ u \to -u) \Bigg\}, \tag{6.92}$$

where $\epsilon = E/T_\nu$. The dependence of the probability (6.92) on the electron–positron gas density $n = n_{e^-} - n_{e^+}$ is defined in terms of its chemical potential [see (6.75)]. Note that the formula for the probability (6.92) holds for hot ($\mu_e \ll T$) and cold ($\mu_e \gg T$) plasmas. For low-density electron–positron and neutrino gases ($T, \mu_e, T_\nu, \mu_\nu \to 0$) formula (6.92) reproduces the result (6.36) for the probability of the process $\nu \to \nu e^- e^+$ in the strong magnetic field limit, $eB \gg E^2 \sin^2\theta$, without a plasma.

In the absence of a neutrino gas, $T_\nu, \mu_\nu \to 0$, the expression for the probability (6.92) for a hot electron–positron plasma ($T \to \infty$) becomes equal to 1/4 of the probability in a pure magnetic field (6.36) as we indicated in Sect. 6.4.2 since the statistical factors for an electron and positron in this limit are 1/2.

6.5.4 The Total Probability of the Neutrino Interaction with Magnetized Electron–Positron Plasma

A correct analysis of the neutrino propagation process in a hot dense plasma in the presence of a strong magnetic field requires to consider the complete set of neutrino-electron processes. Specifically, in addition to the $\nu e^\mp \to \nu e^\mp$ scattering reactions which also take place in the absence of a field, and the $\nu \to \nu e^- e^+$ pair creation process which is only possible in a magnetic field, we also need to take into account the "exotic" process when a neutrino captures an electron–positron pair from the plasma: $\nu e^- e^+ \to \nu$. This process is only allowed when both a magnetic field and a plasma are present. Then only the probability of the process summed over all initial states of the plasma electrons and positrons is physically meaningful. The probability of the $\nu e^\mp \to \nu e^\mp$ scattering channels is defined similarly as the sum over all e^- or e^+ initial states. The total probability of the neutrino interaction with an electron–positron plasma in a magnetic field is made up of the probabilities of these processes. Thus, the probabilities of the scattering processes should be defined as

$$W(\nu e^\mp \to \nu e^\mp) = \frac{1}{\mathcal{T}} \int |\mathcal{S}|^2 \, \mathrm{d}\varGamma_{e^\mp} \; f_{e^\mp} \, \mathrm{d}\varGamma'_{e^\mp} \, (1 - f'_{e^\mp}) \, \mathrm{d}\varGamma'_\nu \, (1 - f'_\nu) \,, \quad (6.93)$$

where $\mathrm{d}\varGamma$ and f functions are defined in Eqs. (6.81)–(6.83). Similarly, the probability for the pair capture process is:

$$W(\nu e^- e^+ \to \nu) = \frac{1}{\mathcal{T}} \int |\mathcal{S}|^2 \, \mathrm{d}\varGamma_{e^-} \; f_{e^-} \, \mathrm{d}\varGamma_{e^+} \; f_{e^+} \, \mathrm{d}\varGamma'_\nu \, (1 - f'_\nu). \quad (6.94)$$

It can be seen from Fig. 6.3 that the scattering and pair capture processes correspond to infinite kinematic regions since the initial electrons and positrons can formally have any energy. Convergence of the integrals is provided by the distribution functions.

The expressions (6.93) and (6.94) are integrated by the same scheme as that described above for the $\nu \to \nu e^- e^+$ pair creation process. An important factor for the integration will be that the energy imparted from the neutrino to the active medium $q_0 = E - E'$ is not positive-definite. For the probability (per unit time) of the neutrino scattering on magnetized plasma electrons we have

$$\begin{aligned}
W(\nu e^- \to \nu e^-) &= \frac{G_\mathrm{F}^2 e B T^2 E}{4\pi^3} \\
&\times \Bigg\{ (C_V + C_A)^2 (1-u)^2 \int\limits_0^{\epsilon\tau\frac{1+u}{2}} \frac{\mathrm{d}\xi}{(1 - \mathrm{e}^{-\xi})(1 + \mathrm{e}^{-\epsilon + \xi/\tau + \eta_\nu})} \ln \frac{1 + \mathrm{e}^{\eta}}{1 + \mathrm{e}^{-\xi + \eta}} \\
&+ (C_V - C_A)^2 (1+u)^2 \int\limits_0^{\epsilon\tau\frac{1-u}{2}} \frac{\mathrm{d}\xi}{(1 - \mathrm{e}^{-\xi})(1 + \mathrm{e}^{-\epsilon + \xi/\tau + \eta_\nu})} \ln \frac{1 + \mathrm{e}^{\eta}}{1 + \mathrm{e}^{-\xi + \eta}} \\
&+ [(C_V + C_A)^2 (1-u)^2 + (C_V - C_A)^2 (1+u)^2] \\
&\times \int\limits_0^{\infty} \frac{\mathrm{d}\xi}{(\mathrm{e}^{\xi} - 1)(1 + \mathrm{e}^{-\epsilon - \xi/\tau + \eta_\nu})} \ln \frac{1 + \mathrm{e}^{\eta}}{1 + \mathrm{e}^{-\xi + \eta}} \Bigg\}. \quad (6.95)
\end{aligned}$$

Taking into account the distribution functions (6.88), the probability of scattering on positrons is obtained from Eq. (6.95) by substituting $\eta \to -\eta$. For the pair capture channel we have

$$W(\nu e^- e^+ \to \nu) = \frac{G_F^2 eBT^2 E}{4\pi^3} \Big[(C_V + C_A)^2 (1-u)^2$$
$$+ (C_V - C_A)^2 (1+u)^2 \Big] \int\limits_0^\infty \frac{\mathrm{d}\xi}{(\mathrm{e}^\xi - 1)(1 + \mathrm{e}^{-\epsilon - \xi/\tau + \eta_\nu})} \ln \frac{\cosh\xi + \cosh\eta}{1 + \cosh\eta}. \tag{6.96}$$

As we have already noted, only the total probability of neutrino interaction with an electron–positron plasma is physically meaningful:

$$\begin{aligned} W(\nu \to \nu) = & \; W(\nu \to \nu e^- e^+) + W(\nu e^- e^+ \to \nu) \\ & + W(\nu e^- \to \nu e^-) + W(\nu e^+ \to \nu e^+). \end{aligned} \tag{6.97}$$

It was found that this quantity had a substantially simpler form:

$$W(\nu \to \nu) = \frac{G_F^2 eBT^2 E}{4\pi^3} \Bigg\{ (C_V + C_A)^2 (1-u)^2$$
$$\times \left[F_1\left(\frac{\epsilon\tau(1+u)}{2}\right) - F_1(-\infty) \right] + (C_A \to -C_A;\; u \to -u) \Bigg\}, \tag{6.98}$$

where $F_1(z)$ is one of the set of functions defined as

$$F_k(z) = \int\limits_0^z \frac{\xi^k \mathrm{d}\xi}{(1 - \mathrm{e}^{-\xi})(1 + \mathrm{e}^{-\epsilon + \eta_\nu + \xi/\tau})}. \tag{6.99}$$

It is interesting that the dependence on the chemical potential of the electron–positron gas μ which was present in the probabilities of the various processes, was cancelled in the total probability.

At first glance, this result seems unusual. Indeed, the chemical potential of a strongly magnetized plasma according to Eq. (6.75) is proportional to its density, while the total probability of the neutrino interaction with the plasma appears to be independent on the density. Turning to individual channels, one can see that for the process of the pair production $\nu \to \nu e^- e^+$, the increase of the plasma density should lead to a decrease of the probability by reducing the number of free energy levels for electrons and positrons. In the process of the pair capture, $\nu e^- e^+ \to \nu$, the probability, respectively, increases with the density. Both of these mechanisms are important for the scattering channels. However, the found effect of the exact cancellation of these mechanisms in the total probability, apparently, could not be predicted in advance, without specific calculations.

For a rarefied neutrino gas the probability (6.98) is expressed in terms of the Euler dilogarithm $\mathrm{Li}_2(x)$:

$$W(\nu \to \nu) = \frac{G_{\mathrm{F}}^2 eB\,T^2 E}{4\pi^3} \Bigg\{ (C_V^2 + C_A^2)\,\frac{E^2 \sin^4\theta}{4T^2} + (C_V + C_A)^2\,(1-\cos\theta)^2\,\mathrm{Li}_2\Big(1 - \mathrm{e}^{-E(1+\cos\theta)/2T}\Big) + (C_V - C_A)^2\,(1+\cos\theta)^2\,\mathrm{Li}_2\Big(1 - \mathrm{e}^{-E(1-\cos\theta)/2T}\Big) + \frac{\pi^2}{3}\Big[(C_V^2 + C_A^2)(1+\cos^2\theta) - 4\,C_V C_A\cos\theta\Big]\Bigg\}. \tag{6.100}$$

We remind that the nth-order polylogarithm $\mathrm{Li}_n(x)$ is defined as

$$\mathrm{Li}_n(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^n}\,. \tag{6.101}$$

The relative contributions of the plasma and the magnetic field to the process of neutrino interaction with the active medium are illustrated in Fig. 6.4 which gives the ratio of the probabilities of neutrino interaction with a magnetized plasma and a pure magnetic field, $R_w = W_{B+pl}/W_B$, for the angle $\theta = \pi/2$ as a function of the ratio of the neutrino energy to the plasma temperature. It can be seen that the interaction probability increases with the temperature increase.

The probability (6.98) determines the partial contribution of these processes to the opacity for neutrino propagation in a medium. An estimate of the mean free path associated with neutrino-electron processes gives

$$\lambda_e = \frac{1}{W} \simeq 170\,\mathrm{km}\left(\frac{10^3 B_e}{B}\right)\left(\frac{5\,\mathrm{MeV}}{T}\right)^3. \tag{6.102}$$

This should be compared with the neutrino mean free path as a result of interaction with nucleons, which is of the order of a kilometer at the density $\rho \sim 10^{12}\,\mathrm{g\,cm^{-3}}$. At first glance the influence of the neutrino-electron reactions on the neutrino propagation process is negligible. However, the mean free path does not exhaust the neutrino physics in a medium. Other important quantities in astrophysical applications are the neutrino energy and momentum losses. Of particular importance is the asymmetry of the neutrino momentum loss caused by the influence of an external magnetic field. Many attempts have been made to calculate these asymmetries caused by neutrino-nucleon processes associated with the problem of the high proper velocities of pulsars (see Sect. 6.5.8). As we shall show, despite the relatively low probability of the neutrino-electron processes, their contribution to the asymmetry may be comparable to the contributions of the neutrino-nucleon processes.

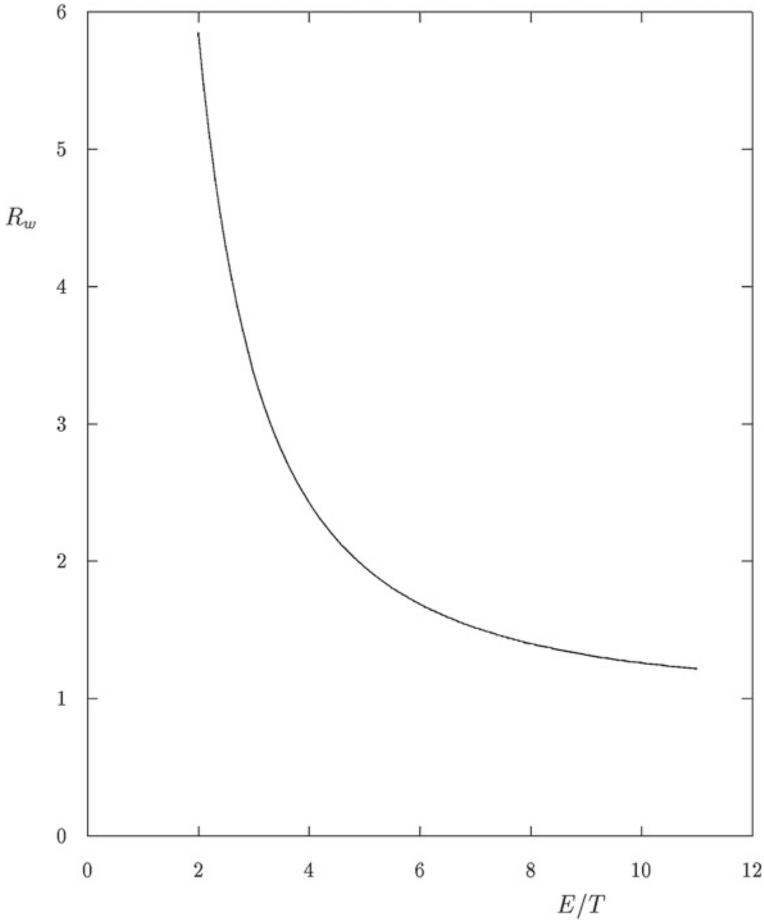

Fig. 6.4 Ratio of the probabilities of neutrino interaction with a magnetized plasma and a pure magnetic field, $R_w = W_{B+pl}/W_B$, for $\theta = \pi/2$ as a function of the ratio of the neutrino energy to the plasma temperature

6.5.5 Mean Losses of the Neutrino Energy and Momentum

In studies of these neutrino-electron interactions in a magnetic field and/or plasma [6–9, 28], the analysis has usually been confined to calculation of the probabilities and cross sections of processes. As we have noted, not only the probabilities of the processes are of practical interest for astrophysics but also the average loss of neutrino energy and momentum in the medium[1] which can be determined by the four-vector Q^α, see Eq. (6.59). The zeroth component Q_0 is associated with the average energy

[1] In general a neutrino can lose and acquire energy and momentum so that we shall subsequently understand “loss” of energy and momentum in the algebraic sense.

lost by a single neutrino per unit time and the spatial components $\mathbf{Q}$ are associated with the loss of the neutrino momentum per unit time.

For a purely magnetic field the four-vector of the losses Q^α was calculated in Sect. 6.4.1. In that case, the losses were caused by the only possible process in the absence of plasma, the pair creation during the motion of a neutrino in a strong magnetic field $\nu \to \nu e^- e^+$. In the strong magnetic field limit for the zeroth and z-components of the vector Q^α we obtained (the field is directed along z)

$$Q_{0,z}^{(B)} = \frac{G_{\mathrm{F}}^2 eBE^5 \sin^4\theta}{48\pi^3} \times \left\{C_V^2 + C_A^2 + 2C_V C_A \cos\theta,\ (C_V^2 + C_A^2)\cos\theta + 2C_V C_A\right\}. \tag{6.103}$$

It can be seen from Eq. (6.103) in particular that even for an isotropic neutrino momentum distribution the average momentum loss will be nonzero (proportional to $C_V C_A$) because of parity nonconservation in weak interaction. As it was shown in 6.4.3, in fields of $\sim 10^3\ B_e$ the integral asymmetry of the neutrino emission caused by the component Q_z and determined by the expression $A = |\sum \mathbf{P}|/\sum|\mathbf{P}|$ could only reach the scale of $\sim 1\ \%$ required to explain the observed pulsar proper velocities as a result of the $\nu \to \nu e^- e^+$ process only.

In the presence of a magnetized plasma our calculations yield the following result for the same components of the loss four-vector:

$$Q_{0,z} = \frac{G_{\mathrm{F}}^2 eBT^3E^2}{4\pi^3} \left\{(C_V + C_A)^2(1-u)^2 \times \left[F_2\left(\frac{\epsilon\tau(1+u)}{2}\right) - F_2(-\infty)\right] \pm (C_A \to -C_A;\ u \to -u)\right\}, \tag{6.104}$$

where the function $F_2(z)$ was determined in Eq. (6.99), and the upper or lower signs correspond to the zeroth and z components. Our result for the loss four-vector obtained for the case of a purely magnetic field (6.103) is reproduced from Eq. (6.104) in the low-density plasma limit $(T,\ T_\nu,\ \mu_\nu \to 0)$.

In order to illustrate the relationship between the contributions of the plasma and the magnetic field to the four-vector of the neutrino energy and momentum losses in an active medium we shall consider the simpler situation of a low-density neutrino gas and rewrite Eq. (6.104) for the angle $\theta = \pi/2$ in the following form:

$$Q_{0,z}(\theta = \pi/2) = \frac{G_{\mathrm{F}}^2 eBE^5}{48\pi^3} \left(C_V^2 + C_A^2, 2C_V C_A\right) \mathcal{F}\left(\frac{E}{T}\right), \tag{6.105}$$

where

$$\mathcal{F}(x) = 1 + \frac{6}{x}\ln\left(1 - \mathrm{e}^{-x/2}\right) - \frac{24}{x^2}\mathrm{Li}_2\left(\mathrm{e}^{-x/2}\right) - \frac{48}{x^3}\mathrm{Li}_3\left(\mathrm{e}^{-x/2}\right). \tag{6.106}$$

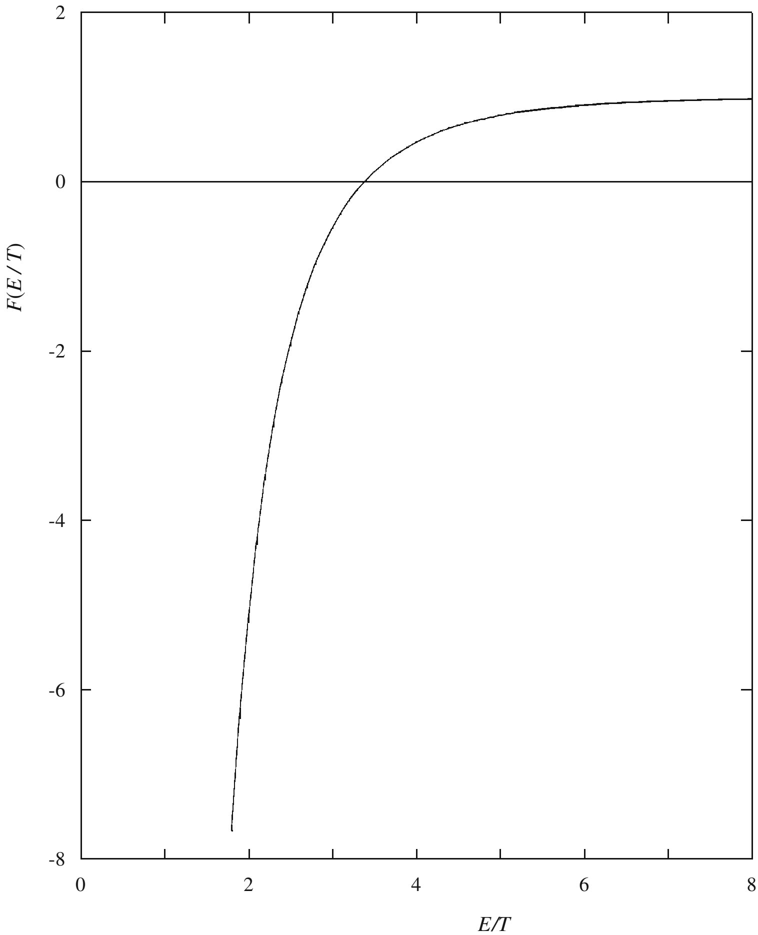

Fig. 6.5 The function $\mathcal{F}(E/T)$ introduced in Eq. (6.105) and determining the dependence of the components of the four-vector of the neutrino energy and momentum losses in a magnetized plasma on the ratio of the neutrino energy to the plasma temperature

It can be seen from a comparison of (6.105) with Eq. (6.103) for $\theta = \pi/2$ that the function $\mathcal{F}(E/T)$ is the ratio of the components of the loss vector in a magnetized plasma and in a purely magnetic field. Figure 6.5 gives a graph of the function $\mathcal{F}(E/T)$. It can be seen that at $E = E_0 \simeq 3.4\ T$ there is a unique "window of transparency" when a neutrino does not exchange energy and momentum with a magnetized plasma. The negative values of the function $\mathcal{F}(E/T)$ at lower energies imply that the neutrino captures energy from the plasma and acquires momentum in the opposite direction to the magnetic field. At energies higher than E_0 the neutrino imparts energy to the plasma and also momentum in the direction of the field. This may have extremely interesting astrophysical consequences.

6.5.6 Integral Action of Neutrinos on a Magnetized Plasma

As an illustration of the application of our results to astrophysical conditions we estimate the volume density of the energy lost by a neutrino per unit time $\dot{\mathcal{E}}$ and the component $\mathcal{F}_z$ (parallel to the field) of the volume density of the force acting on the plasma from neutrinos

$$(\dot{\mathcal{E}}, \mathcal{F}_z) = \int \mathrm{d}n_\nu \, \frac{1}{E} \, Q_{0,z}, \tag{6.107}$$

where $\mathrm{d}n_\nu$ is the initial neutrino density:

$$\mathrm{d}n_\nu = \frac{\mathrm{d}^3 P}{(2\pi)^3} \, \frac{\Phi(\vartheta, R)}{\mathrm{e}^{(E-\mu_\nu)/T_\nu} + 1} \, . \tag{6.108}$$

Here, the angular distribution of the initial neutrinos is taken into account in the function $\Phi(\vartheta, R)$, $\vartheta = \cos\alpha$, α is the angle between the neutrino momentum and the radial direction in the star, and R is the distance from the center of the star. At the same time, the similar function $\Phi(\vartheta', R)$ should be introduced in the statistical factor $(1 - f'_\nu)$ when integrating over the momenta of the final neutrino. In a supernova shell, the neutrino angular distribution is close to isotropic [27] so that in the expansion of the function Φ in terms of ϑ, we can confine ourselves to the lowest Legendre polynomials $P_\ell(\vartheta)$ and this function can be expressed in terms of the average values $\langle\vartheta\rangle$ and $\langle\vartheta^2\rangle$ (which depend on R) as follows:

$$\Phi(\vartheta, R) \simeq 1 + 3\langle P_1(\vartheta)\rangle P_1(\vartheta) + 5\langle P_2(\vartheta)\rangle P_2(\vartheta). \tag{6.109}$$

Neutrinos leaving the central region of a star at high temperature enter the peripheral region where a strong magnetic field is generated and the temperature of the electron–positron gas is lower. In this case the spectral temperatures for different types of neutrino differ [23, 27]:

$$T_{\nu_e} \simeq 4\,\mathrm{MeV}, \quad T_{\bar{\nu}_e} \simeq 5\,\mathrm{MeV}, \quad T_{\nu_{\mu,\tau}} \simeq T_{\bar{\nu}_{\mu,\tau}} \simeq 8\,\mathrm{MeV}\,. \tag{6.110}$$

The action of a neutrino on a plasma leads to the establishment of thermal equilibrium, $\dot{\mathcal{E}}_{tot} = 0$. When analyzing this equilibrium we need to take into account the contributions to $\dot{\mathcal{E}}_{tot}$ made by all processes of neutrino interaction with the medium. As we have noted, the probability of the β processes $\nu_e + n \leftrightarrow e^- + p$ is substantially higher than that for neutrino-electron processes so that these dominate in the energy balance. The energy transferred per unit time per unit plasma volume as a result of these processes involving only electron neutrinos may be expressed in the form

$$\dot{\mathcal{E}}(\beta) \simeq \mathcal{B} \, \frac{T_{\nu_e} - T}{T} \, . \tag{6.111}$$

From this it follows that as a result of neutrino heating the plasma temperature should be very close to the spectral temperature of the electron neutrinos ($T \simeq T_{\nu_e}$). However, the contribution to $\dot{\mathcal{E}}$ made by other types of neutrino whose spectral temperatures exceed T_{ν_e}, has the result that the plasma temperature is slightly higher ($T \gtrsim T_{\nu_e}$). It is therefore meaningful to make separate estimates of the contributions to $(\dot{\mathcal{E}}, \mathcal{F}_z)$ made by neutrino-electron processes involving ν_e and all other neutrinos and antineutrinos.

We stress that the appearance of the force density $\mathcal{F}_z$ in Eq. (6.107) is caused by interference between the vector and axial-vector couplings in the effective Lagrangian (4.66) and is a macroscopic manifestation of parity nonconservation in weak interactions. At first glance, the main contribution to $\mathcal{F}_z$ should be made by electron neutrinos since $C_V(\nu_e) \gg C_V(\nu_{\mu,\tau})$. However, as we shall show below, the main contributions are made by μ and τ neutrinos and antineutrinos (as a result of the conservation of CP parity neutrinos and antineutrinos push the plasma in the same direction). This is because in the vicinity of the ν_e neutrinosphere the spectral temperatures of the other types of neutrinos differ substantially from the plasma temperature $T \simeq T_{\nu_e}$.

6.5.6.1 Processes Involving Electron Neutrinos

We obtained the following expression for the volume density of the neutrino energy losses and the force density (6.107):

$$
\begin{aligned}
(\dot{\mathcal{E}}, \mathcal{F}_z)_{\nu_e} = {} & \frac{G_{\mathrm{F}}^2 eBT^7}{3\pi^5} \left(C_V^2 + C_A^2, 2C_V C_A\right) \\
& \times \left\{ (\tau_e - 1) \int_0^\infty \frac{x^3 \mathrm{d}x}{\mathrm{e}^x - 1} \int_0^\infty \frac{y^3 \mathrm{d}y}{(1 + \mathrm{e}^{-x-y+\eta_\nu})(1 + \mathrm{e}^{y-\eta_\nu})} \right. \\
& \left. + \frac{27}{8} \left(\langle \vartheta^2 \rangle - \frac{1}{3} \right) \int_0^\infty \frac{x^3 \mathrm{d}x}{\mathrm{e}^x - 1} \int_0^\infty \frac{y^3 (3y - x) \mathrm{d}y}{(x + y)^2 (1 + \mathrm{e}^{y-\eta_\nu})} \right\},
\end{aligned}
\tag{6.112}
$$

where $\tau_e = T_{\nu_e}/T$. This formula is written assuming a small deviation from thermal equilibrium between the neutrino gas and the electron–positron plasma $(\tau_e - 1) \ll 1$, and relatively weak asymmetry of the neutrino distribution, $(\langle \vartheta^2 \rangle - 1/3) \ll 1$, is also assumed.

A numerical estimate gives

$$
\begin{aligned}
(\dot{\mathcal{E}}, \mathcal{F}_z)_{\nu_e} \simeq {} & \left(2.0 \times 10^{30} \, \frac{\mathrm{erg}}{\mathrm{cm}^3 \mathrm{s}}, \; 0.57 \times 10^{20} \, \frac{\mathrm{dyne}}{\mathrm{cm}^3} \right) \left(\frac{B}{10^{16} \mathrm{G}} \right) \left(\frac{T}{4 \, \mathrm{MeV}} \right)^7 \\
& \times \mathrm{e}^{\eta_\nu} \left[(\tau_e - 1) + 0.53 \left(\langle \vartheta^2 \rangle - \frac{1}{3} \right) \right].
\end{aligned}
\tag{6.113}
$$

6.5.6.2 Processes Involving $\bar{\nu}_e, \nu_{\mu,\tau}, \bar{\nu}_{\mu,\tau}$

In this case $(T_\nu/T - 1)$ cannot be considered as a small parameter. However, the relative contribution of the asymmetry of the neutrino distribution is small [27] and can be neglected.

For numerical estimates we can conveniently express the values $\dot{\mathcal{E}}$ and $\mathcal{F}_z$ (6.107) in the following form:

$$(\dot{\mathcal{E}}, \mathcal{F})_{\nu_i} \simeq \mathcal{A}\,(C_V^2 + C_A^2, 2C_V C_A)\,\varphi(\eta_i)\,\psi(\tau_i), \tag{6.114}$$

where

$$\begin{aligned}
\mathcal{A} &= \frac{12G_F^2 eBT^7}{\pi^5} = \left(\frac{B}{10^{16}\mathrm{G}}\right)\left(\frac{T}{4\ \mathrm{MeV}}\right)^7 \times \begin{cases} 1.6 \times 10^{30}\ \frac{\mathrm{erg}}{\mathrm{cm^3 s}}\,, \\ 0.55 \times 10^{20}\ \frac{\mathrm{dyne}}{\mathrm{cm^3}}\,, \end{cases} \\
\varphi(\eta_i) &= \frac{\eta_i^4}{24} + \frac{\pi^2\eta_i^2}{12} + \frac{7\pi^4}{360} + \mathrm{Li}_4(-\mathrm{e}^{-\eta_i}), \quad \varphi(0) = \frac{7\pi^4}{720} \simeq 0.947, \\
\psi(\tau_i) &= \frac{\tau_i^7}{6}\int_0^\infty \frac{y^2 \mathrm{d}y}{\mathrm{e}^{\tau_i y} - 1}\left[\mathrm{e}^{(\tau_i - 1)y} - 1\right], \\
\psi(\tau_i)\big|_{\tau_i \to 1} &\simeq \frac{\pi^4}{90}\,(\tau_i - 1)\,.
\end{aligned} \tag{6.115}$$

Formulas (6.112)–(6.115) demonstrate in particular that the action of each individual neutrino fraction on an electron–positron plasma would go to zero when thermodynamic equilibrium is established between this fraction and the plasma $\tau_i = 1,\ \langle\vartheta\rangle = 0,\ \langle\vartheta^2\rangle = 1/3$.

We show that the main contribution to the neutrino action on the plasma is made by μ and τ neutrinos and antineutrinos. In fact the function $\psi(\tau_i)$ (6.115) increases rapidly as the difference between the spectral temperature of the neutrinos and the plasma temperature increases. For example, at temperatures (6.110) we have $\psi(1.25) \simeq 0.824$ for electron antineutrinos and $\psi(2) \simeq 38.47$ for μ and τ neutrinos and antineutrinos. This factor leads to compensation for the smallness of the constant $C_V(\nu_{\mu,\tau})$ and makes the $\nu_{\mu,\tau}, \bar{\nu}_{\mu,\tau}$ contribution not only comparable with the contribution of the electron neutrinos and antineutrinos but even dominant.

As we have noted, the contribution of neutrino-electron processes to the energy action of a neutrino on the plasma is small compared with the contribution of β processes and leads to a small departure from equilibrium between electron neutrinos and the plasma so that the total contribution of β processes and all νe processes to the value of $\dot{\mathcal{E}}$ is zero.

For the force action of a neutrino on the plasma parallel to the magnetic field described by $\mathcal{F}_z$ in formulas (6.112)–(6.115) the total contribution of all types of neutrinos is given by

$$\mathcal{F}_z \simeq 3.6 \times 10^{20}\ \frac{\text{dyne}}{\text{cm}^3} \left(\frac{B}{10^{16}\text{G}}\right) \left(\frac{T}{4\ \text{MeV}}\right)^7 . \tag{6.116}$$

Here we assumed for estimates that the chemical potentials of the neutrinos are zero [23]. Note that the value (6.116) was independent of the chemical potential of an electron–positron plasma.

The force density (6.116) should be compared with the result for a similar force caused by β-processes [29, 30]. Under the same physical conditions our value of the force as a result of neutrino-electron processes is of the same order of magnitude and, which is particularly important, of the same sign as the result of [29, 30]. Thus, the role of neutrino-electron processes in a high-intensity magnetic field may be significant in addition to the contribution of β processes.

The force density (6.116) could lead to a very interesting consequences if a strong toroidal magnetic field [18, 19] is generated in the supernova envelope. This possibility is analyzed in detail below in Sect. 6.5.8.

As we know, in existing systems for numerical modeling of astrophysical cataclysms such as supernova explosions and coalescing of neutron stars, where the physical conditions being studied can be achieved in principle, the neutrino-electron interaction effects studied by us were neglected. However, in detailed analyses of these astrophysical processes it may be important to take into account the influence of an active medium such as a magnetized e^-e^+ plasma, on quantum processes involving neutrinos.

6.5.7 Neutrino-Electron Processes Involving the Contributions of the Excited Landau Levels

In the case when an active medium consists of a magnetic field and very dense plasma, such that the condition is valid: $\mu^2 \gg 2eB$, the plasma electrons could occupy the excited Landau levels. The full set of neutrino-electron processes in such physical conditions of dense magnetized plasma was analyzed in Refs. [31, 32]. In these papers, in contrast to the above-considered situation, the physical conditions were analyzed when the magnetic field was not so strong, whereas the density of plasma was large. Thus, the chemical potential of electrons μ_e was the dominant parameter:

$$\mu_e^2 > 2eB \gg \left(T^2, E^2\right) \gg m_e^2 . \tag{6.117}$$

Here, T is the plasma temperature, E is the typical neutrino energy. Under the conditions (6.117), plasma electrons occupy the excited Landau levels. At the same time it is assumed that the magnetic field strength being relatively weak, see Eq. (6.117), is strong enough, so that the following condition is satisfied:

$$eB \gg \mu_e E . \tag{6.118}$$

In the present astrophysical context, the conditions (6.117) and (6.118) could be realized, for example, in a supernova envelope, where the electron chemical potential is assumed to be $\mu_e \sim 15$ MeV, and the plasma temperature $T \sim 3$ MeV. The magnetic field could be as high as $B \sim 10^{15} - 10^{16}$ G. Under the conditions considered, the approximation of ultrarelativistic plasma is a good one, so we shall neglect the electron mass wherever this causes no complications.

As it was shown in [31], the total set of neutrino-electron processes reduces under the conditions (6.117) and (6.118) to the process of neutrino scattering on plasma electrons. Moreover, both initial and final electrons occupy the same Landau level.

The neutrino-electron scattering in dense magnetized plasma was investigated in [28]. Numerical calculations of the differential cross-section of this process in the limit of a weak magnetic field ($eB < \mu_e E$) were performed. The purpose of this study based on [32], is to calculate analytically not only the probability of the neutrino-electron scattering process, but also the volume density of the neutrino energy and momentum losses under the conditions (6.117) and (6.118).

6.5.7.1 Neutrino-Electron Scattering Probability

The probability of the neutrino-electron scattering per unit time can be obtain by integration over the final and the initial electron states:

$$W(\nu e^- \to \nu e^-) = \sum_{n=0}^{n_{max}} \frac{1}{\mathcal{T}} \int \sum_{s,s'} |\mathcal{S}|^2 \, \mathrm{d}\Gamma_{e^-} \; f_{e^-}(\varepsilon_n) \, \mathrm{d}\Gamma_{e'^-} \; [1 - f_{e'^-}(\varepsilon'_n)] \\ \times \; \mathrm{d}\Gamma'_\nu \, [1 - f'_\nu(E')] \,. \tag{6.119}$$

Here, n_{max} corresponds to the maximal possible Landau level number, which is defined as the integer part of the ratio $\mu_e^2/(2eB) \geqslant 1$, $\varepsilon_n \simeq \sqrt{p_z^2 + 2eBn}$ is the energy of an ultrarelativistic plasma electron occupying the nth Landau level, E' is the final neutrino energy, $\tilde{\mu}_\nu$ and T_ν are the effective chemical potential and the spectral temperature of the neutrino gas respectively. In a general case the neutrino spectral temperature T_ν can differ from the plasma temperature T (we do not assume an equilibrium between neutrino gas and plasma).

The details of integration over the phase space of particles can be found in [31]. The result of the calculation of the probability (6.119) can be presented in a relatively simple form:

$$W(\nu e^- \to \nu e^-) = \frac{G_\mathrm{F}^2 (C_V^2 + C_A^2) \, eB \, T^2 \, E}{4\,\pi^3} \sum_{n=0}^{n_{max}} \frac{1}{z^2} \\ \times \left\{ ((1 + z^2)(1 + u^2) - 4uz) \int\limits_{-a}^{b} \Phi(\xi) \mathrm{d}\xi \right. \tag{6.120}$$

$$+ \frac{1}{zr\tau}(z^2 - 1)(z - u) \int\limits_{-a}^{b} \xi \Phi(\xi) \mathrm{d}\xi \Bigg\} + \ (u \to -u)\,,$$

where $z = \sqrt{1 - 2eBn/\mu_e^2}$, $\Phi(\xi) = \xi[(\mathrm{e}^{\xi} - 1)(\mathrm{e}^{\eta_\nu - r - \xi/\tau} + 1)]^{-1}$, $a = r\tau z(1 + u)/(1 + z)$ and $b = r\tau z(1 - u)/(1 - z)$, $r = E/T_\nu$, $\tau = T_\nu/T$, $\eta_\nu = \tilde{\mu}_\nu/T_\nu$, $u = \cos\theta$, θ is the angle between the initial neutrino momentum $\mathbf{k}$ and the magnetic field direction. The variable ξ defines the spectrum of the probability (6.120) in terms of the final neutrino energy, $\xi = (E' - E)/T$.

In the limit of a very dense plasma ($\mu_e^2 \gg eB$), when a great number of Landau levels are occupied by plasma electrons, one can transform the summation over n to an integration over z:

$$\sum_{n=0}^{[\mu_e^2/(2eB)]} F(z(n)) \simeq \frac{\mu_e^2}{eB} \int\limits_{0}^{1} F(z)\, z \mathrm{d}z\,. \tag{6.121}$$

In this case, the contribution from the lowest Landau levels turns out to be negligibly small, so the main contribution to the probability arises from the highest Landau levels. In this limit, the probability (6.120) can be rewritten in the following form:

$$\begin{aligned} W(\nu e^- \to \nu e^-) = {} & \frac{G_{\mathrm{F}}^2 (C_V^2 + C_A^2)\, \mu_e^2\, T^2\, E}{4\,\pi^3} \int\limits_{0}^{1} \frac{\mathrm{d}z}{z} \\ & \times \Bigg\{ ((1 + z^2)(1 + u^2) - 4uz) \int\limits_{-a}^{b} \Phi(\xi) \mathrm{d}\xi \\ & + \frac{1}{zr\tau}(z^2 - 1)(z - u) \int\limits_{-a}^{b} \xi \Phi(\xi) \mathrm{d}\xi \Bigg\} + \ (u \to -u). \end{aligned} \tag{6.122}$$

As one can see, the probability (6.120) does not depend on the value of the magnetic field strength, but is not isotropic. The dependence on the angle θ manifests this anisotropy of the neutrino-electron process in the presence of a magnetic field. In the limit of a rare neutrino gas when $f_\nu'(E') \ll 1$, the result has a more simple form:

$$W(\nu e^- \to \nu e^-) \simeq \frac{G_{\mathrm{F}}^2 (C_V^2 + C_A^2)\, \mu_e^2\, E^3}{12\,\pi^3}\, I(u)\,, \tag{6.123}$$

$$I(u) = \int\limits_{0}^{1} \frac{z \mathrm{d}z}{(1 + z)^2} \Big(u^4 (3z^2 + 2z + 1) - 12\, u^2 z + z^2 + 2z + 3 \Big)\,.$$

For purposes of comparison, we present here the probability of the neutrino-electron scattering in the absence of field in the same limit of the rare neutrino gas:

$$W_{vac} = \frac{G_{\mathrm{F}}^2 (C_V^2 + C_A^2)\, \mu_e^2\, E^3}{15\, \pi^3}\,. \tag{6.124}$$

The numerical estimate of the ratio of the probabilities (6.123) and (6.124) is presented in Fig. 6.6. It is seen that the probability in a magnetized plasma exceeds the vacuum probability in the vicinity of the point $\theta = \pi/2$ only.

6.5.7.2 Integral Neutrino Action on a Magnetized Plasma

In this section, we calculate the volume density of the neutrino energy and momentum losses per unit time in a medium. According to Eqs. (6.59), (6.107), and (6.108), we can write:

$$(\dot{\mathcal{E}}, \boldsymbol{\mathcal{F}}) = \frac{1}{(2\pi)^3} \int \frac{(q_0, \mathbf{q})\, \mathrm{d}^3 P}{\mathrm{e}^{(E-\tilde{\mu}_\nu)/T_\nu} + 1}\, \mathrm{d}W\,, \tag{6.125}$$

where q_α is the difference between the momenta of the initial and final neutrinos, $q_\alpha = P_\alpha - P'_\alpha$. The zeroth component, $\dot{\mathcal{E}}$, determines the neutrino energy loss from unit volume per unit time. In general, a neutrino propagating through plasma can both lose and capture energy. So, we mean the "loss" of energy in the algebraic sense.

For the neutrino energy loss from unit volume per unit time due to the scattering $\nu e^- \to \nu e^-$ in the limit of a very dense plasma we obtain the following result:

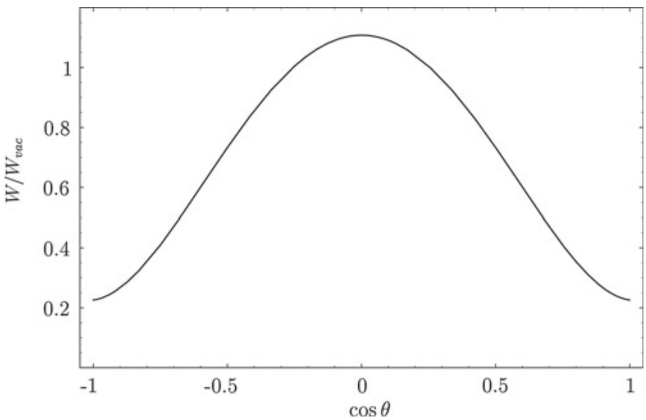

Fig. 6.6 The relative probability of the neutrino-electron scattering in a magnetized plasma as a function of the angle between the initial neutrino momentum and the magnetic field direction. W_{vac} is the probability in a non-magnetized plasma

$$\dot{\mathcal{E}} = \frac{G_F^2 (C_V^2 + C_A^2)}{\pi^3} \, \mu_e^2 \, T^4 \, n_\nu \, J_B(\tau), \tag{6.126}$$

$$J_B(\tau) = \frac{\tau^4}{2} \int_0^1 \frac{dz}{z^2} \int_0^\infty dy \, y^2 \left[y(1 - z^2) + 4 z (1 + z^2) \right]$$
$$\times \frac{1 - e^{y(1-\tau)}}{1 - e^{-y\tau}} \, e^{-y(1+z)/2z}, \tag{6.127}$$

where n_ν is the density of initial neutrinos, the parameter τ has a meaning of a relative neutrino spectral temperature, $\tau = T_\nu / T$. It is interesting to compare this result with the one in a non-magnetized plasma which can be presented in a similar form:

$$\dot{\mathcal{E}}_{B=0} = \frac{G_F^2 (C_V^2 + C_A^2)}{\pi^3} \, \mu_e^2 \, T^4 \, n_\nu \, J_{B=0}(\tau), \tag{6.128}$$

$$J_{B=0}(\tau) = 4\tau^4 \int_0^\infty d\xi \, \xi^2 \, \frac{e^{\xi(\tau-1)} - 1}{e^{\xi\tau} - 1}. \tag{6.129}$$

The functions $J_B(\tau)$ and $J_{B=0}(\tau)$ define the dependence of the neutrino energy losses on the relative neutrino spectral temperature in a magnetized plasma and in a plasma without field respectively. In the limit of a sufficiently large neutrino spectral temperature $\tau \gg 1$ $(T_\nu \gg T)$ the functions take the form:

$$J_B(\tau) \simeq 4.35 \, \tau^4, \quad J_{B=0}(\tau) \simeq 8 \, \tau^4.$$

The graphs of the functions $J_B(\tau)$ and $J_{B=0}(\tau)$ are presented in Fig. 6.7.

As one would expect, at neutrino spectral temperature smaller than the plasma temperature $T_\nu < T$ $(\tau < 1)$ the functions $J_B(\tau)$ and $J_{B=0}(\tau)$ are negative. It implies that a neutrino propagating in a medium picks up energy from the plasma. When $T_\nu > T$ $(\tau > 1)$, the neutrino gives energy to the plasma. When $\tau = 1$ there is a thermal equilibrium when there is no energy exchange between neutrino and electron–positron plasma. It can be seen that the neutrino energy loss in a magnetized plasma is less than the one in a non-magnetized plasma. Hence, under the conditions (6.117) and (6.118), the magnetized plasma becomes more transparent for neutrinos than in the case of plasma without field.

As for the vector $\boldsymbol{\mathcal{F}}$ in Eq. (6.125), it is associated with the volume density of the neutrino momentum loss per unit time, and therefore it defines the neutrino force acting on plasma. Because of the isotropy of plasma in the absence of a magnetic field, one would expect that in the presence of a magnetic field the neutrino force action would be directed along the magnetic field only. However, as it was shown before, the probability of the neutrino-electron scattering (6.122) is symmetric with respect to the substitution $u \to -u$ (or $\theta \to \pi - \theta$). This means that the neutrino scattering on excited electrons does not give a contribution to the neutrino force acting on plasma

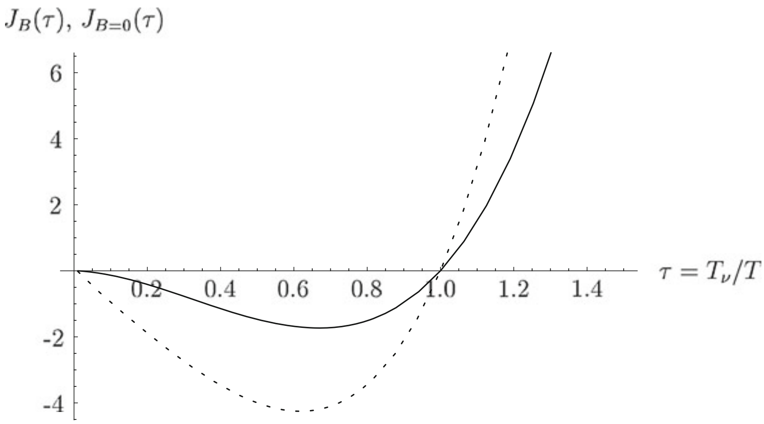

Fig. 6.7 The functions $J_B(\tau)$ (*solid line*) and $J_{B=0}(\tau)$ (*dashed line*) versus the relative spectral neutrino temperature

along the magnetic field. Thus, under the conditions (6.117) and (6.118), there is no neutrino force on plasma at all. Therefore, this force is caused by a contribution of neutrino interactions with ground Landau level electrons only, and the results presented in Eqs. (6.112)–(6.116) have in fact a more general applicability. It may be used even in the limit of dense plasma when chemical potential is considerably greater than the magnetic field strength ($\mu_e^2 \gg eB$).

6.5.8 *Pulsar Natal Kick Via Neutrino-Triggered Magnetorotational Asymmetry*

In this subsection, we will try to apply the results presented above to the well-known problem of large kick velocities of pulsars born in supernova explosions.

6.5.8.1 Pulsar Natal Kick

This problem has been discussed for more than 40 years. The total list of publications with observational data is fairly long. Here, we will point out only the first papers [33, 34], where this problem was formulated and the papers where the data on a sample of 99 pulsars [35] and a sample of 233 pulsars [36] were summarized. In the latter paper, the mean velocity for the sample of 233 pulsars was estimated to be 400 km/s, with more than 15 % of the pulsars having velocities greater than 1000 km/s. The velocities of the two fastest pulsars PSRs B2011+38 and B2224+64 were estimated to be $\sim$1600 km/s.

It is important that a correlation was established between the pulsar velocity directions and rotation axes. Initially, having analyzed a sample of 29 pulsars, Deshpande et al. [37] concluded that the mechanisms predicting a correlation between the pulsar velocity and rotation axis were ruled out. Subsequently, however, Johnston et al. [38] presented strong observational evidence for a relationship between the direction of a pulsar's motion and its rotation axis. A sample of 25 pulsars younger than those used in [37] was analyzed. In particular, for 10 pulsars detected in [38] an offset between the velocity vector and the rotation axis, which is either less than 10° or more than 80°, a fraction that is very unlikely by random chance.

Obviously, an asymmetry in a supernova explosion is responsible for the initial kick, but its nature has not yet been revealed. Various explanations of this asymmetry have been offered in a number of papers.

The attempts to describe the effect only by the hydrodynamics of a supernova explosion without invoking other physical factors could not explain the large velocities. Three-dimensional simulations of the explosion with the assumption of an initial asymmetry in the supernova core before its collapse, which increases during its collapse, lead to a pulsar velocity of no more than 200 km/s [39]. Multidimensional simulations by H.-T. Janka et al. [40], where the explosion anisotropy develops chaotically, yielded a possible pulsar velocity of 10^3 km/s. However, the established correlation between the pulsar velocity direction and rotation axis [38] is not explained in this approach.

In addition to the hydrodynamic approach, there are also other ideas of explaining the pulsar velocities. For example, the pulsar escape was considered during the decay of a close binary system [41]. Another example was the pulsar acceleration within several months after the explosion due to asymmetric electromagnetic radiation caused by the inclination of the magnetic moment with respect to the rotation axis and its displacement relative to the stellar center [42]. However, both of these scenario lead to velocities of the scale of 100 km/s.

In our view, the mechanisms involving neutrinos appear most interesting. Neutrinos are known to carry away about 99 % of the total emitted supernova energy $E \sim 3 \times 10^{53}$ erg. If there is an asymmetry in the neutrino escape of $\sim$3 %, then they would carry away a momentum of $\sim 0.03\, E/c$. The compact explosion remnant, i.e., a neutron star with a mass of $\sim 1.4 M_\odot$, would get the same momentum. In this case, its velocity can be easily estimated to be $\sim$1000 km/s.

An asymmetric neutrino (antineutrino) radiation during a collapse via Urca processes in a strong magnetic field of $10^{14} - 10^{15}$ G in a supernova core was considered [43–48] as a reactive force expelling the neutron star. However, as was subsequently shown [49–52], the neutrinos produced in electroweak processes have small mean free paths in the matter of the central part of a supernova and cannot provide high pulsar velocities.

An interesting mechanism of asymmetry in neutrino radiation during a supernova explosion was considered in Refs. [53–55]. Here, the neutrino flux asymmetry results not from parity violation [43–48], but from an asymmetry in the distribution of the toroidal magnetic field developing during the collapse.

A lively debate was generated by the idea [56], according to which the asymmetry in the neutrino flux from a protoneutron star appears due to neutrino oscillations in matter and an intense magnetic field. The neutrinosphere for ν_τ lies within the neutrinosphere for ν_e, and the resonance transition $\nu_e \to \nu_\tau$ is possible under certain conditions in the region between the neutrinospheres, where ν_e are 'entangled' in the medium, while ν_τ are 'free' to escape. Therefore, the surface of the resonance transition becomes an effective neutrinosphere for ν_τ. In the presence of a magnetic field, this sphere is deformed along the field. Since the temperature depends on the radius, the neutrinosphere deformation results in an anisotropy of the energy flux carried away by neutrinos. This should impart a kick to the nascent neutron star.

However, the idea of an initial pulsar kick due to a deformed neutrinosphere [56] came under serious criticism [57]: after the neutrinosphere deformation, the surfaces of constant temperature will also be deformed, because precisely the neutrinos provide a thermal equilibrium. However, the main problem of this model was soon revealed: it required the existence of neutrinos with a mass of $\sim$100 eV. The established constraint on the neutrino mass, $m_\nu < 2$ eV, 'closed' the model.

Attempts were also made to explain the large space velocities of young pulsars using some possible nonstandard properties of neutrinos. For example, a mechanism was proposed [58] based on the resonant spin-flavor precession of neutrinos with a transition magnetic moment in the magnetic field of a supernova. It was assumed that the asymmetric neutrino radiation could be caused by a distortion of the resonance surface due to matter polarization effects in the supernova magnetic field. The authors [58] argued that the necessary field strength should be 10^{16} G, with the neutrino parameters at the level of existing experimental bounds. However, as was pointed out in [57], the magnetic fields required in the model [58] should actually be more than an order of magnitude stronger.

6.5.8.2 The Initial Pulsar Kick and Sterile Neutrinos

Sterile neutrinos appeared on stage in the paper [59] (see also the review [60] for details). Here, as in [56], the deformation of the neutrinosphere by a magnetic field was discussed, but instead of the oscillations $\nu_{\mu,\tau} \leftrightarrow \nu_e$, the transitions into 'heavy' sterile neutrinos $\nu_{\mu,\tau} \leftrightarrow \nu_s$ were considered. The model was attractive in that the heavy sterile neutrinos (with a mass scale of a few keV) could simultaneously solve two problems: providing an initial velocity of pulsars, they could also play the role of dark matter.

However, when we reproduced the calculations performed in Refs. [59, 60], we found that the asymmetry was overestimated in [60] by a factor of 15. In other words, the necessary magnetic field strength for the declared asymmetry should be a factor of 15 larger: not $\sim 3 \times 10^{16}$ but $\sim 4.6 \times 10^{17}$ G.

Another scenario for using sterile neutrinos to explain the pulsar kick, based on off-resonance transitions was developed in [61]. In this scenario, the fact was used that sterile neutrinos could be produced in beta processes through neutrino mixing, with this process being suppressed due to the smallness of the mixing angle. Nevertheless, they could carry away a significant amount of energy due to two factors:

(1) the neutrinos in the supernova core had energies, $\sim 150\,\mathrm{MeV}$, much greater than those of the active neutrinos, $\sim 20\,\mathrm{MeV}$, emitted from the neutrinosphere;
(2) the emission here originated from the volume, not from the surface.

In the presence of a magnetic field, the neutrinos were emitted asymmetrically and this asymmetry was retained, because the sterile neutrinos were not absorbed but escaped freely, as distinct from the situation considered in Refs. [43–47]. However, as our analysis shows, the authors [61] overestimated the asymmetry at least by a factor of 40. In other words, the magnetic field strengths should be a factor of 40 larger to achieve the asymmetry declared by these authors: not $\sim 10^{16}$ but $\sim 4 \times 10^{17}$ G. In our view, a mistake was made in calculating k_0 defined in Eq. (9) and presented in Fig. 2 of [61]. Note that the authors call k_0 the fraction of electrons in the lowest Landau level, while actually this is the fraction of the electron energy squared in the lowest Landau level. It is this quantity that defines the asymmetry of the neutrino-electron interaction in beta processes. It can be shown that the result [61] is erroneous, both by direct numerical calculations and analytically. Indeed, using Eqs. (9) and (10) from the paper under consideration, the expression for k_0 can be transformed with a good accuracy to

$$k_0 \simeq \frac{eB}{2T^2} \, \frac{J_2(\mu_e/T)}{J_4(\mu_e/T)} \,, \tag{6.130}$$

where B is the magnetic field strength, μ_e and T are the chemical potential and temperature of the electrons, and $J_n(\eta)$ are the Fermi integrals:

$$J_n(\eta) = \int\limits_0^\infty \frac{x^n \, \mathrm{d}x}{\mathrm{e}^{x-\eta} + 1} \,. \tag{6.131}$$

Depending on the electron chemical potential and the magnetic field strength, k_0 was overestimated in Fig. 2 of [61] by a factor from 40 to 90.

In the paper [62], a detailed numerical analysis presented of the transformation of active neutrinos to sterile ones through an MSW-like resonance in a protoneutron star to explain the initial pulsar kick. However, the magnetic field strength needed to achieve the desirable effect should be 10^{17-18} G.

6.5.8.3 Back to Standard Neutrinos?

The following question arises: if we actually need such strong magnetic fields to provide a natal neutron star kick from sterile neutrinos, is it possible to manage with standard neutrinos?

As has already been noted, see Eqs. (6.69) and (6.70), the asymmetry in the emission of standard neutrinos in a strong poloidal magnetic field at the scale of 10^{16} G was not enough to provide the observable neutron star kick.

Note that the mechanism of a significant enhancement in the magnetic field strength during a supernova explosion is known. This is the magnetorotational model for the generation of a toroidal magnetic field in a supernova explosion [18, 19, 63]. A poloidal magnetic field being enhanced during supernova core collapse and frozen in plasma produces a strong toroidal magnetic field due to the differential rotation, which can be greater than the poloidal field by an order of magnitude.

A possible integral effect of neutrinos on a magnetized plasma was evaluated in Sect. 6.5.6, and the combined force action of all types of neutrinos interacting with an electron–positron plasma was obtained, see Eq. (6.116).

The contribution from the neutrino-nucleon processes was estimated in Refs. [29, 30]. For supernova envelope parameters $Y_e \simeq 0.2$ and $\rho \simeq 10^{11-12}$ g cm^{-3}, one can obtain ('νN' means both Urca processes and νN scattering)

$$\mathcal{F}_B^{(\nu N)} \simeq 2.4 \times 10^{20} \left(\frac{B}{10^{16}\mathrm{G}} \right) \frac{\mathrm{dyn}}{\mathrm{cm}^3} \,. \tag{6.132}$$

It is important that the contributions from both neutrino-electron and neutrino-nucleon processes have the same sign. The total neutrino force density is

$$\mathcal{F}_B^{(total)} \simeq 0.6 \times 10^{21} \left(\frac{B}{10^{16}\mathrm{G}} \right) \frac{\mathrm{dyn}}{\mathrm{cm}^3} \,. \tag{6.133}$$

Note that the force density (6.133) is approximately five orders of magnitude lower than the gravitational force density in the same part of the supernova and, consequently, its influence on the radial dynamics of the supernova envelope is negligible. However, when a toroidal magnetic field is generated in the envelope [18, 19, 63], the force (6.133) directed along the field is in no way compensated. It can fairly rapidly (in a time of the order of a second[2]) lead to a significant redistribution of the tangential plasma velocity. In two toroids in which the magnetic fields have opposite directions, the tangential plasma acceleration under the neutrino flux will then have different signs with respect to the direction of rotational plasma motion. This effect can lead to a significant redistribution of the magnetic field lines, concentrating them predominantly in one of the toroids. A similar field configuration was considered in the papers cited above [53–55], where the presence of an initial toroidal field was needed for its appearance. The resulting considerable asymmetry of the magnetic field energy in the two hemispheres can lead to an asymmetry of the supernova explosion and, in particular, can explain the phenomenon of high intrinsic pulsar velocities being discussed. In our view, it would be very interesting to model the

[2] The cooling of a supernova envelope, the so-called Kelvin–Helmholz stage, is known to last for about 10 s.

toroidal magnetic field generation mechanism by taking into account the neutrino force action on plasma via both neutrino-nucleon and neutrino-electron processes.

6.5.8.4 Neutrino-Triggered Magnetorotational Pulsar Natal Kick

The neutrino processes in a toroidal magnetic field frozen in plasma under consideration impart an angular acceleration to a plasma element at distance R from the rotation axis:

$$\dot{\Omega} = \frac{\mathcal{F}}{\rho R} \simeq 1.2 \times 10^3 \left(\frac{B}{10^{16}\mathrm{G}} \right) \frac{1}{\mathrm{s}^2} \, . \tag{6.134}$$

This means that the increase in angular velocity in a time of $\sim$1 s will be

$$\Delta\Omega \sim 10^3 \left(\frac{B}{10^{16}\mathrm{G}} \right) \frac{1}{\mathrm{s}} \, . \tag{6.135}$$

In one hemisphere the angular acceleration coincides with the direction of initial rotation, while in the other hemisphere they are opposite. Pushing the plasma, the neutrino flux curls the toroids in different directions.

Thus, three stages of a pulsar kick can be identified:

(i) the presupernova core collapses with rotation during 0.1 s with the generation of a strong toroidal magnetic field due to the differential rotation;
(ii) pushing the plasma by the tangential force directed along the toroidal magnetic field frozen in plasma, the neutrino outburst leads to a magnetic field asymmetry: the field strength increases in one hemisphere and decreases in the other one, during $\sim$1 s;
(iii) the pressure difference arising in the two hemispheres pushes the core.

According to the momentum conservation law, an energetic plasma jet can be formed in a direction opposite to the pulsar velocity. Such plasma jets being formed in supernova explosions could be gamma-ray burst sources [64]. Of course, a detailed multidimensional numerical simulation of the process is needed. Let us make an order-of-magnitude estimate of the effect that may be expected.

The pressure difference arising in the two hemispheres can be estimated as

$$\Delta p \simeq \frac{B^2}{8\pi} = \frac{(eB)^2}{8\pi\alpha} \, , \tag{6.136}$$

where $\alpha = 1/137$ is the fine-structure constant. The magnetic field pressure causes the compact supernova core, a protoneutron star of mass M, to accelerate:

$$\frac{\mathrm{d}V_{kick}}{\mathrm{d}t} \simeq 1.6 \times 10^5 \left(\frac{B}{10^{16}\mathrm{G}} \right)^2 \left(\frac{R}{20\,\mathrm{km}} \right)^2 \sin 2\theta \; \Delta\theta \left(\frac{1.4\, M_\odot}{M} \right) \frac{\mathrm{km}}{\mathrm{s}^2} \, , \tag{6.137}$$

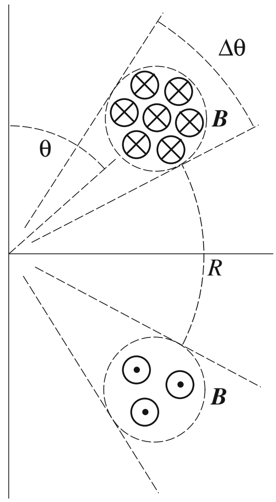

Fig. 6.8 The region of a strong toroidal magnetic field in section by a meridional half-plane. The symbols ⊗ and ⊙ denote the magnetic field directed away from and toward us, respectively. If the magnetic field in the upper hemisphere exceeds that in the lower one by a factor of ~2.3, as is shown in the figure, then the magnetic field pressure in the upper hemisphere will be a factor of ~5.4 larger than that in the lower one

where R, θ and $\Delta\theta$ are the parameters that characterize the region of a strong toroidal magnetic field (see Fig. 6.8).

Taking $\Delta\theta \sim 15^\circ \sim \frac{1}{4}$ and $\theta \sim 45^\circ$ for our estimation, we obtain

$$\frac{\mathrm{d}V_{kick}}{\mathrm{d}t} \simeq 4 \times 10^4 \left(\frac{B}{10^{16}\mathrm{G}}\right)^2 \left(\frac{R}{20\,\mathrm{km}}\right)^2 \left(\frac{1.4\,M_\odot}{M}\right) \frac{\mathrm{km}}{\mathrm{s}^2}\,. \tag{6.138}$$

Actually the acceleration is not constant, because the expansion of the magnetic field volume, which reduces the field strength, should be taken into account. From the magnetic flux conservation we have $p\,V^2 = \mathrm{const}$.

In the same geometry, for the initial pulsar kick velocity we obtain

$$V_{kick} \simeq 600 \left(\frac{B_0}{10^{16}\mathrm{G}}\right) \left(\frac{R}{20\,\mathrm{km}}\right) \left(\frac{\Delta z}{5\,\mathrm{km}}\right)^{1/2} \left(\frac{1.4\,M_\odot}{M}\right)^{1/2} \frac{\mathrm{km}}{\mathrm{s}}\,, \tag{6.139}$$

where B_0 is the maximum toroidal field strength, and Δz is the distance traveled by the compact explosion remnant during the acceleration. It is natural to expect that the field remained after the explosion will be much smaller than the maximum strength B_0.

We emphasize that in our analysis we use the toroidal magnetic fields, which can be greater than the poloidal fields used in other approaches by an order of magnitude.

In our view, a detailed multidimensional numerical simulation of the described mechanism is needed. We hope that it will confirm this effect.

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Chapter 7
Neutrino-Photon Interactions in External Active Media

A strong magnetic field influences essentially on the properties of particles. Firstly, it can induce new interactions between particles—for example, an interaction arising at the one-loop level between electrically neutral neutrinos and photons. Secondly, magnetic fields dramatically change particle kinematics, opening new channels that are forbidden in a vacuum by conservation laws. Among these processes, the radiative transition of a massless neutrino $\nu \to \nu\gamma$, which is also called the neutrino Cherenkov process, has been of great interest for a long time. In this chapter, we analyse this process in external active media: in a magnetic field, and in magnetized plasma. We also consider other neutrino-photon processes, when neutrino interacts with two photons (Compton-like process) and with three photons, under an influence of a magnetic field.

7.1 $\nu\nu\gamma$ Interaction in External Active Media

7.1.1 *The Effective Lagrangian of the $\nu\nu\gamma$ Interaction*

In this section, we present a calculation of the amplitude of the neutrino–photon process due to the $\nu\nu\gamma$ interaction induced by a magnetic field, for a case when the particles involved are, in general, off mass-shell. In other words it means that the effective Lagrangian for the $\nu\nu\gamma$ interaction in a momentum space will be obtained. The calculation is performed within the Standard Model with a possible mixing in the lepton sector. The result is applicable for a magnetic field of any strength when the local limit of the weak interaction is valid.

The effective local Lagrangian of the neutrino–electron interaction (4.66) with a possible lepton mixing taken into account can be rewritten to the form

$$\mathcal{L}_{\nu e} = -\frac{G_{\mathrm{F}}}{\sqrt{2}}\big[\bar{e}\gamma_\alpha(C_V - C_A\gamma_5)e\big]\big[\bar{\nu}_j\gamma^\alpha(1-\gamma_5)\nu_i\big], \tag{7.1}$$

A. Kuznetsov and N. Mikheev, *Electroweak Processes in External Active Media*, Springer Tracts in Modern Physics 252, DOI: 10.1007/978-3-642-36226-2_7,

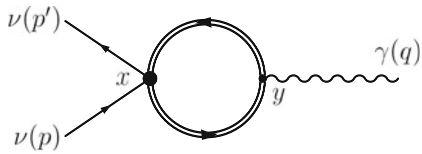

Fig. 7.1 The Feynman diagram describing the vertex $\nu\nu\gamma$ in the local limit of the weak interaction

where C_V, C_A are the vector and axial-vector electroweak constants:

$$C_V = U_{ie}U^*_{je} - \frac{1}{2}\,\delta_{ij}(1 - 4\sin^2\theta_{\mathrm{W}}),$$
$$C_A = U_{ie}U^*_{je} - \frac{1}{2}\,\delta_{ij}.$$

Here, the subscripts i and j label neutrino mass eigenstates, and the matrix elements U_{ie} describe the mixing in the lepton sector. The Feynman diagram describing the vertex $\nu\nu\gamma$ is presented in Fig. 7.1.

It should be recalled that a subtraction procedure is required in calculating the effective Lagrangian of $\nu\nu\gamma$ interaction induced by an external magnetic field. This is because the use of the local limit of weak interaction causes two problems: the amplitude acquires both the ultraviolet divergence and the triangle axial anomaly. It can be readily seen by the expansion of the amplitude of the process $\nu \to \nu\gamma$ in terms of the external magnetic field, as is shown in Fig. 7.2.

The zero term in this expansion,

$$\mathcal{L}^{(0)} = \mathcal{L}(B = 0),$$

involves an ultraviolet divergence, while the term linear in the field,

$$\mathcal{L}^{(1)} = B\,\frac{d\mathcal{L}}{dB}\bigg|_{B=0},$$

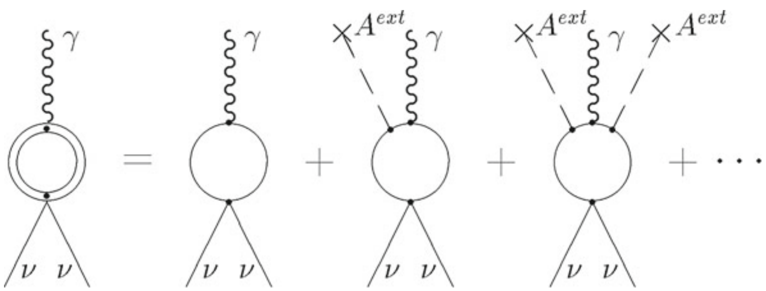

Fig. 7.2 The expansion of the amplitude of the process $\nu \to \nu\gamma$ in terms of the external magnetic field. The *double line* corresponds to the exact propagator of an electron in a magnetic field; the *dashed lines* correspond to the external field

involves the well-known Adler anomaly, because of the presence of the axial-vector interaction in the effective weak Lagrangian. Strictly speaking, both these terms cannot be properly calculated in the local limit, and the correct expression for the effective Lagrangian $\Delta\mathcal{L}_{\nu\nu\gamma}$ induced by an external field can be defined as follows

$$\Delta\mathcal{L}_{\nu\nu\gamma} = \left(\mathcal{L} - \mathcal{L}^{(0)} - \mathcal{L}^{(1)}\right) + \tilde{\mathcal{L}}^{(1)}, \tag{7.2}$$

where the correct term $\tilde{\mathcal{L}}^{(1)}$ linear in the field should be calculated in the electroweak theory without going to the local limit, and with taking into account the contribution from all virtual charged fermions. The expression for $\tilde{\mathcal{L}}^{(1)}$ can be deduced, for example, from the amplitude for the Compton-like process $\nu(p_1) + \gamma^*(q_1) \to \nu(p_2) + \gamma^*(q_2)$ [1, 2] (in general, the photons $\gamma^*(q_1)$ and $\gamma^*(q_2)$ are off the mass shell, and the amplitude has the meaning of an effective Lagrangian in the momentum space) by replacing the field-strength tensor for one of the photons by the strength tensor for a constant uniform magnetic field; that is,

$$q_{1\alpha} \to 0,\ \ f_{1\alpha\beta} \to \mathrm{i}F_{\alpha\beta}, \qquad q_{2\alpha} \to q_\alpha,\ \ f_{2\alpha\beta} \to f_{\alpha\beta},$$

where $f_{\alpha\beta} = q_\alpha\varepsilon_\beta - q_\beta\varepsilon_\alpha$ is the Fourier transform of the photon field-strength tensor, while $F_{\alpha\beta}$ is the strength tensor for an external field. Upon some transformations, the expression for $\tilde{\mathcal{L}}^{(1)}$ can be recast into the form

$$\begin{aligned}\tilde{\mathcal{L}}^{(1)} = \frac{e\,G_{\mathrm{F}}}{\sqrt{2}}\,C_A\,\frac{eB}{4\pi^2}\Bigg[&\frac{(f\tilde{\varphi})(q\varphi\varphi j)}{2q_\parallel^2} - \frac{(q\varphi\varphi f q)(q\tilde{\varphi}j)}{q_\parallel^2 q^2}\\ &+ \frac{(f\tilde{\varphi})(jq)}{2q_\parallel^2}\,\frac{q_\parallel^2 + q^2}{q^2}\Bigg]\mathcal{I}(q^2),\end{aligned} \tag{7.3}$$

where $j^\alpha = \bar{\nu}_j\gamma^\alpha(1-\gamma_5)\nu_i$ is the neutrino current,

$$\mathcal{I}(q^2) = \mathrm{i}\,\frac{q^2}{4}\int_0^1 \mathrm{d}u\,(1-u^2)\int_0^\infty \mathrm{d}t\,\exp\left[-\mathrm{i}t\left(m_e^2 - q^2\frac{1-u^2}{4} - \mathrm{i}\epsilon\right)\right].$$

The effective Lagrangian $\mathcal{L}$ associated with the diagram in Fig. 7.1 is calculated on the basis of conventional Feynman rules, with using the electron propagator in an external constant magnetic field (3.1). We have

$$\mathcal{L} = -\,\mathrm{i}\,\frac{eG_{\mathrm{F}}}{\sqrt{2}}\,j_\alpha\,\varepsilon^*_\beta(q)\int \mathrm{d}^4Z\,\mathrm{Tr}\left[S(-Z)\gamma^\beta\,S(Z)\,\gamma^\alpha\,(C_V - C_A\gamma_5)\right]\mathrm{e}^{-\mathrm{i}qZ}. \tag{7.4}$$

Thus the field-induced part of this Lagrangian can be constructed as the sum of the vector–vector and the vector–axial-vector amplitudes (4.24), $\Delta\mathcal{M}_{VV}$ and $\Delta\mathcal{M}_{VA}$, with the following substitutions of the currents

$$j_{V\beta} \to e\varepsilon^*_\beta(q), \;\; j'_{V\alpha} \to \frac{G_\mathrm{F}}{\sqrt{2}} C_V j_\alpha, \;\; j'_{A\alpha} \to \frac{G_\mathrm{F}}{\sqrt{2}} C_A j_\alpha,$$

and with the further subtraction and restoration of the term linear in the field, as is described above, see (7.2).

The resulting expression for the field-induced effective Lagrangian of the $\nu\nu\gamma$ interaction takes the form

$$\begin{aligned}
\Delta\mathcal{L}_{\nu\nu\gamma} = -\frac{e\,G_\mathrm{F}}{8\pi^2\sqrt{2}} \Bigg\{ C_V \Bigg[& \frac{(f\varphi)\;(q\varphi j)}{q_\perp^2}\; Y_{VV}^{(1)} + \frac{(f\tilde{\varphi})\;(q\tilde{\varphi} j)}{q_\parallel^2}\; Y_{VV}^{(2)} \\
& + 2\,\frac{(q\varphi\varphi f q)}{q_\parallel^2} \left(\frac{(q\varphi\varphi j)}{q_\perp^2} - \frac{(jq)}{q^2} \right) Y_{VV}^{(3)} \Bigg] \\
& + C_A\, eB \Bigg[\frac{(f\tilde{\varphi})\;(q\varphi\varphi j)}{q_\parallel^2} \;\left(Y_{VA}^{(1)} - 1 \right) \\
& + 2\frac{(q\varphi\varphi f q)\;(q\tilde{\varphi} j)}{q_\perp^2\, q_\parallel^2} \left(Y_{VA}^{(2)} + \frac{q_\perp^2}{q^2} \right) \\
& + \frac{(f\tilde{\varphi})(jq)}{q_\parallel^2} \;\left(Y_{VA}^{(3)} - 1 + 2\mathcal{I}(q^2) \right) \Bigg] \Bigg\},
\end{aligned} \tag{7.5}$$

where the functions $Y_{VV}^{(i)}$ and $Y_{VA}^{(i)}$ are defined in (4.31) and (4.33).

The effective Lagrangian (7.5) obtained is manifestly gauge invariant, and is valid for photon and neutrino off-shell. Consequently, it can be used in an analysis of the neutrino electroweak processes in a magnetic field, as the external-field-induced vertex of the $\nu\nu\gamma$ interaction.

However, the kinematics of the processes with photons in a strong magnetic field essentially depends on the photon dispersion properties which were analyzed in Sect. 4.2. A big difference of the 2nd mode photon dispersion properties below and above the threshold $q_\parallel^2 = 4m_e^2$, which is seen in Fig. 4.2, leads to different neutrino processes being possible in the regions of the plot $(q_\perp^2, q_\parallel^2)$, as is shown in Fig. 7.3. A small region depicted by the rectangle where the photon dispersion slightly deviates from the vacuum one, corresponds to the radiative decay of the massive neutrino $\nu_i \to \nu_j\gamma$.

7.1.2 Photon Production by the Massless Neutrino $\nu \to \nu\gamma$

The process $\nu \to \nu\gamma$ in a magnetic field was investigated in the cases of a relatively weak field [3], a strong field [4], and an arbitrary field [5]. In these papers, only the region of relatively small neutrino energies, $E < 2m_e$, was considered. For the case of larger neutrino energies, $E \gtrsim 2m_e$, which is interesting in the light of possible astrophysical applications, large radiative corrections become significant,

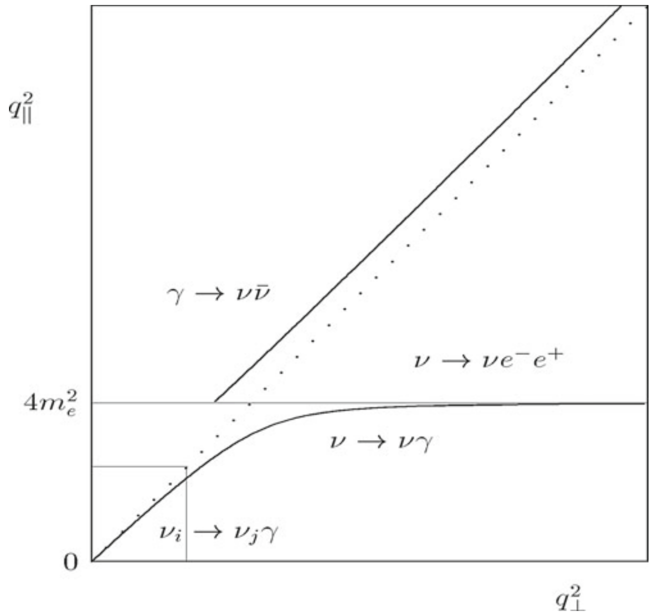

Fig. 7.3 The set of the neutrino processes being kinematically allowed, depending on the 2nd mode photon dispersion properties in a strong magnetic field

which are reduced to the photon wave function renormalization (4.10). One more essential factor is the significant deviation of the 2nd mode photon dispersion from the vacuum one; see Fig. 4.2. Both these factors were taken into account in [6].

The general expression for the effective $\nu\nu\gamma$ vertex is represented in (7.5). We note that the vertex is enhanced substantially in the vicinity of the cyclotron resonances (4.12) as it took place for the photon dispersion operator in a field. The amplitude of the transition $\nu(p) \to \nu'(p') + \gamma(q)$ is simplified essentially in a case of high neutrino energies, $E \gg m_e$, and in the strong field limit where the strength is the maximum physical parameter, $eB > E^2$. The field-induced amplitudes of the processes of $\nu\nu\gamma$ interactions where real photons participate with the polarization vectors defined in (4.10), take the form

$$\begin{aligned} M(\nu\nu\gamma^{(1)}) &= -\frac{eG_{\mathrm{F}}}{4\pi^2\sqrt{2}} \frac{\sqrt{\mathcal{Z}_1}}{\sqrt{q_\perp^2}}\, C_V\ (q\varphi j) Y_{VV}^{(1)}, \\ M(\nu\nu\gamma^{(2)}) &= -\frac{eG_{\mathrm{F}}}{4\pi^2\sqrt{2}} \frac{\sqrt{\mathcal{Z}_2}}{\sqrt{q_\|^2}} \Big[C_V (q\tilde{\varphi} j) Y_{VV}^{(2)} + C_A\, eB(q\varphi\varphi j) \left(Y_{VA}^{(1)} - 1 \right) \Big], \end{aligned} \tag{7.6}$$

where $\mathcal{Z}_1$, $\mathcal{Z}_2$ are the renormalization factors defined in (4.11), and j_α is the neutrino current. The amplitudes (7.6) describe both the photon emission in the neutrino process $\nu \to \nu'\gamma$ (it can be either the radiative decay of massive neutrino or the radiative transition of massless neutrino), and the photon decay into a neutrino pair $\gamma \to \nu\bar{\nu}$.

As was mentioned above, the dispersion of the 1st mode photon slightly deviates from the vacuum law even in a strong field. It means that the collinear kinematics is realized in the process $\nu \to \nu\gamma^{(1)}$:

$$j_\alpha \sim q_\alpha \sim p_\alpha \sim p'_\alpha. \tag{7.7}$$

Consequently, the amplitude $M(\nu\nu\gamma^{(1)})$ has an additional suppression because $(q\varphi j) \simeq (q\varphi q) = 0$. On the other hand, the kinematics is far from collinearity in the transition where the 2nd mode photon participates, especially near the cyclotron resonance where $q_\parallel^2$ tends to $4m_e^2$ from below.

We note, that the amplitude $M(\nu\nu\gamma^{(2)})$ would have the square root singularity in the point $q_\parallel^2 = 4m_e^2$ without taking the renormalization of the photon wave function into account. With the renormalization accounted (the factor $\sqrt{\mathcal{Z}_2}$) the amplitude becomes finite:

$$M(\nu\nu\gamma^{(2)}) \simeq -\frac{eG_{\mathrm{F}}}{4\pi}\,\frac{eB}{\sqrt{q_\perp^2}}\left[C_V(q\tilde{\varphi}j) + C_A(q\varphi\varphi j)\right]. \tag{7.8}$$

The calculation of the process probability is performed in the conventional way for a two-particle decay. In the integration over the phase volume of the final photon, one should keep in mind its dispersion law: $\omega \simeq |q_3|$.

The result for the probability of the process $\nu \to \nu\gamma^{(2)}$ is rather simple in the case $eB \gg E^2\sin^2\theta$,

$$W(\nu \to \nu\gamma^{(2)}) \simeq \frac{\alpha G_{\mathrm{F}}^2}{8\pi^2}(C_V^2 + C_A^2)e^2B^2E\sin^2\theta, \tag{7.9}$$

where E is the energy of the initial neutrino, and θ is the angle between the momentum of the initial neutrino and the magnetic field direction.

The probability of the process $\nu \to \nu\gamma^{(2)}$ is also nonzero in the region of Fig. 4.2 which is above the threshold of the e^-e^+ pair creation, $q_\parallel^2 > 4m_e^2$. This is due to an existence of the imaginary part of the polarization operator which causes an uncertainty of the photon dispersion in a magnetic field in this kinematic region. However, the tree-level channel $\nu \to \nu e^- e^+$ considered earlier dominates in this region.

For the four-vector Q^α (6.59) of the neutrino energy and momentum loss in the considered strong field limit, $eB \gg E^2\sin^2\theta$, one obtains for the process $\nu \to \nu\gamma$:

$$\mathcal{I} = \frac{1}{4}EW\left(1 + \frac{2C_VC_A}{C_V^2 + C_A^2}\cos\theta\right), \tag{7.10}$$

$$\mathcal{F}_z = \frac{1}{4} E W \left(\cos\theta + \frac{2C_V C_A}{C_V^2 + C_A^2} \right), \quad \mathcal{F}_\perp = \frac{1}{2} E W \sin\theta, \tag{7.11}$$

where the probability W should be taken from (7.9).

The asymmetry (6.69) due to the process $\nu \to \nu\gamma$ differs from its value (6.70) obtained for the process $\nu \to \nu e^- e^+$ by the factor $\sim \alpha(eB/E^2)$, or more exactly:

$$A^{(\gamma)} \sim 2\pi\alpha \frac{eB}{E^2} A^{(ee)}, \tag{7.12}$$

where $A^{(ee)}$ is the value defined in (6.70). It is seen that the contributions of the processes $\nu \to \nu\gamma$ and $\nu \to \nu e^- e^+$ into the asymmetry could be comparable in the strong magnetic field despite the suppressing factor α in (7.12).

7.1.3 Photon Decay into the Neutrino Pair $\gamma \to \nu\bar{\nu}$

The process $\gamma \to \nu\bar{\nu}$ is kinematically allowed ($q^2 > 0$) in a magnetic field owing to specific features of photon dispersion. This is so in the region $q_\parallel^2 > 4m_e^2$ for the photon polarization $\varepsilon_\alpha^{(2)}$ and in the region $q_\parallel^2 > (m_e + \sqrt{m_e^2 + 2eB})^2$ for the photon polarization $\varepsilon_\alpha^{(1)}$.

An analysis reveals that, in the considered region ($q^2 > 0$), the photon "mass" induced by a magnetic field is much less than the photon energy ω: $q^2 \ll \omega^2$. This implies that the photon decay $\gamma \to \nu\bar{\nu}$ occurs under the condition of collinear kinematics (7.7), so that the neutrino current can be represented as

$$j_\alpha \simeq 4\sqrt{x(1-x)}\, q_\alpha, \tag{7.13}$$

where $x = E/\omega$ and $1-x = E'/\omega$ are the energy fractions carried by the antineutrino and the neutrino, respectively.

From this and from (7.6), it follows that, in the collinear limit, the amplitude for the decay of the 1st mode photon vanishes and that the expression for the amplitude describing the decay of the 2nd mode photon becomes considerably simpler; that is,

$$\begin{aligned} &M(\nu\nu\gamma^{(1)}) \simeq 0, \\ &M(\nu\nu\gamma^{(2)}) \simeq \frac{2\, e\, G_{\mathrm{F}}\, C_A}{\sqrt{2}\pi^2} \sqrt{x(1-x)}\, [e^2(qFFq)]^{1/2}\, J(q_\parallel^2), \end{aligned} \tag{7.14}$$

where we took into account that $\mathcal{Z}_2 \simeq 1$. The dimensionless field form factor $J(q_\parallel^2)$ has the form

$$
\begin{aligned}
J(q_{\parallel}^2) &= \frac{1}{2}\left(1 - Y_{VA}^{(1)}\right) \\
&\simeq 1 - \mathrm{i}\, m_e^2 \int\limits_0^1 \mathrm{d}u \int\limits_0^\infty \mathrm{d}t\ \exp\left\{ - \mathrm{i}\left[t\left(m_e^2 - q_{\parallel}^2 \frac{1-u^2}{4} \right)\right.\right. \\
&\left.\left. + \frac{q_{\parallel}^2}{2\beta}\, \frac{\cos\beta u t - \cos\beta t}{\sin\beta t} \right]\right\}.
\end{aligned} \tag{7.15}
$$

The process under consideration involves three particles, but its amplitude is not a constant, in contrast to one that occurs in a vacuum. The reason is that the amplitude now depends not only on the 4-momenta of the particles involved but also on the strength tensor of the external field. Therefore, the probability of this process is not merely the product of the amplitude squared and the phase-space volume, but is given by

$$
W(\gamma^{(2)} \to \nu_j \bar{\nu}_i) = \frac{1}{16\pi\omega} \int\limits_{x_1}^{x_2} \mathrm{d}x \left| M(\nu\nu\gamma^{(2)}) \right|^2. \tag{7.16}
$$

The limits of the integration in (7.16), x_1 and x_2, are defined by the ratios of the neutrino masses to the photon "mass", $\mu_i^2 = m_i^2/q^2,\ i = 1, 2$, and can be represented as

$$
\begin{aligned}
x_{1,2} &= \frac{1}{2}(\varepsilon \pm p), \qquad \varepsilon = 1 + \mu_i^2 - \mu_j^2, \\
p &= \sqrt{[1 - (\mu_i + \mu_j)^2][1 - (\mu_i - \mu_j)^2]}.
\end{aligned}
$$

Here, ε and p are, respectively, the energy and the momentum of the ith antineutrino in the ratio to $\sqrt{q^2}/2$ defined in the reference frame comoving with the decaying photon. Substituting expression (7.14) for the amplitude $M(\nu\nu\gamma^{(2)})$ into (7.16), we arrive at

$$
\begin{aligned}
W(\gamma^{(2)} \to \nu_j \bar{\nu}_i) &= \frac{\alpha\, G_{\mathrm{F}}^2\, C_A^2}{12\pi^4 \omega}\, e^2 (qFFq)\, |J(q_{\parallel}^2)|^2 \left[1 + \mu_i^2 + \mu_j^2 - 2(\mu_i^2 - \mu_j^2)^2 \right] \\
&\times \sqrt{[1 - (\mu_i + \mu_j)^2][1 - (\mu_i - \mu_j)^2]}.
\end{aligned} \tag{7.17}
$$

The integral J depends on the variable $q_{\parallel}^2$. The physical meaning of $q_{\parallel}^2$ is seen from the relation

$$
q_{\parallel}^2 \simeq q_{\perp}^2 \simeq \omega^2 \sin^2\theta, \tag{7.18}
$$

where θ is the angle between the momentum $\mathbf{q}$ of the decaying photon and the direction of the magnetic field $\mathbf{B}$.

The expression (7.17) for the probability describes only one channel, the decay of a photon into a neutrino of the type j and an antineutrino of the type i, but only the total decay probability representing the sum over all allowed modes ($\mu_i + \mu_j < 1$) is the quantity of physical interest. Assuming a hierarchy in the neutrino mass spectrum—that is, $m_i^2 \ll q^2$ for $i \leqslant N_L$ and $m_i^2 > q^2$ for $i > N_L$ (thus, N_L is the number of the "light" neutrino species)—we obtain the total probability of the photon decay in the form

$$W = \sum_{i,j=1}^{N_L} W(\gamma^{(2)} \to \nu_j \bar{\nu}_i) = \frac{\alpha\, G_F^2\, \overline{C_A^2}}{12\pi^4 \omega}\, e^2 (qFFq)\, |J(q_\parallel^2)|^2, \tag{7.19}$$

where

$$\overline{C_A^2} = \sum_{i,j=1}^{N_L} C_A^2 = \frac{1}{4} N_L - U^2(1 - U^2),$$

$$U^2 = \sum_{i=1}^{N_L} |U_{ie}|^2 \leqslant 1.$$

If all three neutrino species are "light", $m_i^2 \ll q^2$ ($N_L = 3$, $U^2 = 1$), we have $\overline{C_A^2} = 3/4$, and the probability of the decay $\gamma^{(2)} \to \nu\bar{\nu}$ is independent of the parameters of mixing in the lepton sector.

The function $J(q_\parallel^2)$ is simplified in the two limiting cases.

(i) If the magnetic field is the largest parameter in the problem ($eB \gg q_\parallel^2$), we obtain

$$J(q_\parallel^2) \simeq \frac{1 - v^2}{2v} \left(\ln \frac{1 + v}{1 - v} - \mathrm{i}\pi \right) + 1, \tag{7.20}$$

where $v = \sqrt{1 - 4m_e^2/q_\parallel^2}$.

(ii) In the opposite case of $eB \ll q_\parallel^2$, we arrive at

$$J(q_\parallel^2) \simeq 1. \tag{7.21}$$

At first glance, it may seem that, in view of relation (7.20), the decay probability (7.19) in a strong field has a pole singularity at $q_\parallel^2 \to 4m_e^2$. However, a more accurate solution of the dispersion equation for a photon in this limit shows that

$$|q_\parallel^2 - 4m_e^2|_{\min} = \omega\, \Gamma_{\gamma \to e^- e^+}. \tag{7.22}$$

An apparent singularity like this, but of the square-root type, is known [7] to be encountered in dealing with the photon decay into an electron–positron pair in a magnetic field, $\gamma \to e^+ e^-$. By taking into account the dispersion of the photon in the process $\gamma \to e^+e^-$, it was shown in [8] that the decay width is everywhere finite and that, at $q_\parallel^2 \simeq 4m_e^2$, it reaches a maximum value of

$$\left(\Gamma_{\gamma\to e^-e^+}\right)_{\max} = \frac{\sqrt{3}}{2}\left(\frac{2\,\alpha\,eB}{m_e^2}\right)^{2/3}\frac{m_e^2}{\omega}. \tag{7.23}$$

By virtue of relations (7.22) and (7.23), the probability of the decay process $\gamma \to \nu\bar{\nu}$ is also finite, and its maximum is

$$W_{\max}(\gamma^{(2)} \to \nu\bar{\nu}) = \frac{1}{3\sqrt{3}\pi^2}\left(\frac{2\,\alpha\,eB}{m_e^2}\right)^{1/3}(G_{\mathrm{F}}\,m_e^2)^2\,\overline{C_A^2}\,\frac{eB}{\omega}. \tag{7.24}$$

The probability (7.24) of the electroweak process $\gamma \to \nu\bar{\nu}$ is much less than the probability (7.23) of the process $\gamma \to e^+e^-$ by the factor $(G_{\mathrm{F}}\,m_e^2)^2 \sim 10^{-23}$. However, the former may play the role of an additional source of neutrino cooling in astrophysics.

Let us estimate the energy carried away by neutrinos from a unit volume of the photon gas per unit time. This quantity, referred to as neutrino emissivity, is given by

$$Q = \int \mathrm{d}N_\gamma\,\omega\,W = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\,\frac{1}{\mathrm{e}^{\omega/T}-1}\,\omega\,W. \tag{7.25}$$

Here, we considered that only the 2nd mode photons in (4.10) contribute to the emissivity. In our estimate, we assume that all neutrino species are light: $m_i^2 \ll q^2$, $\overline{C_A^2} = 3/4$.

Substituting the probability given by (7.19) into (7.25), we can recast the expression for the emissivity into the form

$$Q = \frac{\alpha\,(G_{\mathrm{F}}eB)^2}{8\pi^4}\,m_e^5\,\mathcal{F}(T) \simeq 0.96\times 10^{18}\,\frac{\mathrm{erg}}{\mathrm{s\,cm^3}}\left(\frac{B}{B_e}\right)^2\mathcal{F}(T), \tag{7.26}$$

where

$$\mathcal{F}(T) = \frac{8}{\pi^2}\int_0^1 \mathrm{d}u\,(1-u^2)\int_{x_0}^{\infty}\frac{x^4\,\mathrm{d}x}{\mathrm{e}^{x/\tau}-1}\,|J(q_\parallel^2)|^2. \tag{7.27}$$

Here, $\tau = T/2m_e$, the variables of integration are given by $u = \cos\theta$, $x = \omega/2m_e$, and the argument of the function J is $q_\parallel^2 = 4m_e^2\,x^2(1-u^2)$, x_0 is defined from (7.22).

As the analysis shows, the function $\mathcal{F}(T)$ slightly depends on the field strength and has the following form in a wide temperatire region, with the only restriction $T \gtrsim m_e$:

$$\mathcal{F}(T) \simeq \frac{4\zeta(5)}{\pi^2}\left(\frac{T}{m_e}\right)^5, \tag{7.28}$$

where $\zeta(5) \simeq 1.037$ (ζ is the Riemann zeta function).

At low temperatures, $T \ll 2m_e$, the function $\mathcal{F}(T)$ is exponentially small, $\mathcal{F}(T) \sim \exp(-2m_e/T)$.

Finally, we estimate the contribution of the field-induced photon-decay process $\gamma \to \nu\bar{\nu}$ to the neutrino emissivity under the conditions of a supernova explosion. We assume that, in the central region of the explosion of a size of a few hundred kilometers, a strong magnetic field of toroidal type is generated [9, 10]. We then have

$$\frac{\mathrm{d}E}{\mathrm{d}t} \sim 10^{45}\,\frac{\mathrm{erg}}{\mathrm{s}} \left(\frac{B}{10^{15}\,\mathrm{G}}\right)^2 \left(\frac{T}{2\,\mathrm{MeV}}\right)^5 \left(\frac{R}{100\,\mathrm{km}}\right)^3 . \tag{7.29}$$

Recall that the estimated value for the total neutrino emissivity of a supernova is about 10^{52} erg/s. We note that the contribution of the process $\gamma \to \nu\bar{\nu}$ is independent of the neutrino flavors. It can be significant in the low-energy region of the neutrino spectrum.

7.1.4 Radiative Neutrino Transition $\nu \to \nu\gamma$ in Strongly Magnetized Plasma

The process of the radiative massless neutrino transition $\nu \to \nu\gamma$ (neutrino Cherenkov process) is forbidden in vacuum, and it becomes allowed in the presence of plasma and/or magnetic field. There exist several papers where this transition was studied in plasma or magnetic field separately. In plasma the process $\nu \to \nu\gamma$ was firstly investigated in Ref. [11] and later in Refs. [12, 13]. In a pure magnetic field the radiative neutrino transition $\nu \to \nu\gamma$ was studied in the papers [3–6]. In the framework of four-fermion theory the amplitude and the probability of the process were calculated in Refs. [3] and [4] in the crossed field and strong magnetic field respectively. In the Standard Model the amplitude of the neutrino transition $\nu \to \nu\gamma$ was found in [5, 6] for the arbitrary magnetic field strength. In the paper [5] the case of the moderate neutrino energies, $E < 2m_e$ was studied in the kinematical region where the final photon dispersion law was closed to the vacuum one, $q^2 = 0$. The limit of the large neutrino energies and strong magnetic field was investigated in Ref. [6]. There is that case which could be realized at the Kelvin-Helmholz stage of supernova remnant cooling, when the energies of the neutrino are $E \simeq 10-20$ MeV and the magnetic field strength could be as high as $10^{16}-10^{17}$ G [14, 15]. It was shown in [6] that the main contribution into the probability of the neutrino transition $\nu \to \nu\gamma$ was determined from the vicinity of the lowest cyclotron resonance, when the amplitude of the process and photon polarization operator contained simultaneously the square-root singularity.

The purpose of this section is to study the influence of the electron-positron plasma on the process of the radiative massless neutrino transition $\nu \to \nu\gamma$ in a strong magnetic field. The presentation is based on Ref. [16]. This process is considered in the framework of the Standard Model using the effective local Lagrangian of the neutrino-electron interaction (4.66). We investigate the limit of ultrarelativistic strongly magnetized plasma, when the magnetic field strength is the largest physical parameter

$$eB > E^2,\ \mu_e^2,\ T^2 \gg m_e^2\,. \tag{7.30}$$

Here, E is the initial neutrino energy, μ_e is the electron chemical potential, T is the temperature of plasma. Under these conditions electrons and positrons in plasma occupy dominantly the lowest Landau level.

Notice that the amplitude and the probability of the process $\nu \to \nu\gamma$ depend essentially on the polarization of the final photon. In a general case there exist three eigenmodes of the photon polarization operator. The corresponding eigenvectors can be written in the following form:

$$\varepsilon_\mu^{(1)} = \frac{(q\varphi)_\mu}{\sqrt{q_\perp^2}}; \qquad \varepsilon_\mu^{(2)} = \frac{(q\tilde{\varphi})_\mu}{\sqrt{q_\parallel^2}}; \qquad \varepsilon_\mu^{(3)} = \frac{q^2(q\varphi\varphi)_\mu - q_\mu(q\varphi\varphi q)}{\sqrt{q^2 q_\parallel^2 q_\perp^2}}\,. \tag{7.31}$$

Only two of these modes, $\varepsilon_\mu^{(1)}$ and $\varepsilon_\mu^{(2)}$ are the physical one in the pure magnetic field. As the analysis shows, the presence of the strongly magnetized plasma doesn't modify the eigenvectors (7.31) but modifies the eigenvalue corresponding to the vector $\varepsilon_\mu^{(2)}$ only. This is due to the fact that the interaction of the two other eigenmodes with the electrons and positrons which occupy the lowest Landau level is strongly suppressed under the condition (7.30). Hence, only the photon with eigenvector $\varepsilon_\mu^{(2)}$ can be created in the process under consideration, as it takes place in the pure magnetic field, see Sect. 7.1.2.

The process of the radiative neutrino transition is depicted by the Feynman diagram, see Fig. 7.1, where the double line corresponds to the propagator of an electron in the presence of a magnetic field and plasma. Several methods are known in literature describing the process in the background plasma. Here, we use the real-time formalism. The general expression of the real-time propagator in an external field can be found in the paper [17]. In the limit of a strong magnetic field the electron propagator in plasma can be presented in the form:

$$S(x, y) = \mathrm{e}^{\mathrm{i}\Phi(x,y)} \int \frac{\mathrm{d}^4 p}{(2\pi)^4}\, S(p)\, \mathrm{e}^{-\mathrm{i}p(x-y)}, \tag{7.32}$$

where

$$S(p) \simeq 2((\gamma p)_\parallel + m_e)\Pi_- \mathrm{e}^{-p_\perp^2/eB}\left(\frac{1}{p_\parallel^2 - m_e^2 + i\epsilon} - 2\mathrm{i}\pi f_F(p_0)\,\delta(p_\parallel^2 - m_e^2)\right), \tag{7.33}$$

$$f_F(p_0) = f_-(p_0)\Theta(p_0) + f_+(-p_0)\Theta(-p_0), \qquad \Pi_- = \frac{1}{2}(1 - \mathrm{i}\gamma_1\gamma_2)\,.$$

Here, $f_{\mp}(p_0)$ are the distribution functions of electrons and positrons in plasma

$$f_{\mp}(p_0) = \frac{1}{\mathrm{e}^{(p_0 \mp \mu_e)/T} + 1} \,.$$

As it was noticed, in the case of two-point function the noninvariant phase factors $\Phi(x, y)$ were cancelled: $\Phi(x, y) + \Phi(y, x) = 0$. With using the propagator (7.33), the amplitude of the process can be presented in the form:

$$\mathcal{M} = \mathcal{M}_B + \mathcal{M}_{pl}, \tag{7.34}$$

where $\mathcal{M}_B$ is the amplitude of the process $\nu \to \nu\gamma$ corresponding to the pure magnetic field contribution ($T = \mu_e = 0$). Following Ref. [6], it can be expressed in the form

$$\mathcal{M}_B = \frac{eG_{\mathrm{F}}}{2\pi^2\sqrt{2}} \, \frac{eB}{\sqrt{q_{\parallel}^2}} \{C_V(j\tilde{\varphi}q) + C_A(jq)_{\parallel}\} \, H\left(\frac{q_{\parallel}^2}{4m_e^2}\right), \tag{7.35}$$

where the function $H(z)$ is defined in Eq. (4.18). It should be noted that $\mathcal{M}_B$ is the amplitude with the definite photon polarization corresponding to the mode $\varepsilon_{\mu}^{(2)}$ from Eq. (7.31).

The second term in Eq. (7.34), $\mathcal{M}_{pl}$, is induced by the coherent neutrino scattering on plasma electrons and positrons with photon radiation. For $\mathcal{M}_{pl}$ we find

$$\begin{aligned} \mathcal{M}_{pl} = & -\frac{eG_{\mathrm{F}}}{\pi^2\sqrt{2}} \, eB \, m_e^2 \sqrt{q_{\parallel}^2} \{C_V(j\tilde{\varphi}q) + C_A(jq)_{\parallel}\} \\ & \times \int \frac{\mathrm{d}p_z}{\varepsilon} \, \frac{f_-(\varepsilon) + f_+(\varepsilon)}{4(pq)_{\parallel}^2 - (q_{\parallel}^2)^2} \,. \end{aligned} \tag{7.36}$$

As was mentioned above, the amplitude $\mathcal{M}_B$ contains the square-root singularity which is connected with the cyclotron resonance on the lowest Landau level. In the vicinity of the resonance point $q_{\parallel}^2 = 4m_e^2$ it becomes:

$$\mathcal{M}_B \simeq \frac{eG_{\mathrm{F}}}{4\pi\sqrt{2}} \, \frac{eB}{\sqrt{4m_e^2 - q_{\parallel}^2}} \{C_V(j\tilde{\varphi}q) + C_A(jq)_{\parallel}\}. \tag{7.37}$$

It is particularly remarkable that the amplitude $\mathcal{M}_{pl}$ contains the singularity of the same type. In the limit $q_{\parallel}^2 \to 4m_e^2$ the total amplitude (7.34) can be presented in the following form:

$$\mathcal{M} \simeq \mathcal{M}_B \, \mathcal{F}\left(\frac{|q_0|}{2T}\right), \tag{7.38}$$

where

$$\mathcal{F}(x) = \frac{\sinh x}{\cosh x + \cosh \eta}, \qquad \eta = \frac{\mu_e}{T}.$$

It should be stressed that not only the amplitude $\mathcal{M}$ has the singular behaviour but the photon polarization $\Pi^{(2)}$ as well. It can be obtained from Eq. (7.38) by the following replacements

$$\Pi^{(2)} = -\mathcal{M}\left(\frac{G_{\mathrm{F}}}{\sqrt{2}} C_V \to e, C_A \to 0, j_\alpha \to \varepsilon_\alpha^{(2)}\right).$$

For $\Pi^{(2)}$ one has:

$$\Pi^{(2)} \simeq -\frac{2\alpha e B m_e}{\sqrt{4m_e^2 - q_\parallel^2}}\, \mathcal{F}\left(\frac{|q_0|}{2T}\right). \tag{7.39}$$

A large value of $\Pi^{(2)}$ near the resonance requires taking account of large radiative corrections which reduce to a renormalization of the photon wave function:

$$\varepsilon_\alpha^{(2)} \to \varepsilon_\alpha^{(2)}\sqrt{\mathcal{Z}_2}, \quad \mathcal{Z}_2^{-1} = 1 - \frac{\partial \Pi^{(2)}}{\partial q_\parallel^2}. \tag{7.40}$$

Using the formula (7.40) for the amplitude we find

$$\mathcal{M} \to \sqrt{\mathcal{Z}_2}\mathcal{M} \simeq \frac{eG_{\mathrm{F}}}{4\pi}\frac{eB}{\sqrt{q_\perp^2}}\{C_V(j\tilde{\varphi}q) + C_A(jq)_\parallel\}\mathcal{F}\left(\frac{|q_0|}{2T}\right). \tag{7.41}$$

Thus, the photon wave-function renormalization corrects the singular behaviour of the amplitude.

The probability of the process $\nu \to \nu\gamma$ can be obtained by integration of the amplitude over the phase space with taking account of the photon dispersion $q_\parallel^2 - q_\perp^2 = \Pi^{(2)}$.

$$EW = \frac{1}{32\pi^2}\int \left|\mathcal{M}\sqrt{\mathcal{Z}_2}\right|^2 \delta(E - E' - q_0)\,\frac{1}{1 - \mathrm{e}^{-q_0/T}}\,\frac{\mathrm{d}^3 P'}{E' q_0}, \tag{7.42}$$

where the non-trivial photon dispersion law $q_0 = q_0(\mathbf{q})$ should be taken into account. We assume that the neutrino distribution is closed to the Boltzmann one, so one can neglect the deviation of the neutrino statistical factor from the unity. The probability (7.42) is rather complicated in the general case. Here we present the results of our calculation in two limiting cases of the cold plasma, $\mu_e \gg T$, and hot plasma, $T \gg \mu_e$. Notice that in the vicinity of the cyclotron resonance, which gives the main contribution to the probability, the photon dispersion has a rather simple form $q_0 \simeq \sqrt{q_3^2 + 4m_e^2}$.

In the limit of the low temperature, $\mu_e \gg T$, for the probability we obtain:

$$W_{LT} \simeq \frac{\alpha(G_F eB)^2\, E}{16\pi^2}\left\{(C_V - C_A)^2\left[1 - u^2 - \frac{4\mu_e}{E}(1+u)\right]\Theta\left(1 - u - \frac{4\mu_e}{E}\right)\right.$$
$$\left. +\ (C_V + C_A)^2\left[1 - u^2 - \frac{4\mu_e}{E}(1-u)\right]\Theta\left(1 + u - \frac{4\mu_e}{E}\right)\right\}. \tag{7.43}$$

Here, $u = \cos\theta$, θ is the angle between the initial neutrino momentum $\mathbf{P}$ and the magnetic field direction.

In the opposite limit of high temperature, $T \gg \mu_e$, the result for the probability of the process $\nu \to \nu\gamma$ is:

$$W_{HT} \simeq \frac{\alpha(G_F eB)^2\, T}{4\pi^2}\left\{(C_V - C_A)^2(1+u)\, F_1\left(\frac{E(1-u)}{8T}\right)\right.$$
$$\left. + (C_V + C_A)^2(1-u)\, F_1\left(\frac{E(1+u)}{8T}\right)\right\}, \tag{7.44}$$
$$F_1(x) = x + \ln(\cosh x) - \frac{1}{4}\tanh^2 x - \tanh x\,.$$

In the limit of the rarefied plasma both expressions (7.43) and (7.44) provide:

$$W_B \simeq \frac{\alpha(G_F eB)^2}{8\pi^2}\,(C_V^2 + C_A^2)\, E(1 - u^2)\,. \tag{7.45}$$

This result reproduces the formula (7.9) for the radiative neutrino transition probability in the pure strong magnetic field.

Keeping in mind possible applications of our results in astrophysics we calculate the mean values of the neutrino energy and momentum losses. These values were defined earlier by the four-vector Q^α, see Eq. (6.59):

For the zero and third components of Q^α we obtain the following expression in the limit of cold plasma, $T \ll \mu_e$:

$$Q^{0,3} \simeq \frac{\alpha(G_F eB)^2}{64\pi^2} E^3(1-u^2)$$
$$\times\left\{(C_V + C_A)^2\left[1 + u - \frac{16\mu_e^2}{E^2(1+u)}\right]\Theta\left(1 + u - \frac{4\mu_e}{E}\right)\right.$$
$$\left. \pm\ (C_V - C_A)^2\left[1 - u - \frac{16\mu_e^2}{E^2(1-u)}\right]\Theta\left(1 - u - \frac{4\mu_e}{E}\right)\right\}. \tag{7.46}$$

In the opposite case, when $\mu_e \ll T$ we find

$$Q^{0,3} \simeq \frac{\alpha(G_F eB)^2}{2\pi^2} ET^2\left\{(C_V + C_A)^2(1-u)\, F_2\left(\frac{E(1+u)}{4T}\right)\right.$$

$$\pm (C_V + C_A)^2 (1+u)\, F_2\left(\frac{E(1-u)}{4T}\right)\Bigg\}, \tag{7.47}$$

$$F_2(x) = \frac{1}{2}\tanh\frac{x}{2} - \frac{x\mathrm{e}^x(1+2\mathrm{e}^x)}{(1+\mathrm{e}^x)^2} + (2+x)\ln(1+\mathrm{e}^x) + \mathrm{Li}_2(-\mathrm{e}^x) - \ln 4 + \frac{\pi^2}{12},$$

where $\mathrm{Li}_2(x)$ is the polylogarythm function. Notice that in the limit $T \to 0,\ \mu_e \to 0$, both expressions (7.46) and (7.47) reproduce the formula for the four-vector of losses via the process $\nu \to \nu\gamma$ in the pure strong magnetic field, see Eqs. (7.10) and (7.11):

$$Q^{0,3} = \frac{\alpha(G_{\mathrm{F}}eB)^2}{64\pi^2} E^3(1-u^2) \times \Big\{(C_V + C_A)^2(1+u) \pm (C_V - C_A)^2(1-u)\Big\}. \tag{7.48}$$

We note that electron-positron plasma and photon gas make an opposite influence on the process under consideration. On one hand, the electron-positron background decreases the amplitude of the process ($\mathcal{F}(q_0) < 1$). On the other hand, the probability and the mean value of the neutrino energy and momentum loss increases due to the effect of the stimulated photon emission. The numerical analysis, for details see Ref. [16], shows that the combined effect of electron-positron plasma and photon gas leads to the decreasing of the probability in comparison to the result in the strong magnetic field, see Eq. (7.45). The similar supressing plasma influence on four-vector of neutrino energy and momentum losses takes place. Therefore the complex medium plasma + strong magnetic field is more transparent to neutrino with regard to the process $\nu \to \nu\gamma$, than the pure magnetic field.

7.2 Compton-Like Interaction of Neutrinos with Photons

7.2.1 The Amplitude of the Process $\gamma\gamma \to \nu\bar{\nu}$ in Vacuum

7.2.1.1 Standard Weak Interaction

Historically the reaction $\gamma\gamma \to \nu\bar{\nu}$ was one of the first photon–neutrino processes considered in the context of its astrophysical application. In 1959, Pontecorvo suggested that $(e\nu)(e\nu)$ coupling could induce reactions leading to energy loss in stars [18]. One of these processes, $\gamma\gamma \to \nu\bar{\nu}$, caused by this coupling was compared in [19] with other neutrino reactions and a rough estimation of the neutrino energy loss rate was obtained. In both papers the authors used the four-fermion $(V-A)$ Fermi model. The process of conversion of the photon pair into the neutrino-antineutrino pair is described by the two Feynman diagrams with a virtual fermion in the loop and with the photon interchange, see Fig. 7.4.

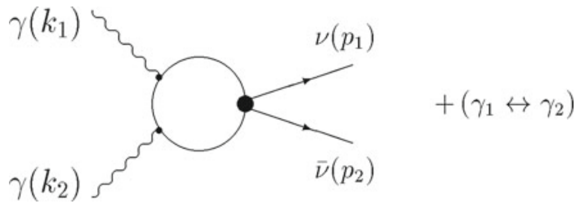

Fig. 7.4 Feynman diagram for the process $\gamma\gamma \to \nu\bar{\nu}$; a *large circle* represents the effective weak interaction of a fermion with a neutrino, see Fig. 4.8

Given the gauge invariance of the electromagnetic interaction, the amplitude of the process can be written in the most general form:

$$\mathcal{M} = \frac{\alpha}{\pi}\frac{G_{\mathrm{F}}}{\sqrt{2}}\left[\bar{\nu}_\ell(p_1)\,T_{\alpha\beta\mu\nu}\,\nu_\ell(-p_2)\right]\,f_1^{\alpha\beta}f_2^{\mu\nu}, \tag{7.49}$$

where the index ℓ defines the neutrino flavors, $\ell = e, \mu, \tau$,[1] $f^{\alpha\beta} = k^\alpha\varepsilon^\beta - k^\beta\varepsilon^\alpha$ is the tensor of the photon electromagnetic field in the momentum space. The tensor $T_{\alpha\beta\mu\nu}$ which is the Dirac matrix, has the dimension of an inverse mass and must be built from the available covariants.

Apparently, the very first correct conclusion about the amplitude was done in Ref. [20]. It is the Gell-Mann theorem: in the case of massless neutrinos, real photons, and in the local limit of the standard weak interaction, the amplitude is exactly zero. Qualitatively, this can be seen from the following argument. In the center-of-mass frame, the left-handed neutrino and the right-handed antineutrino carry the total angular momentum equal to a unit. However, as it was shown by Landau [21] and Yang [22], the system of two photons can not exist in a state with a unit angular momentum (Landau–Yang theorem). In terms of the tensor analysis, this means that with chirality of massless neutrinos and Bose symmetry of photons, there are no covariants to construct the tensor $T_{\alpha\beta\mu\nu}$.

The nonzero amplitude (7.49) arises if any of the Gell-Mann theorem conditions is broken. It may be non-zero neutrino mass, non-locality of the standard weak interaction, non-standard neutrino interaction, or off-shell photons. In the case of massive neutrinos, the process becomes allowed [23, 24] due to the change of a neutrino chirality, with the amplitude being proportional to a neutrino mass. To illustrate the Lorentz structure, we present here an expression for the tensor $T_{\alpha\beta\mu\nu}$ in the case of low-energy photons ($\omega \ll m_e$), where the electron loop gives the maximal contribution to the amplitude:

$$T_{\alpha\beta\mu\nu} = \frac{\mathrm{i}}{12}\left(\delta_{\ell e} - \frac{1}{2}\right)\frac{m_{\nu_\ell}}{m_e^2}\,\gamma_5\,\varepsilon_{\alpha\beta\mu\nu}. \tag{7.50}$$

When the non-locality of the weak interaction via the W boson is taken into account, the momenta of a neutrino and an antineutrino can enter the amplitude not just as

[1] The expression (7.49) can be easily generalized to take into account the lepton mixing.

a sum but separately, providing the following structure [25–27]:

$$T_{\alpha\beta\mu\nu} = \frac{16i}{3}\left(\ln\frac{m_W}{m_e} + \frac{3}{8}\right)\frac{1}{m_W^2}$$
$$\times \left[\gamma_\alpha\, g_{\beta\mu}(p_1 - p_2)_\nu + \gamma_\mu\, g_{\nu\alpha}(p_1 - p_2)_\beta\right](1 - \gamma_5)\,. \tag{7.51}$$

We see that in both cases, the amplitude has a strong suppression either through the small neutrino mass in the numerator or the large W boson mass in the denominator.

7.2.1.2 Model with a Broken Left-Right Symmetry

Another deviation from the conditions of the Gell-Mann theorem, in which the process $\gamma\gamma \to \nu\bar{\nu}$ is also possible, is realized when the neutrino changes its chirality in the effective Lagrangian of the lepton-neutrino interaction. When writing the Lagrangian in the form of the neutral current coupling, the neutrino chirality change is provided if currents are scalar or pseudoscalar. This case considered in Ref. [28] takes place in a model with a broken left-right symmetry [29–36] and with the mixing of vector bosons interacting with the left-handed and right-handed charged weak currents [33]. In this model, the Lagrangian of the $\nu e W$ interaction can be represented as

$$\mathcal{L} = \frac{g}{2\sqrt{2}}\left\{[\bar{e}\gamma_\alpha\,(1-\gamma_5)\,\nu_e]\left(W_1^\alpha\cos\zeta + W_2^\alpha\sin\zeta\right)\right.$$
$$\left. + [\bar{e}\gamma_\alpha\,(1+\gamma_5)\,\nu_e]\left(-W_1^\alpha\sin\zeta + W_2^\alpha\cos\zeta\right) + h.c.\right\}, \tag{7.52}$$

where $W_{1,2}$ are the charged vector W bosons with a definite mass, and ζ is the mixing angle. The existing restrictions on the parameters of the model are obtained in low-energy accelerator experiments, and have the form [37]

$$M_{W_2} > 715\ \text{GeV}\,, \quad \zeta < 0.013\,. \tag{7.53}$$

Due to the smallness of the mixing angle, the state W_2 almost coincides with the right-handed boson W_R.

There also exists a stronger limit on the model parameters, obtained from astrophysical data, namely, from the analysis of neutrino events from the supernova SN1987A. In combination with accelerator data, the limits were obtained [38]:

$$M_{W_R} > 23\ \text{TeV}\,, \quad \zeta < 10^{-5}. \tag{7.54}$$

For realization of the process $\gamma\gamma \to \nu\bar{\nu}$, a part of the effective $\nu\nu ee$ interaction Lagrangian is important, providing a non-standard neutrino or antineutrino chirality. This is possible due to the mixing of the gauge bosons, when the left-handed and right-handed currents from Eq. (7.52) are multiplied in the effective Lagrangian. Given the smallness of the mixing angle and the mass ratio M_{W_1}/M_{W_2}, we can write the Lagrangian of the $\nu\nu ee$ interaction in the form

$$\mathcal{L}_{\text{eff}} \simeq -4\,\zeta\,\frac{G_{\text{F}}}{\sqrt{2}}\left[(\bar{e}e)\,(\bar{\nu}_e\nu_e) - (\bar{e}\gamma_5 e)\,(\bar{\nu}_e\gamma_5\nu_e)\right]. \tag{7.55}$$

There exist two new channels for the conversion of the photon pair into the neutrino-antineutrino pair, if compared with the standard model, namely:

$$\gamma\gamma \to (\nu_e)_L(\bar{\nu}_e)_L\,, \quad \gamma\gamma \to (\nu_e)_R(\bar{\nu}_e)_R\,. \tag{7.56}$$

Here, $(\nu_e)_R$ and $(\bar{\nu}_e)_L$ are the states which are sterile with respect to the standard weak interaction. The total spin of a neutrino pair in both processes (7.56) in the center of mass is zero, and the process $\gamma\gamma \to \nu\bar{\nu}$ is open.

Representing the amplitude of the process caused by the effective Lagrangian (7.55) in the form of (7.49), we have the following expression for the tensor $T_{\alpha\beta\mu\nu}$:

$$\begin{aligned} T_{\alpha\beta\mu\nu} = \frac{4\zeta m_e}{(k_1k_2)}\Bigg\{&\left[1+\frac{1}{2}(1-4\tau)I(\tau)\right]g_{\alpha\nu}g_{\beta\mu}. \\ &+\frac{\mathrm{i}}{4}I(\tau)\gamma_5\varepsilon_{\alpha\beta\mu\nu}\Bigg\}, \end{aligned} \tag{7.57}$$

where

$$\tau = \frac{m_{e^2}}{2(k_1k_2)}, \qquad I(\tau) = \int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y\,\frac{1}{\tau - xy - \mathrm{i}\epsilon}. \tag{7.58}$$

Note that our result (7.57), coinciding in terms of the tensor structure with the one, which can be extracted from Ref. [28], differs from it in numerical coefficients.

The amplitude of the process $\gamma\gamma \to \nu\bar{\nu}$ in this model has also the suppression due to the smallness of the mixing angle ζ.

7.2.1.3 The Case of Virtual Photons

Another case of a non-zero amplitude is realized if one of the photons [39] or both photons [40] are off-shell. In this case, $k_\mu f^{\mu\nu} \neq 0$ and the photon momenta can participate in the construction of the tensor $T_{\alpha\beta\mu\nu}$.

Let us calculate the total amplitude of the process $\nu\gamma^* \to \nu\gamma^*$ in the standard model in the case of virtual photons, with non-zero neutrino mass, and with a possible mixing in the lepton sector [1, 2].

As the analysis shows, the neutrino $(V-A)$ current is factorized in this amplitude which can be presented in the following general form:

$$\mathcal{M} = \frac{\alpha}{\pi} \frac{G_{\rm F}}{\sqrt{2}} \, j_\rho^{(\nu)} \, e_{1\alpha} e^*_{2\beta} \, L_{\alpha\beta\rho}(k_1, k_2), \tag{7.59}$$

where $j_\rho^{(\nu)} = \bar{\nu}_j(p_2)\gamma_\rho \, (1 - \gamma_5)\nu_i(p_1)$, the indices i and j (generally, $i \neq j$) label the neutrino states with definite masses; $e_{1,2}$ are the 4-vectors of polarization and $k_{1,2}$ are the 4-momenta of the photons. As it follows from the above, the tensor $L_{\alpha\beta\rho}$ could contain only two independent momenta k_1 and k_2.

Let us consider in more detail the contribution to the amplitude from the Z boson exchange. For its obtaining, it is necessary to summarize over all fundamental charged fermions f, both leptons and quarks, in the loop. The $L_{\alpha\beta\rho}$ tensor takes the form:

$$L_{\alpha\beta\rho} = \sum_f T_{3f} \; Q_f^2 \; L^{(f)}_{\alpha\beta\rho}, \tag{7.60}$$

where Q_f is the electric charge of a fermion in units of the elementary charge e, T_{3f} is the third component of the weak isospin. For the contribution of a single fermion we obtain the following expression:

$$\begin{aligned} L^{(f)}_{\alpha\beta\rho} = {\rm i} \, \varepsilon_{\lambda\mu\beta\rho} \int\limits_0^1 {\rm d}x \int\limits_0^{1-x} \frac{{\rm d}y}{a_f} \Big\{ & g_{\lambda\alpha} \, k_{1\mu} \, [\, k_1^2 \, x(1-2x) + k_2^2 \, y(1-2y) \\ & - 4(k_1 k_2) \, xy] + 2 \, g_{\lambda\alpha} \, k_{2\mu} \, k_1^2 \, x \\ & + 4 \, k_{1\lambda} \, k_{2\mu} \, x[k_{2\alpha} y - k_{1\alpha}(1-x)] \Big\} + (k_1 \leftrightarrow -k_2 \; , \; \alpha \leftrightarrow \beta), \end{aligned} \tag{7.61}$$

where the notation is used:

$$a_f = m_f^2 + 2(k_1 k_2)xy - k_1^2 \, x(1-x) - k_2^2 \, y(1-y). \tag{7.62}$$

In the formula (7.61), the terms are omitted that do not depend on the mass of a fermion, since, due to the known relation $\sum_f T_{3f} \; Q_f^2 \; = 0$ (for each generation), they do not contribute to the amplitude. The expression (7.61) can be rewritten in such form that the amplitude becomes manifestly gauge invariant. For this, we use the photon electromagnetic field tensor in the momentum space

$$f_{\mu\nu} = k_\mu e_\nu - k_\nu e_\mu, \tag{7.63}$$

and also the dual tensor

$$\tilde{f}_{\mu\nu} = \frac{1}{2} \, \varepsilon_{\mu\nu\alpha\beta} \, f_{\alpha\beta}. \tag{7.64}$$

Introducing the notation

$$R_\rho^{(f)} = e_{1\alpha} e^*_{2\beta} L^{(f)}_{\alpha\beta\rho}, \tag{7.65}$$

we find the expression for the vector $R_\rho^{(f)}$ representing the amplitude in an explicitly gauge invariant form:

$$R_\rho^{(f)} = 4\mathrm{i}\left\{ \tilde{f}_{2\rho\mu}\, f_{1\mu\nu} \int\limits_0^1 x\,\mathrm{d}x \int\limits_0^{1-x} \frac{\mathrm{d}y}{a_f}\,[k_{1\nu}(1-x) - k_{2\nu}y] \right.$$

$$\left. + \tilde{f}_{1\rho\mu} f_{2\mu\nu} \int\limits_0^1 \mathrm{d}x \int\limits_0^{1-x} \frac{y\mathrm{d}y}{a_f}\,[\,k_{1\nu}x - k_{2\nu}(1-y)] \right\}. \tag{7.66}$$

In the transition from Eq. (7.61) to (7.66), the following identity was used:

$$\int\limits_0^1 \mathrm{d}x \int\limits_0^{1-x} \frac{\mathrm{d}y}{a_f} \left[k_1^2 x(1-2x) - k_2^2 y(1-2y) \right] \equiv 0. \tag{7.67}$$

An analysis shows that the contribution to the amplitude of the diagram with a virtual W boson is also expressed through the vector (7.66), where a charged lepton only appears as a virtual fermion. The total amplitude of the process $\nu_i\gamma^* \to \nu_j\gamma^*$ can be represented as

$$\mathcal{M} = \frac{\alpha}{\pi}\,\frac{G_\mathrm{F}}{\sqrt{2}}\, j_\rho^{(\nu)} \left(\sum_\ell U_{i\ell} U_{j\ell}^*\, R_\rho^{(\ell)} + \delta_{ij} \sum_f T_{3f}\, Q_f^2 R_\rho^{(f)} \right), \tag{7.68}$$

where $U_{i\ell}$ is a unitary matrix of the lepton mixing, $\ell = e, \mu, \tau$. The amplitude must satisfy the requirements of the Gell-Mann theorem [20], but in the expression (7.66) it is not obvious yet. Using the following relation for the tensors (7.63) and (7.64):

$$\tilde{f}_{1\rho\mu}\, f_{2\mu\sigma} + \tilde{f}_{2\rho\mu}\, f_{1\mu\sigma} = \frac{1}{2}\, f_{1\mu\nu}\, \tilde{f}_{2\nu\mu}\, g_{\rho\sigma}, \tag{7.69}$$

we write the vector $R_\rho^{(f)}$ in the final form:

$$R_\rho^{(f)} = -4\,\mathrm{i} \left[\frac{1}{2} (f_1 \tilde{f}_2)(k_2 - k_1)_\rho\, A(m_f, k_1, k_2) \right. \tag{7.70}$$

$$\left. - (\tilde{f}_2 f_1 k_1)_\rho\, B(m_f, k_1, k_2) + (\tilde{f}_1 f_2 k_2)_\rho\, B(m_f, k_2, k_1) \right],$$

where the functions are introduced:

$$A(m_f, k_1, k_2) = \int_0^1 x\mathrm{d}x \int_0^{1-x} \frac{y\mathrm{d}y}{m_f^2 + 2(k_1 k_2)xy - k_1^2 x(1-x) - k_2^2 y(1-y)}, \tag{7.71}$$

$$B(m_f, k_1, k_2) = \int_0^1 x\mathrm{d}x \int_0^{1-x} \frac{(1-x-y)\mathrm{d}y}{m_f^2 + 2(k_1 k_2)xy - k_1^2 x(1-x) - k_2^2 y(1-y)}.$$

Thus, for the amplitude in the form of (7.68) and (7.70), the Gell-Mann theorem is satisfied obviously.

The obtained amplitude in special cases coincides with the known results [23, 24, 39, 40]. Thus, the first term in Eq. (7.70) being substituted into Eq. (7.68), gives the divergence of the neutrino current, i.e. it is proportional to the neutrino mass. For photons on mass shell at low energies, $\omega \ll m_e$, imposing $\ell = f = e$ and excluding the lepton mixing, $i = j = \ell'$, $U_{\ell k} = \delta_{\ell k}$, one reproduces from the amplitude of Eqs. (7.68), (7.70) and (7.71) the expression for the tensor (7.50). In another case, when both photons are virtual, $k_{1,2}^2 \neq 0$, the amplitude can be transformed in the case of massless neutrinos to the form which coincides with the result of Ref. [40]. We emphasize that the authors [40] introduced an artificial dependence of the amplitude on the neutrino momenta. It is clear, however, that in this approximation (in fact in the local limit of the weak interaction), the amplitude of the process $\nu\gamma^* \to \nu\gamma^*$ can explicitly depend only on the photon momenta.

In this case at low photon energies, $\omega \ll m_e$, the tensor $T_{\alpha\beta\mu\nu}$ introduced in Eq. (7.49) has the form:

$$T_{\alpha\beta\mu\nu} = \frac{\mathrm{i}}{12 m_e^2} \left(U_{ie} U_{je}^* - \frac{1}{2}\,\delta_{ij} \right) \gamma^\rho (1 - \gamma_5) \left(\varepsilon_{\rho\alpha\mu\nu} k_{1\beta} + \varepsilon_{\rho\mu\alpha\beta} k_{2\nu} \right). \tag{7.72}$$

It should be noted that the total amplitude (7.68), (7.70) allows in particular to obtain the first terms of the expansion over the external field of the amplitudes of the radiative neutrino decay $\nu_i \to \nu_j \gamma$ and of the non-radiative transition $\nu_i \to \nu_j$ in the electromagnetic field of an arbitrary configuration. It is enough to replace in Eq. (7.70) the electromagnetic field tensor of the one or both photons to the external electromagnetic field tensor.

Let us apply the obtained amplitude of the process $\nu\gamma^* \to \nu\gamma^*$ to calculate the probability of the massive neutrino radiative decay $\nu_i \to \nu_j \gamma$ in an external field [41, 42], in the case of relatively weak field. The field tensor of one of the photons is replaced to the tensor of the constant uniform magnetic field:

$$q_{1\alpha} \to 0, \;\; f_{1\alpha\beta} \to \mathrm{i} F_{\alpha\beta}, \qquad q_{2\alpha} \to q_\alpha, \;\; f_{2\alpha\beta} \to f_{\alpha\beta}\,. \tag{7.73}$$

Taking into account that the main contribution comes from the electron loop, and that the photon dispersion in a weak field does not differ from the dispersion in vacuum ($q^2 = 0$), we obtain the amplitude of the process $\nu_i \to \nu_j \gamma$:

$$\mathcal{M} = \frac{e G_{\mathrm{F}} C_A}{48\sqrt{2}\pi^2} \, \frac{B}{B_e} \, (\varphi \tilde{f}^*) \, (j^{(\nu)} q), \tag{7.74}$$

where $C_A = U_{ie} U_{je}^* - \frac{1}{2}\,\delta_{ij}$. The expression (7.74) coincides with the linear in the field term of the amplitude presented in Eq. (4) of Ref. [41].

Assuming for simplicity the finite neutrino to be massless, we find the probability of the decay $\nu_i \to \nu_j \gamma$ in the rest frame of the initial neutrino:

$$W = \frac{\alpha}{18\pi} \, \frac{G_{\mathrm{F}}^2 C_A^2}{192\pi^3} \, m_{\nu i}^5 \left(\frac{B}{B_e}\right)^2. \tag{7.75}$$

The probability (7.75) agrees with Eq. (5) of Ref. [41], but it is 4 times less than the probability obtained from Eq. (32) of Ref. [42] in the weak field limit.

7.2.2 *Neutrino Scattering in the Coulomb Field of a Nucleus*

As one more illustration of the application of the formula (7.68), we consider the scattering of a high-energy neutrino on a nucleus with the photon radiation. In the cited papers [23, 24, 39], only astrophysical manifestations of the process $\nu\gamma \to \nu\gamma$ were studied . Our aim is to explore the possibility to detect this reaction in the laboratory experiment with high-energy neutrinos from the accelerator. From the observational point of view, this process would appear as a bremsstrahlung in the neutrino scattering in the Coulomb field of a nucleus

$$\nu_i + \text{nucleus} \to \nu_j + \gamma + \text{nucleus}. \tag{7.76}$$

The experimental evidence of the reaction should be the detection of a single hard photon without any escort.

The reaction (7.76) amplitude can be obtained from Eqs. (7.68) and (7.70) taking one of the photons (e.g. γ_2) to be real. In this case one has $f_{2\mu\nu} k_{2\nu} = 0$. We shall regard $m_\nu = 0$ and neglect the lepton mixing. Then the amplitude will be defined by the second term in Eq. (7.70). Inserting $(Ze/k_1^2) J_\mu$ instead of $e_{1\mu}$, where J_μ and Ze are the electromagnetic current and the charge of the nucleus, $k_{1\mu}$ and $e_{1\mu}$ are the momentum and the polarization vector of the virtual photon, one obtains

$$\mathcal{M} = 4\,\mathrm{i}\,\frac{Ze\alpha}{\pi}\,\frac{G_{\mathrm{F}}}{\sqrt{2}}\,\varepsilon_{\rho\mu\alpha\beta}\,j_\rho^{(\nu)}\,J_\mu\,k_{2\alpha}\,e^*_{2\beta}\Big[B(m_\ell, k_1, k_2) + \sum_f T_{3f}\,Q_f^2\,B(m_f, k_1, k_2)\Big]. \tag{7.77}$$

Here, m_ℓ is the mass of the charged lepton which is the partner of the neutrino taking part in the reaction. Let us examine the case of small transmitted momenta when the nucleus is still nearly motionless. The momentum modulo $|\mathbf{k}_1|$ is restricted then by the value of k_m which can be estimated as the inverted nucleus radius $1/r \sim k_m \simeq 200 \times A^{-1/3}$ MeV. As the analysis shows, at high energies of the neutrino all the charged fermions contribute to the amplitude (7.77) except t-quark (we still presume $(pk_1) \ll m_W^2 < m_t^2$). In the leading log approximation, the mass of a fermion in the integral $B(m_f, k_1, k_2)$ defined by Eq. (7.71) can be neglected. We get the following expression for the spectrum of radiated photons:

$$\mathrm{d}\sigma = \frac{\alpha}{54\pi}\left(\frac{Z\alpha}{\pi}\right)^2 \frac{G_{\mathrm{F}}^2 k_m^2}{\pi}\,\frac{\mathrm{d}\omega}{\omega}\left[1 - \frac{\omega}{E} + \frac{1}{2}\left(\frac{\omega}{E}\right)^2\right]\ln^3\left(\frac{2\omega}{k_m}\right), \tag{7.78}$$

where ω is the photon energy, E is the initial neutrino energy, k_m is the maximal momentum of the nucleus recoil. For the high energy neutrinos, within the leading log approximation the total cross-section of the process is

$$\sigma \simeq \left(\frac{\alpha}{2\pi}\right)^3 \frac{Z^2}{27}\,\frac{G_{\mathrm{F}}^2 k_m^2}{\pi}\,\ln^4\left(\frac{2E}{k_m}\right). \tag{7.79}$$

For example, for a neutrino energy $E = 100$ GeV we have

$$\sigma \sim \frac{Z^2}{A^{2/3}}\,10^{-46}\,\mathrm{cm}^2\,. \tag{7.80}$$

This small value of the cross-section makes it difficult to observe the bremsstrahlung in the neutrino scattering by the coulombian field of the nucleus. This is true even if one takes into account the distinctive signature of the reaction as the production of a high energy photon without any accompanying particles. It must be noted that the same signature in the neutrino reaction may correspond to the coherent production of photons by nucleons of the nucleus [43, 44]. However, the process we consider has a narrower angular distribution of photons, $\theta < k_m/E$ instead of $\theta < \sqrt{k_m/E}$ [43, 44]. Moreover, it is necessary to distinguish in the neutrino experiment between the electromagnetic showers produced by photons and by recoiled electrons in the process $\nu e \to \nu e$ which has a cross-section 10^4 times larger than (7.79).

Nevertheless, we hope that the experimental difficulties we have pointed out can be overcome in the future. Then the process $\nu\gamma^* \to \nu\gamma$ we have discussed could be accessible to observation. This process (one-loop at the minimum) could be one of the few tests for the validity of higher-order perturbation theory in the standard model of electroweak interaction.

7.2.3 The External Field Influence on the Process $\gamma\gamma \to \nu\bar{\nu}$

As it was mentioned already, a strong magnetic field could enhance this process. Since the electromagnetic tensor field $F_{\mu\nu}$ arises, it opens up a new opportunity to build a tensor $T_{\alpha\beta\mu\nu}$ in the amplitude (7.49). In fact, the field comes into the amplitude in the form of the dimensionless tensor $eF_{\mu\nu}/m_e^2$, providing an extra enhancement if the value of the field exceeds a critical value $B_e = m_e^2/e$.

The process $\gamma\gamma \to \nu\bar{\nu}$ was investigated in Ref. [45] in the framework of the standard model in a relatively weak magnetic field $B \ll B_e$, in the lowest-order expansion over B/B_e, and for the case of low photon energies, $\omega \ll m_e$. Just in this approximation it is appropriate to use the effective Lagrangian obtained in Ref. [46] from the amplitude of the process $\gamma\gamma \to \gamma\nu\bar{\nu}$ and used in Ref. [45]. It follows from Ref. [45], that the amplitude of the process depends linearly on the field. As we show below, this growth takes place only at $B \ll B_e$, but in a strong field $B \gg B_e$, the amplitude becomes a constant in the case of the standard weak interaction.

The process $\gamma\gamma \to \nu\bar{\nu}$ and the crossed channels were also studied in Refs. [47, 48] in a weak magnetic field, and in a wide region of the photon energy, namely, for $\omega < m_W$. In the limit $\omega \ll m_e$, the amplitude obtained in Ref. [48] is consistent with the result of Ref. [45]. Unfortunately, the amplitude is written in Ref. [48] in a very cumbersome form, and just the gauge invariance test is extremely difficult to conduct.

In an earlier paper [49], the process $\gamma\gamma \to \nu\bar{\nu}$ was investigated in a strong magnetic field $B \gg B_e$, for low-energy photons, $\omega \ll m_e$, and without taking into account the contribution of the Z boson.

A general analysis of the three-vertex loop process $\gamma\gamma \to \nu\bar{\nu}$ in a strong magnetic field, based on the asymptotic form of the electron propagator in the field, for arbitrary kinematic conditions was first performed in Ref. [50].

Consider the general case of a three-vertex loop process in a strong magnetic field, which is described by the Feynman diagram shown in Fig. 7.5.

In the process of transformation of the photon pair into a pair of neutrino and antineutrino $\gamma\gamma \to \nu\bar{\nu}$, two vertices are vectors, e.g. $\Gamma_1 = \Gamma_2 = V$, and the third one can be of the vector and axial-vector type in the standard model, $\Gamma_3 = V, A$, and can also be of the scalar and pseudoscalar type when going beyond the standard model, $\Gamma_3 = S, P$. In the case $\Gamma_3 = V$, the diagram of Fig. 7.5 describes also the photon splitting $\gamma \to \gamma\gamma$.

We will use the propagator of the electron in a magnetic field (see Sect. 3.1). The invariant amplitude of the process described by the Feynman diagram in Fig. 7.5, with Eqs. (3.1) and (3.19) in account, has the form

$$\mathcal{M} = e^2 g_3 \int \mathrm{d}^4 X\, \mathrm{d}^4 Y\, \mathrm{Tr}\{(j_3 \Gamma_3) S(Y)(\varepsilon_2 \gamma) S(-X - Y)(\varepsilon_1 \gamma) S(X)\}$$
$$\times\, \mathrm{e}^{-\mathrm{i}e\,(XFY)/2}\, \mathrm{e}^{\mathrm{i}(k_1 X - k_2 Y)} + (\gamma_1 \leftrightarrow \gamma_2), \tag{7.81}$$

where $X = z - x$, $Y = x - y$, Γ_3 is the matrix corresponding to S, P, V or A vertex, g_3 is the coupling constant, j_3 is the corresponding part of the neutrino current in a momentum space, $\varepsilon_1,\ k_1$ and $\varepsilon_2,\ k_2$ are the polarization vectors and the 4-momenta of the initial photons.

Using the propagator of the form (3.1) and (3.2) in the three-vertex loop leads, in general, to very cumbersome expressions. The relatively simple results were obtained only for the process of photon splitting in two cases: in the weak field limit [51] and in a strong field in the approximation of collinear kinematics [52, 53].

To analyze the amplitude of the process (7.81) in a strong field it is advisable to use the asymptotic expression for the electron propagator (3.65). Substituting the propagator into the amplitude, one obtains that two parts of it which differ by the photon interchange, are proportional to the field strength B:

$$\mathcal{M} \simeq -\frac{\mathrm{i}\alpha g_3 e B}{(4\pi)^2} \exp\left(-\frac{{k_1}_\perp{}^2 + k_{2\perp}^2 + (k_1 k_2)_\perp}{2eB}\right) \exp\left(-\mathrm{i}\frac{(k_1 \varphi k_2)}{2eB}\right)$$
$$\times \int \mathrm{d}^2 p\, \mathrm{Tr}\{(j_3 \Gamma_3) S_\parallel(p + k_2)(\varepsilon_2 \gamma) S_\parallel(p)(\varepsilon_1 \gamma) S_\parallel(p - k_1)\}$$
$$+ (\gamma_1 \leftrightarrow \gamma_2), \tag{7.82}$$

where $S_\parallel(p) = 2\Pi_-((p\gamma)_\parallel + m_e)/(p_\parallel^2 - m_e^2)$. It should be noted that the projection operator Π_- selects in the amplitude (7.82) only photons of the one polarization from the two possible, namely, the second mode (see Eq. (4.10)),

$$\varepsilon_\alpha^{(1)} = \frac{F_{\alpha\beta} k_\beta}{\sqrt{(kFFk)}}, \qquad \varepsilon_\alpha^{(2)} = \frac{\widetilde{F}_{\alpha\beta} k_\beta}{\sqrt{(k\widetilde{F}\widetilde{F}k)}}. \tag{7.83}$$

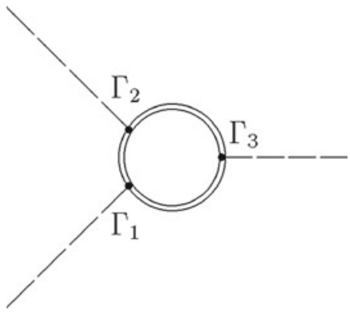

Fig. 7.5 Feynman diagram for a three-vertex loop process in a strong magnetic field: *double lines* correspond to the electron propagators constructed on the base of the exact solutions of the Dirac equation in the external magnetic field

Using the standard procedure, we can transform the trace in the second term of Eq. (7.82) with the interchanged photons to the trace in the first term. This proceeds with the change of sign for $\Gamma_3 = P, V, A$ (and the factor $\sin[(k_1 \varphi k_2)/2eB]$ arises in the resulting amplitude) and without change of sign for $\Gamma_3 = S$ (and the factor $\cos[(k_1 \varphi k_2)/2eB]$ appears after summation).

So, when the magnetic field strength is the maximal physical parameter, $eB \gg k_\perp^2, k_\parallel^2$, only the amplitude with the scalar vertex grows linearly with the field.

7.2.4 A Conversion $\gamma\gamma \to \nu\bar{\nu}$ in the Left–Right Symmetric Extension of the Standard Model

Using the effective Lagrangian of the $\nu\nu ee$ interaction with the scalar coupling (7.55), substituting $\Gamma_3 = 1$, $g_3 = -4\,\zeta\, G_{\mathrm{F}}/\sqrt{2}$ and $j_3 = [\bar{\nu}_e(p_1)\nu_e(-p_2)]$ into the amplitude (7.82) and integrating over the virtual momenta in the strong field limit, we obtain

$$\begin{aligned}
\mathcal{M} = {} & \frac{8\alpha}{\pi} \frac{G_{\mathrm{F}}}{\sqrt{2}} \frac{\zeta}{m_e} \frac{B}{B_e} \, [\bar{\nu}_e(p_1)\, \nu_e(-p_2)] \, \varepsilon_{1\alpha}^{(2)} \varepsilon_{2\beta}^{(2)} \int\limits_0^1 \mathrm{d}x \int\limits_0^{1-x} \frac{\mathrm{d}y}{a^2} \\
& \times \Big\{ \Big[k_{1\parallel}^2 x(1-2x) + k_{2\parallel}^2 y(1-2y) - (k_1 k_2)_\parallel (1-4xy) \Big] \widetilde{\Lambda}^{\alpha\beta} \\
& - (1-2x)(1-2y)\, k_{1\parallel}^\alpha k_{2\parallel}^\beta + (1-4xy)\, k_{2\parallel}^\alpha k_{1\parallel}^\beta \\
& - 2x(1-2x)\, k_{1\parallel}^\alpha k_{1\parallel}^\beta - 2y(1-2y)\, k_{2\parallel}^\alpha k_{2\parallel}^\beta \Big\},
\end{aligned} \tag{7.84}$$

$$a = 1 - \frac{q_\parallel^2}{m_e^2} xy - (1-x-y) \left(\frac{k_{1\parallel}^2}{m_e^2} x + \frac{k_{2\parallel}^2}{m_e^2} y \right), \tag{7.85}$$

where $q_\parallel = k_{1\parallel} + k_{2\parallel}$. The amplitude (7.84) can be rewritten in the explicitly gauge invariant form (7.49):

$$\mathcal{M} = \frac{\alpha}{\pi} \frac{G_{\mathrm{F}}}{\sqrt{2}} \left[\bar{\nu}_e(p_1)\, T_{\alpha\beta\mu\nu}\, \nu_e(-p_2) \right] f_1^{(2)\alpha\beta} f_2^{(2)\mu\nu}, \tag{7.86}$$

where the photon field tensors of the 2nd polarization only enter:

$$f_{\alpha\beta}^{(2)} = k_{\alpha\parallel} \varepsilon_\beta^{(2)} - k_{\beta\parallel} \varepsilon_\alpha^{(2)}.$$

The tensor $T_{\alpha\beta\mu\nu}$ in this case takes the form

$$T_{\alpha\beta\mu\nu} = \frac{4\zeta}{m_e} \frac{B}{B_e} \int\limits_0^1 \mathrm{d}x \int\limits_0^{1-x} \frac{\mathrm{d}y}{a^2} \Big\{ (1-4xy)\, \widetilde{\Lambda}_{\alpha\nu} \widetilde{\Lambda}_{\beta\mu} + 4(1-x-y)(1-2x-2y)\, \frac{1}{q_\parallel^2}\, k_{1\parallel\alpha}\, \widetilde{\Lambda}_{\beta\mu}\, k_{2\parallel\nu} \Big\}. \tag{7.87}$$

To transform the amplitude to the form (7.86), the following non-trivial integral identities were used:

$$\int\limits_0^1 \mathrm{d}x \int\limits_0^{1-x} \mathrm{d}y \frac{Sx(1-2x) - Ty(1-2y)}{A^N} \equiv 0, \tag{7.88}$$

$$\int\limits_0^1 \mathrm{d}x \int\limits_0^{1-x} \mathrm{d}y \frac{Zy(1-2y) + S(1-x-y)(1-2x-2y)}{A^N} \equiv 0, \tag{7.89}$$

$$A = 1 - Zxy - (1-x-y)(Sx + Ty), \tag{7.90}$$

where Z, S, T are the arbitrary parameters, N is an arbitrary integer; in this case, $N = 2$. The identity (7.88) has been already used earlier for the case of $N = 1$ (see Eq. (4.105)).

The expression for the amplitude can be simplified for the two limiting cases (here, the polarization vectors $\varepsilon_{1,2}^{(2)}$ are substituted already, see Eq. (7.83)):

(i) at low photon energies, $\omega \lesssim m_e$:

$$\mathcal{M} \simeq \frac{8\alpha}{3\pi} \frac{G_\mathrm{F}}{\sqrt{2}} \frac{\zeta}{m_e} \frac{B}{B_e} \left[\bar{\nu}_e(p_1)\, \nu_e(-p_2)\right] \sqrt{k_{1\parallel}^2 k_{2\parallel}^2}\,; \tag{7.91}$$

(ii) for high-energy photons, $\omega \gg m_e$, in the leading log approximation:

$$\mathcal{M} \simeq \frac{16\alpha}{\pi} \frac{G_\mathrm{F}}{\sqrt{2}} \zeta \frac{B}{B_e} m_e^3 \left[\bar{\nu}_e(p_1)\, \nu_e(-p_2)\right] \frac{1}{\sqrt{k_{1\parallel}^2 k_{2\parallel}^2}} \ln \frac{\sqrt{k_{1\parallel}^2 k_{2\parallel}^2}}{m_e^2}. \tag{7.92}$$

Calculating the cross-sections of the two processes, $\gamma\gamma \to (\nu_e)_L(\bar{\nu}_e)_L$ and $\gamma\gamma \to (\nu_e)_R(\bar{\nu}_e)_R$, by the standard way, we find that they are equal, $\sigma_{LL} = \sigma_{RR} \equiv \sigma$. In the two limiting cases, the expression for the cross-section takes the form

$$\sigma(\omega \lesssim m_e) \simeq \frac{2\,\alpha^2\,G_{\mathrm{F}}^2\,\zeta^2}{9\pi^3}\,\left(\frac{B}{B_e}\right)^2 \frac{k_{1\|}^2 k_{2\|}^2}{m_e^2}, \tag{7.93}$$

$$\sigma(\omega \gg m_e) \simeq \frac{2\,\alpha^2\,G_{\mathrm{F}}^2\,\zeta^2}{\pi^3}\,\left(\frac{B}{B_e}\right)^2 \frac{m_e^6}{k_{1\|}^2 k_{2\|}^2}\,\ln^2\frac{k_{1\|}^2 k_{2\|}^2}{m_e^4}. \tag{7.94}$$

7.2.5 Possible Manifestations of the $\gamma\gamma \to \nu\bar{\nu}$ Process in Astrophysics

As the observable value in astrophysics, it is considered the stellar energy-loss from unit volume per unit time due to the neutrino escape (neutrino emissivity). For the process $\gamma\gamma \to \nu\bar{\nu}$ enhanced by a magnetic field, considered in the previous section it can be written in the form

$$\begin{aligned} Q = \frac{1}{2}\int\frac{\mathrm{d}^3k_1}{(2\pi)^3}\,\frac{1}{\mathrm{e}^{\omega_1/T}-1}\int\frac{\mathrm{d}^3k_2}{(2\pi)^3}\,\frac{1}{\mathrm{e}^{\omega_2/T}-1}\\ \times\,(\omega_1+\omega_2)\,\frac{(k_1k_2)}{\omega_1\omega_2}\,\sigma\,(\gamma\gamma \to \nu\bar{\nu}), \end{aligned} \tag{7.95}$$

where T is the temperature of the photon gas. It is taken into account in Eq. (7.95) that photons of only one polarization are involved in this process. Since only "sterile" (anti) neutrino of a pair (see Eq. (7.56)) freely departs from hot and dense stellar medium (other neutrino participating in the standard interaction, has a small free path and is trapped) the cross-section should be written as $(\sigma_{LL} + \sigma_{RR})/2 = \sigma$.

(i) The case of low temperatures, $T \lesssim m_e$
   In this case, substituting Eq. (7.93) into Eq. (7.95), we obtain

$$Q_{(B)} \simeq 2.5\times 10^{13}\,\frac{\mathrm{erg}}{\mathrm{s\,cm^3}}\,\left(\frac{\zeta}{0.013}\right)^2\left(\frac{B}{B_e}\right)^2\left(\frac{T}{m_e}\right)^{11}. \tag{7.96}$$

   Let us compare this value with the contributions to the neutrino emissivity through other mechanisms in the $\gamma\gamma \to \nu\bar{\nu}$ process, discussed in this chapter. For example, for the contribution due to the non-zero mass of neutrinos, it was obtained in Ref. [24]:

$$Q_{(m_\nu)} \simeq 0.4\times 10^5\,\frac{\mathrm{erg}}{\mathrm{s\,cm^3}}\,\left(\frac{m_\nu}{1\,\mathrm{eV}}\right)^2\left(\frac{T}{m_e}\right)^{11}. \tag{7.97}$$

   Substituting the cross-section calculated with taking account of the non-locality of the weak interaction [26] into the expression (7.95), one obtains

$$Q_{(\text{nloc})} \simeq 10 \, \frac{\text{erg}}{\text{s}\,\text{cm}^3} \left(\frac{T}{m_e}\right)^{13} . \tag{7.98}$$

It is seen that for $B \gtrsim B_e$, and for mixing at the level of $\zeta \sim 10^{-5}$, the field-induced mechanism of the reaction $\gamma\gamma \to \nu\bar{\nu}$ strongly dominates all the other indicated mechanisms.

(ii) The case of high temperatures, $T \gg m_e$

In the case of high temperatures, substituting Eq. (7.94) into Eq. (7.95), we obtain

$$Q_{(B)} \simeq 0.4 \times 10^{12} \, \frac{\text{erg}}{\text{s}\,\text{cm}^3} \left(\frac{\zeta}{0.013}\right)^2 \left(\frac{B}{B_e}\right)^2 \left(\frac{T}{m_e}\right)^3 \left(\ln \frac{T}{m_e}\right)^5 . \tag{7.99}$$

In order to make a numerical estimation, let us consider the Supernova explosion with generation of very strong magnetic field $B \sim 10^3 \; B_e$, see e.g. [9, 54–56], with the temperature $T \sim 35 \; MeV$ which is believed to be typical for the Supernova core [57], and $V \sim 10^{18}$ cm^3. For the contribution of the considered field-enhanced process $\gamma\gamma \to \nu\bar{\nu}$ into the neutrino luminosity we obtain

$$L \sim 10^{45} \, \frac{\text{erg}}{\text{s}} \left(\frac{\zeta}{0.013}\right)^2 . \tag{7.100}$$

It is too small if compared with the typical Supernova neutrino luminosity 10^{52} erg/sec. Nevertheless, for the field strength $B \gtrsim B_e$ this mechanism could dominate other discussed in the literature mechanisms of the process $\gamma\gamma \to \nu\bar{\nu}$ in the neutrino emissivity of magnetized stars.

As it was noted above, the amplitude (7.82) with the vector and axial-vector vertices, $\Gamma_3 = V, A$, does not contain in the strong field limit, $eB \gg k_\perp^2, k_\parallel^2$, a part linearly increasing with the field growth. This means that the amplitudes both for the process $\gamma\gamma \to \nu\bar{\nu}$ with the standard $\nu\nu ee$ interaction, and for the photon splitting $\gamma \to \gamma\gamma$ do not depend on the field in this limit. It should be also noted that neutrinos produced in the standard interaction, do intensely absorbed by hot and dense stellar environment due to the same interaction. Thus, they can not provide any significant contribution to the cooling of the central region of the exploding supernova.

7.3 Neutrino Photoproduction on a Nuclei in a Strong Magnetic Field

As it was mentioned above, the loop quantum processes whose initial and final states involve only electrically neutral particles such as neutrinos and photons are of special interest. The action of an external field on these processes is caused, first, by the sensitivity of charged virtual fermions to the field. In this case, an electron as a particle with the maximum specific charge e/m_e plays the dominant role. Second,

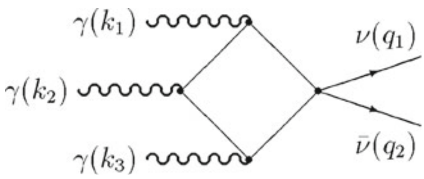

Fig. 7.6 Feynman diagram for the $\gamma + \gamma + \gamma \to \nu + \bar{\nu}$ process

a strong magnetic field gives rise to a considerable change in the dispersion properties of photons and, therefore, in their kinematics.

The contribution of the loop process of neutrino-pair photoproduction on a nucleus

$$\gamma + Ze \to Ze + \gamma + \nu + \bar{\nu} \tag{7.101}$$

in a strong external magnetic field to the cooling of stars was studied in the paper [58] and it was stated that this contribution can compete with the contribution from Urca processes. Therefore, the process (7.101), as one more channel of neutrino energy loss, would be taken into account when describing the cooling of strongly magnetized neutron stars. However, the photon dispersion in the field was ignored in Ref. [58].

In this section, the process of photoproduction of a neutrino pair on a nucleus (7.101) is investigated in a strong magnetic field, with taking account of the photon dispersion in a strong field. The presentation is based on Ref. [59].

The amplitude of neutrino pair photoproduction on a nucleus, Eq. (7.101), can be derived from the amplitude of the interaction between three photons and a neutrino pair, e.g.,

$$\gamma + \gamma + \gamma \to \nu + \bar{\nu}, \tag{7.102}$$

whose Feynman diagram is shown in Fig. 7.6. As is known (see, e.g., [60]), three-photon processes (7.102) in a strong magnetic field are more intense than the corresponding two-photon processes, because the amplitude of processes (7.102) with the vector-axial neutrino current increases linearly with the field, whereas the amplitude of the $\gamma\gamma \to \nu\bar{\nu}$ processes with such a neutrino current is independent of the field.

The amplitude of the process (7.102) in a strong magnetic field can be represented in the covariant form [60]

$$\begin{aligned}\mathcal{M} = &-\frac{8e^3 G_{\mathrm{F}} e B m_e^2}{\sqrt{2}\pi^2} (\varepsilon_1 \tilde{\varphi} k_1)\, (\varepsilon_2 \tilde{\varphi} k_2)\, (\varepsilon_3 \tilde{\varphi} k_3)\, [C_V (j \tilde{\varphi} k_4) + C_A (j \tilde{\varphi} \tilde{\varphi} k_4)] \\ &\times I(k_1, k_2, k_3),\end{aligned} \tag{7.103}$$

Here, C_V and C_A are the vector and axial-vector constants of the effective $\nu\nu ee$ Lagrangian (4.66); $\varepsilon_{1,2,3}$ and $k_{1,2,3}$ are the polarization 4-vectors and photon 4-momenta, respectively; $j_\alpha = [\bar{\nu}(q_1)\gamma_\alpha(1 - \gamma_5)\nu(-q_2)]$ is the Fourier transform of the neutrino current; $k_4 = q_1 + q_2$ is the 4-momentum of a neutrino pair.

The form factor $I(k_1, k_2, k_3)$ has the form of the following triple integral with respect to the Feynman variables:

$$I(k_1,k_2,k_3) = \frac{1}{D} \int\limits_0^1 dx \int\limits_0^x dy \int\limits_0^y dz \Bigg\{ \frac{a(k_1,k_2,k_3)}{[m_e^2 - b(k_1,k_2,k_3)]^3} + \{k_1 \leftrightarrow k_2\} + \{k_2 \leftrightarrow k_3\} \Bigg\}. \quad (7.104)$$

Here,

$$D = k_1^2(k_2k_3) + k_2^2(k_1k_3) + k_3^2(k_1k_2) + 2(k_1k_2)(k_1k_3) + 2(k_1k_2)(k_2k_3) + 2(k_1k_3)(k_2k_3), \quad (7.105)$$

$$a(k_1,k_2,k_3) = k_1^2\,(1-x)^2 - k_2^2\,y(1-y) + k_3^2\,z^2 + (k_1k_2)\,(1-2x)(1-y) + (k_1k_3)\,[1-x-z(1-2x)] - (k_2k_3)\,y(1-2z), \quad (7.106)$$

$$b(k_1,k_2,k_3) = k_1^2\,x(1-x) + k_2^2\,y(1-y) + k_3^2\,z(1-z) + 2(k_1k_2)\,(1-x)y + 2(k_1k_3)\,(1-x)z + 2(k_2k_3)\,(1-y)z. \quad (7.107)$$

where the scalar products $(k_i k_j)$ are the contractions $(k_i \tilde{\varphi}\tilde{\varphi} k_j)$.

For low photon energies, i.e., for $\omega_{1,2,3} \ll m_e$, the integral (7.104) is easily calculated to give

$$I(k_1,k_2,k_3) \simeq \frac{1}{60\,m_e^8}. \quad (7.108)$$

In this case, the amplitude (7.103), in view of Eq. (7.108), corresponds to the effective local $\gamma\gamma\gamma\nu\bar{\nu}$ Lagrangian

$$\mathcal{L}_{eff} = -\frac{e^3\,G_{\mathrm{F}}\,eB}{45\,\sqrt{2}\pi^2\,m_e^6} \left(\frac{\partial A^\alpha}{\partial x_\beta}\,\tilde{\varphi}_{\alpha\beta} \right)^3 \times \frac{\partial}{\partial x_\sigma} [\bar{\nu}\gamma^\rho(1-\gamma_5)\nu]\,[C_V\,\tilde{\varphi}_{\rho\sigma} + C_A\,(\tilde{\varphi}\tilde{\varphi})_{\rho\sigma}]. \quad (7.109)$$

The $\gamma\gamma\gamma\nu\bar{\nu}$ interaction at low energies was previously studied in Ref. [60], where the Lagrangian was overestimated by a factor of two.

An analysis of the dimensionality of the amplitude (7.103) for the limiting values of the characteristic photon energy $|k_1| \sim |k_2| \sim |k_3| \sim \omega$ indicates that the amplitude increases as $\sim \omega^5$ at low energies and decreases as $\sim \omega^{-3}$ at high energies.

When calculating the amplitude of the process (7.101) on a nucleus in the local limit of the effective $\gamma\gamma\gamma\nu\bar{\nu}$ interaction (7.109), it is necessary to take into account the effect of a strong magnetic field on the dispersion properties of real and virtual photons. We will demonstrate that this effect is of crucial importance. We recall that the process (7.101) in a strong magnetic field involves photons of only the 2nd polarizations.

For a virtual photon, it is necessary to use, instead of the vacuum propagator $\sim q^{-2}$, the propagator including the photon polarization tensor eigenvalue $\varPi^{(2)}(q_\parallel^2)$

in a magnetic field:

$$D^{(B)}(q_\parallel^2, q_\perp^2) \;=\; \frac{1}{q^2 - \Pi^{(2)}(q_\parallel^2)}, \tag{7.110}$$

where, $q_\parallel^2 = q_0^2 - q_z^2$, $q_\perp^2 = q_x^2 + q_y^2$, $q^2 = q_\parallel^2 - q_\perp^2$ (the magnetic field is directed along the z axis). For the strong field $B \gg B_e$ and in the approximation $|q_\parallel^2| \ll m_e^2$, this operator takes the simple form [8]

$$\Pi^{(2)}(q_\parallel^2) \simeq -\frac{\alpha}{3\pi}\,\frac{B}{B_e}\,q_\parallel^2. \tag{7.111}$$

It is convenient to introduce the following dimensionless parameter that specifies the field effect in all subsequent expressions:

$$\beta = \frac{\alpha}{3\pi}\,\frac{B}{B_e}. \tag{7.112}$$

The parameter β is equal to 0.77 and 7.7 for fields $10^3\,B_e$ and $10^4\,B_e$, respectively; i.e., it is not small. Taking into account Eqs. (7.111) and (7.112) and that $q_0 = 0$ for the virtual photon connected with a fixed nucleus, we can represent the propagator (7.110) in the form

$$D^{(B)} \;\simeq\; -\,\frac{1}{q_\perp^2 + (1+\beta)q_z^2}. \tag{7.113}$$

At the same time, the strong magnetic field also acts on the real photons involved in process (7.101) and, hence, renormalizes the wave functions:

$$\varepsilon_\alpha \longrightarrow \sqrt{\mathcal{Z}_2}\,\varepsilon_\alpha, \tag{7.114}$$

In view of Eq. (7.111), the renormalization factor $\mathcal{Z}_2$ takes the form

$$\mathcal{Z}_2 = \left(1 - \frac{\partial \Pi(q_\parallel^2)}{\partial q_\parallel^2}\right)^{-1} = \frac{1}{1+\beta}. \tag{7.115}$$

In addition, the kinematic properties of photons change substantially. Taking into account Eqs. (7.111) and (7.112), one can represent the photon dispersion relation $k^2 - \Pi(k_\parallel^2) = 0$ as $\omega^2 = \mathbf{k}^2(1+\beta\cos^2\theta)/(1+\beta)$ and the element of the momentum space in the form

$$\mathrm{d}^3k = (1+\beta)\,\omega^2\mathrm{d}\omega\,\mathrm{d}y\,\mathrm{d}\varphi, \quad y = \cos\theta\sqrt{1+\beta}\Big/\sqrt{1+\beta\cos^2\theta},$$

where θ and φ are the polar and azimuthal angles, respectively.

Using effective Lagrangian (7.109), taking into account the effect of the magnetic field on photon properties (7.110)–(7.115), and substituting the polarization vectors of real photons

$$\varepsilon_\alpha^{(2)} = \frac{(\tilde{\varphi}k)_\alpha}{\sqrt{k_\parallel^2}}, \tag{7.116}$$

we represent the amplitude of the process (7.101) in the form

$$\mathcal{M} = \frac{32\pi\alpha Z\, G_{\mathrm{F}}}{5\sqrt{2}\, m_e^4}\, \frac{\beta}{1+\beta}\, \frac{2m_N\, q_z \sqrt{k_{1\parallel}^2 k_{2\parallel}^2}}{q_\perp^2 + (1+\beta) q_z^2}\, [C_V\, (j\tilde{\varphi}k_4) + C_A (j\tilde{\varphi}\tilde{\varphi}k_4)], \tag{7.117}$$

where m_N is the nuclear mass, $q^\alpha = (0, \mathbf{q})$ is the momentum transfer to the nucleus. This expression for the amplitude differs considerably from that obtained in Ref. [58], where the effect of a strong magnetic field on the dispersion properties of photons was not considered.

The energy carried away by neutrinos from the stellar unit volume per unit time is an important quantity in astrophysical applications. It is defined in terms of the amplitude of the process (7.101) as

$$\begin{aligned} Q_\nu = \frac{(2\pi)^4 n_N}{2m_N} \int |\mathcal{M}|^2\, (\varepsilon_1 + \varepsilon_2)\, \delta^4(k_1 - k_2 - q_1 - q_2 - q) \frac{\mathrm{d}^3 k_1}{(2\pi)^3 2\omega_1} f(\omega_1) \\ \times \frac{\mathrm{d}^3 k_2}{(2\pi)^3 2\omega_2} [1 + f(\omega_2)]\, \frac{\mathrm{d}^3 q_1}{(2\pi)^3 2\varepsilon_1}\, \frac{\mathrm{d}^3 q_2}{(2\pi)^3 2\varepsilon_2}\, \frac{\mathrm{d}^3 q}{(2\pi)^3 2m_N}\,, \end{aligned} \tag{7.118}$$

where n_N is the nuclear density, ε_1 and ε_2 are the energies of neutrino and antineutrino, respectively, and $f(\omega) = [\exp(\omega/T) - 1]^{-1}$ is the distribution function for the equilibrium photon gas at the temperature T.

Substitution of the amplitude (7.117) into Eq. (7.118) leads to the following expression for the neutrino emissivity:

$$Q_\nu = \frac{8\,(2\pi)^9}{225}\, Z^2\, \alpha^2\, G_{\mathrm{F}}^2\, m_e^6\, n_N \left(\frac{T}{m_e}\right)^{14} J\,(\beta)\,. \tag{7.119}$$

The dependence on the field parameter (7.112) is determined by the integral

$$\begin{aligned} J\,(\beta) = \beta^2 \int_{-1}^{1} \mathrm{d}u\,(1-u^2) \int_{-1}^{1} \mathrm{d}v\,(1-v^2) \int_0^1 \mathrm{d}s\, s^3\,(1-s)^8 \int_0^1 \mathrm{d}r\, r^2 \\ \times \int_{-1}^{1} \mathrm{d}x [u - sv - (1-s)rx]^2 (1 - r^2x^2) \left[\overline{C_V^2}(1-r^2) + \overline{C_A^2} r^2 (1-x^2)\right] \\ \times \int_0^{2\pi} \frac{\mathrm{d}\varphi_1}{2\pi} \int_0^{2\pi} \frac{\mathrm{d}\varphi_2}{2\pi}\, \frac{1}{[F(\beta)]^2}\,, \end{aligned} \tag{7.120}$$

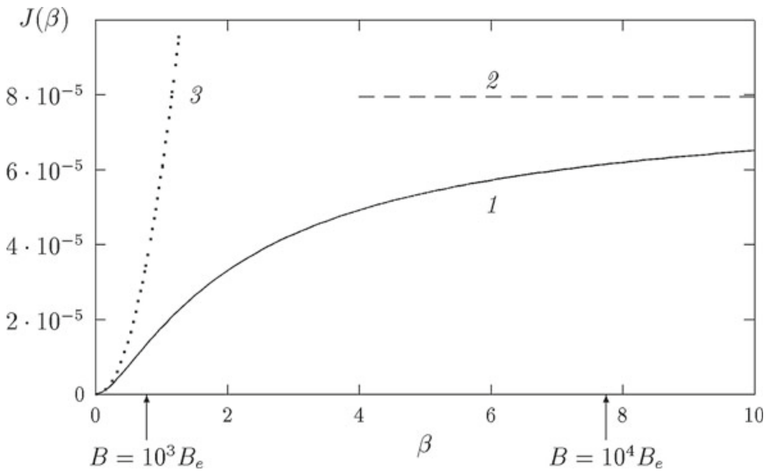

Fig. 7.7 *1* Function $J(\beta)$ [Eq. (7.120)] versus field parameter β; *2* asymptotic behavior of $J(\beta) \to 8 \times 10^{-5}$ at large β; *3* dependence $\sim \beta^2$ obtained disregarding the magnetic field effect on the photon dispersion

where

$$F(\beta) = (1+\beta)\Big\{1 - u^2 + s^2(1-v^2) - 2s\sqrt{1-u^2}\sqrt{1-v^2}\cos\varphi_1 + [u - sv - (1-s)rx]^2\Big\} - 2\sqrt{1+\beta}(1-s)r\sqrt{1-x^2} \times \Big[\sqrt{1-u^2}\cos\varphi_2 - s\sqrt{1-v^2}\cos(\varphi_2 - \varphi_1)\Big] + (1-s^2)r^2(1-x^2), \tag{7.121}$$

and the constants $\overline{C_V^2} = 0.93$ and $\overline{C_A^2} = 0.75$ are obtained by summing over all neutrino production channels for the ν_e, ν_μ and ν_τ neutrinos.

The numerically calculated integral (7.120) is shown in Fig. 7.7. It is seen that taking account of the effect of a strong magnetic field on the photon dispersion changes fundamentally the dependence of the neutrino energy loss on the field magnitude: the quadratic dependence turns into a constant. Taking this behavior into account, we obtain an upper limit for Q_ν in the asymptotically strong field:

$$Q_\nu \lesssim 2.3 \times 10^{27} \left(\frac{T}{m_e}\right)^{14} \left\langle \frac{Z^2}{A} \right\rangle \left(\frac{\rho}{\rho_0}\right) \frac{\text{erg}}{\text{cm}^3\,\text{s}}, \tag{7.122}$$

where Z and A are the charge and mass numbers of the nucleus, the averaging goes over all nuclei, $\rho_0 = 2.8 \times 10^{14}\ \text{g/cm}^3$ is the characteristic nuclear mass density, and ρ is the average mass density of the star.

The result (7.122) should be compared with the power of neutrino energy loss through the standard channel of the modified Urca process [61, 62]:

$$Q_\nu(\text{Urca}) \sim 10^{27} \left(\frac{T}{m_e}\right)^8 \left(\frac{\rho}{\rho_0}\right)^{2/3} \frac{\text{erg}}{\text{cm}^3\,\text{s}}. \tag{7.123}$$

At first glance, the values (7.122) and (7.123) are of the same order of magnitude. However, a more careful analysis of Eq. (7.122) indicates that the conclusion made in Ref. [58] about the competition of the process (7.101) with the Urca processes at magnetic fields $B \sim 10^3 B_e - 10^4 B_e$ is erroneous. The cause is that the large numerical factor arising in Eq. (7.119) and similar formulas in Ref. [58] originates from the integral over the energy ω_1 ($x = \omega_1/T$) of the initial photon:

$$\int_0^\infty \frac{x^{13}\,\mathrm{d}x}{\mathrm{e}^x - 1} = 13!\,\zeta(14) = \frac{(2\pi)^{14}}{24} \simeq 6.2 \times 10^9. \tag{7.124}$$

The main contribution to the integral (7.124) comes from $x \sim 10 \div 20$ ($\omega_1 \sim (10 \div 20)\,T$). Therefore, since the amplitude (7.117) of the process is obtained in the approximation $\omega \lesssim m_e$, the corresponding expression for the neutrino energy loss power is valid for the photon gas temperatures $T \lesssim (1/10)\,m_e$ and is inapplicable at temperatures $T \sim m_e$. Thus, the assumption made in [58] that the factor $(T/m_e)^{14}$ can be taken to be on the order of unity is erroneous. Taking into account the above applicability range, we obtain $(T/m_e)^{14} \lesssim 10^{-14}$.

Thus, the catalyzing effect of a strong magnetic field on the process of the neutrino pair photoproduction on a nucleus decreases considerably if the photon dispersion in the field is taken into account. Therefore, at any field magnitude, neutrino photoproduction cannot compete with the Urca processes.

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Chapter 8
Conclusion

The questions raised in this book, refer to the actual scientific direction, lying at the junction of plasma physics, high magnetic fields, quantum field theory, particle physics and astrophysics. Analysis of problems in the physics of hot dense magnetized plasma, resulting in a detailed quantitative description of the core collapse supernova, definitely points to the need for development of new physics that may be associated with the equation of state of nuclear and subnuclear plasma and weak interactions in the subnuclear regime, as well as the need for further research on the fundamental properties of neutrinos and mechanisms of neutrino interactions in hot dense strongly magnetized plasma, or on the need for the consideration of other, hypothetical, weakly interacting elementary particles.

This branch of science intensively developing for about 40 years, is, of course, far from being complete. There are high expectations both in the further development of a theory, and for new experimental results.

As for the development of a theory, it is impossible to predict an emergence of new productive ideas. However, in the framework of the already developed theoretical apparatus, comprehensive studies surely will continue of hot dense plasma consisting of electron–positron, proton and nucleon components at extreme physical parameters. These are the physical conditions which are realized in the central part of massive stars. At the same time, these conditions are relevant to the characteristics of nuclear and subnuclear matter. Among the factors affecting the astrophysical plasma, which need to be considered, an important role is played by a strong magnetic field and an intensive neutrino flux. In particular, the following questions should be examined:

1. Effect of plasma and magnetic field on the physical characteristics of a neutrino.
2. Neutrino absorption and emission by plasma in a magnetic field.
3. The joint effect of the plasma, magnetic field and the neutrino flux on electromagnetic radiation and its inverse effect on the plasma.
4. Mechanisms of generation of electron-positron plasma by a flux of high-energy photons and electrons.

As for the experimental studies related to the field, they could be divided into two directions. The first group is formed by the ground-based experiments, in particular,

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neutrino beam experiments. The principal expectations are apparently associated with long-baseline neutrino oscillation experiments, aimed at clarifying the mixing parameters and other physical characteristics of a neutrino.

The second area is a group of experiments that can be described as cosmic-terrestrial. It is connected with observations of astrophysical objects, primarily, the remnants of supernova explosions, in a wide range of electromagnetic and other types of radiation. A special class of experiments is formed by neutrino telescopes with the underground, underwater and under the Antarctic ice location, which are focused on the expected explosion of a galactic supernova. An interesting possibility also exists of registration the neutrino signal from the core collapse of a massive star that occurs without disruption of an envelope, that is, without formation of a supernova. Finally, high expectations are associated with gravitational wave detectors, which, according to experts, are very close to the level of sensitivity to the optimistic outlook for the border of the astrophysical intensity and probable frequency of gravitational wave signals expected from the collapse of massive stars.

We still expect a lot of discoveries.

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