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Stefano Bellucci *Editor*

Black Objects in Supergravity

Proceedings of the INFN-Laboratori Nazionali di Frascati School 2011

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Preface

This book is based upon lectures held on 9–13 May 2011 at the INFN-Laboratori Nazionali di Frascati Black Objects in Supergravity School BOSS2011, directed by Stefano Bellucci, with the participation of prestigious lecturers, including G. Lopes Cardoso, W. Chemissany, T. Ortin, J. Perz, O. Vaughan, D. Turton, L. Lusanna, S. Ferrara. All lectures were given at a pedagogical, introductory level, a feature which reflects itself in the specific "flavor" of this volume, which also benefited much from extensive discussions and related reworking of the various contributions.

This is the sixth volume in a series of books on the general topics of supersymmetry, supergravity, black holes and the attractor mechanism. Indeed, based on previous meetings, five volumes were already published:

BELLUCCI S. (2006). Supersymmetric Mechanics - Vol. 1: Supersymmetry, Noncommutativity and Matrix Models. (vol. 698, pp. 1–229). ISBN: 3-540-33313-4. BERLIN HEIDELBERG: Springer Verlag (GERMANY). Springer Lecture Notes in Physics Vol. 698.

BELLUCCI S., S. FERRARA, A. MARRANI. (2006). Supersymmetric Mechanics - Vol. 2: The Attractor Mechanism and Space Time Singularities. (vol. 701, pp. 1–242). ISBN13: 9783540341567. BERLIN HEIDELBERG: Springer Verlag (GERMANY). Springer Lecture Notes in Physics Vol. 701.

BELLUCCI S. (2008). Supersymmetric Mechanics - Vol. 3: Attractors and Black Holes in Supersymmetric Gravity. (vol. 755, pp. 1–373). ISBN-13: 9783540795223. BERLIN HEIDELBERG: Springer Verlag (GERMANY). Springer Lecture Notes in Physics Vol. 755.

BELLUCCI S. (2010). The Attractor Mechanism. Proceedings of the INFN-Laboratori Nazionali di Frascati School 2007. ISSN 0930-8989, ISBN 978-3-642-10735-1, e-ISBN 978-3-642-10736-8. DOI 10.1007/978-3-642-10736-8. Springer Heidelberg Dordrecht London New York. Springer Proceedings in Physics Vol. 134.

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I wish to thank all lecturers and participants to the School for contributing to the success of the School, which prompted the realization of this volume. I wish to thank my wife Gloria and our beloved daughters Costanza, Eleonora, Annalisa, Erica and Maristella for love and inspiration, in want of which I would have never had the strength to complete this effort.

Frascati, Italy, December 2012 Stefano Bellucci

Contents

Chapter 1
Non-holomorphic Deformations of Special Geometry and Their Applications

Gabriel Lopes Cardoso, Bernard de Wit and Swapna Mahapatra

Abstract The aim of these lecture notes is to give a pedagogical introduction to the subject of non-holomorphic deformations of special geometry. This subject was first introduced in the context of $N = 2$ BPS black holes, but has a wider range of applicability. A theorem is presented according to which an arbitrary point-particle Lagrangian can be formulated in terms of a complex function F, whose features are analogous to those of the holomorphic function of special geometry. A crucial role is played by a symplectic vector that represents a complexification of the canonical variables, i.e. the coordinates and canonical momenta. We illustrate the characteristic features of the theorem in the context of field theory models with duality invariances. The function F may depend on a number of external parameters that are not subject to duality transformations. We introduce duality covariant complex variables whose transformation rules under duality are independent of these parameters. We express the real Hesse potential of $N = 2$ supergravity in terms of the new variables and expand it in powers of the external parameters. Then we relate this expansion to the

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one encountered in topological string theory. These lecture notes include exercises which are meant as a guidance to the reader.

1.1 Introduction

As is well known, an abelian $N = 2$ supersymmetric vector multiplet in four dimensions is described by a reduced chiral multiplet, whose gauge covariant degrees of freedom include an (anti-selfdual) field strength $F^{-}_{\mu\nu}$ and a complex scalar field X. The Wilsonian effective Lagrangian for these vector multiplets is encoded in a holomorphic function $F(X)$ which, when coupled to supergravity, is required to be homogeneous of degree two [1]. The abelian vector multiplets may be further coupled to (scalar) chiral multiplets that describe either additional dynamical fields or background fields. The function F will then also depend on holomorphic fields that reside in these chiral multiplets. An example thereof is provided by the coupling of vector multiplets to a conformal supergravity background. The multiplet that describes conformal supergravity is the Weyl multiplet, and the chiral background is given by the square of it [2]. In this case the function F, which now depends on the lowest component field of the chiral background superfield, encodes the couplings of the vector multiplets to the square of the Riemann tensor. These couplings constitute a special class of higher-derivative couplings, namely, they depend on the Riemann tensor but not on derivatives thereof. In this paper we will only consider higher-derivative couplings of this type, i.e. couplings that depend on field strengths but not on their derivatives.[1] We refer to [3] for a discussion on other classes of higher-derivative couplings. When higher-order derivative couplings are absent, we will denote the function F by $F^{(0)}(X)$, which then refers to a Wilsonian action that is at most quadratic in space-time derivatives.

The abelian vector fields in these actions are subject to electric/magnetic duality transformations under which the electric field strengths and their magnetic duals are subjected to symplectic rotations. It is then possible to convert to a different duality frame, by regarding half of the rotated field strengths as the new electric field strengths and the remaining ones as their magnetic duals. The latter are then derivable from a new action. To ensure that the characterization of the new action in terms of a holomorphic function remains preserved, the scalars of the vector multiplets are transformed accordingly. This amounts to rotating the complex fields X^I and the holomorphic derivatives $F_I = \partial F/\partial X^I$ of the underlying function F by the same symplectic rotation as the field strengths and their dual partners [1, 4]. Here the index I labels the vector multiplets (in supergravity it takes the values $I = 0, 1, \ldots, n$). Thus, electric/magnetic duality (which acts on the vector (X^I, F_I)), constitutes an equivalence transformation that relates two Lagrangians (based on two different functions) and gives rise to equivalent sets of equations of

[1] In the language of the theorem that will be presented in Sect. 1.2, this may be rephrased by saying that the Lagrangians we will consider depend on coordinates and velocities, but not on accelerations.

motion and Bianchi identities. A subgroup of these equivalence transformations may constitute a symmetry (an invariance) of the system. For a duality transformation to constitute a symmetry, the substitution $X^I \to \tilde{X}^I$ into F_I must correctly induce the transformation $(X^I, F_I) \to (\tilde{X}^I, \tilde{F}_I)$ [5].

At the Wilsonian level, when coupling the $N = 2$ vector multiplets to supergravity, the scalar fields of the vector multiplets parametrize a non-linear sigma-model whose geometry is called special geometry [6], a name that first arose in the study of the geometry of the effective action of type-II string compactifications on Calabi-Yau threefolds [4]. The sigma-model space is a so-called special-Kähler space, whose Kähler potential is [1],

$$K(z, \bar{z}) = -\ln\left[\frac{\mathrm{i}\big(X^I \bar{F}_I^{(0)} - \bar{X}^I F_I^{(0)}\big)}{|X^0|^2}\right], \tag{1.1}$$

where $F^{(0)}(X)$ is the holomorphic function that determines the supergravity action, which is quadratic in space-time derivatives. Because $F^{(0)}(X)$ is homogeneous of second degree, this Kähler potential depends only on the 'special' holomorphic coordinates $z^i = X^i/X^0$ and their complex conjugates, where $i = 1, \dots, n$, so that we are dealing with a special-Kähler space of complex dimension n. In view of the homogeneity, the symplectic rotations acting on the vector $(X^I, F_I^{(0)})$, induce corresponding (non-linear) transformations on the special coordinates z^i. Up to a Kähler transformation, the Kähler potential transforms as a function under duality.

There actually exist various ways of defining special Kähler geometry. Apart from its definition in terms of special holomorphic coordinates [1], it can also be defined in a coordinate independent way [7]. More recently, the formulation of special geometry in terms of special real instead of special holomorphic coordinates has been emphasized [8–13]. This formulation is based on the real Hesse potential [14–16], which will play an important role below.

In order to pass from the Wilsonian effective action to the 1PI low-energy effective action, one needs to integrate over the massless modes of the model. In the context of $N = 2$ theories this induces non-holomorphic modifications in the gauge and gravitational couplings of the theory that, at the Wilsonian level, are encoded in the holomorphic function F. An early example thereof is provided by the computation of the moduli dependence of string loop corrections to gauge coupling constants in heterotic string compactifications [17]. These non-holomorphic modifications of the coupling functions are crucial to ensure that the low-energy effective action possesses the expected duality symmetries. This is therefore a generic feature of the low-energy effective action of $N = 2$ models with duality symmetries.

Another context where these moduli dependent corrections play an important role is the one of BPS black hole solutions in $N = 2$ models. Their entropy should exhibit the duality symmetries of the underlying model, and this is achieved by taking into account the non-holomorphic modifications of the low-energy effective action. The need for non-holomorphic modifications of the entropy was established in models with exact S-duality [18], and their presence has been confirmed at the semiclassical level from microstate counting [19, 20]. The fact that non-holomorphic modifications can be incorporated into the entropy of BPS black holes gave a first indication that the framework of special geometry can be consistently modified by a class of non-holomorphic deformations, to be described below. This can be understood as follows. The free energy of these BPS black holes turns out to be given by a generalized version of the aforementioned Hesse potential [8, 10, 21]. The Hesse potential is related by a Legendre transformation to the function F that defines the effective action, and thus it can be regarded as the associated 'Hamiltonian'. The Hamiltonian transforms as a function under electric/magnetic duality transformations. If the $N = 2$ model under consideration has a duality symmetry, the Hamiltonian will be invariant under symmetry transformations due to the presence of the aforementioned non-holomorphic modifications. Since the Hamiltonian is related to the function F by an Legendre transformation, these non-holomorphic modifications will also be encoded in F.

This 'Hamiltonian' picture of BPS black holes suggests that special geometry can be consistently modified by a class of non-holomorphic deformations, whereby the holomorphic function $F(X)$ that characterizes the Wilsonian action is replaced by a non-holomorphic function

$$F(X, \bar{X}) = F^{(0)}(X) + 2\mathrm{i}\, \Omega(X, \bar{X}), \tag{1.2}$$

where Ω denotes a real (in general non-harmonic) function. The Wilsonian limit is recovered by taking Ω to be harmonic. In Sect. 1.2 we show that the non-holomorphic deformations of special geometry described by (1.2) occur in a generic setting. There we consider general point-particle Lagrangians (that depend on coordinates and velocities) and their associated Hamiltonians. We present a theorem that shows that the dynamics of these models can be reformulated in terms of a symplectic vector $(X, \partial F/\partial X)$ constructed out of a complex function F of the form (1.2), and whose real part comprises the canonical variables of the associated Hamiltonian. We show that under duality transformations the transformed symplectic vector is again encoded in a non-holomorphic function of the form (1.2). We illustrate the theorem with various field theory examples with higher-derivative interactions. We give a detailed discussion of these examples in order to illustrate the characteristic features of the theorem. One example consists of the Born-Infeld Lagrangian for an abelian gauge field, which we reformulate in the language of the theorem based on (1.2). We subsequently promote the gauge coupling constant to a dynamical field S and discuss the duality symmetries of the resulting model. We then turn to more general models with exact S- and T-duality and discuss the restrictions imposed on Ω by these symmetries.

The function F in (1.2) may depend on a number of external parameters which we denote by η. Under duality transformations, the symplectic vector $(X, \partial F/\partial X)$ transforms into $(\tilde{X}, \partial\tilde{F}/\partial\tilde{X})$, while the parameters η are inert. When expressing the transformed variables $\tilde{X}$ in terms of the X, the relation will depend on η, i.e. $\tilde{X} = \tilde{X}(X, \eta)$. In Sect. 1.3 we introduce covariant complex variables that constitute a complexification of the canonical variables of the Hamiltonian, and whose duality transformation law is independent of η. These variables ensure that when expanding the Hamiltonian in powers of the external parameters, the resulting expansion coefficients transform covariantly under duality transformations. This expansion can also be studied by employing a modified derivative $\mathcal{D}_\eta$, which we construct. The covariant variables introduced in this section have the same duality transformation properties as the ones used in topological string theory and can therefore be identified with the latter. A further indication of the relation with topological string theory is provided by the generating function that relates the canonical variables of the Hamiltonian to the covariant complex variables. This generating function turns out to be the one that is used in the wave function approach to perturbative topological string theory [22–26].

In Sect. 1.4 we turn to supergravity models in the presence of higher-curvature interactions encoded in the square of the Weyl superfield [2, 5]. We consider these models in an $AdS_2 \times S^2$ background and compute the effective action in this background. This is first done at the level of the Wilsonian effective action [27, 28]. Then we assume that the extension to the low-energy effective action can be implemented by replacing the Wilsonian holomorphic function F by the non-holomorphic function (1.2). Next, we perform a Legendre transformation of the low-energy effective action in this background and obtain the associated 'Hamiltonian', which takes the form of the aforementioned generalized Hesse potential. Using the covariant complex variables introduced in Sect. 1.3, we expand the associated Hesse potential (the Hamiltonian) and work out the first few iterations. This reveals a systematic structure. Namely, the Hesse potential decomposes into two classes of terms. One class consists of combinations of terms, constructed out of derivatives of Ω, that transform as functions under electric/magnetic duality. The other class is constructed out of Ω and derivatives thereof. Demanding this second class to also exhibit a proper behavior under duality transformations (as a consequence of the transformation behavior of the Hesse potential) imposes restrictions on Ω. These restrictions are captured by a differential equation that constitutes half of the holomorphic anomaly equation encountered in the context of perturbative topological string theory. The differential equation is a consequence of the tension between maintaining harmonicity of Ω and insisting on a proper behavior under duality transformations [5]. We conclude Sect. 1.4 with a brief discussion of open issues which will be addressed in an upcoming paper. There we will give a detailed discussion of the relation of perturbative topological string theory with the Hesse potential.

In the appendices we have collected various results, as follows. Appendix A discusses the transformation behavior under symplectic transformations of various holomorphic and anti-holomorphic derivatives of F. We use these expressions to give an alternative proof of the integrability of the resulting structures. In addition, we show that when F depends on an external parameter η, its derivative $\partial_\eta F$ transforms as a function under symplectic transformations. In appendix B we show that the modified derivative $\mathcal{D}_\eta$ of Sect. 1.3 acts as a covariant derivative for symplectic transformations. This is done by showing that when given a quantity $G(x, \bar{x}; \eta)$ that transforms as a function under symplectic transformations, also $\mathcal{D}_\eta G$ transforms as a function. In appendix C we review the holomorphic anomaly equation of topological string theory in the big moduli space. Appendix D lists certain combinations that arise in the expansion of the Hesse potential in powers of η and that transform as functions under electric/magnetic duality. In appendix E we list the transformation properties of various derivatives of Ω under duality transformations using the covariant variables of Sect. 1.3.

These lecture notes include exercises which we hope will constitute a guidance to the reader.

1.2 Lecture I: Point-Particle Models and F-Functions

We begin by considering a general point-particle Lagrangian that depends on coordinates ϕ and velocities $\dot{\phi}$. The associated Hamiltonian will depend on the canonical variables ϕ and π, where π denotes the canonical momentum. After briefly reviewing some of the salient features of the Hamiltonian description, such as canonical transformations in phase space, we present a theorem that shows that the dynamics of these models can be reformulated in terms of a symplectic vector that is complex, and whose real part comprises the canonical variables (ϕ, π). This is achieved by introducing a complex function F that depends on complex variables x, with the symplectic vector given by $(x, \partial F/\partial x)$. This reformulation exhibits many of the special geometry features that are typical for $N = 2$ supersymmetric systems. However, it also goes beyond the standard formulation of these systems in that the function F is of the form (1.2), and hence non-holomorphic in general.

We illustrate the theorem with various field theory examples with higher-derivative interactions. We give a detailed discussion of these examples in order to illustrate the characteristic features of the theorem. One example consists of the Born-Infeld Lagrangian for a Maxwell field, which we reformulate in the language of the theorem. We subsequently promote the gauge coupling constant to a dynamical field S and discuss the duality symmetries of the resulting model. We turn to more general models with exact S- and T-duality and discuss the restrictions imposed on Ω by these symmetries.

The reader not interested in the details of these examples may want to proceed to Sect. 1.2.3, where we discuss the form of the Hamiltonian when the function F is such that it transforms homogeneously under a real rescaling of the variables involved.

1.2.1 Theorem

Let us consider a point-particle model described by a Lagrangian L with n coordinates ϕ^i and n velocities $\dot{\phi}^i$. The associated canonical momenta $\partial L/\partial\dot{\phi}^i$ will be denoted by π_i. The Hamiltonian H of the system, which follows from L by Legendre transformation,

$$H(\phi, \pi) = \dot{\phi}^i\,\pi_i - L(\phi, \dot{\phi}), \tag{1.3}$$

depends on (ϕ^i, π_i), which are called canonical variables, since they satisfy the canonical Poisson bracket relations. The variables (ϕ^i, π_i) denote coordinates on a symplectic manifold called the classical phase space of the system. In these coordinates, the symplectic 2-form is $d\pi_i \wedge d\phi^i$. This 2-form is preserved under canonical transformations of (ϕ^i, π_i) given by

$$\begin{pmatrix} \phi^i \\ \pi_i \end{pmatrix} \longrightarrow \begin{pmatrix} \tilde{\phi}^i \\ \tilde{\pi}_i \end{pmatrix} = \begin{pmatrix} U^i{}_j & Z^{ij} \\ W_{ij} & V_i{}^j \end{pmatrix} \begin{pmatrix} \phi^j \\ \pi_j \end{pmatrix}, \tag{1.4}$$

where U, V, Z and W denote $n \times n$ matrices that satisfy the relations

$$\begin{aligned} U^T\, V - W^T\, Z = V^T\, U - Z^T\, W = \mathbb{I}, \\ U^T\, W = W^T\, U \quad , \quad Z^T\, V = V^T\, Z. \end{aligned} \tag{1.5}$$

These relations are precisely such that the transformation (1.4) constitutes an element of $\mathrm{Sp}(2n, \mathbb{R})$. This transformation leaves the Poisson brackets invariant. The Hamiltonian transforms as a function under symplectic transformations, i.e. $\tilde{H}(\tilde{\phi}, \tilde{\pi}) = H(\phi, \pi)$. When the Hamiltonian is invariant under a subset of $\mathrm{Sp}(2n, \mathbb{R})$ transformations, this subset describes a symmetry of the system. This invariance is often called duality invariance. Observe that the Legendre transformation (1.3) also gives rise to the relation $\partial L/\partial \phi^i = -\partial H/\partial \phi^i$ by virtue of $\pi_i = \partial L/\partial \dot{\phi}^i$.

Now we present a theorem that states that the Lagrangian can be reformulated in terms of a complex function $F(x, \bar{x})$ based on complex variables x^i, such that the canonical coordinates (ϕ^i, π_i) coincide with (twice) the real part of (x^i, F_i), where $F_i = \partial F(x, \bar{x})/\partial x^i$.

Theorem Given a Lagrangian $L(\phi, \dot{\phi})$ depending on n coordinates ϕ^i and n velocities $\dot{\phi}^i$, with corresponding Hamiltonian $H(\phi, \pi) = \dot{\phi}^i\, \pi_i - L(\phi, \dot{\phi})$, there exists a description in terms of complex coordinates $x^i = \frac{1}{2}(\phi^i + \mathrm{i}\dot{\phi}^i)$ and a complex function $F(x, \bar{x})$, such that,

$$\begin{aligned} 2\,\mathrm{Re}\, x^i &= \phi^i, \\ 2\,\mathrm{Re}\, F_i(x, \bar{x}) &= \pi_i, \quad \text{where} \quad F_i = \frac{\partial F(x, \bar{x})}{\partial x^i}. \end{aligned} \tag{1.6}$$

The function $F(x, \bar{x})$ is defined up to an anti-holomorphic function and can be decomposed into a holomorphic and a purely imaginary (in general non-harmonic) function,

$$F(x, \bar{x}) = F^{(0)}(x) + 2\mathrm{i}\Omega(x, \bar{x}). \tag{1.7}$$

The relevant equivalence transformations take the form,

$$F^{(0)} \to F^{(0)} + g(x)\,, \qquad \Omega \to \Omega - \mathrm{Im}\, g(x), \tag{1.8}$$

which results in $F(x,\bar{x}) \to F(x,\bar{x}) + \bar{g}(\bar{x})$. The Lagrangian and Hamiltonian can then be expressed in terms of $F^{(0)}$ and Ω,

$$\begin{aligned} L =& 4[\operatorname{Im} F - \Omega], \\ H =& -\mathrm{i}(x^i \bar{F}_{\bar{\imath}} - \bar{x}^{\bar{\imath}} F_i) - 4\operatorname{Im}[F - \tfrac{1}{2}x^i F_i] + 4\,\Omega \\ =& -\mathrm{i}(x^i \bar{F}_{\bar{\imath}} - \bar{x}^{\bar{\imath}} F_i) - 4\operatorname{Im}[F^{(0)} - \tfrac{1}{2}x^i F_i^{(0)}] - 2(2\,\Omega - x^i \Omega_i - \bar{x}^{\bar{\imath}}\Omega_{\bar{\imath}}), \end{aligned} \tag{1.9}$$

with $F_i = \partial F/\partial x^i$, $F_i^{(0)} = \partial F^{(0)}/\partial x^i$, $\Omega_i = \partial\Omega/\partial x^i$, and similarly for $\bar{F}_{\bar{\imath}}$, $\bar{F}_{\bar{\imath}}^{(0)}$ and $\Omega_{\bar{\imath}}$.

Furthermore, a crucial observation is that the $2n$-vector (x^i, F_i) denotes a complexification of the phase space coordinates (ϕ^i, π_i) that transforms precisely as (ϕ^i, π_i) under symplectic transformations, i.e.

$$\begin{pmatrix} x^i \\ F_i(x,\bar{x}) \end{pmatrix} \longrightarrow \begin{pmatrix} \tilde{x}^i \\ \tilde{F}_i(\tilde{x},\bar{\tilde{x}}) \end{pmatrix} = \begin{pmatrix} U^i{}_j & Z^{ij} \\ W_{ij} & V_i{}^j \end{pmatrix} \begin{pmatrix} x^j \\ F_j(x,\bar{x}) \end{pmatrix}. \tag{1.10}$$

Hence, a $\mathrm{Sp}(2n,\mathbb{R})$ transformation of (x^i, F_i) is a canonical transformation of $H(\phi,\pi)$. The Eq. (1.10) are, moreover, integrable: the symplectic transformation yields a new function $\tilde{F}(\tilde{x},\bar{\tilde{x}}) = \tilde{F}^{(0)}(\tilde{x}) + 2\mathrm{i}\tilde{\Omega}(\tilde{x},\bar{\tilde{x}})$, with $\tilde{\Omega}$ real.

Proof The proof of this theorem proceeds as follows. First we introduce the $2n$-vector (x^i, y_i),

$$\begin{aligned} x^i &= \tfrac{1}{2}\Big(\phi^i + \mathrm{i}\frac{\partial H}{\partial \pi_i}\Big), \\ y_i &= \tfrac{1}{2}\Big(\pi_i - \mathrm{i}\frac{\partial H}{\partial \phi^i}\Big), \end{aligned} \tag{1.11}$$

which is constructed out of two canonical pairs, one comprising the variables (ϕ^i, π_i) and the other one comprising derivatives of $H(\phi,\pi)$, namely $(\partial H/\partial\pi_i, -\partial H/\partial\phi^i)$. Both pairs transform in the same way under canonical transformations (1.4). Now we relate the vector (x^i, y_i) to the one given in (1.6), and we show that Lagrangian and the Hamiltonian can be expressed in terms of a complex function $F(x,\bar{x})$ as in (1.9).

The Legendre transformation (1.3) gives $\dot{\phi}^i = \partial H/\partial\pi_i$, where we used $\pi_i = \partial L/\partial\dot{\phi}^i$. This equation establishes that the complex x^i introduced in (1.11) coincide with the x^i defined above (1.6). Then, expressing the Lagrangian in terms of x^i and $\bar{x}^{\bar{\imath}}$, gives

$$\frac{\partial L(x,\bar{x})}{\partial x^i} = -2\mathrm{i}y_i, \tag{1.12}$$

where we used the relation $\partial L/\partial \phi^i = -\partial H/\partial \phi^i$ mentioned below (1.5). Next we write L as the sum of a harmonic and a non-harmonic function (which is always possible),

$$L = -2\mathrm{i}\big[F^{(0)}(x) - \bar{F}^{(0)}(\bar{x})\big] + 4\,\Omega(x,\bar{x}). \tag{1.13}$$

By introducing the combination $F(x,\bar{x}) = F^{(0)}(x) + 2\mathrm{i}\Omega(x,\bar{x})$, we observe that the relation (1.12) can be concisely written as $y_i = \partial F(x,\bar{x})/\partial x^i$, while the Lagrangian (1.13) becomes $L = 4[\mathrm{Im}\, F - \Omega]$. Using this as well as (1.11), we obtain that the Hamiltonian $H(\phi,\pi) = \dot{\phi}^i\,\pi_i - L(\phi,\dot{\phi})$ can be expressed as in (1.9).

Exercise 1 Verify that H can be written as in (1.9).

Thus, we have shown that the vector (x^i, y_i) equals (x^i, F_i), and we have established the validity of (1.9).

Now let us discuss the integrability of (x^i, y_i) under canonical transformations. The vector (x^i, y_i), given in (1.11), consists of two canonical pairs, and hence it transforms as in (1.10) under canonical transformations. We denote the transformed variables by $(\tilde{x}^i, \tilde{y}_i)$. The Hamiltonian transforms as a function, i.e. $\tilde{H}(\mathrm{Re}\,\tilde{x}, \mathrm{Re}\,\tilde{y}) = H(\mathrm{Re}\,x, \mathrm{Re}\,y)$, as already mentioned. Since we are dealing with a canonical transformation, the dual quantities $(\tilde{x}^i, \tilde{y}_i)$ and $\tilde{H}$ will satisfy the same relations as the original quantities (x^i, y_i) and H, so that we can apply the steps (1.11–1.13) to the dual quantities. The dual variables $(\tilde{x}^i, \tilde{y}_i)$ have the decomposition given in (1.11), but now in terms of the dual quantities. The Lagrangian $\tilde{L}$ associated to $\tilde{H}$ is obtained by a Legendre transformation of $\tilde{H}$, i.e. $\tilde{L} = \dot{\tilde{\phi}}^i\,\tilde{\pi}_i - \tilde{H}$. Then, applying the steps given below (1.11) to the dual Lagrangian shows that $\tilde{L} = 4[\mathrm{Im}\,\tilde{F} - \tilde{\Omega}]$, where $\tilde{F}$ is the sum of a holomorphic function $\tilde{F}^{(0)}$ and a real function $\tilde{\Omega}$, i.e. $\tilde{F}(\tilde{x},\bar{\tilde{x}}) = \tilde{F}^{(0)}(\tilde{x}) + 2\mathrm{i}\tilde{\Omega}(\tilde{x},\bar{\tilde{x}})$. This establishes that $(\tilde{x}^i, \tilde{y}_i)$ can be obtained from a new function $\tilde{F}$, and hence ensures the integrability of $(\tilde{x}^i, \tilde{y}_i)$ under symplectic transformations.

To complete the proof of the theorem, we need to discuss one more issue, namely the decompositions of $F(x,\bar{x})$ and $\tilde{F}(\tilde{x},\bar{\tilde{x}})$ and their relation. The decomposition of F into $F^{(0)}$ and Ω suffers from the ambiguity (1.8), and so does the decomposition of $\tilde{F}$. Therefore, to be able to relate both decompositions, we need to fix the ambiguity in the decomposition of $\tilde{F}$, once a decomposition of F has been given. To do so, we proceed as follows.

We consider a symplectic transformation (1.10) which, as we just discussed, yields a new function $\tilde{F}$. Given a decomposition of F, we apply the same transformation to the vector $(x^i, F_i^{(0)})$ alone, where $F_i^{(0)} = \partial F^{(0)}/\partial x^i$. This yields the vector $(\hat{x}^i, \tilde{F}_i^{(0)}(\hat{x}))$, as explained in appendix A. The transformed vector $(\hat{x}^i, \tilde{F}_i^{(0)}(\hat{x}))$ can be integrated, i.e. $\tilde{F}_i^{(0)}$ can be expressed as $\partial\tilde{F}^{(0)}(\hat{x})/\partial\hat{x}^i$, where $\tilde{F}^{(0)}(\hat{x})$ is uniquely determined up to a constant and up to terms linear in $\hat{x}^i$ (see (1.165)) [5]. The expression for $\tilde{F}^{(0)}(\hat{x})$ can be readily obtained by using that the combination $F^{(0)} - \frac{1}{2}\,x^i\,F_i^{(0)}$ transforms as a function under symplectic transformations, i.e. $\delta\left(F^{(0)} - \frac{1}{2}\,x^i\,F_i^{(0)}\right) = \frac{1}{2}\left(\delta x^i\,F_i^{(0)} - x^i\,\delta F_i^{(0)}\right)$. One obtains

$\tilde{F}^{(0)}(\hat{x}) = \frac{1}{2}\,\hat{x}^i\,\tilde{F}_i^{(0)}(\hat{x}) + F^{(0)} - \frac{1}{2}\,x^i\,F_i^{(0)}$, up to a constant and up to terms linear in $\hat{x}^i$. Thus, to relate the decomposition of $\tilde{F}$ to the decomposition of F, we demand that $\tilde{F}^{(0)}$ refers to the combination that follows by applying a symplectic transformation to $(x^i, F_i^{(0)})$, as just described. This in turn determines $\tilde{\Omega} = \frac{1}{4}[\tilde{L} - 4\mathrm{Im}\,\tilde{F}^{(0)}]$. This completes the proof of the theorem.

We finish this subsection with a few comments. First, we note that since both H and $F^{(0)} - \frac{1}{2}\,x^i\,F_i^{(0)}$ transform as functions under symplectic transformations, so does the following combination that appears in (1.9),

$$2\,\Omega - x^i\,\Omega_i - \bar{x}^{\bar{\imath}}\,\Omega_{\bar{\imath}}. \tag{1.14}$$

Second, the transformation law of $2\mathrm{i}\Omega_i = F_i - F_i^{(0)}$ under symplectic transformations is determined by the transformation behavior of F_i and $F_i^{(0)}$, as described above. In appendix A we give an equivalent expression for $\tilde{\Omega}_i$ in terms of a power series in derivatives of Ω, see (1.161). The transformation law of $2\mathrm{i}\Omega_{\bar{\imath}} = F_{\bar{\imath}}$, on the other hand, follows from the reality of $\tilde{\Omega}$,

$$\tilde{\Omega}_{\bar{\imath}} = (\overline{\tilde{\Omega}_i}). \tag{1.15}$$

Third, as mentioned in the introduction, the function $F(x, \bar{x})$ may, in general, depend on a number of external parameters η that are inert under symplectic transformations. Without loss of generality, we may take η to be solely encoded in Ω and, upon transformation, in $\tilde{\Omega}$ (we can use the equivalence relation (1.8) to achieve this). In appendix A we show that $\partial_\eta F = \partial F/\partial\eta$ transforms as a function under symplectic transformations [21]. We will return to this feature in Sect. 1.2.3.

Appendix A also discusses the transformation behavior under symplectic transformations of various holomorphic and anti-holomorphic derivatives of F. We use these expressions to give an alternative proof of the integrability of (1.10).

1.2.2 Examples

We now proceed to illustrate the features of the theorem discussed above in various models that have duality symmetries. To keep the discussion as transparent as possible in all cases, we consider the reduced Lagrangian that is obtained by restricting to spherically symmetric static configurations in flat spacetime. The first model we consider is the Born-Infeld model for an abelian gauge field, which has been known to have an SO(2) duality symmetry for a long time [29]. This symmetry may be enlarged to an $\mathrm{SL}(2,\mathbb{R})$ duality symmetry by coupling the system to a complex scalar field, called the dilaton-axion field [30]. This is the second model we consider. Then we turn to more general models with exact S- and T-duality and discuss the restrictions imposed on Ω by these symmetries. We exhibit how the Born-Infeld-dilaton-axion system fits into this class of models. Finally, we focus on the case when the function $F(x, \bar{x})$ is taken to be homogeneous, and we discuss the form of the associated Hamiltonian.

The Born-Infeld Model

The Born-Infeld Lagrangian[2] for an abelian gauge field in a spacetime with metric $g_{\mu\nu}$ is given by [31]

$$\mathcal{L} = -g^{-2}\left[\sqrt{|\det[g_{\mu\nu} + g\,F_{\mu\nu}]|} - \sqrt{|\det g_{\mu\nu}|}\right]. \tag{1.16}$$

It depends on an external parameter $\eta = g^2$. In the following we consider spherically symmetric static configurations in flat spacetime given by

$$\begin{aligned} ds^2 &= -\,dt^2 + dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right), \\ F_{rt} &= e(r), \quad F_{\theta\varphi} = p\,\sin\theta. \end{aligned} \tag{1.17}$$

Here, the θ-dependence of $F_{\theta\varphi}$ is fixed by rotational invariance, and p is constant by virtue of the Bianchi identity. Evaluating (1.16) for this configuration gives

$$\mathcal{L} = -g^{-2} r^2 \sin^2\theta\left[\sqrt{|1 - g^2 e^2(r)|}\,\sqrt{1 + g^2\,p^2\,r^{-4}} - 1\right]. \tag{1.18}$$

Below we will rewrite (1.18) and bring it into the form (1.9). Since this rewriting does not depend on the angular variables and since it applies to any r-slice, we integrate over the angular variables and pick the r-slice $4\pi r^2 = 1$, for convenience. The resulting reduced Lagrangian reads,

$$\mathcal{L}(e, p) = -g^{-2}\left[\sqrt{1 - g^2 e^2}\,\sqrt{1 + g^2 p^2} - 1\right], \tag{1.19}$$

where we take $g^2 e^2 < 1$.

Exercise 2 Instead of flat spacetime, consider the $AdS_2 \times S^2$ line element $ds^2 = v_1(-r^2 dt^2 + r^{-2} dr^2) + v_2(d\theta^2 + \sin^2\theta\, d\varphi^2)$, where v_1 and v_2 denote constants. Show that the resulting reduced Lagrangian takes the form (1.19) after performing a suitable rescaling of g, e and p.

In the example (1.19), the role of the coordinate ϕ and of the velocity $\dot{\phi}$ introduced above (1.6) is played by p and e, respectively. The associated Hamiltonian $\mathcal{H}$ is obtained by Legendre transforming with respect to $\dot{\phi} = e$. The conjugate momentum π is given by the electric charge q, so that

[2] We will use the notation $\mathcal{L}$ and $\mathcal{H}$ when dealing with Lagrangian and Hamiltonian densities, respectively.

$$\mathcal{H}(p,q) = q\,e - \mathcal{L}(e,p). \tag{1.20}$$

Computing

$$q = \frac{\partial \mathcal{L}}{\partial e} = e\,\sqrt{\frac{1+g^2p^2}{1-g^2e^2}} \quad , \quad f \equiv \frac{\partial \mathcal{L}}{\partial p} = -p\,\sqrt{\frac{1-g^2e^2}{1+g^2p^2}}, \tag{1.21}$$

where we introduced f for later convenience, and substituting in (1.20), we obtain for the Hamiltonian,

$$\mathcal{H}(p,q) = g^{-2}\left[\sqrt{1+g^2(p^2+q^2)} - 1\right]. \tag{1.22}$$

This Hamiltonian is manifestly invariant under SO(2) rotations of p and q and, in particular, under the discrete symmetry that interchanges the electric and magnetic charges. The external parameter $\eta = g^2$ is inert under these transformations. These rotations constitute the only continuous symmetry of the system [29]. Their infinitesimal form can be represented by an Sp$(2,\mathbb{R})$-transformation (1.5) with $U = V = 1$ and $Z = -W = -c$, where $c \in \mathbb{R}$.

Now, following the construction described in the Sect. 1.2.1, we introduce a complex coordinate x in terms of the coordinate $\phi = p$ and the velocity $\dot{\phi} = e$, and a complex function $F(x,\bar{x};g^2)$,

$$x = \tfrac{1}{2}(p + \mathrm{i}e)\,, \qquad F(x,\bar{x};g^2) = F^{(0)}(x) + 2\mathrm{i}\Omega(x,\bar{x};g^2), \tag{1.23}$$

where

$$\begin{aligned} F^{(0)}(x) &= -\tfrac{1}{2}\mathrm{i}\,x^2, \\ \Omega(x,\bar{x};g^2) &= \tfrac{1}{8}\,g^{-2}\left(\sqrt{1+g^2(x+\bar{x})^2} - \sqrt{1+g^2(x-\bar{x})^2}\right)^2. \end{aligned} \tag{1.24}$$

The split into $F^{(0)}$ and Ω is done in such a way that $F^{(0)}$ will encode the contribution at the two-derivative level (which corresponds to the term $\mathcal{L} \approx -\frac{1}{4}F_{\mu\nu}^2 + \mathcal{O}(g^2)$ in (1.16)), while Ω will encode the higher-derivative contributions. Indeed, with these definitions the Lagrangian (1.19) can be written as

$$\mathcal{L} = 4[\mathrm{Im}F - \Omega], \tag{1.25}$$

in agreement with the first equation of (1.9). Next, using the first equation of (1.21), we establish

$$p = 2\,\mathrm{Re}\,x, \quad q = 2\,\mathrm{Re}\,F_x, \tag{1.26}$$

in accordance with (1.6), where we recall that the conjugate momentum π equals q. Then, inserting (1.26) into (1.22) yields

$$\mathcal{H} = \mathrm{i}(\bar{x}\, F_x - x\, \bar{F}_{\bar{x}}) + 4\, g^2 \frac{\partial \Omega}{\partial g^2}, \tag{1.27}$$

where $F_x = \partial F(x, \bar{x}; g^2)/\partial x$. This is in agreement with the second equation of (1.9), since $F^{(0)}$ satisfies $F^{(0)} = \frac{1}{2}\, x F_x^{(0)}$, and Ω obeys the homogeneity relation

$$2\,\Omega = x\,\Omega_x + \bar{x}\,\Omega_{\bar{x}} - 2\, g^2 \frac{\partial \Omega}{\partial g^2}, \tag{1.28}$$

which is a consequence of the behavior of Ω under the real scaling $x \to \lambda\, x$ and $g^2 \to \lambda^{-2}\, g^2$.

Exercise 3 Establish (1.28) by differentiating the relation $\Omega(\lambda\, x, \lambda\, \bar{x}; \lambda^{-2}\, g^2) = \lambda^2\, \Omega(x, \bar{x}; g^2)$.

Exercise 4 Verify (1.25), (1.26) and (1.27).

Rather than performing a Legendre transformation of $\mathcal{L}(e, p)$ with respect to e, we may instead consider performing a Legendre transformation with respect to p. The resulting quantity $\mathcal{S}(e, f)$ will then depend on the canonical pair (e, f), rather than on (p, q). Using the expression for f given in (1.21), we obtain

$$\mathcal{S}(e, f) = f\, p - \mathcal{L}(e, p) = g^{-2} \left[\sqrt{1 - g^2(e^2 + f^2)} - 1 \right], \tag{1.29}$$

which is invariant under SO(2) rotations of e and f. Next, we express $\mathcal{S}(e, f)$ in terms of x and F_x introduced in (1.23). First we establish

$$f = 2\,\mathrm{Im}\, F_x, \tag{1.30}$$

so that

$$x = \tfrac{1}{2}(p + \mathrm{i}e), \qquad F_x = \tfrac{1}{2}(q + \mathrm{i}f). \tag{1.31}$$

Then, using (1.27) and (1.31), we obtain[3]

$$\mathcal{S} = f\, p - q\, e + \mathcal{H} = -i\left(\bar{x}\, F_x - x\, \bar{F}_{\bar{x}}\right) + 4\, g^2 \frac{\partial \Omega}{\partial g^2}. \tag{1.32}$$

Let us now return to the discussion about symplectic transformations alluded to below (1.22). A symplectic transformation (1.10) may either constitute a symmetry (an invariance) of the system or correspond to a symplectic reparametrization of the system giving rise to an equivalent set of equations of motion and Bianchi identities [33]. When a symplectic transformation describes a symmetry, a convenient

[3] In the context of BPS black holes, $\mathcal{H}$ is the Hesse potential, and the double Legendre transform of $\mathcal{H}$ yields the entropy function [8, 32].

method for verifying this consists in performing the substitution $x^i \rightarrow \tilde{x}^i$ in the derivatives F_i, and checking that this correctly induces the symplectic transformation on (x^i, F_i) [5].

To elucidate this, let us consider a particular example, namely the discrete symmetry that interchanges the electric and magnetic charges. It can be implemented by the transformation $(x, F_x) \rightarrow (F_x, -x)$, which operates on the canonical pairs (p, q) and (e, f) through (1.31). This constitutes a symplectic transformation (1.5) with $U = V = 0\,,\ Z = 1\,,\ W = -1$. To verify that the transformation $x \rightarrow \tilde{x} = F_x$ correctly induces the transformation of F_x, we compute

$$F_x = -\mathrm{i}x\ \frac{1 + g^2(x^2 - \bar{x}^2)}{\sqrt{1 + g^2(x + \bar{x})^2}\ \sqrt{1 + g^2(x - \bar{x})^2}}. \tag{1.33}$$

Also, expressing e in terms of p and q (by using the first relation of (1.21)), we may express x in terms of $p = 2\mathrm{Re}\,x$ and $q = 2\mathrm{Re}\,F_x$,

$$x = \tfrac{1}{2}\left(p + \frac{\mathrm{i}q}{\sqrt{1 + g^2(p^2 + q^2)}}\right). \tag{1.34}$$

We leave the following exercise to the reader.

Exercise 5 Using (1.33), show that the transformation $x \rightarrow F_x$ induces the transformation $F_x \rightarrow -x$ by inserting the former on the right hand side of F_x. Similarly, using (1.34), show that the transformation $(\mathrm{Re}\,x, \mathrm{Re}\,F_x) \rightarrow (\mathrm{Re}\,F_x, -\mathrm{Re}\,x)$ induces the transformation $x \rightarrow F_x$.

Next, let us discuss an example of a symplectic transformation that does not constitute a symmetry of the system, but instead describes a reparametrization of it. Namely, consider the following transformation of the canonical pair (p, q),

$$\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 2\mathrm{Re}\,x \\ 2\mathrm{Re}\,F_x \end{pmatrix} \longrightarrow \begin{pmatrix} \tilde{p} \\ \tilde{q} \end{pmatrix} = \begin{pmatrix} 2\mathrm{Re}\,\tilde{x} \\ 2\mathrm{Re}\,\tilde{F}_{\tilde{x}} \end{pmatrix} = \begin{pmatrix} p + \alpha\, q \\ q \end{pmatrix}, \qquad \alpha \in \mathbb{R}. \tag{1.35}$$

This constitutes a symplectic transformation (1.5) given by $U = V = 1\,,\ Z = \alpha\,,\ W = 0$. Since, however, it does not represent an SO(2) rotation of p and q, it does not leave the Hamiltonian (1.22) invariant. To determine the new function $\tilde{F}(\tilde{x}, \bar{\tilde{x}}; g^2)$ associated with this reparametrization, we start on the Hamiltonian side and use the fact that $\mathcal{H}$ transforms as a function under symplectic transformations. Using (1.35) this gives

$$\tilde{\mathcal{H}}(\tilde{p}, \tilde{q}) = \mathcal{H}(p, q) = g^{-2}\left[\sqrt{1 + g^2[(\tilde{p} - \alpha\tilde{q})^2 + \tilde{q}^2]} - 1\right]. \tag{1.36}$$

Now we determine the corresponding Lagrangian by Legendre transformation,

$$\tilde{\mathcal{L}}(\tilde{e}, \tilde{p}) = \tilde{e}\,\tilde{q} - \tilde{\mathcal{H}}(\tilde{p}, \tilde{q}), \tag{1.37}$$

where

$$\tilde{e} = \frac{\partial \tilde{\mathcal{H}}}{\partial \tilde{q}} = \frac{(1+\alpha^2)\tilde{q} - \alpha\,\tilde{p}}{\sqrt{1 + g^2(1+\alpha^2)^{-1}[((1+\alpha^2)\tilde{q} - \alpha\,\tilde{p})^2 + \tilde{p}^2]}}. \tag{1.38}$$

This yields,

$$\tilde{q} = \frac{\alpha\,\tilde{p}}{1+\alpha^2} + \frac{\tilde{e}}{1+\alpha^2}\sqrt{\frac{1+\alpha^2+g^2\,\tilde{p}^2}{1+\alpha^2-g^2\,\tilde{e}^2}}, \tag{1.39}$$

which, when inserted in (1.37), gives

$$\tilde{\mathcal{L}}(\tilde{e}, \tilde{p}) = \frac{\alpha\,\tilde{e}\,\tilde{p}}{1+\alpha^2} - g^{-2}\left[\frac{1}{1+\alpha^2}\sqrt{1+\alpha^2-g^2\,\tilde{e}^2}\,\sqrt{1+\alpha^2+g^2\,\tilde{p}^2} - 1\right]. \tag{1.40}$$

In order to bring the Lagrangian $\tilde{\mathcal{L}}$ into the form $\tilde{\mathcal{L}} = 4\left[\mathrm{Im}\tilde{F} - \tilde{\Omega}\right]$, as in (1.9), we express $\tilde{\mathcal{L}}$ in terms of the complex coordinate

$$\tilde{x} = \tfrac{1}{2}\,(\tilde{p} + \mathrm{i}\tilde{e})\,, \tag{1.41}$$

which is the transformed version of the coordinate x introduced in (1.23). Then, we consider all the terms in $\tilde{\mathcal{L}}$ that are independent of g^2, and we express them in terms of a function $\tilde{F}^{(0)}(\tilde{x})$, as follows,

$$\frac{1}{1+\alpha^2}\left[\alpha\,\tilde{e}\,\tilde{p} + \tfrac{1}{2}\left(\tilde{e}^2 - \tilde{p}^2\right)\right] = 4\,\mathrm{Im}\tilde{F}^{(0)}(\tilde{x}). \tag{1.42}$$

This yields

$$\tilde{F}^{(0)}(\tilde{x}) = \frac{\alpha - \mathrm{i}}{2\,(1+\alpha^2)}\,\tilde{x}^2, \tag{1.43}$$

up to a real constant. It represents the function that is obtained by applying the symplectic transformation (1.35) to $F^{(0)}(x)$, as explained at the end of Sect. 1.2.1. Next, we introduce the function

$$\tilde{F}(\tilde{x}, \bar{\tilde{x}}; g^2) = \tilde{F}^{(0)}(\tilde{x}) + 2\mathrm{i}\tilde{\Omega}(\tilde{x}, \bar{\tilde{x}}; g^2), \tag{1.44}$$

with $\tilde{\Omega}$ real, and we require it to satisfy $\tilde{\mathcal{L}} = 4\left[\mathrm{Im}\tilde{F} - \tilde{\Omega}\right]$. This implies that all the g^2-dependent terms will be encoded in $\tilde{\Omega}(\tilde{x}, \bar{\tilde{x}}; g^2)$. We obtain

$$\tilde{\Omega}(\tilde{x}, \bar{\tilde{x}}; g^2) = \frac{1}{8(1+\alpha^2)\,g^2}\left(\sqrt{1+\alpha^2+g^2(\tilde{x}+\bar{\tilde{x}})^2} - \sqrt{1+\alpha^2+g^2(\tilde{x}-\bar{\tilde{x}})^2}\right)^2. \tag{1.45}$$

This result gives the function $\tilde{F}$ associated with the reparametrization (1.35). We now check that it correctly reproduces the relation $\tilde{q} = 2\,\mathrm{Re}\,\tilde{F}_{\tilde{x}}$, as required by (1.35). We compute $\tilde{F}_{\tilde{x}}$ and obtain,

$$\begin{aligned} \tilde{F}_{\tilde{x}} = {} & \frac{\alpha\,\tilde{x}}{1+\alpha^2} \\ & - \frac{\mathrm{i}}{2\,(1+\alpha^2)}\left\{(\tilde{x}-\bar{\tilde{x}})\sqrt{\frac{1+\alpha^2+g^2(\tilde{x}+\bar{\tilde{x}})^2}{1+\alpha^2+g^2(\tilde{x}-\bar{\tilde{x}})^2}} + (\tilde{x}+\bar{\tilde{x}})\right. \\ & \left.\sqrt{\frac{1+\alpha^2+g^2(\tilde{x}-\bar{\tilde{x}})^2}{1+\alpha^2+g^2(\tilde{x}+\bar{\tilde{x}})^2}}\right\}. \end{aligned} \tag{1.46}$$

We leave the following exercise to the reader.

Exercise 6 Using (1.46), verify explicitly that 2Re $\tilde{F}_{\tilde{x}}$ equals (1.39).

Now we want to see how $\tilde{F}_{\tilde{x}}$ is related to F_x. According to the discussion around (1.10), the symplectic transformation (1.35) of the canonical pair (Re x, Re F_x) induces a corresponding transformation of the vector (x, F_x),

$$\begin{pmatrix} \tilde{x} \\ \tilde{F}_{\tilde{x}} \end{pmatrix} = \begin{pmatrix} x + \alpha\,F_x \\ F_x \end{pmatrix}. \tag{1.47}$$

This is indeed the case, as can be verified explicitly by expressing the transformed variables $(\tilde{p}, \tilde{e})$ in terms of the original variables (p, e) using (1.21), (1.38) and (1.35),

$$\tilde{p} = p + \alpha\,e\,\sqrt{\frac{1+g^2p^2}{1-g^2e^2}}, \qquad \tilde{e} = e - \alpha\,p\,\sqrt{\frac{1-g^2e^2}{1+g^2p^2}}, \tag{1.48}$$

and employing the relation

$$\frac{1+\alpha^2-g^2\tilde{e}^2}{1+\alpha^2+g^2\tilde{p}^2} = \frac{1-g^2e^2}{1+g^2p^2}. \tag{1.49}$$

Exercise 7 Verify (1.47) explicitly using (1.46).

Including a Dilaton-Axion Complex Scalar Field

The Born-Infeld system discussed in the previous section possesses a continuous $SO(2)$ duality symmetry group. It is possible to enlarge this duality symmetry group to Sp(2, $\mathbb{R}$) by coupling the abelian gauge field to a complex scalar field $S = \Phi + \mathrm{i}\, B$ [30]. This is achieved by replacing $g\, F_{\mu\nu}$ in (1.16) with $g\, \Phi^{1/2}\, F_{\mu\nu}$ and adding a term $B F_{\mu\nu} \tilde{F}^{\mu\nu}$ to the Lagrangian, as follows [30]

$$\mathcal{L} = -g^{-2} \left[\sqrt{|\det[g_{\mu\nu} + g\, \Phi^{1/2}\, F_{\mu\nu}]|} - \sqrt{|\det g_{\mu\nu}|} \right] + \tfrac{1}{4}\, B\, F_{\mu\nu} \tilde{F}^{\mu\nu}. \tag{1.50}$$

Then, the combined system of equations of motion and Bianchi identity for $F_{\mu\nu}$ is invariant under Sp(2, $\mathbb{R}$) transformations, provided that S transforms in a suitable fashion. The associated Hamiltonian will then be invariant under these transformations. This will be discussed momentarily. The coupling $g\, \Phi^{1/2}$ replaces the gauge coupling constant with a dynamical field, customarily called the dilaton field, while the term $B F_{\mu\nu} \tilde{F}^{\mu\nu}$ introduces a scalar field degree of freedom called the axion. For this reason, S is also called the dilaton-axion field.

As before, let us consider spherically symmetric static configurations of the form (1.17). Picking again the r-slice $4\pi r^2 = 1$, for convenience, the reduced Lagrangian is now given by

$$\mathcal{L}(e, p, \Phi, B) = -g^{-2} \left[\sqrt{1 - g^2\, \Phi\, e^2}\, \sqrt{1 + g^2\, \Phi\, p^2} - 1 \right] + B\, e\, p, \tag{1.51}$$

where we take $g^2\, \Phi\, e^2 < 1$. This reduces to the previous one in (1.19) when setting $S = 1$. To obtain the associated Hamiltonian $\mathcal{H}$,

$$\mathcal{H}(p, q, \Phi, B) = q\, e - \mathcal{L}(e, p, \Phi, B), \tag{1.52}$$

we first compute $q = \partial\mathcal{L}/\partial e$,

$$q = e\, \Phi\, \sqrt{\frac{1 + g^2\, \Phi\, p^2}{1 - g^2\, \Phi\, e^2}} + B\, p. \tag{1.53}$$

Inverting this relation yields

$$e = \frac{q - B\, p}{\sqrt{\Phi^2 + g^2\, \Phi\, \left[\Phi^2\, p^2 + (q - B\, p)^2\right]}}, \tag{1.54}$$

and substituting in (1.52) gives

$$\mathcal{H}(p, q, \Phi, B) = g^{-2} \left[\sqrt{1 + g^2[\Phi\, p^2 + \Phi^{-1}\, (q - B\, p)^2]} - 1 \right]. \tag{1.55}$$

Then, expressing Φ and B in terms of S and $\bar{S}$ results in

$$\mathcal{H}(p, q, S, \bar{S}) = g^{-2}\left[\sqrt{1 + 2\,g^2\,\Sigma(p, q, S, \bar{S})} - 1\right], \tag{1.56}$$

where

$$\Sigma(p, q, S, \bar{S}) = \frac{q^2 + \mathrm{i}p\,q(S - \bar{S}) + p^2\,|S|^2}{S + \bar{S}}. \tag{1.57}$$

Exercise 8 Verify (1.56).

Now we are in position to discuss the invariance of the Hamiltonian under Sp(2, $\mathbb{R}$) transformations. Consider a general Sp(2, $\mathbb{R}$) transformation of the canonical pair (p, q) given by

$$\begin{pmatrix} p \\ q \end{pmatrix} \longrightarrow \begin{pmatrix} \tilde{p} \\ \tilde{q} \end{pmatrix} = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}, \tag{1.58}$$

where $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$. The latter ensures that the transformation belongs to SL(2, $\mathbb{R}$) $\cong$ Sp(2, $\mathbb{R}$). Then, Σ given in (1.57) is invariant under (1.58) provided that S transforms according to [8]

$$S \to \frac{aS - \mathrm{i}b}{\mathrm{i}cS + d}. \tag{1.59}$$

This explains the role of S in achieving duality invariance. It should be noted that S does not constitute an additional canonical variable, but instead describes a background field. The external parameter g^2 is inert under these transformations.

Exercise 9 Show that Σ is invariant under the combined transformation (1.58) and (1.59).

We observe that $\mathcal{H}$ homogeneously as $\mathcal{H} \to \lambda^2\mathcal{H}$ under the real scaling $(p, q) \to \lambda(p, q)\,,\ g^2 \to \lambda^{-2}\,g^2\,,\ S \to S$, with $\lambda \in \mathbb{R}$.

Let us now return to the reduced Lagrangian (1.51) and recast it in the form $\mathcal{L} = 4\,[\mathrm{Im}F - \Omega]$, where again we introduce the complex variable $x = \frac{1}{2}(p + \mathrm{i}e)$. The function F will now depend on the two complex scalar fields x an S,

$$F(x, \bar{x}, S, \bar{S}; g^2) = F^{(0)}(x, S) + 2\mathrm{i}\Omega(x, \bar{x}, S, \bar{S}; g^2), \tag{1.60}$$

and is determined as follows. The holomorphic function $F^{(0)}$ encodes all the contributions that are independent of g^2, while Ω, which is real, accounts for all the terms in the reduced Lagrangian that depend on g^2. This yields,

$$\begin{aligned} F^{(0)}(x, S) &= -\tfrac{1}{2}\mathrm{i}\,S\,x^2, \qquad\qquad (1.61)\\ \Omega(x, \bar{x}, S, \bar{S}; g^2) &= \tfrac{1}{8}\,g^{-2}\left(\sqrt{1 + \tfrac{1}{2}g^2\,(S + \bar{S})\,(x + \bar{x})^2}\right.\\ &\qquad\left. - \sqrt{1 + \tfrac{1}{2}g^2\,(S + \bar{S})\,(x - \bar{x})^2}\right)^2. \end{aligned}$$

Observe that under the scaling of (p, q) and g^2 discussed below (1.59), e scales as $e \to \lambda e$, and hence x scales as $x \to \lambda x$. This in turn implies that F scales as $F \to \lambda^2 F$.

From (1.6) we infer that the canonical pair (p, q) is given by $(2\mathrm{Re}\, x, 2\mathrm{Re}\, F_x)$. According to the discussion around (1.10), the symplectic transformation (1.58) of the canonical pair $(\mathrm{Re}\, x, \mathrm{Re}\, F_x)$ induces a transformation of the vector (x, F_x) given by $(x, F_x) \to (d\,x - c\,F_x, a\,F_x - b\,x)$. Since (1.58) together with (1.59) constitutes a symmetry of the model, the transformation of F_x must be induced by the transformation of x and S upon substitution. We leave it to the reader to verify this.

Exercise 10 Show that the transformation of x and S (given in (1.58) and (1.59), respectively) induces the transformation $F_x \to a\,F_x - b\,x$ by substituting x and S with $\tilde{x}$ and $\tilde{S}$ in F_x.

The reduced Lagrangian (1.51) describes the system on an r-slice $4\pi r^2 = 1$. Another background leading to a similar reduced Lagrangian, and hence to a similar description in terms of a function F, is provided by an $AdS_2 \times S^2$ spacetime.

Exercise 11 Consider the Born-Infeld-dilaton-axion system in an $AdS_2 \times S^2$ background and show that, after performing a suitable rescaling of g, e and p, the resulting reduced Lagrangian is again encoded in (1.61).

Towards $N = 2$ Supergravity Models

In the Born-Infeld example discussed above, the duality symmetry of the model was enlarged by coupling it to an additional complex scalar field S. This feature is not an accident. In the context of $N = 2$ supersymmetric models, it is well known that the presence of complex scalar fields is crucial in order for the model to have duality symmetries. To explore this in more detail, let us broaden the discussion and consider functions F that depend on three complex scalar fields Y^I (with $I = 0, 1, 2$), as well as on an external parameter η. They will have the form

$$F(Y, \bar{Y}; \eta) = -\tfrac{1}{2}\, \frac{Y^1 (Y^2)^2}{Y^0} + 2\mathrm{i}\Omega(Y, \bar{Y}; \eta). \tag{1.62}$$

The function F describing the Born-Infeld-dilaton-axion system, given in (1.60), is a special case of (1.62). It is obtained by performing the identification $S = -\mathrm{i}\, Y^1/Y^0$, $x = Y^2$ and $\eta = g^2$. This identification is consistent with the scaling properties of x, S and g^2 discussed below (1.59). Namely, by assigning the uniform scaling behavior $Y^I \to \lambda Y^I$ to the Y^I, we reproduce the scalings of x, S and g^2. The function (1.62) may, however, also describe other models, such as genuine $N = 2$ supergravity models and should thus be viewed in a broader context. Depending on the chosen context, the external parameter η will have a different interpretation. Observe that in the description (1.62) based on the Y^I, duality transformations are represented by Sp(6, $\mathbb{R}$) matrices (which are 6×6 matrices of the form (1.5)) acting on (Y^I, F_I), where $F_I = \partial F(Y, \bar{Y}; \eta)/\partial Y^I$. The external parameter η is inert under these transformations.

Let us now assume that a model based on (1.62) has a symmetry associated with a subgroup of Sp(6, $\mathbb{R}$). This will impose restrictions on the form of Ω [18, 21]. For concreteness, we take the symmetry to be an SL(2, $\mathbb{R}$) $\times$ SL(2, $\mathbb{R}$) subgroup. The first SL(2, $\mathbb{R}$) subgroup acts as follows on (Y^I, F_I),

$$\begin{aligned} Y^0 &\to d\,Y^0 + c\,Y^1, & F_0 &\to a\,F_0 - b\,F_1, \\ Y^1 &\to a\,Y^1 + b\,Y^0, & F_1 &\to d\,F_1 - c\,F_0, \\ Y^2 &\to d\,Y^2 - c\,F_2, & F_2 &\to a\,F_2 - b\,Y^2, \end{aligned} \tag{1.63}$$

where a, b, c, d are real parameters that satisfy $ad - bc = 1$. This symmetry is referred to as S-duality. Let us describe its action on two complex scalar fields S and T that are given by the scale invariant combinations $S = -\mathrm{i}Y^1/Y^0$ and $T = -\mathrm{i}Y^2/Y^0$. The field S is the one we encountered above. The S-duality transformation (1.63) acts as

$$S \to \frac{aS - \mathrm{i}b}{\mathrm{i}cS + d}, \quad T \to T + \frac{2\mathrm{i}\,c}{\Delta_\mathrm{S}\,(Y^0)^2}\frac{\partial\Omega}{\partial T}, \quad Y^0 \to \Delta_\mathrm{S}\,Y^0, \tag{1.64}$$

where we view Ω as function of S, T, Y^0 and their complex conjugates, and where

$$\Delta_\mathrm{S} = d + \mathrm{i}c\,S. \tag{1.65}$$

The second SL(2, $\mathbb{R}$) subgroup is referred to as T-duality group. Here we focus on the T-duality transformation that, in the absence of Ω, induces the transformation $T \to 2/T$. It is given by the following Sp(6, $\mathbb{R}$) transformation,

$$\begin{aligned} Y^0 &\to F_1\,, & F_0 &\to -Y^1\,, \\ Y^1 &\to -F_0\,, & F_1 &\to Y^0\,, \\ Y^2 &\to Y^2\,, & F_2 &\to F_2\,, \end{aligned} \tag{1.66}$$

and yields

$$S \to S + \frac{2}{\Delta_\mathrm{T}(Y^0)^2}\left[-Y^0\frac{\partial\Omega}{\partial Y^0} + T\frac{\partial\Omega}{\partial T}\right], \quad T \to \frac{T}{\Delta_\mathrm{T}}, \quad Y^0 \to \Delta_\mathrm{T}\,Y^0, \tag{1.67}$$

where

$$\Delta_\mathrm{T} = \tfrac{1}{2}T^2 + \frac{2}{(Y^0)^2}\frac{\partial\Omega}{\partial S}. \tag{1.68}$$

As already mentioned below (1.32), when a symplectic transformation describes a symmetry of the system, a convenient method for verifying this consists in performing the substitution $Y^I \to \tilde{Y}^I$ in the derivatives F_I, and checking that this substitution correctly induces the symplectic transformation of F_I. This will impose restrictions on the form of F, and hence also on Ω. Imposing that S-duality (1.63) constitutes a symmetry of the model (1.62) results in the following conditions on the transformation behavior of the derivatives of Ω [21],

$$\begin{aligned}\left(\frac{\partial\Omega}{\partial T}\right)'_{\mathrm{S}} &= \frac{\partial\Omega}{\partial T},\\ \left(\frac{\partial\Omega}{\partial S}\right)'_{\mathrm{S}} &= \Delta_{\mathrm{S}}{}^2 \left(\frac{\partial\Omega}{\partial S}\right) + \frac{\partial\left(\Delta_{\mathrm{S}}{}^2\right)}{\partial S}\left[-\tfrac{1}{2}Y^0\,\frac{\partial\Omega}{\partial Y^0} - \frac{\mathrm{i}c}{2\Delta_{\mathrm{S}}\,(Y^0)^2}\left(\frac{\partial\Omega}{\partial T}\right)^2\right],\\ \left(Y^0\,\frac{\partial\Omega}{\partial Y^0}\right)'_{\mathrm{S}} &= Y^0\,\frac{\partial\Omega}{\partial Y^0} + \frac{2\mathrm{i}c}{\Delta_{\mathrm{S}}\,(Y^0)^2}\left(\frac{\partial\Omega}{\partial T}\right)^2,\end{aligned} \tag{1.69}$$

while requiring (1.66) to constitute a symmetry imposes the transformation behavior [21]

$$\begin{aligned}\left(\frac{\partial\Omega}{\partial S}\right)'_{\mathrm{T}} &= \frac{\partial\Omega}{\partial S},\\ \left(\frac{\partial\Omega}{\partial T}\right)'_{\mathrm{T}} &= \left(\Delta_{\mathrm{T}} - T^2\right)\,\frac{\partial\Omega}{\partial T} + T\;Y^0\frac{\partial\Omega}{\partial Y^0},\\ \left(Y^0\frac{\partial\Omega}{\partial Y^0}\right)'_{\mathrm{T}} &= Y^0\frac{\partial\Omega}{\partial Y^0} + \frac{4}{\Delta_{\mathrm{T}}\,(Y^0)^2}\,\frac{\partial\Omega}{\partial S}\left[-Y^0\frac{\partial\Omega}{\partial Y^0} + T\frac{\partial\Omega}{\partial T}\right].\end{aligned} \tag{1.70}$$

These equations allow for various classes of solutions. For instance, if we only impose S-duality invariance, then an exact solution to the S-duality conditions (1.69) is

$$\Omega(S,\bar{S},Y^0,\bar{Y}^0;\eta) = \eta\left[\ln Y^0 + \ln \bar{Y}^0 + \ln(S+\bar{S})\right], \tag{1.71}$$

which is invariant under (1.64). If, on the other hand, we impose both S-duality and T-duality invariance, solutions to both (1.69) and (1.70) may be constructed iteratively by assuming that Ω is analytic in η and power expanding in it, so that

$$\Omega(Y,\bar{Y};\eta) = \sum_{n=1}^{\infty} \eta^n\;\Omega^{(n)}(Y,\bar{Y}). \tag{1.72}$$

Then, at order η, the differential equations (1.69) reduce to

$$\begin{aligned}\left(\frac{\partial\Omega^{(1)}}{\partial T}\right)'_{\mathrm{S}} &= \frac{\partial\Omega^{(1)}}{\partial T},\\ \left(\frac{\partial\Omega^{(1)}}{\partial S}\right)'_{\mathrm{S}} &= \Delta_{\mathrm{S}}{}^2\left(\frac{\partial\Omega^{(1)}}{\partial S}\right) + \frac{\partial\left(\Delta_{\mathrm{S}}{}^2\right)}{\partial S}\left[-\tfrac{1}{2}Y^0\,\frac{\partial\Omega^{(1)}}{\partial Y^0}\right],\end{aligned}$$

$$\left(Y^0\,\frac{\partial\Omega^{(1)}}{\partial Y^0}\right)'_{\rm S} = Y^0\,\frac{\partial\Omega^{(1)}}{\partial Y^0}, \tag{1.73}$$

while the differential equations (1.70) reduce to

$$\begin{aligned} \left(\frac{\partial\Omega^{(1)}}{\partial S}\right)'_{\rm T} &= \frac{\partial\Omega^{(1)}}{\partial S}, \\ \left(\frac{\partial\Omega^{(1)}}{\partial T}\right)'_{\rm T} &= -\tfrac{1}{2}T^2\,\frac{\partial\Omega^{(1)}}{\partial T} + T\;Y^0\frac{\partial\Omega^{(1)}}{\partial Y^0}, \\ \left(Y^0\frac{\partial\Omega^{(1)}}{\partial Y^0}\right)'_{\rm T} &= Y^0\frac{\partial\Omega^{(1)}}{\partial Y^0}. \end{aligned} \tag{1.74}$$

Once a solution $\Omega^{(1)}$ to these equations has been found, the full expression (1.72) can be constructed by solving (1.69) and (1.70) iteratively starting from $\Omega^{(1)}$.

As an application, let us return to the Born-Infeld-dilaton-axion model (1.61) which, as we already mentioned, is a model of the form (1.62) that scales as $F \to \lambda^2 F$ under $Y^I \to \lambda Y^I$ with $\lambda \in \mathbb{R}$ (see below (1.61)). Let us first check that both S- and T-duality constitute invariances of the model. We recall that $x = Y^2$. The S-duality transformation (1.63) precisely induces the transformations (1.58) and (1.59), since $(p, q) = (2\text{Re}\,x, 2\text{Re}\,F_x)$. The T-duality transformation (1.66) leaves (x, F_x) invariant. By expressing Ω given in (1.61) in terms of S, T and Y^0 (and their complex conjugates), we see from (1.67) that also S is invariant under this T-duality transformation, since $Y^0\partial\Omega/\partial Y^0 = T\partial\Omega/\partial T$. Consequently, the Hamiltonian (1.56) is also invariant under (1.66).

Now consider expanding (1.61) in powers of g^2. To first order we obtain

$$\Omega^{(1)} = \tfrac{1}{8}\,|Y^0|^4\,(S+\bar S)^2\,|T|^4. \tag{1.75}$$

It is invariant under both (1.63) and (1.66) to lowest order in g^2, and it is straightforward to check that (1.75) indeed satisfies the differential Eqs. (1.73) and (1.74). We note that under the aforementioned scaling $Y^I \to \lambda\,Y^I$, $\Omega^{(1)}$ scales as $\Omega^{(1)} \to \lambda^4\,\Omega^{(1)}$. This scaling behavior is thus very different from the one encountered in supergravity models, such as those considered in [18, 21], where the function F scaled homogeneously as $F \to \lambda^2\,F$, but the associated $\Omega^{(1)}$ did not scale at all. This difference is due to the fact that in these models, the external parameter η scaled as $\eta \to \lambda^2\,\eta$, while in the Born-Infeld-dilaton-axion model it scales as $\eta \to \lambda^{-2}\,\eta$.

Thus, we see that the actual solutions to (1.69) and (1.70) depend sensitively on the scaling behavior of the Y^I and η. For instance, the solution (1.71) does not exhibit a homogeneous scaling behavior under $Y^I \to \lambda Y^I$. In the next subsection, we further analyze some of the consequences of this scaling behavior.

1.2.3 Homogeneous $F(x, \bar{x}; \eta)$

The theorem in Sect. 1.2.1 did not assume any homogeneity properties for F. Here we will look at the case when F is homogeneous of degree two and discuss some of its consequences. As shown in the previous subsections, an example of a model with this feature is the Born-Infeld-dilaton-axion system.

Let us consider a function $F(x, \bar{x}; \eta) = F^{(0)}(x) + 2\mathrm{i}\Omega(x, \bar{x}; \eta)$ that depends on a real external parameter η, and let us discuss its behavior under the scaling $x \to \lambda x\ ,\ \eta \to \lambda^m \eta$ with $\lambda \in \mathbb{R}$. We take $F^{(0)}(x)$ to be quadratic in x, so that $F^{(0)}$ scales as $F^{(0)}(\lambda x) = \lambda^2 F^{(0)}(x)$. This scaling behavior can be extended to the full function F if we demand that the canonical pair (ϕ, π) given in (1.6) scales uniformly as $(\phi, \pi) \to \lambda (\phi, \pi)$. Then we have

$$F(\lambda x, \lambda \bar{x}; \lambda^m \eta) = \lambda^2 F(x, \bar{x}; \eta), \tag{1.76}$$

which results in the homogeneity relation

$$2 F = x^i F_i + \bar{x}^{\bar{\imath}} F_{\bar{\imath}} + m \eta F_\eta, \tag{1.77}$$

where $F_\eta = \partial F / \partial \eta$. Inspection of (1.11) shows that the associated Hamiltonian H scales with weight two as

$$H(\lambda \phi, \lambda \pi; \lambda^m \eta) = \lambda^2 H(\phi, \pi; \eta), \tag{1.78}$$

so that H satisfies the homogeneity relation,

$$2 H = \phi \frac{\partial H}{\partial \phi} + \pi \frac{\partial H}{\partial \pi} + m \eta \frac{\partial H}{\partial \eta}. \tag{1.79}$$

Using (1.11) as well as $y_i = F_i$, this can be written as

$$H = \mathrm{i}\left(\bar{x}^{\bar{\imath}} F_i - x^i \bar{F}_{\bar{\imath}}\right) + \frac{m}{2} \eta \frac{\partial H}{\partial \eta}. \tag{1.80}$$

Next, using that the dependence on η is solely contained in Ω, we obtain

$$\frac{\partial H}{\partial \eta}|_{\phi, \pi} = -\frac{\partial L}{\partial \eta}|_{\phi, \dot{\phi}} = -4\Omega_\eta, \tag{1.81}$$

where $\Omega_\eta = \partial \Omega / \partial \eta$. Thus, we can express (1.80) as

$$H = \mathrm{i}\left(\bar{x}^{\bar{\imath}} F_i - x^i \bar{F}_{\bar{\imath}}\right) - 2 m \eta \Omega_\eta. \tag{1.82}$$

This relation is in accordance with (1.9) upon substitution of the homogeneity relations $2F^{(0)}(x) = x^i\,F_i^{(0)}$ and $2\,\Omega = x^i\,\Omega_i + \bar{x}^{\bar{\imath}}\,\Omega_{\bar{\imath}} + m\,\eta\,\Omega_\eta$ that follow from (1.77).

The Hamiltonian transforms as a function under symplectic transformations. Since the first term in (1.82) transforms as a function, it follows that Ω_η also transforms as a function. This is in accordance with the general result quoted at the end of Sect. 1.2.1 which states that $\partial_\eta F$ transforms as a function.

An application of the above is provided by the Born-Infeld-dilaton-axion system based on (1.61), whose function F scales according to (1.76) with $m = -2$ (in this example, $\eta = g^2$).

In certain situations, such as in the study of BPS black holes [34], the discussion needs to be extended to an external parameter η that is complex, so that now we consider a function $F(x, \bar{x}; \eta, \bar{\eta}) = F^{(0)}(x) + 2\mathrm{i}\Omega(x, \bar{x}; \eta, \bar{\eta})$ that scales as follows (with $\lambda \in \mathbb{R}$),

$$F(\lambda\,x, \lambda\,\bar{x}; \lambda^m\,\eta, \lambda^m\,\bar{\eta}) = \lambda^2\,F(x, \bar{x}; \eta, \bar{\eta}). \tag{1.83}$$

For instance, in the case of BPS black holes, η is identified with Υ, which is complex and denotes the (rescaled) lowest component of the square of the Weyl superfield. The extension to a complex η results in the presence of an additional term on the right hand side of (1.77) and (1.79),

$$\begin{aligned} 2\,F &= x^i\,F_i + \bar{x}^{\bar{\imath}}\,F_{\bar{\imath}} + m\left(\eta\,F_\eta + \bar{\eta}F_{\bar{\eta}}\right), \\ 2\,H &= \phi\,\frac{\partial H}{\partial\phi} + \pi\,\frac{\partial H}{\partial\pi} + m\left(\eta\,\frac{\partial H}{\partial\eta} + \bar{\eta}\,\frac{\partial H}{\partial\bar{\eta}}\right), \end{aligned} \tag{1.84}$$

and hence

$$H = \mathrm{i}\left(\bar{x}^{\bar{\imath}}\,F_i - x^i\,\bar{F}_{\bar{\imath}}\right) + \frac{m}{2}\left(\eta\,\frac{\partial H}{\partial\eta} + \bar{\eta}\,\frac{\partial H}{\partial\bar{\eta}}\right). \tag{1.85}$$

Then, since the dependence on η and $\bar{\eta}$ is solely contained in Ω, we obtain

$$H = \mathrm{i}\left(\bar{x}^{\bar{\imath}}\,F_i - x^i\,\bar{F}_{\bar{\imath}}\right) - 2\,m\left(\eta\,\Omega_\eta + \bar{\eta}\,\Omega_{\bar{\eta}}\right). \tag{1.86}$$

This is in accordance with (1.9) upon substitution of the homogeneity relations $2F^{(0)}(x) = x^i\,F_i^{(0)}$ and $2\,\Omega = x^i\,\Omega_i + \bar{x}^{\bar{\imath}}\,\Omega_{\bar{\imath}} + m\,(\eta\,\Omega_\eta + \bar{\eta}\,\Omega_{\bar{\eta}})$ that follow from (1.84). The case of BPS black holes mentioned above corresponds to $m = 2$ [8, 21].

The above extends straightforwardly to the case of multiple real external parameters.

1.3 Lecture II: Duality Covariant Complex Variables

As already discussed, the function $F(x, \bar{x})$ may depend on a number of external parameters η. Under duality transformations (1.10), the symplectic vector $(x^i, F_i(x, \bar{x}))$ transforms into $(\tilde{x}^i, \tilde{F}_i(\tilde{x}, \bar{\tilde{x}}))$, while the parameters η are inert. When expressing the transformed variables $\tilde{x}^i$ in terms of the original x^i, the relation will depend on η, i.e. $\tilde{x}^i = \tilde{x}^i(x, \bar{x}, \eta)$. In this section we introduce duality covariant complex variables t^i whose duality transformation law is independent of η. These variables constitute a complexification of the canonical variables of the Hamiltonian and ensure that when expanding the Hamiltonian in powers of the external parameters, the resulting expansion coefficients transform covariantly under duality transformations. This expansion can also be organized by employing a suitable covariant derivative, which we construct. The covariant variables introduced here have the same duality transformation properties as the ones used in topological string theory and can therefore be identified with the latter.

We begin by writing the Hamiltonian H given in (1.9) in the form

$$\begin{aligned} H = & -\mathrm{i}(x^i\,\bar{F}^{(0)}_{\bar{\imath}} - \bar{x}^{\bar{\imath}}\,F^{(0)}_i) - 4\,\mathrm{Im}[F^{(0)} - \tfrac{1}{2}x^i\,F^{(0)}_i] \\ & -2\Big[2\,\Omega - (x^i - \bar{x}^{\bar{\imath}})(\Omega_i - \Omega_{\bar{\imath}})\Big], \end{aligned} \tag{1.87}$$

where we made use of (1.7). We take $\Omega(x, \bar{x}; \eta)$ to depend on a single real parameter η that is inert under symplectic transformations. The discussion given below can be extended to the case of multiple real external parameters in a straightforward manner. For later convenience, we introduce the notation $\Omega_\eta = \partial\Omega/\partial\eta$, $F_{\eta j} = \partial^2 F/\partial\eta\partial x^j$, etc.

The Hamiltonian (1.87) is given in terms of complex fields x^i and $\bar{x}^{\bar{\imath}}$ whose transformation law under duality depends on the external parameter η. Now we define complex variables t^i whose transformation law does not depend on η, as follows. We introduce the complex vector $(t^i, F^{(0)}_i(t))$ and equate its real part with the vector comprising the canonical variables (ϕ^i, π_i) [10],

$$\begin{aligned} 2\mathrm{Re}\,t^i &= \phi^i, \\ 2\mathrm{Re}\,F^{(0)}_i(t) &= \pi_i. \end{aligned} \tag{1.88}$$

This definition ensures that the vector $(t^i, F^{(0)}_i(t))$ describes a complexification of (ϕ^i, π_i) that transforms in the same way as (ϕ^i, π_i) under duality transformations, namely as in (1.4). This yields the transformation law

$$\tilde{t}^i = U^i{}_j\,t^j + Z^{ij}\,F^{(0)}_j(t), \tag{1.89}$$

which, differently from the one for the $\tilde{x}^i$, is independent of η.

Using (1.6), the new variables t^i are related to the x^i by

$$
\begin{aligned}
2\mathrm{Re}\, t^i &= 2\mathrm{Re}\, x^i , \\
2\mathrm{Re}\; F_i^{(0)}(t) &= 2\mathrm{Re}\; F_i(x, \bar{x}; \eta).
\end{aligned} \tag{1.90}
$$

Now we consider the series expansion of H in powers of η. If the expansion is performed keeping x^i and $x^{\bar{\imath}}$ fixed, the resulting coefficients functions in the expansion do not have a nice behavior under sympletic transformations because of the aforementioned dependence of $\tilde{x}^i$ on η. This implies that the coefficient functions at a given order in η will transform into coefficient functions at higher order. This can be avoided by performing an expansion in powers of η keeping t^i and $t^{\bar{\imath}}$ fixed instead. We obtain

$$
H = \sum_{n=0}^{\infty} \frac{\eta^n}{n!}\; f^{(n)}(t, \bar{t}), \tag{1.91}
$$

where the coefficient functions

$$
f^{(n)} = \partial_\eta^n H(t, \bar{t}; \eta)\Big|_{\eta=0} \tag{1.92}
$$

transform as functions under symplectic transformations, i.e. $\tilde{f}^{(n)}(\tilde{t}, \bar{\tilde{t}}) = f^{(n)}(t, \bar{t})$. Viewing them as as functions of $\mathrm{Re}\, t^i$ and of $\mathrm{Re}\; F_i^{(0)}(t)$, we can re-express them in terms of x^i and $\bar{x}^{\bar{\imath}}$ using (1.90), as follows. First we introduce a modified derivative $\mathcal{D}_\eta$ [5, 33] that has the feature that it annihilates the canonical variables (ϕ^i, π_i), so that

$$
\mathcal{D}_\eta \left(\mathrm{Re}\, x^i\right) = 0, \qquad \mathcal{D}_\eta \left(\mathrm{Re}\; F_i\right) = 0. \tag{1.93}
$$

We then use $\mathcal{D}_\eta$ to expand H in powers of η while keeping $\mathrm{Re}\, x^i$ and $\mathrm{Re}\; F_i$ fixed,

$$
H = \sum_{n=0}^{\infty} \frac{\eta^n}{n!}\; H^{(n)}, \tag{1.94}
$$

where the coefficient functions are given by

$$
H^{(n)} = \mathcal{D}_\eta^n H(x, \bar{x}; \eta)\Big|_{\eta=0}. \tag{1.95}
$$

By comparing (1.91) with (1.94), we conclude that $f^{(n)} = H^{(n)}$, so that the symplectic coefficient functions $f^{(n)}$ can be expressed as

$$
f^{(n)} = \partial_\eta^n H(t, \bar{t}; \eta)\Big|_{\eta=0} = \mathcal{D}_\eta^n H(x, \bar{x}; \eta)\Big|_{\eta=0}. \tag{1.96}
$$

The modified derivative $\mathcal{D}_\eta$ used in the expansion is given by

$$
\mathcal{D}_\eta = \partial_\eta + \mathrm{i}\, \hat{N}^{ij} \left(F_{\eta j} + \bar{F}_{\eta \bar{j}}\right) (\partial_i - \partial_{\bar{\imath}}) , \tag{1.97}
$$

where $\hat{N}^{ij}$ denotes the inverse of

$$\hat{N}_{ij} = -\mathrm{i}\left[F_{ij} - \bar{F}_{\bar{\imath}\bar{\jmath}} - F_{i\bar{\jmath}} + \bar{F}_{\bar{\imath}j}\right]. \tag{1.98}$$

Using (1.7), the above can also be written as

$$\mathcal{D}_\eta = \partial_\eta - 2\,\hat{N}^{ij}\left(\Omega_{\eta j} - \Omega_{\eta\bar{\jmath}}\right)(\partial_i - \partial_{\bar{\imath}}), \tag{1.99}$$

with

$$\begin{aligned}\hat{N}_{ij} &= N_{ij} + 4\mathrm{Re}\left(\Omega_{ij} - \Omega_{i\bar{\jmath}}\right), \\ N_{ij} &= -\,\mathrm{i}\left[F^{(0)}_{ij} - \bar{F}^{(0)}_{\bar{\imath}\bar{\jmath}}\right].\end{aligned} \tag{1.100}$$

Observe that $\hat{N}_{ij}$ is a real symmetric matrix.

Exercise 12 Verify (1.93).

We now give the first few terms in the expansion of H. We choose to evaluate them using (1.95). Expanding Ω in a power series[4] in η,

$$\Omega(x,\bar{x};\eta) = \sum_{n=1}^{\infty}\frac{\eta^n}{n!}\,\Omega^{(n)}(x,\bar{x}), \tag{1.101}$$

we obtain

$$\begin{aligned}
f^{(0)} &= -\,\mathrm{i}(x^i\,\bar{F}^{(0)}_{\bar{\imath}} - \bar{x}^{\bar{\imath}}\,F^{(0)}_i) - 4\,\mathrm{Im}[F^{(0)} - \tfrac{1}{2}x^i\,F^{(0)}_i], \\
f^{(1)} &= -\,4\,\Omega^{(1)}, \\
f^{(2)} &= -\,4\left[\Omega^{(2)} - 2N^{ij}\left(\Omega^{(1)}_i - \Omega^{(1)}_{\bar{\imath}}\right)\left(\Omega^{(1)}_j - \Omega^{(1)}_{\bar{\jmath}}\right)\right], \\
f^{(3)} &= -\,4\Big[\Omega^{(3)} - 6N^{ij}\left(\Omega^{(2)}_i - \Omega^{(2)}_{\bar{\imath}}\right)\left(\Omega^{(1)}_j - \Omega^{(1)}_{\bar{\jmath}}\right) \\
&\quad + 12N^{ik}N^{jl}\left(\Omega^{(1)}_{ij} - \Omega^{(1)}_{i\bar{\jmath}} + \mathrm{c.c.}\right)\left(\Omega^{(1)}_k - \Omega^{(1)}_{\bar{k}}\right)\left(\Omega^{(1)}_l - \Omega^{(1)}_{\bar{l}}\right) \\
&\quad + 4\mathrm{i}\,N^{ip}N^{jl}N^{km}\left(\Omega^{(1)}_i - \Omega^{(1)}_{\bar{\imath}}\right)\left(\Omega^{(1)}_j - \Omega^{(1)}_{\bar{\jmath}}\right)\left(\Omega^{(1)}_k - \Omega^{(1)}_{\bar{k}}\right) \\
&\quad \times\left(F^{(0)}_{plm} + \bar{F}^{(0)}_{\bar{p}\bar{l}\bar{m}}\right)\Big].
\end{aligned} \tag{1.102}$$

Observe that at any given order in η, there is no distinction between x^i and t^i, so that in (1.102) we may replace x^i everywhere by t^i.

The expansion (1.94) yields expansion functions that are symplectic functions. This implies that $\mathcal{D}_\eta$ acts as a covariant derivative for symplectic transformations.

[4] Note that here we have chosen a different normalization for the $\Omega^{(n)}$ compared to the one in (1.72).

This can be verified explicitely and is done in appendix B, where we show that if a quantity $G(x, \bar{x}; \eta)$ transforms as a function under symplectic transformations, then so does $\mathcal{D}_\eta G$. In particular, applying $\mathcal{D}_\eta$ to H yields the relation

$$\partial_\eta H(t, \bar{t}; \eta) = \mathcal{D}_\eta H(x, \bar{x}; \eta), \tag{1.103}$$

where the right-hand side defines a symplectic function. More generally, applying multiple derivatives $\mathcal{D}_\eta^n$ on any symplectic function depending on x^i and $\bar{x}^{\bar{\imath}}$, will again yield a symplectic function. As an example, consider applying $\mathcal{D}_\eta$ and $\mathcal{D}_\eta^2$ on (1.87),

$$\begin{aligned} \mathcal{D}_\eta H(x, \bar{x}; \eta) &= -4\, \partial_\eta \Omega(x, \bar{x}; \eta), \\ \mathcal{D}_\eta^2 H(x, \bar{x}; \eta) &= -4 \left[\partial_\eta^2 \Omega - 2\, \hat{N}^{ij} \partial_\eta \omega_i\, \partial_\eta \omega_j \right], \end{aligned} \tag{1.104}$$

where $\omega_i = \Omega_i - \Omega_{\bar{\imath}}$. According to the above, both these expressions transform as functions under symplectic transformations. For the first expression this is confirmed by the result (1.177) which shows that $\partial_\eta \Omega$ transforms as a function. The second expression shows that, while $\partial_\eta^2 \Omega$ does not transform as a function, there exists a modification that can be included such that the result does again transform as a function. Expressions like these were derived earlier in a holomorphic setup [5, 33]. Furthermore, we note that the differential operators $\mathcal{D}^i$, defined by

$$\mathcal{D}^i = \hat{N}^{ij} \left(\frac{\partial}{\partial x^j} - \frac{\partial}{\partial \bar{x}^{\bar{\jmath}}} \right), \tag{1.105}$$

are mutually commuting, and they also commute with $\mathcal{D}_\eta$,

$$[\mathcal{D}^i, \mathcal{D}^j] = [\mathcal{D}^i, \mathcal{D}_\eta] = 0. \tag{1.106}$$

Exercise 13 Verify (1.106).

As already mentioned, it is possible to extend the above to the case of several independent real parameters $\eta, \eta', \eta'', \ldots$. In that case the additional operators, $\mathcal{D}_{\eta'}$, etc., will also commute with the operators considered in (1.106).

Obviously, when imposing the restriction $\eta = 0$ on the functions $\mathcal{D}_\eta^n H$, they reduce to the expressions for the $f^{(n)}$ obtained in (1.102). This can be explicitly verified for the functions given in (1.104) by comparing them to the expressions in (1.102).

Let us return to the relation (1.88) and discuss it in the light of phase space variables. As mentioned in Sect. 1.2.1, we view (ϕ^i, π_i) as coordinates on a classical phase space equipped with the symplectic form $d\pi_i \wedge d\phi^i$. Let us express the symplectic form in terms of the t^i using (1.88),

$$d\pi_i \wedge d\phi^i = \mathrm{i}\, N_{ij}\, dt^i \wedge d\bar{t}^{\bar{j}}, \tag{1.107}$$

with N_{ij} given in (1.100). This relation may be interpreted as a canonical transformation from variables (ϕ^i, π_i) to $(t^i, \bar{t}^{\bar{i}})$ which is generated by a function S that depends on half of all the coordinates. We take S to depend on ϕ^i and t^i. We determine it in the linearized approximation by expanding N_{ij} around a background value t^i_B. Performing the shift

$$t^i \to t^i_B + t^i, \qquad \bar{t}^{\bar{i}} \to \bar{t}^{\bar{i}}_B + \bar{t}^{\bar{i}}, \tag{1.108}$$

and keeping only terms linear in the fluctuations t^i and $\bar{t}^{\bar{i}}$, we obtain from (1.88),

$$\begin{aligned} \phi^i &= t^i + \bar{t}^{\bar{i}}, \\ \pi_i &= F^{(0)}_{ij}(t_B)\, t^j + \bar{F}^{(0)}_{\bar{i}\bar{j}}(\bar{t}_B)\, \bar{t}^{\bar{j}}, \end{aligned} \tag{1.109}$$

where we absorbed the fluctuation independent pieces into the definition of (ϕ^i, π_i). Then, expressing π_i in terms of t^i and ϕ^i,

$$\pi_i = \mathrm{i} N_{ij}(t_B, \bar{t}_B)\, t^j + \bar{F}^{(0)}_{\bar{i}\bar{j}}(\bar{t}_B)\, \phi^j, \tag{1.110}$$

and introducing the combination

$$P_i = -\mathrm{i}\, N_{ij}(t_B, \bar{t}_B) \left(\phi^j - t^j\right), \tag{1.111}$$

yields

$$d\pi_i \wedge d\phi^i = \mathrm{i}\, N_{ij}(t_B, \bar{t}_B)\, dt^i \wedge d\bar{t}^{\bar{j}} = dP_i \wedge dt^i. \tag{1.112}$$

Hence, the 1-form $\pi_i\, d\phi^i - P_i\, dt^i$ is closed, so that locally,

$$\pi_i\, d\phi^i - P_i\, dt^i = dS, \tag{1.113}$$

where $S(\phi, t)$ is called the generating function of the canonical transformation. Then, integrating this relation yields the following expression for the generating function $S(\phi, t; t_B, \bar{t}_B)$ [23–25],

$$\begin{aligned} S(\phi, t; t_B, \bar{t}_B) = \tfrac{1}{2} \bar{F}^{(0)}_{\bar{i}\bar{j}}(\bar{t}_B)\, \phi^i \phi^j + \mathrm{i}\, N_{ij}(t_B, \bar{t}_B)\, \phi^i t^j - \tfrac{1}{2} \mathrm{i}\, N_{ij}(t_B, \bar{t}_B)\, t^i t^j \\ + c(t_B, \bar{t}_B), \end{aligned} \tag{1.114}$$

where c denotes a background dependent integration constant. Observe that $S(\phi, t; t_B, \bar{t}_B)$ is holomorphic in the fluctuation t and non-holomorphic in the background t_B. The generating function $S(\phi, t; t_B, \bar{t}_B)$ plays a crucial role in the wave function approach to perturbative topological string theory. This approach represents a concise framework [22–26] for deriving the holomorphic anomaly equation of topological string theory [35, 36], and will be reviewed in appendix C.

1.4 Lecture III: The Hesse Potential and the Topological String

In the previous sections we showed that the dynamics of a general class of Lagrangians is encoded in a non-holomorphic function F of the form given in (1.2). This function F may depend on a number of external parameters η. We expressed the associated Hamiltonian in terms of duality covariant complex variables and showed that in these variables, the expansion of the Hamiltonian in a power series in η yields expansion coefficients that transform as functions under duality. In this section we apply these techniques to supergravity models in the presence of higher-curvature interactions encoded in the square of the Weyl superfield [2, 5]. We consider these models in an $AdS_2 \times S^2$ background. The Hamiltonian (1.9) associated to the reduced Lagrangian is a (generalized) Hesse potential. The Hesse potential plays a central role in the formulation of special geometry in terms of real variables [7, 14–16]. The external parameter η, which is now complex, is identified with the lowest component field of the square of the Weyl superfield.

We begin by reviewing the computation of the Wilsonian effective Lagrangian in an $AdS_2 \times S^2$ background [27, 28] and relate it to the presentation of Sect. 1.2. We then generalize the discussion to the case of a function F of type (1.7) with a non-harmonic Ω. We express the Hesse potential in terms of the aforementioned duality covariant complex variables, and expand it in powers of η and $\bar{\eta}$. This reveals a systematic structure. Namely, the Hesse potential decomposes into two classes of terms. One class consists of combinations of terms, constructed out of derivatives of Ω, that transform as functions under electric/magnetic duality. The other class is constructed out of Ω and derivatives thereof. Demanding this second class to also exhibit a proper behavior under duality transformations (as a consequence of the transformation behavior of the Hesse potential) imposes restrictions on Ω. These restrictions are captured by a differential equation that equals half of the holomorphic anomaly equation encountered in perturbative topological string theory.

1.4.1 The Reduced Wilsonian Lagrangian in an $AdS_2 \times S^2$ Background

We consider the coupling of $N = 2$ vector multiplets to $N = 2$ supergravity in the presence of higher-curvature interactions encoded in the square of the Weyl superfield [2, 5]. We use the conventions of $N = 2$ supergravity, whereby the vector multiplets are labelled by a capital index $I = 0, \ldots, n$ (instead of the index i used in the previous sections). The degrees of freedom of a vector multiplet include an abelian gauge field and a complex scalar field, and these will thus carry an index I. We denote the complex scalar fields by X^I. The square of the Weyl superfield has various component fields. The highest component field contains the square of the anti-selfdual components of the Riemann tensor, while the lowest one, denoted by

$\hat{A}$, equals the square of an anti-selfdual tensor field. Below we will find it convenient to work with rescaled complex fields Y^I and Υ, which are related to the X^I and $\hat{A}$ by a complex rescaling [34].

First we evaluate the Wilsonian effective Lagrangian of these models on a field configuration consistent with the SO(2, 1) × SO(3) isometry of an $AdS_2 \times S^2$ background. The spacetime metric $g_{\mu\nu}$ and the field strengths $F_{\mu\nu}{}^I$ of the abelian gauge fields are given by

$$\begin{aligned} ds^2 &= v_1\Big(- r^2\, dt^2 + \frac{dr^2}{r^2}\Big) + v_2\Big(d\theta^2 + \sin^2\theta\, d\varphi^2\Big), \\ F_{rt}{}^I &= e^I, \qquad F_{\theta\varphi}{}^I = p^I \sin\theta. \end{aligned} \tag{1.115}$$

The θ-dependence of $F_{\theta\varphi}{}^I$ is fixed by rotational invariance and the p^I denote the magnetic charges. The quantities v_1, v_2, e^I and p^I are all constant by virtue of the SO(2, 1) × SO(3) symmetry.

It is well-known [2] that the Wilsonian Lagrangian $\mathcal{L}$ is encoded in a holomorphic function $F(X, \hat{A})$, which is homogeneous of degree two under the scaling discussed in (1.76), i.e. $F(\lambda X, \lambda^2 \hat{A}) = \lambda^2 F(X, \hat{A})$. Evaluating the Wilsonian Lagrangian in the background (1.115) and integrating over S^2 [32],

$$\mathcal{F} = \int d\theta\, d\varphi\, \sqrt{|g|}\, \mathcal{L}, \tag{1.116}$$

yields the reduced Wilsonian Lagrangian which depends on e^I and p^I, on the rescaled fields Y^I and Υ, and on v_1 and v_2 through the ratio $U = v_1/v_2$.

In the following, we will restrict to supersymmetric backgrounds, for simplicity, in which case $U = 1$ and $\Upsilon = -64$ [34]. Then, the reduced Wilsonian Lagrangian reads [27, 28],

$$\begin{aligned} \mathcal{F}(e, p, Y, \bar{Y}; \Upsilon, \bar{\Upsilon}) = &- \tfrac{1}{8}\mathrm{i}\left(F_{IJ} - \bar{F}_{\bar{I}\bar{J}}\right)\Big(e^I e^J - p^I p^J\Big) - \tfrac{1}{4}\left(F_{IJ} + \bar{F}_{\bar{I}\bar{J}}\right) e^I p^J \\ &+ \tfrac{1}{2}\mathrm{i} e^I \Big(F_I + F_{IJ}\bar{Y}^{\bar{J}} - \text{h.c.}\Big) - \tfrac{1}{2} p^I \Big(F_I - F_{IJ}\bar{Y}^{\bar{J}} + \text{h.c.}\Big) \\ &+ \mathrm{i}\Big(F - Y^I F_I + \tfrac{1}{2}\bar{F}_{\bar{I}\bar{J}} Y^I Y^J - \text{h.c.}\Big), \end{aligned} \tag{1.117}$$

where $\Upsilon = \bar{\Upsilon} = -64$ and $F_I = \partial F/\partial Y^I$, $F_{IJ} = \partial^2 F/\partial Y^I \partial Y^J$, etc. Introducing the complex scalar fields $x^I = \frac{1}{2}(p^I + \mathrm{i}e^I)$ of Sect. 1.2.2 (see (1.31)), the reduced Lagrangian becomes a function of two types of complex scalar fields, namely the x^I that incorporate the electromagnetic information, and the moduli fields Y^I.

Now we recall that in an $AdS_2 \times S^2$ background the electro/magnetic quantities appearing in (1.115) are related to the moduli fields Y^I. When the background is supersymmetric, the relation takes the form [37]

$$x^I = \mathrm{i}\bar{Y}^I. \tag{1.118}$$

In the context of BPS black holes, the real part of this equation yields the magnetic attractor equation. Then, using (1.118), the reduced Wilsonian Lagrangian becomes equal to

$$\mathcal{F}(Y,\bar{Y};\Upsilon,\bar{\Upsilon}) = -2\,\mathrm{Im}F(Y,\Upsilon), \tag{1.119}$$

with $\Upsilon = \bar{\Upsilon} = -64$.

Exercise 14 Verify (1.119).

Let us reformulate the reduced Lagrangian (1.119), which is based on a holomorphic functions $F(Y,\Upsilon)$, in terms of the function $F(Y,\bar{Y};\Upsilon,\bar{\Upsilon}) = F^{(0)}(Y) + 2\mathrm{i}\Omega(Y,\bar{Y};\Upsilon,\bar{\Upsilon})$ introduced in Sect. 1.2. This is achieved by using the equivalence transformation (1.8). Writing the holomorphic function $F(Y,\Upsilon)$ as $F(Y,\Upsilon) = F^{(0)}(Y) - g(Y,\Upsilon)$ and applying (1.8), we obtain $\Omega = -\mathrm{Im}\,g(Y,\Upsilon)$. Thus, at the Wilsonian level, Ω is a harmonic function, and the reduced Lagrangian can be expressed as

$$\begin{aligned}\mathcal{F}(Y,\bar{Y};\Upsilon,\bar{\Upsilon}) &= -2\left[\mathrm{Im}F^{(0)}(Y) + \Omega(Y,\bar{Y};\Upsilon,\bar{\Upsilon})\right] \\ &= -2\left[\mathrm{Im}F(Y,\bar{Y};\Upsilon,\bar{\Upsilon}) - \Omega(Y,\bar{Y};\Upsilon,\bar{\Upsilon})\right],\end{aligned} \tag{1.120}$$

with $\Upsilon = \bar{\Upsilon} = -64$. Both $F^{(0)}$ and Ω are homogeneous functions of degree two, so that $\mathcal{F}(\lambda Y,\lambda\bar{Y};\lambda^2\Upsilon,\lambda^2\bar{\Upsilon}) = \lambda^2\,\mathcal{F}(Y,\bar{Y};\Upsilon,\bar{\Upsilon})$.

The reduced Lagrangian (1.120) agrees with the one in (1.9), up to an overall normalization factor of -2. In the following, we rescale (1.120) by this factor, so that from now on

$$\mathcal{F}(Y,\bar{Y};\Upsilon,\bar{\Upsilon}) = 4\left[\mathrm{Im}F(Y,\bar{Y};\Upsilon,\bar{\Upsilon}) - \Omega(Y,\bar{Y};\Upsilon,\bar{\Upsilon})\right]. \tag{1.121}$$

Using (1.118), we infer that $p^I = -\mathrm{i}\left(Y^I - \bar{Y}^I\right)$ and $e^I = Y^I + \bar{Y}^I$. According to (1.6), on the other hand, the real part of Y^I plays the role of the canonical variable ϕ^I, so that we have $\phi^I = e^I$. We may thus view $\mathcal{F}$ as a function of p^I and ϕ^I, and consider its Legendre transformation either with respect to p^I or with respect to ϕ^I. Performing the Legendre transformations with respect to the p^I, i.e. $\mathcal{H} = \mathcal{F} - p^I\,\pi_I$, results in

$$\pi_I = \frac{\partial\mathcal{F}}{\partial p^I} = F_I + \bar{F}_{\bar{I}}, \tag{1.122}$$

and hence

$$\begin{aligned}\mathcal{H} &= \mathrm{i}\left[Y^I\,\bar{F}_{\bar{I}} - \bar{Y}^{\bar{I}}\,F_I\right] + 2\left[2\,\Omega - Y^I\Omega_I - \bar{Y}^{\bar{I}}\Omega_{\bar{I}}\right] \\ &= \mathrm{i}\left[Y^I\,\bar{F}^{(0)}_{\bar{I}} - \bar{Y}^{\bar{I}}\,F^{(0)}_I\right] + 2\left[2\,\Omega - (Y^I - \bar{Y}^{\bar{I}})(\Omega_I - \Omega_{\bar{I}})\right],\end{aligned} \tag{1.123}$$

which is the analogue of the Hamiltonian (1.9) (up to an overall sign difference in the definition of both quantities). In the context of BPS black holes, $\mathcal{H}$ denotes the

BPS free energy of the black hole. When viewed as a function of ϕ^I and π_I, $\mathcal{H}(\phi, \pi)$ is called the Hesse potential.

Exercise 15 Verify (1.123).

On the other hand, performing the Legendre transformations with respect to the ϕ^I, i.e. $\mathcal{S} = \mathcal{F} - \phi^I q_I$, results in

$$q_I = \frac{\partial \mathcal{F}}{\partial \phi^I} = -\mathrm{i}\left(F_I - \bar{F}_{\bar{I}}\right), \tag{1.124}$$

and hence

$$\begin{aligned} \mathcal{S} &= -\mathrm{i}\left[Y^I \bar{F}_{\bar{I}} - \bar{Y}^{\bar{I}} F_I\right] + 2\left[2\,\Omega - Y^I \Omega_I - \bar{Y}^{\bar{I}} \Omega_{\bar{I}}\right] \\ &= -\mathrm{i}(Y^I \bar{F}^{(0)}_{\bar{I}} - \bar{Y}^{\bar{I}} F^{(0)}_I) + 2\left[2\,\Omega - (Y^I + \bar{Y}^{\bar{I}})(\Omega_I + \Omega_{\bar{I}})\right]. \end{aligned} \tag{1.125}$$

In the context of BPS black holes, (1.124) is the electric attractor equation, and $\mathcal{S}$ denotes the black hole entropy when viewed as function of p^I and q_I [34].

Exercise 16 Verify (1.125).

The entropy $\mathcal{S}$ can be obtained from the Hesse potential by a double Legendre transformation with respect to (ϕ^I, π_I) [8], i.e.

$$\mathcal{S}(p, q) = \mathcal{H}(\phi, \pi) + \pi_I\, p^I - \phi^I\, q_I \tag{1.126}$$

with $p^I = -\partial\mathcal{H}/\partial\pi_I$ and $q_I = \partial\mathcal{H}/\partial\phi^I$.

1.4.2 The Reduced Low-Energy Effective Action in an $AdS_2 \times S^2$ Background

When passing from the Wilsonian to the low-energy effective action, non-holomorphic terms emerge that are crucial for maintaining duality invariances [17], and that therefore need to be incorporated into the framework of the previous subsection. In the following, we assume that these terms can be incorporated into Ω by giving up the requirement that Ω is harmonic. We take the reduced low-energy effective Lagrangian and the associated Hesse potential to be given by (1.121) and (1.123), respectively, but now based on a non-harmonic Ω.

The Hesse potential (1.123) is given in terms of complex scalar fields Y^I and $\bar{Y}^I$. Under duality transformations, the scalar fields Y^I transform into $\tilde{Y}^I = \tilde{Y}^I(Y, \bar{Y}, \Upsilon, \bar{\Upsilon})$ (and similarly for the $\bar{\tilde{Y}}^I$), as discussed in Sect. 1.3. In order to obtain expansion coefficients that have a proper behavior under duality when expanding $\mathcal{H}$ in powers of Υ and Υ, we first express $\mathcal{H}$ in terms of the duality covariant complex coordinates introduced in Sect. 1.3. This can be achieved by iteration, and the result for the Hesse potential in the new coordinates then takes the form of an infinite power series in terms of Ω and its derivatives. We explicitly evaluate the first terms in this expansion up to order Ω^5. This suffices for appreciating the general structure of the full result. The actual calculations are rather cumbersome, and we have relegated some relevant material to appendices D and E. The expression for the Hesse potential, given in (1.143), consists of a sum of contributions $\mathcal{H}_i^{(a)}$, each of which transforms as a function under symplectic transformations. The function $\mathcal{H}^{(1)}$ is the only one that contains Ω, while all the other $\mathcal{H}_i^{(a)}$ contain derivatives of Ω. Using that $\mathcal{H}^{(1)}$ transforms as a function under symplectic transformations, we determine the transformation law of Ω, which is given in (1.146). In the following, we present a detailed derivation of these results. We suppress the superscript in $F^{(0)}$ for the most part, for simplicity.

The Hesse potential $\mathcal{H}$ is defined in terms of the real variables (ϕ^I, π_I), whose definition depends on the full effective action. These may be expressed in terms of the duality covariant variables introduced in (1.88), and which will be denoted by $\mathcal{Y}^I$ in the following. Inspection of (1.90) shows that these new variables are such that they coincide precisely with the fields Y^I that one would obtain from (ϕ^I, π_I) by using only the lowest-order holomorphic function $F^{(0)}$,

$$\begin{aligned} 2\,\mathrm{Re}\,\mathcal{Y}^I &= \phi^I = 2\,\mathrm{Re}\,Y^I, \\ 2\,\mathrm{Re}\,F_I^{(0)}(\mathcal{Y}) &= \pi_I = 2\,\mathrm{Re}\,F_I(Y, \bar{Y}; \Upsilon, \bar{\Upsilon}). \end{aligned} \tag{1.127}$$

Since the relation between the new variables and the real variables (ϕ^I, π_I) depends only on $F^{(0)}$, their duality transformations will not depend on the the details of the full effective action. Under symplectic transformations they transform according to,

$$\tilde{\mathcal{Y}}^I = U^I{}_J \mathcal{Y}^J + Z^{IJ}\, F_J^{(0)}(\mathcal{Y}) = \mathcal{S}_0{}^I{}_J(\mathcal{Y})\, \mathcal{Y}^J, \tag{1.128}$$

where

$$\mathcal{S}_0{}^I{}_J(\mathcal{Y}) = U^I{}_J + Z^{IK}\, F_{KJ}^{(0)}(\mathcal{Y}). \tag{1.129}$$

At the two-derivative level, where $\Omega = 0$, we have $\mathcal{Y}^I = Y^I$, but in higher orders the relation between these moduli is complicated and will depend on Ω. Hence we write $\mathcal{Y}^I = Y^I + \Delta Y^I$, where ΔY^I is purely imaginary. Writing $F = F^{(0)} + 2\mathrm{i}\Omega$, we will express (1.128) in terms of $F^{(0)}$ and Ω, so that we can henceforth suppress the superscript in $F^{(0)}$. Hence, in the following, F will denote a holomorphic homogeneous function of degree two. Therefore it is not necessary to make a distinction between holomorphic and anti-holomorphic derivatives of this function. The Eq. (1.127) can then be written as,

$$\begin{aligned} &F_I(\mathcal{Y} - \Delta Y) + \bar{F}_I(\bar{\mathcal{Y}} + \Delta Y) - F_I(\mathcal{Y}) - \bar{F}_I(\bar{\mathcal{Y}}) \\ &\quad = -2\mathrm{i}\left[\Omega_I(\mathcal{Y} - \Delta Y, \bar{\mathcal{Y}} + \Delta Y) - \Omega_{\bar{I}}(\mathcal{Y} - \Delta Y, \bar{\mathcal{Y}} + \Delta Y)\right]. \end{aligned} \tag{1.130}$$

Upon Taylor expanding, this equation will lead to an infinite power series in ΔY^I. Retaining only the term of first order in ΔY^I shows that it is proportional to the first derivative of Ω. Proceeding by iteration will then lead to an expression for ΔY^I involving increasing powers of Ω and its derivatives taken at $Y^I = \mathcal{Y}^I$. Here it suffices to give the result of this iteration up to fourth order in Ω,

$$\begin{aligned}\Delta Y^I =&\, 2\,(\Omega^I - \Omega^{\bar{I}}) \\ &- 2\mathrm{i}(F+\bar{F})^{IJK}(\Omega_J - \Omega_{\bar{J}})(\Omega_K - \Omega_{\bar{K}}) - 8\,\mathrm{Re}(\Omega^{IJ} - \Omega^{I\bar{J}})\,(\Omega_J - \Omega_{\bar{J}}) \\ &+ \tfrac{4}{3}\mathrm{i}\Big[(F-\bar{F})^{IJKL} + 3\mathrm{i}(F+\bar{F})^{IJM}(F+\bar{F})_M{}^{KL}\Big] \\ &\times(\Omega_J - \Omega_{\bar{J}})(\Omega_K - \Omega_{\bar{K}})(\Omega_L - \Omega_{\bar{L}}) \\ &+ 8\mathrm{i}\Big[2\,(F+\bar{F})^{IJ}{}_K\mathrm{Re}(\Omega^{KL} - \Omega^{K\bar{L}}) + \mathrm{Re}(\Omega^{IK} - \Omega^{I\bar{K}})(F+\bar{F})_K{}^{JL}\Big] \\ &\times(\Omega_J - \Omega_{\bar{J}})(\Omega_L - \Omega_{\bar{L}}) \\ &+ 32\,\mathrm{Re}(\Omega^{IJ} - \Omega^{I\bar{J}})\,\mathrm{Re}(\Omega_{JK} - \Omega_{J\bar{K}})\,(\Omega^K - \Omega^{\bar{K}}) \\ &+ 8\mathrm{i}\,\mathrm{Im}(\Omega^{IJK} - 2\,\Omega^{IJ\bar{K}} + \Omega^{I\bar{J}\bar{K}})(\Omega_J - \Omega_{\bar{J}})\,(\Omega_K - \Omega_{\bar{K}}) + \mathcal{O}(\Omega^4).\end{aligned} \tag{1.131}$$

Here indices have been raised by making use of N^{IJ}, which denotes the inverse of

$$N_{IJ} = 2\,\mathrm{Im}F_{IJ}, \tag{1.132}$$

where we stress that all the derivatives of F and Ω are taken at $Y^I = \mathcal{Y}^I$ and $\bar{Y}^I = \bar{\mathcal{Y}}^I$.

Furthermore, we obtain the following expression for the Hesse potential (1.123),

$$\begin{aligned}\mathcal{H}(\mathcal{Y},\bar{\mathcal{Y}}) =& -\mathrm{i}[\bar{\mathcal{Y}}^I F_I(\mathcal{Y}) - \mathcal{Y}^I \bar{F}_I(\bar{\mathcal{Y}})] + 4\,\Omega(\mathcal{Y},\bar{\mathcal{Y}}) \\ &-\mathrm{i}\Big[\mathcal{Y}^I(F_I(Y) - F_I(\mathcal{Y})) + \Delta Y^I F_I(Y) - \text{h.c.}\Big] \\ &+4\Big[\Omega(Y,\bar{Y}) - \Omega(\mathcal{Y},\bar{\mathcal{Y}}) + \Delta Y^I\left(\Omega_I(Y,\bar{Y}) - \Omega_{\bar{I}}(Y,\bar{Y})\right)\Big].\end{aligned} \tag{1.133}$$

Here we made use of (1.130) at an intermediate stage of the calculation. Again this result must be Taylor expanded upon writing $Y^I = \mathcal{Y}^I - \Delta Y^I$ and $\bar{Y}^I = \bar{\mathcal{Y}}^I + \Delta Y^I$. The last two lines of (1.133) then lead to a power series in ΔY, starting at second order in the ΔY,

$$\begin{aligned}\mathcal{H}(\mathcal{Y},\bar{\mathcal{Y}}) =& -\mathrm{i}[\bar{\mathcal{Y}}^I F_I(\mathcal{Y}) - \mathcal{Y}^I \bar{F}_I(\bar{\mathcal{Y}})] + 4\,\Omega(\mathcal{Y},\bar{\mathcal{Y}}) \\ &- N_{IJ}\Delta Y^I\Delta Y^J - \tfrac{2}{3}\mathrm{i}(F+\bar{F})_{IJK}\Delta Y^I\Delta Y^J\Delta Y^K \\ &- 4\,\mathrm{Re}(\Omega_{IJ} - \Omega_{I\bar{J}})\Delta Y^I\Delta Y^J + \tfrac{1}{4}\mathrm{i}(F-\bar{F})_{IJKL}\Delta Y^I\Delta Y^J\Delta Y^K\Delta Y^L \\ &+ \tfrac{8}{3}\mathrm{i}\,\mathrm{Im}(\Omega_{IJK} - 3\Omega_{IJ\bar{K}})\Delta Y^I\Delta Y^J\Delta Y^K + \cdots.\end{aligned} \tag{1.134}$$

Inserting the result of the iteration (1.131) into the expression above leads to the following expression for the Hesse potential, up to terms of order Ω^5,

$$\begin{aligned}\mathcal{H}(\mathcal{Y},\bar{\mathcal{Y}}) = &-\mathrm{i}[\bar{\mathcal{Y}}^I F_I(\mathcal{Y}) - \mathcal{Y}^I \bar{F}_I(\bar{\mathcal{Y}})] + 4\,\Omega(\mathcal{Y},\bar{\mathcal{Y}}) \\ &- 4\,\hat{N}^{IJ}\omega_I\,\omega_J + \tfrac{8}{3}\mathrm{i}(F+\bar{F})_{IJK}\hat{N}^{IL}\hat{N}^{JM}\hat{N}^{KN}\omega_L\,\omega_M\,\omega_N \\ &- \tfrac{4}{3}\mathrm{i}[(F-\bar{F})_{IJKL} + 3\mathrm{i}(F+\bar{F})_{IJR}\hat{N}^{RS}(F+\bar{F})_{SKL}] \\ &\times \hat{N}^{IM}\hat{N}^{JN}\hat{N}^{KP}\hat{N}^{LQ}\omega_M\,\omega_N\,\omega_P\,\omega_Q \\ &- \tfrac{32}{3}\mathrm{i}\,\mathrm{Im}(\Omega_{IJK} - 3\Omega_{IJ\bar{K}})\hat{N}^{IL}\hat{N}^{JM}\hat{N}^{KN}\omega_L\,\omega_M\,\omega_N + \mathcal{O}(\Omega^5),\end{aligned} \tag{1.135}$$

where $\omega_I = \Omega_I - \Omega_{\bar{I}}$, and where we also made use of $\hat{N}^{IJ}$, which is the inverse of the real, symmetric matrix $\hat{N}_{IJ}$ given in (1.100), namely

$$\hat{N}_{IJ} = N_{IJ} + 4\,\mathrm{Re}(\Omega_{IJ} - \Omega_{I\bar{J}}). \tag{1.136}$$

Upon expanding $\hat{N}^{IJ}$ we straightforwardly determine the contributions to the Hesse potential up to fifth order in Ω,

$$\begin{aligned}\mathcal{H} = &\,\mathcal{H}|_{\Omega=0} + 4\,\Omega - 4\,N^{IJ}(\Omega_I\Omega_J + \Omega_{\bar{I}}\Omega_{\bar{J}}) + 8\,N^{IJ}\Omega_I\Omega_{\bar{J}} \\ &+ 16\,\mathrm{Re}(\Omega_{IJ} - \Omega_{I\bar{J}})N^{IK}N^{JL}\big(\Omega_K\Omega_L + \Omega_{\bar{K}}\Omega_{\bar{L}} - 2\,\Omega_K\Omega_{\bar{L}}\big) \\ &- \tfrac{16}{3}(F+\bar{F})_{IJK}N^{IL}N^{JM}N^{KN}\,\mathrm{Im}(\Omega_L\Omega_M\Omega_N - 3\,\Omega_L\Omega_M\Omega_{\bar{N}}) \\ &- 64N^{IP}\mathrm{Re}\left(\Omega_{PQ} - \Omega_{P\bar{Q}}\right)N^{QR}\mathrm{Re}\left(\Omega_{RK} - \Omega_{R\bar{K}}\right)N^{KJ} \\ &\times \left(\Omega_I\Omega_J + \Omega_{\bar{I}}\Omega_{\bar{J}} - 2\Omega_I\Omega_{\bar{J}}\right) \\ &+ 64(F+\bar{F})_{IJK}N^{IL}N^{JM}N^{KP}\,\mathrm{Re}\left(\Omega_{PQ} - \Omega_{P\bar{Q}}\right)N^{QN} \\ &\times \mathrm{Im}(\Omega_L\Omega_M\Omega_N - 3\,\Omega_L\Omega_M\Omega_{\bar{N}}) \\ &- \tfrac{8}{3}\mathrm{i}[(F-\bar{F})_{IJKL} + 3\mathrm{i}(F+\bar{F})_{R(IJ}N^{RS}(F+\bar{F})_{KL)S}]N^{IM}N^{JN}N^{KP}N^{LQ} \\ &\times \mathrm{Re}\left(\Omega_M\Omega_N\Omega_P\Omega_Q - 4\Omega_M\Omega_N\Omega_P\Omega_{\bar{Q}} + 3\Omega_M\Omega_N\Omega_{\bar{P}}\Omega_{\bar{Q}}\right) \\ &+ \tfrac{64}{3}\,\mathrm{Im}(\Omega_{IJK} - 3\Omega_{IJ\bar{K}})N^{IL}N^{JM}N^{KN}\,\mathrm{Im}(\Omega_L\Omega_M\Omega_N - 3\,\Omega_L\Omega_M\Omega_{\bar{N}}) \\ &+ \mathcal{O}(\Omega^5).\end{aligned} \tag{1.137}$$

We stress once more that this expression is taken at $Y^I = \mathcal{Y}^I$.

The expression (1.137) gives the Hesse potential in terms of the duality covariant variables $\mathcal{Y}^I$ and $\bar{\mathcal{Y}}^I$, up to order Ω^5. It takes a rather complicated form, even at this order of approximation. Nevertheless, it will turn out that there is some systematics here. First of all, the Hesse potential (1.137) transforms as a function under duality transformations acting on the fields $\mathcal{Y}^I$. This in turn enables one to determine how Ω should transform. Clearly, when $\Omega = 0$ the Hesse potential transforms manifestly as a function. In general the transformation behaviour of Ω must be rather complicated in view of the non-linear dependence of the Hesse potential on Ω. To evaluate this transformation, we have to perform yet another iteration procedure.

To demonstrate how this iteration proceeds, let us have a look at the first few steps. Consider the expression (1.137) at first order in Ω. At this order, Ω must transform as a function, since both $\mathcal{H}$ and $\mathcal{H}|_{\Omega=0}$ transform as functions. This implies that

$$\begin{aligned} \tilde{\Omega}(\tilde{\mathcal{Y}}, \bar{\tilde{\mathcal{Y}}}) &= \Omega(\mathcal{Y}, \bar{\mathcal{Y}}) \,, \\ \tilde{\Omega}_I(\tilde{\mathcal{Y}}, \bar{\tilde{\mathcal{Y}}}) &= [\mathcal{S}_0^{-1}]^J{}_I(\mathcal{Y})\, \Omega_J(\mathcal{Y}, \bar{\mathcal{Y}}). \end{aligned} \tag{1.138}$$

Now consider the terms of order Ω^2 in (1.137). Applying the transformation given in the second line of (1.138) to these terms and demanding $\mathcal{H}$ to transform as a function, shows that the result given in the first line of (1.138) must be modified to

$$\tilde{\Omega} = \Omega - \mathrm{i}\big(\mathcal{Z}_0^{IJ}\, \Omega_I \Omega_J - \bar{\mathcal{Z}}_0^{\bar{I}\bar{J}}\, \Omega_{\bar{I}} \Omega_{\bar{J}}\big) + \mathcal{O}(\Omega^3), \tag{1.139}$$

which in turn gives rise to the following result for derivatives of Ω,

$$\begin{aligned} \tilde{\Omega}_I &= [\mathcal{S}_0^{-1}]^J{}_I \Big[\Omega_J + \mathrm{i} F_{JKL}\, \mathcal{Z}_0^{KM} \Omega_M\, \mathcal{Z}_0^{LN} \Omega_N - 2\mathrm{i}\Omega_{JK} \mathcal{Z}_0^{KL} \Omega_L \\ &\qquad + 2\mathrm{i}\Omega_{J\bar{K}} \bar{\mathcal{Z}}_0^{\bar{K}\bar{L}} \Omega_{\bar{L}}\Big] + \mathcal{O}(\Omega^3), \\ \tilde{\Omega}_{IJ} &= [\mathcal{S}_0^{-1}]^K{}_I [\mathcal{S}_0^{-1}]^L{}_J \Big[\Omega_{KL} - F_{KLM}\, \mathcal{Z}_0^{MN}\, \Omega_N\Big] + \mathcal{O}(\Omega^2), \\ \tilde{\Omega}_{I\bar{J}} &= [\mathcal{S}_0^{-1}]^K{}_I [\bar{\mathcal{S}}_0^{-1}]^{\bar{L}}{}_{\bar{J}}\, \Omega_{K\bar{L}} + \mathcal{O}(\Omega^2), \end{aligned} \tag{1.140}$$

where the symmetric matrix $\mathcal{Z}_0^{IJ}$ is defined by[5]

$$\mathcal{Z}_0^{IJ} = [\mathcal{S}_0^{-1}]^I{}_K\, Z^{KJ}. \tag{1.141}$$

Here we made use of the relations,

$$\begin{aligned} [\mathcal{S}_0^{-1}]^I{}_K\, [\bar{\mathcal{S}}_0]^{\bar{K}}{}_{\bar{J}} &= \delta^I{}_J - \mathrm{i}\mathcal{Z}_0^{IK} N_{KJ}, \\ \tilde{N}_{IJ} &= [\mathcal{S}_0^{-1}]^K{}_I [\bar{\mathcal{S}}_0^{-1}]^{\bar{L}}{}_{\bar{J}}\, N_{KL}, \\ \delta\mathcal{Z}_0^{IJ} &= -\,\mathcal{Z}_0^{IK}\, \delta F_{KL}\, \mathcal{Z}_0^{LJ}, \end{aligned} \tag{1.142}$$

which are independent of Ω.

This iteration can be continued by including the terms of order Ω^3, making use of (1.140) for derivatives of Ω, to obtain the expression for $\tilde{\Omega}$ up terms of order Ω^4. In the next iterative step one then derives the effect of a duality transformation on Ω

[5] This quantity was first defined in [5]. It appeared later in [25], where it was denoted by Δ.

up to terms of order Ω^5. Before presenting this result, we wish to observe that terms transforming as a proper function under duality, will not contribute to this result. This is precisely what already happened to the Ω-independent contribution to the Hesse potential, which decouples from the above equations. As it turns out there actually exists an infinite set of contributions to the Hesse potential that transform as functions under duality. By separating those from (1.137), we do not change the transformation behaviour of Ω, but we can extract certain functions from the Hesse potential in order to simplify its structure. We obtain

$$\begin{aligned}\mathcal{H} =& \mathcal{H}^{(0)} + \mathcal{H}^{(1)} + \mathcal{H}^{(2)} + \big(\mathcal{H}_1^{(3)} + \mathcal{H}_2^{(3)} + \text{h.c.}\big) + \mathcal{H}_3^{(3)} + \mathcal{H}_1^{(4)} + \mathcal{H}_2^{(4)} + \mathcal{H}_3^{(4)} \\ &+ \big(\mathcal{H}_4^{(4)} + \mathcal{H}_5^{(4)} + \mathcal{H}_6^{(4)} + \mathcal{H}_7^{(4)} + \mathcal{H}_8^{(4)} + \mathcal{H}_9^{(4)} + \text{h.c.}\big) \ldots ,\end{aligned} \tag{1.143}$$

where the $\mathcal{H}_i^{(a)}$ are certain expressions to be defined below, whose leading term is of order Ω^a. For higher values of a it turns out that there exists more than one functions with the same value of a, and those will be labeled by $i = 1, 2, \ldots$. Of all the combinations $\mathcal{H}_i^{(a)}$ appearing in (1.143), $\mathcal{H}^{(1)}$ is the only that contains Ω, while all the other combinations contain derivatives of Ω. Obviously, $\mathcal{H}^{(0)}$ equals,

$$\mathcal{H}^{(0)} = -\mathrm{i}[\bar{\mathcal{Y}}^I F_I(\mathcal{Y}) - \mathcal{Y}^I \bar{F}_I(\bar{\mathcal{Y}})], \tag{1.144}$$

whereas $\mathcal{H}^{(1)}$ at this level of iteration is given by,

$$\begin{aligned}\mathcal{H}^{(1)} =& 4\,\Omega - 4\,N^{IJ}(\Omega_I\Omega_J + \Omega_{\bar{I}}\Omega_{\bar{J}}) \\ &+ 16\,\mathrm{Re}\Big[(\Omega_{IJ})(N\Omega)^I(N\Omega)^J\Big] + 16\Omega_{I\bar{J}}\,(N\Omega)^I(N\bar{\Omega})^J \\ &- \tfrac{16}{3}\,\mathrm{Im}\Big[F_{IJK}(N\Omega)^I(N\Omega)^J(N\Omega)^K\Big] \\ &- \tfrac{4}{3}\mathrm{i}\Big[\Big(F_{IJKL} + 3\mathrm{i}F_{R(IJ}N^{RS}F_{KL)S}\Big)(N\Omega)^I(N\Omega)^J(N\Omega)^K(N\Omega)^L - \text{h.c.}\Big] \\ &- \tfrac{16}{3}\Big[\Omega_{IJK}(N\Omega)^I(N\Omega)^J(N\Omega)^K + \text{h.c.}\Big] \\ &- 16\Big[\Omega_{IJ\bar{K}}(N\Omega)^I(N\Omega)^J(N\bar{\Omega})^K + \text{h.c.}\Big] \\ &- 16\mathrm{i}\Big[F_{IJK}N^{KP}\,\Omega_{PQ}(N\Omega)^I(N\Omega)^J(N\Omega)^Q - \text{h.c.}\Big] \\ &- 16\Big[(N\Omega)^P\,\Omega_{PQ}\,N^{QR}\Omega_{RK}\,(N\Omega)^K + \text{h.c.}\Big] \\ &- 32\Big[(N\Omega)^P\,\Omega_{PQ}\,N^{QR}\Omega_{R\bar{K}}\,(N\bar{\Omega})^K + \text{h.c.}\Big] \\ &- 16\Big[(N\Omega)^P\,\Omega_{P\bar{Q}}\,N^{QR}\Omega_{\bar{R}K}\,(N\Omega)^K + \text{h.c.}\Big] \\ &- 16\mathrm{i}\Big[F_{IJK}N^{KP}\,\Omega_{P\bar{Q}}(N\Omega)^I(N\Omega)^J(N\bar{\Omega})^Q - \text{h.c.}\Big] + \mathcal{O}(\Omega^5).\end{aligned} \tag{1.145}$$

Here we have used the notation $(N\Omega)^I = N^{IJ}\Omega_J$, $(N\bar{\Omega})^I = N^{IJ}\Omega_{\bar{J}}$. The symmetrization $F_{R(IJ}N^{RS}F_{KL)S}$ is defined with a symmetrization factor $1/(4!)$.

The expressions for the higher-order functions $\mathcal{H}_i^{(a)}$ with $a = 2, 3, 4$ are given in appendix D. Each of these higher-order functions transforms as a function under symplectic transformations. Demanding $\mathcal{H}^{(1)}$ to also transform as a function under these transformations determines the transformation behavior of Ω. Proceeding as already explained below (1.138) we obtain for the transformation law of Ω (up to order Ω^5),

$$
\begin{aligned}
\tilde{\Omega} = \Omega & - \mathrm{i}\big(\mathcal{Z}_0^{IJ}\,\Omega_I\Omega_J - \bar{\mathcal{Z}}_0^{\bar{I}\bar{J}}\,\Omega_{\bar{I}}\Omega_{\bar{J}}\big) \\
& + \tfrac{2}{3}\big(F_{IJK}\,\mathcal{Z}_0^{IL}\Omega_L\,\mathcal{Z}_0^{JM}\Omega_M\,\mathcal{Z}_0^{KN}\Omega_N + \text{h.c.}\big) \\
& - 2\big(\Omega_{IJ}\,\mathcal{Z}_0^{IK}\Omega_K\mathcal{Z}_0^{JL}\Omega_L + \text{h.c.}\big) + 4\,\Omega_{I\bar{J}}\,\mathcal{Z}_0^{IK}\Omega_K\,\bar{\mathcal{Z}}_0^{\bar{J}\bar{L}}\Omega_{\bar{L}} \\
& + \Big[- \tfrac{\mathrm{i}}{3}F_{IJKL}(\mathcal{Z}_0\Omega)^I(\mathcal{Z}_0\Omega)^J(\mathcal{Z}_0\Omega)^K(\mathcal{Z}_0\Omega)^L \\
& + \tfrac{4\mathrm{i}}{3}\Omega_{IJK}(\mathcal{Z}_0\Omega)^I(\mathcal{Z}_0\Omega)^J(\mathcal{Z}_0\Omega)^K \\
& + \mathrm{i}\,F_{IJR}\,\mathcal{Z}_0^{RS}\,F_{SKL}\,(\mathcal{Z}_0\Omega)^I(\mathcal{Z}_0\Omega)^J(\mathcal{Z}_0\Omega)^K(\mathcal{Z}_0\Omega)^L \\
& - 4\mathrm{i}\Omega_{IJ\bar{K}}\,(\mathcal{Z}_0\Omega)^I\,(\mathcal{Z}_0\Omega)^J(\bar{\mathcal{Z}}_0\bar{\Omega})^K \\
& - 4\mathrm{i}\,F_{IJK}\mathcal{Z}_0^{KP}\,\Omega_{PQ}\,(\mathcal{Z}_0\Omega)^I(\mathcal{Z}_0\Omega)^J\,(\mathcal{Z}_0\Omega)^Q \\
& + 4\mathrm{i}\,F_{IJK}\mathcal{Z}_0^{KP}\,\Omega_{P\bar{Q}}(\mathcal{Z}_0\Omega)^I(\mathcal{Z}_0\Omega)^J(\bar{\mathcal{Z}}_0\bar{\Omega})^Q \\
& + 4\mathrm{i}\,(\mathcal{Z}_0\Omega)^P\,\Omega_{PQ}\,\mathcal{Z}_0^{QR}\Big(\Omega_{RK}\,(\mathcal{Z}_0\Omega)^K - 2\Omega_{R\bar{K}}\,(\bar{\mathcal{Z}}_0\bar{\Omega})^K\Big) \\
& - 4\mathrm{i}\,(\mathcal{Z}_0\Omega)^P\,\Omega_{P\bar{Q}}\,\bar{\mathcal{Z}}_0^{\bar{Q}\bar{R}}\Omega_{\bar{R}K}\,(\mathcal{Z}_0\Omega)^K + \text{h.c.}\Big] + \mathcal{O}(\Omega^5).
\end{aligned} \tag{1.146}
$$

The transformation laws of the derivatives of Ω, such as those in (1.140), are summarized in appendix E.

The transformation law (1.146), which is entirely encoded in $\mathcal{Z}_0$ and in $\bar{\mathcal{Z}}_0$, suggest a systematic pattern, which we now explore. First we observe that (1.146) simplifies when taking Ω to be harmonic both in $\mathcal{Y}^I$ and Υ,

$$
\Omega(\mathcal{Y}, \bar{\mathcal{Y}}; \Upsilon, \bar{\Upsilon}) = f(\mathcal{Y}, \Upsilon) + \text{h.c.}. \tag{1.147}
$$

We obtain

$$
\begin{aligned}
\tilde{\Omega} = \Omega + \Big[& - \mathrm{i}\mathcal{Z}_0^{IJ}\,\Omega_I\Omega_J \\
& + \tfrac{2}{3}F_{IJK}\,\mathcal{Z}_0^{IL}\Omega_L\,\mathcal{Z}_0^{JM}\Omega_M\,\mathcal{Z}_0^{KN}\Omega_N \\
& - 2\,\Omega_{IJ}\,\mathcal{Z}_0^{IK}\Omega_K\mathcal{Z}_0^{JL}\Omega_L \\
& - \tfrac{\mathrm{i}}{3}F_{IJKL}(\mathcal{Z}_0\Omega)^I(\mathcal{Z}_0\Omega)^J(\mathcal{Z}_0\Omega)^K(\mathcal{Z}_0\Omega)^L \\
& + \tfrac{4\mathrm{i}}{3}\,\Omega_{IJK}(\mathcal{Z}_0\Omega)^I(\mathcal{Z}_0\Omega)^J(\mathcal{Z}_0\Omega)^K \\
& + \mathrm{i}\,F_{IJR}\,\mathcal{Z}_0^{RS}\,F_{SKL}\,(\mathcal{Z}_0\Omega)^I(\mathcal{Z}_0\Omega)^J(\mathcal{Z}_0\Omega)^K(\mathcal{Z}_0\Omega)^L
\end{aligned}
$$

$$
\begin{aligned}
&- 4\mathrm{i}\, F_{IJK}\, \mathcal{Z}_0^{KP}\, \Omega_{PQ}\, (\mathcal{Z}_0\Omega)^I (\mathcal{Z}_0\Omega)^J\, (\mathcal{Z}_0\Omega)^Q \\
&+ 4\mathrm{i}\, \mathcal{Z}_0^{IP}\, \Omega_{PQ}\, \mathcal{Z}_0^{QR}\, \Omega_{RK}\, (\mathcal{Z}_0\Omega)^K\, \Omega_I + \text{h.c.}\Big] + \mathcal{O}(\Omega^5),
\end{aligned} \tag{1.148}
$$

which shows that $\tilde{\Omega}$ also is harmonic. Hence, the harmonicity of Ω is preserved under symplectic transformations. The transformation law (1.148) has a certain resemblance with the one encountered in the context of perturbative topological string theory, where $\mathcal{Z}_0^{IJ}$ plays the role of a propagator [25]. The relation with topological string theory will be discussed below. Next, inserting (1.147) into (1.145), we find that $\mathcal{H}^{(1)}$ is also almost harmonic, i.e. it equals the real part of a function that contains only purely holomorphic derivatives of F and Ω, contracted with the non-holomorphic tensor N^{IJ},

$$
\begin{aligned}
\mathcal{H}^{(1)} = \Big[&4\, f(\mathcal{Y}, \Upsilon) - 4\, N^{IJ}\, \Omega_I \Omega_J \\
&+ 8(\Omega_{IJ})(N\Omega)^I (N\Omega)^J + \tfrac{8}{3}\mathrm{i}\, F_{IJK}(N\Omega)^I (N\Omega)^J (N\Omega)^K \\
&- \tfrac{4}{3}\mathrm{i}\left(F_{IJKL} + 3\mathrm{i} F_{R(IJ} N^{RS} F_{KL)S}\right)(N\Omega)^I (N\Omega)^J (N\Omega)^K (N\Omega)^L \\
&- \tfrac{16}{3}\Omega_{IJK}(N\Omega)^I (N\Omega)^J (N\Omega)^K \\
&- 16\mathrm{i}\, F_{IJK} N^{KP}\, \Omega_{PQ}(N\Omega)^I (N\Omega)^J (N\Omega)^Q \\
&- 16(N\Omega)^P\, \Omega_{PQ}\, N^{QR} \Omega_{RK}\, (N\Omega)^K + \text{h.c.}\Big] + \mathcal{O}(\Omega^5).
\end{aligned} \tag{1.149}
$$

Thus, when Ω is of the form (1.147), $\mathcal{H}^{(1)}$ is given in terms of the real part of a function that is holomorphic in Υ. Moreover, since N^{IJ} is homogeneous of degree zero, this function is homogeneous of degree two in $\mathcal{Y}^I$ and homogeneous of degree zero in $\bar{\mathcal{Y}}^I$.

Let us now elucidate the relation of $\mathcal{H}^{(1)}$ given in (1.149) with topological string theory. We write $\mathcal{H}^{(1)}$ as

$$
\mathcal{H}^{(1)} = h(\mathcal{Y}, \bar{\mathcal{Y}}, \Upsilon) + \text{h.c.}, \tag{1.150}
$$

and we consider two expansions of $h(\mathcal{Y}, \bar{\mathcal{Y}}, \Upsilon)$, namely one in powers of Ω and the other one in powers of Υ. First we consider the expansion in powers of Ω. Expanding h as

$$
h = \sum_{g=1}^{\infty} h^{(g)} \tag{1.151}
$$

and comparing with (1.149), we obtain

$$
\begin{aligned}
h^{(1)} &= 4 f, \quad h^{(2)} = -4 N^{IJ}\, \Omega_I \Omega_J, \\
h^{(3)} &= 8\Omega_{IJ}(N\Omega)^I (N\Omega)^J + \tfrac{8}{3}\mathrm{i} F_{IJK}(N\Omega)^I (N\Omega)^J (N\Omega)^K, \\
h^{(4)} &= -\tfrac{4}{3}\mathrm{i}\left(F_{IJKL} + 3\mathrm{i} F_{R(IJ} N^{RS} F_{KL)S}\right)(N\Omega)^I (N\Omega)^J (N\Omega)^K (N\Omega)^L
\end{aligned}
$$

$$
\begin{aligned}
&- \tfrac{16}{3}\Omega_{IJK}(N\Omega)^I (N\Omega)^J (N\Omega)^K \\
&- 16\mathrm{i}\, F_{IJK} N^{KP}\, \Omega_{PQ}(N\Omega)^I (N\Omega)^J (N\Omega)^Q \\
&- 16(N\Omega)^P\, \Omega_{PQ}\, N^{QR}\Omega_{RK}\, (N\Omega)^K ,
\end{aligned} \tag{1.152}
$$

where $(N\Omega)^I = N^{IJ} f_J$. This shows that all the $h^{(g)}$ are non-holomorphic in $\mathcal{Y}^I$ with the exception of $h^{(1)}$. Using these expressions, one finds by direct calculation that the following relation holds,

$$
\partial_{\bar{I}} h^{(g)} = \tfrac{1}{4}\mathrm{i}\, \bar{F}_{\bar{I}}{}^{JK} \sum_{r=1}^{g-1} \partial_J h^{(r)}\, \partial_K h^{(g-r)}, \quad g \geqslant 2, \tag{1.153}
$$

where $\bar{F}_{\bar{I}}{}^{JK} = \bar{F}_{\bar{I}\bar{L}\bar{M}} N^{LJ} N^{MK}$.

Exercise 17 Verify (1.153) for $g = 2, 3$.

Equation (1.153) captures the $\bar{\mathcal{Y}}^I$-dependence of $h^{(g)}$ (for $g \geqslant 2$). This dependence is a consequence of requiring $\mathcal{H}^{(1)}$ to have a proper behavior under symplectic transformations [5]. The differential Eq. (1.153) resembles the holomorphic anomaly equation of perturbative topological string theory. The latter arises in a specific setting, namely in the study of the non-holomorphicity of the genus-g topological free energies $F^{(g)}$ [36]. To exhibit the relation with the holomorphic anomaly equation, we turn to the second expansion and expand both $f(\mathcal{Y}, \Upsilon)$ and $h(\mathcal{Y}, \bar{\mathcal{Y}}, \Upsilon)$ in powers of Υ,

$$
\begin{aligned}
f(\mathcal{Y}, \Upsilon) &= -\tfrac{1}{2}\mathrm{i} \sum_{g=1}^{\infty} \Upsilon^g\, f^{(g)}(\mathcal{Y}), \\
h(\mathcal{Y}, \bar{\mathcal{Y}}, \Upsilon) &= -2\mathrm{i} \sum_{g=1}^{\infty} \Upsilon^g\, F^{(g)}(\mathcal{Y}, \bar{\mathcal{Y}}).
\end{aligned} \tag{1.154}
$$

Then we obtain

$$
\begin{aligned}
F^{(1)}(\mathcal{Y}) &= f^{(1)}(\mathcal{Y}), \quad F^{(2)}(\mathcal{Y}, \bar{\mathcal{Y}}) = f^{(2)}(\mathcal{Y}) + \tfrac{1}{2}\mathrm{i}\, N^{IJ} F^{(1)}_I F^{(1)}_J , \\
F^{(3)}(\mathcal{Y}, \bar{\mathcal{Y}}) &= f^{(3)}(\mathcal{Y}) + \mathrm{i}\, N^{IJ} f^{(2)}_I F^{(1)}_J - \tfrac{1}{2} F^{(1)}_{IJ} (NF^{(1)})^I (NF^{(1)})^J \\
&\quad - \tfrac{1}{6}\mathrm{i} F_{IJK} (NF^{(1)})^I (NF^{(1)})^J (NF^{(1)})^K , \\
F^{(4)}(\mathcal{Y}, \bar{\mathcal{Y}}) &= f^{(4)}(\mathcal{Y}) + \mathrm{i}\, N^{IJ} f^{(3)}_I F^{(1)}_J + \tfrac{1}{2}\mathrm{i}\, N^{IJ} f^{(2)}_I f^{(2)}_J \\
&\quad - \tfrac{1}{2} f^{(2)}_{IJ} (NF^{(1)})^I (NF^{(1)})^J - F^{(1)}_{IJ} (Nf^{(2)})^I (NF^{(1)})^J \\
&\quad - \tfrac{1}{2}\mathrm{i} F_{IJK} (Nf^{(2)})^I (NF^{(1)})^J (NF^{(1)})^K \\
&\quad + \tfrac{1}{24} \Big(F_{IJKL} + 3\mathrm{i} F_{R(IJ} N^{RS} F_{KL)S} \Big) (NF^{(1)})^I (NF^{(1)})^J \\
&\qquad \times (NF^{(1)})^K (NF^{(1)})^L
\end{aligned}
$$

$$
\begin{aligned}
&- \tfrac{1}{6}\mathrm{i}\, F^{(1)}_{IJK} (NF^{(1)})^I (NF^{(1)})^J (NF^{(1)})^K \\
&+ \tfrac{1}{2}\, F_{IJK} N^{KP}\, F^{(1)}_{PQ} (NF^{(1)})^I (NF^{(1)})^J (NF^{(1)})^Q \\
&- \tfrac{1}{2}\mathrm{i}\, (NF^{(1)})^P\, F^{(1)}_{PQ}\, N^{QR} F^{(1)}_{RK}\, (NF^{(1)})^K ,
\end{aligned}
\qquad (1.155)
$$

where $(NF^{(1)})^I = N^{IJ}\, F^{(1)}_J$ and $(Nf^{(2)})^I = N^{IJ}\, f^{(2)}_J$. Observe that all the $F^{(g)}$ are non-holomorphic except $F^{(1)}$. Using these expressions, one again finds by direct calculation,

$$
\partial_{\bar I} F^{(g)} = \tfrac{1}{2}\, \bar F_{\bar I}{}^{JK} \sum_{r=1}^{g-1} \partial_J F^{(r)}\, \partial_K F^{(g-r)} , \qquad g \geqslant 2. \qquad (1.156)
$$

This is similar to (1.153), except that now the relation holds order by order in Υ, whereas (1.153) holds order by order in Ω. Both expansions are, nevertheless, related. Namely, taking f in (1.154) to consist of only $f^{(1)}$, the expansion (1.155) coincides with the expansion (1.152).

Summarizing, we have found the following. When expressing the Hesse potential, which is a symplectic function, in terms of the duality covariant complex variables (1.127), we obtain an infinite set of contributions $\mathcal{H}^{(a)}_i$, all of which transform as functions under symplectic transformations. One of them, namely $\mathcal{H}^{(1)}$, has a structure that arises in topological string theory. $\mathcal{H}^{(1)}$ is the only contribution that contains Ω, while all the other combinations contain derivatives of Ω. When Ω is taken to be harmonic in all the variables (i.e. in both $\mathcal{Y}^I$ and Υ), the resulting $\mathcal{H}^{(1)}$ is given in terms of the real part of a function that is holomorphic in Υ, homogeneous of degree two in $\mathcal{Y}^I$ and homogeneous of degree zero in $\bar{\mathcal{Y}}^I$. Then, expanding $\mathcal{H}^{(1)}$ in powers of Υ yields expansion functions $F^{(g)}$, given in (1.155), that transform as functions under symplectic transformations. The $F^{(g)}$ are all non-holomorphic, with the exception of $F^{(1)}$, and the non-holomorphicity is governed by (1.156). This differential equation equals half of the holomorphic anomaly equation of perturbative topological string theory, which reads [38]

$$
\partial_{\bar I} F^{(g)} = \tfrac{1}{2}\, \bar F_{\bar I}{}^{JK} \left(D_J \partial_K F^{(g-1)} + \sum_{r=1}^{g-1} \partial_J F^{(r)}\, \partial_K F^{(g-r)} \right), \quad g \geqslant 2, \qquad (1.157)
$$

where $D_L V_M = \partial_L V_M + \mathrm{i} N^{PI} F_{ILM} V_P$. This is the holomorphic anomaly equation in the so-called big moduli space [38], and its derivation is reviewed in appendix C following [25]. In the context of topological string theory, the $F^{(g)}$ denote free energies that arise in the perturbative expansion of the topological free energy $F_{\rm top}$ in powers of the topological string coupling $g_{\rm top}$, i.e. $F_{\rm top} = \sum_{g=0}^{\infty} g_{\rm top}^{2g-2}\, F^{(g)}$. Whereas $F^{(0)}$ is holomorphic (it only depends on $\mathcal{Y}$), all the higher $F^{(g)}$ (with $g \geqslant 1$) are non-holomophic. For $g \geqslant 2$ this non-holomorphicity is captured by (1.157).

The fact that the first term on the right hand side of (1.157) is missing in (1.156) is due to the holomorphic nature of the expansion function $F^{(1)}$ appearing in (1.155).

Were it to be non-holomorphic, it would induce a modification of the relation (1.156). The required modification arises by replacing the holomorphic quantity $F_I^{(1)} = f_I^{(1)}$ with the non-holomorphic combination $F_I^{(1)} = f_I^{(1)} + \frac{1}{2}\,\mathrm{i}\,F_{IJK}\,N^{JK}$ (see (1.206)). This will be addressed in an upcoming paper.

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A Symplectic Reparametrizations

In Sect. 1.2.1 we introduced the $2n$-vector (x^i, F_i) and discussed its behavior under symplectic transformations. Here we consider derivatives of F_i and show how they transform under symplectic transformations. We use the resulting expressions to give an alternative proof of integrability of the Eq. (1.10). In addition, we show that $\partial_\eta F$ transforms as a function under symplectic transformations.

We begin by recalling some of the elements of Sect. 1.2.1. The $2n$-vector (x^i, F_i) is constructed using $F(x,\bar{x}) = F^{(0)}(x) + 2\mathrm{i}\Omega(x,\bar{x})$. Under symplectic transformations, it transforms as,

$$\begin{aligned}\tilde{x}^i &= U^i{}_j\, x^j + Z^{ij}[F_j^{(0)}(x) + 2\mathrm{i}\Omega_j(x,\bar{x})],\\ \tilde{F}_i(\tilde{x},\bar{\tilde{x}}) &= V_i{}^j\,[F_j^{(0)}(x) + 2\mathrm{i}\Omega_j(x,\bar{x})] + W_{ij}\,x^j,\end{aligned} \tag{1.158}$$

where U, V, Z and W are the $n\times n$ submatrices (1.5) that define a symplectic transformation belonging to $\mathrm{Sp}(2n,\mathbb{R})$. Without loss of generality, we decompose $\tilde{F}_i$ as

$$\tilde{F}_i(\tilde{x},\bar{\tilde{x}}) = \tilde{F}_i^{(0)}(\tilde{x}) + 2\mathrm{i}\tilde{\Omega}_i(\tilde{x},\bar{\tilde{x}}). \tag{1.159}$$

This decomposition, which a priori is arbitrary, can be related to the decomposition of $F_i = F_i^{(0)} + 2\mathrm{i}\Omega_i$ in the following way. The symplectic transformation (1.158) is specified by the matrices U, V, W and Z. Consider applying the same transformation (specified by these matrices) to the vector $(x^i, F_i^{(0)})$ alone. This yields the vector $(\hat{x}^i, \tilde{F}_i^{(0)}(\hat{x}))$, which is expressed in terms of $\hat{x}^i = \tilde{x}^i - 2\mathrm{i}Z^{ij}\,\Omega_j(x,\bar{x})$ instead of $\tilde{x}^i$,

$$
\begin{aligned}
\hat{x}^i &= U^i{}_j\, x^j + Z^{ij}\, F^{(0)}_j(x), \\
\tilde{F}^{(0)}_i(\hat{x}) &= V_i{}^j\, F^{(0)}_j(x) + W_{ij}\, x^j.
\end{aligned} \tag{1.160}
$$

Thus, by demanding that $\tilde{F}^{(0)}_i$ follows from the same symplectic transformation applied on $F^{(0)}_i$ alone, we relate the decomposition of $\tilde{F}_i$ to the decomposition of F_i. Then, the second equation of (1.158) can be written as

$$
\begin{aligned}
\tilde{\Omega}_i(\tilde{x}, \bar{\tilde{x}}) &= V_i{}^j\, \Omega_j(x, \bar{x}) - \tfrac{1}{2}\mathrm{i}[\tilde{F}^{(0)}_i(\hat{x}) - \tilde{F}^{(0)}_i(\tilde{x})] \qquad (1.161)\\
&= V_i{}^j\, \Omega_j(x, \bar{x}) \\
&\quad + \tfrac{1}{2}\mathrm{i} \sum_{m=1}^{\infty} \frac{(2\mathrm{i})^m}{m!} Z^{j_1 k_1}\, \Omega_{k_1}(x, \bar{x}) \cdots Z^{j_m k_m}\, \Omega_{k_m}(x, \bar{x})\; \tilde{F}^{(0)}_{i j_1 \cdots j_m}(\hat{x}),
\end{aligned}
$$

where the $\tilde{F}^{(0)}_{ij_1\cdots j_m}(\hat{x})$ denote multiple derivatives of $\tilde{F}^{(0)}_i(\tilde{x})$ evaluated at $\hat{x}$. The right-hand side of (1.161) can be written entirely in terms of functions of x and $\bar{x}$, upon expressing $\tilde{F}^{(0)}_{ij_1\cdots j_m}(\hat{x})$ in terms of derivatives of $F^{(0)}_i(x)$ using (1.160). We give the first few derivatives,

$$
\begin{aligned}
\tilde{F}^{(0)}_{ij}(\hat{x}) &= (V_i{}^l\, F^{(0)}_{lk} + W_{ik})\, [\mathcal{S}_0^{-1}]^k{}_j, \qquad (1.162)\\
\tilde{F}^{(0)}_{ijk}(\hat{x}) &= [\mathcal{S}_0^{-1}]^l{}_i\, [\mathcal{S}_0^{-1}]^m{}_j\, [\mathcal{S}_0^{-1}]^n{}_k\, F^{(0)}_{lmn}, \\
\tilde{F}^{(0)}_{ijkl}(\hat{x}) &= [\mathcal{S}_0^{-1}]^m{}_i\, [\mathcal{S}_0^{-1}]^n{}_j\, [\mathcal{S}_0^{-1}]^p{}_k\, [\mathcal{S}_0^{-1}]^q{}_l \Big[F^{(0)}_{mnpq} - 3\, F^{(0)}_{r(mn} \mathcal{Z}_0^{rs}\, F^{(0)}_{pq)s} \Big],
\end{aligned}
$$

where we used the definitions

$$
\begin{aligned}
\mathcal{S}_0{}^i{}_j &= U^i{}_j + Z^{ik} F^{(0)}_{kj}, \\
\mathcal{Z}_0^{ij} &= [\mathcal{S}_0^{-1}]^i{}_k\, Z^{kj}.
\end{aligned} \tag{1.163}
$$

Let us consider the first expression of (1.162). While $F^{(0)}_{ij}$ is manifestly symmetric in i, j, this appears not to be the case for $\tilde{F}^{(0)}_{ij}$. However, using the properties (1.5) of the matrices U, V, W and Z, it follows that $\tilde{F}^{(0)}_{ij}$ is symmetric in i, j. Using this, we obtain

$$
\tilde{F}^{(0)}_{ij}(\hat{x})\, Z^{jk} = V_i{}^k - [\mathcal{S}_0^{-1,T}]_i{}^k. \tag{1.164}
$$

Exercise 18 Verify (1.164) by computing $V^T \mathcal{S}_0$.

The symmetry of $\tilde{F}^{(0)}_{ij}$ implies that $\tilde{F}^{(0)}_i(\hat{x})$ can be integrated, i.e. $\tilde{F}^{(0)}_i(\hat{x}) = \partial \tilde{F}^{(0)}(\hat{x})/\partial \hat{x}^i$, with $\tilde{F}^{(0)}(\hat{x})$ given by the well-known expression [5],

$$\tilde{F}^{(0)}(\hat{x}) = F^{(0)}(x) - \tfrac{1}{2}x^i F_i^{(0)} + \tfrac{1}{2}(U^T W)_{ij}\, x^i x^j + \tfrac{1}{2}(U^T V + W^T Z)_i{}^j x^i F_j^{(0)} \\ + \tfrac{1}{2}(Z^T V)^{ij} F_i^{(0)} F_j^{(0)}, \tag{1.165}$$

up to a constant and up to terms linear in $\hat{x}^i$.

In addition to (1.163), we will also need the combinations $\mathcal{S}$ and $\hat{\mathcal{S}}$ given in (1.167) and (1.169) below, which are related to $\mathcal{S}_0$ by

$$\begin{aligned} \mathcal{S}^i{}_j &= \mathcal{S}_0{}^i{}_j + 2\mathrm{i} Z^{ik}\Omega_{kj}, \\ \hat{\mathcal{S}}^i{}_j &= \mathcal{S}_0{}^i{}_j + Z^{ik}\big[2\mathrm{i}\Omega_{kj} - 4\,\Omega_{k\bar{l}}\bar{\mathcal{Z}}^{\bar{l}\bar{m}}\Omega_{\bar{m}j}\big], \\ \mathcal{Z}^{ij} &= [\mathcal{S}^{-1}]^i{}_k\, Z^{kj}. \end{aligned} \tag{1.166}$$

Observe that the matrices $\mathcal{Z}_0$, $\mathcal{Z}$ and $\hat{\mathcal{Z}} = \hat{\mathcal{S}}^{-1} Z$ are symmetric matrices by virtue of the fact that ZU^T is a symmetric matrix [5].

Next we consider the transformation behavior of the derivatives $F_{ij} = \partial F_i/\partial x^j$ and $F_{i\bar{j}} = \partial F_i/\partial \bar{x}^{\bar{j}}$. First we observe that

$$\frac{\partial \tilde{x}^i}{\partial x^j} \equiv \mathcal{S}^i{}_j = U^i{}_j + Z^{ik} F_{kj}\,, \qquad \frac{\partial \tilde{x}^i}{\partial \bar{x}^{\bar{j}}} \equiv Z^{ik} F_{k\bar{j}}. \tag{1.167}$$

Applying the chain rule to (1.158) yields the relation

$$F_{ij} \to \tilde{F}_{ij} = \Big(V_i{}^l \hat{F}_{lk} + W_{ik}\Big)\, [\hat{\mathcal{S}}^{-1}]^k{}_j, \tag{1.168}$$

where $\tilde{F}_{ij} = \partial \tilde{F}_i/\partial \tilde{x}^j$ and

$$\begin{aligned} \hat{F}_{ij} &= F_{ij} - F_{i\bar{k}}\,\mathcal{Z}^{\bar{k}\bar{l}}\,\bar{F}_{\bar{l}j} = F_{ij}^{(0)} + 2\mathrm{i}\Omega_{ij} - 4\,\Omega_{i\bar{k}}\,\bar{\mathcal{Z}}^{\bar{k}\bar{l}}\,\Omega_{\bar{l}j}, \\ \hat{\mathcal{S}}^i{}_j &= U^i{}_j + Z^{ik}\hat{F}_{kj}. \end{aligned} \tag{1.169}$$

Exercise 19 Derive (1.168) by differentiating the second equation of (1.158) with respect to either x or $\bar{x}$. Then combine the two resulting equations to arrive at (1.168).

Then, using the first equation of (1.162) as well as (1.164) in (1.168) yields,

$$\begin{aligned} \tilde{\Omega}_{ij}(\tilde{x},\bar{\tilde{x}}) = {}& \tfrac{1}{2}\mathrm{i}\big[\tilde{F}_{ij}^{(0)}(\tilde{x}) - \tilde{F}_{ij}^{(0)}(\tilde{x}^k - 2\mathrm{i}Z^{kl}\Omega_l(x,\bar{x}))\big] \\ & + [\hat{\mathcal{S}}^{-1}]^k{}_i\,[\hat{\mathcal{S}}^{-1}]^l{}_j\Big[\Omega_{kl} + 2\mathrm{i}\Omega_{k\bar{m}}\bar{\mathcal{Z}}^{\bar{m}\bar{n}}\Omega_{\bar{n}l} \\ & + 2\mathrm{i}(\Omega_{km} + 2\mathrm{i}\Omega_{k\bar{p}}\bar{\mathcal{Z}}^{\bar{p}\bar{r}}\Omega_{\bar{r}m})\mathcal{Z}_0^{mn}(\Omega_{nl} + 2\mathrm{i}\Omega_{n\bar{q}}\bar{\mathcal{Z}}^{\bar{q}\bar{s}}\Omega_{\bar{s}l})\Big], \end{aligned} \tag{1.170}$$

which is symmetric by virtue of the symmetry of $\tilde{F}_{ij}^{(0)}$, Ω_{ij}, $\mathcal{Z}^{mn}$ and $\mathcal{Z}_0^{mn}$.

Subsequently we derive the following result from (1.161) [21],

$$\tilde{\Omega}_{i\bar{j}} = [\hat{\mathcal{S}}^{-1}]^k{}_i\, [\bar{\mathcal{S}}^{-1}]^{\bar{l}}{}_{\bar{j}}\, \Omega_{k\bar{l}} = [\mathcal{S}^{-1}]^k{}_i [\bar{\hat{\mathcal{S}}}^{-1}]^{\bar{l}}{}_{\bar{j}}\, \Omega_{k\bar{l}}. \tag{1.171}$$

Exercise 20 Deduce (1.171) by taking the first line of (1.161) and differentiating it with respect to $\bar{x}$. Use the relation (1.164) in the form

$$V_i{}^j = [\mathcal{S}_0^{-1,T}]_i{}^k + (V_i{}^l F^{(0)}_{lk} + W_{ik})\mathcal{Z}_0^{kj}, \tag{1.172}$$

together with (1.166).

The relation (1.171) establishes that $\tilde{\Omega}_{i\bar{j}} = \overline{(\tilde{\Omega}_{j\bar{i}})}$. Using this as well as (1.15), and recalling that $\tilde{\Omega}_{i\bar{j}} = \partial\tilde{\Omega}_i/\partial\bar{\tilde{x}}^{\bar{j}}$, we obtain $\tilde{\Omega}_{i\bar{j}} = \overline{(\tilde{\Omega}_{j\bar{i}})} = \overline{(\partial\tilde{\Omega}_j/\partial\bar{\tilde{x}}^{\bar{i}})} = \partial\overline{(\tilde{\Omega}_j)}/\partial\tilde{x}^i = \partial\tilde{\Omega}_{\bar{j}}/\partial\tilde{x}^i \equiv \tilde{\Omega}_{\bar{j}i}$. This, together with the symmetry of $\tilde{\Omega}_{ij}$, ensures the integrability of (1.158), as follows.

We consider the 1-form $\tilde{A} = \tilde{\Omega}_i\, d\tilde{x}^i + \tilde{\Omega}_{\bar{i}}\, d\bar{\tilde{x}}^{\bar{i}}$, which is real by virtue of $\tilde{\Omega}_{\bar{i}} = \overline{(\tilde{\Omega}_i)}$. Its field strength reads $\tilde{\mathfrak{F}} = d\tilde{A} = \tilde{\Omega}_{ij}\, d\tilde{x}^j \wedge d\tilde{x}^i + \left(\tilde{\Omega}_{i\bar{j}} - \tilde{\Omega}_{\bar{j}i}\right) d\bar{\tilde{x}}^{\bar{j}} \wedge d\tilde{x}^i + \tilde{\Omega}_{\bar{i}\bar{j}}\, d\bar{\tilde{x}}^{\bar{j}} \wedge d\bar{\tilde{x}}^{\bar{i}}$. Then, using $\tilde{\Omega}_{ij} = \tilde{\Omega}_{ji}$ as well as $\tilde{\Omega}_{i\bar{j}} = \tilde{\Omega}_{\bar{j}i}$, we conclude that $\tilde{\mathfrak{F}} = 0$, which establishes that locally $\tilde{A} = d\tilde{\Omega}$, with a real $\tilde{\Omega}$.

Hence we conclude that the Eq. (1.158) are integrable and the decomposition (1.7) is preserved, i.e. the transformed $2n$-vector $(\tilde{x}^i, \tilde{F}_i)$ is constructed from a new function $\tilde{F}(\tilde{x}, \bar{\tilde{x}}) = \tilde{F}^{(0)}(\tilde{x}) + 2\mathrm{i}\tilde{\Omega}(\tilde{x}, \bar{\tilde{x}})$ with a real $\tilde{\Omega}(\tilde{x}, \bar{\tilde{x}})$. This was established in Sect. 1.2.1 by relying on the Hamiltonian.

Next, let us assume that the function F depends on a auxiliary real parameter η that is inert under symplectic transformation, i.e. $F(x, \bar{x}; \eta)$, and let us consider partial derivatives with respect to it. A little calculation shows that $\partial_\eta F_i$ transforms in the following way,

$$\partial_\eta \tilde{F}_i = [\hat{\mathcal{S}}^{-1}]^j{}_i \left[\partial_\eta F_j - F_{j\bar{k}}\, \bar{\mathcal{Z}}^{\bar{k}\bar{l}}\, \partial_\eta \bar{F}_{\bar{l}}\right], \tag{1.173}$$

where $\tilde{x}$ and $\bar{\tilde{x}}$ are kept fixed under the η-derivative in $\partial_\eta \tilde{F}_i(\tilde{x}, \bar{\tilde{x}}; \eta)$, while in $\partial_\eta F_i(x, \bar{x}; \eta)$ the arguments x and $\bar{x}$ are kept fixed.

Exercise 21 Verify (1.173) by differentiating the second equation of (1.158) with respect to η, keeping x and $\bar{x}$ fixed. Subsequently, use (1.159), (1.168) and (1.171) to arrive at (1.173).

Let us first consider (1.173) in the case of a holomorphic function F, so that $\Omega = 0$. In that case (1.173) implies that the derivative with respect to x^i of $\partial_\eta \tilde{F} - \partial_\eta F$ must vanish. Therefore it follows that $\partial_\eta F$ transforms as a function under symplectic transformations (possibly up to an x-independent expression, which is irrelevant in view of the same argument that led to the equivalence (1.8)).

When $\Omega \neq 0$ one derives the following result using (1.173),

$$\frac{\partial \tilde{x}^j}{\partial x^i}\,\partial_\eta \tilde{F}_j - \frac{\partial \bar{\tilde{x}}^{\bar{j}}}{\partial x^i}\,\partial_\eta(\overline{\tilde{F}_j}) = \partial_\eta F_i . \tag{1.174}$$

Exercise 22 Deduce (1.174) by suitably combining (1.173) with its complex conjugate, and using the relation

$$\bar{\mathcal{Z}}^{\bar{\imath}\bar{\jmath}}\,\bar{F}_{\bar{\jmath}k}\,[\hat{\mathcal{S}}^{-1}\mathcal{S}]^k{}_l = [\bar{\hat{\mathcal{S}}}^{-1}\bar{\mathcal{S}}]^{\bar{\imath}}{}_{\bar{\jmath}}\,\bar{\mathcal{Z}}^{\bar{\jmath}\bar{k}}\,\bar{F}_{\bar{k}l}. \tag{1.175}$$

Next, we assume without loss of generality that the dependence of $\tilde{F}$ on η is entirely contained in $\tilde{\Omega}$. Then, using (1.15), it follows that

$$\partial_\eta(\overline{\tilde{F}_j}) = -\partial_\eta \tilde{F}_{\bar{\jmath}}, \tag{1.176}$$

and the relation (1.174) simplifies. Namely, the left hand side of (1.174) becomes equal to $\partial(\partial_\eta \tilde{F})/\partial x^i$, where we used the existence of the new function $\tilde{F}$. Thus, we obtain from (1.174),

$$\frac{\partial}{\partial x^i}\left(\partial_\eta \tilde{F} - \partial_\eta F\right) = 0. \tag{1.177}$$

This equation, together with its complex conjugate equation, implies that $\partial_\eta \tilde{F} - \partial_\eta F$ vanishes upon differentiation with respect to x and $\bar{x}$, so that $\partial_\eta F$ transforms as a function under symplectic transformations (possibly up to an irrelevant term that is independent of x and $\bar{x}$).

B The Covariant Derivative $\mathcal{D}_\eta$

The modified derivative (1.97) acts as a covariant derivative for symplectic transformations. Here we verify this explicitly by showing that, given a quantity $G(x, \bar{x}; \eta)$ that transforms as a function under symplectic transformations, also $\mathcal{D}_\eta G$ transforms as a function.

To establish this, we need the transformation law of $\hat{N}^{ij}$ that enters in (1.97). Under symplectic transformations, $\hat{N}_{ij}$ given in (1.98) transforms as

$$\begin{aligned}
\tilde{\hat{N}}_{ij} = {} & [\hat{\mathcal{S}}^{-1}]^k{}_i\,[\bar{\hat{\mathcal{S}}}^{-1}]^{\bar{l}}{}_{\bar{\jmath}}\Big[\hat{N}_{kl} + \mathrm{i}\,F_{k\bar{m}}\,\bar{\mathcal{Z}}^{\bar{m}\bar{n}}\,\bar{F}_{\bar{n}p}\left(\delta_l^p - \mathcal{Z}^{pq}\,F_{q\bar{l}}\right) \\
& - \mathrm{i}\;\bar{F}_{\bar{k}m}\,\mathcal{Z}^{mn}\,F_{n\bar{p}}\left(\delta_{\bar{l}}^{\bar{p}} - \bar{\mathcal{Z}}^{\bar{p}\bar{q}}\,\bar{F}_{\bar{q}l}\right)\Big] \\
& + \mathrm{i}\,[\hat{\mathcal{S}}^{-1}]^k{}_i\,[\bar{\hat{\mathcal{S}}}^{-1}]^{\bar{l}}{}_{\bar{\jmath}}\,\bar{F}_{\bar{k}m}\,[\mathcal{S}^{-1}\,\hat{\mathcal{S}}]^m{}_l - \mathrm{i}[\bar{\hat{\mathcal{S}}}^{-1}]^{\bar{k}}{}_{\bar{\imath}}\,\bar{F}_{\bar{k}l}\,[\mathcal{S}^{-1}]^l{}_j,
\end{aligned} \tag{1.178}$$

where $\mathcal{S}$, $\hat{\mathcal{S}}$ and $\mathcal{Z}$ are defined in (1.166).

Exercise 23 Verify (1.178) using (1.168) and (1.171).

Then, it follows that the inverse matrix $\hat{N}^{ij}$ transforms as

$$\tilde{\hat{N}}^{ij} = \left(\mathcal{S}^i{}_l - Z^{in}\, F_{n\bar{l}}\right)\, \hat{N}^{lk}\, \left(\mathcal{S}^j{}_k - Z^{jm}\, F_{m\bar{k}}\right) - \mathrm{i}\, \mathcal{S}^i{}_k\, \mathcal{Z}^{kl}\, \mathcal{S}^j{}_l. \tag{1.179}$$

Since the matrix $\mathcal{Z} = \mathcal{S}^{-1}\, Z$ is symmetric [5], so is $\tilde{\hat{N}}^{ij}$. Observe that it can also be written as

$$\tilde{\hat{N}}^{ij} = \left(\bar{\mathcal{S}}^{\bar{\imath}}{}_{\bar{l}} - Z^{in}\, \bar{F}_{\bar{n}l}\right)\, \hat{N}^{lk}\, \left(\mathcal{S}^j{}_k - Z^{jm}\, F_{m\bar{k}}\right) - \mathrm{i}\, Z^{il}\, Z^{jm}\, F_{l\bar{m}}. \tag{1.180}$$

Establishing the transformation behavior (1.179) turns out to be a tedious exercise, which we relegate to end of this appendix.

Now consider a quantity $G(x, \bar{x}; \eta)$ that transforms as a function under symplectic transformations, i.e. $G(x, \bar{x}; \eta) = \tilde{G}(\tilde{x}, \bar{\tilde{x}}; \eta)$. We then calculate the behavior of $\mathcal{D}_\eta G$ under symplectic transformations. First we establish

$$G_\eta = \tilde{G}_\eta + \tilde{G}_i\, Z^{ij}\, F_{\eta j} + \tilde{G}_{\bar{\imath}}\, Z^{ij}\, \bar{F}_{\eta\bar{j}}, \tag{1.181}$$

where, on the right hand side, the tilde quantities are differentiated with respect to the tilde variables, while those without a tilde are differentiated with respect to the original variables. Similarly,

$$G_i - G_{\bar{\imath}} = \left(\tilde{G}_j - \tilde{G}_{\bar{j}}\right) \left(\mathcal{S}^j{}_i - Z^{jk}\, F_{k\bar{\imath}}\right) + \mathrm{i}\, \tilde{G}_{\bar{j}}\, Z^{jk}\, \hat{N}_{ki}, \tag{1.182}$$

as well as

$$F_{\eta j} = \tilde{F}_{\eta i}\, \mathcal{S}^i{}_j + \tilde{F}_{\eta\bar{\imath}}\, Z^{ik}\, \bar{F}_{\bar{k}j}, \tag{1.183}$$

where we used that F_η transforms as a symplectic function, as established in (1.177).

Exercise 24 Verify (1.181) and (1.182) using $G(x, \bar{x}; \eta) = \tilde{G}(\tilde{x}, \bar{\tilde{x}}; \eta)$.

Then, inserting (1.181) and (1.182) into (1.97) yields,

$$\mathcal{D}_\eta G = \tilde{G}_\eta + \left(\tilde{G}_i - \tilde{G}_{\bar{\imath}}\right) Z^{ij}\, F_{\eta j} + \mathrm{i}\, \hat{N}^{ij}\, \left(F_{\eta j} + \bar{F}_{\eta\bar{j}}\right) \left(\tilde{G}_k - \tilde{G}_{\bar{k}}\right) \left(\mathcal{S}^k{}_i - Z^{kl}\, F_{l\bar{\imath}}\right). \tag{1.184}$$

Next, using (1.183), we compute

$$\begin{aligned} \left(F_{\eta j} + \bar{F}_{\eta\bar{j}}\right) = & \left(\tilde{F}_{\eta k} + \bar{\tilde{F}}_{\eta\bar{k}}\right) \left(\mathcal{S}^k{}_j - Z^{kl}\, F_{l\bar{j}}\right) - \mathrm{i}\, \bar{\tilde{F}}_{\eta\bar{l}}\, Z^{lk}\, \hat{N}_{kj} \\ & + \left(\tilde{F}_{\eta l} + \bar{\tilde{F}}_{\eta l}\right) Z^{lk}\, F_{k\bar{j}} + \left(\tilde{F}_{\eta\bar{l}} + \bar{\tilde{F}}_{\eta\bar{l}}\right) Z^{lk}\, \bar{F}_{\bar{k}j}. \end{aligned} \tag{1.185}$$

Using that $\tilde{F}$ has the decomposition

$$\tilde{F}(\tilde{x}, \bar{\tilde{x}}; \eta) = \tilde{F}^{(0)}(\tilde{x}) + 2\mathrm{i}\, \tilde{\Omega}(\tilde{x}, \bar{\tilde{x}}; \eta) \tag{1.186}$$

with $\tilde{\Omega}$ real, it follows that the second line of (1.185) vanishes. Inserting the first line of (1.185) into (1.184) and using $F_{i\bar{j}} = -\bar{F}_{\bar{j}i}$ as well as $\mathcal{S}\,Z^T = Z\,\mathcal{S}^T$, we obtain

$$\mathcal{D}_\eta G = \tilde{G}_\eta + \mathrm{i}\,\tilde{N}^{ij}\left(\tilde{F}_{\eta j} + \bar{\tilde{F}}_{\eta\bar{j}}\right)\left(\tilde{G}_i - \tilde{G}_{\bar{i}}\right) = \widetilde{(\mathcal{D}_\eta G)}, \tag{1.187}$$

which shows that $\mathcal{D}_\eta G$ transforms as a function under symplectic transformations.

Now we return to the transformation behavior of $\hat{N}^{ij}$ given in (1.179) and verify that it is the inverse of (1.178), i.e. $\tilde{\hat{N}}^{-1}\,\tilde{\hat{N}} = \mathbb{I}$. We use the decomposition $F(x,\bar{x};\eta) = F^{(0)}(x) + 2\mathrm{i}\Omega(x,\bar{x};\eta)$. We find it useful to introduce the following matrix notation,

$$\begin{aligned}
\bar{\mathcal{S}}^{-1}\,\mathcal{S} &= \mathbb{I} + \bar{\mathcal{Z}}\left(F_{\cdot\cdot} - \bar{F}_{--}\right),\\
\mathcal{S}^{-1}\,\hat{\mathcal{S}} &= \mathbb{I} - X \quad , \quad X = \mathcal{Z}\,F_{\cdot -}\,\bar{\mathcal{Z}}\bar{F}_{-\cdot} = 4\,\mathcal{Z}\,\Omega_{\cdot -}\,\bar{\mathcal{Z}}\,\Omega_{-\cdot},\\
\hat{\mathcal{S}}^{-1}\mathcal{S} &= (\mathbb{I} - X)^{-1} = \sum_{n=0}^{\infty} X^n,\\
\bar{\hat{\mathcal{S}}} &= \mathcal{S}\left[\mathbb{I} - X - \mathcal{Z}\left(\hat{F}_{\cdot\cdot} - \bar{\hat{F}}_{--}\right)\right] = \left[\mathbb{I} - \mathcal{Z}\left(F_{\cdot\cdot} - \bar{F}_{--}\right) - 4\,\mathcal{Z}\,\Omega_{-\cdot}\,\mathcal{Z}\,\Omega_{\cdot -}\right],\\
\mathcal{Z} - \bar{\mathcal{Z}} &= -\,\bar{\mathcal{Z}}\left(F_{\cdot\cdot} - \bar{F}_{--}\right)\mathcal{Z} = -\mathcal{Z}\left(F_{\cdot\cdot} - \bar{F}_{--}\right)\bar{\mathcal{Z}},
\end{aligned} \tag{1.188}$$

where we assume that the power series expansion of $\mathcal{S}^{-1}\,\hat{\mathcal{S}}$ is convergent. Here $F_{\cdot\cdot}$, F_{--} , $F_{\cdot -}$ denote entries of the type F_{ij}, $F_{\bar{i}\bar{j}}$, $F_{i\bar{j}}$, respectively. Then, using (1.178), we compute

$$\begin{aligned}
\mathcal{S}^T\,\tilde{\hat{N}}\,\bar{\hat{\mathcal{S}}} = &\sum_{n=0}^{\infty}\left(X^n\right)^T\left(\hat{N} + 4\mathrm{i}\,\Omega_{\cdot -}\,\bar{\mathcal{Z}}\,\Omega_{-\cdot} - 4\mathrm{i}\Omega_{-\cdot}\,\mathcal{Z}\,\Omega_{\cdot -} + 2\Omega_{\cdot -}\,\bar{X} + 2\Omega_{-\cdot}\right)\\
&- 2\left(\bar{\mathcal{S}}^{-1}\,\mathcal{S}\right)^T\sum_{n=0}^{\infty}\left(\bar{X}^n\right)^T\ \Omega_{-\cdot}\left[\mathbb{I} - \mathcal{Z}\left(F_{\cdot\cdot} - \bar{F}_{--}\right) - 4\,\mathcal{Z}\,\Omega_{-\cdot}\,\mathcal{Z}\,\Omega_{\cdot -}\right].
\end{aligned} \tag{1.189}$$

Multiplying this with $\tilde{\hat{N}}^{-1}\,\mathcal{S}^{-1,T}$ from the left and requiring the resulting expression to equal $\bar{\hat{\mathcal{S}}}$ yields the relation

$$\begin{aligned}
&\left[\hat{N}^{-1} - 2\mathrm{i}\,\hat{N}^{-1}\,\Omega_{-\cdot}\,\mathcal{Z} - 2\mathrm{i}\,\mathcal{Z}\,\Omega_{\cdot -}\,\hat{N}^{-1} - 4\mathcal{Z}\,\Omega_{\cdot -}\,\hat{N}^{-1}\,\Omega_{-\cdot}\,\mathcal{Z} - \mathrm{i}\mathcal{Z}\right]\\
&\qquad\left[\sum_{n=0}^{\infty}\left(X^n\right)^T\left(\hat{N} + 4\mathrm{i}\,\Omega_{\cdot -}\,\bar{\mathcal{Z}}\,\Omega_{-\cdot} - 4\mathrm{i}\Omega_{-\cdot}\,\mathcal{Z}\,\Omega_{\cdot -} + 2\Omega_{\cdot -}\,\bar{X} + 2\Omega_{-\cdot}\right)\right.
\end{aligned}$$

$$
\begin{aligned}
& -2\left(\bar{\mathcal{S}}^{-1}\,\mathcal{S}\right)^T \sum_{n=0}^{\infty}\left(\bar{X}^n\right)^T\ \Omega_{-.}\left[\mathbb{I}-\mathcal{Z}\left(F_{..}-\bar{F}_{--}\right)-4\,\mathcal{Z}\,\Omega_{-.}\,\mathcal{Z}\,\Omega_{.-}\right]\Bigg] \\
&= \left[\mathbb{I}-\mathcal{Z}\left(F_{..}-\bar{F}_{--}\right)-4\,\mathcal{Z}\,\Omega_{-.}\,\mathcal{Z}\,\Omega_{.-}\right]. \qquad (1.190)
\end{aligned}
$$

Thus, checking $\hat{\tilde{N}}^{-1}\,\hat{\tilde{N}}=\mathbb{I}$ amounts to verifying the relation (1.190). To do so, we write (1.190) as a power series in $\mathcal{Z}$ by converting $\bar{\mathcal{Z}}$ into $\mathcal{Z}$ using the last relation in (1.188). Introducing the expressions

$$
\begin{aligned}
\sigma &= 4\,\Omega_{-.}\,\mathcal{Z}\,\Omega_{.-}\,\mathcal{Z}, \\
\Delta &= \sum_{n=1}^{\infty}\left[\left(F_{..}-\bar{F}_{--}\right)\mathcal{Z}\right]^n, \qquad (1.191)
\end{aligned}
$$

we obtain

$$
\begin{aligned}
\bar{X} &= 4\,\mathcal{Z}\,(\mathbb{I}+\Delta)\ \Omega_{-.}\,\mathcal{Z}\,\Omega_{.-}, \\
\bar{X}^T\ \Omega_{-.} &= \Omega_{-.}\,X, \\
\sum_{n=0}^{\infty}\left(\bar{X}^n\right)^T\ \Omega_{-.} &= \Omega_{-.}\sum_{n=0}^{\infty}X^n, \\
X^n &= 4\,\mathcal{Z}\,\Omega_{.-}\,\mathcal{Z}\,[(\mathbb{I}+\Delta)\,\sigma]^{n-1}\,(\mathbb{I}+\Delta)\,\Omega_{-.}, \qquad n\geqslant 1, \\
\left(X^n\right)^T &= 4\,\Omega_{.-}(\mathbb{I}+\Delta^T)\left[\sigma^T(\mathbb{I}+\Delta^T)\right]^{n-1}\mathcal{Z}\,\Omega_{-.}\,\mathcal{Z}, \qquad n\geqslant 1, \\
\left(\bar{\mathcal{S}}^{-1}\,\mathcal{S}\right)^T &= \mathbb{I}+\left(F_{..}-\bar{F}_{--}\right)\mathcal{Z}\,(\mathbb{I}+\Delta)\,. \qquad (1.192)
\end{aligned}
$$

Then, (1.190) becomes

$$
\begin{aligned}
& \left[\mathbb{I}-2\mathrm{i}\,\Omega_{-.}\,\mathcal{Z}-2\mathrm{i}\,\hat{N}\,\mathcal{Z}\,\Omega_{.-}\,\hat{N}^{-1}-4\,\hat{N}\,\mathcal{Z}\,\Omega_{.-}\,\hat{N}^{-1}\,\Omega_{-.}\,\mathcal{Z}-\mathrm{i}\,\hat{N}\,\mathcal{Z}\right] \\
& \Bigg[\sum_{n=0}^{\infty}\left(X^n\right)^T\Big[\hat{N}+4\mathrm{i}\,\Omega_{.-}\,\mathcal{Z}\ (\mathbb{I}+\Delta)\,\Omega_{-.}-4\mathrm{i}\Omega_{-.}\,\mathcal{Z}\,\Omega_{.-} \\
& +8\Omega_{.-}\,\mathcal{Z}\,(\mathbb{I}+\Delta)\,\Omega_{-.}\,\mathcal{Z}\,\Omega_{.-}+2\Omega_{-.}\Big] \\
& -2\left[\mathbb{I}+\left(F_{..}-\bar{F}_{--}\right)\mathcal{Z}\,(\mathbb{I}+\Delta)\right]\Omega_{-.}\sum_{n=0}^{\infty}X^n\left[\mathbb{I}-\mathcal{Z}\left(F_{..}-\bar{F}_{--}\right)\right. \\
& \left.-4\,\mathcal{Z}\,\Omega_{-.}\,\mathcal{Z}\,\Omega_{.-}\right]\Bigg] = \hat{N}\left[\mathbb{I}-\mathcal{Z}\left(F_{..}-\bar{F}_{--}\right)-4\,\mathcal{Z}\,\Omega_{-.}\,\mathcal{Z}\,\Omega_{.-}\right], \qquad (1.193)
\end{aligned}
$$

where X^n (with $n\geqslant 1$) is expressed in terms of $\mathcal{Z}$ according to (1.192). Now we proceed to check that (1.193) is indeed satisfied, order by order in $\mathcal{Z}$. Observe that the right hand side of (1.193) is quadratic in $\mathcal{Z}$, so first we check the cancellation of the terms up to order $\mathcal{Z}^2$. Then we proceed to check the terms at order n with $n \geqslant 3$. Here we use the relations

$$\begin{aligned} F_{\cdot\cdot} - \bar{F}_{--} &= \mathrm{i}\hat{N} + 2\mathrm{i}\Omega_{\cdot -} + 2\mathrm{i}\Omega_{-\cdot}, \\ \Delta^T\, \mathcal{Z} &= \mathcal{Z}\, \Delta, \\ \left[\sigma^T \left(\mathbb{I} + \Delta^T\right)\right]^n \mathcal{Z} &= \mathcal{Z}\left[\sigma\left(\mathbb{I} + \Delta\right)\right]^n, \end{aligned} \tag{1.194}$$

and we organize the terms at order n into those that end on either N (introduced in (1.100)), $\Omega_{\cdot -}$ or $\Omega_{-\cdot}$. It is then straightforward, but tedious, to check that at order n in $\mathcal{Z}$ all these terms cancel out. This establishes the validity of the transformation law (1.179).

C The Holomorphic Anomaly Equation in Big Moduli Space

The holomorphic anomaly Eq. (1.157) of perturbative topological string theory [35, 36] can be suscintly derived in the wave function approach [22] to the latter [23–26]. In this approach, the topological string partition function Z is represented by a wavefunction,

$$Z(t; t_B, \bar{t}_B) = \int d\phi\, \mathrm{e}^{-S(\phi, t; t_B, \bar{t}_B)/\hbar}\, Z(\phi), \tag{1.195}$$

where $S(\phi, t; t_B, \bar{t}_B)$ denotes the generating function (1.114) of canonical transformations[6]. We take the background dependent constant $c(t_B, \bar{t}_B)$ appearing in S to be given by [23–26]

$$c(t_B, \bar{t}_B) = -\frac{\hbar}{2} \ln\det N_{IJ}(t_B, \bar{t}_B), \tag{1.196}$$

with N_{IJ} as in (1.132).

Differentiating (1.195) with respect to the background field $\bar{t}_B$ on the one hand, and with respect to the fluctuations t on the other hand, yields the relation [24],

$$\frac{\partial Z(t; t_B, \bar{t}_B)}{\partial \bar{t}_B^L} = \frac{\hbar}{2}\, \bar{F}_{\bar{L}}{}^{IJ}\, \frac{\partial}{\partial t^I} \frac{\partial}{\partial t^J} Z(t; t_B, \bar{t}_B). \tag{1.197}$$

Here $\bar{F}_{\bar{L}}{}^{IJ}$ is evaluated on the background, and is given by $\bar{F}_{\bar{L}}{}^{IJ} = \bar{F}_{\bar{L}\bar{M}\bar{O}} N^{MI} N^{OJ}$. Assigning scaling dimension 1 to both t_B and t (and to their complex conjugates) and scaling dimension 2 to $\hbar$, we see that (1.197) has scaling dimension -1. Setting

$$Z(t; t_B, \bar{t}_B) = \mathrm{e}^{W(t; t_B, \bar{t}_B)/\hbar}, \tag{1.198}$$

[6] We use the conventions of Sect. 1.4 and suppress the superscript of $F^{(0)}$.

we obtain from (1.197)

$$\frac{\partial W(t;t_B,\bar t_B)}{\partial \bar t^L_B} = \frac{1}{2}\,\bar F_{\bar L}{}^{IJ}\left(\hbar\,\frac{\partial^2 W}{\partial t^I\,\partial t^J} + \frac{\partial W}{\partial t^I}\,\frac{\partial W}{\partial t^J}\right), \tag{1.199}$$

which has scaling dimension 1. The BCOV-solution [36] is obtained by making the ansatz [38]

$$W = \sum_{g=0,n=0}^{\infty} \frac{\hbar^g}{n!}\, C^{(g)}_{I_1\ldots I_n}(t_B,\bar t_B)\, t^{I_1}\ldots t^{I_n}, \tag{1.200}$$

with

$$C^{(g)}_{I_1\ldots I_n} = 0, \qquad 2g-2+n \leqslant 0. \tag{1.201}$$

The $C^{(g)}_{I_1\ldots I_n}$ are symmetric in $I_1,\ldots,I_n$ and have scaling dimension $2-2g-n$. Inserting the ansatz (1.200) into (1.199), equating the terms of order $\hbar^g$ for $g \geqslant 2$ and setting $t = 0$ gives,

$$\partial_{\bar L} C^{(g)}(t_B,\bar t_B) = \frac{1}{2}\,\bar F_{\bar L}{}^{IJ}\left(C^{(g-1)}_{IJ} + \sum_{r=1}^{g-1} C^{(r)}_I\, C^{(g-r)}_J\right), \qquad g \geqslant 2. \tag{1.202}$$

Exercise 25 Verify (1.202).

Now we set [38]

$$C^{(g)}_{I_1\ldots I_n} = D_{I_1}\ldots D_{I_n} F^{(g)}, \qquad g \geqslant 1, \tag{1.203}$$

where D_L is given by

$$D_L V_M = \partial_L V_M + \mathrm{i}\, N^{PI} F_{ILM} V_P. \tag{1.204}$$

D_L acts as a covariant derivative for symplectic reparametrizations $V_M \to \left(\mathcal{S}_0^{-1}\right)^P{}_M V_P$, since N^{IJ} transforms as $N^{IJ} \to [\mathcal{S}_0\, N^{-1}\, \mathcal{S}_0]^{IJ} - \mathrm{i}[\mathcal{S}_0\, \mathcal{Z}_0\, \mathcal{S}_0]^{IJ}$ (see (1.179)). The $F^{(g)}$ have scaling dimension $2-2g$ and transform as functions under symplectic transformations. Inserting (1.203) into (1.202) yields the holomorphic anomaly equation in big moduli space [38],

$$\partial_{\bar L} F^{(g)}(t_B,\bar t_B) = \frac{1}{2}\,\bar F_{\bar L}{}^{IJ}\left(D_I\partial_J F^{(g-1)} + \sum_{r=1}^{g-1}\partial_I F^{(r)}\,\partial_J F^{(g-r)}\right), \qquad g \geqslant 2. \tag{1.205}$$

As an example, consider solving (1.205) for $g = 2$. We need $F_I^{(1)} = \partial_I F^{(1)}(t_B, \bar{t}_B)$, which is non-holomorphic and given by[7]

$$\partial_I F^{(1)}(t_B, \bar{t}_B) = \partial_I f^{(1)}(t_B) + \tfrac{1}{2}\,\mathrm{i}\, F_{IJK}\, N^{JK}. \tag{1.206}$$

Then, solving (1.205) for $F^{(2)}$ yields [25, 38]

$$\begin{aligned} F^{(2)}(t_B, \bar{t}_B) = f^{(2)}(t_B) &+ \tfrac{1}{2}\,\mathrm{i}\, N^{IJ}\left(D_I F_J^{(1)} + F_I^{(1)} F_J^{(1)}\right) \\ &+ \tfrac{1}{2} N^{IJ} N^{KL}\left(\tfrac{1}{4} F_{IJKL} + \tfrac{1}{3}\mathrm{i}\, N^{MN} F_{IKM} F_{JLN} + F_{IJK} F_L^{(1)}\right). \end{aligned} \tag{1.207}$$

In this expression, all the terms are evaluated on the background $(t_B, \bar{t}_B)$.

Exercise 26 Verify that (1.207) solves (1.205).

Observe that (1.206) transforms covariantly under symplectic transformations, provided that $f^{(1)}$ transforms as $f^{(1)} \longrightarrow f^{(1)} - \frac{1}{2} \ln\det \mathcal{S}_0$ in order to compensate for the transformation behavior $N_{IJ} \longrightarrow N_{KL}\,[\bar{\mathcal{S}}_0^{-1}]^K{}_I\,[\mathcal{S}_0^{-1}]^L{}_J$ [5], so that

$$\begin{aligned} f_I^{(1)} &\longrightarrow \left(f_J^{(1)} - \tfrac{1}{2}\mathcal{Z}_0^{PQ}\, F_{PQJ}\right)\left(\mathcal{S}_0^{-1}\right)^J{}_I, \\ f_{IJ}^{(1)} &\longrightarrow \left(\mathcal{S}_0^{-1}\right)^Q{}_J\, \partial_Q\left[\left(f_L^{(1)} - \tfrac{1}{2}\mathcal{Z}_0^{PQ}\, F_{PQL}\right)\left(\mathcal{S}_0^{-1}\right)^L{}_I\right]. \end{aligned} \tag{1.208}$$

Exercise 27 Determine the transformation behavior of $f^{(2)}(t_B)$ under symplectic transformations (1.128) that ensures that $F^{(2)}(t_B, \bar{t}_B)$ transforms as a function. A useful transformation law is,

$$\begin{aligned} F_{IJKL} \longrightarrow\; & \left(\mathcal{S}_0^{-1}\right)^M{}_I\, \partial_M\left[F_{NOP}\left(\mathcal{S}_0^{-1}\right)^N{}_J\left(\mathcal{S}_0^{-1}\right)^O{}_K\left(\mathcal{S}_0^{-1}\right)^P{}_L\right] \\ &= \left(\mathcal{S}_0^{-1}\right)^M{}_I\left(\mathcal{S}_0^{-1}\right)^N{}_J\left(\mathcal{S}_0^{-1}\right)^O{}_K\left(\mathcal{S}_0^{-1}\right)^P{}_L\Big[F_{MNOP} \\ &\qquad - F_{MPS}\mathcal{Z}_0^{SR}F_{RNO} - F_{OPS}\mathcal{Z}_0^{SR}F_{RMN} - F_{NPS}\mathcal{Z}_0^{SR}F_{ROM}\Big]. \end{aligned} \tag{1.209}$$

[7] $F^{(1)}$ contains an additional term proportional to the Kähler potential (1.1), but this term drops out of (1.205) due to the special geometry relation $\bar{F}_{\bar{I}\bar{J}\bar{K}}\, \bar{t}^{\bar{K}} = 0$.

D The Functions $\mathcal{H}_i^{(a)}$ for $a \geqslant 2$

Here we collect the explicit results for the various functions $\mathcal{H}_i^{(a)}$ (with $a \geqslant 2$) that appear in (1.143). These functions can be determined by iteration. We present the functions up to order $\mathcal{O}(\Omega^4)$. We use the notation $(N\Omega)^I = N^{IJ}\Omega_J$, $(N\bar{\Omega})^I = N^{IJ}\Omega_{\bar{J}}$. The symmetrization $F_{R(IJ}N^{RS}F_{KL)S}$ is defined with a symmetrization factor $1/(4!)$.

$$
\begin{aligned}
\mathcal{H}^{(2)} = {} & 8\, N^{IJ}\Omega_I\Omega_{\bar{J}} - 16\big[\Omega_{IJ}(N\bar{\Omega})^I(N\Omega)^J + \Omega_{I\bar{J}}(N\bar{\Omega})^I(N\bar{\Omega})^J + \text{h.c.}\big] \\
& - 8\mathrm{i}\big[F_{IJK}(N\bar{\Omega})^I(N\Omega)^J(N\Omega)^K - \text{h.c.}\big] \\
& + \tfrac{16}{3}\mathrm{i}\Big[\Big(F_{IJKL} + 3\mathrm{i}F_{IJR}N^{RS}F_{SKL}\Big)\,(N\Omega)^I(N\Omega)^J(N\Omega)^K(N\bar{\Omega})^L \\
& - \text{h.c.}\ \Big] + 16\big[\Omega_{IJK}\,(N\Omega)^I(N\Omega)^J(N\bar{\Omega})^K + \text{h.c.}\big] \\
& + 16\Big[\,\Big(\Omega_{IJ\bar{K}} + \mathrm{i}F_{IJP}N^{PQ}\Omega_{Q\bar{K}}\Big)\,\Big((N\Omega)^I(N\Omega)^J(N\Omega)^K \\
& + 2\,(N\Omega)^I(N\bar{\Omega})^J(N\bar{\Omega})^K\Big) + \text{h.c.}\Big] \\
& + 32\Big[\Omega_{IQ}\,N^{QR}\,\Omega_{RJ}\,(N\Omega)^I(N\bar{\Omega})^J + \text{h.c.}\Big] \\
& + 32\,\Omega_{IQ}\,N^{QR}\,\Omega_{\bar{R}\bar{J}}\,(N\Omega)^I(N\bar{\Omega})^J \\
& + 16\mathrm{i}\Big[F_{IJK}\,N^{KP}\,\Omega_{PQ}\,\Big((N\Omega)^I(N\Omega)^J(N\bar{\Omega})^Q \\
& + 2\,(N\Omega)^Q(N\Omega)^I(N\bar{\Omega})^J\Big) - \text{h.c.}\Big] \\
& + 16\mathrm{i}\Big[F_{IJK}\,N^{KP}\,\Omega_{\bar{P}\bar{Q}}\,(N\Omega)^I(N\Omega)^J(N\bar{\Omega})^Q - \text{h.c.}\Big] \\
& + 8\,(N\Omega)^I\,(N\Omega)^J\,F_{IJQ}N^{QR}\bar{F}_{\bar{R}\bar{K}\bar{L}}(N\bar{\Omega})^K(N\bar{\Omega})^L \\
& + 32\Big[(N\Omega)^I\,\Omega_{IJ}\,N^{JK}\Omega_{K\bar{L}}(N\Omega)^L + \text{h.c.}\Big] \\
& + 32\Big[(N\bar{\Omega})^I\,\Omega_{IJ}\,N^{JK}\Omega_{K\bar{L}}(N\bar{\Omega})^L + \text{h.c.}\Big] \\
& + 32\Big[(N\Omega)^I\,\Omega_{IJ}\,N^{JK}\Omega_{\bar{K}L}(N\Omega)^L + \text{h.c.}\Big] \\
& + 16\mathrm{i}\Big[(N\Omega)^I(N\Omega)^J F_{IJK}N^{KL}\Omega_{\bar{L}P}(N\Omega)^P - \text{h.c.}\Big] \\
& + 32\Big[(N\Omega)^I\Omega_{I\bar{J}}\,N^{JK}\,\Omega_{\bar{K}L}\,(N\bar{\Omega})^L + \text{h.c.}\Big] \\
& + 32\Big[(N\Omega)^I\Omega_{I\bar{J}}\,N^{JK}\,\Omega_{K\bar{L}}\,(N\bar{\Omega})^L\Big],
\end{aligned} \tag{1.210}
$$

$$
\begin{aligned}
\mathcal{H}_1^{(3)} = {} & -\tfrac{8}{3}\mathrm{i}F_{IJK}(N\bar{\Omega})^I(N\bar{\Omega})^J(N\bar{\Omega})^K \\
& + 8\mathrm{i}\,F_{IJK}(N\bar{\Omega})^J(N\bar{\Omega})^K\,N^{IP}\Big[2\Omega_{\bar{P}\bar{Q}}(N\bar{\Omega})^Q + 2\Omega_{\bar{P}Q}(N\Omega)^Q \\
& - \mathrm{i}\,\bar{F}_{\bar{P}\bar{Q}\bar{R}}(N\bar{\Omega})^Q(N\bar{\Omega})^R\Big],
\end{aligned} \tag{1.211}
$$

$$
\begin{aligned}
\mathcal{H}_2^{(3)} &= 8\left(\Omega_{IJ} + \mathrm{i}F_{IJK}(N\Omega)^K\right)(N\bar{\Omega})^I(N\bar{\Omega})^J \\
&\quad - \tfrac{4}{3}\mathrm{i}\left(F_{IJKL} + 3\mathrm{i}F_{R(IJ}N^{RS}F_{KL)S}\right)\Big(6(N\Omega)^I(N\Omega)^J(N\bar{\Omega})^K(N\bar{\Omega})^L \\
&\quad - 4\,(N\bar{\Omega})^I(N\bar{\Omega})^J(N\bar{\Omega})^K(N\Omega)^L\Big)\Big] \\
&\quad - \tfrac{16}{3}\,\Omega_{IJK}\left(3\,(N\bar{\Omega})^I(N\bar{\Omega})^J(N\Omega)^K - (N\bar{\Omega})^I(N\bar{\Omega})^J(N\bar{\Omega})^K\right) \\
&\quad - 16\,\Omega_{IJ\bar{K}}(N\bar{\Omega})^I(N\bar{\Omega})^J(N\bar{\Omega})^K \\
&\quad - 16\mathrm{i}\,F_{IJK}N^{KP}\,\Omega_{PQ}\big[-(N\bar{\Omega})^I(N\bar{\Omega})^J(N\bar{\Omega})^Q \\
&\quad + (N\bar{\Omega})^I(N\bar{\Omega})^J(N\Omega)^Q + 2(N\bar{\Omega})^I(N\Omega)^J(N\bar{\Omega})^Q\big] \\
&\quad - 16\,(N\bar{\Omega})^P\,\Omega_{PQ}\,N^{QR}\Omega_{RK}\,(N\bar{\Omega})^K \\
&\quad - 32\,(N\Omega)^I\left(\Omega_{IJ} + \mathrm{i}F_{IJP}(N\Omega)^P\right)N^{JK}\left(\Omega_{\bar{K}\bar{L}} - \mathrm{i}\bar{F}_{\bar{K}\bar{L}\bar{M}}(N\bar{\Omega})^M\right)(N\Omega)^L \\
&\quad + 16\mathrm{i}(N\Omega)^I(N\Omega)^J F_{IJP}N^{PK}\left(\Omega_{\bar{K}\bar{L}} - \mathrm{i}\bar{F}_{\bar{K}\bar{L}\bar{Q}}(N\bar{\Omega})^{\bar{Q}}\right)(N\Omega)^L \\
&\quad - 16\,(N\Omega)^P\,\Omega_{\bar{P}Q}N^{QR}\Omega_{R\bar{K}}(N\Omega)^K \\
&\quad - 32\,(N\bar{\Omega})^I\left(\Omega_{IJ} + \mathrm{i}F_{IJK}(N\Omega)^K\right)N^{JL}\Omega_{\bar{L}M}(N\Omega)^M \\
&\quad - 16\mathrm{i}\,(N\bar{\Omega})^I\,(N\bar{\Omega})^J F_{IJK}N^{KP}\Omega_{P\bar{Q}}(N\bar{\Omega})^Q,
\end{aligned}
\tag{1.212}
$$

$$
\begin{aligned}
\mathcal{H}_3^{(3)} &= 16\,\Omega_{I\bar{J}}(N\bar{\Omega})^I(N\Omega)^J \\
&\quad - 16\Big[2(N\bar{\Omega})^K(N\Omega)^L\left(\Omega_{KM}N^{MN}\Omega_{N\bar{L}} + \Omega_{K\bar{L}Q}(N\Omega)^Q\right) \\
&\quad + (N\bar{\Omega})^K\Omega_{K\bar{L}}N^{LP}\Big(\mathrm{i}F_{PMN}(N\Omega)^M(N\Omega)^N + 2\Omega_{PJ}(N\Omega)^J \\
&\quad + 2\,\Omega_{P\bar{J}}(N\bar{\Omega})^J\Big) + 2\mathrm{i}(N\bar{\Omega})^I(N\Omega)^J F_{IJK}N^{KP}\Omega_{P\bar{Q}}(N\Omega)^Q + \text{h.c.}\Big],
\end{aligned}
\tag{1.213}
$$

$$
\mathcal{H}_1^{(4)} = 32\,(N\bar{\Omega})^I\left(\Omega_{IJ} + \mathrm{i}F_{IJK}(N\Omega)^K\right)N^{JP}\left(\Omega_{\bar{P}\bar{Q}} - \mathrm{i}\bar{F}_{\bar{P}\bar{Q}\bar{R}}(N\bar{\Omega})^R\right)(N\Omega)^Q,
\tag{1.214}
$$

$$
\mathcal{H}_2^{(4)} = 32\,(N\Omega)^P\,\Omega_{\bar{P}Q}\,N^{QR}\,\Omega_{\bar{R}K}\,(N\bar{\Omega})^K
\tag{1.215}
$$

$$
\mathcal{H}_3^{(4)} = 8\,F_{IJR}N^{RS}\bar{F}_{\bar{S}\bar{K}\bar{L}}\,(N\bar{\Omega})^I(N\bar{\Omega})^J(N\Omega)^K(N\Omega)^L,
\tag{1.216}
$$

$$
\mathcal{H}_4^{(4)} = -\tfrac{4}{3}\mathrm{i}\left(F_{IJKL} + 3\mathrm{i}\,F_{IJR}N^{RS}F_{SKL}\right)(N\bar{\Omega})^I(N\bar{\Omega})^J(N\bar{\Omega})^K(N\bar{\Omega})^L,
\tag{1.217}
$$

$$
\mathcal{H}_5^{(4)} = -16\mathrm{i}\,F_{IJK}N^{KL}\Omega_{\bar{L}Q}\,(N\bar{\Omega})^Q(N\bar{\Omega})^I(N\bar{\Omega})^J,
\tag{1.218}
$$

$$
\mathcal{H}_6^{(4)} = -16\mathrm{i}\,F_{IJK}N^{KP}\left(\Omega_{\bar{P}\bar{Q}} - \mathrm{i}\bar{F}_{\bar{P}\bar{Q}\bar{R}}(N\bar{\Omega})^R\right)(N\bar{\Omega})^I(N\bar{\Omega})^J(N\Omega)^Q,
\tag{1.219}
$$

$$
\mathcal{H}_7^{(4)} = 16\left(\Omega_{IJ\bar{K}} + \mathrm{i}F_{IJP}\,N^{PQ}\,\Omega_{Q\bar{K}}\right)(N\bar{\Omega})^I(N\bar{\Omega})^J(N\Omega)^K,
\tag{1.220}
$$

$$
\mathcal{H}_8^{(4)} = 32\,(N\bar{\Omega})^I\left(\Omega_{IJ} + \mathrm{i}F_{IJK}(N\Omega)^K\right)N^{JP}\Omega_{\bar{P}Q}\,(N\bar{\Omega})^Q,
\tag{1.221}
$$

$$
\mathcal{H}_9^{(4)} = -16\mathrm{i}\,(N\bar{\Omega})^I(N\bar{\Omega})^J F_{IJK}N^{KL}\Omega_{\bar{L}P}(N\bar{\Omega})^P.
\tag{1.222}
$$

E Transformation Laws by Iteration

The Hesse potential in Sect. 1.4 depends on Ω, whose behavior under symplectic transformations can be determined by iteration. Here we summarize the result for the transformation behavior of derivatives of Ω (expressed in terms of the covariant variables of Sect. 1.3), up to a certain order. We use the conventions of Sect. 1.4 and suppress the superscript of $F^{(0)}$.

$$
\begin{aligned}
\tilde{\Omega}_I = [\mathcal{S}_0^{-1}]^J{}_I \Big[& \Omega_J + \mathrm{i}F_{JKL}\,(\mathcal{Z}_0\Omega)^K\,(\mathcal{Z}_0\Omega)^L - 2\mathrm{i}\Omega_{JK}(\mathcal{Z}_0\Omega)^K \\
& + 2\mathrm{i}\Omega_{J\bar{K}}(\bar{\mathcal{Z}}_0\bar{\Omega})^{\bar{K}} + \tfrac{2}{3}F_{JKLP}(\mathcal{Z}_0\Omega)^K(\mathcal{Z}_0\Omega)^L(\mathcal{Z}_0\Omega)^P \\
& + 2F_{KLP}(\mathcal{Z}_0\Omega)^K{}_J(\mathcal{Z}_0\Omega)^L(\mathcal{Z}_0\Omega)^P \\
& + 4F_{JKL}(\mathcal{Z}_0\Omega)^K(\mathcal{Z}_0\Omega)^L{}_P(\mathcal{Z}_0\Omega)^P \\
& - 4F_{JKL}(\mathcal{Z}_0\Omega)^K(\mathcal{Z}_0\Omega)^L{}_{\bar{P}}(\bar{\mathcal{Z}}_0\bar{\Omega})^{\bar{P}} \\
& - 2F_{JKL}\mathcal{Z}_0^{LP}F_{PQS}(\mathcal{Z}_0\Omega)^K(\mathcal{Z}_0\Omega)^Q(\mathcal{Z}_0\Omega)^S \\
& + 2\bar{F}_{\bar{K}\bar{L}\bar{P}}(\bar{\mathcal{Z}}_0\bar{\Omega})^{\bar{K}}{}_J(\bar{\mathcal{Z}}_0\bar{\Omega})^{\bar{L}}(\bar{\mathcal{Z}}_0\bar{\Omega})^{\bar{P}} \\
& - 2\Omega_{JKL}(\mathcal{Z}_0\Omega)^K(\mathcal{Z}_0\Omega)^L - 4\Omega_{KL}(\mathcal{Z}_0\Omega)^K{}_J(\mathcal{Z}_0\Omega)^L \\
& - 2\Omega_{J\bar{K}\bar{L}}(\bar{\mathcal{Z}}_0\bar{\Omega})^{\bar{K}}(\bar{\mathcal{Z}}_0\bar{\Omega})^{\bar{L}} - 4\Omega_{\bar{K}\bar{L}}(\bar{\mathcal{Z}}_0\bar{\Omega})^{\bar{K}}{}_J(\bar{\mathcal{Z}}_0\bar{\Omega})^{\bar{L}} \\
& + 4\Omega_{JK\bar{L}}(\mathcal{Z}_0\Omega)^K(\bar{\mathcal{Z}}_0\bar{\Omega})^{\bar{L}} + 4\Omega_{K\bar{L}}(\mathcal{Z}_0\Omega)^K{}_J(\bar{\mathcal{Z}}_0\bar{\Omega})^{\bar{L}} \\
& + 4\Omega_{K\bar{L}}(\mathcal{Z}_0\Omega)^K(\bar{\mathcal{Z}}_0\bar{\Omega})^{\bar{L}}{}_J\Big] + \mathcal{O}(\Omega^4)\,, \\
\tilde{\Omega}_{IJ} = [\mathcal{S}_0^{-1}]^K{}_I[\mathcal{S}_0^{-1}]^L{}_J\Big[& \Omega_{KL} - F_{KLM}\,\mathcal{Z}_0^{MN}\Omega_N \\
& - \mathrm{i}F_{KLP}\mathcal{Z}_0^{PM}F_{MQR}(\mathcal{Z}_0\Omega)^Q(\mathcal{Z}_0\Omega)^R + 2\mathrm{i}F_{KLP}(\mathcal{Z}_0\Omega)^P{}_Q(\mathcal{Z}_0\Omega)^Q \\
& - 2\mathrm{i}F_{KLP}(\mathcal{Z}_0\Omega)^P{}_{\bar{Q}}(\bar{\mathcal{Z}}_0\bar{\Omega})^{\bar{Q}} + \mathrm{i}F_{KLMN}(\mathcal{Z}_0\Omega)^M(\mathcal{Z}_0\Omega)^N \\
& + \mathrm{i}F_{KMN}(\mathcal{Z}_0\Omega)^M{}_L(\mathcal{Z}_0\Omega)^N + \mathrm{i}F_{KMN}(\mathcal{Z}_0\Omega)^N{}_L(\mathcal{Z}_0\Omega)^M \\
& - 2\mathrm{i}F_{KMN}\mathcal{Z}_0^{MP}F_{PQL}(\mathcal{Z}_0\Omega)^Q(\mathcal{Z}_0\Omega)^N \\
& - 2\mathrm{i}\Omega_{KLP}(\mathcal{Z}_0\Omega)^P - 2\mathrm{i}\Omega_{KP}(\mathcal{Z}_0\Omega)^P{}_L + 2\mathrm{i}\Omega_{KP}\mathcal{Z}_0^{PQ}F_{QLS}(\mathcal{Z}_0\Omega)^S \\
& + 2\mathrm{i}\Omega_{KL\bar{P}}(\bar{\mathcal{Z}}_0\bar{\Omega})^{\bar{P}} + 2\mathrm{i}\Omega_{K\bar{P}}(\bar{\mathcal{Z}}_0\bar{\Omega})^{\bar{P}}{}_L\Big] + \mathcal{O}(\Omega^3)\,, \\
\tilde{\Omega}_{I\bar{J}} = [\mathcal{S}_0^{-1}]^K{}_I[\bar{\mathcal{S}}_0^{-1}]^{\bar{L}}{}_{\bar{J}}\Big[& \Omega_{K\bar{L}} + 2\mathrm{i}F_{KMN}(\mathcal{Z}_0\Omega)^M{}_{\bar{L}}(\mathcal{Z}_0\Omega)^N \\
& - 2\mathrm{i}\bar{F}_{\bar{L}\bar{P}\bar{N}}(\bar{\mathcal{Z}}_0\bar{\Omega})^{\bar{N}}{}_K(\bar{\mathcal{Z}}_0\bar{\Omega})^{\bar{P}} \\
& - 2\mathrm{i}\Omega_{KM\bar{L}}(\mathcal{Z}_0\Omega)^M - 2\mathrm{i}\Omega_{KM}(\mathcal{Z}_0\Omega)^M{}_{\bar{L}} + 2\mathrm{i}\Omega_{K\bar{L}\bar{M}}(\bar{\mathcal{Z}}_0\bar{\Omega})^{\bar{M}} \\
& + 2\mathrm{i}\Omega_{K\bar{M}}(\bar{\mathcal{Z}}_0\bar{\Omega})^{\bar{M}}{}_{\bar{L}}\Big] + \mathcal{O}(\Omega^3)\,,
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\Omega}_{IJL} &= [\mathcal{S}_0^{-1}]^M{}_I [\mathcal{S}_0^{-1}]^N{}_J [\mathcal{S}_0^{-1}]^K{}_L \Big[\Omega_{MNK} - F_{MNKP} (\mathcal{Z}_0 \Omega)^P \\
&\quad - F_{MNP}\ (\mathcal{Z}_0\Omega)^P{}_K - F_{KMP}(\mathcal{Z}_0\Omega)^P{}_N - F_{NKP}(\mathcal{Z}_0\Omega)^P{}_M \\
&\quad + F_{MNP}\mathcal{Z}_0^{PQ} F_{KQR}(\mathcal{Z}_0\Omega)^R + F_{KMP}\mathcal{Z}_0^{PQ} F_{QNR}(\mathcal{Z}_0\Omega)^R \\
&\quad + F_{NKP}\mathcal{Z}_0^{PQ} F_{QMR}(\mathcal{Z}_0\Omega)^R \Big] + \mathcal{O}(\Omega^2)\,, \\
\tilde{\Omega}_{IJ\bar{K}} &= [\mathcal{S}_0^{-1}]^M{}_I [\mathcal{S}_0^{-1}]^N{}_J [\bar{\mathcal{S}}_0^{-1}]^{\bar{L}}{}_{\bar{K}} \Big[\Omega_{MN\bar{L}} - F_{MNQ}(\mathcal{Z}_0\Omega)^Q{}_{\bar{L}} \Big] \\
&\quad + \mathcal{O}(\Omega^2),
\end{aligned}
\tag{1.223}
$$

where $(\mathcal{Z}_0\Omega)^M = \mathcal{Z}_0^{MN}\Omega_N$, $(\bar{\mathcal{Z}}_0\bar{\Omega})^{\bar{M}} = \bar{\mathcal{Z}}_0^{\bar{M}\bar{N}}\Omega_{\bar{N}}$, $(\mathcal{Z}_0\Omega)^M{}_{\bar{L}} = \mathcal{Z}_0^{MN}\Omega_{N\bar{L}}$, $(\bar{\mathcal{Z}}_0\bar{\Omega})^{\bar{P}}{}_L = \bar{\mathcal{Z}}_0^{\bar{P}\bar{N}}\Omega_{\bar{N}L}$, $(\mathcal{Z}_0\Omega)^L{}_{\bar{P}} = \mathcal{Z}_0^{LK}\Omega_{K\bar{P}}$, $(\bar{\mathcal{Z}}_0\bar{\Omega})^{\bar{P}}{}_{\bar{L}} = \bar{\mathcal{Z}}_0^{\bar{P}\bar{N}}\Omega_{\bar{N}\bar{L}}$.

References

1. B. de Wit, A. Van Proeyen, Potentials and symmetries of general gauged N = 2 supergravity: Yang-Mills models. Nucl. Phys. **B245**, 89 (1984)
2. E. Bergshoeff, M. de Roo, B. de Wit, Extended conformal supergravity. Nucl. Phys. **B182**, 173 (1981)
3. B. de Wit, S. Katmadas, M. van Zalk, New supersymmetric higher-derivative couplings: full N = 2 superspace does not count! JHEP **1101**, 007 (2011). http://xxx.lanl.gov/abs/1010.2150
4. S. Cecotti, S. Ferrara, L. Girardello, Geometry of type II superstrings and the moduli of superconformal field theories. Int. J. Mod. Phys. **A4**, 2475 (1989)
5. B. de Wit, N = 2 electric-magnetic duality in a chiral background. Nucl. Phys. Proc. Suppl. **49**, 191–200 (1996). http://xxx.lanl.gov/abs/hep-th/9602060
6. A. Strominger, Special geometry. Commun. Math. Phys. **133**, 163–180 (1990)
7. D.S. Freed, Special Kahler manifolds. Commun. Math. Phys. **203**, 31–52 (1999). http://xxx.lanl.gov/abs/hep-th/9712042
8. G.L. Cardoso, B. de Wit, J. Käppeli, T. Mohaupt, Black hole partition functions and duality. JHEP **0603**, 074 (2006). http://xxx.lanl.gov/abs/hep-th/0601108
9. S. Ferrara, O. Maciá, Real symplectic formulation of local special geometry. Phys. Lett. **B637**, 102–106 (2006). http://xxx.lanl.gov/abs/hep-th/0603111
10. G.L. Cardoso, B. de Wit , S. Mahapatra, BPS black holes, the Hesse potential, and the topological string. JHEP **1006**, 052 (2010). http://xxx.lanl.gov/abs/1003.1970
11. D. Van den Bleeken, BPS dyons and Hesse flow. JHEP **1202**, 067 (2012). http://xxx.lanl.gov/abs/1111.6979
12. T. Mohaupt, O. Vaughan, The Hesse potential, the c-map and black hole solutions. http://xxx.lanl.gov/abs/1112.2876
13. T. Mohaupt, O. Vaughan, Developments in special geometry. J. Phys. Conf. Ser. **343**, 012078 (2012). http://xxx.lanl.gov/abs/1112.2873
14. N.J. Hitchin, The moduli space of complex Lagrangian submanifolds. Asian J. Math. **3**, 77 (1999). http://xxx.lanl.gov/abs/math.DG/9901069
15. D.V. Alekseevsky, V. Cortés, C. Devchand, Special complex manifolds. J. Geom. Phys. **42**, 85 (2002). http://xxx.lanl.gov/abs/math.DG/9910091
16. V. Cortés, A holomorphic representation formula for parabolic hyperspheres. http://xxx.lanl.gov/abs/math.DG/0107037
17. L.J. Dixon, V. Kaplunovsky, J. Louis, Moduli dependence of string loop corrections to gauge coupling constants. Nucl. Phys. **B355**, 649–688 (1991)

18. G.L. Cardoso, B. de Wit, T. Mohaupt, Macroscopic entropy formulae and nonholomorphic corrections for supersymmetric black holes. Nucl. Phys. **B567**, 87–110 (2000). http://xxx.lanl.gov/abs/hep-th/9906094
19. G.L. Cardoso, B. de Wit, J. Käppeli, T. Mohaupt, Asymptotic degeneracy of dyonic N = 4 string states and black hole entropy. JHEP **0412**, 075 (2004). http://xxx.lanl.gov/abs/hep-th/0412287
20. D.P. Jatkar, A. Sen, Dyon spectrum in CHL models. JHEP **0604**, 018 (2006). http://xxx.lanl.gov/abs/hep-th/0510147
21. G.L. Cardoso, B. de Wit, S. Mahapatra, Subleading and non-holomorphic corrections to N = 2 BPS black hole entropy. JHEP **02**, 006 (2009). http://xxx.lanl.gov/abs/0808.2627
22. E. Witten, Quantum background independence in string theory. http://xxx.lanl.gov/abs/hep-th/9306122
23. R. Dijkgraaf, E.P. Verlinde, M. Vonk, On the partition sum of the NS five-brane. http://xxx.lanl.gov/abs/hep-th/0205281
24. E.P. Verlinde, Attractors and the holomorphic anomaly. http://xxx.lanl.gov/abs/hep-th/0412139
25. M. Aganagic, V. Bouchard, A. Klemm, Topological strings and (almost) modular forms. Commun. Math. Phys. **277**, 771–819 (2008). http://xxx.lanl.gov/abs/hep-th/0607100
26. M. Günaydin, A. Neitzke, B. Pioline, Topological wave functions and heat equations. JHEP **0612**, 070 (2006). http://xxx.lanl.gov/abs/hep-th/0607200
27. B. Sahoo, A. Sen, Higher derivative corrections to non-supersymmetric extremal black holes in N = 2 supergravity. JHEP **0609**, 029 (2006). http://xxx.lanl.gov/abs/hep-th/0603149
28. G.L. Cardoso, B. de Wit, S. Mahapatra, Black hole entropy functions and attractor equations. JHEP **0703**, 085 (2007). http://xxx.lanl.gov/abs/hep-th/0612225
29. E. Schrödinger, Contribution to borns new theory of the electromagnetic field. Proc. Roy. Soc. (London) **A150**, 465–477 (1935)
30. G.W. Gibbons, D.A. Rasheed, Sl(2, R) invariance of nonlinear electrodynamics coupled to an axion and a dilaton. Phys. Lett. **B365**, 46–50 (1996). http://xxx.lanl.gov/abs/hep-th/9509141
31. M. Born, L. Infeld, Foundations of the new field theory. Proc. Roy. Soc. (London) **A144**, 425–451 (1934)
32. A. Sen, Black hole entropy function and the attractor mechanism in higher derivative gravity. JHEP **0509**, 038 (2005). http://xxx.lanl.gov/abs/hep-th/0506177
33. B. de Wit, N = 2 symplectic reparametrizations in a chiral background. Fortsch. Phys. **44**, 529–538 (1996). http://xxx.lanl.gov/abs/hep-th/9603191
34. G.L. Cardoso, B. de Wit, T. Mohaupt, Corrections to macroscopic supersymmetric black hole entropy. Phys. Lett. **B451**, 309–316 (1999). http://xxx.lanl.gov/abs/hep-th/9812082
35. M. Bershadsky, S. Cecotti, H. Ooguri, C. Vafa, Holomorphic anomalies in topological field theories. Nucl. Phys. **B405**, 279–304 (1993). http://xxx.lanl.gov/abs/hep-th/9302103
36. M. Bershadsky, S. Cecotti, H. Ooguri, C. Vafa, Kodaira-Spencer theory of gravity and exact results for quantum string amplitudes. Commun. Math. Phys. **165**, 311–428 (1994). http://xxx.lanl.gov/abs/hep-th/9309140
37. G.L. Cardoso, B. de Wit, J. Käppeli, T. Mohaupt, Stationary BPS solutions in N = 2 supergravity with R^2 interactions. JHEP **0012**, 019 (2000). http://xxx.lanl.gov/abs/hep-th/0009234
38. T.W. Grimm, A. Klemm, M. Mariño, M. Weiss, Direct integration of the topological string. JHEP **0708**, 058 (2007). http://xxx.lanl.gov/abs/hep-th/0702187

Chapter 2
Black Holes in String Theory

Iosif Bena, Sheer El-Showk and Bert Vercnocke

Abstract These lectures notes provide a fast-track introduction to modern developments in black hole physics within string theory, including microscopic computations of the black hole entropy as well as construction and quantization of microstates using supergravity. These notes are largely self-contained and should be accessible to students at an early PhD or Masters level. Topics covered include the black holes in supergravity, D-branes, Strominger-Vafa's computation of the black hole entropy via D-branes, AdS-CFT and its applications to black hole phyisics, multicenter solutions, and the geometric quantization of the latter.

2.1 Why Black Holes?

This is a lecture series about black holes, but that does not mean that every little detail about what a black hole is will be explained. Our purpose is not to give a comprehensive review of the subject, but rather to fast-track interested students and researchers to the "juicy" aspects of the field using as little sophistication as possible. Students who wish to devote the rest of their life to the study of black holes in string theory, while they may find this overview useful, are urged to follow the "classical" route of learning first all the gory details of string theory, then all the gory details of black holes in general relativity, then read ten or fifteen foundational articles from the glorious nineties, as well as a few more recent ones in their preferred sub-area of research.

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S. Bellucci (ed.), *Black Objects in Supergravity*, Springer Proceedings in Physics 144,
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We start these notes in this section by reviewing the main questions of black hole physics. For more details on the (GR) aspects of black holes, see for instance the Course de Physique at IPhT by Nathalie Deruelle in 2009 [1] and references therein.

Note: In these lecture notes, we choose for a pedagogical referencing style. We refer to useful books, lectures and reviews as much as possible, and we will give only the most relevant original papers when appropriate. More references can be found in the reviews and pedagogical papers we refer to.

2.1.1 Classical Black Holes

Black holes are *classical* solutions that appear naturally in GR. The first black hole metric was written down for the first time almost a century ago by Karl Schwarzschild (although at that point it was only used to model the geometry outside of a spherically symmetric object as the Sun or the Earth). It is a solution to the Einstein equations determined by one parameter, the mass.

Very crudely, we can picture such a black hole as a region of spacetime in which things can fall, or be thrown in, but nothing comes out, see Fig. 2.1 for a cartoon. The boundary from which no round-trip tickets are available any more, is called the event horizon. The name "black hole" fits very well: classically, a black hole does not emit anything, not even light.

We can say more than just drawing cartoons. In GR, there is a very well-defined picture one can make of a spacetime that showcases its causal properties, while it still fits on a page: the Penrose diagram. It can be obtained by performing a conformal transformation (scaling) on the metric. The Penrose diagram is then a two-dimensional picture of the conformal metric. The key feature is that time-like surfaces (light-rays) are still at 45° angles and we can therefore easily infer the causal structure of the spacetime. The Penrose diagram for the Schwarzschild black hole is shown in Fig. 2.2.

Any object travels on a causal curve: it has to stay within its future lightcone. We see that once something falls into the horizon, it can never get out again. From

Fig. 2.1 A *classical black hole* is the ultimate solution for those smelly diapers of your one-year-old daughter, nagging mother-in-laws or ageing national monuments: you can throw things in, but nothing comes out

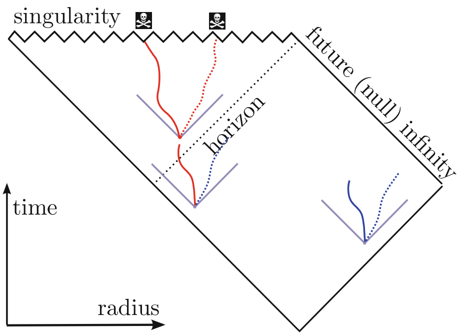


Fig. 2.2 The Penrose diagram for the Schwardschild metric. Some lightcones and particle trajectories are drawn outside and inside the *black hole* horizon. Note that the singularity (*sawtooth line*) is in the causal future of any object that falls behind the horizon

the Penrose diagram, we also see that anything that falls in will further collapse and eventually hit the singularity.

Two important observations where made by Carter, Hawking, Penrose…from the 1960s onwards:

- No memory in horizon region of what the black hole is made of, this region is smooth and has no special features:
  "Black holes have no hair"

- Black hole uniqueness theorems (1960s–1970s):
  A static black hole is fully characterized by its mass.[1]

A black hole of a certain mass could thus be made up out of anything: ipods, elephants, grad students…from the outside it will look the same.

2.1.2 A Little Bit of Quantum Mechanics

What happens if we add quantum mechanics to the game? The region of spacetime around the horizon of a black hole has a curvature and hence a certain energy density. We know that in QFT, energy can decay into a particle-antiparticle pair. This idea has led Hawking to perform a semiclassical analysis of QFT in a black hole background. Through the Hawking process, pairs will be created and once in a while one of the two falls into the black hole horizon, while the other escapes off to spatial infinity.

[1] The more general time-independent solution, a stationary black hole, is fully determined by its mass ánd angular momentum. When GR is coupled to an electromagnetic field, a black hole can have an electric and a magnetic charge as well. However, there is no additional memory of what formed the black hole: there are no higher multipole moments etc.

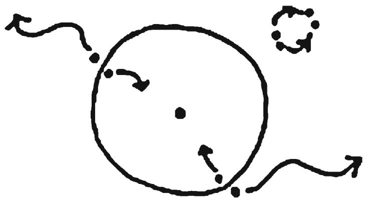


Fig. 2.3 A cartoon of the Hawking process. The *black hole* geometry is pictured as a point, the singularity, surrounded by a horizon. A QFT calculation in the *black hole* spacetime leads to pairwise particle creation such that close to the horizon, one of these particles can fall into the horizon, the other escaping to infinity

The net result is that the black hole mass is lowered and energy, under the form of thermal radiation, escapes to infinity, see Fig. 2.3.

The black hole behaves as a black body, with a temperature proportional to the strength of the gravitational field at the horizon. One finds this temperature is inversely proportional to the black hole mass:

$$T = \frac{\hbar c^3}{k_B} \frac{1}{8\pi G_4 M} \simeq 6 \times 10^{-8} \left(\frac{M}{M_\odot} \right) \text{Kelvin} \tag{2.1}$$

where $M_\odot \simeq 2 \times 10^{30}$ kg is the mass of the sun. The bigger the black hole is (more mass), the lower its gravitational field a the horizon and hence how lower its temperature. For a typical astrophysical black hole, ranging from several to several million solar masses, this is a very small temperature.

By the laws of black hole thermodynamics, a black hole also has an entropy. It was first conjectured by Bekenstein [2] and later proven by Hawking [3] that this entropy is proportional to the area of the black hole horizon:

$$S_{BH} = \frac{A_H}{4G_N}, \tag{2.2}$$

where G_N is Newton's constant, related to the Planck length as $G_N \sim l_P^2$. In Planck units, we thus have $S_{BH} = A_H/4l_P^2$ with $l_P \simeq 1.6 \times 10^{-35}$ m. The entropy of a typical black hole will thus be very large. For a Schwarzschild black hole, we find that the Bekenstein-Hawking entropy is proportional to the square of the black hole mass:

$$S_{Schw} \simeq 10^{76} \times \left(\frac{M}{M_\odot} \right)^2 . \tag{2.3}$$

This is a huge entropy! For a solar mass black hole (which would have a radius of about 3 km) we find 10^{76}, for the black hole in the center of our galaxy of several million solar masses, we find about $S_{Gal} \simeq 10^{90}$.

How should we understand this entropy? Boltzmann has taught us that the entropy is related to a number N of microstates, microscopic configurations with the same macroscopic properties:

$$S = \log(N). \tag{2.4}$$

We would hence conclude that the quantum mechanics of black holes leads to an incredibly large amounts of microstates: $N_{QM} \sim e^{10^{76}}$. However, in the classical GR picture we do not understand this number, as there is only one stationary solution with the black hole mass (the macroscopic parameter of the configuration): $N_{GR} = 1$. This numerical discrepancy is the largest unexplained number in theoretical physics.[2]

2.1.3 Problems

- **Where are the microstates?** Maybe the $N = \exp(S_{BH})$ states live in the region of the singularity, and GR just does not see them? Recent arguments by Mathur and others point out that this would not solve the information paradox (second point), and black hole microstates should differ from the black hole significantly also at horizon scales. Such 'microstate geometries' do not exist within general relativity.
- **Information paradox**. The Hawking radiation process has positive feedback: as a black hole radiates, it loses mass, increasing its temperature, which increases the rate of radiation. If we wait long enough, by the Hawking process a black hole will continue radiating until all of its mass is radiated away and we are left with only thermal radiation. This leads to a problem: where has the information of the initial state gone? Once a black hole forms, the spacetime is completely determined by the mass. All other information of the initial state that went into the black hole seems gone: whether we make a black hole out of 2 seven-ton elephants, or 200 seventy-kilogram graduate students, the classical black hole geometry is indistinguishible. As the black hole evaporates, only the thermal radiation comes out, there is no information about the initial state in the Hawking radiation neither.

Note that a black hole we start from that goes to a universe without black hole, but filled with thermal radiation cannot be obtained by unitary evolution. People have come up with many ideas to solve this problem: maybe physics is not unitary, or the black hole does not evaporate completely and there is a remnant with high entropy, and other explanations. Not one has proven satisfactory. Currently, the most popular viewpoint among string theorists is that the physics is nuitary, the information paradox is just an artefact of semiclassical gravitational physics.

We would like to solve these problems. The solution is in the study of black holes in a quantum gravity theory, that can unify classical GR with quantum mechanics.

[2] For comparison, the famous cosmological constant problem is the large ratio $\Lambda_{QFT}/\Lambda_{obs} \sim 10^{120}$ between the "expected" value Λ_{QFT} and the observed value Λ_{obs}. This number is peanuts compared to the required number of black hole microstates!

String theory is a powerful mathematical framework that does exactly this. We do not have to believe that this theory describes the real world. As a quantum gravity theory, string theory can be tested by its answers to the issues related to black holes (information problem, entropy problem). If it does not pass this test, and cannot solve these problems, we throw it to the garbage as a quantum theory of gravity. If it does, we can start thinking about other tests and problems to attack—and maybe start believing it describes the real world after all.

2.2 Building Blocks

In this section, we provide the tools to construct black hole solutions of string theory. It is not our intention to give a lecture series on string theory: We will not tell you how to build the computer, but how to *programme* it. For further information on string theory basics, see the textbooks [4–9], and for supergravity, the low-energy limit of string theory, see [10].

2.2.1 Caught in the Web

String theory is a framework that has grown dynamically over the past thirty or so years. Various limits of this theory have been studied, see Fig. 2.4. Historically all the corners of this diagram were constructed as different theories and only about 15 years ago it was realized that they were all related through various dualities, and can be seen as limits of one theory. We reserve the term "string theory" for the encompassing framework.[3]

In these lectures, we will only consider M-theory, type IIA and type IIB string theory. The natural geometric interpretation of M-theory is 11-dimensional, while the type II strings live in ten dimensions. We will mainly study the low-energy limits of string theory. "Low energy" is relative. We mean that we stick to the zero mass

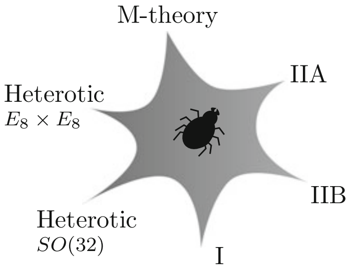


Fig. 2.4 We should view string theory as a web, of which we understand several corners, where perturbative and other techniques can be used. In these lectures, we will only consider M-theory, type IIA and type IIB string theory

[3] Often people refer to the entire framework as "M-theory". We like to view this eleven-dimensional theory as one of the corners of the string web instead.

Table 2.1 The theories we work in

Theory	Low-energy limit
M	11d supergravity
IIA	10d IIA supergravity
IIB	10d IIB supergravity

fields of the string spectrum. The low-energy limits of string theories are supergravity theories: gravity theories that are extensions of general relativity with other fields, whose couplings are fixed by the requirement of supersymmetry. See Table 2.1.

2.2.2 An Analogy for M Theory

To get a grip on the field content of these higher-dimensional beasts, we first make an analogy with Maxwell theory in four dimensions.

Maxwell Theory

The action for Maxwell theory coupled to gravity is:

$$S = \int d^4x \sqrt{-g}\,(R + \tfrac{1}{4} F_{\mu\nu} F^{\mu\nu}), \tag{2.5}$$

where $F_{\mu\nu}$ is the electromagnetic field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \equiv 2\partial_{[\mu} A_{\nu]}$.
What are the fundamental objects in this theory?

- **Electrons**. An electrically charged particle with electric charge e couples to the electric field as
$$S_{el} = e \int \left[A_\mu \frac{dx^\mu}{d\tau} \right] d\tau, \tag{2.6}$$
where τ parameterizes the world-line of the particle and $x^\mu(\tau)$ describes the embedding of the particle's world-line in space-time (Fig. 2.5). A particle that is not moving in a certain reference frame, couples to the time component of the electric field as $e \int A_0 dx^0$ with $x^0 = \tau$. The electric field profile sourced by such a field is
$$A_0 = \frac{e}{r}, \qquad \boldsymbol{E} = \nabla A_0 = -\frac{e}{r^2}\boldsymbol{u}_r, \tag{2.7}$$
where $\boldsymbol{u}_r$ is a unit vector in the radial direction. Note that a moving electron couples to magnetic components A_i of the gauge field as well.
The electric field of a charged particle solves Maxwell's equation's (the equations of motion for the field A_μ) with a delta-function source:
$$\nabla^2 A_0 = e\delta(\boldsymbol{r}). \tag{2.8}$$

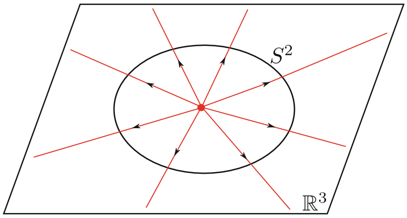


Fig. 2.5 Magnetic field lines from a magnetic monopole. The total charge is measured by integrating the flux over a surface (for instance a two-sphere) surrounding the source

- **Magnetic monopoles**. In theory, there can also be magnetically charged particles in four dimensions. These are monopole sources of the magnetic field. The charge of these particles can be measured by integrating the magnetic field lines over a two-sphere surrounding the charge (see Fig. 2.6a):

$$g_M = \frac{1}{4\pi} \int_{S^2} F_{\mu\nu} dx^\mu dx^\nu. \tag{2.9}$$

The magnetic monopole sources a profile for the magnetic field. For a flat metric $g_{\mu\nu} = \eta_{\mu\nu}$, we have:

$$F_{ij} = -\epsilon_{ijk} B^k. \tag{2.10}$$

The coupling to the electromagnetic field is found in an indirect way. Just as the electron couples to the gauge field, the magnetic monopole couples to the (Hodge) *dual* electric field:

$$\tilde{F}_{\mu\nu} = \frac{1}{2}\sqrt{-g}\epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}, \tag{2.11}$$

as

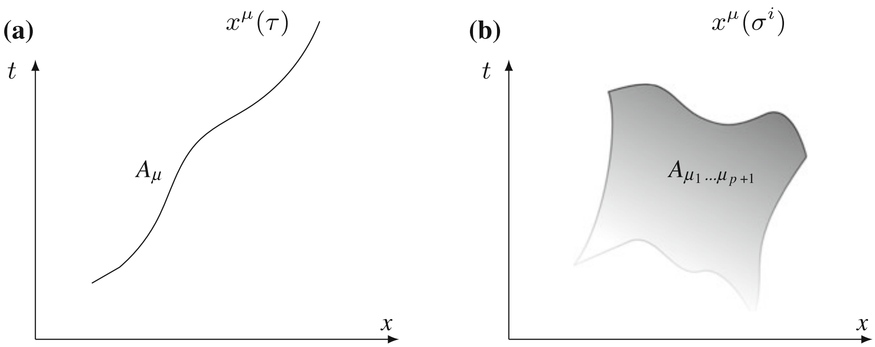


Fig. 2.6 A charged particle traces out a one-dimensional world line, its higher dimensional analogue (a p-brane) traces out a $(p+1)$-dimensional world volume, sourcing a $(p+1)$-form potential. For a p-brane, we parametrize the world volume in terms of σ^i ($i = 0 \ldots p$). **a** A charged particle. **b** A p-brane

$$S_{mag} = g_M \int \left[\tilde{A}_\mu \frac{dx^\mu}{d\tau} \right] d\tau. \tag{2.12}$$

The dual field sourced by a static magnetic monopole is then

$$\tilde{A}_0 = \frac{g_M}{r}. \tag{2.13}$$

In flat space with metric $ds^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$, this gives the magnetic field in polar coordinates (using 2.11)

$$F_{\theta\phi} = g_M \sin\theta\,, \qquad \text{or} \qquad \boldsymbol{B} = -\frac{g_M}{r^2}\boldsymbol{u}_r. \tag{2.14}$$

where $\boldsymbol{u}_r$ is a unit vector in the radial direction.

Exercise 2.2.1 *Show that the magnetic monopole field solves the Bianchi identity up to a delta-function source:*

$$\partial_r F_{\theta\phi} + \partial_\theta F_{\phi r} + \partial_\phi F_{r\theta} = g_M \delta(\boldsymbol{r}). \tag{2.15}$$

Hint: integrate the equation on a ball of arbitrary radius R centered at $\boldsymbol{r} = 0$ (ball means a 'filled' two-sphere here). You can use the integral $\int_{r=0}^{r=R} \sqrt{g} dr\theta d\phi$ with the metric

$$ds^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{2.16}$$

Eleven-Dimensional Supergravity

The features of eleven-dimensional supergravity (the low-energy limit of M-theory) are very similar to those of four-dimensional Einstein-Maxwell theory. The bosonic fields are again the metric and a gauge field, which is now a *three*-form potential $A_{\mu\nu\rho}$, instead of the one-form of Maxwell theory. These fields and their couplings are dictated by supersymmetry: supergravity theories are theories of gravity that are (locally) supersymmetric, and due to this extra symmetry, the possible fields and their couplings are constrained.

The three-form has a four-form field strength. We will often use form notation instead of writing everything out in components. The four-form field strength of M-theory is written as

$$F_4 = \frac{1}{4!} F_{\mu\nu\rho\sigma} dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma. \tag{2.17}$$

where $F_{\mu\nu\rho\sigma}$ are the components of a four-form gauge field

$$F_{\mu\nu\rho\sigma} = 4!\partial_{[\mu} A_{\nu\rho\sigma]}. \tag{2.18}$$

The Lagrangian for eleven-dimensional supergravity is [11]

$$S = \int d^{11}x \sqrt{-g} \left(R + \frac{1}{2}\frac{1}{4!} F_{\mu\nu\rho\sigma} F^{\mu\nu\rho\sigma} \right) + \frac{1}{3} \int A_3 \wedge F_4 \wedge F_4, \tag{2.19}$$

The last term does not contain the metric, it is topological. For single electric or magnetic sources this so-called Chern-Simons term does not contribute. We focus only on the other terms in the action, which are the straightforward generalization of Einstein-Maxwell theory.

What are the fundamental charged objects of this theory?

- **Electric object: M2-brane**. The counterpart of the electron (which couples to the gauge field component A_0) is an object that couples to the electric component of the three-form potential C_{0ij}. Because of the additional directions, this potential couples naturally to a two-dimensional extended object or membrane, with a three-dimensional world volume Σ (generalizing the particle with a one-dimensional world volume). This membrane of M-theory is also called M2-brane. For a membrane extended along the directions x^1, x^2 we have:

$$S_{M2} = Q_{M2} \int_{\Sigma} C_{012} dx^0 dx^1 dx^2. \tag{2.20}$$

  where Q_{M2} is proportional to the charge of the M2-branes.
- **Magnetic object: M5-brane**. In analogy with the magnetic particle, we can also consider a magnetic monopole charge for the field strength $F_{\mu\nu\rho\sigma}$. To measure its charge, we have to integrate the field strength over a *four*-sphere, see Fig. 2.7:

$$Q_{M5} = \frac{1}{\mathrm{vol}(S^4)} \int_{S^4} F_{\mu\nu\rho\sigma} dx^\mu dx^\nu dx^\rho dx^\sigma, \tag{2.21}$$

  From Fig. 2.7 we can also find the dimensionality of the magnetic monopole of M-theory. The field lines run in a five-dimensional transverse plane (directions

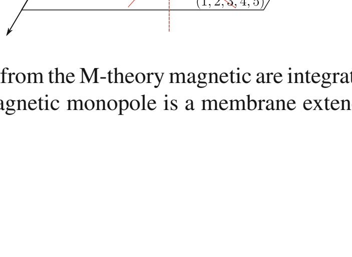


Fig. 2.7 Magnetic field lines from the M-theory magnetic are integrated over an S^4 in the transverse $\mathbb{R}^5(x^1 \ldots x^5)$. Hence, this magnetic monopole is a membrane extending in five space dimensions $(x^6 \ldots x^{10})$

1,2,3,4,5) and the magnetic monopole takes up the remaining five dimensions (6,7,8,9,11).[4] This object is called the M5-brane.

2.2.3 *Type II String Theory*

We relate ten-dimensional string theories and M-theory. See Chap. 8 of [8] for a more detailed account.

Type IIA Supergravity from Dimensional Reduction

Consider eleven-dimensional M-theory. We imagine making the direction x^{11} small and 'compactifying' it on a circle. See Fig. 2.8. What happens to the objects of M-theory? There are two distinct possibilities for each fundamental object: either the world-volume of the object is wrapped on x^{11}, meaning that one of its directions shrinks away, or the world-volume is completely inside the ten large dimensions of space-time. We summarize the possibilities for M-theory objects in Table 2.2.

An important new object is the momentum wave. Because we compactify on a circle, momentum along x^{11} is quantized and momentum waves excitations have a discrete mass spectrum:

$$m = \frac{1}{\ell_{11}}, \frac{2}{\ell_{11}}, \frac{3}{\ell_{11}} \dots \tag{2.22}$$

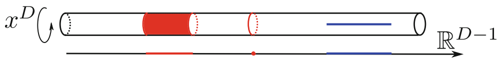


Fig. 2.8 Curling up one out of D dimensions makes a space-time look essentially $(D-1)$-dimensional. An object that is wrapped on the compact dimension has a world-volume of one dimension lower (a membrane becomes a string, a string becomes a point etc.), an unwrapped object remains of the same dimension

Table 2.2 Objects in IIA after compactifying M-theory on a circle

M-theory		IIA supergravity	
Object	Directions	Object	Directions
M2	0, 1, 11	String	0, 1
	0, 1, 2	Membrane	0, 1, 2
M5	0, 1, 2, 3, 4, 11	4d membrane	0, 1, 2, 3, 4
	0, 1, 2, 3, 4, 5	5d membrane	0, 1, 2, 3, 4, 5
Mom. wave	0, 11	Particle	0

[4] We choose to write the time directions as x^0 and space time directions $x^1, x^2, \dots$. However, we choose the 'eleventh' dimensions to be x^{11} and skip x^{10}.

where ℓ_{11} is the radius of the circle. Upon compactification, momentum waves have quantized excitations and become point particles.

We can now interpret all these new objects after compactification. The resulting ten-dimensional theory is called IIA string theory.[5] Its low-energy limit is IIA supergravity. It was found independently in the 1980s and only in the mid-1990s people realized its connection to eleven-dimensional supergravity and M-theory through compactification. The objects of IIA string theory, which were found earlier through quantization of the IIA string, correspond exactly to what we found above from compactifying M-theory (see [7] and references therein to guide you to the original works on the quantization of the IIA string). These are organized in two sectors[6]:

The NS–NS Sector

It contains the following objects:

- F1: the fundamental quantized string of IIA string theory. It comes from an M2-brane wrapped on x^{11}.
- NS5-brane: this is not a D-brane, but is in fact the 'magnetic monopole' associated to the 'electric' F1. It descends from the non-wrapped M5-brane.

The R–R Sector

These are Dirichlet branes, or D-branes for short. They arise from possible Dirichlet boundary conditions one can put on an open fundamental string. One finds that, depending on the type of string theory, only certain dimensionalities of submanifolds of space-time can provide such Dirichlet-boundary conditions while remaining stable objects. These are the allowed D-branes. In IIA one only finds stable D-branes of even dimensions (D0, D2, D4...). Surprisingly, one finds that these D-branes not only describe boundary conditions for strings, but they can also have a dynamics of their own. We will expand on this as we go on.

The relation of the D-branes to M-theory is:

- D0-brane: or D-particle, coming from a momentum wave along the compact eleventh dimension (eleven-dimensional metric degree of freedom).
- D2-brane: the D2-brane is an M2-brane that is not wrapped on the compact direction.

---

[5] Type IIA supergravity is one of the two possible ten-dimensional supergravity theories invariant under $\mathcal{N} = 2$ supersymmetry, namely the one for which the two supersymmetry generators (spinors) have opposite chirality. The other $\mathcal{N} = 2$ supergravity in ten dimensions is type IIB supergravity, the low-energy limit of IIB superstring theory, which has two supersymmetry generators with the same chirality.

[6] Different boundary conditions for the fermionic fields living on the world-volume of the type II string give different possible fields in the string spectrum. In the massless spectrum we observe that Neveu-Schwarz-boundary conditions (anti-periodic) give the NS-fields: metric $g_{\mu\nu}$, B-field $B_{\mu\nu}$, and dilaton ϕ. Ramond boundary conditions (periodic) give RR fields $C^{(0)}$, $C^{(2)}$, $C^{(4)}$.

- D4-brane: an M5-brane wrapped on x^{11}. It is the magnetic monopole associated to the D2-brane.
- D6-brane: yet another D-brane in the string spectrum. It descends from a certain smooth type of geometry in M-theory known as the Kaluza-Klein monopole, and is the magnetic equivalent of the D0-brane.

IIA Supergravity Action

We have seen what are the objects that appear in IIA string theory. Let us summarize the fields they couple to, and give the low-energy effective action of type IIA string theory. ('Low energy' is relative and means energies E well below the scale set by the string length $E \ll 1/l_s$. The energies reached in present-day accelerators are 'low' in this terminology.) In this limit, the only vibration modes of the string that are of relevance are the massless modes. They are described by type IIA supergravity. We are only concerned with the bosonic content of the theory, given by the following fields.

- The ten-dimensional *metric*, with components $g_{\mu\nu}$. Its excitations are gravitons.
- The *dilaton* ϕ. This is a scalar field. Its vacuum expectation value sets the value of the string coupling as $g_s = \langle e^{\phi} \rangle$. In eleven-dimensional M-theory, it is a metric component that sets the size of the eleventh direction. It plays an important role in string theory – it sets the value of the string coupling and determines the validity of perturbative string theory. When the eleventh dimension is small, we get weakly coupled IIA string theory and conversely, the strongly coupled limit of IIA theory opens up an extra space-time dimension giving M-theory. We will not consider the dilaton further.
- An *antisymmetric two-form* field with components B_{MN}. This is the gauge field three-form potential C of M-theory with one compactified direction:

$$B_{\mu\nu} \equiv C_{\mu\nu 11}. \tag{2.23}$$

  This field couples electrically to the F1 string and magnetically to the NS5-brane.
- *Higher-form gauge fields*. These are generalizations of the Maxwell field A_μ of four dimensions. We have a one-form potential with components C_μ and a three-form with components $C_{\mu\nu\rho}$.[7] The gauge field C_μ for the D0-brane is related to the eleven-dimensional metric $g^{(11)}_{\mu\nu}$ as $C_\mu = g^{(11)}_{\mu 11}$ (up to a factor involving the dilaton). Its magnetic monopole source is the D6-brane. In a similar fashion, the components of the three-form gauge field $C_{\mu\nu\rho} \equiv A_{\mu\nu\rho}$ in ten dimensions define the Ramond-Ramond three-form gauge field and they couple electrically to the D2 branes and magnetically to the D4-branes.

From now on, we use differential form notation and write B_2, C_1, C_3 (for instance $C_1 = C_M dx^M$ and $B_2 = \frac{1}{2} B_{MN} dx^M \wedge dx^N$) with associated field strengths

[7] We adopt common notation C for the Ramon-Ramond gauge field in ten dimensions that couple to D-branes, and A for the gauge field in elven dimensions that couples to M-branes.

$H_3 = dB_2$, $F_2 = dC_1$, $F_4 = dC_3$. All the fields above form the bosonic content of the type IIA supergravity action, which is, up to two derivatives, completely determined by supersymmetry to have the form [7]

$$S = \frac{1}{16\pi G_{10}} \int d^{10}x \, e^{-2\phi}\sqrt{-g}\left(R - \frac{1}{2}|H_{(3)}|^2 - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}|F_{(2)})|^2 - \frac{1}{2}|\tilde{F}_{(4)}|^2\right) \\ - \frac{1}{16\pi G_{10}} \int \frac{1}{2} B_{(2)} \wedge F_{(4)} \wedge F_{(4)}. \tag{2.24}$$

where G_{10} is Newton's constant in ten dimensions, we introduced $\tilde{F}_{(4)} \equiv F_{(4)} - C_{(1)} \wedge H_{(3)}$ and we have the notation $|F_{(n)}|^2 = \frac{1}{n!} F_{\mu_1 \dots \mu_n} F^{\mu_1 \cdots \mu_n}$ and likewise for $|H_{(3)}|^2$.

Historically, all these higher-form gauge fields were first found in the spectrum of string theory, but people had at that point (the 1980s) no idea what objects they coupled to. It took until the mid-1990s ago before it was realized that the objects the R–R fields couple to are in fact the Dirichlet-branes.

In a similar way, IIB string theory has a plethora of higher-dimensional objects. The NS-sector (including the F1 string and the NS5 brane) also appears, but IIB has only stable branes of *uneven* dimensionality, versus the *even* branes of IIA. See Table 2.3.

Dualities

One may wonder how to relate IIB to IIA and M-theory, since at this point we wrote down the fields in a rather ad hoc way. The clue lies in several dualities of the string spectrum.

S-duality

We first focus on a symmetry of the spectrum of the IIB string. We observe that the spectrum can be organised in pairs of the same dimensions: F1–D1, NS5–D5 (we

Table 2.3 Coupling of branes to n-form potentials. In ten dimensions, an $n = (p + 1)$-form potential couples to a p-brane through an electric coupling and to a $(6 - p)$ through a magnetic coupling

Potential	IIA			IIB			
	B_2	C_1	C_3	B_2	C_0	C_2	C_4
Electric	F1	D0	D2	F1	D(−1)	D1	D3
Magnetic	NS5	D6	D4	NS5	D7	D5	D3

We give the brane couplings of the NS–NS sector (F1 stands for fundamental string, NS5 for the magnetically dual five-brane) and R–R sector of type IIA and type IIB string theory. (We do not consider the IIA (magnetic) D8-brane and its electric counterpart. The D(−1) brane should be seen as an instanton.)

also have NS7–D7, but that example is a little special so we ignore it further and refer the interested reader to the literature [12, 13] and [14–16]). This corresponds to the pairing of the B-field $B_{\mu\nu}$ with the RR two-form $C_{\mu\nu}$ and the same for their magnetic dual fields $\tilde{B}$ and $\tilde{C}$ (which are in fact 6-forms as Exercise 2.2.2 asks you to show).

Exercise 2.2.2 *Generalize the dualization rule (2.11) for two-forms in four space-time dimensions to arbitrary dimensions D and arbitrary p-forms (you need the inverse metric to raise indices). This operation is called Hodge duality (see for instance []). Use this to write down which form couples to which brane in both IIA and IIB theory.*

What about the D3 brane? What does it pair up with? The D3-brane couples electrically to a four-form potential, with a five-form field strength F_5. In fact, in IIB supergravity, F_5 obeys the property

$$F_{\mu_1\ldots\mu_5} = \tilde{F}_{\mu_1\ldots\mu_5} \equiv \frac{1}{5!}\sqrt{-g}\epsilon_{\mu_1\ldots\mu_5\ \mu_6\ldots\mu_{10}}F^{\mu_6\ldots\mu_{10}} \tag{2.25}$$

and therefore, using Exercise 2.2.2, the five-form field strength that couples to the D3 brane is self-dual $F_5 = \tilde{F}_5$. Hence the D3 brane ‘pairs up with itself’: the D3-brane is a dyon, it is both an electrically charged brane and a magnetic monopole! We will see below that this dyonic nature separates the D3-brane from the other branes.

There exists a clean symmetry interchanging the fields B_2 with C_2, while leaving F_5 unaltered. This transformations is called S-duality and it interchanges F1’s with D1’s, D5’s with NS5’s and leaves the D3 brane as it is. It is a very useful transformation in navigating through the zoo of brane solutions.[8]

T-duality

There is another symmetry that maps the string spectra of *different* string theories onto each other. Imagine wrapping the IIA string on a circle of radius R. A string wrapped on the compact dimension has a mass proportional to its tension T_{F1} times the radius R of the string. The string length ℓ_s is related to the string tension as $T_{F1} = 1/2\pi(\ell_s)^2$, so this mass comes in fundamental units of R/ℓ_s^2. The number of units is a topological number and describes how many times the string winds along the compactified dimensions. We call them (string) winding modes.

We can also put momentum modes on the string. These momentum modes should be viewed as oscillations travelling on the string. Again, these fundamental string excitations come in quanta, proportional to $1/R$; for larger radius R, the energy cost of a momentum mode goes down. We can play the same game for IIB string theory compactified on a circle of radius $\tilde{R}$. See Table 2.4 and Fig. 2.9.

[8] In the near-horizon geometry of a D3 brane, which is $AdS_5 \times S^5$ as we will see below, S-duality becomes the strong-weak coupling duality of $N = 4$ super Yang-mills, the theory dual to the $AdS_5 \times S^5$ background through the AdS/CFT duality.

Table 2.4 The mass m of winding and momentum modes of IIA string theory compactified on a circle of radius R and IIB theory compactified on a circle of radius $\tilde{R}$

IIA	Winding	Momentum	IIB	Winding	Momentum
$m =$	$\frac{R}{(\ell_s)^2}$	$\frac{1}{R}$	$m =$	$\frac{\tilde{R}}{(\ell_s)^2}$	$\frac{1}{\tilde{R}}$
	$\frac{2R}{(\ell_s)^2}$	$\frac{2}{R}$		$\frac{2\tilde{R}}{(\ell_s)^2}$	$\frac{2}{\tilde{R}}$
	$\frac{3R}{(\ell_s)^2}$	$\frac{3}{R}$		$\frac{3\tilde{R}}{(\ell_s)^2}$	$\frac{3}{\tilde{R}}$
	$\dots$	$\dots$		$\dots$	$\dots$

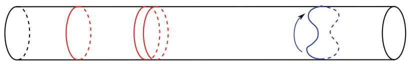


Fig. 2.9 *Left* (in *red*): a string winding one or several times around the compact dimensions, *right* (*blue*): a vibrational or momentum mode of the string

It turns out that the spectra of IIA and IIB compactified on circles of radius R and $\tilde{R} = (\ell_s)^2/R$ are *exactly* mapped into each other under T-duality: momentum modes map to winding modes and vice versa. See also Table 2.4. We reserve p for the units of momentum charge and F1 for the amount of string winding. Schematically, T-duality thus acts as:

IIA		**IIB**
F1	$\longleftrightarrow$	p
p	$\longleftrightarrow$	F1

The symmetry of the string spectra in these two different string theories opens up a huge portion of parameter space where we can actually have a geometric interpretation of string theory. Say we consider type IIA string theory. As long as R is large compared to the string scale, we have a pretty good control because string excitations behave as particles and we can use the supergravity approximation (action contains no more than two space-time derivatives). However, when the size of the circle is small compared to the string length scale, corrections due to the stringy nature are huge and we lose this control. Then T-duality makes it possible to go to type IIB theory with $\tilde{R} \gg \ell_s$. (Note that for circle radius $R \simeq \ell_s$, we still cannot say too much.)

Dualities for D-branes

Consider the setup of Fig. 2.10. We compactify string theory on a circle. A brane that is wrapped on this circle, will no longer extend along this direction after T-duality. Conversely, a D-brane that does not wrap the T-duality circle, will become a D-brane of one dimension higher wrapping the circle after T-duality.

To get the gist of it, we apply T-duality on the (supersymmetric) intersection of two species of D-branes. Let us start from a D3-D3 brane intersection in type IIB

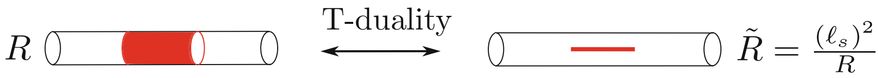


Fig. 2.10 Under T-duality, a D-brane wrapping the *circle* is mapped to a D-brane of one dimension lower and vice versa

$$\begin{array}{r|l} \text{IIB : D3} & 0\ 1\ 2\ 3 \\ \text{D3} & 0\ \ \ \ \ \ \ \ 3\ 4\ 5 \end{array}$$

Say we compactify the 3-direction. Under a T-duality to IIA, we get the branes:

$$\begin{array}{r|l} \text{IIA D2} & 0\ 1\ 2 \\ \text{D2} & 0\ \ \ \ \ \ \ \ 4\ 5 \end{array}$$

We can continue on this, see Exercise 2.2.3.

Exercise 2.2.3 *Show that three additional T-dualities on the two orthogonal D2-branes, along directions 1, 2 and 3 give the D1-D5 brane intersection:*

$$\begin{array}{r|l} \text{IIB : D1} & 0\ \ \ \ \ \ \ \ 3 \\ \text{D5} & 0\ 1\ 2\ 3\ 4\ 5 \end{array}$$

We will use this brane setup ('D1-D5 system') a lot in the study of black holes and their entropy.

Similarly, we can consider S-dualities. For instance, the D1-D5 setup of Exercise 2.2.3 becomes after S-duality:

$$\begin{array}{r|l} \text{IIB : F1} & 0\ \ \ \ \ \ \ \ 3 \\ \text{NS5} & 0\ 1\ 2\ 3\ 4\ 5 \end{array}$$

We see that the dualities give some insight in an entire zoo of complicated D-brane configurations. On the level of supergravity, they form a solution generating tool (see the next section). We can interpret all these two-brane intersections as really one solution, which takes on different forms in different 'duality frames'. We can get the supergravity solution in any frame in no time from the T-duality rules. This applies equally well to any other brane solution.

We will make extensive use of T- and S-dualities on black hole solutions. This will map to black holes which may look a bit different, but all have the same physical properties (entropy, temperature…). We will always work in the duality frame most adapted to the questions we are asking at that moment. In particular, we will often work in the D1-D5 duality frame of Exercise 2.2.3.

2.2.4 *p-brane Supergravity Solutions*

Let us consider some actual supergravity limits of D*p*-brane solutions. For further references, see the complete, but extremely short account of [17], some more information for instance in [18, 19] or [20] for a more black hole oriented D*p*-brane review (p runs over the allowed integers).

2-Brane Solution

For concreteness, we discuss the D2-brane solution of IIA supergravity, extending along directions 0,1,2 (time and two space directions). As in the analogy with electromagnetism, this brane sources a three-form potential C_{012}. It has a non-zero tension or mass density and hence it also couples to the metric. There is a third field it sources, the dilaton.

The exact way the D2 brane source affects those fields, is through one function of the space-time coordinates. We call that function Z. One finds the metric

$$ds^2 \equiv g_{\mu\nu} dx^\mu dx^\nu = Z^{-1/2}(-dx_0^2 + dx_1^2 + dx_2^2) + Z^{1/2}(dx_3^2 + \ldots + dx_9^2) \quad (2.26)$$

and the other non-zero fields are the three-form that couples to the 2-brane and the dilaton:

$$C_{012} = Z^{-1}\,, \qquad e^\phi = Z^{1/4}. \quad (2.27)$$

We will not consider the dilaton ϕ any further. Concentrating on the other fields, we see that the solution has Lorentz invariance along the D2 brane directions 0, 1, 2 and Euclidean symmetry in the transverse directions.

The D2 brane behaves as a point particle in the transverse $\mathbb{R}^7$. The function Z plays the role of the Maxwell potential in the transverse $\mathbb{R}^7$. From the supergravity equations of motion, one finds that it obeys the Laplace equation on $\mathbb{R}^7$:

$$\Delta_7 Z = 0. \quad (2.28)$$

In the presence of sources, this is modified to

$$\Delta_7 Z = \rho_{D2}. \quad (2.29)$$

For a stack of N_{D2} D2-branes sitting at the origin of our coordinate system, the source is a delta function $\rho_{D2} = N_{D2}\delta(\boldsymbol{r}_7)$ and we find the solution

$$Z = 1 + \frac{N_{D2}}{r^5}, \quad (2.30)$$

where r is the radius of the transverse space $r^2 \equiv x_3^2 + \dots x_9^2$. The integration constant can always be set to one by a constant rescaling of the coordinates.

As $r \to 0$, we approach the D2-brane source and $Z \to \infty$. From the expression for the metric, we see that the $\mathbb{R}^{1,2}$ factor shrinks, while the $\mathbb{R}^7$ blows up. This is not just a coordinate singularity, but $r = 0$ is a singular locus in space-time. This can be seen from the three-form potential C_{012}. It goes to zero at $r = 0$ but the energy of the C-field

$$E = \frac{1}{4!} F_{\mu\nu\rho\sigma} F_{\mu'\nu'\rho'\sigma'} g^{\mu\mu'} g^{\nu\nu'} g^{\rho\rho'} g^{\sigma\sigma'} \tag{2.31}$$

blows up as $r \to 0$ and we conclude that the D2 brane solution contains a singularity, which is not shielded by a horizon ('naked singularity').[9]

Note that the equations of motion are linear in the sense that we can add multiple (singular) D2-brane sources:

$$\Delta_7 Z = N_a \delta(\boldsymbol{r} - \boldsymbol{r}_a) + N_b \delta(\boldsymbol{r} - \boldsymbol{r}_b) + \dots . \tag{2.32}$$

We consider all D2-branes of the same 'species', with the world volume along the 0, 1, 2 directions.

Then the only thing that changes is that the function Z becomes a sum of harmonic functions, sourced at different locations:

$$Z = 1 + \frac{N_b}{|\boldsymbol{r} - \boldsymbol{r}_b|^5} + \frac{N_a}{|\boldsymbol{r} - \boldsymbol{r}_b|^5} + \dots . \tag{2.33}$$

We see that this solution can describe any density of D2-branes, even a continuous one.

D3-Brane from T-duality

Start from a continuous distribution of D2 branes along a line in the transverse space (this is also called 'smearing' the D-brane charge). Say that we put this smeared D2-branes along the x_7 direction, see Fig. 2.11.

Because the solution is now homogeneous in x_7, we can compactify this direction. Then the solution for such a continuous distribution of D2-brane charge on a finite line segment goes as:

$$Z = 1 + \frac{N_{D2}/L_7}{r^4}, \tag{2.34}$$

[9] Although there is a naked singularity in the supergravity solution, as a solution to string theory, a D-brane is well-defined. As $r \to 0$, the dilaton ϕ blows up. Since it sets the length of the eleven-dimensional compactification circle of M-theory, the eleventh dimension decompactifies near $r \to 0$. We hence get the near-M2-brane solution of eleven-dimensional M-theory, which is well-defined in all of space-time.

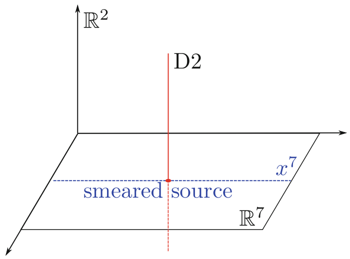


Fig. 2.11 A D2 brane smeared along x_7

with L_7 the length of the compactified direction. Next we perform a T-duality along x_7 to a D3-brane solution of type IIB string theory. What does this solution look like? Remember that the size of the compact circle is inverted after this duality transformation $\sqrt{g_{77}} \to (\ell_s)^2/\sqrt{g_{77}}$, and hence the metric of the resulting solution is (we set $\ell_s = 1$ for simplicity):

$$ds^2 = Z^{-1/2}(-dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2 + dx_7^2) + Z^{1/2}ds_{6\mathrm{d}}^2. \tag{2.35}$$

The three-form gets an additional leg to become the IIB four-form:

$$C_{0127} = Z^{-1}. \tag{2.36}$$

and the solution for the function Z is

$$Z = 1 + \frac{N_{D3}}{r^4}. \tag{2.37}$$

Near-Solution and Brane Throat

What does the geometry look like close to the D3-brane? We approach the D3-brane as we take $r \to 0$. This means that in the function Z, we can effectively drop the constant and write $Z = N_{D3}/r^4$ as $r \to 0$.

To reinstate the correct dimensions, we write $Z = R^4/r^4$, with R some reference radius. First write the transverse six-dimensional space in terms of polar coordinates as

$$ds_{6\mathrm{d}}^2 = dr^2 + r^2 d\Omega_5^2, \tag{2.38}$$

Then the near-geometry of the D3-brane is

$$ds_{\mathrm{near}}^2 = \frac{r^2}{R^2}(-dt^2 + dx_1^2 + dx_2^2 + dx_7^2) + R^2\frac{dr^2}{r^2} + R^2 d\Omega_5^2. \tag{2.39}$$

What has the D3-brane done? It has opened up a "throat": as we approach $r \to 0$ from infinity, the S^5 will get smaller and smaller. But near the D3 brane it attains a finite size, set by the radius R. Note that the metric distance to $r = 0$ from any

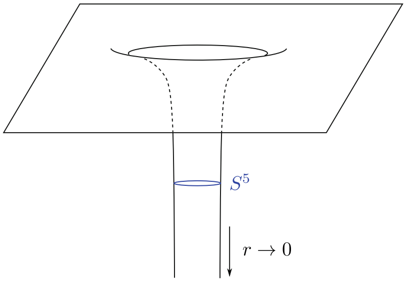


Fig. 2.12 A cartoon of the D3-brane geometry. As we approach the D3 brane, an infinity throat opens with constant transverse S^5 size

other point in space-time with $r > 0$ is actually infinite and the D3-brane throat is infinitely deep.

Physically, the D3-brane solution *forces* the $AdS_5 \times S^5$ geometry to appear.[10] This is a special feature of the D3-brane that the other D-branes do not possess (in fact, all the D0, D1…D6-branes have a naked singularity if we consider them in ten-dimensional supergravity). The origin lies in the dyonic nature of the D3 brane: it is both an electric and a magnetic charge for the four-form potential C_{0127} (Fig. 2.12).

The $AdS_5 \times S^5$ geometry is the riding horse of holography. Classical gravitational physics on this background is dual, through the AdS/CFT correspondence, to strongly coupled conformal field theory in $N = 4$ Super-Yang-Mills. We will come back to this later.

BPS Property: Mass = Charge

The charge of a D3 brane is given by integrating the gauge field that couples magnetically to it over a surface surrounding the brane (as for the magnetic monopole of electromagnetism)

$$Q_{D3} = \frac{1}{5!} \int_{S^5} F_{ijklm}\, dx^i dx^j dx^k dx^l dx^m, \tag{2.40}$$

where the field strength is $F_{ijklm} = 5! \partial_{[i} C_{jklm]}$. So far, we have only given the electric component of the gauge field C_{0123}. Exercise 2.2.4 asks you to derive the magnetic component of the field strength: since the five-form F_5 of IIB string theory is self-dual, it *must* have magnetic components as well.

Exercise 2.2.4 *Derive, using the duality*

$$F_{\nu_1 \ldots \nu_5} = \tilde{F}_{\nu_1 \ldots \nu_5} \equiv \frac{1}{5!} \sqrt{-g} \epsilon_{\nu_1 \ldots \nu_5\, \mu_1 \mu_2 \mu_3 \mu_4} g^{\mu_1 \mu_1'} g^{\mu_2 \mu_2'} g^{\mu_3 \mu_3'} g^{\mu_4 \mu_4'} g^{\mu_5 \mu_5'} F_{\mu_1' \mu_2' \mu_3' \mu_4' \mu_5'}. \tag{2.41}$$

and the expression for the electric components of the field strength

[10] For the aficionados: this is the same mechanism that forces the extremal Reissner-Nordstrom black hole to have a near-horizon region of the form $AdS_2 \times S^2$.

$$F_{0123r} = \partial_r Z^{-1}. \tag{2.42}$$

the form of the magnetic components F_{45678}.

With this result, we find that from integrating over an S^5 at $r \to \infty$ to cover the entire flux emanating from the D3 brane, that the charge of the D-brane is

$$Q_{D3} = N_{D3}, \tag{2.43}$$

up to some numerical coefficient that we set to one for simplicity's sake.

The mass of the D3-brane can be derived from the component g_{tt} of the metric, following the prescription of Arnowit, Deser and Misner (ADM) (see [21] for more details on the ADM formalism in GR, and [20] for a discussion in p-brane space-times). In particular, when expanding this component for large r, in asymptotically flat D-dimensional space-time the leading terms for a point-like source are:

$$g_{tt} = -1 + \frac{16\pi G_N}{(D-2)\Omega_{D-2}} \frac{M}{r_{D-3}} \tag{2.44}$$

where G_N is Newton's constant and Ω_n is the area of the n-sphere of unit radius S^n. A D3-brane is effectively like a point in $D = 7$ and we see from (2.35) and (2.37) that M is proportional to the number N_{D3} of D-branes. We have not been too careful about prefactors in the expression for the metric, so we only state the dependence on g_s of the end result:

$$M_{D3} = \frac{N_{D3}}{g_s}, \tag{2.45}$$

where g_s ("g-string") is the string coupling constant. This is an interesting feature: the masses of all D-branes are inversely proportional to the string coupling constant. This should be contrasted with electromagnetism. The mass of the electron, the fundamental object, is independent of the coupling (let's call it g). On the other hand, the mass of a soliton in field theory goes as $1/g^2$. The magnetic monopole's mass has this behaviour. So we see that the D-brane is neither a fundamental object nor a soliton of string theory.

The mass of the fundamental string, the fundamental object of string theory, is independent of g_s (we have seen that the string tension, or mass density, is $T_{F1} = 1/2\pi\ell_s^2$). One finds that the mass of the NS5 brane goes as

$$M_{NS5} \sim \frac{1}{g_s^2}, \tag{2.46}$$

and the NS5 brane is really a soliton of string theory. The different dependence on g_s of the masses of all these objects shows up in the 'warp factor' Z of the supergravity solutions. We track the dependence on g_s and drop other proportionality factors, such as the string length ℓ_s. Newton's constant G_N goes as $G_N \sim g_s^2$ (this follows from the low-energy supergravity action of ten-dimensional string theories). In general we have

$$Z = 1 + \frac{G_N M}{r^{\#}}. \tag{2.47}$$

where # is the appropriate power. For a D-brane, this gives

$$Z_{\mathrm{D-brane}} = 1 + \frac{N_{\mathrm{D}} g_s}{r^{\#}}, \tag{2.48}$$

for an NS5 and a string we have

$$\begin{aligned} Z_{\mathrm{NS5}} &= 1 + \frac{N_{\mathrm{NS5}}}{r^{\#}}, \\ Z_{\mathrm{F1}} &= 1 + \frac{N_{\mathrm{F1}} g_s^2}{r^{\#}}. \end{aligned} \tag{2.49}$$

Going back to the D3 brane, we find in ‘dimensionless’ units that

$$Q_{D3} = M_{D3}. \tag{2.50}$$

We interpret this as: “the mass (density) of a D3-brane is equal to its charge (density)”.

What does this mean physically? The gravitational attraction and the electric repulsion are exactly balanced, even though both forces are huge. This is why we can have D-brane solutions with sources at many points and still remain stable. This is different in electromagnetism, where two electrons would fly apart; the electric repulsion always takes the upper hand and we cannot build multi-center electron-solutions.

Note that there is an underlying physical bound $M \geq Q$ for any charged object. When the mass is smaller than the charge, then the solution is unphysical (more on this in the next section). This bound is called BPS bound after Bogomol’nyi, Prasad and Sommerfield. Note that this bound typically appears in supersymmetric theories. In the real world, supersymmetry is not manifest and elementary particles such as electrons do not satisfy the BPS bound: $e > m_e$ in the units we are using here.

We call the equal mass and charge of the D3 brane a BPS property. The BPS-ness of the D3-brane and all the other D-branes is a consequence of supersymmetry. All D-brane solutions (and the F1 and NS5) are invariant under a set of supersymmetry transformations, and the mass of any supersymmetric object is equal to its charge in natural units.

General D*p*-Brane Solution

For completeness, we give the solution for a Dp-brane to the IIA action given in (2.24) with general p. It has the non-zero fields:

$$
\begin{aligned}
ds^2 &= H^{-1/2}(-dt^2 + dx^m dx^m) + H^{1/2}(dr^2 + r^2 d\Omega^2_{(q-1)}) \\
e^{2\phi} &= H^{-\frac{1}{2}(p-3)} \\
C_{t\,\mu_1\ldots\mu_p r} &= Z^{-1}
\end{aligned}
\tag{2.51}
$$

where H is a harmonic function

$$
Z(r) = 1 + \frac{Q_p}{r^{7-p}}. \tag{2.52}
$$

Also the Dp-branes of type IIB supergravity have this form for uneven p.

2.3 Black Hole Solutions

We discuss how to obtain black hole solutions from D-branes that are wrapped on compact spaces. For completeness, we first show how to make a black D-brane. We will later focus on supersymmetric black holes, because these are easier to construct and understand microscopically.

2.3.1 Non-extremal Black Holes

Let us forget about supersymmetry for a moment, and see if we can make a black hole with $M > Q$. We do not try to make a multi-D-brane solution or anything like that, but just want to make a black hole, or a black object, with more mass than charge. An easy such solution is a black D3-brane. Its metric is given by

$$
ds^2 = -Z^{-1/2}\left(-f(r)dt^2 + dx_1^2 + dx_2^2 + dx_3^2\right) + Z^{1/2}\left(\frac{dr^2}{f(r)} + r^2 d\Omega_5^2\right). \tag{2.53}
$$

and the gauge field takes the same form as for the ordinary D3-brane

$$
C_{0123} = Z^{-1}. \tag{2.54}
$$

When the function $f(r) = 1$, this is just the supersymmetric D3 brane we have encountered before. By adding the function $f(r)$, the D3-brane is turned into a "black brane". The function f obeys the same Laplace equation in transverse space as Z:

$$
\Delta f = 0. \tag{2.55}
$$

Typically, one considers the solution

$$
f(r) = 1 - \frac{\Delta M}{r^4}. \tag{2.56}
$$

Table 2.5 Near-horizon geometry of black D3-brane and thermal physics in field theory

String theory on $AdS_5 \times S^5$	$N = 4$ Super-Yang-Mills (4d)
Weak coupling	Strong coupling
Black hole in $AdS_5 \times S^5$ at temperature T	$N = 4$ SYM at temperature T

The warp factor $Z = c + N/r^4$, where c is a constant. When $c = 1$ the metric describes a black *membrane* in ten dimensions, with flat asymptotics. When $c = 0$, the metric describes a black *hole* in the AdS_5 factor of the $AdS_5 \times S^5$ near-horizon geometry of D3-branes (2.39). We consider the former. The charge for this solution is still given as for the normal D3 brane

$$Q = \int F_5 = N. \tag{2.57}$$

The mass (obtained from g_{tt} as before) is now

$$M = Q + \Delta M \tag{2.58}$$

We make two remarks. First note that when $\Delta M < 0$, this describes a singular solution with a naked singularity. Hence we consider $\Delta M > 0$ for physical reasons. Also, we see that unlike the supersymmetric D3 brane, two (or more) of these objects are not in equilibrium any more. Two black branes will attract and eventually collapse to a single black object, because the gravitational attraction is larger than the electrostatic repulsion.

A black hole, or black brane, that saturates the BPS bound $M = Q$ is also called extremal. Such a black object has zero Hawking temperature and does not emit radiation. When $M > Q$, the black object has a non-zero temperature and is called non-extremal. For small ΔM, the temperature is proportional to the mass excess:

$$T \sim \Delta M. \tag{2.59}$$

We see that by the $f(r)$ "black deformation", we can create a solution with non-trivial mass, charge and temperature.

This solution is very useful for holography. In the near-brane region $r \to 0$, we have $Z \sim 1/r^4$ and the black brane metric describes a black hole in $AdS_5 \times S^5$. Following the AdS/CFT correspondence, this maps to turning on a temperature in $N = 4$ Super-Yang Mills theory in four dimensions, see Table 2.5. So a black hole corresponds to warming up the field theory. Conversely, a temperature in field theory gives a black hole in AdS_5.

In conclusion, we see that by warming up the D3 brane with $f(r)$, we can study the dual field theory and its properties (conductivity, transport coefficients ...) from weakly coupled strings in the $AdS_5 \times S^5$ black hole background. We could call this field "applied string theory". A lot of people nowadays use string theory no longer as a theory that describes the real world, but as a sort of calculator that we can use to teach us valuable information in other, strongly coupled, systems.

- What about quantum effects? Quantum effects are controlled by g_s, the string coupling. In the limit we consider (horizon area A_H and charge Q very large in Planck units, for instance $A_H \gg \ell_P^2$ with ℓ_P the Planck length) such that supergravity is a good description, we expect quantum effects to not destroy the geometry. Of course, when we only consider one D-brane, this limit does not hold and the question of quantum corrections becomes really important. More on this in Sect. 2.4.1.

Note that other D-branes also have such a non-extremal version. We can get for instance a black D2-brane very easily by T-duality. See Exercise 2.3.5. Black p-branes all have

$$M > Q \qquad T > 0, \tag{2.60}$$

and they are found from a deformation of the metric by one function $f(r)$ determined by

$$\Delta_d f = 0, \tag{2.61}$$

where d is the number of transverse dimensions.

Exercise 2.3.5 *T-dualize the metric (2.53) and four-form potential (2.54) of the black D3 brane along direction x^3. Show that this becomes a smeared black 2-brane. In particular, repeat the mass calculation from the g_{tt} metric component and show that $M = Q + \Delta M$.*

2.3.2 Supersymmetric Black Holes in Four Dimensions

For the largest part of these lectures, we concentrate on supersymmetric black holes. The reason is that when a black hole solution preserves supersymmetry, it can be constructed more easily and even be understood microscopically.

Consider again the orthogonal D2-brane system

$$\begin{array}{lllllll} \mathrm{IIA}: & \mathrm{D2} & 0 & 1 & 2 & & \\ & \mathrm{D2} & 0 & & & 3 & 4 \end{array}$$

Normally any two objects that we put together would either attract or repel. However, this combination of D2-branes is supersymmetric and feels a flat potential: supersymmetry exactly amounts to canceling forces and we can put the branes together in a stable fashion.

We can even do more. It turns out to be possible to add an extra D2-brane and even a D6 brane, while still preserving supersymmetry, in the following configuration:

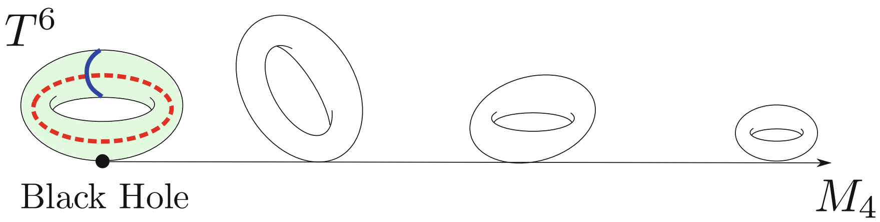


Fig. 2.13 Four-dimensional *black hole* from compactification on a six-torus. The T^6 has a different size and shape at each point of four-dimensional spacetime M_4. At the position of the *black hole*, there are branes present that are wrapped on T^6

IIA :	$D2_1$	0	1	2	–	–	–	–
	$D2_2$	0	–	–	3	4	–	–
	$D2_3$	0	–	–	–	–	5	6
	$D6_4$	0	1	2	3	4	5	6

This combination of branes experiences a flat potential and is stable. This follows from the supersymmetry it preserves. To show this, we would need to check the supersymmetry algebra; we will not do this in these lectures. For the black hole discussion, we smear the D2 branes on their transverse directions inside $x^1 \dots x^6$, which we denote by "–" and we number the branes from 1 to 4 for practical reasons.

The solution for the D2-D6 system can actually be written down! The metric takes a very intuitive form:

$$\begin{aligned} ds^2 = &-(Z_1Z_2Z_3Z_4)^{-1/2}dt^2 + (Z_1Z_2Z_3Z_4)^{1/2}(dx_7^2 + dx_8^2 + dx_9^2) \\ &+ \frac{(Z_2Z_3)^{1/2}}{(Z_1Z_4)^{1/2}}(dx_1^2 + dx_2^2) + \frac{(Z_1Z_3)^{1/2}}{(Z_2Z_4)^{1/2}}(dx_3^2 + dx_4^2) + \frac{(Z_1Z_2)^{1/2}}{(Z_3Z_4)^{1/2}}(dx_5^2 + dx_6^2). \end{aligned} \tag{2.62}$$

This solution reduces to any single brane solution when only one of the four branes is present (for say Z_1 non-trivial, and the other ones constant, $Z_2 = Z_3 = Z_4 = 1$, we retrieve the metric of the D2-brane). Amazingly, this D2-D2-D2-D6 solution, which is constructed from the same "harmonic function rule" we had for single Dp-branes ($Z^{-1/2}$ metric component when the brane worldvolume is along that direction, $Z^{1/2}$ when it is orthogonal) applies to all of the Z_i individually, regardless of the presence of the other branes. This is a very non-trivial feature and would not happen for a generic solution; it is only for such a special class of supersymmetric solutions, that we get such a nice structure at the end of the day. For more information, see the original references [22].

Exercise 2.3.6 *Consider the directions $x^1 \dots x^6$ to be compact and to describe a six-torus, or T^6. T-dualize the D2-D2-D2-D6 metric 6 times along $x^1, \dots, x^6$. Write down the resulting D4-D4-D4-D0 metric.*

In order to write down the explicit solution, we need to smear the D2 branes uniformly along the transverse directions in T^6 (the compact directions $x^1 \dots x^6$).[11] This means we have to smear $D2_1$ along directions 3456, $D2_2$ along 1256 and $D2_3$ along 1234. Then all branes are points in three-dimensional space spanned by x^7, x^8, x^9. Therefore the warp factors Z_i obey (Fig. 2.13):

$$\Delta_3 Z_i = 0 \qquad \rightarrow \qquad Z_i = 1 + \frac{Q_i}{r}. \tag{2.63}$$

We will show in the next section how the dimensionful charges Q_i are related to the integers N_i counting the number of D-branes.

Exercise 2.3.7 *Convince yourself that smearing a D-brane along a spacelike direction changes the radial dependence of Z in the correct way. For example, take a D2 brane along directions x^1, x^2 and smear it along the circular dimension x^3 with a homogeneous density $\rho_{\text{smear}} \sim 1/R_3$, where R_3 is the radius of the 3-circle. Show that in this process, the solution to the Laplace equation becomes $Z = 1 + \tilde{Q}_{D2}/r^4$ rather than $Z = 1 + Q_{D2}/r^5$, with $\tilde{Q}_{D2} = Q_{D2}/R_3$.*

What is our solution? We study the asymptotic limits.

- At $r \to \infty$: The metric becomes that of four-dimensional Minkowski spacetime times a flat torus with fixed radii:

$$ds^2_{r\to\infty} = -dt^2 + ds^2(\mathbb{R}^3) + ds^2(T^6) \tag{2.64}$$

  This means we have compactified string theory on a six-torus to a four-dimensional theory. The leading terms in the $1/r$ expansion of the g_{tt} metric component are

$$g_{tt} = 1 - \frac{1}{2}\frac{Q_1 + Q_2 + Q_3 + Q_4}{r}, \tag{2.65}$$

  and we see that the mass of this solution is (up to factors we do not care about at this point)

$$M = Q_1 + Q_2 + Q_3 + Q_4. \tag{2.66}$$

  Again, this solution saturates the BPS bound and is extremal (the minimal amount of gravitational mass for given charges and angular momenta): its mass is the sum of the charges of the individual branes; there is no binding energy. This is a consequence of supersymmetry.
- At $r \to 0$: all the 1's drop out of the warp factors Z_i and the metric becomes

[11] If we did not smear the individual D-branes making up the black hole solution, then the metric would depend on some of the internal coordinates as well. We only want dependence on four-dimensionsional space-time. In addition, if we T-dualize one D-brane, then the result becomes smeared along the dualization direction. To get a four-dimensional black hole that looks the same in all duality frames, we need to work in a duality frame where the branes are smeared on orthogonal compact directions.

$$ds^2_{r\to 0} = -\frac{r^2}{R^2}dt^2 + \frac{R^2}{r^2}(dx_7^2 + dx_8^2 + dx_9^2) + \left(\frac{Q_2 Q_3}{Q_1 Q_4}\right)^{1/2}(dx_1^2 + dx_2^2)$$
$$+ \left(\frac{Q_1 Q_3}{Q_2 Q_4}\right)^{1/2}(dx_3^2 + dx_4^2) + \left(\frac{Q_1 Q_2}{Q_3 Q_4}\right)^{1/2}(dx_5^2 + dx_6^2). \quad (2.67)$$

The six-torus spanned by the directions $x_1 \dots x_6$ has constant radii. If we go to polar coordinates in $\mathbb{R}^3$ spanned by x_7, x_8, x_9, then the metric is

$$ds^2 = -\frac{r^2}{R^2}dt^2 + R^2\frac{dr^2}{r^2} + R^2 d\Omega_2^2 + ds^2(T^6). \quad (2.68)$$

The first two terms describe AdS_2. The other terms describe an S^2 and a T^6 of constant radii. Therefore, the $r \to 0$ limit of the D2-D2-D2-D6 spacetime describes a compactification of string theory on T^6 to the four-dimensional $AdS_2 \times S^2$ geometry.
We also observe that g_{tt} vanishes as $r \to 0$ and hence $r \to 0$ describes an event horizon.

From these facts we conclude that the metric of this D2-D2-D2-D6 brane system describes a real black hole in four dimensions. We will refer to this four-dimensional black hole as the "four-charge black hole".

Note that all the Q_i appearing in the metric are positive. Only the gauge fields (which we have not given) are sensitive to the sign of the charges. The gravitational field sourced by a positive or a negative charge is exactly the same. An anti-D2 brane would have the same metric and mass as a D2-brane, but opposite electric field.

To understand the full spacetime, we make our lives a bit easier and restrict to all charges equal:

$$Q_i \equiv Q\,, \qquad Z_i \equiv Z = 1 + \frac{Q}{r}. \quad (2.69)$$

The black hole metric (2.62) becomes

$$ds^2 = -\left(1 + \frac{Q}{r}\right)^{-2} dt^2 + \left(1 + \frac{Q}{r}\right)^2 (dr^2 + r^2 d\Omega_2^2) + ds^2(T^6). \quad (2.70)$$

The T^6 has a constant metric and does not play a role in the physics. We further concentrate only on the four-dimensional part of the geometry.

Our claim is that this metric represents a black hole. A very special one even, with $M = Q$. Let us go over the textbook black hole teachings to see if our claim is valid.

1. The first black hole you learn about in classical GR, is the Schwarzschild black hole. It is a solution to vacuum gravity, described by the metric

$$ds^2_{\mathrm{S}} = -\left(1 - \frac{2M}{\rho}\right)dt^2 + \left(1 - \frac{2M}{\rho}\right)^{-1} d\rho^2 + \rho^2 d\Omega_2^2. \quad (2.71)$$

This is clearly not the same as our solution. We need to include a charge for the black hole.

2. Luckily there is also the second black hole you encounter in your favourite classical GR course. It is the Reissner-Nordström black hole. This black hole is a solution to Einstein-Maxwell theory (the Lagrangian (2.5)). It is given by

$$ds^2_{\mathrm{RN}} = -\left(1 - \frac{2M}{\rho} + \frac{Q}{\rho^2}\right) dt^2 + \left(1 - \frac{2M}{\rho} + \frac{Q}{\rho^2}\right)^{-1} d\rho^2 + \rho^2 d\Omega_2^2. \quad (2.72)$$

It has a very interesting limit

$$M = Q. \quad (2.73)$$

Then the metric becomes

$$ds^2_{\mathrm{RN}} = -\left(1 - \frac{Q}{\rho}\right)^2 dt^2 + \left(1 - \frac{Q}{\rho}\right)^{-2} d\rho^2 + \rho^2 d\Omega_2^2. \quad (2.74)$$

What does this have to do with our black hole metric, which has $g_{tt} = Z^{-2} = 1/(1 + Q/r)^2$? If we redefine

$$r = \rho - Q, \quad (2.75)$$

then we find the D-brane black hole solution on the nose!

These "$M = Q$" black holes are the ones we can construct most easily in string theory. They are extremal and are frozen at zero temperature:

$$T_{BH} = 0, \quad (2.76)$$

but have a non-zero mass M and entropy S_{BH}. The Bekenstein-Hawking entropy is given by the horizon area (we ignore numerical factors)

$$S_{BH} = \frac{A_H}{4G_N} = \pi R^4 = 2\pi\sqrt{Q_1 Q_2 Q_3 Q_4}, \quad (2.77)$$

or

$$S_{BH} = 2\pi Q^2 \quad (2.78)$$

when all $Q_i = Q$. This entropy comes from some microscopic states. Who makes this entropy? We will answer this in the sections to come.

In ρ-coordinates, this is clearly a black hole. The horizon is at $\rho = Q$, where $g_{tt} = 0$. The coordinate r we used for the string theory metric is only well-defined outside the horizon ($r > 0$).[12] Note that in the single D2-brane solution, the coordinate r is a measure for the distance from the brane at $r = 0$ (the same goes for D0,

[12] At the position of the horizon, we have a degenerate coordinate system, but there is no physical singularity at $r = 0$.

D1, D4, D5 and D6 brane solutions). For the supersymmetric black hole, space is "created" behind the $r = 0$ point and a large $AdS_2 \times S^2$ throat develops. The way to see this is by passing through a set of coordinates for which the metric extends beyond the horizon to the black hole singularity.

We come back to the regime of validity of supergravity. One can perform a calculation to show that the curvature of a D-brane goes as $1/Q$. In terms of the number N of D-branes, this gives a curvature proportional to $1/g_s N$. Therefore the solutions we have considered are only valid when $g_s N \gg 1$ (small curvature, we can trust classical physics). When the number of branes is too small and $g_s N \ll 1$, supergravity can no longer be used to describe the solution. The large curvature of spacetime takes us out of the low-energy description and higher energy modes should be taken into account. Note that this does *not* mean there is no D-brane any more. Think of classical electromagnetism. The electron is also a singular solution, but this gets resolved in the quantum theory. Similarly, string theory takes over for the quantum description of D-branes when $g_s N \sim 1$: string loops are suppressed by $g_s N$, rather than g_s. We discuss this in Sect. 2.4.

2.3.3 Supersymmetric Black Holes in Five Dimensions

For phenomenological and other existential reasons, we like four dimensions. Nonetheless, we make the switch to five dimensions, because five-dimensional black holes are easier to construct and analyze. Using dualities and dimensional reduction, a lot of what we do can be connected to four-dimensional physics.

Consider eleven-dimensional supergravity, with three orthogonal M2 branes as:

$$\begin{array}{llllllll} \mathrm{M2}_1 & 0 & 1 & 2 & - & - & - & - \\ \mathrm{M2}_2 & 0 & - & - & 3 & 4 & - & - \\ \mathrm{M2}_3 & 0 & - & - & - & - & 5 & 6 \end{array}$$

We consider the directions $x^1 \dots x^6$ to be compactified such that they again form a T^6. As for the four-dimensional black hole, the branes are smeared on their transverse directions on T^6, denoted by "–" in the table. Therefore the M2-branes are all pointlike in the transverse $\mathbb{R}^4$ spanned by x^7, x^8, x^9, x^{10}. The solution is determined by three functions:

$$Z_1 = 1 + \frac{Q_1}{\rho^2}, \qquad Z_2 = 1 + \frac{Q_2}{\rho^2}, \qquad Z_3 = 1 + \frac{Q_3}{\rho^2}. \tag{2.79}$$

We will use ρ for the radius for black holes in five-dimensional space-time, to distinguish from r for four-dimensional black holes. Note the power $1/\rho^2$ for harmonic functions on in five dimensional space-time.

It turns out that for an eleven-dimensional supergravity solution, we can play the same game with the harmonic functions. The only difference is that different powers appear in the metric. When an M2 brane is wrapped along a direction, we multiply the corresponding metric component with an extra factor $Z^{-2/3}$, when the brane is transverse, we multiply it with $Z^{1/3}$. In particular, the supergravity solution for the (supersymmetric) M2-M2-M2 brane system is

$$\begin{aligned} ds^2 = &-(Z_1Z_2Z_3)^{-2/3}dt^2 + (Z_1Z_2Z_3)^{1/3}(d\rho^2 + \rho^2 d\Omega_3^2) \\ &+ \left(\frac{Z_2Z_3}{Z_1^2}\right)^{1/3}(dx_1^2 + dx_2^2) + \left(\frac{Z_1Z_3}{Z_2^2}\right)^{1/3}(dx_3^2 + dx_4^2) \\ &+ \left(\frac{Z_1Z_2}{Z_3^2}\right)^{1/3}(dx_5^2 + dx_6^2). \end{aligned} \tag{2.80}$$

This solution describes a black hole in five spacetime dimensions. This can be seen from the limits

- $r \to \infty$: The metric describes a direct product of five-dimensional flat Minkowski spacetime with a six-torus with constant radii. This is a compactification of flat eleven-dimension spacetime to five dimensions.
- $r \to 0$: This is the horizon of the black hole. Write the transverse $\mathbb{R}^4$ metric in polar coordinates $dx_{78910}^2 = dr^2 + r^2 d\Omega_3^2$. Then for $r \to 0$, the metric looks like

$$ds^2 = -\frac{r^4}{R^4}dt^2 + R^2\frac{dr^2}{r^2} + R^2 d\Omega_3^2 + ds^2(T^6), \tag{2.81}$$

where $R^2 = (Q_1Q_2Q_3)^{1/3}$ and the last term describes a torus of constant radii. By the coordinate redefinition $\rho = r^2$, we see that the first two terms form the metric of AdS_2 ($g_{tt} \to 0$ and $g_{rr} \to \infty$ in such a way to give AdS_2) and the S^3 has constant radius. Hence the near-horizon geometry is $AdS_2 \times S^3 \times T^6$.

We have encountered many examples of $AdS_p \times S^q$ geometries from D-branes: $AdS_5 \times S^5$ from the D3 brane, $AdS_2 \times S^2 \times T^6$ from D2-D2-D2-D6, $AdS_2 \times S^3 \times T^6$ from M2-M2-M2. We will later also encounter $AdS_3 \times S^3 \times T^4$.

Entropy in Gory Detail

We want to give the exact expression for the Bekenstein-Hawking entropy of the black hole. This entropy is proportional to the horizon area in Planck units, or more precisely

$$S_{BH} = \frac{A_H}{4G_N}. \tag{2.82}$$

Note that this looks independent of the dimension. However, if we use the horizon area in D dimensions, we should also use Newton's constant in D dimensions.

Let us get our hands dirty and derive this beast. The horizon area of the eleven-dimensional metric is really the area of $S^3 \times T^6$ at $r = 0$:

$$A_H = \int_{S^3 \times T^6} \sqrt{g} = \int_{S^3} \sqrt{g_{S^3}} \int_{T^6} \sqrt{g_{T^6}} \Big|_{r \to 0}. \tag{2.83}$$

The area of the S^3 in the metric (2.81) is:

$$\int_{S^3} \sqrt{g_{S^3}} = \sqrt{R^6}\, \Omega_3 = 2\pi^2 \sqrt{Q_1 Q_2 Q_3}, \tag{2.84}$$

where $\Omega_3 = 2\pi^2$ is the area of a three-sphere with unit radius. The area of the T^6 for the metric (2.81) is

$$\begin{aligned} \int_{T^6} \sqrt{g_{T^6}} &= \int dx_1 \dots dx_6 \sqrt{\left(\frac{Z_2 Z_3}{Z_1^2}\right)^{1/3} \left(\frac{Z_1 Z_3}{Z_2^2}\right)^{1/3} \left(\frac{Z_1 Z_2}{Z_3^2}\right)^{1/3}} \\ &= \prod_{i=1}^{6} (2\pi L_i), \end{aligned} \tag{2.85}$$

where L_i are the radii of the x_i circles.[13]

We want to express the entropy in terms of a dimensionless number that can be related to a number of microstates. Before we can continue, we have to find the exact relation of the supergravity charges Q_i to the actual integer numbers that count the number of M2 branes of type i that source the supergravity solution. So far, we have been sloppy with the distinction between the supergravity charges Q_i (with dimensions of length squared and appearing in the functions Z_i) and the actual brane numbers N_i. All numerical factors in the exact relation $Q_i = (\dots) N_i$ are extremely important: these will become prefactors in the entropy, which is exponentiated to get the number of black hole microstates. A mistake of a factor of 2 in a number as e^N or e^{2N} has huge consequences!

To find the relation between Q_i and N_i, we first consider the gauge fields of the solution. These are given by

$$C_{012} = Z_1^{-1}, \qquad C_{034} = Z_2^{-1}, \qquad C_{056} = Z_3^{-1}. \tag{2.86}$$

Remember that Q_i represent *densities* of M2-branes, smeared on some directions. For instance, Q_1 describes the density of N_1 M2-branes smeared on the directions x^3, x^4, x^5, x^6. Hence on general grounds, we expect that such a density should scale as

$$Q_1 = \frac{N_1}{L_3 L_4 L_5 L_6} (\dots). \tag{2.87}$$

The exact coefficient $(\dots)$ is left to as a homework problem in Exercise 2.3.8.

[13] The T^6 radii L_i are defined by identifying the x_i periodically as $x_i = x_i + 2\pi L_i$.

Exercise 2.3.8 *The number of M2-branes can be read off by integrating the magnetic gauge field strength over a surface that surrounds the M2-branes as*[14]*:*

$$(2\pi\ell_P)^6 N_{M2} = \int_{\Sigma_7} F_7, \tag{2.88}$$

where ℓ_P *is Planck's constant (in eleven dimensions) and* Σ_7 *is the surface surrounding the M2-branes. For the two-torus* T^2_{12} *spanned by* x_1, x_2*, this is:*

$$\Sigma^7 = T^2_{34} \times T^2_{56} \times S^3, \tag{2.89}$$

The magnetic seven-form gauge field is found from the Hodge dualization relation $F_7 = \star_{11} F_4$*, which is written out as*

$$F_{i_1\ldots i_7} = \frac{1}{4!}\sqrt{-g}\epsilon_{i_1\ldots i_7;\, j_8 j_9 j_{10} j_{11}} g^{j_8 j_8'} g^{j_9 j_9'} g^{j_{10} j_{10}'} g^{j_{11} j_{11}'} F_{j_8 j_9 j_{10} j_{11}}. \tag{2.90}$$

Take the metric (2.81) and the four-form field strength $F_4 = dC$ *with components*

$$F_{012r} = \partial_r C_{012} = \partial_r (Z_1^{-1}), \tag{2.91}$$

and analogously for F_{034r} *and* F_{056r}*. Calculate the dual seven-form and use (2.88) to express the charges* Q_i *in terms of the integers* N_i *that count the number of branes. Show that the exact relation is*

$$Q_1 = \frac{N_1(\ell_P)^6}{L_3 L_4 L_5 L_6}, \qquad Q_2 = \frac{N_2(\ell_P)^6}{L_1 L_2 L_5 L_6}, \qquad Q_3 = \frac{N_3(\ell_P)^6}{L_1 L_2 L_3 L_4}, \tag{2.92}$$

where L_i *are the radii of the circles at infinity.*

We continue with the horizon area (2.83). It is given in terms of the charges as

$$A_H = 2\pi^2 \sqrt{Q_1 Q_2 Q_3} \prod_{i=1}^{6} (2\pi L_i). \tag{2.93}$$

By substituting the result from Exercise 2.3.8, Eq. (2.92), we get

$$A_H = 2\pi^2 (2\pi)^6 (\ell_P)^9 \sqrt{N_1 N_2 N_3}. \tag{2.94}$$

We want to we evaluate the entropy $S_{BH} = A_H/4G_N$. The definition of the Planck length in terms of Newton's constant for any dimension D is:

[14] Remember the analogy with electromagnetism, for a magnetic monopole we find the quantized monopole charge N is $N \propto \int_{S^2} F_2$.

$$16\pi G_N = (2\pi)^{D-3}(\ell_P)^{D-2}. \tag{2.95}$$

Plugging this into $S_{BH} = A_H/4G_N$ for $D = 11$, we conclude that the Bekenstein-Hawking entropy of the black hole is

$$\boxed{S_{BH} = 2\pi\sqrt{N_1 N_2 N_3}}. \tag{2.96}$$

A few remarks are in order. Note that we began with D-branes on a torus. As the torus gets smaller or larger (by changing L_i), the solution changes drastically. We get different black holes because the Q_i's change. But: the entropy does not care whether the torus is of diameter 1 mm or 1 Mpc. This is a very interesting fact: S_{BH} does not change as you change the torus radii. For a non-supersymmetric solution, you would expect that the entropy depends on the parameters of the torus; the invariance of S_{BH} under variations of the torus radii is a feature due to supersymmetry.

We can use this feature to do dualities on the internal torus. The five-dimensional black hole will have the same entropy, but the black hole can be made up out of different branes in some other string theory. Take for instance the duality chain: (1) reduce along x_6 to IIA, (2) two T-dualities along x_1, x_2, (3) a T-duality along x_5:

				IIA:										**IIA:**										**IIB:**		
D2	0	1	2	–	–	–	T	D0	0	–	–	–	–	–	T	D1	0	–	–	–	–	5
D2	0	–	–	3	4	–	$\rightarrow$	D4	0	1	2	3	4	–	$\rightarrow$	D5	0	1	2	3	4	5
F1	0	–	–	–	–	5	$x_{1,2}$	F1	0	–	–	–	–	5	x_5	p	0	–	–	–	–	5

For this T-dualization, you need to know that for F1's that do not wrap the T-duality circle, nothing happens at all: they remain F1's.) The end result of this little exercise is an intersection of D5 branes with D1 branes and momentum along the common direction. This is the celebrated D1-D5-P system. We will mainly study the three-charge black hole in five-dimensions in this duality frame.

Exercise 2.3.9 *Use dimensional reduction and the T-duality chain from the M2-M2-M2 system to the D1-D5-P frame to show that the metric becomes*

$$\begin{aligned} ds^2 = {} & -(Z_1 Z_5)^{-1/2}(dt^2 + dz^2) + (Z_1 Z_5)^{-1/2}(Z_p^{-1} - 1)(dz - dt)^2 \\ & + (Z_1 Z_5)^{1/2} dx_{78910}^2 + (Z_1 Z_5)^{-1/2} dx_{1234}^2. \end{aligned} \tag{2.97}$$

You will need to perform a minor change of coordinates and use that the reduction ansatz from M-theory to IIA supergravity is

$$ds_{11}^2 = e^{2\phi/3} ds_{10}^2 + e^{-4\phi/3} dx_{10}^2, \tag{2.98}$$

with ϕ the dilaton and ds_{10}^2 the ten-dimensional metric.

When $p = 0$, there is no momentum charge. The metric only depends on the functions

$$Z_{1,5} = 1 + \frac{g_s N_{1,5}}{r^2}, \qquad Z_p = 1 + \frac{(g_s)^2 N_p}{r^2}. \tag{2.99}$$

In the limit $r \to 0$, we can drop the 1's in the harmonic functions and the metric becomes $AdS_3 \times S^3 \times T^4$, see Exercise 2.3.10. Also this geometry is very useful for holography. String theory on this background is dual to a $(1+1)$-dimensional CFT.

Exercise 2.3.10 *Show that for $r \to 0$, the metric (2.97) with $p = 0$ ($Z_p = 1$), the metric becomes*

$$ds^2 = r^2(-dt^2 + dx_5^2) + \frac{dr^2}{r^2} + d\Omega_3^2 + ds^2(T^4), \tag{2.100}$$

where the last term describes the metric on a T^4 with constant radii.

But …there is a "but": for $Z_p = 1$ (no momentum charge) the horizon area is zero. This can be seen from the metric. At $r = 0$ it is singular and one can show that the Ricci scalar in five-dimensional spacetime blows up at $r = 0$ and the horizon coincides with a curvature singularity.

When $p \neq 0$, the entropy is

$$S_{BH} = 2\pi\sqrt{N_1 N_5 N_p}. \tag{2.101}$$

String theory on the near-horizon region is dual to the same $(1+1)$-dimensional CFT as for the $p = 0$ solution. Now there is a non-trivial momentum in the game, which translates into an extra charge of that the CFT states that are dual to the black hole can have.

The D1-D5-P black hole entropy comes from the many ways in the which the CFT can carry this momentum p. This result is proven in the next section. It is *most amazing*: the entropy of a black hole is recovered from counting states in a $(1+1)$-dimensional CFT!

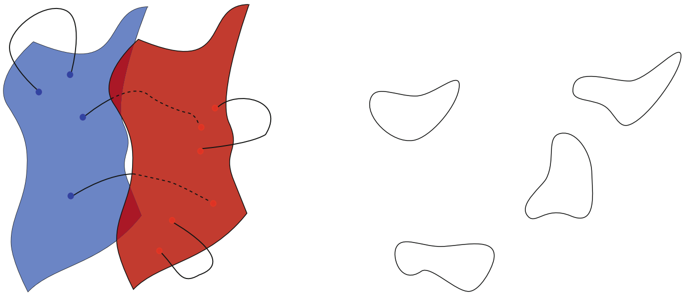
2.4 Black Hole Microscopics

To properly account for the entropy of the black hole we first have to learn some very basic string theory. In the spirit of the rest of the lectures we'll eschew any details we don't need and ask the reader to trust us since we're supposed to be experts.

We explain how to derive the black hole entropy from a microscopic counting of states for a:

1. **D1-D5-P black hole** (also "three-charge black hole") with entropy:

$$S_{BH} = 2\pi\sqrt{N_1 N_5 N_p}. \tag{2.102}$$



(a) **(b)**

Fig. 2.14 Strings can be of two types, depending on the boundary conditions we put on the string: closed or open strings. The end-point of open strings are confined to D-branes. **a** Open strings end on D-branes. **b** Closed strings propagating in spacetime

2. **D6-D2-D2-D2 black hole** (also: "four-charge black hole") with entropy:

$$S_{BH} \sim \sqrt{N_{\mathrm{D2}} N_{\mathrm{D2}} N_{\mathrm{D2}} N_{\mathrm{D6}}}. \tag{2.103}$$

When all the charges above are equal this black hole has a very nice interpretation as the extremal Reissner-Nordtström black hole in four dimensions.

We will discuss the three-charge black hole first. Historically, this was the first black hole for which a microscopic counting was done that could explain the entropy (by Strominger and Vafa [23]). We will treat the four-charge black hole in four dimensions afterwards. It is the latter one which may have more appeal, as it describes the extremal black hole of Einstein-Maxwell theory in four dimensions ('extremal Reissner-Nordström black hole').[15]

2.4.1 A Brief Review of Open and Closed String Theory

String theory is a theory of (surprise, surprise:) strings. Strings come in two types: closed strings form closed loops in spacetime with no end-points (imagine rubber bands floating around in spacetime), while open strings have two ends (imagine a strand of rope stretched between…between what?), see Fig. 2.14.

In general the ends of open strings are not free to move in all directions of space-time but are constrained to lie along higher dimensional "membranes". It turns out that these membranes are nothing other than the D-branes we found before as solu-

[15] This does not mean that it is a realistic astrophysical black hole. In nature, black holes will shed (almost) all their charge and be charge neutral. Supersymmetric black holes are extremal; they have the maximum amount of charge allowed for their mass and are hence not the black holes we observe in the sky.

tions to supergravity! Although it is hard to see why this is so, we will try to argue it briefly later.

Scales and Limits

One of the nice features of string theory is that it very naturally introduces a new length scale, ℓ_s, the string length. This is because fundamental strings (like all strings) have a tension, τ_{F1}, and this can be defined in terms of the string length, ℓ_s, a new fundemental length scale defined by this tension,

$$\tau_{F1} = \frac{1}{\ell_s^2}. \tag{2.104}$$

Note that the length dimension of τ_{F1} is defined so that integrating the tension over a one-dimensional volume yields a unit of mass, namely the mass of a string.

Oscillations on a world-volume of a string have an energy cost dependent on the string tension just like a regular guitar string. The mass of the harmonic modes is quantized in units of the string mass

$$M_s \propto \frac{1}{\ell_s}. \tag{2.105}$$

When this value is large then stringy modes are very massive and we can, to a good approximation, restrict ourselves to only the lowest lying sector corresponding to massless strings, see Fig. 2.15. In this limit when the string mass is very large and only a few modes remain strings essentially look like point particles and (owing to the various possible massless oscillations possible) generate a spectrum of fields in spacetime, see Table 2.6.

Even though we will not generally need the details of this spectrum, it is important to realize that the closed string spectrum generates supergravity with the associated fields. Open strings, on the other hand, are described at low energy by a gauge theory since A_μ has the degrees of freedom to be a gauge field (coupled to matter and

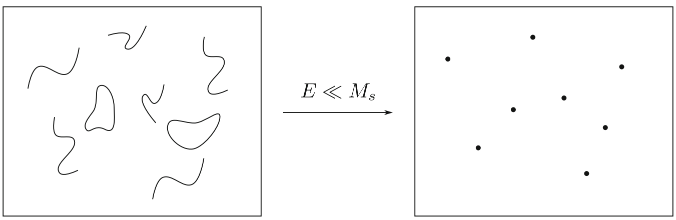


Fig. 2.15 When we probe strings at energy scales far below the string scale, $E \ll M_s$, then we can't excite oscillators on the strings so they look and act like point particles

Table 2.6 The massless spectrum of closed and open strings

String type	Spacetime fields generated
Closed	$g_{\mu\nu}, B_{\mu\nu}, C^{(3)}_{\mu_1\mu_2\mu_3}, \psi^\alpha_\mu, \ldots$
Open	$A_\mu, \phi, \psi_\alpha, \ldots$

fermions). This theory however does not live on all of spacetime but only on the D-branes on which the open strings are restricted to end.

In any gravity theory, including string theory, there is a fundamental length scale related to Newton's constant and the strength of gravitational interactions: the Planck scale. This is set by the Planck length ℓ_P, through the relation with Newton's constant (in D dimensions of spacetime):

$$G_N = (2\pi)^{D-3}(\ell_P)^{D-2}. \tag{2.106}$$

The introduction of a second fundamental length scale in string theory, ℓ_s, means that string theory has an associated dimensionless constant, the string coupling or g-string

$$g_s = f\left(\frac{\ell_s}{\ell_P}\right). \tag{2.107}$$

The exact dependence can be derived using the fact that the low-energy limit of closed string scattering (which depends on g_s and ℓ_s, as explained below) can be related to graviton scattering (which depends on G_N). A graviton propagator is controlled by Newton's constant. If we interpret this as a string exchange, we get two factors of g_s, one for emitting and one for absorbing a closed string. This gives

$$G_N \propto \ell_P^{D-2} \propto g_s^2\, \ell_s^{D-2}, \tag{2.108}$$

and we find

$$g_s \propto \frac{\ell_s}{\ell_P}. \tag{2.109}$$

From this we see that g_s controls the hiarachy of scales in string theory. When $g_s \ll 1$ we have $\ell_s \ll \ell_P$ so stringy excitations are much less massive than the Planck scale and we can do "classical string theory". On the other hand when $g_s \gg 1$ then any stringy excitations is more massive than the Planck scale and thus highly quantum. Therefore g_s acts as a *dimensionless* coupling in string theory telling us when the theory can be treated classically versus when it is necessarily strongly coupled.

While tuning g_s puts us in a theory with a certain string and Planck scale we have a further freedom to choose the energy scale at which we probe this theory. In any given physical process there is an associated dimensionalful energy scale such as e.g. the mass of the heaviest particle we consider, the energy of a scattering process, etc.... Thus even if we choose $g_s \ll 1$ we have the further freedom to consider only processes with E restricted to

$$E \ll M_s \ll M_P, \tag{2.110}$$

which means that the scale of our physics is smaller than the string scale (which in turn is lower than the Planck scale). The limit $M_s \ll M_P$ (or $g_s \ll 1$) means that string perturbation theory is valid and that we can look at classical string theory. The limit $E \ll M_s$ means that strings effectively look like point particles (we only look at those excitations that have very low energy compared to the scale set by the string length and we cannot distinguish the stringy nature of the string). This limit, in which semi-classical particle physics is a good approximation, is one in which we will often find ourselves.

String Perturbation Theory

Above we motivated g_s as a dimensionless coupling emerging from comparing the dimensionful ℓ_s and ℓ_P but within string theory this can actually be derived. String perturbation theory is described in terms of the mathematical "genus" of the string world-sheet (the two dimensional submanifold describing the strings path in space-time). Let's take a look at the loop expansion of a string process. As a Feynman diagram represents the worldlines of in- and outgoing particles and intermediate processes (propagators, loops), a string diagram represents the worldvolume of a string.

We represent perturbation theory for an ingoing closed string to an outgoing closed string in Fig. 2.16, which explaines visually the genus expansion. For every number of loops, there is exactly one type (topology) of string worldvolume.

Every loop in a closed string diagram introduces an extra factor of $(g_s)^2$. The limit where $g_s \to 0$ suppressed the loops and hence also quantum effects: this is the classical limit. If we further impose the extra "low-energy limit" $E \ll M_s$, such that the strings look like particles then the string diagrams reduce to standard Feynman diagrams because in this limit we send $\ell_s \to 0$ so the worldsheet compresses down to a world-line (see also Fig. 2.16).

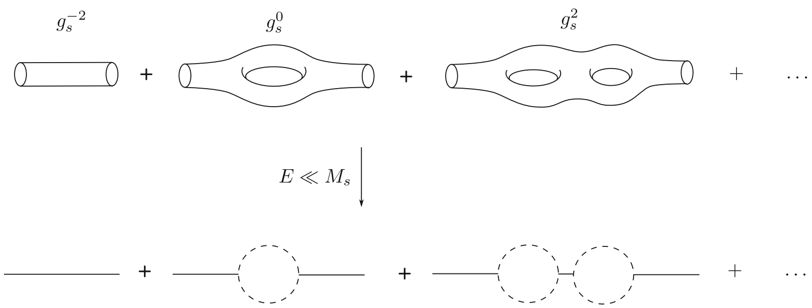


Fig. 2.16 String perturbation theory is a genus (# holes) expansion of string world-sheets. For closed strings, every hole introduces a factor of $(g_s)^2$ in the expansion. For excitations well below the string scale, strings behave like particles and we recover ordinary Feynman diagrams

We will generally work in this regime and keep only the zero-mass excitations of the string.[16] Then the closed string gives exactly the fields of supergravity, see Table 2.6: the metric, dilaton and B-field and the gauge potentials that we have seen when discussing D-branes (Ramond-Ramond fields). Thus one can think of supergravity as the *low energy limit* of *weakly coupled* string theory (and indeed this is where we will mostly be working).

If we consider Fig. 2.16 with in- and out-going graviton states (in the $E \ll M_s \ll M_P$ limit) then the prefactor for the first loop diagram (in the bottom row) is G_N. Computing this same diagram in string theory one finds a pre-factor $g_s^2\, \ell_s^{D-2}$ where the g_s factors come from the genus-counting and the ℓ_s dependence must follow from dimensional analysis (ℓ_s is the only length scale in string perturbation theory). This is the origin of Eq. (2.108).

What about open string perturbation theory? Open strings strech between D-branes and their end-points are labelled by the branes they end on. Thus open string perturbation theory gains an additional factor, N, the number of D-branes, from the degeneracy of open string considered in any scattering process (see Fig. 2.17). Thus the perturbative series is a power series in $g_s N$. This is similar to the expansion in a gauge theory with N_c colors, where we get an expansion in powers of gN_c, and indeed as we will see below this resemblance is no accident.

The low lying (massless) sector of the open strings are a vector field A_μ, a number of spinors ψ^α (fermions) and scalar fields ϕ^i, see Table 2.6. These fields are bound to the brane, because the open string endpoints are. The gauge fields can be interpreted as describing the D-brane dynamics: the scalars describe the transverse motion of the brane (there is one scalar for every direction transverse to the brane worldvolume), the vector (which has only directions on the worldvolume) describes a gauge theory living on the brane and the fermions are needed for supersymmetry. Note that if we only consider open strings, we cannot get a metric: a metric (gravitons) sits only in the closed string spectrum.

Questions from the audience:

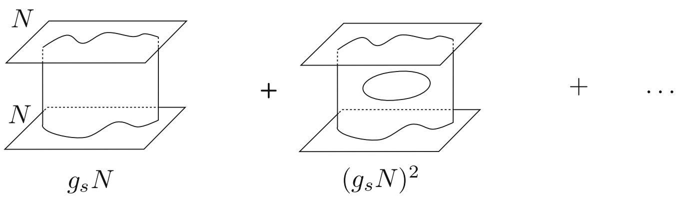


Fig. 2.17 Open string perturbation theory is an expansion in $g_s N$ where N is the number of branes. This is because each loop increases the genus by one (another factor of g_s) and also generates an additional trace over the N gauge factors (another factor of N). Note there is an additional overall factor of g_s above; we show only the relative g_s factors

[16] These are not the lowest-energy modes of the string. Those are tachyonic (negative energy) modes, that can be consistently projected out of the spectrum of string theory.

- *Have we not introduced a cut-off E by restricting our energies to $E \ll M_s$.* No because what we mean by $E \ll M_s$ is that we consider only massless excitations so the cuttoff is actually $E \sim 0$. Or said better we are sending $M_s/E \to \infty$ so we decouple stringy excitations. We assume that any dynamics or additional scales we introduce will be small with respect to M_s unless we explicitly state otherwise. Note that the number of massless excitations can be very large: for open strings on N D-branes, we get a $U(N)$ gauge theory, which has many (massless) fields.
- *Why and how do open strings leave a D-brane?* We have not yet said what closed strings do with a D-brane. Figure 2.19 shows the process by which a closed string leaves a D-brane.

The gauge/gravity duality we mentioned before, is really an open/closed string duality. The theory living on the worldvolume of a string (the so-called worldsheet theory) which describes the propagation of a string in spacetime has a symmetry allowing us to interchange proper time (τ) and proper length (σ) on the worldsheet (it is a symmetry of the string itself). Then a loop diagram in open string theory, looks like a tree level diagram describing the exchange of closed strings between two D-branes, see Fig. 2.18. We will return to this later.

From Fig. 2.19, we see that a process of a closed string interacting with a D-brane has a factor of $(g_s)^2$: we can see this as a graviton exchange. This is another way to see why $G_N \sim (g_s)^2$.

For the discussion of the black hole entropy, we will take the limit $g_s \to 0$. In this limit, open and closed strings naively decouple, since their interaction (Fig. 2.19) is suppressed. Note however that the open-closed diagram receives an enhancement from the degeneracy of open strings so the final effective coupling controlling the interaction of closed and open strings will be $g_s N$, the same coupling that governs interactions between open strings. Thus by taking $g_s \to 0$ but with $g_s N$ fixed we can supress quantum gravity effects but still allow open-strings, or D-branes, to source closed strings (yielding the supergravity solutions described in previous sections). We will return to this later.

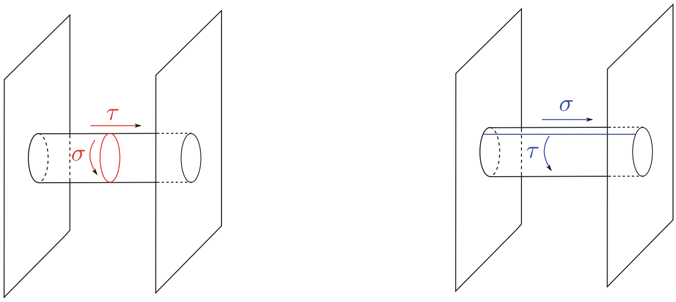


Fig. 2.18 By exchanging the role of string time (τ) and length σ, we can interpret this diagram as an exchange of closed string between D-branes (*left*), or a loop diagram in open string theory (*right*)

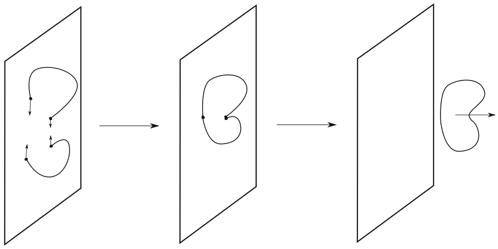


Fig. 2.19 Interpretation of a closed string leaving from a D-brane from open string interaction. Note that each interaction (each pair of end points joining) introduces a factor of g_s in the amplitude of this process

The Stringy D1-D5-P Black Hole

We consider the D1-D5-P system along the following directions. The D5 branes are on compact directions in spacetime, the D1 and the momentum are along one of the directions of the D5:

$$\begin{array}{c|c|cccc|c|cccc}
 & & \multicolumn{4}{c|}{T^4} & S^1 & \multicolumn{4}{c}{\mathbb{R}^4} \\
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
D5 & \times & \times & \times & \times & \times & \times & & & & \\
D1 & \times & & & & & \times & & & & \\
P & \times & & & & & \times & & & &
\end{array}$$

We can picture this as in Fig. 2.20.

Question from the audience:

- *What is "P", the momentum, exactly?* This can be thought of as a gravitational wave propagating along the S^1 direction. We can see this by a manipulation of the metric (2.97). By changing coordinates, $x_- \to x_5 - t$, the metric looks like

$$\begin{aligned} ds^2 &= -(Z_1 Z_5)^{-1/2}\mathbf{dt}^2 + (Z_1 Z_5)^{1/2} dx_-^2 + Z_p^{-1} dt dx_- \\ &\quad + (Z_1 Z_5)^{1/2}(\mathbf{dr}^2 + r^2 d\Omega_3^2) + ds^2(T^4), \qquad Z_i = 1 + \frac{Q_i}{r^2}. \end{aligned} \tag{2.111}$$

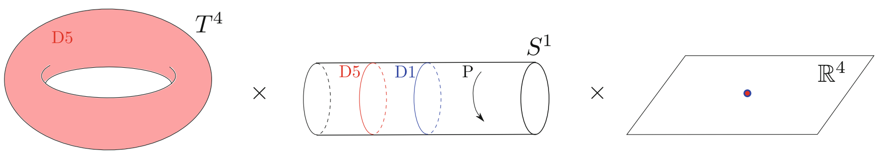


Fig. 2.20 The D1-D5-P system. The D5's are wrapped on $T^4 \times S^1$, along the S^1 we also wrap D1's and we put gravitational waves (momentum), denoted P

The angular momentum of this solution is related to the mixed time-space components of the metric: in this case $p \sim \partial/\partial x^5$ is given by the $1/r^2$ term $Z_p^{-1} = 1 - Q_p/r^2 + \ldots$, so Q_p is indeed the momentum charge.

Remember that the supergravity charges are actually charge densities (we omit numerical factors):

$$Q_1 \sim g_s(\ell_s)^2 N_1 \qquad Q_5 \sim g_s(\ell_s)^2 N_5\,, \qquad Q_p \sim g_s^2 N_p. \tag{2.112}$$

The horizon area depends on the string length and the string coupling:

$$A_H \sim \sqrt{Q_1 Q_5 Q_p} \sim g_s^2(\ell_s)^3\sqrt{N_1 N_5 N_p}, \tag{2.113}$$

but the Bekenstein-Hawking entropy is independent of the coupling and length scales:

$$S_{BH} = \frac{A_H}{4G_N} = 2\pi\sqrt{N_1 N_5 N_p}. \tag{2.114}$$

From D-branes to Black Holes

Let us now use the observation that S_{BH} is independent of g_s to our advantage. Namely we will argue that by tuning g_s we can interpolate between a regime where the system is described by open strings ending on D-branes to a regime where the system is a black hole with a horizon area which is large in string units $A_H/\ell_s^3 \gg 1$ (i.e. a regular looking supergravity black hole). To do this let us recall:

- g_s is the perturbative parameter in both string theory and gravity. $g_s \ll 1$ is the (semi)classical regime while $g_s \sim 1$ is the quantum regime.
- The coupling between closed and open strings, on the other hand, is controlled by $g_s N$ so if we fix $g_s N$ to be large then D-branes back-react on closed strings (giving geometry) even if we send $g_s \to 0$. This is analogous to saying we can send $G_N \to 0$ (the exactly classical limit of GR) while keeping $G_N M$ fixed for some source so we have a reasonable non-trivial limit giving classical GR solutions.
- Thus our approach will be to fix the entropy by fixing the N's (number of branes) to some very large value but then tune g_s such that we vary from $g_s N \sim 0$ to $g_s N \gg 1$. At $g_s N \sim 0$ we can describe the system in terms of weakly coupled *open* strings on stacks of N D-branes. Closed and open strings decouple in this regime and we can neglect gravity. At $g_s N \gg 1$, on the other hand, the D-branes back react and form a large black hole

Let us see this all in more detail.

Black Holes at $g_s N \gg 1$

The scale of the supergravity solution is set by the charges Q_i in the warp factors. Remember that the supergravity charges appear as $Z = 1 + Q_i/r^2$ and they determine

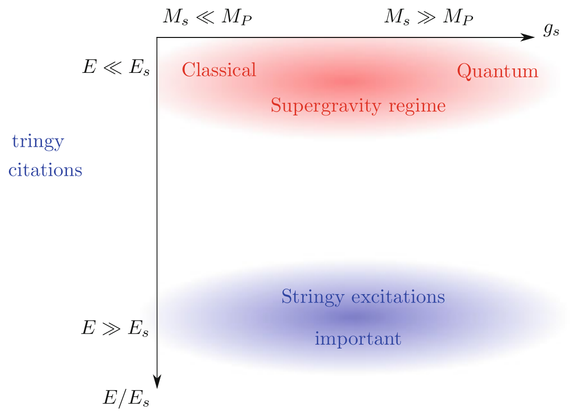


Fig. 2.21 Tuning the coupling in closed string perturbation theory. Kepping only the low energy (zero mass) modes, we have a theory of particles, supergravity. We restrict to small g_s, and only consider *classical* supergravity

the size of the solution. In general, we have $Q_i = G_N M_i$, see Eqs. (2.47–2.49). For a D-brane, we have $M_D = N/g_s$ and hence $Z \sim g_s N_D$, while for the momentum excitations, we have $M_p = N_p$ (just an excitation) and hence $Z \sim g_s^2 N_p$. If $g_s N$ is small (order 1), the area of the black hole is small in string units. Hence we cannot use supergravity to describe it: massive string modes become important, and supergravity only describes the *massless* modes. This violates our earlier physical requirement $E \ll M_s$ (put another way such black holes would involve curvature of the order of the inverse of the string length and thus probing them would involve energies at this scale). We see that we need the horizon to be large in string units to describe (super)gravity black holes and thus we consider instead the regime:

$$g_s N \gg 1. \tag{2.115}$$

We further impose $g_s \to 0$. Closed string theory is non-interacting in this regime as this limit suppresses quantum gravity corrections. This is true whether you are in string theory or in gravity since at low energy a closed string loop looks like a graviton loop, see Fig. 2.21. Thus in the limit $g_s N \gg 1$ with $g_s \to 0$ the D1-D5-P system resembles a large supergravity black hole.

Open Strings at $g_s N \ll 1$

For a large (semi-classical) black hole, we need A_H to be large both in string units, $A_H \gg (\ell_s)^3$, and in Plank units so, via (2.113–2.114), we must take $N_1 N_5 N_p$ to be very large; thus we take the "$N \to \infty$" where this is understood to apply to all the N's.

But as we may still vary g_s we can dial the coupling $g_s N$ allowing us to interpolate between the large black hole like description above (at large $g_s N$) and a weakly-coupled open string description where the open strings end on the branes and don't

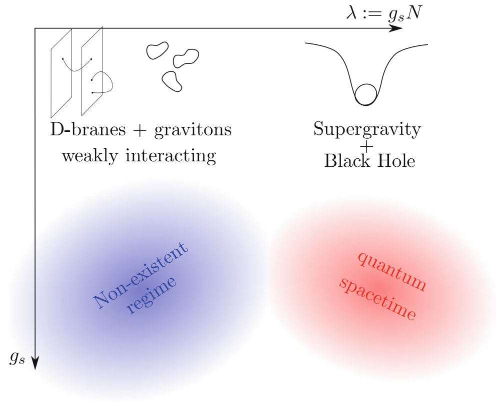


Fig. 2.22 By dialling the coupling $g_s N$ (while keeping g_s small), we can interpret the D1-D5-P system as a *black hole* or as open strings stretching between D-branes. Since the torus volume goes as $V_{T^4} \sim Q_1/Q_5$ in string units, it disappears from the picture and we only retain the five-dimensional geometry. Note that the lower *left* region is non-existent (since we always have that $g_s N > g_s$)

interact with closed strings and gravity (and open string perturbation theory is valid since $g_s N \ll 1$). This tuning is depicted in Fig. 2.22.

Because the entropy is independent of the coupling, g_s, we expect to be able to reproduce the entropy from a counting of supersymmetric states in the weakly coupled open string picture. Note that we take $g_s \to 0$ throughout this diagram so closed strings and gravity are always semi-classical but the open string coupling is $g_s N$ so if we also take $g_s N \to 0$ open strings become weakly coupled and furthermore there is no interaction between the closed and open string sector (even though closed string perturbation theory goes with powers of g_s the couplings to N D-branes goes as $g_s N$ so only in this limit do D-branes not source gravitons). Thus the limit $g_s \to 0$ with $g_s N \to 0$ gives weakly coupled open strings on D-branes in flat spacetime.

We summarize:

- If $g_s \to 0$, you always suppress closed string loop effects (quantum gravity effects)
- $g_s N$ tells you how much closed strings (and gravitons) feel the source. From an open string perspective tuning $g_s N$ is turning open string loop effects on/off.
- If $g_S N \ll 1$, you can count the number of states of these strings stretching between the D-branes, because essentially we get a free (open string) theory (loop effects suppressed). This is reminiscent of holography, where we have $g_s N \ll 1$ giving Yang-Mills weakly coupled, no gravity, and $g_s N \gg 1$ giving Yang-Mills strongly coupled, or AdS_5 gravity.

Note that if the entropy did depend on g_s, then none of this would make sense. A toy model will follow with a rigourous proof that it is g_s is independent.[17]

A question from the audience:

- *Can we get the gravity solution from open string calculations?* Yes you can, but it's a pain. Say we want to find the metric. You can expand the gravitational solution

[17] Extrapolating from toy models is many a string theoriest's idea of a mathematical proof of complicated string theory effects.

in the open string coupling $g_s N$

$$g_{tt} = (Z_1 Z_5)^{-1/2} \sim 1 + g_s N + (g_s N)^2 + g_s^3 + \dots \tag{2.116}$$

One can then try to match this to an open string loop expansion. The one-loop computation is doable and has been done (Stefano Giusto, a former postdoc at IPhT is doing this). Higher loops are extremely tough; solving supergravity equations of motion is much simpler.

2.4.2 Supersymmetric Indices

We have just argued that the D1-D5-P system looks like a black hole for $g_s N \gg 1$, and like a system of very weakly coupled strings for $g_s N \ll 1$. We want to count the states that make up the entropy in the weakly coupled theory. Why can we trust such a computation? The answer is that in supersymmetric theories certain quantities are protected and cannot depend on continuous parameters such as g_s. Although we will not give a proof of this for the D1-D5-P system we illustrate the idea with a simpler toy model.

Consider supersymmetric quantum mechanics. It is defined by the Hamiltonian

$$H = \{Q, Q^\dagger\} \equiv Q^\dagger Q + Q Q^\dagger. \tag{2.117}$$

The operator Q is fermionic, and anticommutes with itself:

$$\{Q, Q\} = 2Q^2 = 0 \tag{2.118}$$

We define BPS states (or "supersymmetric states") as states that are annihilated by Q, but are not given by acting with Q on another state (Q-closed but not Q-exact):

$$|\psi\rangle_{\rm BPS} : \qquad Q|\psi\rangle_{\rm BPS} = 0\,, \quad |\psi\rangle_{\rm BPS} \neq Q|\psi'\rangle. \tag{2.119}$$

Exercise 2.4.11 *Prove the following properties:*

1. *The Hamiltonian H has only positive eigenvalues. Show that BPS states are states of minimal (zero) energy:*
$$H|\psi\rangle_{\rm BPS} = 0. \tag{2.120}$$

2. *Let $|\phi\rangle$ be a non-BPS state. Prove that ϕ is degenerate to*

$$|\phi'\rangle = Q|\phi\rangle, \qquad E_\phi = E_{\phi'}. \tag{2.121}$$

Introduce the operator $(-1)^F$, defined through its action on bosonic and fermionic states as:

$$(-1)^F|\text{boson}\rangle = |\text{boson}\rangle, \qquad (-1)^F|\text{fermion}\rangle = -|\text{fermion}\rangle. \tag{2.122}$$

This operator $\mathbb{Z}_2$-grades the Hilbert space. Note that it anticommutes with the operator Q:

$$\{(-1)^F, Q\} = 0. \tag{2.123}$$

Define the Witten index

$$Z = \mathrm{Tr}\,[(-1)^F e^{-\beta H}], \tag{2.124}$$

where β is a number.

3. *Show that*

$$Z = (\#\text{ bosonic BPS states}) - (\#\text{ fermionic BPS states}) \tag{2.125}$$

4. *Show that*

$$\frac{\partial Z}{\partial \beta} = 0. \tag{2.126}$$

5. *Redo the calculation with the Hamilatonian*

$$H = H_0 + gH_1, \tag{2.127}$$

   where both the original Hamiltonian H_0 and the perturbed Hamiltonian H obey the supersymmetry property

$$H_0 = \{Q_0, Q_0^\dagger\}, \qquad H = \{Q, Q^\dagger\} \tag{2.128}$$

   for two different fermionic operators Q_0, Q. Show that the function Z is independent of g.

In this exercise, you have proven that the Witten index, which counts the difference in the number of bosonic and fermionic ground states, is independent of the coupling g. The key thing to note is that at strong coupling, the total number of ground states is equal to the Witten index. By its independence on the coupling g, we can calculate the Witten index at weak coupling to count the number of ground states at strong coupling.

We rephrase that in a more mathematical language. Define the trace over the BPS Hilbert space:

$$Z_{\mathrm{BPS}} = \mathrm{Tr}_{\mathrm{BPS}}\,(e^{-\beta H}) = \mathrm{Tr})\mathrm{BPS}1 = \#\text{bosons} + \#\text{fermions}. \tag{2.129}$$

This counts the total number of ground states (in the exercise you have proven that the BPS states are exactly the ground states of the Hamiltonian). Note that this is always larger than the Witten index:

$$Z_{\rm BPS} > Z_{\rm Witten}. \tag{2.130}$$

At weak coupling, we expect that this is much larger. But at large values of the coupling, you expect that the number of BPS states will match the index because "Anything that can lift, will lift"; i.e. perturbing the system enough will lift degenerate boson/fermions pairs until we have only one species or the other left (i.e. the minimum necessary to preserve the Witten index which cannot vary as we mess around with the couplings). Thus at strong coupling we expect the number of states to match the Witten index. Since the latter is independent of the value of the coupling, we can calculate it at weak coupling and use it to know the number of BPS states at strong coupling.

Question:

- *Are there any restrictions on the validity of the extrapolation to strong coupling?* One way it could break down, is because of a phase transition or discontinuity. There are no walls of marginal stability for this index, so that does not pose a problem. However for extended-supersymmetry theories, where you have several operators Q_i:

$$H = \sum_{i,j=1}^{n} \epsilon^{ij} \{Q_i, Q_j^\dagger\}, \tag{2.131}$$

  the counting of $1/n$ BPS states (that are only annihilated by 1 of the n operators Q_i), is a lot more subtle. And the black hole states are exactly of this form—but we will not go into the details.

2.4.3 Counting States for the Three-Charge Black Hole

We study the D1-D5-P system of Fig. 2.20 in the limit $R^4 \gg V_{T^4}$, which means in terms of the charges

$$\frac{Q_p}{Q_1 Q_5} \gg 1. \tag{2.132}$$

In this regime the S^1 is much larger than the other compact directions on which the branes are wrapped so the theories on the D1 and D5 reduce to a theory living on the S^1 with radius R as depicted in Fig. 2.23. The rotation (momentum along x^5) of the D1 and D5 will translate into rotation of the open strings, so we put momentum on the strings to account for Q_p.

We motivate everything from the open string picture. It is not easy to show that D1/D5 momentum follows from F1 with momentum, so you will have to take our word for it. In principle, we can divide momentum over all possibilities: open F1, closed F1, D1's, D5's one or several branes, combinations, single wrapping, multiple wrapping etc: everything can carry momentum. We are interested in the typical, dominant contributions. We will find that we get the most entropy by putting all of

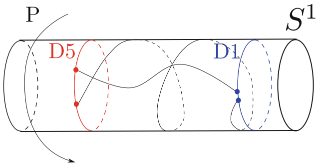


Fig. 2.23 In the limit $R^4 \gg V_{T^4}$ we can largely ignore excitation on the torus and the physics is effectively described by an open string stretched between the D1- and D-5 branes which wrap the *circle*. The open string also carries momentum along the S^1

the momentum in the open string sector because of fractional momentum quantization described on the next page.

To arrive at this picture of the black whole we have to go to weak coupling by tuning $g_s \to 0$ such that $g_s N \ll 1$; in this regime the D-branes are heavy static objects (their mass goes as N/g_s) but they decouple from gravity and are entirely described weakly interacting open strings ending on them. Moreover because we are interested in supersymmetric configurations (as our black hole is supersymmetric) it suffices to restrict to the ground states of the open strings as excited modes break more supersymmetry (recall from the exercises above that supersymmetry tends to require minimal energy). Thus the open strings essentially become point particles connecting two coincident branes. Moreover, at very small $g_s N$ the open strings are essentially free so their wavefunctions are momentum eigenstates on the S^1

$$\psi(x_5) = \sum_n e^{-\frac{2\pi n}{R} x_5}. \tag{2.133}$$

The wave function of a particle normally has to be single valued as we go around a circle but, because these particles carry additional labels, corresponding to the D-brane they're ending on, this is no longer the case. For instance a string ending on a D1 that wraps twice around the circle carries a coordinate, $x^{(1)}$, its location on the D1 and this coordinate itself is not single-valued on the S^1 (i.e. the coordinate length is 4π). This lack of single-valuedness may be familiar from fermions which need not be periodic on a circle because they carry internal (spinorial) indices. Here the additional internal data is just the coordinate on the brane the string endpoint is attached to.

Let us now consider a string with two endpoints going around the circle several times. Take for example a string stretched between a D1 brane that wraps the circle twice, and a D5 brane that wrap the circle three times.[18] If we unwrap the circle, this configuration looks like Fig. 2.23.

The open string wave function depends on the string coordinate x_5 and has two labels, coordinates on the D1-branes and D5-branes (Fig. 2.24):

[18] Note that for "2 D-branes" on a compact circle, we have either 2 distinct D-branes or a D-brane wrapping the circle 2 times.

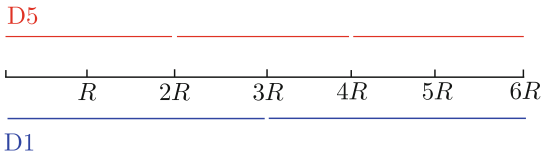


Fig. 2.24 A D1 brane wrapping the *circle* twice and a D5 brane wrapping the *circle* three times. We need to go six times around the *circle* before we reach the same point again

$$\psi(x^{(1)}, x^{(5)}). \tag{2.134}$$

Depending on the label, we have the periodicities:

$$x^{(1)} \sim x^{(1)} + 2R\,, \qquad x^{(5)} \sim x^{(5)} + 3R. \tag{2.135}$$

The wave function of the string then, depending on both $x^{(1)}$ and $x^{(5)}$ is not periodic in R, but rather has a periodicity of $6R$:

$$\psi(x^{(1)}, x^{(5)}) = \psi(x^{(1)} + 6R, x^{(5)} + 6R). \tag{2.136}$$

For a general number of branes, we conclude that the string wave function is periodic in $N_1 N_5 R$ (at least if N_1 and N_5 are coprime). Thus we can expand any such wavefunction in a set of modes with this periodicity:

$$\psi(x_5) \sim e^{-2\pi \frac{n}{N_1 N_5 R} x_5} \tag{2.137}$$

The number n denotes the number of momentum units; momentum on such D1-D5-string is quantized in units of $1/N_1 N_5 R$ rather than $1/R$. This phenomena is referred to as momentum *fractionalization* because momenta can now come in fractional units.

Note that the total spacetime momentum, N_p, as measured e.g. at infinity in black hole solution, is still quantized in units of $1/R$ because metric modes (which carry the momentum) are single valued around the S^1. But the individual open strings carrying the momentum can carry fractional momentum – it is only the sum of all the momenta that must be integrally quantized (in units of $1/R$).

What about non-coprime N_1, N_5? We can always consider the nearest-coprime number by subtracting a small number $m \ll N_{1,5}$ such that $N_1 - m$ and N_5 are coprime. Then the leading contribution to the entropy is still $N_1 N_5 N_p$ as any difference will be suppressed by powers of m/N_1. As we will explain below it is always entropically favourable to be in the configuration with maximal fractionalization so this configuration will dominate.

We want to put N_p units of momentum on the D1-D5-string system but there are many ways of doing this by putting different amounts of momenta on different open strings. Thus the entropy of the system is given by considering:

In how many ways can we get the momentum $p = N_p/R$ from partitioning the momentum over the D1-D5 open strings (with wave function (2.137)?

We can translate this to counting the number of partitions

$$\sum_{m=1}^{\infty} \frac{n_m m}{N_1 N_5 R} = \frac{N_p}{R}. \tag{2.138}$$

The number m counts the momentum in units of $1/N_1 N_5 R$ added by n_m strings of this type. For instance, the easiest (but not most entropic) way to get such a partitioning is to take one string with $m = N_p N_1 N_5$ units of momenta.

We count the number of different ways to form free strings (free excitations)

$$M \equiv N_1 N_5 N_p = \sum_{m=1}^{\infty} n_m m. \tag{2.139}$$

This is a counting of partitions of integers. We claim that this is counted by the partition function

$$Z = (1 + q + q^2 + \ldots)(1 + q^2 + q^4 + \ldots)(1 + q^3 + q^6 + \ldots)(\ldots). \tag{2.140}$$

The first contributions are

$$Z = 1 + q + 2q^2 + 3q^3 + \ldots \tag{2.141}$$

and the coefficients of q^n indeed count the partitions of the numbers n: one partitioning of 1, two of the number 2 ($1 + 1$ and 2), three for 3 ($1 + 2$, $2 + 1$ and 3) and so on. If we write the partition function as

$$Z = \sum_{n=0}^{\infty} d_n q^n, \tag{2.142}$$

then d_n counts the number of partitions of the integer n.

Using our knowledge of a geometric series for $q < 1$:

$$\sum_{n=0}^{\infty} q^n = \frac{1}{1-q}. \tag{2.143}$$

we see that the partition function can be written as the product

$$Z = \prod_{n=1}^{\infty} \frac{1}{1-q^n}. \tag{2.144}$$

How can we evaluate this partition function? We perform a calculation in the canonical ensemble: rather than fixing M we fix a dual "effective inverse temperature" β and we will go to a "high temperature"-limit. First we write q as

$$q = e^{-\beta}. \tag{2.145}$$

We calculate the average occupation number

$$\langle n \rangle = \frac{1}{Z} \sum_n n\, d_n e^{-\beta n} = \frac{\partial}{\partial \beta} \log Z. \tag{2.146}$$

This number will give us the leading contribution to the entropy.

First we evaluate the logarithm of the partition function:

$$\begin{aligned}
\log Z &= -\sum_{n=1}^{\infty} \log(1-q^n) \\
&= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{(q^n)^m}{m} \\
&= \sum_{m=1}^{\infty} \frac{1}{m} \sum_{n=1}^{\infty} (q^m)^n \\
&= \sum_{m=1}^{\infty} \frac{1}{m} \sum_{n=1}^{\infty} \left(\frac{1}{1-q^m} - 1 \right) \\
&= \sum_{m=1}^{\infty} \frac{1}{m} \frac{q^m}{1-q^m}.
\end{aligned} \tag{2.147}$$

In the second to last line we used (2.143), and compensated for the over counting for $n = 0$.

Now we take a "high temperature"-limit, by taking $\beta \ll 1$:

$$q \lesssim 1. \tag{2.148}$$

Then $\langle n \rangle$ will be large because we get the large n contributions of the sum $Z = \sum_n d_n q^n$. The leading terms in this limit are

$$q = 1 - \beta + \mathcal{O}(\beta^2)\,, \qquad q^m = 1 - m\beta + \mathcal{O}(\beta^2). \tag{2.149}$$

The logarithm of the partition function becomes

$$\log Z = \frac{1}{\beta} \sum_m m^{-2} + \mathcal{O}(\beta^0). \tag{2.150}$$

We can rewrite this in terms of $\zeta(n)$, Riemann's ζ function, which gives for n an integer:

$$\zeta(n) \equiv \sum_{m=1}^{\infty} \frac{1}{m^n}. \tag{2.151}$$

Then the average particle number is

$$\langle n \rangle = \frac{\zeta(2)}{\beta^2}. \tag{2.152}$$

Standard thermodynamics gives us that the entropy in the canonical ensemble is

$$S = \log Z + \beta \langle n \rangle, \tag{2.153}$$

and this gives

$$S = \frac{2}{\beta}\zeta(2). \tag{2.154}$$

To express the entropy in terms of the number $M \equiv \langle n \rangle$, we invert the relation (2.152), $\beta = \sqrt{\zeta(2)/M}$, and we use that $\zeta(z) = \pi^2/6$. This gives the entropy:

$$S = 2\pi\sqrt{\frac{M}{6}} = 2\pi\sqrt{\frac{N_1 N_5 N_p}{6}}. \tag{2.155}$$

There is a factor of 6 off in the square root compared to the supergravity result! Did we make a counting mistake?

Some remarks:

- Why do we count in canonical ensemble in terms of $\langle n \rangle$ instead of counting the d_n directly for $d = M$ (i.e. working in the microcanonical ensemble)? Recall that for large occupation numbers the canonical and microconincal ensemble are equivalent and we are interested in the large M asymptotics. This is exactly what we do in standard statistical mechanics: E in the canonical ensemble is replaced by $\langle H \rangle$, the expectation value of the Hamiltonian.
- We assumed $\beta \to 0$. We need to check this was a valid assumption. By

$$\beta = \sqrt{\frac{\zeta(2)}{M}}, \tag{2.156}$$

this gives $M \to \infty$: this is exactly the regime we are interested in from the validity of the supergravity solution.

Let us get back to this factor of 6. With the results of Exercise 2.4.12 , we find that the entropy for a "supersymmetric system" (equal number of fermionic and bosonic excitations) is

$$S = 2\pi\sqrt{\frac{cM}{4}}, \tag{2.157}$$

with c the number of bosons. We count the number of massless modes on S^1, but the entropically dominant strings are those stretching between the D1 and the D5-branes. Those 1–5 strings have four bosonic degrees of freedom from their freedom of moving around in T^4 (and these modes have 4 fermionic superpartners justifying the use of the supersymmetric counting formula).[19]

Therefore, the D1-D5-P system has $c = 4$ and we reproduce the black hole entropy on the nose:

$$S = 2\pi\sqrt{N_1 N_5 N_p}. \tag{2.158}$$

Hooray to string theory!

Exercise 2.4.12 *Prove the following statements:*

- *For the partition function*

$$Z_c = \left(\prod_{n=1}^{\infty} \frac{1}{1-q^n}\right)^c, \qquad c \in \mathbb{N}, \tag{2.159}$$

the entropy in the large temperature limit is

$$S = 2\pi\sqrt{\frac{cN}{6}}. \tag{2.160}$$

In a free theory, this formula is easy to show. This partition function is nothing but the partition function of c free bosonic oscillators.

- *Z_c was the partition function for c bosons. For fermions, which have either occupation number 0 or 1, we need to put in something extra. Using similar reasoning as for c = 1 boson partition function, show that the partition function for fermions is*

$$Z_{\text{fermions}} = \prod_{n=1}(1+q^n). \tag{2.161}$$

Show that for the partition function for c bosonic and fermionic string excitations is

$$Z = \left[\prod_{n=1}^{\infty}\left(\frac{1+q^n}{1-q^n}\right)\right] \tag{2.162}$$

and that in the high temperature limit, this gives the entropy

[19] There are also contributions from 1-1 and 5-5 strings, but these have momentum quantized in units of $p \sim 1/N_1$ and $- \sim 1/N_5$ and are hence subleading.

$$S = 2\pi\sqrt{\frac{cN}{4}}. \tag{2.163}$$

2.5 AdS/CFT

In this section we will "formalize" the counting arguments of the previous section by putting it in the much larger context of AdS/CFT, a very deep duality between gauge theory and gravity (or between open and closed strings), discussed first in [24–26].

We have seen that g_s, the string coupling, and the number of D-branes N allow us to interpolate between different regimes, see Figs. 2.21 and 2.22. The coupling g_s sets the "quantum" nature of closed string interactions. When $g_s \ll 1$: we have $M_s \ll M_P$ and string theory is classical. Low-lying string excitations are not so massive as to require quantum gravity to understand them. When $g_s \gg 1$ on the other hand, any massive stringy excitation (except the point-like ground states) are in the quantum gravity regime and there is no such thing as classical string theory.

Recall that in the previous section we very heuristically suggested that there is a general duality between open and closed strings: in the presence of a D-brane tree-level closed string diagrams can alternately be interpreted as an open string loop diagrams (see e.g. Fig. 2.18). While we believe this duality holds in general it is quite hard to study because its rather difficult to study excited stringy states. What has been studied and demonstrated in great detail however is a very particular low-energy limit of this duality: AdS/CFT.

In this section, we wish to motivate and study this particular limit and the associated duality. We consider string theory with N D-branes and take a low-energy limit by fixing the energy *at asymptotic infinity* such that $E \ll M_s$ (in a sense we will describe in more detail below). In this low-energy limit we want to consider the regimes:

- $g_s N \ll 1$: Open string theory reduces to a weakly coupled gauge theory describing the system. As we will explain below the description in terms of closed strings is not very tractable in this regime because the near-brane geometry has string-scale curvature and would require the full complex machinery of closed string theory to describe it (i.e. a reduction to massless supergravity modes is not sufficient).
- $g_s N \gg 1$: The same gauge theory above is now strongly coupled and while we can still think of it in terms of open strings this description is not very traceable. Rather a more tractable description is the dual closed string or supergravity picture with D-branes back-reacting and giving a near-brane geometry with a low curvature scale.

The main point is that open-closed duality implies either picture is valid but one may be more computationally tractable in a certain regime than the other. Here we will motivate this duality primarily in its low-energy limit where it becomes the AdS/CFT correspondence.

2.5.1 'Deriving' AdS/CFT

For simplicity in the exposition below we will take $N_1 = N_5 = N$. We further define

$$\lambda = g_s N. \tag{2.164}$$

We represent the small λ and large λ system in Fig. 2.22.

An Open String Perspective ($\lambda \ll 1$)

Let us start by considering the weak coupling picture, $g_s N \ll 1$, where we have a description in terms of perturbative closed and open strings with the latter ending on infinitely heavy, static D-branes. We will restrict ourselves to low energy excitations in this regime as we explain in more detail below.

The spacetime geometry at $\lambda = g_s N \ll 1$ is:

$$M_{1,10} = \mathbb{R}^{1,4} \times S^1 \times T^4. \tag{2.165}$$

In flat space there is a globally defined notion of energy which is the same for an observer near the brane as for an asymptotic observer:

$$E_0 = E_\infty. \tag{2.166}$$

Here E_0 is the energy of an observer in the bulk, or near the brane (this distinction will become important at strong coupling where warp factors shift energies measured at infinity with respect to those near the brane).

How does a process where open strings interact with closed strings depend on the this characteristic energy scale? Such a process was depicted in Fig. 2.19. At low energy, we have gravitons leaving the brane. The amplitude for such a process is proportional to:

$$g_s^2 \ell_s^{D-2} N^2 = G_N N^2. \tag{2.167}$$

From a closed string perspective this is just a gravitational interaction that must be proportional to the masses and G_N. From an open perspective there is one factor of $g_s N$ for each open string endpoint on N D-branes. For instance, for the D1-D5 system, we have to sum over all the ways we can get this process, and there are $N_1 N_5$ possible ways of making 1–5 strings, see Fig. 2.25. The g_s factors follow because as evident from Fig. 2.18 closed emission from a brane looks like an open loop diagram.

Let us take a six-dimensional emission perspective, focusing on $\mathbb{R}^{1,4} \times S^1$ (dropping the T^4 part of the geometry). The dimensionless rate is fixed, on dimensional grounds to depend on the energy E of the process as

$$E^4 G_N = (g_s N)^2 \ell_s^{D-2} E^4 = \lambda^2 (E \ell_s)^4. \tag{2.168}$$

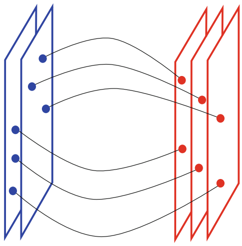


Fig. 2.25 Strings stretching between N_1 D1 branes and N_5 D5 branes. There are $N_1 N_5$ ways of making D1-D5 strings

We would like to work at low enough energies so this process is highly suppressed and the physics of the brane effectively decouples from that of the rest of spacetime. Thus we need to consider energies such that

$$E\ell_s \ll 1/\sqrt{\lambda}$$

In the limit above the open string physics on the brane decouples from interactions with bulk closed strings but open string theory is still rather complicated so let us consider a further limit $E\ell_s \ll 1$. In this limit stringy excitations are very massive and can be integrated out and open string theory on the brane reduces to gauge theory. Thus the limit we really want to consider is

$$E\ell_s = E^{\infty}\ell_s \ll \min\,(1, 1/\sqrt{\lambda}). \tag{2.169}$$

In this limit the physics of the D-brane "decouples" from that of the bulk and gives, at $\lambda = g_s N \ll 1$, a weakly coupled gauge theory living on the brane. The gauge theory is weakly coupled because λ is nothing other than the 't Hooft parameter—the natural coupling constant of a large N gauge theory (see [27] for a pedagogical exposition of large N gauge theories). But notice that we could also have taken the same limit at large λ and this should in principle describe the strongly coupled version of this gauge theory. Because we restrict to energies satisfying both $E\ell_s \ll 1$ and $E\ell_s \ll 1/\sqrt{\lambda}$ for any value of λ the decoupling of the brane from the rest of the geometry should remain valid as should the "gauge theory limit" of the open strings. The only thing that changes is that the gauge theory becomes strongly coupled. Can we understand this from the closed string perspective?

A Closed String Perspective ($\lambda \gg 1$)

Let's now move to the closed string perspective at $\lambda \gg 1$. Take the metric of the D1-D5-P system

$$ds^2 = \frac{1}{\sqrt{Z_1 Z_5}}(-dt^2 + dx_5^2 + Z_p dx_-^2) + \sqrt{Z_1 Z_5}\, d\boldsymbol{x}_4^2, \tag{2.170}$$

with the light-cone coordinate

$$x_- = t - x_5. \tag{2.171}$$

This metric describes a momentum excitation along one direction, because the light-cone coordinate x_+ is absent.

Remember that this metric has the following regions:

- Asymptotically flat $\mathbb{R}^{1,4} \times S^1 \times T^4$.
- Near horizon region. There is an $AdS_3 \times S^3 \times T^4$ throat and the black hole horizon sits at the bottom of this throat. The quick way to get this decoupled region is to drop the constants in the Z_1 and Z_5 harmonic function but keeping the constant in the Z_p harmonic function. See e.g. [28] for a more detailed exposition of this limit.

so the metric and spacetime at infinity look the same as in the weak coupling limit; only the region near the branes changes.

From the metric, we know that the charges Q in the harmonic functions $Z = 1 + Q/r^2$ go as $Q \sim g_s N (\ell_s)^2 =: \lambda(\ell_s)^2$. Therefore the scale of the throat is set by

$$L \sim \sqrt{\lambda}\ell_s. \tag{2.172}$$

Low Energy Excitations

As above we want to work with "low energy excitations". But: what is energy in this setup? Let us start with the energy E^∞ measured by an observer at infinity in the black hole spacetime and let us restrict, once more, to[20]

$$E^\infty \ell_s \ll 1. \tag{2.173}$$

This means that no strings are excited and we only see gravity modes. Asymptotically, string theory reduces to just (super)gravity.

On the other hand, the throat also has a characteristic energy scale set by $E_{\text{throat}} = 1/L \sim 1/\sqrt{\lambda}\ell_s$. Any asymptotic excitation with an energy lower than

$$E^\infty \ell_s < \frac{1}{\sqrt{\lambda}} \tag{2.174}$$

decouples from the throat: its wave length is larger than the scale of the throat and any such mode shot in from infinity will fly by and not be absorbed by the throat.

Thus at this scale asymptotic excitations decouple from excitations in the throat just as they did in the open string analysis at $g_s N \ll 1$. The low energy limit (2.169) thus has the effect of isolating the "near-horizon" physics down the throat from what happens further away. This "decoupling" is an essential feature of AdS/CFT so we will always work, for all values of λ, in the limit

[20] In natural units $\hbar = c = 1$, energy is measured in dimensions of inverse length $[E] = L^{-1}$.

$$E^{\infty}\ell_s \ll \min(1, 1/\sqrt{\lambda}). \tag{2.175}$$

Another way to phrase this is that we consider the theory defined by excitations whose (asymptotic) energy remains finite as we send $\ell_s \to \infty$.

So far we have phrased this limit in terms of the energy measured at infinity and shown that asymptotically stringy excitations become infinitely massive and can be ignored in this limit. What about the near-horizon throat region?

In a gravitational theory energy can only be defined locally. The redshift relates the energy between two observers at r_1 and r_2 as $\int_{r_1}^{r_2}\sqrt{g_{tt}}$. Approximating this integral by its value down the throat, the energy E_0 of a local observer at say $r = 1$ in the throat is related to the asymptotically measured energy as

$$E^{\infty} \sim (Z_1 Z_5)^{1/4} E_0 = \sqrt{\lambda} E_0. \tag{2.176}$$

What does this imply about the energy of excitations down the throat in our limit (2.175)? Consider the two cases:

1. $\lambda > 1$: Then by (2.175) we have $\sqrt{\lambda}E_0\ell_s \ll 1/\sqrt{\lambda}$, which can be written as:

$$E_0\ell_s \ll 1/\lambda < 1. \tag{2.177}$$

   There are no stringy excitations down the throat.
2. $\lambda \ll 1$; Then by (2.175) we have $\sqrt{\lambda}E_0\ell_s \ll 1$, or:

$$E_0\ell_s \ll 1/\sqrt{\lambda}, \tag{2.178}$$

   but we also have $1/\sqrt{\lambda} \gg 1$ and thus we can have stringy modes down the throat. This happens because the energy of these modes is so red-shifted that, at infinity, we still have $E^{\infty} = \sqrt{\lambda}E_0 \ll 1$ even if we consider excitations with e.g. $E_0\ell_s \sim n \gg 1$ so long as $n\sqrt{\lambda} \ll 1$.

We conclude that there *can* be stringy excitations down the throat only when $\lambda \ll 1$. These are decoupled from the asymptotic region due to the redshift.

Thus the closed string picture we arrive at is one where the spacetime has a throat region and an asymptotically flat region and, at low energies, these regions are decoupled from each other. As in the open picture we are interested in the throat region near the brane itself let us examine what that region looks like in more detail (Fig. 2.26).

Throat Geometry

Let us consider the geometry of the throat. First we put $Z_p = 1$ effectively setting $Q_p = 0$. We can later add the momentum as excitations on the throat geometry. Deep in the throat we have $r \ll \sqrt{\lambda}\,\ell_s$ and hence

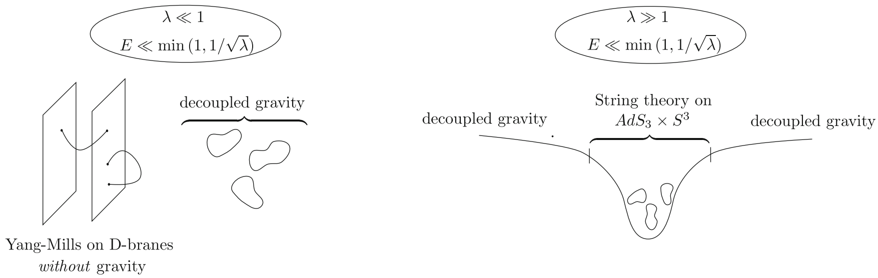


Fig. 2.26 We consider low-energy excitations $E^{\infty}\ell_s < \min(1, 1/\sqrt{\lambda})$. *Left* in the regime $\lambda \gg 1$, we have a field theory describing open string theory, *right* for $\lambda \ll 1$, we can have a "stringy" *black hole*, with (open) string excitations and gravitons down the throat, which decouple from the asymptotic geometry

$$Z_{1,5} \sim \frac{\lambda(\ell_s)^2}{r^2}. \tag{2.179}$$

The geometry becomes

$$ds^2 = \frac{r^2}{\lambda \ell_s^2}(-dt^2 + dx_5^2 + \ldots) + (\lambda \ell_s^2)\frac{dr^2}{r^2} + \lambda \ell_s^2 d\Omega_3^2 + ds^2(T^4). \tag{2.180}$$

This is the geometry of $AdS_3 \times S^3$ (times a constant volume T^4). The radius of anti-de Sitter space and the three-sphere are equal and set by λ in string units:

$$R_{AdS} = R_S = \sqrt{\lambda}\ell_s. \tag{2.181}$$

Note that the geometry $AdS_3 \times S^3 \times T^4$ is a solution to the equations of motion itself, essentially because the equations for the warp factors

$$\Delta Z_i = 0, \tag{2.182}$$

are insensitive to the presence or absence of the integration constant, h, in the harmonic functions $Z = h + Q/r^2$ and it is this feature which distinguishes the $AdS_3 \times S^3$ solution from the flat-space one.

2.5.2 Putting it All Together: AdS/CFT

Let us now put together the various pieces we have assembled. Recall that we claim that there is an open-closed duality meaning that we are free to use open or closed strings to describe a given system. The system we are interested in is the D1-D5-P system. We study this system in the particular low-energy limit (2.169). Note that this limit is phrased in terms of dimensionless parameters $E\ell_s$ so it is a consistent "decoupling" limit; we can formally take a limit sending $E^{\infty}\ell_s \to 0$ and this defines a completely independent subsector of the original string theory.

When taking this limit what we find is that:

- In the *open description* the open strings on the brane decouple from the physics off the brane and furthermore only the massless open strings survive. Thus open string theory reduces to supersymmetric Yang-Mills on the D-brane. At $\lambda \ll 1$ this theory is weakly coupled and can be studied. When $\lambda \gg 1$ this becomes a strongly coupled gauge theory and it is hard to compute anything.
- In the *closed description* the closed strings near the horizon (down the throat) decouple from those asymptotically far away so there is a self-contained closed string theory living on $AdS_3 \times S^3 \times T^4$. When $\lambda \gg 1$ only light excitations survive the low-energy limit so we are left with supergravity on the aforementioned spacetime but when $\lambda \ll 1$ stringy modes *can* be excited so the theory really is a full string theory.

The statement of AdS/CFT, which we see is just a low-energy limit of open-closed duality, is that the two descriptions listed above are in fact equivelent! Put another way supersymmetric Yang-Mills on a D-brane is equivelant to a string theory on an AdS spacetime. When the gauge theory is weakly coupled ($\lambda \ll 1$) the AdS is very stringy and thus its hard to study it (many massive string modes are excited). On the other hand, when the gauge theory is strongly coupled ($\lambda \gg 1$) the closed string theory on AdS reduces to supergravity leading to the remarkable observation that we can understand strongly coupled gauge theories by studying semi-classical supergravity! This is the primary reason why AdS/CFT has been so fruitful in the last years.

It should be emphasized that all the statements made above were made in the limit of sending $N \to \infty$ and $g_s \to 0$ while keeping the combination $\lambda = g_s N$ as a free parameter. Thus the gauge theories above always have very large gauge groups $SU(N)$ with $N \to \infty$. The duality between gauge theory and closed string theory is believed to hold even for finite N and there are numerous computations checking $1/N$ corrections to the above. This regime is much harder to study however as making N and λ finite means that g_s can no longer be zero and we need to consider higher loop diagrams in string theory or supergravity and this is quite hard.

2.5.3 AdS/CFT Dictionary

In Table 2.7 we collect the various equivalences implied by AdS/CFT. When the field theory is weakly coupled, the AdS space has a very small radius L and string theory corrections are important (strongly coupled string theory on AdS). When the field theory is strongly coupled, the AdS space is large and well described by classical supergravity. In terms of couplings this means that for small λ, we have good control

Table 2.7 Equivalence between open string and closed string theory for various values of λ

Yang-Mills on a D-brane	Closed string theory on AdS
Decoupled sector:	Closed strings
gauge theory on a brane	down the throat
No strings/no gravity	*Full, closed string theory*
λ: gauge ('t Hooft) coupling constant	$\lambda = L/\ell_s$: size of AdS in string units
N: rank of gauge group	$N = L/\ell_p$: size of AdS in Planck units
λ small: weakly coupled	AdS small $\rightarrow$ stringy
Control	*No control*
λ large: strongly coupled	AdS large $\rightarrow$ Supergravity
No control	*Control*

of the gauge theory, whereas for large λ, we have good control of the gravitational anti-de Sitter physics.

Note that unlike string theory in flat space where the only parameter is $g_s \sim f(\ell_s/\ell_p)$ in AdS there is an additional dimensionful scale, L, the AdS radius, allowing us to define two independent parameters: N and λ. Following the discussion above we see that in gauge theory, N is the rank of the gauge group while in gravity, N is the size of the AdS space in Planck units (while λ is the size of AdS in string units). The inverse AdS radius measured in Plank units, $1/N$, provides the natural perturbative parameter for quantum gravity in the bulk; i.e. this parameter enters in loop corrections for both gravity and string theory. Thus the limit of an infinite number of colors, $N \rightarrow \infty$ is nothing other than the classical limit in the AdS theory! While this may seem like a somewhat strange statement it in fact parallels a well known statement in gauge theory that at large N the dynamics of the gauge theory become much simpler (see [27] for an explanation).

Exercise 2.5.13 *Show that the AdS length (size of the D1-D5 black hole throat) in string units is set by λ, and in Planck length by N:*

$$\lambda = L/\ell_s, \qquad N = (L/\ell_P)^n \tag{2.183}$$

for some number n. Find n.

Because supergravity is only valid at large N, we only understand large N gauge groups from supergravity. On the other hand, we could invert this to maybe learn quantum gravity from small N gauge groups. For instance, for $N = 2, 3$ the size of AdS space is a few Planck units and gravity is strongly coupled.

Note that the AdS/CFT correspondence is a conjecture. We haven't proven anything, we have just given motivation! It is very hard to prove: a proof would require a detailed knowledge of strongly coupled field theories. However, it is very well established as many very non-trivial computations (not necessarily protected by symmetry) have been found to match on both sides and thus most string theorists hold it to be true. In some sense it is nothing more than the low energy limit of the much more powerful open/closed string duality hinted at by Fig. 2.18. The closed string exchange between D-branes, which can be interpreted as a tree level open string diagram, has all the massive modes implicit. For AdS/CFT, we only consider the massless, non-oscillatory modes.

Formal AdS/CFT Duality

The correspondence can be formalized by equating the path integrals of the two theories:

$$\boxed{Z_{\mathrm{CFT}}(\lambda, N) = Z_{\mathrm{IIB}}^{\mathrm{string}}(\lambda, N)|_{\text{on asympt. AdS space}}}. \tag{2.184}$$

This equality summarizes the AdS/CFT conjecture.

We often restrict to λ very large, and then we get an equivalence between large 't Hooft coupling CFT and IIB supergravity on an asymptotically AdS space:

$$Z_{\mathrm{CFT}}(\lambda \to \infty, N) = Z_{\mathrm{IIB}}^{\mathrm{sugra}}(N)|_{\mathrm{AdS}}, \tag{2.185}$$

where sugra stands for supergravity. Schematically, we can write the supergravity path integral as

$$Z_{\mathrm{IIB}}^{\mathrm{sugra}}(N) = \int \mathcal{D}g \, \exp\left(-\int \sqrt{-g}(\text{gravitons} + \ldots)\right), \tag{2.186}$$

there are other fields besides the metric g, but let's just forget about them for the sake of the argument. When N is large, we are doing classical supergravity: at fixed $\lambda = g_s N$, loops are suppressed because g_s is small. Then we can perform a saddle point approximation around the minima of the action (the classical solutions to the equations of motion), and the large N approximation is

$$Z_{\mathrm{IIB}}^{\mathrm{sugra}}(N \to \infty) = \sum_i e^{-S_i}, \tag{2.187}$$

The sum runs over solutions to the equations of motion (saddle points) and it is actually possible to calculate its main contributions. In the limit $\lambda \to \infty$ (large 't Hooft coupling) and $N \to \infty$ (planar limit), states in the CFT are hence related to classical solutions in AdS.

The left-hand side of (2.185) is always well-defined because CFTs are formally well-defined objects. Thus, as a consequence of the AdS/CFT correspondence, the right-hand side is also well defined: quantum gravity on AdS spaces is hence better defined than a generic QFT!

2.5.4 Entropy Counting

A black hole solution has an entropy and a temperature. Thus the natural candidate dual in the CFT is an ensemble of states corresponding to a thermal density matrix with the same quantum numbers as the black hole (in particular the mass). Such a density matrix has the following form

$$\rho_{BH} = \sum_{\psi} e^{-\beta H} |\psi\rangle\langle\psi|. \tag{2.188}$$

At high temperature there is no difference between the microcanonical and the canonical ensemble. Therefore we can work with the temperature, the thermodynamic dual of the mass, rather than with the mass itself.

Remember the set-up of the D1-D5-P system wrapped on $T^4 \times S^1$ of Fig. 2.20. The CFT that describes this system lives on the two-dimensional spacetime formed by the common circle on which the branes are wrapped and the time direction: $S^1 \times \mathbb{R}_t$. (This is the CFT dual to the AdS_3 near-horizon geometry of the D1-D5 black hole.)

Cardy gave us a formula for the entropy in a CFT at high temperature, irrespective of the coupling:

$$S \sim \sqrt{\frac{cL_0}{6}}, \tag{2.189}$$

where L_0 is the momentum along one direction, and c is the central charge. Although we will not justify this formula (it is a standard result in the study of 2d CFTs) let us note that it gives the number of states at a given level, L_0, in a CFT with central charge c. Because we are assuming the black hole to correspond to a thermal ensemble which is essentially a sum over all states we can use this formula and simply substitute in the black hole quantum numbers that give c and L_0 via AdS/CFT.

Note that Cardy's formula has the same form as the entropy computed using our a simple free oscillator counting. There c was the "entropy density". For a boson in a free theory, $c = 1$, for a free fermion one has $c = 1/2$. But the CFT we are considering here is strongly coupled since we want a large classical black hole so $\lambda \gg 1$ (as is N). Thus we cannot simply model the system using free fields but the great virtue of Cardy's formula is that it holds for any CFT, even a strongly coupled one. Moreover, it does not rely on any assumption of supersymmetry so this is a qualitatively different way of computing the degeneracy (recall that we were able to use a "free" open string picture in our previous counting because we argued, via supersymmetry, that we could work in the small $\lambda = g_s N$ regime and then simply tune λ to large values without changing the number of supersymmetric states).

For gravity on an AdS space, the central charge of the dual CFT is the AdS length in Planck units (we will motivate this partially below):

$$c = \frac{L}{\ell_P}, \tag{2.190}$$

Note, as expected (for the entropy to be invariant), this quantity is independent of the coupling but depends only on the D1-D5 charges:

$$c = N_1 N_5. \tag{2.191}$$

On the other hand this would not be the case if the central charge was the AdS length in string units, because then c would be equal to $\sqrt{g_s}\sqrt{N_1 N_5}$ and hence coupling-dependent. The fact that c is independent of the string coupling g_s is very important, because it assures that the entropy (through the Cardy formula) is independent of the coupling as well.

If we put the momentum excitations on the D1-D5 AdS_3 throat to match the full D1-D5-P black hole solution then in the dual CFT this corresponds adding light-like momentum along the string that the dual CFT lives on. Although we will not review this in detail it simply follows because the quantum numbers in the CFT can be matched to those in AdS and under this identification momentum waves in the bulk simply correspond to momenta along the CFT worldsheet. Thus, like the spacetime momentum[21] the momentum in the CFT is chiral and thus corresponds to a state with non-vanishing $L_0 \sim N_p$. Thus Cardy's formula gives the entropy

$$S \sim \sqrt{\frac{N_1 N_5 N_p}{6}}. \tag{2.193}$$

This does not rely on weak coupling but rather is valid for any value of g_s.

We have now argued, via AdS/CFT correspondence, that a thermal ensemble in a strongly coupled CFT is dual to a black hole geometry, and that we can use the Cardy formula to compute the entropy. Let us briefly motivate the identification of the central charge which we recall is

$$c = N_1 N_5. \tag{2.194}$$

From the original brane theory this is not hard to believe as the dominant degrees of freedom are the 1-5 strings and there are $N_1 N_5$ of them (recall that the central charge of a CFT is some measure of the degrees of freedom). This can be seen another way: in a 2d CFT, the partition function at high temperature goes as

$$Z_{\mathrm{CFT}} \sim e^{cT}, \tag{2.195}$$

and hence the entropy goes as

$$S \sim \log Z_{\mathrm{CFT}} \sim cT. \tag{2.196}$$

[21] Remember that the metric of the D1-D5-P system looks like

$$ds^2 = -dt^2 + dx_5^2 + Z_p dx_-^2, \tag{2.192}$$

with $dx_- = dt - dx_5$. This fixes a particular chirality of the plane wave.

Table 2.8 For supersymmetric black holes, we can match the Bekenstein-Hawking entropy from a weak coupling computation

$S_{\mathrm{BH}}^{\mathrm{micro}}$	$=$	$S_{\mathrm{BH}}^{\mathrm{macro}}$
$\downarrow$		$\downarrow$
$\log(N)$		$A_H/4G_N$
Weak coupling		**Strong coupling**

This also shows why we can interpret c as the entropy density.

Questions from the audience:

- *The black hole is extremal. How can there be a (CFT) temperature?* In CFT, there is a left and a right temperature, related to the total amount of left- and right moving excitations. Using the null circle x_- (or x_+ if we would have that coordinate in the metric), gives a length of this thermal circle that gives a temperature T_L (T_R for x_+). The total temperature of a thermal ensemble of states is related to those temperatures as
$$\frac{1}{T} = \frac{1}{T_R} + \frac{1}{T_L}. \tag{2.197}$$
T_L and T_R are in fact chemical potentials for the quantum numbers L_0 and $\bar{L}_0$ in the CFT; these measure the number of left and right moving light-light momentum waves. In the extremal D1-D5 setup, we only have left-moving excitations and hence $T_L \neq 0$, but still $T_R = 0$. Therefore the BH temperature T is zero, even though there is a "left-moving temperature" T_L.
- *We have treated AdS/CFT. Here we had AdS_3 of the near-horizon plus the dual CFT. What happens if you insert a black hole inside an asymptotically AdS space?* Consider AdS with a black hole inside it. This corresponds to a CFT at a non-zero temperature T (so both T_L and T_R are non-zero), see Table 2.5.

2.5.5 Non-supersymmetric Black Holes

For supersymmetric black holes, we have seen that the microscopic entropy matches the macroscopic one as in Table 2.8.

We have seen two arguments why the weak-coupling, microscopic calculation gives the correct result for the entropy of the black hole at strong coupling:

- An index which is protected by supersymmetry: it can be calculated at weak coupling and continued to strong coupling.
- AdS/CFT correspondence. The result for the entropy uses the Cardy formula and can be calculated regardless of the coupling, as long as we high temperature states in the CFT (here temperature includes left or right moving temperature).

Both these arguments rely on supersymmetry but in different ways. The first argument requires supersymmetry by construction whereas Cardy's formula holds in any 2d CFT, even one without supersymmetry. Unfortunately only supersymmetric black holes have near-horizon AdS_3 factor which allow us to use $\mathrm{AdS}_3/\mathrm{CFT}_2$ and invoke Cardy's formula. What about non-supersymmetric solutions in asymptotically flat spacetime? The index will no longer be protected, and we cannot rely on the AdS/CFT correspondence anymore, because the near-horizon solution of a non-extremal black hole does not have an AdS factor.

On the other hand we can consider non-supersymmetric asymptotically AdS black holes (black holes embedded in an AdS spacetime rather than flat space). We can put a non-extremal black hole (black hole with a non-zero temperature) in $AdS_5 \times S^5$. Without the black hole, the geometry is dual to a conformal field theory, namely $\mathcal{N} = 4$ Super-Yang Mills theory. It is a supersymmetric and conformal (there is no dimensionful scale) field theory that is very similar to QCD.

When we put a black hole in spacetime, this is dual by the AdS/CFT correspondence to heating up the CFT, and hence introducing a scale. This is the setup of Table 2.5.

A high temperature excites the many states of this field theory (gluons, fermions…), and therefore you get an entropy, a number of states that are excited at a given temperature. The temperature breaks both conformal invariance and supersymmetry in the field theory and we get a non-supersymmetric state corresponding to the black hole.

We can repeat the counting of the previous section and find the entropy, both in the field theory (a non-trivial calculation involving fermions, $SU(N)$ gauge groups and so on) and in gravity (an easy calculation using the horizon entropy). One finds:

$$\begin{array}{ll} \mathcal{N} = 4 \text{ SYM} & \left\{ \begin{array}{l} S^{\text{micro}} = a(N)T^3 \\ \\ S^{\text{macro}} = \frac{3}{4}a(N)T^3 \end{array} \right. \\ \text{Supergravity} & \end{array}$$

with

$$a(N) = \frac{2\pi^2}{3} N^2. \tag{2.198}$$

A pedagogical derivation of this result can be found in [29].

The supergravity entropy only sees three quarters of the entropy of the microscopic counting. We can interpret this as the degrees of freedom that are changing. The field theory computation above is done at weak coupling ($\lambda \ll 1$) where we can easily compute whereas the black hole, which must be large in string units, corresponds to large values of λ. Thus there is some less of states as the spectrum shifts about from weak to strong coupling in Fig. 2.27. Note that this does not happen in the supersymmetric case because of the supersymmetric index is a protected quantity.

The fact that the entropy at a fixed energy changes as we vary the coupling should not be too surprising. As λ is increased various states will receive quantum corrections to their energy and the spectrum will shift about in a complicated way.

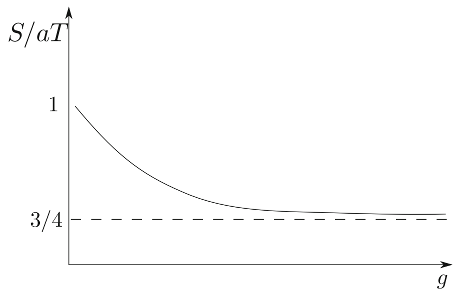


Fig. 2.27 A sketch of the entropy as a function of the coupling for the *black hole* in $AdS_5 \times S^5$ (see Fig. 2 in [30] for a detailed graph)

It turns out that there are other quantities which are also relatively robust so we may hope to compute them using AdS/CFT. That is to say there are quantities which are shared by a large class of theories—a *universality class*—which we may hope contains both $\mathcal{N} = 4$ SYM (the CFT which is dual to string theory on AdS_5) and other more physically relevant theories like QCD (or perhaps all Yang-Mills like theories). Since such quantities don't depend strongly on the detailed structure of the theory we can try to apply AdS/CFT to compute them even if we do not yet know the dual of QCD. Another way of thinking of this is that the gravity dual of $\mathcal{N} = 4$ SYM captures the strong coupling dynamics of a gauge theory and it may be that at strong coupling gauge theories display certain universal behavior.

As an example, take two fundamental properties of fluids in such theories: the entropy density s and the viscosity η. The entropy to viscosity ratio η/s for the quark gluon plasma of QCD can be observed experimentally. In the large N limit the value of η/s can be found exactly in $\mathcal{N} = 4$ SYM, from a weakly coupled gravity computation, and this value is of the same order as the observed value in the RHIC collider, see Table 2.9. Moreover, any calculation in the string theory ballpark always gives the same value of $\eta/s = 1/4\pi$. This is all the more intriguing because existing QCD theories (in which it is difficult to compute strongly coupled quantities) find a number which is off by an order of magnitude.

For this reason, people use AdS/CFT to describe strongly coupled QCD, and also strongly coupled condensed matter theories (so-called AdS/CMT, for instance for superconductors at strong coupling). In fact, this has been the main use of the AdS/CFT correspondence so far and this entire field can be put under the name "holography". There are many articles which can lead you in this direction, see for instance the previous courses on holography at IPhT [31, 32] (see also [33]).

Table 2.9 Entropy to viscosity ratio

η/s	Theory/Experiment
$1/4\pi \cong 0.0796$	$\mathcal{N} = 4$ SYM
$0.12 \pm \ldots$	QCD (Experiment)
$\mathcal{O}(1)$	QCD (Theory)

2.6 The Fuzzball Proposal and Black Hole Hair

In this section, we elucidate the idea that black hole entropy is explained by the existence of a large number of 'black hole microstate' solutions. These are geometries that are solutions to the equations of motion of string theory, have no horizon themselves, but should come in large enough numbers to account for the black hole entropy.

Let us get back to the main problem. We have a microscopic and a microscopic entropy, which agree numerically, but both are valid in different regimes. As an example, think about the air in a room. It is made up out of many molecules. Still, we can extract the entropy without reference to the microscopic state of the molecules through equations of state:

$$\begin{aligned} pV &= nRT, \\ dE &= TdS + pdV. \end{aligned} \tag{2.199}$$

We can determine the entropy S without knowing what air is made of—thermody namically, the entropy is a measure of the energy change in a system on which we have no control or understanding (in contrast to the work term pdV, which we control very well).

So much for thermodynamics, on to statistical mechanics. Boltzmann has taught us that for a given energy E and temperature T, all N different states of the molecules in the room make up the entropy as:

$$S^{\text{micro}} = \log(N). \tag{2.200}$$

This connection between statistical mechanics and thermodynamics is already 150 years old. Does it work for a black hole too? Can we find a number of microstates N for a black hole with a given set of mass and charges, such that $S_{BH} = \log(N)$?

At this point, the programme we followed so far is incomplete. The microscopic calculation ("statistical mechanics") takes place in one regime, but this statistical description is not valid when $g_s N \ll 1$. We have the following question:

- Say you take a state that makes up the entropy in the microscopic calculation. What happens if you follow such states one by one and bring them over to strong coupling?

People believed for a long time that as gravity grows stronger, a horizon forms around the D-branes and the objects end up "being" the black hole [34–38], see Fig. 2.28. Because gravity is always attractive, you expect that as you make Newton's constant larger, increasing the gravitational attraction, "normal" objects only becomes smaller. Only a black hole grows with increasing G_N, as the horizon radius for a (Schwarzschild) black hole scales with Newton's constant as

$$r_H = 2G_N M, \tag{2.201}$$

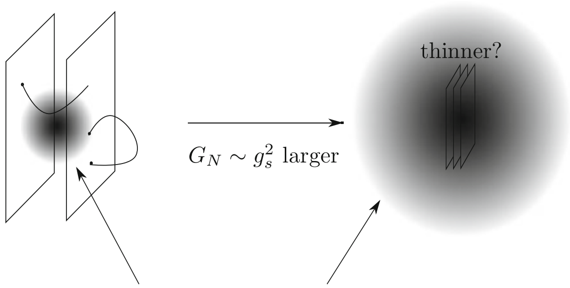


Fig. 2.28 At low G_N ($g_s \ll 1$), the would-be *black hole* horizon is of smaller or equal size as the brane system. For large G_N, the *black hole* horizon is much bigger than the size of the D-brane system at weak coupling

with M the mass of the black hole. Thus the horizon actually *grows* when you make gravity stronger. Take for instance a neutron star. This is a charge neutral object. Imagine a thought experiment in which we scale up Newton's constant G_N. The horizon radius of a black hole that has the same mass as the neutron star will become larger until for a certain large value of G_N, the neutron star collapses into a black hole. This intuition caused people to think for a long time that whatever state you take out of the $\exp(2\pi\sqrt{N_1 N_2 N_3})$ black hole microstates in the weak coupling description, all of them become a black hole with a singularity in the middle.

We can represent this pictorially. Say we have three microstates made up out of open strings on D1-D5 branes in the decoupled regime, as in Fig. 2.29. As we make gravity stronger, all of these would seem to fall behind the horizon and the information of the state making up the black hole is in the region near the singularity.

We discussed earlier the information paradox: We can throw anything into the black hole, but within GR, this information gets lost and never comes out, as the black hole evaporates into thermal radiation. Since the Hawking radiation process deals with the region around the black hole horizon, the intuitive picture of what happens to a brane microstate does not solve the problem. The horizon region is in the causal past of the singularity and physics in this region has no idea of what happens at the

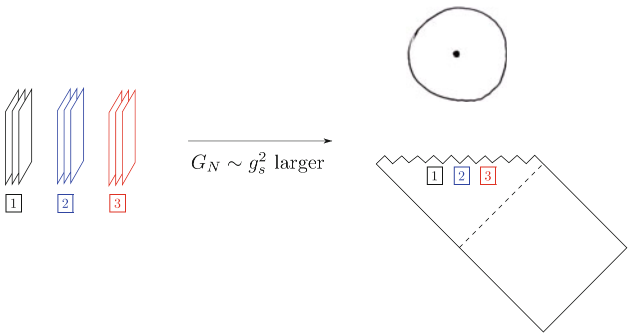


Fig. 2.29 In the naive picture, cranking up G_N puts the information of the microstate 1,2 or 3 into the garbage near the singularity

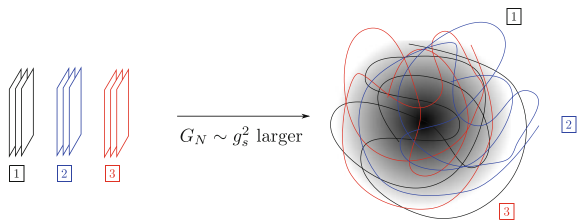


Fig. 2.30 The 'fuzzball proposal': cranking up g_s gives a complicated state of strings and branes of horizon size

singularity. All information still sits near the singularity and the information paradox is still there.[22] In fact, if we want to evade the information problem, arguments by Mathur show that one needs large corrections to the black hole geometry near the horizon [39].

Through the D1-D5-P black hole and the AdS/CFT duality, we should be able to find the CFT process dual to Hawking radiation. In CFT, we can actually address this problem.

In recent years, it has become clear that certain black hole microstates actually grow with G_N just as the black hole does! Look at a microstate. As g_s grows large, they actually become *bigger* and will be of the *same size* as the would-be black hole horizon, see Fig. 2.30. It is an ongoing task to find the actual geometries describing the strong coupling equivalent of the D-brane microstates. For such microstates that are of a size comparable to the black hole's, Hawking evaporation will know about what information made the black hole.

The main problem with this proposal is that you need to explicitly construct 'microstates' of the same size as the black hole horizon. The black hole horizon grows as G_N, but most things get smaller for increasing G_N. We need some very special objects. We will show how to build such growing states that correspond to the CFT we counted at g_s small. These will not have a horizon at large g_s.

The largest success of this proposal has been in the constructing of supersymmetric microstate geometries, see [40–45] for reviews. However, supersymmetric black holes do not radiate, and there is no comparison of the Hawking process. For non-supersymmetric radiating black holes, some large G_N microstates ('microstate geometries') have been constructed [46, 47]. They radiate and the Hawking radiation rate of the black hole agrees nicely with the decay of these states [45, 48–51].

We will review how to count the number of microstate geometries for supersymmetric black holes, using an appropriate quantization technique. So far, the number of microstate geometries found is subleading when compared to the black hole entropy. Ongoing research tries to construct more microstate geometries, see [52–54]. For work on non-supersymmetric multi-center solutions and microstate geometries, see [55–58].

[22] The information paradox leads to a breakdown of unitarity in quantum theory and hence a breakdown of quantum mechanics itself. If we want to save quantum mechanics, we need to make sure there is no information loss.

Note that we have come at the frontier of research: we have some hints about it, but people do not know yet if the proposal is generally true or not. In the next section we will show how to build (certain) fuzzball solutions for the supersymmetric 3-charge black hole.

2.7 Multi-Center Solutions

In this section, we show how to construct five-dimensional multi-center solutions that generalize the string theory black holes we have seen earlier. The microstate geometries for the black hole, or classical fuzzballs, will be in this class.

2.7.1 Preliminaries

In this section, we discuss some necessary basics on differential forms and their application in electromagnetism, and we explain the notation we use in the remainder of the text. We also give some exercises that illustrate an important new term (as opposed to Maxwell theory) in the supergravity action, the Chern-Simons term. This new terms allows for solutions with 'charge dissolved in fluxes', a crucial ingredient for the construction of microstate geometries. The reader familiar with these concepts can skip to the next section on the construction of multi-center solutions.

Differential Forms, Einstein-Maxwell, Sources

We review the following notions, by means of exercises:

- Differential forms, form notation and the definition of the Hodge star operator $\star$.
- 'True' magnetic sources (monopoles) versus 'moving electrons'.
- Sourced electromagnetism in flat space and in curved space (Einstein-Maxwell).

Consider electromagnetism. The anti-symmetric two-form is related to the electric field $\boldsymbol{E}$ and magnetic field $\boldsymbol{B}$ as

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix}. \tag{2.202}$$

In terms of this matrix, the Maxwell equations in vacuum are:

$$\partial_\mu F^{\mu\nu} = 0,$$
$$\partial_{[\mu} F_{\mu\nu]} = 0. \tag{2.203}$$

In form notation they are equivalent to

$$d \star F = 0,$$
$$dF = 0. \tag{2.204}$$

The first expression is the equation of motion that follows from the Lagrangian of electromagnetism:

$$S = \frac{1}{2} \int \star F \wedge F = \frac{1}{4} \int F_{\mu\nu} F^{\mu\nu}. \tag{2.205}$$

The second equation is the Bianchi identity, which just says that locally F is the exterior derivative of a potential $F = dA$, or $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ in form notation.

Exercise 2.7.14 *If you are not familiar with the expressions (2.204) (exterior derivative, Hodge star operator $\star$), read up on it in a book on differential geometry and show that the Eqs. (2.203) and (2.204) are equivalent.*

In particulate, we normalize m-forms as

$$A = \frac{1}{m!} A_{\mu_1 \ldots \mu_m} dx^{\mu_1} \wedge \ldots dx^{\mu_m}. \tag{2.206}$$

The exterior derivative acts on an m-form to produce an $(m+1)$-form as

$$dA_m = \frac{\partial A_{\mu_1 \ldots \mu_m}}{\partial x^\nu} dx^\nu \wedge dx^{\mu_1} \wedge \ldots dx^{\mu_m}, \tag{2.207}$$

and in d dimensions the Hodge star $\star$ takes an m-form to an $n = d - m$ form as follows

$$(\star\lambda)_{\mu_1 \ldots \mu_n} := \frac{1}{m!} \sqrt{g}\, \epsilon_{\mu_1 \ldots \mu_n \nu_1 \ldots \nu_m}\, g^{\nu_1 \rho_1} \ldots g^{\nu_m \rho_m} \lambda_{\rho_1 \ldots \rho_m}. \tag{2.208}$$

Here ϵ is the totally antisymmetric tensor.

Recall that in electromagnetism we can generate a magnetic field by accelerating an electron. However, while a speeding electron generates a magnetic field it does not generate a *magnetic charge*. This is because electric charge only appears in the equation

$$d \star F = q\, \delta(x), \tag{2.209}$$

whereas the magnetic charge sources the Bianchi identity

$$dF = m\, \delta(x). \tag{2.210}$$

The difference between these two is the following. If $m = 0$ then $dF = 0$ everywhere. In flat space this implies there exists a globally defined one-form, $A = A_\mu dx^\mu$, the vector potential, such that $F = dA$. If on the other hand $m \neq 0$ then at the origin F is not closed. Hence there is no globally defined object A such that $F = dA$. However, we can still define an object A everywhere away from the origin (or define it patch-wise). As a side note one might object that solving the electric equation requires something like $A_0 = q/r$, which is singular at the origin. However, we can always smoothen this singular source by allowing a charge distribution (for instance by replacing $q\delta(x)$ with a Gaussian qe^{-qr^2}). The same trick will *not* work for m because the Eq. (2.210) has $dF = ddA$ which is identically zero if A is globally defined.

To write a general field strength that includes both electric and magnetic charge we can do the following. We write

$$F = dA + \Theta, \tag{2.211}$$

with A a global one form encoding the electric charge (and perhaps some part of the magnetic field) via $d\star dA = q\delta(x)$. The two-form Θ on the other hand is *not* globally of the form $d(something)$ but rather satisfies $d\Theta = m\delta(x)$ and hence encodes the part of the field strength coming from the magnetic charge. To see this recall that the definition of the magnetic charge is the integral of the flux through an S^2 around the origin:

$$m = \frac{1}{4\pi}\int_{S^2} F = \frac{1}{4\pi}\int_{S^2}(dA + \Theta) = \frac{1}{4\pi}\int_{S^2}\Theta, \tag{2.212}$$

where the last equality follows because S^2 is a compact manifold without boundary and dA is a total derivative of a globally defined object. Hence the integral $\int dA$ vanishes by Stokes' theorem.

The 'electric part' of the gauge field, A, solves $d \star dA = q\delta(\boldsymbol{x})$. It can be found by thinking of A as harmonic $\nabla^2 A = \delta(\boldsymbol{x})$. This equation has solutions of the form $A = \frac{q}{r}dt$ (actually there is a larger class of solutions constructed of polynomials of the coordinates but the latter are not normalizable). For the solution of Θ, we refer to Exercise 2.7.15.

Exercise 2.7.15 *Write $\Theta = dB$ where B is only locally defined such that the integral (2.212) gives the magnetic charge m. (Find the form of B first).* Hint: *Explicitly construct*

$$B = f(\theta)d\theta \wedge d\phi, \tag{2.213}$$

using polar coordinates for the flat metric $ds^2_{3,flat} = dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$, such that $\int dB = 4\pi m$, with m a constant.

Note that when we couple electromagnetism to gravity (Einstein-Maxwell theory), the equation $d \star F = \delta(\boldsymbol{x})$ involves the metric via the Hodge star. Hence the solution becomes more complicated. It turns out that the metrics of the D-brane type have solutions that look like

$$A = H^{-1} dt, \tag{2.214}$$

where H is some harmonic function that determines the solutions and appears in the metric. Typically, in four dimensions harmonic functions are $H = 1 + q/r$, and asymptotically ($r \to \infty$), we recover the flat space solution $A = -q/r\, dt$.

Important Exercises: Chern-Simons Action

We show how the appearance of new terms in the supergravity Lagrangians (compared to electromagnetism) can allow for 'fuzzball' solutions.

The Lagrangian of electromagnetism coupled to gravity in four dimensions is

$$\mathcal{L}_4 = \tfrac{1}{4}\sqrt{-g} F_{\mu\nu} F_{\rho\sigma} g^{\mu\rho} g^{\nu\sigma} \tag{2.215}$$

$$= \tfrac{1}{2} \star F \wedge F. \tag{2.216}$$

This is the gauge and Lorentz invariant action for the Maxwell field A_μ. In five dimensions, an extra term is possible:

$$\begin{aligned}\mathcal{L}_5 &= \tfrac{1}{4}\sqrt{-g} F_{\mu\nu} F^{\mu\nu} + \tfrac{1}{12}\epsilon^{\mu\nu\rho\sigma\tau} A_\mu F_{\nu\rho} F_{\sigma\tau} \\ &= \tfrac{1}{2} \star F \wedge F + \tfrac{1}{3} A \wedge F \wedge F.\end{aligned} \tag{2.217}$$

This new term seems to be breaking gauge invariance. Consider the gauge transformation:

$$A_\mu \to A_\mu + \partial_\mu \lambda, \tag{2.218}$$

with λ a function. The field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is clearly gauge invariant. The second term in the five-dimensional Lagrangian has a "naked" A_μ and you might expect it is gauge non-invariant. The exercise asks you to prove this intuition wrong.

Exercise 2.7.16 *Show that in five dimensions, the Chern-Simons action*

$$S_{CS} = \int \epsilon^{\mu\nu\rho\sigma\tau} A_\mu F_{\nu\rho} F_{\sigma\tau}. \tag{2.219}$$

is invariant under gauge transformations (2.218). It suffices to show that the integrand is invariant up to a total derivative.

Most extensions of general relativity based on string theory (in particular supergravity) have such a term. So it is important to study its physical consequences.[23]

[23] It is also important for confinement in supersymmetric holographically dual gauge theories through the AdS/CFT correspondence, but that is another matter. See [59, 60].

Choose coordinates x^0, x^1, x^2, x^3, x^4 in five dimensions. Remember that a static electron couples to the gauge field as

$$\int A_0 dt. \tag{2.220}$$

Because of the term (2.219), a non-trivial A_0 is sourced by magnetic terms $F_{12}F_{34}$ through the equations of motion, which schematically have the form $\partial_i F_{0i} = F_{12}F_{34}$ (see Exercise 2.7.17). Even if you don't have electrons, but just magnetic fields of two different kinds, you can have electric fields!

Exercise 2.7.17 *Derive the equations of motion for A_μ following from the action (2.217):*

$$d \star F = F \wedge F. \tag{2.221}$$

Show that you can source electric fields with magnetic fields along different directions, by working this out in components (including the metric components involved in the Hodge star operation).

In the literature, one refers to solutions with this mechanism (magnetic fluxes giving a net electric charge) as solutions with charges dissolved in fluxes.

We will use this kind of solutions with charge dissolved in flux to build microstate geometries. In fact, this mechanism is crucial for the existence of microstate geometries. The absence of such a term in regular electromagnetism is also the reason people had not found black hole microstate geometries before the advent of string theory. This mechanism is widely used in other solutions as well, such as flux compactifications used for the construction of string vacua, see [61] for a review.

2.7.2 Building General Solutions

We discuss how to obtain new solutions with 'charge dissolved in fluxes'. We do this in a stepwise fashion: first we discuss the five-dimensional black hole (without and with rotation), and then we show how to put in magnetic charges.

M2-M2-M2 Black Hole

Let us write down a five dimensional electrically charged black hole by starting in M-theory (11-dimensions) and writing a solution down that involves a compact six-torus. Recall in particular, the supergravity solution for the (supersymmetric) M2-M2-M2 brane system

$$
\begin{aligned}
ds^2 = & -(Z_1Z_2Z_3)^{-2/3}dt^2 + (Z_1Z_2Z_3)^{1/3}ds^2(\mathbb{R}^4) \\
& + \frac{(Z_2Z_3)^{1/3}}{Z_1^{1/3}}(dx_1^2+dx_2^2) + \frac{(Z_1Z_3)^{1/3}}{Z_2^{1/3}}(dx_3^2+dx_4^2) + \frac{(Z_1Z_2)^{1/3}}{Z_3^{1/3}}(dx_5^2+dx_6^2).
\end{aligned}
\tag{2.222}
$$

This solution describes five space-time dimensions because we actually take the coordinates $x_1, \ldots, x_6$ to be compact ($x_i \sim x_i + 2\pi L_i$ for $i = 1, \ldots, 6$. They describe a six-torus T^6. We write the T^6 as the product of three two-tori T^2.

The M2-branes are all point-like in the transverse $\mathbb{R}^4$ spanned by x^7, x^8, x^9, x^{10} which we can write in radial coordinates

$$
ds_4^2 = d\rho^2 + \rho^2 d\Omega_3^2 \tag{2.223}
$$

and the five-dimensional black hole is determined by the functions:

$$
Z_1 = 1 + \frac{Q_1}{\rho^2}, \qquad Z_2 = 1 + \frac{Q_2}{\rho^2}, \qquad Z_3 = 1 + \frac{Q_3}{\rho^2}. \tag{2.224}
$$

The unusual power 2 rather than 1 in the denominator is because we are solving this equation in four rather than three space dimensions. Note that we refer to the radius in $\mathbb{R}^4$ as ρ, to avoid confusion with r for the radius of $\mathbb{R}^3$.

These functions are defined simply by requiring them to solve the equation:

$$
\Box_4 Z_I(x) = Q_I \delta(\rho) \tag{2.225}
$$

where $\Box_4 \cdot = \sqrt{g_4}^{-1}\partial_i(\sqrt{g_4}g_4^{ij}\partial_j \ \cdot)$ is defined with respect to the four-dimensional flat metric in the solution above (on $\mathbb{R}^4$). This equation says that we have M2 sources sitting at the origin of $\mathbb{R}^4$ with charges Q_I. The 1 in the equation above is simply a homogeneous solution we are free to add to any given solution to the Eq. (2.225). Since this equation is linear we are free to superimpose solutions (adding delta function sources). Hence the most general solution corresponds to an arbitrary number of M2 sources at various positions $\boldsymbol{\rho}_p \in \mathbb{R}^4$ and p labels the "centers":

$$
Z_I = \text{constant} + \sum_p \frac{Q_p}{|\rho - \rho_p|^2} \tag{2.226}
$$

See Fig. 2.31.

Recall that in M-theory we have a 3-form gauge potential and for the solution above it has the following form

$$
C_{012} = Z_1^{-1}, \qquad C_{034} = Z_2^{-1}, \qquad C_{056} = Z_3^{-1}. \tag{2.227}
$$

By "compactifying" on the $x_1, \ldots, x_6$ directions we can think of this as a five-dimensional solution times T^6 and one can show that this six-torus is actually small

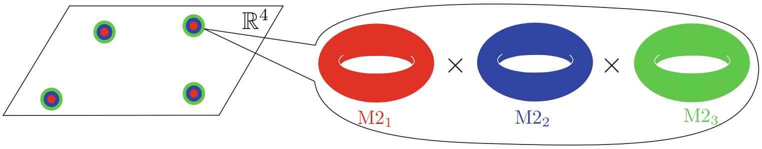


Fig. 2.31 Multiple M2-brane sources in $\mathbb{R}^4$. Each source can correspond to three types of M2-branes wrapped on a T^2 inside T^6, and smeared in the other torus directions

(the length of each cycle is order 1 in string units) so at low energies this space-time looks five-dimensional. In this case the different components of the three-form C_3 reduce to three independent gauge fields A^I_μ in five dimensions:

$$A^{(1)}_\mu = C_{\mu 12}, \qquad A^{(2)}_\mu = C_{\mu 34}, \qquad A^{(3)}_\mu = C_{\mu 56} \tag{2.228}$$

And likewise there are three field-strengths, $F^{(I)} = dA^{(I)}$ with $I = 1, 2, 3$. In form notation, the four-form $F_4 = dC_3$ of M-theory is then given by

$$F_4 = F^{(I)} \wedge \omega_I = d(Z_I^{-1} dt) \wedge \omega_I = (\partial_\rho Z_I^{-1}) d\rho \wedge dt \wedge \omega_I, \tag{2.229}$$

where we defined the volume forms on eqch two-torus:

$$\omega_1 = dx_1 \wedge dx_2\,, \qquad \omega_2 = dx_3 \wedge dx_4\,, \qquad \omega_3 = dx_5 \wedge dx_6. \tag{2.230}$$

In five dimensions the solution given by the functions Z_I of (2.224) is a spherically symmetric, electrically charged black hole in $\mathbb{R}^{1,4}$. We can generalize this solution in three ways, and we will do so in the remainder of this section, by:

- Adding angular momentum
- Adding magnetic charge
- Adding a more complicated base space (instead of $\mathbb{R}^4$)
- (Adding a more general internal space that preserves supersymmetry in five dimensions: a Calabi-Yau manifold instead of a T^6. We will not do this explicitly in these lectures.)

Multi-center solutions with these ingredients can describe black hole microstate geometries.

Adding Angular Momentum

The first generalization is to add angular momentum to this solution. We do this by replacing dt in the metric with $dt + k$ where $k = k_i(x)dx^i$ $(i = 7, 8, 9, 10)$ is a one-form in the four-dimensional base space:

$$ds_5^2 = -(Z_1Z_2Z_3)^{-2/3}(dt+k)^2 + (Z_1Z_2Z_3)^{1/3}ds^2(\mathbb{R}^4) \tag{2.231}$$

We will only consider the five non-compact directions from now on. Since the gauge field and metric are coupled via the equations of motion, adding angular momentum to the metric modifies the gauge field as well:

$$F^{(I)} = d(Z_I^{-1}(dt+k)) = d(Z_I^{-1}) \wedge (dt+k) + Z_I^{-1}dk. \tag{2.232}$$

Note this field strength has magnetic $F_{ij}^{(I)}$ components (from $\partial_i k_j$), because we have a moving charge. Remember from Sect. 2.7.1 that this does not represent a genuine magnetic monopole charge. This setup allows to describe a rotating supersymmetric black hole [62].

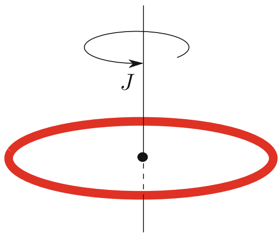
By adding a $k = k_i(x)dx^i$ term to the metric we get non-vanishing g_{ti} cross-terms in the metric. Such terms imply that the space-time itself carries angular momentum. This is *not* to be confused with being time-dependent. None of the fields above, including the metric, contains any explicit dependence on the time coordinate. A rather good analogy is to consider a *featureless* spinning ring in for instance $\mathbb{R}^3$, see Fig. 2.32. Since the ring is featureless nothing changes in time: the ring is always just sitting there spinning and from one instance to the next everything looks identical. Nonetheless, this solution carries angular momentum. In GR, such solutions with mixed g_{ti} components but no time-dependence are referred to as *stationary*. Solutions with no time dependence and $g_{ti} = 0$ are *static*.[24]

In $\mathbb{R}^4$ there are two independent angular momenta, because we can think of $\mathbb{R}^4$ as $\mathbb{R}^2 \times \mathbb{R}^2$: we have one independent angular momentum in each plane. For a single centered black hole, supersymmetry, is only preserved if we force these two angular momenta to be equal. This condition can be generalized as

$$(1 + \star_4)dk = 0 \tag{2.233}$$

which implies k is self-dual. Here $\star_4$ is the Hodge dual defined on the flat $\mathbb{R}^4$ given by $x_7, \ldots, x_{10}$. Note that acting on this with d we find $d \star dk = 0$, meaning k is

Fig. 2.32 A uniformly spinning ring with angular momentum J around its symmetry axis



[24] Stated without reference to a set of coordinates, 'static' means that the metric admits a global, nowhere zero, time-like hypersurface orthogonal Killing vector field. A generalization are the 'stationary' space-times, which admit a global, nowhere zero time-like Killing vector field.

a harmonic one-form. We will see later that by turning on additional fields, we can relax the condition of equal angular momenta for supersymmetric solutions.

Exercise 2.7.18 *Show that Eq. (2.233) is solved by (2.235). The constant J is proportional to the angular momentum of space-time. See for instance Sect. 2.2 in [20] for more information on asymptotic charges.*

Recall that without k we had the entropy $S_{BH} = \sqrt{Q_1 Q_2 Q_3}$ (up to numerical factors). When we turn on k we get an asymptotic angular momentum J. It can be read off from the asymptotic expansion of k in terms of the angles ϕ_1 and ϕ_2 in the two orthogonal $\mathbb{R}^2$-planes.

If we write the metric on $\mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2$ as

$$ds^2 = d\rho^2 + \rho^2(d\theta^2 + \sin^2\theta d\phi_1^2 + \cos^2\theta d\phi_2^2), \tag{2.234}$$

the asymptotically leading terms of the momentum one-form k are

$$k = \frac{J}{\rho^2}\sin^2\theta d\phi_1 + \frac{J}{\rho^2}\cos^2\theta d\phi_2, \tag{2.235}$$

with J a constant.

One can compute the horizon area to be (up to a numerical prefactor)

$$S = \sqrt{Q_1 Q_2 Q_3 - J^2}, \tag{2.236}$$

We see that angular momentum reduces the entropy. From a macroscopic point of view this is not hard to understand as the horizon is spinning very fast and this causes it to Lorentz contract and shrink. If we try to spin it up too fast, to the point that $J^2 = Q_1 Q_2 Q_3$, the horizon shrinks to zero size and we cannot go further (at least not with this ansatz). Although we will not say much about it, it is possible to reproduce this entropy using techniques quite similar to those of Sect. 2.4 (and indeed this was done shortly after the $J = 0$ entropy was first reproduced in [62]). The supersymmetric black hole with rotation is often called BMPV black hole after the authors of [62]. The interested reader can read more on microstate counting for these rotating black holes in [20].

Magnetic Charges

Above we added angular momentum to the metric. Even though this sourced magnetic components of the field strength, this was only so in much the same way as a moving electron generates a magnetic field. While a speeding electron generates a magnetic field it does not generate a *magnetic charge* as discussed in the preliminaries of Sect. 2.7.1.

If we want magnetic charges we need to add a closed but not exact term to each of the electromagnetic fields $F^{(I)}$ which we denote by $\Theta^{(I)}$. The field strengths becomes

$$F^{(I)} = d\left(Z_I^{-1}(dt + k)\right) + \Theta^{(I)}. \tag{2.237}$$

Of course this would not be consistent without modifying the form of the metric as well. It turns out this modification is rather straightforward. Recall that in the original metric, the Z_I were potentials sourced by delta-function sources at the locations of the M2's:

$$\Box_4 Z_I(x) = \sum_p Q_I \delta(x_p) \tag{2.238}$$

This source naturally corresponds to an electric field which must satisfy

$$d \star F^{(I)} = \sum_p Q_I \delta(x_p), \qquad dF^{(I)} = 0. \tag{2.239}$$

Recall that in string theory we have peculiar terms in the action such as

$$S = \frac{1}{2}\int F \wedge \star F + \frac{1}{3}\int A \wedge F \wedge F, \tag{2.240}$$

which implies that magnetic flux in this theory can source electric charge via the equation of motion

$$d \star F = F \wedge F. \tag{2.241}$$

This equation translates, in this setting, into a constraint on the functions Z_I which now are no longer simply sourced by a delta-function but look like

$$\Box_4 Z_I(x) = Q_I \delta(x) + \left| \star_4[\Theta^{(J)} \wedge \Theta^{(K)}] \right|, \tag{2.242}$$

with I, J, K all different.

It is important to realize that what is happening here is that if we have two pairs of magnetic charges in this theory they can induce *electric* charge. Thus even if our solution has no explicit electric source (no delta function on the right-hand side of (2.242)) there can be non-trivial electric charge carried by the fields $F^{(I)}$ themselves. Note that this phenomenon, and even the equation above, should look very familiar from non-abelian gauge theories where the gauge field sources itself and carries electric charge (think of glueballs in QCD). The difference is that here we are dealing with an *abelian* theory, and the non-linear interactions arise because of the strange second term in the action (2.240).

While it is obvious that $\Theta^{(I)}$ must be closed away from sources this is not the only constraint they must satisfy. It is harder to show but it turns out that supersymmetry also imposes that the Θ's appearing above are self-dual so that

$$\Theta^{(I)} = \star_4 \Theta^{(I)}. \tag{2.243}$$

Angular Momentum from Crossed Fields

Recall from electromagnetism that when the electromagnetic field has both an electric and magnetic component it carries angular momentum in the form of a Poynting vector

$$\boldsymbol{J} = \boldsymbol{E} \times \boldsymbol{B}. \tag{2.244}$$

While the original solution given above had angular momentum coming from the metric (2.231) encoded in the mixed metric components $g_{ti} \sim k_i$, the addition of a magnetic field changes the angular momentum. This comes from the supergravity equation

$$(1 + \star_4)dk = Z_1\Theta^{(1)} + Z_2\Theta^{(2)} + Z_3\Theta^{(3)}, \tag{2.245}$$

which modifies (2.233) in a way that is essentially analogous to (2.244) with Z_I encoding the electric field and $\Theta^{(I)}$ the magnetic.

Exercise 2.7.19 *For a flavour of why a constraint like (2.243) might follow from supersymmetry consider the action for electromagnetism in four space-time dimensions*

$$S = \int F \wedge \star F \tag{2.246}$$

and decompose $F = F^+ + F^-$ *into self-dual and anti-self-dual parts* $F^{\pm} = \frac{1}{2}(1 \pm \star)F$. *Rewriting the action in terms of* $F^{\pm}$ *show that it takes the form*

$$S = \int \left(F^+ \wedge F^+ - F^- \wedge F^-\right). \tag{2.247}$$

If we put $F = F^+$ *(or put otherwise* $F^- = 0$*) then the action is a positive definite perfect square. This is related, morally, to supersymmetry because the latter has a Hamiltonian* $H = \{Q^\dagger, Q\}$ *which is also a sum of squares implying that the energy is always greater an zero. In both cases solving the quadratic equations can be reduced to solving linear ones:*

$$F^+ = 0, \qquad vs. \qquad Q|\phi\rangle = 0, \tag{2.248}$$

and the solutions are minimal action and *minimal energy configurations.*

Overview Before Continuing

We have derived the following system of equations that describes a solution with 3 electric charges, 3 magnetic charges and angular momentum:

$$\Theta^{(I)} = \star_4 \Theta^{(I)},$$
$$\Box_4 Z_I(x) = \frac{1}{2} C_{IJK} \left| \star_4 [\Theta^{(J)} \wedge \Theta^{(K)}] \right|,$$
$$(1 + \star) dk = Z_I \Theta^{(I)}, \tag{2.249}$$

where $C_{IJK} = 1$ when all I, J, K are different and zero otherwise and the sum over repeated indices is implied. On the right-hand side of the last two equations, we silently assume the possibility of delta-function sources as well.

We wrote the equations in a suggestive order. To solve these equations, we first have to find a set of self-dual two-forms $\Theta^{(I)}$ on $\mathbb{R}^4$. Then we can solve the functions Z_I in terms of those two-forms. Finally, we need to construct the momentum k from Z_I and $\Theta^{(I)}$. Amazingly, this is a solution of supergravity, the low-energy limit of string theory, and a solution to these equations is a supersymmetric supergravity solution.

Before we solve this system in the specified order, we extend the four-dimensional space $\mathbb{R}^4$ to a non-trivial base space.

Non-trivial Base Space

So far we have taken the four-dimensional metric ds_4^2 to be flat. However, it turns out that supersymmetry does not require this space to be trivial but to be a more general metric of hyperkähler type [63].

An interesting and pretty general class of four-dimensional metrics that are hyperkähler are the Gibbons-Hawking and Taub-NUT metrics which take the form of a circle fibre (coordinate ψ) over flat three-dimensional $\mathbb{R}^3$:

$$ds_4^2 = V^{-1}(d\psi + A)^2 + V(dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)), \tag{2.250}$$

where V depends only on the three-dimensional coordinates r, θ, ϕ and the one-form A satisfies

$$\nabla \times A = \nabla V. \tag{2.251}$$

The fibre coordinate is periodically identified as $\psi \sim \psi + 4\pi$.

The harmonic V on this space has the general form

$$V = \epsilon_0 + \sum_i \frac{q_i^0}{r_i} \tag{2.252}$$

where now $r_i = |\boldsymbol{r} - \boldsymbol{r}_i|$ and $\boldsymbol{r}_i \in \mathbb{R}^3$. When working on $\mathbb{R}^3$ space instead of $\mathbb{R}^4$ we will use the Hodge dual $\star_3$ and radial coordinate r instead of $\star_4$ and ρ.

Near a pole of V, the Gibbons-Hawking metric looks like $\mathbb{R}^4$, as Exercise (2.7.20) asks you to show. Asymptotically, at large r, the four-dimensional space is $\mathbb{R}^3 \times S^1$.

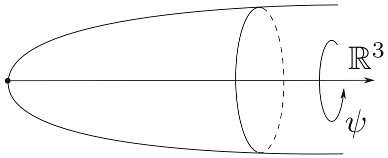


Fig. 2.33 Taub-NUT space with the harmonic function $V = 1 + n/r$ looks like a cigar. Near $r \to 0$, the ψ *circle* shrinks to zero size smoothly and space is locally $\mathbb{R}^4/\mathbb{Z}_n$. Asymptotically, the ψ *circle* is of constant radius and space-time asymptotes to $\mathbb{R}^3 \times S^1$

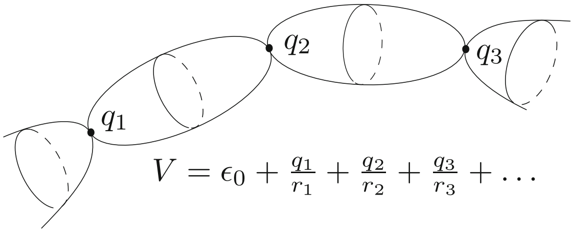


Fig. 2.34 Multi-center taub-NUT space is a "bubbled geometry". At each center, the size of the ψ *circle* goes to zero and the geometry looks like smooth $\mathbb{R}^4/\mathbb{Z}_n$. Asymptotically, the geometry is $\mathbb{R}^3 \times S^1$

We can read the radius of S^1 from the asymptotic expansion of the metric as the constant $1/\sqrt{\epsilon_0}$. By varying ϵ_0, we can thus interpolate between a compactification to three dimensional flat space, and $\mathbb{R}^4$ asymptotics by taking ϵ_0 to be zero. See Figs. 2.33 and 2.34 for depictions of single and multi-centered Taub-NUT spaces.

Exercise 2.7.20 *Show that if we choose $V = 1/r$ (with r the radial distance in the $\mathbb{R}^3$) we recover the trivial metric on $\mathbb{R}^4$ globally.* Hint: *Change coordinates to $\rho = 2\sqrt{r}$ and show that the metric for small ρ becomes*

$$ds_4^2 = d\rho^2 + \rho^2 d\Omega_3^2, \tag{2.253}$$

with $d\Omega_3^2$ the metric on an S^3 of unit radius. In doing so you show that the ψ circle shrinks to zero size smoothly *at the location of any pole in V since, whatever the form of V, near a pole it looks like $V = 1/r$. Hence* **the space-time is smooth at the location of the poles.** *(In fact, near a pole the function V looks like n/r for some charge n. This leads to an orbifold singularity $S^3/\mathbb{Z}_n$. Since string theory is well-defined on orbifold backgrounds, we still consider this as a regular space-time.)*

Exercise 2.7.21 invites you to explore the full eleven- and ten-dimensional solution with a Taub-NUT center and no M2-branes. They give respectively the eleven-dimensional Kaluza-Klein monopole and the 6-brane of IIA supergravity.

Exercise 2.7.21 *If we set $Z_I = 1$ and $V = 1 + \frac{n}{r}$ and we take the product of the space-time (2.6) with $\mathbb{R}^{1,6}$ then we get an 11-dimensional metric that is a solution* *of M-theory. As shown in the previous exercise this metric is* smooth *since the poles in V actually do not give any singularities in space-time. Now check that we can reduce on ψ and get a 10-dimensional solution corresponding to a D6-brane in IIA supergravity (*Hint: *see Sect. 2.4 of Amanda Peet's lecture notes [20] or Polchinski [6] Chap. 8 to see how to do the dimensional reduction). As a consequence, D6-branes in M-theory lift to smooth geometries in M-theory since the D6-brane poles correspond to poles in the V function which are smooth in 11-dimensional space-time.*

2.7.3 Solutions to the Equations of Motion and Supersymmetry

We specify how to find the complete solution to the equations of motion and the supersymmetry equations. These five-dimensional solutions were first described in [64, 65]. First we repeat the ansatz for a torus compactification of M-theory to a five-dimensional supersymmetric solution:

$$ds^2 = -(Z_1Z_2Z_3)^{-2/3}dt^2 + (Z_1Z_2Z_3)^{1/3}ds_4^2 + \sum_{I=1}^{3} \frac{(Z_1Z_2Z_3)^{1/3}}{Z_I^{2/3}} ds_I^2, \quad (2.254)$$

where ds_I, $I = 1, 2, 3$ are the metrics on three T^2's (for example $ds_1^2 = dx_1^2 + dx_2^2$). The four-form field strength decomposes into three two-form field strengths as:

$$F_4 = F^{(I)} \wedge \omega_I, \qquad F^{(I)} = d\left(Z_I^{-1}(dt+k)\right) + \Theta^{(I)}. \quad (2.255)$$

For the four-dimensional base space, we take the general class of Gibbons-Hawking or multi-centered Taub-NUT metrics[25]

$$ds_4^2 = V^{-1}(d\psi + A)^2 + V\underbrace{(dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2))}_{\mathbb{R}^3}. \quad (2.256)$$

For the rest of this chapter we work directly in five dimensions and will no longer consider the compact part of the geometry (though that is easy to add in).

The solutions above involve unknowns $k = k_i dx^i$, Z_I and $\Theta^{(I)} = \frac{1}{2}\Theta_{ij}^{(I)} dx^i \wedge dx^j$. They only depend on the coordinates of the three-dimensional flat base space (the Taub-NUT angle ψ is an isometry of the solution). We take the base space to be fixed but of course this means we should specify a V and then fix A via $\nabla \times A = \nabla V$. When the base space is Taub-NUT (asymptotically $\mathbb{R}^3 \times S^1$), the five-dimensional solutions can be compactified to the four-dimensional solutions found in [66, 67].

[25] In fact, the requirement of supersymmetry only requires the base space to be hyperkähler. The additional constraint of a Taub-NUT of Gibbons-Hawking metric makes it possible to solve for the metric explicitly. For more information, see [41] and reference therein.

Supersymmetry and the equations of motion can be simply repackaged into the following conditions

$$\Theta^I = \star_4 \Theta^{(I)}, \tag{2.257}$$

$$\nabla^2 Z_I = \tfrac{1}{2} C_{IJK} \left| \star_4 [\Theta^{(J)} \wedge \Theta^{(K)}] \right|, \tag{2.258}$$

$$(1 + \star_4) dk = Z_I \Theta^{(I)}, \tag{2.259}$$

where C_{IJK} is a completely symmetric tensor. For a more general supersymmetry-preserving compactification of M-theory on a six-dimensional Calabi-Yau manifold, C_{IJK} is given by the triple intersection products of a basis of two-cycles on the Calabi-Yau. We restrict to T^6 compactifications, for which $C_{IJK} = |\epsilon_{IJK}|$. Note that in the second equation we write no longer $\Box_4 Z_I$ but $\nabla^2 Z_I$, since the solution does not depend on the Gibbons-Hawking coordinate ψ. We will also omit the explicit possible delta function sources from now on.

As we noted before, now that we have specified the base space, we can solve this system in three steps: first we need to give the self-dual closed two-forms $\Theta^{(I)}$, then we solve functions Z_I, and then we can solve k. Note that in every step, the procedure is linear in the "new" unknown; hence this is a very tractable problem. We follow the three steps now.

1. Self-Dual Two-Forms

First we construct the $\Theta^{(I)}$. On Taub-NUT space, like $\mathbb{R}^4$, it is not hard to solve $\Theta = \star_4 \Theta$. First define the vielbeins

$$e^0 = V^{-1/2}(d\psi + A), \quad e^i = V^{1/2} dy_i, \tag{2.260}$$

such that the four-dimensional Taub-NUT metric (2.6) is written as a sum of squares:

$$ds^2 = (e^0)^2 + (e^1)^2 + (e^2)^2 + (e^3)^2. \tag{2.261}$$

Then one can check that the two-form

$$\Omega = (e^0 \wedge e^1 + e^2 \wedge e^3) \tag{2.262}$$

is self-dual ($\Omega = \star_4 \Omega$). There are actually three such self-dual Ω's we can construct by permuting the indices on the first term (the sign of the permuted second term is fixed by self-duality).

Exercise 2.7.22 *Check the above statement. First prove that*

$$\star_4 \, (e^A \wedge e^B) = \frac{1}{2} \epsilon^{ABCD} (e^C \wedge e^D) \tag{2.263}$$

for A, B, C, D from 0 to 3. Then prove that the three Ω^a defined as

$$\Omega^1 = e^0 \wedge e^1 + e^2 \wedge e^3, \quad \Omega^2 = e^0 \wedge e^2 + e^3 \wedge e^1, \quad \Omega^3 = e^0 \wedge e^3 + e^1 \wedge e^2, \tag{2.264}$$

are self-dual two-forms under $\star_4$.

The two-forms $\Theta^{(I)}$ must not only be self-dual but also locally closed (and hence co-closed because they are harmonic). Thus we start with Ω^a, $a = 1, 2, 3$ and construct a closed self-dual two-form Θ as

$$\Theta = \partial_a \left(\frac{K}{V} \right) \Omega^a \tag{2.265}$$

Exercise 2.7.23 asks you to prove that Θ is closed only if K is harmonic on the flat three-dimensional space.

Exercise 2.7.23 *Show that Θ defined in (2.265) is closed if K is harmonic on $\mathbb{R}^3$ ($\nabla^2 K = 0$).*

Recall that a harmonic function K on $\mathbb{R}^3$ satisfies $\nabla^2 K = 0$ which has the general solution

$$K = h + \sum_q \frac{p_q}{|\boldsymbol{r} - \boldsymbol{r}_q|}. \tag{2.266}$$

where $\boldsymbol{r}_p$ are arbitrary vectors in $\mathbb{R}^3$ at which H can be singular and the charges p_p and asymptotic value h are constants. In fact $\nabla^2 H = 0$ only holds away from $\boldsymbol{r}_p$ and this equation should be understood as $\nabla^2 K = p_p\, \delta(r - r_p)$. We see that our solution can have an arbitrary number of centers ('sources') on $\mathbb{R}^3$.

Hence the magnetic fluxes of the solution are the self-dual and closed two-forms

$$\Theta^{(I)} = \partial_a \left(\frac{K^I}{V} \right) \Omega^a, \tag{2.267}$$

with K^I three harmonic functions. We will write the harmonic function K^I in terms of charges and asymptotic constants as:

$$K^I = h^I + \sum_{q=1}^{N} \frac{p_q^I}{|\boldsymbol{r} - \boldsymbol{r}_q|}. \tag{2.268}$$

2. Warp Factors

The system of Eq. (2.257) is essentially *linear* if solved in the right order (there are no quadratic interactions or fields sourcing *themselves* quadratically). So once we have Θ we can plug it into (2.258) and solve for the 'warp factors' Z_I. The solution must be sourced by the right-hand side of (2.258) but can also include a homogeneous contribution that solves the equation $\nabla^2 Z_I = 0$. Combining these we get

$$Z_I = \frac{C_{IJK} K^J K^K}{V} + L_I, \tag{2.269}$$

where L_I are three more independent harmonic functions (on $\mathbb{R}^3$) satisfying $\nabla^2 L_I = 0$:

$$L_I = h_I + \sum_{p=1}^{N} \frac{q_{I,p}}{|\boldsymbol{r} - \boldsymbol{r}_p|} \tag{2.270}$$

Exercise 2.7.24 *Check that Z_I given in Eq. (2.269) satisfies (2.258).*

3. Rotation One-Form

The final Eq. (2.259) simply reproduces the (anti-)self-duality condition we mentioned above ($dk = - \star dk$) in the absence of explicit magnetic source ($\Theta = 0$). When such sources are turned on we solve this equation by decomposing k through the following ansatz:

$$k = \mu(d\psi + A) + \omega, \tag{2.271}$$

with $\omega = \omega_i dx^i$ a form on $\mathbb{R}^3$ and μ a function of the three-dimensional coordinates.

Exercise 2.7.25 *Show that plugging the ansatz (2.271) into (2.259) yields an equation for ω and μ:*

$$\boldsymbol{\nabla} \times \boldsymbol{\omega} = (V\boldsymbol{\nabla}\mu - \mu\boldsymbol{\nabla}V) - V\,Z_I \boldsymbol{\nabla}\left(\frac{K^I}{V}\right) \tag{2.272}$$

where as always we sum over $I = 1, \ldots, 3$.

To solve the Eq. (2.272) for ω we take a further divergence and use $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{\omega}) = 0$ to obtain

$$V\nabla^2\mu = \boldsymbol{\nabla} \cdot \left(V\,Z_I \boldsymbol{\nabla}\left(\frac{K^I}{V}\right)\right). \tag{2.273}$$

Exercise 2.7.26 *Show that this can be solved as*

$$\mu = \frac{1}{6} C_{IJK} \frac{K^I K^J K^K}{V^2} + \frac{1}{2}\frac{K^I L_I}{V} + M, \tag{2.274}$$

with M a harmonic function. The corresponding solution for ω satisfies

$$\boldsymbol{\nabla} \times \boldsymbol{\omega} = V\boldsymbol{\nabla}M - M\boldsymbol{\nabla}V + \tfrac{1}{2}(K^I\boldsymbol{\nabla}L_I - L_I\boldsymbol{\nabla}K^I) \tag{2.275}$$

There is a nice and clean way of writing the solution for ω in terms of the harmonic functions. Write the harmonic functions as a vector

$$H \equiv (V, L_1, L_2, L_3; M, K_1, K_2, K_3). \tag{2.276}$$

Then the right-hand side of (2.275) defines a symplectic product of such matrices:

$$\nabla \times \omega = \langle H, \nabla H \rangle. \tag{2.277}$$

While it is possible to get an explicit form for ω in simple examples, one generally has to resort to patches to specify the solution for ω given the harmonics V, K^I, L_I and M.

Exercise 2.7.27 *Show that on a flat base in absence of magnetic charges ($\Theta^{(I)} = 0$), you reproduce the earlier expression for k of Eq. (2.235). Use Exercise 2.7.20 for the coordinate transformation to flat space*

$$ds_4^2 = d\rho^2 + \rho^2(d\theta^2 + \sin^2\theta\, d\phi_1^2 + \cos^2\theta\, d\phi_2^2), \tag{2.278}$$

and take a single center with $M = m/r$. Determine the relation between J and m.

2.7.4 Physical Solution and Fuzzballs

Above we have shown that the solution can be specified in terms of eight harmonic functions V, K^I, L_I and M. We started with a black hole with harmonic functions $Z_I = L_I$, encoding three electric charges, and angular momentum encoded by the harmonic function M. In terms of eleven-dimensional M-theory, we have the brane interpretation:

M2's: L_1, L_2, L_3	Angular Momentum: M

Now we have also 3 magnetic fields, given by the harmonic functions K^I, and a magnetic geometric charge (of the Gibbons-Hawking space), encoded by V. The black hole charge can be dissolved in the magnetic fields. In M-theory language, these correspond to

M5's: K^1, K^2, K^3	Kaluza-Klein monopole: V

For concreteness, we fix a notation for the charges and constants of the harmonic functions. We organize the harmonic functions in a symplectic vector H:

$$H = (H^0, H^I, H_I, H_0) \equiv (V, K^I, L_I, M). \tag{2.279}$$

The symplectic vector of harmonic functions is written in terms of a symplectic array of constants h and charges Γ at each center:

$$H = h + \sum_{q=1}^{N} \frac{\Gamma_q}{|\boldsymbol{r} - \boldsymbol{r}_q|}, \tag{2.280}$$

with

$$h = (h^0, h^I; h_I, h_0), \qquad \Gamma \equiv (p^0, p^I, q_I, q_0). \tag{2.281}$$

For later use, we define the symplectic product of any two symplectic vectors A, B as:

$$\langle A, B\rangle = A^0 B_0 - A_0 B^0 + \tfrac{1}{2}(A^I B_I - A_I B^I). \tag{2.282}$$

In the remainder of this section, we give the physical requirements one has to impose on the solutions, and we show how we can construct microstate geometries.

Physical Requirements

At this point, getting the solution from harmonic functions is like blindly using a computer. We still have many questions: Are these solution physical? What are their properties? Are there singularities? We will answer these questions now.

We start with the vector $\boldsymbol{\omega}$ that describes the angular momentum of the metric in $\mathbb{R}^3$. To have it well-defined in space-time, the divergence of (2.277) should be zero:

$$\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{\omega}) = 0. \tag{2.283}$$

This gives a condition on the harmonic functions. First we write them as the symplectic product of the vector of harmonic functions H:

$$H = h + \sum_{i} \frac{\Gamma_i}{|\boldsymbol{r} - \boldsymbol{r}_i|}. \tag{2.284}$$

Then (2.283) gives the condition:

$$\langle H, \nabla^2 H\rangle = V\nabla^2 M - M\nabla^2 V + \tfrac{1}{2}(K^I \nabla^2 L_I - L_I \nabla^2 K^I) = 0. \tag{2.285}$$

The leading terms are those at the positions of the centers. Writing the charges for a harmonic functions at each center as $\Gamma_i = (p_i^0, p_i^I, q_{I,i}, q_{0,i})$, we have

$$\sum_j \langle H, \Gamma_j\rangle \delta(\boldsymbol{r}_j) = 0. \tag{2.286}$$

Demanding that each delta function contribution is zero gives one condition for each center $\boldsymbol{r}_i$:

$$0 = \langle \Gamma_i, h \rangle + \sum_j \frac{\langle \Gamma_i, \Gamma_j \rangle}{r_{ij}}, \tag{2.287}$$

with the relative distances

$$r_{ij} = |\boldsymbol{r}_i - \mathbf{r}_j|. \tag{2.288}$$

The physical interpretation of these equations is to assure there are no Dirac-Misner strings in the geometry (such that there is no source on the right-hand side of (2.283)).

Once the charges are fixed, the Eq. (2.286) then give constraints on the center positions $\boldsymbol{r}_i$: these equations tell you where the points are. We call these 'bubble equations' (giving r_p's in terms of Q's), because the resulting geometries have 'bubbles' (non-trivial two-cycles). Other names for these equations are 'integrability equations' (term coined by the original discoverer, Denef [66, 67]) and 'Denef equations', in the context of the related four-dimensional solutions.

Two-Center Solution Space

What is the space of solutions of the bubble equations? For simplicity, we restrict to two centers first. Then there is only one equation:

$$\frac{\langle \Gamma_1, \Gamma_2 \rangle}{r_{12}} + \langle \Gamma_1, h \rangle = 0. \tag{2.289}$$

We should have $\langle \Gamma_1, h \rangle \langle \Gamma_1, \Gamma_2 \rangle < 0$ to find a solution. This equation then fixes the distance r_{12}. The space of solutions is given by 2 points fixed by a rigid rod. The system has two degrees of freedom: two points in space-time have three degrees of freedom in $\mathbb{R}^3$ (three for each point, minus three for the center of mass), and the bubble equation fixes one. The solution space is the S^2 of possible positions of the second point at a distance r_{12} of the first one.

The vector of constants, h, determines the asymptotics of the harmonic functions through $H_{r\to\infty} = h$ and it determines what the space looks like asymptotically (for instance it contains a constant h^0 for the harmonic function $V = h^0 + p^0/r$ in the metric). For fixed charges Γ_1, Γ_2, the constants h also describe an interesting moduli space. Fix the charges such that $\langle \Gamma_1, \Gamma_2 \rangle > 0$. The value of h then determines if we can find a solution to the bubble Eq. (2.289). Take for instance a geometry with constants h such that $\langle \Gamma_1, h \rangle < 0$ and the bubble Eq. (2.289) have a solution. By tuning the asymptotic parameters h, we could go from $\langle \Gamma_1, h \rangle < 0$ to $\langle \Gamma_1, h \rangle = 0$ and even $\langle \Gamma_1, h \rangle > 0$: the solution disappears. It is no longer a valid physical solution. If we look at the solution space in function of the asymptotic parameters, the boundary $\langle \Gamma_1, h \rangle = 0$ determines a "wall of marginal stability". When crossing a wall of marginal stability ("wall-crossing"), these states just disappear. When $\langle \Gamma_1, h \rangle < 0$, the solution is part of the solution space, and we have an entropy associated to them (the 'number' of such states). When we cross the wall of marginal stability in the moduli space of allowed constant parameters h, the solution is gone and the entropy that counts all allowed solutions jumps.

Three-Center Solution Space

We turn to the more interesting solution space for three centers. The vector of harmonic functions is.

$$H = \frac{\Gamma_1}{|\boldsymbol{r} - \boldsymbol{r}_1|} + \frac{\Gamma_2}{|\boldsymbol{r} - \boldsymbol{r}_2|} + \frac{\Gamma_3}{|\boldsymbol{r} - \boldsymbol{r}_3|} + h. \tag{2.290}$$

From (2.286), we get three equations, one at each center (from the $\delta(\boldsymbol{r}_i)$-contributions)

$$\begin{aligned} \frac{\langle \Gamma_1, \Gamma_2 \rangle}{r_{12}} + \frac{\langle \Gamma_1, \Gamma_3 \rangle}{r_{13}} + \langle \Gamma_1, h \rangle = 0, \\ \frac{\langle \Gamma_2, \Gamma_1 \rangle}{r_{12}} + \frac{\langle \Gamma_2, \Gamma_3 \rangle}{r_{23}} + \langle \Gamma_2, h \rangle = 0, \\ \frac{\langle \Gamma_1, \Gamma_3 \rangle}{r_{13}} + \frac{\langle \Gamma_2, \Gamma_3 \rangle}{r_{23}} + \langle \Gamma_3, h \rangle = 0. \end{aligned} \tag{2.291}$$

These equations can be thought of as describing a balance of forces. The symplectic products pairs electric with magnetic charges (M, L_I are electric, K^I, V magnetic). We get a huge angular momentum forcing the points away from each other. But because of supersymmetry, all forces cancel and any solution is perfectly stable.

Define

$$A_{ij} \equiv \langle \Gamma_i, \Gamma_j \rangle. \tag{2.292}$$

Note that the symplectic product is antisymmetric and hence so is the matrix A. By a cyclic permutation of charges at the different centers, we can always take

$$A_{12} > 0, \qquad A_{23} > 0, \qquad A_{31} > 0. \tag{2.293}$$

Then the bubble equations are

$$\begin{aligned} \frac{A_{12}}{r_{12}} - \frac{A_{31}}{r_{13}} + h_1 = 0, \\ -\frac{A_{12}}{r_{12}} + \frac{A_{23}}{r_{23}} + h_2 = 0, \\ \frac{A_{12}}{r_{12}} - \frac{A_{23}}{r_{23}} + h_3 = 0, \end{aligned} \tag{2.294}$$

where the constants h_i are defined as $h_i = \langle \Gamma_i, h \rangle$. Only two of these equations are independent (for instance the sum of the first two gives the third one), and they leave only one of the distances r_{ij} unfixed. In total, three centers in $\mathbb{R}^3$ have 6 degrees of freedom (or "dof's"), three for each center minus three for the center of mass (only relative positions are important). The bubble equations fix two more. We thus have 4 degrees of freedom left. We can take these to be

- The radius r_{13} (1 dof)
- The orientation of r_{13} (2 dof's)
- The $U(1)$ angle around r_{13} (1 dof)

When we would consider n points instead of 3, the bubble equations allow for a $2(n-1)$-dimensional space of solutions (Fig. 2.35).

Scaling Solutions

One solution looks very interesting. If the triangle inequalities are satisfied:

$$|A_{12}| + |A_{23}| \geq |A_{31}|, \tag{2.295}$$

(and cyclic), there is a limit where the radii go to zero:

$$\begin{aligned} r_{12} &= |A_{12}|\epsilon + \mathcal{O}(\epsilon^2), \\ r_{13} &= |A_{13}|\epsilon + \mathcal{O}(\epsilon^2), \\ r_{23} &= |A_{23}|\epsilon + \mathcal{O}(\epsilon^2). \end{aligned} \tag{2.296}$$

As $\epsilon \to 0$, the bubble equations are satisfied up to first order, because the constants h_i can be suitable 'eaten up' by order $\mathcal{O}(\epsilon)$ terms in $\frac{A_{ij}}{r_{ij}} = \frac{1}{\epsilon} + \mathcal{O}(\epsilon)$. The r_{ij}'s are the lengths of the sides of a triangle and always satisfy triangle inequalities. The limit $\epsilon \to 0$ can only be done when also the $|A_{ij}|$ satisfy the triangle inequalities. We then have a limit where all radii go to zero. The points sit on a fixed triangle which gets smaller and smaller. If the triangle inequalities are not satisfied, we cannot have such a scaling limit.

Scaling Solutions

What is so special about these solutions? We have stated before the idea to replace the black hole geometry with some other object. In this section, we have made this more concrete. We can find an object with the same (electric/M2) charges as the black hole, but which also has magnetic dipole charges. The black hole is replaced

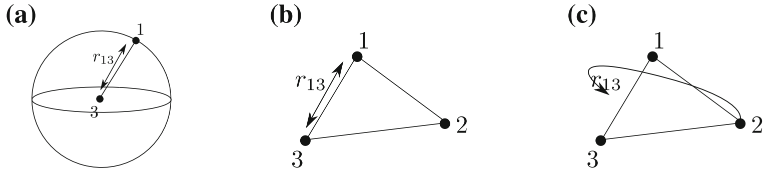


Fig. 2.35 A three-center configuration has 4 free parameters by the bubble equations. **a** S^2 of orientations of r_{13}. **b** Scale of r_{13}. **c** $U(1)$ angle around r_{13}

by a solution with many centers and magnetic charges, by finding the solution from the harmonic functions $H = (V, K^I, L_I, M)$.[26]

The solutions can 'go scaling', such that the several centers can come closer and closer, by sending some control parameter $\epsilon \to 0$, as in Eq. (2.296). When $\epsilon = 0$ and the centers are on top of each other, we recover the black hole (Fig. 2.36).

Remember that we were considering extremal black holes. These have an infinitely deep throat.[27] A scaling solution with scaling size ϵ, has a throat of length $L \propto -\ln \epsilon$. As $\epsilon \to 0$, you get a throat with a cap that gets longer and longer. These solutions form an infinite family, see Fig. 2.37 for an illustration.

A paradox

The scaling solutions form an infinite family: we can make ϵ smaller and smaller, we always find good solutions. But from AdS/CFT, we know that there is a finite entropy

$$S = \sqrt{Q_1 Q_2 Q_3}, \tag{2.297}$$

which tells us there is a finite number of states. This is a puzzle [68]:

- $N_{\text{micro}} = e^{S_{BH}}$ is large but finite.
- $N_{\text{class. grav.}}$ (number of smooth solutions) is infinite.[28]

How to reconcile these pictures? That's for the next section!

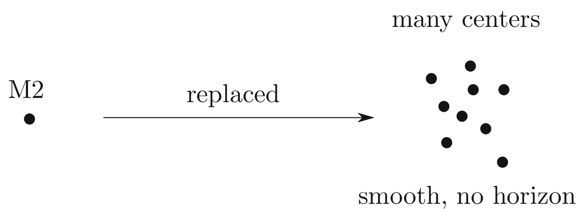


Fig. 2.36 Replacing the *black hole* with a multi-center configuration

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[26] In fact, there are certain conditions the harmonic functions H have to obey such that the multi-center geometry is also smooth and horizonless at each center. We will not dwell on that, see [41] for more information.

[27] By 'infinite throat', people mean that the spatial metric distance $\int ds$ to the horizon from any point outside the horizon blows up.

[28] Note: only a subset of this multi-center solutions are actual fuzzballs. We need some more information to discuss them, we will leave it at this for the moment.

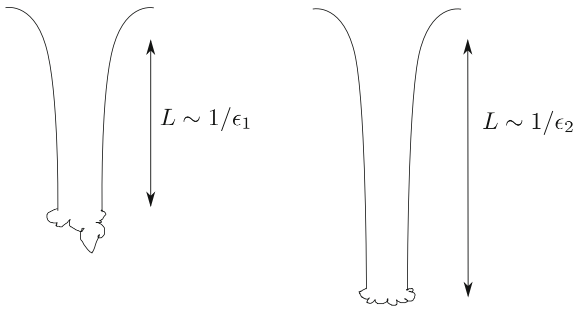


Fig. 2.37 For every value of ϵ we find a scaling solution with a deep throat. As $\epsilon \to 0$, we recover the infinitely deep *black hole* throat

2.8 Quantizing Geometries

So far we have studied a large class of supersymmetric multi-centered solutions and have suggested that they are related to the microstates of large supersymmetric black holes. But to make this connection between classical geometries and quantum states we have to "quantize". Since these are gravitational solutions quantizing them seems rather daunting and certainly we do not know how to do this in full generality. Rather here we will introduce a powerful covariant formalism for quantizing systems without resorting to a Hamiltonian formulation (which would be tedious in this case). In particular we will show how the solution space of a system is formally equivalent to the phase space and how we can thus construct states directly on this space. This construction usually goes under the name of "geometric quantization" but we will eschew many of the mathematical technicalities that usually are associated with this. Rather we will focus on explaining why this makes sense.

Note that we will make heavy use of supersymmetry as we do not have access to the full solution space of the theory but rather only some supersymmetric truncation of the latter. Quantizing a sub-space of a system is not necessarily a consistent thing to do but in this case we can rely on supersymmetry-based arguments (and explicit matching with expectations) to see that the Hilbert spaces we generate are a good approximation to the actual Hilbert space of the system.

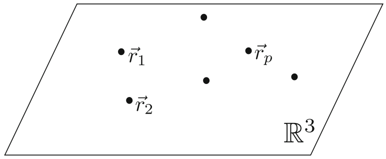
2.8.1 Constraint Equations and Solution Space

To keep this chapter well-contained, we choose to recall the necessary background material discussed in previous sections.

We start in eleven-dimensions from the metric and gauge field

$$\begin{aligned} ds^2 &= (Z_1 Z_2 Z_3)^{-2/3} dt^2 + (Z_1 Z_2 Z_3)^{1/3} ds_4^2 + ds^2(T^6), \\ F_4 &= [d(Z_1^{-1}(dt+k)) + \Theta^I] \wedge dx_1 \wedge dx_2 + \cdots \end{aligned} \tag{2.298}$$

Fig. 2.38 The multi-center solutions are sourced on multiple positions in the $\mathbb{R}^3$ base of Taub-NUT space



with the four-dimensional multi-center Taub-NUT metric

$$ds_4^2 = V^{-1}(d\psi + A) + V ds^2(\mathbb{R}^3). \tag{2.299}$$

The functions Z_I, one-form k and two-forms Θ^I that determine the solution are found from the harmonic functions

$$H \equiv (V, K^I, L_I, M), \tag{2.300}$$

as explained in the previous section (Fig. 2.38).

The harmonic functions satisfy a sourced harmonic equation:

$$\nabla^2 H = \sum_i \Gamma_p \delta(\boldsymbol{r} - \boldsymbol{r}_p). \tag{2.301}$$

The solution is

$$H = \sum_{p=1}^{N} \frac{\Gamma_p}{|\boldsymbol{r} - \boldsymbol{r}_p|} + h_0, \tag{2.302}$$

where $\boldsymbol{r}_p$ are the position vectors of the different centers in $\mathbb{R}^3$ and h_0 is a vector of constants for the different harmonic functions. The charges at each center give poles in the harmonic functions, corresponding to multiple sources, and each may or may not have a horizon (depending on the charge Γ_p at the center).

Given a set of asymptotic charges $\Gamma = \sum_{p=1}^{N} \Gamma_p$ the space of all possible solutions with N centers is given by all the possible ways of arranging these centers in $\mathbb{R}^3$.

At first glance, we would think this space is $\mathbb{R}^{3N-3}$, the space of locations of N centers on $\mathbb{R}^3$.[29]

However, the positions of the centers are constrained in terms of the charges, by the bubble or Denef equations introduced in the last section:

[29] Only the relative positions are of importance, hence the degrees of freedom of one of the centers do not count and we get $3N - 3$ coordinates that specify a physical solution with N centers.

$$\forall p: \qquad \sum_{\substack{q=1 \\ q \neq p}}^{N} \frac{\langle \Gamma_p, \Gamma_q \rangle}{|\boldsymbol{r}_p - \boldsymbol{r}_q|} + \langle \Gamma_p, h \rangle = 0. \tag{2.303}$$

We write the harmonic functions and charges as symplectic vectors:

$$H = (\underbrace{V,\ K^I}_{\text{elec.}},\ \underbrace{L_I,\ M}_{\text{magn.}}), \qquad \Gamma = (p^0, p^I, q_I, q_0). \tag{2.304}$$

with $I = 1, 2, 3$ giving us either possible charges at each center.

Given two symplectic vectors of harmonic functions H and H' recall that there exists a symplectic inner product that couples electric and magnetic components

$$\langle H, H' \rangle = VM' - MV' + K^I L'_I - L_I K'^I. \tag{2.305}$$

Note that this pairing is antisymmetric. You should think of it as giving momentum from crossed electric and magnetic fields, similar to the Poynting vector in electromagnetism:

$$\boldsymbol{J} = \boldsymbol{E} \times \boldsymbol{B}. \tag{2.306}$$

The constraints (2.303) have a clear physical meaning. The first way to understand them is through supersymmetry. Each individual center breaks $\mathcal{N} = 2$ supersymmetry of the supergravity theory to a *particular* $\mathcal{N} = 1$ subgroup. Generically all the centers break $\mathcal{N} = 2$ to a different residual $\mathcal{N} = 1$ (encoded in a $U(1)$ valued phase) but when the distances between the centers satisfy the Eq. (2.303) the $\mathcal{N} = 1$ supersymmetry preserved by all the centers are compatible and thus the combined system preserves an overall $\mathcal{N} = 1$ supersymmetry.

There is a second interpretation of the constraints (2.303). Consider for concreteness a solution with two centers. The Poynting vector gives an angular momentum "binding". For electromagnetism in flat space, we get for a magnetic charge m and an electric charge q that

$$J = \frac{qm}{2}, \tag{2.307}$$

no matter what the distance is between the two centers. With gravity, the angular momentum depends on the distance between the centers:

$$J = \frac{qm}{r}, \tag{2.308}$$

and there is a non-zero force. The constraint equations can be interpreted as the condition for all those forces to balance.

Exercise 2.8.28 *Show that the sum over p (from 1 tot N) of (2.303) is zero.*

From Exercise 2.8.28, we see that there are in fact only $N - 1$ independent constraints. Therefore, the solution space is a $(2N - 2)$ dimensional submanifold of $\mathbb{R}^{3N-3}$:

$$M_{2N-2} \subset \mathbb{R}^{3N-3}. \tag{2.309}$$

For instance, for two centers we get

$$M_2 \subset \mathbb{R}^3. \tag{2.310}$$

The constraint fixes the distance $r_{12} = |\boldsymbol{r}_1 - \boldsymbol{r}_2|$ so M_2 corresponds to the possible rotations of the position $\boldsymbol{r}_2$ around $\boldsymbol{r}_1$ with fixed inter-center separation r_{12}. This is of course nothing but a two-sphere

$$M_2 = S^2. \tag{2.311}$$

The constraint equations should be understood as follows. When we fix the asymptotic charges, there is still a continuous family of positions we can vary. Hence the solution space itself is a function of the charges $M_{2N-2}(\Gamma_p)$.

Our goal here will be to calculate the "number of states" in a fixed solution space. The reason to undertake such a computation is the following. For a given charge vector Γ, if we consider all possible decompositions in to multiple centers $\Gamma = \sum_p \Gamma_p$ and compute the states from each such solution space, we may hope that this can reproduce the entropy of a single center black hole with total charge Γ. If so then we have a found a good *supergravity* realization of the black hole microstates. But to convert the solutions above into "microstates" we have to quantize the solution space. Therefore we first give some basic quantum mechanics to see how to get a quantum space out of a classical solution space.

2.8.2 *Basic Quantum Mechanics*

We recall classical mechanics in the Hamiltonian symplectic formalism, its quantization and the concepts of phase space and its relation to the space of solutions.

Hamiltonian Formulation

Let us recall the basic simple formulation of quantum mechanics (which is not covariant) and then try to modify it to make it more covariant. If we start with a Lagrangian of a system with positions q:

$$L(q^i, \dot{q}^i). \tag{2.312}$$

with $i = 1, \cdots, n$ then the generalized momenta are

$$p_i = \frac{\partial L}{\partial \dot{q}^i}. \tag{2.313}$$

From this Lagrangian we can derive an associated Hamiltonian which is a function of the positions and generalized momenta only (for ease of notation we will mostly suppress indices on position and momentum vectors)

$$H(q, p) = p\dot{q} - L. \tag{2.314}$$

In terms of which the equations of motion are

$$\begin{aligned} \dot{p} &= -\frac{\partial H}{\partial q}, \\ \dot{q} &= \frac{\partial H}{\partial p}, \end{aligned} \tag{2.315}$$

Of course we could have foregone a Lagrangian and simply postulated a Hamiltonian system directly but the connection with a Lagrangian formulation will be important in what follows. The Hamiltonian formulation is based on the phase space which is the space of positions q and momenta p on a fixed time slice. It is this dependence of a choice of time slice (and direction) that makes the formulation non-covariant.

An essential ingredient in the Hamiltonian formulation of classical mechanics is the Poisson bracket, defined on any functions on the phase space, via

$$\{f, g\} = \frac{\partial f}{\partial p}\frac{\partial g}{\partial q} - \frac{\partial f}{\partial q}\frac{\partial g}{\partial q}. \tag{2.316}$$

In the simple systems first encountered in physics we often have $\{q, p\} = 1$ but this need not always be the case and this is one of the reasons a more general formulation is necessary. More generally we expect some bivector ω such that

$$\{q^i, p^j\} = \omega^{ij}. \tag{2.317}$$

While locally we can find coordinates such that ω is diagonal this need not hold globally. It is very important, however, that ω^{ij} be invertible as this allows us to find a symplectic two-form:

$$\omega \equiv \omega_{ij} dq^i \wedge dp^j. \tag{2.318}$$

which defines a symplectic structure on the phase space. Thus in general the Hamiltonian formulation requires the set of data (p, q, H, ω^{ij}).

We have tacitly assumed above that there is some natural choice of p's and q's on the entire phase space but if the latter is some non-trivial manifold then we need to cover it with patches. How then does one define, on each patch, which local coordinates should be thought of as positions and which momenta?

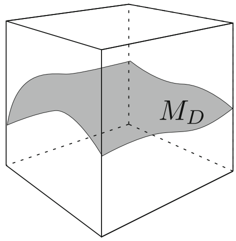


Fig. 2.39 A n-dimensional subspace M_n

A more covariant way to do this is to consider n-dimensional subspace M_n of the $2n$-dimensional phase space, as in Fig. 2.39, on which the pullback of the symplectic form vanishes:

$$\omega|_{M_n} = 0, \tag{2.319}$$

Such subspaces are referred to as Lagrangian submanifolds and they are interesting because if we consider any local coordinates, x^i, on them then by virtue of (2.319) we have

$$\{x^i, x^j\} = 0, \tag{2.320}$$

This is non-trivial because the x may be some non-trivial combination of p and q. The fact that they nonetheless have vanishing Poisson brackets mean they can be thought of as a new set of canonical positions. Thus Lagrangians in phase space are a covariant generalization of the splitting of phase space coordinates into canonical position and momenta.

So far we have used classical notions such as Poisson brackets but this discussion generalizes to quantum mechanics. To quantize a classical system we replace the Poisson bracket by a commutator (or anti-commutator for fermions)

$$[q, p] = i\hbar. \tag{2.321}$$

Thus the p's and q's can no longer correspond to $2n$ numbers but rather half of them are now operators. Normally, we take the q's to be commuting numbers, and p are their derivatives

$$p = \frac{\hbar}{i}\frac{\partial}{\partial q}. \tag{2.322}$$

Thus we see a Lagrangian subspace is nothing other than a space of mutually commuting variables

$$[x^i, x^j] = 0, \tag{2.323}$$

Once more such manifolds define (in a covariant way) natural slices of phase space that we can think of as position spaces.

This notion is quite important because in quantum mechanics states must be functions of only one set of canonical variables – the position or the momenta but not both. Thus Lagrangian submanifolds allow us to define the Hilbert space of states in a nice covariant way as the space of (wave) functions on a Lagrangian submanifolds

$$\mathcal{H} = \{\psi(x) \in \mathcal{L}^2(M_n, \mathbb{C})\}. \tag{2.324}$$

The advantage here over the usual formulation is that we have covariantized our approach as the Eq. (2.319) is a coordinate-invariant statement. Moreover this approach generalizes to more complex systems where the phase space (the space of (q^i, p^i)) is not merely $\mathbb{R}^{2d}$ but some more complex manifold. Of course we are implicitly assuming there is some nice foliation of the phase space into time slice $M_n(t)$ where t is some parameterization of time.

A consequence of this more formal description of the quantum phase space is that it yields another way to compute the number of states. This is simply the symplectic volume of the phase space: (up to some subtleties that we can neglect)

$$\#\,\text{states} = \int_{\text{phase space}} \omega^n. \tag{2.325}$$

where we note that ω^n is a $2n$-form that we can integrate over the entire space. Classically this does not count states because it is not integer quantized. In quantum mechanics, however, we think of ω as partitioning the phase space into Plank-sized cells. As a consequence its volume must be normalized such that the volume is integrally quantized (Fig. 2.40).

Mathematically, this can be justified because the wave functions are actually sections of a bundle defined on M_n and associated with ω (which is essentially its curvature). Thus the integral above computes (again, up to some subtleties) the index of an operator D associated with this bundle:

$$\text{ind}\, D = \int (\ldots). \tag{2.326}$$

Recall that an index counts the number of (chiral) zero modes of a particular operator and this is an integral quantity. In our setup, things are simple enough that the $(\ldots)$ are just ω^n.

The current treatment raises an important question:

- *Classically, we expect an infinite number of states (everything is continuous). Hence we should be able to go anywhere in phase space and have an infinite number of allowed states. But $\int \omega^n$ should be finite? Is there a clash?*

We will answer this question explicitly in an example below. Yes, classically the number of states is infinite, but the *volume* of phase space is finite. Only in quantum mechanics, the volume is the number of states.

Exercise 2.8.29 *Consider a particle in a box of length L.*

1. *Compute the number of quantum states: calculate the integral*

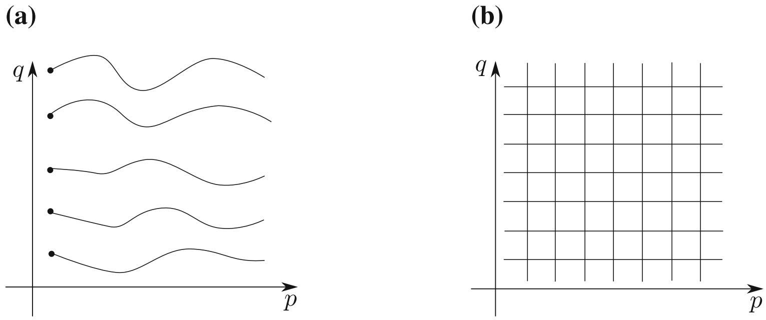


Fig. 2.40 Classical versus quantum phase space. The volume of classical phase space can be a real number, in quantum mechanics it is an integer. **a** Classically, we can continuously integrate histories. **b** In quantum mechanics, phase space is a discrete grid of points

$$\int_0^L \int_0^{p_{\max}} \omega, \tag{2.327}$$

with

$$[x, p] = \omega^{-1} \tag{2.328}$$

and $p_{\max}$ should be allowed quantum values (see a textbook on quantum mechanics). Convince yourself this integral counts the number of states.

2. *Repeat the calculation for a two-dimensional box.*

Let us consider a simple example to get a better feel for this formalism. Take the Hamiltonian of a free particle

$$H = \tfrac{1}{2}p^2. \tag{2.329}$$

Given q and p, we can always define the complex coordinates on phase space:

$$z = q + ip\,, \qquad \bar{z} = q - ip. \tag{2.330}$$

Then we have the commutation relation

$$[z, \bar{z}] = 1. \tag{2.331}$$

In terms of $z, \bar{z}$ it is no longer obvious which coordinate is a "position" and which a "momentum" and we must make an arbitrary choice. We can, for instance, take wave functions to depend only on z:

$$\psi(z). \tag{2.332}$$

Now the number of states is counted by an index

$$\text{ind}\,(\bar{\partial}) = \#\,\text{states}, \tag{2.333}$$

with $\bar{\partial}$ the Dolbeault operator because clearly $\bar{\partial}\psi(z) = 0$ so wave functions are simply functions annihilated by $\bar{\partial}$. Note that this method needs a complex structure on phase space, which can not always be defined. For a simple manifold like $\mathbb{R}^{2n}$ it can be done. If there is a complex structure, then it turns out that the above gives a good way to quantize.

Consider now a slight extension of the free particle model. Couple it to an electromagnetic field. The Lagrangian is

$$L = \tfrac{1}{2}(\dot{q} + Aq)^2. \tag{2.334}$$

The canonical momentum is

$$p = \dot{q} + Aq. \tag{2.335}$$

This is very different from previous examples! Even if there is no velocity, $\dot{q} = 0$, there is still a non-vanishing momentum. When there are space components of the gauge field

$$A_i \neq 0, \tag{2.336}$$

the position themselves no longer commute:

$$\omega^{ij} = [q^i, p^j] = A_j[q^i, q^j] \neq 0. \tag{2.337}$$

The non-commutativity of phase space becomes a non-commutativity of the physical space due to the magnetic field A_i.

From Phase Space to Solution Space

So far we have reformulated quantum mechanics in a slightly more covariant and general language but let us see what this is useful for. Here we will try to prove the following claims:

1. The number of states is the symplectic volume of phase space.
2. Phase space is isomorphic to solution space (up to some caveats).

and hence:

- **The number of states is the symplectic volume of solution space.**

The first claim we have already argued in the previous section. The last one follows trivially from the other two. Thus we are left with demonstrating the validity of our second claim above.

Given any initial point in phase space $\{q_0, p_0\}$ there is a prescription to generate an entire "history": namely we integrate using the equations of motion with initial conditions $\{q_0, p_0\}$. The p's act morally as velocities, and they allow us to integrate

$q(t_0)$ for any t_0 to a further time step (see Fig. 2.41). Thus any point in phase space corresponds to a full solution to the equations of motion (a "history" of the particle or system).

Conversely, given a solution $q(t)$ to the equations of motion and a choice of time slice at for instance t_0, we can unique extract a point in the phase space by simply reading off $\{q(t_0), p(t_0)\}$ evaluated on the solution $q(t)$ at time t_0. Thus, once a time-slice is fixed, each solution uniquely maps to a point in the phase space (Fig. 2.41). Combining these observations we have now proved our second claim above.

What's more there is a natural way to compute the symplectic form directly in the Lagrangian formulation. This allows us to use the solution space to compute both the number of states and their explicit form without ever needing to use a Hamiltonian formulation (going to the phase space and formulating everything in terms of conjugate variables).

An important subtlety, however, is that the arguments made above apply to the full solution space and phase space—it is these full spaces that are isomorphic. It is not clear, if we restrict to a subspace of the solution space, whether this maps to a proper phase space. This is important in this situation because the supersymmetric solution space is exactly such a truncation.

2.8.3 *Intermezzo: From QM to QFT and GR*

We want to go from quantum mechanics (QM) to Quantum Field Theory (QFT). In QM, the points at time t are unconstrained, and the wave function $\psi(x)$ is a function of the unconstrained positions. In QFT, the points on each time slice are now fields ϕ that are constrained by the equations of motion, and the wave functional $\Psi(\phi)$ is a function of those constrained fields. Note that we use the formulation of time slices and evolution of the fields from one to the other defining wave functions on each slice. This is equivalent to the path integral formulation

$$\langle\psi'|e^{iHt}|\psi\rangle = \int \mathcal{D}e^{-S}. \tag{2.338}$$

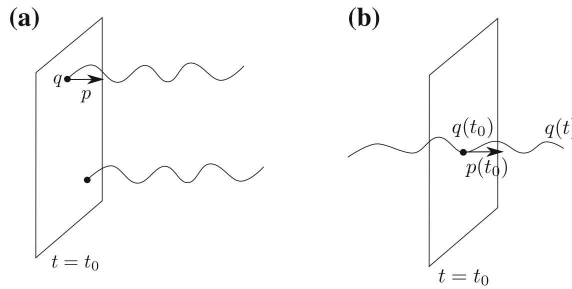


Fig. 2.41 *Left* given an initial configuration at $t = t_0$, we can integrate the equations of motion to obtain the full solution $q(t)$, $p(t)$. *Right* given a solution $q(t)$, we have a phase space at every t

In field theory, the coordinates and momenta are replaced by fields:

$$\begin{aligned} q &\to \phi(x) \\ p &\to \Pi(x) = \frac{\partial L}{\partial \dot{\phi}}. \end{aligned} \tag{2.339}$$

As before for quantum mechanics, in field theory we consider the fields on a spatial slice such as the one in Fig. 2.42.

In GR, things are a little more tricky than in field theory because the background is not fixed. We will not address these subtleties here but will simply assume we find a nice foliation of all the space-times we consider. We define spatial slices Σ such as the one in Fig. 2.43 and we use a metric adapted to the slices

$$ds^2 = (N^2 + \beta_k \beta^k)dt^2 + 2\beta_k dx^k dt + h_{ij} dx^i dx^j, \tag{2.340}$$

in terms of the data

$$(h_{ij}, \beta_k, N). \tag{2.341}$$

where now h_{ij} is a metric on the spatial slice.

One finds that β_k and N are non-dynamical variables as their momenta are zero:

$$\Pi^\beta = 0, \qquad \Pi^N = 0. \tag{2.342}$$

These equations can be interpreted as constraints on the other fields. The only dynamical variables are then the three-dimensional metric h_{ij} and its momenta Π_h:

$$\Pi_h^{ij} \equiv \frac{\delta L}{\delta \partial_t h_{ij}}. \tag{2.343}$$

What terms contribute to the momentum Π_h? These are terms in the Lagrangian of the form:

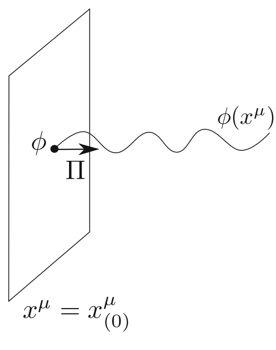


Fig. 2.42 Fields on a spatial slice of constant t

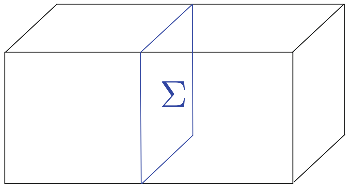


Fig. 2.43 GR on a spatial slices Σ

$$L = \cdots + \partial_t h_{ij} \Omega^{t,ij} + \cdots \tag{2.344}$$

Assume first that $\beta_k = 0$. Then the metric has no mixed spatial-temporal components:

$$g_{\mu\nu} = g_{ij} + g_{tt}, \tag{2.345}$$

and $\partial_t h_{ij}$ can only talk to something else ($\Omega^{t,ij}$) with another time derivative and hence

$$\Pi^{ij} \sim \dot{h}^{ij}. \tag{2.346}$$

For time-independent solutions we would thus have $\Pi^{ij} = 0$. Thus if we consider families of static solutions (time-independent and no mixed terms in the metric) they cannot map to a full phase space as they contain no momentum-like variables. Instead such solutions map to a Lagrangian submanifold of the full phase space (they form a "configuration space" rather than a phase space).

If, on the other hand, $\beta_k \neq 0$ then $\partial_t h_{ij}$ can couple to terms like $\partial^i g^{tj}$ etc., with *spatial* derivatives. Therefore,

$$\Pi^{ij} \sim \text{time independent terms}, \tag{2.347}$$

which means $\Pi^{ij} \neq 0$ even for time-independent solutions. Remember that the multi-center metrics we were looking are of this sort since they are stationary (time-independent with mixed terms $g_{ti} \sim k_i$ terms coming from a $(dt + k_i dx^i)^2$).

Therefore, the commutation relations go as

$$[h_{ij}, \Pi^{kl}] \sim [h_{ij}, h^{kl}], \tag{2.348}$$

analogous to the previous example of a particle in a magnetic field with

$$[q_i, p_j] \sim [q_i, q_j]. \tag{2.349}$$

The spatial metrics no longer commute on the phase space. This will be very important for getting the number of states.

Crnkovic-Witten-Zuckerman Formalism

Since we are working with solution spaces we want a covariant formalism rather than the non-covariant GR Hamiltonian formalism we discussed above. Let us see how to arrive at this. Consider a class of solutions with a spatial foliation with each time slice being a Cauchy surface

$$\Sigma = \text{Cauchy surface}. \tag{2.350}$$

Define

$$\omega := \int_{\Sigma} d\Sigma_{\ell} J^{\ell}, \tag{2.351}$$

Here J^{ℓ} is the "symplectic current" associated with the action (see below). We have introduced the $(D-1)$-form

$$d\Sigma_{\ell} = \Sigma_{\mu_1 \ldots \mu_{D-1} \ell} dx^1 \wedge \ldots \wedge dx^{D-1}. \tag{2.352}$$

which is just the volume form on the Cauchy surface. Then ω is a two-form on the space of fields. The symplectic current is

$$J_{\ell} = \delta \left[\frac{\delta L}{\delta \partial_{\ell} \phi_k} \right] \wedge \delta \phi^k, \tag{2.353}$$

where ϕ^k runs over the fields. If $\ell = 0$, we get $J_0 = d\Pi \wedge d\phi$, reminiscent of the symplectic form in mechanics $dp \wedge dq$. But unlike the standard formulation this is covariant as we have not fixed a coordinatized notion of time. Rather by using spacelike foliation we get a covariant notion of time as the direction normal to the slices (but with no reference to a coordinate system).

Exercise 2.8.30 *Play around with ω:*

1. *Show that ω is closed under a field variation*

$$\delta_{\phi} \omega = 0. \tag{2.354}$$

2. *Show that the symplectic current is conserved*

$$\partial_{\ell} J^{\ell} = 0. \tag{2.355}$$

You need to impose the equations of motion for one of these.

From the exercise we see that ω does not vary from slice to slice (because it is conserved).

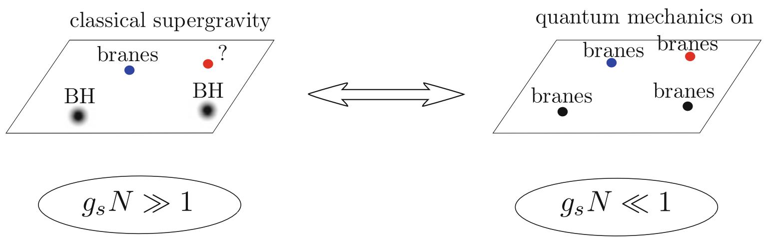


Fig. 2.44 At large $g_s N$, we have the supergravity multi-center solution. Each center can be either a black hole (with a horizon), or some horizonless singularity, or a smooth center etc. For small $g_s N$, we just have non-back-reacting branes at several positions in flat space-time

2.8.4 Back to Solution Space

Now we have the pieces in place to quantize our space of solutions. We begin by evaluating the symplectic form for the Lagrangian of M-theory. The fields are the metric and the four-form and are evaluated at the positions on solution space:

$$\phi_\ell = \{g_{\mu\nu}[\boldsymbol{r}_p], F_{\mu\nu\rho\sigma}[\boldsymbol{r}_p]\}. \tag{2.356}$$

The symplectic form looks like

$$J^\ell = \delta\left[\frac{\delta L}{\delta\partial_\ell g[\boldsymbol{r}_p]}\right] \wedge \delta g[\boldsymbol{r}_p] + \text{ four-form term.} \tag{2.357}$$

The two-form ω will be something like

$$(\ldots) \wedge d\boldsymbol{r}_p, \tag{2.358}$$

where each $\{\boldsymbol{r}_p\}$ parametrizes a metric; these are the "coordinates" of our solution.

How to do this? Remember that the constraint equations come from the integrability condition of the defining equation for $\boldsymbol{\omega}$ (which is part of the metric $g_{\mu\nu}$):

$$\boldsymbol{\nabla} \times \boldsymbol{\omega} = V\boldsymbol{\nabla} M + \cdots . \tag{2.359}$$

We need to find $\boldsymbol{\omega}(\boldsymbol{r}_p)$, construct $g(\boldsymbol{\omega})$ and then we can find J^ℓ. This is very difficult because inverting Eq. (2.359) cannot be easily done.

We will follow the lazy string theorist approach and use supersymmetry to our advantage. The back-reacted supergravity system is valid for $g_s N \gg 1$. As we discussed in previous sections, when $g_s N \ll 1$, we just have a quantum mechanical theory on branes at the positions of the centers on eleven-dimensional flat space-time $\mathbb{R}^3 \times T^6 \times \mathbb{R}_t$, see Fig. 2.44.

It can be shown that on each $g_s N$ side the solution space and the symplectic form are protected because of supersymmetry (the proof uses the fact that both are determined by the certain terms in the Lagrangian whose form is fixed by supersymmetry and thus cannot change even as we vary $g_s N$). Moreover one can check by explicit computation that the solution spaces at strong and weak coupling are exactly the same. For instance, for 2 centers, we still find S^2 as the solution space. Thus we are free to compute the symplectic form directly in the brane quantum mechanics which is a much easier computation.

The result we get from the $g_s N \ll 1$ quantum-mechanics-on-branes calculation is

$$\omega = \tfrac{1}{4} \sum_{p,q} \langle \Gamma_p, \Gamma_q \rangle \frac{r^i_{pq}}{|r_{pq}|^2} \epsilon_{ijk} \delta r^j_{pq} \wedge \delta r^k_{pq}, \tag{2.360}$$

and we defined

$$\boldsymbol{r}_{pq} = \boldsymbol{r}_p - \boldsymbol{r}_q. \tag{2.361}$$

The real coordinates in this calculation are the $\boldsymbol{r}_{pq}$, vectors between the centers. While we do not show the detailed derivation of this formula here (the interested reader can find it in [69]) its origin is very easy to understand. Recall from the discussion in the previous section that an electrically charge particle in the background of a magnetic field has a coupling $(\dot{q} + eAq)$, with e the electric charge, and this leads to a canonical momentum of the form

$$p = eA(q)\, q. \tag{2.362}$$

The symplectic form (2.360) is exactly of this form: each center feels, via $\langle \Gamma_p, \Gamma_q \rangle$ an electric-magnetic coupling to the gauge field generated by any other center which is "magnetically" charged with respect to it. So (2.360) is really just of the form $\omega = A(q)\delta q \wedge \delta q$ where we have plugged in the appropriate value for $A(q)$.

Morally, the $\delta r^j_{pq} \wedge \delta r^k_{pq}$ are like the $dx^i \wedge dx^j$ contributions in quantum mechanics. As before, this means that coordinates do not commute:

$$[r^i_{pq}, r^j_{pq}] = \omega^{ij} \neq 0. \tag{2.363}$$

Note that the r^i_{pq} only talk with the $r^j_{p'q'}$ when $p = p', q = q'$: the several components of a the vector between the p^{th} and q^{th} centers are non-commutative, but they commute with all the other components of all the other inter-center vectors. There is only pairwise non-commutativity.

The angular momentum is:

$$J = \frac{1}{2} \sum_{p,q} \langle \Gamma_p, \Gamma_q \rangle \frac{\boldsymbol{r}_{pq}}{|r_{pq}|}. \tag{2.364}$$

It is a sum of contributions from each pair of points. Each individual contribution is a vector along the line connecting two points (unit vectors $\frac{\boldsymbol{r}_{pq}}{|r_{pq}|}$) with size the angular momentum from the crossed electric and magnetic fields $\langle \Gamma_p, \Gamma_q \rangle$.

Two-Center Solutions

Let us make things more clear using an explicit example with two centers. Write $J = \langle \Gamma_1, \Gamma_2 \rangle$, then the volume form on phase space is

$$\omega = J \sin\theta \, d\theta \wedge d\phi, \tag{2.365}$$

the standard symplectic form on a two-sphere. (Remember that the solution space for two center is the S^2 of orientations of the fixed rod $\boldsymbol{r}_{12}$.) The normalization of the two-form is the angular momentum between the two centers.

The number of states is then

$$\int_{S^2} \omega = 2|J| + 1. \tag{2.366}$$

We get $2|J| + 1$ rather than $2|J|$ because of subtleties with fermions. This is exactly the number of states for an angular momentum multiplet (Fig. 2.45).

Exercise 2.8.31 *"Meaningless algebra" for the two-center solution space:*

- *Check that*

$$d\omega = 0 \tag{2.367}$$

- *Check that ω_{S^2} defined as (2.360) evaluates to (2.365).*

Three-Center Solutions

Solution space is $2N - 2$ dimensional. For $N = 3$, we get a four-dimensional solution space M_4. The bubble equations fix two distances in terms of the third, say $r_{23}(r_{12})$ and $r_{13}(r_{12})$. The four remaining parameters are

- The distance r_{12}.
- The $U(1)$ of orientations around segment r_{12}.
- The orientation of r_{12} in space (an S^2 as for the two-center solution space).

Therefore the solution space is:

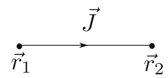


Fig. 2.45 Two-center solution

$$M_4 = I \times U(1) \times S^2, \tag{2.368}$$

where I is the line segment of r_{12}. The second product is a non-trivial fibration.

Note that the size of the angular momentum is a function of the distance r_{12} as well:

$$J(r_{12}). \tag{2.369}$$

By the bubble equations the interval I of allowed r_{12} values is constrained

$$I = [r_{12}^{\min}, r_{12}^{\max}]. \tag{2.370}$$

Hence also the angular momentum is bounded between $J_{\min}$ and $J_{\max}$, see Fig. 2.46.

We can see the system as a whole range of angular momentum multiplets, see Fig. 2.47. Let us note an important caveat here when discussing entropy. We are referring here only to the configuration entropy coming from the different ways of arranging the centers. Each individual center, if it has a horizon, may have additional entropy associated with that horizon. In our discussion of entropy above we neglect this because we are mostly interested in looking for black hole microstates. That is to say we want to find a realization of the black hole entropy via horizonless smooth solutions. If the centers are themselves black holes with horizons, we are not counting the horizon entropy of a single black hole with the total charge of all the centers.

More Centers?

Let us fix the total charge, Γ, and consider an N-center decomposition

$$\Gamma = \sum_{N=1}^{\infty} \left(\sum_{q=1}^{N} \Gamma_q \right). \tag{2.371}$$

For large charge Γ the number of centers N can be quite large and we can also arrange the centers all to be horizonless. What are all possible states corresponding to these charges? We fix Γ first, then we fix the sectors we want to divide over, and we divide the charges. All these states are in one Hilbert space, of total charge Γ. Are all these possible states reproducing the black hole entropy of a single black

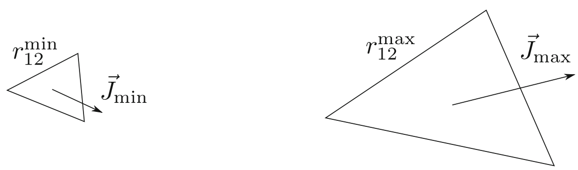


Fig. 2.46 The angular momentum is a function of the size of r_{12}

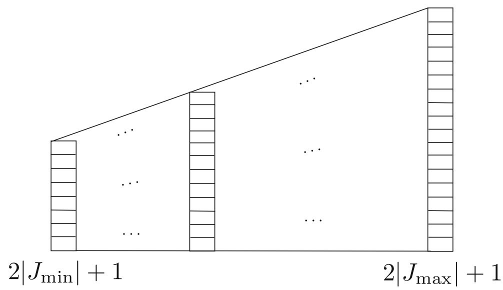


Fig. 2.47 The angular momentum is a function of the size of r_{12}. The states are divided into one angular momentum multiplet for each allowed value of J

hole with charge Γ? Should we use smooth centers? How many can we put? Can we reproduce the entropy?

The result in the literature so far is:

- For fully interacting centers ($\langle \Gamma_p, \Gamma_q \rangle \neq 0$), this counting has only been done in full generality for 2 and 3 centers. It has been extended to $N + 1$ centers, where the first N have all charges equal $\Gamma_1 = \ldots = \Gamma_N$ and the other center has non-vanishing $\langle \Gamma_p, \Gamma_{N+1} \rangle$ with all the others.

Note that classically, there can be a problem due to configurations with runaway behaviour. One of the centers can go off to infinity in the bubble equations, and this screws up the asymptotics, see Fig. 2.48.

After quantization, there is a density on $M_4 = \mathbb{R} \times U(1) \times S^2$. This gives a finite volume. There is no more runaway, because the wave function for the positions of the centers has no support at infinity, 'the tail is vanishing'. This renders $\langle \boldsymbol{r}_p \rangle$ finite. See Fig. 2.49.

2.8.5 Scaling Solutions

Let us go to solutions where the centers can come arbitrarily close. We stay in the three-center example. Remember that the bubble equations look like

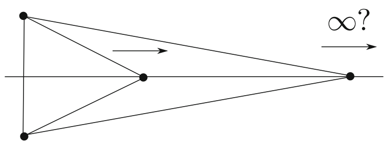


Fig. 2.48 Classically, one of the centers can run of to infinity

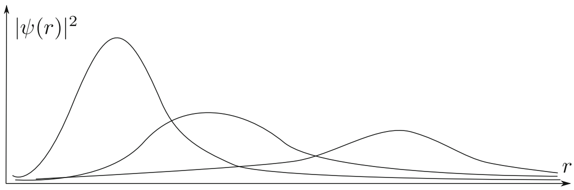


Fig. 2.49 In quantum mechanics, the wave function has no support at infinity

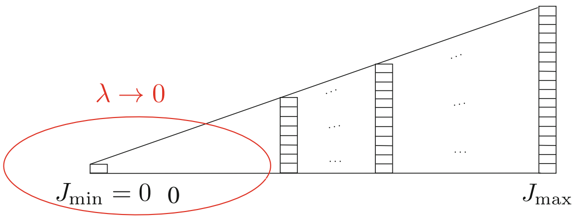


Fig. 2.50 The angular momentum multiplet triangle is completed for scaling solutions, since the solution space contains the limit $\lambda \to 0$, such that $J_{\min} = 0$

$$\frac{\langle\Gamma_1, \Gamma_2\rangle}{r_{12}} + \frac{\langle\Gamma_1, \Gamma_3\rangle}{r_{13}} = c_1,$$
$$\frac{\langle\Gamma_2, \Gamma_1\rangle}{r_{12}} + \frac{\langle\Gamma_2, \Gamma_3\rangle}{r_{13}} = c_2,$$
$$\frac{\langle\Gamma_3, \Gamma_1\rangle}{r_{13}} + \frac{\langle\Gamma_3, \Gamma_2\rangle}{r_{23}} = c_3, \tag{2.372}$$

with $c_p = -\langle\Gamma_p, h\rangle$. We look for solutions with

$$r_{pq} = \lambda\langle\Gamma_p, \Gamma_q\rangle + \mathcal{O}(\lambda^2), \tag{2.373}$$

such that we can send $\lambda \to 0$. Then we find that $\langle\Gamma_p, \Gamma_q\rangle = \alpha\, r_{pq}$ for some constant α. Hence we can only take this limit when the $\Gamma_{pq} \equiv \langle\Gamma_p, \Gamma_q\rangle$ satisfy the triangle inequalities.

As a consequence, the angular momentum is zero when $\lambda = 0$:

$$\boldsymbol{J} = \sum \Gamma_{pq} \frac{\boldsymbol{r}_{pq}}{r_{pq}} = \alpha \sum \boldsymbol{r}_{pq} = 0, \tag{2.374}$$

where the last equality follows because the $\boldsymbol{r}_{pq}$ form a closed triangle. Therefore, near $\lambda \to 0$, we have $J \to 0$. This means that we 'complete' the triangle of states in the angular momentum multiplets of Fig. 2.47 to that of Fig. 2.50. We can parametrize the region near $J_{\min} = 0$ by the scaling parameter λ.

When the inter-center distance $r_{pq} \sim \lambda \to 0$, the geometry develops a very deep throat of size proportional to $1/\lambda$, see Fig. 2.51. As the centers come closer and closer, the throat becomes deeper and deeper.

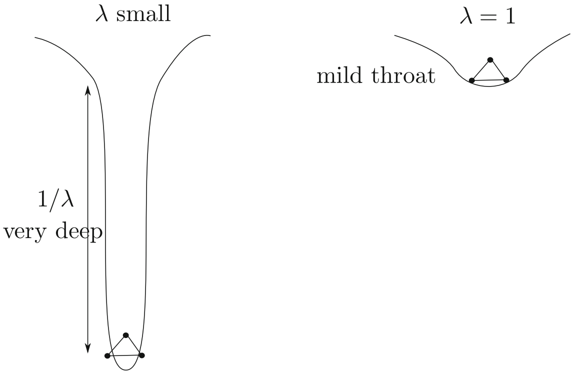


Fig. 2.51 By scaling down the distances between the centers as $\lambda \to 0$, the geometry develops a very deep throat whose size is inversely proportional to λ. When λ is of order 1 on the other hand, we only have a very mild throat

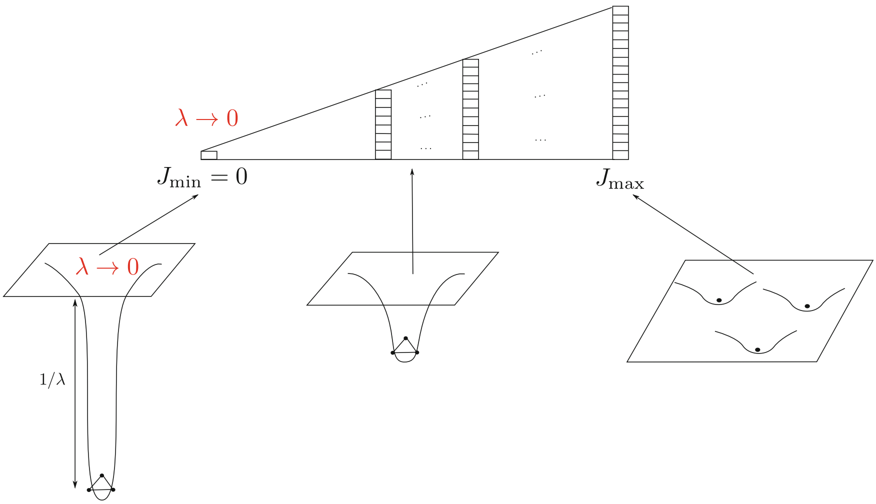


Fig. 2.52 The correspondence of scaling solutions of a certain size to angular momentum multiplets in the quantized solutions space

Putting these things together, gives a situation of the states in solution space as in Fig. 2.52. This reveals a paradox. As $\lambda \to 0$, we get deeper and deeper microstates and we can continue like this forever. On the other hand, the number of states associated to the region of small λ of Fig. 2.52, gives a finite number of states. Stated in a different way, in quantum mechanics, it is meaningless to put states in a cell smaller than $\hbar$-size. Remember that on solution space, we had non-commuting coordinates r^i_{pq} and r^j_{pq}. This translates to the impossibility of localizing r^i_{pq} and r^j_{pq} with a resolution

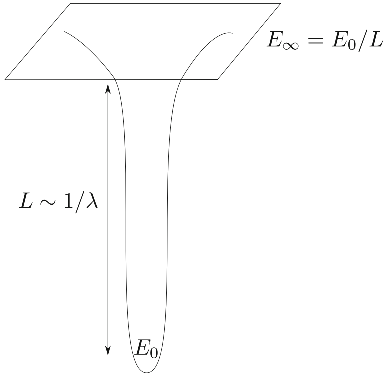


Fig. 2.53 The energy E_0 of an excitation down the throat is redshifted to $E_\infty \sim E_0/L$, with L the throat length

smaller than $\hbar$. Therefore there is some cut-off, and all deeper and deeper microstates must correspond to one quantum state.

Hence even though we can make the throats as deep as we want classically, all these deep throats do not exists after quantization. This is related to the earlier puzzle, that due to redshift, the energy E_∞ would have a continuous spectrum for deeper and deeper throats, see Fig. 2.53: a string stretching between two centers remains massless at spatial infinity.

On the other hand, the CFT should have a discrete spectrum, otherwise the counting of microstates would not give a finite number. So the question is whether there is a cut-off in the throat, and what it is.

While the exact answer to this question depends on the state we consider and is somewhat complicated, a simple order of magnitude estimate can be gleaned as follows. We consider the geometry of the throat up to the scale where λ takes its expectation value in the lowest angular momentum state (the state at $J = J_{min}$; see Fig. 2.53 above). That is, we compute $\langle\lambda\rangle$ in the state $|j = 0\rangle$ and then plug this into the harmonics to yield a solution. This gives a cutoff on the throat and we can determine the mass gap by putting a scalar field on this background and computing the gap in its spectrum (this is analogous to a standard computation to determine the mass gap in global AdS and essentially measures the "size of the box" provided by the gravitational potential).

This computation yields a mass gap that, when measured in AdS units $1/L_{AdS}$, scales as $1/c$. Here c is a dimensionless number given by comparing the AdS length to the plank length $c = L_{AdS}/\ell_P$. Thus the mass gap is

$$\frac{1}{c\,L_{AdS}}. \tag{2.375}$$

whereas the mass gap in global AdS is just

$$\frac{1}{L_{AdS}}. \tag{2.376}$$

The suggestive terminology c alludes to the fact that this number is the central charge of the dual CFT. For example in the case where the AdS_3 is the near horizon of the D1-D5 black hole c is proportional to $Q_1 Q_5$ and is the central charge of the dual D1-D5 CFT.

This is a very significant result. Recall that in our derivation of the black hole entropy in earlier sections a very important role was played by the so called "long string picture" where the entropically dominant sector of the CFT came from a string with a winding numer that is proportional to $Q_1 Q_5$ as well. Consequently the momentum of this string was quantized in units $\frac{1}{Q_1 Q_5 R}$ with R the dimensionful length of the CFT circle $R = 2\pi L_{AdS}$.

This computation thus suggests that the quantum corrections to the deep throat microstates not only discretize the spectrum, hence resolving the issue of a continuous spectrum, but also do this by giving them a mass gap corresponding to the most entropic sector of the CFT. This suggests these states at least occupy the "typical" sector of the CFT and hence are potentially the kind of states that account for the black hole entropy.

References

1. N. Deruelle, *Black Holes in General Relativity*, Course de Physique Théorique at IPhT, CEA Saclay,ipht.cea.fr/Docspht//articles/t09/346/public/Cours2009.pdfipht.cea.fr/Docspht//articles/t09/346/public/Cours2009.pdf
2. J.D. Bekenstein, Black holes and the second law. Nuovo Cim. Lett. **4**, 737–740 (1972)
3. S.W. Hawking, Particle creation by black holes. Commun. Math. Phys. **43**, 199–220 (1975)
4. M.B. Green, J.H. Schwarz, E. Witten, *Superstring Theoy, Introduction*, Cambridge Monographs On Mathematical Physics, vol 1 (Cambridge University Press, UK, 1987), p. 469
5. M.B. Green, J.H. Schwarz, E. Witten, *Superstring Theory, Loop qamplitudes, anomalies and phenomenology*, Cambridge Monographs On Mathematical Physics, vol 2 (Cambridge University Press, UK, 1987), p. 596
6. J. Polchinski, *String Theory, An Introduction to the Bosonic String*, vol 1 (Cambridge University Press, UK, 1998), p. 402
7. J. Polchinski, *String Theory, Superstring Theory and Beyond*, vol 2 (Cambridge University Press, UK, 1998), p. 531
8. K. Becker, M. Becker, J.H. Schwarz, *String Theory and M-theory: A Modern Introduction* (Cambridge University Press, UK, 2007)
9. E. Kiritsis, *String Theory in a Nutshell* (Princeton University Press, USA 2007)
10. D.Z. Freedman, A. Van Proeyen, *Supergravity* (Cambridge University Press, Cambridge, UK, 2012), p. 607
11. E. Cremmer, B. Julia, J. Scherk, Supergravity theory in eleven-dimensions. Phys. Lett. **B76**, 409–412 (1978)
12. G.W. Gibbons, M.B. Green, M.J. Perry, Instantons and seven-branes in type IIB superstring theory. Phys. Lett. **B370**, 37–44 (1996). http://www.arXiv.org/abs/hep-th/9511080 hep-th/9511080
13. F. Denef, *Les Houches Lectures on Constructing String Vacua*, http://www.arXiv.org/abs/0803.1194 0803.1194
14. E.A. Bergshoeff, J. Hartong, T. Ortin, D. Roest, Seven-branes and supersymmetry. JHEP **0702**, 003 (2007). http://www.arXiv.org/abs/hep-th/0612072 hep-th/0612072

15. E. Bergshoeff, J. Hartong, D. Sorokin, Q7-branes and their coupling to IIB supergravity. JHEP **0712**, 079 (2007). http://www.arXiv.org/abs/0708.2287 0708.2287
16. J. Raeymaekers, D. Van den Bleeken, B. Vercnocke, Chronology protection and the stringy exclusion principle. JHEP **1104**, 037 (2011). http://www.arXiv.org/abs/1011.5693 1011.5693
17. E. Bergshoeff, *p-branes, D-branes and M-branes*, http://www.arXiv.org/abs/hep-th/9611099 hep-th/9611099
18. K.S. Stelle, *BPS Branes in Supergravity*, http://www.arXiv.org/abs/hep-th/9803116 hep-th/9803116
19. R. Argurio, *Brane Physics in M-theory*, http://www.arXiv.org/abs/hep-th/9807171 hep-th/9807171
20. A.W. Peet, *TASI Lectures on Black Holes in String Theory*, http://www.arXiv.org/abs/hep-th/0008241 hep-th/0008241
21. C.W. Misner, K. Thorne, J. Wheeler, *Gravitation*
22. A.A. Tseytlin, Harmonic superpositions of M-branes. Nucl. Phys. **B475**, 149–163 (1996). http://www.arXiv.org/abs/hep-th/9604035 hep-th/9604035
23. A. Strominger, C. Vafa, Microscopic origin of the Bekenstein-Hawking entropy. Phys. Lett. **B379**, 99–104 (1996). http://www.arXiv.org/abs/hep-th/9601029 hep-th/9601029
24. J.M. Maldacena, The Large N limit of superconformal field theories and supergravity. Adv. Theor. Math. Phys. **2**, 231–252 (1998). http://www.arXiv.org/abs/hep-th/9711200 hep-th/9711200
25. E. Witten, Anti-de Sitter space and holography. Adv. Theor. Math. Phys. **2**, 253–291, (1998). http://www.arXiv.org/abs/hep-th/9802150 hep-th/9802150
26. S. Gubser, I.R. Klebanov, A.M. Polyakov, Gauge theory correlators from noncritical string theory. Phys. Lett. **B428**, 105–114 (1998). http://www.arXiv.org/abs/hep-th/9802109 hep-th/9802109
27. G. 't Hooft, *Large N*, http://www.arXiv.org/abs/hep-th/0204069 hep-th/0204069
28. I. Bena, M. Berkooz, J. de Boer, S. El-Showk, D. Van den Bleeken, Scaling BPS solutions and pure-Higgs states. JHEP **1211**, 171 (2012). http://www.arXiv.org/abs/1205.5023 1205.5023
29. I.R. Klebanov, *TASI lectures: Introduction to the AdS/CFT Correspondence*, http://www.arXiv.org/abs/hep-th/0009139 hep-th/0009139
30. J.-P. Blaizot, E. Iancu, U. Kraemmer and A. Rebhan, Hard thermal loops and the entropy of supersymmetric Yang-Mills theories, JHEP **0706**, 035 (2007). http://www.arXiv.org/abs/hep-ph/0611393 hep-ph/0611393
31. A. Bernamonti and R. Peschanski, Time-dependent AdS/CFT correspondence and the Quark-Gluon plasma. Nucl. Phys. Proc. Suppl. **216**, 94–120 (2011). http://www.arXiv.org/abs/1102.0725 1102.0725
32. M. Petrini, *The AdS/CFT correspondence*, Course de Physique Théorique at IPhT, CEA Saclay, http://ipht.cea.fr/Docspht/search/article.php?id=t10/312http://ipht.cea.fr/Docspht/search/article.php?id=t10/312
33. J.R. Peschanski Robi, *The dynamics of Quark-Gluon Plasma and AdS/CFT correspondence*, Course de Physique Théorique at IPhT, CEA Saclay, http://ipht.cea.fr/Docspht/search/article.php?id=t10/310http://ipht.cea.fr/Docspht/search/article.php?id=t10/310
34. L. Susskind, *Some Speculations About Black Hole Entropy in String Theory*, http://www.arXiv.org/abs/hep-th/9309145 hep-th/9309145
35. E. Halyo, A. Rajaraman, L. Susskind, Braneless black holes. Phys. Lett. **B392**, 319–322 (1997). http://www.arXiv.org/abs/hep-th/9605112 hep-th/9605112
36. E. Halyo, B. Kol, A. Rajaraman, L. Susskind, Counting Schwarzschild and charged black holes. Phys. Lett. **B401**, 15–20 (1997). http://www.arXiv.org/abs/hep-th/9609075 hep-th/9609075
37. G.T. Horowitz, J. Polchinski, A Correspondence principle for black holes and strings. Phys. Rev. **D55**, 6189–6197 (1997). http://www.arXiv.org/abs/hep-th/9612146 hep-th/9612146
38. T. Damour, G. Veneziano, Selfgravitating fundamental strings and black holes. Nucl. Phys. **B568**, 93–119 (2000). http://www.arXiv.org/abs/hep-th/9907030 hep-th/9907030
39. S.D. Mathur, The Information paradox: A Pedagogical introduction. Class. Quant. Grav. **26**, 224001 (2009). http://www.arXiv.org/abs/0909.1038 0909.1038

40. S.D. Mathur, The fuzzball proposal for black holes: An elementary review. Fortsch. Phys. **53**, 793–827 (2005). http://www.arXiv.org/abs/hep-th/0502050 hep-th/0502050
41. I. Bena, N.P. Warner, Black holes, black rings and their microstates. Lect. Notes Phys. **755**, 1–92 (2008). http://www.arXiv.org/abs/hep-th/0701216 hep-th/0701216
42. S.D. Mathur, *Fuzzballs and the Information Paradox: A Summary and Conjectures*, http://www.arXiv.org/abs/0810.4525 0810.4525
43. V. Balasubramanian, J. de Boer, S. El-Showk and I. Messamah, Black holes as effective geometries. Class. Quant. Grav. **25**, 214004 (2008). http://www.arXiv.org/abs/0811.0263 0811.0263
44. K. Skenderis, M. Taylor, The fuzzball proposal for black holes. Phys. Rept. **467**, 117–171 (2008). http://www.arXiv.org/abs/0804.0552 0804.0552
45. B.D. Chowdhury, A. Virmani, *Modave Lectures on Fuzzballs and Emission from the D1-D5 System*, http://www.arXiv.org/abs/1001.1444 1001.1444
46. V. Jejjala, O. Madden, S.F. Ross, G. Titchener, Non-supersymmetric smooth geometries and D1-D5-P bound states. Phys. Rev. **D71**, 124030 (2005). http://www.arXiv.org/abs/hep-th/0504181 hep-th/0504181
47. S. Giusto, S.F. Ross, A. Saxena, Non-supersymmetric microstates of the D1-D5-KK system. JHEP **12**, 065 (2007). http://www.arXiv.org/abs/0708.3845 0708.3845
48. V. Cardoso, O.J. Dias, J.L. Hovdebo, R.C. Myers, Instability of non-supersymmetric smooth geometries. Phys. Rev. **D73**, 064031 (2006). http://www.arXiv.org/abs/hep-th/0512277 hep-th/0512277
49. B.D. Chowdhury, S.D. Mathur, Radiation from the non-extremal fuzzball. Class. Quant. Grav. **25**, 135005 (2008). http://www.arXiv.org/abs/0711.4817 0711.4817
50. B.D. Chowdhury, S.D. Mathur, Non-extremal fuzzballs and ergoregion emission. Class. Quant. Grav. **26**, 035006 (2009). http://www.arXiv.org/abs/0810.2951 0810.2951
51. B.D. Chowdhury, S.D. Mathur, Pair creation in non-extremal fuzzball geometries. Class. Quant. Grav. **25**, 225021 (2008). http://www.arXiv.org/abs/0806.2309 0806.2309
52. I. Bena, J. de Boer, M. Shigemori, N.P. Warner, Double, Double Supertube Bubble. JHEP **1110**, 116 (2011). http://www.arXiv.org/abs/1107.2650 1107.2650
53. I. Bena, S. Giusto, M. Shigemori, N.P. Warner, Supersymmetric solutions in six dimensions: a linear structure. JHEP **1203**, 084 (2012). http://www.arXiv.org/abs/1110.2781 1110.2781
54. S. Giusto, R. Russo, D. Turton, New D1-D5-P geometries from string amplitudes. JHEP **1111**, 062 (2011). http://www.arXiv.org/abs/1108.6331 1108.6331
55. I. Bena, S. Giusto, C. Ruef, N.P. Warner, Supergravity solutions from floating branes. JHEP **1003**, 047 (2010). http://www.arXiv.org/abs/0910.1860 0910.1860
56. G. Dall'Agata, S. Giusto, C. Ruef, U-duality and non-BPS solutions. JHEP **1102**, 074 (2011). http://www.arXiv.org/abs/1012.4803 1012.4803
57. G. Bossard, C. Ruef, Interacting non-BPS black holes. Gen. Rel. Grav. **44**, 21–66 (2012). http://www.arXiv.org/abs/1106.5806 1106.5806
58. G. Bossard, S. Katmadas, Duality covariant non-BPS first order systems. JHEP **1209**, 100 (2012). http://www.arXiv.org/abs/1205.5461 1205.5461
59. I.R. Klebanov, A.A. Tseytlin, Gravity duals of supersymmetric SU(N) x SU(N+M) gauge theories. Nucl. Phys. **B578**, 123–138 (2000). http://www.arXiv.org/abs/hep-th/0002159 hep-th/0002159
60. I.R. Klebanov, M.J. Strassler, Supergravity and a confining gauge theory: Duality cascades and chi SB resolution of naked singularities. JHEP **0008**, 052 (2000). http://www.arXiv.org/abs/hep-th/0007191 hep-th/0007191
61. M. Grana, Flux compactifications in string theory: a comprehensive review. Phys. Rept. **423**, 91–158 (2006). http://www.arXiv.org/abs/hep-th/0509003 hep-th/0509003
62. J. Breckenridge, R.C. Myers, A. Peet, C. Vafa, D-branes and spinning black holes. Phys. Lett. **B391**, 93–98 (1997). http://www.arXiv.org/abs/hep-th/9602065 hep-th/9602065
63. J.B. Gutowski, H.S. Reall, General supersymmetric AdS(5) black holes. JHEP **0404**, 048 (2004). http://www.arXiv.org/abs/hep-th/0401129 hep-th/0401129
64. J.P. Gauntlett, J.B. Gutowski, Concentric black rings. Phys. Rev. **D71**, 025013 (2005). http://www.arXiv.org/abs/hep-th/0408010 hep-th/0408010

65. I. Bena, P. Kraus, N.P. Warner, Black rings in Taub-NUT. Phys. Rev. **D72**, 084019 (2005). http://www.arXiv.org/abs/hep-th/0504142 hep-th/0504142
66. F. Denef, Supergravity flows and D-brane stability. JHEP **0008**, 050 (2000). http://www.arXiv.org/abs/hep-th/0005049 hep-th/0005049
67. B. Bates, F. Denef, Exact solutions for supersymmetric stationary black hole composites. JHEP **1111**, 127 (2011). http://www.arXiv.org/abs/hep-th/0304094 hep-th/0304094
68. I. Bena, C.-W. Wang, N.P. Warner, Plumbing the Abyss: Black ring microstates. JHEP **0807**, 019 (2008). http://www.arXiv.org/abs/0706.3786 0706.3786
69. J. de Boer, S. El-Showk, I. Messamah, D. Van den Bleeken, Quantizing N = 2 multicenter solutions. JHEP **0905**, 002 (2009). http://www.arXiv.org/abs/0807.4556 0807.4556

Chapter 3
From Black Strings to Lifshitz Black Branes

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We review the status of the construction of the asymptotically Lifshitz black brane solutions in Supergravity and String theory. We propose a general method to construct analytic $z = 2$ Lifshitz black brane solutions. The method is based on deforming 5-dimensional AdS black strings by an axion wave and reducing to 4-dimensions. We illustrate this method with examples.

3.1 Introduction

The standard application of gauge/gravity duality is the AdS/CFT correspondence, which gives rise to a relativistically scale invariant boundary theory. However, more recently, attention has centered on systems having more general scaling properties, such as non-relativistic field theories, in particular, a non-relativistic Lifshitz scaling

$$t \to \lambda^z t, \qquad x_i \to \lambda x_i, \qquad r \to \lambda r. \tag{3.1}$$

In order to produce such a dynamical scaling, the spacetime metric must be taken of the form

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$$\mathrm{d}s^2 = L^2\Big(-\frac{1}{r^{2z}}\mathrm{d}t^2 + \frac{1}{r^2}\mathrm{d}r^2 + \frac{1}{r^2}(\mathrm{d}x^2 + \mathrm{d}y^2)\Big) \tag{3.2}$$

which explicitly respects the scaling (3.1).The subject of Lifshitz holography was initiated by [1]. For more details about the basic holographic properties of Lifshitz space-times see [2]. By now we have at our disposal embeddings into string theory of Lifshitz geometries. They were recently found in [3, 4].

Black holes possessing Lifshitz asymptotics have been extensively studied in various models [5–8]. Despite all this considerable effort, a proper embedding of Lifshitz black holes into string theory was still missing. All the solutions that have been constructed in string theory so far are only known numerically (see [9, 10] for recent work). This note can be viewed as a step towards analytic black brane solutions. More details and explanations will be provided in a forthcoming publication [11].

In accordance with the results of [3, 12, 13], a 4-dimensional Lifshitz space with $z = 2$ can be attained starting from a model in five dimensions containing an axion field and admitting an AdS_5 vacuum. More explicitly, one can obtain Lagrangians supporting $z = 2$ Lifshitz space-times by Scherk-Schwarz reduction of Lagrangians supporting $z = 0$ Schrödinger space-times.[1] In other words, take some 5-dimensional supergravity theory that possesses AdS solutions and that contains an axion and then look for asymptotically AdS_5 black string solutions with a null Killing vector on its world-volume and deform them by an axion wave.

This short note is concerned with the study of asymptotically AdS_5 black string solutions to the following $5D$ Lagrangian

$$\begin{aligned} S_{\text{trunc.}} = \frac{1}{2\kappa_5^2}\int \mathrm{d}^5x\sqrt{-g}\Big[& R + 12 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}e^{2\phi}(\partial\chi)^2 \\ & - \frac{1}{12\sqrt{3}}\epsilon^{\mu\nu\rho\sigma\lambda}F_{\mu\nu}F_{\rho\sigma}A_\lambda\Big] \end{aligned} \tag{3.4}$$

and the $z = 2$ four-dimensional asymptotically Lifshitz black branes solutions to the $4D$ Lagrangian can be found after Scherk-Schwarz reduction (for more details see [13]). The action (3.4) is a very simple model that comes from a consistent truncation of type IIB supergravity and it is suitable for the embedding of Lifshitz-like solutions into string theory.[2] Our results can be seen as a step forward in extending our analysis of [13] to charged black strings, and hence charged Lifshitz black branes. The field equations derived from (3.4) are

[1] Five-dimensional Schrödinger space-time has the following metric

$$ds^2 = -\frac{dt^2}{r^{2z}} + \frac{1}{r^2}(-2dtd\zeta + dr^2 + dx^2 + dy^2). \tag{3.3}$$

[2] This 5-dimensional gauged $N = 2$ supergravity can also be seen as a consistent truncation of the minimal supergravity coupled to the universal hypermultiplet.

$$R_{\mu\nu} = -4g_{\mu\nu} + \frac{1}{2}F_{\rho\mu}F^{\rho}{}_{\nu} - \frac{1}{12}g_{\mu\nu}F^2 + \frac{1}{2}\partial_\mu\phi\partial_\nu\phi + \frac{1}{2}e^{2\phi}\partial_\mu\chi\partial_\nu\chi,$$

$$\partial_\mu(\sqrt{-g}F^{\mu\nu}) = \frac{1}{4\sqrt{3}}\epsilon^{\mu\rho\sigma\lambda\nu}F_{\mu\rho}F_{\sigma\lambda}, \tag{3.5}$$

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\partial^\mu\phi) = e^{2\phi}(\partial\chi)^2, \qquad \partial_\mu(\sqrt{-g}e^{2\phi}\partial^\mu\chi) = 0. \tag{3.6}$$

3.2 Lifshitz Black Branes from Black Strings

The five-dimensional black string Ansatz is [13]

$$\mathrm{d}s_5^2 = r^2(2A_1\mathrm{d}t\mathrm{d}u + \mathrm{d}x^2 + \mathrm{d}y^2) + \frac{1}{r^2F}\mathrm{d}r^2 \tag{3.7}$$

$$F_{xy} = P(r) \neq \text{const.} \qquad \text{(other components are zero)} \tag{3.8}$$

$$\phi = \phi(r), \qquad \chi = \chi(r). \tag{3.9}$$

The scalar field equation gives

$$\frac{d}{dr}(r^3A_1F^{-1/2}r^2F\phi') = 0 \Rightarrow r^5A_1F^{1/2}\phi' = \text{Const.} \tag{3.10}$$

From Einstein equations we find

$$-rFF' - r^2FF'' + r^2F'^2 - 32F - 12rF' + 32 + \frac{10}{3}P^2r^{-4}F + P^2r^{-4}rF' - \frac{16}{3}P^2r^{-4} + \frac{2}{9}P^4r^{-8} - \frac{4}{3}\frac{PP'}{r^3}F = 0. \tag{3.11}$$

For a magnetic Ansatz P must be a constant. Imposing asymptotically AdS boundary conditions enforces $P = 0$. When $P = 0$, we have $F = 1 - \frac{M}{r^4}$. The full black string solution deformed by an axion wave reads [13]

$$F = 1 - Mr^{-4}, \quad A_1 = F^{1/2}, \quad e^\phi = g_sF^{\pm 1/2}, \quad \chi = \ell u, \tag{3.12}$$

$$A_{2(-)} = -\frac{g_s^2\ell^2}{4r^2} + \frac{g_s^2\ell^2}{2M^{1/2}}A_1 \arcsin\frac{M^{1/2}}{r^2}, \tag{3.13}$$

$$A_{2(+)} = \frac{g_s^2\ell^2}{4M^{1/2}}A_1\left[(1-\log 2)\arcsin\frac{M^{1/2}}{r^2} - \left(\arcsin\frac{M^{1/2}}{r^2}\right)\log A_1 + \frac{1}{2}\mathrm{Cl}_2\left(2\arcsin A_1\right)\right]. \tag{3.14}$$

The subscript ($\pm$) in A_2 stands for the sign in (3.12). The function $\mathrm{Cl}_2(x)$ is the Clausen function defined in [13]. The metric has a curvature singularity at $r = M^{1/4}$. We now compactify the u direction and reduce to four dimensions, writing the metric (3.7) in the form of a Kaluza-Klein reduction Ansatz

$$ds_5^2 = e^{-\Phi} ds^2 + e^{2\Phi}(du + \mathcal{A})^2 \tag{3.15}$$

where

$$ds^2 = \frac{1}{\mathfrak{r}} A_2^{1/2} \left[-\frac{1}{\mathfrak{r}^2 A_2} F dt^2 + \frac{1}{\mathfrak{r}^2} \left(dx^2 + dy^2 \right) + \frac{d\mathfrak{r}^2}{\mathfrak{r}^2 F} \right], \tag{3.16}$$

$$A = \frac{F^{1/2}}{A_2} dt, \qquad e^{2\Phi} = \frac{1}{r^2} A_2, \tag{3.17}$$

where $\mathfrak{r} = 1/r$. The asymptotic form is given by the Lifshitz invariant solution. The 4-dimensional result is particularly sensitive to the zeros of A_2. In fact these points form curvature singularities (see [13] for more details).

Next, we want to look for more general Lifshitz black branes solutions than those of [13], that are preferably non-singular. A first thing one can do is to consider the following Ansatz

$$\mathrm{d}s_5^2 = 2R^2 A_1 \mathrm{d}t \mathrm{d}u + \frac{1}{R^2 F} \mathrm{d}R^2 + R^2 \mathrm{d}\Omega_k \tag{3.18}$$

where $\mathrm{d}\Omega_k^2$ denotes the metric of 2-manifold $\mathcal{M}_2$ of constant curvature k, with $k = 0, \pm 1$; $\mathcal{M}_2$ is a quotient space of the universal coverings S^2 ($k = 1$), hyperbolic space $\mathbb{H}^2$ ($k = -1$) or flat E^2 ($k = 0$). Explicitly we write

$$\mathrm{d}\Omega_k^2 = d\theta^2 + \Sigma(\theta)^2 d\varphi^2, \qquad \Sigma(\theta) = \begin{cases} \sin\theta, & k = 1, \\ 0, & k = 0, \\ \sinh\theta, & k = -1. \end{cases} \tag{3.19}$$

The black string has a magnetic field

$$F_{\theta\varphi} = kP\Sigma(\theta), \qquad A_\varphi = kP \int \Sigma(\theta) \mathrm{d}\theta. \tag{3.20}$$

The gauge field Eq. (3.5) are trivially satisfied for this choice of $U(1)$ gauge field. We plug the Ansatz into the Einstein equations. We obtain (setting $\chi = 0$)

$$R_{tu} = -4r^2 A_1 - \frac{1}{6} r^{-2} A_1 k^2 P^2 \tag{3.21}$$

$$R_{rr} = -4r^{-2} F^{-1} - \frac{1}{6} r^{-6} F^{-1} k^2 P^2 + \frac{1}{2} (\phi')^2 \tag{3.22}$$

$$R_{\theta\theta} = -4r^2 + \frac{1}{3}r^{-2}k^2P^2 \tag{3.23}$$

$$R_{\varphi\varphi} = \Sigma^2 R_{\theta\theta} \tag{3.24}$$

From the first and third equations we obtain

$$FF'' - F'^2 + \frac{FF'}{r} + F\left(-\frac{10k^2P^2}{3r^6} + \frac{8k}{r^4} + \frac{32}{r^2}\right) + F'\left(-\frac{k^2P^2}{r^5} + \frac{3k}{r^3} + \frac{12}{r}\right)$$
$$-\frac{2k^4P^4}{9r^{10}} + \frac{4k^3P^2}{3r^8} + \frac{16k^2P^2}{3r^6} - \frac{2k^2}{r^6} - \frac{16k}{r^4} - \frac{32}{r^2} = 0. \tag{3.25}$$

This equation can be rewritten as

$$F''F - 3F'f + \frac{1}{r}F'F - F'^2 + 2\Big(f' + \frac{k}{r^4} + \frac{12}{r^2}\Big)F = 2f^2 \tag{3.26}$$

where

$$f = \frac{k^2P^2}{3r^5} - \frac{k}{r^3} - \frac{4}{r}. \tag{3.27}$$

The same type of equation can be found in [8]. Finding the most general solution of this equation is a formidable task. However, it is possible to obtain a particular solution for which k is arbitrary and $P^2 = \frac{1}{3}$

$$F = \left(1 + \frac{k}{3r^2}\right)^2, \quad A_1 = F^{3/2}, \quad \phi = \phi_0. \tag{3.28}$$

Now, we deform the black string solution (3.28) by the axion wave. We take (for $k = -1$)

$$e^{\phi} = g_s = \text{constant}, \qquad \chi = \ell u. \tag{3.29}$$

From the R_{rr} equation, and after some algebraic manipulation, we obtain

$$-4(\mathfrak{r}-1)\mathfrak{r}^2 g'' - 4(2-3\mathfrak{r})\mathfrak{r}g' - 27(\mathfrak{r}-1)g + 3g_s^2\ell^2 = 0. \tag{3.30}$$

where we have defined $A_2 = g(\mathfrak{r})$ with $\mathfrak{r} = F^{\frac{1}{2}}$. For $\ell = 0$, once can find an analytic solution for $g(\mathfrak{r})$ being expressed as a combination of hypergeometric functions. This clearly indicates that uplifting the rather-difficult $4D$ differential equations to $D = 5$ allows one to decouple these differential equations.

We next compactify along the u-direction and we obtain the following asymptotically Lifshitz black brane solution (for $\ell \neq 0$)

$$ds^2 = \frac{\sqrt{g}}{R}\Big(-\frac{F^3}{R^2 g}dt^2 + \frac{1}{R^2}\mathrm{d}\Omega_k + \frac{dR^2}{R^2 F}\Big), \tag{3.31}$$

$$\mathcal{A} = \frac{F^{3/2}}{g}dt, \qquad e^{2\Phi} = \frac{g}{R^2}, \tag{3.32}$$

where $R = 1/r$. Whether the obtained black brane solution is singular or not can be determined once the zeroes of g are found. This turns out to be a bit cumbersome and we hope to report on it in the near future. It has been shown in some cases that the addition of hypermultiplets might affect the (non)-singular behavior of black brane solutions. This definitely deserves further investigation.

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References

1. S. Kachru, X. Liu, M. Mulligan, Phys. Rev. D **78**, 106005 (2008). [arXiv:0808.1725 [hep-th]]
2. M. Taylor, Non-relativistic holography. arXiv:0812.0530 [hep-th]
3. D. Cassani, A.F. Faedo, Constructing Lifshitz solutions from AdS. JHEP **1105**, 013 (2011). [arXiv:1102.5344 [hep-th]]
4. A. Donos, J.P. Gauntlett, Lifshitz solutions of D=10 and D=11 supergravity. JHEP **1012**, 002 (2010). [arXiv:1008.2062 [hep-th]]
5. J. Tarrio, S. Vandoren, Black holes and black branes in Lifshitz spacetimes. JHEP **1109**, 017 (2011). [arXiv:1105.6335 [hep-th]]
6. R.B. Mann, Lifshitz topological black holes. JHEP **0906**, 075 (2009). [arXiv:0905.1136 [hep-th]]
7. G. Bertoldi, B.A. Burrington, A. Peet, Black holes in asymptotically Lifshitz spacetimes with arbitrary critical exponent. Phys. Rev. D **80**, 126003 (2009). [arXiv:0905.3183 [hep-th]]
8. A. Bernamonti, M.M. Caldarelli, D. Klemm, R. Olea, C. Sieg, E. Zorzan,JHEP **0801**, 061 (2008). [arXiv:0708.2402 [hep-th]]
9. L. Barclay, R. Gregory, S. Parameswaran, G. Tasinato, I. Zavala, Lifshitz black holes in IIA supergravity. arXiv:1203.0576 [hep-th]
10. I. Amado, A.F. Faedo, Lifshitz black holes in string theory. JHEP **1107**, 004 (2011). [arXiv:1105.4862 [hep-th]]
11. W. Chemissany, J. Hartong, B. Vercnocke [to appear]
12. N. Halmagyi, M. Petrini, A. Zaffaroni, Non-relativistic solutions of N=2 gauged supergravity. JHEP **1108**, 041 (2011). [arXiv:1102.5740 [hep-th]]
13. W. Chemissany, J. Hartong, From D3-branes to Lifshitz space-times.Class. Quant. Grav. **28**, 195011 (2011). [arXiv:1105.0612 [hep-th]]

Chapter 4
Non-extremal Black-Hole Solutions of $\mathcal{N} = 2,\ d = 4,\ 5$ Supergravity

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4.1 Introduction

Black holes have been intensely studied in the framework of string theory for the last 20 years. They are described by classical solutions of the supergravity theories that describe effectively the low-energy dynamics of different string compactifications. Being solutions of theories with local supersymmetry one can distinguish among them the particular class of those that preserve some unbroken supersymmetries (called supersymmetric or, less precisely, BPS).

The special properties enjoyed by these black-hole solutions makes them very interesting (they are the ones for which the entropy was first computed by counting their microstates [1] and they are among those for which there is an attractor mechanism at work [2–5]) and easier to construct. For instance, it is known how to construct, systematically, all the black-hole solutions of any theory of ungauged $\mathcal{N} = 2,\ d = 4$ [6–9][1] and $d = 5$ [11, 12] supergravity coupled to any number vector supermultiplets and some general results are also known for higher-$\mathcal{N}$, $d = 4$ supergravities [13]. Some supersymmetric black-hole solutions of non-Abelian gauged $\mathcal{N} = 2,\ d = 4$ supergravity are also known in fully analytic form [14–16].

In spite of their interest, non-extremal black-hole solutions of these theories are much less known. Here we are going to review recent progress in the construction of non-extremal black holes and branes, particularly in ungauged theories of $\mathcal{N} = 2$, $d = 4$ and $d = 5$ supergravity coupled to vector supermultiplets [17–25].

This progress is based, first of all, in the use of the FGK formalism [26], conveniently generalized in [23] to arbitrary spacetime dimension $d \geq 4$ and worldvolume dimension $p \geq 0$. Usually, only the results of [26] concerning extremal black holes

[1] See also [10] for a review.

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and attractors are used, but the formalism provides a setting which simplifies the task of finding explicit solutions. We review this formalism in Sect. 4.2.

A second ingredient is the general ansatz for single, static, non-extremal black holes of $\mathcal{N} = 2$, $d = 4, 5$ supergravity presented in [19, 20]. This ansatz, which we review in Sect. 4.3, can be understood as a deformation of the general supersymmetric solution of [6–9] in which the harmonic functions (traditionally denoted by H^i) are replaced by linear combinations of hyperbolic sines and cosines, but the physical fields have the same form in terms of those functions as they had in terms of the H^i.

Several arguments in support of the generality of this ansatz were given in [19], but the main assumption that the functional form of the physical fields in terms of the functions H^i can be given stronger foundations. In [17, 18] for the $\mathcal{N} = 2,\ d = 5$ case and in [21, 22] for the $\mathcal{N} = 2,\ d = 4$ case, it was shown that the H^i can be used as dynamical variables in the reduced action of the FGK formalism. The change of variables from the physical fields to the H^i assumes the same functional dependence of the former on the later both for extremal (supersymmetric and non supersymmetric) and non-extremal black holes, proving the assumption. This allows the use of these variables in more complex settings, such as rotating black holes or black-holes in gauged supergravities, as had been observed before.

The use of these variables in combination with the FGK formalism (a combination that we call H-FGK formalism) simplifies considerably the task of finding general, explicit extremal and non-extremal solutions and also general results about families of solutions (see [24] and [25] for the 5- and 4-dimensional cases, respectively), as we will review in Sect. 4.4.

4.2 FGK Formalism

In this Section we will review the generalization presented in [23] to arbitrary spacetime dimension $d \geq 4$ and worldvolume dimension $p \geq 0$ of the formalism introduced in [26] for 4-dimensional black holes. The generalization to $d \geq 4$ is necessary to study the black holes of $\mathcal{N} = 2,\ d = 5$ theories and the generalization to black p-branes will allow us to study the black string solutions of those theories. The black holes of $\mathcal{N} = 2,\ d = 4$ can be obtained by direct dimensional reduction of the 5-dimensional black holes and double dimensional reduction of the black strings, hence the interest in these objects.

4.2.1 Derivation of the Effective Action

The main ingredients of the FGK formalism are a generic action which can describe the relevant bosonic sectors of most (or all) the ungauged supergravities and a generic metric and coordinate choice which can describe the exterior of all the single, static, black p-brane solutions of those theories. The generic action is then reduced using as reduction ansatz the generic metric, which has only one undetermined function that will remain a variable of the dimensionally reduced equations of motion. The staticity of the ansatz leaves us with only one parameter on which the physical fields (metric function plus scalar fields) can depend in the dimensionally reduced equations of motion. it is, then, possible to find an effective 1-dimensional (mechanical) action for the remaining variables (which has to be supplemented by a constraint) from which one can derive the equations of motion and general results concerning the black p-brane solutions of those theories.

The generic action that we propose is

$$\begin{aligned}\mathcal{I}[g, A^{\Lambda}_{(p+1)}, \phi^i] = \int d^d x \sqrt{|g|}\, \Big\{ R + \mathcal{G}_{ij}(\phi) \partial_\mu \phi^i \partial^\mu \phi^j \\ + 4 \tfrac{(-1)^p}{(p+2)!} I_{\Lambda\Sigma}(\phi) F^{\Lambda}_{(p+2)} \cdot F^{\Sigma}_{(p+2)} \Big\},\end{aligned} \tag{4.1}$$

where the scalar fields ϕ^i parametrize a non-linear σ-model with metric $\mathcal{G}_{ij}(\phi)$, $I_{\Lambda\Sigma}(\phi)$ is a scalar-dependent, negative-definite (kinetic) matrix that describes the coupling of the scalars to the $(p+1)$-forms $A^{\Lambda}_{(p+1)}$ to which the p-branes couple electrically,

$$F^{\Lambda}_{(p+2)\,\mu_1\cdots\mu_{p+2}} = (p+2)\partial_{[\mu_1|} A^{\Lambda}_{(p+1)\,|\mu_2\cdots\mu_{p+2}]}, \tag{4.2}$$

are their field strengths and we have used the notation

$$F^{\Lambda}_{(p+2)} \cdot F^{\Sigma}_{(p+2)} \equiv F^{\Lambda}_{(p+2)\,\mu_1\cdots\mu_{p+2}} F^{\Sigma}_{(p+2)}{}^{\mu_1\cdots\mu_{p+2}}. \tag{4.3}$$

We define, as usual, the worldvolume dimension of the dual brane $\tilde{p} \equiv d - p - 4$. In general $p \neq \tilde{p}$ and neither the dual $\tilde{p}$-brane can couple to the $(p+1)$-forms $A^{\Lambda}_{(p+1)}$ nor the electric p-branes can couple to the dual $(\tilde{p}+1)$-forms $A_{\Lambda\,(\tilde{p}+1)}$. Thus, the above model is generically sufficient.

However, there are particular cases in which the above model is too simple: when $p = \tilde{p} = (d-4)/2$ one should consider additional terms in the action of the form

$$+ 4\xi^2 \tfrac{(-1)^p}{(p+2)!} R_{\Lambda\Sigma}(\phi) F^{\Lambda}_{(p+2)} \cdot \star F^{\Sigma}_{(p+2)}. \tag{4.4}$$

Here $R_{\Lambda\Sigma}(\phi)$ is a scalar-dependent matrix such that

$$R_{\Lambda\Sigma} = -\xi^2 R_{\Sigma\Lambda}, \tag{4.5}$$

where

$$\xi^2 \equiv \star^2 = -(-1)^{d/2} = (-1)^{p+1}, \tag{4.6}$$

and $\star$ is the operator that relates $(p+2)$-form field strengths to their $(\tilde{p}+2)$-form Hodge duals. In these cases our ansatz must take into account that the same brane can also be magnetically charged i.e. they can be dyonic.

There is yet another particular case: when $d = 4n + 2$ the dyonic branes can also be self- or anti-self-dual because the $(p+2)$-form field strengths can also be self- or anti-self-duals. When this is the case, the electric and magnetic charges must be equal up to a sign that depends on the self- or anti-self-duality and on the conventions used.

The second ingredient mentioned at the beginning of this section is a generic ansatz for the metric of charged, static, flat[2], black p-brane in $d = p + \tilde{p} + 4$ dimensions, with a transverse radial coordinate ρ chosen in such a way that the event horizon is at $\rho \to \infty$.

We have arrived at this ansatz[3] by studying (see the Appendix in Ref. [23]) the metrics of well-known families of p-brane solutions, such as those originally found in Ref. [28] and reproduced in Ref. [29] whose conventions and notations we follow here.

The ansatz depends on two independent functions of the radial (in the $(\tilde{p}+3)$-dimensional transverse space) coordinate ρ $\tilde{U}(\rho)$ and $W(\rho)$ to be found by solving the equations of motion and a "background" metric in the $(\tilde{p}+3)$-dimensional transverse space $\gamma_{(\tilde{p}+3)\,\underline{mn}}$ which has the fixed form

$$\gamma_{(\tilde{p}+3)\,\underline{mn}}dx^m dx^n = \left(\frac{\omega/2}{\sinh\left(\frac{\omega}{2}\rho\right)}\right)^{\frac{2}{\tilde{p}+1}} \left[\left(\frac{\omega/2}{\sinh\left(\frac{\omega}{2}\rho\right)}\right)^2 \frac{d\rho^2}{(\tilde{p}+1)^2} + d\Omega^2_{(\tilde{p}+2)}\right], \tag{4.7}$$

where, $d\Omega^2_{(\tilde{p}+2)}$ is the metric of the round $(\tilde{p}+2)$-sphere of unit radius and ω is the *non-extremality parameter*, denoted by r_0 in the 4-dimensional case considered in Refs. [19, 26].[4] Furthermore, the worldvolume of the p-brane is parametrized by the time coordinate t and the p spacelike coordinates $(y^1, \cdots, y^p)$ that we denote collectively by $\mathbf{y}_{(p)}$.

With all these elements, the generic metric takes the form

$$ds^2_{(d)} = e^{\frac{2}{p+1}\tilde{U}}\left[W^{\frac{p}{p+1}}dt^2 - W^{-\frac{1}{p+1}}d\mathbf{y}^2_{(p)}\right] - e^{-\frac{2}{\tilde{p}+1}\tilde{U}}\gamma_{(\tilde{p}+3)\,\underline{mn}}dx^m dx^n. \tag{4.8}$$

Some comments are in order. First, observe that this metric reduces in the $p = 0$ case to the metrics used in $d = 4$ and arbitrary d-dimensional black holes in Refs. [20, 26] respectively (W disappears and $\tilde{U}$ is just U in the notation used in those references). Secondly, observe that, for general values of p, we have two independent functions $\tilde{U}$ and W instead of just one, as in the black-hole case which should be recovered after dimensional reduction. the presence of W cannot be "gauged away": while it is possible to redefine $\tilde{U}$ and the transverse metric $\gamma_{(\tilde{p}+3)\,\underline{mn}}$ so as to totally absorb W in some components of the metric, it is not possible to do it simultaneously in all of them.

---

[2] By this we mean that the metric of the spatial part of its worldvolume is Euclidean. As we are going to see, the metric of the full worldvolume is not flat.

[3] This metric has also been derived from the equations of motion in Refs. [27].

[4] In higher dimensions ω is not a length, hence the change in notation.

Although the presence of one additional independent function is somewhat unexpected, is should be clear that there is nothing wrong with using it as long as we perform the reduction substituting the ansatz for the metric directly in the equations of motion. The reduced equations of motion will then tell us whether we have two independent functions or just one and what is the relation between them in the second case. We will also use normalization and regularity conditions to further constrain these functions.

The ansatz for $(p+1)$-form potentials $A^{\Lambda}_{(p+1)}$ for electrically-charged p-branes is

$$A^{\Lambda}_{(p+1)\,ty_1\cdots y_p} = \psi^{\Lambda}(\rho), \tag{4.9}$$

(all the other components vanish).

In the particular case $p = \tilde{p} = (d-4)/2$, in which the branes can also be magnetically charged with respect to the dual (*magnetic*) $(p+1)$-form potentials that we are going to denote by $A_{(p+1)\,\Lambda}$, we have to start by giving a proper definition of these dual potentials. The starting point are the equations of motion of the *electric* $(p+1)$-form potentials which are the only ones that appear in the original action Eq. (4.1). As mentioned before, in this particular case the action has to be supplemented by the term in Eq. (4.4). Taking all this into account the equations of motion can be written as

$$dG_{(p+2)\,\Lambda} = 0, \tag{4.10}$$

where the $(p+2)$-form $G_{(p+2)\,\Lambda}$ (*magnetic field strengths*) is defined by

$$G_{(p+2)\,\Lambda} \equiv R_{\Lambda\Sigma} F^{\Sigma}_{(p+2)} + I_{\Lambda\Sigma} \star F^{\Sigma}_{(p+2)}. \tag{4.11}$$

As it is well known, these differential equations imply the local existence of the magnetic $(p+1)$-form potentials $A_{(p+1)\,\Lambda}$ satisfying

$$G_{(p+2)\,\Lambda} = dA_{(p+1)\,\Lambda}. \tag{4.12}$$

Then, in this particular cases, we also make the following ansatz for the magnetic potentials

$$A_{(p+1)\,\Lambda\,ty_1\cdots y_p} = \chi_{\Lambda}(\rho). \tag{4.13}$$

This ansatz implies that some of the spatial components of the electric potentials $A^{\Lambda}_{(p+1)}$ do not vanish and, actually, have complicated dependencies on the angular coordinates of the transverse $(p+2)$ sphere. The magnetic potentials codify very efficiently these complicated dependencies and their use (the relevant spatial components of the electric potentials can be expressed quite easily in terms of the time component of the magnetic ones in the static case that we are considering) simplifies the reduction of the equations of motion.

It is convenient to arrange all the electric and magnetic $(p+2)$-form field strengths and electrostatic and magnetostatic potentials into single vectors whose components are labeled by $M, N, \ldots$

$$\left(\mathcal{F}^M\right) \equiv \begin{pmatrix} F^\Lambda \\ G_\Lambda \end{pmatrix}, \qquad \left(\Psi^M\right) \equiv \begin{pmatrix} \psi^\Lambda \\ \chi_\Lambda \end{pmatrix}. \tag{4.14}$$

In terms of the vector of field strengths so the Bianchi identities and Maxwell equations can be written as

$$d\mathcal{F}^M = 0, \tag{4.15}$$

which is covariant under linear transformations

$$\begin{pmatrix} F' \\ G' \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} F \\ G \end{pmatrix}, \tag{4.16}$$

where $A,\ B,\ C,\ D$ are constant matrices, but not all of them are consistent with the definition of the magnetic field strengths in terms of the electric ones Eq. (4.12). This definition must be preserved by the linear transformations if they are going to be symmetries of the equations of motion of the theory and this requires that the scalar-dependent matrices R, I transform according to

$$N' = (C + DN)\,(A + BN)^{-1}\,, \tag{4.17}$$

where we have defined the matrix N by

$$N \equiv R + \xi I. \tag{4.18}$$

In $d = 4$ $\xi = i$ and $N_{\Lambda\Sigma} \equiv \mathcal{N}_{\Lambda\Sigma}$ is the complex *period matrix.*

On the other hand, using the above-defined vectors, the contribution of the $(p+1)$-form potentials to the energy-momentum tensor can be written in the compact form

$$\Omega_{MN} \star \mathcal{F}^M{}_{\mu\alpha_1\cdots\alpha_{p+1}} \mathcal{F}^N{}_\nu{}^{\alpha_1\cdots\alpha_{p+1}}, \tag{4.19}$$

where

$$(\Omega_{MN}) \equiv \begin{pmatrix} 0 & \mathbb{I} \\ \xi^2\mathbb{I} & 0 \end{pmatrix}, \tag{4.20}$$

plays the rôle of a metric that we will use to raise and lower M, N indices. This implies that, in order to preserve the Einstein equations, the linear transformations of the n electric and n magnetic field strengths must be restricted to the group $\mathrm{O}(n, n)$ when $\xi^2 = +1$ and to the group $\mathrm{Sp}(2n+2, \mathbb{R})$ when $\xi^2 = -1$ (in particular, for $d = 4$ dimensions).

There is an alternative expression for this contribution to the energy-momentum tensor which turns out to be very useful when we perform the reduction of the Einstein equations with the above ansatz:

$$\mathcal{M}_{MN}(N)\mathcal{F}^M{}_{\mu\alpha_1\cdots\alpha_{p+1}} \mathcal{F}^N{}_\nu{}^{\alpha_1\cdots\alpha_{p+1}}, \tag{4.21}$$

where the symmetric matrix $\mathcal{M}_{MN}(N)$ is given by

$$
\begin{aligned}
(\mathcal{M}_{MN}(N)) &\equiv \begin{pmatrix} I - \xi^2 R I^{-1} R & \xi^2 R I^{-1} \\ -I^{-1} R & I^{-1} \end{pmatrix}, \\
\left(\mathcal{M}^{MN}(N)\right) &= \begin{pmatrix} I^{-1} & -\xi^2 I^{-1} R \\ R I^{-1} & I - \xi^2 R I^{-1} R \end{pmatrix} = (\mathcal{M}_{NP}(N))^{-1}.
\end{aligned} \tag{4.22}
$$

These formulae are only relevant in the particular case $p = \tilde{p} = (d-4)/2$. However, we can use them in any dimension including the additional terms (matrix $R_{\Lambda\Sigma}$, magnetic charges p^{Λ} etc.) in the understanding that they vanish whenever the condition is not satisfied (and $R_{\Lambda\Sigma} = p^{\Lambda} = 0$).

The last piece of our ansatz is the assumption that the scalar fields ϕ^i only depend on the radial coordinate ρ.

Plugging this ansatz into the equations of motion derived from the action Eq. (4.1) supplemented by the term in Eq. (4.4) we get five sets of equations for $\tilde{U}$, W, the potentials Ψ^M and the scalars ϕ^i. We consider first the following two equations:

$$\frac{d^2 \ln W}{d\rho^2} = 0, \tag{4.23}$$

$$\frac{d}{d\rho}\left[e^{-2\tilde{U}}\, \mathcal{M}_{MN}\, \dot{\Psi}^N\right] = 0. \tag{4.24}$$

(overdots denoting derivatives w.r.t. ρ) that can be integrated immediately. The result is, normalizing $W(0) = 1$ at spatial infinity, introducing the integration constants γ and $\mathcal{Q}_M$, and the normalization constant α

$$W = e^{\gamma\rho}, \tag{4.25}$$

$$\dot{\Psi}^M = \alpha e^{2\tilde{U}} \mathcal{M}^{MN} \mathcal{Q}_N. \tag{4.26}$$

The constants $\mathcal{Q}_M$ are, up to the global normalization represented by the constant α, just the electric and magnetic charges of the dyonic p-brane with respect to the $(p+1)$-form potentials

$$\mathcal{Q}_M \sim \int_{S^{\tilde{p}+2}} \star \mathcal{M}_{MN} \mathcal{F}^N, \quad (\mathcal{Q}^M) \equiv \begin{pmatrix} p^{\Lambda} \\ q_{\Lambda} \end{pmatrix}, \quad \mathcal{Q}_M \equiv \Omega_{MN} \mathcal{Q}^N. \tag{4.27}$$

With $W = e^{\gamma\rho}$ the metric ansatz takes the form

$$ds^2_{(d)} = e^{\frac{2}{p+1}\tilde{U}}\left[e^{\frac{p}{p+1}\gamma\rho} dt^2 - e^{-\frac{1}{p+1}\gamma\rho} d\mathbf{y}^2_{(p)}\right] - e^{-\frac{2}{\tilde{p}+1}\tilde{U}} \gamma_{(\tilde{p}+3)\,\underline{mn}} dx^m dx^n, \tag{4.28}$$

and only depends on one undetermined function, $\tilde{U}$, as expected. It, however, depends on two constants, ω and γ which are, a priori, independent. We only expect one constant in the metric (since we should be able to reduce it to a black-hole's) and, actually, we can eliminate one of them by requiring the regularity of the black brane's horizon.

Let us study the near-horizon limit of the above metric. In this limit, the angular part of the transverse metric behaves as

$$\sim e^{\frac{1}{\tilde{p}+1}\omega\rho}(-\omega)^{\frac{2}{\tilde{p}+1}}d\Omega^2_{(\tilde{p}+2)}, \tag{4.29}$$

which means in that black p-branes with regular horizons $\tilde{U}$ must behave as

$$\tilde{U} \sim C + \frac{\omega}{2}\rho, \tag{4.30}$$

for the angular part of the complete metric to be regular in that limit. Defining the entropy density by unit (world-) volume $\tilde{S}$ by

$$\tilde{S} \equiv \frac{A_{\mathrm{h}\,(\tilde{p}+2)}}{\omega_{(\tilde{p}+2)}}, \tag{4.31}$$

where $A_{\mathrm{h}\,(\tilde{p}+2)}$ is the volume of the $(\tilde{p}+2)$-dimensional constant worldvolume sections of the horizon and $\omega_{(\tilde{p}+2)}$ is the volume of the round $(\tilde{p}+2)$-sphere of unit radius[5] we find that the above behavior of $\tilde{U}$ leads to the entropy density

$$\tilde{S} = \left(-e^{-C}\omega\right)^{\frac{\tilde{p}+2}{\tilde{p}+1}}, \quad \Rightarrow \quad e^{C} = -\omega\tilde{S}^{-\frac{\tilde{p}+1}{\tilde{p}+2}}. \tag{4.32}$$

Then, in order for the worldvolume metric to be regular in this limit, $\tilde{U}$ and W must behave as[6]

$$e^{\tilde{U}} \sim (-\omega)\tilde{S}^{-\frac{\tilde{p}+1}{\tilde{p}+2}}e^{\frac{\omega}{2}\rho}, \quad W \sim e^{\omega\rho}, \tag{4.33}$$

where have chosen arbitrarily a normalization constant. Since we have just seen that $W = e^{\gamma\rho}$, we conclude that in black branes with regular horizons $\omega = \gamma$ and the general metric for regular p-branes is, therefore, given by

$$ds^2_{(d)} = e^{\frac{2}{\tilde{p}+1}\tilde{U}}\left[e^{\frac{p}{p+1}\omega\rho}dt^2 - e^{-\frac{1}{p+1}\omega\rho}d\mathbf{y}^2_{(p)}\right] - e^{-\frac{2}{\tilde{p}+1}\tilde{U}}\gamma_{(\tilde{p}+3)\,\underline{mn}}dx^m dx^n. \tag{4.34}$$

We can now consider the near-horizon limit of the time-radial part of general metric and find that it can always (for $\omega \neq 0$) can be brought into the Rindler-like form

[5] Not to be mistaken for the non-extremality parameter ω.

[6] This is true for $\omega \neq 0$. The near-horizon behavior for $\omega = 0$ is given by Eq. (4.46).

$$\sim e^{\frac{2}{p+1}C} \exp\left(-\frac{(\tilde{p}+1)e^{Cc}}{(-\omega)^{\frac{1}{\tilde{p}+1}}}\varrho\right)\left[dt^2 - d\varrho^2\right] = e^{-\frac{4\pi}{\beta}\rho}\left[dt^2 - d\varrho^2\right], \tag{4.35}$$

where

$$c \equiv \frac{d-2}{(p+1)(\tilde{p}+1)}, \tag{4.36}$$

and the inverse temperature is

$$\beta = \frac{4\pi(-\omega)^{\frac{1}{\tilde{p}+1}}}{(\tilde{p}+1)e^{Cc}}. \tag{4.37}$$

This result for the temperature and the above result for the entropy density lead to the following relation between them and the non-extremality parameter

$$(-\omega)^{\frac{1}{p+1}} = \frac{4\pi}{\tilde{p}+1} T \tilde{S}^{\frac{(d-2)}{(p+1)(\tilde{p}+2)}}, \tag{4.38}$$

which generalizes the relation obtained in Ref. [30] for 4-dimensional black holes and justifies in part the definition of the extremality parameter since it shows that ω will vanish whenever the brane's temperature vanishes if the entropy density does not diverge in this limit.

We can use the first integrals of the two equations of motion above to eliminate W and Ψ^M (which only occurs through $\dot{\Psi}^M$) from the remaining three equations of motion, which only involve the variables $\tilde{U}$ and ϕ^i and take the form:

$$\ddot{\tilde{U}} + e^{2\tilde{U}} V_{\mathrm{BB}} = 0, \tag{4.39}$$

$$\ddot{\phi}^i + \Gamma_{jk}{}^i \dot{\phi}^j \dot{\phi}^k + \frac{d-2}{2(\tilde{p}+1)(p+1)} e^{2\tilde{U}} \partial^i V_{\mathrm{BB}} = 0, \tag{4.40}$$

$$(\dot{\tilde{U}})^2 + \frac{(p+1)(\tilde{p}+1)}{d-2} \mathcal{G}_{ij} \dot{\phi}^i \dot{\phi}^j + e^{2\tilde{U}} V_{\mathrm{BB}} = (\omega/2)^2, \tag{4.41}$$

where $\Gamma_{jk}{}^i(\phi)$ are the components of the Levi-Civita connection of the scalar metric $\mathcal{G}_{ij}(\phi)$, we have defined the negative semidefinite *black-brane potential* (a generalization of the black-hole potential of Ref. [26])

$$V_{\mathrm{BB}}(\phi, \mathcal{Q}) \equiv 2\alpha^2 \frac{(p+1)(\tilde{p}+1)}{(d-2)} \mathcal{M}_{MN} \mathcal{Q}^M \mathcal{Q}^N. \tag{4.42}$$

The first two equations of motion can be derived from the effective action

$$\mathcal{I}[\tilde{U}, \phi^i] = \int d\rho \left\{(\dot{\tilde{U}})^2 + \frac{(p+1)(\tilde{p}+1)}{d-2} \mathcal{G}_{ij} \dot{\phi}^i \dot{\phi}^j - e^{2\tilde{U}} V_{\mathrm{BB}} + \hat{\mathcal{B}}^2\right\}. \tag{4.43}$$

The third equation is the Hamiltonian constraint (which follows from the ρ-independence of the Lagrangian) with a particular value for the integration constant related to the non-extremality parameter and the integration constant γ.

Let us summarize the results of this section. We have shown that we can use consistently the ansatz

$$\begin{aligned} ds^2_{(d)} &= e^{\frac{2}{p+1}\tilde{U}}\left[e^{\frac{p}{p+1}\omega\rho}dt^2 - e^{-\frac{1}{p+1}\omega\rho}d\mathbf{y}^2_{(p)}\right] - e^{-\frac{2}{\tilde{p}+1}\tilde{U}}\gamma_{(\tilde{p}+3)\,\underline{mn}}dx^m dx^n. \\ A^M_{(p+1)} &= \Psi^M(\rho)\, dt\wedge dy^1\wedge\cdots\wedge dy^p, \quad \dot{\Psi}^M = \alpha e^{2\tilde{U}}\mathcal{M}^{MN}\mathcal{Q}_N, \\ \phi^i &= \phi^i(\rho), \end{aligned} \tag{4.44}$$

where $\tilde{U}$ is a function of ρ; γ, $\mathcal{Q}_M$ are constants and $\gamma_{(\tilde{p}+3)\,\underline{mn}}$ is the transverse space metric given in Eq. (4.8) to describe flat, static, regular black-brane solutions of the theories defined by generic family of actions Eq. (4.1). We have also shown that the above ansatz gives these theories if Eqs. (4.39–4.41) are satisfied.[7]

4.2.2 FGK Theorems for Static Flat Branes

The formalism presented in the previous section can be used t derive generalizations of the results obtained in Refs. [20, 26] for black holes.

Let us first consider extremal black branes $\omega = 0$. The general form of their metrics can be obtained by taking the $\omega \longrightarrow 0$ limit of the general metric Eq. (4.34):

$$ds^2_{(d)} = e^{\frac{2\tilde{U}}{p+1}}\left[dt^2 - d\mathbf{y}^2_{(p)}\right] - \frac{e^{-\frac{2\tilde{U}}{\tilde{p}+1}}}{\rho^{\frac{2}{\tilde{p}+1}}}\left[\frac{1}{\rho^2}\frac{d\rho^2}{(\tilde{p}+1)^2} + d\Omega^2_{(\tilde{p}+2)}\right]. \tag{4.45}$$

For the near-horizon ($\rho\to\infty$) limit of this metric to be regular, $\tilde{U}$ must behave as

$$e^{\tilde{U}} \sim \tilde{S}^{-\frac{\tilde{p}+1}{\tilde{p}+2}}\rho^{-1}, \tag{4.46}$$

where $\tilde{S}$ is the entropy density per unit worldvolume defined in Eq. (4.31). Therefore, the near-horizon limit of Eq. (4.45) is the metric of the direct product $AdS_{p+2}\times S^{\tilde{p}+2}$, both with radii equal to $\tilde{S}^{\frac{1}{\tilde{p}+2}}$:

[7] The same result can be obtained by reducing first the action Eq. (4.1) to $(d-p) = (\tilde{p}+4)$ dimensions in such a way that the action only contains the Einstein-Hilbert term, scalars and 1-forms and then by using the FGK formalism of Ref. [20] for d-dimensional black holes ($p=0$). See the Appendix in Ref. [19].

$$ds^2_{(d)} = \rho^{\frac{-2}{p+1}} \tilde{S}^{-\frac{2(\tilde{p}+1)}{(p+1)(\tilde{p}+2)}} \left[dt^2 - d\mathbf{y}^2_{(p)}\right] - \tilde{S}^{\frac{2}{\tilde{p}+2}} \left[\frac{1}{\rho^2} \frac{d\rho^2}{(\tilde{p}+1)^2} + d\Omega^2_{(\tilde{p}+2)}\right]. \tag{4.47}$$

To make further progress we need to impose a regularity condition on the scalars which generalizes the one used in Ref. [26] for 4-dimensional black holes. We require that

$$\lim_{\rho\to\infty} \frac{(p+1)(\tilde{p}+1)}{d-2} \mathcal{G}_{ij} \dot{\phi}^i \dot{\phi}^j e^{2\tilde{U}} \rho^4 \equiv \mathcal{X} < \infty. \tag{4.48}$$

from which it follows that the near-horizon limit $\rho \to \infty$ of Eq. (4.41) (the Hamiltonian constraint) is

$$1 + \mathcal{X} \tilde{S}^{\frac{2(\tilde{p}+1)}{\tilde{p}+2}} + \tilde{S}^{-\frac{\tilde{p}+1}{\tilde{p}+2}} V_{\rm BB}(\phi_H, \mathcal{Q}) = 0. \tag{4.49}$$

Assuming that the near-horizon limit is regular, which implies that the entropy density $\tilde{S}$ does not vanish and that the values of the scalars on the horizon $\phi^i_{\rm h}$ do not diverge $\phi^i_{\rm h} < \infty$ so

$$\lim_{\rho\to\infty} \rho \frac{d\phi^i}{d\rho} = 0, \quad \forall i, \tag{4.50}$$

then it can be shown that

$$\mathcal{X} = 0, \tag{4.51}$$

and from Eqs. (4.49) and (4.51) we obtain

$$\tilde{S} = \left[-V_{\rm BB}(\phi_{\rm h}, \mathcal{Q})\right]^{\frac{\tilde{p}+2}{2(\tilde{p}+1)}}, \tag{4.52}$$

so the entropy of an extremal brane is given by (a power of) the value of the black-brane potential at the horizon.

Furthermore, under the same assumptions, we deduce, from the near-horizon limit of the equations of the scalars, that the values of the scalars on the horizon $\phi^i_{\rm h}$ are such that

$$\mathcal{G}^{ij}(\phi_{\rm h}) \partial_i V_{\rm BB}(\phi_{\rm h}, \mathcal{Q}) = 0, \tag{4.53}$$

and, if the metric of the scalar manifold $\mathcal{G}_{ij}$ is regular and the values of the scalars on the horizon are admissible so $\mathcal{G}_{ij}(\phi_{\rm h})$ is also regular, then

$$\partial_i V_{\rm BB}(\phi_{\rm h}, \mathcal{Q}) = 0, \tag{4.54}$$

which generalizes the usual attractor mechanism for static extremal black holes to the case of static extremal flat branes.

We would like to stress the fact that the black-brane potential on the horizon does not depend on the moduli (the asymptotic values of the scalars at spatial infinity) even if the values of the scalars on the horizon do (which is what happens in general.[8]

Finally, if we consider the Hamiltonian constraint Eq. (4.41) at spatial infinity ($\rho \to 0^+$) we obtain the generalization of the so-called extremality (or antigravity) bound for black holes

$$\tilde{u}^2 + \frac{(p+1)(\tilde{p}+1)}{d-2}\mathcal{G}_{ij}(\phi_\infty)\Sigma^i\Sigma^j + V_{BB}(\phi_\infty, \mathcal{Q}) = (\omega/2)^2, \tag{4.55}$$

where Σ^i are the scalar charges and we have defined the constant

$$\tilde{u} = -\tilde{U}'(0). \tag{4.56}$$

This constant is a combination of the black p-brane's tension T_p and the non-extremality parameter ω. The relation comes from the definition of the brane's tension:

$$T_p = -\frac{1}{(p+1)(\tilde{p}+2)}\left[(d-2)\tilde{u} + p(\tilde{p}+1)\omega/2\right], \tag{4.57}$$

Then, the brane's antigravity bound differs from the black hole's by terms proportional to $p\omega$ which vanish in the black-hole case $p = 0$.

4.2.3 Inner Horizons

The general metric Eq. (4.34) is designed to cover the exterior of the black brane's event horizon. In general, though, we expect charged black branes to have two horizons that will coincide in the extremal limit, as it happens for charged black holes. The inner horizon, which appears as another place at which the g_{tt} component vanishes, is not an event horizon. In the 4-dimensional Reissner-Nordström black hole, which is the best studied example, the inner horizon is actually a Cauchy horizon.

In Ref. [19] it was shown that, in the 4-dimensional black-hole case ($p = 0$) the same general metric covers the interior of the inner horizon (the region between the curvature singularity and the inner horizon) for the range of the radial coordinate[9]

[8] Only the supersymmetric attractors, that is, r the values of the scalars of supersymmetric black-brane solutions, are guaranteed to depend only on the charges. The general situation for extremal non-supersymmetric black branes is that the scalars on the horizon keep some dependence on their values at spatial infinity. This situation is sometimes referred to as the existence of a *moduli space of attractors* parametrized by some numbers whose physical meaning (i.e. their expressions in terms of the physical constants) is seldom given in the literature. These parameters are functions of the moduli, as shown explicitly in the $\overline{\mathbb{CP}}^n$ model studied in [19].

Of course, there are some moduli-independent non-supersymmetric attractors but it is important to realize that this is what happens in general.

[9] In that reference the coordinate used was $\tau = -\rho$.

$\rho \in (-\infty, \rho_{\rm sing})$, where $\rho_{\rm sing}$ denotes the location of the curvature singularity and the inner horizon is placed at $\rho = -\infty$.

In the 5-dimensional black-hole case studied in Ref. [20] it was observed that the general metric Eq. (4.34) is not well defined for negative values of ρ, from which it was concluded that the same metric could not cover the interior of the inner horizon. However, as it has been realized in Ref. [24], one can obtain from a metric of the form Eq. (4.34), regular for $\rho \in (0, +\infty)$ and covering the exterior of the black brane's event horizon, another metric by the simple transformation[10]

$$\rho \longrightarrow -\varrho, \qquad e^{-\tilde{U}(\rho)} \longrightarrow -e^{-U(-\varrho)}. \tag{4.58}$$

The new metric has the same general form, but describes the interior of the inner horizon for $\varrho \in (\varrho_{\rm sing}, +\infty)$. Observe that, if the original function $e^{-\tilde{U}}$ is always finite[11] for positive values of ρ, the transformed metric will generically hit a singularity before ϱ reaches 0 because, after the transformation, $e^{\tilde{U}}$ one will have a zero for some finite positive value of ϱ, as the explicit examples worked out in the references show.

The reasons to believe that the transformed metric is the metric that covers the interior of the horizon of the same black-brane spacetime are the same that make us believe that the region covered by standard Reissner-Nordström metric for $r < r_-$ corresponds to the interior of the black hole whose exterior is described by that metric for $r > r_+$. Since the Reissner-Nordström metric is singular at $r = r_+$ and $r = r_-$, the standard solution is actually giving us three different metrics which we interpret as covering three different regions of the same black-hole spacetime.

The upshot of this discussion is that the above transformation will allow us to compute the "temperature" and "entropy" of the inner horizon and check the *geometric mean property*. This property has been observed to hold for many different solutions and it has been proven for the charged, rotating, asymptotically flat or anti-De Sitter black-hole solutions of a wide class of theories in [31], following earlier work [32–36].[12] The property consists in the mass-independence (and moduli-independence, when there are scalars present in the theory) of the product of the "entropies" of all the horizons of a black-hole solution. In the asymptotically-flat cases that we are considering, in which the solutions usually have only two horizons, if we denote by $\tilde{S}_+$ the entropy density of the outer (event) horizon by $\tilde{S}_-$ the entropy density of the inner (Cauchy) horizon, the geometry mean property says that $\tilde{S}_+\tilde{S}_-$ is mass and moduli-independent, which means that it only depends on the electric and magnetic charges, which are quantized. This implies that the product only depends on integer numbers, which is a very suggestive property of entropy-related quantities.

[10] It would be stressed that this a transformation that relates solutions, and not a coordinate transformation.

[11] In the construction of the solution this is achieved by requiring the positivity of certain constants that appear in it, such as the mass or the entropy.

[12] A related result valid for horizons of arbitrary topology has been recently found in [37].

4.2.4 FGK Formalism for the Black Holes of $\mathcal{N} = 2,\ d = 5$ Theories

Let us see how the general formalism developed in the previous sections works in the particular case of the black-hole solutions of theories of $\mathcal{N} = 2,\ d = 5$ supergravity coupled to vector supermultiplets Refs. [38, 39]. We will use the conventions of Refs. [40, 41].

For black-hole solutions (which will only be electrically charged with respect to the vector fields) we can safely ignore the Chern-Simons term in the bosonic action and work with

$$\mathcal{I}[g_{\mu\nu}, A^{I}{}_{\mu}, \phi^{x}] = \int d^5x \left\{R + \tfrac{1}{2}g_{xy}\partial_{\mu}\phi^{x}\partial^{\mu}\phi^{y} - \tfrac{1}{8}a_{IJ}F^{I}{}_{\mu\nu}F^{J\,\mu\nu}\right\}, \tag{4.59}$$

where $I, J = 0, 1, \cdots, n$ and $x, y = 1, \cdots, n$. The scalar target spaces are determined by the existence of $n+1$ functions $h^{I}(\phi)$ of the n physical scalars ϕ^{x} subject to the constraint

$$C_{IJK}h^{I}h^{J}h^{K} = 1, \tag{4.60}$$

where C_{IJK} is a completely symmetric constant tensor that defines the model. These functions, like the vector fields themselves, transform linearly under the duality group, which must be embedded in SO$(n+1)$, while the physical scalars transform non-linearly, in general. This helps to make the symmetry manifest and it is the main reason why these objects (like the symplectic section of the theories of $\mathcal{N} = 2, d = 4$ supergravity coupled to vector supermultiplets) are introduced.

We also define

$$h_{I} \equiv C_{IJK}h^{J}h^{K} \quad (\text{so } h_{I}h^{I} \ = \ 1), \tag{4.61}$$

$$a_{IJ} \equiv -2C_{IJK}h^{K} + 3h_{I}h_{J}. \tag{4.62}$$

a_{IJ} can be used to raise and lower the index of the functions h^{I}, h_{I} and its derivatives

$$h^{I}{}_{x} \ \equiv \ -\sqrt{3}\partial_{x}h^{I}, \quad h_{Ix} \ \equiv \ a_{IJ}h^{J} \ = \ +\sqrt{3}\partial_{x}h_{I}. \tag{4.63}$$

These are orthogonal which are orthogonal to the h^{I} with respect to the metric a_{IJ}:

$$h^{I}h_{Ix} = h_{I}h^{I}{}_{x} = 0. \tag{4.64}$$

Finally, the target-space metric is related to the matrix a_{IJ} by

$$g_{xy} \ \equiv \ a_{IJ}h^{I}{}_{x}h^{J}{}_{y}, \tag{4.65}$$

which leads to

$$a^{IJ} = h^I h^J + g^{xy} h^I{}_x h^J{}_y. \tag{4.66}$$

The general FGK formalism constructed in the previous section leads, for this particular case and conventions to the general metric ($\tilde{U} \to U$)

$$ds^2 = e^{2U} dt^2 - e^{-U} \left(\frac{\omega/2}{\sinh\left(\frac{\omega}{2}\rho\right)} \right) \left[\left(\frac{\omega/2}{\sinh\left(\frac{\omega}{2}\rho\right)} \right)^2 \frac{d\rho^2}{4} + d\Omega^2_{(3)} \right], \tag{4.67}$$

the effective action

$$\mathcal{I}[U, \phi^x] = \int d\rho \left\{ (\dot{U})^2 + \tfrac{1}{3} g_{xy} \dot{\phi}^x \dot{\phi}^y - e^{2U} V_{\text{bh}} \right\}, \tag{4.68}$$

and the Hamiltonian constraint becomes

$$(\dot{U})^2 + \tfrac{1}{3} g_{xy} \dot{\phi}^x \dot{\phi}^y + e^{2U} V_{\text{bh}} = (\omega/2)^2, \tag{4.69}$$

where the black-brane potential (renamed here black-hole potential) is given by[13]

$$-V_{\text{bh}}(\phi, q) = a^{IJ} q_I q_J = \mathcal{Z}_{\text{e}}^2 + 3 g^{xy} \partial_x \mathcal{Z}_{\text{e}} \partial_y \mathcal{Z}_{\text{e}}, \tag{4.70}$$

where

$$\mathcal{Z}_{\text{e}}(\phi, q) \equiv h^I(\phi) q_I, \tag{4.71}$$

is the *(electric) black-hole central charge* and we have used Eq. (4.65).

A special feature of the FGK formalism for this and other supergravity theories is that the black-brane potential can be written in terms of central charges and that one can prove that the black-brane potential is extremized when the central charge is also extremized:

$$\partial_x \mathcal{Z}|_{\phi_{\text{h}}} = 0, \quad \Rightarrow \quad \partial_x V_{\text{bh}}|_{\phi_{\text{h}}} = 0. \tag{4.72}$$

The converse is not true. The extrema of the central charge are the supersymmetric attractors, the values towards which the scalar fields are attracted when we approach the horizon of supersymmetric extremal black holes.

In some cases the black-hole potential can be written in a similar fashion for other functions of the scalars and charges called *superpotentials* in the literature. the extremization of these superpotentials also leads to the extremization of the black-hole potential, but the extrema are not the supersymmetric attractors and it is not guaranteed that they will only be functions of the charges, as discussed before. Extremal non-supersymmetric black holes are related to these superpotentials, as we will discuss later.

[13] We have chosen, for convenience, the normalization $\alpha^2 = 3/32$.

4.2.5 FGK Formalism for the Black Strings of $\mathcal{N} = 2,\ d = 5$ Theories

The theories of $\mathcal{N} = 2,\ d = 5$ supergravity coupled to vector supermultiplets also admit black string solutions charged with respect to the 2-forms $B_{I\,\mu\nu}$ dual to the vector fields $A^I{}_\mu$. Due to the Chern-Simons term, it is not possible to dualize completely the action, replacing everywhere the vectors by the 2-forms. However, for purely magnetic (string) solutions, electrically charged only with respect to the 2-forms, the Chern-Simons term is, again, irrelevant, and one can work with the bosonic action

$$\mathcal{I} = \int \sqrt{g}\left\{R + \tfrac{1}{2}g_{xy}\partial_\mu\phi^x\partial^\mu\phi^y + \tfrac{1}{2\cdot 3!}a^{IJ}G_{I\,\mu\nu\kappa}G_J{}^{\mu\nu\kappa}\right\}, \tag{4.73}$$

where $G_I = dB_I$. Observe that the kinetic matrix is in this case the inverse of the kinetic matrix of the black-hole case.

The general formalism can be applied straightforwardly and we arrive to the general for of the metric for non-extremal black strings ($d = 5$, $p = 1$)

$$ds^2 = e^{\tilde{U}}\left[e^{\frac{\omega}{2}\rho}dt^2 - e^{-\frac{\omega}{2}\rho}dy^2\right] - e^{-2\tilde{U}}\left(\frac{\omega/2}{\sinh\left(\frac{\omega}{2}\rho\right)}\right)^2\left[\left(\frac{\omega/2}{\sinh\left(\frac{\omega}{2}\rho\right)}\right)^2 d\rho^2 + d\Omega^2_{(2)}\right], \tag{4.74}$$

to the effective action

$$\mathcal{I}[\tilde{U}, \phi^x] = \int d\rho\left\{(\dot{\tilde{U}})^2 + \tfrac{1}{3}g_{xy}\dot{\phi}^x\dot{\phi}^y - e^{2U}V_{\rm st}\right\}, \tag{4.75}$$

plus the Hamiltonian constraint

$$\dot{\tilde{U}}^2 + \tfrac{1}{3}g_{xy}\dot{\phi}^x\dot{\phi}^y + e^{2U}V_{\rm st} = (\omega/2)^2, \tag{4.76}$$

where we have defined the *black-string potential* as

$$-V_{\rm st}(\phi, p) \equiv a_{IJ}\,p^I p^J = \mathcal{Z}_{\rm m}^2 + 3\partial_x\mathcal{Z}_{\rm m}\partial^x\mathcal{Z}_{\rm m}. \tag{4.77}$$

Here we have introduced the *(magnetic) string central charge*

$$\mathcal{Z}_{\rm m}(\phi, p) = h_I(\phi)p^I, \tag{4.78}$$

which for supersymmetric extremal strings plays the same rôle as the electric one plays for supersymmetric extremal black holes.

4.2.6 FGK Formalism for $\mathcal{N} = 2,\ d = 4$ Theories

The black-hole solutions of the theories of ungauged $\mathcal{N} = 2,\ d = 4$ supergravity coupled to n vector supermultiplets[14] have been the ,most studied of all. As mentioned above, they can be electric and magnetically charged with respect to the $\bar{n} = n + 1$ vector fields $A^{\Lambda}{}_{\mu}$, $\Lambda = 0, 1, \ldots, n$, and the n complex scalars of these theories, denoted by Z^i, $i = 1, \ldots, n$, which parametrize a special Kähler manifold with Kähler metric $\mathcal{G}_{ij^*} = \partial_i \partial_{j^*} \mathcal{K}$, where $\mathcal{K}(Z, Z^*)$ is the Kähler potential, can have non-trivial profiles.

The bosonic action of these theories is always of the form

$$\begin{aligned} \mathcal{I} = \int d^4x \sqrt{|g|}\, \Big\{ & R + 2\mathcal{G}_{ij^*} \partial_\mu Z^i \partial^\mu Z^{*\,j^*} \\ & + 2\Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma} F^{\Lambda}{}_{\mu\nu} F^{\Sigma\,\mu\nu} - 2\Re\mathrm{e}\mathcal{N}_{\Lambda\Sigma} F^{\Lambda}{}_{\mu\nu} \star F^{\Sigma\,\mu\nu} \Big\}, \end{aligned} \tag{4.79}$$

where $\mathcal{N}_{\Lambda\Sigma}(Z, Z^*)$ is the period matrix mentioned before and which is related to the Kähler metric by the structure of special Kähler geometry.

The general form of the black-hole metrics of these theories is ($\tilde{U} \to U$)

$$ds^2 = e^{2U} - e^{-2U} \left(\frac{\omega/2}{\sinh\left(\frac{\omega}{2}\rho\right)} \right)^2 \left[\left(\frac{\omega/2}{\sinh\left(\frac{\omega}{2}\rho\right)} \right)^2 d\rho^2 + d\Omega^2_{(2)} \right], \tag{4.80}$$

the effective action takes the form

$$\mathcal{I}[U, Z^i] = \int d\rho \left\{ (U')^2 + \mathcal{G}_{ij^*} \dot{Z}^i \dot{Z}^{*\,j^*} - e^{2U} V_{\rm bh} \right\}, \tag{4.81}$$

and the Hamiltonian constraint is given by

$$(\dot{U})^2 + \mathcal{G}_{ij^*} \dot{Z}^i \dot{Z}^{*\,j^*} + e^{2U} V_{\rm bh} = (\omega/2)^2. \tag{4.82}$$

In these theories the black-hole potential takes the simple form

$$-V_{\rm bh}(Z, Z^*, \mathcal{Q}) = |\mathcal{Z}|^2 + \mathcal{G}^{ij^*} \mathcal{D}_i \mathcal{Z} \mathcal{D}_{j^*} \mathcal{Z}^*, \tag{4.83}$$

where

$$\mathcal{Z} = \mathcal{Z}(Z, Z^*, \mathcal{Q}) \equiv \langle \mathcal{V} \mid \mathcal{Q} \rangle = -\mathcal{V}^M \mathcal{Q}^N \Omega_{MN} = p^{\Lambda} \mathcal{M}_{\Lambda} - q_{\Lambda} \mathcal{L}^{\Lambda}, \tag{4.84}$$

[14] For more information on these theories see, for instance, Ref. [42], the review [43], and the original works [44, 45].

is the *central charge* of the theory, $(\mathcal{V}^M) = \begin{pmatrix} \mathcal{L}^\Lambda \\ \mathcal{M}_\Lambda \end{pmatrix}$ is the covariantly holomorphic symplectic section, $(\Omega_{MN}) = \begin{pmatrix} 0 & \\ - & 0 \end{pmatrix}$ is the symplectic metric, and

$$\mathcal{D}_i \mathcal{Z} = e^{-\mathcal{K}/2}\partial_i \left(e^{\mathcal{K}/2}\mathcal{Z}\right), \tag{4.85}$$

is the Kähler-covariant derivative.

The supersymmetric attractors of these theories extremize the absolute value of the central charge

$$\partial_i |\mathcal{Z}||_{Z_{\rm h}} = 0. \tag{4.86}$$

4.3 General Solutions and General Ansatzs

The general ansatzs that we are going to use to construct non-extremal black-hole solutions are based on the structure of the supersymmetric extremal ones which are been found in full generality for theories of $\mathcal{N} = 2,\ d = 4, 5$ supergravity coupled to vector supermultiplets using the method pioneered by Tod [46, 47]. Therefore, we are going to start by reviewing them.

4.3.1 General Supersymmetric Solutions

Black Holes of $\mathcal{N} = 2,\ d = 5$

All the supersymmetric solutions of the theories of ungauged $\mathcal{N} = 2,\ d = 5$ supergravity coupled to vector supermultiplets only[15] were found in Refs. [11, 12] We use here the conventions and prescription of Ref. [40], specializing it for the static case.

The supersymmetric, extremal, static black-hole solutions of these theories with n vector supermultiplets are constructed as follows:

1. With the metric function e^U and the scalar functions h_I we define the $\bar{n} = n + 1$ combinations
$$H_I(\rho) \equiv e^{-U} h_I. \tag{4.87}$$
2. These combinations are single-pole harmonic functions in the 4-dimensional transverse space of the general extremal metric Eq. (4.45) which we rewrite here

[15] These results have been extended to theories with hypermultiplets and tensor multiplets in Refs. [40, 48, 49] but these only include regular black-hole solutions when the additional fields vanish and, therefore, we will not consider them here.

for convenience[16] for $d = 5$, $p = 0$ ($\tilde{U} \to U$):

$$ds^2 = e^{2U} dt^2 - e^{-U} \frac{1}{\rho} \left[\frac{1}{4\rho^2} d\rho^2 + d\Omega^2_{(3)} \right]. \tag{4.88}$$

In other words: they are linear functions of the radial coordinate ρ:

$$H_I = A_I + B_I \rho, \tag{4.89}$$

for some constants A_I, B_I.

3. The solutions are completely determined by these harmonic functions. All the physical fields can be constructed in terms of them:

   (a) The vector field strengths are given by

   $$F^I = -\sqrt{3} d(e^U h^I) \wedge dt \tag{4.90}$$

   from which one can identify (in the same normalization we have chosen before) the coefficients B_I with the electric charges q_I:

   $$B_I = q_I. \tag{4.91}$$

   This identification is an important part of the structure of the supersymmetric extremal solutions of these theories.

   (b) The scalar fields can be written, for instance, in the form

   $$\phi^x = h_x / h_0 = H_x / H_0. \tag{4.92}$$

   (c) In order to write the metric in terms of the harmonic functions we first need to solve the (5-dimensional equivalent of the) *stabilization equations*, i.e. we need to find how to write the h^I in terms of the h_I and, therefore, in terms of the H_I and e^U. Then, the constraint $C_{IJK} h^I h^J h^K = 1$ gives a relation between e^U and the harmonic functions,

   $$e^U(H)$$

   which we will describe in detail when we review the H-FGK formalism.

4. The expressions of the physical fields can be use to determine completely the constants A_I in terms of their asymptotic values (basically only the moduli since the metric function is normalized to 1 at spatial infinity), as explained in Ref. [24].

[16] The change $\rho = r^{-2}$ brings the metric to the standard form.

Black Strings of $\mathcal{N} = 2,\ d = 5$

The supersymmetric, extremal, static black-string solutions of these theories with n vector supermultiplets are constructed following a very similar recipe [12, 48, 50]:

1. With the metric function e^U and the scalar functions h^I we define the $\bar{n} = n + 1$ combinations
$$K^I(\rho) \equiv e^{-U} h^I. \tag{4.93}$$
2. These combinations are single-pole harmonic functions in the 3-dimensional transverse space of the general extremal metric Eq. (4.45) which we rewrite here for convenience[17] for $d = 5$, $p = 1$ ($\tilde{U} \to U$):
$$ds^2 = e^U[dt^2 - dy^2] - e^{-2U}\frac{1}{\rho^2}\left[\frac{1}{\rho^2}d\rho^2 + d\Omega^2_{(2)}\right]. \tag{4.94}$$
In other words: again, they are linear functions of the radial coordinate ρ:
$$K^I = A^I + B^I\rho, \tag{4.95}$$
for some constants A^I, B^I.
3. The solutions are completely determined by these harmonic functions. All the physical fields can be constructed in terms of them:
   (a) The field strengths are given by
   $$F^I = \sqrt{3} \star_{(3)} dH^I. \tag{4.96}$$
   from which one can identify the coefficients B^I with the magnetic charges (the electric charges of the dual field strengths) p^I:
   $$B^I = p^I. \tag{4.97}$$
   Again, this identification is a feature of the supersymmetric extremal solutions.
   (b) The scalar fields can be written as in the black-hole case, which requires that we solve the stabilization equations, or we can use a different parametrization f the scalar manifold and write
   $$\phi^x = h^x/h^0 = K^x/K^0. \tag{4.98}$$
   (c) The metric function e^U is found by substituting the definition of the variables K^I in the constraint $C_{IJK}h^I h^J h^K = 1$, which yields

[17] The change $\rho = r^{-1}$ brings the metric to the standard form.

$$e^{-3U}(K) = C_{IJK} K^I K^J K^K. \tag{4.99}$$

4. The expressions of the physical fields can be use to determine completely the constants A^I in terms of their asymptotic values (basically only the moduli since the metric function is normalized to 1 at spatial infinity ($\rho = 0$)):

$$\phi^x_\infty = A^x/A^0, \quad e^{-3U}(A) = 1, \quad (A^0)^{-3} = e^{-3U}(A/A^0) = e^{-3U}(\phi_\infty), \tag{4.100}$$

where we defined, for convenience $\phi^0 \equiv 1$. Then

$$A^0 = e^U(\phi_\infty), \quad A^x = \phi^x_\infty e^U(\phi_\infty). \tag{4.101}$$

Black Holes of $\mathcal{N} = 2,\ d = 4$

All the timelike[18] supersymmetric solutions of the most general, gauged matter-coupled theories have been classified in Refs. [9, 14–16, 46, 51–54]. The supersymmetric extremal black holes of the ungauged theories[19] were constructed in [6–9] and we are going to give the recipe of Ref. [9] to construct the static ones: all the supersymmetric solutions of a theory of $\mathcal{N} = 2,\ d = 4$ supergravity coupled to vector supermultiplets and defined by the covariantly-holomorphic section $\mathcal{V}^M$ can be constructed as follows:

1. We introduce an auxiliary function of Kähler weight 1 (like $\mathcal{V}$) which, as we will see later (and we can safely ignore here) is related to the metric function e^U by $e^{2U} = 2|X|^2$.
2. We define the Kähler-neutral real symplectic vectors $\mathcal{R}^M$ and $\mathcal{I}^M$

$$\mathcal{R}^M + i\mathcal{I}^M \equiv \mathcal{V}^M/X. \tag{4.102}$$

   No Kähler gauge-fixing are necessary with this construction.
3. The components of $\mathcal{I}^M$ are real functions H^M which are single-pole harmonic functions in the 3-dimensional transverse space of the general extremal metric Eq. (4.45) which we rewrite here for convenience[20] for $d = 4$, $p = 0$ ($\tilde{U} \to U$):

$$ds^2 = e^{2U}dt^2 - e^{-2U}\frac{1}{\rho^2}\left[\frac{1}{\rho^2}d\rho^2 + d\Omega^2_{(2)}\right]. \tag{4.103}$$

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[18] These are the supersymmetric solutions such that the vector constructed as a bilinear from its Killing spinor is timelike. In particular, it is a timelike Killing vector. The other possible class is the null class. The supersymmetric extremal black-hole solutions belong to the timelike class.

[19] In the gauged theories there are asymptotically-AdS black holes [55] and also asymptotically-flat, regular black holes with non-Abelian hair [14–16], but here we are not going to consider these cases.

[20] The change $\rho = r^{-1}$ brings the metric to the standard form.

Yet again, they are linear functions of the radial coordinate ρ:

$$H^M = A^M + B^M \rho, \tag{4.104}$$

for some constants A^M, B^M.

4. In this case, the constants must satisfy the constraint

$$A^M B_M = \langle\, A \mid B \rangle = 0. \tag{4.105}$$

This constraint is equivalent to the requirement that there is no NUT charge [56]. A solution with NUT charge is, first of all, not static, and second of all, it would generically have either singularities or closed timelike curves.

5. The solutions are completely determined by these harmonic functions. All the physical fields can be constructed in terms of them. The construction requires finding the $\mathcal{R}^M$s in terms of the $\mathcal{I}^M$s, and, hence, of the harmonic functions H^M. This is always possible due to the redundancy of the description provided by $\mathcal{V}$ which implies the existence of relations between $\mathcal{R}$s and $\mathcal{I}$s known as *stabilization equations*. These h may be very difficult to solve in practice.

   (a) The vector field strengths are given by

   $$\mathcal{F}^M = -\sqrt{2} d(\mathcal{R}^M |X|^2) \wedge dt - \sqrt{2}|X|^2 \star (dt \wedge d\mathcal{I}^M), \tag{4.106}$$

   which allows us to identify the constants B^M with the electric and magnetic charges collected in the symplectic vector $\mathcal{Q}^M$:

   $$(B^M) = (\mathcal{Q}^M) = \begin{pmatrix} p^\Lambda \\ q_\Lambda \end{pmatrix}. \tag{4.107}$$

   This is a characteristic feature of the supersymmetric extremal solutions.

   (b) The physical scalars Z^i are given by the quotients

   $$Z^i = \frac{\mathcal{V}^i / X}{\mathcal{V}^0 / X} = \frac{\mathcal{R}^i + i\mathcal{I}^i}{\mathcal{R}^0 + i\mathcal{I}^0}. \tag{4.108}$$

   (c) The metric function is given by

   $$e^{-2U} = \frac{1}{2|X|^2} = \langle\, \mathcal{R} \mid \mathcal{I} \,\rangle. \tag{4.109}$$

6. The expressions of the physical fields can be use to determine completely the constants A^M in terms of their asymptotic values (basically only the moduli since the metric function is normalized to 1 at spatial infinity $\rho = 0$). The asymptotic conditions take the form

$$e^{2U}(A) = 1, \tag{4.110}$$

$$Z^i_\infty = \frac{\mathcal{R}^i(A) + iA^i}{\mathcal{R}^0(A) + iA^0}. \tag{4.111}$$

Now, let us write X as

$$X = \tfrac{1}{\sqrt{2}} e^{U+i\alpha}, \tag{4.112}$$

where α is some function. Then, the definition of $\mathcal{I}^M$ implies that

$$H^M = \sqrt{2} e^{-U} \Im\mathrm{m}(e^{-i\alpha}\mathcal{V}^M), \tag{4.113}$$

and, at spatial infinity $\rho = 0$, using the asymptotic flatness conditions Eq. (4.110), we find

$$A^M = \sqrt{2}\,\Im\mathrm{m}(e^{-i\alpha_\infty}\mathcal{V}^M_\infty). \tag{4.114}$$

α_∞ can be found using Eq. (4.105) and the definition of the central charge Eq. (4.84). Observe that

$$A_M B^M = \langle\, H \mid B\,\rangle = \Im\mathrm{m}\langle\, \mathcal{V}/X \mid B\,\rangle = \Im\mathrm{m}(\tilde{\mathcal{Z}}/X) = e^{-U}\Im\mathrm{m}(e^{-i\alpha}\tilde{\mathcal{Z}}) = 0. \tag{4.115}$$

Then

$$e^{i\alpha} = \pm\tilde{\mathcal{Z}}/|\tilde{\mathcal{Z}}|, \tag{4.116}$$

and the general expression of the constants A^M as functions of the charges $\mathcal{Q}^M$ and the asymptotic values of the scalar fields Z^i_∞ is

$$A^M = \pm\sqrt{2}\,\Im\mathrm{m}\left(\frac{\mathcal{Z}^*}{|\mathcal{Z}|}\mathcal{V}^M_\infty\right). \tag{4.117}$$

It can be seen that only the upper sign gives a positive value of the mass and a regular black-hole metric.

4.3.2 General Ansatzs for Non-extremal Solutions

In the previous section we have seen how, the supersymmetric extremal black-hole and black-string solutions of $\mathcal{N} = 2,\ d = 4, 5$ theories can be constructed by following a simple recipe. The main ingredient in the recipe is the expression of the physical fields (the metric function e^U and the scalar fields Z^i in $d = 4$ and ϕ^x in $d = 5$, which are the only fields that need to be determined in the FGK formalism) in terms of some functions H^M, H_I and K^I in the different theories and cases. In the supersymmetric solutions these functions are linear in the radial coordinate ρ.

Based on these recipes we can make the following ansatz for the non-extremal solutions: the physical fields are given by the same expressions in terms of the functions H^M, H_I and K^I as in the supersymmetric case but, now, these function are no longer linear in ρ. It seems that in all cases [19, 20, 24] these functions are linear combinations of hyperbolic sines and cosines of $\frac{\omega}{2}\rho$:

$$H^M = A^M \cosh\left(\frac{\omega}{2}\rho\right) + \frac{2B^M}{\omega} \sinh\left(\frac{\omega}{2}\rho\right), \tag{4.118}$$

etc.

We are assuming that there is a universal way to express the physical fields of this kind of solutions in terms of the variables H^M, H_I and K^I, and this probably needs some justification, beyond the examples for which this seems to be the case.[21] It can be argued that the duality-invariance of e^U and the duality-covariance of the scalars can only be achieved by very specific combinations of functions and that we roughly expect as many independent functions as electric and magnetic charges can be carried by the black objects. There is, however, a better argument: for the cases considered, the functions H^M, H_I and K^I can be used as independent variables in the FGK formalism. In other words: the general expressions for e^U and the scalars as functions of the H^M, H_I or K^I can always be used to change the variables in the FGK effective action and Hamiltonian constraints. In this way, one gets an equivalent formulation of the FGK system in which the fundamental variables are the functions H^M, H_I or K^I that we have called *H-FGK formalism* [22].[22] Solving the new equations of motion and Hamiltonian constraint for the new variables one can reconstruct the physical fields using always the same expressions.

This proves the first assumption. As for the second assumption in our ansatz (the hyperbolicity of the functions H^M, H_I and K^I in the non-extremal cases), there is no complete proof, although in the H-FGK formalism it arises as a most natural possibility.

In the next section we review the H-FGK formalism.

4.4 A Better Framework: The H-FGK Formalism

4.4.1 For the Black-Hole Solutions of $\mathcal{N} = 2,\ d = 5$

Here we are going to show how the metric function e^U and the n real scalars ϕ^x can be replaced in the FGK action by the $\bar{n} = n + 1$ variables denoted by H_I. We will also need to define $\bar{n}$ dual variables $\tilde{H}^I$ for intermediate calculations.

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[21] Apart from the examples studied in Refs. [19, 20], the assumption is true in all the supersymmetric solutions of $\mathcal{N} = 2,\ d = 4, 5$ theories, for all matter couplings and gaugings.

[22] A similar, more general, formalism that reduces to the H-FGK one for single, static, spherically-symmetric black holes of $\mathcal{N} = 2,\ d = 4, 5$ has been given in Refs. [17, 18, 21]. The $\mathcal{N} = 2,\ d = 5$ string case has not been treated with this method.

A very important ingredient of the ensuing calculations will be the homogeneity of the functions that occur in the supergravity theories and in the formalism. To start with, we define $\mathcal{V}(h^{\cdot})$, homogeneous of third degree in the h^Is

$$\mathcal{V}(h^{\cdot}) \equiv C_{IJK} h^I(\phi) h^J(\phi) h^K(\phi). \tag{4.119}$$

This function defines the scalar manifold as the hypersurface $\mathcal{V} = 1$. The dual scalar functions h_I, defined in Eq. (4.61) can also be defined by

$$h_I(h^{\cdot}) \equiv \tfrac{1}{3} \frac{\partial \mathcal{V}}{\partial h^I}. \tag{4.120}$$

They are, obviously, homogenous of second degree in the h^I. This relation can be inverted to express the h^I as functions of the h_I, $h^I(h_{\cdot})$ (finding this relations is the same as solving the *stabilization equations*). It is evident that $h^I(h_{\cdot})$ is homogeneous of degree 1/2 which implies that, in its turn, $\mathcal{V}(h_{\cdot})$ is homogeneous of degree 3/2.

It is useful to define the Legendre transform of $\mathcal{V}(h^{\cdot})$ $\mathcal{W}(h_{\cdot})$ by

$$\mathcal{W}(h_{\cdot}) \;\equiv\; 3 h_I h^I(h_{\cdot}) - \mathcal{V}(h^{\cdot}) = 2\mathcal{V}[h^{\cdot}(h_{\cdot})], \tag{4.121}$$

which is homogenous of degree 3/2. From the standard properties of the Legendre transform we get

$$h^I \;\equiv\; \tfrac{1}{3} \frac{\partial \mathcal{W}}{\partial h_I}. \tag{4.122}$$

The next step in this construction is the introduction of two sets of variables H_I and $\tilde{H}^I$ which are related to the physical fields (U, ϕ^x) by

$$H_I \equiv e^{-U} h_I(\phi), \tag{4.123}$$

$$\tilde{H}^I \equiv e^{-U/2} h^I(\phi), \tag{4.124}$$

and two new functions V and W, which have the same form in the new variables as $\mathcal{V}$ and $\mathcal{W}$ had in the old ones, that is

$$\mathsf{V}(\tilde{H}) \equiv C_{IJK} \tilde{H}^I \tilde{H}^J \tilde{H}^K, \tag{4.125}$$

$$\mathsf{W}(H) \equiv 3\tilde{H}^I H_I \;-\; \mathsf{V}(\tilde{H}) \;=\; 2\mathsf{V}. \tag{4.126}$$

These functions are not constrained as $\mathcal{V}$ and $\mathcal{W}$ are.

The properties that we proved for $\mathcal{V}$ and $\mathcal{W}$ (in particular, the homogeneity properties) implies the following properties for $\mathcal{V}$ and $\mathcal{W}$:

$$H_I \equiv \tfrac{1}{3} \frac{\partial \mathsf{V}}{\partial \tilde{H}^I}, \tag{4.127}$$

$$\tilde{H}^I \equiv \tfrac{1}{3}\frac{\partial \mathsf{W}}{\partial H_I} \equiv \tfrac{1}{3}\,\partial^I \mathsf{W}, \tag{4.128}$$

$$e^{-\frac{3}{2}U} = \tfrac{1}{2}\mathsf{W}(H), \tag{4.129}$$

$$h_I = (\mathsf{W}/2)^{-2/3} H_I, \tag{4.130}$$

$$h^I = (\mathsf{W}/2)^{-1/3} \tilde{H}^I. \tag{4.131}$$

Having defined the $\bar{n}$ variables H_I in terms of the metric function e^U and the n scalar fields ϕ^x (through h_I), we can view the above formulae Eqs. (4.129) and (4.130) as the inverse relations and we can use these relations and the rest of the auxiliary formulae to rewrite the FGK action Eq. (4.68) in terms of the new variables H^I.

First, we rewrite that action in the equivalent form

$$\mathcal{I}_{\text{FGK}}[U, \phi^x] = \int d\rho \left\{(\dot{U})^2 + a^{IJ}\dot{h}_I\dot{h}_J + e^{2U}a^{IJ}q_I q_J\right\}, \tag{4.132}$$

so that we only need to express U, h_I and a^{IJ} in terms of the new variables. For U and h_I this is, trivial, using the above formulae. For the inverse metric a^{IJ} one can show that the relation between a^{IJ} and the new variables is

$$a^{IJ} = -\tfrac{2}{3}\,(\mathsf{W}/2)^{4/3}\,\partial^I\partial^J \log \mathsf{W}, \tag{4.133}$$

and, therefore, after the change of variables, the effective FGK action becomes

$$-\tfrac{3}{2}\mathcal{I}_{\text{H}-\text{FGK}}[H] = \int d\rho \left\{\partial^I\partial^J \log\,\mathsf{W}\left(\dot{H}_I\dot{H}_J + q_I q_J\right)\right\}, \tag{4.134}$$

while the Hamiltonian constraint becomes

$$\mathcal{H} \equiv \partial^I\partial^J \log\,\mathsf{W}\left(\dot{H}_I\dot{H}_J - q_I q_J\right) = -\tfrac{3}{2}(\omega/2)^2. \tag{4.135}$$

Observe that $\partial^I\partial^J \log \mathrm{W}$ plays the role of a metric in a σ-model with coordinates H_I. Manifolds whose metrics can be written as the Hessian of a function are called *Hessian manifolds* and the function ($\log \mathsf{W}$ in this case) is known as *Hessian potential*. The problem of finding black-hole solutions becomes, thus, a mechanical problem on a Hessian manifold.

The equations of motion derived from the effective action (4.134) are

$$\partial^K\partial^I\partial^J \log\,\mathsf{W}\left(\dot{H}_I\dot{H}_J - q_I q_J\right) + 2\partial^K\partial^I \log \mathsf{W}\,\ddot{H}_I = 0. \tag{4.136}$$

Multiplying by H_K and using the homogeneity properties of W and the Hamiltonian constraint we get

$$\partial^I \log \mathsf{W}\, \ddot{H}_I \;=\; \tfrac{3}{2}(\omega/2)^2. \tag{4.137}$$

This is just the equation of motion of U after the change of variables.

Observe that in the extremal case $\mathcal{B} = 0$, the equations of motion can be always satisfied by harmonic functions $\dot{H}_I = q_I$. This proves that the supersymmetric configurations constructed according to the recipe give in previous sections are always solutions of the equations of motion.

On the other hand, observe that, since W is homogenous of degree $3/2$ on the H_I

$$H_I \partial^I \log \mathsf{W} = \frac{3}{2}, \tag{4.138}$$

we can rewrite the Eq. (4.137) in the form

$$\partial^I \log \mathsf{W} \left[\ddot{H}_I - (\omega/2)^2 H_I\right], \tag{4.139}$$

which is generically solved by functions H_I satisfying

$$\ddot{H}_I - (\omega/2)^2 H_I = 0, \tag{4.140}$$

that is: by linear combinations of hyperbolic sines and cosines of $\frac{\omega}{2}\rho$. Thus justifies our ansatz for non-extremal black holes.

The application of this formalism to extremal non-supersymmetric and non-extremal black-hole solutions has been studied in detail in Ref. ([22]) showing the power of this formalism to obtain general results concerning the entropy, first-order flow equations for extremal and non-extremal black holes etc.

4.4.2 For the Black-String Solutions of $\mathcal{N} = 2,\ d = 5$

An analogous formalism can be developed for string-like solutions, taking into account that, even though we are in the same theory, we are interested in different solutions which are naturally given in terms of different variables: the functions K^I [12, 48, 50], related to the $h^I(\phi)$. We will also introduce auxiliary dual functions $\tilde{K}_I$.

The new variables are defined by

$$K^I \equiv e^{-U} h^I(\phi), \tag{4.141}$$

and we also define the function

$$\mathsf{V}(K) \equiv C_{IJK} K^I K^J K^K, \tag{4.142}$$

which is homogenous of third degree on the K^I. The equation that defines the scalar manifold implies that the metric function is related to the new variables by

$$e^{-3U} = \mathsf{V}(K). \tag{4.143}$$

The dual variables $\tilde{K}_I$ can be defined either by

$$\tilde{K}_I \equiv e^{-2U} h_I(\phi) \tag{4.144}$$

or by

$$\tilde{K}_I \equiv \tfrac{1}{3}\partial_I \mathsf{V}(K). \tag{4.145}$$

Following essentially the same steps as in the black-hole case, we arrive to the H-FGK action

$$-3\,\mathcal{I}_{\text{H-FGK}}[K] = \int d\rho \left\{\partial_I\partial_J \mathsf{V}\left(\dot{K}^I \dot{K}^J + p^I p^J\right)\right\}, \tag{4.146}$$

and the Hamiltonian constraint

$$\mathcal{H} \equiv \partial_I\partial_J \mathsf{V}\left(\dot{K}^I \dot{K}^J - p^I p^J\right) = -3(\omega/2)^2. \tag{4.147}$$

The equations of motion that follow from the H-FGK action are

$$\partial_I\partial_K\partial_L \mathsf{V}\left(\dot{K}^K \dot{K}^L - K^K \ddot{K}^L - p^K p^L\right) = 0. \tag{4.148}$$

Contracting these equations with K^I one gets

$$\ddot{K}^I \partial_I \log \mathsf{V} = 3(\omega/2)^2, \tag{4.149}$$

which can be rewritten in the form

$$\partial_I \mathsf{V}\left[\ddot{K}^I - (\omega/2)^2 K^I\right] = 0, \tag{4.150}$$

which is, again, solved generically by linear combinations of hyperbolic sines and cosines of $\frac{\omega}{2}\rho$.

4.4.3 $\mathcal{N} = 2,\ d = 4$

The 4-dimensional case is more complicated. To start with, there is a mismatch between the number of original variables in the FGK formalism: e^U and the Z^i represent $2n + 1$ real degrees of freedom and the variables of the H-FGK formalism

H^M are $2n + 2$. This should not be a problem, because we can always perform a change of variables that increases the number of variables, since the change will introduce constraints in the system. However, defining the change of variables will be more complicated. It is convenient to start with the complex variable of Kähler weight one (as the covariantly holomorphic symplectic section)

$$X = \tfrac{1}{\sqrt{2}} e^{U+i\alpha}, \tag{4.151}$$

where the phase α is a variable that does not occur in the original FGK formalism.

As in the 5-dimensional case, the homogeneity properties of the functions that appear in the supergravity theory are essential in this construction. They are simpler to find if we assume that the theory is specified by the *prepotential* $\mathcal{F}$ which is a homogeneous function of second degree in the complex coordinates $\mathcal{X}^\Lambda$. Defining

$$\mathcal{F}_\Lambda \equiv \frac{\partial \mathcal{F}}{\partial \mathcal{X}^\Lambda}, \quad \mathcal{F}_{\Lambda\Sigma} \equiv \frac{\partial^2 \mathcal{F}}{\partial \mathcal{X}^\Lambda \partial \mathcal{X}^\Sigma}, \tag{4.152}$$

we find that

$$\mathcal{F}_\Lambda = \mathcal{F}_{\Lambda\Sigma} \mathcal{X}^\Sigma. \tag{4.153}$$

The coordinates $\mathcal{X}^\Lambda$ and the dual coordinates $\mathcal{F}_\Lambda$ are related to the components of t he covariantly holomorphic section by

$$\left(\mathcal{V}^M\right) = \begin{pmatrix} \mathcal{L}^\Lambda \\ \mathcal{M}_\Lambda \end{pmatrix} = e^{\mathcal{K}/2} \begin{pmatrix} \mathcal{X}^\Lambda \\ \mathcal{F}_\Lambda \end{pmatrix}, \tag{4.154}$$

where $\mathcal{K}$ is the Kähler potential. Then, the above relation implies this relation between the components of $\mathcal{V}^M$ (dividing by X):

$$\frac{\mathcal{M}_\Lambda}{X} = \mathcal{F}_{\Lambda\Sigma} \frac{\mathcal{L}^\Sigma}{X}. \tag{4.155}$$

Splitting this relation into its real and imaginary parts and using the definitions Eq. (4.102) we get

$$\mathcal{R}_M = -\mathcal{M}_{MN}(\mathcal{F}) \mathcal{I}^N, \tag{4.156}$$

where the $2\bar{n} \times 2\bar{n}$ symmetric symplectic matrix $\mathcal{M}_{MN}(\mathcal{A})$ is defined for any complex symmetric $\bar{n} \times \bar{n}$ matrix $\mathcal{A}_{\Lambda\Sigma}$ with non-degenerate imaginary part by

$$\mathcal{M}(\mathcal{A}) \equiv \begin{pmatrix} \Im\mathfrak{m}\, \mathcal{A}_{\Lambda\Sigma} + \Re\mathfrak{e}\, \mathcal{A}_{\Lambda\Omega}\, \Im\mathfrak{m}\, \mathcal{A}^{-1|\,\Omega\Gamma}\, \Re\mathfrak{e}\, \mathcal{A}_{\Gamma\Sigma} & -\Re\mathfrak{e}\, \mathcal{A}_{\Lambda\Omega}\, \Im\mathfrak{m}\, \mathcal{A}^{-1|\,\Omega\Sigma} \\ -\Im\mathfrak{m}\, \mathcal{A}^{-1|\,\Lambda\Omega}\, \Re\mathfrak{e}\, \mathcal{A}_{\Omega\Sigma} & \Im\mathfrak{m}\, \mathcal{A}^{-1|\,\Lambda\Sigma} \end{pmatrix}. \tag{4.157}$$

In the above expression $\mathcal{A}_{\Lambda\Sigma} = \mathcal{F}_{\Lambda\Sigma}$. Later on we will use the matrix $\mathcal{M}_{MN}(\mathcal{N})$ where $\mathcal{N}_{\Lambda\Sigma}$ is the period matrix. Both matrices are related by[23]

$$-\mathcal{M}_{MN}(\mathcal{N}) = \mathcal{M}_{MN}(\mathcal{F}) + 4\mathcal{V}_{(M}\mathcal{V}^*_{N)}. \tag{4.158}$$

The inverse of $\mathcal{M}_{MN}$, denoted by $\mathcal{M}^{MN}$, can be obtained by raising the indices with the inverse symplectic metric.

It is also immediate to prove the relation

$$d\mathcal{R}_M = -\mathcal{M}_{MN}(\mathcal{F})\, d\mathcal{I}^N, \tag{4.159}$$

from which one can derive the following relation between partial derivatives [56]:

$$\frac{\partial \mathcal{I}^M}{\partial \mathcal{R}_N} = \frac{\partial \mathcal{I}^N}{\partial \mathcal{R}_M} = -\frac{\partial \mathcal{R}^M}{\partial \mathcal{I}_N} = -\frac{\partial \mathcal{R}^N}{\partial \mathcal{I}_M} = -\mathcal{M}^{MN}(\mathcal{F}). \tag{4.160}$$

We are now ready to introduce two dual sets of variables H^M and $\tilde{H}_M$ and replace the original $\bar{n}$ complex fields X, Z^i by the $2\bar{n}$ real variables H^M:

$$H^M \equiv \mathcal{I}^M(X, Z, X^*, Z^*), \tag{4.161}$$

$$\tilde{H}_M \equiv \mathcal{R}^M(H). \tag{4.162}$$

Observe that the definition of the dual variables $\tilde{H}_M(H)$ implies that the stabilization equations have been solved. Knowing both sets of variables, we can reconstruct the physical fields :

$$e^{-2U} = \frac{1}{2|X|^2} = \mathcal{R}_M\mathcal{I}^M, \tag{4.163}$$

$$Z^i = \frac{\tilde{H}^i(H) + iH^i}{\tilde{H}^0(H) + iH^0}. \tag{4.164}$$

The phase of X (α) can be found[24] by solving the differential equation (*cf.* Eqs. (3.8), (3.28) in Ref. [58])

$$\dot{\alpha} = 2|X|^2\dot{H}^M H_M - \mathcal{Q}_\star, \tag{4.165}$$

where

$$\mathcal{Q}_\star = \tfrac{1}{2i}\dot{Z}^i\partial_i\mathcal{K} + \text{c.c.}, \tag{4.166}$$

is the pullback of the Kähler connection 1-form.

[23] This relation can be derived from the identities in Ref. [57].

[24] Observe that we do not really need it, since it does not appear in the original FGK action anyway.

We are now almost ready to perform the change of variables in the FGK action. First, we need to introduce the function $\mathsf{W}(H)$

$$\mathsf{W}(H) \equiv \tilde{H}_M(H)H^M = e^{-2U} = \frac{1}{2|X|^2}, \tag{4.167}$$

which is homogenous of second degree in the H^M. Using the properties (4.160) one can show that

$$\partial_M \mathsf{W} \equiv \frac{\partial \mathsf{W}}{\partial H^M} = 2\tilde{H}_M, \tag{4.168}$$

$$\partial^M \mathsf{W} \equiv \frac{\partial \mathsf{W}}{\partial \tilde{H}_M} = 2H^M, \tag{4.169}$$

$$\partial_M \partial_N \mathsf{W} = -2\mathcal{M}_{MN}(\mathcal{F}), \tag{4.170}$$

$$\mathsf{W}\partial_M \partial_N \log \mathsf{W} = 2\mathcal{M}_{MN}(\mathcal{N}) + 4\mathsf{W}^{-1} H_M H_N, \tag{4.171}$$

where the last property is based on Eq. (4.158).

We also need the special geometry identity

$$\mathcal{G}_{ij^*} = -i\mathcal{D}_i \mathcal{V}_M \mathcal{D}_{j^*} \mathcal{V}^{*\,M} \tag{4.172}$$

to deal with the scalars' kinetic term.

Using all these results, after some work, we can rewrite the FGK effective action in the form

$$-I_{\text{H-FGK}}[H] = \int d\tau \left\{ \tfrac{1}{2}\partial_M \partial_N \log \mathsf{W} \left(\dot{H}^M \dot{H}^N + \tfrac{1}{2}\mathcal{Q}^M \mathcal{Q}^N \right) - \Lambda \right\}, \tag{4.173}$$

where we have defined

$$\Lambda \equiv \left(\frac{\dot{H}^M H_M}{\mathsf{W}} \right)^2 + \left(\frac{\mathcal{Q}^M H_M}{\mathsf{W}} \right)^2, \tag{4.174}$$

and the Hamiltonian constraint in the form

$$\mathcal{H} \equiv -\tfrac{1}{2}\partial_M \partial_N \log \mathsf{W} \left[\dot{H}^M \dot{H}^N - \tfrac{1}{2}\mathcal{Q}^M \mathcal{Q}^N \right] + \left(\frac{\dot{H}^M H_M}{\mathsf{W}} \right)^2 - \left(\frac{\mathcal{Q}^M H_M}{\mathsf{W}} \right)^2 = r_0^2, \tag{4.175}$$

where we are using the more conventional form of the non-extremality parameter $r_0 = \omega/2$ in $d = 4$.

The equations of motion for the H^P can be written in the form

$$\tfrac{1}{2}\partial_P\partial_M\partial_N \log \mathsf{W}\, \left[\dot{H}^M\dot{H}^N - \tfrac{1}{2}\mathcal{Q}^M\mathcal{Q}^N\right] + \partial_P\partial_M \log \mathsf{W}\, \ddot{H}^M - \frac{d}{d\tau}\left(\frac{\partial \Lambda}{\partial \dot{H}^P}\right) + \frac{\partial \Lambda}{\partial H^P} = 0. \tag{4.176}$$

Contracting these equations with H^P and using the homogeneity properties of the different terms as well as the Hamiltonian constraint above, we find the equation (*cf.* Eq. (3.31) of Ref. [58] for the stationary extremal case)

$$\tfrac{1}{2}\partial_M \log \mathsf{W} \left(\ddot{H}^M - r_0^2 H^M\right) + \left(\frac{\dot{H}^M H_M}{\mathsf{W}}\right)^2 = 0, \tag{4.177}$$

which corresponds to the equation of motion of the variable U in the standard FGK formulation.

If we impose the constraint

$$\dot{H}^M H_M = 0, \tag{4.178}$$

which implies the absence of NUT charge in the supersymmetric extremal case, we find that the above equation is solved, quite naturally, by H^Ms which are linear combinations of hyperbolic sines and cosines of $r_0\rho$. Furthermore, in the extremal case ($r_0 = 0$) the equations of motion are solved by linear functions of ρ such that $\dot{H}^M = \mathcal{Q}^M$ [56]. We recover, in this way, the supersymmetric extremal functions reviewed before. A more general study of the extremal non-supersymmetric and non-extremal solutions will be presented elsewhere [25].

4.5 Conclusions

As promised in the introduction, we have constructed a formalism that justifies the general ansatzs proposed in Refs. [19, 20] to find non-extremal black-hole and black-string solutions in theories of ungauged $\mathcal{N} = 2$, $d = 4, 5$ supergravity coupled to vector supermultiplets. The formalism turns out to be most useful in the study of general classes of solutions [24, 25] and, to a certain extent, closes the problem of finding the most general static, spherically symmetric black-hole and black-string solutions of those theories. At this point, the use of this formalism to find solutions of complicated theories that have resisted other methods remains a challenge, since more of the examples studied so far correspond to simple theories.

The extension of this formalism to handle Abelian gaugings via Fayet-Iliopoulos terms is straightforward and will be studied in [25].[25]

How about other 4- and 5-dimensional supergravities? It might seem that, since they are quite different (in particular the relations between the numbers of scalar fields and the possible electric and magnetic charges) an H-FGK formulation is

[25] It has also been studied via the analogous method mentioned before in Ref. [59].

simply not possible. However, the general form of the black-hole solutions of all the 4-dimensional supergravities is known [13] and structures similar to those of the $\mathcal{N} = 2$ case arise quite naturally there and these similarities have been recently used to construct the metrics and the vector field strengths of the supersymmetric, extremal, (single- or multi-center) black holes of $\mathcal{N} = 8,\ d = 4$ supergravity have been constructed in terms of a set of harmonic functions H^M [60]. It is not known how to construct explicitly the scalar fields, though. However, as we have seen, the explicit expressions of the fields are not always needed to perform a change of variables, since they tend to appear in combinations that we do know how to express in the new variables. Therefore, it is not ruled out that such formulations are possible. If found, they would give us a handle on the non-supersymmetric extremal solutions and on the non-extremal ones that we are now missing. Work in this direction is under way.

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References

1. A. Strominger, C. Vafa, Microscopic origin of the Bekenstein-Hawking entropy. Phys. Lett. **B379**, 99–104 (1996). [arXiv: hep-th/9601029]
2. S. Ferrara, R. Kallosh, A. Strominger, N = 2 extremal black holes. Phys. Rev. **D52**, 5412–5416 (1995). [arXiv: hep-th/9508072]
3. A. Strominger, Macroscopic entropy of N = 2 extremal black holes. Phys. Lett. **B383**, 39–43 (1996). [arXiv: hep-th/9602111]
4. S. Ferrara, R. Kallosh, Supersymmetry and attractors. Phys. Rev. **D54**, 1514–1524 (1996). [arXiv: hep-th/9602136]
5. S. Ferrara, R. Kallosh, Universality of supersymmetric attractors. Phys. Rev. **D54**, 1525–1534 (1996). [arXiv: hep-th/9603090]
6. K. Behrndt, D. Lüst, W.A. Sabra, Stationary solutions of N = 2 supergravity. Nucl. Phys. **B510**, 264–288 (1998). [arXiv: hep-th/9705169]
7. G.L. Cardoso, B. de Wit, J. Käppeli, T. Mohaupt, Stationary BPS solutions in N = 2 supergravity with R**2 interactions. JHEP **0012**, 019 (2000). [arXiv: hep-th/0009234]
8. F. Denef, Supergravity flows and D-brane stability. JHEP **0008**, 050 (2000). [arXiv: hep-th/0005049]
9. P. Meessen, T. Ortín, The supersymmetric configurations of N = 2, D = 4 supergravity coupled to vector supermultiplets. Nucl. Phys. **B749**, 291–324 (2006). [arXiv: hep-th/0603099]
10. T. Mohaupt, Black hole entropy, special geometry and strings. Fortsch. Phys. **49**, 3 (2001). [arXiv: hep-th/0007195]
11. J.P. Gauntlett, J.B. Gutowski, C.M. Hull, S. Pakis, H.S. Reall, All supersymmetric solutions of minimal supergravity in five dimensions. Class. Quant. Grav. **20**, 4587 (2003). [arXiv: hep-th/0209114]
12. J.P. Gauntlett, J.B. Gutowski, General concentric black rings. Phys. Rev. D **71**, 045002 (2005). [arXiv: hep-th/0408122]

13. P. Meessen, T. Ortín, S. Vaulà, All the timelike supersymmetric solutions of all ungauged d = 4 supergravities. JHEP **1011**, 072 (2010). [arXiv:1006.0239 [hep-th]]
14. M. Hübscher, P. Meessen, T. Ortín, S. Vaulà, Supersymmetric N = 2 Einstein-Yang-Mills monopoles and covariant attractors. Phys. Rev. D **78**, 065031 (2008). [arXiv:0712.1530]
15. P. Meessen, Supersymmetric coloured/hairy black holes. Phys. Lett. B **665**, 388 (2008). [arXiv:0803.0684 [hep-th]]
16. M. Hübscher, P. Meessen, T. Ortín, S. Vaulà, N = 2 Einstein-Yang-Mills's BPS solutions. JHEP **0809**, 099 (2008). [arXiv:0806.1477]
17. T. Mohaupt, K. Waite, Instantons, black holes and harmonic functions. JHEP **0910**, 058 (2009). [arXiv:0906.3451]
18. T. Mohaupt, O. Vaughan, Non-extremal black holes, harmonic functions, and attractor equations, class. Quant. Grav. **27**, 235008 (2010). [arXiv:1006.3439 [hep-th]]
19. P. Galli, T. Ortín, J. Perz, C.S. Shahbazi, Non-extremal black holes of N = 2, d = 4 supergravity. JHEP **1107**, 041 (2011). [arXiv:1105.3311]
20. P. Meessen, T. Ortín, Non-extremal black holes of N = 2, d = 5 supergravity. Phys. Lett. **B707**, 178–183 (2012). [arXiv:1107.5454]
21. T. Mohaupt, O. Vaughan, The Hesse potential, the c-map and black hole solutions. [arXiv:1112.2876]
22. P. Meessen, T. Ortín, J. Perz and C.S. Shahbazi, H-FGK-formalism for black-hole solutions of N = 2, d = 4 and d = 5 supergravity, Phys. Lett. **B709**, 260 (2012). [arXiv:1112.3332]
23. A. de Antonio Martín, T. Ortín, C.S. Shahbazi, The FGK formalism for black p-branes in d dimensions. JHEP **1205**, 045 (2012). [arXiv:1203.0260 [hep-th]]
24. P. Meessen, T. Ortín, J. Perz, C.S. Shahbazi, Black holes and black strings of N = 2, d = 5 supergravity in the H-FGK formalism. [arXiv:1204.0507 [hep-th]]
25. P. Galli, T. Ortín, J. Perz, C. S. Shahbazi, in preparation
26. S. Ferrara, G.W. Gibbons, R. Kallosh, Black holes and critical points in moduli space. Nucl. Phys. **B500**, 75–93 (1997). [arXiv: hep-th/ 9702103]
27. B. Janssen, P. Smyth, T. Van Riet, B. Vercnocke, A first-order formalism for timelike and spacelike brane solutions. JHEP **0804**, 007 (2008). [arXiv:0712.2808 [hep-th]]
28. G.T. Horowitz, A. Strominger, Black strings and P-branes. Nucl. Phys. B **360**, 197 (1991)
29. T. Ortín, *Gravity and Strings* (Cambridge University Press, Cambridge Unversity, 2004)
30. G.W. Gibbons, R. Kallosh, B. Kol, Moduli, scalar charges, and the first law of black hole thermodynamics. Phys. Rev. Lett. **77**, 4992 (1996). [arXiv: hep-th/ 9607108]
31. M. Cvetič, G. W. Gibbons, C.N. Pope, Universal area product formulae for rotating and charged black holes in four and higher dimensions. Phys. Rev. Lett. **106**, 121301 (2011). [arXiv:1011.0008]
32. M. Cvetič, D. Youm, Entropy of nonextreme charged rotating black holes in string theory. Phys. Rev. **D54**, 2612 (1996). [arXiv: hep-th/ 9603147]
33. F. Larsen, A String model of black hole microstates. Phys. Rev. **D56**, 1005 (1997). [arXiv: hep-th/ 9702153]
34. M. Cvetič, F. Larsen, General rotating black holes in string theory: Grey body factors and event horizons. Phys. Rev. **D56**, 4994 (1997). [arXiv: hep-th/ 9705192]
35. M. Cvetič, F. Larsen, Grey body factors for rotating black holes in four-dimensions. Nucl. Phys. **B506**, 107 (1997). [arXiv: hep-th/ 9706071]
36. M. Cvetič, F. Larsen, Greybody factors and charges in Kerr/CFT. JHEP **0909**, 088 (2009). [arXiv:0908.1136]
37. A. Castro, M.J. Rodríguez, Universal properties and the first law of black hole inner mechanics. [arXiv:1204.1284]
38. M. Gunaydin, G. Sierra, P.K. Townsend, The Geometry of N = 2 Maxwell-Einstein Supergravity and Jordan Algebras. Nucl. Phys. **B242**, 244 (1984)
39. B. de Wit, F. Vanderseypen, A. Van Proeyen, Symmetry structure of special geometries. Nucl. Phys. **B400**, 463–524 (1993). [arXiv: hep-th/ 9210068]
40. J. Bellorín, P. Meessen, T. Ortín, All the supersymmetric solutions of N = 1, d = 5 ungauged supergravity. JHEP **0701**, 020 (2007). [arXiv: hep-th/ 0610196]

41. E. Bergshoeff, S. Cucu, T. de Wit, J. Gheerardyn, S. Vandoren, A. Van Proeyen, N = 2 supergravity in five dimensions revisited. Class. Quant. Grav. **21**, 3015 (2004). [arXiv: hep-th/ 0403045]
42. L. Andrianopoli, M. Bertolini, A. Ceresole, R. D'Auria, S. Ferrara, P. Fré, T. Magri, N = 2 supergravity and N = 2 super Yang-Mills theory on general scalar manifolds: symplectic covariance, gaugings and the momentum map. J. Geom. Phys. **23**, 111 (1997). [arXiv: hep-th/ 9605032]
43. A. Van Proeyen, N = 2 supergravity in d = 4, 5, 6 and its matter couplings. Lectures given at the Institut Henri Poincaré, Paris, November 2000
44. B. de Wit, A. Van Proeyen, Potentials and symmetries of general gauged N = 2 supergravity—Yang-Mills models. Nucl. Phys. B **245**, 89 (1984)
45. B. de Wit, P.G. Lauwers, A. Van Proeyen, Lagrangians Of N = 2 supergravity—matter systems. Nucl. Phys. B **255**, 569 (1985)
46. K.P. Tod, All metrics admitting supercovariantly constant spinors. Phys. Lett. **B121**, 241–244 (1983)
47. J.P. Gauntlett, S. Pakis, The geometry of D = 11 killing spinors. JHEP **0304**, 039 (2003). [arXiv: hep-th/ 0212008]
48. J. Bellorín, T. Ortín, Characterization of all the supersymmetric solutions of gauged N = 1, d = 5 supergravity. JHEP **0708**, 096 (2007). [arXiv:0705.2567]
49. J. Bellorín, Supersymmetric solutions of gauged five-dimensional supergravity with general matter couplings. Class. Quant. Grav. **26**, 195012 (2009). [arXiv:0810.0527]
50. A.H. Chamseddine, W.A. Sabra, Calabi-Yau black holes and enhancement of supersymmetry in five-dimensions. Phys. Lett. **B460**, 63 (1999). [arXiv: hep-th/ 9903046]
51. G.W. Gibbons, C.M. Hull, A bogomolny bound For general relativity and solitons In N = 2 supergravity. Phys. Lett. B **109**, 190 (1982)
52. M.M. Caldarelli, D. Klemm, All supersymmetric solutions of N = 2, D = 4 gauged supergravity. JHEP **0309**, 019 (2003). [arXiv: hep-th/ 0307022]
53. M. Hübscher, P. Meessen, T. Ortín, Nucl. Phys. **B759**, 228–248 (2006). [arXiv: hep-th/ 0606281]
54. P. Meessen, T. Ortín, Supersymmetric solutions to gauged N = 2 d = 4 sugra: the full timelike shebang. Nucl. Phys. B **863**, 65 (2012). [arXiv:1204.0493 [hep-th]]
55. S. L. Cacciatori, D. Klemm, Supersymmetric AdS(4) black holes and attractors. JHEP **1001**, 085 (2010). [arXiv:0911.4926 [hep-th]]
56. J. Bellorín, P. Meessen, T. Ortín Supersymmetry, attractors and cosmic censorship. Nucl. Phys. **B762**, 229–255 (2007). [arXiv: hep-th/ 0606201]
57. A. Ceresole, R. D'Auria, S. Ferrara, The symplectic structure of N = 2 supergravity and its central extension. Nucl. Phys. Proc. Suppl. **46**, 67 (1996). [arXiv: hep-th/ 9509160]
58. P. Galli, K. Goldstein, S. Katmadas, J. Perz, First-order flows and stabilisation equations for non-BPS extremal black holes. JHEP **1106**, 070 (2011). [arXiv:1012.4020]
59. D. Klemm, O. Vaughan, Nonextremal black holes in gauged supergravity and the real formulation of special geometry. [arXiv:1207.2679 [hep-th]]
60. T. Ortín, C.S. Shahbazi, The supersymmetric black holes of N = 8 supergravity. [arXiv:1206.3190 [hep-th]]

Chapter 5
Non-supersymmetric Extremal Black Holes: First-Order Flows and Stabilisation Equations

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We review the results of [1, 2] on reducing the second-order equations of motion for stationary extremal black holes in four-dimensional $N = 2$ supergravity to first-order flow equations and further to non-differential stabilisation equations.

5.1 Introduction

Supergravity theories, being extensions of general relativity, admit black hole solutions. Finding them, as indeed any type of solutions in any theory, can be greatly simplified by a judicious exploitation of symmetries. One example, which would be valid also in Einstein's relativity, concerns space-time symmetries: for solutions translationally invariant in time, we can take the metric to be stationary, i.e. to have

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a timelike Killing vector. In the simplest case, without rotation or NUT charge, the ansatz for the line element can be chosen to be static (invariant under time reversal) and spherically symmetric.

In supergravity, we can in addition take advantage of supersymmetry, as long as the solution is also, at least partly, supersymmetric (or 'BPS'), i.e. when it is invariant under some of the supersymmetry transformations (a supersymmetric variation of the fields vanishes). If we look for classical, purely bosonic solutions (the expectation value of anticommuting fields must be zero, in other words fermions have no classical limit), the only non-trivial conditions (Killing spinor equations) come from the supersymmetry transformations of the fermionic fields, as the result of the supersymmetry transformations acting on bosons is fermionic and this by our assumption is automatically zero. Since the Killing spinor equations are of first order in derivatives of the fields, they are usually easier to solve than the second-order equations of motion.

Finally, we can make use of other internal symmetries of the theory, especially its invariance under duality transformations. We shall return to this point later, but already now let us mention that it may be helpful if the existence of these symmetries can be reflected in the description of the theory itself.

5.2 Special Geometry

In this review we restrict our attention to the bosonic sector of $N = 2$ ungauged supergravity in four spacetime dimensions [3, 4], which is one of the most widely encountered settings in the study of supergravity black holes, for it is sufficiently complicated to have physically interesting solutions, yet simple enough to be tractable. We consider only vector multiplets, because hypermultiplets do not couple to them and can be taken as constants in a self-consistent solution.

The remaining part of the action contains the familiar Einstein–Hilbert term for gravity, kinetic terms for n_{v} neutral complex scalars and $n_{\mathrm{v}} + 1$ abelian gauge fields (where n_{v} is the number of vector multiplets; the extra gauge field is the graviphoton of the gravity multiplet):

$$
\begin{aligned}
I_{4\mathrm{D}} = \frac{1}{16\pi} \int \Big(& R \star 1 - 2\, g_{a\bar{b}}(z, \bar{z})\, \mathrm{d}z^a \wedge \star\, \mathrm{d}\bar{z}^{\bar{b}} \\
& - \mathrm{Im}\, \mathcal{N}_{IJ}(z, \bar{z})\, F^I \wedge \star F^J - \mathrm{Re}\, \mathcal{N}_{IJ}(z, \bar{z})\, F^I \wedge F^J \Big).
\end{aligned} \tag{5.1}
$$

Importantly for our subsequent discussion, both the scalar manifold metric $g_{a\bar{b}}$ and the gauge kinetic matrix $\mathcal{N}$ depend on the scalars. This dependence, however, is not totally arbitrary. The scalar manifold is special Kähler, which means that its metric follows from the (real) Kähler potential K

$$
g_{a\bar{b}} = \partial_a \partial_{\bar{b}} K(z, \bar{z}), \tag{5.2}
$$

and that the Kähler potential itself can be determined by specifying a prepotential F (not to be confused with the field strengths $F^I = \mathrm{d}A^I$).[1] The prepotential is usually displayed in homogeneous (projective) coordinates $X^I(z)$, rather than the affine coordinates $z^a = X^a(z)/X^0(z)$ corresponding to physical scalars, and we shall take it to be cubic (so-called 'very special geometry'):

$$F = -\frac{1}{6} D_{abc} \frac{X^a(z)X^b(z)X^c(z)}{X^0(z)}. \tag{5.3}$$

The Kähler potential can be calculated as the symplectic invariant

$$K = -\ln\left[\mathrm{i}\begin{pmatrix} X^I(z) & F_I(z)\end{pmatrix}\begin{pmatrix} 0 & -\mathbb{I} \\ \mathbb{I} & 0\end{pmatrix}\overline{\begin{pmatrix} X^I(z) \\ F_I(z)\end{pmatrix}}\right] \tag{5.4}$$

$$=: -\ln\left(\mathrm{i}\langle \Omega_{\mathrm{hol}}(z), \overline{\Omega_{\mathrm{hol}}(z)}\rangle\right), \tag{5.5}$$

where $F_I = \partial_I F = \partial F/\partial X^I(z)$ and $\Omega_{\mathrm{hol}}(z)$ is the holomorphic section of special geometry. (In practice it will be more convenient to use the covariantly holomorphic section $\Omega = \mathrm{e}^{K/2}\Omega_{\mathrm{hol}}$.) The detailed form of the matrix $\mathcal{N}$ will not be needed in our considerations, but it too can be obtained from the prepotential.

5.3 Static Supersymmetric Black Holes

The static, spherically symmetric case is the simplest to analyse, because all spacetime-dependent quantities become functions of the radial coordinate τ only. For the line element one can then take [6]

$$\mathrm{d}s^2 = -\mathrm{e}^{2U(\tau)}\mathrm{d}t^2 + \mathrm{e}^{-2U(\tau)}\left(\frac{c^4}{\sinh^4(c\tau)}\mathrm{d}\tau^2 + \frac{c^2}{\sinh^2(c\tau)}\mathrm{d}\Omega^2_{(2)}\right), \tag{5.6}$$

where c is a non-extremality parameter and $\mathrm{d}\Omega^2_{(2)}$ is the metric of a unit two-sphere. (The horizon is located at $\tau \to \pm\infty$, depending on conventions, and $\tau \to 0$ corresponds to spatial infinity, where we shall also require $U = 0$ for asymptotic flatness.) The vector fields can be expressed in terms of the magnetic and electric charges

$$\Gamma := \begin{pmatrix} p^I \\ q_I \end{pmatrix} = \frac{1}{4\pi}\int_{S^2}\begin{pmatrix} F^I \\ G_I\end{pmatrix}, \tag{5.7}$$

[1] Strictly speaking, the existence of a prepotential is not guaranteed in every symplectic frame, but a frame where a prepotential exists can always be found by a symplectic transformation [5].

where $G_I = \operatorname{Im}\mathcal{N}_{IJ} \star F^J + \operatorname{Re}\mathcal{N}_{IJ}\, F^J$ is the dual field strength. Upon substitution one obtains [7] the action[2]

$$I_{\text{eff}} = -\frac{1}{4\pi}\int \mathrm{d}t \int \mathrm{d}\tau \left(\dot{U}^2 + g_{a\bar{b}}\dot{z}^a\dot{\bar{z}}^{\bar{b}} + \mathrm{e}^{2U} V_{\text{bh}}\right) \tag{5.8}$$

with an effective 'black hole potential'

$$V_{\text{bh}} = -\tfrac{1}{2}\Gamma^{\mathrm{T}}\mathcal{M}(\mathcal{N})\Gamma, \tag{5.9}$$

which is a function of the charges and (through the gauge couplings) of the scalars (again, the detailed form of the matrix $\mathcal{M}$ is immaterial for our purposes). This action reproduces the original equations of motion, except for one component of the Einstein equations, which is referred to as the Hamiltonian constraint:

$$\dot{U}^2 + g_{a\bar{b}}\dot{z}^a\dot{\bar{z}}^{\bar{b}} - \mathrm{e}^{2U} V_{\text{bh}} = c^2. \tag{5.10}$$

The black hole potential can be expressed in terms of the central charge[3] $Z = \langle\Gamma, \Omega\rangle$ as

$$V_{\text{bh}} = |Z|^2 + 4g^{a\bar{b}}\partial_a|Z|\partial_{\bar{b}}|Z|. \tag{5.11}$$

This, together with the Hamiltonian constraint and the identification of the mass with the asymptotic value of the tt component of the metric differentiated with respect to τ (which, by asymptotic flatness, equals $\dot{U}$ here) implies that solutions that saturate the supersymmetric BPS bound

$$M = |Z(z_\infty, \Gamma)| \tag{5.12}$$

must be extremal: $c = 0$. In that case the line element simplifies to

$$\mathrm{d}s^2 = -\mathrm{e}^{2U(\tau)}\mathrm{d}t^2 + \mathrm{e}^{-2U(\tau)}\left(\frac{1}{\tau^4}\mathrm{d}\tau^2 + \frac{1}{\tau^2}\mathrm{d}\Omega^2_{(2)}\right) \tag{5.13}$$

with $\tau = \pm 1/|\mathbf{x}|$, so the terms in parentheses represent the flat metric on $\mathbb{R}^3$.

Formula (5.11) also makes it possible to rewrite the effective Lagrangian (up to a total derivative term, which can be neglected) as a sum of squares:

$$\mathcal{L}_{\text{eff}} \propto \left(\dot{U} + \mathrm{e}^U|Z|\right)^2 + \left\|\dot{z}^a + 2\mathrm{e}^U g^{a\bar{b}}\partial_{\bar{b}}|Z|\right\|^2. \tag{5.14}$$

[2] We adopt the plus sign for the potential term, as is customary in the black hole potential literature.

[3] More precisely, Z is a function of the radial coordinate. It coincides with the central charge of the supersymmetry algebra at spatial infinity.

Since $\delta(\ldots)^2 = 2(\ldots)\delta(\ldots)$, the action attains a stationary value when the two brackets vanish separately, which immediately leads to first-order gradient flow equations that by construction satisfy the (second-order) equations of motion:

$$\dot{U} = -\mathrm{e}^{U}|Z|, \tag{5.15}$$

$$\dot{z}^a = -2\mathrm{e}^{U} g^{a\bar{b}} \partial_{\bar{b}} |Z|. \tag{5.16}$$

The flow generated by these equations terminates at $\dot{z}^a = 0$ in the scalar manifold and on the horizon in spacetime. This means that the critical points of $|Z(z, \bar{z}, \Gamma)|$ determine the horizon values of the scalars in terms of the charges (attractor mechanism).

The conditions for critical points can be brought to the form known as the attractor equations [8]

$$2\mathrm{Im}(\bar{Z}\Omega) = \Gamma. \tag{5.17}$$

When the scalars are constant in spacetime (so-called 'doubly extremal black holes'), the attractor values are taken everywhere. The non-constant solutions are given in terms of harmonic functions by the similar stabilisation equations[4] [9]

$$2\mathrm{Im}(\Omega_{\mathrm{hol}}) = \mathcal{H}, \qquad \mathcal{H} = \Gamma\tau + h, \tag{5.18}$$

where h is the vector of constants related to the asymptotic values of the scalars.

5.4 Beyond the Static, Supersymmetric Case

For supersymmetric black holes, as we indicated in the introduction, the fact that the equations of motion can be reduced to first-order equations is not surprising. More unexpectedly, it turns out [10] that non-supersymmetric and even non-extremal black holes can enjoy a like description. In the non-supersymmetric extremal case this becomes evident [11] when the black hole potential remains invariant under rotations of the charge vector by a matrix S, $\Gamma \mapsto \tilde{\Gamma} = S\Gamma$:

$$S^{\mathrm{T}}\mathcal{M}S = \mathcal{M} \quad \Longrightarrow \quad V_{\mathrm{bh}} = W^2 + 4g^{a\bar{b}}\partial_a W \partial_{\bar{b}} W, \quad W = |\langle\tilde{\Gamma}, \Omega\rangle|. \tag{5.19}$$

Flow equations based on the superpotential[5] W (or 'fake central charge', as it is built from the 'fake charges' $\tilde{\Gamma}$) are entirely analogous to (5.15), (5.16), except that the respective attractor equations may be underdetermined and may not fix the horizon

[4] Some authors interchange the meaning of 'stabilisation equations' and 'attractor equations' relative to our nomenclature or use the term 'generalised stabilisation equations' for those involving harmonic functions. Occasionally the name 'stabilisation equations' is given to the relation, implied by Eq. (5.18), between the real and imaginary parts of the symplectic section.

[5] In some papers called the prepotential.

values of scalars completely in terms of the charges (this is due to flat directions of the effective potential; the entropy, being related to the stationary value of the potential, is therefore still independent of the asymptotic values of the scalars [12]). In the non-extremal case [13] one seeks an expansion of the whole potential term (including the warp factor) into squares of partial derivatives of a (generalised) superpotential Y:

$$\mathrm{e}^{2U} V_{\mathrm{bh}} = (\partial_U Y)^2 + 4g^{a\bar{b}}\partial_a Y \partial_{\bar{b}} Y - c^2. \tag{5.20}$$

This again leads to gradient flow equations, but not to the attractor mechanism, because unlike Z or W, Y depends also on the metric function U.

What about extremal, but not necessarily static and spherically symmetric black holes (for instance, rotating stationary solutions or multicentre configurations)? Could one still reduce the equations of motion to first-order equations and integrate them to non-differential stabilisation equations? To answer these questions we proceed in a similar manner to the static case, but using a different formalism, originally devised by Denef [14] for supersymmetric solutions.

Let us first reinterpret or redefine the quantities that we have already encountered from the geometrical perspective, when the supergravity considered is viewed as a low-energy approximation of type IIB string theory compactified on a Calabi–Yau three-fold M_{CY}. Specifically, the number of vector multiplets is equal to one of the Hodge numbers, $n_{\mathrm{v}} = h^{2,1}$, the holomorphic symplectic section Ω_{hol} can be identified with the unique up to rescaling (3, 0)-form that characterises the compactification manifold and the symplectic product becomes the intersection product, for instance

$$\mathrm{e}^{-K} = \mathrm{i}\int_{M_{\mathrm{CY}}} \Omega_{\mathrm{hol}} \wedge \bar{\Omega}_{\mathrm{hol}}. \tag{5.21}$$

In the canonical basis $\{\alpha_I, \beta^I\}$ of the third integral cohomology $H^3(M_{\mathrm{CY}}, \mathbb{Z})$:

$$\Omega_{\mathrm{hol}} = X^I(z)\alpha_I - F_I(z)\beta^I, \qquad \Gamma = p^I\alpha_I - q_I\beta^I \tag{5.22}$$

and the electromagnetic fields can be seen as the components of the IIB five-form

$$\mathcal{F} = F^I \otimes \alpha_I - G_I \otimes \beta^I, \tag{5.23}$$

which should be self-dual in 10 dimensions. Decomposing this condition into $4+6$ dimensions as $\star_{10}\mathcal{F} = (\star \otimes \diamond)\mathcal{F} = \mathcal{F}$ we can relate the Hodge operator on the internal manifold to matrix $\mathcal{M}$ appearing in the black hole potential:

$$\diamond \begin{pmatrix} \beta^I \\ \alpha_I \end{pmatrix} = \begin{pmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix} \mathcal{M} \begin{pmatrix} \beta^I \\ \alpha_I \end{pmatrix}. \tag{5.24}$$

For the spacetime line element we take [15]

$$\mathrm{d}s^2 = -\mathrm{e}^{2U(\mathbf{x})}\big(\mathrm{d}t + \omega(\mathbf{x})\big)^2 + \mathrm{e}^{-2U(\mathbf{x})}\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j, \quad \omega(\mathbf{x}) = \omega_i(\mathbf{x})\mathrm{d}x^i \tag{5.25}$$

and this time we trade the manifest Lorentz invariance of the action for the duality invariance by writing the Lagrangian in terms of the spatial components of field strengths $\boldsymbol{\mathcal{F}}$ (boldface signals here and below the spatial part) and in terms of the product

$$(\boldsymbol{\mathcal{B}}, \boldsymbol{\mathcal{C}}) = \frac{\mathrm{e}^{2U}}{1-w^2}\int_{M_{\mathrm{CY}}} \boldsymbol{\mathcal{B}} \wedge \big[\star_0(\diamond\boldsymbol{\mathcal{C}}) - \star_0(w\wedge\diamond\boldsymbol{\mathcal{C}})\,w + \star_0(w\wedge\star_0\boldsymbol{\mathcal{C}})\big], \tag{5.26}$$

defined for any $\boldsymbol{\mathcal{B}}, \boldsymbol{\mathcal{C}} \in \Omega^2(\mathbb{R}^3) \otimes H^3(M_{\mathrm{CY}})$, where $\star_0$ is the Hodge dual with respect to the flat metric δ_{ij} in space and $\Omega^2(\mathbb{R}^3)$ is the set of two-forms on space. The result is

$$\begin{aligned} I_{4\mathrm{D\ eff}} = -\frac{1}{16\pi}\int \mathrm{d}t \int_{\mathbb{R}^3} \Big[& 2\mathbf{d}U \wedge \star_0\mathbf{d}U - \tfrac{1}{2}\mathrm{e}^{4U}\mathbf{d}\omega\wedge\star_0\mathbf{d}\omega \\ &+ 2g_{a\bar b}\,\mathbf{d}z^a \wedge \star_0\mathbf{d}\bar z^{\bar b} + (\boldsymbol{\mathcal{F}}, \boldsymbol{\mathcal{F}})\Big]. \end{aligned} \tag{5.27}$$

Analogously to the derivation in the previous section, we need to rewrite the effective Lagrangian in a form that yields first-order equations. The suitable pairing between the scalars and the gauge fields is afforded by the combination

$$\boldsymbol{\mathcal{G}} = \boldsymbol{\mathcal{F}} - 2\mathrm{Im}\,\star_0\mathbf{D}(\mathrm{e}^{-U}\mathrm{e}^{-\mathrm{i}\alpha}\Omega) + 2\mathrm{Re}\,\mathbf{D}(\mathrm{e}^{U}\mathrm{e}^{-\mathrm{i}\alpha}\Omega\,\omega), \tag{5.28}$$

where

$$\mathbf{D} = \mathbf{d} + \mathrm{i}(\mathbf{Q} + \mathbf{d}\alpha + \tfrac{1}{2}\mathrm{e}^{2U}\star_0\mathbf{d}\omega), \qquad \mathbf{Q} = \mathrm{Im}(\partial_a K\,\mathbf{d}z^a) \tag{5.29}$$

and $\alpha(\mathbf{x})$ is at this stage an arbitrary function. The Lagrangian becomes

$$\begin{aligned} \mathcal{L}_{\mathrm{eff}} = (\boldsymbol{\mathcal{G}}, \boldsymbol{\mathcal{G}}) &- 4\,(\mathbf{Q} + \mathbf{d}\alpha + \tfrac{1}{2}\mathrm{e}^{2U}\star_0\mathbf{d}\omega) \wedge \mathrm{Im}\langle\boldsymbol{\mathcal{G}}, \mathrm{e}^{U}\mathrm{e}^{-\mathrm{i}\alpha}\Omega\rangle \\ &+ \mathbf{d}\Big[2w\wedge(\mathbf{Q}+\mathbf{d}\alpha) + 4\mathrm{Re}\langle\boldsymbol{\mathcal{F}}, \mathrm{e}^{U}\mathrm{e}^{-\mathrm{i}\alpha}\Omega\rangle\Big]. \end{aligned} \tag{5.30}$$

Neglecting the total derivative term and requiring that the variations of the remaining two terms vanish separately, directly leads to the first-order equations

$$\boldsymbol{\mathcal{F}} - 2\mathrm{Im}\,\star_0\mathbf{D}(\mathrm{e}^{-U}\mathrm{e}^{-\mathrm{i}\alpha}\Omega) + 2\mathrm{Re}\,\mathbf{D}(\mathrm{e}^{U}\mathrm{e}^{-\mathrm{i}\alpha}\Omega\,\omega) = 0, \tag{5.31}$$

$$\mathbf{Q} + \mathbf{d}\alpha + \tfrac{1}{2}\mathrm{e}^{2U}\star_0\mathbf{d}\omega = 0. \tag{5.32}$$

The second relation implies $\mathbf{D} = \mathbf{d}$, so taking the differential of the first yields the Laplace equation

$$2\,\mathbf{d}\star_0\mathbf{d}(\mathrm{e}^{-U}\mathrm{e}^{-\mathrm{i}\alpha}\Omega) = 0 \tag{5.33}$$

on account of the closure of the field strength ($\mathbf{d}\mathcal{F} = 0$). The resulting supersymmetric stabilisation equations have the same form [16] as those for static, spherically black holes (in the gauge $K = 2U$, $\alpha = 0$), but now the harmonic functions can be multicentred, with the constituents located at $\mathbf{x}_n$:

$$2\mathrm{Im}(\mathrm{e}^{-U}\mathrm{e}^{-\mathrm{i}\alpha}\Omega) = \mathcal{H}, \qquad \mathcal{H} = \sum_{n=1}^{N} \Gamma_n \tau_n + h_n, \tag{5.34}$$

where $\tau_n = \pm 1/|\mathbf{x} - \mathbf{x}_n|$. The poles of harmonic functions must be the physical charges, as mandated by Eq. (5.31), and the constants h_n dictate the asymptotic values of the scalars. Once the stabilisation equations have been solved, all unknowns, in particular the metric warp factor and the one-form ω, can be calculated.

A non-supersymmetric generalisation [1, 2] inspired by the superpotential approach can be achieved through the replacement of $\mathcal{F}$ by a fake field strength $\tilde{\mathcal{F}}$ that reproduces the same gauge term: $(\tilde{\mathcal{F}}, \tilde{\mathcal{F}}) = (\mathcal{F}, \mathcal{F})$. If these field strengths are related by a constant matrix, the above derivation can be repeated without any other adjustments, but the relevant multicentre configurations, with centres at arbitrary distances from each other, have vanishing angular momentum and purely imaginary scalars z^a.

Allowing the relation between the fake and actual field strengths to be arbitrary causes a number of complications, due to the fact that $\tilde{\mathcal{F}}$ may not be closed. The condition of reproducing the original gauge term in the Lagrangian has to be relaxed by adding a possible three-form deviation $\boldsymbol{\Xi}$

$$(\tilde{\mathcal{F}}, \tilde{\mathcal{F}}) = (\mathcal{F}, \mathcal{F}) + \boldsymbol{\Xi}, \tag{5.35}$$

and a new term $\boldsymbol{\eta}$ appears in the rewriting:

$$\begin{aligned}\mathcal{L} = (\tilde{\mathcal{G}}, \tilde{\mathcal{G}}) - 4\,(\mathbf{Q} + \mathbf{d}\alpha + \boldsymbol{\eta} + \tfrac{1}{2}\mathrm{e}^{2U}\star_0\mathbf{d}\omega) \wedge \mathrm{Im}\langle\tilde{\mathcal{G}}, \mathrm{e}^{U}\mathrm{e}^{-\mathrm{i}\alpha}\Omega\rangle \\ + \mathbf{d}\,[\,2w \wedge (\mathbf{Q} + \mathbf{d}\alpha) + 4\mathrm{Re}\langle\tilde{\mathcal{F}}, \mathrm{e}^{U}\mathrm{e}^{-\mathrm{i}\alpha}\Omega\rangle\,],\end{aligned} \tag{5.36}$$

where

$$\tilde{\mathcal{G}} = \tilde{\mathcal{F}} - 2\mathrm{Im}\star_0\mathbf{D}(\mathrm{e}^{-U}\mathrm{e}^{-\mathrm{i}\alpha}\Omega) + 2\mathrm{Re}\,\mathbf{D}(\mathrm{e}^{U}\mathrm{e}^{-\mathrm{i}\alpha}\Omega\omega). \tag{5.37}$$

The two new quantities $\boldsymbol{\Xi}$ and $\boldsymbol{\eta}$ are constrained by consistency requirements to satisfy:

$$\boldsymbol{\Xi} = -2\mathbf{d}\boldsymbol{\eta} \wedge w, \tag{5.38}$$

$$\boldsymbol{\eta} \wedge \mathrm{Im}\langle\tilde{\mathcal{G}}, \mathrm{e}^{U}\mathrm{e}^{-\mathrm{i}\alpha}\Omega\rangle = \langle\mathbf{d}\tilde{\mathcal{F}}, \mathrm{Re}(\mathrm{e}^{U}\mathrm{e}^{-\mathrm{i}\alpha}\Omega)\rangle + \tfrac{1}{4}\boldsymbol{\Xi}. \tag{5.39}$$

Evidently, the general stationary extremal case can be reduced to first-order equations $\tilde{\mathcal{G}} = 0$ and $\mathbf{D} = \mathbf{d} - \mathrm{i}\boldsymbol{\eta}$, but the description is much more involved than before: even after elimination of $\boldsymbol{\Xi}$ we are left with an additional unknown object $\boldsymbol{\eta}$, which has to obey a complicated equation. Crucially, the combination $\mathrm{e}^{-U}\mathrm{e}^{-\mathrm{i}\alpha}\Omega$ is generically no longer determined by Laplace's equation, thus the solutions will no longer be given in terms of pure harmonic functions.

5.5 An Ansatz for Stabilisation Equations

Although in the non-supersymmetric case we are unable to integrate the equations of motion directly to non-differential stabilisation equations, we can still try to find a suitable ansatz [2]. To that end it is useful to look at the known 'almost-BPS' [17] seed solution [18] for single-centred under-rotating extremal black holes in four-dimensional theories with cubic prepotentials[6]:

$$2\mathrm{Im}\,\hat{\Omega} = \tilde{\mathcal{H}} + \tilde{\mathcal{R}}, \qquad \boldsymbol{\mathcal{F}} = \star_0 \mathbf{d}\mathcal{H} - 2\,\mathbf{d}(\mathrm{e}^{2U}\mathrm{Re}\,\hat{\Omega}\,\omega), \tag{5.40}$$

where $\hat{\Omega} = \mathrm{e}^{-U}\mathrm{e}^{-\mathrm{i}\alpha}\Omega$ and

$$\mathcal{H} = (H^0,\ 0;\ 0,\ H_a), \quad \tilde{\mathcal{H}} = (-H^0,\ 0;\ 0,\ H_a), \quad \tilde{\mathcal{R}} = \left(0,\ 0;\ \frac{M}{H^0},\ 0\right). \tag{5.41}$$

The harmonic functions are:

$$H^0 = h^0 + p^0\tau, \qquad H_a = h_a + q_a\tau, \qquad M = b + J\tau^2\cos\theta, \tag{5.42}$$

where J denotes the angular momentum.

Besides the familiar harmonic part, in this case $\tilde{\mathcal{H}}$, the stabilisation equations involve ratios of harmonic functions, $\tilde{\mathcal{R}}$. As one can verify, the new object that we had to introduce to compensate for the possible non-closure of $\tilde{\boldsymbol{\mathcal{F}}}$ does not vanish if $M \neq 0$:

$$\boldsymbol{\eta} = \mathrm{e}^{2U}\langle \mathbf{d}\tilde{\mathcal{R}}, \tilde{\mathcal{H}}\rangle = -\mathrm{e}^{2U}H^0\,\mathbf{d}\left(\frac{M}{H^0}\right). \tag{5.43}$$

Interestingly, the anharmonic part of the stabilisation equations persists even without the angular momentum, as long as the constant b responsible for the non-trivial flow of the axions $\mathrm{Re}\,z^a$ is non-zero.

Under duality transformations $\tilde{\mathcal{H}}$ and $\tilde{\mathcal{R}}$ will change: in particular the relation between $\tilde{\mathcal{H}}$ and $\mathcal{H}$ will be more complicated than merely a switch of sign. The structure of the ansatz for the stabilisation equations stays nonetheless the same. Since the starting configuration is a seed solution, we postulate that in general the right-hand side of the stabilisation equations for extremal black holes is given by a sum of harmonic functions and ratios of harmonic functions. Whereas the anharmonic part may vanish, as it did in the supersymmetric case, the harmonic part must be always present, to ensure the correct near-horizon behaviour.

[6] The notation has been slightly changed in comparison with [2].

5.6 Conclusions

As we have seen, it is possible to derive first-order equations not only for supersymmetric black holes. In the stationary extremal case this can be accomplished by a merger of the superpotential approach with Denef's duality-covariant formalism. The essential step in the rewriting is a suitable pairing between the scalar degrees of freedom and the gauge fields.

Unfortunately, the procedure applied to stationary non-supersymmetric extremal black holes does not offer nearly as much simplification as in the BPS case. This is partly due to the fact that the superpotential approach itself requires a new quantity (originally: the superpotential, here: the fake field strength or the vector of fake charges), whose relation to the physical parameters needs to be determined separately, and partly due to the anharmonicity of the stabilisation equations, reflecting the non-closure of the fake field strength. Nevertheless, the combinations of the variables entering the stabilisation equations are universal, in the sense that when one uses them as the new degrees of freedom, the equations of motion expressed in terms of them take the same form for all black hole solutions, irrespective of supersymmetry or extremality [19, 20].

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References

1. P. Galli, J. Perz, Non-supersymmetric extremal multicenter black holes with superpotentials. JHEP **02**, 102 (2010). arXiv:0909.5185 [hep-th]
2. P. Galli, K. Goldstein, S. Katmadas, J. Perz, First-order flows and stabilisation equations for non-BPS extremal black holes. JHEP **06**, 070 (2011). arXiv:1012.4020 [hep-th]
3. B. de Wit, A. Van Proeyen, Potentials and symmetries of general gauged $N = 2$ supergravity: Yang-Mills models. Nucl. Phys. **B245**, 89 (1984)
4. B. de Wit, P.G. Lauwers, A. Van Proeyen, Lagrangians of $N = 2$ supergravity-matter systems. Nucl. Phys. **B255**, 569 (1985)
5. B. Craps, F. Roose, W. Troost, A. Van Proeyen, What is special Kähler geometry? Nucl. Phys. **B503**, 565–613 (1997). arXiv:hep-th/9703082
6. G. Gibbons, Antigravitating black hole solitons with scalar hair in $N = 4$ supergravity. Nucl. Phys. **B207**, 337–349 (1982)
7. S. Ferrara, G.W. Gibbons, R. Kallosh, Black holes and critical points in moduli space. Nucl. Phys. **B500**, 75–93 (1997). arXiv:hep-th/9702103
8. S. Ferrara, R. Kallosh, Supersymmetry and attractors. Phys. Rev. **D54**, 1514–1524 (1996). arXiv:hep-th/9602136

9. W. Sabra, General static $N = 2$ black holes. Mod. Phys. Lett. **A12**, 2585–2590 (1997). arXiv:hep-th/9703101 [hep-th]
10. C.M. Miller, K. Schalm, E.J. Weinberg, Nonextremal black holes are BPS. Phys. Rev. **D76**, 044001 (2007). arXiv:hep-th/0612308
11. A. Ceresole, G. Dall'Agata, Flow equations for non-BPS extremal black holes. JHEP **03**, 110 (2007). arXiv:hep-th/0702088
12. A. Sen, Entropy function for heterotic black holes. JHEP **03**, 008 (2006). arXiv:hep-th/0508042
13. J. Perz, P. Smyth, T. Van Riet, B. Vercnocke, First-order flow equations for extremal and non-extremal black holes. JHEP **03**, 150 (2009). arXiv:0810.1528 [hep-th]
14. F. Denef, Supergravity flows and D-brane stability. JHEP **08**, 050 (2000). arXiv:hep-th/0005049
15. K.P. Tod, All metrics admitting supercovariantly constant spinors. Phys. Lett. **B121**, 241–244 (1983)
16. K. Behrndt, D. Lüst, W.A. Sabra, Stationary solutions of $N = 2$ supergravity. Nucl. Phys. **B510**, 264–288 (1998). arXiv:hep-th/9705169
17. K. Goldstein, S. Katmadas, Almost BPS black holes. JHEP **05**, 058 (2009). arXiv:0812.4183 [hep-th]
18. I. Bena, G. Dall'Agata, S. Giusto, C. Ruef, N. P. Warner, Non-BPS black rings and black holes in Taub-NUT. JHEP **06**, 015 (2009). arXiv:0902.4526 [hep-th]
19. T. Mohaupt, O. Vaughan, Non-extremal black holes, harmonic functions, and attractor equations. Class. Quant. Grav. **27**, 235008 (2010). arXiv:1006.3439 [hep-th]
20. P. Meessen, T. Ortín, J. Perz, C. Shahbazi, H-FGK formalism for black-hole solutions of $N = 2$, $d = 4$ and $d = 5$ supergravity. Phys. Lett. **B709**, 260–265 (2012). arXiv:1112.3332 [hep-th]

Chapter 6
Non-extremal Black Holes from the Generalised R-map

Thomas Mohaupt and Owen Vaughan

Abstract We review the timelike dimensional reduction of a class of five-dimensional theories that generalises $5D$, $\mathcal{N} = 2$ supergravity coupled to vector multiplets. As an application we construct instanton solutions to the four-dimensional Euclidean theory, and investigate the criteria for solutions to lift to static non-extremal black holes in five dimensions. We focus specifically on two classes of models: STU-like models, and models with a block diagonal target space metric. For STU-like models the second order equations of motion of the four-dimensional theory can be solved explicitly, and we obtain the general solution. For block diagonal models we find a restricted class of solutions, where the number of independent scalar fields depends on the number of blocks. When lifting these solutions to five dimensions we show, by explicit calculation, that one obtains static non-extremal black holes with scalar fields that take finite values on the horizon only if the number of integration constants reduces by exactly half.

6.1 Introduction

Black holes provide an important testing ground for string theory and other theories of quantum gravity. Theories with extended supersymmetry allow for extremal BPS black hole solutions, and for certain examples the microscopic and macroscopic entropy has been calculated with agreement to leading order [1, 2], and even to higher orders when including R^2 corrections [3–5]. Interestingly, the entropy of certain near-extremal black holes can also be calculated [6–10], with at least leading

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order agreement. In order to improve on this analysis it is critical to have a systematic understanding of non-extremal black hole solutions of lower-dimensional supergravity theories. This naturally leads one to consider maps between the various special geometries of $\mathcal{N} = 2$ supergravity through dimensional reduction, which are also interesting mathematically. These go by the names of the r-map and c-map, and although they have been known for some time [11–14], they have also seen much recent interest, a small sample of which is given by [15–23]. Dimensional reduction over time need not be restricted to supersymmetric theories [24–27], with the standard reference for non-linear sigma models coupled to vector fields and gravity being the seminal paper [28]. Since static, single-centred black hole solutions correspond to geodesics in the target manifold of the image of these maps, there exists a rich interplay between physical objects and geometrical constructions.

We will review the procedure presented in [29] for producing non-extremal static black hole solutions to a large class of five-dimensional theories, which includes $\mathcal{N} = 2$ supergravity coupled to vector multiplets as a subclass. The method is based on [30], and uses dimensional reduction (the r-map) over a timelike direction followed by a specific field redefinition, which can be understood as follows: The physical scalar fields parametrise a hypersurface in a larger ambient space (a d-conical affine special real manifold). The field redefinition combines the physical scalar fields with the Kaluza-Klein scalar, which can be used to parametrise the direction orthogonal to the hypersurface. The new scalar fields then parametrise the whole of the ambient space. After this procedure the effective Lagrangian for static, spherically symmetric and purely electric backgrounds takes the particularly simple form:

$$\mathrm{e}_4^{-1}\mathcal{L}_4 = \tfrac{1}{2}R_4 - \tfrac{3}{4}a_{IJ}(\sigma)\left(\partial_\mu\sigma^I\partial^\mu\sigma^J - \partial_\mu b^I\partial^\mu b^J\right).$$

Here σ^I are the scalars fields which combine the original five-dimensional physical scalars with the Kaluza-Klein scalar. The axionic scalar fields b^I descend from the gauge sector, and represent the electric potentials.

Solving the equations of motion corresponds to constructing harmonic maps from reduced spacetime into a target manifold, which becomes enlarged due to the dimensional reduction procedure. We focus on STU-like models, for which the general solution to the full second order equations of motion can be found. This is a class of models that contains the STU model along with specific generalisations that share the same feature of having a diagonal target space metric. We also consider models with block diagonal target space metrics, where a restricted class of solutions can be found that is based on the solutions to STU-like models. We will see that the number of independent scalar fields in these solutions depends on the number of blocks in the metric. For all models this provides one universal solution with constant scalar fields, because all metrics can be thought of as having at least one block (the whole metric).

We then investigate the criteria for solutions to correspond to static, non-extremal black holes in the five-dimensional theory with scalar fields that take finite values on the horizon. We find that the number of integration constants must reduce by half, which is suggestive of a first order rewriting. While first order equations governing examples of non-extremal black holes have been known for some time [17, 26, 27, 31–35], it has previously been used (to our knowledge) only as an ansatz for obtaining specific non-extremal solutions. The logic presented here is different. We consider the most general type of solution and then restrict it to solutions that describe non-extremal black holes. For STU-like models all calculations are performed explicitly, and actually rather simply.

Since this method does not rely critically on supersymmetry, we are able to consider a larger class of theories than $5D$, $\mathcal{N} = 2$ supergravity coupled to vector multiplets. This is achieved by generalising the geometry of the target manifold of the scalar fields in two ways: first, we do not require that the Hesse potential (often called the prepotential) is a homogeneous polynomial, but just a homogeneous function. Second, we allow the degree of homogeneity not just to be three, but to be arbitrary. Mathematically, this means that we replace the projective special real target manifold, which is required for $5D$, $\mathcal{N} = 2$ supergravity [36], with a generalised projective special real manifold. The generalisation is captured in the degree of homogeneity of the Hesse potential of the corresponding d-conical affine special real manifold [37]. The kinetic terms of the gauge fields also get modified in an appropriate fashion. We refer to the dimensional reduction of such a theory as the generalised r-map. Various geometrical aspects of this map have been discussed in [37], and the analogous generalisation of the rigid r-map has been also been considered in [38].

6.1.1 The Reissner-Nordström Black Hole

Let us first briefly review the five-dimensional Reissner-Nordström black hole, which will be our guiding example. This is a static, spherically symmetric and purely electric solution to a five-dimensional theory of gravity coupled to a single U(1) gauge field. The line element for this solution can be written as

$$ds_5^2 = -\frac{W}{\mathcal{H}^2}dt^2 + \mathcal{H}\left(W^{-1}dr^2 + r^2 d\Omega_{(3)}^2\right), \tag{6.1}$$

where the functions $\mathcal{H}$ and W are given by

$$\mathcal{H} = 1 + \frac{q}{r^2}, \qquad W = 1 - \frac{2c}{r^2},$$

and are harmonic functions with respect to the flat metric on $\mathbb{R}^4$, i.e.

$$\Delta_4 \mathcal{H} = \Delta_4 W = 0.$$

The parameter q is the electric charge, and c is the non-extremality parameter. The mass is given by $m^2 = q^2 - c^2$. In these coordinates the solution has an outer event horizon at $r = 2c$ and an inner Cauchy horizon at $r = 0$. One can analytically continue these coordinates to the singularity, which is located at $r = -\sqrt{q}$. The extremal limit in given by $c \to 0$, in which case $W \to 1$. It will be useful later to decompose the five-dimensional metric according to

$$ds_5^2 = -e^{2\bar{\sigma}\phi}dt^2 + e^{-\bar{\sigma}}ds_4^2,$$

which for the Reissner-Nordström metric corresponds to

$$e^{\bar{\sigma}} = \frac{\sqrt{W}}{\mathcal{H}}, \qquad ds_4^2 = \frac{dr^2}{\sqrt{W}} + \sqrt{W}r^2 d\Omega_{(3)}^2. \tag{6.2}$$

The simple example of the Reissner-Nordström black hole gives us some important clues about non-extremal solutions:

(i) The solution is built from harmonic functions on $\mathbb{R}^4$.
(ii) The four-dimensional line element is flat in the extremal limit.
(iii) The non-extremal solution is obtained by dressing the extremal solution with one additional harmonic function W.

We will see that these key features of the Reissner-Nordström black hole are also true of more complicated non-extremal solutions.

6.2 Generalising 5D, $\mathcal{N} = 2$ Supergravity

Before we write down the Lagrangian of the class of theories under consideration, we will first give a mathematical description of generalised projective special real geometry, which is a generalisation of the geometry of 5D vector multiplets. This is based in part on [37], work in progress with Vicente Cortés and the first author, and a summary given in [39]. The less mathematically inclined reader may skip this section and move directly to Sect. 6.2.2.

6.2.1 Generalising Special Real Geometry

A d-conical affine special real manifold (M, g, ∇, ξ) is a pseudo-Riemannian manifold (M, g) of $\dim_{\mathbb{R}} M = (n+1)$ equipped with a flat, torsion free 'special' connection ∇ and vector field ξ such that

(i) ∇g is completely symmetric.
(ii) $D\xi = \frac{d}{2}\mathbb{1}$, where D is the Levi-Civita connection.
(iii) $\nabla\xi = \mathbb{1}$.

Let us discuss each condition in turn. Firstly one can define a natural set of special coordinates h^I that are flat with respect to ∇, i.e.

$$\nabla dh^I = 0\,, \qquad \Rightarrow \qquad \nabla_I = \partial_I.$$

With respect to these coordinates the condition (i) ensures that

$$\frac{\partial}{\partial h^I} g_{JK}(h) = \frac{\partial}{\partial h^J} g_{IK}(h),$$

and, hence, the metric g is given by the second derivatives of a function

$$g = \partial^2 H.$$

Such a function is referred to a Hesse potential, and it is not unique. For condition (ii) we follow a similar analysis to [40], which deals with the particular case $d = 2$. This condition implies that ξ is a homothetic Killing vector field of weight d

$$\mathcal{L}_\xi g = dg.$$

Moreover it ensures that the manifold has the property of being d-conical, which means there always exists a coordinate system (r, x^i), with $r^d = g(\xi, \xi)$, such that the metric decomposes as

$$g = r^{d-2} dr^2 + r^d \bar{g}(x^i).$$

In these coordinates $\xi = r\frac{\partial}{\partial r}$. One can then define the new coordinates $y^I = (r, rx^i)$, for which the homothetic Killing vector ξ becomes an Euler vector field

$$\xi = y^I \frac{\partial}{\partial y^I}.$$

In such coordinates the metric components are homogeneous functions of degree $(d-2)$

$$\xi g_{IJ}(y) = (d-2) g_{IJ}(y),$$

which can be deduced from the fact that $\left[\xi, \frac{\partial}{\partial y^I}\right] = -\frac{\partial}{\partial y^I}$. The last condition (iii) can be seen as a compatibility condition between the previous two conditions. It ensures that ξ is the Euler field associated with the special coordinates

$$\xi = h^I \frac{\partial}{\partial h^I},$$

and, hence, the metric components are homogeneous functions of degree $(d-2)$ with respect to the special coordinates h^I. It follows that one can always choose a unique Hesse potential that is homogeneous of degree d, which is given by

$$H = \frac{1}{d(d-1)} g_{IJ} h^I h^J .$$

In order to obtain physically relevant signatures we will require this Hesse potential to be strictly positive.

It is convenient to introduce a second metric on M, given by

$$a = \partial^2 \tilde{H},$$

where $\tilde{H} := -\frac{1}{d} \log H$. We can write this metric in a basis of special coordinates as

$$a = a_{IJ} dh^I \otimes dh^J = -\frac{1}{d} \left(\frac{H_{IJ}}{H} - \frac{H_I H_J}{H^2} \right) dh^I \otimes dh^J , \tag{6.3}$$

where H_I, H_{IJ} are the first and second derivatives of the Hesse potential. If the metric g has signature $(+ - \ldots -)$, which is the case for supergravity, then a is strictly positive definite. The vector field ξ acts as an isometry of the metric a

$$\mathcal{L}_\xi a = 0.$$

We define a generalised special real manifold $(\bar{M}, \bar{g})$ as a hypersurface of constant H in a d-conical affine special real manifold, with metric induced from a. If $\dim_{\mathbb{R}} M = (n+1)$ then $\dim_{\mathbb{R}} \bar{M} = n$. It is particularly convenient to consider the hypersurface defined by $H = 1$

$$\bar{M} \simeq \{H = 1\} \subset M,$$

and we denote the embedding of $\bar{M}$ into M given by the hypersurface $H = 1$ by $i : \bar{M} \to M$. For this embedding both the pull-back of $-\frac{1}{d} g$ and a give the same metric on $\bar{M}$

$$\bar{g} = i^* \left(-\frac{1}{d} \partial^2 H \right) = i^* \left(\partial^2 \tilde{H} \right) .$$

Let ϕ^x denote local coordinates on $\bar{M}$, which therefore parametrise the hypersurface $H = 1$. The metric can be written as

$$\bar{g} = \bar{g}_{xy} d\phi^x \otimes d\phi^y = \left(a_{IJ} \frac{\partial h^I}{\partial \phi^x} \frac{\partial h^J}{\partial \phi^y} \right) \Bigg|_{H=1} d\phi^x \otimes d\phi^y .$$

A particularly natural set of coordinates is given by

$$\phi^x = \frac{h^x}{h^0}, \qquad h^0 = \hat{H}(\phi^1, \ldots, \phi^n)^{-\frac{1}{d}} := H \left(1, \frac{h^1}{h^0}, \ldots, \frac{h^n}{h^0} \right)^{-\frac{1}{d}} . \tag{6.4}$$

These are analogous to the inhomogeneous special coordinates $z^i = X^i/X^0$ on a projective special Kähler manifold. It is worth noting that one can also realise $\bar{M}$ as the quotient manifold $M/\mathbb{R}^{>0}$ with quotient metric obtained from (M, a).

For the special case that $d = 3$ and the Hesse potential is a polynomial then $(\bar{M}, \bar{g})$ represents the target manifold of $5D$, $\mathcal{N} = 2$ supergravity coupled to vector multiplets [36]. The matrix a_{IJ} restricted to the hypersurface $H = 1$ provides the kinetic term for the gauge fields. We will make the same identifications when considering more general Lagrangians, only we no longer require that $d = 3$ or the Hesse potential is a polynomial.

6.2.2 Generalising the Lagrangian

We are now ready to generalise the Lagrangian of five-dimensional $\mathcal{N} = 2$ supergravity coupled to n abelian vector multiplets. Our starting point is the Lagrangian of a five-dimensional theory of gravity coupled to n scalar fields and $(n+1)$ abelian gauge fields

$$\mathrm{e}_5^{-1}\mathcal{L}_5 = \frac{1}{2}R_5 - \frac{3}{4}\bar{g}_{xy}\partial_{\hat{\mu}}\phi^x\partial^{\hat{\mu}}\phi^y - \frac{1}{4}a_{IJ}F^I_{\hat{\mu}\hat{\nu}}F^{J\hat{\mu}\hat{\nu}}, \tag{6.5}$$

We could also have included a Chern-Simons term, however this will not be relevant for solutions which are static and purely electric. Likewise for fermionic terms. Spacetime indices run from $\hat{\mu} = 0, \ldots, 4$, and target space indices from $x = 1, \ldots n$, $I = 0, \ldots, n$. The coupling matrices $\bar{g}_{xy}$ and a_{IJ} depend only on ϕ^x.

The scalar fields form a non-linear sigma model with values in an n-dimensional target manifold that we require to be generalised projective special real (as defined in the previous section). The matrix a_{IJ} are the components of the tensor field (6.3) on the corresponding d-conical affine special real manifold. We will require that g_{IJ} has signature $(+-\ldots-)$, and, hence, a_{IJ} is positive definite. One obtains a projective special real manifold, and therefore $5D$, $\mathcal{N} = 2$ supergravity, for the special case when $d = 3$ and the Hesse potential is a polynomial.

We prefer not to work with the n physical scalar fields ϕ^x but rather the $(n+1)$ special coordinates h^I, which are subject to the hyper-surface constraint

$$H(h) = 1. \tag{6.6}$$

Here H is a smooth homogeneous function of degree d, and represents the Hesse potential of the corresponding d-conical affine special real manifold. It is often convenient to choose the parametrisation given by (6.4), where ϕ^x and h^I can be related explicitly. In the Lagrangian one must make the replacement

$$\bar{g}_{xy}(\phi)\partial_{\hat{\mu}}\phi^x\partial^{\hat{\mu}}\phi^y \to a_{IJ}(h)\partial_{\hat{\mu}}h^I\partial^{\hat{\mu}}h^J\Big|_{H=1},$$

and, hence, the Lagrangian can be written as

$$\mathrm{e}_5^{-1}\mathcal{L}_5 = \tfrac{1}{2}R_5 - \tfrac{3}{4}a_{IJ}\partial_{\hat{\mu}}h^I\partial^{\hat{\mu}}h^J - \tfrac{1}{4}a_{IJ}F^I_{\hat{\mu}\hat{\nu}}F^{J\hat{\mu}\hat{\nu}}, \tag{6.7}$$

where it is understood that the scalar fields h^I are now subject to the constraint (6.6). Two advantages of using the special coordinates h^I are immediately clear: we now have the same number of scalar fields as gauge fields, and the coupling matrices are the same. The coupling matrix a_{IJ} can be written in these coordinates as

$$a_{IJ}(h) = \partial^2_{I,J}\tilde{H}(h),$$

where as in the previous section $\tilde{H} := -\frac{1}{d}\log H$. The details of the model are completely determined by the Hesse potential H.

6.3 Dimensional Reduction and Equations of Motion

We now impose that backgrounds are static, and make the following decomposition of the five-dimensional metric:

$$ds_5^2 = -e^{2\tilde{\sigma}}dt^2 + e^{-\tilde{\sigma}}ds_4^2,$$

We impose further that backgrounds are purely electric, so the gauge vector and field strength decompose as

$$\mathcal{A}^I = \sqrt{\frac{3}{2}}\,b^I dt + C^I_\mu dx^\mu, \qquad C^I_\mu = \text{const.}$$

Choosing C^I_μ to be constant ensures that the magnetic components of the field strengths $F^I_{\hat{\mu}\hat{\nu}}$ vanish, and we can write

$$F^I_{\hat{\mu}\hat{\nu}}F^{J\hat{\mu}\hat{\nu}} = -3e^{-2\tilde{\sigma}}\partial_\mu b^I\partial^\mu b^J.$$

The scalar fields b^I represent the electric potentials. After integrating out the redundant timelike dimension, the four-dimensional Lagrangian takes the form

$$\mathrm{e}_4^{-1}\mathcal{L}_4 = \tfrac{1}{2}R_4 - \tfrac{3}{4}\partial_\mu\tilde{\sigma}\partial^\mu\tilde{\sigma} - \tfrac{3}{4}a_{IJ}(h)\left(\partial_\mu h^I\partial^\mu h^J - e^{-2\tilde{\sigma}}\partial_\mu b^I\partial^\mu b^J\right).$$

We now combine the KK-scalar $\tilde{\sigma}$ and the constrained scalar fields h^I into the new scalar fields σ^I

$$\sigma^I := e^{\tilde{\sigma}}h^I. \tag{6.8}$$

The $(n+1)$ scalar fields σ^I are unconstrained, as the KK-scalar absorbs the hypersurface constraint (6.6), which now becomes

$$H(\sigma) = e^{d\tilde{\sigma}}.$$

One can therefore interpret h^I and the KK-scalar $\tilde{\sigma}$ as fields that depend on σ^I, which are a set of independent fields. Since $a_{IJ}(h)$ is homogeneous of degree -2 and $a_{IJ}(h)h^I\partial_\mu h^J = 0$ we have

$$a_{IJ}(h)\partial_\mu h^I\partial^\mu h^J = a_{IJ}(\sigma)\partial_\mu\sigma^I\partial^\mu\sigma^J - \partial_\mu\tilde{\sigma}\partial^\mu\tilde{\sigma},$$

The four-dimensional Lagrangian can now be written as

$$\mathrm{e}_4^{-1}\mathcal{L}_4 = \tfrac{1}{2}R_4 - \tfrac{3}{4}a_{IJ}(\sigma)\left(\partial_\mu\sigma^I\partial^\mu\sigma^J - \partial_\mu b^I\partial^\mu b^J\right). \tag{6.9}$$

This Lagrangian encodes all the information about the theory for static and purely electric backgrounds. It will be useful later to note that the scalar fields σ^I satisfy the relation

$$a_{IJ}(\sigma)\sigma^I\sigma^J = 1. \tag{6.10}$$

We will now impose that backgrounds are spherically symmetric. This is in fact enough to completely determine the four-dimensional metric[1]

$$ds_4^2 = \frac{c^3}{\sinh^3(2c\tau)}d\tau^2 + \frac{c}{\sinh(2c\tau)}d\Omega_{(3)}^2. \tag{6.11}$$

Here τ is an affine parameter in the radial direction, which is related to the standard radial coordinate through

$$r^2 = \frac{ce^{2c\tau}}{\sinh(2c\tau)}. \tag{6.12}$$

Subbing in r to the four-dimensional metric (6.11) one finds that it is nothing other than the spatial part of the Reissner-Nordström metric with respect to the decomposition (6.2)

$$ds_4^2 = \frac{dr^2}{\sqrt{W}} + \sqrt{W}r^2 d\Omega_{(3)}^2,$$

where

$$W = 1 - \frac{2c}{r^2} = e^{-4c\tau}\ . \tag{6.13}$$

The effective one-dimensional Lagrangian for spherically symmetric backgrounds is given by

[1] For a derivation see [29], and see [41] for a general formula for $d \geq 4$ dimensions.

$$\mathcal{L}_1 = \tfrac{1}{4} a_{IJ}(\sigma) \left(\dot{\sigma}^I \dot{\sigma}^J - \dot{b}^I \dot{b}^J \right), \tag{6.14}$$

which must be supplemented by the Hamiltonian constraint

$$\tfrac{1}{4} a_{IJ}(\sigma) \left(\dot{\sigma}^I \dot{\sigma}^J - \dot{b}^I \dot{b}^J \right) = c^2. \tag{6.15}$$

The equations of motion for the one-dimensional Lagrangian (6.14) are

$$\begin{aligned} \frac{d}{d\tau}\left(a_{IJ}(\sigma)\dot{\sigma}^J \right) - \tfrac{1}{2}\partial_I a_{JK}(\sigma)\left(\dot{\sigma}^J \dot{\sigma}^K - \dot{b}^J \dot{b}^K \right) &= 0, \\ \frac{d}{d\tau}\left(a_{IJ}(\sigma)\dot{b}^J \right) &= 0. \end{aligned}$$

The equations of motion for b^I can be solved immediately

$$a_{IJ}(\sigma)\dot{b}^J = Q_I,$$

where the Q_I are constant electric charges that correspond to the isometry $b^I \to b^I + C^I$.

The remaining second order equation of motion for σ^I becomes much simpler if one introduces a natural set of dual coordinates σ_I, defined by

$$\sigma_I := \partial_I \tilde{H} = -a_{IJ}(\sigma)\sigma^J.$$

It is clear that both coordinates σ^I and dual coordinates σ_I are related algebraically. The derivative of σ_I can by written using the chain rule as

$$\dot{\sigma}_I = \frac{d}{d\tau}\sigma_I = a_{IJ}(\sigma)\dot{\sigma}^J.$$

Plugging the dual coordinates into the second order equations of motion and Hamiltonian constraint we find

$$\ddot{\sigma}_I + \tfrac{1}{2}\partial_I a^{JK}(\sigma)\left(\dot{\sigma}_J \dot{\sigma}_K - Q_J Q_K \right) = 0, \tag{6.16}$$

$$\tfrac{1}{4} a^{IJ}(\sigma)\left(\dot{\sigma}_I \dot{\sigma}_J - Q_I Q_J \right) = c^2. \tag{6.17}$$

We are left to solve these equations of motion.

Extremal instanton solutions correspond to the choice $c = 0$. In this case the equations of motion can be solved for arbitrary models by[2]

$$\dot{\sigma}_I = \pm Q_I, \qquad \Rightarrow \qquad \sigma_I = A_I \pm Q_I \tau.$$

[2] These solutions necessarily lift to BPS black holes. If the metric of the target manifolds allows for a field rotation matrix $R^I{}_K$ that satisfies $a_{IJ} R^I{}_K R^J{}_L = a_{KL}$ then one can generalise this ansatz to produce solutions which lift to non-BPS black holes [30, 42, 43].

Note that the number of possible independent integration constants from $(n+1)$ second order differential equations should be $(2n+2)$, but in the extremal solution above we only have $(n+1)$ integration constants. This is because extremal solutions must satisfy the first order attractor equations, of which much has already been explained in the literature, see for example [30, 34].

We will now investigate non-extremal solutions where $c \neq 0$. This turns out to be considerably more difficult, as the non-extremality parameter entangles the second order equations of motion in a highly non-trivial manner, and we can only find the most general solution for specific models.

6.4 Instanton Solutions

6.4.1 General Solution of STU-like Models

Let us fix that we have n physical scalar fields ϕ^x and a generalised projective special real target manifold. We will consider STU-like models, where the Hesse potential on the corresponding d-conical affine special real manifold takes the form

$$H(h) = \left(h^0 h^1 \dots h^n\right)^{\frac{d}{(n+1)}},$$

or models that can be brought to this form by a linear transformation. We will only consider patches where h^I are pointwise non-zero, and note that by construction the Hesse potential is strictly positive. This class of models actually generalises the class of models for which solutions were found in [29], where only the special case $d = (n+1)$ was considered. The supergravity STU model is given by the special case $n = 2$ and $d = 3$. Using the formula (6.4) the hypersurface $H = 1$ can be parametrised by the n physical scalar fields ϕ^x through

$$\phi^x = \frac{h^x}{h^0}, \qquad h^0 = (\phi^1 \dots \phi^n)^{-\frac{1}{(n+1)}}. \tag{6.18}$$

We now need to calculate the equations of motion (6.16) and (6.17) for this class of models. The matrix a^{IJ} and its derivative can be calculated using (6.3), and are given in terms of dual coordinates σ_I by

$$\begin{aligned} a^{IJ} &= \mathrm{diag}\left(\frac{1}{(n+1)\,\sigma_0^2}, \dots, \frac{1}{(n+1)\,\sigma_n^2}\right), \\ \partial_I a^{JK} &= \mathrm{diag}\left(-\frac{2}{\sigma_0}, \dots, -\frac{2}{\sigma_n}\right). \end{aligned}$$

The equations of motion then take the form

$$\ddot{\sigma}_I - \frac{\left[(\dot{\sigma}_I)^2 - (Q_I)^2\right]}{\sigma_I} = 0, \tag{6.19}$$

$$\sum_I \frac{\left[(\dot{\sigma}_I)^2 - (Q_I)^2\right]}{(n+1)\,\sigma_I^2} = 4c^2. \tag{6.20}$$

The second order Eqs. (6.19) for each coordinate σ_I completely decouple from one-another, and can be explicitly integrated to find the general solution

$$\sigma_I = \pm \frac{Q_I}{B_I} \sinh\left(B_I \tau + B_I \frac{A_I}{Q_I}\right). \tag{6.21}$$

The constraint (6.20) then relates the integration constants with the non-extremality parameter

$$\frac{1}{(n+1)}(B_0)^2 + \ldots + \frac{1}{(n+1)}(B_n)^2 = 4c^2. \tag{6.22}$$

One can either interpret c as a dependent parameter, or see this as a restriction on the integration constants. Either way, after solving all equations of motion we are left with $(2n+2)$ free parameters. Since the solution is invariant under $B_I \to -B_I$ we can assume without loss of generality that the B_I are non-negative. The Kaluza-Klein scalar can be written in terms of the dual coordinates as

$$e^{-\tilde{\sigma}} = (-1)^{(n+1)}(n+1)(\sigma_0 \ldots \sigma_n)^{\frac{1}{(n+1)}}.$$

Note that upon setting $c \to 0$ we immediately have $B_I \to 0$ due to (6.22). The general solution then reduces to the extremal solution.

6.4.2 Block Diagonal Models

For models in which the matrix a_{IJ} splits into distinct blocks, or can be made to do so be a linear transformation, we find a restricted class of solutions in which the number of independent scalar fields is the same as the number of blocks. Solutions to each block are given again by the general solution (6.21). We will demonstrate this with an example that has two blocks.

Consider a model with n physical scalar fields and a generalised projective special real target manifold with a corresponding Hesse potential that is homogeneous of degree d. For a general Hesse potential the physical scalar fields can be written using (6.4) as

$$\phi^x = \frac{h^x}{h^0}, \qquad h^0 = \hat{H}(\phi^1, \ldots, \phi^n)^{-\frac{1}{d}} := H\left(1, \frac{h^1}{h^0}, \ldots, \frac{h^n}{h^0}\right)^{-\frac{1}{d}}.$$

We will assume that the metric a_{IJ} decomposes into precisely two blocks

$$a_{IJ} = \begin{pmatrix} * \dots * & 0 & 0 & 0 \\ \vdots \ddots \vdots & 0 & 0 & 0 \\ * \dots * & 0 & 0 & 0 \\ 0 & 0 & 0 & * \dots * \\ 0 & 0 & 0 & \vdots \ddots \vdots \\ 0 & 0 & 0 & * \dots * \end{pmatrix}.$$

Let us denote the size of the first block by $k \times k$ and the second block by $l \times l$, so that $k + l = (n + 1)$. A Hesse potential that produces such a block diagonal metric is given by

$$H(\sigma^0, \dots, \sigma^n) = H^1(\sigma^0, \dots, \sigma^{k-1}) H^2(\sigma^k, \dots, \sigma^n).$$

We now set all scalar fields within each block to be proportional to one another

$$\sigma^0 \propto \dots \propto \sigma^{k-1}, \qquad \sigma^k \propto \dots \propto \sigma^n,$$

which implies that the dual coordinates σ_I are proportional to one-another

$$\sigma_{(0)} := \sigma_0 \propto \dots \propto \sigma_{k-1}, \qquad \sigma_{(1)} := \sigma_k \propto \dots \propto \sigma_n.$$

The solution is characterised by just two independent scalar fields $\sigma_{(0)}$ and $\sigma_{(1)}$ and two electric charges $Q_{(0)}$ and $Q_{(1)}$, where

$$Q_{(0)} := Q_0 = \frac{\sigma_1}{\sigma_0} Q_1 = \dots = \frac{\sigma_{k-1}}{\sigma_0} Q_{k-1},$$
$$Q_{(1)} := Q_k = \frac{\sigma_{k+1}}{\sigma_k} Q_k = \dots = \frac{\sigma_n}{\sigma_k} Q_n.$$

There is only one independent physical scalar field

$$\phi^{(1)} := \phi^k = \frac{\sigma^k}{\sigma^{k+1}} \phi^{k+1} = \dots = \frac{\sigma^k}{\sigma^n} \phi^n,$$

and the other physical scalars are constant

$$\phi^1 = \frac{\sigma^1}{\sigma^2} \phi^2 = \dots = \frac{\sigma^1}{\sigma^{k-1}} \phi^{k-1} = \text{const.}$$

The equations of motion reduce to

$$\ddot{\sigma}_{(0)} - \frac{\left[(\dot{\sigma}_{(0)})^2 - (Q_{(0)})^2\right]}{\sigma_{(0)}} = 0, \tag{6.23}$$

$$\ddot{\sigma}_{(1)} - \frac{\left[(\dot{\sigma}_{(1)})^2 - (Q_{(1)})^2\right]}{\sigma_{(1)}} = 0, \tag{6.24}$$

$$\psi_0 \frac{\left[(\dot{\sigma}_{(0)})^2 - (Q_{(0)})^2\right]}{\sigma_{(0)}^2} + \psi_1 \frac{\left[(\dot{\sigma}_{(1)})^2 - (Q_{(1)})^2\right]}{\sigma_{(1)}^2} = 4c^2, \tag{6.25}$$

where ψ_0, ψ_1 are fixed constants that depend on the ratios $\frac{\sigma_x}{\sigma_0}$, and from (6.10) they must satisfy the identity

$$\psi_0 + \psi_1 = 1.$$

Just as for STU-like models, we can find the general solution to the second order Eqs. (6.23), (6.24), which is given by

$$\sigma_{(0)} = \pm \frac{Q_{(0)}}{B_{(0)}} \sinh\left(B_{(0)}\tau + B_{(0)}\frac{A_{(0)}}{Q_{(0)}}\right), \tag{6.26}$$

$$\sigma_{(1)} = \pm \frac{Q_{(1)}}{B_{(1)}} \sinh\left(B_{(1)}\tau + B_{(1)}\frac{A_{(1)}}{Q_{(1)}}\right), \tag{6.27}$$

and the constraint (6.25) places one restriction on the integration constants

$$\psi_0 \left(B_{(0)}\right)^2 + \psi_1 \left(B_{(1)}\right)^2 = 4c^2 \ . \tag{6.28}$$

The solution naturally generalises to models with more than two blocks. For a metric with two blocks we obtained solutions characterised by one non-constant scalar field. With three blocks solutions will be characterised by two independent non-constant scalar fields, etc. We can write the Kaluza-Klein scalar as

$$e^{-\tilde{\sigma}} = \mu \left(\sigma_{(0)}\right)^{\frac{k}{(n+1)}} \left(\sigma_{(1)}\right)^{\frac{l}{(n+1)}} ,$$

where μ is a fixed constant that depends on the ratios $\frac{\sigma_x}{\sigma_0}$.

Since every matrix can be thought of as having one block (the whole matrix), this method provides at least one universal instanton solution for any model. In this case all the physical scalar fields are constant. We will see in the next section that when we lift the universal solution to five dimensions we obtain the Reissner-Nordström black hole.

6.5 Non-extremal Black Hole Solutions

The four-dimensional instanton solutions in the previous section can be lifted to static solutions of the five dimensional theory by retracing the steps of dimensional reduction. However, for these solutions to correspond to black holes they need to satisfy certain criteria:

1. An event horizon with finite area must exist.
2. The physical scalar fields ϕ^x must take finite values on the horizon.

We will show that these two requirement force us to make restrictions on the integration constants that reduce the number by exactly half—just like the extremal case—which suggests a first order rewriting. The fact that certain non-example black holes are governed by first order equations has been known in the literature for some time [17, 26, 27, 31–35]. But here we present the argument differently. For STU-like models we start with the most general solution to the equations of motion truncated to static, spherically symmetric and purely electric backgrounds. We then impose the above criteria on the general solution, and by doing so find the most general type of non-extremal black hole solution using the parametrisation of the physical scalar fields given by (6.4). The fact that the number of integration constants reduces by half is interesting because there is no reason a priori that non-extremal solutions should be governed by first order equations. Since we see no reason why the STU-like models should be a privileged with respect to the number of integration constants, it is reasonable to suspect that this is a feature of non-extremal black hole solutions to all models.

6.5.1 STU-like Models

We can lift the instanton solutions found in the previous section to a static solution to the five-dimensional theory

$$ds_5^2 = -\frac{1}{(n+1)^2\,(\sigma_0\ldots\sigma_n)^{\frac{2}{(n+1)}}}dt^2 + (-1)^{(n+1)}(n+1)\,(\sigma_0\ldots\sigma_n)^{\frac{1}{(n+1)}}\left(\frac{c^3}{\sinh^3 2c\tau}d\tau^2 + \frac{c}{\sinh 2c\tau}d\Omega^2_{(3)}\right),$$

where one should note that $(-1)^{(n+1)}(n+1)(\sigma_0\ldots\sigma_n)^{\frac{1}{(n+1)}}$ is positive between radial infinity and the outer horizon $\tau \in (0, +\infty)$. The area A of the outer event horizon is given by

$$A = \lim_{\tau\to+\infty}(-1)^{(n+1)}(n+1)\,(\sigma_0\ldots\sigma_n)^{\frac{1}{(n+1)}}\,\frac{c}{\sinh 2c\tau}.$$

The highest order term in the numerator is proportional to $e^{\frac{1}{(n+1)}(B_0+\ldots+B_n)\tau}$ (recall that the B_I are non-negative), which must exactly cancel with the highest order term in the denominator $e^{2c\tau}$. We can conclude that in order to obtain a finite area we must have

$$\frac{1}{(n+1)}(B_0+\ldots+B_n)=2c. \tag{6.29}$$

Next, we turn our attention to the physical scalar fields ϕ^x. These can be written in terms of the dual scalars σ_I simple by

$$\phi^x=\frac{\sigma_0}{\sigma_x}.$$

In the limit $\tau\to+\infty$ the physical scalars ϕ^x will not take finite values[3] for generic choices of B_I. The only way to ensure that they take finite values is to impose

$$B_0=B_1=\ldots=B_n.$$

Combining this with (6.29) we conclude that in order to have a finite horizon and finite scalar fields the integration constants must satisfy

$$B_0=\ldots=B_n=2c. \tag{6.30}$$

The solution (6.21) therefore reduces to

$$\sigma_I=\pm\frac{Q_I}{2c}\sinh\left(2c\tau+2c\frac{A_I}{Q_I}\right). \tag{6.31}$$

Lastly, in order for the solution to be Minkowski space at radial infinity it must satisfy $e^{\tilde{\sigma}}\to 1$, which places one further constraint on the integration constants

$$(-1)^{(n+1)}(n+1)\left[\pm\frac{Q_0}{2c}\sinh\left(2c\frac{A_0}{Q_0}\right)\ldots\pm\frac{Q_n}{2c}\sinh\left(2c\frac{A_n}{Q_n}\right)\right]^{\frac{1}{(n+1)}}=1. \tag{6.32}$$

Due to the constraints (6.30) and (6.32) the number of integration constants reduces by precisely one half, from $(2n+2)$ to $(n+1)$. This is suggestive of a first order rewriting, and indeed this can be achieved by first defining the generating function $\mathcal{W}=\mathcal{W}(\sigma^I,Q_I,c)$ by

$$\mathcal{W}:=\pm\frac{1}{(n+1)}\sum_I\Bigg[\sqrt{4c^2+(n+1)^2Q_I^2{\sigma^I}^2}+c\log\left(\frac{\sqrt{4c^2+(n+1)^2Q_I^2{\sigma^I}^2}-2c}{\sqrt{4c^2+(n+1)^2Q_I^2{\sigma^I}^2}+2c}\right)\Bigg].$$

[3] By finite values we mean $\phi^x\not\to 0,\pm\infty$.

This is of a similar form to the generating function for the four-dimensional STU model [44]. We can therefore write the solution as first order flow equations

$$\begin{aligned}\dot{\sigma}_I &= \frac{\partial}{\partial \sigma^I}\mathcal{W},\\ &= \pm\sqrt{Q_I^2 + 4c^2\sigma_I^2},\end{aligned}$$

which is clearly solved by (6.31). These are first order differential equations (in τ), which relate $\dot{\sigma}_I$ to the gradient of a function. They can alternatively be written as $\dot{\sigma}^I = a^{IJ}\partial_J\mathcal{W}$.

Collecting everything together, we find that the most general static black hole solution for STU-like models is given by

$$ds_5^2 = -\frac{W}{(\mathcal{H}_0\ldots\mathcal{H}_n)^{\frac{2}{(n+1)}}}dt^2 + (\mathcal{H}_0\ldots\mathcal{H}_n)^{\frac{1}{(n+1)}}\left(\frac{dr^2}{W} + r^2 d\Omega_{(3)}^2\right),$$

where

$$\begin{aligned} W &= 1 - \frac{2c}{r^2}, & \mathcal{H}_I &= \mp(n+1)\left[\frac{Q_I}{2c}\sinh\left(2c\frac{A_I}{Q_I}\right) + \frac{Q_I e^{-2c\frac{A_I}{Q_I}}}{2}\frac{1}{r^2}\right],\\ &= e^{-4c\tau}, & &= \mp(n+1)\left[\frac{1}{4c}Q_I e^{2c\frac{A_I}{Q_I}} - \frac{1}{4c}Q_I e^{-2c\frac{A_I}{Q_I}}e^{-4c\tau}\right],\end{aligned}$$

and the scalar fields are given by

$$\phi^x = \frac{\sigma_0}{\sigma_x}, \qquad \sigma_I = \frac{-1}{(n+1)}\frac{\mathcal{H}_I}{\sqrt{W}} = \pm\frac{Q_I}{2c}\sinh\left(2c\tau + 2c\frac{A_I}{Q_I}\right).$$

For the case where $n = 2$ and $d = 3$ this reproduces the non-extremal black hole solutions of $5D$, $\mathcal{N} = 2$ supergravity originally found in [9, 45].

6.5.2 Block Diagonal Models

Let us now lift the instanton solutions to models with block diagonal matrix a_{IJ}, described in the previous section, to static solutions in five dimensions. Again we will focus on an example with two blocks of size $k \times k$ and $l \times l$. The line element is given by

$$ds_5^2 = -\frac{1}{\mu\left(\sigma_{(0)}\right)^{\frac{2k}{(n+1)}}\left(\sigma_{(1)}\right)^{\frac{2l}{(n+1)}}}dt^2 + \mu\left(\sigma_{(0)}\right)^{\frac{k}{(n+1)}}\left(\sigma_{(1)}\right)^{\frac{l}{(n+1)}}\left(\frac{c^3}{\sinh^3 2c\tau}d\tau^2 + \frac{c}{\sinh 2c\tau}d\Omega_{(3)}^2\right).$$

The area A of the outer event horizon is given by

$$A = \lim_{\tau\to+\infty}\mu\left(\sigma_{(0)}\right)^{\frac{k}{(n+1)}}\left(\sigma_{(1)}\right)^{\frac{l}{(n+1)}}\frac{c}{\sinh 2c\tau}.$$

The highest order term in the numerator is proportional to $e^{\left(\frac{k}{(n+1)}B_{(0)}+\frac{l}{(n+1)}B_{(1)}\right)\tau}$, which must exactly cancel with the highest order term in the denominator $e^{2c\tau}$. We can conclude that in order to obtain a finite area we must have

$$\frac{k}{(n+1)}B_{(0)} + \frac{l}{(n+1)}B_{(1)} = 2c.$$

The physical scalar field $\phi^{(1)}$ can be written in terms of the dual scalars $\sigma_{(0,1)}$ as

$$\phi^{(1)} \sim \frac{\sigma_{(0)}}{\sigma_{(1)}}.$$

In the limit $\tau \to +\infty$ the physical scalar $\phi^{(1)}$ will not take finite values for generic choices of $B_{(0,1)}$. The only way to ensure that they take finite values is to impose

$$B_{(0)} = B_{(1)}.$$

Combining this with (6.5.2) we conclude that in order to have a finite horizon and finite scalar fields the integration constants must satisfy

$$B_{(0)} = B_{(1)} = 2c.$$

Ensuring that the solution is Minkowski space at radial infinity $e^{\tilde{\sigma}} \to 1$ places one further constraint on the integration constants

$$\mu\left(\pm\frac{Q_{(0)}}{2c}\sinh\left(2c\frac{A_{(0)}}{Q_{(0)}}\right)\right)^{\frac{k}{(n+1)}}\left(\pm\frac{Q_{(1)}}{2c}\sinh\left(2c\frac{A_{(1)}}{Q_{(1)}}\right)\right)^{\frac{l}{(n+1)}} = 1.$$

Collecting everything together, we find that our solution for the block diagonal models can be written as

$$ds_5^2 = -\frac{W}{\left(\mathcal{H}_{(0)}\right)^{\frac{2k}{(n+1)}}\left(\mathcal{H}_{(1)}\right)^{\frac{2l}{(n+1)}}}dt^2 + \left(\mathcal{H}_{(0)}\right)^{\frac{k}{(n+1)}}\left(\mathcal{H}_{(1)}\right)^{\frac{l}{(n+1)}}\left(\frac{dr^2}{W} + r^2 d\Omega_{(3)}^2\right),$$

where

$$
\begin{aligned}
W &= 1 - \frac{2c}{r^2} = e^{-4c\tau}, \\
\mathcal{H}_{(0,1)} &= \pm\mu \left[\frac{Q_{(0,1)}}{2c} \sinh\left(2c\frac{A_{(0,1)}}{Q_{(0,1)}}\right) + \frac{Q_{(0,1)} e^{-2c\frac{A_{(0,1)}}{Q_{(0,1)}}}}{2} \frac{1}{r^2} \right], \\
&= \pm\mu \left[\frac{1}{4c} Q_{(1,2)} e^{2c\frac{A_{(0,1)}}{Q_{(0,1)}}} - \frac{1}{4c} Q_{(0,1)} e^{-2c\frac{A_{(0,1)}}{Q_{(0,1)}}} e^{-4c\tau} \right].
\end{aligned}
$$

and the scalar fields are given by

$$
\begin{aligned}
\phi^{(1)} &\sim \frac{\sigma_{(0)}}{\sigma_{(1)}}, \\
\sigma_{(0,1)} &= \frac{1}{\mu} \frac{\mathcal{H}_{(0,1)}}{\sqrt{W}} = \pm \frac{Q_{(0,1)}}{2c} \sinh\left(2c\tau + 2c\frac{A_{(0,1)}}{Q_{(0,1)}}\right).
\end{aligned}
$$

6.6 Conclusion and Outlook

We have discussed the notion of a d-conical affine special real manifolds and correspondingly generalised projective special real manifolds. The latter generalises the geometry of projective special real manifolds, which appear as the target manifolds of $5D$, $\mathcal{N} = 2$ supergravity coupled to vector multiplets. We used this to construct a class of five-dimensional gravity-scalar-vector theories that generalises $\mathcal{N} = 2$ supergravity coupled to vector multiplets.

Through dimensional reduction and the specific field redefinition (6.8) one can obtain a particularly simple effective Lagrangian for static, spherically symmetric and purely electric solutions (6.9). One key feature was that we worked always at the level of the 'larger' moduli space: the d-conical affine special real manifold. We then focused on STU-like models, where we found the general solution to the equations of motion, and models that are block diagonal, where we found solution with as many independent scalar fields as there are blocks. Since the metrics of all models contain at least one block, this also provides a universal solution to all models.

We then investigated which solutions correspond to non-extremal black holes solutions of the five-dimensional theory. In order to obtain a finite horizon area and finite scalar fields the number of integration constants must halve, suggesting a first order rewriting of the equations of motion. For STU-like models all calculations were performed explicitly, and at every stage we can set $c \to 0$ to obtain the extremal solution. Since we see no reason STU-like models should be privileged in their number of integration constants, we conjecture that all non-extremal black hole solutions should have half the number of integration constants one would expect from the second order equations of motion.

One obvious extension to this work is to investigate solutions of more complicated models. However, it was shown in [29] that the hyperbolic-sine form of the solution to STU-like models (6.31) does not give the most general solution for generic models. One must therefore replace the hyperbolic-sine function with something more complicated. It is an open question whether one can find a general formula for such a function, e.g. [41], or whether one can only find explicit formulas for specific models. At this point it is still not even clear in the literature that every extremal black hole solution admits a non-extremal generalisation [16, 17].

One may also wonder whether this analysis can be repeated for four-dimensional theories. In [21] it was shown that the effective action for static solutions to $4D$, $\mathcal{N} = 2$ supergravity coupled to vector multiplets can be brought to the same simple form as (6.9) for general static spacetime metrics (see p. 51 of [21]). One can then follow exactly the same logic for axion-free solutions to the four-dimensional STU model as we have present here for the five-dimensional STU model: one can find the general solution to the equations of motion, and show that these correspond to black hole solutions with finite scalar fields only when the number of integration constants reduces by half. This will be presented in future work [46].

Another natural extension is to consider various other types of solutions. These include solutions with a cosmological constant or Taub-NUT charge, rotating solutions, domain walls, black strings and cosmological solutions. Reduction over time has previously been used to construct black ring solutions [24, 47, 48], and in [49] black ring solutions were constructed based on [50]. Cosmological solutions may also be particularly interesting as the non-extremal black hole solutions we have discussed can be continued beyond the horizon where the Killing vector is spacelike. This provides a natural starting point for the construction of S-brane cosmological solutions [51, 52].

Theories of gauged supergravity are also applicable to the analysis presented in this paper. In [53] it was shown that the same procedure can be used to find new non-extremal solutions to four-dimensional Fayet-Iliopoulos gauged supergravity. It would also be interesting to investigate non-extremal solutions of five-dimensional gauged supergravity, though solutions to the STU model have previously been found by other methods [54].

Lastly, one may wonder whether special Kähler geometry, which corresponds to $4D$, $\mathcal{N} = 2$ supergravity coupled to vector multiplets, can be generalised in a way similar to the generalisation of special real geometry considered in this paper. At present there does not exist a well defined generalisation of special Kähler geometry. However, the dimensional reduction of d-conical affine special real geometry suggests that generalising the degree of homogeneity of the holomorphic prepotential may provide one consistent generalisation of conic affine special Kähler geometry. This would be interesting to investigate in the future.

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References

1. S. Ferrara, R. Kallosh, A. Strominger, N=2 extremal black holes. Phys. Rev. D **52**, 5412 (1995). hep-th/9508072
2. A. Strominger, C. Vafa, Microscopic origin of the Bekenstein-Hawking entropy. Phys. Lett. B **379**, 99 (1996). hep-th/9601029
3. J. M. Maldacena, A. Strominger, E. Witten, JHEP **9712**, 002 (1997). [hep-th/9711053]
4. C. Vafa, Black holes and Calabi-Yau threefolds. Adv. Theor. Math. Phys. **2**, 207 (1998). [hep-th/9711067]
5. G. Lopes Cardoso, B. de Wit, T. Mohaupt, Corrections to macroscopic supersymmetric black hole entropy. Phys. Lett. B **451**, 309 (1999). [hep-th/9812082]
6. C.G. Callan, J.M. Maldacena, D-brane approach to black hole quantum mechanics. Nucl. Phys. B **472**, 591 (1996). hep-th/9602043
7. G.T. Horowitz, A. Strominger, Counting states of near-extremal black holes. Phys. Rev. Lett. **77**, 2368 (1996). hep-th/9602051
8. M. Cvetic, D. Youm, Entropy of non-extreme charged rotating black holes in string theory. Phys. Rev. D **54**, 2612 (1996). hep-th/9603147
9. J.M. Maldacena, A. Strominger, Black hole greybody factors and D-brane spectroscopy. Phys. Rev. D **55**, 861 (1997). hep-th/9609026
10. K. Behrndt, M. Cvetic, W.A. Sabra, The entropy of near-extreme N=2 black holes. Phys. Rev. D **58**, 084018 (1998). hep-th/9712221
11. S. Cecotti, S. Ferrara, L. Girardello, Geometry of type II superstrings and the moduli of superconformal field theories. Int. J. Mod. Phys. A **4**, 2475 (1989)
12. S. Ferrara, S. Sabharwal, Quaternionic manifolds for type II superstring vacua of Calabi-Yau spaces. Nucl. Phys. B **332**, 317 (1990)
13. B. de Wit, A. Van Proeyen, Special geometry, cubic polynomials and homogeneous quaternionic spaces. Commun. Math. Phys. **149**, 307 (1992). [hep-th/9112027]
14. B. de Wit, A. Van Proeyen, Isometries of special manifolds. (1995). hep-th/9505097
15. M. Gunaydin, A. Neitzke, B. Pioline, A. Waldron, BPS black holes, quantum attractor flows and automorphic forms. Phys. Rev. D **73**, 084019 (2006). [hep-th/0512296]
16. G. Bossard, H. Nicolai, K.S. Stelle, Universal BPS structure of stationary supergravity solutions. JHEP **0907**, 003 (2009). [arXiv:0902.4438 [hep-th]]
17. W. Chemissany, J. Rosseel, M. Trigiante, T. Van Riet, The Full integration of black hole solutions to symmetric supergravity theories. Nucl. Phys. B **830**, 391 (2010). [arXiv:0903.2777 [hep-th]]
18. G. Bossard, Y. Michel, B. Pioline, Extremal black holes, nilpotent orbits and the true fake superpotential. JHEP **1001**, 038 (2010). [arXiv:0908.1742 [hep-th]]
19. M. Azreg-Ainou, G. Clement, D.V. Gal'tsov, All extremal instantons in Einstein-Maxwell-dilaton-axion theory. Phys. Rev. D **84**, 104042 (2011). [arXiv:1107.5746 [hep-th]]
20. V. Cortés, J. Louis, P. Smyth, H. Triendl, On certain Kähler quotients of quaternionic Kähler manifolds. (2011). arXiv:1111.0679 [math.DG]
21. T. Mohaupt, O. Vaughan, The Hesse potential, the c-map and black hole solutions. JHEP **1207**, 163 (2012). [arXiv:1112.2876 [hep-th]]
22. S. Alexandrov, c-map as c=1 string. Nucl. Phys. B **863**, 329 (2012). [arXiv:1201.4392 [hep-th]]
23. D.V. Alekseevsky, V. Cortés, T. Mohaupt, Conification of Kähler and hyper-Kähler manifolds. (2012). arXiv:1205.2964 [math.DG]
24. S.S. Yazadjiev, Generating dyonic solutions in 5D Einstein-dilaton gravity with antisymmetric forms and dyonic black rings. Phys. Rev. D **73**, 124032 (2006). [hep-th/0512229]
25. D. Gaiotto, W. Li, M. Padi, Non-Supersymmetric Attractor Flow in Symmetric Spaces. JHEP **0712**, 093 (2007). [arXiv:0710.1638 [hep-th]]
26. B. Janssen, P. Smyth, T. Van Riet, B. Vercnocke, A first-order formalism for timelike and spacelike brane solutions. JHEP **04**, 007 (2008). arXiv:0712.2808
27. J. Perz, P. Smyth, T. Van Riet, B. Vercnocke, First-order flow equations for extremal and non-extremal black holes. JHEP **0903**, 150 (2009). arXiv:0810.1528

28. P. Breitenlohner, D. Maison, G.W. Gibbons, Four-dimensional black holes from Kaluza-Klein theories. Commun. Math. Phys. **120**, 295 (1988)
29. T. Mohaupt, O. Vaughan, Non-extremal black holes, harmonic functions, and attractor equations. Class. Quant. Grav. **27**, 235008 (2010). [arXiv:1006.3439 [hep-th]]
30. T. Mohaupt, K. Waite, Instantons, black holes and harmonic functions. JHEP **0910**, 058 (2009). [arXiv:0906.3451 [hep-th]]
31. H. Lu, C.N. Pope, J.F. Vazquez-Poritz, From AdS black holes to supersymmetric flux branes. Nucl. Phys. B **709**, 47 (2005). hep-th/0307001
32. C.M. Miller, K. Schalm, E.J. Weinberg, Nonextremal black holes are BPS. Phys. Rev. D **76**, 044001 (2007). hep-th/0612308
33. M.R. Garousi, A. Ghodsi, On attractor mechanism and entropy function for non-extremal black holes. JHEP **05**, 043 (2007). hep-th/0703260
34. L. Andrianopoli, R. D'Auria, E. Orazi, First order description of black holes in moduli space. JHEP **11**, 032 (2007). arXiv:0706.0712
35. G.L. Cardoso, V. Grass, On five-dimensional non-extremal charged black holes and FRW cosmology. Nucl. Phys. B **803**, 209 (2008). arXiv:0803.2819
36. M. Gunaydin, G. Sierra, P.K. Townsend, The geometry of N=2 Maxwell-Einstein supergravity and Jordan algebras. Nucl. Phys. B **242**, 244 (1984)
37. V. Cortes, T. Mohaupt, H. Xu, Completeness in supergravity constructions. (2011). arXiv:1101.5103 [hep-th]
38. D. V. Alekseevsky, V. Cortes, Geometric construction of the r-map: from affine special real to special Kähler manifolds. Comm. Math. Phys. **291**, 579–590 (2009). [arXiv:0811.1658 [math.DG]]
39. T. Mohaupt, O. Vaughan, Developments in special geometry. J. Phys. Conf. Ser. **343**, 012078 (2012). [arXiv:1112.2873 [hep-th]]
40. G.W. Gibbons, P. Rychenkova, Cones, triSasakian structures and superconformal invariance. Phys. Lett. B **443**, 138 (1998). [hep-th/9809158]
41. P. Meessen, T. Ortin, Non-extremal black holes of N=2, d=5 supergravity. Phys. Lett. B **707**, 178 (2012). [arXiv:1107.5454 [hep-th]]
42. A. Ceresole, G. Dall'Agata, Flow equations for non-BPSextremal black holes. JHEP **0703**, 110 (2007). [hep-th/0702088]
43. G. Lopes Cardoso, A. Ceresole, G. Dall'Agata, J.M. Oberreuter, J. Perz, First-order flow equations for extremal black holes in very special geometry. JHEP **0710**, 063 (2007). [arXiv:0706.3373 [hep-th]]
44. P. Galli, T. Ortin, J. Perz, C.S. Shahbazi, Non-extremal black holes of N=2, d=4 supergravity. JHEP **1107**, 041 (2011). [arXiv:1105.3311 [hep-th]]
45. J. M. Maldacena, Black holes in string theory. (1996). hep-th/9607235
46. T. Mohaupt, O. Vaughan, to appear
47. S.S. Yazadjiev, Completely integrable sector in 5-D Einstein-Maxwell gravity and derivation of the dipole black ring solutions. Phys. Rev. D **73**, 104007 (2006). [hep-th/0602116]
48. S.S. Yazadjiev, Solution generating in 5D Einstein-Maxwell-dilaton gravity and derivation of dipole black ring solutions. JHEP **0607**, 036 (2006). [hep-th/0604140]
49. P. Meessen, T. Ortin, J. Perz, C. S. Shahbazi, Black holes and black strings of N=2, d=5 supergravity in the H-FGK formalism. (2012). arXiv:1204.0507 [hep-th]
50. P. Meessen, T. Ortin, J. Perz, C.S. Shahbazi, H-FGK formalism for black-hole solutions of N=2, d=4 and d=5 supergravity. Phys. Lett. B **709**, 260 (2012). [arXiv:1112.3332 [hep-th]]
51. K. Behrndt, S. Forste, String Kaluza-Klein cosmology. Nucl. Phys. B **430**, 441 (1994). [hep-th/9403179]
52. M. Gutperle, A. Strominger, Spacelike branes. JHEP **0204**, 018 (2002). [hep-th/0202210]
53. D. Klemm, O. Vaughan, Nonextremal black holes in gauged supergravity and the real formulation of special geometry. arXiv:1207.2679 [hep-th]
54. K. Behrndt, M. Cvetic, W.A. Sabra, Nonextreme black holes of five-dimensional N=2 AdS supergravity. Nucl. Phys. B **553**, 317 (1999). [hep-th/9810227]

Chapter 7
Black Hole Microstate Geometries from String Amplitudes

David Turton

Abstract In this talk we review recent calculations of the asymptotic supergravity fields sourced by bound states of D1 and D5-branes carrying travelling waves. We compute disk one-point functions for the massless closed string fields. At large distances from the branes, the effective open string coupling is small, even in the regime of parameters where the classical D1-D5-P black hole may be considered. The fields sourced by the branes differ from the black hole solution by various multipole moments, and have led to the construction of a new $1/8$-BPS ansatz in type IIB supergravity.

7.1 Introduction

Black holes provide (at least) two major challenges for any theory of quantum gravity: to give a microscopic interpretation of the Bekenstein-Hawking entropy [1, 2], and to resolve the information paradox [3]. String theory promises to pass both tests: the microscopic interpretation of the Bekenstein-Hawking entropy is provided by enumerating microstates of the black hole [4–6], and studying the properties of these microstates promises to resolve the information paradox.

The information paradox states, roughly, that if a classical black hole metric with a horizon is a valid description of a physical black hole in Nature, then Hawking radiation leads to a breakdown of unitarity or exotic remnant objects (for a recent rigorous treatment, see [7]). A conservative way of avoiding these pathologies is to ask whether the physics of individual black hole microstates modifies the process of Hawking radiation.

The study of the gravitational description of individual microstates has motivated a 'fuzzball' picture of a black hole [8, 9]. The fuzzball conjecture is composed of two

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parts: firstly that quantum effects are important at the would-be horizon of a black hole, making Hawking radiation a unitary process; secondly, that the mechanism underlying this is that the quantum bound state of matter making up the black hole has a macroscopic size, of order the horizon scale.

To investigate this conjecture, one must understand the characteristic size of (the wavefunctions of) individual bound states. A fruitful line of inquiry has been to construct and analyze classical supergravity solutions describing the gravitational fields sourced by semiclassical/coherent states of the Hilbert space of the black hole (for reviews, see [10–13]). Supergravity solutions which describe individual microstates have been found not to have horizons themselves.

Given such a supergravity solution however, it may not always be clear whether it corresponds to a black hole microstate (see e.g. [14, 15]). In this talk we describe calculations which directly associate supergravity fields with the microscopic bound states they describe. We consider particular bound states of D-branes, and derive the asymptotic supergravity fields from worldsheet amplitudes. The amplitudes are disk-level one-point functions for the emission of massless closed string fields.

We first derive the fields sourced by a D1-brane with a travelling wave and relate them to the previously known two-charge supergravity fields [16, 17]. We then derive the fields sourced by a D1-D5 bound state with a travelling wave and find a new set of three-charge supergravity fields, more general than previously considered [18]. The results reviewed here appeared in the papers [19] and [20].

This talk is structured as follows. In Sect. 7.2 we introduce the calculation and discuss its regime of validity. In Sect. 7.3 we review the D1-P calculation, and in Sect. 7.4 we review the D1-D5-P calculation.

7.2 The Calculation and its Regime of Validity

The procedure we follow for calculating the asymptotic fields sourced by D-brane bound states was developed in [21–24]. First, we calculate the momentum-space amplitude $\mathcal{A}(k)$ for the emission of a massless closed string. We then extract the field of interest (e.g. graviton), multiply by a free propagator, and Fourier transform to obtain the spacetime one-point function.

For applications to black holes, given N D-branes we are interested in the regime $g_s N \gg 1$, where a classical black hole solution might be relevant. The naive open string coupling is also $g_s N$, so it seems we are out of the regime of open string perturbation theory (see e.g. [5]).

However if one considers the above calculation for the fields at a distance r from the bound state (Fig. 7.1), one finds that the effective open string coupling is in fact

$$\epsilon \;=\; g_s N \left(\frac{\alpha'}{r^2}\right)^{\frac{7-p}{2}} . \tag{7.1}$$

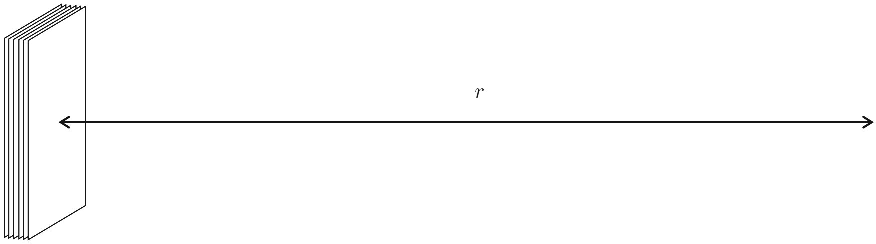


Fig. 7.1 For the calculation of the fields at large distances r from a bound state of N Dp-branes, the effective open string coupling is small if $r^{7-p} \gg g_s N \sqrt{\alpha'}^{7-p}$

This effective open string coupling may be understood as follows. The next order in open string perturbation theory corresponds to adding an extra border to the string worldsheet. The factor of N comes from the N choices of which Dp-brane the open string endpoints can end on. The extra border on the worldsheet also introduces a loop momentum integral, two extra propagators, and reduces the background superghost charge by two units. This results in the above powers of $\frac{\alpha'}{r^2}$, as discussed in detail in [20].

The next order in closed string perturbation theory corresponds to adding handles to the closed string propagator, which we suppress by working at $g_s \ll 1$. Thus we work in the following regime of parameters:

$$g_s \ll 1, \qquad g_s N \left(\frac{\alpha'}{r^2}\right)^{\frac{7-p}{2}} \ll 1. \tag{7.2}$$

Thus one can simultaneously consider $g_s N \gg 1$, provided r is sufficiently large.

One can rephrase the second condition above as saying that disk amplitudes give the leading contribution to the fields at lengthscales much greater than the characteristic size of the D-brane bound state, $r^{7-p} \gg g_s N \sqrt{\alpha'}^{7-p}$. A similar perturbative expansion was made some time ago in the field theory analogue of our calculation [25].

Since the fields in which we are interested are massless, the emitted closed string state has non-zero momentum only in the four non-compact directions of the $\mathbb{R}^4$, i.e. a spacelike momentum. The momentum-space amplitude $\mathcal{A}(k)$ mentioned above is defined by analytically continuing k to complex values such that we impose $k^2 = 0$, i.e. the emitted string state is treated as on-shell [23].

One can ask whether this procedure fails to capture any physics relevant to the calculation. For example, one could add to the amplitude $\mathcal{A}(k)$ a contribution proportional to any positive power of k^2, which would vanish if $k^2 = 0$. Suppose we add a term proportional to k^2; then multiplying by a free propagator $1/k^2$ and Fourier transforming gives a Dirac delta-function in position space. Similarly, higher powers of k^2 correspond to derivatives of the delta-function in position space. This signifies that these terms are relevant for physics very close to the location of the D-brane, and do not affect the large distance behaviour of the supergravity fields.

7.3 The Two-Charge D1-P Amplitude

We consider type IIB string theory on $\mathbb{R}^{1,4} \times S^1 \times T^4$. We denote the 10D coordinates (x^μ, ψ^μ) by $\mu, \nu = t, y, 1, \ldots, 8$. We use $(i, j, \ldots)$ and $x^1, \ldots, x^4$ for the $\mathbb{R}^4$ directions, we use $(a, b, \ldots)$ and $x^5, \ldots, x^8$ for the T^4 directions and we use $(I, J, \ldots)$ to refer to the combined $\mathbb{R}^{1,4} \times S^1$ directions. We work in the light-cone coordinates

$$v = (t + y), \qquad u = (t - y) \tag{7.3}$$

constructed from the time and S^1 directions. We consider a D1-brane wrapped around y and carrying a v-dependent travelling wave:

$$D1 \begin{vmatrix} v & u \\ \mathrm{x} & \mathrm{x} \end{vmatrix} \begin{matrix} \mathbb{R}^4 \\ f_i(v) \end{matrix} \begin{vmatrix} T^4 \\ f_a(v) = 0 \end{matrix} \tag{7.4}$$

Here "x" denotes a Neumann direction and f indicates the (v-dependent) position of the D-brane in the Dirichlet directions. From the start we set the profile along the T^4 directions to be trivial, $f_a = 0$. The D1-P amplitude is depicted in (Fig. 7.2). An analogous calculation may be performed for the case of a D5-brane wrapped on the $T^4 \times S^1$ directions, and both these amplitudes contribute to the D1-D5-P amplitude that we discuss in the next section.

The boundary conditions on the worldsheet fields in the open string picture may be expressed in terms of a reflection matrix R as

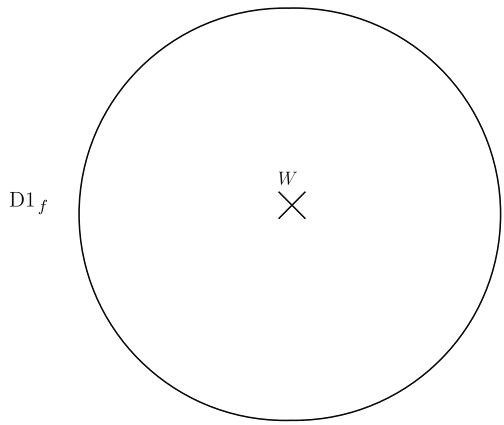


Fig. 7.2 The one-point function for emission of the closed string state W from a disk ending on a D1 brane with profile f

$$\widetilde{\psi}^{\mu} = \eta\, R^{\mu}_{\nu}(\mathbf{v})\psi^{\nu} \tag{7.5}$$

$$\overline{\partial} X^{\mu}_{R} = R^{\mu}_{\nu}(\mathbf{v})\delta X^{\nu}_{L} - \delta^{\mu}_{u}\, 4\alpha' \ddot{f}_{j}(\mathbf{v})\psi^{j}\psi^{v} \tag{7.6}$$

where η can be set to 1 at $\sigma = 0$, while at $\sigma = \pi$ we have $\eta = 1$ or $\eta = -1$ corresponding to the NS and R sectors respectively. In the above we use a bold letter for the string field corresponding to the coordinates

$$\mathbf{x}^{\mu}(z, \bar{z}) \;=\; \frac{1}{2}\Big[X_L(z) + X_R(\bar{z})\Big]. \tag{7.7}$$

The holomorphic and the anti-holomorphic world-sheet fields are then identified with the reflection matrix R where (see [19] and references within)

$$R^{\mu}{}_{\nu} \;=\; \begin{pmatrix} 1 & 0 & 0 & 0 \\ 4|\dot{f}(\mathbf{v})|^2 & 1 & -4\dot{f}_i(\mathbf{v}) & 0 \\ 2\dot{f}_i(\mathbf{v}) & 0 & \mathbb{1} & 0 \\ 0 & 0 & 0 & \mathbb{1} \end{pmatrix}, \tag{7.8}$$

where $\mathbb{1}$ denotes the four-dimensional unit matrix and the indices follow the ordering (v, u, i, a).

The most direct way to derive the one-point functions in the current setup is to use the boundary state formalism [23]. The calculation we now review was carried out in [19] by using the boundary state for a D-brane with a null wave derived in [26–28].

The wrapped D1-brane may be viewed as a set of n_w different D-brane strands, with a non-trivial holonomy gluing these strands together. Each strand carries a segment of the full profile $f^i_{(s)}$, with $s = 1, \ldots, n_w$. The boundary state describing the wrapped D1-brane can be expanded in terms of the closed string perturbative states. The first terms of this expansion are

$$\begin{aligned} |D1; f\rangle = &-i\frac{\kappa\,\tau_1}{2}\sum_{s=1}^{n_w}\int du \int_0^{2\pi R} dv \int \frac{d^4 p_i}{(2\pi)^4}\, e^{-ip_i f^i_{(s)}(v)}\,\frac{c_0 + \widetilde{c}_0}{2} \\ &\times\; c_1\widetilde{c}_1\Big[-\psi^{\mu}_{-\frac{1}{2}}({}^{\mathrm{t}}R)_{\mu\nu}\widetilde{\psi}^{\nu}_{-\frac{1}{2}} + \gamma_{-\frac{1}{2}}\widetilde{\beta}_{-\frac{1}{2}} - \beta_{-\frac{1}{2}}\widetilde{\gamma}_{-\frac{1}{2}} + \ldots\Big] \\ &\times\; |u, v, p_i, 0\rangle_{-1,\widetilde{-1}} \end{aligned} \tag{7.9}$$

where $\tau_1 = [2\pi\alpha' g_s]^{-1}$ is the physical tension of a D1-brane, and where ${}^{\mathrm{t}}R$ is the transpose of R. The ket in (7.9) represents a closed string state obtained by acting on the $SL(2, C)$ invariant vacuum with an $e^{ip_i x^i}$ in the $\mathbb{R}^4$ directions. We wrote the delta functions on the p_u and p_v momenta as integrals in configuration space $du,\ dv$. The boundary state enforces the identification (7.5), which in the approximation (7.9) holds just for the first oscillator $\widetilde{\psi}^{\mu}_{-1/2}$.

The second line of (7.9) contains all the massless NS-NS states; we can separate the irreducible contributions by taking the scalar product with each state. Having done this, the contribution to the NS-NS couplings is

$$\mathcal{A}_{\rm dil}(k) = -\,i\frac{\kappa\,\tau_1}{2}V_u\sqrt{2}\hat{\phi}\int_0^{L_T} d\hat{v}\,e^{-ik\cdot f(\hat{v})}\,, \tag{7.10}$$

$$\begin{aligned}\mathcal{A}_{\rm gra}(k) = -\,i\frac{\kappa\,\tau_1}{2}V_u\int_0^{L_T} d\hat{v}\,e^{-ik\cdot f(\hat{v})}\Big[&-\frac{3}{2}(-\hat{h}_{tt}+\hat{h}_{yy})\\ &+\frac{1}{2}(\hat{h}_{ii}+\hat{h}_{aa})-2\hat{h}_{vv}|\dot{f}|^2+4\hat{h}_{vi}\dot{f}^i\Big]\end{aligned} \tag{7.11}$$

where V_u is the (divergent) volume along the u direction and the integrals over v in each strand in (7.9) have become a single integral over the multi-wound worldvolume coordinate $\hat{v}$ which runs from 0 to $L_T = 2\pi n_w R$.

For the R-R coupling, we simply recall the results of [19]:

$$\mathcal{A}_{\rm RR}(k) = -i\sqrt{2}\kappa\,\tau_1 V_u\int_0^{L_T} d\hat{v}\,e^{-ik\cdot f(\hat{v})}\Big[2\hat{C}^{(2)}_{uv}+\hat{C}^{(2)}_{vi}\dot{f}^i\Big]. \tag{7.12}$$

The next step is to multiply by a free propagator and Fourier transform to find the position-space massless fields. After doing this, one finds agreement with the known D1-P solutions obtained by an S-duality of the solutions of [16, 17]. Further details may be found in [19].

7.4 The Three-Charge D1-D5-P Amplitude

We next consider a D1-D5 bound state carrying a travelling wave; the black hole solution with the same charges has a macroscopic horizon [5], and so this case is more interesting and richer than that of the previous section.

We consider a D1-D5 bound state with a common travelling-wave profile $f_i(v)$ along the branes. The D5-brane is wrapped on the $T^4 \times S^1$.

$$\begin{array}{c|cc|c|c} & v & u & \mathbb{R}^4 & T^4 \\ D1 & \text{x} & \text{x} & f_i(v) & f_a(v)=0 \\ D5 & \text{x} & \text{x} & f_i(v) & \text{x} \end{array} \tag{7.13}$$

The disk amplitude of most interest in this setup is the one where the disk has half its boundary on a D1 and the other half on the D5, and two twisted vertex operator insertions, as studied in [29, 24].

The vertex operators take the form

$$V_\mu = \mu^A e^{-\frac{\varphi}{2}} S_A \,\Delta, \qquad\qquad V_{\bar\mu} = \bar\mu^A e^{-\frac{\varphi}{2}} S_A \,\Delta \tag{7.14}$$

where μ^A and $\bar\mu^A$ are Chan-Paton matrices with $n_1 \times n_5$ and $n_5 \times n_1$ components respectively, S_A are the $SO(1,5)$ spin fields, φ is the free boson appearing in the bosonized language of the worldsheet superghost (β, γ), and Δ is the bosonic twist operator with conformal dimension $\frac{1}{4}$ which acts along the four mixed ND directions and changes the boundary conditions from Neumann to Dirichlet and vice versa.

We focus on open string condensates involving the Ramond sector states only. These states break the $SO(4)$ symmetry of the DD directions $\mathbb{R}^4$, and are invariant under the $SO(4)$ acting on the compact T^4 torus. The most general condensate of Ramond open strings can be written as:

$$\bar\mu^A \,\mu^B = v_I (C\Gamma^I)^{[AB]} + \frac{1}{3!}\, v_{IJK} (C\Gamma^{IJK})^{(AB)}, \tag{7.15}$$

where the parentheses indicate that the first term is automatically antisymmetric, while the second is symmetric. The open string bispinor condensate is thus specified by a one-form v_I and an self-dual three-form v_{IJK}. The self-duality of v_{IJK} follows from $\bar\mu^A$ and μ^B having definite 6D chirality and can be written as

$$v_{IJK} = \frac{1}{3!} \epsilon_{IJKLMN} v^{LMN} \,. \tag{7.16}$$

In this talk we consider only the components of v_{IJK} which have one leg in the t, y directions and two legs in the $\mathbb{R}^4$; this choice of components was associated to considering profiles only in the $\mathbb{R}^4$ directions in [24]. Since the spinors $\bar\mu^A$ and μ^B carry $n_5 \times n_1$ and $n_1 \times n_5$ Chan-Paton indices, the condensate $\bar\mu^A \mu^B$ must be thought of as the vev for the sum

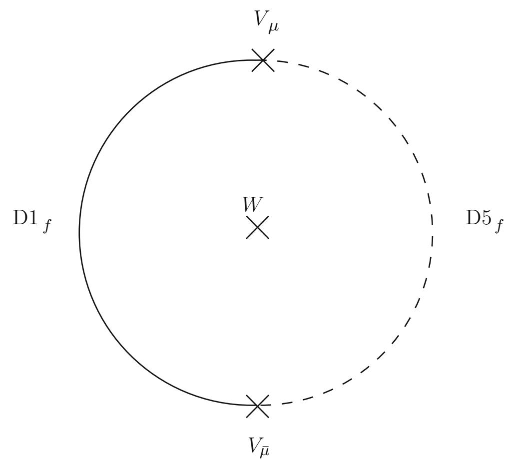
$$\sum_{m=1}^{n_1} \sum_{n=1}^{n_5} \bar\mu^A_{mn} \,\mu^B_{nm} \,, \tag{7.17}$$

which, for generic choices of the Chan-Paton factors, is of order $n_1 n_5$.

The open string insertions (7.14) are related to the vevs of the strings stretched between the D1 and D5 branes, which we are treating perturbatively. The microstates for which we might expect a gravitational description have large open string vevs, so in principle we should resum amplitudes with many twisted vertices. However each pair of open string insertions (7.14) comes with a factor of $1/r$ in the large distance expansion of the corresponding gravity solution [24, 20].

Thus in the following we focus only on the leading contributions at large distances which are induced by the amplitudes with one border and one pair of open vertices

Fig. 7.3 The simplest amplitude involving all three charges of the D1-D5-P microstate: the worldsheet topology is that of a mixed disk diagram where part of the border lies on the D1 brane and part on the D5 brane



V_μ, $V_{\bar{\mu}}$ (see Fig. 7.3). This should be sufficient to derive the sourced supergravity fields up to order $1/r^4$.

Thus the amplitude we now calculate is

$$\mathcal{A}^{\text{D1-D5}}_{NS,R} = \int \frac{\prod_{i=1}^{4} dz_i}{dV_{\text{CKG}}} \left\langle V_\mu(z_1)\, W^{(-k)}_{NS,R}(z_2, z_3)\, V_{\bar{\mu}}(z_4) \right\rangle_f , \tag{7.18}$$

where the subscript f is to remind that, in this disk correlator, the identification between holomorphic and anti-holomorphic components depends on the profile of the D-branes through (7.5–7.8).

In order to have a non-trivial correlator we must saturate the superghost charge (-2) of the disk. The two open string vertices together contribute -1, thus in the NS sector we use the closed string vertex operator in the $(0, -1)$ picture,

$$W^{(k)}_{NS} = \mathcal{G}_{\mu\nu} \left(\partial X^\mu_L - i\, \frac{k}{2} \cdot \psi\, \psi^\mu \right) \mathrm{e}^{i\frac{k}{2}\cdot X_L}(z)\, \widetilde{\psi}^\nu \mathrm{e}^{-\widetilde{\varphi}} \mathrm{e}^{i\frac{k}{2}\cdot X_R}(\bar{z}) + \ldots , \tag{7.19}$$

where the dots stand for other terms that ensure the BRST invariance of the vertex, but that do not play any role in the correlator under analysis.

We will not review the intermediate steps of the calculation here; details are given in [20]. We move on to discuss the spacetime fields which result from the calculation.

7.4.1 New D1-D5-P Geometries

The fields obtained from the calculation of the previous section fit into the following ansatz, which solves the supergravity equations perturbatively[1] in $1/r$ up to $1/r^4$. Using the short-hand notation

$$d\hat{t} = dt + k, \quad d\hat{y} = dy + dt - \frac{dt + k}{Z_3} + a_3\,, \tag{7.20}$$

the ansatz (in the string frame) is

$$\begin{aligned}
ds^2 &= \frac{1}{\sqrt{Z_1 Z_2}}\Big[-\frac{1}{Z_3}\,d\hat{t}^2 + Z_3\,d\hat{y}^2\Big] + \sqrt{Z_1 Z_2}\,ds_4^2 + \sqrt{\frac{Z_1}{Z_2}}\,ds^2_{T^4},\\
B &= -Z_4\,d\hat{t}\wedge d\hat{y} + a_4\wedge(d\hat{t} + d\hat{y}) + \delta_2\,,\\
e^{2\phi} &= \frac{Z_1}{Z_2} \qquad C^{(0)} \;=\; Z_4,\\
C^{(2)} &= -\frac{1}{Z_1}d\hat{t}\wedge d\hat{y} + a_1\wedge(d\hat{t} + d\hat{y}) + \gamma_2\,,\\
F^{(5)} &= dZ_4\wedge dz^4 + \frac{Z_2}{Z_1}\ *_4\,dZ_4\wedge d\hat{t}\wedge d\hat{y}\,,
\end{aligned} \tag{7.21}$$

where ds_4^2 is a generic Euclidean metric on $\mathbb{R}^4$; $ds^2_{T^4}$ is the flat metric on T^4; Z_I are 0-forms, k, a_I are 1-forms, and γ_2, δ_2 are 2-forms on $\mathbb{R}^4$. The above quantities are subject to the conditions

$$d\delta_2 = *_4 da_4\,, \quad d\gamma_2 = *_4 dZ_2\,, \tag{7.22}$$

and we take the asymptotic boundary conditions

$$\begin{aligned}
Z_1,\ Z_2,\ Z_3 &= 1 + O(r^{-2}), \qquad & Z_4 \;&=\; O(r^{-4}),\\
k,\ a_1,\ a_3,\ a_4 &= O(r^{-3}), & ds_4^2 \;&=\; dx_i dx_i + O(r^{-4}).
\end{aligned} \tag{7.23}$$

The above fields satisfy the approximate supergravity Killing spinor equations up to order $1/r^4$. It turns out, however, that one can keep the full r dependence of the string results and still satisfy the approximate supergravity Killing spinor equations, to linear order in the condensate v_{IJK}.

The full r dependence of the supergravity fields describes the small $g_s N$ and small $\mathbf{v}_{IJK}$ limit, i.e. the weak gravity regime and the region of the Higgs branch infinitesimally close to its intersection with the Coulomb branch. If one is interested

[1] This ansatz was later extended to a full non-linear supergravity ansatz in [30]. Solutions have been studied in [31].

in the black hole regime (large $g_s N$ and finite $\mathbf{v}_{IJK}$), one should keep only the large r limit (up to $1/r^4$ order) of the following results.

We next review the most interesting features of the string amplitude; for the full set of fields see [20]. If we set Z_4 and a_4 to zero, the ansatz reduces to that in [18]. These 'new' fields are thus the most interesting. The disk amplitude gives

$$Z_4 = -\mathbf{v}_{ujk}\,\partial_j \left[\frac{1}{L_T} \int_0^{L_T} d\hat{v}\, \frac{\dot{f}_k}{|x^i - f^i|^2} \right], \tag{7.24}$$

$$a_4 = \mathbf{v}_{uij}\,\partial_j \left[\frac{1}{L_T} \int_0^{L_T} d\hat{v}\, \frac{|\dot{f}|^2}{|x^i - f^i|^2} \right] dx^i \tag{7.25}$$

where we have absorbed some factors multiplying the open string condensate,

$$\mathbf{v}_{IJK} \;=\; -\frac{2\sqrt{2} n_w \kappa}{\pi V_4} v_{IJK} \,. \tag{7.26}$$

Note that the new fields above vanish in either of the two-charge limits in which we set either v_{IJK} or f to zero.

Another interesting outcome of our calculation is that it predicts that the 4D base metric ds_4^2, which is simply the flat metric on $\mathbb{R}^4$ in the 2-charge case, is a *non-trivial* hyper-Kähler metric when all three charges are present. The base metric which arises from the string amplitude is

$$ds_4^2 = \Big(\delta_{ij} + \mathbf{v}_{uli}\,\partial_l \mathcal{I}_j + \mathbf{v}_{ulj}\,\partial_l \mathcal{I}_i - \delta_{ij}\,\mathbf{v}_{ulk}\,\partial_l \mathcal{I}_k\Big)\, dx^i dx^j \,, \tag{7.27}$$

where

$$\mathcal{I}_j = \frac{1}{L_T} \int_0^{L_T} d\hat{v}\, \frac{\dot{f}_j}{|x^i - f^i|^2} \,. \tag{7.28}$$

The non-flatness of the base metric for 3-charge microstate geometries was previously observed in the particular solution of [32], but had remained until now largely unexplained. It is nice to see that the disk amplitudes lead directly to this feature.

7.5 Summary

In this talk we have seen how disk amplitudes can be used to derive the asymptotic supergravity fields sourced by bound states of D-branes. At large distances from the bound state, the effective open string coupling is small, even in the regime of parameters in which there is a classical black hole solution with the same charges.

The supergravity fields differ from the black hole solution by various multipole moments, suggesting that the D1-D5-P black hole solution is not an exact description of the gravitational fields sourced by individual microstates. Rather the black hole solution is likely to be an approximate thermodynamic description of the entire system. Thus the results reviewed here support the fuzzball proposal.

It would be interesting to apply the techniques reviewed here to other D-brane bound states, and we hope that this will lead to an improved understanding of the physics of black holes in string theory.

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References

1. J.D. Bekenstein, Black holes and entropy. Phys. Rev. **D7**, 2333–2346 (1973)
2. S.W. Hawking, Particle creation by black holes. Commun. Math. Phys. **43**, 199–220 (1975)
3. S.W. Hawking, Breakdown of predictability in gravitational collapse. Phys. Rev. **D14**, 2460–2473 (1976)
4. A. Sen, Extremal black holes and elementary string states. Mod. Phys. Lett. **A10**, 2081–2094 (1995). arXiv:hep-th/9504147
5. A. Strominger, C. Vafa, Microscopic origin of the Bekenstein-Hawking entropy. Phys. Lett. **B379**, 99–104 (1996). arXiv:hep-th/9601029
6. C.G. Callan, J.M. Maldacena, D-brane approach to black hole quantum mechanics. Nucl. Phys. **B472**, 591–610 (1996). arXiv:hep-th/9602043
7. S.D. Mathur, The information paradox: a pedagogical introduction. Class. Quant. Grav. **26**, 224001 (2009). arXiv:0909.1038 [hep-th]
8. S.D. Mathur, The fuzzball proposal for black holes: an elementary review. Fortsch. Phys. **53**, 793–827 (2005). arXiv:hep-th/0502050
9. S.D. Mathur, Fuzzballs and the information paradox: a summary and conjectures. (2008). arXiv:0810.4525 [hep-th]
10. I. Bena, N.P. Warner, Black holes, black rings and their microstates. Lect. Notes Phys. **755**, 1–92 (2008). arXiv:hep-th/0701216
11. K. Skenderis, M. Taylor, The fuzzball proposal for black holes. Phys. Rept. **467**, 117–171 (2008). arXiv:0804.0552 [hep-th]
12. V. Balasubramanian, J. de Boer, S. El-Showk, I. Messamah, Black Holes as Effective Geometries. Class. Quant. Grav. **25**, 214004 (2008). arXiv:0811.0263 [hep-th]
13. B.D. Chowdhury, A. Virmani, Modave lectures on fuzzballs and emission from the D1-D5 system. (2010). arXiv:1001.1444 [hep-th]
14. I. Bena, C.-W. Wang, N.P. Warner, Mergers and typical black hole microstates. JHEP **11**, 042 (2006). arXiv:hep-th/0608217
15. A. Sen, Two charge system revisited: small black holes or horizonless solutions?. (2009). arXiv:0908.3402 [hep-th]
16. A. Dabholkar, J.P. Gauntlett, J.A. Harvey, D. Waldram, Strings as solitons and black holes as strings. Nucl. Phys. **B474**, 85–121 (1996). arXiv:hep-th/9511053
17. C.G. Callan, J.M. Maldacena, A.W. Peet, Extremal black holes as fundamental strings. Nucl. Phys. **B475**, 645–678 (1996). arXiv:hep-th/9510134
18. I. Bena, N.P. Warner, One ring to rule them all...and in the darkness bind them? Adv. Theor. Math. Phys. **9**, 667–701 (2005). arXiv:hep-th/0408106
19. W. Black, R. Russo, D. Turton, The supergravity fields for a D-brane with a travelling wave from string amplitudes. Phys. Lett. **B694**, 246–251 (2010). arXiv:1007.2856 [hep-th]

20. S. Giusto, R. Russo, D. Turton, New D1–D5-P geometries from string amplitudes. JHEP **11**, 062 (2011). arXiv:1108.6331 [hep-th]
21. M.R. Garousi, R.C. Myers, Superstring scattering from D-branes. Nucl. Phys. **B475**, 193–224 (1996). arXiv:hep-th/9603194 [hep-th]
22. A. Hashimoto, I.R. Klebanov, Scattering of strings from D-branes., Nucl. Phys. Proc. Suppl. **55B**, 118–133 (1997). arXiv:hep-th/9611214 [hep-th]
23. P. Di Vecchia et al., Classical p-branes from boundary state. Nucl. Phys. **B507**, 259–276 (1997). arXiv:hep-th/9707068
24. S. Giusto, J.F. Morales, R. Russo, D1D5 microstate geometries from string amplitudes. JHEP **03**, 130 (2010). arXiv:0912.2270 [hep-th]
25. M.J. Duff, Quantum tree graphs and the Schwarzschild solution. Phys. Rev. **D7**, 2317–2326 (1973)
26. Y. Hikida, H. Takayanagi, T. Takayanagi, Boundary states for D-branes with traveling waves. JHEP **04**, 032 (2003). arXiv:hep-th/0303214
27. J.D. Blum, Gravitational radiation from travelling waves on D-strings. Phys. Rev. **D68**, 086003 (2003). arXiv:hep-th/0304173
28. C.P. Bachas, M.R. Gaberdiel, World-sheet duality for D-branes with travelling waves. JHEP **03**, 015 (2004). arXiv:hep-th/0310017
29. M. Billo et al., Classical gauge instantons from open strings. JHEP **02**, 045 (2003). arXiv:hep-th/0211250
30. S. Giusto, R. Russo, Adding new hair to the 3-charge black ring. (2012). arXiv:1201.2585 [hep-th]
31. O. Vasilakis, Bubbling the newly grown black ring hair. (2012). arXiv:1202.1819 [hep-th]
32. S. Giusto, S.D. Mathur, Geometry of D1-D5-P bound states. Nucl. Phys. **B729**, 203–220 (2005). arXiv:hep-th/0409067

Chapter 8
From Clock Synchronization to Dark Matter as a Relativistic Inertial Effect

Luca Lusanna

8.1 Introduction

One of the main open problems in astrophysics is the dominance of *dark* entities, the dark matter and the dark energy, in the existing description of the universe given by the standard ΛCDM cosmological model [1, 2] based on the cosmological principle (homogeneity and isotropy of the space-time), which selects the class of Friedmann-Robertson-Walker (FWR) space-times. After the transition from quantum cosmology to classical astrophysics, with the Heisenberg cut roughly located at a suitable cosmic time ($\approx 10^5$ years after the big bang) and at the recombination surface identified by the cosmic microwave background (CMB), one has a description of the universe in which the known forms of baryonic matter and radiation contribute only with a few percents of the global budget. One has a great variety of models trying to explain the composition of the universe in accelerated expansion (based on data on high red-shift supernovae, galaxy clusters and CMB): WIMPS (mainly super-symmetric particles), $f(R)$ modifications of Einstein gravity (with a modified Newton potential), MOND (with a modification of Newton law),... for dark matter; cosmological constant, string theory, back-reaction (spatial averages, non-linearity of Einstein equations), inhomogeneous space-times (Lemaitre-Tolman-Bondi, Szekeres), scalar fields (quintessence, k-essence, phantom), fluids (Chaplygin fluid), ... for dark energy.

Most of these developments rest on a description based on a family of FRW space-times with nearly flat 3-spaces (as required by CMB data) as the reference space-times where to interpret the astronomical data (luminosity, light spectrum, angles) on the 2-dimensional sky vault. Therefore, the starting point is the extension of the standards of relativistic metrology near the Earth and in the Solar System to astronomy: to reconstruct a 4-dimensional space-time one needs new standards of time and length

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like the cosmic time and the luminosity distance (or any other astrometric definition, see Ref. [3, 4]) allowing to define an International Celestial Reference System (ICRS) [5–9], namely a 4-coordinate system describing a 3-universe evolving in time, where the astronomical data have to be dynamically interpreted according to Einstein gravity or some of its extensions.

The aim of this Lecture is to suggest a new viewpoint on the origin of dark matter, and maybe also of dark energy, starting from a re-reading of the general covariance of Einstein general relativity (GR), which could be also applied to every generally covariant extension of this theory if needed. It is an extended version of the review paper [10]. In this Introduction one will delineate the framework of our approach and then in the subsequent Sections one will give more details of the various topics.

The gauge group of the Lagrangian formulation of Einstein GR, the diffeomorphism group, implies that the 4-coordinates of the space-time are *gauge variables*. As a consequence, the search of GR observables is restricted to *4-scalars* and at the theoretical level one tries to describe gravitational dynamical properties in term of them. However, inside the Solar System the experimental localization of macroscopic classical objects is unavoidably done by choosing some *convention* for the local 4-coordinates of space-time. Atomic physicists, NASA engineers and astronomers have chosen a series of reference frames and standards of time and length suitable for the existing technology [11–13]. These conventions determine certain Post-Minkowskian (PM) 4-coordinate systems of an asymptotically Minkowskian space-time, in which the instantaneous 3-spaces are not strictly Euclidean. Then these reference frames are seen as a local approximation of a reference frame in ICRS, where however the space-time has become a cosmological FWR one, which is only conformally asymptotically Minkowskian at spatial infinity. A search of a consistent patching of the 4-coordinates from inside the Solar System to the rest of the universe will start when the data from the future GAIA mission [14] for the cartography of the Milky Way will be available. This will allow a PM definition of a Galactic Reference System containing at leat our galaxy. Let us remark that notwithstanding the FRW instantaneous 3-spaces are not strictly Euclidean, all the books on galaxy dynamics describe the galaxies by means of Kepler theory in Galilei space-time.

This state of affairs requires to revisit Einstein GR to see whether it is possible to identify which components of the 4-metric tensor are connected with the gauge freedom in the choice of the 4-coordinates and which ones describe the dynamical degrees of freedom of the gravitational field. Since this cannot be done at the Lagrangian level, one must restrict himself to the class of globally hyperbolic, asymptotically flat space-times allowing a Hamiltonian description starting from the description of Einstein GR in terms of the ADM action [15] instead than in terms of the Einstein-Hilbert one. In canonical ADM gravity one can use Dirac theory of constraints [16, 17] to describe the Hamiltonian gauge group, whose generators are the first-class constraints of the model. The basic tool of this approach is the possibility to find so-called Shanmugadhasan canonical transformations [18–21], which identify special canonical bases adapted to the first-class constraints (and also to the second-class ones when present). In these special canonical bases the vanishing of certain momenta (or of certain configurational coordinates) corresponds to the vanishing of well defined Abelianized combinations of the first-class constraints (Abelianized because the new constraints have exactly zero Poisson brackets even if the original constraints were not in strong involution). As a consequence, the variables conjugate to these Abelianized constraints are inertial Hamiltonian gauge variables describing the Hamiltonian gauge freedom. The remaining $2+2$ conjugate variables describe the dynamical tidal degrees of freedom of the gravitational field (the two polarizations of gravitational waves in the linearized theory). If one would be able to include all the constraints in the Shanmugadhasan canonical basis, these $2+2$ variables would be the *Dirac observables* of the gravitational field, invariant under the Hamiltonian gauge transformations. However such Dirac observables are not known: one only has statements about their existence [22–26]. Moreover, in general they are not 4-scalar observables. The problem of the connection between the 4-diffeomorphism group and the Hamiltonian gauge group was studied in Ref. [27–32] by means of the inverse Legendre transformation and of the notion of dynamical symmetry. The conclusion is that on the space of solutions of Einstein equations there is an overlap of the two types of observables: there should exists special Shanmugadhasan canonical bases in which the $2+2$ Dirac observables become 4-scalars when restricted to the space of solutions of the Einstein equations. In any case the identification of the inertial gauge components of the 4-metric is what is needed to make a fixation of 4-coordinates as required by relativistic metrology.

Another problem is that asymptotically flat space-times have the SPI group of asymptotic symmetries (direction-dependent asymptotic Killing symmetries) [33] and this is an obstruction to the existence of asymptotic Lorentz generators for the gravitational field [34, 35]. However if one restricts the class of space-times to those *not containing super-translations* [36], then the SPI group reduces to the asymptotic ADM Poincaré group [37–39]: these space-times are *asymptotically Minkowskian*, they contain an asymptotic Minkowski 4-metric (to be used as an *asymptotic background* at spatial infinity in the linearization of the theory) and they have asymptotic inertial observers at spatial infinity whose spatial axes may be identified by means of the fixed stars of star catalogues.[1] Moreover, in the limit of vanishing Newton constant ($G = 0$) the asymptotic ADM Poincaré generators become the generators of the special relativistic Poincaré group describing the matter present in the space-time. This is an important condition for the inclusion into GR of the classical version of the standard model of particle physics, whose properties are all connected with the representations of this group in the inertial frames of Minkowski space-time. In absence of matter a sub-class of these space-times is the (singularity-free) family of Chrstodoulou-Klainermann solutions of Einstein equations [40] (they are near to Minkowski space-time in a norm sense and contain gravitational waves).

[1] The fixed stars can be considered as an empirical definition of spatial infinity of the observable universe.

Moreover, in this restricted class of space-times the canonical Hamiltonian is the ADM energy [41–44], so that there is no *frozen picture* like in the "spatially compact space-times without boundaries" used in loop quantum gravity.[2]

To take into account the fermion fields present in the standard particle model one must extend ADM gravity to ADM tetrad gravity. Since our class of space-times admits orthonormal tetrads and a spinor structure [46], the extension can be done by simply replacing the 4-metric in the ADM action with its expression in terms of tetrad fields, considered as the basic 16 configurational variables substituting the 10 metric fields.

To study ADM tetrad gravity the preliminary problem is to choose a coordinatization of the space-time compatible with relativistic metrology. This requires a definition of global non-inertial frames, because the equivalence principle forbids the existence of global inertial frames in GR. Due to the Lorentz signature of the space-time this is a non-trivial task already in special relativity (SR): there is no notion of instantaneous 3-space, because the only intrinsic structure is the conformal one, i.e. the light-cone as the locus of incoming and outgoing radiation. The existing coordinatizations, like either Fermi or Riemann-normal coordinates, hold only locally. They are based on the *1+3 point of view*, in which only the world-line of a time-like observer is given. In each point of the world-line the observer 4-velocity determines an orthogonal 3-dimensional space-like tangent hyper-plane, which is identified with an instantaneous 3-space. However, these tangent planes intersect at a certain distance from the world-line (the so-called acceleration length depending upon the 4-acceleration of the observer [47, 48]), where 4-coordinates of the Fermi type develop a coordinate singularity. Another type of coordinate singularity is developed in rigidly rotating coordinate systems at a distance r from the rotation axis where $\omega\, r = c$ (ω is the angular velocity and c the two-way velocity of light). This is the so-called "horizon problem of the rotating disk": a time-like 4-velocity becomes a null vector at $\omega\, r = c$, like it happens on the horizon of a black-hole. See Ref. [49–52] for a classification of the possible pathologies of non-inertial frames and on how to avoid them.

In this Lecture one will review the way out from these problems based on the $3+1$ *point of view* in which, besides the world-line of a time-like observer, one gives a global nice foliation of the space-time with instantaneous 3-spaces. Then a metrology-oriented notion of 4-coordinates, the so-called *radar 4-coordinates* first introduced by Bondi [53, 54], is introduced in these global non-inertial frames. One will give the conditions for a foliation to be nice, i.e. for the absence of pathologies like the ones of the rotating disk and of the Fermi coordinates.

[2] In these space-times the canonical Hamiltonian vanishes and the Dirac Hamiltonian is a combination of first-class constraints, so that it only generates Hamiltonian gauge transformations. In the reduced phase space, quotient with respect the Hamilonian gauge group, the reduced Hamiltonian is zero and one has a *frozen picture* of dynamics. This class of space-times fits well with Machian ideas (no boundary conditions) and with interpretations in which there is no physical time like the one in Ref. [45]. However, it is not clear how to include in this framework the standard model of particle physics.

Let us remark that the theory of global non-inertial frames is also needed to speak of predictability in a (either classical or quantum) theory in which the basic equations of motion are partial differential equations (PDE). To be able to use the existence and unicity theorem for the solutions of PDE's, one needs a well-posed Cauchy problem, whose prerequisite is a sound definition of an instantaneous 3-space (i.e. of a clock synchronization convention) where the Cauchy data are given. To give the data on a space-like surface is *not factual*, but with the data on the backward light-cone of an observer it is not yet possible to demonstrate the theorem. However, also the $1 + 3$ point of view is non factual, because it requires the knowledge of a world-line from the whole past to all the future.

A Section of this Lecture will be devoted to the developments in relativistic particle mechanics made possible by the $3 + 1$ point of view in SR [49–52, 55–57]. By means of *parametrized Minkowski theories* [49–52, 55], one can get the description of arbitrary isolated systems (particles, strings, fluids, fields) admitting a Lagrangian formulation in arbitrary non-inertial frames with the transition among non-inertial frames described as a "gauge transformation" (general covariance under the frame-preserving diffeomorphisms of Ref. [58]). Moreover this framework allows us to define the *inertial and non-inertial rest frames* of the isolated systems, where to develop the rest-frame instant form of the dynamics and to build the explicit form of the Lorentz boosts for interacting systems. This makes possible to study the problem of the relativistic center of mass [59–61], relativistic bound states [62–67], relativistic kinetic theory and relativistic micro-canonical ensemble [68] and various other systems [69–75]. Moreover a Wigner-covariant relativistic quantum mechanics [76], with a solution of all the known problems introduced by SR, has been developed after some preliminary work done in Ref. [77, 78]. This will allow us to study relativistic entanglement.

After this digression in SR one defines global non-inertial frames with radar 4-coordinates in the asymptotically Minkowskian space-times of GR[3] and one gives the parametrization of the tetrads and of the 4-metric in them. The absence of super-translations implies that these non-inertial frames are non-inertial rest frames of the 3-universe. Starting from the ADM action for tetrad gravity one defines the Hamiltonian formalism in a phase space containing 16 configurational field variables and 16 conjugate moments. One identifies the 14 first-class constraints of the system and one finds that the canonical Hamiltonian is the weak ADM energy (it is given as a volume integral over 3-space). The existence of these 14 first-class constraints implies that 14 components of the tetrads (or of the conjugate momenta) are Hamiltonian gauge variables describing the *inertial* aspects of the gravitational field (6 of these inertial variables describe the extra gauge freedom in the choice of the tetrads and in their transport along world-lines). Therefore there are only $2+2$ degrees of freedom for the description of the *tidal* dynamical aspects of the gravitational field. The asymptotic ADM Poincaré generators can be evaluated explicitly. Till now the type of matter

[3] While in SR Minkowski space-time is an absolute notion, unifying the absolute notions of time and 3-space of the non-relativistic Galilei space-time, in GR there is no absolute notion: space-time becomes dynamical [79–81] with its metric structure satisfying Einstein equations.

studied in this framework consists of the electro-magnetic field and of N charged scalar particles, whose signs of the energy and electric charges are Grassmann-valued to regularize both the gravitational and electro-magnetic self-energies (it is both a ultraviolet and an infrared regularization),

Then it will be shown that there is a Shanmugadhasan canonical transformation [82] (implementing the so-called York map [83] and diagonalizing the York-Lichnerowics approach [84]) to a so-called York canonical basis adapted to 10 of the 14 first-class constraints. Only the super-Hamiltonian and super-momentum constraints, whose general solution is not known, are not included in the basis, but it is clarified which variables are to be determined by their solution. Among the inertial gauge variables there is the York time [85–87] ${}^3K$, i.e. the trace of the extrinsic curvature of the 3-spaces as 3-manifolds embedded into the space-time. It is the only gauge variable which is a momentum in the York canonical basis[4]: this is due to the Lorentz signature of space-time, because the York time and three other inertial gauge variables can be used as 4-coordinates of the space-time (see Ref. [79–81] for this topic and for its relevance in the solution of the hole argument). Therefore an identification of the inertial gauge variables to be fixed to get a 4-coordinate system in relativistic metrology was found. In the first paper of Ref. [88–90] there is the expression of the Hamilton equations for all the variables of the York canonical basis.

An important remark is that in the framework of the York canonical basis the natural family of gauges is not the harmonic one, but the family of 3-orthogonal Schwinger time gauges in which the 3-metric in the 3-spaces is diagonal.

Both in SR and GR an admissible $3+1$ splitting of space-time has two associated congruences of time-like observers [49–52], geometrically defined and not to be confused with the congruence of the world-lines of fluid elements, when relativistic fluids are added as matter in GR [91–94]. One of the two congruences, with zero vorticity, is the congruence of the Eulerian observers, whose 4-velocity field is the field of unit normals to the 3-spaces. This congruence allows us to re-express the non-vanishing momenta of the York canonical basis in terms of the expansion ($\theta = -{}^3K$) and of the shear of the Eulerian observers. This allows us to compare the Hamilton equations of ADM canonical gravity with the usual first-order non-Hamiltonian ADM equations deducible from Einstein equations given a $3+1$ splitting of space-time but without using the Hamiltonian formalism. As a consequence, one can extend our Hamiltonian identification of the inertial and tidal variables of the gravitational field to the Lagrangian framework and use it in the cosmological (conformally asymptotically flat) space-times: in them it is not possible to formulate the Hamiltonian formalism but the standard ADM equations are well defined. The time inertial gauge variable needed for relativistic metrology is now the expansion of the Eulerian observers of the given $3+1$ splitting of the globally hyperbolic cosmological space-time.

The next step (see the second paper of Ref. [88–90]) is the definition of a PM linearization of ADM tetrad gravity in the family of 3-orthogonal Schwinger time gauges in which one chooses 3-coordinates diagonalizing the 3-metric in the 3-spaces

[4] Instead in Yang-Mills theory all the gauge variables are configurational.

and an arbitrarily given numerical function for the York time ${}^3K$. The cosmic time τ_{cosm} has to be chosen so that the 3-spaces $\tau_{cosm} = const.$ have an extrinsic curvature with the given value of ${}^3K$. This PM linearization uses the asymptotic Minkowski 4-metric as an *asymptotic background*, so that one never splits the 4-metric with respect to a fixed Minkowski metric in the bulk like in the standard approach to gravitational waves. A ultraviolet cutoff on the matter is needed.

This leads to a PM formulation of gravitational waves in non-harmonic 3-orthogonal gauges. All the constraints can be solved, an explicit expression of the PM 4-metric can be given and the explicit form of the Hamilton equations for the tidal degrees of freedom of the gravitational field and the matter can be obtained. It is non-trivial to show that all the standard results about gravitational waves in harmonic gauges [95] can be reproduced in the 3-orthogonal gauges with the help of the formalism of Ref. [96]. As shown in the third paper of Ref. [88–90] (where the matter is restricted only to scalar particles), all the 4- and 3- curvature tensors of the space-time can be explicitly evaluated and the time-like and null geodesics can be studied. It is also possible to evaluate the red-shift of light rays and the luminosity distance finding their dependence on the York time and verify the old Hubble red-shift-distance law (see Ref. [97]), which becomes the usual Hubble law (a velocity-distance relation) when one uses the standard cosmological model. In the Solar System the results in the 3-orthogonal gauges are compatible with the ones in the harmonic gauges used in relativistic metrology [11–13].

The main important result or this lecture are the PM Hamilton equations and the implied PM second-order equations of motion for the particles. Their PN limit identifies the Newton forces acting on the particles at the lowest order augmented with 1PN forces compatible with the known results on binaries in harmonic gauges [98, 99]. However, there are extra 0.5PN forces, depending linearly on the non-local York time ${}^3\mathcal{K} = \frac{1}{\triangle}\,{}^3K$ ($\triangle$ is the asymptotic Laplacian of the 3-space), representing either a friction or an anti-friction force according to the sign of ${}^3\mathcal{K}$. These 0.5PN forces are our main result, because their effect can be re-interpreted as an extra *effective* (time-, position- and velocity-dependent) *contribution to the inertial mass* of the particles in the equations used in the three main signatures for the existence of dark matter: the rotation curves of spiral galaxies [100–102] and the masses of clusters of galaxies from the virial theorem [103–105] and from weak gravitational lensing [103–107].

While gravitational and inertial masses are equal in Einstein GR, the PM limit, followed by the PN one, shows that the non-Euclidean nature of the 3-spaces implies a breaking of the Newtonian equality of the two types of masses, which holds only in the absolute Euclidean 3-space of Galilei space-time.

As a consequence the data on dark matter can be re-read as a partial fixation of the non-local York time ${}^3\mathcal{K}$. However to fix the York time ${}^3K = \triangle\,{}^3\mathcal{K}$ one needs a global information on ${}^3\mathcal{K}$ on the whole 3-space, in particular in the voids among galaxy clusters.

Therefore one has an indication that (at least part of) dark matter could be re-absorbed in a PM extension of the conventions in the existing ICRS, such that the 3-spaces $\tau_{ICRS} = const.$ determined by a suitable ICRS time have a York time ${}^3K$ such that the derived non-local York time reproduces the data for the signatures of dark matter.

In the Conclusions it will be suggested that also the open problem of dark energy could be rephrased as the determination of a suitable York time in inhomogeneous cosmological space-times. Therefore there is the possibility of an understanding of the "dark" aspects of the universe in terms of relativistic metrology.

8.2 Relativistic Metrology

As shown in Ref. [108] modern relativistic metrology is not only deeply rooted in Maxwell theory and its quantization but is also beginning to take into account GR.

The basic metrological conventions on the Earth surface are:

(a) An atomic clock as a standard of time. The fundamental conceptual time scale is the *SI atomic second*: it is the duration of 9 192 631 770 periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of the cesium 133 atom. This definition refers to a cesium atom at rest at a temperature of 0^o. However the practical time standard is the International Atomic Time (TAI), which is defined as a suitable weighted average of the SI time kept by over 200 atomic clocks in about 70 national laboratories worldwide. Time scales connected with TAI are the GPS Station Time and the Universal Time (UC) based on Earth's rotation.[5] All the other existing time scales inside the Solar System are connected to this standard by fixed conventions.

(b) The 2-way velocity of light (only one clock is involved in its definition), fixed to the value $c = 299\,792\,458\,\mathrm{m\,s^{-1}}$, in place of the standard of length.[6] To measure the 3-distance between two objects in an inertial frame one sends a ray of light from the first object, to which is associated an atomic clock, to the second one, where it is reflected and then reabsorbed by the first object. The measure of the flight time and the 2-way velocity of light determine the 3-distance between the objects. This convention is compatible with the Euclidean 3-space of inertial frames in Minkowski space-time. When the technology will allow one to measure the deviations from Euclidean 3-space implied by PN gravity one will need a modified convention taking into account a general relativistic notion of length.

Given these standards one can think to the Global Positioning System (GPS) as a local standard of space-time. To define GPS one needs a conventional reference frame centered on a given time-like observer. Inside the Solar System one has well defined conventions for the following reference frames:

[5] It is based on Very Long Baseline Interferometry (VLBI) observations of distant quasars, on Lunar Laser Ranging (LLR) and on determination of GPS satellite orbits.

[6] The *meter* is the length of the path traveled by light in vacuum during a time interval of $1/c$ of a second.

(A) The description of satellites around the Earth is done by means of NASA coordinates either in the International Terrestrial Reference System (ITRS; it is a frame fixed on the Earth surface) or in the Geocentric Celestial Reference System (GCRS) centered on the world-line of the Earth center (see Ref. [11–13]). Both of them use a *geocentric coordinate time* t_G connected to TAI.

(B) The description of planets and other objects in the Solar System uses the Barycentric Celestial Reference System (BCRS), centered in the barycenter of the Solar System (see Ref. [11–13]). It uses a *barycentric coordinate time* t_B connected to t_G and TAI.

While ITRS is essentially realized as a non-relativistic non-inertial frame in Galilei space-time, BCRS is defined as a *quasi-inertial frame*, *non-rotating* with respect to some selected fixed stars, in Minkowski space-time with nearly-Euclidean 3-spaces (one ignores the perturbations induced from the Milky Way). It can also be considered as a PM space-time with 3-spaces having a very small extrinsic curvature of order c^{-2}. GCRS is obtained from BCRS by taking into account Earth's rotation around the Sun with a suitable Lorentz boost with corrections from PN gravity.[7] By taking into account the extension of the geoid and Earth revolution around its axis one goes from the quasi-Minkowskian GCRS to the quasi-Galilean ITRS.

New problems emerge by going outside the Solar System. In astronomy the positions of stars and galaxies are determined from the data (luminosity, light spectrum, angles) on the sky, i.e. on a 2-dimensional spherical surface around the Earth with the relations between it and the observatory on the Earth done with GPS. To get a description of stars and galaxies as living in a 4-dimensional space-time one introduces the International Celestial Reference System ICRS (see Refs. [5–9]). Its time scale is a "second" connected to GPS, TAI and SI and therefore to t_G and t_B. ICRS has the origin in the solar system barycenter, which is considered as quasi-inertial observer carrying a quasi-inertial (essentially non-relativistic) reference frame with rectangular 3-coordinates in a *nearly Galilei space-time* whose 3-spaces are nearly Euclidean. The directions of the spatial axes are effectively defined by the adopted coordinates of 212 extragalactic radio sources observed by VLBI . These radio sources (quasars and AGN, active galactic nuclei) are assumed to have no observable intrinsic angular momentum. Thus, the ICRS is a *space-fixed* system, more precisely a *kinematically non-rotating* system, which provides the orientation of BCRS.

In astronomy the unit of length is the *astronomical unit AU*, approximately equal to the mean Earth-Sun distance. Measurements of the relative positions of planets in the Solar System are done by radar: one measures the time taken for light to be reflected from an object using the conventional value of the velocity of light c. Both for objects inside the Solar System and for the nearest stars one measure the distance with the *trigonometric parallax* by using the propagation of light and its velocity c in inertial frames. One measures the difference (the inclination angle) in the apparent position of an object viewed along two different lines of sight at two different times and then uses Euclidean geometry to evaluate the distance. The used unit in astronomy is the parsec, which is 3.26 light-years or 3.26×10^{16} m.

[7] See Ref. [109] for possible gravitational anomalies inside the Solar System.

This convention cannot be used for more distant either galactic or extra-galactic objects. New notions like standard candles, dynamical parallax, spectroscopic parallax, luminosity distance,... are needed [3, 4]. These notions involve both aspects of light propagation in curved space-times and cosmological assumptions like the Hubble law (velocity-red-shift linear relation).

However if one takes into account the description of the universe given by cosmology, the actual cosmological space-time cannot be a nearly Galilei space-time but it must be a cosmological space-time with some theoretical *cosmic time*. In the standard cosmological model [1, 2] it is a homogeneous and isotropic FRW space-time whose instantaneous 3-spaces have nearly vanishing internal 3-curvature, so that they may locally be replaced with Euclidean 3-spaces as it is done in galactic dynamics. However they have a time-dependent *conformal factor* (it is one in Galilei space-time) responsible for the Hubble constant regulating the expansion of the universe. As a consequence the transition from the astronomical ICRS to an astrophysical description taking into account cosmology is far from being understood.

What is still lacking is a PM extension of the celestial frame such that the PM BCRS frame is its restriction to the solar system inside our galaxy. In particular one needs the definition of a coordinate time t_{ICRS} connected to t_B such that the 3-spaces $t_{ICRS} = const.$ have a very small internal 3-curvature and a suitable extrinsic curvature as sub-manifolds of the space-time connected with the Hubble constant. In this way this astronomical PM ICRS would be consistent with the FRW cosmological space-times used in astrophysics except for the conformal factor determining the accelerated expansion of the universe and creating problems in the metrological use of fixed stars.

Hopefully at least an PM extension of ICRS including our galaxy (with the definition of a galactic coordinate system) will be achieved with the ESA GAIA mission devoted to the cartography of the Milky Way [14].

8.3 Clock Synchronization and Global Non-inertial Frames in Minkowski Space-Time

Since in the Minkowski space-time of SR time is not absolute, there is no intrinsic notion of 3-space and of synchronization of clocks: both of them have to be defined with some convention. As a consequence the *1-way velocity of light* from one observer A to an observer B has a meaning only after a choice of a convention for synchronizing the clock in A with the one in B. Therefore the crucial quantity in special relativity is the *2-way (or round trip) velocity of light c* involving only one clock. It is this velocity (a kind of mean velocity) which is isotropic and constant in SR and replaces the standard of length in relativistic metrology.

Einstein convention for the synchronization of clocks in Minkowski space-time uses the 2-way velocity of light to identify the Euclidean 3-spaces of the inertial frames centered on an inertial observer A by means of only its clock. The inertial observer A sends a ray of light at x_i^o towards the (in general accelerated) observer B; the ray is reflected towards A at a point P of B world-line and then reabsorbed by A at x_f^o; by convention P is synchronous with the mid-point between emission and absorption on A's world-line, i.e. $x_P^o = x_i^o + \frac{1}{2}(x_f^o - x_i^o) = \frac{1}{2}(x_i^o + x_f^o)$. This convention selects the Euclidean instantaneous 3-spaces $x^o = ct = const.$ of the inertial frames centered on A. Only in this case the one-way velocity of light between A and B coincides with the two-way one, c. However, as said in the Introduction, if the observer A is accelerated, the convention can breaks down due the possible appearance of coordinate singularities.

As a consequence, a theory of global non-inertial frames in Minkowski space-time has to be developed in a metrology-oriented way to overcame the pathologies of the $1+3$ point of view. This has been done in the papers of Ref. [49–52] based on the $3+1$ *point of view* and on the use of observer-dependent Lorentz scalar radar 4-coordinates. This theory and its implications for the description of isolated systems in SR will be reviewed in this Section.

8.3.1 3 + 1 Splittings of Minkowski Spacetime and Radar 4-Coordinates

Assume that the world-line $x^\mu(\tau)$ of an arbitrary time-like observer carrying a standard atomic clock is given: τ is an arbitrary monotonically increasing function of the proper time of this clock. Then one gives an admissible $3+1$ splitting of Minkowski space-time, namely a nice foliation with space-like instantaneous 3-spaces Σ_τ. It is the mathematical idealization of a protocol for clock synchronization: all the clocks in the points of Σ_τ sign the same time of the atomic clock of the observer.[8] On each 3-space Σ_τ one chooses curvilinear 3-coordinates σ^r having the observer as origin. These are the Lorentz-scalar and observer-dependent *radar 4-coordinates* $\sigma^A = (\tau; \sigma^r)$.

If $x^\mu \mapsto \sigma^A(x)$ is the coordinate transformation from the Cartesian 4-coordinates x^μ of a reference inertial observer to radar coordinates, its inverse $\sigma^A \mapsto x^\mu = z^\mu(\tau, \sigma^r)$ defines the *embedding* functions $z^\mu(\tau, \sigma^r)$ describing the 3-spaces Σ_τ as embedded 3-manifold into Minkowski space-time. The induced 4-metric on Σ_τ is the following functional of the embedding ${}^4g_{AB}(\tau, \sigma^r) = [z^\mu_A\, \eta_{\mu\nu}\, z^\nu_B](\tau, \sigma^r)$, where $z^\mu_A = \partial z^\mu / \partial \sigma^A$ and ${}^4\eta_{\mu\nu} = \epsilon\,(+ - - -)$ is the flat metric.[9] While the 4-vectors $z^\mu_r(\tau, \sigma^u)$ are tangent to Σ_τ, so that the unit normal $l^\mu(\tau, \sigma^u)$ is proportional to $\epsilon^\mu{}_{\alpha\beta\gamma}\, [z^\alpha_1\, z^\beta_2\, z^\gamma_3](\tau, \sigma^u)$, one has $z^\mu_\tau(\tau, \sigma^r) = [N\, l^\mu + N^r\, z^\mu_r](\tau, \sigma^r)$ with $N(\tau, \sigma^r) = \epsilon\, [z^\mu_\tau\, l_\mu](\tau, \sigma^r) = 1 + n(\tau, \sigma^r)$ and $N_r(\tau, \sigma^r) = -\epsilon\, g_{\tau r}(\tau, \sigma^r)$ being the lapse and shift functions.

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[8] It is the non-factual idealization required by the Cauchy problem generalizing the existing protocols for building coordinate system inside the future light-cone of a time-like observer.

[9] $\epsilon = \pm 1$ according to either the particle physics $\epsilon = 1$ or the general relativity $\epsilon = -1$ convention.

As a consequence, the components of the 4-metric ${}^4g_{AB}(\tau, \sigma^r)$ have the following expression

$$\begin{aligned}
\epsilon\, {}^4g_{\tau\tau} &= N^2 - N_r\, N^r, \qquad -\epsilon\, {}^4g_{\tau r} = N_r = {}^3g_{rs}\, N^s,\\
{}^3g_{rs} &= -\epsilon\, {}^4g_{rs} = \sum_{a=1}^{3} {}^3e_{(a)r}\, {}^3e_{(a)s}\\
&= \tilde{\phi}^{2/3} \sum_{a=1}^{3} e^{2\, \sum_{\bar{b}=1}^{2} \gamma_{\bar{b}a}\, R_{\bar{b}}}\, V_{ra}(\theta^i)\, V_{sa}(\theta^i),
\end{aligned} \tag{8.1}$$

where ${}^3e_{(a)r}(\tau, \sigma^u)$ are cotriads on Σ_τ, $\tilde{\phi}^2(\tau, \sigma^r) = det\, {}^3g_{rs}(\tau, \sigma^r)$ is the 3-volume element on Σ_τ, $\lambda_a(\tau, \sigma^r) = [\tilde{\phi}^{1/3}\, e^{\sum_{\bar{b}=1}^{2} \gamma_{\bar{b}a}\, R_{\bar{b}}}](\tau, \sigma^r)$ are the positive eigenvalues of the 3-metric ($\gamma_{\bar{a}a}$ are suitable numerical constants) and $V(\theta^i(\tau, \sigma^r))$ are diagonalizing rotation matrices depending on three Euler angles.

Therefore starting from the *four* independent embedding functions $z^\mu(\tau, \sigma^r)$ one obtains the *ten* components ${}^4g_{AB}$ of the 4-metric (or the quantities N, N_r, $\tilde{\phi}$, $R_{\bar{a}}$, θ^i), which play the role of the *inertial potentials* generating the relativistic apparent forces in the non-inertial frame. It can be shown [49–52] that the usual non-relativistic Newtonian inertial potentials are hidden in the functions N, N_r and θ^i. The extrinsic curvature tensor ${}^3K_{rs}(\tau, \sigma^u) = [\frac{1}{2N}\, (N_{r|s} + N_{s|r} - \partial_\tau\, {}^3g_{rs})](\tau, \sigma^u)$, describing the *shape* of the instantaneous 3-spaces of the non-inertial frame as embedded 3-sub-manifolds of Minkowski space-time, is a secondary inertial potential, functional of the ten inertial potentials ${}^4g_{AB}$.

The foliation is nice and admissible if it satisfies the conditions:

(1) $N(\tau, \sigma^r) > 0$ in every point of Σ_τ so that the 3-spaces never intersect, avoiding the coordinate singularity of Fermi coordinates;
(2) $\epsilon\, {}^4g_{\tau\tau}(\tau, \sigma^r) > 0$, so to avoid the coordinate singularity of the rotating disk, and with the positive-definite 3-metric ${}^3g_{rs}(\tau, \sigma^u) = -\epsilon\, {}^4g_{rs}(\tau, \sigma^u)$ having three positive eigenvalues (these are the Møller conditions [110, 111]);
(3) all the 3-spaces Σ_τ must tend to the same space-like hyper-plane at spatial infinity with a unit normal ϵ^μ_τ, which is the time-like 4-vector of a set of asymptotic ortho-normal tetrads ϵ^μ_A. These tetrads are carried by asymptotic inertial observers and the spatial axes ϵ^μ_r are identified by the fixed stars of star catalogues. At spatial infinity the lapse function tends to 1 and the shift functions vanish.

8.3.2 Global Non-inertial Frames in Minkowski Spacetime

By using the asymptotic tetrads ϵ^μ_A one can give the following parametrization of the embedding functions

$$z^\mu(\tau, \sigma^r) = x^\mu(\tau) + \epsilon^\mu_A F^A(\tau, \sigma^r), \quad F^A(\tau, 0) = 0,$$

$$x^\mu(\tau) = x^\mu_o + \epsilon^\mu_A f^A(\tau), \tag{8.2}$$

where $x^\mu(\tau)$ is the world-line of the observer. The functions $f^A(\tau)$ determine the 4-velocity $u^\mu(\tau) = \dot{x}^\mu(\tau)/\sqrt{\epsilon\, \dot{x}^2(\tau)}$ ($\dot{x}^\mu(\tau) = \frac{dx^\mu(\tau)}{d\tau}$) and the 4-acceleration $a^\mu(\tau) = \frac{du^\mu(\tau)}{d\tau}$ of the observer.

The Møller conditions are non-linear differential conditions on the functions $f^A(\tau)$ and $F^A(\tau, \sigma^r)$, so that it is very difficult to construct explicit examples of admissible $3 + 1$ splittings. When these conditions are satisfied Eqs. (8.2) describe a global non-inertial frame in Minkowski space-time.

Till now the solution of Møller conditions is known in the following two cases in which the instantaneous 3-spaces are parallel Euclidean space-like hyper-planes not equally spaced due to a linear acceleration.

(A) *Rigid non-inertial reference frames with translational acceleration*. An example are the following embeddings

$$z^\mu(\tau, \sigma^u) = x^\mu_o + \epsilon^\mu_\tau f(\tau) + \epsilon^\mu_r \sigma^r,$$
$$g_{\tau\tau}(\tau, \sigma^u) = \epsilon \left(\frac{df(\tau)}{d\tau}\right)^2, \quad g_{\tau r}(\tau, \sigma^u) = 0, \quad g_{rs}(\tau, \sigma^u) = -\epsilon\, \delta_{rs}. \tag{8.3}$$

This is a foliation with parallel hyper-planes with normal $l^\mu = \epsilon^\mu_\tau = const.$ and with the time-like observer $x^\mu(\tau) = x^\mu_o + \epsilon^\mu_\tau f(\tau)$ as origin of the 3-coordinates. The hyper-planes have translational acceleration $\ddot{x}^\mu(\tau) = \epsilon^\mu_\tau \ddot{f}(\tau)$, so that they are not uniformly distributed like in the inertial case $f(\tau) = \tau$.

(B) *Differentially rotating non-inertial frames* without the coordinate singularity of the rotating disk. The embedding defining this frames is

$$z^\mu(\tau, \sigma^u) = x^\mu(\tau) + \epsilon^\mu_r R^r{}_s(\tau, \sigma)\, \sigma^s \ \to_{\sigma\to\infty}\ x^\mu(\tau) + \epsilon^\mu_r \sigma^r,$$
$$R^r{}_s(\tau, \sigma) = R^r{}_s(\alpha_i(\tau, \sigma)) = R^r{}_s(F(\sigma)\, \tilde{\alpha}_i(\tau)),$$
$$0 < F(\sigma) < \frac{1}{A\,\sigma}, \qquad \frac{dF(\sigma)}{d\sigma} \neq 0 \ (Moller\ conditions),$$

$$z^\mu_\tau(\tau, \sigma^u) = \dot{x}^\mu(\tau) - \epsilon^\mu_r R^r{}_s(\tau, \sigma)\, \delta^{sw}\, \epsilon_{wuv}\, \sigma^u\, \frac{\Omega^v(\tau, \sigma)}{c},$$
$$z^\mu_r(\tau, \sigma^u) = \epsilon^\mu_k R^k{}_v(\tau, \sigma) \left(\delta^v_r + \Omega^v_{(r)u}(\tau, \sigma)\, \sigma^u\right), \tag{8.4}$$

where $\sigma = |\boldsymbol{\sigma}|$ and $R^r{}_s(\alpha_i(\tau, \sigma))$ is a rotation matrix satisfying the asymptotic conditions $R^r{}_s(\tau, \sigma) \to_{\sigma\to\infty} \delta^r_s$, $\partial_A R^r{}_s(\tau, \sigma) \to_{\sigma\to\infty} 0$, whose Euler angles have the expression $\alpha_i(\tau, \boldsymbol{\sigma}) = F(\sigma)\,\tilde{\alpha}_i(\tau)$, $i = 1, 2, 3$. The unit normal is $l^\mu = \epsilon^\mu_\tau = const.$ and the lapse function is $1 + n(\tau, \sigma^u) = \epsilon\left(z^\mu_\tau\, l_\mu\right)(\tau, \sigma^u) = \epsilon\,\epsilon^\mu_\tau\,\dot{x}_\mu(\tau) > 0$. In Eq. (8.4) one uses the notations $\Omega_{(r)}(\tau, \sigma) = R^{-1}(\tau, \boldsymbol{\sigma})\,\partial_r\,R(\tau, \sigma)$ and $\left(R^{-1}(\tau, \sigma)\,\partial_\tau\,R(\tau, \sigma)\right)^u{}_v = \delta^{um}\,\epsilon_{mvr}\,\frac{\Omega^r(\tau,\sigma)}{c}$, with $\Omega^r(\tau, \sigma) = F(\sigma)\,\tilde{\Omega}(\tau, \sigma)\,\hat{n}^r(\tau, \sigma)$[10] being the angular velocity. The angular velocity vanishes at spatial infinity and has an upper bound proportional to the minimum of the linear velocity $v_l(\tau) = \dot{x}_\mu\, l^\mu$ orthogonal to the space-like hyper-planes. When the rotation axis is fixed and $\tilde{\Omega}(\tau, \sigma) = \omega = const.$, a simple choice for the function $F(\sigma)$ is $F(\sigma) = \frac{1}{1+\frac{\omega^2\sigma^2}{c^2}}$.[11]

To evaluate the non-relativistic limit for $c \to \infty$, where $\tau = c\,t$ with t the absolute Newtonian time, one chooses the gauge function $F(\sigma) = \frac{1}{1+\frac{\omega^2\sigma^2}{c^2}} \to_{c\to\infty} 1 - \frac{\omega^2\sigma^2}{c^2} + O(c^{-4})$. This implies that the corrections to rigidly-rotating non-inertial frames coming from Møller conditions are of order $O(c^{-2})$ and become important at the distance from the rotation axis where the horizon problem for rigid rotations appears.

As shown in the first paper in Refs. [49–52], *global rigid rotations are forbidden in relativistic theories*, because, if one uses the embedding $z^\mu(\tau, \sigma^u) = x^\mu(\tau) + \epsilon^\mu_r\, R^r{}_s(\tau)\,\sigma^s$ describing a global rigid rotation with angular velocity $\Omega^r = \Omega^r(\tau)$, then the resulting $g_{\tau\tau}(\tau, \sigma^u)$ violates Møller conditions, because it vanishes at $\sigma = \sigma_R = \frac{1}{\Omega(\tau)}\left[\sqrt{\dot{x}^2(\tau) + [\dot{x}_\mu(\tau)\,\epsilon^\mu_r\, R^r{}_s(\tau)\,(\hat{\sigma}\times\hat{\Omega}(\tau))^r]^2} - \dot{x}_\mu(\tau)\,\epsilon^\mu_r\, R^r{}_s(\tau)\,(\hat{\sigma}\times\hat{\Omega}(\tau))^r\right]$ ($\sigma^u = \sigma\,\hat{\sigma}^u$, $\Omega^r = \Omega\,\hat{\Omega}^r$, $\hat{\sigma}^2 = \hat{\Omega}^2 = 1$). At this distance from the rotation axis the tangential rotational velocity becomes equal to the velocity of light. This is the *horizon problem* of the rotating disk (the horizon is often named the *light cylinder*). Let us remark that even if in the existing theory of rotating relativistic stars [112] one uses differential rotations, notwithstanding that in the study of the magnetosphere of pulsars often the notion of light cylinder is still used.

The search of admissible 3 + 1 splittings with non-Euclidean 3-spaces is much more difficult. The simplest case is the following parametrization of the embeddings (8.1) in terms of Lorentz matrices $\Lambda^A{}_B(\tau, \sigma) \to_{\sigma\to\infty} \delta^A_B$[12] with $\Lambda^A{}_B(\tau, 0)$ finite. The Lorentz matrix is written in the form $\Lambda = \mathcal{B}\,\mathcal{R}$ as the product of a boost $\mathcal{B}(\tau, \sigma)$ and a rotation $\mathcal{R}(\tau, \sigma)$ like the one in Eq. (8.4) ($\mathcal{R}^\tau{}_\tau = 1, \mathcal{R}^\tau{}_r = 0, \mathcal{R}^r{}_s = R^r{}_s$). The components of the boost are $\mathcal{B}^\tau{}_\tau(\tau, \sigma) = \gamma(\tau, \sigma) = 1/\sqrt{1 - \boldsymbol{\beta}^2(\tau, \sigma)}$, $\mathcal{B}^\tau{}_r(\tau, \sigma) = \gamma(\tau, \sigma)\,\beta_r(\tau, \sigma)$, $\mathcal{R}^r{}_s(\tau, \sigma) = \delta^r_s + \frac{\gamma\,\beta^r\,\beta_s}{1+\gamma}(\tau, \sigma)$, with $\beta^r(\tau, \sigma) = G(\sigma)\,\beta^r(\tau)$, where

---

[10] $\hat{n}^r(\tau, \sigma)$ defines the instantaneous rotation axis and $0 < \tilde{\Omega}(\tau, \sigma) < 2\,max\left(\dot{\tilde{\alpha}}(\tau), \dot{\tilde{\beta}}(\tau), \dot{\tilde{\gamma}}(\tau)\right)$.

[11] Nearly rigid rotating systems, like a rotating disk of radius σ_o, can be described by using a function $F(\sigma)$ approximating the step function $\theta(\sigma - \sigma_o)$.

[12] It corresponds to the *locality hypothesis* of Ref. [47, 48], according to which at each instant of time the detectors of an accelerated observer give the same indications as the detectors of the instantaneously comoving inertial observer.

$\beta^r(\tau)$ is defined by the 4-velocity of the observer $u^\mu(\tau) = \epsilon^\mu_A\, \beta^A(\tau)/\sqrt{1-\boldsymbol{\beta}^2(\tau)}$, $\beta^A(\tau) = (1; \beta^r(\tau))$. The Møller conditions are restrictions on $G(\sigma) \;\rightarrow_{\sigma\rightarrow\infty}\; 0$ with $G(0)$ finite, whose explicit form is still under investigation.

See the second paper of Ref. [49–52] for the description of the electro-magnetic field and of phenomena like the Sagnac effect and the Faraday rotation in this framework for non-inertial frames. Moreover the embedding (8.4) has been used in the first paper of Ref. [77, 78] on quantum mechanics in non-inertial frames.

8.3.3 Congruences of Timelike Observers Associated with a 3 + 1 Splitting

Each admissible 3 + 1 splitting of space-time allows one to define two associated congruences of time-like observers.

(i) The congruence of the Eulerian observers with the unit normal $l^\mu(\tau,\sigma^r) = z^\mu_A(\tau,\sigma^r)\, l^A(\tau,\sigma^r)$ to the 3-spaces embedded in Minkowski space-time as unit 4-velocity. The world-lines of these observers are the integral curves of the unit normal and in general are not geodesics. In adapted radar 4-coordinates the contro-variant orthonormal tetrads carried by the Eulerian observers are $l^A(\tau,\sigma^r)$, ${}^4\overset{\circ}{\bar{E}}{}^A_{(a)}(\tau,\sigma^r) = (0; {}^3e^u_{(a)}(\tau,\sigma^r))$, where ${}^3e^u_{(a)}(\tau,\sigma^r)$ $(a = 1,2,3)$ are triads on the 3-space.

If ${}^4\nabla$ is the covariant derivative associated with the 4-metric ${}^4g_{AB}(\tau,\sigma^r)$ induced by a 3 + 1 splitting, the equation

$$ {}^4\nabla_A\, \epsilon\, l_B = \epsilon\, l_A\, {}^3a_B + \sigma_{AB} + \frac{1}{3}\,\theta\, {}^3h_{AB} - \omega_{AB}, \quad \left({}^3h_{AB} = {}^4g_{AB} - \epsilon\, l_A\, l_B\right), \tag{8.5} $$

defines the *acceleration* ${}^3a^A$ (${}^3a^A\, l_A = 0$), the *expansion* θ, the *shear* $\sigma_{AB} = \sigma_{BA}$ ($\sigma_{AB}\, l^B = 0$) and the *vorticity or twist* $\omega_{AB} = -\omega_{BA}$ ($\omega_{AB}\, l^B = 0$) of the Eulerian observers with $\omega_{AB} = 0$ since they are surface-forming by construction. They will be useful in GR as shown in Section 7.

(ii) The skew congruence with unit 4-velocity $v^\mu(\tau,\sigma^r) = z^\mu_A(\tau,\sigma^r)\, v^A(\tau,\sigma^r)$ (in general it is not surface-forming, i.e. it has a non-vanishing vorticity, like the one of a rotating disk). The observers of the skew congruence have the world-lines (integral curves of the 4-velocity) defined by $\sigma^r = const.$ for every τ, because the unit 4-velocity tangent to the flux lines $x^\mu_{\boldsymbol{\sigma}_o}(\tau) = z^\mu(\tau,\sigma^r_o)$ is $v^\mu_{\boldsymbol{\sigma}_o}(\tau) = z^\mu_\tau(\tau,\sigma^r_o)/\sqrt{\epsilon\, {}^4g_{\tau\tau}(\tau,\sigma^r_o)}$ (there is no horizon problem because it is everywhere time-like in admissible 3 + 1 splittings). They carry contro-variant orthonormal tetrads, given in Ref. [91], not adapted to the foliation, connected in each point by a Lorentz transformation to the ones of the Eulerian observer present in this point.

8.3.4 Parametrized Minkowski Theories

In the global non-inertial frames of Minkowski space-time it is possible to describe isolated systems (particles, strings, fields, fluids) admitting a Lagrangian formulation by means of *parametrized Minkowski theories* [49–52, 55–57].

The existence of a Lagrangian, which can be coupled to an external gravitational field, makes possible the determination of the matter energy-momentum tensor and of the ten conserved Poincaré generators P^μ and $J^{\mu\nu}$ (assumed finite) of every configuration of the isolated system.

First of all one must replace the matter variables of the isolated system with new ones knowing the clock synchronization convention defining the 3-spaces Σ_τ. For instance a Klein-Gordon field $\tilde{\phi}(x)$ will be replaced with $\phi(\tau, \sigma^r) = \tilde{\phi}(z(\tau, \sigma^r))$; the same for every other field. Instead for a relativistic particle with world-line $x^\mu(\tau)$ one must make a choice of its energy sign: then the positive- (or negative-) energy particle will be described by 3-coordinates $\eta^r(\tau)$ defined by the intersection of its world-line with Σ_τ: $x^\mu(\tau) = z^\mu(\tau, \eta^r(\tau))$. Differently from all the previous approaches to relativistic mechanics, the dynamical configuration variables are the 3-coordinates $\eta^r(\tau)$ and not the world-lines $x^\mu(\tau)$ (to rebuild them in an arbitrary frame one needs the embedding defining that frame). This fact eliminates the possibility to have time-like excitations in the spectrum of relativistic bound states: inside each 3-space only space-like correlations among the particles are possible.

Then one replaces the external gravitational 4-metric in the coupled Lagrangian with the 4-metric ${}^4g_{AB}(\tau, \sigma^r)$, which is a functional of the embedding defining an admissible $3+1$ splitting of Minkowski space-time, and the matter fields with the new ones knowing the instantaneous 3-spaces Σ_τ.

Parametrized Minkowski theories are defined by the resulting Lagrangian depending on the given matter and on the embedding $z^\mu(\tau, \sigma^r)$. The resulting action is invariant under the *frame-preserving diffeomorphisms* $\tau \mapsto \tau^{'}(\tau, \sigma^u), \sigma^r \mapsto \sigma^{'r}(\sigma^u)$ firstly introduced in Ref. [58]. As a consequence, there are four first-class constraints with exactly vanishing Poisson brackets (an Abelianized analogue of the super-Hamiltonian and super-momentum constraints of canonical gravity) determining the momenta conjugated to the embeddings in terms of the matter energy-momentum tensor. This implies that the embeddings $z^\mu(\tau, \sigma^r)$ are *gauge variables*, so that *all the admissible non-inertial or inertial frames are gauge equivalent*, namely physics does *not* depend on the clock synchronization convention and on the choice of the 3-coordinates σ^r: only the appearances of phenomena change by changing the notion of instantaneous 3-space.[13]

Even if the gauge group is formed by the frame-preserving diffeomorphisms, the matter energy-momentum tensor allows the determination of the ten conserved Poincaré generators P^μ and $J^{\mu\nu}$ (assumed finite) of every configuration of the system

[13] In the first paper of Ref. [77, 78] there is the definition of *parametrized Galilei theories*, non relativistic limit of the parametrized Minkowski theories. Also the inertial and non-inertial frames in Galilei space-time are gauge equivalent in this formulation.

(in non-inertial frames they are asymptotic generators at spatial infinity like the ADM ones in GR).

As an example one may consider N free scalar particles with masses m_i and sign of the energy $\eta_i = \pm$, whose world-lines are identified by the configurational variables $\eta_i^r(\tau)$: $x_i^\mu(\tau) = z^\mu(\tau, \eta_i^r(\tau))$, $i = 1, \dots, N$. In parametrized Minkowski theories they are described by the following action depending on the configurational variables $\eta_i^r(\tau)$ and $z^\mu(\tau, \sigma^r)$

$$
\begin{aligned}
S &= \int d\tau\, d^3\sigma\, \mathcal{L}(\tau, \sigma^u) = \int d\tau\, L(\tau),\\
\mathcal{L}(\tau, \sigma^u) &= -\sum_{i=1}^N \delta^3(\sigma^u - \eta_i^u(\tau))\\
&\quad m_i c\, \eta_i \sqrt{\epsilon\, [{}^4g_{\tau\tau}(\tau, \sigma^u) + 2\, {}^4g_{\tau r}(\tau, \sigma^u)\, \dot{\eta}_i^r(\tau) + {}^4g_{rs}(\tau, \sigma^u)\, \dot{\eta}_i^r(\tau)\, \dot{\eta}_i^s(\tau)]}.
\end{aligned}
\tag{8.6}
$$

The resulting canonical momenta $\kappa_{ir}(\tau) = \frac{\partial L(\tau)}{\partial \eta_i^r}$, $\rho_\mu(\tau, \sigma^u) = -\epsilon\, \frac{\partial \mathcal{L}(\tau, \sigma^u)}{\partial z_\tau^\mu(\tau, \sigma^u))}$ satisfy the Poisson brackets $\{\eta_i^r(\tau), \kappa_{js}(\tau)\} = -\delta_s^r\, \delta_{ij}$, $\{z^\mu(\tau, \sigma^u), \rho_\nu(\tau, \sigma^{'u})\} = -\delta_\nu^\mu\, \delta^3(\sigma^u - \sigma^{'u})$. The Poincaré generators and the energy-momentum tensor of this system are ($h^{rs} = -\epsilon\, \gamma^{rs}$ with $\gamma^{ru}\, {}^4g_{us} = \delta_s^r$; $\gamma = -\epsilon\, det\, {}^4g_{rs}$)

$$
\begin{aligned}
P^\mu &= \int d^3\sigma \rho^\mu(\tau, \sigma^u), \qquad J^{\mu\nu} = \int d^3\sigma (z^\mu \rho^\nu - z^\nu \rho^\mu)(\tau, \sigma^u),\\
T^{AB}(\tau, \sigma^u) &= -\frac{2}{\sqrt{-det\, {}^4g_{CD}(\tau, \sigma^u)}}\, \frac{\delta S}{\delta\, {}^4g_{AB}(\tau, \sigma^u)}, \qquad T^{\mu\nu} = z_A^\mu\, z_B^\nu\, T^{AB},\\
T_{\perp\perp}(\tau, \sigma^u) &= \Big(l_\mu\, l_\nu\, T^{\mu\nu}\Big)(\tau, \sigma^u)\\
&= \sum_{i=1}^N \frac{\delta^3(\sigma^u - \eta_i^u(\tau))}{\sqrt{\gamma(\tau, \sigma^u)}} \eta_i \sqrt{m_i^2 c^2 + h^{rs}(\tau, \sigma^u)\, \kappa_{ir}(\tau)\, \kappa_{is}(\tau)},\\
T_{\perp r}(\tau, \sigma^u) &= \Big(l_\mu\, z_{r\nu}\, T^{\mu\nu}\Big)(\tau, \sigma^u) = \sum_{i=1}^N \frac{\delta^3(\sigma^u - \eta_i^u(\tau))}{\sqrt{\gamma(\tau, \sigma^u)}}\, \kappa_{ir}(\tau),\\
T_{rs}(\tau, \sigma^u) &= \Big(z_{r\mu}\, z_{s\nu}\, T^{\mu\nu}\Big)(\tau, \sigma^u)\\
&= \sum_{i=1}^N \frac{\delta^3(\sigma^u - \eta_i^u(\tau))}{\sqrt{\gamma(\tau, \sigma^u)}} \eta_i\, \frac{\kappa_{ir}(\tau)\, \kappa_{is}(\tau)}{\sqrt{m_i^2 c^2 + h^{vw}(\tau, \sigma^u)\, \kappa_{iv}(\tau)\, \kappa_{iw}(\tau)}}.
\end{aligned}
\tag{8.7}
$$

The four first-class constraints implying the gauge nature of the embedding and the gauge equivalence of the description in different non-inertial frames are

$$\rho_\mu(\tau, \sigma^u) - \sqrt{\gamma(\tau, \sigma^u)}\Big[l_\mu\, T_{\perp\perp} - z_{r\mu}\, h^{rs}\, T_{\perp s}\Big](\tau, \sigma^u) \approx 0. \tag{8.8}$$

The same description can be given for the Klein-Gordon [113] and Dirac [114] fields and for the electro-magnetic field [49–52].

To describe the physics in a given admissible non-inertial frame described by an embedding $z^\mu_F(\tau, \sigma^u)$ one must add the gauge-fixings $z^\mu(\tau, \sigma^u) - z^\mu_F(\tau, \sigma^u) \approx 0$.

8.3.5 The Instant Form of Dynamics in the Inertial Rest Frames and the Problem of the Relativistic Center of Mass

If one restricts himself to inertial frames, one can define the *inertial rest-frame instant form of dynamics for isolated systems* by choosing the $3+1$ splitting corresponding to the intrinsic inertial rest frame of the isolated system centered on an inertial observer: the instantaneous 3-spaces, named *Wigner 3-spaces* due to the fact that the 3-vectors inside them are Wigner spin-1 3-vectors [55–57], are orthogonal to the conserved 4-momentum P^μ (assumed time-like, $\epsilon\, P^2 > 0$) of the configuration.

In this framework one can give the final solution to the old problem of the relativistic extension of the Newtonian center of mass of an isolated system. In its rest frame there are *only* three notions of collective variables, which can be built by using *only* the Poincaré generators:

the canonical non-covariant Newton-Wigner center of mass (or center of spin) $\tilde{x}^\mu(\tau)$,
the non-canonical covariant Fokker-Pryce center of inertia $Y^\mu(\tau)$
the non-canonical non-covariant Møller center of energy $R^\mu(\tau)$.

While $Y^\mu(\tau)$ is a 4-vector, $\tilde{x}^\mu(\tau)$ and $R^\mu(\tau)$ are not 4-vectors. All of them tend to the Newtonian center of mass in the non-relativistic limit. Since the Poincaré generators know the whole Σ_τ, they and therefore also these three collective variables are *non-local* quantities: as a consequence they are *non measurable* with local means [49–52, 59–61, 65, 66, 76].

If one centers the inertial rest frame on the world-line of the Fokker-Planck center of inertia thought as an inertial observer, then the corresponding embedding has the expression [55–57, 65, 66]

$$z^\mu_W(\tau, \boldsymbol{\sigma}) = Y^\mu(\tau) + \epsilon^\mu_r(\boldsymbol{h})\, \sigma^r, \tag{8.9}$$

where $Y^\mu(\tau)$ is the Fokker-Pryce center-of-inertia 4-vector, $\boldsymbol{h} = \boldsymbol{P}/\sqrt{\epsilon\, P^2}$ and $\epsilon^\mu{}_{A=\nu}(\boldsymbol{h}) = L^\mu{}_{A=\nu}(P, \overset{\circ}{P})$ are the columns of the standard Wigner boost for time-like orbits sending $P^\mu = \sqrt{\epsilon P^2}\, (\sqrt{1+\boldsymbol{h}^2}; \boldsymbol{h})$ to $\overset{\circ}{P}{}^\mu = \sqrt{\epsilon P^2}\, (1; 0)$. Their expression is $\epsilon^\mu_\tau(\boldsymbol{h}) = h^\mu = (\sqrt{1+\boldsymbol{h}^2}; \boldsymbol{h})$ and $\epsilon^\mu_r(\boldsymbol{h}) = \Big(h_r; \delta^i_r + \frac{h^i\, h_r}{1+\sqrt{1+\boldsymbol{h}^2}}\Big)$ as shown in Appendix B of Ref. [115].

As shown in Ref. [59–61, 65–67, 76], the three collective variables can be expressed as known functions of the Lorentz-scalar rest time $\tau = c\, T_s = h \cdot \tilde{x} = h \cdot Y = h \cdot R$, of canonically conjugate Jacobi data (frozen Cauchy data) $\boldsymbol{z} = Mc\, \boldsymbol{x}_{NW}(0)$ and $\boldsymbol{h} = \boldsymbol{P}/Mc$,[14] of the invariant mass $Mc = \sqrt{\epsilon P^2}$ of the system and of its rest spin $\bar{\boldsymbol{S}}$.

While the world-line of the non-canonical covariant external Fokker-Pryce 4-center of inertia is

$$
\begin{aligned}
Y^\mu(\tau) &= z^\mu_W(\tau, \boldsymbol{0}) = \Big(\tilde{x}^o(\tau); \boldsymbol{Y}(\tau)\Big) \\
&= \left(\sqrt{1+\boldsymbol{h}^2}\, \Big(\tau + \frac{\boldsymbol{h}\cdot\boldsymbol{z}}{Mc}\Big); \frac{\boldsymbol{z}}{Mc} + \Big(\tau + \frac{\boldsymbol{h}\cdot\boldsymbol{z}}{Mc}\Big)\, \boldsymbol{h} + \frac{\boldsymbol{S}\times\boldsymbol{h}}{Mc\, \Big(1+\sqrt{1+\boldsymbol{h}^2}\Big)}\right),
\end{aligned}
\tag{8.10}
$$

the pseudo-world-line of the canonical non-covariant external 4-center of mass is ($\tilde{\boldsymbol{\sigma}} = \frac{-\boldsymbol{S}\times\boldsymbol{h}}{Mc\, (1+\sqrt{1+\boldsymbol{h}^2})}$ from Ref. [59–61])

$$
\begin{aligned}
\tilde{x}^\mu(\tau) &= \Big(\tilde{x}^o(\tau); \tilde{\boldsymbol{x}}(\tau)\Big) = z^\mu_W(\tau, \tilde{\boldsymbol{\sigma}}) = Y^\mu(\tau) + \left(0; \frac{-\boldsymbol{S}\times\boldsymbol{h}}{Mc\, (1+\sqrt{1+\boldsymbol{h}^2})}\right) \\
&= \left(\sqrt{1+\boldsymbol{h}^2}\, \Big(\tau + \frac{\boldsymbol{h}\cdot\boldsymbol{z}}{Mc}\Big); \frac{\boldsymbol{z}}{Mc} + \Big(\tau + \frac{\boldsymbol{h}\cdot\boldsymbol{z}}{Mc}\Big)\, \boldsymbol{h}\right).
\end{aligned}
\tag{8.11}
$$

The world-lines of the positive-energy particles are parametrized by the Wigner 3-vectors $\boldsymbol{\eta}_i(\tau)$, $i = 1, 2, \ldots, N$, and are given by

$$
x^\mu_i(\tau) = z^\mu_W(\tau, \boldsymbol{\eta}_i(\tau)) = Y^\mu(\tau) + \epsilon^\mu_r(\tau)\, \eta^r_i(\tau). \tag{8.12}
$$

The world-lines $x^\mu_i(\tau)$ of the particles are derived (interaction-dependent) quantities. Also the standard particle 4-momenta are derived quantities, whose expression is $p^\mu_i(\tau) = \epsilon^\mu_A(\boldsymbol{h})\, \kappa^A_i(\tau) = h^\mu\, \sqrt{m_i^2 c^2 + \boldsymbol{\kappa}_i^2(\tau)} - \epsilon^\mu_r(\boldsymbol{h})\, \kappa_{ir}(\tau)$ with $\epsilon\, p_i^2 = m_i^2 c^2$ in the free case.

[14] Their Poisson brackets are $\{z^i, h^j\} = \delta^{ij}$. $\boldsymbol{x}_{NW}(\tau)$ is the standard Newton-Wigner non-covariant 3-position, classical counterpart of the corresponding position operator; the use of $\boldsymbol{z}$ avoids to take into account the mass spectrum of the isolated system in the description of the center of mass. The non-covariance of $\boldsymbol{z}$ under Poincaré transformations (a, Λ) has the following form [76, 115] $z^i \mapsto z'^{\,i} = \Big(\Lambda^i{}_j - \frac{\Lambda^i{}_\mu\, h^\mu}{\Lambda^o{}_\nu\, h^\nu}\, \lambda^o{}_j\Big)\, z^j + \Big(\Lambda^i{}_\mu - \frac{\Lambda^i{}_\nu\, h^\nu}{\Lambda^o{}_\rho\, h^\rho}\, \Lambda^o{}_\mu\Big)\, (\Lambda^{-1}\, a)^\mu$.

In the case of interacting particles the reconstruction of the world-lines requires a complex interaction-dependent procedure delineated in Ref. [67], where there is also a comparison of the present approach with the other formulations of relativistic mechanics developed for the study of the problem of *relativistic bound states*. See Ref. [49–52] for the extension to non-inertial frames.

In general the world-lines $x_i^\mu(\tau)$ do not satisfy vanishing Poisson brackets (they are relativistic predictive coordinates, see Ref. [67]): already at the classical level a *non-commutative structure* emerges due to the Lorentz signature of the space-time [76].

In each Lorentz frame one has different pseudo-world-lines describing R^μ and $\tilde{x}^\mu$: the canonical 4-center of mass $\tilde{x}^\mu$ *lies in between* Y^μ and R^μ in every (non rest)-frame. As discussed in Subsection IIF of Ref. [65, 66], this leads to the existence of the *Møller non-covariance world-tube*, around the world-line Y^μ of the covariant non-canonical Fokker-Pryce 4-center of inertia Y^μ. The *invariant radius* of the tube is $\rho = \sqrt{-\epsilon\, W^2}/p^2 = |\boldsymbol{S}|/\sqrt{\epsilon\, P^2}$ where ($W^2 = -P^2\, \boldsymbol{S}^2$ is the Pauli-Lubanski invariant when $\epsilon\, P^2 > 0$). This classical intrinsic radius is a non-local effect of Lorentz signature absent in Euclidean spaces and delimits the non-covariance effects (the pseudo-world-lines) of the canonical 4-center of mass $\tilde{x}^\mu$.[15] They are not detectable because the Møller radius is of the order of the Compton wave-length: an attempt to test its interior would mean to enter in the quantum regime of pair production. The Møller radius ρ is also a remnant of the energy conditions of general relativity in flat Minkowski space-time [55].

Finally Eqs.(8.7) can be used to extend the multipolar expansions of Ref. [116–118] to this framework for relativistic isolated systems as it is shown in the third paper of Refs. [59–61].

8.3.6 The Description of Isolated Systems in the Rest Frame and Their Poincaré Generators

In the inertial rest frame of an isolated system Eq. (8.7) are the starting point to get the explicit form of its Poincaré generators, in particular of the Lorentz boosts, which, differently from the Galilei ones, are interaction dependent.

As shown in Ref. [65, 66], every isolated system (i.e. a closed universe) can be visualized as a decoupled non-covariant collective (non-local) pseudo-particle (the *external* center of mass), described by the frozen Jacobi data $\boldsymbol{z}$, $\boldsymbol{h}$, carrying a *pole-dipole structure*, namely the invariant mass $M\,c$ (the monopole) and the rest spin $\bar{\boldsymbol{S}}$

[15] In the rest-frame the world-tube is a cylinder: in each instantaneous 3-space there is a disk of possible positions of the canonical 3-center of mass orthogonal to the spin. In the non-relativistic limit the radius ρ of the disk tends to zero and one recovers the non-relativistic center of mass.

(the dipole) of the system, and with an associated *external* realization of the Poincaré group[16]:

$$P^\mu = M\,c\,h^\mu = M\,c\,\Big(\sqrt{1+\,\boldsymbol{h}^2};\boldsymbol{h}\Big),$$
$$J^{ij} = z^i\,h^j - z^j\,h^i + \epsilon^{ijk}\,S^k, \qquad K^i = J^{oi} = -\sqrt{1+\boldsymbol{h}^2}\,z^i + \frac{(\boldsymbol{S}\times\,\boldsymbol{h})^i}{1+\sqrt{1+\boldsymbol{h}^2}}. \tag{8.13}$$

The universal breaking of Lorentz covariance is connected to this decoupled non-local collective variable and is irrelevant because all the dynamics of the isolated system leaves inside the Wigner 3-spaces and is Wigner-covariant. The invariant mass and the rest spin are built in terms of the Wigner-covariant variables of the given isolated system ($\boldsymbol{\eta}_i(\tau)$ and $\boldsymbol{\kappa}_i(\tau)$) inside the Wigner 3-spaces [49–52, 59–61, 65, 66, 76].

In each Wigner 3-space Σ_τ there is a *unfaithful internal* realization of the Poincaré algebra, whose generators are built by using the energy-momentum tensor (8.7) of the isolated system. While the internal energy and angular momentum are Mc and $\bar{\boldsymbol{S}}$ respectively, the internal 3-momentum vanishes: it is the rest frame condition. Also the internal (interaction dependent) Lorentz boost vanishes: this condition identifies the covariant non-canonical Fokker-Pryce center of inertia as the natural inertial observer origin of the 3-coordinates σ^r in each Wigner 3-space.

For N free particles the internal Poincaré generators have the following expression

$$M\,c = \frac{1}{c}\,\mathcal{E}_{(int)} = \sum_{i=1}^N \sqrt{m_i^2\,c^2+\boldsymbol{\kappa}_i^2},$$
$$\boldsymbol{\mathcal{P}}_{(int)} = \sum_{i=1}^N \boldsymbol{\kappa}_i \approx 0,$$
$$\boldsymbol{S} = \boldsymbol{\mathcal{J}}_{(int)} = \sum_{i=1}^N \boldsymbol{\eta}_i \times \boldsymbol{\kappa}_i,$$
$$\boldsymbol{\mathcal{K}}_{(int)} = -\sum_{i=1}^N \boldsymbol{\eta}_i\sqrt{m_i^2\,c^2+\boldsymbol{\kappa}_i^2} \approx 0. \tag{8.14}$$

Since one is in an instant form of the dynamics, in the interacting case only Mc and $\boldsymbol{\mathcal{K}}_{(int)}$ become interaction dependent.

[16] The last term in the Lorentz boosts induces the Wigner rotation of the 3-vectors inside the Wigner 3-spaces.

The three pairs of second-class (interaction dependent) constraints $\boldsymbol{\mathcal{P}}_{(int)} \approx 0$, $\boldsymbol{\mathcal{K}}_{(int)} \approx 0$, eliminate the *internal* 3-center of mass and its conjugate momentum inside the Wigner 3-spaces[17]: this avoids a double counting of the collective variables (external and internal center of mass). As a consequence the dynamics inside the Wigner 3-spaces is described in terms of internal Wigner-covariant relative variables. In the case of N relativistic particles one defines the following canonical transformation [76] (see Ref. [59–61] for other variants) ($m = \sum_{i=1}^N m_i$)

$$\begin{aligned}
&\boldsymbol{\eta}_+ = \sum_{i=1}^N \frac{m_i}{m} \boldsymbol{\eta}_i, \quad \boldsymbol{\kappa}_+ = \boldsymbol{\mathcal{P}}_{(int)} = \sum_{i=1}^N \boldsymbol{\kappa}_i,\\
&\boldsymbol{\rho}_a = \sqrt{N} \sum_{i=1}^N \gamma_{ai}\, \boldsymbol{\eta}_i, \qquad \boldsymbol{\pi}_a = \frac{1}{\sqrt{N}} \sum_{i=1}^N \Gamma_{ai}\, \boldsymbol{\kappa}_i, \quad a = 1, \dots, N-1,\\
&\boldsymbol{\eta}_i = \boldsymbol{\eta}_+ + \frac{1}{\sqrt{N}} \sum_{a-1}^{N-1} \Gamma_{ai}\, \boldsymbol{\rho}_a, \qquad \boldsymbol{\kappa}_i = \frac{m_i}{m} \boldsymbol{\kappa}_+ + \sqrt{N} \sum_{a=1}^{N-1} \gamma_{ai}\, \boldsymbol{\pi}_a,
\end{aligned} \tag{8.15}$$

with the following canonicity conditions[18]

$$\begin{aligned}
&\sum_{i=1}^N \gamma_{ai} = 0, \qquad \sum_{i=1}^N \gamma_{ai}\, \gamma_{bi} = \delta_{ab}, \qquad \sum_{a=1}^{N-1} \gamma_{ai}\, \gamma_{aj} = \delta_{ij} - \frac{1}{N},\\
&\Gamma_{ai} = \gamma_{ai} - \sum_{k=1}^N \frac{m_k}{m} \gamma_{ak}, \qquad \gamma_{ai} = \Gamma_{ai} - \frac{1}{N} \sum_{k=1}^N \Gamma_{ak},\\
&\sum_{i=1}^N \frac{m_i}{m} \Gamma_{ai} = 0, \qquad \sum_{i=1}^N \gamma_{ai}\, \Gamma_{bi} = \delta_{ab}, \qquad \sum_{a=1}^{N-1} \gamma_{ai}\, \Gamma_{aj} = \delta_{ij} - \frac{m_i}{m}.
\end{aligned} \tag{8.16}$$

Since Eq. (8.14) imply $\boldsymbol{\kappa}_+(\tau) = \boldsymbol{\mathcal{P}}_{(int)} \approx 0$ and $\boldsymbol{\eta}_+(\tau) \approx \boldsymbol{f}_+(\boldsymbol{\rho}_a(\tau), \boldsymbol{\pi}_a(\tau))$ due to $\boldsymbol{\mathcal{K}}_{(int)} \approx 0$, the invariant mass Mc and the rest spin $\bar{\boldsymbol{S}}$ become functions only of the $N-1$ pairs of relative canonical variables.

As a consequence, Eqs. (8.10), (8.12) and (8.15) imply that the world-lines $x_i^\mu(\tau)$ can be expressed in terms of the Jacobi data $\boldsymbol{z}, \boldsymbol{h}$, and of the relative variables $\boldsymbol{\rho}_a(\tau)$, $\boldsymbol{\pi}_a(\tau)$, $a = 1, \dots, N-1$. See Ref. [113] for the collective and relative variables of the Klein-Gordon field and the second paper in Ref. [65, 66] for such variables for

---

[17] One can show [59–61, 65, 66] that one has $\boldsymbol{\mathcal{K}}_{(int)} = -M\, \boldsymbol{R}_+$, where $\boldsymbol{R}_+$ is the internal Møller 3-center of energy inside the Wigner 3-spaces. The rest frame condition $\boldsymbol{\mathcal{P}}_{(int)} \approx 0$ implies $\boldsymbol{R}_+ \approx \boldsymbol{q}_+ \approx \boldsymbol{y}_+$, where $\boldsymbol{q}_+$ is the internal 3-center of mass and $\boldsymbol{y}_+$ the internal Fokker-Pryce 3-center of inertia.

[18] Equations (8.15) describe a family of canonical transformations, because the γ_{ai}'s depend on $\frac{1}{2}(N-1)(N-2)$ free independent parameters.

the electro-magnetic field in the radiation gauge. For these systems one can give for the first time the explicit closed form of the interaction-dependent Lorentz boosts.

One finds that disregarding the unobservable external center of mass all the dynamics is described only by relative variables: this is a form of *weak relationism* without the heavy foundational problem of approaches like the one in Ref. [119, 120].

The non-relativistic limit of this description [76] is Newton mechanics with the Newton center of mass decoupled from the relative variables and moreover after a canonical transformation to the frozen Hamilton-Jacobi description of the center of mass.

An important remark is that the internal space of relative variables is independent from the reference inertial frame used for the description of the isolated system. As shown in Ref. [76], the formalism is built in such a way that the Wigner rotation induced on the relative variables by a Lorentz transformation connecting two reference inertial frames is the identity, i.e. the space of the relative variables in an *abstract internal space* insensitive to Lorentz transformations carried by the external center of mass (or in a more covariant description carried by the Fokker-Pryce center-of-inertia 4-vector origin of the embedding (8.9)).

Finally in Ref. [49–52] there is the extension of the formalism to admissible *non-inertial rest frames*, where P^μ is orthogonal to the asymptotic space-like hyperplanes to which the instantaneous 3-spaces tend at spatial infinity. In these non-inertial rest frames the internal Poincaré generators are asymptotic (constant of the motion) symmetry generators like the asymptotic ADM ones in the asymptotically Minkowskian space-times.

8.4 Implications for Relativistic Mechanics and Classical Field Theory in Special Relativity and the Multi-Temporal Quantization Approach

In the rest-frame instant form of the dynamics it has been possible to find the explicit form of the internal Poincaré generators (in particular of the interaction-dependent invariant mass and Lorentz boosts) not only for the Klein-Gordon [113] and Dirac [69–71, 114] fields, but also for the electro-magnetic field in the radiation gauge (the only one suitable for the Shanmugadhasan canonical transformations of constraint theory [121]) [49–52, 65, 66], for relativistic fluids [91–93], spinning particles [62–64, 69–71] and for massless particles, the Nambu string and the two-level atom [72–75].

In this Section some other developments in SR will be reviewed.

8.4.1 Relativistic Atomic Physics

Standard atomic physics [122, 123] is a semi-relativistic treatment of quantum electro-dynamics (QED) in which the matter fields are approximated by scalar (or spinning) particles, the relevant energies are below the threshold of pair production and the electro-magnetic field is described in the Coulomb gauge at the order $1/c$.

In Refs. [62–66] a fully relativistic formulation of classical atomic physics in the rest-frame instant form of dynamics was given with the electro-magnetic field in the radiation gauge and with the electric charges Q_i of the positive-energy particles being Grassmann-valued ($Q_i^2 = 0$, $Q_i\, Q_j = Q_j\, Q_i$ for $i \neq j$) to regularize the electro-magnetic self-energies on the world-lines of particles. In the language of QED this is both a ultraviolet regularization (no loop contributions) and an infrared one (no brehmstrahlung), so that only the one-photon exchange diagram contributes and its static and non-static effects are replaced by potentials in a formulation based on the Cauchy problem. Therefore the starting point is a parametrized Minkowski theory with N charged positive-energy particles mutually interacting with a Coulomb potential and coupled to a dynamical transverse electro-magnetic field described by the canonical variables $\boldsymbol{A}_\perp(\tau, \sigma^r)$ and $\boldsymbol{\pi}_\perp(\tau, \sigma^r) = \boldsymbol{E}_\perp(\tau, \sigma^r)$.

In the first paper of Ref. [62–64] (the second paper is devoted to spinning particles) it is shown that the use of the Lienard-Wiechert solution (see the third paper in Ref. [62–64]) with “no incoming radiation field” allows one to arrive at a description of N charged particles dressed with a Coulomb cloud and mutually interacting through the Coulomb potential augmented with the full relativistic Darwin potential. This happens independently from the choice of the Green function (retarded, advanced, symmetric,..) due to the Grassmann regularization. The quantization allows one to recover the standard instantaneous approximation for relativistic bound states, which till now had only been obtained starting from QED (either in the instantaneous approximations of the Bethe-Salpeter equation or in the quasi-potential approach). In the case of spinning particle the relativistic Salpeter potential was identified.

Moreover in Ref. [65, 66] it is shown that by using the previous results one can find a canonical transformation from the canonical basis $\boldsymbol{\eta}_i(\tau)$, $\boldsymbol{\kappa}_i(\tau)$, $\boldsymbol{A}_\perp(\tau, \sigma^r)$, $\boldsymbol{\pi}_\perp(\tau, \sigma^r)$, in which the internal Poincaré generators have the expression in the case $\mathrm{N} = 2$ ($\boldsymbol{B} = \boldsymbol{\partial} \times \boldsymbol{A}_\perp$, $c(\boldsymbol{\sigma}) = -1/4\pi\, |\boldsymbol{\sigma}|$)

$$
\begin{aligned}
\mathcal{E}_{(int)} &= M\, c^2 = c \sum_{i=1}^{N} \sqrt{m_i^2\, c^2 + \Big(\boldsymbol{\kappa}_i(\tau) - \frac{Q_i}{c}\, \boldsymbol{A}_\perp(\tau, \boldsymbol{\eta}_i(\tau))\Big)^2} \\
&\quad + \sum_{i \neq j} \frac{Q_i\, Q_j}{4\pi\, \mid \boldsymbol{\eta}_i(\tau) - \boldsymbol{\eta}_j(\tau) \mid} + \frac{1}{2} \int d^3\sigma\, [\boldsymbol{\pi}_\perp^2 + \boldsymbol{B}^2](\tau, \boldsymbol{\sigma}), \\
\mathcal{P}_{(int)} &= \sum_{i=1}^{N} \boldsymbol{\kappa}_i(\tau) + \frac{1}{c} \int d^3\sigma\, [\boldsymbol{\pi}_\perp \times \boldsymbol{B}](\tau, \boldsymbol{\sigma}) \approx 0,
\end{aligned}
$$

$$\bar{S}^r = \sum_{i=1}^{N} \Big(\boldsymbol{\eta}_i(\tau) \times \boldsymbol{\kappa}_i(\tau)\Big)^r + \frac{1}{c} \int d^3\sigma (\boldsymbol{\sigma} \times \Big([\boldsymbol{\pi}_\perp \times \boldsymbol{B}]\Big)^r (\tau, \boldsymbol{\sigma}),$$

$$\begin{aligned}
\mathcal{K}^r_{(int)} &= -\sum_{i=1}^{N} \eta_i^r(\tau) \sqrt{m_i^2\, c^2 + \Big(\, \boldsymbol{\kappa}_i(\tau) - \frac{Q_i}{c} \boldsymbol{A}_\perp(\tau, \boldsymbol{\eta}_i(\tau))\Big)^2} \\
&\quad + \frac{1}{c} \sum_{i=1}^{N} \Big[\sum_{j\neq i}^{1...N} Q_i\, Q_j \,[\, \frac{1}{\triangle_{\boldsymbol{\eta}_j}} \frac{\partial}{\partial \eta_j^r} c(\boldsymbol{\eta}_i(\tau) - \boldsymbol{\eta}_j(\tau)) \\
&\quad - \eta_j^r(\tau)\, c(\, \boldsymbol{\eta}_i(\tau) - \boldsymbol{\eta}_j(\tau))] + Q_i \int d^3\sigma\, \pi^r_\perp(\tau, \boldsymbol{\sigma})\, c(\boldsymbol{\sigma} -\ \boldsymbol{\eta}_i(\tau))\Big] \\
&\quad - \frac{1}{2c} \int d^3\sigma\, \sigma^r\, (\boldsymbol{\pi}^2_\perp + \boldsymbol{B}^2)(\tau, \boldsymbol{\sigma}),
\end{aligned} \tag{8.17}$$

to a new canonical basis $\hat{\boldsymbol{\eta}}_i(\tau), \hat{\boldsymbol{\kappa}}_i(\tau),\ \boldsymbol{A}_{\perp rad}(\tau, \sigma^r), \boldsymbol{\pi}_{\perp rad}(\tau, \sigma^r)$ so that in the rest frame there is a decoupled free radiation transverse field and a system of charged particles mutually interacting with Coulomb plus Darwin potential. See the first paper in Ref. [62–64] for the explicit form of the relativistic Darwin potential. The new internal Poincaré generators in the $N = 2$ case are

$$\begin{aligned}
\boldsymbol{\mathcal{E}}_{(int)} &= M\, c^2 = c \sum_{i=1}^{2} \sqrt{m_i^2\, c^2 + \hat{\boldsymbol{\kappa}}_i^2(\tau)} + \frac{Q_1\, Q_2}{4\pi\, |\, \hat{\boldsymbol{\eta}}_1(\tau) - \hat{\boldsymbol{\eta}}_2(\tau)|} \\
&\quad + V_{DARWIN}(\hat{\boldsymbol{\kappa}}_1(\tau), \hat{\boldsymbol{\kappa}}_2(\tau), \hat{\boldsymbol{\eta}}_1(\tau) - \hat{\boldsymbol{\eta}}_2(\tau)) \\
&\quad + \frac{1}{2} \int d^3\sigma \Big(\boldsymbol{\pi}^2_{\perp rad} + \boldsymbol{B}^2_{rad}\Big)(\tau, \boldsymbol{\sigma}) = \mathcal{E}_{matter} + \mathcal{E}_{rad}, \\
\boldsymbol{\mathcal{P}}_{(int)} &= \sum_{i=1}^{2} \hat{\boldsymbol{\kappa}}_i(\tau) + \frac{1}{c} \int d^3\sigma \Big(\boldsymbol{\pi}_{\perp rad} \times\ \boldsymbol{B}_{rad}\Big)(\tau, \boldsymbol{\sigma}) = \boldsymbol{\mathcal{P}}_{matter} + \boldsymbol{\mathcal{P}}_{rad} \approx 0, \\
\bar{\boldsymbol{S}} &= \sum_i \hat{\boldsymbol{\eta}}_i \times \hat{\boldsymbol{\kappa}}_i + \frac{1}{c} \int d^3\sigma\, \boldsymbol{\sigma} \times \Big(\boldsymbol{\pi}_{\perp rad} \times \boldsymbol{B}_{rad}\Big)(\tau, \boldsymbol{\sigma}) = \bar{\boldsymbol{S}}_{matter} + \bar{\boldsymbol{S}}_{rad},
\end{aligned}$$

$$\begin{aligned}
\mathcal{K}_{(int)} &= -\sum_{i=1}^{2} \hat{\boldsymbol{\eta}}_i \sqrt{m_i^2\, c^2 + \hat{\boldsymbol{\kappa}}_i^2} \\
&\quad - \frac{1}{2} \frac{Q_1\, Q_2}{c} \left[\hat{\boldsymbol{\eta}}_1 \frac{\hat{\boldsymbol{\kappa}}_1 \cdot \Big(\frac{1}{2}\, \frac{\partial\, \hat{\mathcal{K}}_{12}(\hat{\boldsymbol{\kappa}}_1, \hat{\boldsymbol{\kappa}}_2, \hat{\boldsymbol{\rho}}_{12})}{\partial\, \hat{\boldsymbol{\rho}}_{12}} - 2\, \boldsymbol{A}_{\perp S2}(\hat{\boldsymbol{\kappa}}_2, \hat{\boldsymbol{\rho}}_{12})\Big)}{\sqrt{m_1^2\, c^2 + \hat{\boldsymbol{\kappa}}_1^2}} \right.
\end{aligned}$$

$$
\begin{aligned}
&+ \hat{\boldsymbol{\eta}}_2 \, \frac{\hat{\boldsymbol{\kappa}}_2 \cdot \Big(\frac{1}{2}\, \frac{\partial \hat{\mathcal{K}}_{12}(\hat{\boldsymbol{\kappa}}_1, \hat{\boldsymbol{\kappa}}_2, \hat{\boldsymbol{\rho}}_{12})}{\partial \hat{\boldsymbol{\rho}}_{12}} - 2\, \boldsymbol{A}_{\perp S1}(\hat{\boldsymbol{\kappa}}_1, \hat{\boldsymbol{\rho}}_{12}) \Big)}{\sqrt{m_2^2\, c^2 + \hat{\boldsymbol{\kappa}}_2^2}} \Bigg] \\
&- \frac{1}{2}\, \frac{Q_1\, Q_2}{c} \Big(\sqrt{m_1^2\, c^2 + \hat{\boldsymbol{\kappa}}_1^2}\, \frac{\partial}{\partial \hat{\boldsymbol{\kappa}}_1} + \sqrt{m_2^2\, c^2 + \hat{\boldsymbol{\kappa}}_2^2}\, \frac{\partial}{\partial \hat{\boldsymbol{\kappa}}_2} \Big)\, \hat{\mathcal{K}}_{12}(\hat{\boldsymbol{\kappa}}_1, \hat{\boldsymbol{\kappa}}_2, \hat{\boldsymbol{\rho}}_{12}) \\
&- \frac{Q_1\, Q_2}{4\pi\, c} \int d^3\sigma \Big(\frac{\hat{\boldsymbol{\pi}}_{\perp S1}(\boldsymbol{\sigma} - \hat{\boldsymbol{\eta}}_1, \hat{\boldsymbol{\kappa}}_1)}{|\boldsymbol{\sigma} - \hat{\boldsymbol{\eta}}_2|} + \frac{\hat{\boldsymbol{\pi}}_{\perp S2}(\boldsymbol{\sigma} - \hat{\boldsymbol{\eta}}_2, \hat{\boldsymbol{\kappa}}_2)}{|\boldsymbol{\sigma} - \hat{\boldsymbol{\eta}}_1|} \Big) \\
&- \frac{Q_1\, Q_2}{c} \int d^3\sigma\, \boldsymbol{\sigma}\, \Big(\hat{\boldsymbol{\pi}}_{\perp S1}(\boldsymbol{\sigma} - \hat{\boldsymbol{\eta}}_1, \hat{\boldsymbol{\kappa}}_1) \cdot \hat{\boldsymbol{\pi}}_{\perp S2}(\boldsymbol{\sigma} - \hat{\boldsymbol{\eta}}_2, \hat{\boldsymbol{\kappa}}_2) \\
&+ \hat{\boldsymbol{B}}_{S1}(\boldsymbol{\sigma} - \hat{\boldsymbol{\eta}}_1, \hat{\boldsymbol{\kappa}}_1) \cdot \hat{\boldsymbol{B}}_{S2}(\boldsymbol{\sigma} - \hat{\boldsymbol{\eta}}_2, \hat{\boldsymbol{\kappa}}_2) \Big) \\
&- \frac{1}{2\, c} \int d^3\sigma\, \boldsymbol{\sigma}\, \Big(\boldsymbol{\pi}_{\perp rad}^2 + \boldsymbol{B}_{rad}^2 \Big)(\tau, \boldsymbol{\sigma}) = \boldsymbol{\mathcal{K}}_{matter} + \boldsymbol{\mathcal{K}}_{rad} \approx 0. \qquad (8.18)
\end{aligned}
$$

The only restriction on the two decoupled systems is the elimination of their overall internal 3-center of mass inside the Wigner 3-spaces. Therefore, at the classical level there is a way out from the Haag theorem forbidding the existence of the interaction picture in QED, so that there is no unitary evolution based on interpolating fields from the "in" states to the "out" ones in scattering processes. While the extension of these results to the non-inertial rest frame is done in Ref. [49–52], the quantization of this framework is under investigation.

In the first paper of Ref. [72–75] there is the formulation in the rest-frame instant form of the relativistic quark model in the radiation gauge for the SU(3) Yang-Mills fields with scalar quarks having Grassmann-valued color charges. While in Eq. (101) of that paper there is the rest-frame condition, in Eq. (97) there is the invariant mass Mc^2 for a quark-antiquark system. In it the electro-magnetic Coulomb potential of Eq. (17) is replaced with a potential, given in Eq. (95), depending on the color transverse vector potential through the Green function of the SU(3) covariant divergence. The non-linearity of the problem does not allow to evaluate a Lienard-Wiechert solution and to find the analogue of Eq. (8.18).

8.4.2 Relativistic Kinetic Theory and Relativistic Micro-Canonical Ensemble

In the rest-frame instant form of dynamics it is also possible to give a finally consistent treatment of relativistic kinetic theory and relativistic statistical mechanics [68]. In particular one can give a definition of the relativistic micro-canonical ensemble for an isolated system of N interacting particles with fixed internal energy $\mathcal{E}$ and rest spin $\boldsymbol{\mathcal{S}}$ only in terms of the internal Poincaré generators in the Wigner 3-spaces by means of the partition function (V is the volume defined by the function $\chi(V)$ vanishing outside it)

$$\tilde{Z}(\mathcal{E}, \mathcal{S}, V, N) = \frac{1}{N!} \int \prod_i^{1...N} d^3\eta_i\, \chi(V) \int \prod_j^{1...N} d^3\kappa_j\, \delta(M\, c^2 - \mathcal{E})$$
$$\delta^3(\bar{\mathbf{S}} - \bar{\mathbf{S}})\, \delta^3(\boldsymbol{\mathcal{P}}_{(int)})\, \delta^3\left(\frac{\boldsymbol{\mathcal{K}}_{(int)}}{Mc}\right). \tag{8.19}$$

Also it extension to non-inertial rest frames can be given by using the results of Ref. [49–52] with the result that notwithstanding the presence of long-range inertial forces one has still an equilibrium distribution.

8.4.3 Relativistic Quantum Mechanics and Relativistic Entanglement

A new formulation of *relativistic quantum mechanics* in the Wigner 3-spaces of the inertial rest frame is developed in Ref. [76] in absence of the electro-magnetic field. It englobes all the known results about relativistic bound states (absence of relative times) and avoids the causality problems of the Hegerfeldt theorem [124, 125] (the instantaneous spreading of wave packets).

In it one quantizes the frozen Jacobi data $\boldsymbol{z}$ and $\boldsymbol{h}$ of the canonical non-covariant decoupled external center of mass and the relative variables in the Wigner 3-spaces. Since the center of mass is decoupled, its non-covariance is irrelevant: like for the wave function of the universe, who will observe it?

The resulting Hilbert space has the following tensor product structure: $H = H_{com,HJ} \otimes H_{rel}$, where $H_{com,HJ}$ is the Hilbert space of the external center of mass (in the Hamilton-Jacobi formulation due to the use of frozen Jacobi data) while H_{rel} is the Hilbert space of the relative variables in the abstract internal space living in the Wigner 3-spaces. While at the non-relativistic level this presentation of the Hilbert space is unitarily equivalent to the tensor product of the Hilbert spaces H_i of the individual particles $H = H_1 \otimes H_2 \otimes \cdots$, this is not true at the relativistic level.

If one considers two scalar quantum particles with Klein-Gordon wave functions belonging to Hilbert spaces $\mathcal{H}_{x_i^o}$, in the tensor-product Hilbert space $(\mathcal{H}_1)_{x_1^o} \otimes (\mathcal{H}_2)_{x_2^o} \otimes \cdots$ there is no correlation among the times of the particles (their clocks are not synchronized) so that in most of the states there are some particles in the absolute future of the others. As a consequence the two types of Hilbert spaces lead to unitarily inequivalent descriptions and have different scalar products (compare Refs. [76] and [115]).

As a consequence, at the relativistic level the zeroth postulate of non-relativistic quantum mechanics does not hold: the Hilbert space of composite systems is not the tensor product of the Hilbert spaces of the sub-systems. Contrary to Einstein's notion of separability (separate objects have their independent real states) [126, 127] one gets a *kinematical spatial non-separability* induced by the need of clock synchronization for eliminating the relative times and to be able to formulate a well-posed relativistic Cauchy problem.

Moreover one has the *non-locality* of the non-covariant external center of mass which implies its *non-measurability* with local instruments.[19] While its conjugate momentum operator must be well defined and self-adjoint, because its eigenvalues describe the possible values for the total momentum of the isolated system (the momentum basis is therefore a *preferred basis* in the Hilbert space), it is not clear whether it is meaningful to define center-of-mass wave packets.

These *non-locality* and *kinematical spatial non-separability* are due to the Lorentz signature of Minkowski space-time and this fact reduce the relevance of *quantum non-locality* in the study of the foundational problems of quantum mechanics [126, 127] which have to be rephrased in terms of relative variables.

The quantization defined in Ref. [76] leads to a first formulation of a theory for *relativistic entanglement*, which is deeply different from the non-relativistic entanglement due to these kinematical non-locality and spatial non-separability. To have control on the Poincaré group one needs an isolated systems containing all the relevant entities (for instance both Alice and Bob) of the experiment under investigation and also the environment when needed. One has to learn to reason in terms of relative variables adapted to the experiment like molecular physicists do when they look to the best system of Jacobi coordinates adapted to the main chemical bonds in the given molecule. This theory has still to be developed together with its extension to non-inertial rest frames.

8.4.4 Multitemporal Quantization in Non-Inertial Frames

This quantization of relativistic mechanics can be extended to the class of global non-inertial frames with space-like hyper-planes as 3-spaces and differentially rotating 3-coordinates defined in Ref. [49–52] by using the *multi-temporal quantization* approach developed in Ref. [129, 130].

As shown in Ref. [77, 78], in this type of quantization one quantizes only the 3-coordinates $\eta_i^r(\tau)$ of the particles and *not* the inertial effects (like the Coriolis and centrifugal ones): they remain c-numbers describing the appearances of phenomena. The known results in atomic and nuclear physics are reproduced.

8.4.5 Open Problem

The main open problem in SR is the quantization of *fields* in non-inertial frames due to the no-go theorem of Ref. [131, 132] showing the existence of obstructions to the unitary evolution of a massive quantum Klein-Gordon field between two space-like surfaces of Minkowski space-time. It turns out that the Bogoljubov transformation

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[19] In Ref. [128] it was shown that the quantum Newton-Wigner position should not be a self-adjoint operator, but only a symmetric one, with an implication of bad localization.

connecting the creation and destruction operators on the two surfaces is not of the Hilbert-Schmidt type, i.e. that the Tomonaga-Schwinger approach in general is not unitary. One must reformulate the problem using the nice foliations of the admissible $3+1$ splittings of Minkowski space-time and to try to identify all the $3+1$ splittings allowing unitary evolution. This will be a prerequisite to any attempt to quantize canonical gravity taking into account the equivalence principle (global inertial frames do not exist) with the further problem that in general the Fourier transform does not exist in Einstein space-times.

8.5 Non-Inertial Frames in Asymptotically Minkowskian Einstein Space-Times and ADM Tetrad Gravity

After this description of SR induced by the metrology-oriented problem of clock synchronization, one has to face the same problems in the globally hyperbolic, topologically trivial, asymptotically Minkowskian space-times without super-translations[20] of GR. As shown in the first paper of Ref. [41–44], in the chosen class of space-times the ten *strong* asymptotic ADM Poincaré generators P^A_{ADM}, J^{AB}_{ADM} (they are fluxes through a 2-surface at spatial infinity) are well defined functionals of the 4-metric fixed by the boundary conditions at spatial infinity.

While in SR Minkowski space-time is an absolute notion, in Einstein GR also the space-time is a dynamical object [79–81] and the gravitational field is described by the metric structure of the space-time, namely by the ten dynamical fields ${}^4g_{\mu\nu}(x)$ (x^μ are world 4-coordinates). The 4-metric ${}^4g_{\mu\nu}(x)$ tends in a suitable way to the flat Minkowski 4-metric ${}^4\eta_{\mu\nu}$ at spatial infinity [41–44]: having an *asymptotic Minkowskian background* the usual splitting of the 4-metric in the bulk in a background plus perturbations in the weak field limit can be avoided as shown in Sect. VII.

The ten dynamical fields ${}^4g_{\mu\nu}(x)$ are not only a (pre)potential for the gravitational field (like the electro-magnetic and Yang-Mills fields are the potentials for electromagnetic and non-Abelian forces) but also determines the *chrono-geometrical structure of space-time* through the line element $ds^2 = {}^4g_{\mu\nu}\, dx^\mu\, dx^\nu$. Therefore the 4-metric teaches relativistic causality to the other fields: it says to massless particles like photons and gluons which are the allowed world-lines in each point of space-time. This basic property is lost in every quantum field theory approach to gravity with a fixed background 4-metric.[21]

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[20] These space-times must also be without Killing symmetries, because, otherwise, at the Hamiltonian level one should introduce complicated sets of extra Dirac constraints for each existing Killing vector.

[21] The ACES mission of ESA [133–135] will give the first precision measurement of the gravitational red-shift of the geoid, namely of the $1/c^2$ deformation of Minkowski light-cone caused by the geopotential. In every quantum field theory approach to gravity, where the definition of the Fock space requires the use of the Fourier transform on a fixed background space-time with a fixed light-cone, this is a non-perturbative effect requiring the re-summation of the perturbative expansion.

In these space-times one can define global non-inertial frames by using the same admissible $3+1$ splittings, centered on a time-like observer, and the observer-dependent radar 4-coordinates $\sigma^A = (\tau; \sigma^r)$ employed in SR. This will allow to separate the *inertial* (gauge) degrees of freedom of the gravitational field (playing the role of inertial potentials) from the dynamical *tidal* ones at the Hamiltonian level.

In GR the dynamical fields are the components ${}^4g_{\mu\nu}(x)$ of the 4-metric and not the embeddings $x^\mu = z^\mu(\tau, \sigma^r)$ defining the admissible $3+1$ splittings of space-time like in the parametrized Minkowski theories of SR. Now the gradients $z^\mu_A(\tau, \sigma^r)$ of the embeddings give the transition coefficients from radar to world 4-coordinates, so that the components ${}^4g_{AB}(\tau, \sigma^r) = z^\mu_A(\tau, \sigma^r)\, z^\nu_B(\tau, \sigma^r)\, {}^4g_{\mu\nu}(z(\tau, \sigma^r))$ of the 4-metric will be the dynamical fields in the ADM action. Like in SR the 4-vectors $z^\mu_A(\tau, \sigma^r)$, tangent to the 3-spaces Σ_τ, are used to define the unit normal $l^\mu(\tau, \sigma^r) = z^\mu_A(\tau, \sigma^r)\, l^A(\tau, \sigma^r)$ to Σ_τ, while the 4-vector $z^\mu_\tau(\tau, \sigma^r)$ has the lapse function as component along the unit normal and the shift functions as components along the tangent vectors.

Since the world-line of the time-like observer can be chosen as the origin of a set of the spatial world coordinates, i.e. $x^\mu(\tau) = (x^o(\tau); 0)$, it turns out that with this choice the space-like surfaces of constant coordinate time $x^o(\tau) = const.$ coincide with the dynamical instantaneous 3-spaces Σ_τ with $\tau = const.$. By using asymptotic flat tetrads $\epsilon^\mu_A = \delta^\mu_o\, \delta^\tau_A + \delta^\mu_i\, \delta^i_A$ (with ϵ^A_μ denoting the inverse flat cotetrads) and by choosing a coordinate world time $x^o(\tau) = x^o_o + \epsilon^o_\tau\, \tau = x^o_o + \tau$, one gets the following preferred embedding corresponding to these given world 4-coordinates

$$x^\mu = z^\mu(\tau, \sigma^r) = x^\mu(\tau) + \epsilon^\mu_r\, \sigma^r = \delta^\mu_o\, x^o_o + \epsilon^\mu_A\, \sigma^A. \tag{8.20}$$

This choice implies $z^\mu_A(\tau, \sigma^r) = \epsilon^\mu_A$ and ${}^4g_{\mu\nu}(x = z(\tau, \sigma^r)) = \epsilon^A_\mu\, \epsilon^B_\nu\, {}^4g_{AB}(\tau, \sigma^r)$.

As shown in Ref. [79–81], the dynamical nature of space-time implies that each solution (i.e. an Einstein 4-geometry) of Einstein's equations (or of the associated ADM Hamilton equations) dynamically selects a preferred $3+1$ splitting of the space-time, namely in GR the instantaneous 3-spaces are dynamically determined in the chosen world coordinate system. Equation (8.20) can be used to describe this $3+1$ splitting and then by means of 4-diffeomorphisms the solution can be written in an arbitrary world 4-coordinate system in general not adapted to the dynamical $3+1$ splitting. This gives rise to the 4-geometry corresponding to the given solution.

To define the canonical formalism the Einstein-Hilbert action for metric gravity (depending on the second derivative of the metric) must be replaced with the ADM action (the two actions differ for a surface tern at spatial infinity). As shown in the first paper of Refs. [41–44], the Legendre transform and the definition of a consistent canonical Hamiltonian require the introduction of the DeWitt surface term at spatial infinity: the final canonical Hamiltonian turns out to be the *strong* ADM energy (a flux through a 2-surface at spatial infinity), which is equal to the *weak* ADM energy (expressed as a volume integral over the 3-space) plus constraints. Therefore there is not a frozen picture but an evolution generated by a Dirac Hamiltonian equal to the weak ADM energy plus a linear combination of the first class constraints. Also the other strong ADM Poincaré generators are replaced by their weakly equivalent weak form $\hat{P}^A_{ADM}$, $\hat{J}^{AB}_{ADM}$.

In the first paper of Ref. [41–44] it is also shown that the boundary conditions on the 4-metric required by the absence of super-translations imply that the only admissible $3+1$ splittings of space-time (i.e. the allowed global non-inertial frames) are the *non-inertial rest frames*: their 3-spaces are asymptotically orthogonal to the weak ADM 4-momentum. Therefore one gets $\hat{P}^r_{ADM} \approx 0$ as the rest-frame condition of the 3-universe with a mass and a rest spin fixed by the boundary conditions. Like in SR the 3-universe can be visualized as a decoupled non-covariant (non-measurable) external relativistic center of mass plus an internal non-inertial rest-frame 3-space containing only relative variables (see the first paper in Ref. [88–90]).

8.5.1 The Parametrization of Tetrads for ADM Tetrad Gravity

To take into account the coupling of fermions to the gravitational field metric gravity has to be replaced with tetrad gravity. This can be achieved by decomposing the 4-metric on cotetrad fields (by convention a sum on repeated indices is assumed)

$$ {}^4g_{AB}(\tau,\sigma^r) = E^{(\alpha)}_A(\tau,\sigma^r)\, {}^4\eta_{(\alpha)(\beta)}\, E^{(\beta)}_B(\tau,\sigma^r), \tag{8.21}$$

by putting this expression into the ADM action and by considering the resulting action, a functional of the 16 fields $E^{(\alpha)}_A(\tau,\sigma^r)$, as the action for ADM tetrad gravity. In Eq. (8.21) (α) are flat indices and the cotetrad fields $E^{(\alpha)}_A$ are the inverse of the tetrad fields $E^A_{(\alpha)}$, which are connected to the world tetrad fields by $E^\mu_{(\alpha)}(x) = z^\mu_A(\tau,\sigma^r)\, E^A_{(\alpha)}(z(\tau,\sigma^r))$ by the embedding of Eq. (8.20).

This leads to an interpretation of gravity based on a congruence of time-like observers endowed with orthonormal tetrads: in each point of space-time the time-like axis is the unit 4-velocity of the observer, while the spatial axes are a (gauge) convention for observer's gyroscopes. This framework was developed in the second and third paper of Refs. [41–44].

Even if the action of ADM tetrad gravity depends upon 16 fields, the counting of the physical degrees of freedom of the gravitational field does not change, because this action is invariant not only under the group of 4-difeomorphisms but also under the O(3,1) gauge group of the Newman-Penrose approach [136] (the extra gauge freedom acting on the tetrads in the tangent space of each point of space-time).

The cotetrads $E^{(\alpha)}_A(\tau,\sigma^r)$ are the new configuration variables. They are connected to cotetrads ${}^4\overset{\circ}{E}{}^{(\alpha)}_A(\tau,\sigma^r)$ adapted to the $3+1$ splitting of space-time, namely such that the inverse adapted time-like tetrad ${}^4\overset{\circ}{E}{}^A_{(o)}(\tau,\sigma^r)$ is the unit normal to the 3-space Σ_τ, by a standard Wigner boosts for time-like Poincaré orbits with parameters $\varphi_{(a)}(\tau,\sigma^r)$, $a=1,2,3$

$$E^{(\alpha)}_A = L^{(\alpha)}{}_{(\beta)}(\varphi_{(a)})\, \overset{o}{E}{}^{(\beta)}_A, \qquad {}^4g_{AB} = {}^4\overset{\circ}{E}{}^{(\alpha)}_A\, {}^4\eta_{(\alpha)(\beta)}\, {}^4\overset{\circ}{E}{}^{(\beta)}_B,$$
$$L^{(\alpha)}{}_{(\beta)}(\varphi_{(a)}) \overset{def}{=} L^{(\alpha)}{}_{(\beta)}(V(z(\sigma));\ \overset{\circ}{V}) = \delta^{(\alpha)}_{(\beta)} + 2\epsilon\, V^{(\alpha)}(z(\sigma))\, \overset{\circ}{V}_{(\beta)} -$$
$$- \epsilon\, \frac{(V^{(\alpha)}(z(\sigma)) + \overset{\circ}{V}{}^{(\alpha)})\,(V_{(\beta)}(z(\sigma)) + \overset{\circ}{V}_{(\beta)})}{1 + V^{(o)}(z(\sigma))}. \tag{8.22}$$

In each tangent plane to a point of Σ_τ this point-dependent standard Wigner boost sends the unit future-pointing time-like vector $\overset{o}{V}{}^{(\alpha)} = (1; 0)$ into the unit time-like vector $V^{(\alpha)} = {}^4E^{(\alpha)}_A\, l^A = \Big(\sqrt{1 + \sum_a \varphi^2_{(a)}};\, \varphi^{(a)} = -\epsilon\, \varphi_{(a)}\Big)$. As a consequence, the flat indices (a) of the adapted tetrads and cotetrads and of the triads and cotriads on Σ_τ transform as Wigner spin-1 indices under point-dependent SO(3) Wigner rotations $R_{(a)(b)}(V(z(\sigma));\ \Lambda(z(\sigma))\,)$ associated with Lorentz transformations $\Lambda^{(\alpha)}{}_{(\beta)}(z)$ in the tangent plane to the space-time in the given point of Σ_τ. Instead the index (o) of the adapted tetrads and cotetrads is a local Lorentz scalar index.

The adapted tetrads and cotetrads have the expression

$$ {}^4\overset{\circ}{E}{}^A_{(o)} = \frac{1}{1+n}\left(1; -\sum_a n_{(a)}\, {}^3e^r_{(a)}\right) = l^A, \qquad {}^4\overset{\circ}{E}{}^A_{(a)} = \left(0; {}^3e^r_{(a)}\right),$$
$$ {}^4\overset{\circ}{E}{}^{(o)}_A = (1+n)\,(1; \mathbf{0}) = \epsilon\, l_A, \qquad {}^4\overset{\circ}{E}{}^{(a)}_A = \left(n_{(a)}; {}^3e_{(a)r}\right), \tag{8.23}$$

where ${}^3e^r_{(a)}$ and ${}^3e_{(a)r}$ are triads and cotriads on Σ_τ and $n_{(a)} = n_r\, {}^3e^r_{(a)} = n^r\, {}^3e_{(a)r}$[22] are adapted shift functions. In Eq. (8.23) $N(\tau, \boldsymbol{\sigma}) = 1 + n(\tau, \boldsymbol{\sigma}) > 0$, with $n(\tau, \boldsymbol{\sigma})$ vanishing at spatial infinity (absence of super-translations), so that $N(\tau, \boldsymbol{\sigma})\, d\tau$ is positive from Σ_τ to $\Sigma_{\tau+d\tau}$, is the lapse function; $N^r(\tau, \boldsymbol{\sigma}) = n^r(\tau, \boldsymbol{\sigma})$, vanishing at spatial infinity (absence of super-translations), are the shift functions.

The adapted tetrads ${}^4\overset{\circ}{E}{}^A_{(a)}$ are defined modulo SO(3) rotations ${}^4\overset{\circ}{E}{}^A_{(a)} = \sum_b R_{(a)(b)}(\alpha_{(e)})\, {}^4\overset{\circ}{\bar{E}}{}^A_{(b)}$, ${}^3e^r_{(a)} = \sum_b R_{(a)(b)}(\alpha_{(e)})\, {}^3\bar{e}^r_{(b)}$, where $\alpha_{(a)}(\tau, \boldsymbol{\sigma})$ are three point-dependent Euler angles. After having chosen an arbitrary point-dependent origin $\alpha_{(a)}(\tau, \boldsymbol{\sigma}) = 0$, one arrives at the following adapted tetrads and cotetrads $[\bar{n}_{(a)} = \sum_b n_{(b)}\, R_{(b)(a)}(\alpha_{(e)})\,, \sum_a n_{(a)}\, {}^3e^r_{(a)} = \sum_a \bar{n}_{(a)}\, {}^3\bar{e}^r_{(a)}]$

$$ {}^4\overset{\circ}{\bar{E}}{}^A_{(o)} = {}^4\overset{\circ}{E}{}^A_{(o)} = \frac{1}{1+n}\left(1; -\sum_a \bar{n}_{(a)}\, {}^3\bar{e}^r_{(a)}\right) = l^A, \qquad {}^4\overset{\circ}{\bar{E}}{}^A_{(a)} = (0; {}^3\bar{e}^r_{(a)}),$$
$$ {}^4\overset{\circ}{\bar{E}}{}^{(o)}_A = {}^4\overset{\circ}{E}{}^{(o)}_A = (1+n)\,(1; \mathbf{0}) = \epsilon\, l_A, \qquad {}^4\overset{\circ}{\bar{E}}{}^{(a)}_A = \left(\bar{n}_{(a)}; {}^3\bar{e}_{(a)r}\right), \tag{8.24}$$

[22] Since one uses the positive-definite 3-metric $\delta_{(a)(b)}$, one will use only lower flat spatial indices. Therefore for the cotriads one uses the notation ${}^3e^{(a)}_r \overset{def}{=} {}^3e_{(a)r}$ with $\delta_{(a)(b)} = {}^3e^r_{(a)}\, {}^3e_{(b)r}$.

which one will use as a reference standard.

The expression for the general tetrad

$$
\begin{aligned}
{}^4E^A_{(\alpha)} &= {}^4\overset{\circ}{E}{}^A_{(\beta)}\, L^{(\beta)}{}_{(\alpha)}(\varphi_{(a)}) \\
&= {}^4\overset{\circ}{\bar{E}}{}^A_{(o)}\, L^{(o)}{}_{(\alpha)}(\varphi_{(c)}) + \sum_{ab} {}^4\overset{\circ}{\bar{E}}{}^A_{(b)}\, R^T_{(b)(a)}(\alpha_{(c)})\, L^{(a)}{}_{(\alpha)}(\varphi_{(c)}),
\end{aligned}
\tag{8.25}
$$

shows that every point-dependent Lorentz transformation Λ in the tangent planes may be parametrized with the (Wigner) boost parameters $\varphi_{(a)}$ and the Euler angles $\alpha_{(a)}$, being the product $\Lambda = R\,L$ of a rotation and a boost.

The future-oriented unit normal to Σ_τ and the projector on Σ_τ are $l_A = \epsilon\,(1+n)\,\Big(1;\, 0\Big)$, ${}^4g^{AB}\, l_A\, l_B = \epsilon$, $l^A = \epsilon\,(1+n)\, {}^4g^{A\tau} = \frac{1}{1+n}\Big(1;\, -n^r\Big) = \frac{1}{1+n}\Big(1;\, -\sum_a \bar{n}_{(a)}\, {}^3\bar{e}^r_{(a)}\Big)$, ${}^3h^B_A = \delta^B_A - \epsilon\, l_A\, l^B$.

The 4-metric has the following expression

$$
\begin{aligned}
{}^4g_{\tau\tau} &= \epsilon\,\Big[(1+n)^2 - {}^3g^{rs}\, n_r\, n_s\Big] = \epsilon\,\Big[(1+n)^2 - \sum_a \bar{n}^2_{(a)}\Big],\\
{}^4g_{\tau r} &= -\epsilon\, n_r = -\epsilon\, \sum_a \bar{n}_{(a)}\, {}^3\bar{e}_{(a)r},\\
{}^4g_{rs} &= -\epsilon\, {}^3g_{rs} = -\epsilon\, \sum_a {}^3e_{(a)r}\, {}^3e_{(a)s} = -\epsilon\, \sum_a {}^3\bar{e}_{(a)r}\, {}^3\bar{e}_{(a)s},\\
{}^4g^{\tau\tau} &= \frac{\epsilon}{(1+n)^2}, \qquad {}^4g^{\tau r} = -\epsilon\, \frac{n^r}{(1+n)^2} = -\epsilon\, \frac{\sum_a {}^3\bar{e}^r_{(a)}\, \bar{n}_{(a)}}{(1+n)^2},\\
{}^4g^{rs} &= -\epsilon\, \Big({}^3g^{rs} - \frac{n^r\, n^s}{(1+n)^2}\Big) = -\epsilon\, \sum_{ab} {}^3\bar{e}^r_{(a)}\, {}^3\bar{e}^s_{(b)}\, \Big(\delta_{(a)(b)} - \frac{\bar{n}_{(a)}\, \bar{n}_{(b)}}{(1+n)^2}\Big),\\
\sqrt{-g} &= \sqrt{|{}^4g|} = \frac{\sqrt{{}^3g}}{\sqrt{\epsilon\, {}^4g^{\tau\tau}}} = \sqrt{\gamma}\,(1+n) = {}^3e\,(1+n),\\
{}^3g &= \gamma = ({}^3e)^2, \quad {}^3e = det\, {}^3e_{(a)r}.
\end{aligned}
\tag{8.26}
$$

The 3-metric ${}^3g_{rs}$ has signature $(+++)$, so that one may put all the flat 3-indices *down*. One has ${}^3g^{ru}\, {}^3g_{us} = \delta^r_s$.

8.5.2 The ADM Phase Space and the ADM Hamilton Equations

The given parametrization of the cotetrad fields leads to rewrite the action of ADM tetrad gravity in terms of the following 16 fields as configuration variables: three boost parameters $\varphi_{(a)}(\tau, \sigma^u)$; the lapse $N(\tau, \sigma^u) = 1 + n(\tau, \sigma^u)$ and shift $n_{(a)}(\tau, \sigma^u)$ functions; the nine components of cotriad fields ${}^3e_{(a)r}(\tau, \sigma^u)$ on the 3-spaces Σ_τ.

As shown in the second and third paper of Ref. [41–44], in Ref. [82] and in the first paper of Ref. [88–90], the ADM action for the gravitational field has the expression

$$\begin{aligned} S_{grav} = \frac{c^3}{16\pi G} \int d\tau d^3\sigma \Big[(1+n)\, {}^3e\, \epsilon_{(a)(b)(c)}\, {}^3e^r_{(a)}\, {}^3e^s_{(b)}\, {}^3\Omega_{rs(c)} \\ + \frac{{}^3e}{2(1+n)} ({}^3G_o^{-1})_{(a)(b)(c)(d)}\, {}^3e^r_{(b)} \Big(n_{(a)|r} - \partial_\tau\, {}^3e_{(a)r} \Big) \\ {}^3e^s_{(d)} (n_{(c)|s} - \partial_\tau\, {}^3e_{(c)\,s}) \Big] (\tau, \sigma^u). \end{aligned} \tag{8.27}$$

In it ${}^3\Omega_{rs(a)} = \partial_r\, {}^3\omega_{s(a)} - \partial_s\, {}^3\omega_{r(a)} - \epsilon_{(a)(b)(c)}\, {}^3\omega_{r(b)}\, {}^3\omega_{s(c)}$ is the field strength associated with the 3-spin connection ${}^3\omega_{r(a)} = \frac{1}{2}\, \epsilon_{(a)(b)(c)}\, [{}^3e^u_{(b)}\, (\partial_r\, {}^3e_{(c)u} - \partial_u\, {}^3e_{(c)r}) + \frac{1}{2}\, {}^3e^u_{(b)}\, {}^3e^v_{(c)}\, {}^3e_{(d)r}\, (\partial_v\, {}^3e_{(d)u} - \partial_u\, {}^3e_{(d)v})]$ and $({}^3G_o^{-1})_{(a)(b)(c)(d)} = \delta_{(a)(c)}\delta_{(b)(d)} + \delta_{(a)(d)}\delta_{(b)(c)} - 2\delta_{(a)(b)}\delta_{(c)(d)}$ is the flat (with lower indices) inverse of the flat Wheeler-DeWitt super-metric ${}^3G_{o(a)(b)(c)(d)} = \delta_{(a)(c)}\, \delta_{(b)(d)} + \delta_{(a)(d)}\, \delta_{(b)(c)} - \delta_{(a)(b)}\, \delta_{(c)(d)}$, ${}^3G_{o(a)(b)(e)(f)}\, {}^3G^{-1}_{o(e)(f)(c)(d)} = 2\, (\delta_{(a)(c)}\, \delta_{(b)(d)} + \delta_{(a)(d)}\, \delta_{(b)(c)})$.

The canonical momenta $\pi_{\varphi_{(a)}}(\tau, \sigma^u)$, $\pi_n(\tau, \sigma^u)$, $\pi_{n_{(a)}}(\tau, \sigma^u)$, ${}^3\pi^r_{(a)}(\tau, \sigma^u)$, conjugate to the configuration variables satisfy 14 first-class constraints: the ten primary constraints (the last three constraints generate rotations on quantities with flat indices (a) like the cotriads)

$$\begin{aligned} &\pi_{\varphi_{(a)}}(\tau, \sigma^u) \approx 0, \qquad \pi_n(\tau, \sigma^u) \approx 0, \quad \pi_{n_{(a)}}(\tau, \sigma^u) \approx 0, \\ &{}^3M_{(a)}(\tau, \sigma^u) = \epsilon_{(a)(b)(c)}\, {}^3e_{(b)r}(\tau, \sigma^u)\, {}^3\pi^r_{(c)}(\tau, \sigma^u) \approx 0, \end{aligned} \tag{8.28}$$

and the secondary super-Hamiltonian and super-momentum constraints

$$\begin{aligned} \mathcal{H}(\tau, \sigma^u) &= \Big[\frac{c^3}{16\pi G}\, {}^3e\, \epsilon_{(a)(b)(c)}\, {}^3e^r_{(a)}\, {}^3e^s_{(b)}\, {}^3\Omega_{rs(c)} \\ &\quad - \frac{2\pi G}{c^3\, {}^3e}\, {}^3G_{o(a)(b)(c)(d)}\, {}^3e_{(a)r}\, {}^3\pi^r_{(b)}\, {}^3e_{(c)s}\, {}^3\pi^s_{(d)} \Big] (\tau, \sigma^u) \\ &\quad + \mathcal{M}(\tau, \sigma^u) \approx 0, \\ \mathcal{H}_{(a)}(\tau, \sigma^u) &= \Big[\partial_r\, {}^3\pi^r_{(a)} - \epsilon_{(a)(b)(c)}\, {}^3\omega_{r(b)}\, {}^3\pi^r_{(c)} + {}^3e^r_{(a)} \mathcal{M}_r \Big] (\tau, \sigma^u) \approx 0. \end{aligned} \tag{8.29}$$

The functions $\mathcal{M}(\tau, \sigma^u)$ and $\mathcal{M}_r(\tau, \sigma^u)$ describe the matter present in the space-time: $\mathcal{M}(\tau, \sigma^u)$ is the (matter- and metric-dependent) internal mass density, while $\mathcal{M}_r(\tau, \sigma^u)$ is the universal (metric-independent) internal momentum density. If the action of matter is added to Eq. (8.27), one can evaluate the energy-momentum tensor $T^{AB}(\tau, \sigma^u) = -\Big[\frac{2}{\sqrt{-{}^4g}}\, \frac{\delta S_{matter}}{\delta\, {}^4g_{AB}} \Big](\tau, \boldsymbol{\sigma})$ of the matter[23] and determine these functions

[23] The Hamilton equations imply ${}^4\nabla_A\, T^{AB} \equiv 0$ in accord with Einstein's equations and the Bianchi identity.

from the following parametrization

$$T^{\tau\tau}(\tau,\sigma^u) = \frac{\mathcal{M}(\tau,\sigma^u)}{[{}^3e\,(1+n)^2](\tau,\sigma^u)},$$
$$T^{\tau r}(\tau,\sigma^u) = \frac{{}^3e^r_{(a)}\Big[(1+n)\,{}^3e^s_{(a)}\,\mathcal{M}_s - n_{(a)}\,\mathcal{M}\Big]}{{}^3e\,(1+n)^2}(\tau,\sigma^u). \tag{8.30}$$

The extrinsic curvature tensor of the 3-spaces Σ_τ as 3-manifolds embedded into the space-time has the following expression in terms of the barred cotriads of Eq. (8.24) and their conjugate barred momenta

$$ {}^3K_{rs} = -\frac{4\pi G}{c^3\,{}^3\bar{e}}\sum_{abu}\Big[\Big({}^3\bar{e}_{(a)r}\,{}^3\bar{e}_{(b)s} + {}^3\bar{e}_{(a)s}\,{}^3\bar{e}_{(b)r}\Big)\,{}^3\bar{e}_{(a)u}\,\bar{\pi}^u_{(b)}$$
$$ -{}^3\bar{e}_{(a)r}\,{}^3\bar{e}_{(a)s}\,{}^3\bar{e}_{(b)u}\,\bar{\pi}^u_{(b)}\Big]. \tag{8.31}$$

Therefore the basis of canonical variables for this formulation of tetrad gravity, naturally adapted to 7 of the 14 first-class constraints, is

$\varphi_{(a)}$	n	$n_{(a)}$	${}^3e_{(a)r}$
$\pi_{\varphi_{(a)}} \approx 0$	$\pi_n \approx 0$	$\pi_{n_{(a)}} \approx 0$	${}^3\pi^r_{(a)}$

(8.32)

The behavior of these fields at spatial infinity (compatible with the absence of super-translations) is given in Eq. (5.5) of the third paper in Refs. [41–44]; in particular for the cotriads one has ${}^3e_{(a)r}(\tau,\sigma^r) \rightarrow_{r\rightarrow\infty} \Big(1+\frac{const.}{2r}\Big)\,\delta_{ar} + O(r^{-3/2})$ ($r=\sqrt{\sum_r(\sigma^r)^2}$).

From the action (8.29), after having added the matter action, one can obtain the standard non-Hamiltonian ADM equations ($|r$ denotes the 3-covariant derivative in the 3-space Σ_τ with 3-metric ${}^3g_{rs}$; ${}^3R_{rs}$ is the 3-Ricci tensor of Σ_τ)

$$\partial_\tau\,{}^3g_{rs} \overset{\circ}{=} n_{r|s} + n_{s|r} - 2\,(1+n)\,{}^3K_{rs},$$
$$\partial_\tau\,{}^3K_{rs} \overset{\circ}{=} (1+n)\,\Big({}^3R_{rs} + {}^3K\,{}^3K_{rs} - 2\,{}^3K_{ru}\,{}^3K^u{}_s\Big)$$
$$ - n_{|s|r} + n^u{}_{|s}\,{}^3K_{ur} + n^u{}_{|r}\,{}^3K_{us} + n^u\,{}^3K_{rs|u}, \tag{8.33}$$

with the quantities appearing in these equations re-expressed in terms of the configurational variables of Eq. (8.32).

Instead at the Hamiltonian level one can get the Hamilton equations for all the variables of the canonical basis (8.32), as shown in the first paper of Ref. [88–90], by using the Dirac Hamiltonian. As shown in Refs. [41–44], the Dirac Hamiltonian has the form (if the matter contains the electro-magnetic field there are extra terms with the electro-magnetic first-class constraints)

$$
\begin{aligned}
H_D = \frac{1}{c}\,\hat{E}_{ADM} &+ \int d^3\sigma\,\Big[n\,\mathcal{H} - n_{(a)}\,\mathcal{H}_{(a)}\Big](\tau,\sigma^u)\\
&+ \int d^3\sigma\,\Big[\lambda_n\,\pi_n + \lambda_{n_{(a)}}\,\pi_{n_{(a)}} + \lambda_{\varphi_{(a)}}\,\pi_{\varphi_{(a)}} + \mu_{(a)}\,{}^3M_{(a)}\Big](\tau,\sigma^u),
\end{aligned}
\tag{8.34}
$$

where $\hat{E}_{ADM}$ is the weak ADM energy and the λ's are arbitrary Dirac multipliers.

See Eqs. (2.22), (3.43) and (3.47) of the first paper of Ref. [88–90] for the expression of the ten weak asymptotic ADM Poincaré generators $\hat{E}_{ADM}$, $\hat{P}^r_{ADM}$, $\hat{J}^r_{adm}$, $\hat{\mathcal{K}}^r_{ADM}$. Since one is in a *non-inertial rest frame* (due to the absence of super-translations), one has the rest-frame conditions $\hat{P}^r_{ADM} \approx 0$ like in SR. Then one has to add the conditions $\hat{\mathcal{K}}^r_{ADM} \approx 0$ to eliminate the internal 3-center of mass of the 3-universe like in SR [49–52]. Therefore the 3-universe can be seen as a decoupled external canonical non-covariant center of mass carrying a pole-dipole structure: the invariant mass $Mc = \frac{1}{c}\,\hat{E}_{ADM}$ and the rest spin $\hat{J}^{rs}_{ADM}$. This view is in accord with an old suggestions of Dirac [16, 17].

In Ref. [88–90] the study of ADM canonical tetrad gravity was done with the following type of matter: N charged scalar particles (described by the canonical variables $\eta^r_i(\tau)$, $\kappa_{ir}(\tau)$) and the electro-magnetic field in the non-covariant radiation gauge (described by the canonical variables $A^r_\perp(\tau,\sigma^u)$, $\pi^r_\perp(\tau,\sigma^u)$ as shown in Ref. [49–52]). The particles (described by an action like the one in Eq. (8.6)) have not only Grassmann-valued electric charges Q_i ($Q_i^2 = 0$, $Q_i\,Q_j = Q_j\,Q_i$ for $i \neq j$) to regularize the electro-magnetic self-energies, but also Grassmann-valued signs of the energy ($\eta_i^2 = 0$, $\eta_i\,\eta_j = \eta_j\,\eta_i$ for $i \neq j$) to regularize the gravitational self-energies.[24]

Instead in Ref. [91] the matter is a perfect fluid described by the action of Ref. [94] re-expressed in the $3+1$ point of view in Refs. [92, 93].

In the case of N particles the functions $\mathcal{M}$ and $\mathcal{M}_r$ have the expression (see Ref. [88–90] for their form in presence of the electro-magnetic field)

$$
\begin{aligned}
\mathcal{M}(\tau,\sigma^u) &= \sum_{i=1}^N \delta^3(\sigma^u,\eta^u_i(\tau))\,\eta_i\sqrt{m_i^2\,c^2 + {}^3e^r_{(a)}(\tau,\sigma^u)\kappa_{ir}(\tau)\,{}^3e^s_{(a)}(\tau,\sigma^u)\kappa_{is}(\tau)},\\
\mathcal{M}_r(\tau,\sigma^u) &= \sum_{i=1}^N \eta_i\,\kappa_{ir}(\tau),
\end{aligned}
\tag{8.35}
$$

while in the case of dust [91], described by canonical coordinates $\alpha^i(\tau,\sigma^u)$, $\Pi_i(\tau,\sigma^u)$, $i = 1,2,3$, they have the expression

[24] Both quantities are two-valued. The elementary electric charges are $Q = \pm e$, with e the electron charge. Analogously the sign of the energy of a particle is a topological two-valued number (the two branches of the mass-shell hyperboloid). The formal quantization of these Grassmann variables gives two-level fermionic oscillators. At the classical level the self-energies make the classical equations of motion ill-defined on the world-lines of the particles. The ultraviolet and infrared Grassmann regularization allows to cure this problem and to get consistent solution of regularized equations of motion. See Refs. [65, 66] for the electro-magnetic case.

$$\mathcal{M}(\tau, \sigma^u) = \sqrt{\mu^2\, [det\, (\partial_s\, \alpha^j)]^2 + \tilde{\phi}^{-2/3} \sum_{arsij} Q_a^{-2}\, V_{ra}\, V_{sa}\, \partial_r\, \alpha^i\, \partial_s\, \alpha^j\, \Pi_i\, \Pi_j} \times (\tau, \sigma^u),$$
$$\mathcal{M}_r(\tau, \sigma^u) = \sum_i \partial_r\, \alpha^i(\tau, \sigma^u)\, \Pi_i(\tau, \sigma^u). \tag{8.36}$$

8.6 The York Canonical Basis and the Inertial and Tidal Degrees of Freedom of the Gravitational Field

The presence of 14 first-class constraints in the phase space having the 32 fields of Eq. (8.32) as a canonical basis implies that there are 14 gauge variables describing *inertial effects* and 2 canonical pairs of physical degrees of freedom describing the *tidal effects* of the gravitational field (namely gravitational waves in the weak field limit). To disentangle the inertial effects from the tidal ones one needs a canonical transformation to a new canonical basis adapted to all the ten primary constraints (8.28) and containing the barred variables defined in Eq. (8.24). This is the topic of this Section.

8.6.1 The York Canonical Basis

A canonical transformation adapted to the ten primary constraints (8.28) was found in Ref. [82]. It implements the York map of Ref. [83] in the cases in which the 3-metric ${}^3g_{rs}$ has three distinct eigenvalues and diagonalizes the York-Lichnerowicz approach (see Ref. [84] for a review).

As said before Eq. (8.24), one can decompose the cotriads on Σ_τ in the product of a rotation matrix, belonging to the subgroup SO(3) of the tetrad gauge group and depending on three Euler angles $\alpha_{(a)}(\tau, \sigma^r)$, and of barred cotriads depending only on six independent fields. The canonical transformation Abelianizes the constraints ${}^3M_{(a)}(\tau, \sigma^u) \approx 0$ of Eqs. (8.28), satisfying $\{{}^3M_{(a)}(\tau, \sigma^u), {}^3M_{(b)}(\tau, \sigma^{'u})\} = \epsilon_{(a)(b)(c)}\, {}^3M_{(c)}(\tau, \sigma^u)\delta^3(\sigma^u, \sigma^{'u})$, and replaces them with the vanishing of the three momenta $\pi^{(\alpha)}_{(a)}(\tau, \sigma^r) \approx 0$ conjugate to the Euler angles.

The new canonical basis, named York canonical basis, is ($a = 1, 2, 3$; $\bar{a} = 1, 2$)

$\varphi_{(a)}$	$\alpha_{(a)}$	n	$\bar{n}_{(a)}$	θ^r	$\tilde{\phi}$	$R_{\bar{a}}$
$\pi_{\varphi_{(a)}} \approx 0$	$\pi^{(\alpha)}_{(a)} \approx 0$	$\pi_n \approx 0$	$\pi_{\bar{n}_{(a)}} \approx 0$	$\pi^{(\theta)}_r$	$\pi_{\tilde{\phi}} = \frac{c^3}{12\pi G}\, {}^3K$	$\Pi_{\bar{a}}$

(8.37)

In it the cotriads and the components of the 4-metric have the following expression

$$
\begin{aligned}
{}^3e_{(a)r} &= \sum_b R_{(a)(b)}(\alpha_{(c)})\,{}^3\bar{e}_{(b)r} = \sum_b R_{(a)(b)}(\alpha_{(c)})\,V_{rb}(\theta^i)\,\tilde{\phi}^{1/3}\,e^{\sum_{\bar{a}}^{1,2}\gamma_{\bar{a}a}R_{\bar{a}}},\\
{}^4g_{\tau\tau} &= \epsilon\left[(1+n)^2 - \sum_a \bar{n}_{(a)}^2\right],\\
{}^4g_{\tau r} &= -\epsilon\sum_a n_{(a)}\,{}^3e_{(a)r} = -\epsilon\sum_a \bar{n}_{(a)}\,{}^3\bar{e}_{(a)r},\quad \tilde{\phi} = \phi^6 = \sqrt{det\,{}^3g_{rs}},\\
{}^4g_{rs} &= -\epsilon\,{}^3g_{rs} = -\epsilon\,\tilde{\phi}^{2/3}\sum_a V_{ra}(\theta^i)\,V_{sa}(\theta^i)\,Q_a^2,\quad Q_a = e^{\sum_{\bar{a}}^{1,2}\gamma_{\bar{a}a}R_{\bar{a}}},
\end{aligned}
\tag{8.38}
$$

The set of numerical parameters $\gamma_{\bar{a}a}$ appearing in Q_a satisfies [41–44] $\sum_u \gamma_{\bar{a}u} = 0$, $\sum_u \gamma_{\bar{a}u}\,\gamma_{\bar{b}u} = \delta_{\bar{a}\bar{b}}$, $\sum_{\bar{a}} \gamma_{\bar{a}u}\,\gamma_{\bar{a}v} = \delta_{uv} - \frac{1}{3}$. Each solution of these equations defines a different York canonical basis.

This canonical basis has been found due to the fact that the 3-metric ${}^3g_{rs}$ is a real symmetric 3×3 matrix, which may be diagonalized with an *orthogonal* matrix $V(\theta^r)$, $V^{-1} = V^T$ ($\sum_u V_{ua}\,V_{ub} = \delta_{ab}$, $\sum_a V_{ua}\,V_{va} = \delta_{uv}$, $\sum_{uv} \epsilon_{wuv}\,V_{ua}\,V_{vb} = \sum_c \epsilon_{abc}\,V_{cw}$), $det\,V = 1$, depending on three parameters θ^i ($i = 1, 2, 3$),[25] whose conjugate momenta $\Pi_i^{(\theta)}$ are to be determined as solutions of the super-momentum constraints. If one chooses these three gauge parameters to be Euler angles $\hat{\theta}^i(\tau, \boldsymbol{\sigma})$, one gets a description of the 3-coordinate systems on Σ_τ from a local point of view, because they give the orientation of the tangents to the three 3-coordinate lines through each point. However, for the calculations (see Refs. [88–90]) it is more convenient to choose the three gauge parameters as first kind coordinates $\theta^i(\tau, \boldsymbol{\sigma})$ ($-\infty < \theta^i < +\infty$) on the O(3) group manifold, so that by definition one has $V_{ru}(\theta^i) = \left(e^{-\sum_i \hat{T}_i\,\theta^i}\right)_{ru}$, where $(\hat{T}_i)_{ru} = \epsilon_{rui}$ are the generators of the o(3) Lie algebra in the adjoint representation, and the Euler angles may be expressed as $\hat{\theta}^i = f^i(\theta^n)$. The Cartan matrix has the form $A(\theta^n) = \frac{1-e^{-\sum_i \hat{T}_i\,\theta^i}}{\sum_i \hat{T}_i\,\theta^i}$ and satisfies $A_{ri}(\theta^n)\,\theta^i = \delta_{ri}\,\theta^i$; $B(\theta^i) = A^{-1}(\theta^i)$.

From now on for the sake of notational simplicity the symbol V will mean $V(\theta^i)$.

The extrinsic curvature tensor of the 3-space Σ_τ has the expression

$$
\begin{aligned}
{}^3K_{rs}(\tau, \sigma^u) = -\frac{4\pi\,G}{c^3}\Big[\tilde{\phi}^{-1/3}\Big(\sum_a Q_a^2\,V_{ra}\,V_{sa}\left[2\sum_{\bar{b}}\gamma_{\bar{b}a}\,\Pi_{\bar{b}} - \tilde{\phi}\,\pi_{\tilde{\phi}}\right]\\
+\sum_{ab} Q_a\,Q_b\,(V_{ra}\,V_{sb} + V_{rb}\,V_{sa})\sum_{twi}\frac{\epsilon_{abt}\,V_{wt}\,B_{iw}\,\pi_i^{(\theta)}}{Q_b\,Q_a^{-1} - Q_a\,Q_b^{-1}}\Big)\Big](\tau, \sigma^u).
\end{aligned}
\tag{8.39}
$$

[25] Due to the positive signature of the 3-metric, one defines the matrix V with the following indices: V_{ru}. Since the choice of Shanmugadhasan canonical bases breaks manifest covariance, one will use the notation $V_{ua} = \sum_v V_{uv}\,\delta_{v(a)}$ instead of $V_{u(a)}$.

This canonical transformation realizes a *York map* because the gauge variable $\pi_{\tilde{\phi}}$ (describing the freedom in the choice of the trace of the extrinsic curvature of the instantaneous 3-spaces Σ_τ) is proportional to *York internal extrinsic time* ${}^3K$. It is the only gauge variable among the momenta: this is a reflex of the Lorentz signature of space-time, because $\pi_{\tilde{\phi}}$ and θ^n can be used as a set of 4-coordinates for the space-time [79–81]. The York time describes the effect of gauge transformations producing a deformation of the shape of the 3-space along the 4-normal to the 3-space as a 3-sub-manifold of space-time.

Its conjugate variable, to be determined by the super-Hamiltonian constraint (interpreted as the Lichnerowicz equation), is $\tilde{\phi} = \phi^6 = {}^3\bar{e} = \sqrt{det\, {}^3g_{rs}}$, which is proportional to *Misner's internal intrinsic time*; moreover $\tilde{\phi}$ is the *3-volume density* on Σ_τ: $V_R = \int_R d^3\sigma\, \tilde{\phi}$, $R \subset \Sigma_\tau$. Since one has ${}^3g_{rs} = \tilde{\phi}^{2/3}\, {}^3\hat{g}_{rs}$ with $det\, {}^3\hat{g}_{rs} = 1$, $\tilde{\phi}$ is also called the *conformal factor* of the 3-metric.

The two pairs of canonical variables $R_{\bar{a}}$, $\Pi_{\bar{a}}$, $\bar{a} = 1, 2$, describe the generalized *tidal effects*, namely the independent physical degrees of freedom of the gravitational field. They are 3-scalars on Σ_τ and the configuration tidal variables $R_{\bar{a}}$ parametrize *the two eigenvalues of the 3-metric* ${}^3\hat{g}_{rs}$ *with unit determinant*. They are Dirac observables *only* with respect to the gauge transformations generated by 10 of the 14 first class constraints. Let us remark that, if one fixes completely the gauge and one goes to Dirac brackets, then the only surviving dynamical variables $R_{\bar{a}}$ and $\Pi_{\bar{a}}$ become two pairs of *non canonical* Dirac observables for that gauge: the two pairs of canonical Dirac observables have to be found as a Darboux basis of the copy of the reduced phase space identified by the gauge and they will be (in general non-local) functionals of the $R_{\bar{a}}$, $\Pi_{\bar{a}}$ variables.

Therefore, the 14 arbitrary gauge variables are $\varphi_{(a)}(\tau, \sigma^u)$, $\alpha_{(a)}(\tau, \sigma^u)$, $n(\tau, \sigma^u)$, $\bar{n}_{(a)}(\tau, \sigma^u)$, $\theta^i(\tau, \sigma^u)$, $\pi_{\tilde{\phi}}(\tau, \sigma^u)$: they describe the following generalized *inertial effects* [82]:

(a) $\alpha_{(a)}(\tau, \sigma^u)$ and $\varphi_{(a)}(\tau, \sigma^u)$ are the 6 configuration variables parametrizing the O(3,1) gauge freedom in the choice of the tetrads in the tangent plane to each point of Σ_τ and describe the arbitrariness in the choice of a tetrad to be associated to a time-like observer, whose world-line goes through the point $(\tau, \boldsymbol{\sigma})$. They fix *the unit 4-velocity of the observer and the conventions for the orientation of three gyroscopes and their transport along the world-line of the observer*. The *Schwinger time gauges* are defined by the gauge fixings $\alpha_{(a)}(\tau, \sigma^u) \approx 0$, $\varphi_{(a)}(\tau, \sigma^u) \approx 0$.

(b) $\theta^i(\tau, \sigma^u)$ (depending only on the 3-metric) describe the arbitrariness in the choice of the 3-coordinates in the instantaneous 3-spaces Σ_τ of the chosen non-inertial frame centered on an arbitrary time-like observer. Their choice will induce a pattern of *relativistic inertial forces* for the gravitational field, whose potentials are the functions $V_{ra}(\theta^i)$ present in the weak ADM energy $\hat{E}_{ADM}$.

(c) $\bar{n}_{(a)}(\tau, \sigma^u)$, the shift functions, describe which points on different instantaneous 3-spaces have the same numerical value of the 3-coordinates. They are the inertial potentials describing the effects of the non-vanishing off-diagonal components

$^4g_{\tau r}(\tau, \sigma^u)$ of the 4-metric, namely they are the *gravito-magnetic potentials*[26] responsible of effects like the dragging of inertial frames (Lens-Thirring effect) in the post-Newtonian approximation. The shift functions are determined by the τ-preservation of the gauge fixings determining the gauge variables $\theta^i(\tau, \sigma^u)$.

(d) $\pi_{\tilde{\phi}}(\tau, \sigma^u)$, i.e. the York time $^3K(\tau, \sigma^u)$, describes the non-dynamical arbitrariness in the choice of the convention for the synchronization of distant clocks which remains in the transition from SR to GR. Since the York time is present in the Dirac Hamiltonian, it is a *new inertial potential* connected to the problem of the relativistic freedom in the choice of the *shape of the instantaneous 3-space*, which has no Newtonian analogue (in Galilei space-time time is absolute and there is an absolute notion of Euclidean 3-space). Its effects are completely unexplored. Instead the other components of the extrinsic curvature of Σ_τ are dynamically determined once a 3-coordinate system has been chosen in the 3-space.

(e) $1 + n(\tau, \sigma^u)$, the lapse function appearing in the Dirac Hamiltonian, describes the arbitrariness in the choice of the unit of proper time in each point of the simultaneity surfaces Σ_τ, namely how these surfaces are packed in the 3 + 1 splitting. The lapse function is determined by the τ-preservation of the gauge fixing for the gauge variable $^3K(\tau, \sigma^u)$.

As shown in Ref. [79–81], the dynamical nature of space-time implies that each solution (i.e. an Einstein 4-geometry) of Einstein's equations (or of the associated ADM Hamilton equations) dynamically selects a preferred 3 + 1 splitting of the space-time, namely in GR the instantaneous 3-spaces are dynamically determined modulo only one inertial gauge function (the gauge freedom in clock synchronization in GR). In the York canonical basis this function is the *York time*, namely the trace of the extrinsic curvature of the 3-space. Instead in SR the gauge freedom in clock synchronization depends on four basic gauge functions, the embeddings $z^\mu(\tau, \sigma^r)$, and both the 4-metric and the whole extrinsic curvature tensor were derived inertial potentials. Instead in GR the extrinsic curvature tensor of the 3-spaces is a mixture of dynamical (tidal) pieces and inertial gauge variables playing the role of inertial potentials.

[26] In the post-Newtonian approximation in harmonic gauges they are the counterpart of the electro-magnetic vector potentials describing magnetic fields [84]: (A) $N = 1 + n$, $n \stackrel{def}{=} -\frac{4\epsilon}{c^2}\Phi_G$ with Φ_G the *gravito-electric potential*; (B) $n_r \stackrel{def}{=} \frac{2\epsilon}{c^2} A_{Gr}$ with A_{Gr} the *gravito-magnetic* potential; (C) $E_{Gr} = \partial_r \Phi_G - \partial_\tau (\frac{1}{2} A_{Gr})$ (the *gravito-electric field*) and $B_{Gr} = \epsilon_{ruv}\, \partial_u A_{Gv} = c\, \Omega_{Gr}$ (the *gravito-magnetic field*). Let us remark that in arbitrary gauges the analogy with electro-magnetism breaks down.

8.6.2 3-Orthogonal Schwinger Time Gauges and Hamilton Equations

As shown in the first paper in Refs. [88–90], in the York canonical basis the Dirac Hamiltonian (8.34) becomes (the λ's are arbitrary Dirac multipliers; the Dirac multiplier $\lambda_r(\tau)$ implements the rest frame condition $\hat{P}^r_{ADM} \approx 0$)

$$
\begin{aligned}
H_D &= \frac{1}{c}\,\hat{E}_{ADM} + \int d^3\sigma \Big[n\,\mathcal{H} - n_{(a)}\,\mathcal{H}_{(a)}\Big](\tau, \sigma^u) + \lambda_r(\tau)\,\hat{P}^r_{ADM} \\
&\quad + \int d^3\sigma \Big[\lambda_n\,\pi_n + \lambda_{\bar{n}_{(a)}}\,\pi_{\bar{n}_{(a)}} + \lambda_{\varphi_{(a)}}\,\pi_{\varphi_{(a)}} + \lambda_{\alpha_{(a)}}\,\pi^{(\alpha)}_{(a)}\Big](\tau, \sigma^u),
\end{aligned}
\tag{8.40}
$$

with the following expression for the weak ADM energy

$$
\begin{aligned}
\hat{E}_{ADM} &= c \int d^3\sigma \Bigg[\check{\mathcal{M}} - \frac{c^3}{16\pi\,G}\,\mathcal{S} + \frac{2\pi\,G}{c^3}\,\tilde{\phi}^{-1}\Bigg(-3\,(\tilde{\phi}\,\pi_{\tilde{\phi}})^2 + 2\sum_{\bar{b}} \Pi^2_{\bar{b}} \\
&\quad + 2 \sum_{abtwiuvj} \frac{\epsilon_{abt}\,\epsilon_{abu}\,V_{wt}\,B_{iw}\,V_{vu}\,B_{jv}\,\pi^{(\theta)}_i\,\pi^{(\theta)}_j}{\Big[Q_a\,Q_b^{-1} - Q_b\,Q_a^{-1}\Big]^2}\Bigg)\Bigg](\tau, \sigma^u).
\end{aligned}
\tag{8.41}
$$

In it $\mathcal{S}(\tau, \sigma^u)$ is a function of $\tilde{\phi}$, θ^i and $R_{\bar{a}}$ (given in Eq. (B8) of the first paper in Ref. [88–90]), which play the role of an inertial potential depending on the choice of the 3-coordinates in the 3-space (it is the $\Gamma - \Gamma$ term in the scalar 3-curvature of the 3-space).

Equation (8.41) shows that the kinetic term, quadratic in the momenta, is *not positive definite*. While the kinetic energy of the tidal variables and the last term[27] are positive definite, there is the negative kinetic terms (vanishing only in the gauges ${}^3K(\tau, \sigma^u) = 0$) $-\frac{c^4}{24\pi G}\int D^3\sigma\,\tilde{\phi}(\tau, \sigma^u)\,{}^3K^2(\tau, \sigma^u)$. It is an inertial potential associated with the inertial gauge variable York time, which is a momentum due to the Lorentz signature of space-time. It was known that this quadratic form is not definite positive, but only in the York canonical basis this can be made explicit.

In the York canonical basis it is possible to follow the procedure for the fixation of a gauge natural from the point of view of constraint theory when there are chains of first-class constraints [18–21]. This procedure implies that one has to add six gauge fixings to the primary constraints without without secondaries ($\pi_{\varphi_{(a)}}(\tau, \sigma^u) \approx 0$, $\pi_{\alpha_{(a)}}(\tau, \sigma^u) \approx 0$) and four gauge fixings to the secondary super-Hamiltonian and super-momentum constraints. These ten gauge fixings must be preserved in time, namely their Poisson brackets with the Dirac Hamiltonian must vanish. The τ-

[27] It describes gravito-magnetic effects.

preservation of the six gauge fixings determining the gauge variables $\alpha_{(a)}(\tau, \sigma^u)$ and $\varphi_{(a)}(\tau, \sigma^u)$ produces the equations determining the six Dirac multipliers $\lambda_{\varphi_{(a)}}(\tau, \sigma^u)$, $\lambda_{\alpha_{(a)}}(\tau, \sigma^u)$. The τ-preservation of the other four gauge fixings, determining the gauge variables $\theta^i(\tau, \sigma^u)$ and the York time ${}^3K(\tau, \sigma^u)$, produces four secondary gauge fixing constraints for the determination of the lapse and shift functions. Then the τ-preservation of these secondary gauge fixings determines the four Dirac multipliers $\lambda_n(\tau, \sigma^u)$, $\lambda_{\bar{n}_{(a)}}(\tau, \sigma^u)$. Instead in numerical gravity one gives independent gauge fixings for both the primary and secondary gauge variables in such a way to minimize the computer time.

In Section V of the first paper in Refs. [88–90] there is a review of the gauges usually used in canonical gravity. It is shown that the commonly used family of the harmonic gauges is not natural according to the above procedure. The harmonic gauge fixings imply hyperbolic PDE for the lapse and shift functions, to be added to the hyperbolic PDE for the tidal variables. Therefore in harmonic gauges both the tidal variables and the lapse and shift functions depend (in a retarded way) from the *no-incoming radiation* condition on the Cauchy surface in the past (so that the knowledge of ${}^3K$ from the initial time till today is needed).

Instead the natural gauge fixings in the York canonical basis of ADM tetrad gravity are the family of *Schwinger time gauges*, where the O(3,1) gauge freedom of the tetrads is eliminated with the gauge fixings (implying $\lambda_{\varphi_{(a)}}(\tau, \sigma^u) = \lambda_{\alpha_{(a)}}(\tau, \sigma^u) = 0$)

$$\alpha_{(a)}(\tau, \sigma^u) \approx 0, \quad \varphi_{(a)}(\tau, \sigma^u) \approx 0, \tag{8.42}$$

and the subfamily of the *3-orthogonal gauges*

$$\theta^i(\tau, \sigma^u) \approx 0, \quad {}^3K(\tau, \sigma^u) \approx F(\tau, \sigma^u) = \textit{numerical function}, \tag{8.43}$$

in which the 3-coordinates are chosen in such a way that the 3-metric in the 3-spaces Σ_τ is diagonal. The τ-preservation of Eq. (8.43) gives four coupled elliptic PDE for the lapse and shift functions. Therefore in these gauges only the tidal variables (the gravitational waves after linearization), and therefore only the eigenvalues of the 3-metric with unit determinant inside Σ_τ, depend (in a retarded way) on the no-incoming radiation condition. The solutions $\tilde{\phi}$ and $\pi_i^{(\theta)}$ of the constraints and the lapse $1+n$ and shift $\bar{n}_{(a)}$ functions depend only on the 3-space Σ_τ with fixed τ. If the matter consists of positive energy particles (with a Grassmann regularization of the gravitational self-energies) [88–90] these solutions will contain action-at-a-distance gravitational potentials (replacing the Newton ones) and gravito-magnetic potentials.

In the family of 3-orthogonal gauges the weak ADM energy and the super-Hamiltonian and super-momentum constraints (they are coupled *elliptic* PDE for their unknowns) have the expression (see Eq. (3.47) of the first paper in Ref. [88–90] for the other weak ADM Poincaré generators)

$$
\begin{aligned}
\hat{E}_{ADM}|_{\theta^i=0} &= c \int d^3\sigma \Bigg[\mathcal{M}|_{\theta^i=0} - \frac{c^3}{16\pi\, G}\, \mathcal{S}|_{\theta^i=0} \\
&\quad + \frac{2\pi\, G}{c^3}\, \tilde{\phi}^{-1} \Bigg(-3\, (\tilde{\phi}\, \pi_{\tilde{\phi}})^2 + 2 \sum_{\bar{b}} \Pi_{\bar{b}}^2 \\
&\quad + 2 \sum_{abij} \frac{\epsilon_{abi}\, \epsilon_{abj}\, \pi_i^{(\theta)}\, \pi_j^{(\theta)}}{\left[Q_a\, Q_b^{-1} - Q_b\, Q_a^{-1} \right]^2} \Bigg) \Bigg] (\tau, \sigma^u), \\
\mathcal{H}(\tau, \sigma^u)|_{\theta^i=0} & \\
&= \frac{c^3}{16\pi\, G}\, \tilde{\phi}^{1/6}(\tau, \sigma^u) \left[8\, \hat{\triangle}\, \tilde{\phi}^{1/6} - {}^3\hat{R}|_{\theta^i=0}\, \tilde{\phi}^{1/6} \right] (\tau, \sigma^u) \\
&\quad + \mathcal{M}|_{\theta^i=0}(\tau, \sigma^u) + \frac{2\pi\, G}{c^3}\, \tilde{\phi}^{-1} \Bigg[-3\, (\tilde{\phi}\, \pi_{\tilde{\phi}})^2 + 2 \sum_{\bar{b}} \Pi_{\bar{b}}^2 \\
&\quad + 2 \sum_{abij} \frac{\epsilon_{abi}\, \epsilon_{abj}\, \pi_i^{(\theta)}\, \pi_j^{(\theta)}}{\left[Q_a\, Q_b^{-1} - Q_b\, Q_a^{-1} \right]^2} \Bigg] \\
&\quad \times (\tau, \sigma^u), \\
\tilde{\bar{\mathcal{H}}}_{(a)}|_{\theta^i=0}(\tau, \sigma^u) &= \phi^{-2}(\tau, \boldsymbol{\sigma}) \Big[\sum_{b\neq a} \sum_i \frac{\epsilon_{abi}\, Q_b^{-1}}{Q_b\, Q_a^{-1} - Q_a\, Q_b^{-1}}\, \partial_b\, \pi_i^{(\theta)} \\
&\quad + 2 \sum_{b\neq a} \sum_i \frac{\epsilon_{abi}\, Q_a^{-1}}{\left(Q_b\, Q_a^{-1} - Q_a Q_b^{-1} \right)^2} \sum_{\bar{c}} (\gamma_{\bar{c}a} - \gamma_{\bar{c}b})\, \partial_b\, R_{\bar{c}}\ \pi_i^{(\theta)} \\
&\quad + Q_a^{-1} \Big(\phi^6\, \partial_a \pi_{\tilde{\phi}} + \sum_{\bar{b}} (\gamma_{\bar{b}a}\, \partial_a\, \Pi_{\bar{b}} - \partial_a\, R_{\bar{b}}\, \Pi_{\bar{b}}) + \mathcal{M}_a \Big) \Big] (\tau, \sigma^u), \\
\hat{\triangle} &= \sum_r Q_r^{-2} \Big[\partial_r^2 + 2 \sum_{\bar{a}} \gamma_{\bar{a}r}\, \partial_r\, R_{\bar{a}}(\tau, \sigma^u)\, \partial_r \Big], \\
\mathcal{S}_{\theta^i=0}(\tau, \sigma^u) &= \Big(\tilde{\phi}^{1/3} \sum_a Q_a^{-2} \Big[\frac{2}{9}\, (\tilde{\phi}^{-1}\, \partial_a\, \tilde{\phi})^2 \\
&\quad + \sum_{\bar{b}} \Big(\sum_{\bar{c}} (2\, \gamma_{\bar{b}a}\, \gamma_{\bar{c}a} - \delta_{\bar{b}\bar{c}})\, \partial_a\, R_{\bar{c}} \\
&\quad - \frac{2}{3}\, \gamma_{\bar{b}a}\, \tilde{\phi}^{-1}\, \partial_a\, \tilde{\phi} \Big) \partial_a\, R_{\bar{b}} \Big] \Big) (\tau, \sigma^u). \qquad (8.44)
\end{aligned}
$$

In the first paper in Refs. [88–90] there is the explicit form of the Hamilton equations for all the canonical variables of the gravitational field and of the matter replacing the standard 12 ADM equations and the matter equations ${}^4\nabla_A\, T^{AB} = 0$ in the Schwinger time gauges and their restriction to the 3-orthogonal gauges. They could also be obtained from the effective Dirac Hamiltonian of the 3-orthogonal gauges, which is evaluated by means of a τ-dependent canonical transformation sending the gauge momentum $\pi_{\tilde{\phi}}(\tau, \sigma^u)$ in the gauge-fixing conditions $\pi^{'}_{\tilde{\phi}}(\tau, \sigma^u) = \frac{c^3}{12\pi G}\Big({}^3K(\tau, \sigma^u) - F(\tau, \sigma^u)\Big) \approx 0$ and which is given in Eq. (4.39) of the second paper of Ref. [88–90].

These equations are divided in five groups:

(A) The *contracted Bianchi identities*, namely the evolution equations for the solutions $\tilde{\phi}(\tau, \sigma^u)$ and $\pi_i^{(\theta)}(\tau, \sigma^u)$ of the super-Hamiltonian and super-momentum constraints: they are identities saying that, given a solution of the constraints on a Cauchy surface, it remains a solution also at later times.
(B) The evolution equation for the four basic gauge variables $\theta^i(\tau, \sigma^u)$ and ${}^3K(\tau, \sigma^u)$ (the equation for the York time is the Raychaudhuri equation[28]): these equations determine the lapse and the shift functions once four gauge-fixings for the basic gauge variables are given.
(C) The equations $\partial_\tau\, n(\tau, \sigma^u) = \lambda_n(\tau, \sigma^u)$ and $\partial_\tau\, \bar{n}_{(a)}(\tau, \sigma^u) = \lambda_{\bar{n}_{(a)}}(\tau, \sigma^u)$. Once the lapse and shift functions of the chosen gauge have been found, they determine the associated Dirac multipliers.
(D) The *hyperbolic* evolution PDE for the tidal variables $R_{\bar{a}}(\tau, \sigma^u)$, $\Pi_{\bar{a}}(\tau, \sigma^u)$. When the equations for $\partial_\tau\, R_{\bar{a}}(\tau, \sigma^u)$ is inverted to get $\Pi_{\bar{a}}(\tau, \sigma^u)$ in terms of $R_{\bar{a}}(\tau, \sigma^u)$ and its derivatives, then the Hamilton equations for $\Pi_{\bar{a}}(\tau, \sigma^u)$ become hyperbolic PDE for the evolution of the physical tidal variable $R_{\bar{a}}(\tau, \sigma^u)$.
(E) The Hamilton equations for matter, when present.

Given a solution of the super-momentum and super-Hamiltonian constraints and the Cauchy data for the tidal variables on an initial 3-space, one can find a solution of Einstein's equations in radar 4-coordinates adapted to a time-like observer in the chosen gauge.

8.6.3 The Congruence of Eulerian Observers and the Non-Hamiltonian First-Order ADM Equations of Cosmological Spacetimes

Like in SR one can consider the congruence of the Eulerian observers with zero vorticity associated with the 3 + 1 splitting of space-time, whose properties are

[28] This equation is relevant for studying the developments of caustics in a congruence of time-like geodesics for converging values of the expansion θ and of singularities in Einstein space-times [137–139]. However the boundary conditions of asymptotically Minkowskian space-times without super-translations should avoid the singularity theorems as it happens with their subfamily without matter of Ref. [40].

described by Eq. (8.5). In the first paper of Ref. [88–90] it is shown that in ADM tetrad gravity the congruence has the following properties in each point $(\tau; \sigma^r)$[29]:

(a) The *acceleration* ${}^3a^A = l^B\, {}^4\nabla_B\, l^A = {}^4g^{AB}\, {}^3a_B$ has the components ${}^3a^\tau = 0$, ${}^3a^r = \epsilon\, \tilde{\phi}^{-2/3}\, Q_a^{-2}\, V_{ra}\, V_{sa}\, \partial_s\, ln(1+n)$, ${}^3a_\tau = -\tilde{\phi}^{-1/3}\, Q_a^{-1}\, V_{ra}\, \bar{n}_{(a)}\, \partial_r\, ln(1+n)$, ${}^3a_r = -\partial_r\, ln\,(1+n)$.

(b) The *expansion*[30] coincides with the *York time*:

$$\theta = {}^4\nabla_A\, l^A = -\epsilon\, {}^3K = -\epsilon\, \frac{12\pi\, G}{c^3}\, \pi_{\tilde{\phi}}. \tag{8.45}$$

In cosmology the expansion is proportional to the Hubble constant and the dimensionless *cosmological deceleration parameter* is $q = 3\, l^A\, {}^4\nabla_A\, \frac{1}{\theta} - 1 = -3\, \theta^{-2}\, l^A\, \partial_A\, \theta - 1$.

(c) By using Eq. (8.24) it can be shown that the *shear*[31] $\sigma_{AB} = \sigma_{BA} = -\frac{\epsilon}{2}\, ({}^3a_A\, l_B + {}^3a_B\, l_A) + \frac{\epsilon}{2}\, ({}^4\nabla_A\, l_B + {}^4\nabla_B\, l_A) - \frac{1}{3}\, \theta\, {}^3h_{AB} = \sigma_{(\alpha)(\beta)}\, {}^4\overset{\circ}{\bar{E}}{}^{(\alpha)}_A\, {}^4\overset{\circ}{\bar{E}}{}^{(\beta)}_B$ has the following components $\sigma_{(o)(o)} = \sigma_{(o)(r)} = 0$, $\sigma_{(a)(b)} = \sigma_{(b)(a)} = ({}^3K_{rs} - \frac{1}{3}\, {}^3g_{rs}\, {}^3K)\, {}^3\bar{e}^r_{(a)}\, {}^3\bar{e}^s_{(b)}$, $\sum_a \sigma_{(a)(a)} = 0$. $\sigma_{(a)(b)}$ depends upon the canonical variables θ^r, $\tilde{\phi}$, $R_{\bar{a}}$, $\pi_i^{(\theta)}$ and $\Pi_{\bar{a}}$.

By using Eq. (8.39) for the extrinsic curvature tensor one finds that the diagonal elements $\sigma_{(a)(a)}$ of the shear are also connected with the tidal momenta $\Pi_{\bar{a}}$, while the non-diagonal elements $\sigma_{(a)(b)}|_{a\neq b}$ are connected with the momenta $\pi_i^{(\theta)}$ (the unknowns in the super-momentum constraints)

$$\begin{aligned}
\Pi_{\bar{a}} &= -\frac{c^3}{8\pi\, G}\, \tilde{\phi}\, \sum_a \gamma_{\bar{a}a}\, \sigma_{(a)(a)},\\
\pi_i^{(\theta)} &= \frac{c^3}{8\pi\, G}\, \tilde{\phi}\, \sum_{wtab} A_{wi}\, V_{wt}\, Q_a\, Q_b^{-1}\, \epsilon_{tab}\, \sigma_{(a)(b)}|_{a\neq b},\\
{}^3K_{rs} &= \tilde{\phi}^{2/3}\, \sum_{ab} \Big(-\frac{\epsilon}{3}\, \theta\, \delta_{ab} + \sigma_{(a)(b)}\Big)\, Q_a\, Q_b\, V_{ra}\, V_{sb}\\
&\to_{\theta^i \to 0}\, \tilde{\phi}^{2/3}\, Q_r\, Q_s\, \Big(-\frac{\epsilon}{3}\, \theta + \sigma_{(a)(b)}\Big).
\end{aligned} \tag{8.46}$$

[29] See the "$1+3$ point of view" of Ref. [140] for a discussion of gravity in terms of the second non-surface-forming congruence of time-like observers associated with a $3+1$ splitting of space-time.

[30] It measures the average expansion of the infinitesimally nearby world-lines surrounding a given world-line in the congruence.

[31] It measures how an initial sphere in the tangent space to the given world-line, which is Lie-transported along the world-line tangent l^μ (i.e. it has zero Lie derivative with respect to $l^\mu\, \partial_\mu$), is distorted towards an ellipsoid with principal axes given by the eigenvectors of $\sigma^\mu{}_\nu$, with rate given by the eigenvalues of $\sigma^\mu{}_\nu$.

Therefore the Eulerian observers associated to the $3+1$ splitting of space-time induce a geometrical interpretation of some of the momenta of the York canonical basis:

(1) *their expansion θ is the gauge variable York time ${}^3K = \frac{12\pi G}{c^3}\,\pi_{\tilde{\phi}}$ determining the non-dynamical gauge part of the shape of the instantaneous 3-spaces Σ_τ as a sub-manifold of space-time*;

(2) *the diagonal elements of their shear describe the tidal momenta $\Pi_{\bar{a}}$, while the non-diagonal elements are connected to the variables $\pi_i^{(\theta)}$*, determined by the super-momentum constraints.

In Eq. (8.44), valid in the 3-orthogonal gauges, the term quadratic in the momenta $\pi_i^{(\theta)}$ in the weak ADM energy and in the super-Hamiltonian constraint can be written as $\frac{c^3}{16\pi G}\,\tilde{\phi}\,\sum_{ab, a\neq b}\,\sigma^2_{(a)(b)}$, while the super-momentum constraints can be written in the form of PDE for the non-diagonal elements of the shear

$$\begin{aligned}\bar{\mathcal{H}}_{(a)}|_{\theta^i=0}(\tau,\sigma^u) = &-\frac{c^3}{8\pi\,G}\,\tilde{\phi}^{2/3}(\tau,\sigma^u)\,\Big(\sum_{b\neq a}\,Q_b^{-1}\,\Big[\partial_b\,\sigma_{(a)(b)}\\ &+\Big(\tilde{\phi}^{-1}\,\partial_b\,\tilde{\phi}+\sum_{\bar{b}}\,(\gamma_{\bar{b}a}-\gamma_{\bar{b}b})\,\partial_b\,R_{\bar{b}}\Big)\,\sigma_{(a)(b)}\Big]\\ &-\frac{8\pi\,G}{c^3}\,\tilde{\phi}^{-1}\,Q_a^{-1}\,\Big[\tilde{\phi}\,\partial_a\,\pi_{\tilde{\phi}}+\sum_{\bar{b}}\,(\gamma_{\bar{b}a}\,\partial_a\,\Pi_{\bar{b}}-\partial_a\,R_{\bar{b}}\,\Pi_{\bar{b}})\\ &+\mathcal{M}_a\Big]\Big)(\tau,\sigma^u)\approx 0.\end{aligned}\tag{8.47}$$

As a consequence, by using ${}^3g_{rs}$ of Eq. (8.38) and ${}^3K_{rs}$ of Eq. (8.46), the first-order non-Hamiltonian ADM equations (8.33) can be re-expressed in terms of the configurational variables n, $\bar{n}_{(a)}$, $\tilde{\phi}$, θ^i, $R_{\bar{a}}$, and of the expansion θ and shear $\sigma_{(a)(b)}$ of the Eulerian observers. Then the 12 equations can be put in the form of equations determining $\partial_\tau\,\tilde{\phi}$, $\partial_\tau\,R_{\bar{a}}$, $\partial_\tau\,\theta^i$, $\partial_\tau\,\theta$ and $\partial_\tau\,\sigma_{(a)(b)}$. In Eq. (2.17) of the first paper in Ref. [88–90] this manipulation is explicitly done for the first six equations (8.33).

These results are important for extending the identification of the inertial and tidal variables of the gravitational field, achieved with the York canonical basis, to cosmological space-times. Since these space-times are only conformally asymptotically flat, the Hamiltonian formalism is not defined. However, they are globally hyperbolic and admit $3+1$ splittings with the associated congruence of Eulerian observers. As a consequence, in them Einstein's equations are usually replaced with the non-Hamiltonian first-order ADM equations plus the super-Hamiltonian and super-momentum constraints. Our analysis implies that, since the 4-metric can always be put in the form of Eq. (8.38), the inertial gauge variables of the cosmological space-times are n, $\bar{n}_{(a)}$, θ^i and the expansion $\theta = -\epsilon\,{}^3K$, while the physical tidal variables are $R_{\bar{a}}$ and the diagonal components of the shear $\sigma_{(a)(a)}$ ($\sum_a\,\sigma_{(a)(a)}=0$). The unknown in the super-Hamiltonian constraint is the conformal factor $\tilde{\phi}$ of the 3-metric in Σ_τ, while the unknowns in the super-Hamiltonian constraints are the non-diagonal components of the shear $\sigma_{(a)(b)}|_{a \neq b}$.

8.7 Post-Minkowskian Linearization in Non-harmonic 3-Orthogonal Gauges and Post-Minkowskian Gravitational Waves

In the second paper of Ref. [88–90] it was shown that in the family of non-harmonic 3-orthogonal Schwinger gauges it is possible to define a consistent *linearization* of ADM canonical tetrad gravity plus matter (N charged scalar particles of masses m_i, Grassmann-valued signs of energy η_i, Grassmann-valued electric charges Q_i, plus the electro-magnetic field in the radiation gauge) in the weak field approximation, to obtain a formulation of *Hamiltonian Post-Minkowskian (HPM) gravity with non-flat Riemannian 3-spaces and asymptotic Minkowski background*.

In the standard linearization one introduces a *fixed Minkowski background space-time*, introduces the decomposition ${}^4g_{\mu\nu}(x) = {}^4\eta_{\mu\nu} + {}^4h_{\mu\nu}(x)$ in an inertial frame and studies the linearized equations of motion for the small Minkowskian fields ${}^4h_{\mu\nu}(x)$. The approximation is assumed valid over a *big enough characteristic length L interpretable as the reduced wavelength $\lambda/2\pi$ of the resulting gravitational waves (GW)* (only for distances higher of L the linearization breaks down and curved space-time effects become relevant). For the Solar System there is a PN approximation in harmonic gauges, which is adopted in the BCRS [11–13] and whose 3-spaces $t_B = const.$ have deviations of order c^{-2} from Euclidean 3-spaces.

See Refs. [95, 141] and Appendix A of the second paper in Refs. [88–90] for a review of all the results of the standard approach and of the existing points of view on the subject [142–157].

In the class of asymptotically Minkowskian space-times without super-translations the 4-metric tends to an asymptotic Minkowski metric at spatial infinity, ${}^4g_{AB} \rightarrow {}^4\eta_{AB}$, which can be used as an *asymptotic background*. The decomposition ${}^4g_{AB} = {}^4\eta_{AB} + {}^4h_{(1)AB}$, with a first order perturbation ${}^4h_{(1)AB}$ vanishing at spatial infinity, is defined in a global non-inertial rest frame of an asymptotically Minkowskian space-time deviating for first order effects from a global inertial rest frame of an abstract Minkowski space-time $M_{(\infty)}$. The non-Euclidean 3-spaces Σ_τ will deviate by first order effects from the Euclidean 3-spaces $\Sigma_{\tau(\infty)}$ of the inertial rest frame of $M_{(\infty)}$ coinciding with the limit of Σ_τ at spatial infinity. When needed differential operators like the Laplacian in Σ_τ will be approximated with the flat Laplacian in $\Sigma_{\tau(\infty)}$.

If $\zeta << 1$ is a small a-dimensional parameter, a consistent Hamiltonian linearization implies the following restrictions on the variables of the York canonical basis in the family of 3-orthogonal gauges with ${}^3K(\tau, \boldsymbol{\sigma}) = F(\tau, \boldsymbol{\sigma}) =$ *numerical function* (in this Section one uses the notation $\boldsymbol{\sigma}$ for the curvilinear 3-coordinates σ^r)

$$R_{\bar{a}}(\tau,\boldsymbol{\sigma}) = R_{(1)\bar{a}}(\tau,\boldsymbol{\sigma}) = O(\zeta) << 1,$$
$$\Pi_{\bar{a}}(\tau,\boldsymbol{\sigma}) = \Pi_{(1)\bar{a}}(\tau,\boldsymbol{\sigma}) = \frac{1}{L\,G}\,O(\zeta),$$
$$\tilde{\phi}(\tau,\boldsymbol{\sigma}) = \sqrt{det\,{}^3g_{rs}(\tau,\boldsymbol{\sigma})} = 1 + 6\,\phi_{(1)}(\tau,\boldsymbol{\sigma}) + O(\zeta^2),$$
$$N(\tau,\boldsymbol{\sigma}) = 1 + n(\tau,\boldsymbol{\sigma}) = 1 + n_{(1)}(\tau,\boldsymbol{\sigma}) + O(\zeta^2),$$
$$\epsilon\,{}^4g_{\tau\tau}(\tau,\boldsymbol{\sigma}) = 1 + \epsilon\,{}^4h_{(1)\tau\tau}(\tau,\boldsymbol{\sigma}) = 1 + 2\,n_{(1)}(\tau,\boldsymbol{\sigma}) + O(\zeta^2),$$
$$\bar{n}_{(a)}(\tau,\boldsymbol{\sigma}) = -\epsilon\,{}^4g_{\tau a}(\tau,\boldsymbol{\sigma}) = -\epsilon\,{}^4h_{(1)\tau r}(\tau,\boldsymbol{\sigma}) = \bar{n}_{(1)(a)}(\tau,\boldsymbol{\sigma}) + O(\zeta^2),$$

$$\begin{aligned}
{}^3K(\tau,\boldsymbol{\sigma}) &= \frac{12\pi\,G}{c^3}\,\pi_{\tilde{\phi}}(\tau,\boldsymbol{\sigma}) = {}^3K_{(1)}(\tau,\boldsymbol{\sigma}) = \frac{12\pi\,G}{c^3}\,\pi_{(1)\tilde{\phi}}(\tau,\boldsymbol{\sigma})\\
&= \frac{1}{L}\,O(\zeta),\, \sigma_{(a)(b)}|_{a\neq b}(\tau,\boldsymbol{\sigma}) = \sigma_{(1)(a)(b)}|_{a\neq b}(\tau,\boldsymbol{\sigma})\\
&= \frac{1}{L}\,O(\zeta),\, {}^3g_{rs}(\tau,\boldsymbol{\sigma}) = -\epsilon\,{}^4g_{rs}(\tau,\boldsymbol{\sigma}) = \delta_{rs} - \epsilon\,{}^4h_{(1)rs}(\tau,\boldsymbol{\sigma})\\
&= [1 + 2\,(\Gamma_r^{(1)}(\tau,\boldsymbol{\sigma}) + 2\,\phi_{(1)}(\tau,\boldsymbol{\sigma}))]\,\delta_{rs} + O(\zeta^2),\, \Gamma_a^{(1)}(\tau,\boldsymbol{\sigma})\\
&= \sum_{\bar{a}r=1}^{2} \gamma_{\bar{a}a}\,R_{\bar{a}}(\tau,\boldsymbol{\sigma}), \qquad R_{\bar{a}}(\tau,\boldsymbol{\sigma}) = \sum_{a=1}^{3} \gamma_{\bar{a}a}\,\Gamma_a^{(1)}(\tau,\boldsymbol{\sigma}).
\end{aligned} \tag{8.48}$$

The tidal variables $R_{\bar{a}}(\tau,\boldsymbol{\sigma})$ are slowly varying over the length L and times L/c; one has $(\frac{L}{{}^4\mathcal{R}})^2 = O(\zeta)$, where ${}^4\mathcal{R}$ is the mean radius of curvature of space-time.

The consistency of the Hamiltonian linearization requires the introduction of a *ultra-violet cutoff M for matter*. For the particles, described by the canonical variables $\boldsymbol{\eta}_i(\tau)$ and $\boldsymbol{\kappa}_i(\tau)$, this implies the conditions $\frac{m_i}{M}$, $\frac{\kappa_i}{M} = O(\zeta)$. With similar restrictions on the electro-magnetic field one gets that the energy-momentum tensor of matter is $T^{AB}(\tau,\boldsymbol{\sigma}) = T^{AB}_{(1)}(\tau,\boldsymbol{\sigma}) + O(\zeta^2)$. Therefore also the mass and momentum densities have the behavior $\mathcal{M}(\tau,\boldsymbol{\sigma}) = \mathcal{M}_{(1)}(\tau,\boldsymbol{\sigma}) + O(\zeta^2)$, $\mathcal{M}_r(\tau,\boldsymbol{\sigma}) = \mathcal{M}_{(1)r}(\tau,\boldsymbol{\sigma}) + O(\zeta^2)$. This approximation is not reliable at distances from the point particles less than the gravitational radius $R_M = \frac{M\,G}{c^2} \approx 10^{-29}\,M$ determined by the cutoff mass. *The weak ADM Poincaré generators become equal to the Poincaré generators of this matter in the inertial rest frame of the Minkowski space-time $M_{(\infty)}$ plus terms of order $O(\zeta^2)$ containing GW and matter*. Finally the GW described by this linearization must have wavelengths satisfying $\lambda/2\pi \approx L >> R_M$. If all the particles are contained in a compact set of radius l_c (the source), one must have $l_c >> R_M$ for particles with relativistic velocities and $l_c \geq R_M$ for slow particles (like in binaries). See Ref. [95] for more details.

With this Hamiltonian linearization one can avoid to make PN expansions: one gets fully relativistic expressions, i.e. a HPM formulation of gravity.

The effective Hamiltonian adapted to the 3-orthogonal gauges and replacing the weak ADM energy is $\frac{1}{c}\left(\hat{E}_{ADM(1)} + \hat{E}_{ADM(2)}\right) + \frac{c^3}{12\pi G}\int d^3\epsilon\left(\partial_\tau\,{}^3K_{(1)}\Big[1 + \frac{6}{\triangle}\left(\frac{1}{4}\sum_a \partial_a^2\,\Gamma_a^{(1)} - \frac{2\pi G}{c^3}\,\mathcal{M}_{(1)}\right)\Big]\right)(\tau,\sigma^u) + O(\zeta^3)$ in the PM linearized theory.

In the second paper of Refs. [88–90] one has found the solutions of the super-momentum and super-Hamiltonian constraints and of the equations for the lapse and shift functions with the Bianchi identities satisfied. Therefore one knows the first order quantities $\pi^{(\theta)}_{(1)i}(\tau, \boldsymbol{\sigma})$, $\tilde{\phi}(\tau, \boldsymbol{\sigma}) = 1 + 6\, \phi_{(1)}(\tau, \boldsymbol{\sigma})$, $1 + n_{(1)}(\tau, \boldsymbol{\sigma})$, $\bar{n}_{(1)(a)}(\tau, \boldsymbol{\sigma})$ (the quantities containing the *action-at-a-distance part* of the gravitational interaction in the 3-orthogonal gauges) with an explicit expression for the *PM Newton and gravito-magnetic potentials*. In absence of the electro-magnetic field they are (the terms in $\Gamma^{(1)}_a(\tau, \boldsymbol{\sigma})$ describe the contribution of GW)[32]

$$
\begin{aligned}
\tilde{\phi}(\tau, \boldsymbol{\sigma}) &= 1 + 6\, \phi_{(1)}(\tau, \boldsymbol{\sigma}) \\
&= 1 + \frac{3\, G}{c^3} \sum_i \eta_i \, \frac{\sqrt{m_i^2\, c^2 + \boldsymbol{\kappa}_i^2(\tau)}}{|\boldsymbol{\sigma} - \boldsymbol{\eta}_i(\tau)|} \\
&\quad - \frac{3}{8\pi} \int d^3\sigma_1 \, \frac{\sum_a \partial^2_{1a}\, \Gamma^{(1)}_a(\tau, \boldsymbol{\sigma}_1)}{|\boldsymbol{\sigma} - \boldsymbol{\sigma}_1|}, \\
\epsilon^4 g_{\tau\tau}(\tau, \boldsymbol{\sigma}) &= 1 + 2\, n_{(1)}(\tau, \boldsymbol{\sigma}) = 1 - 2\, \partial_\tau\, {}^3\mathcal{K}_{(1)}(\tau, \boldsymbol{\sigma}) \\
&\quad - \frac{2\, G}{c^3} \sum_i \eta_i \, \frac{\sqrt{m_i^2\, c^2 + \boldsymbol{\kappa}_i^2(\tau)}}{|\boldsymbol{\sigma} - \boldsymbol{\eta}_i(\tau)|} \left(1 + \frac{\boldsymbol{\kappa}_i^2}{m_i^2\, c^2 + \boldsymbol{\kappa}_i^2} \right), \\
-\, \epsilon^4 g_{\tau a}(\tau, \boldsymbol{\sigma}) &= \bar{n}_{(1)(a)}(\tau, \boldsymbol{\sigma}) = \partial_a\, {}^3\mathcal{K}_{(1)}(\tau, \boldsymbol{\sigma}) \\
&\quad - \frac{G}{c^3} \sum_i \frac{\eta_i}{|\boldsymbol{\sigma} - \boldsymbol{\eta}_i(\tau)|} \Big(\frac{7}{2} \kappa_{ia}(\tau) \\
&\quad - \frac{1}{2} \frac{(\sigma^a - \eta_i^a(\tau))\, \boldsymbol{\kappa}_i(\tau) \cdot (\boldsymbol{\sigma} - \boldsymbol{\eta}_i(\tau))}{|\boldsymbol{\sigma} - \boldsymbol{\eta}_i(\tau)|^2} \Big) \\
&\quad - \int \frac{d^3\sigma_1}{4\pi\, |\boldsymbol{\sigma} - \boldsymbol{\sigma}_1|}\, \partial_{1a}\, \partial_\tau \Big[2\, \Gamma^{(1)}_a(\tau, \boldsymbol{\sigma}_1) \\
&\quad - \int d^3\sigma_2 \, \frac{\sum_c \partial^2_{2c}\, \Gamma^{(1)}_c(\tau, \boldsymbol{\sigma}_2)}{8\pi\, |\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2|} \Big], \\
\sigma_{(1)(a)(b)}|_{a \neq b}(\tau, \boldsymbol{\sigma}) &= \frac{1}{2} \Big(\partial_a\, \bar{n}_{(1)(b)} + \partial_b\, \bar{n}_{(1)(a)} \Big)|_{a \neq b}(\tau, \boldsymbol{\sigma}).
\end{aligned}
\tag{8.49}
$$

Instead the linearization of the Hamilton equations for the tidal variables $R_{\bar{a}}(\tau, \boldsymbol{\sigma})$ implies that they satisfy the following wave equation[33] ($\triangle$ and $\Box$ are the flat Laplacian and the flat D'Alambertian on $\Sigma_{\tau(\infty)}$)

---

[32] Quantities like $|\boldsymbol{\eta}_i(\tau) - \boldsymbol{\eta}_j(\tau)|$ are the Euclidean 3-distance between the two particles in the asymptotic 3-space $\Sigma_{\tau(\infty)}$, which differs by quantities of order $O(\zeta)$ from the real non-Euclidean 3-distance in Σ_τ as shown in Eq. (3.3) of the third paper in Ref. [88–90].

[33] For the tidal momenta one gets $\frac{8\pi G}{c^3}\, \Pi_{\bar{a}}(\tau, \boldsymbol{\sigma}) = [\partial_\tau R_{\bar{a}} - \sum_a \gamma_{\bar{a}a}\, \partial_a\, \bar{n}_{(1)(a)}](\tau, \boldsymbol{\sigma}) + O(\zeta^2)$, so that the diagonal elements of the shear are $\sigma_{(1)(a)(a)}(\tau, \boldsymbol{\sigma}) = [-\sum_{\bar{a}} \gamma_{\bar{a}a}\, \partial_\tau R_{\bar{a}} + \bar{n}_{(1)(a)} - \frac{1}{3} \sum_b \bar{n}_{(1)(b)}](\tau, \boldsymbol{\sigma}) + O(\zeta^2)$.

$$\partial_\tau^2 R_{\bar a}(\tau,\boldsymbol{\sigma}) = \triangle R_{\bar a}(\tau,\boldsymbol{\sigma}) + \sum_a \gamma_{\bar a a}\Big[\partial_\tau\,\partial_a\,\bar n_{(1)(a)} + \partial_a^2\, n_{(1)} + 2\,\partial_a^2\,\phi_{(1)} - 2\,\partial_a^2\,\Gamma_a^{(1)} + \frac{8\pi\,G}{c^3}\,T^{aa}_{(1)}\Big](\tau,\boldsymbol{\sigma}). \tag{8.50}$$

By using Eq. (8.49) this wave equation becomes

$$\begin{aligned}
&\Box \sum_{\bar b} M_{\bar a\bar b}\,R_{\bar b}(\tau,\boldsymbol{\sigma}) = E_{\bar a}(\tau,\boldsymbol{\sigma}),\\
&M_{\bar a\bar b} = \delta_{\bar a\bar b} - \sum_a \gamma_{\bar a a}\,\frac{\partial_a^2}{\triangle}\Big(2\,\gamma_{\bar b a} - \frac{1}{2}\sum_b \gamma_{\bar b b}\,\frac{\partial_b^2}{\triangle}\Big),\\
&E_{\bar a}(\tau,\boldsymbol{\sigma}) = \frac{4\pi\,G}{c^3}\sum_a \gamma_{\bar a a}\Big[\frac{\partial_\tau\,\partial_a}{\triangle}\Big(4\,\mathcal{M}_{(1)a} - \frac{\partial_a}{\triangle}\sum_c \partial_c\,\mathcal{M}_{(1)c}\Big) + 2\,T^{aa}_{(1)} + \frac{\partial_a^2}{\triangle}\sum_b T^{bb}_{(1)}\Big](\tau,\boldsymbol{\sigma}),\\
&\Downarrow\\
&\Box \sum_b \tilde M_{ab}\,\Gamma_b^{(1)}(\tau,\boldsymbol{\sigma}) = \sum_{\bar a}\gamma_{\bar a a}\,E_{\bar a}(\tau,\boldsymbol{\sigma}),\\
&\tilde M_{ab} = \sum_{\bar a\bar b}\gamma_{\bar a a}\,\gamma_{\bar b b}\,M_{\bar a\bar b} = \delta_{ab}\Big(1 - 2\,\frac{\partial_a^2}{\triangle}\Big) + \frac{1}{2}\Big(1 + \frac{\partial_a^2}{\triangle}\Big)\frac{\partial_b^2}{\triangle},\\
&\sum_a \tilde M_{ab} = 0, \qquad M_{\bar a\bar b} = \sum_{ab}\gamma_{\bar a a}\,\gamma_{\bar b b}\,\tilde M_{ab}.
\end{aligned} \tag{8.51}$$

To understand the meaning of the spatial operators $M_{\bar a\bar b}$ and $\tilde M_{ab}$, one must consider the perturbation ${}^4h_{(1)rs}(\tau,\boldsymbol{\sigma}) = -2\,\epsilon\,\delta_{rs}\,(\Gamma_r^{(1)} + 2\,\phi_{(1)})(\tau,\boldsymbol{\sigma})$ of Eq. (8.48) and apply to it the following decomposition, given in Ref. [96],

$$\begin{aligned}
{}^4h_{(1)rs}(\tau,\boldsymbol{\sigma}) = \Big({}^4h^{TT}_{(1)rs} + \frac{1}{3}\,\delta_{rs}\,H_{(1)} + \frac{1}{2}\,(\partial_r\,\epsilon_{(1)s} + \partial_s\,\epsilon_{(1)r})\\
+ (\partial_r\,\partial_s - \frac{1}{3}\,\delta_{rs}\,\triangle)\,\lambda_{(1)}\Big)(\tau,\boldsymbol{\sigma}),
\end{aligned} \tag{8.52}$$

with $\sum_r \partial_r\,\epsilon_{(1)r} = 0$ and ${}^4h^{TT}_{(1)rs}$ traceless and transverse (TT), i.e. $\sum_r {}^4h^{TT}_{(1)rr} = 0$, $\sum_r \partial_r\,{}^4h^{TT}_{(1)rs} = 0$. Since one finds $H_{(1)}(\tau,\boldsymbol{\sigma}) = -12\,\epsilon\,\phi_{(1)}(\tau,\boldsymbol{\sigma})$, $\lambda_{(1)}(\tau,\boldsymbol{\sigma}) = -3\,\epsilon\,\sum_u \frac{\partial_u^2}{\triangle^2}\,\Gamma_u^{(1)}(\tau,\boldsymbol{\sigma})$ and $\epsilon_{(1)r}(\tau,\boldsymbol{\sigma}) = -4\,\epsilon\,\frac{\partial_r}{\triangle}\Big(\Gamma_r^{(1)} - \sum_u \frac{\partial_u^2}{\triangle}\,\Gamma_u^{(1)}\Big)(\tau,\boldsymbol{\sigma})$, it turns out that the TT part of the spatial metric is independent from $\phi_{(1)}$ and has the expression

$$
\begin{aligned}
{}^4h^{TT}_{(1)rs}(\tau,\boldsymbol{\sigma}) =& -\epsilon\Big[\Big(2\,\Gamma^{(1)}_r + \sum_u \frac{\partial_u^2}{\triangle}\,\Gamma^{(1)}_u\Big)\,\delta_{rs} - \\
&- 2\,\frac{\partial_r\,\partial_s}{\triangle}\,(\Gamma^{(1)}_r + \Gamma^{(1)}_s) + \frac{\partial_r\,\partial_s}{\triangle}\sum_u \frac{\partial_u^2}{\triangle}\,\Gamma^{(1)}_u\Big](\tau,\boldsymbol{\sigma}), \\
&\Rightarrow {}^4h^{TT}_{(1)aa}(\tau,\boldsymbol{\sigma}) = -2\,\epsilon\sum_b \tilde{M}_{ab}\,\Gamma^{(1)}_b(\tau,\boldsymbol{\sigma}).
\end{aligned} \tag{8.53}
$$

Therefore the spatial operator $\tilde{M}_{ab}$ connects the tidal variables $R_{\bar{a}}(\tau,\boldsymbol{\sigma})$ of the York canonical basis to the TT components of the 3-metric. By applying the decomposition (8.52) to the spatial part $T^{rs}_{(1)}(\tau,\boldsymbol{\sigma})$ of the energy-momentum one verifies that like in the harmonic gauges [95] the TT part of the 3-metric satisfies the wave equation $\Box\,{}^4h^{TT}_{rs}(\tau,\boldsymbol{\sigma}) = -\epsilon\,\frac{16\pi G}{c^3}\,T^{(TT)}_{(1)rs}(\tau,\boldsymbol{\sigma})$.

The retarded solution of the wave equation with a no-incoming radiation condition gives the following expression for the tidal variables (the HPM-GW)

$$
\begin{aligned}
R_{\bar{a}}(\tau,\boldsymbol{\sigma}) =& -\sum_a \gamma_{\bar{a}a}\,\Gamma^{(1)}_a(\tau,\boldsymbol{\sigma}) \stackrel{\circ}{=} \sum_{ab}\gamma_{\bar{a}a}\,\tilde{M}^{-1}_{ab}(\tau,\boldsymbol{\sigma}) \\
&\frac{2\,G}{c^3}\int d^3\sigma_1\,\frac{T^{(TT)bb}_{(1)}(\tau - |\boldsymbol{\sigma}-\boldsymbol{\sigma}_1|;\boldsymbol{\sigma}_1)}{|\boldsymbol{\sigma}-\boldsymbol{\sigma}_1|}, \\
\frac{8\pi\,G}{c^3}\,\Pi_{\bar{a}}(\tau,\boldsymbol{\sigma}) =& \Big(\sum_{\bar{b}} M_{\bar{a}\bar{b}}\,\partial_\tau\,R_{\bar{b}} - \sum_a \gamma_{\bar{a}a}\Big[\frac{4\pi\,G}{c^3}\,\frac{1}{\triangle}\,(4\,\partial_a\,\mathcal{M}_{(1)a} \\
&- \frac{\partial_a^2}{\triangle}\sum_c \partial_c\,\mathcal{M}_{(1)c}) + \partial_a^2\,{}^3\mathcal{K}_{(1)}\Big]\Big)(\tau,\boldsymbol{\sigma}).
\end{aligned} \tag{8.54}
$$

The explicit form of the inverse operator is given in the second paper of Ref. [88–90]. By using the multipolar expansion of the energy-momentum $T^{AB}_{(1)}$ of Ref. [116–118] in the HPM version adapted to the rest-frame instant form of dynamics of Ref. [59–61], one gets

$$
R_{\bar{a}}(\tau,\boldsymbol{\sigma}) = -\frac{G}{c^3}\sum_{ab}\gamma_{\bar{a}a}\,\tilde{M}^{-1}_{ab}\,\frac{\partial_\tau^2\,q^{(TT)aa|\tau\tau}(\tau - |\boldsymbol{\sigma}|)}{|\boldsymbol{\sigma}|} + (\textit{higher multipoles}), \tag{8.55}
$$

where $q^{(TT)aa|\tau\tau}(\tau)$ is the *TT mass quadrupole* with respect to the center of energy (put in the origin of the radar 4-coordinates). An analogous result holds for ${}^4h^{TT}_{rs}(\tau,\boldsymbol{\sigma})$ and this implies a *HPM relativistic version of the standard mass quadrupole emission formula*.

Moreover, notwithstanding there is no gravitational self-energy due to the Grassmann regularization, the energy, 3-momentum and angular momentum balance equations in HPM-GW emission are verified by *using the conservation of the asymptotic ADM Poincaré generators* (the same happens with the asymptotic Larmor formula of the electro-magnetic case with Grassmann regularization as shown in the last paper of Ref. [62–64]). See Refs. [95, 155–163] for the use of the self-energy in the standard derivation of this result by means of PN expansions.

Equations (8.49) and (8.54) show that the HPM linearization with no-incoming radiation condition and Grassmann regularization is a theory with only dynamical matter interacting through suitable action-at-a-distance and retarded effective potentials. Instead in relativistic atomic physics in SR the no-incoming radiation condition and the Grassmann regularization kill also the retardation leaving only the action-at-a-distance inter-particle Coulomb plus Darwin potentials. See Eq. (7.22) of the second paper of Ref. [88–90] for the expression of the weak ADM energy till order $O(\zeta^3)$.

Moreover it can be shown that the coordinate transformation $\bar{\tau} = \tau$, $\bar{\sigma}^r = \sigma^r + \frac{1}{2}\frac{\partial_r}{\triangle}\Big(4\,\Gamma_r^{(1)} - \sum_c \frac{\partial_c^2}{\triangle}\,\Gamma_c^{(1)}\Big)(\tau,\boldsymbol{\sigma})$, introducing new τ-dependent radar 3-coordinates on the 3-space Σ_τ, allows one to make a transition from the 3-orthogonal gauge with the 4-metric given by Eqs. (8.48) and (8.49) to a *generalized non-3-orthogonal TT gauge* containing the TT 3-metric (8.53)

$$
\begin{aligned}
&{}^4g_{(1)AB} = {}^4\eta_{AB}\\
&+\epsilon\begin{pmatrix} -2\,\frac{\partial_\tau}{\Delta}\,{}^3K_{(1)} + \alpha(matter) & -\frac{\partial_r}{\Delta}\,{}^3K_{(1)} + A_r(\Gamma_a^{(1)}) + \beta_r(matter)\\ -\frac{\partial_s}{\Delta}\,{}^3K_{(1)} + A_s(\Gamma_a^{(1)}) + \beta_s(matter) & \Big[B_r(\Gamma_a^{(1)}) + \gamma(matter)\Big]\,\delta_{rs} \end{pmatrix}\\
&+O(\zeta^2),\\
&\Downarrow\\
&{}^4\bar{g}_{AB} = {}^4\eta_{AB} + \epsilon\begin{pmatrix} -2\,\frac{\partial_\tau}{\Delta}\,{}^3K_{(1)} + \alpha(matter) & -\frac{\partial_r}{\Delta}\,{}^3K_{(1)} + \beta_r(matter)\\ -\frac{\partial_s}{\Delta}\,{}^3K_{(1)} + \beta_s(matter) & \epsilon\,{}^4h^{TT}_{(1)rs} + \delta_{rs}\,\gamma(matter) \end{pmatrix}\\
&\qquad + O(\zeta^2).
\end{aligned}
\tag{8.56}
$$

The functions appearing in Eq. (8.56) are: $A_r(\Gamma_a^{(1)}) = -\frac{1}{2}\,\partial_\tau\,\frac{\partial_r}{\triangle}\Big(4\,\Gamma_r^{(1)} - \sum_c \frac{\partial_c^2}{\triangle}\,\Gamma_c^{(1)}\Big)$, $B_r(\Gamma_a^{(1)}) = -2\Big(\Gamma_r^{(1)} + \frac{1}{2}\sum_c \frac{\partial_c^2}{\triangle}\,\Gamma_c^{(1)}\Big)$, $\alpha(matter) = \frac{8\pi G}{c^3}\,\frac{1}{\triangle}\Big(\mathcal{M}_{(1)} + \sum_c T^{cc}_{(1)}\Big)$, $\beta_r(matter) = -\frac{4\pi G}{c^3}\,\frac{1}{\triangle}\Big(4\,\mathcal{M}_{(1)r} - \frac{\partial_r}{\triangle}\sum_c \partial_c\,\mathcal{M}_{(1)c}\Big)$, $\gamma(matter) = \frac{8\pi G}{c^3}\,\frac{1}{\triangle}\,\mathcal{M}_{(1)}$.

Also in absence of matter this TT gauge differs from the usual harmonic ones for the non-spatial terms depending upon the inertial gauge variable *non-local York time*

$$
{}^3\mathcal{K}_{(1)}(\tau,\boldsymbol{\sigma}) = \frac{1}{\triangle}\,{}^3K_{(1)}(\tau,\boldsymbol{\sigma}),
\tag{8.57}
$$

describing the HPM form of the gauge freedom in clock synchronization.

If one uses the coordinate system of the generalized TT gauge, one can introduce the standard polarization pattern of GW for ${}^4h^{TT}_{rs}$ (see Refs. [95, 96, 164]) and then the inverse transformation gives the polarization pattern of HPM-GW in the family of 3-orthogonal gauges.

If the matter sources have a compact support and if the matter terms $\frac{1}{\triangle}\, \mathcal{M}_{(1)}(\tau, \boldsymbol{\sigma})$ and $\frac{1}{\triangle}\, \mathcal{M}_{(1)r}(\tau, \boldsymbol{\sigma})$ are negligible in the radiation zone far away from the sources, then Eq. (8.56) gives a *spatial TT-gauge* with still the explicit dependence on the inertial gauge variable ${}^3\mathcal{K}_{(1)}(\tau, \boldsymbol{\sigma})$ (non existing in Newtonian gravity), which determines the non-Euclidean nature of the instantaneous 3-spaces. Then one can study the *far field* of compact matter sources: the restriction to the Solar System of the resulting HPM 4-metric[34] is compatible with the harmonic PN 4-metric of BCRS [11–13] if the non-local York time is of order c^{-2}. The resulting shift function should be used for the HPM description of gravito-magnetism (see Refs. [84, 165–171] for the Lense-Thirring and other associated effects).

The TT gauge allows one to reproduce the various descriptions of the GW detectors and of the reference frames used in GW detection in terms of HPM-GW: this is done in Subsection VIID of the second paper of Ref. [88–90], where the effect of a HPM-GW on a test mass is given in terms of the proper distance between two nearby geodesics.

The HPM-GW propagate in non-Euclidean instantaneous 3-spaces Σ_τ differing from the inertial asymptotic Euclidean 3-spaces $\Sigma_{\tau(\infty)}$ at the first order. In the family of 3-orthogonal gauges with York time ${}^3K_{(1)}(\tau, \boldsymbol{\sigma}) \approx F_{(1)}(\tau, \boldsymbol{\sigma}) =$ *numerical function*, the dynamically determined 3-spaces Σ_τ have an intrinsic 3-curvature ${}^3\hat{R}|_{\theta^i=0} = 2\,\sum_a \partial_a^2\, \Gamma_a^{(1)}$ determined only by the HPM-GW (and therefore by the matter energy-momentum tensor in the past as shown by Eq. (8.54)). Their extrinsic curvature tensor as sub-manifolds of space-time is

$$
{}^3K_{(1)rs} \;\approx\; \sigma_{(1)(r)(s)}|_{r\neq s} + \delta_{rs}\, \Big(\frac{1}{3}\, F_{(1)} - \partial_\tau\, \Gamma_r^{(1)} + \partial_r\, \bar{n}_{(1)(r)} - \sum_a\, \partial_a\, \bar{n}_{(1)(a)}\Big), \quad (8.58)
$$

with $\bar{n}_{(1)(r)}$, $\sigma_{(1)(r)(s)}|_{r\neq s}$ and $\Gamma_r^{(1)}$ given in Eqs. (8.49) and (8.54). The York time appears only in Eq. (8.58): all the other PM quantities depend on the non-local York time ${}^3\mathcal{K}_{(1)}(\tau, \boldsymbol{\sigma}) \approx \frac{1}{\triangle}\, F_{(1)}(\tau, \boldsymbol{\sigma})$

In the third paper of Refs. [88–90], where the matter is restricted only to the particles,[35] one evaluates all the properties of these HPM space-times:

(a) the 3-volume element, the 3-distance and the intrinsic and extrinsic 3-curvature tensors of the 3-spaces Σ_τ;
(b) the proper time of a time-like observer;

[34] See Eq. (7.20) of the second paper in Ref. [88–90], where ${}^4g_{\tau\tau}(\tau, \boldsymbol{\sigma})$ and ${}^4g_{\tau r}(\tau, \boldsymbol{\sigma})$ are explicitly depending on the non-local York time.

[35] The properties of HPM transverse electro-magnetic fields have still to be explored.

(c) the time-like and null 4-geodesics (they are relevant for the definition of the radial velocity of stars as shown in the IAU conventions of Ref. [172] and in study of *time delays* [167–171, 173–175]);
(d) the red-shift and luminosity distance. In particular one finds that the recessional velocity of a star is proportional to its luminosity distance from the Earth at least for small distances. This is in accord with the Hubble old red-shift-distance relation which is formalized in the Hubble law (velocity-distance relation) when the standard cosmological model is used (see for instance Ref. [97] on these topics). These results have a dependence on the non-local York time, which could play a role in giving a different interpretation of the data from super-novae, which are used as a support for dark energy [1, 2].

Finally, in Subsection IIIB of the second paper in Refs. [88–90] it is shown that this HPM linearization can be interpreted as the first term of a HPM expansion in powers of the Newton constant G in the family of 3-orthogonal gauges. This expansion has still to be studied. In particular it will be useful to check whether in the HPM formulation there are phenomena (appearing at high orders in the standard PN expansions) like the hereditary tails starting from 1.5PN $[O((\frac{v}{c})^3)]$ and the non-linear (Christodoulou) memory starting from 3PN (see Ref. [176] for a review).[36] This would allow one to make a comparison with all the results of the PN expansions, in which today there is control on the GW solution and on the matter equations of motion till order 3.5PN $[O((\frac{v}{c})^7)]$ (for binaries see the review in chapter 4 of Ref. [95]) and well established connections with numerical relativity (see the review in Ref. [177]) especially for the binary black hole problem (see the review in Ref. [178]).

8.8 Post-Minkowskian Hamilton Equations for Particles, their Post-Newtonian Limit and Dark Matter as a Relativistic Inertial Effect

The PM Hamilton equations and their PN limit in 3-orthogonal gauges for a system of N scalar particles of mass m_i and Grassmann-valued signs of energy η_i is discussed in this Section by using the results of the third paper in Ref. [88–90]. See Refs. [141, 164, 179, 180] for classical texts on the motion of particles in gravitational fields and Refs. [146–149, 181–184] for more recent developments.[37]

[36] They imply that GW propagate not only on the flat light-cone but also inside it (i.e. with all possible speeds $0 \le v \le c$): there is an instantaneous wavefront followed by a tail traveling at lower speed (it arrives later and then fades away) and a persistent zero-frequency non-linear memory.

[37] In this approach point particles are considered as independent matter degrees of freedom with a Grassmann regularization of the self-energies to get well defined world-lines (see also Ref. [183, 184]): they are not considered as point-like singularities of solutions of Einstein's equations (the point of view of Ref. [179]). Solutions of this type have to be described with distributions and, as shown in Ref. [185], the most general class of such solutions under mathematical control includes singularities simulating matter shells, but not either strings or particles. See also Ref. [186].

The treatment in the 3-orthogonal gauges of the PM Hamilton equations for the electro-magnetic field in the radiation gauge is given in the second paper of Ref. [88–90], while the PM Hamilton equations for perfect fluids are given in Ref. [91].

With only particles the PM approximation with the ultraviolet cutoff M implies $\boldsymbol{\kappa}_i(\tau) = \frac{m_i c\, \dot{\boldsymbol{\eta}}_i(\tau)}{\sqrt{1-\dot{\boldsymbol{\eta}}_i^2(\tau)}} + Mc\, O(\zeta)$, $\mathcal{M}_{(1)}(\tau, \boldsymbol{\sigma}) = \sum_i \delta^3(\boldsymbol{\sigma}, \boldsymbol{\eta}_i(\tau))\, \eta_i\, \frac{m_i c}{\sqrt{1-\dot{\boldsymbol{\eta}}_i^2(\tau)}} + O(\zeta^2)$, $\mathcal{M}_{(1)r}(\tau, \boldsymbol{\sigma}) = \sum_i \delta^3(\boldsymbol{\sigma}, \boldsymbol{\eta}_i(\tau))\, \eta_i\, \frac{m_i c\, \dot{\boldsymbol{\eta}}_{ir}(\tau)}{\sqrt{1-\dot{\boldsymbol{\eta}}_i^2(\tau)}} + O(\zeta^2)$. Moreover one has $\ddot{\eta}_i(\tau) = O(\zeta)$. The notation $\dot{a}(\tau) = \frac{da(\tau)}{d\tau}$ is used.

One can make a equal time development of the retarded kernel in Eq. (8.54) like in Ref. [62–64] for the extraction of the Darwin potential from the Lienard-Wiechert solution (see Eqs. (5.1)–(5.21) of Ref. [62–64] with $\sum_s P_\perp^{rs}(\boldsymbol{\sigma})\, \dot{\eta}_i^s(\tau) \mapsto \sum_{uv} \mathcal{P}_{bbuv}(\boldsymbol{\sigma})\, \frac{\dot{\eta}_i^u(\tau)\, \dot{\eta}_i^v(\tau)}{\sqrt{1-\dot{\eta}_i^2(\tau)}}$). In this way one gets the following expression of the HPM GW from point masses

$$
\begin{aligned}
\Gamma_a^{(1)}(\tau, \boldsymbol{\sigma}) \overset{\circ}{=} & - \frac{2\,G}{c^2} \sum_b \tilde{M}_{ab}^{-1}(\boldsymbol{\sigma}) \sum_i \eta_i\, m_i \sum_{uv} \mathcal{P}_{bbuv}(\boldsymbol{\sigma})\, \frac{\dot{\eta}_i^u(\tau)\, \dot{\eta}_i^v(\tau)}{\sqrt{1-\dot{\eta}_i^2(\tau)}} \\
& \Big[|\boldsymbol{\sigma} - \boldsymbol{\eta}_i(\tau)|^{-1} + \sum_{m=1}^{\infty} \frac{1}{(2m)!} \Big(\dot{\boldsymbol{\eta}}_i(\tau) \cdot \frac{\partial}{\partial\, \boldsymbol{\sigma}} \Big)^{2m} |\boldsymbol{\sigma} - \boldsymbol{\eta}_i(\tau)|^{2m-1} \Big] \\
& + O(\zeta^2), \\
\mathcal{P}_{rsuv} = & \frac{1}{2}\, (\delta_{ru}\, \delta_{sv} + \delta_{rv}\, \delta_{su}) \\
& - \frac{1}{2} \Big(\delta_{rs} - \frac{\partial_r\, \partial_s}{\triangle} \Big) \delta_{uv} + \frac{1}{2} \Big(\delta_{rs} + \frac{\partial_r\, \partial_s}{\triangle} \Big) \frac{\partial_u\, \partial_v}{\triangle} \\
& - \frac{1}{2} \Big[\frac{\partial_u}{\triangle}\, (\delta_{rv}\, \partial_s + \delta_{sv}\, \partial_r) + \frac{\partial_v}{\triangle}\, (\delta_{ru}\, \partial_s + \delta_{su}\, \partial_r) \Big],
\end{aligned}
\tag{8.59}
$$

with the retardation effects pushed to order $O(\zeta^2)$.

If the lapse and shift functions are rewritten in the form $n_{(1)} = \check{n}_{(1)} - \partial_\tau\, {}^3\mathcal{K}_{(1)}$, $\bar{n}_{(1)(r)} = \check{\bar{n}}_{(1)(r)} + \partial_r\, {}^3\mathcal{K}_{(1)}$, to display their dependence on the inertial gauge variable non-local York time, it can be shown that the PM Hamilton equations for the particles imply the following form of the PM Grassmann regularized second-order equations of motion showing explicitly the equality of the inertial and gravitational masses of the particles

$$
\begin{aligned}
m_i\, \eta_i\, \ddot{\eta}_i^r(\tau) \overset{\circ}{=} & \eta_i \sqrt{1-\dot{\boldsymbol{\eta}}_i^2(\tau)} \Big(\mathcal{F}_i^r - \dot{\eta}_i^r(\tau)\, \dot{\boldsymbol{\eta}}_i(\tau) \cdot \boldsymbol{F}_i \Big) (\tau | \boldsymbol{\eta}_i(\tau) | \boldsymbol{\eta}_{k\neq i}(\tau)) \\
& \overset{def}{=} \eta_i\, F_i^r(\tau | \boldsymbol{\eta}_i(\tau) | \boldsymbol{\eta}_{k\neq i}(\tau)),
\end{aligned}
$$

$$
\begin{aligned}
&\eta_i\, \mathcal{F}_i^r(\tau|\boldsymbol{\eta}_i(\tau)|\boldsymbol{\eta}_{k\neq i}(\tau))\\
&\quad = \frac{m_i\,\eta_i}{\sqrt{1-\dot{\boldsymbol{\eta}}_i^2(\tau)}}\Big(-\frac{\partial\,\check{n}_{(1)}(\tau,\boldsymbol{\eta}_i(\tau))}{\partial\,\eta_i^r}\\
&\qquad + \frac{\dot{\eta}_i^r(\tau)}{1-\dot{\boldsymbol{\eta}}_i^2(\tau)}\sum_u \Big[\dot{\eta}_i^u(\tau)\,\frac{\partial\,\check{n}_{(1)}}{\partial\,\eta_i^u} + \sum_{j\neq i}\dot{\eta}_j^u(\tau)\,\frac{\partial\,\check{n}_{(1)}}{\partial\,\eta_j^u}\Big](\tau,\boldsymbol{\eta}_i(\tau))\\
&\qquad + \Big(\sum_u \dot{\eta}_i^u(\tau)\Big[\frac{\partial\,\check{\bar{n}}_{(1)(u)}}{\partial\,\eta_i^r} - \frac{\partial\,\check{\bar{n}}_{(1)(r)}}{\partial\,\eta_i^u}\Big] - \sum_{j\neq i}\sum_u \dot{\eta}_j^u(\tau)\,\frac{\partial\,\check{\bar{n}}_{(1)(r)}}{\partial\,\eta_j^u}\\
&\qquad - \frac{\dot{\eta}_i^r(\tau)}{1-\dot{\boldsymbol{\eta}}_i^2(\tau)}\sum_u \dot{\eta}_i^u(\tau)\sum_s\Big[\dot{\eta}_i^s(\tau)\,\frac{\partial\,\check{\bar{n}}_{(1)(u)}}{\partial\,\eta_i^s}\\
&\qquad + \sum_{j\neq i}\dot{\eta}_j^s(\tau)\,\frac{\partial\,\check{\bar{n}}_{(1)(u)}}{\partial\,\eta_j^s}\Big]\Big)(\tau,\boldsymbol{\eta}_i(\tau)) + \Big(\sum_u (\dot{\eta}_i^u(\tau))^2\,\frac{\partial\,(\Gamma_u^{(1)}+2\,\phi_{(1)})}{\partial\,\eta_i^r}\\
&\qquad - \dot{\eta}_i^r(\tau)\sum_u\Big[\dot{\eta}_i^u(\tau)\Big(2\,\frac{\partial\,(\Gamma_r^{(1)}+2\,\phi_{(1)})}{\partial\,\eta_i^u} + \sum_c \frac{(\dot{\eta}_i^c(\tau))^2}{1-\dot{\boldsymbol{\eta}}_i^2(\tau)}\,\frac{\partial\,(\Gamma_c^{(1)}+2\,\phi_{(1)})}{\partial\,\eta_i^u}\Big)\\
&\qquad + \sum_{j\neq i}\dot{\eta}_j^u(\tau)\Big(2\,\frac{\partial\,(\Gamma_r^{(1)}+2\,\phi_{(1)})}{\partial\,\eta_j^u}\\
&\qquad + \sum_c \frac{(\dot{\eta}_i^c(\tau))^2}{1-\dot{\boldsymbol{\eta}}_i^2(\tau)}\,\frac{\partial\,(\Gamma_c^{(1)}+2\,\phi_{(1)})}{\partial\,\eta_j^u}\Big)\Big]\Big)(\tau,\boldsymbol{\eta}_i(\tau))\\
&\qquad - \frac{\dot{\eta}_i^r(\tau)}{1-\dot{\boldsymbol{\eta}}_i^2(\tau)}\Big[\partial_\tau^2|_{\boldsymbol{\eta}_i}\,{}^3\mathcal{K}_{(1)} + 2\sum_s \dot{\eta}_i^s(\tau)\,\frac{\partial\,\partial_\tau|_{\boldsymbol{\eta}_i}\,{}^3\mathcal{K}_{(1)}}{\partial\,\eta_i^s}\\
&\qquad + \sum_{su}\dot{\eta}_i^s(\tau)\,\dot{\eta}_i^u(\tau)\,\frac{\partial^2\,{}^3\mathcal{K}_{(1)}}{\partial\,\eta_i^u\,\partial\,\eta_i^s}\Big](\tau,\boldsymbol{\eta}_i(\tau))\Big) + O(\zeta^2).
\end{aligned}
\tag{8.60}
$$

The effective action-at-a-distance force $\boldsymbol{F}_i(\tau)$ contains

(a) the contribution of the lapse function $\check{n}_{(1)}$, which generalizes the Newton force;
(b) the contribution of the shift functions $\check{\bar{n}}_{(1)(r)}$, which gives the gravito-magnetic effects;
(c) the retarded contribution of HPM GW, described by the functions $\Gamma_r^{(1)}$ of Eq. (8.59);
(d) the contribution of the volume element $\phi_{(1)}$ ($\tilde{\phi} = 1 + 6\,\phi_{(1)} + O(\zeta^2)$), always summed to the HPM GW, giving forces of Newton type;
(e) the contribution of the inertial gauge variable (the non-local York time) ${}^3\mathcal{K}_{(1)} = \frac{1}{\triangle}\,{}^3K_{(1)}$.

In the electro-magnetic case in SR [65, 66] the regularized coupled second-order equations of motion of the particles obtained by using the Lienard-Wiechert solutions for the electro-magnetic field are independent by the type of Green function (retarded or advanced or symmetric) used. The electro-magnetic retardation effects, killed by the Grassmann regularization, are connected with QED radiative corrections to the one-photon exchange diagram. This is not strictly true in the gravitational case. The effect of retardation is not killed by the Grasmann regularization but only pushed to $O(\zeta^2)$: at this order it should give extra contributions to the second-order equations of motion. This shows that our semi-classical approximation, obtained with our Grassmann regularization, of a unspecified "quantum gravity" theory does not take into account only a "one-graviton exchange diagram": in the spin 2 case there is an extra retardation effect showing up only at higher HPM orders.[38]

8.8.1 The Center-of-Mass Problem in General Relativity and in the HPM Linearization

As said in Sect. 5, the 3-universe is described in a non inertial rest frame with non-Euclidean 3-spaces Σ_τ tending to Euclidean inertial ones $\Sigma_{\tau(\infty)}$ at spatial infinity. Both matter and gravitational degrees of freedom live inside Σ_τ and their internal 3-center of mass is eliminated by the rest-frame condition $\hat{P}^r_{ADM} \approx 0$ (implied by the absence of super-translations) if also the condition $\hat{K}^r_{ADM} = \hat{J}^{\tau r}_{ADM} \approx 0$ is added like in SR. The 3-universe may be described as an external decoupled center of mass carrying a pole-dipole structure: $\hat{E}_{ADM}$ is the invariant mass and $\hat{J}^{rs}_{ADM}$ the rest spin. As in SR the condition $\hat{K}^r_{ADM} \approx 0$ selects the Fokker-Pryce center of inertia as the natural time-like observer origin of the radar coordinates: it follows a non-geodetic straight world-line like the asymptotic inertial observers existing in these space-times.

This is a way out from the the problem of the center of mass in general relativity and of its world-line, a still open problem in generic space-times as can be seen from Refs. [116–118, 187–193] (and Refs. [146–149] for the PN approach). Usually, by means of some supplementary condition, the center of mass is associated to the monopole of a multipolar expansion of the energy-momentum of a small body (see Refs. [59–61] for the special relativistic case).

In SR the elimination of the internal 3-center of mass leads to describe the dynamics inside Σ_τ only in terms of relative variables (see Eq. (8.15) in the case of particles). However relative variables *do not exist* in the non-Euclidean 3-spaces of curved space-times, where flat objects like $\boldsymbol{r}_{ij}(\tau) = \boldsymbol{\eta}_i(\tau) - \boldsymbol{\eta}_j(\tau)$ have to be replaced with a quantity proportional to the tangent vector to the space-like 3-geodesics joining the two particles in the non-Euclidean 3-space Σ_τ (see Ref. [194] for an implementation of this idea). Quantities like $\boldsymbol{r}^2_{ij}(\tau)$ have to be replaced with the Synge world

[38] In the electro-magnetic case the Grassmann regularization implies $Q_i\, \dot{\eta}^r_i(\tau - |\boldsymbol{\sigma}|) = Q_i\, \dot{\eta}_i(\tau)$ and equations of motion of the type $\ddot{\eta}^r_i(\tau) = Q_i\, \dots$ with $Q_i^2 = 0$. In the gravitational case the equations of motion are of the type $\eta_i\, \ddot{\eta}^r_i(\tau) = \eta_i\, \dots$ with $\eta_i^2 = 0$, but the Grassmann regularized retardation in Eq. (8.54) gives Eq. (8.59) only at the lowest order in ζ and has contributions of every order $O(\zeta^k)$.

function [173–175, 180, 183, 184].[39] This problem is another reason why extended objects tend to be replaced with point-like multipoles, which, however, do not span a canonical basis of phase space (see Refs. [59–61] for SR).

However, at the level of the HPM approximation one can introduce relative variables for the particles, like the SR ones of Eq. (8.15), defined as 3-vectors in the asymptotic inertial rest frame $\Sigma_{\tau(\infty)}$ by putting $\boldsymbol{\eta}_i(\tau) = \boldsymbol{\eta}_{(o)i}(\tau) + \boldsymbol{\eta}_{(1)i}(\tau)$ and $\boldsymbol{\kappa}_i(\tau) = \boldsymbol{\kappa}_{(o)i}(\tau) + \boldsymbol{\kappa}_{(1)i}(\tau)$ with $\boldsymbol{\eta}_{(o)i}(\tau)$, $\boldsymbol{\kappa}_{(o)i}(\tau) = O(\zeta)$. This allows one to define HPM collective and relative canonical variables for the particles, with the collective variables eliminated by the conditions $\hat{P}^r_{ADM} \approx 0$ and $\hat{K}^r_{ADM} \approx 0$ (at the lowest order they become the SR conditions).

In the case of two particles (with total and reduced masses $M = m_1 + m_2$ and $\mu = \frac{m_1\, m_2}{M}$) one puts $\boldsymbol{\eta}_1(\tau) = \boldsymbol{\eta}_{12}(\tau) + \frac{m_2}{M}\, \boldsymbol{\rho}_{12}(\tau)$, $\boldsymbol{\eta}_2(\tau) = \boldsymbol{\eta}_{12}(\tau) - \frac{m_1}{M}\, \boldsymbol{\rho}_{12}(\tau)$, $\boldsymbol{\kappa}_1(\tau) = \frac{m_1}{M}\, \boldsymbol{\kappa}_{12}(\tau) + \boldsymbol{\pi}_{12}(\tau)$, $\boldsymbol{\kappa}_2(\tau) = \frac{m_2}{M}\, \boldsymbol{\kappa}_{12}(\tau) - \boldsymbol{\pi}_{12}(\tau)$ and goes to the new canonical basis $\boldsymbol{\eta}_{12}(\tau) = \frac{m_1\, \boldsymbol{\eta}_1(\tau) + m_2\, \boldsymbol{\eta}_2}{M}$, $\boldsymbol{\rho}_{12}(\tau) = \boldsymbol{\eta}_1(\tau) - \boldsymbol{\eta}_2(\tau)$, $\boldsymbol{\kappa}_{12}(\tau) = \boldsymbol{\kappa}_1(\tau) + \boldsymbol{\kappa}_2(\tau)$, $\boldsymbol{\pi}_{12}(\tau) = \frac{m_2\, \boldsymbol{\kappa}_1(\tau) - m_1\, \boldsymbol{\kappa}_2(\tau)}{M}$.

It can be shown that the conditions $\hat{P}^r_{ADM} \approx 0$ and $\hat{K}^r_{ADM} \approx 0$ imply

$$
\begin{aligned}
\boldsymbol{\eta}_1(\tau) &\approx \Big(\frac{m_2}{M} - A_{(o)}(\tau)\Big)\, \boldsymbol{\rho}_{(o)12}(\tau) + \frac{m_2}{M}\, \boldsymbol{\rho}_{(1)12}(\tau) + \boldsymbol{f}_{(1)}(\tau)[rel.var., GW],\\
\boldsymbol{\eta}_2(\tau) &\approx -\Big(\frac{m_1}{M} + A_{(o)}(\tau)\Big)\, \boldsymbol{\rho}_{(o)12}(\tau) - \frac{m_1}{M}\, \boldsymbol{\rho}_{(1)12}(\tau) + \boldsymbol{f}_{(1)}(\tau)[rel.var., GW],\\
A_{(o)}(\tau) &= \frac{\frac{m_2}{M}\, \sqrt{m_1^2\, c^2 + \boldsymbol{\pi}^2_{(1)12}(\tau)} - \frac{m_1}{M}\, \sqrt{m_2^2\, c^2 + \boldsymbol{\pi}^2_{(1)12}(\tau)}}{\sqrt{m_1^2\, c^2 + \boldsymbol{\pi}^2_{(1)12}(\tau)} + \sqrt{m_2^2\, c^2 + \boldsymbol{\pi}^2_{(1)12}(\tau)}},
\end{aligned}
\tag{8.61}
$$

for some function $\boldsymbol{f}_{(1)}[rel.var., GW](\tau) \approx \boldsymbol{\eta}_{(1)12}(\tau)$ depending on the relative variables and the HPM GW of Eq. (8.59) in absence of incoming radiation. Then the equations of motion (8.60) imply

$$
\mu\, \ddot{\rho}^r_{(o)12}(\tau) \stackrel{\circ}{=} \frac{m_2}{M}\, F^r_1\, \big(\tau | \boldsymbol{\eta}_{(o)1}(\tau) | \boldsymbol{\eta}_{(o)2}(\tau)\big) - \frac{m_1}{M}\, F^r_2\, \big(\tau | \boldsymbol{\eta}_{(o)2}(\tau) | \boldsymbol{\eta}_{(o)1}(\tau)\big), \tag{8.62}
$$

for the relative configurational variable. The collective configurational variable has $\boldsymbol{\eta}_{(o)12}(\tau) \approx -A_{(o)}(\tau)\, \boldsymbol{\rho}_{(o)12}(\tau)$ at the lowest order, while at the first order there is an equation of motion equivalent to $\ddot{\eta}^r_{(1)12}(\tau) \approx \frac{d^2}{d\tau^2}\, \boldsymbol{f}_{(1)}[rel.var., GW]\, (\tau)$.

[39] It is a bi-tensor, i.e. a scalar in both the points $\boldsymbol{\eta}_i(\tau)$ and $\boldsymbol{\eta}_j(\tau)$, defined in terms of the space-like geodesic connecting them in Σ_τ. See Eq. (3.13) of the third paper in Ref. [88–90].

8.8.2 The Post-Newtonian Expansion at all Orders in the Slow Motion Limit

If all the particles are contained in a compact set of radius l_c, one can add a *slow motion condition* in the form $\sqrt{\epsilon} = \frac{v}{c} \approx \sqrt{\frac{R_{m_i}}{l_c}}$, $i = 1, \dots, N$ ($R_{m_i} = \frac{2\,G\,m_i}{c^2}$ is the gravitational radius of particle i) with $l_c \geq R_M$ and $\lambda >> l_c$ (see the Introduction). In this case one can do the PN expansion of Eqs. (8.60).

After having put $\tau = c\,t$, one makes the following change of notation

$$\begin{aligned}
\boldsymbol{\eta}_i(\tau) &= \tilde{\boldsymbol{\eta}}_i(t), \qquad \boldsymbol{v}_i(t) = \frac{d\tilde{\boldsymbol{\eta}}_i(t)}{dt}, \qquad \boldsymbol{a}_i(t) = \frac{d^2\tilde{\boldsymbol{\eta}}_i(t)}{dt^2},\\
\dot{\boldsymbol{\eta}}_i(\tau) &= \frac{\boldsymbol{v}_i(t)}{c}, \qquad \ddot{\boldsymbol{\eta}}_i(\tau) = \frac{\boldsymbol{a}_i(t)}{c^2}.
\end{aligned} \tag{8.63}$$

For the non-local York time one uses the notation ${}^3\tilde{\mathcal{K}}_{(1)}(t, \boldsymbol{\sigma}) = {}^3\mathcal{K}_{(1)}(\tau, \boldsymbol{\sigma})$.

Then one studies the PN expansion of the equations of motion (8.60) with the result (kPN means of order $O(c^{-2k})$)

$$\begin{aligned}
m_i\,\frac{d^2\,\tilde{\eta}_i^r(t)}{dt^2} &= m_i\Big[-G\,\frac{\partial}{\partial\,\tilde{\eta}_i^r}\sum_{j\neq i}\frac{m_j}{|\tilde{\boldsymbol{\eta}}_i(t)-\tilde{\boldsymbol{\eta}}_j(t)|} - \frac{1}{c}\,\frac{d\tilde{\eta}_i^r(t)}{dt}\Big(\partial_t^2|_{\tilde{\eta}i}\,{}^3\tilde{\mathcal{K}}_{(1)}\\
&\quad + 2\sum_u v_i^u(t)\,\frac{\partial\,\partial_t|_{\tilde{\boldsymbol{\eta}}_i}\,{}^3\tilde{\mathcal{K}}_{(1)}}{\partial\,\tilde{\eta}_i^u} + \sum_{uv} v_i^u(t)\,v_i^v(t)\,\frac{\partial^2\,{}^3\tilde{\mathcal{K}}_{(1)}}{\partial\,\tilde{\eta}_i^u\,\partial\,\tilde{\eta}_i^v}\Big)(t, \tilde{\boldsymbol{\eta}}_i(t))\Big]\\
&\quad + F^r_{i(1PN)}(t) + (\textit{higher PN orders}).
\end{aligned} \tag{8.64}$$

At the lowest order one finds the standard Newton gravitational force $\boldsymbol{F}_{i(Newton)}(t) = -m_i\,G\,\frac{\partial}{\partial\,\tilde{\eta}_i^r}\,\sum_{j\neq i}\frac{m_j}{|\tilde{\boldsymbol{\eta}}_i(t)-\tilde{\boldsymbol{\eta}}_j(t)|}$.

The unexpected result is a 0.5 PN force term containing all the dependence upon the non-local York time. The (arbitrary in these gauges) double rate of change in time of the trace of the extrinsic curvature creates a 0.5 PN damping (or anti-damping since the sign of the inertial gauge variable ${}^3\mathcal{K}_{(1)}$ of Eq. (8.57) is not fixed) effect on the motion of particles. This is a inertial effect (hidden in the lapse function) not existing in Newton theory where the Euclidean 3-space is absolute.

Then there are all the other kPN terms with $k = 1, 1.5, 2, \dots$ Since these results have been obtained without introducing ad hoc Lagrangians for the particles, are not in the harmonic gauge and do not contain terms of order $O(\zeta^2)$ and higher, it is not possible to make a comparison with the standard PN expansion (whose terms are known till the order 3.5PN [95]). Therefore only the 1PN and 0.5PN terms will be considered in the next two Subsections.

8.8.3 The HPM Binaries at the 1PN Order

Since in the next Subsection the 0.5PN term depending on the non-local York time will be connected with dark matter at the level of galaxies and clusters of galaxies and since there is no convincing evidence of dark matter in the Solar System and near the galactic plane of the Milky Way [195], it is reasonable to assume ${}^3\mathcal{K}_{(1)}(\tau, \boldsymbol{\sigma}) = \frac{1}{\triangle} F_{(1)}(\tau, \boldsymbol{\sigma}) \approx 0$ near a star with planets and near a binary.

In the description of Subsection 8.1 of the 1PN two-body problem, which is relevant for the treatment of binary systems[40] as shown in Chapter VI of Refs. [95] based on Ref. [98, 99, 157, 196–199], it can be shown that the relative momentum in the rest frame has the 1PN expression $\pi_{12}(\tau) = \pi_{(1)12}(\tau) \approx \mu\, \boldsymbol{v}_{(rel)(o)12}(t) \Big[1 + \frac{m_1^3 + m_2^3}{2M^3} (\frac{\boldsymbol{v}_{(rel)(o)12}(t)}{c})^2\Big]$, where $\boldsymbol{v}_{(rel)(o)12}(t) = \frac{d\, \boldsymbol{\rho}_{(o)12}(t)}{dt}$ is the velocity of the lowest order $\boldsymbol{\rho}_{(o)12}(\tau)$ of the relative variable.

If one ignores the York time and considers only positive energy particles ($\eta_1, \eta_2 \mapsto +1$), the 1PN equations of motion for the relative variable of the binary implied by Eqs. (8.62) and 1PN expression of the weak ADM energy $\hat{E}_{ADM}$ and of the rest spin $\hat{J}^{rs}_{ADM}$ (being determined by the boundary conditions they are constants of the motion implying planar motion in the plane orthogonal to the rest spin) can be shown to be

$$
\begin{aligned}
\frac{d\, \boldsymbol{v}_{(rel)(o)12}(t)}{dt} &= -\, G M\, \frac{\boldsymbol{\rho}_{(o)12}(t)}{|\boldsymbol{\rho}_{(o)12}(t)|^3} \Big[1 + \Big(1 + 3\frac{\mu}{M}\Big) \frac{v^2_{(rel)(o)12}(t)}{c^2} \\
&\quad - \frac{3\mu}{2M} \Big(\frac{\boldsymbol{v}_{(rel)(o)12}(t) \cdot \boldsymbol{\rho}_{(o)12}(t)\big)}{|\boldsymbol{\rho}_{(o)12}(t)|}\Big)^2\Big] \\
&\quad - \frac{G M}{|\boldsymbol{\rho}_{(o)12}(t)|^3} \Big(4 - \frac{2\mu}{M}\Big) \boldsymbol{v}_{(rel)(o)12}(t)\, \frac{\boldsymbol{v}_{(rel)(o)12}(t) \cdot \boldsymbol{\rho}_{(o)12}(t)\big)}{|\boldsymbol{\rho}_{(o)12}(t)|}. \\
\hat{E}_{ADM(1PN)} &= \sum_i m_i\, c^2 + \mu \Big(\frac{1}{2}\, \boldsymbol{v}^2_{(rel)(o)12}(t) \Big[1 + \frac{m_1^3 + m_2^3}{M^3} \Big(\frac{\boldsymbol{v}_{(rel)(o)12}(t)}{c}\Big)^2\Big] \\
&\quad - \frac{G M}{|\boldsymbol{\rho}_{(o)12}(t)|} \Big[1 + \frac{1}{2} \Big(\Big(3 + \frac{\mu}{M}\Big) \frac{\boldsymbol{v}^2_{(rel)(o)12}(t)}{c^2} + \\
&\quad + \frac{\mu}{M} \Big(\frac{\boldsymbol{v}_{(rel)(o)12}(t)}{c} \cdot \frac{\boldsymbol{\rho}_{(o)12}(t)}{|\boldsymbol{\rho}_{(o)12}(t)|}\Big)^2\Big)\Big]\Big), \\
\hat{J}^{rs}_{ADM(1PN)} &= \Big(\rho^r_{(o)12}(t)\, v^s_{(rel)(o)12}(t) - \rho^s_{(o)12}(t)\, v^r_{(rel)(o)12}(t)\Big) \\
&\quad \Big[1 + \frac{m_1^3 + m_2^3}{2M^3} \Big(\frac{\boldsymbol{v}_{(rel)(o)12}(t)}{c}\Big)^2\Big]. \qquad (8.65)
\end{aligned}
$$

[40] For binaries one assumes $\frac{v}{c} \approx \sqrt{\frac{R_m}{l_c}} << 1$, where $l_c \approx |\tilde{\boldsymbol{r}}|$ with $\tilde{\boldsymbol{r}}(t)$ being the relative separation after the decoupling of the center of mass. Often one considers the case $m_1 \approx m_2$. See chapter 4 of Ref. [95] for a review of the emission of GW's from circular and elliptic Keplerian orbits and of the induced inspiral phase.

Our 1PN equations (8.65) in the 3-orthogonal gauges coincide with Eqs. (2.5), (2.13) and (2.14) of the first paper in Ref. [98, 99] (without G^2 terms since they are $O(\zeta^2)$), which are obtained in the family of harmonic gauges starting from an ad hoc 1PN Lagrangian for the relative motion of two test particles (first derived by Infeld and Plebanski [179]).[41] These equations are the starting point for studying the post-Keplerian parameters of the binaries, which, together with the Roemer, Einstein and Shapiro time delays (both near Earth and near the binary) in light propagation, allow one to fit the experimental data from the binaries (see the second paper in Ref. [98, 99] and Chapter VI of Ref. [95]). Therefore these results are reproduced also in our 3-orthogonal gauge with ${}^3\mathcal{K}_{(1)}(\tau, \boldsymbol{\sigma}) = 0$.

8.8.4 From the Three Signatures for Dark Matter Reinterpreted as Relativistic Inertial Effects Induced by the York Time to the Need of a PM ICRS

To study the effects induced by the 0.5PN velocity-dependent (friction or anti-friction) force term in Eq. (8.64), depending on the inertial gauge variable non-local York time ${}^3\tilde{\mathcal{K}}_{(1)}(t, \boldsymbol{\sigma}) = \frac{1}{\triangle}\, {}^3\tilde{K}_{(1)}(\tau, \boldsymbol{\sigma}) \approx \frac{1}{\triangle} F_{(1)}(\tau, \boldsymbol{\sigma})$ with $F_{(1)}(\tau, \boldsymbol{\sigma})$ arbitrary numerical function, it is convenient to rewrite such equations in the form

$$\begin{aligned} &\frac{d}{dt}\Big[m_i \Big(1 + \frac{1}{c}\, \frac{d}{dt}\, {}^3\tilde{\mathcal{K}}_{(1)}(t, \tilde{\boldsymbol{\eta}}_i(t)) \Big) \frac{d\, \tilde{\eta}_i^r(t)}{dt} \Big] \\ &\quad \overset{\circ}{=} - G\, \frac{\partial}{\partial\, \tilde{\eta}_i^r} \sum_{j \neq i} \eta_j\, \frac{m_i\, m_j}{|\tilde{\boldsymbol{\eta}}_i(t) - \tilde{\boldsymbol{\eta}}_j(t)|} + \mathcal{O}(\zeta^2), \end{aligned} \tag{8.66}$$

because the damping or anti-damping factors in Eq. (8.64) are $\gamma_i(t, \tilde{\boldsymbol{\eta}}_i(t)) = \frac{d^2}{dt^2}\, {}^3\tilde{\mathcal{K}}_{(1)}(t, \tilde{\boldsymbol{\eta}}_i(t))$ and $\ddot{\tilde{\boldsymbol{\eta}}}_i(t) = O(\zeta)$.

As a consequence the velocity-dependent force can be *reinterpreted* as the introduction of an *effective (time-, velocity- and position-dependent) inertial mass term* for the kinetic energy of each particle:

$$m_i \;\mapsto m_i \Big(1 + \frac{1}{c}\, \frac{d}{dt}\, {}^3\tilde{\mathcal{K}}_{(1)}(t, \tilde{\boldsymbol{\eta}}_i(t)) \Big) = m_i + (\Delta\, m)_i(t, \tilde{\boldsymbol{\eta}}_i(t)), \tag{8.67}$$

in each instantaneous 3-space. Instead in the Newton potential there are the gravitational masses of the particles, equal to the inertial ones in the 4-dimensional space-time due to the equivalence principle. Therefore the effect is due to a modification of the effective inertial mass in each non-Euclidean 3-space depending on its shape as a 3-sub-manifold of space-time: *it is the equality of the inertial and gravitational*

[41] This is also the starting point of the effective one body description of the two-body problem of Refs. [181, 182].

masses of Newtonian gravity to be violated! In Galilei space-time the Euclidean 3-space is an absolute time-independent notion like Newtonian time: the non-relativistic non-inertial frames live in this absolute 3-space differently from what happens in SR and GR, where they are (in general non-Euclidean) 3-sub-manifolds of the space-time.

Equations (8.64), (8.66) and (8.67) can be applied to the three main signatures of the existence of dark matter in the observed masses of galaxies and clusters of galaxies, where the 1PN forces are not important, namely the virial theorem [103–105], the weak gravitational lensing [103–107] and the rotation curves of spiral galaxies (see Ref. [100–102] for a review), to give a *reinterpretation of dark matter as a relativistic inertial effect*.

A) *Masses of Clusters of Galaxies from the Virial Theorem*. For a bound system of N particles of mass m (N equal mass galaxies) at equilibrium, the virial theorem relates the average kinetic energy $< E_{kin} >$ in the system to the average potential energy $< U_{pot} >$ in the system: $< E_{kin} > = -\frac{1}{2} < U_{pot} >$ assuming Newton gravity. For the average kinetic energy of a galaxy in the cluster one takes $< E_{kin} > \approx \frac{1}{2} m < v^2 >$, where $< v^2 >$ is the average of the square of the radial velocity of single galaxies with respect to the center of the cluster (measured with Doppler shift methods; the velocity distribution is assumed isotropic). The average potential energy of the galaxy is assumed of the form $< U_{pot} > \approx -G \frac{m M}{\mathcal{R}}$, where $M = Nm$ is the total mass of the cluster and $\mathcal{R} = \alpha R$ is a "effective radius" depending on the cluster size R (the angular diameter of the cluster and its distance from Earth are needed to find R) and on the mass distribution on the cluster (usually $\alpha \approx 1/2$). Then the virial theorem implies $M \approx \frac{\mathcal{R}}{G} < v^2 >$. It turns out that this mass M of the cluster is usually at least an order of magnitude bigger that the baryonic matter of the cluster $M_{bar} = N m$ (spectroscopically determined). By applying Eq. (8.64) to the equilibrium condition for a self-gravitating system, i.e. $\frac{d^2}{dt^2} \sum_i m_i \mid \tilde{\boldsymbol{\eta}}_i(t) \mid^2 = 0$ with $m_i = m$, one gets $\sum_i m_i v_i^2(t) - G \sum_{i>j} \frac{m_i m_j}{|\tilde{\boldsymbol{\eta}}_i(t) - \tilde{\boldsymbol{\eta}}_j(t)|} - \frac{1}{c} \sum_i m_i \Big(\tilde{\boldsymbol{\eta}}_i(t) \cdot \boldsymbol{v}_i(t)\Big) \gamma_i(t, \tilde{\boldsymbol{\eta}}_i(t)) = 0$ with $m_i = m_j = m$. Therefore one can write $\langle U_{pot} \rangle = -\frac{1}{N} \sum_{i>j} \frac{G m^2}{|\tilde{\boldsymbol{\eta}}_i(t) - \tilde{\boldsymbol{\eta}}_j(t)|} \approx G \frac{m M_{bar}}{\mathcal{R}}$ (with $\mathcal{R} = R/2$) and $\frac{1}{2} m \langle v^2 \rangle = -\frac{1}{2} \langle U_{pot} \rangle + \frac{m}{2c} \langle \Big(\tilde{\boldsymbol{\eta}} \cdot \boldsymbol{v}\Big) \gamma(t, \tilde{\boldsymbol{\eta}}) \rangle$ with the notation $\langle \Big(\tilde{\boldsymbol{\eta}} \cdot \boldsymbol{v}\Big) \gamma(t, \tilde{\boldsymbol{\eta}}) \rangle = \frac{1}{N} \sum_i \Big(\tilde{\boldsymbol{\eta}}_i(t) \cdot \boldsymbol{v}_i(t)\Big) \gamma_i(t, \tilde{\boldsymbol{\eta}}_i(t))$ (it contains the non-local York time). Therefore for the measured mass M (the effective inertial mass in 3-space) one has

$$M = \frac{\mathcal{R}}{G} < v^2 > = M_{bar} + \frac{\mathcal{R}}{G\,c} \Big\langle \Big(\tilde{\boldsymbol{\eta}} \cdot \boldsymbol{v}\Big) \gamma(t, \tilde{\boldsymbol{\eta}}) \Big\rangle \stackrel{def}{=} M_{bar} + M_{DM}, \tag{8.68}$$

and one sees that the non-local York time can give rise to a dark matter contribution $M_{DM} = M - M_{bar}$.

B) *Masses of galaxies or clusters of galaxies from weak gravitational lensing*. Usually one considers a galaxy (or a cluster of galaxies) of big mass M behind which a distant, bright object (often a galaxy) is located. The light from the distant object is bent by the massive one (the lens) and arrives on the Earth deflected from the original propagation direction. As shown in Refs. [106, 107] one has to evaluate Einstein deflection of light, emitted by a source S at distance d_S from the observer O on the Earth, generated by the big mass at a distance d_D from the observer O. The mass M, at distance d_{DS} from the source S, is considered as a point-like mass generating a 4-metric of the Schwarzschild type (Schwarzschild lens). The ray of light is assumed to propagate in Minkowski space-time till near M, to be deflected by an angle α by the local gravitational field of M and then to propagate in Minkowski space-time till the observer O. The distances d_S, d_D, d_{DS}, are evaluated by the observer O at some reference time in some nearly-inertial Minkowski frame with nearly Euclidean 3-spaces (in the Euclidean case $d_{DS} = d_S - d_D$). If $\xi = \theta\, d_D$ is the impact parameter of the ray of light at M and if $\xi >> R_s = \frac{2\, G M}{c^2}$ (the gravitational radius), Einstein's deflection angle is $\alpha = \frac{2\, R_s}{\xi} = \frac{4\, G M}{c^2\, \xi}$ and the so-called Einstein radius (or characteristic angle) is $\alpha_o = \sqrt{2\, R_s\, \frac{d_{DS}}{d_D\, d_S}} = \sqrt{\frac{4\, G M}{c^2}\, \frac{d_{DS}}{d_D\, d_S}}$. A measurement of the deflection angle and of the three distances allows to get a value for the mass M of the lens, which usually turns out to be much larger of its mass inferred from the luminosity of the lens. For the calculation of the deflection angle one considers the propagation of ray of light in a stationary 4-metric of the BCRS type and uses a version of the Fermat principle containing an effective index of refraction n. One has $n = {}^4g_{\tau\tau} = \epsilon\, [1 - \frac{2w}{c^2} - 2\, \partial_\tau\, {}^3\mathcal{K}]$ in the PM approximation. Since one has $\frac{2\, w}{c^2} = -\frac{G M_{bar}}{c^2\, |\boldsymbol{\sigma}|}$, the definition $2\, \partial_\tau\, {}^3\mathcal{K}_{(1)} \stackrel{def}{=} -\frac{G M_{DM}}{c^2\, |\boldsymbol{\sigma}|}$ leads to an Einstein deflection angle

$$\alpha = \frac{4\, G M}{c^2\, \xi} \qquad with \quad M \stackrel{def}{=} M_{bar} + M_{DM}. \tag{8.69}$$

Therefore also in this case the measured mass M is the sum of a baryonic mass M_{bar} and of a dark matter mass M_{DM} induced by the non-local York time at the location of the lens.

C) *Masses of Spiral Galaxy Masses from Their Rotation Curves.* In this case one considers a two-body problem (a point-like galaxy and a body circulating around it) described in terms of an internal center of mass $\tilde{\boldsymbol{\eta}}_{12}(t) \approx \tilde{\boldsymbol{\eta}}_{(1)12}(t)$ ($\tilde{\boldsymbol{\eta}}_{(o)12}(t) = 0$ is the origin of the 3-coordinates) and a relative variable $\tilde{\boldsymbol{\rho}}_{12}(t)$. Then the sum and difference of Eqs. (8.64) imply the equations of motion for $\tilde{\boldsymbol{\eta}}_{(1)12}(t)$ and $\tilde{\boldsymbol{\rho}}_{12}(t)$. While the first equation implies a small motion of the overall system, the second one has the form

$$\begin{aligned}
\frac{d^2\, \tilde{\rho}^r_{(1)12}(t)}{dt^2} &\stackrel{\circ}{=} -G M\, \frac{\tilde{\rho}^r_{12}(t)}{|\tilde{\boldsymbol{\rho}}_{12}(t)|^3} - \frac{1}{c}\, \frac{d\tilde{\rho}^r_{12}(t)}{dt}\, \gamma_+(t, \tilde{\boldsymbol{\rho}}_{12}(t), \boldsymbol{v}(t)),\\
\gamma_+\,(t, \tilde{\boldsymbol{\rho}}_{1212}(t), \boldsymbol{v}(t)) &= \frac{m_1}{M}\, \gamma_1\left(t, \frac{m_2}{M}\, \tilde{\boldsymbol{\rho}}_{1212}(t), \boldsymbol{v}(t)\right) + \frac{m_2}{M}\, \gamma_2\\
&\quad \times \left(t, -\frac{m_1}{M}\, \tilde{\boldsymbol{\rho}}_{1212}(t), \boldsymbol{v}(t)\right),
\end{aligned} \tag{8.70}$$

where γ_i are the damping or anti-damping factors defined after Eq. (8.66). Equation (8.70) gives the two-body Kepler problem with an extra perturbative force. Without it a Keplerian solution with circular trajectory such that $\mid \tilde{\boldsymbol{\rho}}_{12}(t) \mid = R = const.$ implies that the Keplerian velocity $\boldsymbol{v}_o(t) = v_o\, \hat{n}(t)$ has the modulus vanishing at large distances, $v_o = \sqrt{\frac{GM}{R}} \rightarrow_{R\rightarrow\infty} 0$. Instead the rotation curves of spiral galaxies imply that the relative 3-velocity goes to constant for large R, i.e. $v = \sqrt{\frac{G\,(M_{bar} + \Delta\, M(r))}{r}} \rightarrow_{R\rightarrow\infty} const.$ (M_{bar} is the spectroscopically determined baryon mass), so that the extra required term $\Delta\, M(r)$ is interpreted as the mass M_{DM} of a dark matter halo.

The presence of the extra force term implies that the velocity must be written as $\boldsymbol{v}(t) = \boldsymbol{v}_o(t) + \boldsymbol{v}_1(t)$ with $v_1(t)$ a first order perturbative correction satisfying $\frac{dv_1^r(t)}{dt} = -\frac{v_o^r}{c}\, \hat{n}(t)\, \gamma_+(t, \tilde{\boldsymbol{\rho}}_{12}(t), \boldsymbol{v}_o^r(t))$. Therefore at the first order in the perturbation one gets $v^2(t) = v_o^2 \Big(1 - \frac{2}{c}\, \hat{n}(t) \cdot \int_o^t dt_1\, \hat{n}(t_1)\, \gamma_+(t_1, \tilde{\boldsymbol{\rho}}_{12}(t_1),\, \boldsymbol{v}_o(t_1)) \Big)$. Therefore, after having taken a mean value over a period T (the time dependence of the mass of a galaxy is not known) the effective mass of the two-body system is

$$\begin{aligned} M_{eff} &= \frac{\langle v^2 \rangle\, R}{G} = M\, \left(1 - \left\langle \frac{2}{c}\, \hat{n}(t) \cdot \int^t dt_1\, \hat{n}(t_1)\, \gamma_+(t_1, \tilde{\boldsymbol{\rho}}_{12}(t_1), \boldsymbol{v}_o(t_1)) \right\rangle \right) \\ &= M_{bar} + M_{DM}. \end{aligned} \tag{8.71}$$

with a $\Delta\, M(r) = M_{DM}$ function only of the mean value of the total time derivative of the non-local ${}^3\mathcal{K}_{(1)}$ to be fitted to the experimental data.

Therefore, the existence of the inertial gauge variable York time, a property of the non-Euclidean 3-spaces as 3-sub-manifolds of Einstein space-times (connected only to the general relativistic remnant of the gauge freedom in clock synchronization, independently from cosmological assumptions) implies the possibility of describing part (or maybe all) dark matter as a *relativistic inertial effect in Einstein gravity* without alternative explanations using:

(1) the non-relativistic MOND approach [200] (where one modifies Newton equations);
(2) modified gravity theories like the $f(R)$ ones (see for instance Refs. [201]; here one gets a modification of the Newton potential);
(3) the assumption of the existence of WIMP particles [202].

Let us also remark that the 0.5PN effect has origin in the lapse function and not in the shift one, as in the gravito-magnetic elimination of dark matter proposed in Ref. [203].

The open problem with this explanation of dark matter is the determination of the non-local York time from the data on dark matter. From what is known about dark matter in the Solar System and inside the Milky Way near the galactic plane, it seems that ${}^3\mathcal{K}_{(1)}(\tau, \boldsymbol{\sigma})$ is negligible near the stars inside a galaxy. Instead the non-local York time (or better a mean value in time of its total time derivative) should be relevant around the galaxies and the clusters of galaxies, where there are big concentrations of mass and well defined signatures of dark matter. Instead there is no indication on its value in the voids existing among the clusters of galaxies.

Therefore the known data on dark matter do not allow one to get an experimental determination of the York time ${}^3K_{(1)}(\tau, \boldsymbol{\sigma}) = \triangle\, {}^3\mathcal{K}_{(1)}(\tau, \boldsymbol{\sigma})$, because to do it one needs *to know the non-local York time on all the 3-universe* at a given τ.

Since, as said in the Introduction, at the experimental level *the description of matter is intrinsically coordinate-dependent*, namely is connected with the conventions used by physicists, engineers and astronomers for the modeling of space-time, one has to choose a gauge (i.e. a 4-coordinate system) in non-modified Einstein gravity which is in agreement with the observational conventions in astronomy. This way out from the gauge problem in GR requires a choice of 3-coordinates on the instantaneous 3-spaces identified by a choice of time and by a clock synchronization convention, i.e. a fixation of the York time ${}^3K_{(1)}(\tau, \boldsymbol{\sigma})$. The convention resulting by one set of such choices would give a *PM extension of ICRS*, with BCRS being its quasi-Minkowskian approximation for the Solar System. Since the existing ICRS [5–9, 11–13] has diagonal 3-metric, 3-orthogonal gauges are a convenient choice.

The real problem is the extraction of an indication of which kind of function of time and 3-coordinates to use for the York time ${}^3K_{(1)}(\tau, \boldsymbol{\sigma})$ from astrophysical data different from the ones giving information about dark matter. Once one would have a phenomenological parametrization of the York time, then the data on dark matter would put restrictions on the induced phenomenological parametrization of the non-local York time ${}^3\mathcal{K}_{(1)}(\tau, \boldsymbol{\sigma}) = \frac{1}{\triangle}\, {}^3K_{(1)}(\tau, \boldsymbol{\sigma})$. As it will be delineated in the final Section, to implement this program one has to look at the astrophysical data on dark energy after having succeeded to interpret also it as a relativistic inertial effect in suitable cosmological space-times in which one can induce the distinction between inertial and tidal degrees of freedom of the gravitational field from the previously discussed Hamiltonian framework.

8.9 Dark Energy and Other Open Problems

This Lecture contains a full review of an approach to SR and to asymptotically Minkowskian classical canonical Einstein GR based on a description of global non-inertial frames centered on a time-like observer which is suggested by relativistic metrology. The *gauge freedom in clock synchronization*, which does not exist in Galilei space-time (Newton time and Euclidean 3-spaces are absolute) and is not restricted in Minkowski space-time (it spans the class of the admissible 3+1 splittings of this absolute space-time), is restricted in GR to the gauge freedom connected with the inertial gauge variable ${}^3K$, the York time, which determines the shape of the instantaneous non-Euclidean 3-spaces as 3-sub-manifolds of the space-time.

The study of canonical ADM tetrad gravity in asymptotically Minkowskian space-times without super-translations (so that they admit an asymptotic ADM Poincaré algebra at spatial infinity) in the York canonical basis allowed one to disentangle the tidal degrees of freedom of the gravitational field from the inertial gauge ones (they include the York time), to find the family of non-harmonic 3-orthogonal Schwinger time gauges and to define a HPM linearization in them. The main properties of these non-harmonic gauges are that only the HPM-GW (but not the lapse and shift functions) are retarded quantities with a no-incoming radiation condition and that one can naturally find which quantities depend upon the York time.

Relativistic particle mechanics, coupled to the electro-magnetic field in the radiation gauge, has been studied both in SR and GR with a suitable Grassmann regularization of the self-energies so to get well defined equations of motion.

In SR, after a clarification of the problem of the relativistic center of mass and the definition of inertial and non-inertial rest frames of isolated systems, it was possible to develop a formulation, the parametrized Minkowski theories, in which the transitions among global non-inertial frames are gauge transformations. Then isolated systems were described in the rest-frame instant form of dynamics and the structure of their Poincaré generators and of their relative variables in the instantaneous Wigner 3-spaces was clarified. With this approach it was possible to give a new formulation of the micro-canonical ensemble in relativistic kinetic theory and to develop a formulation of relativistic quantum mechanics and relativistic entanglement taking into account the known results about relativistic bound states and the spatial non-separability and non-locality induced by the Lorentz signature of Minkowski space-time.

In GR it was possible to derive regularized equations of motion of the particles in the non-inertial rest frame and to study their PM limit in the HPM linearization in the 3-orthogonal gauges and the emission of HPM GW (with the energy balance under control even in absence of self-forces). Then the PN limit of these PM equations allows one to recover the known 1PN results of harmonic gauges. The more surprising result is that in the PN expansion of the PM equations of motion there is a 0.5PN term in the forces depending upon the York time. This opens the possibility to describe dark matter as a relativistic inertial effect implying that the effective inertial mass of particles in the 3-spaces is bigger of the gravitational mass because it depends on the York time (i.e. on the shape of the 3-space as a 3-sub-manifold of the space-time: this is impossible in Newton gravity in Galilei space-time and leads to a violation of the Newtonian equivalence principle).

The proposed solution to the gauge problem in GR based on the conventions of relativistic metrology for ICRS and the results of the last Section on the re-interpretation of dark matter as a relativistic inertial effect arising as a consequence of a convention on the York time in an extended PM ICRS push toward the necessity of similar re-interpretation also of dark energy in cosmology [1, 2, 204–212]. As it has been shown, the identification of the tidal and inertial degrees of freedom of the gravitational field can be reformulated in the framework of the non-Hamiltonian first-order ADM equations by means of the replacement of the Hamiltonian momenta with the expansion and the shear of the Eulerian observers associated with the $3+1$ splitting of the space-time. Therefore this identification can also be applied to the cosmological space-times which do not admit a Hamiltonian formulation: also in them the identification of the instantaneous 3-spaces Σ_τ, now labeled by a cosmic time, requires a conventional choice of clock synchronization, i.e. a convention on the

York time ${}^3K$ defining the shape of the 3-spaces as 3-sub-manifolds of the space-time, and of 3-coordinates (the 3-orthogonal ones are acceptable also in cosmology).

In the standard ΛCDM cosmological model the class of cosmological solutions of Einstein equations is restricted to Friedmann-Robertson-Walker (FRW) space-times with nearly Euclidean 3-spaces (i.e. with a small internal 3-curvature). In them the Killing symmetries connected with homogeneity and isotropy imply (τ is the cosmic time, $a(\tau)$ the scale factor) that the York time is no more a gauge variable but coincides with the Hubble constant: ${}^3K(\tau) = -\frac{\dot{a}(\tau)}{a(\tau)} = -H(\tau)$. However at the first order in cosmological perturbations (see Ref. [213] for a review) one has ${}^3K = -H + {}^3K_{(1)}$ with ${}^3K_{(1)}$ being again an inertial gauge variable to be fixed with a metrological convention. Therefore the York time has a central role also in cosmology and one needs to know the dependence on it of the main quantities, like the red-shift and the luminosity distance from supernovae, which require the introduction of the notion of dark energy to explain the 3-universe and its accelerated expansion in the framework of the standard ΛCDM cosmological model.

Instead in inhomogeneous space-times without Killing symmetries like the Szekeres ones [214–218] the York time remains an arbitrary inertial gauge variable. Therefore the main open problem of the present approach is to see whether it is possible to find a 3-orthogonal gauge in a inhomogeneous Einstein space-time (at least in a PM approximation) in which the convention on the inertial gauge variable York time allows one to accomplish the following two tasks simultaneously: (a) to eliminate both dark matter and dark energy through the choice of a 4-coordinate system (suggested by astrophysical data) to be used in a consistent PM reformulation of ICRS and (b) to save the main good properties of the standard ΛCDM cosmological model due to the inertial and dynamical properties of the space-time. As matter one will take the dust, whose description in the York canonical basis is given in Ref. [91].

Also in the back-reaction approach [219–224] to cosmology, according to which dark energy is a byproduct of the non-linearities of GR when one considers spatial averages of 3-scalar quantities in the 3-spaces on large scales to get a cosmological description of the universe taking into account its observed inhomogeneity, one gets that the spatial average of the product of the lapse function and of the York time (a 3-scalar gauge variable) gives the effective Hubble constant. Since this approach starts from the Hamiltonian description of an asymptotically flat space-time and since all the canonical variables in the York canonical basis, except the angles θ^i, are 3-scalars, the formalism presented in this Lecture will allow to study the spatial average of nearly all the Hamilton equations and not only of the super-Hamiltonian constraint and of the Hamilton equation for the York time as in the existing formulation of the approach. This will be done by using the perfect fluids of Ref. [91] as matter.

Also the recent point of view of Ref. [225], taking into account the relevance of the voids among the clusters of galaxies, has to be reformulated in terms of the York time.

Finally one should find the dependence upon he York time of the Landau-Lifschitz energy-momentum pseudo-tensor and re-express it as the effective energy-momentum tensor of a viscous pseudo-fluid. One will have to check whether for certain choices of the York time the resulting effective equation of state of the fluid has negative pressure, realizing also in this way a simulation of dark energy.

Other open problems in GR under investigation are:

(A) Find the second order of the HPM expansion to see whether in PM space-times there is the emergence of hereditary terms [95, 176] like the ones present in harmonic gauges.
(B) Study the PM equations of motion of the transverse electro-magnetic field trying to find Lienard-Wiechert-type solutions in GR. Study astrophysical problems where the electro-magnetic field is relevant.
(C) Find the expression in the York canonical basis of the Weyl scalars of the Newman-Penrose approach [136] and then of the four Weyl eigenvalues, which are tetrad-independent 4-scalar invariants of the gravitational field. Is it possible to find a canonical transformation replacing the 3-scalar tidal variables with four 4-scalar functions of the Weyl eigenvalues? Are Weyl eigenvalues Dirac observables?
(D) Try to make a multi-temporal quantization (see Refs. [77, 78, 129, 130]) of the linearized HPM theory over the asymptotic Minkowski space-time, in which, after a Shanmugadhasan canonical transformation to a new York canonical basis adapted to all the constraints, only the tidal variables are quantized but not the inertial gauge ones. After this type of quantization, in which the lapse and shift functions remain c-numbers, the space-time would still be a classical 4-manifold: only the two eigenvalues of the 3-metric describing GW are quantized and therefore only 3-metric properties like 3-distances, 3-areas, 3-volumes become quantum properties. After having re-expressed the Ashtekar variables [226–229] for asymptotically Minkowskian space-times (see Appendix B of Ref. [10]) in this final York canonical basis it will be possible to compare the outcomes of this new type of quantization with loop quantum gravity.

References

1. M. Bartelmann, The dark universe. Rev. Mod. Phys. **82**, 331 (2010) (arXiv 0906.5036)
2. R. Bean, TASI 2009. Lectures on Cosmic Acceleration (arXiv 1003.4468).
3. UCLA 2007, The ABC's of Distances (http://www.astro.ucla.edu/wright/distance.htm
4. B.W. Carroll, D.A. Ostlie, *An Introduction to Modern Astrophysics*, 2nd edn. (Pearson Education Inc., New Jersey, 2007)
5. J. Kovalevski, I.I. Mueller, B. Kolaczek, *Reference Frames in Astronomy and Geophysics* (Kluwer, Dordrecht, 1989)
6. O.J. Sovers, J.L. Fanselow, Astrometry and geodesy with radio interferometry: experiments, models, results. Rev. Mod. Phys. **70**, 1393 (1998)
7. C. Ma, E.F. Arias, T.M. Eubanks, A.L. Fey, A.M. Gontier, C.S. Jacobs, O.J. Sovers, B.A. Archinal, P. Charlot, The international celestial reference frame as realized by very long baseline interferometry. AJ **116**, 516 (1998)
8. K.J. Johnstone, Chr. de Vegt, Reference frames in astronomy. Annu. Rev. Astron. Astrophys. **37**, 97 (1999)

9. A. Fey, G. Gordon, C. Jacobs (eds.), The Second Realization of the International Celestial Reference Frame by Very Long Baseline Interferometry, 2009 IERS Technical, Notes 35.
10. L. Lusanna, Canonical Gravity and Relativistic Metrology: From Clock Synchronization to Dark Matter as a Relativistic Inertial Effect (arXiv 1108.3224)
11. M.Soffel, S.A. Klioner, G. Petit, P. Wolf, S.M. Kopeikin, P. Bretagnon, V.A. Brumberg, N. Capitaine, T. Damour, T. Fukushima, B. Guinot, T. Huang, L. Lindegren, C. Ma, K. Nordtvedt, J. Ries, P.K. Seidelmann, D. Vokroulicky', C. Will, Ch. Xu, The IAU 2000 resolutions for astrometry, celestial mechanics and metrology in the relativistic rramework: explanatory supplement. Astron. J., **126**, 2687–2706 (2003) (arXiv astro-ph/0303376).
12. D.D. McCarthy, G. Petit, IERS TN 32 (2004), Verlag des BKG. Kaplan G.H., The IAU Resolutions on Astronomical Reference Systems, Time Scales and Earth Rotation Models, IERS Conventions (2003). U.S.Naval Observatory circular No. 179 (2005) (arXiv astro-ph/0602086)
13. T.D. Moyer, *Formulation for Observed and Computed Values of Deep Space Network Data Types for Navigation* (John Wiley, New York, 2003)
14. S. Jordan, The GAIA project: technique, performance and status. Astron. Nachr. **329**, 875 (2008). doi:10.1002/asna.200811065
15. R. Arnowitt, S. Deser, C.W. Misner, in *The Dynamics of General Relativity, ch.7 of Gravitation: An Introduction to Current Research*, ed. by L. Witten (Wiley, New York, 1962) (arXiv gr-qc/0405109)
16. P.A.M. Dirac, *Lectures on Quantum Mechanics, Belfer Graduate School of Science, Monographs Series* (Yeshiva University, New York, 1964)
17. M. Henneaux, C. Teitelboim, *Quantization of Gauge Systems* (Princeton University Press, Princeton, 1992)
18. S. Shanmugadhasan, Canonical formalism for degenerate Lagrangians. J. Math. Phys. **14**, 677 (1973)
19. L. Lusanna, The Shanmugadhasan canonical transformation, function groups and the second Noether Theorem. Int. J. Mod. Phys. **A8**, 4193 (1993)
20. L. Lusanna, The Relevance of Canonical Transformations in Gauge Theories and General Relativity, Lecture Notes of "Seminario Interdisciplinare di Matematica" (Basilicata Univ.) 5, 125 (2006)
21. M. Chaichian, D. Louis Martinez, L. Lusanna, Dirac's constrained systems: the classification of second-class constraints, Ann. Phys. (N.Y.) **232**, 40 (1994)
22. B. Dittrich, Partial and complete observables for hamiltonian constrained systems. Gen. Rel. Grav. **39**, 1891 (2007) (arXiv gr-qc/0411013)
23. B. Dittrich, Partial and complete observables for canonical general relativity. Gen. Rel. Grav. **23**, 6155 (2006) (arXiv gr-qc/0507106)
24. T. Thiemann, Reduced phase space quantization and Dirac observables. Class. Quantum Grav. **23**, 1163 (2006) (arXiv gr-qc/0411031)
25. J.M. Pons, D.C. Salisbury, K.A. Sundermeyer, Revisiting observables in generally covariant theories in the light of gauge fixing methods. Phys. Rev. **D80**, 084015 (2009) (0905.4564)
26. J.M. Pons, D.C. Salisbury, K.A. Sundermeyer, Observables in classical canonical gravity: folklore demystified (arXiv 1001.2726)
27. V. Moncrief, Space-time symmetries and linearization stability of the Einstein equations, I. J. Math. Phys. **16**, 493 (1975)
28. V. Moncrief, Decompositions of gravitational perturbations. J. Math. Phys. **16**, 1556 (1975)
29. V. Moncrief, Space-time symmetries and linearization stability of the Einstein equations, II. J. Math. Phys. **17**, 1893 (1976)
30. V. Moncrief, Invariant states and quantized gravitational perturbations. Phys. Rev. **D18**, 983 (1978)
31. Y. Choquet-Bruhat, A.E. Fischer, J.E. Marsden, *Maximal hypersurfaces and positivity of mass*, LXVII E. Fermi Summer School of Physics Isolated Gravitating Systems in General Relativity, ed. by J. Ehlers (North-Holland, Amsterdam, 1979)
32. A.E. Fisher, J.E. Marsden, *The Initial Value Problem and the Dynamical Formulation of General Relativity*, in *General Relativity. An Einstein Centenary Survey*, ed. by S.W. Hawking, W. Israel (Cambridge University Press, Cambridge, 1979)

33. A. Ashtekar, *Asymptotic Structure of the Gravitational Field at Spatial Infinity*, in *General Relativity and Gravitation*, vol 2, ed. by A. Held (Plenum, New York, 1980)
34. P.J. McCarthy, Asymptotically flat space-times and elementary particles. Phys. Rev. Lett. **29**, 817 (1972)
35. P.J. McCarthy, Structure of the Bondi-Metzner-Sachs group. J. Math. Phys. **13**, 1837 (1972)
36. R. Beig, Ó. Murchadha, The Poincaré Group as the Symmetry Group of Canonical General Relativity. Ann. Phys. (N.Y.) **174**, 463 (1987)
37. L. Fatibene, M. Ferraris, M. Francaviglia, L. Lusanna, ADm pseudotensors, conserved quantities and covariant conservation laws in general relativity. Ann. Phys. Ann.Phys. **327**, 1593 (2012) (arXiv 1007.4071)
38. J.L. Jaramillo, E. Gourgoulhon, *Mass and Angular Momentum in General Relativity, in Mass and Motion in General Relativity*, ed. by L. Blanchet, A. Spallicci, B. Whiting, Fundamental Theories in Physics, vol 162 (Springer, Berlin, 2011), p. 87 (arXiv 1001.5429)
39. L.B. Szabados, On the roots of Poincaré structure of asymptotically flat spacetimes. Class. Quant. Grav. **20**, 2627 (2003) (arXiv gr-qc/0302033)
40. D. Christodoulou, S. Klainerman, *The Global Nonlinear Stability of the Minkowski Space* (Princeton University Press, Princeton, 1993)
41. L. Lusanna, The rest-frame instant form of metric gravity. Gen. Rel. Grav. **33**, 1579 (2001) (arXiv gr-qc/0101048)
42. L. Lusanna, S. Russo, A new parametrization for tetrad gravity. Gen. Rel. Grav. **34**, 189 (2002) (arXiv gr-qc/0102074)
43. R. DePietri, L. Lusanna, L. Martucci, S. Russo, Dirac's observables for the rest-frame instant form of tetrad gravity in a completely fixed 3-orthogonal gauge. Gen. Rel. Grav. **34**, 877 (2002) (arXiv gr-qc/0105084)
44. J. Agresti, R. De Pietri, L. Lusanna, L. Martucci, Hamiltonian linearization of the rest-frame instant form of tetrad gravity in a completely fixed 3-orthogonal gauge: a radiation gauge for background-independent gravitational waves in a post-Minkowskian Einstein spacetime. Gen. Rel. Grav. **36**, 1055 (2004) (arXiv gr-qc/0302084)
45. C. Rovelli, *Quantum Gravity* (Cambridge University Press, Cambridge, 2004)
46. R. Geroch, Spinor structure of space-times in general relativity I. J. Math. Phys. **9**, 1739 (1968)
47. B. Mashhoon, *The Hypothesis of Locality and its Limitations, in Relativity in Rotating Frames*, ed. by G. Rizzi, M.L. Ruggiero (Kluwer, Dordrecht, 2003) (arXiv gr-qc/0303029)
48. B. Mashhoon, U. Muench, Length measurements in accelerated systems. Ann. Phys. (Leipzig) **11**, 532 (2002)
49. D. Alba, L. Lusanna, Charged particles and the electro-magnetic field in non-inertial frames: I. Admissible 3+1 splittings of minkowski spacetime and the non-inertial rest frames. Int. J. Geom. Meth. Phys. **7**, 33 (2010) (arXiv 0908.0213)
50. D. Alba, L. Lusanna, Charged particles and the electro-magnetic field in non-inertial frames: II. Applications: rotating frames, Sagnac effect, Faraday rotation, wrap-up effect. Int. J. Geom. Meth. Phys. **7**, 185 (2010) (arXiv 0908.0215)
51. D. Alba, L. Lusanna, Generalized radar 4-coordinates and equal-time cauchy surfaces for arbitrary accelerated observers (2005). Int. J. Mod. Phys. **D16**, 1149 (2007) (arXiv gr-qc/0501090)
52. D. Bini, L. Lusanna, B. Mashhoon, Limitations of radar coordinates. Int. J. Mod. Phys. **D14**, 1 (2005) (arXiv gr-qc/0409052)
53. H. Bondi, *Assumption and Myth in Physical Theory* (Cambridge University Press, Cambridge, 1967)
54. R. D'Inverno, *Introducing Einstein Relativity* (Oxford University Press, Oxford, 1992)
55. L. Lusanna, The N- and 1-time classical descriptions of N-body relativistic kinematics and the electromagnetic interaction. Int. J. Mod. Phys. **A12**, 645 (1997)
56. L. Lusanna, *The Chrono-Geometrical Structure of Special and General Relativity: A Re-Visitation of Canonical Geometrodynamics*. Lectures at 42nd Karpacz Winter School of Theoretical Physics: Current Mathematical Topics in Gravitation and Cosmology, Ladek, Poland, 6–11 Feb 2006. Int. J. Geom. Methods in Mod. Phys. **4**, 79 (2007). (arXiv gr-qc/0604120)

57. L. Lusanna, *The Chronogeometrical Structure of Special and General Relativity: towards a Background-Independent Description of the Gravitational Field and Elementary Particles (2004)*, in General Relativity Research Trends, ed. by A. Reiner, Horizon in World Physics, vol. 249 (Nova Science, New York, 2005) (arXiv gr-qc/0404122)
58. E. Schmutzer, J. Plebanski, Quantum mechanics in noninertial frames of reference. Fortschr. Phys. **25**, 37 (1978)
59. D. Alba, L. Lusanna, M. Pauri, *New Directions in Non-Relativistic and Relativistic Rotational and Multipole Kinematics for N-Body and Continuous Systems (2005)*, in *Atomic and Molecular Clusters: New Research*, ed. by Y.L. Ping (Nova Science, New York, 2006) (arXiv hep-th/0505005)
60. D. Alba, L. Lusanna, M. Pauri, Centers of mass and rotational kinematics for the relativistic N-body problem in the rest-frame instant form. J. Math. Phys. **43**, 1677–1727 (2002) (arXiv hep-th/0102087)
61. D. Alba, L. Lusanna, M. Pauri, Multipolar expansions for closed and open systems of relativistic particles. J. Math. Phys. **46**, 062505, 1–36 (2004) (arXiv hep-th/0402181)
62. H.W. Crater, L. Lusanna, The rest-frame Darwin potential from the Lienard-Wiechert solution in the radiation gauge. Ann. Phys. (N.Y.) **289**, 87 (2001)
63. D. Alba, H.W. Crater, L. Lusanna, The semiclassical relativistic Darwin potential for spinning particles in the rest frame instant form: two-body bound states with spin 1/2 constituents. Int. J. Mod. Phys. **A16**, 3365–3478 (2001) (arXiv hep-th/0103109)
64. D. Alba, L. Lusanna, The Lienard-Wiechert potential of charged scalar particles and their relation to scalar electrodynamics in the rest-frame instant form. Int. J. Mod. Phys. **A13**, 2791 (1998) (arXiv hep-th/0708156)
65. D. Alba, H.W. Crater, L. Lusanna, Towards relativistic atom physics. I. The rest-frame instant form of dynamics and a canonical transformation for a system of charged particles plus the electro-magnetic field. Can. J. Phys. **88**, 379 (2010) (arXiv 0806.2383)
66. D. Alba, H.W. Crater, L. Lusanna, Towards relativistic atom physics. II. Collective and relative relativistic variables for a system of charged particles plus the electro-magnetic field. Can. J. Phys. **88**, 425 (2010) (arXiv 0811.0715)
67. D. Alba, H.W. Crater, L. Lusanna, Hamiltonian relativistic two-body problem: center of mass and orbit reconstruction. J. Phys. **A40**, 9585 (2007) (arXiv gr-qc/0610200)
68. D. Alba, H.W. Crater, L. Lusanna, On the relativistic micro-canonical ensemble and relativistic kinetic theory for N relativistic particles in inertial and non-inertial rest frames (arXiv 1202.4667)
69. D. Bini, L. Lusanna, Spin-rotation couplings: spinning test particles and Dirac fields. Gen. Rel. Grav. **40**, 1145 (2008) (arXiv 0710.0791)
70. F. Bigazzi, L. Lusanna, Spinning particles on spacelike hypersurfaces and their rest-frame description. Int. J. Mod. Phys. **A14**, 1429 (1999) (arXiv hep-th/9807052)
71. F. Bigazzi, L. Lusanna, Dirac fields on spacelike hypersurfaces, their rest-frame description and Dirac observables. Int. J. Mod. Phys. **A14**, 1877 (1999) (arXiv hep-th/9807054)
72. D. Alba, L. Lusanna, The classical relativistic quark model in the rest-frame Wigner-covariant Coulomb gauge. Int. J. Mod. Phys. **A13**, 3275 (1998) (arXiv hep-th/9705156)
73. D. Alba, H.W. Crater, L. Lusanna, Massless particles plus matter in the rest-frame instant form of dynamics. J. Phys. **A 43** 405203 (arXiv 1005.5521)
74. D. Alba, H.W. Crater, L. Lusanna, The rest-frame instant form and Dirac observables for the open Nambu string. Eur. Phys. J. Plus **126**, 26 (2011)(arXiv 1005.3659)
75. D. Alba, H.W. Crater, L. Lusanna, A relativistic version of the two-level atom in the rest-frame instant form of dynamics. J. Phys. **A46**, 195303 (2013) (arXiv 1107.1669)
76. D. Alba, H.W. Crater, L. Lusanna, Relativistic quantum mechanics and relativistic entanglement in the rest-frame instant form of dynamics. J. Math. Phys. **52**, 062301 (2011) (arXiv 0907.1816)
77. D. Alba, L. Lusanna, Quantum mechanics in noninertial frames with a multitemporal quantization scheme: I. Relativistic particles. Int. J. Mod. Phys. **A21**, 2781 (2006) (arXiv hep-th/0502194)

78. D. Alba, Quantum mechanics in noninertial frames with a multitemporal quantization scheme: II. Nonrelativistic particles. Int. J. Mod. Phys. **A21**, 3917 (arXiv hep-th/0504060)
79. L. Lusanna, M. Pauri, Explaining Leibniz equivalence as difference of non-inertial appearances: dis-solution of the Hole argument and physical individuation of point-events. Hist. Phil. Mod. Phys. **37**, 692 (2006) (arXiv gr-qc/0604087)
80. L. Lusanna, M. Pauri, The physical role of gravitational and gauge degrees of freedom in general relativity. I: Dynamical synchronization and generalized inertial effects; II: Dirac versus Bergmann observables and the objectivity of space-time. Gen. Rel. Grav. **38**, 187 and 229 (2006) (arXiv gr-qc/0403081 and 0407007)
81. L. Lusanna, M. Pauri, in *Dynamical Emergence of Instantaneous 3-Spaces in a Class of Models of General Relativity*, ed. by A. van der Merwe. Relativity and the Dimensionality of the World (Springer Series Fundamental Theories of Physics) (to appear) (arXiv gr-qc/0611045)
82. D. Alba, L. Lusanna, The York map as a Shanmugadhasan canonical transformation in tetrad gravity and the role of non-inertial frames in the geometrical view of the gravitational field. Gen. Rel. Grav. **39**, 2149 (2007) (arXiv gr-qc/0604086, v2)
83. J. Isenberg, J.E. Marsden, The York map is a canonical transformation. J. Geom. Phys. **1**, 85 (1984)
84. I. Ciufolini, J.A. Wheeler, *Gravitation and Inertia* (Princeton University Press, Princeton, 1995)
85. J.W. York Jr., *Kinematics and Dynamics of General Relativity, in Sources of Gravitational Radiation*, ed. by L.L. Smarr, Battelle-Seattle Workshop 1978 (Cambridge University Press, Cambridge, 1979)
86. A. Quadir, J.A. Wheeler, *York's Cosmic Time versus Proper Time*, in *From SU(3) to Gravity*, Y.Ne'eman;s festschrift, ed. by E. Gotsma, G. Tauber (Cambridge University Press, Cambridge, 1985)
87. R. Beig, *The Classical Theory of Canonical General Relativity*, in *Canonical Gravity: from Classical to Quantum*, Bad Honnef 1993, ed. by J. Ehlers, H. Friedrich, Lecture note in Physics 434 (Springer, Berlin, 1994)
88. D. Alba, L. Lusanna, The Einstein-Maxwell-Particle system in the York canonical basis of ADM tetrad gravity: I) The equations of motion in arbitrary Schwinger time gauges. Can. J. Phys. **90**, 1017 (2012) (arXiv 0907.4087)
89. D. Alba, L. Lusanna, The Einstein-Maxwell-Particle system in the York canonical basis of ADM tetrad gravity: II) The weak field approximation in the 3-orthogonal gauges and Hamiltonian post-Minkowskian gravity: the N-body problem and gravitational waves with asymptotic background. Can. J. Phys. **90**, 1077 (2012) (arXiv 1003.5143)
90. D. Alba, L. Lusanna, The Einstein-Maxwell-Particle system in the York canonical basis of ADM tetrad gravity: III) The post-Minkowskian N-body problem, its post-Newtonian limit in non-harmonic 3-orthogonal gauges and dark matter as an inertial effect. Can. J. Phys. **90**, 1131 (2012) (arXiv 1009.1794)
91. D. Alba, L. Lusanna, *Dust in the York Canonical Basis of ADM Tetrad Gravity: the Problem of Vorticity* (arXiv 1106.0403)
92. L. Lusanna, D. Nowak-Szczepaniak, The rest-frame instant form of relativistic perfect fluids with equation of state $\rho = \rho(\eta, s)$ and of nondissipative elastic materials. Int. J. Mod. Phys. **A15**, 4943 (2000)
93. D. Alba, L. Lusanna, Generalized Eulerian coordinates for relativistic fluids: hamiltonian rest-frame instant form, relative variables, rotational kinematics. Int. J. Mod. Phys. **A19**, 3025 (2004) (arXiv hep-th/020903)
94. J.D. Brown, Action functionals for relativistic perfect fluids. Class. Quantum Grav. **10**, 1579 (1993)
95. M. Maggiore, *Gravitational Waves* (Oxford University Press, Oxford, 2008)
96. E.E. Flanagan, S.A. Hughes, The basics of gravitational wave theory. New J. Phys. **7**, 204 (2005) (arXiv gr-qc/0501041)
97. E. Harrison, The redshift-distance and velocity-distance laws. Astrophys. J. **403**, 28 (1993)

98. T. Damour, N. Deruelle, General relativistic celestial mechanics of binary systems. I. The post-Newtonian motion. Ann. Inst.H.Poincare' **43**, 107 (1985)
99. T. Damour, N. Deruelle, General relativistic celestial mechanics of binary systems. II. The post-Newtonian timing formula. Ann. Inst.H.Poincare' **44**, 263 (1986)
100. E. Battaner, E. Florido, The rotation curve of spiral galaxies and its cosmological implications. Fund. Cosmic Phys. **21**, 1 (2000) (arXiv astro-ph/0010475)
101. D.G. Banhatti, Disk galaxy rotation curves and dark matter distribution. Curr. Sci. **94**, 986 (2008) (arXiv astro-ph/0703430)
102. W.J.G. de Blok, A. Bosma, High resolution rotation curves of low surface brightness galaxies. Astron. Astrophys. **385**, 816 (2002) (arXiv astro-ph/0201276)
103. M.S. Longair, *Galaxy Formation* (Springer, Berlin, 2008)
104. M. Ross, *Dark Matter: the Evidence from Astronomy, Astrophysics and Cosmology* (arXiv 1001.0316)
105. K. Garret, G. Duda, Dark matter: a primer. Adv. Astron. **2011**, 968283 (2011) (arXiv 1006.2483)
106. P. Schneider, J. Ehlers, E.E. Falco, *Gravitational Lenses* (Springer, Berlin, 1992)
107. M. Bartelmann, P. Schneider, *Weak Gravitational Lensing* (arXiv astro-ph/9912508)
108. B. Guinot, Application of general relativity to metrology. Metrologia **34**, 261 (1997)
109. S.G. Turyshev, V.T. Toth, The pioneer anomaly. Liv. Rev. Rel. **13**, 4 (2010) (arXiv 1001.3686)
110. C.M. Moller, *The Theory of Relativity* (Oxford University Press, Oxford, 1957)
111. C.M. Moller, Sur la dinamique des syste'mes ayant un moment angulaire interne. Ann. Inst.H.Poincare' **11**, 251 (1949)
112. N. Stergioulas, Rotating stars in relativity. Liv. Rev. Rel. **6**, 3 (2003)
113. L. Lusanna, M. Materassi, A canonical decomposition in collective and relative variables of a Klein-Gordon field in the rest-frame Wigner-covariant instant form. Int. J. Mod. Phys. **A15**, 2821 (2000) (arXiv hep-th/9904202)
114. F. Bigazzi, L. Lusanna, Dirac fields on spacelike hypersurfaces, their rest-frame description and Dirac observables. Int. J. Mod. Phys. **A14**, 1877 (1999) (arXiv hep-th/9807054)
115. G. Longhi, L. Lusanna, Bound-state solutions, invariant scalar products and conserved currents for a class of two-body relativistic systems. Phys. Rev. **D34**, 3707 (1986)
116. W.G. Dixon, *Extended Objects in General Relativity: their Description and Motion, in Isolated Gravitating Systems in General Relativity*, ed. by J. Ehlers. Proc. Int. School of Phys. Enrico Fermi LXVII, (North-Holland, Amsterdam, 1979), p. 156
117. W.G. Dixon, Mathisson's new mechanics: its aims and realisation. Acta Physica Polonica B Proc. Suppl. **1**, 27 (2008)
118. W.G. Dixon, Description of extended bodies by multipole moments in special relativity. J. Math. Phys. **8**, 1591 (1967)
119. C. Rovelli, Relational quantum mechanics. Int. J. Theor. Phys. **35**, 1637 (1996) (quant-ph/9609002)
120. C. Rovelli, M. Smerlak, Relational EPR. Found. Phys. **37**, 427 (2007) (quant-ph/0604064)
121. P.M.A. Dirac, Gauge invariant formulation of quantum electrodynamics. Can. J. Phys. **33**, 650 (1955)
122. C. Cohen-Tannoudji, J. Dupont-Roc, G. Grynberg, *Atom-Photon Interactions. Basic Processes and Applications* (Wiley, New York, 1992)
123. C. Cohen-Tannoudji, J. Dupont-Roc, G. Grynberg, *Photons and Atoms. Introduction to Quantum Electrodynamics* (Wiley, New York, 1989)
124. G.C. Hegerfeldt, Remark on causality and particle localization. Phys. Rev. **D10**, 3320 (1974)
125. G.C. Hegerfeldt, Violation of causality in relativistic quantum theory? Phys. Rev. Lett. **54**, 2395 (1985)
126. A. Einstein, B. Podolski, N. Rosen, Can quantum-mechanical description of physical reality be considered complete? Phys. Rev. **47**, 777 (1935)
127. H.R. Brown, *Physical Relativity. Space-Time Structure from a Dynamical Perspective* (Oxford University Press, Oxford, 2005)

128. J. Butterfield, G. Fleming, *Strange Positions, in From Physics to Philosophy*, ed. by J. Butterfield, C. Pagonis, (Cambridge University Press, Cambridge, 1999), pp. 108–165
129. L. Lusanna, Classical observables of gauge theories from the multitemporal approach. Contemp. Math. **132**, 531 (1992)
130. L. Lusanna, *From Relativistic Mechanics towards Green's Functions: Multitemporal Dynamics*. Proc. VII Seminar on Problems of High Energy Physics and Quantum Field Theory, Protvino 1984 (Protvino University Press, Protvino, 1984)
131. C.G. Torre, M. Varadarajan, Functional evolution of free quantum fields. Class. Quantum Grav. **16**, 2651 (1999)
132. C.G. Torre, M. Varadarajan, Quantum fields at any time. Phys. Rev. **D58**, 064007 (1998)
133. L. Cacciapuoti, C. Salomon, *ACES: Mission Concept and Scientific Objective*, 28/03/2007, ESA document, Estec (ACES_Science_v1_printout.doc)
134. L. Blanchet, C. Salomon, P. Teyssandier, P. Wolf, Relativistic theory for time and frequency transfer to order $1/c^3$. Astron. Astrophys. **370**, 320 (2000)
135. L. Lusanna, Dynamical Emergence of 3-Space in General Relativity: Implications for the ACES Mission, in *Proceedings of the 42th Rencontres de Moriond Gravitational Waves and Experimental Gravity*, La Thuile, Italy, 11–18 March 2007
136. J. Stewart, *Advanced General Relativity* (Cambridge University Press, Cambridge, 1993)
137. R.M. Wald, *General Relativity* (Chicago University Press, Chicago, 1984)
138. J.M.M. Senovilla, *A Singularity Theorem based on Spatial Averages* (arXiv gr-qc/0610127)
139. J.M.M. Senovilla, singularity theorems, their consequences. Gen. Rel. Grav. **30**, 701 (1998)
140. H. van Elst, C. Uggla, General relativistic $1+3$ orthonormal frame approach. Class. Quantum Grav. **14**, 2673 (1997)
141. H. Stephani, *General Relativity*, 2nd edn. (Cambridge University Press, Cambridge, 1996)
142. R. Arnowitt, S. Deser, C.W. Misner, Wave zone in general relativity. Phys. Rev. **121**, 1556 (1961)
143. K. Thorne, in *Gravitational Radiation*, ed. by S. Hawking, W. Israel. Three Hundred Years of Gravitation (Cambridge University Press, Cambridge, 1987), pp. 330–458
144. K. Thorne, in *The Theory of Gravitational Radiation: An Introductory Review*, ed. by N. Deruelle, T. Piran. Gravitational Radiation (North-Holland, Amsterdam, 1983), p. 1
145. K. Thorne, Multipole expansions of gravitational radiation. Rev. Mod. Phys. *52*, 299 (1980)
146. L. Blanchet, Gravitational radiation from post-Newtonian sources and inspiralling compact binaries. Living Rev. Rel. **9**, 4 (2006)
147. L. Blanchet, *Post-Newtonian Theory and the Two-Body Problem* (arXiv 0907.3596)
148. T. Damour, Gravitational Radiation and the Motion of Compact Bodies, in *Gravitational Radiation*, ed. by N. Deruelle, T. Piran (North-Holland, Amsterdam, 1983), pp. 59–144
149. T. Damour, The Problem of Motion in Newtonian and Einsteinian Gravity, in *Three Hundred Years of Gravitation*, ed. by S. Hawking, W. Israel (Cambridge University Press, Cambridge, 1987), pp. 128–198
150. M.E. Pati, C.M. Will, Post-Newtonian gravitational radiation and equations of motion via direct integration of the relaxed Einstein equations: foundations. Phys. Rev. **D62**, 124015 (2002)
151. M.E. Pati, C.M. Will, II. Two-body equations of motion to second post-Newtonian order and radiation reaction to 3.5 post-Newtonian order. Phys. Rev. **D65**, 104008 (2002)
152. C. Will, III. Radiaction reaction for binary systems with spinning bodies. Phys. Rev. **D71**, 084027 (2005)
153. H. Wang, C. Will IV, Radiation reaction for binary systems with spin-spin coupling. Phys. Rev. **D75**, 064017 (2007)
154. T. Mitchell, C. Will, V. Evidence for the strong equivalence principle in second post-Newtonian order. Phys. Rev. **D75**, 124025 (2007)
155. G. Schaefer, The gravitational quadrupole radiation-reaction force and the canonical formalism of ADM. Ann. Phys. (N.Y.) **161**, 81 (1985)
156. G. Schaefer, The ADM Hamiltonian and the postlinear approximation. Gen. Rel. Grav. **18**, 255 (1985)

157. T. Ledvinka, G. Schaefer, J. Bicak, Relativistic closed-form Hamiltonian for many-body gravitating systems in the post-Minkowskian approximation. Phys. Rev. Lett. **100**, 251101 (2008) (arXiv 0807.0214)
158. Y. Mino, M. Sasaki, T. Tanaka, Gravitational radiation reaction to a particle motion. Phys. Rev. **D55**, 3457 (1997) (arXiv gr-qc/9606018)
159. T.C. Quinn, R.M. Wald, Phys. Rev. **D56**, 3381 (1997) (arXiv gr-qc/9610053)
160. S. Detweiler, B.F. Whiting, Self-force via a Green's function decomposition. Phys. Rev. **D67**, 024025 (2003) (arXiv gr-qc/0202086)
161. R.M. Wald, *Introduction to Gravitational Self-Force* (arXiv 0907.0412)
162. S.E. Gralla, R.M. Wald, Derivation of Gravitational Self-Force, in *Mass and its Motion*, proceedings of the 2008 CNRS School in Orléans/France, ed. by L. Blanchet, A. Spallicci, B. Whiting (Springer, Berlin, 2011) (arXiv 0907.0414)
163. A. Pound, *A New Derivation of the Gravitational Self-Force* (2009) (arXiv 0907.5197)
164. B.F. Schutz, *A First Course in General Relativity* (Cambridge University Press, Cambridge, 1985)
165. I. Ciufolini, E.C. Pavlis, A confirmation of the general relativistic prediction of the Lense-Thirring effect. Nature **431**, 958 (2004)
166. C.W.F. Everitt, B.W. Parkinson, Gravity Probe B Science Results-NASA Final Report, (http://einstein.stanford.edu/content/final-report/GPB-Final-NASA-Report-020509-web.pdf)
167. S.M. Kopeikin, E.B. Fomalont, Gravitomagnetism, causality and aberration of gravity in the gravitational light-ray deflection experiments. Gen. Rel. Grav. **39**, 1583 (2007) (arXiv gr-qc/0510077)
168. S.M. Kopeikin, E.B. Fomalont, Aberration and the fundamental speed of gravity in the Jovian deflection experiment. Found. Phys. **36**, 1244 (2006) (arXiv astro-ph/0311063)
169. S.M. Kopeikin, V.V. Makarov, Gravitational bending of light by planetary multipoles and its measurement with microarcsecond astronomical interferometers. Phys. Rev. **D75**, 062002 (2007) (arXiv astro-ph/0611358)
170. S. Carlip, Model-dependence of Shapiro time delay and the "speed of gravity/speed of light" controversy. Class. Quantum Grav. **21**, 3803 (2004) (arXiv gr-qc/0403060)
171. C. Will, Propagation speed of gravity and the relativistic time delay. Astrophys. J. **590**, 683 (2003) (arXiv astro-ph/0301145)
172. L. Lindegren, D. Dravins, The fundamental definition of 'radial velocity'. Astron. Astrophys. **401**, 1185 (2003) (arXiv astro-ph/0302522)
173. P. Teyssandier, C. Le Poncin-Lafitte, B. Linet, A universal tool for determining the time delay and the frequency shift of light: Synge's world function. Astrophys. Space Sci. Libr. **349**, 153 (2007) (arXiv 0711.0034)
174. P. Teyssandier, C. Le Poncin-Lafitte, General Post-Minkowskian expansion of time transfer functions. Class. Quantum Grav. **25**, 145020 (2008) (arXiv 0803.0277)
175. M.H. Bruegmann, Light deflection in the postlinear gravitational filed of bounded pointlike masses. Phys. Rev. **D72**, 024012 (2005) (arXiv gr-qc/0501095)
176. M. Favata, *Post-Newtonian Corrections to the Gravitational-Wave Memory for Quasi-Circular, Inspiralling Compact Binaries* (2009) (arXiv 0812.0069)
177. J.L. Jaramillo, J.A.V. Kroon, E. Gourgoulhon, From geometry to numerics: interdisciplinary aspects in mathematical and numerical relativity. Class. Quantum Grav. **25**, 093001 (2008)
178. I. Hinder, *The Current Status of Binary Black Hole Simulations in Numerical Relativity* (2010) (arXiv 1001.5161)
179. L. Infeld, J. Plebanski, *Motion and Relativity* (Pergamon, Oxford, 1960)
180. J.L. Synge, *Relativity: The General Theory* (North Holland, Amsterdam, 1964)
181. T. Damour, A. Nagar, *The Effective One Body Description of the Two-Body Problem* (arXiv 0906.1769)
182. T. Damour, *Introductory Lectures on the Effective One Body Formalism* (arXiv 0802.4047)
183. E. Poisson, The motion of point particles in curved space-time. Living Rev. Rel. **7**, 6(2004) (arXiv gr-qc/0306052)

184. E. Poisson, Constructing the Self-Force, in *Mass and its Motion*, Proceedings of the 2008 CNRS School in Orléans/France, ed. by L. Blanchet, A. Spallicci, B. Whiting (Springer, Berlin, 2011)
185. R. Geroch, J. Traschen, Strings and other distributional sources in general relativity. Phys. Rev. **D36**, 1017 (1987)
186. R. Steinbauer, J.A. Vickers, The use of generalized functions and distributions in general relativity. Class. Quantum Grav. **23**, R91 (2006) (arXiv gr-qc/0603078)
187. J. Ehlers, E. Rundolph, Dynamics of extended bodies in general relativity: center-of-mass description and quasi-rigidity. Gen. Rel. Grav. **8**, 197 (1977)
188. W. Beiglboeck, The center of mass in Einstein's theory of gravitation. Commun. Math. Phys. **5**, 106 (1967)
189. R. Schattner, The center of mass in general relativity. Gen. Rel. Grav. **10**, 377 (1978)
190. R. Schattner, The uniqueness of the center of mass in general relativity. Gen. Rel. Grav. **10**, 395 (1979)
191. J. Ehlers, R. Geroch, Equation of motion of small bodies in relativity. Ann. Phys. **309**, 232 (2004)
192. S. Kopeikin, I. Vlasov, Parametrized post-Newtonian theory of reference frames, multipolar expansions and equations of motion in the N-body problem. Phys. Rep. **400**, 209 (2004) (arXiv gr-qc/0403068)
193. J. Steinhoff, D. Puetzfeld, *Multipolar Equations of Motion for Extended Test Bodies in General Relativity* (arXiv 0909.3756)
194. A. Barducci, R. Casalbuoni, L. Lusanna, Energy-momentum tensor of extended relativistic systems. Nuovo Cim. **54A**, 340 (1979)
195. G. Schaefer, Post-Newtonian Methods: Analytic Results on the Binary Problem, in *Mass and Motion in General Relativity*, Proceedings of the 2008 CNRS School in Orleans/France, ed. by L. Blanchet, A. Spallicci, B. Whiting (Springer, Berlin, 2011) (arXiv 0910.2857)
196. G. Schaefer, The gravitational quadrupole radiation-reaction force and the canonical formalism of ADM. Ann. Phys. (N.Y.) **161**, 81 (1985)
197. G. Schaefer, The ADM Hamiltonian and the postlinear approximation. Gen. Rel. Grav. **18**, 255 (1985)
198. C.M. Will, *On the Unreasonable Effectiveness of the Post-Newtonian Approximation in Gravitational Physics* (arXiv 1102.5192)
199. C. Moni Bidin, G. Carraro, G.A. Me'ndez, W.F. van Altena, *No Evidence for a Dark Matter Disk within 4 kpc from the Galactic Plane* (arXiv 1011.1289)
200. M. Milgrom, *New Physics at Low Accelerations (MOND): An Alternative to Dark Matter* (arXiv 0912.2678)
201. S. Capozziello, V.F. Cardone, A. Troisi, Low surface brightness galaxies rotation curves in the low energy limit of R^n gravity : no need for dark matter? Mon. Not. R. Astron. Soc. **375**, 1423 (2007) (arXiv astro-ph/0603522)
202. J.L. Feng, *Dark Matter Candidates from Particle Physics and Methods of Detection* (arXiv 1003.0904)
203. F.I. Cooperstock, S. Tieu, General relativistic velocity: the alternative to dark matter. Mod. Phys. Lett. **A23**, 1745 (2008) (arXiv 0712.0019)
204. G.F.R. Ellis, H. van Elst, Cosmological Models, Cargese Lectures 1998, NATO Adv. Stud. Inst. Ser. C. Math. Phys. Sci. **541**, 1 (1999) (arXiv gr-qc/9812046)
205. G.F.R. Ellis, *Inhomogeneity Effects in Cosmology* (arXiv 1103.2335)
206. C. Clarkson, G. Ellis, A. Faltenbacher, R. Maartens, O. Umeh, J.P. Uzan,*(Mis-)Interpreting Supernovae Observations in a Lumpy Universe* (arXiv 1109.2484)
207. T. Buchert, G.F.R. Ellis, The universe seen at different scales. Phys. Lett. **A347**, 38 (2005) (arXiv gr-qc?0506106)
208. R. Maartens, *Is the Universe Homogeneous?* (arXiv 1104.1300)
209. C.G. Tsagas, A. Challinor, R. Maartens, Relativistic cosmology and large-scale structure. Phys. Rep. **465**, 61 (2008) (arXiv 0705.4397)

210. C. Clarkson, R. Maartens, Inhomogeneity and the foundations of concordance cosmology. Class. Quantum Grav. **27**, 124008 (arXiv 1005.2165)
211. R. Durrer, *What do we really Know about Dark Energy?* (arXiv 1103.5331)
212. C. Bonvin, R. Durrer, *What Galaxy Surveys Really Measure* (arXiv 1105.5280)
213. K. Nakamura, Second-order gauge-invariant cosmological perturbation theory: current status. Adv. Astron. **2010**, 576273 (2010) (arXiv 1001.2621)
214. W.B. Bonnor, A.H. Sulaiman, N. Tanimura, Szekeres's space-times have no Killing vectors. Gen. Rel. Grav. **8**, 549 (1977)
215. B.K. Berger, D.M. Eardley, D.W. Olson, Notes on the spacetimes of Szekeres. Phys. Rev. **D16**, 3086 (1977)
216. A. Nwankwo A., M. Ishak, J. Thomson, *Luminosity Distance and Redshift in the Szekeres Inhomogeneous Cosmological Models* (arXiv 1005.2989)
217. J. Pleban'ski, A. Krasin'ski, *An Introduction to General Relativity and Cosmology* (Cambridge University Press, Cambridge, 2006)
218. M.N. Celerier, *Effects of Inhomogeneities on the Expansion of the Universe: a Challenge to Dark Energy?* (arXiv 1203.2814)
219. T. Buchert, Dark energy from structure: a status report. Gen. Rel. Grav. *40*, 467 (2008) (arXiv 0707.2153)
220. A. Wiegand, T. Buchert, Multiscale cosmology and structure-emerging dark energy: a plausibility analysis. Phys. Rev. *D82*, 023523 (2010) (arXiv 1002.3912)
221. J. Larena, Spatially averaged cosmology in an arbitrary coordinate system. Phys. Rev. **D79**, 084006 (2009) (arXiv 0902.3159)
222. R.J. van den Hoogen, *Averaging Spacetime: Where do we go from here?* (arXiv 1003.4020)
223. S. Räsänen, Backreaction: directions of progress. Class. Quantum Grav. *28*, 164008 (2011) (arXiv 1102.0408)
224. T. Buchert, S. Räsänen, Backreaction in late-time cosmology. Ann. Rev. Nucl. Particle Sci. (invited review) (arXiv 1112.5335)
225. D.L. Wiltshire, *What is Dust? Physical Foundations of the Averaging Problem in Cosmology* (arXiv 1106.1693)
226. A. Ashtekar, *New Perspectives in Canonical Gravity* (Bibliopolis, Napoli, 1988)
227. M. Henneaux, J.E. Nelson, C. Schomblond, Derivation of Ashtekar variables from tetrad gravity. Phys. Rev. **D39**, 434 (1989)
228. T. Thiemann, *Modern Canonical Quantum General Relativity* (Cambridge University Press, Cambridge, 2007)
229. P. Dona', S. Speziale, *Introductory Lectures to Loop Quantum Gravity* (arXiv 1007.0402)

Chapter 9
On Symmetries of Extremal Black Holes with One and Two Centers

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Abstract After a brief introduction to the *Attractor Mechanism*, we review the appearance of groups *of type* E_7 as generalized electric-magnetic duality symmetries in locally supersymmetric theories of gravity, with particular emphasis on the symplectic structure of fluxes in the background of extremal black hole solutions, with one or two centers. In the latter case, the role of an "*horizontal*" symmetry $SL_h(2, \mathbb{R})$ is elucidated by presenting a set of two-centered relations governing the structure of two-centered invariant polynomials.

9.1 Introduction

The *Attractor Mechanism* (AM) [1–5] governs the dynamics in the scalar manifold of Maxwell-Einstein (super)gravity theories. It keeps standing as a crucial fascinating key topic. Along the last years, a number of papers have been devoted to the investigation of attractor configurations of extremal black p-branes in diverse space-time dimensions; for some lists of Refs., see e.g. [6–16].

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The AM is related to dynamical systems with fixed points, describing the equilibrium state and the stability features of the system under consideration.[1] When the AM holds, the particular property of the long-range behavior of the dynamical flows in the considered (dissipative) system is the following: in approaching the fixed points, properly named *attractors*, the orbits of the dynamical evolution lose all memory of their initial conditions, but however the overall dynamics remains completely deterministic.

The first example of AM in supersymmetric systems was discovered in the theory of static, spherically symmetric, asymptotically flat extremal dyonic black holes in N = 2 Maxwell-Einstein supergravity in d = 4 and 5 space-time dimensions (see the first two Refs. of [1–5]). In the following, we will briefly present some basic facts about the d = 4 case.

The multiplet content of a completely general N = 2, d = 4 supergravity theory is the following (see e.g. [17], and Refs. therein):

1. the *gravitational* multiplet
$$\left(V_{\mu}^{a}, \psi^{A}, \psi_{A}, A^{0}\right), \tag{9.1.1}$$
described by the *Vielbein* one-form V^a ($a = 0, 1, 2, 3$) (together with the spin-connection one-form ω^{ab}), the $SU(2)$ doublet of gravitino one-forms ψ^A, ψ_A ($A = 1, 2$, with the upper and lower indices respectively denoting right and left chirality, i.e. $\gamma_5\psi_A = -\gamma_5\psi^A$), and the graviphoton one-form A^0;
2. n_V *vector* supermultiplets
$$\left(A^{I}, \lambda^{iA}, \overline{\lambda}_{A}^{\overline{i}}, z^{i}\right), \tag{9.1.2}$$
each containing a gauge boson one-form A^I ($I = 1, \ldots, n_V$), a doublet of gauginos (zero-form spinors) λ^{iA}, $\overline{\lambda}_{A}^{\overline{i}}$, and a complex scalar field (zero-form) z^i ($i = 1, \ldots, n_V$). The scalar fields z^i can be regarded as coordinates on a complex manifold $\mathcal{M}_{n_V}$ ($dim_{\mathbb{C}}\mathcal{M}_{n_V} = n_V$), which is actually a *special Kähler* manifold;
3. n_H *hypermultiplets*
$$\left(\zeta_{\alpha}, \zeta^{\alpha}, q^{u}\right), \tag{9.1.3}$$

[1] We recall that a point x_{fix} where the phase velocity $v\left(x_{fix}\right)$ vanishes is called a *fixed* point, and it gives a representation of the considered dynamical system in its equilibrium state,
$$v\left(x_{fix}\right) = 0.$$
The fixed point is said to be an *attractor* of some motion $x\,(t)$ if
$$lim_{t\to\infty}x(t) = x_{fix}.$$

each formed by a doublet of zero-form spinors, that is the hyperinos $\zeta_\alpha, \zeta^\alpha$ ($\alpha = 1, \ldots, 2n_H$), and four real scalar fields q^u ($u = 1, \ldots, 4n_H$), which can be considered as coordinates of a quaternionic manifold $\mathcal{H}_{n_H}$ ($dim_{\mathbb{H}}\mathcal{H}_{n_H} = n_H$).

At least in absence of gauging and without quantum corrections, the n_H hypermultiplets are spectators in the AM. This can be understood by looking at the transformation properties of the Fermi fields: the hyperinos $\zeta_\alpha, \zeta^\alpha$'s transform independently on the vector fields, whereas the gauginos' supersymmetry transformations depend on the Maxwell vector fields. Consequently, the contribution of the hypermultiplets can be dynamically decoupled from the rest of the physical system; in particular, it is also completely independent from the evolution dynamics of the complex scalars z^i's coming from the vector multiplets (i.e. from the evolution flow in $\mathcal{M}_{n_V}$). Indeed, disregarding for simplicity's sake the fermionic and gauging terms, the supersymmetry transformations of gauginos and hyperinos respectively read (see e.g. [17], and Refs. therein)

$$\delta\lambda^{iA} = i\partial_\mu z^i \gamma^\mu \varepsilon^A + G^{-i}_{\mu\nu}\gamma^{\mu\nu}\varepsilon_B \epsilon^{AB}; \tag{9.1.4}$$

$$\delta\zeta_\alpha = i\mathcal{U}^{B\beta}_u \partial_\mu q^u \gamma^\mu \varepsilon^A \epsilon_{AB}\mathbb{C}_{\alpha\beta}. \tag{9.1.5}$$

(9.1.5) implies that the asymptotical configurations of the quaternionic hypermultiplets' scalars are *unconstrained*, and therefore they can vary continuously in the manifold $\mathcal{H}_{n_H}$ of the related quaternionic non-linear sigma model.

Thus, as far as ungauged theories are concerned, for the treatment of AM one can restrict to consider N = 2, d = 4 Maxwell-Einstein supergravity, in which n_V vector multiplets (9.1.2) are coupled to the gravity multiplet (9.1.1). The relevant dynamical system to be considered is the one related to the radial evolution of the configurations of complex scalar fields of such n_V vector multiplets. When approaching the event horizon of the black hole, the scalars dynamically run into fixed points, taking values which are only function (of the ratios) of the electric and magnetic charges associated to Abelian Maxwell vector potentials under consideration.

The inverse distance to the event horizon is the fundamental evolution parameter in the dynamics towards the fixed points represented by the *attractor* configurations of the scalar fields. Such near-horizon configurations, which "attracts" the dynamical evolutive flows in $\mathcal{M}_{n_V}$, are completely independent on the initial data of such an evolution, i.e. on the spatial asymptotical configurations of the scalars. Consequently, for what concerns the scalar dynamics, the system completely loses memory of its initial data, because the dynamical evolution is "attracted" by some fixed configuration points, purely depending on the electric and magnetic charges.

In the framework of supergravity theories, extremal black holes can be interpreted as BPS (Bogomol'ny-Prasad-Sommerfeld)-saturated [18] interpolating metric singularities in the low-energy effective limit of higher-dimensional superstrings or M-theory [19]. Their asymptotically relevant parameters include the ADM mass [20], the electrical and magnetic charges (defined by integrating the fluxes of related field strengths over the 2-sphere at infinity), and the asymptotical values of the (dynamically relevant set of) scalar fields. The AM implies that the class of black holes under consideration loses all its "scalar hair" within the near-horizon geometry. This means that the extremal black hole solutions, in the near-horizon limit in which they approach the Bertotti-Robinson $AdS_2 \times S^2$ conformally flat metric [21, 22], are characterized only by electric and magnetic charges, but not by the continuously-varying asymptotical values of the scalar fields.

An important progress in the geometric interpretation of the AM was achieved in the last Refs. of [1–5], in which the attractor near-horizon scalar configurations were related to the critical points of a suitably defined black hole effective potential function V_{BH}. In general, V_{BH} is a positive definite function of scalar fields and electric and magnetic charges, and its non-degenerate critical points in $\mathcal{M}_{n_V}$

$$\forall i = 1, \ldots, n_V, \ \frac{\partial V_{BH}}{\partial z^i} = 0 : \ \ V_{BH}|_{\frac{\partial V_{BH}}{\partial z}=0} > 0, \tag{9.1.6}$$

fix the scalar fields to depend only on electric and magnetic fluxes (charges). In the Einstein two-derivative approximation, the (semi)classical Bekenstein-Hawking entropy (S_{BH})—area (A_H) formula [23–27] yields the (purely charge-dependent) black hole entropy S_{BH} to be

$$S_{BH} = \pi \frac{A_H}{4} = \pi \ V_{BH}|_{\frac{\partial V_{BH}}{\partial z}=0} = \pi \sqrt{|\mathcal{I}_4|}, \tag{9.1.7}$$

where $\mathcal{I}_4$ is the unique independent invariant homogeneous polynomial (quartic in charges) in the relevant representation $\mathbf{R}$ of G in which the charges sit (see Eq. (9.1.9) and discussion below). The last step of (9.1.7) does not apply to d $= 4$ supergravity theories with quadratic charge polynomial invariant, namely to the N $= 2$ *minimally coupled* sequence [28] and to the $N = 3$ [29] theory; in these cases, in (9.1.7) $\sqrt{|\mathcal{I}_4|}$ gets replaced by $|\mathcal{I}_2|$.

In presence of $n = n_V + 1$ Abelian vector fields, the charge vector ($\Lambda = 0, 1, \ldots, n_V$)

$$Q \equiv \left(p^\Lambda, q_\Lambda\right) \tag{9.1.8}$$

of magnetic (p^Λ) and electric (q_Λ) fluxes sits in a $2n$-dimensional representation $\mathbf{R}$ of the U-duality[2] group G, defining the Gaillard-Zumino embedding [33] of G itself into $Sp\,(2n, \mathbb{R})$, which is the largest group acting linearly on the fluxes themselves:

$$G \overset{\mathbf{R}}{\subsetneq} Sp\,(2n, \mathbb{R}). \tag{9.1.9}$$

We consider here the (semi-)classical limit of large charges, also indicated by the fact that we consider $Sp\,(2n, \mathbb{R})$, and not $Sp\,(2n, \mathbb{Z})$ (no Dirac-Schwinger-Zwanziger quantization condition is implemented on the fluxes themselves).

[2] Here U-duality is referred to as the "continuous" symmetries of [30, 31]. Their discrete versions are the U-duality non-perturbative string theory symmetries introduced by Hull and Townsend [32].

After [34–36], the **R**-representation space of the U-duality group is known to exhibit a *stratification* into disjoint classes of orbits, which can be defined through invariant sets of constraints on the (lowest order, actually unique) G-invariant $\mathcal{I}$ built out of the symplectic representation **R**; this will be reported in Sect. 9.3 It is here worth remarking the crucial distinction between the "large" orbits and "small" orbits. While the former have $\mathcal{I} \neq 0$ and support an attractor behavior of the scalar flow in the near-horizon geometry of the extremal black hole background [1–5], for the latter the Attractor Mechanism does not hold, they have $\mathcal{I} = 0$ and thus they correspond to solutions with vanishing Bekenstein-Hawking [23–27] entropy (*at least* at the Einsteinian two-derivative level).

9.2 U-Duality and Groups of Type E_7

From the treatment above, the black hole entropy S_{BH} is invariant under the electric-magnetic duality, in which the non-compact U-duality group has a symplectic action both on the charge vector Q (9.1.8) and on the scalar fields (through the definition of a *flat* symplectic bundle [37] over the scalar manifold itself); see e.g. [38] for a review. The latter property makes relevant the mathematical notion of groups of type E_7.

The first axiomatic characterization of groups of type E_7 through a module (irrep.) was given in 1967 by Brown [39]. A group G of type E_7 is a Lie group endowed with a representation **R** such that:

1. **R** is *symplectic*, i.e.:
$$\exists! \mathbb{C}_{[MN]} \equiv \mathbf{1} \in \mathbf{R} \times_a \mathbf{R}; \tag{9.2.1}$$
(the subscripts "s" and "a" stand for symmetric and skew-symmetric throughout) in turn, $\mathbb{C}_{[MN]}$ defines a non-degenerate skew-symmetric bilinear form (*symplectic product*); given two different charge vectors Q_1 and Q_2 in **R**, such a bilinear form is defined as
$$\langle Q_1, Q_2 \rangle \equiv Q_1^M Q_2^N \mathbb{C}_{MN} = - \langle Q_2, Q_1 \rangle ; \tag{9.2.2}$$
2. **R** admits a unique rank-4 completely symmetric primitive G-invariant structure, usually named K-tensor
$$\exists! \mathbb{K}_{(MNPQ)} \equiv \mathbf{1} \in [\mathbf{R} \times \mathbf{R} \times \mathbf{R} \times \mathbf{R}]_s ; \tag{9.2.3}$$
thus, by contracting the K-tensor with the same charge vector Q in **R**, one can construct a rank-4 homogeneous G-invariant polynomial, named $\mathcal{I}_4$:
$$\mathcal{I}_4 (Q) \equiv \frac{1}{2} \mathbb{K}_{MNPQ} Q^M Q^N Q^P Q^Q , \tag{9.2.4}$$

which corresponds to the evaluation of the rank-4 symmetric form $\mathbf{q}$ induced by the K-tensor on four identical modules $\mathbf{R}$:

$$\mathcal{I}_4(Q) = \frac{1}{2}\,\mathbf{q}(Q_1, Q_2, Q_3, Q_4)|_{Q_1=Q_2=Q_3=Q_4\equiv Q}$$
$$\equiv \frac{1}{2}\left[\mathbb{K}_{MNPQ} Q_1^M Q_2^N Q_3^P Q_4^Q\right]_{Q_1=Q_2=Q_3=Q_4\equiv Q}. \quad (9.2.5)$$

A famous example of *quartic* invariant in $G = E_7$ is the *Cartan-Cremmer-Julia* invariant [40], constructed out of the fundamental irrep. $\mathbf{R} = \mathbf{56}$.

3. if a trilinear map T: $\mathbf{R} \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ is defined such that

$$\langle T(Q_1, Q_2, Q_3), Q_4\rangle = \mathbf{q}(Q_1, Q_2, Q_3, Q_4), \quad (9.2.6)$$

then it holds that

$$\langle T(Q_1, Q_1, Q_2), T(Q_2, Q_2, Q_2)\rangle = \langle Q_1, Q_2\rangle\,\mathbf{q}(Q_1, Q_2, Q_2, Q_2). \quad (9.2.7)$$

This last property makes the group of type E_7 amenable to a treatment in terms of (rank-3) Jordan algebras and related Freudenthal triple systems.

Remarkably, groups of type E_7, appearing in $D = 4$ supergravity as U-duality groups, admit a $D = 5$ uplift to groups of type E_6, as well as a $D = 3$ downlift to groups of type E_8; see [41]. It should also be recalled that split form of exceptional Lie groups appear in the exceptional Cremmer-Julia [42, 43] sequence $E_{D(D)}$ of U-duality groups of M-theory compactified on a D-dimensional torus, in $D = 3, 4, 5$.

It is intriguing to notice that the first paper on groups of type E_7 was written about a decade before the discovery of extended (N $= 2$) supergravity [44], in which electromagnetic duality symmetry was observed [45, 46]. The connection of groups of type E_7 to supergravity can be summarized by stating that all $2 \le N \le 8$-extended supergravities in $D = 4$ with symmetric scalar manifolds $\frac{G}{H}$ have G of type E_7 [47, 48], with the exception of N $= 2$ group $G = U(1, n)$ and $N = 3$ group $G = U(3, n)$. These latter in fact have a quadratic invariant Hermitian form $\left(Q_1, \overline{Q}_2\right)$, whose imaginary part is the symplectic (skew-symmetric) product and whose real part is the symmetric quadratic invariant $\mathcal{I}_2(Q)$ defined as follows

$$\mathcal{I}_2(Q) \equiv \left[\mathrm{Re}\left(Q_1, \overline{Q}_2\right)\right]_{Q_1=Q_2}; \quad (9.2.8)$$

$$\left\langle Q_1, \overline{Q}_2\right\rangle = -\mathrm{Im}\left(Q_1, \overline{Q}_2\right). \quad (9.2.9)$$

Thus, the fundamental representations of pseudo-unitary groups $U(p, n)$, which have a Hermitian quadratic invariant form, do not strictly qualify for groups of type E_7.

In theories with groups of type E_7, the Bekenstein-Hawking black hole entropy is given by

$$S = \pi\sqrt{|\mathcal{I}_4(Q)|}, \quad (9.2.10)$$

Table 9.1 $N \geq 3$ supergravity sequence of groups G of the corresponding $\frac{G}{H}$ symmetric spaces, and their symplectic representations $\mathbf{R}$

N	G	R
3	$U(3,\ n)$	$(\mathbf{3}+\mathbf{n})$
4	$SL(2,\mathbb{R}) \times SO(6,\ n)$	$(\mathbf{2},\ \mathbf{6}+\mathbf{n})$
5	$SU(1,\ 5)$	$\mathbf{20}$
6	$SO^*(12)$	$\mathbf{32}$
8	$E_{7(7)}$	$\mathbf{56}$

Table 9.2 N = 2 choices of groups G of the $\frac{G}{H}$ symmetric spaces and their symplectic representations $\mathbf{R}$. The last four lines refer to "magic" N = 2 supergravities

G	R
$U(1,n)$	$(\mathbf{1}+\mathbf{n})$
$SL(2,\ \mathbb{R}) \times SO(2,\ n)$	$(\mathbf{2},\ \mathbf{2}+\mathbf{n})$
$SL(2,\ \mathbb{R})$	$\mathbf{4}$
$Sp(6,\ \mathbb{R})$	$\mathbf{14}'$
$SU(3,\ 3)$	$\mathbf{20}$
$SO^*(12)$	$\mathbf{32}$
$E_{7(-25)}$	$\mathbf{56}$

as it was proved for the case of $G = E_{7(7)}$ (corresponding to $N = 8$ supergravity) in [49]. For N = 2 group $G = U(1,n)$ and $N = 3$ group $G = U(3,n)$ the analogue of (9.2.10) reads

$$S = \pi \left|\mathcal{I}_2(Q)\right| . \tag{9.2.11}$$

For $3 < N \leq 8$ the following groups of type E_7 are relevant: $E_{7(7)}$, $SO^*(12)$, $SU(1,5)$, $SL(2,\mathbb{R}) \times SOs(6,n)$; see Table 9.1. In N = 2 cases of *symmetric* vector multiplets' scalar manifolds, there are 6 groups of type E_7 [50] : $E_{7(-25)}$, $SO^*(12)$, $SU(3,3)$, $Sp(6,\mathbb{R})$, $SL(2,\mathbb{R})$, and $SL(2,\mathbb{R}) \times SO(2,n)$; see Table 9.2. Here n is the integer describing the number of matter (vector) multiplets for $N = 4, 3, 2$.

9.3 Duality Orbits

We here report some results on the stratification of the $\mathbf{R}$ irrep. space of simple groups G E_7. For a recent account, with a detailed list of Refs., see e.g. [51].

In supergravity, this corresponds to U-duality invariant constraints defining the charge orbits of a single-centered extremal black hole, namely of the G-invariant conditions defining the *rank* of the dyonic charge vector Q (9.1.8) in $\mathbf{R}$ as an element of the corresponding Freudenthal triple system (FTS) (see [52, 53], and Refs. therein). The symplectic indices $M = 1, \ldots, \mathbf{f}$ ($\mathbf{f} \equiv \dim_{\mathbb{R}} \mathbf{R}(G)$) are raised and lowered with the symplectic metric $\mathbb{C}_{MN}$ defined by (9.2.1). By recalling the definition (9.2.4) of the unique primitive rank-4 G-invariant polynomial constructed with Q in $\mathbf{R}$, the *rank* of a non-null Q as an element of FTS(G) ranges from four to one, and it is manifestly G-invariantly characterized as follows:

1. *rank*$(Q) = 4$. This corresponds to "large" extremal black holes, with non-vanishing area of the event horizon (exhibiting *Attractor Mechanism* [1–5]):

$$\mathcal{I}_4(Q) < 0, \quad or \quad \mathcal{I}_4(Q) > 0 \tag{9.3.1}$$

2. *rank*$(Q) = 3$. This corresponds to "small" *lightlike* extremal black holes, with vanishing area of the event horizon:

$$\begin{aligned} &\mathcal{I}_4(Q) = 0; \\ &T(Q, Q, Q) \neq 0. \end{aligned} \tag{9.3.2}$$

3. *rank*$(Q) = 2$. This corresponds to "small" *critical* extremal black holes:

$$\begin{aligned} &T(Q, Q, Q) = 0; \\ &3T(Q, Q, P) + \langle Q, P\rangle Q \neq 0. \end{aligned} \tag{9.3.3}$$

4. *rank*$(Q) = 1$. This corresponds to "small" *doubly-critical* extremal BHs [34, 36]:

$$3T(Q, Q, P) + \langle Q, P\rangle Q = 0, \ \forall P \in \mathbf{R}. \tag{9.3.4}$$

Let us consider the doubly-criticality condition (9.3.4) more in detail. *At least* for *simple* groups of type E_7, the following holds:

$$\mathbf{R} \times_s \mathbf{R} = \mathbf{Adj} + \mathbf{S}; \tag{9.3.5}$$

$$\mathbf{R} \times_a \mathbf{R} = \mathbf{1} + \mathbf{A}, \tag{9.3.6}$$

where $\mathbf{S}$ and $\mathbf{A}$ are suitable irreps.. For example, for $G = E_7$ ($\mathbf{R} = \mathbf{56}$, $\mathbf{Adj} = \mathbf{133}$) one gets

$$(\mathbf{56} \times \mathbf{56})_s = \mathbf{133} + \mathbf{1463}; \tag{9.3.7}$$

$$(\mathbf{56} \times \mathbf{56})_a = \mathbf{1} + \mathbf{1539}. \tag{9.3.8}$$

For such groups, one can construct the projection operator on $\mathbf{Adj}(G)$:

$$\mathcal{P}_{AB}{}^{CD} = \mathcal{P}_{(AB)}{}^{(CD)}; \tag{9.3.9}$$

$$\mathcal{P}_{AB}{}^{CD} \frac{\partial^2 \mathcal{I}_4}{\partial Q^C \partial Q^D} = \left.\frac{\partial^2 \mathcal{I}_4}{\partial Q^A \partial Q^B}\right|_{\mathbf{Adj}(G)}; \tag{9.3.10}$$

$$\mathcal{P}_{AB}{}^{CD} \mathcal{P}_{CD}{}^{EF} \frac{\partial^2 \mathcal{I}_4}{\partial Q^E \partial Q^F} = \mathcal{P}_{AB}{}^{EF} \frac{\partial^2 \mathcal{I}_4}{\partial Q^E \partial Q^F}, \tag{9.3.11}$$

where (recall (9.3.5))

$$\frac{\partial^2 \mathcal{I}_4}{\partial Q^A \partial Q^B} = \left.\frac{\partial^2 \mathcal{I}_4}{\partial Q^A \partial Q^B}\right|_{\mathbf{Adj}(G)} + \left.\frac{\partial^2 \mathcal{I}_4}{\partial Q^A \partial Q^B}\right|_{\mathbf{S}(G)}; \tag{9.3.12}$$

$$\left.\frac{\partial^2 \mathcal{I}_4}{\partial Q^A \partial Q^B}\right|_{\mathbf{Adj}(G)} = 2\,(1-\tau)\,(3\mathbb{K}_{ABCD} + \mathbb{C}_{AC}\mathbb{C}_{BD})\,Q^C Q^D; \tag{9.3.13}$$

$$\left.\frac{\partial^2 \mathcal{I}_4}{\partial Q^A \partial Q^B}\right|_{\mathbf{S}(G)} = 2\,[3\tau\mathbb{K}_{ABCD} + (\tau-1)\,\mathbb{C}_{AC}\mathbb{C}_{BD}]\,Q^C Q^D, \tag{9.3.14}$$

where $\tau \equiv 2\mathbf{d}/\,[\mathbf{f}\,(\mathbf{f}+1)]$, $\mathbf{d} \equiv \dim_{\mathbb{R}}(\mathbf{Adj}\,(G))$. The explicit expression of $\mathcal{P}_{AB}^{\ \ CD}$ reads[3] ($\alpha = 1, \ldots, \mathbf{d}$):

$$\mathcal{P}_{AB}^{\ \ CD} = \tau\left(3\mathbb{C}^{CE}\mathbb{C}^{DF}\mathbb{K}_{EFAB} + \delta^C_{(A}\delta^D_{B)}\right) = -t^{\alpha|CD}t_{\alpha|AB}, \tag{9.3.15}$$

where the relation [56] (see also [57])

$$\mathbb{K}_{MNPQ} = -\frac{1}{3\tau}t^{\alpha}_{(MN}t_{\alpha|PQ)} = -\frac{1}{3\tau}\left[t^{\alpha}_{MN}t_{\alpha|PQ} - \tau\mathbb{C}_{M(P}\mathbb{C}_{Q)N}\right], \tag{9.3.16}$$

where

$$t^{\alpha}_{MN} = t^{\alpha}_{(MN)}; \quad t^{\alpha}_{MN}\mathbb{C}^{MN} = 0 \tag{9.3.17}$$

is the symplectic representation of the generators of the Lie algebra $\mathfrak{g}$ of G. Notice that $\tau < 1$ is nothing but the ratio of the dimensions of the adjoint **Adj** and rank-2 symmetric $\mathbf{R} \times_s \mathbf{R}$ (9.3.5) reps. of G, or equivalently the ratio of upper and lower indices of t^{α}_{MN}'s themselves.

9.4 From One to Two Centers

In multi-centered black hole solutions [58–66], a charge vector Q_a can be associated to each center, with the index $a = 1, \ldots, p$, with p denoting the number of centers. This index transforms in the fundamental representation $\mathbf{p}$ of the so-called "*horizontal*" symmetry $SL_h\,(p, \mathbb{R})$ introduced in [67] (see also [68]).

We will here focus on the simplest case $p = 2$, presenting a number of fundamental relations defining the structure of electric-magnetic fluxes of two-centered black hole solutions [69].

From [67, 70], we define the symmetric $\mathbf{I}_{abcd}$ tensor, sitting in the spin $s = 2$ irrep. **5** of $SL_h(2, \mathbb{R})$, as

[3] For related results in terms of a map formulated in the "$4D/5D$ special coordinates" symplectic frame (and thus manifestly covariant under the $d = 5$ U-duality group G_5), see e.g. [54, 55].

$$\mathbf{I}_{abcd} \equiv \frac{1}{2}\mathbb{K}_{MNPQ} Q_a^M Q_b^N Q_c^P Q_d^Q. \tag{9.4.1}$$

Thus, its first derivative reads

$$\widetilde{Q}_{M|abc} \equiv \frac{1}{4}\frac{\partial \mathbf{I}_{abcd}}{\partial Q_d^M} = \frac{1}{2}\mathbb{K}_{MNPQ} Q_a^N Q_b^P Q_c^Q = \widetilde{Q}_{M|(abc)}, \tag{9.4.2}$$

sitting in the spin $s = 3/2$ irrep. **4** of $SL_h(2, \mathbb{R})$ (the horizontal indices $a = 1, 2$ are raised and lowered with ϵ^{ab}, with $\epsilon^{12} \equiv 1$). For clarity's sake, we report the explicit expressions of the various components of $\mathbf{I}_{abcd}$ (9.4.1), as well as their relations with the components of $\widetilde{Q}_{abc}$ (9.4.2) (the subscripts "+2, +1, 0, −1, −2" denote the horizontal helicity of the various components [67, 70]):

$$\mathbf{I}_{+2} \equiv \mathcal{I}_4(Q_1) \equiv \mathbf{I}_{1111} = \left\langle \widetilde{Q}_{111}, Q_1 \right\rangle; \tag{9.4.3}$$

$$\mathbf{I}_{+1} \equiv \mathbf{I}_{1112} = \left\langle \widetilde{Q}_{111}, Q_2 \right\rangle = \left\langle \widetilde{Q}_{112}, Q_1 \right\rangle; \tag{9.4.4}$$

$$\mathbf{I}_{0} \equiv \mathbf{I}_{1122} = \left\langle \widetilde{Q}_{112}, Q_2 \right\rangle = \left\langle \widetilde{Q}_{122}, Q_1 \right\rangle; \tag{9.4.5}$$

$$\mathbf{I}_{-1} \equiv \mathbf{I}_{1222} = \left\langle \widetilde{Q}_{122}, Q_2 \right\rangle = \left\langle \widetilde{Q}_{222}, Q_1 \right\rangle; \tag{9.4.6}$$

$$\mathbf{I}_{-2} \equiv \mathcal{I}_4(Q_2) \equiv \mathbf{I}_{2222} = \left\langle \widetilde{Q}_{222}, Q_2 \right\rangle. \tag{9.4.7}$$

Thus, one can consider the following symplectic product of spin 3/2 horizontal charge tensors:

$$\left\langle \widetilde{Q}_{abc}, \widetilde{Q}_{def} \right\rangle \equiv \widetilde{Q}_{M|abc}\widetilde{Q}_{N|def}\mathbb{C}^{MN}. \tag{9.4.8}$$

A priori, $\left\langle \widetilde{Q}_{abc}, \widetilde{Q}_{def} \right\rangle$ should project onto spin $s = 3, 2, 1, 0$ irreps. of $SL_h(2, \mathbb{R})$ itself; however, due to the complete symmetry of the K-tensor (and to the results of [39, 56]), the projections on spin $s = 3$ and 1 do vanish:

$$s = 3 : \left\langle \widetilde{Q}_{(abc}, \widetilde{Q}_{def)} \right\rangle = 0; \tag{9.4.9}$$

$$s = 2 : \left\langle \widetilde{Q}_{(ab|c}, \widetilde{Q}_{d|ef)} \right\rangle \epsilon^{cd} = \frac{2}{3}\mathcal{W}\mathbf{I}_{abef}; \tag{9.4.10}$$

$$s = 1 : \left\langle \widetilde{Q}_{(a|bc}, \widetilde{Q}_{de|f)} \right\rangle \epsilon^{bd}\epsilon^{ce} = 0; \tag{9.4.11}$$

$$s = 0 : \left\langle \widetilde{Q}_{abc}, \widetilde{Q}_{def} \right\rangle \epsilon^{ad}\epsilon^{be}\epsilon^{cf} = 8\mathbf{I}_6, \tag{9.4.12}$$

where the symplectic product $\mathcal{W}$ and the sextic horizontal polynomial $\mathbf{I}_6$ [70] are respectively defined as (also cfr. (9.2.2))

$$\mathcal{W} \equiv \langle Q_1, Q_2 \rangle = \frac{1}{2}\mathbb{C}_{MN}\epsilon^{ab} Q_a^M Q_b^N; \tag{9.4.13}$$

$$\mathbf{I}_6 \equiv \frac{1}{8}\left\langle \widetilde{Q}_{abc}, \widetilde{Q}_{def} \right\rangle \epsilon^{ad}\epsilon^{be}\epsilon^{cf} = \frac{1}{4}\left\langle \widetilde{Q}_{111}, \widetilde{Q}_{222} \right\rangle + \frac{3}{4}\left\langle \widetilde{Q}_{122}, \widetilde{Q}_{112} \right\rangle. \tag{9.4.14}$$

The complementary relation to (9.4.14), namely $\frac{1}{4}\left\langle \widetilde{Q}_{111}, \widetilde{Q}_{222} \right\rangle - \frac{3}{4}\left\langle \widetilde{Q}_{122}, \widetilde{Q}_{112} \right\rangle$ consistently turns out to be proportional (through $\mathcal{W}$) to the zero helicity component of $\mathbf{I}_{abcd}$; indeed, by setting $(a, b, e, f) = (1, 1, 2, 2)$ in (9.4.10), one obtains:

$$\frac{1}{2}\mathbf{I}_0\mathcal{W} = \frac{1}{4}\left\langle \widetilde{Q}_{111}, \widetilde{Q}_{222} \right\rangle - \frac{3}{4}\left\langle \widetilde{Q}_{122}, \widetilde{Q}_{112} \right\rangle. \tag{9.4.15}$$

We conclude by pointing out some consequences of the *rank* of a charge vector, say Q_1, on the set of two-centered invariant polynomials defined above [69]:

$$rank\,(Q_1) = 3 \Rightarrow \mathbf{I}_{+2} = 0; \tag{9.4.16}$$

$$rank\,(Q_1) = 2 \Rightarrow \widetilde{Q}_{111} = 0 \Rightarrow \begin{cases} \mathbf{I}_{+2} = \mathbf{I}_{+1} = 0; \\ \mathbf{I}_6 = -\frac{1}{2}\mathbf{I}_0\mathcal{W}; \end{cases} \tag{9.4.17}$$

$$rank\,(Q_1) = 1 \Rightarrow \begin{cases} \mathbf{I}_{+2} = \mathbf{I}_{+1} = 0; \\ \mathbf{I}_0 = -\frac{1}{6}\mathcal{W}^2; \\ \mathbf{I}_6 = -\frac{1}{2}\mathbf{I}_0\mathcal{W} = \frac{1}{12}\mathcal{W}^3. \end{cases} \tag{9.4.18}$$

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References

1. S. Ferrara, R. Kallosh, A. Strominger, N=2 extremal black holes. Phys. Rev. **D52**, 5412 (1995). hep-th/9508072
2. A. Strominger, Macroscopic entropy of N=2 extremal black holes. Phys. Lett. **B383**, 39 (1996). hep-th/9602111
3. S. Ferrara, R. Kallosh, Supersymmetry and attractors. Phys. Rev. **D54**, 1514 (1996). hep-th/9602136
4. S. Ferrara, R. Kallosh, Universality of supersymmetric attractors. Phys. Rev. **D54**, 1525 (1996). hep-th/9603090
5. S. Ferrara, G.W. Gibbons, R. Kallosh, Black holes and critical points in moduli space. Nucl. Phys. **B500**, 75 (1997). hep-th/9702103
6. S. Bellucci, S. Ferrara, A. Marrani, Supersymmetric mechanics. Vol. 2: the attractor mechanism and space-time singularities. *Lecture Notes in Physics*, vol. 701 (Springer-Verlag, Heidelberg, 2006)
7. L. Andrianopoli, R. D'Auria, S. Ferrara, U duality and central charges in various dimensions revisited. Int. J. Mod. Phys. **A13**, 431 (1998). hep-th/9612105

8. B. Pioline, Lectures on black holes, topological strings and quantum attractors, Class. Quant. Grav. **23**, S981 (2006). hep-th/0607227
9. L. Andrianopoli, R. D'Auria, S. Ferrara, M. Trigiante, Extremal black holes in supergravity. *Lecture Notes in Physics*, vol. 737, (Springer-Verlag, Heidelberg, 2008), p. 661. hep-th/0611345
10. A. Sen, Black hole entropy function, attractors and precision counting of microstates. Gen. Rel. Grav. **40**, 2249 (2008). arXiv:0708.1270
11. S. Bellucci, S. Ferrara, R. Kallosh, A. Marrani, Extremal black hole and flux vacua attractors. *Lecture Notes in Physics*, vol. 755, (Springer-Verlag, Heidelberg, 2008), p. 115. arXiv:0711.4547
12. S. Ferrara, K. Hayakawa, A. Marrani, Erice Lectures on black holes and attractors. Fortsch. Phys. **56**, 993 (2008). arXiv:0805.2498
13. S. Bellucci, S. Ferrara, M. Günaydin, A. Marrani, SAM lectures on extremal black holes in d=4 extended supergravity. in *The Attractor Mechanism Springer Proceedings in Physics*, vol. 134, 1–30 (2010). arXiv:0905.3739
14. M. Günaydin, Lectures on spectrum generating symmetries and U-duality in supergravity, extremal black holes, quantum attractors and harmonic superspace. in *The Attractor Mechanism Springer Proceedings in Physics*, vol. 134, 31–84 (2010). arXiv:0908.0374
15. G. Dall'Agata, Black holes in supergravity: flow equations and duality. in *Supersymmetric Gravity and Black Holes Springer Proceedings in Physics*, vol. 142, 1–45, (2013). arXiv:1106.2611 [hep-th]
16. A. Ceresole, Extremal black hole flows and duality. Fortsch. Phys. **59**, 545 (2011)
17. L. Andrianopoli, M. Bertolini, A. Ceresole, R. D'Auria, S. Ferrara, P. Frè, T. Magri, N=2 supergravity and N=2 super Yang-Mills theory on general scalar manifolds: symplectic covariance, gaugings and the momentum map. J. Geom. Phys. **23**, 111 (1997). hep-th/9605032
18. G.W. Gibbons, C.M. Hull, A bogomol'ny bound for general relativity and solitons in N=2 supergravity. Phys. Lett. **B109**, 190 (1982)
19. G.W. Gibbons, P.K. Townsend, Vacuum interpolation in supergravity via super-p-branes. Phys. Rev. Lett. **71**, 3754 (1993). hep-th/9307049
20. R. Arnowitt, S. Deser, C.W. Misner, The dynamics of general relativity. in *Gravitation: An Introduction to Current Research*, ed. by L. Witten (Wiley, New York, 1962)
21. B. Bertotti, Uniform electromagnetic field in the theory of general relativity. Phys. Rev. **116**, 1331 (1959)
22. I. Robinson, Bull. Acad. Polon. **7**, 351 (1959)
23. J.D. Bekenstein, Phys. Rev. **D7**, 2333 (1973)
24. S.W. Hawking, Phys. Rev. Lett. **26**, 1344 (1971)
25. C. DeWitt, B.S. DeWitt, *Black Holes* (Les Houches 1972) (Gordon and Breach, New York, 1973)
26. S.W. Hawking, Nature **248**, 30 (1974)
27. S.W. Hawking, Comm. Math. Phys. **43**, 199 (1975)
28. J.F. Luciani, Coupling of O(2) supergravity with several vector multiplets. Nucl. Phys. **B132**, 325 (1978)
29. L. Castellani, A. Ceresole, S. Ferrara, R. D'Auria, P. Fré, E. Maina, The complete N=3 matter coupled supergravity. Nucl. Phys. **B268**, 317 (1986)
30. E. Cremmer, B. Julia, The N=8 supergravity theory. 1. The Lagrangian. Phys. Lett. **B80**, 48 (1978)
31. E. Cremmer, B. Julia, The SO(8) supergravity. Nucl. Phys. **B159**, 141 (1979)
32. C. Hull, P.K. Townsend, Unity of superstring dualities. Nucl. Phys. **B438**, 109 (1995). hep-th/9410167
33. M.K. Gaillard, B. Zumino, Duality rotations for interacting fields. Nucl. Phys. **B193**, 221 (1981)
34. S. Ferrara, M. Günaydin, Orbits of exceptional groups, duality and BPS states in string theory. Int. J. Mod. Phys. **A13**, 2075 (1998). hep-th/9708025
35. R. D'Auria, S. Ferrara, M.A. Lledó, On central charges and Hamiltonians for 0-brane dynamics. Phys. Rev. *D60*, 084007 (1999). hep-th/9903089

36. S. Ferrara, J.M. Maldacena, Branes, central charges and U-duality invariant BPS conditions. Class. Quant. Grav. **15**, 749 (1998). hep-th/9706097
37. A. Strominger, Special geometry. Commun. Math. Phys. **133**, 163 (1990)
38. B. de Wit, Introduction to black hole entropy and supersymmetry. (2005). hep-th/0503211
39. R.B. Brown, Groups of type E_7. J. Reine Angew. Math. **236**, 79 (1969)
40. E. Cartan, *Œuvres complètes* (Editions du Centre National de la Recherche Scientifique, Paris, 1984)
41. P. Truini, Exceptional Lie Algebras, SU(3) and Jordan Pairs. arXiv:1112.1258 [math-ph]
42. B. Julia, Group disintegrations in *Superspace and Supergravity*, eds. by S.W. Hawking, M. Rocek (Cambridge University Press, Cambridge, 1981)
43. E. Cremmer, Supergravities in 5 dimensions. in *Superspace and Supergravity*, eds. by S.W.Hawking, M. Rocek (Cambridge University Press, Cambridge, 1981)
44. S. Ferrara, P. Van Nieuwenhuizen, Phys. Rev. Lett. **17**, 1669 (1976)
45. S. Ferrara, C. Savoy, B. Zumino, Nucl. Phys. **B121**, 393 (1977)
46. E. Cremmer, B. Julia, The SO(8) supergravity. Nucl. Phys. **B159**, 141 (1979)
47. L. Borsten, D. Dahanayake, M.J. Duff, W. Rubens, Black holes admitting a Freudenthal dual. Phys. Rev. **D80**, 026003 (2009). [arXiv:0903.5517 [hep-th]]
48. S. Ferrara, A. Marrani, A. Yeranyan, Freudenthal duality and generalized special geometry. Phys. Lett. **B701**, 640–645 (2011). [arXiv:1102.4857 [hep-th]]
49. R. Kallosh, B. Kol, E(7) symmetric area of the black hole horizon. Phys. Rev. D **53**, 5344 (1996). [arXiv:hep-th/9602014]
50. M. Gunaydin, G. Sierra, P.K. Townsend, Exceptional supergravity theories and the magic square. Phys. Lett. **B133**, 72 (1983)
51. L. Borsten, M.J. Duff, S. Ferrara, A. Marrani, W. Rubens, Small orbits. Phys. Rev. **D85**, 086002 (2012). arXiv:1108.0424 [hep-th] (to appear in Phys. Rev. D.)
52. J.C. Ferrar, Strictly regular elements in Freudenthal triple systems. Trans. Amer. Math. Soc. **174**, 313 (1972)
53. S. Krutelevich, Jordan algebras, exceptional groups, and higher composition laws, J. Algebra **314**, 924 (2007), arXiv:math/0411104
54. O. Shukuzawa, Explicit classification of orbits in Jordan algebra and Freudenthal vector space over the exceptional lie groups. Commun. Algebra **34**1, 197 (2006)
55. K. Yokota, Exceptional lie groups. (2009). arXiv:0902.0431 [math.DG]
56. A. Marrani, E. Orazi, F. Riccioni, Exceptional reductions. J. Phys. **A44**, 155207 (2011). arXiv:1012.5797 [hep-th]
57. A. Galperin, O. Ogievetsky, Harmonic potentials for quaternionic symmetric sigma models. Phys. Lett. **B301**, 67 (1993). hep-th/9210153
58. F. Denef, Supergravity flows and D-brane stability. JHEP **0008**, 050 (2000) hep-th/0005049
59. B. Bates, F. Denef, Exact solutions for supersymmetric stationary black hole, composites. hep-th/0304094
60. D. Gaiotto, W.W. Li, M. Padi, Non-supersymmetric attractor flow in symmetric spaces. JHEP **0712**, 093 (2007). arXiv:0710.1638 [hep-th]
61. K. Goldstein, S. Katmadas, Almost BPS black holes. JHEP **0905** 058 (2009). arXiv:0812.4183 [hep-th]
62. E.G. Gimon, F. Larsen, J. Simon, Constituent model of extremal non-BPS black holes. JHEP **0907**, 052 (2009). arXiv:0903.0719 [hep-th]
63. I. Bena, S. Giusto, C. Ruef, N. P. Warner, Multi-center non-BPS black holes: the solution. JHEP **0911**, 032 (2009). arXiv:0908.2121 [hep-th]
64. G. Bossard, C. Ruef, Interacting non-BPS black holes. Gen. Rel. Grav. **44**, 21 (2012). arXiv:1106.5806 [hep-th]
65. G. Bossard, Octonionic black holes. J. High Energy Phy. **113**, (2012). arXiv:1203.0530 [hep-th]
66. P. Fré, A.S. Sorin, Extremal multicenter black holes: nilpotent orbits and tits satake universality classes. J. High Energy Phys. **3**, (2013). arXiv:1205.1233 [hep-th]
67. S. Ferrara, A. Marrani, E. Orazi, R. Stora, A. Yeranyan, Two-center black holes duality-invariants for stu model and its lower-rank descendants. J. Math. Phys. **52**, 062302 (2011). arXiv:1011.5864 [hep-th]

68. P. Levay, Two-center black holes, qubits and elliptic curves. Phys. Rev. **D84**, 025023 (2011). arXiv:1104.0144 [hep-th]
69. A. Ceresole, S. Ferrara, A. Marrani, A. Yeranyan, Small black hole constituents and horizontal symmetry. JHEP **1106**, 078 (2011). arXiv:1104.4652 [hep-th]
70. L. Andrianopoli, R.D'Auria, S. Ferrara, A. Marrani, M. Trigiante, Two-centered magical charge orbits. JHEP **1104**, 041 (2011). arXiv:1101.3496 [hep-th]